Conformal covariance of the Liouville quantum gravity metric for $\gamma \in (0,2)$

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Abstract

For $\gamma \in (0,2)$, $U \subset \mathbb{C}$, and an instance h of the Gaussian free field (GFF) on U, the γ-Liouville quantum gravity (LQG) surface associated with (U,h) is formally described by the Riemannian metric tensor $e^{\gamma h}(dx^2+dy^2)$ on U. Previous work by the authors showed that one can define a canonical metric (distance function) D_h on U associated with a γ-LQG surface. We show that this metric is conformally covariant in the sense that it respects the coordinate change formula for γ-LQG surfaces. That is, if $U, \widetilde{U}$ are domains, $\phi \colon U \to \widetilde{U}$ is a conformal transformation, $Q = 2/\gamma + \gamma/2$, and $\widetilde{h} = h \circ \phi^{-1} + Q \log |(\phi^{-1})'|$, then $D_h(z,w) = D_{\widetilde{h}}(\phi(z),\phi(w))$ for all $z,w \in U$. This proves that D_h is intrinsic to the quantum surface structure of (U,h), i.e., it does not depend on the particular choice of parameterization.

Keywords: Liouville quantum gravity, Gaussian free field, coordinate change, conformal covariance, LQG metric

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1 Introduction

1.1 Overview

Fix $\gamma \in (0,2)$, suppose that $U \subseteq \mathbb{C}$ is a domain, and let h be an instance of (some form of) the Gaussian free field (GFF) on U. The γ-*Liouville quantum gravity* (LQG) surface described by h formally corresponds to

$$e^{\gamma h(z)}(dx^2+dy^2), \quad z = x+iy \tag{1.1}$$

where $dx^2 + dy^2$ denotes the Euclidean metric on U. This expression does not make literal sense since h is a distribution and not a function so does not take values at points. Previously, the volume form associated with (1.1) was constructed by Duplantier and Sheffield in [DS11] using a regularization procedure. Namely, for each $\epsilon > 0$ and $z \in U$ so that $B_\epsilon(z) \subseteq U$ let $h_\epsilon(z)$ denote the average of h on $\partial B_\epsilon(z)$. Then the volume form μ_h is given by the limit as $\epsilon \to 0$ of

$$\epsilon^{\gamma^2/2} e^{\gamma h_\epsilon(z)} dxdy, \quad z = x + iy \tag{1.2}$$

where $dxdy$ denotes Lebesgue measure on U. The factor $\epsilon^{\gamma^2/2}$ is necessary for the limit to exist and be non-trivial. It is also possible to use a similar procedure to make sense of the lengths of certain types of curves [DS11, She16]. See [Kah85, RV14] for a more general theory of random measures of this type.

The LQG measure satisfies a certain change of coordinates formula [DS11, Proposition 2.1]. Suppose that $\widetilde{U} \subseteq \mathbb{C}$ is another domain, $\phi \colon U \to \widetilde{U}$ is a conformal transformation, and

$$\widetilde{h} = h \circ \phi^{-1} + Q \log |(\phi^{-1})'|, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}. \tag{1.3}$$

Then a.s. $\mu_h(A) = \mu_{\widetilde{h}}(\phi(A))$ for all Borel sets $A \subseteq U$. Two domain/field pairs (U, h), $(\widetilde{U}, \widetilde{h})$ are said to be *equivalent as LQG surfaces* if they are related as in (1.3). An *LQG surface* is an equivalence class of domain/field pairs with respect to this equivalence relation. We think of two equivalent pairs as being two embeddings of the same surface.

In a previous series of papers [MS20, MS16a, MS16b], a metric (distance function) associated with a $\sqrt{8/3}$-LQG surface was constructed in the special case when $\gamma = \sqrt{8/3}$. These works also showed that a certain special $\sqrt{8/3}$-LQG surface is equivalent, as a metric measure space, to the Brownian map of Le Gall [Le 13] and Miermont [Mie13].

This work is part of a larger project which is focused on constructing for all $\gamma \in (0, 2)$ the metric space structure of γ-LQG, i.e., the Riemannian distance function associated with (1.1), and proving its basic properties. We now explain the construction of the metric, which was carried out in the previous works [DDDF19, GM19b, DFG$^+$20, GM20, GM19a]. It is shown in [DG18, DZZ19] that for each $\gamma \in (0, 2)$, there is an exponent $d_\gamma > 2$ which can be defined in several equivalent ways, e.g., as the ball volume growth exponent for certain random planar maps in the γ-LQG universality class. It is shown in [GP19b] that d_γ is the Hausdorff dimension of a γ-LQG surface, viewed as a metric space. The value of d_γ is not known explicitly except that $d_{\sqrt{8/3}} = 4$, but see [DG18, GP19a, Ang19] for reasonably sharp bounds on d_γ.

We define

$$\xi = \xi_\gamma := \frac{\gamma}{d_\gamma}. \tag{1.4}$$

The significance of the parameter ξ is as follows: for a smooth function f, the Riemannian distance function associated with the metric tensor $e^f(dx^2 + dy^2)$ is obtained by integrating $e^{f/2}$ with respect to the Euclidean length measure on smooth paths. This makes it so that scaling the volume form by a factor of C corresponds to scaling distances by a factor of $C^{1/2}$. The Hausdorff dimension of the γ-LQG metric is d_γ, rather than 2, so scaling the volume form (i.e., the LQG area measure) by a factor of C should correspond to scaling distances by a factor of C^{1/d_γ}. This is achieved by defining the distance function using a regularization of $e^{\xi h}$ instead of $e^{\gamma h/2}$.

Suppose for simplicity that h is a whole-plane GFF. In light of the preceding paragraph, a natural way to approximate the distance function associated with (1.1) is via the random metrics

$$D_h^\varepsilon(z, w) := \inf_{P: z \to w} \int_0^1 e^{\xi h_\varepsilon(P(t))} |P'(t)| \, dt, \tag{1.5}$$

where the infimum is over all piecewise continuously differentiable paths from z to w and $\{h_\varepsilon\}_{\varepsilon>0}$ is a certain family of continuous functions which approximate the GFF as $\varepsilon \to 0$ (for technical reasons convergence has only been shown when we take h_ε to be the convolution of h with the heat kernel).

It is shown in [DDDF19] that the family of random metrics (1.5) (suitably re-scaled) is tight w.r.t. the local uniform topology on $\mathbb{C} \times \mathbb{C}$, and every possible subsequential limit is a metric which induces the Euclidean topology. See also [DD19, DF20, DD20] for earlier tightness results for approximations of the LQG metric, preceding [DDDF19].

Subsequently, it was shown in [GM19a], building on [GM19b, DFG+20, GM20], that the subsequential limit is unique, and in fact the metrics D_h^ε, suitably re-scaled, converge in probability as $\varepsilon \to 0$ to a metric D_h on $\mathbb{C}$. This D_h is defined to be the γ-LQG metric. The metric D_h is characterized by a list of axioms including the metric version of the coordinate change formula (1.3) for all complex affine functions. These conditions are listed just below. In particular, the metric for $\gamma = \sqrt{8/3}$ is the same as the one in [MS20, MS16a, MS16b]. By the local dependence of D_h on h, it follows that one can measurably associate an LQG metric for $\gamma \in (0,2)$ with the GFF on any planar domain (see [GM19a, Remark 1.5]).

The purpose of this work is to show that the resulting metric satisfies the metric analog of (1.3) for general conformal maps. Consequently, the metric constructed in [GM19a] is intrinsic to the quantum surface structure of an LQG surface, i.e., the particular choice of embedding does not change the definition of the metric. As we will see, establishing (1.3) for general conformal maps from the case of just complex affine maps is trickier than one might expect.

Although this work builds on [DDDF19, GM19b, DFG+20, GM20, GM19a], it can be read without any knowledge of these works, or even any knowledge about LQG beyond basic properties of the GFF. The reason for this is that we take the axiomatic definition of the whole-plane γ-LQG metric from [GM19a] as our starting point, and deduce our results from these axioms.

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1.2 Main results

We will now define a notion of a γ-LQG metric for arbitrary open domains $U \subset \mathbb{C}$. The definition of the γ-LQG metric in [GM19a] is the special case when $U = \mathbb{C}$. We first need some preliminary definitions. Throughout, (X, D) denotes a metric space.

For a continuous curve $P : [a,b] \to X$ (here $[a,b]$ is equipped with the Euclidean metric and X is equipped with the metric D), the *D-length* of P is defined by

$$\mathrm{len}(P; D) := \sup_T \sum_{i=1}^{\#T} D(P(t_i), P(t_{i-1}))$$

where the supremum is over all partitions $T : a = t_0 < \cdots < t_{\#T} = b$ of $[a,b]$. Note that the D-length of a curve may be infinite.

For $Y \subset X$, the *internal metric of D on Y* is defined by

$$D(x, y; Y) := \inf_{P \subset Y} \mathrm{len}(P; D), \quad \forall x, y \in Y \tag{1.6}$$

where the infimum is over all paths P in Y from x to y. Then $D(\cdot,\cdot;Y)$ is a metric on Y, except that it is allowed to take the value ∞.

We say that (X,D) is a *length space* if for each $x,y\in X$ and each $\varepsilon>0$, there exists a curve of D-length at most $D(x,y)+\varepsilon$ from x to y.

A *continuous metric* on an open domain $U\subset\mathbb{C}$ is a metric D on U which induces the Euclidean topology on U. We equip the space of such metrics with the local uniform topology for functions from $U\times U$ to $[0,\infty)$. We allow a continuous metric to satisfy $D(u,v)=\infty$ if u and v are in different connected components of U. In this case, in order to have $D^n\to D$ w.r.t. the local uniform topology we require that for large enough n, $D^n(u,v)=\infty$ if and only if $D(u,v)=\infty$.

A *GFF plus a continuous function* on an open domain $U\subset\mathbb{C}$ is a random distribution h on U which can be coupled with a random continuous function f in such a way that $h-f$ has the law of the (zero-boundary or whole-plane, as appropriate) GFF on U. We emphasize that f is not required to extend continuously to $\overline{U}$.

For $U\subset\mathbb{C}$, let $\mathcal{D}'(U)$ be the space of distributions (in the sense of Schwartz) on $\mathbb{C}$, equipped with the usual weak topology.

Definition 1.1. A *γ-LQG metric* is a collection of functions $h\mapsto D_h$, one for each open set $U\subset\mathbb{C}$, from $\mathcal{D}'(U)$ to the space of continuous metrics on U with the following properties. Let $U\subset\mathbb{C}$ and let h be a GFF plus a continuous function on U.[1] Then the associated metric D_h satisfies the following axioms.

I. **Length space.** Almost surely, (U,D_h) is a length space, i.e., the D_h-distance between any two points of U is the infimum of the D_h-lengths of D_h-continuous paths (equivalently, Euclidean continuous paths) in U between the two points.

II. **Locality.** Let $V\subset U$ be a deterministic open set. The D_h-internal metric $D_h(\cdot,\cdot;V)$ is a.s. equal to $D_{h|_V}$ (so in particular it is a.s. determined by $h|_V$).

III. **Weyl scaling.** Let ξ be as in (1.4). For a continuous function $f:U\to\mathbb{R}$, define

$$(e^{\xi f}\cdot D_h)(z,w):=\inf_{P:z\to w}\int_0^{\mathrm{len}(P;D_h)} e^{\xi f(P(t))}\,dt,\quad \forall z,w\in U, \tag{1.7}$$

where the infimum is over all continuous paths from z to w in U parameterized by D_h-length. Then a.s. $e^{\xi f}\cdot D_h=D_{h+f}$ for every bounded continuous function $f:U\to\mathbb{R}$.

IV. **Conformal coordinate change.** Let $\widetilde{U}\subset\mathbb{C}$ and let $\phi:U\to\widetilde{U}$ be a deterministic conformal map. Then with Q as in (1.3), a.s.

$$D_h(z,w)=D_{h\circ\phi^{-1}+Q\log|(\phi^{-1})'|}(\phi(z),\phi(w)),\quad \forall z,w\in U. \tag{1.8}$$

It is shown in [GM19a, Theorem 1.2 and Corollary 1.3] (building on [DDDF19, GM19b, DFG+20, GM20]) that there is a unique LQG metric in the special case when $U=\mathbb{C}$, i.e., one has the following statement.

[1] Our axioms for a γ-LQG metric only concern a.s. properties of D_h when h is a GFF plus a continuous function. So, once we have defined D_h a.s. when h is a GFF plus a continuous function, we can take D to be any measurable mapping $\mathcal{D}'(U)\to\{\text{continuous metrics on } U\}$ which is a.s. consistent with our given definition when h is a GFF plus a continuous function. In fact, the construction of the metric in [DDDF19, DFG+20, GM20, GM19a] only gives an explicit definition of D_h in the case when h is a GFF plus a continuous function.

Theorem 1.2 ([GM19a]). *There is a measurable function $h \mapsto D_h$ from $\mathcal{H}'(\mathbb{C})$ to the space of continuous metrics on $\mathbb{C}$ which satisfies the above axioms with $U = \widetilde{U} = \mathbb{C}$. In this restricted setting, Axiom II is replaced by the requirement that $D_h(\cdot,\cdot;V)$ is a.s. determined by $h|_V$ and Axiom IV reads as follows.*

*IV′. **Coordinate change for complex affine maps.** For each fixed deterministic $a, b \in \mathbb{C}$, $a \neq 0$, a.s.*

$$D_h(az+b, aw+b) = D_{h(a\cdot+b)+Q\log|a|}(z,w), \quad \forall z,w \in \mathbb{C}. \tag{1.9}$$

Furthermore, if $h \mapsto D_h$ and $h \mapsto \widetilde{D}_h$ are two such measurable functions, then there is a deterministic constant $C > 0$ such that a.s. $D_h = C\widetilde{D}_h$ whenever h is a GFF plus a continuous function.

We call D_h from Theorem 1.2 the *γ-LQG metric associated with h*. Following [GM20, Remark 1.2], it is not hard to extend the definition of D_h from Theorem 1.2 to GFF-type distributions on proper subdomains of $\mathbb{C}$, as we now explain. Suppose that h is a whole-plane GFF. For each deterministic open set $U \subset \mathbb{C}$, the metric $D_h(\cdot,\cdot;U)$ is a.s. determined by $h|_U$ so we can simply define $D_{h|_U} := D_h(\cdot,\cdot;U)$. We can write $h|_U = \mathring{h}^U + \mathfrak{h}^U$, where $\mathring{h}^U$ is a zero-boundary GFF on U and $\mathfrak{h}^U$ is a random harmonic function on U independent from $\mathring{h}^U$. In the notation (1.7), we define

$$D_{\mathring{h}^U} := e^{-\xi\mathfrak{h}^U} \cdot D_{h|_U}. \tag{1.10}$$

It is easily seen from Axioms II and III that $D_{\mathring{h}^U}$ is a measurable function of $\mathring{h}^U$ (see [GM20, Remark 1.2]). In light of Axiom III, we can then define $D_{\mathring{h}^U+f}$ as a measurable function of $\mathring{h}^U + f$ for any random continuous function $f : U \to \mathbb{R}$. By inspection, this function from distributions to metrics satisfies Axioms I through III above. The main result of this paper is the following theorem which verifies that the above metric satisfies Axiom IV. This completes the program to define the γ-LQG metric for all $\gamma \in (0,2)$ on an arbitrary planar domain.

Theorem 1.3. *Let $U \subset \mathbb{C}$ be an open domain and let $\phi : \mathbb{C} \to U$ be a conformal map. Also let h^U be a GFF on U plus a continuous function. Almost surely, the γ-LQG metric satisfies the coordinate change formula*

$$D_{h^U}(z,w) = D_{h^U\circ\phi^{-1}+Q\log|(\phi^{-1})'|}(\phi(z),\phi(w)), \quad \forall z,w \in U. \tag{1.11}$$

That is, the mapping $h^U \mapsto D_{h^U}$ constructed in [GM19a] is a γ-LQG metric in the sense of Definition 1.1.

As noted above, Theorem 1.3 says that the LQG metric depends intrinsically on the γ-LQG surface (U, D_h), i.e., it does not depend on the particular choice of parameterization for this surface. Hence a γ-LQG surface with any choice of underlying conformal structure makes sense as a metric space.

1.3 Outline

Throughout most of the proof of Theorem 1.3, we will work with a whole-plane GFF h restricted to a domain in $\mathbb{C}$. We will transfer to other variants of the GFF at the very end of the argument using Axiom III.

For an open set $U \subset \mathbb{C}$ and a conformal map $\phi : U \to \phi(U)$, we define

$$h^\phi := h\circ\phi^{-1} + Q\log|(\phi^{-1})'| \quad \text{and} \quad D_h^\phi(z,w) := D_{h^\phi}(\phi(z),\phi(w)), \quad \forall z,w \in U. \tag{1.12}$$

By the conformal invariance of the GFF, h^ϕ is the sum of a zero-boundary GFF and a harmonic function on $\phi(U)$. Therefore D_{h^ϕ} is defined as explained before the statement of Theorem 1.3. Furthermore, from the locality of D_h it is easily seen that D_h^ϕ is a local metric for $h|_U$ and is a.s. determined by $h|_U$. We want to show that a.s. $D_h^\phi = D_{h|_U}$.

As one might expect, the basic idea of the proof is to use that the conformal map ϕ looks approximately like a complex affine map in a small neighborhood of a typical point, then apply Axiom IV′. However, there are a number of complications in making this argument work which make the proof of Theorem 1.3 more difficult than one might expect at first glance.

The first main step of the proof, which is carried out in **Section 2**, is to show that if $z \in U$ and $r > 0$ is small, then D_h and D_h^ϕ are close on $B_r(z)$, in the sense that with high probability $\sup_{u,v\in B_r(z)} |D_h^\phi(u,v) - D_h(u,v)|$ is of smaller order than the D_h-diameter of $B_r(z)$ (which by Axioms III and IV′ is typically of order $r^{\xi Q} e^{\xi h_r(z)}$). See Proposition 2.1 for a precise statement. Here we note that when $r > 0$ is small, the D_h-diameter of $B_r(z)$ is smaller than its D_h-distance to ∂U, so the restrictions to $B_r(z)$ of $D_{h|_U} = D_h(\cdot,\cdot;U)$ and D_h agree.

The main difficulty in this step is that we do not know a priori that $\phi \mapsto D_h^\phi$ depends continuously on the conformal map ϕ in the almost sure sense. This is because we do not know that $\phi \mapsto D_{h\circ\phi^{-1}}$ is continuous. Rather, we only know that if ϕ^{-1} is uniformly close to the linear map $z \mapsto \alpha z$ (which will be the case if we start with an arbitrary conformal map ϕ and zoom in on a sufficiently small neighborhood of any given point) then the law of $h \circ \phi^{-1}$ is close to the law of $h(\alpha\cdot)$ in the total variation sense (Lemma 2.4). This tells us that the *marginal laws* of $D_{h\circ\phi^{-1}}$ and $D_{h(\alpha\cdot)}$ are close.

We will show in Lemma 2.3 that *joint law* of $D_{h\circ\phi^{-1}}$ and $D_{h(\alpha\cdot)}$ is close to the joint law of two copies of the same instance of $D_{h(\alpha\cdot)}$. The basic idea of the argument is as follows. If $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of conformal maps such that ϕ_n^{-1} converges uniformly on compact subsets of $\mathbb{C}$ to $z \mapsto \alpha z$, then using basic facts about the GFF we can establish the convergence of joint laws

$$(h \circ \phi_n^{-1}, D_{h\circ\phi_n^{-1}}) \to (h(\alpha\cdot), D_{h(\alpha\cdot)}) \quad \text{and} \quad (h, h\circ\phi_n^{-1}) \to (h, h(\alpha\cdot)). \tag{1.13}$$

This implies that the joint laws of the 4-tuples $(h, D_h, h\circ\phi_n^{-1}, D_{h\circ\phi_n^{-1}})$ are tight, and moreover allows us to show that any possible subsequential limit is of the form $(h, D_h, h(\alpha\cdot), D_{h(\alpha\cdot)})$.

By re-scaling and applying Axiom IV′, the preceding paragraph allows us to show that if $\phi : U \to \phi(U)$ is a conformal map, then the metric $D_{h|_U}$ and the metric D_h^ϕ appearing in (1.11) are close at small scales in the desired sense.

In **Section 3**, we upgrade from the statement that $D_{h|_U}$ and D_h^ϕ are close with high probability in a small neighborhood of any point to the statement that $D_{h|_U}$ and D_h^ϕ are close with high probability everywhere. This will be carried out in two steps. In Section 3.1, we show that $D_{h|_U}$ and D_h^ϕ are a.s. bi-Lipschitz equivalent using a general criterion for bi-Lipschitz equivalence of two local metrics for the same GFF (Theorem 3.4). We then show that the optimal bi-Lipschitz constant is 1 in Section 3.2 using a "good annulus covering" argument similar to the one in [GM19a, Section 3].

The reason why we need to use a two-step argument of this form is as follows. Even though we know that $D_{h|_U}$ and D_h^ϕ are close at small scales, our estimates are not sharp enough to say directly that a quantity of the form $\sum_{j=1}^N |D_{h|_U}(P(t_{j-1}), P(t_j)) - D_h^\phi(P(t_{j-1}), P(t_j))|$ is small when $P : [0,T] \to U$ is a D_h-rectifiable path and $0 = t_0 < t_1 < \cdots < t_N = T$ is a fine partition of $[0,T]$. The arguments of Section 3 allow us to restrict attention to "good" scales where we can say that the ratios of certain $D_{h|_U}$-distances and D_h^ϕ-distances are close to 1.

1.4 Basic notation

We write $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

For $a < b$, we define the discrete interval $[a,b]_{\mathbb{Z}} := [a,b] \cap \mathbb{Z}$.

If $f : (0,\infty) \to \mathbb{R}$ and $g : (0,\infty) \to (0,\infty)$, we say that $f(\varepsilon) = O_\varepsilon(g(\varepsilon))$ (resp. $f(\varepsilon) = o_\varepsilon(g(\varepsilon))$) as $\varepsilon \to 0$ if $f(\varepsilon)/g(\varepsilon)$ remains bounded (resp. tends to zero) as $\varepsilon \to 0$. We similarly define $O(\cdot)$ and $o(\cdot)$ errors as a parameter goes to infinity. We will often specify any requirements on the dependencies on rates of convergence in $O(\cdot)$ and $o(\cdot)$ errors in the statements of lemmas/propositions/theorems, in which case we implicitly require that errors, implicit constants, etc., appearing in the proof satisfy the same dependencies.

For $z \in \mathbb{C}$ and $r > 0$, we write $B_r(z)$ for the Euclidean ball of radius r centered at z. We also define the open annulus

$$\mathbb{A}_{r_1,r_2}(z) := B_{r_2}(z) \setminus \overline{B_{r_1}(z)}, \quad \forall 0 < r_r < r_2 < \infty. \tag{1.14}$$

2 Comparison of D_h and D_h^ϕ at small scales

The goal of this section is to show that in the notation (1.12), the metrics D_{h^ϕ} and $D_{h|_U}$ are close with high probability at small scales (see Proposition 2.1 just below).

We will be working with conformal maps, so since circles are not preserved under conformal maps it is sometimes convenient to use a slightly different normalization for the GFF than the usual $h_1(0) = 0$. In particular, we fix a smooth compactly supported, radially symmetric bump function $\mathbb{f} : \mathbb{C} \to [0,1]$ with $\int_{\mathbb{C}} f(z)\,dz = 1$ and for $z \in \mathbb{C}$ and $r > 0$ we define

$$h_{\mathbb{f},r}(z) := (h(r \cdot + z), \mathbb{f}) = (h, r^{-2}\mathbb{f}(r^{-1}(\cdot - z))). \tag{2.1}$$

We will often normalize h by requiring $h_{\mathbb{f},1}(0) = 0$ instead of $h_1(0) = 0$. The advantage of this is that, since $\mathbb{f}$ is smooth, the smoothed average $h_{\mathbb{f},1}(0)$ depends continuously on h in the distributional topology. This fact is needed in the proof of Lemma 2.5 below.

The main result of this section is the following proposition.

Proposition 2.1. *Let h be a whole-plane GFF normalized so that $h_{\mathbb{f},1}(0) = 0$. For each fixed $\delta > 0$ and compact set $K \subset U$,*

$$\lim_{r\to 0} \inf_{z \in K} \mathbb{P}\left[\sup_{u,v \in B_r(z)} |D_h^\phi(u,v) - D_h(u,v)| \leq \delta r^{\xi Q} e^{\xi h_{\mathbb{f},r}(z)}\right] = 1. \tag{2.2}$$

Proposition 2.1 involves the smooth averages $h_{\mathbb{f},r}(z)$ instead of circle averages, but it is easy to convert to statements which do not depend on the choice of normalization for the field. For example, we have the following consequence of Proposition 2.1 which will be used in Section 3.2.

Lemma 2.2. *Let h be a whole-plane GFF, with any choice of normalization. Fix $\delta, b \in (0,1)$. For each fixed compact set $K \subset U$,*

$$\lim_{r\to 0} \inf_{z \in K} \mathbb{P}\left[1 - \delta \leq \frac{D_h^\phi(u,v)}{D_h(u,v)} \leq 1 + \delta,\ \forall u,v \in B_r(z) \text{ with } |u - v| \geq br\right] = 1. \tag{2.3}$$

Proof. By Axiom III, changing the normalization of h (i.e., adding a constant to h) does not affect the value of $D_h^\phi(u,v)/D_h(u,v)$. Therefore, we can assume without loss of generality that h is normalized so that $h_{\mathbb{f},1}(0) = 0$. By Axioms III and IV$'$ and the scale and translation invariance of the law of h, modulo additive constant, for each $\varepsilon > 0$ we can find $c > 0$ depending only on b such that with probability at least $1-\varepsilon$,

$$D_h(u,v) \geq c r^{\xi Q} e^{\xi h_{\mathbb{f},r}(z)}, \ \forall u,v \in B_r(z) \text{ with } |u-v| \geq br.$$

The lemma statement follows by combining this with Proposition 2.1 with $c\delta$ in place of δ, then sending $\varepsilon \to 0$. □

2.1 D_h^ϕ converges to D_h as ϕ converges to a linear map

Throughout this section we let h be a whole-plane GFF normalized so that $h_{\mathbb{f},1}(0) = 0$, with $\mathbb{f}$ as in (2.1). The main step in the proof of Proposition 2.1 is the following lemma, which we will prove in this section.

Lemma 2.3. *Let $\phi_n : U_n \to \phi_n(U_n)$ be a sequence of conformal maps such that $\phi_n(0) = 0$, $\phi_n'(0) \to 1/\alpha \in \mathbb{C} \setminus \{0\}$ as $n \to \infty$, and each fixed compact subset of $\mathbb{C}$ is contained in U_n for large enough n. Then*

$$\left(h, D_h, h^{\phi_n}, D_h^{\phi_n}\right) \to (h, D_h, h(\alpha\cdot), D_h) \tag{2.4}$$

in law with respect to the distributional topology and the local uniform topology on $\mathbb{C}\times\mathbb{C}$, as appropriate.

The main difficulty in the proof of Lemma 2.3 is comparing the metrics $D_{h\circ\phi_n^{-1}}$ and $D_{h(\alpha\cdot)}$. We will accomplish this using the outline discussed in Section 1.3. We first need the following elementary lemma for the GFF.

Lemma 2.4. *Let $\phi_n : U_n \to \phi_n(U_n)$ be as in Lemma 2.3. Then for each compact set $K \subset \mathbb{C}$,*

$$h \circ \phi_n^{-1}|_K \to h(\alpha\cdot)|_K, \quad \text{almost surely in the distributional sense} \tag{2.5}$$

and

$$\left(h \circ \phi_n^{-1} - (h\circ\phi_n^{-1})_{\mathbb{f},|\alpha|}(0)\right)|_K \to h(\alpha\cdot)|_K, \quad \text{in the total variation sense.} \tag{2.6}$$

Proof. By the Koebe distortion theorem, $\phi_n'(z) \to 1/\alpha$ uniformly on compact subsets of $\mathbb{C}$. It follows that $\phi_n(z) \to z/\alpha$ uniformly on compact subsets of $\mathbb{C}$. By the Cauchy integral formula, all of the higher-order derivatives of ϕ_n converge to zero uniformly on compact subsets of $\mathbb{C}$. Furthermore, $\phi_n^{-1} \to \alpha z$, $(\phi_n^{-1})'(z) \to \alpha$, and all of the higher-order derivatives of ϕ_n^{-1} converge to zero uniformly on compact subsets of $\mathbb{C}$.

Consequently, if $f : \mathbb{C} \to \mathbb{R}$ is a smooth, compactly supported function then $f\circ\phi_n$ and all of its derivatives of all orders converge uniformly to $f(\alpha\cdot)$ and its corresponding derivatives as $n \to \infty$. Therefore,

$$(h\circ\phi_n^{-1}, f) = (h, |\phi_n'|^2(f\circ\phi_n)) \to (h, |\alpha|^{-2} f(\alpha^{-1}\cdot)) = (h(\alpha\cdot), f). \tag{2.7}$$

This gives (2.5).

To prove (2.6), write $h|_{U_n} = \mathring{h}_n + \mathfrak{h}_n$, where $\mathring{h}_n$ is a zero-boundary GFF on U_n and $\mathfrak{h}_n$ is an independent random harmonic function on U_n. Then $\mathring{h}_n \circ \phi_n^{-1}$ is a zero-boundary GFF on $\phi(U_n)$. By [MS17, Proposition 2.10], we have $\mathring{h}_n\circ\phi_n^{-1}|_K \to h|_K$ in the total variation sense if we view both $\mathring{h}_n$ and h as being defined modulo a global additive constant. The field $h(\alpha\cdot)$ is normalized so that $(h(\alpha\cdot))_{\mathbb{f},|\alpha|}(0)$ is zero. Therefore,

$$\left(\mathring{h}_n\circ\phi_n^{-1} - (\mathring{h}_n\circ\phi_n^{-1})_{\mathbb{f},|\alpha|}(0)\right)|_K \to h(\alpha\cdot)|_K \tag{2.8}$$

in total variation, *without* having to view the distributions as being defined modulo additive constant.

On the other hand, basic estimates for the harmonic part of the GFF (see the proof of [MS17, Proposition 2.10]) combined with the aforementioned convergence of ϕ_n^{-1} to $z \mapsto \alpha z$ shows that for any fixed compact set $K' \subset \mathbb{C}$, the Dirichlet energy of $\mathfrak{h}_n|_{K'}$ tends to zero in probability as $n \to \infty$. Combining this with the convergence of ϕ_n^{-1} and all of its derivatives mentioned above, we get that the same is true with $\mathfrak{h}_n \circ \phi_n^{-1}$ in place of $\mathfrak{h}_n$.

Recall that if f is a smooth compactly supported bump function on U_n, then the laws of $\mathring{h}_n \circ \phi_n^{-1}$ and $\mathring{h}_n \circ \phi_n^{-1} + f$ are mutually absolutely continuous, and the Radon-Nikodym derivative of the latter with respect to the former is $\exp\Big((\mathring{h}_n \circ \phi_n^{-1}, f)_\nabla - \frac{1}{2}(f,f)_\nabla\Big)$, where $(g_1, g_2)_\nabla := \frac{1}{2\pi}\int_{U_n} \nabla g_1(z) \cdot \nabla g_2(z)\, d^2 z$ denotes the Dirichlet inner product. By applying this formula with f equal to a smooth, compactly supported bump function times $\mathfrak{h}_n \circ \phi_n^{-1} - (\mathfrak{h}_n \circ \phi_n^{-1})_{\mathbb{f},|\alpha|}$, we obtain (2.6) from (2.8) and the preceding paragraph. □

We can now establish the convergence of the second two coordinates in Lemma 2.3.

Lemma 2.5. *Let $\phi_n : U_n \to \phi_n(U_n)$ be as in Lemma 2.3. Then*

$$\left(h^{\phi_n}, D_h^{\phi_n}\right) \to (h(\alpha\cdot) + Q\log|\alpha|, D_h) \tag{2.9}$$

in law with respect to the distributional topology on the first coordinate and the local uniform topology on $\mathbb{C} \times \mathbb{C}$ on the second coordinate.

Proof. Consider a large bounded open set $V \subset \mathbb{C}$. Since $D_h(\cdot,\cdot;V)$ is a deterministic functional of $h|_V$ (Axiom II), the total variation convergence in Lemma 2.4 implies that

$$\left(\big(h \circ \phi_n^{-1} - (h \circ \phi_n^{-1})_{\mathbb{f},|\alpha|}(0)\big)|_V, D_{h\circ\phi_n^{-1} - (h\circ\phi_n^{-1})_{\mathbb{f},|\alpha|}(0)}(\cdot,\cdot;V)\right) \to \big(h(\alpha\cdot)|_V, D_{h(\alpha\cdot)}(\cdot,\cdot;V)\big) \tag{2.10}$$

in the total variation sense. Since the function $\mathbb{f}$ of (2.1) is smooth and compactly supported, we can apply (2.5) of Lemma 2.4 to get that

$$(h \circ \phi_n^{-1})_{\mathbb{f},|\alpha|}(0) \to (h(\alpha\cdot))_{\mathbb{f},|\alpha|}(0) = h_{\mathbb{f},1}(0) = 0 \tag{2.11}$$

in law as $n \to \infty$.

By combining (2.10) and (2.11) (and using Axiom III to get that the map $c \mapsto D_{h+c}$ is continuous), then letting V increase to all of $\mathbb{C}$, we obtain

$$\left(h \circ \phi_n^{-1}, D_{h\circ\phi_n^{-1}}\right) \to \big(h(\alpha\cdot), D_{h(\alpha\cdot)}\big) \tag{2.12}$$

in law.

Recall that $h^{\phi_n} = h \circ \phi_n^{-1} + Q\log|(\phi_n^{-1})'|$ and $D_h^{\phi_n} = D_{h^{\phi_n}}(\phi_n(\cdot), \phi_n(\cdot))$. We have $\phi_n(z) \to \alpha^{-1} z$ and $Q\log|(\phi_n^{-1})'| \to Q\log|\alpha|$ uniformly on compact subsets of $\mathbb{C}$ (see the beginning of the proof of Lemma 2.4). Combining this with (2.12) and using Axiom III to deal with the convergence of the metrics shows that

$$\left(h^{\phi_n}, D_h^{\phi_n}\right) \to \big(h(\alpha\cdot) + Q\log|\alpha|, D_{h(\alpha\cdot)+Q\log|\alpha|}(\alpha^{-1}\cdot, \alpha^{-1}\cdot)\big) \tag{2.13}$$

in law. The right side of (2.13) equals $(h(\alpha\cdot) + Q\log|\alpha|, D_h)$ by Axiom IV′. □

Proof of Lemma 2.3. By Lemma 2.5 and the Prokhorov theorem, for any sequence of n's tending to ∞, there is a subsequence $\mathcal{N}$ and a coupling $(h, D_h, h', D_{h'})$ of two whole-plane GFF's and their associated metrics such that as $\mathcal{N} \ni n \to \infty$,

$$\left(h, D_h, h^{\phi_n}, D_h^{\phi_n}\right) \to \left(h, D_h, h'(\alpha\cdot) + Q\log|\alpha|, D_{h'}\right),$$

in law. By the a.s. convergence part of Lemma 2.4, we have $(h, h^{\phi_n}) \to (h, h(\alpha\cdot) + Q\log|\alpha|)$ in law. Hence $h' = h$ a.s., so also $D_{h'} = D_h$ a.s. Therefore our subsequential limit is given by the right side of (2.4). Since our initial choice of subsequence was arbitrary, we obtain the statement of the lemma. $\square$

2.2 Uniform comparison of D_h and D_h^ϕ

Continue to assume that h is a whole-plane GFF normalized so that $h_{\mathbb{f},1}(0) = 0$. To deduce Proposition 2.1 from Lemma 2.3, we need to re-scale to convert from a statement about conformal maps at small scales to a statement about conformal maps which are close to linear at constant-order scales; and we need to ensure that the estimate we obtain is uniform over all $z \in U$. The re-scaling will be accomplished by means of the following basic calculation.

Lemma 2.6. *Fix $r > 0$ and $z \in \mathbb{C}$ and let $\widetilde{h} := h(r\cdot + z) - h_{\mathbb{f},r}(z)$, so that $\widetilde{h} \stackrel{d}{=} h$. Also let $\widetilde{\phi} : r^{-1}(U - z) \to r^{-1}(\phi(U) - z)$ be defined by $\widetilde{\phi}(w) = r^{-1}(\phi(rw + z) - z)$. Then*

$$D_{\widetilde{h}}^{\widetilde{\phi}}(u, v) = r^{-\xi Q} e^{-\xi h_{\mathbb{f},r}(z)} D_h^\phi(ru + z, rv + z), \quad \forall u, v \in r^{-1}(U - z). \tag{2.14}$$

Proof. Recall the definition of h^ϕ from (1.12). We apply Axiom III and then Axiom IV′ to h^ϕ to get that for $u, v \in r^{-1}(U - z)$,

$$\begin{aligned}
D_{\widetilde{h}}^{\widetilde{\phi}}(u, v) &= D_{h(\phi^{-1}(r\cdot+z)) + Q\log|(\phi^{-1})'(r\cdot+z)| - h_{\mathbb{f},r}(z)}\left(r^{-1}(\phi(ru+z) - z), r^{-1}(\phi(rv+z) - z)\right) \\
&= e^{-\xi h_{\mathbb{f},r}(z)} D_{h^\phi(r\cdot+z)}\left(r^{-1}(\phi(ru+z) - z), r^{-1}(\phi(rv+z) - z)\right) \quad \text{(Axiom III)} \\
&= r^{-\xi Q} e^{-\xi h_{\mathbb{f},r}(z)} D_{h^\phi}(\phi(ru+z), \phi(rv+z)) \quad \text{(Axiom IV}'\text{)} \\
&= r^{-\xi Q} e^{-\xi h_{\mathbb{f},r}(z)} D_h^\phi(ru+z, rv+z) \quad \text{(by the definition (1.12) of } D_h^\phi).
\end{aligned}$$

$\square$

In what follows, we fix $\delta > 0$ and for $z \in U$ and $r > 0$ such that $B_r(z) \subset U$, we let

$$F_r(z) := \left\{ \sup_{u,v \in B_r(z)} |D_h^\phi(u, v) - D_h(u, v)| \le \delta r^{\xi Q} e^{\xi h_{\mathbb{f},r}(z)} \right\} \tag{2.15}$$

be the event of Proposition 2.1.

Lemma 2.7. *Consider a sequence of points $\{z_n\}_{n\in\mathbb{N}} \subset U$ and radii $\{r_n\}_{n\in\mathbb{N}}$ such that $z_n \to z \in U$ and $r_n \to 0$. In the notation of Proposition 2.1, we have $\lim_{n\to\infty} \mathbb{P}[F_{r_n}(z_n)] = 1$.*

Proof. For $n \in \mathbb{N}$, define the conformal map

$$\phi_n : r_n^{-1}(U_n - z_n) \to r_n^{-1}(\phi(U_n) - z_n) \quad \text{by} \quad \phi_n(w) = r_n^{-1}(\phi(r_n w + z_n) - z_n)$$

Then $\phi_n(0) = 0$, $\phi_n'(0) = \phi'(z_n) \to \phi'(z)$, and (since $r_n \to 0$ and z lies at positive distance from ∂U) every compact subset of $\mathbb{C}$ is contained in $r_n^{-1}(U_n - z_n)$ for large enough n.

Define the field $h_n := h(r_n \cdot + z_n) - h_{\mathfrak{f},r_n}(z_n) \stackrel{d}{=} h$. By Lemma 2.3 applied with h_n in place of h and $\alpha = 1/\phi_n'(z)$, we have the convergence of joint laws

$$\left(h_n, D_{h_n}, h_n^{\phi_n}, D_{h_n}^{\phi_n}\right) \to \left(h, D_h, h(\cdot/\phi'(z)) + Q\log|1/\phi'(z)|, D_h\right). \tag{2.16}$$

In particular, it holds with probability tending to 1 as $n \to \infty$ that

$$|D_{h_n}^{\phi_n}(u,v) - D_{h_n}(u,v)| \le \delta, \quad \forall u,v \in \mathbb{D}. \tag{2.17}$$

By Lemma 2.6 along with Axioms III and IV′ for h, if r_n is sufficiently large that $\mathbb{D} \subset r_n^{-1}(U - z_n)$, then

$$\begin{aligned} D_{h_n}(u,v) &= r_n^{-\xi Q} e^{-\xi h_{\mathfrak{f},r_n}(z_n)} D_h(r_n u + z_n, r_n v + z_n) \quad \text{and} \\ D_{h_n}^{\phi_n}(u,v) &= r_n^{-\xi Q} e^{-\xi h_{\mathfrak{f},r_n}(z_n)} D_h^{\phi}(r_n u + z_n, r_n v + z_n), \quad \forall u,v \in \mathbb{D}. \end{aligned} \tag{2.18}$$

Combining (2.17) with (2.18) gives the statement of the lemma. ☐

Proof of Proposition 2.1. Assume by way of contradiction that for some compact set $K \subset U$, the relation (2.2) fails. Then there is an $\varepsilon > 0$, a sequence $r_n \to 0$, and a sequence of points $z_n \in K$ such that $\mathbb{P}[F_{r_n}(z_n)] \le 1 - \varepsilon$ for every $n \in \mathbb{N}$. By possibly passing to a subsequence, we can assume without loss of generality that $z_n \to z \in K$. Then Lemma 2.7 shows that $\lim_{n\to\infty} \mathbb{P}[F_{r_n}(z_n)] = 1$, which is a contradiction. ☐

3 Proof of Theorem 1.3

Recall the notation D_h^ϕ from (1.12). To prove Theorem 1.3, we want to upgrade from the statement that $D_{h|_U}$ and D_h^ϕ are close with high probability at small scales (Proposition 2.1) to the statement that these two metrics are a.s. close globally. This will be done using various local independence properties of the GFF. In this section we will mostly use circle averages rather than the smoothed average $h_{\mathfrak{f},r}(z)$ of Section 2.

3.1 Bi-Lipschitz equivalence of D_h and D_h^ϕ

Before establishing that $D_{h|_U} = D_h^\phi$, we will show that the two metrics are bi-Lipschitz equivalent.

Proposition 3.1. *Let h be a whole-plane GFF, with any choice of additive constant. For each conformal map $\phi : U \to \phi(U)$, there is a constant $C \ge 1$ such that a.s.*

$$C^{-1} D_{h|_U}(z,w) \le D_h^\phi(z,w) \le C D_{h|_U}(z,w), \quad \forall z,w \in U. \tag{3.1}$$

Proposition 3.1 will be a consequence of Proposition 2.1 together with a general criterion for metrics coupled with the same GFF to be bi-Lipschitz equivalent which is proven in [GM19b]. To state the criterion, we need a couple of preliminary definitions.

Definition 3.2 (Jointly local metrics)**.** Let $U \subset \mathbb{C}$ be a connected open set and let $(h, D_1, \ldots, D_n)$ be a coupling of a GFF on U and n random continuous length metrics. We say that $D_1, \ldots, D_n$ are *jointly local metrics* for h if for any open set $V \subset U$, the collection of internal metrics $\{D_j(\cdot,\cdot;V)\}_{j=1,\ldots,n}$ is conditionally independent from $(h|_{U\setminus V}, \{D_j(\cdot,\cdot;U \setminus \overline{V})\}_{j=1,\ldots,n})$ given $h|_V$.

We note that if each of $D_1, \dots, D_n$ is a local metric for h and is determined by h, then $D_1, \dots, D_n$ are jointly local for h. This is a consequence of [GM19b, Lemma 1.4]. In particular, $D_{h|_U}$ and D_h^ϕ are jointly local for $h|_U$.

Definition 3.3 (Additive local metrics)**.** Let $U \subset \mathbb{C}$ be a connected open set and let $(h^U, D_1, \dots, D_n)$ be a coupling of a GFF on U and n random continuous length metrics which are jointly local for h. For $\xi \in \mathbb{R}$, we say that $D_1, \dots, D_n$ are *ξ-additive* for h^U if for each $z \in U$ and each $r > 0$ such that $B_r(z) \subset U$, the metrics $(e^{-\xi h_r^U(z)} D_1, \dots, e^{-\xi h_r^U(z)} D_n)$ are jointly local metrics for $h^U - h_r^U(z)$ (where, as per usual, $h_r^U(z)$ denotes the circle average).

It is clear from Axiom III that the metrics $D_{h|_U}$ and D_h^ϕ are ξ-additive for $h|_U$. The following theorem is [GM19b, Theorem 1.6]. For the statement, we recall the notation for Euclidean annuli from (1.14).

Theorem 3.4. *Let $\xi \in \mathbb{R}$, let h be a whole-plane GFF normalized so that $h_1(0) = 0$, let $U \subset \mathbb{C}$, and let $(h, D, \widetilde{D})$ be a coupling of h with two random continuous metrics on U which are jointly local and ξ-additive for $h|_U$. There is a universal constant $p \in (0,1)$ such that the following is true. Suppose there is a constant $C > 0$ such that for each compact set $K \subset U$, there exists $r_K > 0$ such that*

$$\mathbb{P}\left[\sup_{u,v \in \partial B_r(z)} \widetilde{D}\big(u, v; \mathbb{A}_{r/2,2r}(z)\big) \le C D(\partial B_{r/2}(z), \partial B_r(z))\right] \ge p, \quad \forall z \in K, \quad \forall r \in (0, r_K]. \tag{3.2}$$

Then a.s. $\widetilde{D}(z,w) \le C D(z,w)$ for all $z, w \in \mathbb{C}$.

Let us now check the condition (3.2) for the metrics $D_{h|_U}$ and D_{h^ϕ} using Proposition 2.1.

Lemma 3.5. *Let h be a whole-plane GFF, with any choice of additive constant. For each $p \in (0,1)$, there exists a constant $C = C(p, \gamma) > 0$ such that for each choice of conformal map $\phi : U \to \phi(U)$ and each compact set $K \subset U$, there exists $r_K = r_K(\phi) > 0$ such that*

$$\mathbb{P}\left[\sup_{u,v \in \partial B_r(z)} D_h^\phi\big(u, v; \mathbb{A}_{r/2,2r}(z)\big) \le C D_{h|_U}(\partial B_{r/2}(z), \partial B_r(z))\right] \ge p, \quad \forall z \in K, \quad \forall r \in (0, r_K]; \tag{3.3}$$

and the same is true with D_h^ϕ and D_h interchanged.

Proof. As in the proof of Lemma 2.2, due to Axiom III we can assume without loss of generality that h is normalized so that $h_{\mathbb{f},1}(0) = 0$. By Axioms III and IV$'$, the fact that D_h induces the Euclidean topology, and the scale and translation invariance of the law of h, modulo additive constant, we can find small constants $a, b > 0$ such that for each $z \in \mathbb{C}$ and each $r > 0$, it holds with probability at least $1 - (1-p)/2$ that the following is true.

1. $D_h(\partial B_r(z), \partial \mathbb{A}_{r/2,2r}(z)) \ge a r^{\xi Q} e^{\xi h_{\mathbb{f},r}(z)}$.
2. For each $u, v \in \partial B_r(z)$ with $|u - v| \le br$, we have $D_h(u,v) \le (a/100) r^{\xi Q} e^{\xi h_{\mathbb{f},r}(z)}$.

By Proposition 2.1, for each compact set $K \subset U$ there exists $r_K = r_K(\phi) > 0$ such that for each $z \in K$ and each $r \in (0, r_K]$, it holds with probability at least $1 - (1-p)/2$ that

$$\sup_{u,v \in B_{2r}(z)} |D_h^\phi(u,v) - D_h(u,v)| \le \frac{a}{100} r^{\xi Q} e^{\xi h_{\mathbb{f},r}(z)}. \tag{3.4}$$

Combining these estimates shows that for each $z \in K$ and each $r \in (0, r_K]$, it holds with probability at least p that

1. $D_h^\phi\big(\partial B_r(z), \partial \mathbb{A}_{r/2,2r}(z)\big) \geq (a/2) r^{\xi Q} e^{\xi h_{\mathbb{f},r}(z)}$.

2. For each $u,v \in \partial B_r(z)$ with $|u-v| \leq br$, we have $D_h^\phi(u,v) \leq (a/2) r^{\xi Q} e^{\xi h_{\mathbb{f},r}(z)}$.

If $u,v \in \partial B_r(z)$ such that $D_h(u,v) \leq D_h(\partial B_r(z), \partial \mathbb{A}_{r/2,2r}(z))$, then $D_h(u,v) = D_h(u,v;\mathbb{A}_{r/2,2r}(z))$. By applying condition 2 for D_h and the triangle inequality at most $2\pi/b$ times, we therefore have that

$$\sup_{u,v\in\partial B_r(z)} D_h\big(u,v;\mathbb{A}_{r/2,2r}(z)\big) \leq \frac{a\pi}{50b} r^{\xi Q} e^{\xi h_{\mathbb{f},r}(z)}. \tag{3.5}$$

Similarly, by condition 2 for D_h^ϕ and applying the triangle inequality at most $2\pi/b$ times, we have that

$$\sup_{u,v\in\partial B_r(z)} D_h^\phi\big(u,v;\mathbb{A}_{r/2,2r}(z)\big) \leq \frac{a\pi}{b} r^{\xi Q} e^{\xi h_{\mathbb{f},r}(z)}. \tag{3.6}$$

Combining the preceding two estimates with condition 1 for each of D_h and D_h^ϕ gives the statement of the lemma with $C = \pi/b$. □

Proof of Proposition 3.1. The metrics D_h and D_h^ϕ are each local metrics for $h|_U$. Moreover, these metrics are determined by $h|_U$ so they are jointly local for $h|_U$. By Axiom III, for $z\in\mathbb{C}$ and $r>0$, the metrics $e^{-\xi h_r(z)} D_h$ and $e^{-\xi h_r(z)} D_h^\phi$ are each local for $h|_U - h_r(z)$, so in particular these metrics are ξ-additive for $h|_U$. Therefore, the proposition statement follows from Lemma 3.5 and Theorem 3.4. □

3.2 The bi-Lipschitz constant is 1

We will now show that in fact the constant C in Proposition 3.1 can be taken to be one.

Proposition 3.6. *Let h be a whole-plane GFF (with any choice of normalization). Let $U\subset\mathbb{C}$ be an open domain and let $\phi: U\to\phi(U)$ be a conformal map. Almost surely, we have $D_h^\phi(z,w) \leq D_{h|_U}(z,w)$ for each $z,w\in U$.*

Before proving Proposition 3.6, we explain why it implies our main result.

Proof of Theorem 1.3, assuming Proposition 3.6. For a whole-plane GFF h and a domain $U\subset\mathbb{C}$, we can write $h|_U$ as the sum of a zero-boundary GFF on U and an independent random harmonic function on U. Therefore, if h^U is a random distribution on U as in Theorem 1.3, then we can couple h^U with h in such a way that $g := h|_U - h^U$ is a random continuous function on U. By Proposition 3.6 and Axiom III, it follows that a.s.

$$\begin{aligned} D_{h^U}(z,w) = (e^{\xi g}\cdot D_{h|_U})(z,w) &\geq (e^{\xi g}\cdot D_h^\phi)(z,w) \\ &= (e^{\xi(g\circ\phi^{-1})}\cdot D_{h^\phi})(\phi(z),\phi(w)) \\ &= D_{h^U\circ\phi^{-1} + Q\log|(\phi^{-1})'|}(\phi(z),\phi(w)), \quad \forall z,w\in U. \end{aligned}$$

This is a one-sided version of (1.11). By the conformal invariance of the law of the zero-boundary GFF, we can apply this one-sided statement with $h^U\circ\phi^{-1} + Q\log|(\phi^{-1})'|$ in place of h and ϕ^{-1} in place of ϕ to get the opposite inequality in (1.11). □

The proof of Proposition 3.6 is similar to the proof of [GM19a, Proposition 3.6]. Fix a small $\delta\in(0,1)$ (which we will eventually send to zero) and a parameter $\alpha\in(1/2,1)$, a little bit less than 1. We will use Lemma 2.2 together with a general independence result for events for the GFF restricted to annuli (Lemma 3.7 just below) to cover a given compact set $K \subset U$ by Euclidean balls of the form $B_{r/2}(z)$ such that $D_h^\phi(u,v) \le (1+\delta) D_{h|_U}(u,v)$ for each $u \in \partial B_{\alpha r}(z)$ and each $v \in \partial B_r(z)$ which can be joined by a $D_{h|_U}$-geodesic contained in $\overline{\mathbb{A}_{\alpha r,r}(z)}$. If we assume that $D_h^\phi \le C D_{h|_U}$ for some $C > 1$, then by considering the times when a $D_{h|_U}$-geodesic between two fixed points $\mathbb{z}, \mathbb{w} \in \mathbb{C}$ crosses the annulus $\mathbb{A}_{\alpha r,r}(z)$ for such a z and r, we will be able to show that $D_h^\phi(\mathbb{z},\mathbb{w}) \le C_\delta D_{h|_U}(\mathbb{z},\mathbb{w})$ for a constant C_δ which is strictly smaller than C if δ is chosen to be sufficiently small. This shows that one has to have $D_h^\phi \le C D_{h|_U}$ for $C = 1$, since if the optimal constant for which this holds is strictly bigger than 1, then this optimal constant can be improved, which is a contradiction.

The following annulus iteration lemma, which is [GM19b, Lemma 3.1], a generalization of a result from [MQ20], will be used to produce the desired covering by balls of the form $B_{r/2}(z)$.

Lemma 3.7. *Fix $0 < s_1 < s_2 < 1$. Let $\{r_k\}_{k\in\mathbb{N}}$ be a decreasing sequence of positive real numbers such that $r_{k+1}/r_k \le s_1$ for each $k \in \mathbb{N}$ and let $\{E_{r_k}\}_{k\in\mathbb{N}}$ be events such that $E_{r_k} \in \sigma\Big((h - h_{r_k}(0))|_{\mathbb{A}_{s_1 r_k, s_2 r_k}(0)}\Big)$ for each $k \in \mathbb{N}$. For $K \in \mathbb{N}$, let $N(K)$ be the number of $k \in [1,K]_{\mathbb{Z}}$ for which E_{r_k} occurs.*

For each $a > 0$ and each $b \in (0,1)$, there exists $p = p(a,b,s_1,s_2) \in (0,1)$ and $c = c(a,b,s_1,s_2) > 0$ such that if

$$\mathbb{P}[E_{r_k}] \ge p, \quad \forall k \in \mathbb{N}, \tag{3.7}$$

then

$$\mathbb{P}[N(K) < bK] \le c e^{-aK}, \quad \forall K \in \mathbb{N}. \tag{3.8}$$

Let us now define the events to which we will apply Lemma 3.7. For $z \in U$, $r > 0$ such that $B_r(z) \subset U$, and parameters $\alpha \in (1/2,1)$, $A > 1$, and $\delta \in (0,1)$ and let $E_r(z) = E_r(z;\alpha,A,\delta)$ be the event that the following is true.

1. For each $u \in \partial B_{\alpha r}(z)$ and each $v \in \partial B_r(z)$ such that there is a D_h-geodesic from u to v which is contained in $\overline{\mathbb{A}_{\alpha r,r}(z)}$, we have $D_h^\phi(u,v) \le (1+\delta) D_h(u,v)$.

2. If $u \in \partial B_{\alpha r}(z)$ and $v \in \partial B_r(z)$ such that either $D_h(u,v) > D_h(u, \partial \mathbb{A}_{r/2,2r}(z))$ or $D_h^\phi(u,v) > D_h^\phi(u, \partial\mathbb{A}_{r/2,2r}(z))$, then each path from u to v which stays in $\overline{\mathbb{A}_{\alpha r,r}(z)}$ has D_h-length strictly larger than $D_h\big(u,v;\mathbb{A}_{r/2,2r}(z)\big)$ (see Figure 1 for an illustration).

3. There is a path in $\mathbb{A}_{\alpha r,r}(z)$ which disconnects the inner and outer boundaries of $\mathbb{A}_{\alpha r,r}(z)$ and has D_h-length at most $A D_h(\partial B_{\alpha r}(z), \partial B_r(z))$.

Condition 1 is the main point of the event $E_r(z)$, as discussed just above. The purpose of condition 2 is to ensure that $E_r(z)$ is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$. Indeed, as we will see in the proof of Lemma 3.8 just below, on the event that this condition is satisfied $h|_{\mathbb{A}_{r/2,2r}(z)}$ determines which paths in $\overline{\mathbb{A}_{\alpha r,r}(z)}$ are D_h-geodesics. The purpose of condition 3 is to ensure that the annuli $\mathbb{A}_{\alpha r,r}(z)$ for which $E_r(z)$ occurs (rather than just the balls $B_r(z)$ for which $E_r(z)$ occurs) cover a positive fraction of the D_h-length of a D_h-geodesic. Indeed, if a D_h-geodesic between two points outside of $B_r(z)$ enters $B_{\alpha r}(z)$, then it must cross the path from condition 3 twice. Since geodesics are length minimizing, this means that it can spend at most $A D_h(\partial B_{\alpha r}(z), \partial B_r(z))$ units of time in $B_{\alpha r}(z)$: otherwise the path from condition 3 would provide a shortcut.

We want to use Lemma 3.7 to argue that with high probability we can cover any given compact subset of U by balls $B_{r/2}(z)$ for which $E_r(z)$ occurs. We first check the measurability condition in Lemma 3.7

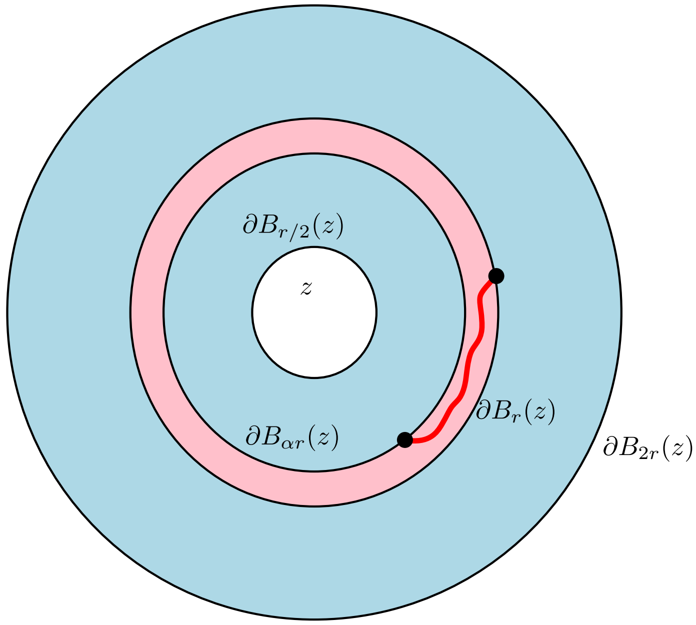

Figure 1: Illustration of condition 2 in the definition of $E_r(z)$. Suppose that $u \in \partial B_{\alpha r}(z)$ and $v \in \partial B_r(z)$ are "far apart" in the sense that v is further from u than the boundary of the large blue annulus $\mathbb{A}_{r/2,r}(z)$, w.r.t. either D_h or D_h^ϕ. Then condition 2 says that any path from u to v which is contained in $\overline{\mathbb{A}_{\alpha r,r}(z)}$ (such as the red path in the figure) has to have D_h-length strictly larger than $D_h(u,v;\mathbb{A}_{r/2,2r}(z))$, so in particular such a path cannot be a D_h-geodesic. To obtain that this condition holds with high probability, we will choose α to be close to 1 and use the fact paths which are "close" to a circular arc have large D_h-lengths (see Lemma 3.10).

Lemma 3.8. *For each $z \in \mathbb{C}$ and $r > 0$,*

$$E_r(z) \in \sigma\Big((h - h_{4r}(z))|_{\mathbb{A}_{r/2,2r}(z)}\Big). \tag{3.9}$$

Proof. By Axiom III subtracting $h_{4r}(0)$ from h results in scaling each of D_h and D_h^ϕ by $e^{\xi h_{4r}(0)}$, so does not affect the occurrence of $E_r(z)$. Hence it suffices to show that $E_r(z) \in \sigma(h|_{\mathbb{A}_{r/2,2r}(z)})$. By Axiom II, condition 3 in the definition of $E_r(z)$ is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$.

For $u \in \partial B_{\alpha r}(z)$ and $v \in \partial B_r(z)$, we can determine whether $D_h(u,v) > D_h(u,\partial\mathbb{A}_{r/2,2r}(z))$ from the internal metric $D_h\big(\cdot,\cdot;\mathbb{A}_{r/2,2r}(z)\big)$: indeed, $D_h(u,\partial\mathbb{A}_{r/2,2r}(z))$ is clearly determined by this internal metric and $D_h(u,v) \le D_h(u,\partial\mathbb{A}_{r/2,2r}(z))$ if and only if v is contained in the D_h-ball of radius $D_h(u,\partial\mathbb{A}_{r/2,2r}(z))$ centered at v, which is contained in $\overline{\mathbb{A}_{r/2,2r}(z)}$. Similar considerations hold with D_h^ϕ in place of D_h. By the locality of the metrics D_h and D_h^ϕ, it follows that condition 2 in the definition of $E_r(z)$ is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$.

If P is a path from $u \in \partial B_{\alpha r}(z)$ to $v \in \partial B_r(z)$ which stays in $\overline{\mathbb{A}_{\alpha r,r}(z)}$, then P is a D_h-geodesic if and only if $\mathrm{len}(P;D_h) = D_h(u,v)$. Therefore, if condition 2 holds, then in order for P to be a D_h-geodesic we must have $D_h(u,v) \le D_h(u,\partial\mathbb{A}_{r/2,2r}(z))$ and $D_h^\phi(u,v) \le D_h^\phi(u,\partial\mathbb{A}_{r/2,2r}(z))$ (note that $D_h(u,v;\mathbb{A}_{r/2,2r}(z)) \ge D_h(u,v)$). If this is the case, then we can tell whether P is a D_h-geodesic from the restriction of h to the D_h-metric ball of radius $D_h(u,\partial\mathbb{A}_{r/2,2r}(z))$ centered at u. We know this restriction is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$ by Axiom II.

On the event that $D_h(u,v) \le D_h(u,\partial\mathbb{A}_{r/2,2r}(z))$ and $D_h^\phi(u,v) \le D_h^\phi(u,\partial\mathbb{A}_{r/2,2r}(z))$, both

$D_h(u,v)$ and $D_h^\phi(u,v)$ are determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$. Therefore, the intersection of conditions 1 and 2 in the definition of $E_r(z)$ is determined by $h|_{\mathbb{A}_{r/2,2r}(z)}$. Hence we have proven (3.9). ☐

We now use Lemma 2.2 to prove a lower bound for the probability that $E_r(z)$ occurs for at least one small value of r.

Lemma 3.9. *For each $q>1$, there exist parameters $\alpha\in(1/2,1)$ and $A>1$, depending only on q, such that for each compact set $K\subset U$ and each $\delta\in(0,1)$, we have*

$$\inf_{z\in K}\mathbb{P}\Big[E_r(z)\text{ occurs for at least one } r\in[\varepsilon^2,\varepsilon]\cap\{2^{-k}:k\in\mathbb{N}\}\Big]\geq 1-O_\varepsilon(\varepsilon^q). \tag{3.10}$$

To deal with condition 2 in the definition of $E_r(z)$, we will use the following lower bound for the D_h-lengths of paths in a narrow Euclidean annulus, which is [GM19a, Lemma 2.11] (note that the number $\mathfrak{c}_r$ from [GM19a] is equal to $r^{\xi Q}$, see [GM19a, Section 1.4]).

Lemma 3.10 ([GM19a])**.** *For each $S>s>0$ and each $p\in(0,1)$, there exists $\alpha_*=\alpha_*(s,S,p)\in(1/2,1)$ such that for each $\alpha\in[\alpha_*,1)$, each $z\in\mathbb{C}$, and each $r>0$,*

$$\mathbb{P}\Big[\inf\Big\{D_h(u,v;\mathbb{A}_{\alpha r,r}(z)):u,v\in\mathbb{A}_{\alpha r,r}(z),D_h(u,v)\geq sr^{\xi Q}e^{\xi h_r(z)}\Big\}\geq Sr^{\xi Q}e^{\xi h_r(z)}\Big]\geq p. \tag{3.11}$$

Proof of Lemma 3.9. By Lemma 3.8, we can apply Lemma 3.7 to find that there exists $p=p(q)\in(0,1)$ such that if (3.12) just below holds, then (3.10) holds:

$$\exists r_0=r_0(K,\delta)>0\text{ such that }\mathbb{P}[E_r(z)]\geq p,\quad\forall z\in K,\quad\forall r\in(0,r_0]. \tag{3.12}$$

It therefore suffices to choose α and A in a manner depending only on p in such a way that (3.12) holds.

We first deal with condition 2. By Axioms III and IV′, the fact that D_h induces the Euclidean topology, and the scale and translation invariance of the law of h, modulo additive constant, we can find $S>s>0$ depending on p such that for each sufficiently small $r>0$ (depending only on K), for each $z\in K$ it holds with probability at least $1-(1-p)/4$ that

$$D_h\big(\partial\mathbb{A}_{3r/4,r}(z),\partial\mathbb{A}_{r/2,2r}(z)\big)\geq sr^{\xi Q}e^{\xi h_r(z)}\quad\text{and}\quad\sup_{u,v\in\mathbb{A}_{3r/4,r}(z)}D_h\big(u,v;\mathbb{A}_{r/2,2r}(z)\big)\leq Sr^{\xi Q}e^{\xi h_r(z)}. \tag{3.13}$$

Since ϕ is nearly linear at small scales, after possibly decreasing s and increasing S we can arrange that the same is true with D_h^ϕ in place of D_h. Since $\mathbb{A}_{\alpha r,r}(z)\subset\mathbb{A}_{3r/4,r}(z)$ for any choice of $\alpha\in[3/4,1)$, Lemma 3.10 with the above choice of s and S gives an $\alpha\in[3/4,1)$ depending on p such that for each sufficiently small $r>0$, it holds for each $z\in K$ that the probability of condition 2 in the definition of $E_r(z)$ is at least $1-(1-p)/3$.

By applying Axioms III and IV′ as above, we can find $A>1$ depending on p such that for each sufficiently small $r>0$, it holds for each $z\in K$ that the probability of condition 3 in the definition of $E_r(z)$ is at least $1-(1-p)/3$.

By Lemma 2.2 applied with $b=1-\alpha$, for each sufficiently small $r>0$, it holds for each $z\in K$ that the probability of condition 1 in the definition of $E_r(z)$ is at least $1-(1-p)/3$. Combining the three preceding paragraphs shows that (3.12) holds. ☐

Lemma 3.11. *There is a universal constant $q>1$ such that if α and A are chosen as in Lemma 3.9 for this choice of q, then for each compact set $K\subset U$ and each $\delta\in(0,1)$, it holds with probability tending to 1 as $\varepsilon\to0$ that the following is true. For each $z\in K$, there exists $r\in[\varepsilon^2,\varepsilon]\cap\{2^{-k}:k\in\mathbb{N}\}$ and $w\in\left(\frac{\varepsilon^2}{4}\mathbb{Z}^2\right)\cap B_\varepsilon(K)$ such that $z\in B_{r/2}(w)$ and $E_r(w)$ occurs.*

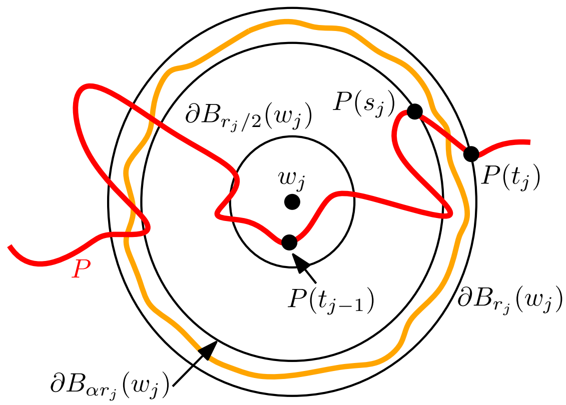

Figure 2: Illustration of the proof of Proposition 3.6. The D_h-geodesic P from $\mathbb{z}$ to $\mathbb{w}$ along with one of the balls $B_{r_j}(w_j)$ hit by P for which $E_{r_j}(w_j)$ occurs are shown. The time t_j is the first time after t_{j-1} at which P exits $B_{r_j}(w_j)$ and the time s_j is the last time before t_j at which P hits $\partial B_{\alpha r_j}(w_j)$. Condition 1 in the definition of $E_{r_j}(w_j)$ tells us that $D_h^\phi(P(s_j), P(t_j)) \leq (1+\delta)(t_j - s_j)$. The orange path comes from condition 3 in the definition of $E_{r_j}(w_j)$. It has D_h-length is at most $AD_h(\partial B_{\alpha r_j}(w_j), \partial B_{r_j}(w_j)) \leq A(t_j - s_j)$. Since P is a D_h-geodesic and P crosses this orange path both before time t_{j-1} and after time s_j, it follows that $s_j - t_{j-1} \leq A(t_j - s_j)$. This allows us to show that the "good" intervals $[s_j, t_j]$ occupy a uniformly positive fraction of the total D_h-length of P. This then allows us to show that $D_h^\phi(\mathbb{z}, \mathbb{w}) \leq C_\delta D_{h|_U}(\mathbb{z}, \mathbb{w})$ for a constant $C_\delta > 0$ which is strictly smaller than C_* if we assume that $C_* > 1$ and δ is chosen to be sufficiently small.

Proof. Upon choosing q sufficiently large, this follows from Lemma 3.9 and a union bound over all $w \in \left(\frac{\varepsilon^2}{4}\mathbb{Z}^2\right) \cap B_\varepsilon(K)$. ☐

Proof of Proposition 3.6. See Figure 2 for an illustration of the proof.

Step 1: setup. Let α and A be chosen as in Lemma 3.11. Also let

$$C_* := \inf\left\{ C > 1 : \mathbb{P}\left[\sup_{z,w \in U, z \neq w} \frac{D_h^\phi(z,w)}{D_{h|_U}(z,w)} \leq C \right] = 1 \right\}. \tag{3.14}$$

Proposition 3.1 implies that $C_* < \infty$. We want to show that $C_* \leq 1$.

To this end, we will show that a.s.

$$D_h^\phi(\mathbb{z}, \mathbb{w}) \leq C_\delta D_{h|_U}(\mathbb{z}, \mathbb{w}), \quad \forall \mathbb{z}, \mathbb{w} \in U, \quad \text{where} \quad C_\delta := 1 + \delta + \frac{A}{A+1}(C_* - 1 - \delta). \tag{3.15}$$

If $C_* > 1$ and $\delta > 0$ is chosen sufficiently small (depending on C_* and A), then $C_\delta < C_*$. This contradicts the definition of C_*, so we infer that $C_* \leq 1$. It remains only to prove (3.15).

Step 2: regularity event. The idea of the proof of (3.15) is to use Lemma 3.11 and conditions 1 and 3 in the definition of $E_r(z)$ to show that if $P : [0, D_h(\mathbb{z}, \mathbb{w})] \to \mathbb{C}$ is a D_h-geodesic, then with high probability the following is true. We can cover a $1 - \frac{A}{A+1}$-fraction of the interval $[0, D_h(\mathbb{z}, \mathbb{w})]$ by intervals of the form $[s,t]$ such that $D_h^\phi(P(s), P(t)) \leq (1+\delta)(t-s)$. In order for the comparison of D_h-lengths and D_h^ϕ-lengths to make sense, we need to make sure that our D_h-geodesic P stays in U. We now introduce a regularity event on which we can force certain D_h-geodesics to stay in U.

Fix a compact set $K \subset U$ and let $\zeta \in (0,1)$ be a small parameter which we will eventually send to zero. By the continuity of D_h, we can find a small parameter $\rho \in (0,1)$ and a compact set K' satisfying $K \subset K' \subset U$, depending on K and ζ, such that with probability at least $1-\zeta$, we have

$$D_h(\mathbb{z}, \mathbb{w}) \leq D_h(\mathbb{z}, \mathbb{C} \setminus K'), \quad \forall \mathbb{z}, \mathbb{w} \in K \quad \text{with} \quad |\mathbb{z} - \mathbb{w}| \leq \rho. \tag{3.16}$$

If (3.16) holds, then $D_h(\mathbb{z}, \mathbb{w}) = D_{h|_U}(\mathbb{z}, \mathbb{w})$ for each pair of points $\mathbb{z}, \mathbb{w} \in K$ with $|\mathbb{z} - \mathbb{w}| \leq \rho$ and moreover every D_h-geodesic between two such points is contained in K', so in particular is also a $D_{h|_U}$-geodesic.

For $\varepsilon > 0$, let $F^\varepsilon_{K'}$ be the event that (3.16) holds and the event of Lemma 3.11 occurs with the above choices of α, A, δ and with K' in place of K, so that $\mathbb{P}[F^\varepsilon_{K'}] \geq 1 - \zeta - o_\varepsilon(1)$.

Step 3: reducing to an estimate for nearby points. We claim that on $F^\varepsilon_{K'}$, it is a.s. the case that

$$D_h^\phi(\mathbb{z}, \mathbb{w}) \leq C_\delta D_{h|_U}(\mathbb{z}, \mathbb{w}) + o_\varepsilon(1), \quad \forall \mathbb{z}, \mathbb{w} \in K \quad \text{with} \quad |\mathbb{z} - \mathbb{w}| \leq \rho, \tag{3.17}$$

where the $o_\varepsilon(1)$ is a random error which tends to zero in probability as $\varepsilon \to 0$, uniformly over all $\mathbb{z}, \mathbb{w} \in K$. Before proving (3.17), we explain why it implies (3.15). Applying (3.17) and sending $\varepsilon \to 0$ shows that with probability at least $1 - \zeta$, we have $D_h^\phi(\mathbb{z}, \mathbb{w}) \leq C_\delta D_{h|_U}(\mathbb{z}, \mathbb{w})$ for each $\mathbb{z}, \mathbb{w} \in K$ with $|\mathbb{z} - \mathbb{w}| \leq \rho$. This implies that with probability least $1 - \zeta$, the D_h^ϕ-length of any path contained in K is at most C_δ times its $D_{h|_U}$-length. Since $D_{h|_U}$ and D_h^ϕ are length metrics, sending $\zeta \to 0$ and letting K increase to all of U gives (3.15).

Step 4: decomposition of a D_h-geodesic into segments. Assume that $F^\varepsilon_{K'}$ occurs, let $\mathbb{z}, \mathbb{w} \in K$ with $|\mathbb{z} - \mathbb{w}| \leq \rho$, and let P be a D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. As noted after (3.16), we have $P \subset K'$. We will define several objects which depend on P and ε, but to lighten notation we will not make P and ε explicit in the notation. See Figure 2 for an illustration of the definitions.

Let $t_0 = 0$ and inductively let t_j for $j \in \mathbb{N}$ be the smallest time $t \geq t_{j-1}$ at which P exits a Euclidean ball of the form $B_r(w)$ for $w \in \left(\frac{\varepsilon^2}{4}\mathbb{Z}^2\right) \cap B_\varepsilon(K)$ and $r \in [\varepsilon^2, \varepsilon] \cap \{2^{-k} : k \in \mathbb{N}\}$ such that $P(t_{j-1}) \in B_{r/2}(w)$ and $E_r(w)$ occurs; or let $t_j = D_h(\mathbb{z}, \mathbb{w})$ if no such t exists. If $t_j < D_h(\mathbb{z}, \mathbb{w})$, let w_j and r_j be the corresponding values of w and r. Also let s_j be the last time before t_j at which P exits $B_{\alpha r_j}(w)$. Note that $s_j \in [t_{j-1}, t_j]$ and $P([s_j, t_j]) \subset \overline{\mathbb{A}_{\alpha r_j, r_j}(w_j)}$.

Define the indices

$$\underline{J} := \max\{j \in \mathbb{N} : |\mathbb{z} - P(t_{j-1})| < 2\varepsilon\} \quad \text{and} \quad \overline{J} := \min\{j \in \mathbb{N} : |\mathbb{w} - P(t_{j+1})| < 2\varepsilon\}. \tag{3.18}$$

Since $r_j \leq \varepsilon$ and $P(t_j) \in B_{r_j}(w_j)$ for each j, we have $\mathbb{z}, \mathbb{w} \notin B_{r_j}(w_j)$ for $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$. By the definition of $F^\varepsilon_{K'}$, on this event we have $t_j < D_h(\mathbb{z}, \mathbb{w})$ and $|P(t_{j-1}) - P(t_j)| \leq 2\varepsilon$ whenever $|\mathbb{w} - P(t_{j-1})| \geq \varepsilon$. Therefore, on $F^\varepsilon_{K'}$,

$$P(t_{\underline{J}}) \in B_{4\varepsilon}(\mathbb{z}) \quad \text{and} \quad P(t_{\overline{J}}) \in B_{4\varepsilon}(\mathbb{w}). \tag{3.19}$$

Since P is a D_h-geodesic, for $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$ also $P|_{[s_j, t_j]}$ is a D_h-geodesic from $P(s_j) \in \partial B_{\alpha r_j}(w_j)$ to $P(t_j) \in \partial B_{r_j}(w_j)$. By definition, this D_h-geodesic stays in $\overline{\mathbb{A}_{\alpha r_j, r_j}(w_j)}$. Combining this with condition 1 in the definition of $E_{r_j}(w_j)$ (applied with $u = P(s_j)$ and $v = P(t_j)$) and the definition (3.14) of C_*, we obtain

$$D_h^\phi(P(s_j), P(t_j)) \leq (1+\delta)(t_j - s_j) \quad \text{and} \quad D_h^\phi(P(t_{j-1}), P(s_j)) \leq C_*(s_j - t_{j-1}), \quad \forall j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}. \tag{3.20}$$

Step 5: comparing $s_j - t_{j-1}$ to $t_j - s_j$. In order to extract a non-trivial upper bound for $D_h^\phi(\mathbb{z},\mathbb{w})$ from (3.20), we need to show that the "good" intervals $[s_j,t_j]$ occupy a positive fraction of the total length of the time interval $[0, D_h(\mathbb{z},\mathbb{w})]$. To this end, we will now argue that $s_j - t_{j-1}$ is not too much larger than $t_j - s_j$.

If $j \in [\underline{J}, \overline{J}]_{\mathbb{Z}}$, then since $r_j \leq \varepsilon$ and $|P(t_j) - \mathbb{z}| \wedge |P(t_j) - \mathbb{w}| \geq 2\varepsilon$, the geodesic P must cross the annulus $\mathbb{A}_{\alpha r_j, r_j}(w_j)$ at least once before time t_{j-1} and at least once after time s_j. By the definition of $E_{r_j}(w_j)$, there is a path disconnecting the inner and outer boundaries of this annulus with D_h-length at most $AD_h\big(\partial B_{\alpha r_j}(w_j), \partial B_{r_j}(w_j)\big)$. The geodesic P must hit this path at least once before time t_{j-1} and at least once after time s_j. Since P is a geodesic and $P(s_j) \in \partial B_{\alpha r_j}(w_j)$, $P(t_j) \in \partial B_{r_j}(w_j)$, it follows that

$$s_j - t_{j-1} \leq AD_h\big(\partial B_{\alpha r_j}(w_j), \partial B_{r_j}(w_j)\big) \leq A(t_j - s_j).$$

Adding $A(s_j - t_{j-1})$ to both sides of this inequality, then dividing by $A+1$, gives

$$s_j - t_{j-1} \leq \frac{A}{A+1}(t_j - t_{j-1}). \tag{3.21}$$

Step 6: upper bound for D_h^ϕ. As above, we assume that $F_{K'}^\varepsilon$ occurs, we let $\mathbb{z},\mathbb{w} \in K$ with $|\mathbb{z}-\mathbb{w}| \leq \rho$ and we let P be the D_h-geodesic from $\mathbb{z}$ to $\mathbb{w}$. In the notation above, it holds for each $j \in [\underline{J}+1, \overline{J}]_{\mathbb{Z}}$ that

$$\begin{aligned} D_h^\phi(P(t_{j-1}), P(t_j)) &\leq D_h^\phi(P(t_{j-1}), P(s_j)) + D_h^\phi(P(s_j), P(t_j)) \quad \text{(triangle inequality)} \\ &\leq C_*(s_j - t_{j-1}) + (1+\delta)(t_j - s_j) \quad \text{(by (3.20))} \\ &= (1+\delta)(t_j - t_{j-1}) + (C_* - 1 - \delta)(s_j - t_{j-1}) \\ &= C_\delta(t_j - t_{j-1}) \quad \text{(by (3.21) and the definition of } C_\delta). \end{aligned} \tag{3.22}$$

We now apply (3.19) and sum the estimate (3.22) to get

$$\begin{aligned} D_h^\phi(B_{4\varepsilon\mathrm{r}}(\mathbb{z}), B_{4\varepsilon\mathrm{r}}(\mathbb{w})) &\leq D_h^\phi\big(P(t_{\underline{J}}), P(t_{\overline{J}})\big) \quad \text{(by (3.19))} \\ &\leq \sum_{j=\underline{J}+1}^{\overline{J}} D_h^\phi(P(t_{j-1}), P(t_j)) \\ &\leq C_\delta(t_{\overline{J}} - t_{\underline{J}}) \quad \text{(by (3.22))} \\ &\leq C_\delta D_{h|_U}(\mathbb{z},\mathbb{w}) \quad \text{(since } P \text{ is a } D_{h|_U}\text{-geodesic)}. \end{aligned} \tag{3.23}$$

By the continuity of $(z,w) \mapsto D_h^\phi(z,w)$ and the triangle inequality, a.s.

$$D_h^\phi(\mathbb{z},\mathbb{w}) \leq D_h^\phi(B_{4\varepsilon\mathrm{r}}(\mathbb{z}), B_{4\varepsilon\mathrm{r}}(\mathbb{w})) + o_\varepsilon(1) \tag{3.24}$$

where the $o_\varepsilon(1)$ tends to 0 in probability as $\varepsilon \to 0$, uniformly over all $\mathbb{z},\mathbb{w} \in K$. Combining this with (3.23) gives (3.17). □

References

[Ang19] M. Ang. Comparison of discrete and continuum Liouville first passage percolation. *Electron. Commun. Probab.*, 24:Paper No. 64, 12, 2019, 1904.09285. MR4029433

[DD19] J. Ding and A. Dunlap. Liouville first-passage percolation: Subsequential scaling limits at high temperature. *Ann. Probab.*, 47(2):690–742, 2019, 1605.04011. MR3916932

[DD20] J. Ding and A. Dunlap. Subsequential scaling limits for Liouville graph distance. *Comm. Math. Phys.*, 376(2):1499–1572, 2020, 1812.06921. MR4103974

[DDDF19] J. Ding, J. Dubédat, A. Dunlap, and H. Falconet. Tightness of Liouville first passage percolation for $\gamma \in (0, 2)$. *ArXiv e-prints*, Apr 2019, 1904.08021.

[DF20] J. Dubédat and H. Falconet. Liouville metric of star-scale invariant fields: tails and Weyl scaling. *Probab. Theory Related Fields*, 176(1-2):293–352, 2020, 1809.02607. MR4055191

[DFG$^+$20] J. Dubédat, H. Falconet, E. Gwynne, J. Pfeffer, and X. Sun. Weak LQG metrics and Liouville first passage percolation. *Probability Theory and Related Fields*, 178(1):369–436, 2020, 1905.00380.

[DG18] J. Ding and E. Gwynne. The fractal dimension of Liouville quantum gravity: universality, monotonicity, and bounds. *Communications in Mathematical Physics*, 374:1877–1934, 2018, 1807.01072.

[DS11] B. Duplantier and S. Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011, 1206.0212. MR2819163 (2012f:81251)

[DZZ19] J. Ding, O. Zeitouni, and F. Zhang. Heat kernel for Liouville Brownian motion and Liouville graph distance. *Comm. Math. Phys.*, 371(2):561–618, 2019, 1807.00422. MR4019914

[GM19a] E. Gwynne and J. Miller. Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$. *Inventiones Mathematicae*, to appear, 2019, 1905.00383.

[GM19b] E. Gwynne and J. Miller. Local metrics of the Gaussian free field. *Annales de l'Institut Fourier*, to appear, 2019, 1905.00379.

[GM20] E. Gwynne and J. Miller. Confluence of geodesics in Liouville quantum gravity for $\gamma \in (0, 2)$. *Ann. Probab.*, 48(4):1861–1901, 2020, 1905.00381. MR4124527

[GP19a] E. Gwynne and J. Pfeffer. Bounds for distances and geodesic dimension in Liouville first passage percolation. *Electronic Communications in Probability*, 24:no. 56, 12, 2019, 1903.09561.

[GP19b] E. Gwynne and J. Pfeffer. KPZ formulas for the Liouville quantum gravity metric. *Transactions of the American Mathematical Society*, to appear, 2019.

[Kah85] J.-P. Kahane. Sur le chaos multiplicatif. *Ann. Sci. Math. Québec*, 9(2):105–150, 1985. MR829798 (88h:60099a)

[Le 13] J.-F. Le Gall. Uniqueness and universality of the Brownian map. *Ann. Probab.*, 41(4):2880–2960, 2013, 1105.4842. MR3112934

[Mie13] G. Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.*, 210(2):319–401, 2013, 1104.1606. MR3070569

[MQ20] J. Miller and W. Qian. The geodesics in Liouville quantum gravity are not Schramm-Loewner evolutions. *Probab. Theory Related Fields*, 177(3-4):677–709, 2020, 1812.03913.

[MS16a] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding. *ArXiv e-prints*, May 2016, 1605.03563.

[MS16b] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map III: the conformal structure is determined. *ArXiv e-prints*, August 2016, 1608.05391.

[MS17] J. Miller and S. Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields*, 169(3-4):729–869, 2017, 1302.4738. MR3719057

[MS20] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map I: the QLE$(8/3, 0)$ metric. *Invent. Math.*, 219(1):75–152, 2020, 1507.00719. MR4050102

[RV14] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and applications: A review. *Probab. Surv.*, 11:315–392, 2014, 1305.6221. MR3274356

[She16] S. Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.*, 44(5):3474–3545, 2016, 1012.4797. MR3551203