Buoyancy-driven exchange flows in inclined ducts

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Buoyancy-driven exchange flows arise in the natural and built environment wherever bodies of fluids at different densities are connected by a narrow constriction. In this paper we study these flows in the laboratory using the canonical stratified inclined duct experiment, which sustains an exchange flow in an inclined duct of rectangular cross-section over long time periods (Meyer & Linden, *J. Fluid Mech.*, vol. 753, 2014). We study the behaviour of these sustained stratified shear flows by focusing on three dependent variables of particular interest: the qualitative flow regime (laminar, wavy, intermittently turbulent, or fully turbulent), the mass flux (net transport of buoyancy between reservoirs), and the interfacial thickness (thickness of the layer of intermediate density between the two counter-flowing layers). Dimensional analysis reveals five non-dimensional independent input parameters: the duct aspect ratios in the longitudinal direction A and spanwise direction B, the tilt angle θ, the Reynolds number Re (based on the initial buoyancy difference driving the flow), and the Prandtl number Pr (we consider both salt and temperature stratifications). After reviewing the literature and open questions on the scaling of regimes, mass flux, and interfacial thickness with A, B, θ, Re, Pr, we present the first extensive, unified set of experimental data where we varied systematically all five input parameters and measured all three output variables with the same methodology. Our results in the (θ, Re) plane for five sets of (A, B, Pr) reveal a variety of scaling laws, and a non-trivial dependence of all three variables on all five parameters, in addition to a sixth elusive parameter. We further develop three classes of candidate models to explain the observed scaling laws: (i) the recent volume-averaged energetics of Lefauve, Partridge & Linden, *J. Fluid Mech.*, 2019; (ii) two-layer frictional hydraulics; (iii) turbulent mixing models. While these models provide significant qualitative and quantitative descriptions of the experimental results, they also highlight the need for further progress on shear-driven turbulent flows and their interfacial waves, layering, intermittency, and mixing properties.

Key words:

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1. Introduction

Buoyancy-driven exchange flows naturally arise where relatively large bodies of fluid have different densities on either side of a relatively narrow constriction. In a gravitational field, this difference in buoyancy, usually in the horizontal direction, results in a horizontal hydrostatic pressure gradient along the constriction, of opposite sign above and below a 'neutral level', a height at which the pressures on either side of the constriction are equal. This pressure gradient drives a counter-flow through the constriction, in which fluid from the negatively-buoyant reservoir flows below the neutral level towards the positively-buoyant reservoir, and conversely, with equal magnitude. Such buoyancy-driven exchange

ows result in little to no net volume transport, but crucially, in a net buoyancy transport between the reservoirs which tends to homogenise buoyancy di erences in the system (i.e. towards equilibrium). In addition, irreversible mixing often occurs across the interface between the two counter- owing layers of uid, creating an intermediate layer of partially mixed uid, and partially reducing the buoyancy transport. The net transport and mixing of the active scalar eld (e.g. heat, salt, or other solutes) and of other potential passive scalar elds having di erent concentrations in either reservoirs (e.g. pollutants or nutrients) have a wide range of consequences of interest. For this reason, the study of buoyancy-driven exchange ows has a rich history. (The primary role of buoyancy being implicit throughout the paper, we will simply refer to these ows as `exchange ows'.)

Aristotle o ered the rst recorded explanation of the movement of salty water within the Mediterranean Sea (Deacon 1971, pp. 8-9). Ever since, exchange ows through the straits of Gibraltar and the Bosphorus have driven much speculation and research, due to their crucial roles in the water and salt balances of the Mediterranean Sea, countering its evaporation by net volume transport and allowing its very existence (as rst demonstrated experimentally by Marsigli in the 1680s (Deacon 1971, Chap. 7)). More recently, it has been recognised that nutrient transport from the Atlantic partially supported primary production in Mediterranean ecosystems (Estrada 1996). The quanti cation, modelling, and discussion of the past and current impact of exchange ows in straits, estuaries, or between lakes continues to generate a vast literature.

Exchange ows of gases also have a great variety of perhaps even more tangible and ancient applications to society in the `natural ventilation' of buildings (Linden 1999). It would be surprising indeed if some ice-age prehistoric *Homo Sapiens* did not ponder the in ow of cold outside air and the out ow of heat or re combustion products when choosing a cave suitable for living. More recently, engineering problems of air ow through open doorways or ventilation ducts, or the escape of gases from ruptured industrial pipes, have stimulated further research.

More fundamentally, exchange ows are stably-strati ed shear ows, a canonical class of ows widely used in the mathematical study of strati ed turbulence, dating back at least to (Reynolds 1883, § 12) and Taylor (1931). Multi-layered strati ed shear ows have complex hydrodynamic stability and turbulent mixing properties (Caul eld 1994; Peltier & Caul eld 2003). The straightforward and steady forcing of exchange ows make them ideal laboratory strati ed shear ows because of the ability to sustain, over long time periods, high levels of turbulent intensity and mixing representative of large-scale natural ows.

The aim of this paper is to carry out a thorough review and exploratory study of buoyancy-driven exchange ows in inclined ducts. To do this, we will focus on the behaviour of three key variables:

(i) the qualitative ow regime (e.g. laminar, wavy, intermittently or fully turbulent);

(ii) the mean buoyancy transport;

(iii) the mean thickness of any potential interfacial mixing layer.

The above three variables are particularly relevant in applications to predict exchange rates of active or passive scalars (e.g. salt, heat, pollutants, nutrients) between two di erent uid bodies (e.g. rooms in a building, seas or lakes on either sides of a strait).

However, our primary motivation in this paper is to contribute to a larger research e ort into the fundamental properties of turbulence in sustained strati ed shear ows of geophysical relevance. The above three variables have thus been chosen for their particular ability to be readily captured by simple laboratory techniques while encapsulating several key ow features that are currently the subject of active research, such as:

Figure 1: (a) The Strati ed Inclined Duct (SID), in which an exchange ow takes place through a rectangular duct connecting two reservoirs at densities $_0$ = 2 and inclined at an angle from the horizontal. (b) Notation (in dimensional units). The x and z axes are respectively aligned with the horizontal and vertical of the duct (hence z makes an angle with gravity, here > 0). The duct has dimensions L W H. The streamwise velocity u has typical peak-to-peak magnitude U . The density strati cation has magnitude , with an interfacial layer of typical thickness .

interfacial `Holmboe' waves (Salehipour et al. 2016; Lefauve et al. 2018); spatio-temporal turbulent intermittency (de Bruyn Kops 2015; Portwood et al. 2016; Taylor et al. 2016); and layering and mixing (Salehipour & Peltier 2015; Zhou et al. 2017; Lucas et al. 2017; Salehipour et al. 2018).

To achieve this aim, the remainder of the paper is organised as follows. In § 2 we introduce a canonical experiment ideally suited to study the rich dynamics of exchange ows, and analyse the a priori importance of its non-dimensional input parameters. In § 3 we review the current state of knowledge on the behaviour of our three key variables in order to motivate our study. In § 4 we present our experimental results and scaling laws. In § 5 we explain some of these results with a variety of models, and we summarise and conclude in § 6.

2. The experiment

2.1. Setup and notation

The strati ed inclined duct experiment (hereafter abbreviated `SID') is sketched in gure 1(a). This conceptually simple experiment consists of two reservoirs initially lled with aqueous solutions of di erent densities $_0$ = 2, connected by a long rectangular duct that can be tilted at an angle from the horizontal. At the start of the experiment, the duct is opened, initiating a brief transient gravity current. Shortly after, at t = 0, an exchange ow starts and is sustained through the duct for long periods of time, until the accumulation of uid of a di erent density from the other reservoir reaches the ends of the duct and the experiment is stopped at t = T (typically after several minutes and many duct transit times). This exchange ow has at least four qualitatively di erent ow regimes, based on the experimental input parameters, as we discuss later in the paper.

Our notation is shown in gure 1(b) and largely follows that of Lefauve et al. (2018, 2019). The duct has length L, height H, and width W. The streamwise x axis is aligned along the duct and the spanwise y axis is aligned across the duct, making the z axis tilted

at an angle from the vertical (resulting in a non-zero streamwise projection of gravity $g\sin$). The angle is de ned to be positive when the bottom end of the duct sits in the reservoir of lower density, as shown here. The velocity vector eld is $\mathbf{u}(x;y;z;t) = (u;v;w)$ along $x;y;z$, and the density eld is (x;y;z;t). All spatial coordinates are centred in the middle of the duct, such that $(x;y;z;t)$ 2 [L=2;L=2] [W=2;W=2] [H=2;H=2] [0;T].

Next, we de ne two integral scalar quantities of particular interest in exchange ows:

(i) Q the volume ux as the volumetric ow rate averaged over the duration of an experiment

$$Q \quad \langle j\ uji_{x;y;z;t} \ ; \tag{2.1}$$

where $\langle juji_{x;y;z;t} \quad 1=(LWHT) \int_0^T \int_{H=2}^{H=2} \int_{W=2}^{W=2} \int_{L=2}^{L=2} juj\,dx\,dy\,dz\,dt$. The volume ux $Q > 0$ measures the magnitude of the exchange ow between the two reservoirs. It is di erent from the net (or `barotropic') volume ux $\langle ui_{x;y;z;t} \quad 0$, since, to a good approximation, the volume of uid in each reservoirs is conserved during an experiment (assuming the levels of the free surface in each reservoir are carefully set before the start of the experiment).

(ii) Q_m the mass ux as the net ow rate of mass averaged of the duration of an experiment

$$Q_m \quad \frac{2}{\ }\langle(\quad {}_0)ui_{x;y;z;t} \ ; \tag{2.2}$$

which is equivalent to a buoyancy ux up to a multiplicative constant g. By de nition $0 < Q_m \ 6\ Q$. The rst inequality holds since, in our notation, negatively-buoyant uid (${}_0 <$ 6 ${}_0 + \ = 2$) ows on average to the right ($u > 0$) and conversely. The second inequality would be an equality in the absence of molecular di usion inside the duct (i.e. if all uid moving right had density ${}_0 + \ = 2$ and conversely). In any real ow, laminar (and potentially turbulent) di usion at the interface are responsible for an interfacial layer of intermediate density $j \quad {}_0j < \ = \ 2$ of nite thickness > 0 (gure 1(b)).

2.2. Non-dimensionalisation

A total of seven parameters are believed to play important roles in the SID: four geometrical parameters: L, H, W, , and three dynamical parameters: the reduced gravity $g^0 \quad g = {}_0$ (under the Boussinesq approximation $0 < \ = {}_0 \quad 1$), the kinematic viscosity of water ($= 1{:}05 \quad 10^{\ 6}\ m^2\ s^{\ 1}$) and the molecular di usivity of the stratifying agent (active scalar) . In this paper, we will primarily consider salt strati cation (${}_S = 1{:}50 \quad 10^{\ 9}\ m^2\ s^{\ 1}$), but will also discuss temperature strati cation (${}_T = 1{:}50 \quad 10^{\ 7}\ m^2\ s^{\ 1}$). From these seven parameters having two dimensions (of length and time), we construct ve independent non-dimensional parameters below.

The rst three non-dimensional parameters are geometrical: , and the aspect ratios of the duct in the longitudinal and spanwise direction, respectively:

$$A \quad \frac{L}{H} \quad \text{and} \quad B \quad \frac{W}{H}; \tag{2.3}$$

We choose to non-dimensionalise lengths by the length scale H=2, de ning the non-dimensional position vector as $\tilde{x} \quad x=(H=2)$ such that $(\tilde{x};\tilde{y};\tilde{z})$ 2 [A;A] [B;B] [1;1]. As an exception, we choose to non-dimensionalise the typical thickness of the interfacial density layer by H, for consistency with other de nitions in the literature on mixing in exchange ows: $\tilde{\ } \quad =H$, such that $\tilde{\ }$2 [0;1].

The last two non-dimensional parameters are dynamical. We de ne an `input' Reynolds number based on the velocity scale $\sqrt{g'H}$ and length scale $H/2$:

$$Re \quad \frac{\sqrt{g'H}H}{2} = \frac{\sqrt{gH^3}}{2} \sqrt{\frac{\quad}{\quad_0}}: \tag{2.4}$$

Consequently, we non-dimensionalise the velocity vector as $\tilde{u} = u/\sqrt{g'H}$, and time by the advective time unit $\tilde{t} \quad 2\sqrt{g'/H}t$ (hereafter abbreviated ATU). We de ne our last parameter, the Prandtl number (or Schmidt number), as the ratio of the momentum to active scalar di usivity:

$$Pr \quad \frac{}{}: \tag{2.5}$$

where takes the value $_S$ or $_T$ depending on the type of strati cation (salt or temperature, giving respectively $Pr = 700$ and $Pr = 7$). Finally, we de ne the non-dimensional Boussinesq density eld as $\tilde{} \quad (\quad _0)/(\quad = 2)$, such that $\tilde{} \quad 2 [\quad 1; 1]$.

We now reformulate the aim of this paper (introduced in § 1) more speci cally as: exploring the behaviour of ow regimes, mass ux $\tilde{Q}_m$, and interfacial layer thickness $\tilde{}$ in the ve-dimensional space of non-dimensional input parameters $(A; B; ; Re; Pr)$.

In the next section we address the dimensional scaling of the velocity in the experiment. By discussing the a priori in uence of the input parameters identi ed above on the velocity scale in this problem, we will provide a basis for subsequent scaling arguments in the paper.

2.3. Scaling of the velocity

Having constructed our Reynolds number (2.4) using the velocity scale $\sqrt{g'H}$, here we show that it is the relevant velocity scale to use in such exchange ows. As sketched in gure 1(b), we de ne the typical peak-to-peak velocity as U. This velocity scale is not set by the experimenter as an input parameter, rather it is chosen by the ow as an output parameter. From dimensional analysis, we write

$$\frac{U}{2} = \sqrt{g'H} f_U (A; B; ; Re; Pr): \tag{2.6}$$

In order to show that our Reynolds number (2.4) and our non-dimensionalisation the velocity by $\sqrt{g'H}$ are relevant (and such that $\tilde{u} \quad 2 [\quad 1; 1]$), we will show below that we indeed expect $U = 2\sqrt{g'H}$ and $f_U (A; B; ; Re; Pr) \quad 1$. Although some aspects of this discussion can be found in Lefauvre et al. (2018, 2019), the importance of this dimensional analysis for this paper justi es the more detailed discussion that we o er below.

The velocity scale U in quasi-steady state results from a dynamical balance in the steady, horizontal momentum equation under the Boussinesq approximation (in dimensional units)

$$\underbrace{u \quad r \quad u}_{\text{inertial (I)}} = \underbrace{(1/_0)@p}_{\text{hydrostatic (H)}} + \underbrace{g \sin (\quad _0)/_0}_{\text{gravitational (G)}} + \underbrace{\quad r^2 u}_{\text{viscous (V)}}; \tag{2.7}$$

In addition to the standard inertial (I) and viscous (V) terms, this equation highlights the two distinct `forcing' mechanisms in SID ows:

(H) a hydrostatic longitudinal pressure gradient, the minimal ingredient for exchange ow, resulting from each end of the duct sitting in reservoirs at di erent densities. This hydrostatic pressure in the duct increases linearly with depth $@p =$

$g\cos\ = (4L)z$, driving a ow in opposite directions on either side of the neutral level $z = 0$: $(1=\ _0)@p = g^0\cos\ =(4L)z$;

(G) a gravitational body force reinforcing the ow by the the acceleration of the positively-buoyant layer upward (to the left in gure 1) and of the negatively-buoyant layer downward (to the right) when the tilt angle is positive $g\sin\ > 0$ (the focus of this paper), and conversely when the tilt angle is negative.

Rewriting (2.7) in non-dimensional form and ignoring multiplicative constants, we obtain

$$\underbrace{(\ U\ )^2 u\ r\sim u}_{I} \quad \underbrace{(g^0H\cos\ )z}_{H} + \underbrace{(\ g^0L\sin\ )\sim}_{G} + \underbrace{(\ \ U`\ \ ^2L)r\sim^2 u}_{V}; \tag{2.8}$$

where $`$ is the smallest length scale of density gradients $`(=$ in laminar ows, and $`$ in turbulent ows).

To simplify this complex `four-way' balance, it is instructive to consider the four possible `two-way' dominant balances to deduce four possible scalings for U (ignoring constants and assuming $\cos\ \ 1$ since the focus of this paper is on small angles).

(IH) The inertial-hydrostatic balance. First, we can neglect the gravitational (G) term with respect to the hydrostatic (H) term if $g^0H\cos\ \ g^0L\sin\ $, i.e. when the tilt angle of the duct is much smaller than its `geometrical' angle:

$$0 < \tag{2.9}$$

where we de ne the geometrical angle as

$$\tan\ ^1(A\ \ ^1) \tag{2.10}$$

Second, we can neglect the viscous (V) term if $g^0H\ \ \ U`\ \ ^2L$, i.e. if the Reynolds number is larger than $\mathrm{Re}\ \ HL=`^2$. This corresponds to

$$\mathrm{Re}\ \ A \tag{2.11}$$

in laminar ow (ignoring the case $B\ \ 1$ for simplicity), and to a larger lower bound in turbulent ows. Under these conditions, balancing I and H gives the scaling $U\ \ \sqrt{g^0H}$, i.e. $f_U\ \ 1$, which corresponds to our choice in § 2.1.

(IG) The inertial-gravitational balance. Using analogous arguments, if $\ \ \ $ and $\mathrm{Re}\ \ HL=`^2$, we expect the scaling $U\ \ \sqrt{g^0L\sin\ }$, i.e. $f_U\ (A;\ \ ) \ \ \sqrt{A\sin\ }\ \ 1$.

(HV) The hydrostatic-viscous balance. If $\ \ \ $ and $\mathrm{Re}\ \ A$, we expect $f_U\ (A;B;\mathrm{Re})\ \ A\ ^1\mathrm{Re}\ \ 1$ (some dependence on B being unavoidable in such a viscous ow).

(GV) The gravitational-viscous balance. If $\ \ \ $ and $\mathrm{Re}\ \ A$, we expect $f_U\ (B;\ ;\mathrm{Re})\ \ \sin\ \mathrm{Re}\ \ A$.

Figure 2 summarises the above analysis and the following conclusions.

(i) The parameters A, $\ $ and Re play particularly important roles in SID ows, since the variation of $\ $ and Re above or below thresholds set by A can alter the scaling of U (i.e. f_U). The parameter B appears less important in this respect (except in narrow ducts where $B\ \ 1$ and the Re threshold becomes $AB\ \ ^2$).

(ii) At low tilt angles $0 < \ \ $, f_U increases from $\ \ 1$ when $\mathrm{Re}\ \ A$ to $\ \ 1$ when $\mathrm{Re}\ \ A$. At high enough Re, f_U likely retains a dependence on $A;B;\mathrm{Re}$ due to turbulence (the constant `IH' scaling being a singular limit for $\mathrm{Re}\ !\ 1$).

(iii) At high tilt angles $\ \ \ $ and Reynolds number $\mathrm{Re}\ \ A$, f_U should increase well

Figure 2: Summary of the scaling analysis of U based on the four two-way dominant balances of the streamwise momentum equation (2.8). In each corner of the $(\,;\,Re)$ plane, the IH, IG, HV, and GV balances predict the scaling of f_U $U/(2\sqrt{g'H})$ on either extreme side of $=$ $\tan^{-1}(A^{-1})$ and $Re = A$. The region of practical interest studied in this paper is shown in blue. Although no a priori `two-way' balance allows us to determine accurately the scaling of $f_U(A; B; \,; Re; Pr)$ in this region, hydraulic control requires that f_U 1, as in the IH scaling (see text).

above 1, and likely retains a dependence on $A; B; \,; Re$ (the `IG' scaling being a singular limit for $Re \to 1$).

(iv) The blue rectangle in gure 2 represents the region of interest in most exchange ows of practical interest and in this paper. In this region, three or four physical mechanisms must be considered simultaneously (IHV, IGHV or IGV). Since few ows ever satisfy or , we consider that in general $f_U = f_U(A; B; \,; Re; Pr)$ (the Pr dependence re ects the fact that the active scalar can no longer be neglected at high Re due to its e ect on turbulence and mixing). The existence and value of the upper edge of this region, i.e. the Re value at which viscous and di usive e ects are negligible (the `practical $Re = 1$ limit') are a priori unknown.

Although the above `two-way' balances do not allow us to con dently guess the scaling of f_U in the blue region, theoretical arguments and empirical evidence on hydraulic control support $f_U(A; B; \,; Re; Pr)$ 1 for IHV, IGHV and IGV ows.

Hydraulic control of two-layer exchange ows dates back to Stommel & Farmer (1953); Wood (1968, 1970) and was formalised mathematically by Armi (1986); Lawrence (1990); Dalziel (1991). In steady, inviscid, irrotational, hydrostatic (i.e. `IH') exchange ows, the `composite Froude number' G is unity, which using our notation and assuming streamwise invariance of the ow ($\partial/\partial x = 0$), reads:

$$G^2 = 4\,\frac{\langle u^2\rangle_{x;y;z;t}}{g'H} = 1 \quad \Rightarrow \quad \langle u\rangle_{x;y;z;t} = \bar{Q} = \frac{1}{2}. \tag{2.12}$$

Such exchange ows are called *maximal*: the phase speed of long interfacial gravity waves $\sqrt{g'H}$ `controls' the ow at sharp changes in geometry (on either ends of the duct), and sets the maximal non-dimensional volume ux to $\bar{Q} = 1/2$.

In `plug-like' hydraulic ows ($Re \to 1$), the velocity in each layer $U/2$ is equal to its layer-average Q, giving an upper bound $f_U = \bar{Q} = 1/2$. By contrast, in real-life

nite- Re ows, the peak U= 2 is larger than the average Q (typically by a factor 2), such that the upper bound is f U 2Q 1. This upper bound remains approximately valid throughout the blue region of gure 2. We thus answer the question motivating this section: our choice of non-dimensionalising u by p g0H U= 2 in order to have j+j . 1 is indeed relevant to SID ows.

Henceforth, we drop the tildes and, unless explicitly stated otherwise, use non-dimensional variables throughout.

3. Background

We sketch the current state of knowledge on the behaviour of ow regimes, mass ux, and interfacial layer thickness with input parameters in § 3.1. We highlight the limitations of previous studies and the current open questions to motivate our study in § 3.2. A more thorough review of the literature supporting these conclusions is given in appendix A, and a synthesis is given in table A.1.

3.1. Current state of knowledge

The ow regimes have been observed and classi ed in a relatively consistent way in the literature. Throughout this paper, we adopt the nomenclature of Meyer & Linden (2014): L (laminar ow with at interface) H (interfacial Holmboe waves), I (intermittently turbulent), T (fully turbulent). The consensus is that the ow becomes increasingly disorganised and turbulent with A, and Re. At a xed > 0 , all ow regimes (L; H; I; T) can be visited by increasing Re, and conversely at xed Re and increasing (Macagno & Rouse 1961; Wilkinson 1986; Kiel 1991; Meyer & Linden 2014; Lefauve et al. 2019) (hereafter MR61, W86, K91, ML14 and LPL19, respectively). Both K91 and ML14 observed regime transitions scaling with A tan = tan = tan (or A for small angles), i.e. A controls the scaling. However, the scaling in Re is subject to debate, and may change on either side of (LPL19). These conclusions are illustrated schematically in gure 3(a) (the interrogation marks denote open questions).

The mass ux has a complex non-monotonic behaviour in A; ; Re sketched in gure 3(b). While the dependence on Re is clear at Re < 500A (MR61, W86, ML14, LPL19) due to the in uence of viscous boundary layers, it is still debated at Re > 500A: Mercer & Thompson (1975) (hereafter MT75) and ML14 argued in favour of this dependence on Re even above 500A whereas Leach & Thompson (1975) (heareafter LT75) and K91 argued against it. The mass ux reaches a maximum Q_m 0:4 0:5 at = 2 and `high enough' Re (MT75, K91, ML14, LPL19) and decays for smaller/larger and Re (W86, LPL19) in a poorly-studied fashion.

The interfacial layer thickness has only been studied experimentally in K91, who observed monotonic increase of with both A and , good collapse with A tan (reaching its maximum = 1 at & 2), and independence on Re (gure 3 (c)). The behaviour of at low Re < 500A remains unknown.

3.2. Limitations of previous studies

Many aspects of the scaling of regimes Q_m and with A; B; ; Re; P r remain open questions. For example, the e ects of Re on , and the e ects of B and P r on all three variables have not been studied at all. Moreover, despite our e orts to unify their ndings in § 3.1 and appendix A, these past studies of the SID experiment inherently provide a fragmented view of the problem due to the following limitations (made clear by table A.1):

Figure 3: Illustration of the current state of knowledge on the idealised behaviour of the (a) ow regimes, (b) mass ux, and (c) interfacial layer thickness with respect to A; ; Re (the axes have logarithmic scale). Interrogation marks refer to open questions. For more details, see the literature review in appendix A.

(i) they used slightly di erent setups and geometries (e.g. presence vs absence of free surfaces in the reservoirs, rectangular ducts vs circular pipes), and slightly di erent measuring methodologies (e.g. for Q_m);

(ii) only one study (K91) addressed the interdependence of the three variables of interest (regime, Q_m,), while the remaining studies measured either only regimes (MR61), only Q_m (LT75, MT75), or both (ML14, LPL19);

(iii) they focused on the variation of a single parameter (MR61), two parameters (W86, K91, LPL19), or at most three parameters (MT75, ML14) in which case the third parameter took only two di erent values;

(iv) they studied limited regions of the parameter space, and it is di cult to con dently interpolate results obtained by di erent setups in di erent regions (such as $Re < 500A$ and $> 500A$).

The experimental results and models in the next two sections attempt to overcome the above limitations by providing a more uni ed view of the problem.

4. Experimental results

In order to make progress on the scaling of ow regimes, Q_m and with $A; B; ; Re; Pr$, we obtained a comprehensive set of experimental data using an identical setup, measuring all three dependent variables with the same methodology (described in appendix B), and varying all ve independent parameters in a systematic fashion. We introduce the di erent duct geometries and data sets used in § 4.1, and present our results on ow regimes in § 4.2, on mass ux in § 4.3, and on interfacial layer thickness in § 4.4.

4.1. Data sets

All experimental data presented in the following were obtained in the strati ed inclined duct (SID) setup sketched in gure 1. We used four di erent duct geometries and two types of strati cation (salt and temperature) to obtain the following ve distinct data sets, listed in table 1:

LSID (L for Large) with height $H = 100$ mm, and $A = 30$, $B = 1$;

Table 1: The ve data sets used in this paper, using four duct geometries (abbreviated LSID, HSID, mSID, tSID) with di erent dimensional heights H, lengths $L = AH$ and widths $W = BH$, and two types of strati cation (salt and temperature). We emphasise in bold the resulting di erences in the `xed' non-dimensional parameters $A; B; Pr$ with respect to the `control' geometry (top row). We also emphasise the di erence in H between LSID and mSID, to test whether or not H plays a role other than through the non-dimensional parameters $A; B; Re$. We also list the range of $; Re$ explored, and the number of regime, Q_m and data points obtained in the $(; Re)$ plane. Some of these data have been published or discussed in some form in ML14 (denoted by ^{y}) and LPL19 (denoted by ^{y}) and are reused here with their permission for further analysis. Measurements of Q_m and were not practical with heat strati cation (hence the - symbol, see text for more details). Total: 886 individual experiments and 1545 data points.

	Duct scale		Fixed params.			Varied params.		Number of data points		
Name	H (mm)	Cross-section	A	B	Pr	$( )$	Re (10^3)	regime	Q_m	
LSID	100		30	1	700	$[1; 4]$	$[2; 20]$	173	20	115
HSID	100		15	1	700	$[0; 4]$	$[1; 20]$	74	34	58
mSID	45		30	1	700	$[1; 6]$	$[0.3; 6]$	360^y	162^y	91
tSID	90		15	$^1 4$	700	$[1; 3]$	$[3; 15]$	131	92	87
mSIDT	45		30	1	7	$[0; 10]$	$[0.3; 1.5]$	148	-	-

HSID (H for Half) which only di ers from the LSID (the `control' geometry) in that it is half the length: $A = 15$ (highlighted in bold in table 1);

mSID (m for mini) which only di ers from the LSID in its height $H = 45$ mm, but keeps $A; B; Pr$ identical (this is done by scaling down $H; W; L$ by the same factor $100/45$ such that the mSID and LSID ducts remain geometrically similar). Note that the mSID and LSID con gurations should yield identical data at identical Re since H should only play a role through the non-dimensional parameters $A; B; Re$. However, we will see in § 4.2-4.4 that this hypothesis is challenged by our data.

tSID (t for tall) which di ers from the HSID primarily in its tall spanwise aspect ratio $B = 1/4$ (and, secondarily, in a marginally smaller height $H = 90$ mm);

mSIDT (m for mini and T for temperature) which di ers from the mSID in that the strati cation was achieved by di erent reservoir water temperatures (hence $Pr = 7$), as opposed to di erent salinities in the above data sets (where $Pr = 700$). This limited the density di erence achieved, re ected in the lower Re.

Table 1 also lists, for each data set, the range of variation of and Re, and the number of data points, i.e. distinct (; Re) couples for which we have data on regime, Q_m , and .

Note that the regime and Q_m data of the top three data sets have already been published in some form by Meyer & Linden (2014) (ML14, denoted by) and Lefauve et al. (2019) (LPL19, denoted by ^{y}), as discussed in Appendix A.1-A.2. However, ML14 plotted their LSID and HSID data together (see their gure 7-8) and did not investigate their potential di erences, while LPL19 only commented in passing on a t of the Q_m data in the (; Re) plane (see their gure 9). The individual reproduction and thorough discussion of these data alongside more recent data using a uni ed non-dimensional approach will be key to this paper. All ve data sets have been used in the PhD thesis of Lefauve (2018) (especially in Chapters 3 and 5), and the detailed parameters of all experiments are tabulated in his appendix A for completeness. Most of the raw and processed data used in this paper are available on the repository doi.org/10.17863/CAM.48821 (more details in Appendix B).

Our focus on long ducts, evidenced by our choice of A = 15 and 30, re ects our focus on ows relevant to geophysical and environmental applications, which are typically largely horizontal (0) and stably strati ed in the vertical (as opposed to the di erent case of vertical exchange ow with = 90). The SID experiment conveniently exhibits all possible ow regimes, including high levels of turbulence and mixing, between = 0 and a few at most (§ 3.1). In long ducts (large A), tan $^{1}(A^{1})$ is therefore small enough to allow us to study all the key dynamics of sustained strati ed ows while keeping small enough for these ows to remain largely horizontal, and thus geophysically relevant.

As a result of this focus on long ducts, in the remainder of the paper we make the approximation that

$$\cos \quad 1 \quad \text{and} \quad \sin \quad ; \tag{4.1}$$

This approximation is accurate to better than 2 % for the angles considered in our data sets (6 10). Unless explicitly speci ed by the symbol, will now be expressed in radians (typically in scaling laws).

4.2. Flow regimes

The L; H; I; T ow regimes were determined following the ML14 nomenclature as in appendix B.1 (except for a new regime which we discuss in the next paragraph). Figure 4 shows the resulting regime maps in the (; Re) plane corresponding to the ve data sets.

First, we note the introduction of a `new' W regime in the tSID and mSIDT data (panels (d,e)). This W (wave) regime is similar to the H (Holmboe) regime, but describes interfacial waves which were not recognised as Holmboe waves in shadowgraphs. These waves were of two types. First, in the tSID geometry at positive angles > 0, the waves did not exhibit the distinctive `cusped' shape of Holmboe waves and the waves appeared to be generated at the ends of the duct and to decay as they travel inside the duct. Second, in the mSIDT larger-amplitude, tilde-shaped internal waves were observed across most of the height of the duct, contrary to Holmboe waves which are typically con ned to a much thinner interfacial region. Further discussion of these waves falls outside the scope of this paper, but can be found in (Lefauve 2018, §§ 3.2.3-3.2.4) (hereafter abbreviated L18). This new observation highlights the richness of the ow dynamics in the SID experiment. However, for the purpose of this paper, the H and W regimes are su ciently similar in their characteristics (mostly laminar ow with interfacial waves) that we group them under the same regime for the purpose of discussing regime transitions.

The main observation of gure 4 is that the transitions between regimes can be described as simple curves in the (Re) plane that do not overlap (or `collapse') between

(a) LSID

(b) HSID

(c) mSID

(d) tSID

(e) mSIDT

Figure 4: Regime diagrams in the $(;\mathrm{Re})$ plane (lin-log scale) using the ve data sets of table 1 (the scaled cross-section of each duct is sketched for comparison in the top right corner of each panel). The error in is of order $0{:}2$ and is slightly larger than the symbol size, whereas the error in Re is much smaller than the symbol size, except in (e) at small Re.

(a) LSID

(b) HSID

(c) mSID

(d) tSID

(e) mSIDT

Figure 5: Regime and Q_m in the (; Re) plane (log-log scale, thus only containing the regime and Q_m data of gure 4 for which > 0). The dashed and dotted lines represent the power law scalings Re = const. and Re^2 = const., respectively. The gray shadings represent the special threshold values of interest = and Re = 50 A. The ML14 arrow in panel (a) denotes the I ! T transition curve identi ed by ML14. Black contours in panels (a-d) represent the t to the Q_m data (see § 4.3), representing (a) 20 data points (coe cient of determination $R^2 = 0:56$), (b) 34 points ($R^2 = 0:81$), (c) 162 points ($R^2 = 0:80$), and (d) 92 points ($R^2 = 0:86$)