George Spencer-Brown's laws of form fifty years on: why we should be giving it more attention in mathematics education

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Abstract: *George Spencer-Brown's Laws of Forms was first published in 1969. In the fifty years since its publication, it has influenced mathematicians, scientists, philosophers, and sociologists. Its influence on mathematics education has been negligible. In this paper, I present a brief introduction to the theory and its philosophical underpinnings. And I set out an argument why Laws of Form should be given more attention in mathematics education.*

Introduction

Last year (2019) marked the 50th anniversary of the publication of *Laws of Form (LoF),* written by George Spencer-Brown (1969). Bertrand Russell described *Laws of Form* as, "a calculus of great power and simplicity. Not since Euclid's Elements have we seen anything like it" (Homes, 2016). Heinz von Foerster, pioneer of second order cybernetics, offered no reserve in his praise for Spencer-Brown's *Laws of Form*.

> The laws of form have finally been written! With a "Spencer-Brown" transistorized power razor (a Twentieth Century model of Occam's razor) G. Spencer-Brown cuts smoothly through two millennia of growth of the most prolific and persistent of semantic weeds, presenting us with his superbly written *Laws of Form*. This Herculean task which now, in retrospect, is of profound simplicity rests on his discovery of the form of laws. Laws are not descriptions, they are commands, injunctions: "Do!" Thus the first constructive proposition in this book (page 3) is the injunction: "Draw a distinction!" an exhortation to perform the primordial creative act (von Foerster, 1971, p. 12).

Despite such positive endorsements, the book itself took ten years to write and get published. It was rejected by Longman who published Spencer-Brown's earlier work on probability. Unwin refused to publish it until Bertrand Russell told them they should and the first printing sold out before it reached the shops (Spencer-Brown, 2008). The reception then and subsequently was not always so encouraging Williams (2019) recounts how the journalist George Goodman gave a copy *of Laws of Form* to some of his mathematician friends at the Institute for Advanced Study in Princeton. It was a "'nice exercise in Boolean algebra,' they responded, 'but what was all this about changing consciousness?'" (Smith, 1975, p. 297 cited in Williams, 2019, p. 8). Banaschewski (1977) argued that *Laws of Form* present a primary algebra that is nothing more than new notation for Boolean algebra.

While Spencer-Brown's work has influenced thinking in a number of fields and disciplines including cybernetics, physics, mathematics, sociology, philosophy, biology and computer science, with people like Heinz von Foerster, Louis Kauffman, Niklas Luhmann, Humberto Maturana, Francisco Varela and William Bricken, citing Spencer-Brown's work as either an epistemological foundation for their theories or that they have interpreted and developed Spencer-Brown's algebra. Yet, while von Glasersfeld (1995) refers to Spencer-Brown in *Radical Constructivism* and *Laws of Form* is cited by Mason (2002) in the *Researching your Own Practice: The Discipline of Noticing*, this theory appears to have had little influence on thinking in mathematics education. The purpose of this paper is to provide an overview of the mathematics and underpinning philosophy in *Laws of Form* and finally, I begin to consider the possible implications and uses of *Laws of Form* in the mathematics classroom.

George Spencer-Brown

Spencer-Brown[12][13] (1923-2016) was born in Grimsby, Lincolnshire, England. He served in the Royal Navy as a radio operator and communications engineer during the Second World War

[12] R. John Williams (2019) notes the difficulty in citing Spencer Brown's name. Sometimes he is referred to as G. Spencer-Brown or George Spencer Brown, this is because Spencer Brown himself altered his name and also went by the names 'James', 'David' 'Maxwell' and others. In this paper, I will use Spencer Brown.

[13] For a further account of George Spencer Brown's name and personal reflections on him, see Ellsbury (2017).

(Tydecks, 2017). He studied at Oxford and Cambridge, he worked with Ludwig Wittgenstein from 1950 to 1951, and with Bertrand Russell on the foundations of mathematics in 1960, He also taught mathematics at Oxford, Cambridge and the University of London (Bakken, 2014). His professional career was varied, he held positions, among others, as a mathematician, psychologist, pilot, educational consultant, author and poet, adviser in military communications (coding and code-breaking) and football correspondent to the *Daily Express* (Bakken, 2014). Between 1959-1961, he was chief logic designer at Mullard Equipment and advisor at British Railways from 1963-1968. As Tydecks (2017) notes his work from an early stage transcends the fields of engineering, mathematics and psychology. It is suggested that Spencer-Brown's thinking was influenced while developing a counting machine for British Railways, he asked why formal logic could not deal with imaginary values in calculations that were used by engineers (Baecker, 1999). His epistemological and ontological perspective that is at the heart of LoF was well developed by the mid-1950s (Williams, 2019). In *Probability and scientific inference* (Spencer-Brown, 1957), he questions whether a scientist can prove a law through experiment or is it that *x* and *y* happen a certain way so far? He presents the question: "Why are there any worlds at all?" (Spencer-Brown, 1957, p. 5). He goes on to consider why there should be this particular universe rather than others and invites us to imagine that there is nothing at all[14] (Williams, 2019). The void is an essential starting point for the LoF, as I will show in the next section.

The Laws of Form

The 1972 edition of Laws of Form (LoF) begins:

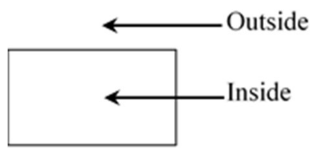
> The theme of this book is that a universe comes into being when a space is severed or taken apart. The skin of a living organism cuts off an outside from an inside. So does a circumference of a circle in a plane. By tracing the way we represent such a severance, we can begin to reconstruct, with an accuracy and coverage that appear almost uncanny, the basic forms underlying

[14] For an expressive articulation of this from an in interview in 2013 see https://vimeo.com/181216140

> linguistic, mathematical, physical and biological science, and can begin to see how the familiar laws of our own experience follow inexorably from the original act of severance (Spencer-Brown, 1969/1972, p. v).

Spencer-Brown's notation is based on one symbol, the mark:

The mark is shorthand for a rectangle drawn in the plane and dividing the plane into the regions inside and outside the rectangle and this is the starting point for Spencer-Brown's mathematical system. As Kauffman explains, "*A distinction is instantiated in the distinction that the mark makes in the plane"* (Kauffman, n.d.).



This distinction corresponds to a fundamental characteristic of consciousness, the capacity to compare and contrast, to make comparative judgements (Thurstone, 1927) and to observe difference (Bateson, 1972). It is tempting at this stage to see these early moves in LoF as trivial, but it is important to recognise the significance of severance and distinction since 'difference' appears to be an essential aspect of consciousness: As Bateson suggests, "… a difference which makes a difference is an *idea* or unit of information" (Bateson, 1972, p. 321).

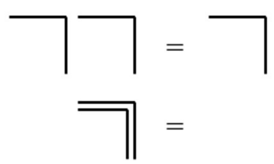
Unusually and probably uniquely, LoF begins with an explanation of its own typography (Kauffman, 2017). This is central to the iconicity of the mathematics as pointed out by Bricken (2017, 2019). This is not a representational or symbolic mathematics, it begins with a typography that has as close a connection to the world as, say, the tally notch, but from which we can see arithmetic, algebra and logic in new and interesting ways. I often think the recalcitrant and reluctant learner of mathematics question, "what is this for, or when will I need this?" in reference to some aspects of the school mathematics curriculum is a more profound expression and even exasperation with the disconnect between the symbols of mathematics and the world.

Lewin explains how LoF "introduced a new arithmetic with binary value in unary form" (Lewin, 2020, p. 9). Kauffman (2001) shows the lineage from C S Peirce's (1839–1914) *Sign of Illation* to LoF. But there is also common thinking in LoF with Leibniz (1645-1716) and the Pythagoreans (Lewin, 2018). What Spencer-Brown is doing is dispensing with the representational and beginning with the iconic as a foundation to mathematics and logic.

From the initial move of making a distinction, Spencer-Brown goes on to introduce a primary arithmetic and primary algebra.

The primary arithmetic and primary algebra

The primary arithmetic (Spencer-Brown, 1969/1972, Chapter 4) develops from the initial distinction and two important initial laws:



I1 *Law of calling*

I2 *Law of crossing*

The *law of calling* is where two adjacent marks condense to a single mark, or a single mark expands to form two adjacent marks. In the second equation, the *law of crossing*, two marks, one inside the other, disappears to form the unmarked state indicated by nothing at all. Alternatively, the unmarked state is equivalent to two nested marks. The first can be explained in terms of the observation that once a thing has been named, to name it again is tautology. The second considers what happens if you cross a mark twice and it turns out, obviously, that the previous crossing is negated by the second crossing.

Tydecks (2017) usefully explains the laws of calling and crossing in terms of switching, which is where Spencer-Brown is likely have to developed these ideas. The mark can be represented by a single switch, where opening the switch changes the state. If there are multiple switches that open at the same time, they just have the same effect as a single switch. Closing the switch returns the system to the original state and this represents the law of crossing. LoF extends this idea spatially.

The *primary arithmetic* (Spencer-Brown, 1969/1972, Chapter 4) culminates in the derivation and proof of two theorems: *invariance* (J1) and theorem *variance* (J2). These, Spencer-Brown refers to, as the initial theorems of the primary algebra. In the *primary algebra* (Spencer-Brown, 1969/1972, Chapter 6) builds on the theorems developed in the primary arithmetic in chapter 4 (two examples are shown below C1 and C2).

$$\overline{A\ \ \overline{A}|}| \quad = \qquad\qquad \textbf{J1}$$

$$\overline{\overline{A}|\ \overline{B}|}|\ C \quad = \quad \overline{\overline{AC}|\ \overline{BC}|}| \qquad \textbf{J2}$$

$$\overline{\overline{A}|}| \quad = \quad A \qquad \textbf{C1}$$

$$\overline{AB}|\ B \quad = \quad \overline{A}|\ B \qquad \textbf{C2}$$

Proof of C2

$$\overline{AB}|\ B \quad = \quad \overline{\overline{\overline{A}|}|\ \overline{\overline{B}|}|}|\ B \qquad \text{(C1)}$$

$$= \quad \overline{\overline{\overline{A}|\ B}|\ \overline{\overline{B}|\ B}|}| \qquad \text{(J2)}$$

$$= \quad \overline{\overline{\overline{A}|\ B}|}| \qquad \text{(J1)}$$

$$= \quad \overline{A}|\ B \qquad \text{(C1)}$$

The primary algebra can also be interpreted in terms of logic (we make TRUE the marked state and FALSE the unmarked state). The 'mark' can be interpreted as NOT in Boolean algebra terms

If ***a*** is TRUE then $\overline{a}|$ is FALSE

If ***a*** is FALSE then $\overline{a}|$ is TRUE

AND can be written $\overline{\overline{a}|\ \overline{b}|}|$

The truth table using Spencer-Brown's notation is shown in Table 1.

Table 1: Logic table for AND in Spencer-Brown notation

a	***b***	***a*** ⌉	***b*** ⌉	***a*** ⌉ ***b*** ⌉⌉
FALSE (unmarked)	FALSE (unmarked)	TRUE (marked)	TRUE (marked)	TRUE
TRUE (marked)	FALSE (unmarked)	FALSE (unmarked)	TRUE (marked)	FALSE
FALSE (unmarked)	TRUE (marked)	TRUE (marked)	FALSE (unmarked)	FALSE
TRUE (marked)	TRUE (marked)	FALSE (unmarked)	FALSE (unmarked)	TRUE

Arithmetic in the form

In this section, I am going to introduce how LoF can be interpreted in relation to the traditions of arithmetic. What I include here gives a sense of this, but there are further possibilities in thinking about how school curricula might be re-considered in relation to the teaching of arithmetic and pre-algebra/ early algebra (see, for example, Kauffman, 1995; Lewin, 2018, 2020 for an elaboration on the foundations of arithmetic in LoF). While my purpose here, in this paper, is, primarily, to consider LoF strategically and philosophically concerning mathematics education, I include a brief introduction to the arithmetic.

Natural numbers can be considered as a row of marks surrounded by a mark (Kauffman, 1995).

3 = 

It is interesting how this way of seeing number resonates with Bergson (1913/2001): "Every number is one, since it is brought before the mind by a simple intuition" (Bergson, 1913/2001, pp. 75–76). Indeed from the laws of calling and crossing every 'number' is reducible to the marked or unmarked state. Bergson strongly hints at the significance of severance and distinction in the process of consciousness as well as in the development of arithmetical capabilities:

> What properly belongs to the mind is the indivisible process by which it concentrates attention successively on the different parts of a given space; but the parts which have thus been isolated remain in order to join with the others, and, once the addition is made they may be broken up in any way whatever. They are therefore parts of space, and space is, accordingly, the material with which the mind builds up number, the medium in which the mind places it (Bergson, 1913/2001, p. 84).

Kauffman (1995) developed an arithmetic of natural numbers from the LoF and suggests the following interpretations for addition and multiplication

The law of crossing tells us that $A = \overline{\overline{A\rvert}\rvert}$ which allows the removal boundaries and a definition of addition:

$$A + B = \overline{\overline{A\rvert}\,\overline{B\rvert}\rvert}$$

Multiplication requires the copying of a number:

$$\times\ \overline{\overline{\ \ \rvert}\,\overline{\ \ \rvert}\,\overline{\ \ \rvert}\rvert}$$

$$\begin{aligned} A \times 3 &= A \times \overline{\overline{\ \ \rvert}\,\overline{\ \ \rvert}\,\overline{\ \ \rvert}\rvert} \\ &= \overline{\overline{A\rvert}\,\overline{A\rvert}\,\overline{A\rvert}\rvert} \end{aligned}$$

Even by this stage the more skeptical reader might still not be convinced by the significance of LoF, and that is understandable, I have already outlined the skepticism articulated by mathematicians after its publication. While the starting point of 'the mark', distinction, the primary arithmetic and primary algebra represent a departure from the axiomatic foundations of much of the traditions of Western mathematics, it is how LoF addresses issues of paradox and self-referentiality where some of the most important contributions to philosophy and science lie.

Self-referential paradox

The most important development with LoF is its confrontation with self-referentiality and paradox. Spencer-Brown refers to the following self-referential paradox, the so-called liar paradox: "This statement is false" (Spencer-Brown, 1969/1972, p. x). He says that statements such as *this statement is false* can considered to be either true, false or meaningless. If a statement is meaningful and not true, then it must be false or if it is not false it must be true. If *this statement is false* is meaningful then it must be true or false. But it leads to a paradoxical situation that if it is true then it must be false.

Spencer-Brown goes on to observe that there is "…an equally vicious paradox in ordinary equation in ordinary equation theory" which he explains has not been noticed because we do not express in a way that is analogous to "This statement is false". By analogy, as Spencer-Brown proceeds, it is assumed that a number can be positive, negative or zero. Then a non-zero that is not positive must be negative and that if it is not negative, it must be positive. Spencer-Brown asks us to consider the following.

$$x^2 + 1 = 0$$

After transposition

$$x^2 = -1$$

$$x = \frac{-1}{x}$$

The latter form of the equation is self-referential; like, as Spencer-Brown suggests "This statement is false". By inspection, x must be either +1 or –1. If $x = +1$:

$$+1 = \frac{-1}{+1} = -1$$

Which is paradoxical, and so is $x = -1$

$$-1 = \frac{-1}{-1} = +1$$

The solution to this is to introduce imaginary numbers, of course, as Spencer-Brown reminds us, but he goes further with arithmetic, algebra and in logic in confronting self-referential paradox. Indeed, I argue the very foundations of LoF are in this area. This is in contrast to the traditions of mathematics where paradox and self-referentiality are either ignored or disguised. This Spencer-Brown deals with in Chapters 11 and 12 (and notes in the appendix). The paradox or ambiguity between two states, the marked and unmarked, is the profound contradiction alongside the capacity of human beings to observe and experience difference. This idea is captured in Derrida's concept and process of *différance* (Derrida, 1972/1982). Merleau-Ponty (1945/2002) repeatedly points to the ambiguity at the heart of a phenomenological perception of the objects. One of the most profound expressions of this comes from Wittgenstein in *Philosophical investigations* (1953/1968) with the 'duckrabbit', where it is possible to see an object in two distinct ways.

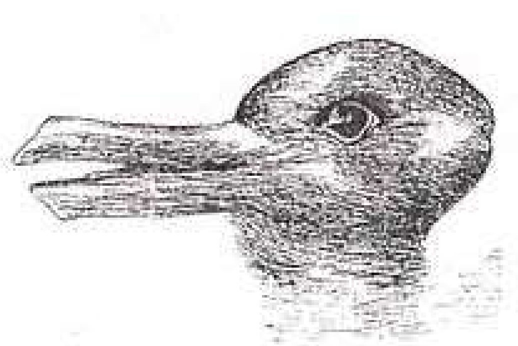


Figure 1: Duckrabbit ambiguity

The ambiguity and contingency of 'objects' in the world became apparent to me in the mid-1990s. I was looking at a tree in the grounds of Wolfson College, Cambridge and it occurred to me then that there was an ambiguity between the 'tree' and what was not a 'tree'. While it is clear that I can distinguish, i.e. I make a distinction between what is the tree and what is not the tree, there is an ambiguity. In making a distinction in the world, those distinctions remain logically paradoxical. This is beautifully expressed by the surrealist painter René Magritte (1898-1967) in the *Treachery of Images* (see Figure 2).

Figure 2: *Treachery of Images* René Magritte

When we make a distinction in the abstract, it becomes a paradox.

The notion of the 'object' is constantly in dispute and reducible to a paradox. Spencer-Brown expresses this self-referential paradox with a circle representing the marked state and as being 'equal' to the unmarked state. Spencer-Brown treats the equals sign as meaning 'can be confused with'. Thus, the marked state can be confused with the unmarked state.

$$\bigcirc =$$

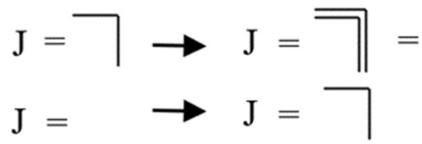
This is not dissimilar to the symbolism of yin and yang in ancient Chinese philosophy.

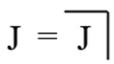
Kauffman (n.d.) represents the self-referential paradox as follows:

$$J = \overline{J\,}|$$

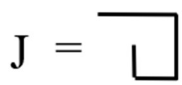
Here we get an apparent paradoxical equation where the marked case is equal to the unmarked state, where both states are marked J.

One way around this ambiguity is to introduce imaginary numbers as already stated, but another way of looking at this is through recursion and re-entry. If J is the marked state, then J can be marked again. Or if J is unmarked then it becomes marked as Kauffman shows in the following:

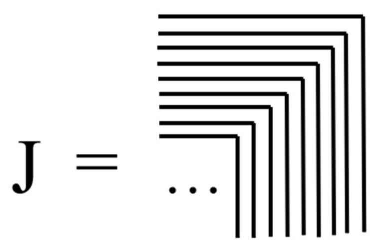


 therefore, can be seen as infinitely recursive. Or that J repeatedly re-enters itself.

Kauffman uses Spencer-Brown's notation to show this re-entry as:



Or it may be denoted by an infinite nesting of marked states:



It is the oscillation between a marked and an unmarked state that is the basis of time (see Kauffman & Varela, 1980). The oscillation created by re-entry and negation is inherent in a self-referential system. The implications are considerable in explaining how open systems like living things or social systems of meaning can react to contingency (Hui, 2019).

> Recursivity is not mere mechanical repetition; it is characterized by the looping movement of returning to itself in order to determine itself, while

> every movement is open to contingency, which in turn determines its singularity (Hui, 2019, p. 4).

While Spencer-Brown presents an abstracted, even idealized form of self-referential paradox, real systems interact with each other in complex and contingent ways allowing for variation and new forms. Although a genome maintains a continuity of form, Maturana and Varela (1980), treat this as homeostasis, not as static equilibrium, where the genome maintains continuity, but through a dynamic process of *autopoiesis* that involves feedback from the environment through recursion or re-entry. The formulation of this stems from the ideas of Spencer-Brown. This has also influenced the development of a systems theory of society (see, for example, Luhmann, 2013).

To see this alternative perspective in the mathematics presented in LoF, we can begin by looking at second order equations in a way that has become less familiar in school mathematics. The equation, $x^2 = ax + b$ is a quadratic equation with a solution that is sometimes imaginary in the sense that it utilizes complex numbers of the form $R + S\boldsymbol{i}$ where $\boldsymbol{i}^2 = -1$. This hides the self-referentiality of second order equation, but it can be re-written in the form, $x = a + b/x$, and it can be expanded to an infinitely recursive form.

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{\dots}}}}}}}$$

From continued recursion (re-entry) we can represent irrational numbers such as $\sqrt{2}$ as a continued fraction:

$$\sqrt{2} = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2+\dots}}}}}}$$

A further example in nature is the Golden Ratio. Two quantities, *a* and *b*, are in the Golden Ratio if $a > b > 0$ and their ratio is the same as ration of their sum i.e. $\frac{a+b}{b} = \frac{a}{b} = \varphi$

This is the same as the solution to the quadratic equation $x^2 - x - 1 = 0$, where $x = \frac{1 \pm \sqrt{5}}{2}$. The golden ratio is the positive solution $\varphi = \frac{1+\sqrt{5}}{2}$.

We can consider this as a continued fraction or self-referential recursive system by rearranging the quadratic equation as before.

$$x = 1 + \frac{1}{x}$$

$$\varphi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}}}$$

From the elements of each of these systems of recursion, there are emergent properties, in the first example, $\sqrt{2}$, and the second φ, the Golden Ratio. Emergence is the idea, simply, that an entity made up of elements has emergent properties that are likely to be different to those elements. Humphreys (2016) refers to this as *generative atomism,* that from the elements or atoms of a system there are generated distinct features of the whole. This is illustrated in the image created by a Mandelbrot set is an example of emergence (see Figure 3).

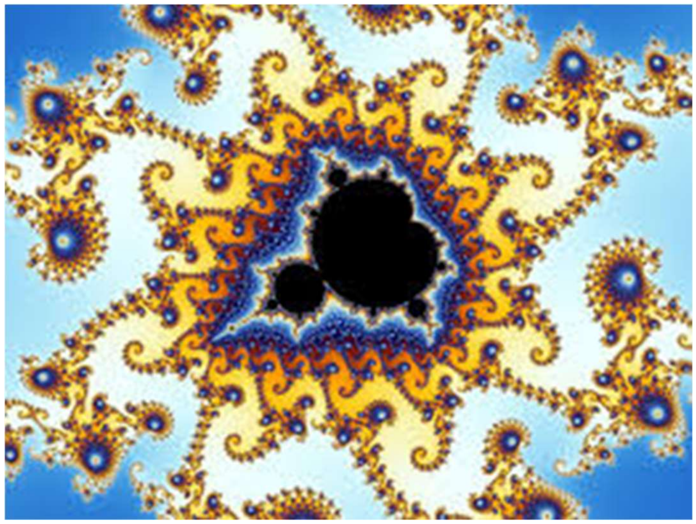


Figure 3: Mandelbrot set

The rules that govern the set are simple, each point is described by one number which defines its horizontal and vertical position. The vertical is mapped to imaginary space and the horizontal to real space i.e. the co-ordinates are mapped to complex space using the following equation where z and c represent complex numbers:

$$z_{n+1} = z_n^2 + c$$

The complex number z can be represented by:

$z = a + \boldsymbol{i}b$**,** where a and b are real numbers and ***i*** represents $\sqrt{-1}$

The square of a complex number is:

$$(a + \boldsymbol{i}b)^2 = a^2 - b^2 + 2\boldsymbol{i}ab$$

The real component is $a^2 - b^2$ and the imaginary component is $2\boldsymbol{i}ab$. The Mandelbrot set is generated when the number of iterations for the value of z_{n+1} is calculated to be greater than some threshold value for each pixel. The colour variation seen on the diagram depends on the number of iterations for the prior calculation for that starting value (Mandelbrot, 1982). The point being in this is that the Mandelbrot set has emergent properties based on the relationships between its elements.

The essence of the patterns that we observe and experience in the world and the order in nature seem likely to emerge through systems of self-referentiality and recursion from paradox and ambiguity. Indeed, theoretical physics is pushing in this direction with theoretical physicists Louis Kauffman (Kauffman & Lomonaco, 2018) and Peter Rowlands (Rowlands, 2007) employing LoF to explain the behaviour of fermions and the emergence of space and time. So we may yet realise Galileo's claim that "mathematics is the language in which God has written the universe" but not in the way that the mainstream of mathematics has been headed through the European Enlightenment.

The starting point for the LoF, based on the mark and the distinction, represents a contrast from mathematical traditions. It has been characterized as emanationist (Lewin, 2018) or emergentist (Robertson, 1999). In the following section, I contrast the emanationist perspective with the main traditions in the philosophy of mathematics.

Traditions and orthodoxy in the philosophy of mathematics

This section begins with a historical account of mathematics to highlight the development of traditions of thinking in mathematics. This is, principally, Eurocentric, since Western mathematics, or mathematics that might be ascribed contemporarily as being from the Global North, has come to dominate academic discourse (Nyoni, 2020). LoF represents an alternative perspective, and even a challenge, to the Western Enlightenment tradition of mathematics. In spite of this justification for presenting a Eurocentric account of the history and philosophy of mathematics, it remains important to be cognisant of the claim that Western mathematics is an instrument of colonialism (Raju, 2017) or, in the least, Western mathematics is a "secret weapon of cultural imperialism" (Bishop, 1995). We should see mathematics from the perspective of ethnomathematics (d'Ambrosio, 1985) and with a global (and colonial) history (Raju, 2017). As Bishop observed:

> Clearly, it is now possible to put forward the thesis that all cultures have generated mathematical ideas, just as all cultures have generated language, religion, morals, customs and kinship systems. (Bishop, 1995, p. 72).

Joseph (2011) makes an important distinction between the traditions of Western mathematics and other cultures.

> The concept of mathematics found outside the Graeco-European praxis was very different. The aim was not to build an imposing edifice on a few self-evident axioms but to validate a result by any suitable method. Some of the most impressive work in Indian and Chinese mathematics examined in later chapters, such as the summations of mathematical series, or the use of Pascal's triangle in solving higher-order numerical equations, or the derivations of infinite series, or "proofs" of the so-called Pythagorean theorem, involve computations and visual demonstrations that were not formulated with reference to any formal deductive system (Joseph, 2011, p. xiii).

The distinction being alluded to is between a Western tradition, which Raju (2017) and Joseph (2011) characterises as based on deduction and proof, compared with other ('Eastern') traditions which don't make use of deductive proof, but we could attribute as *iconic* in Bricken's (2019) terms.

Rorty's (1988) characterisation of Western mathematics as 'representational' is developed by Lewin (2018) who argues that *representationalism* can be attributed to John Locke (1632-1704) whose approach was based on an extreme form of empiricism. Locke argued that conscious beings, human beings, have no innate ideas and there is no *a priori* knowledge and a new-born child's mind is *table rasa*, a clean sheet (Locke, 1689/1993). For Locke, all our ideas and knowledge are representations derived from reality. It is during the European Renaissance and into the beginning of the Enlightenment, mathematics became oriented toward the description and explanation of the motion of physical objects (Ravn & Skovsmose, 2018). This form of representationalism is central to the thinking of Isaac Newton (1643-1727), where the universe can be represented by mathematical objects that can be calculated and, therefore, we can predict the future. "In philosophical dispositions we ought to abstract from our senses and consider things themselves, distinct from what are only sensible measures of them" (Para V Scholium to the Definitions in Philosophiae Naturalis Principia Mathematica, Bk. 1 (1689); trans. Andrew Motte (1729), rev. Florian Cajori, Berkeley: University of California Press, 1934. pp. 6-12, cited in Rynasiewicz, 2014). From Hume, Mill and Kant mathematics becomes part of the "human apparatus of cognition" (Ravn & Skovsmose, 2018, p. 2). Mathematics becomes abstracted from nature and takes on a "structural capacity" (ibid.) in the minds of human beings. And while much of this reflects Kantian transcendental idealism (something quite different on the face of it to representationalism or indeed the scientific realism that Locke aspired to), what is driving this is a primary distinction between mind and body, the binary that was introduced by Descartes and then appropriated and embedded into culture and philosophy. The abstraction from our senses and the consideration of the things themselves is an expression of the representationalist creed. From representations, the principles of logic and arithmetic, for example, can be used as the foundation of reasoning; to predict, induct and deduce based on the manipulations of the representations of objects.

The idea that things correspond to the things they represent is central to Russell's advocation of the Correspondence Theory of Truth (Russell, 2009). While Lewin (2018) acknowledges the value of the term representationalism and uses it extensively to argue the need for an alternative anti-representational emergent mathematics, he recognises the complexities underlying a general characterisation of representationalism in mathematics. Representationalism in terms of a correspondence theory is not empiricism as such, as was Locke's aspiration, since experience as a source of knowledge does not imply a correspondence to external objects. And neither, as Lewin further suggests is representationalism limited to materialism which again was Locke's foundation, the existence of the external material world (Lewin, 2018). In other words, by the time of Bertrand Russell and his collaborator in Cambridge, George Moore, representation has become a more complex philosophical concept than it was at the beginning of the Enlightenment. But still there is a legitimate claim that representation is a key aspect of Western culture, society and political economy as well as being influential on the history and philosophy of science and mathematics. It belies the belief in the triumph of reason and objectivity, over nature and indeed over human affairs.

Orthodox mathematics is based on a philosophy of mathematics with the following features: Firstly, that it is *a priori*, it does not rely on experience of the world, where truths are derived entirely from reason (Linnebo, 2017). Secondly, that "the knowledge is concerned with truths that are *necessary,* in the sense that things could not have been otherwise" (Linnebo, 2017, p. 12). This means that mathematical processes are independent of the world. And thirdly, mathematical knowledge involves objects which are not in space or time, that are not involved in causal relationships and are therefore abstract (Linnebo, 2017). This perspective often coincides with a Platonist view, where mathematical objects are real and exist independently of us (J. R. Brown, 2008). The *a priori* perspective on mathematics can be further subdivided into three schools of thought:

Logicism - "… that the concepts and objects of mathematics, such as 'number', can be defined from logical terminology; and with these definitions, the theorems of mathematics can be derived from principles of logic" (Shapiro, 2000, p. 108).

Formalism - where the mathematics is characterized by or its "essence" (Shapiro, 2000, p. 140) is the "manipulation of characters". In other words mathematics is about "…typographical characters and the rules for manipulating the them" (Shapiro, 2000, p. 140).

Intuitionism – is quite different from logicism and formalism but for many of its proponents the Platonist view that mathematical objects are abstract and independent of the world. Its departure is through a rejection of the binary logic position generally taken up with logicism (Linnebo, 2017; Shapiro, 2000), that there is a binary, true and false, for example.

Kauffman (2017) argues that Spencer-Brown's approach based on the mark and distinction is distinguishable from the logicist and formalist traditions which work, "…exclusively with standard typography" (p. 9). Bricken (2017) characterizes Spencer-Brown's approach as *iconic* in contrast to a *symbolic* orthodoxy. Bricken describes 'iconic' as "…forms that look like what they convey"

(p. 30) and 'symbolic' where "[i]ts expressions are formally disconnected from what they mean" (p. 30). Bricken (2019) argues that this "symbolic formalization" represents "concepts using encoded symbols that bear no resemblance to the concepts they identify" (p. 7). This is consistent with Lewin's (2018) attribution of representationalism. This, however, is likely to be rejected by mathematicians who subscribe to Platonic idealism, where the underlying philosophy is based on abstract objects that are independent of the world. The challenge to this is in any attempt to explain "…how human beings in a seemingly *a priori* way acquire knowledge of necessary truths concerned with abstract objects" (Linnebo, 2017, p. 18). The problem, according to Quine (1960/2013), is any argument must start in the midst of the science, *in medias res.* Linnebo (2017) refers to *the integration challenge*, of how to integrate the metaphysics of mathematics (what it is about) with its epistemology and from this to decide on the first principles or axioms that prove theorems or results. This is where the representational issue arises, it is not about the proposition that mathematical objects are abstract and the *a priori* nature of mathematical knowledge, it is concerned with supporting this assertion. It relies on language, symbols and therefore representationalism for the justification of itself and its theories. Linnebo concedes a circularity in this, a "mild circularity" (p. 20): "We are presupposing that *our* perceptual beliefs are reliable in order to explain *why* they are reliable" (Linnebo, 2017, p. 20). Therefore, it is reasonable to attribute the characteristic of the 'symbolic' or representational to the traditions of Western mathematical practices, because the representations and symbols we use for this purpose come from our experience of the world.

The limits of a self-contained axiomatic mathematics are effectively expressed in Gödel's (1906-1978) *incompleteness theorem;* the *first incompleteness theorem* can be expressed as follows:

> …that any consistent formal system F in which a certain amount of elementary arithmetic can be carried out is incomplete. That is, there are statements of the language of F which can neither be proved or disproved in F (Linnebo, 2017, p. 66).

Independently Gödel and John von Neumann identified a corollary, the *second incompleteness theorem,* "Let F be as above. Then F cannot prove its own consistency" (Linnebo, 2017, p. 66). The implications of this are that it is not possible to prove consistency without relying on assumptions. Therefore, it is necessary to go beyond platonic idealism and make assumptions. This necessitates phenomenology, that the distinction between the mind (the subject) and the objective world is no longer viable but necessarily connected. Kauffman argues a formal system whose foundations are based on axioms can never fulfill the need for an ever-expanding system, and that Spencer-Brown derives a novel and original approach to create a diagrammatic epistemology and which is generative (Kauffman, 2017). Indeed, if we are to draw implications from the recursive nature of self-referential systems, then mathematics (as a rational system), as Skovsmose argues, provides a certainty that is "completely internal" (Skovsmose, 2008, p. 71).

Orthodox approaches to mathematics have steered away from fundamental paradoxes and ambiguities. Logical positivism disavows paradox, as Russell acknowledged (Lewin, 2018). On the other hand, Spencer-Brown's LoF addresses these issues head on; it emanates, through self-referentiality, from the void, paradox, ambiguity and contradiction.

Implications for mathematics education

To conclude, I consider some possible implications for the school mathematics curriculum. And while I say that there has been little influence of *Laws of Form* in the mathematics classroom, there are exceptions. The mathematician, Moshe Klein, has produced materials and apps for very young children in Israel, based on *Laws of Form* (Kulikov, Klein, & Pelz, 2019). Burnett-Stuart (2016) has created a website with very accessible explanations of aspects of the primary algebra of *Laws of Form* which could be adapted and used in school contexts.

Overall, I agree with Bricken, it is important that we replace the symbolic with the iconic.

> So we arrive at the pregnant question: what arithmetic feel like in this century? Exploring and playing with and getting the feel of iconic arithmetic can be astonishingly familiar, it is how arithmetic was before universal schooling sucked the life out of it. If we replace abstraction by embodiment, will mathematics return to Earth? (Bricken, 2019, p. xxix).

Bricken argues that it was around the time of Lagrange (1736-1813) that visual arguments were dispensed with in preference for analytic arguments based on verbal-logic reason, since this was believed to be more secure.

> The math that we teach in schools lurched into the twentieth century on the back of this crisis in confidence. Those in the mathematical community who wondered about the rigor of mathematics discovered, after thousands of years, that they did not really understand arithmetic or geometry or badly behaving functions or even rigorous thinking.And so the community adopted a radical plan to put mathematics on a firm foundation. The hot idea was **symbolic formalization** [original emphasis], representing concepts using encoded symbols that bear no resemblance to the concepts they identif (Bricken, 2019, p. 7).

What would that look like in the classroom? Do we simply replace all previous traditions of curriculum and educational practice with one based on LoF. While there are many criticisms that could be made against the orthodoxy of mathematics, and the mathematics practices taught in schools, these practices are deeply embedded within a society's culture. A revolutionary change is not something that could be realized. However, the introduction of aspects of LoF might be a more appropriate strategy for enriching mathematical understanding and promoting not only a stronger epistemological foundation to the mathematics found in schools but also promote philosophical thinking. As Prediger (2007) observes, the importance and value of philosophical reflection in the mathematics classroom is often recognized, but classroom practices infrequently afford the space or time for such reflections. A reason for this may be that the representational or symbolic approach to mathematics disguises some of the profound philosophical questions, like around paradox or the axiomatic assumptions in mathematics. By introducing the 'mark' or distinction and by introducing the idea of re-entry and recursion, important philosophical dimensions about the nature of the world we are in are introduced. Also, much of the manipulation and theorems that are present in LoF still require abstract reasoning and the manipulation of symbols, but the iconic foundation means that the abstract is never too far from lived experience.

The next stage would involve developing and evaluating tasks. There are a range of materials already developed (see, for example, Bricken, 2019; Burnett-Stuart, 2016; Kulikov et al., 2019; Lewin, 2018). The approach to take, I suggest, should use design experiments (A. L. Brown, 1992) or later developments of design-based research (Barab & Squire, 2004; Swan, 2008) which are like action research approaches but involve the development of teaching approaches, materials as well as the development of theory.

The implications of *Laws of Form* go along way beyond this brief introduction. The exercise of generating the primary arithmetic and algebra and their manipulation is a valid mathematical exercise. However, when children are asking (as they frequently did in my math classes), "what is this for, when will I use any of this stuff?", we often answer in terms of the necessity and the value of mathematics as a tool, we talk less about the philosophical foundations of mathematics probably because of the profound logical inconsistencies. But we do have a moral responsibility to ensure that children do have opportunity to encounter the philosophical basis of the mathematics that we use. *Laws of Form* gives us the tools to do that.

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