

## Iterated $\phi^4$ kinks

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**ABSTRACT:** A first order equation for a static  $\phi^4$  kink in the presence of an impurity is extended into an iterative scheme. At the first iteration, the solution is the standard kink, but at the second iteration the kink impurity generates a kink-antikink solution or a bump solution, depending on a constant of integration. The third iterate can be a kink-antikink-kink solution or a single kink modified by a variant of the kink's shape mode. All equations are first order ODEs, so the  $n$ th iterate has  $n$  moduli, and it is proposed that the moduli space could be used to model the dynamics of  $n$  kinks and antikinks. Curiously, fixed points of the iteration are  $\phi^6$  kinks.

**KEYWORDS:** Nonperturbative Effects, Solitons Monopoles and Instantons, Effective Field Theories

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## Contents

<b>1</b>	<b><math>\phi^4</math> kinks and impurities</b>	<b>1</b>
<b>2</b>	<b>Iterated kinks</b>	<b>5</b>
<b>3</b>	<b>Space-deformed kinks</b>	<b>7</b>
<b>4</b>	<b><math>\phi^6</math> kink as fixed point</b>	<b>10</b>
<b>5</b>	<b>Energy function for iterated kinks</b>	<b>11</b>
<b>6</b>	<b>Summary</b>	<b>12</b>
<b>A</b>	<b>Iterated polynomials</b>	<b>13</b>

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## 1 $\phi^4$ kinks and impurities

The  $\phi^4$  scalar field theory in one spatial dimension has Lagrangian

$$L = \frac{1}{2} \int_{-\infty}^{\infty} \left\{ \left( \frac{\partial \phi}{\partial t} \right)^2 - \left( \frac{\partial \phi}{\partial x} \right)^2 - (1 - \phi^2)^2 \right\} dx \quad (1.1)$$

and dynamical field equation

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} - 2(1 - \phi^2)\phi = 0. \quad (1.2)$$

The vacuum solutions are  $\phi = \pm 1$  and kinks and antikinks are solutions interpolating between these vacua [1, 2]. The kink satisfies boundary conditions  $\phi \rightarrow -1$  as  $x \rightarrow -\infty$  and  $\phi \rightarrow 1$  as  $x \rightarrow \infty$ , and for the antikink the boundary conditions are reversed. Small and moderate amplitude field oscillations around either vacuum are interpreted as radiation, and tend to disperse.

As is well known, a static kink obeys the first order differential equation

$$\frac{d\phi}{dx} = 1 - \phi^2, \quad (1.3)$$

and the family of kink solutions is  $\phi(x) = \tanh(x - a)$ . The constant of integration  $a$  is the centre of the kink, and we refer to it as the kink's modulus. The manifold of allowed values of  $a$  (the whole real line) is the moduli space of kink solutions. The equation

$$\frac{d\phi}{dx} = -(1 - \phi^2) \quad (1.4)$$

has the antikink solution  $-\tanh(x - b)$ , and its centre  $b$  is the antikink's modulus.

In kink-antikink dynamics one studies the time-evolution of a field that is initially close to a kink centred at  $a$  joined to an antikink centred at  $b$ , where  $b \gg a$ . For this configuration,  $\phi \rightarrow -1$  as  $x \rightarrow \pm\infty$ , but between  $a$  and  $b$ ,  $\phi$  is initially close to 1. Even at rest, the kink and antikink attract, but the force is exponentially small in  $b - a$ . If the kink and antikink are given initial velocities toward each other, they approach more rapidly. The evolution is complicated during the collision. The kink and antikink can completely annihilate into radiation (a rather slow process), or they can quasi-elastically scatter, emitting less radiation. What happens depends sensitively on the initial velocities [3–6].

Ideally, one would like to model kink-antikink dynamics in terms of a finite number of degrees of freedom, coupled to radiation. To do this it is helpful to have a moduli space of field configurations with at least two moduli — one representing the kink-antikink separation, and the other the centre of mass. Further to these moduli one can consider oscillations of the shapes of the kink and antikink. But there is no obvious moduli space available within the original  $\phi^4$  theory. There are no static fields representing kink and antikink together, because of the attractive force between them.

One idea is to use the gradient flow curve connecting a well separated kink-antikink to the vacuum  $\phi = -1$ . This consists of the instantaneous field configurations obtained by replacing  $\frac{\partial^2 \phi}{\partial t^2}$  by  $\frac{\partial \phi}{\partial t}$  in the dynamical field equation, and evolving from a well separated kink-antikink configuration to the vacuum [7]. These field configurations form a moduli space which is fairly closely followed in the true, second order dynamics, but the vacuum configuration is an endpoint of this moduli space, whereas the true dynamics conserves energy and smoothly passes through the vacuum, or close by it, into field configurations where  $\phi$  is everywhere less than  $-1$ . The field then continues to evolve, oscillating and emitting some radiation in the process. Gradient flow therefore fails to produce a satisfactory moduli space in this case.

A promising resolution of this difficulty has recently been identified [8], based on consideration of the modified static, first order equation

$$\frac{d\phi}{dx} = -(1 - \phi^2)\chi(x). \tag{1.5}$$

$\chi$  is referred to as an impurity field, and eq. (1.5) as the kink equation in the presence of an impurity [9, 10]. We need to analyse eq. (1.5) in some detail. Throughout, we assume that  $\chi \rightarrow -1$  as  $x \rightarrow -\infty$ , with the approach sufficiently rapid that the integral

$$\int_{-\infty}^x (1 + \chi(x')) dx' \tag{1.6}$$

converges. We also assume that  $\chi \rightarrow \pm 1$  as  $x \rightarrow \infty$ , and that if  $\chi \rightarrow -1$  then the integral

$$\int_{-\infty}^{\infty} (1 + \chi(x')) dx' \tag{1.7}$$

also converges. Only impurities satisfying these conditions occur in the context of the iterated kinks that will be introduced in section 2.

Linearising eq. (1.5), we see that  $\phi = -1$  is an attractor as  $x \rightarrow -\infty$ , and  $\phi = 1$  a repeller. We can therefore impose the boundary condition  $\phi \rightarrow -1$  as  $x \rightarrow -\infty$ , which

excludes the vacuum solution  $\phi(x) = 1$ . As  $x \rightarrow \infty$ ,  $\phi = 1$  is an attractor and  $\phi = -1$  a repeller in the case that  $\chi \rightarrow -1$ , so for generic solutions,  $\phi \rightarrow 1$  as  $x \rightarrow \infty$ . Similarly,  $\phi = 1$  is a repeller and  $\phi = -1$  an attractor in the case that  $\chi \rightarrow 1$ , so  $\phi \rightarrow -1$  as  $x \rightarrow \infty$ . Solutions cannot cross  $\phi = \pm 1$  so, apart from the vacuum solution  $\phi(x) = -1$ , either  $\phi$  is trapped between  $-1$  and  $1$ , or  $\phi$  is everywhere less than  $-1$ .

A general impurity field  $\chi$  that oscillates between  $-1$  and  $1$  can make eq. (1.5) resemble the original equations (1.3) and (1.4) in different regions, thus allowing for solutions having several kinks and antikinks. We stress that these are static solutions of a first order equation.

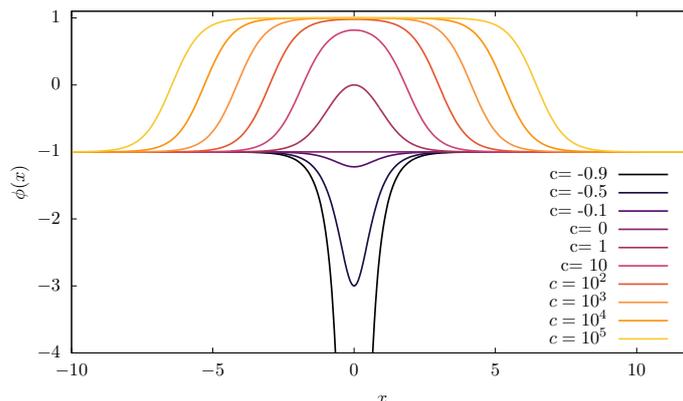
We can make some more precise statements about the solutions trapped between  $-1$  and  $1$  by exploiting Rolle's theorem. Let us define kink and antikink locations to be precisely the points  $x$  where  $\phi(x) = 0$ , with  $\frac{d\phi}{dx}$  positive for a kink, and negative for an antikink. The non-generic situation where zeros of  $\phi$  coalesce and  $\frac{d\phi}{dx} = 0$  is where a kink-antikink pair is about to be produced or annihilated. Let us focus on the generic case where  $\chi$  and  $\phi$  have simple zeros. By Rolle's theorem, between any pair of distinct zeros of  $\phi$  there is a point where  $\frac{d\phi}{dx}$  is zero. Suppose, then, that the impurity  $\chi$  has  $N$  zeros. These zeros split the real line into  $N + 1$  intervals (two of which extend to  $\pm\infty$ ), and there can be at most one kink or antikink in each of these intervals.  $\phi$  therefore has at most  $N + 1$  kinks and antikinks. There can be fewer, by a multiple of 2, and the number varies as the constant of integration in the solution of eq. (1.5) varies. For our choice of boundary condition they must alternate as kink-antikink-kink-...

A simple impurity is the  $\phi^4$  kink itself,  $\chi(x) = \tanh x$ , with its zero at the origin. The precise solution of eq. (1.5) for this impurity is given below, but let us describe a subset of the solutions more heuristically here. As  $\tanh x$  is close to  $-1$  in the region  $x \ll 0$ , eq. (1.5) resembles equation (1.3) here, and allows a kink solution centred at  $-A$ , with  $A \gg 0$ . For  $x \gg 0$ ,  $\tanh x$  is close to  $1$ , so eq. (1.5) in this region resembles the sign-reversed equation (1.4), which admits an antikink solution centred at  $B$ , with  $B \gg 0$ . Solving for all  $x$ , one finds a kink-antikink configuration, where the kink is at  $-A$  and the antikink is at  $B = A$ . The locations are related, because a first order equation has solutions that depend on only one free parameter.

We see that the impurity  $\chi(x) = \tanh x$  acts as a mirror. The kink part of the solution, around  $-A$ , is reflected in the impurity as an antikink around  $A$ . If the impurity is  $\chi(x) = \tanh(x - a)$ , then there is a solution with a kink at  $a - A$  and antikink at  $a + A$ . There are now two moduli — one is the centre of the impurity, and the other the distance of the kink and antikink from the impurity. We propose that the moduli space of these solutions could be used to model the kink-antikink fields that occur in the original  $\phi^4$  theory dynamics. The metric on the moduli space has been calculated [8], but there is also a potential energy, that has not yet been worked out. Both are needed to define a dynamics on moduli space.

The exact solutions of eq. (1.5), for  $\chi(x) = \tanh x$ , are

$$\phi(x) = \frac{c - \cosh^2 x}{c + \cosh^2 x}. \tag{1.8}$$



**Figure 1.** Kink-antikink and bump solutions  $\phi$ .

The allowed range of the modulus, the constant of integration  $c$ , is  $c > -1$ . Outside this range,  $\phi$  has singularities. All solutions satisfy the boundary conditions  $\phi \rightarrow -1$  as  $x \rightarrow \pm\infty$ . The kink-antikink configurations, described earlier heuristically, occur for  $c$  considerably greater than 1. Then the zeros of  $\phi$  are approximately where  $e^{2x} = 4c$  and  $e^{-2x} = 4c$ , that is, at  $x = \pm\frac{1}{2}\log(2c)$ . These are the locations  $\pm A$  of the antikink and kink. We can check the field profile near  $x = -\frac{1}{2}\log(2c)$ . Just keeping the dominant exponential term in  $\cosh x$ , we find the kink  $\phi(x) \simeq \tanh(x + \frac{1}{2}\log(2c))$ .

When  $c = 1$  the kink and antikink annihilate, and for  $c < 1$  there is no kink or antikink, as  $\phi$  is nowhere zero. The solution that remains we call a bump. For  $c$  small, it is a small positive or negative bump around  $\phi = -1$  of the form

$$\phi(x) \simeq -1 + \frac{2c}{\cosh^2 x}, \tag{1.9}$$

and for  $c = 0$ , it reduces to the vacuum  $\phi(x) = -1$ . For  $c$  near  $-1$  the bump is large and negative, with  $\phi \ll -1$  near the origin. This set of solutions, over the whole allowed range of  $c$ , forms a good moduli space for kink-antikink annihilation (with centre of mass at the origin), better than what is obtained using gradient flow, because it interpolates from well separated kink and antikink, through the vacuum, to a large negative bump. See figure 1. All these configurations occur in kink-antikink dynamics.

Another impurity that has been considered in [11] is of the bump shape (1.9),

$$\chi(x) = -1 + \frac{2c}{\cosh^2 x}, \tag{1.10}$$

with  $c$  not necessarily small. For  $c = 0$ , one solution of eq. (1.5) is the standard kink centred at the origin, but for  $c$  small and non-zero, the kink becomes deformed by a variant of the shape mode [12]. For  $c > \frac{1}{2}$  the impurity (1.10) has two zeros. This allows the kink to be sufficiently deformed that it becomes a kink-antikink-kink configuration.

Recall that the shape mode is a small, normalisable deformation of the kink with frequency of oscillation  $\omega = \sqrt{3}$  according to the linearised dynamical equation (1.2) for  $\phi$ . The continuum of radiation modes have frequencies  $\omega \geq 2$ , and the kink's translation zero

mode has frequency  $\omega = 0$ . A kink distorted by both a zero mode of amplitude  $\alpha$  and a shape mode of amplitude  $\beta$  has the form

$$\phi(x) = \tanh x + \alpha \frac{1}{\cosh^2 x} + \beta \frac{\sinh x}{\cosh^2 x}. \quad (1.11)$$

For the impurity (1.10), with  $c$  small, there are solutions of eq. (1.5) close to the standard kink that are similar to (1.11). To see this, set  $\phi(x) = \tanh x + \eta(x)$  and work to linear order in both  $\eta$  and  $c$ . Eq. (1.5) then reduces to

$$\frac{d\eta}{dx} = -2 \tanh x \eta - \frac{2c}{\cosh^4 x}, \quad (1.12)$$

and this linear inhomogeneous equation has the general solution

$$\eta(x) = \alpha \frac{1}{\cosh^2 x} - 2c \frac{\sinh x}{\cosh^3 x}, \quad (1.13)$$

combining a zero mode of arbitrary amplitude with a modified shape mode where the power of  $\cosh x$  in the denominator is 3 not 2.

This is interesting. The shape mode usually arises through oscillations of the kink, but here a variant arises independently through the effect of a small-amplitude bump impurity which itself arises (approximately) as a solution of the kink equation with kink impurity (1.5). This hints at an exact iterative scheme that could capture more of the degrees of freedom needed to model kink-antikink dynamics using a finite-dimensional moduli space. It has long been recognised that an effective model for kink-antikink dynamics should allow not only for the kink-antikink separation, but also for the shapes of the kink and antikink to be deformed [3, 5, 6, 13]. The shape mode also plays a role in (symmetric) kink-antikink-kink dynamics [14]. A kink-antikink-kink configuration can annihilate into a single kink, emitting radiation, and the approach towards annihilation is approximately tangent to the shape mode of the surviving kink.

All this suggests that useful moduli spaces of multiple kink-antikink configurations can be found as exact solutions of an iterated kink equation with impurity. We describe this next.

## 2 Iterated kinks

Our proposed iterated kink equation is

$$\frac{d\phi_n}{dx} = -(1 - \phi_n^2)\phi_{n-1}, \quad n = 1, 2, 3, \dots, \quad (2.1)$$

where we fix  $\phi_0(x) = -1$ .

We impose the boundary condition  $\phi_n \rightarrow -1$  as  $x \rightarrow -\infty$ , for all  $n$ , and also require that  $\phi_n$  has no singularities. This allows the vacuum solution  $\phi_n(x) = -1$  for any  $n$ , but excludes  $\phi_n(x) = 1$ . The boundary condition appears to be consistent, by the following inductive argument. Obviously  $\phi_0$  satisfies the boundary condition, and linearisation of

eq. (2.1) about  $\phi_n = -1$  shows that if  $\phi_{n-1}$  satisfies the boundary condition, then  $\phi_n(x) \sim -1 + \mu e^{2x}$  for  $x \ll 0$  and some constant  $\mu$ , and hence  $\phi_n$  satisfies the boundary condition.

The iteration can go on indefinitely, introducing one extra modulus each time. On the other hand, for each  $n$  there is always the vacuum solution  $\phi_n(x) = -1$ , whatever the form of  $\phi_{n-1}$ , and one can iterate this repeatedly and get the vacuum for all larger  $n$ . The iteration has then effectively stopped at the  $(n - 1)$ th step.

Iterating the argument in section 1 concerning the attractive and repulsive natures of  $\phi = -1$  and  $\phi = 1$ , we deduce that for generic solutions of eq. (2.1),  $\phi_n \rightarrow 1$  ( $-1$ ) as  $x \rightarrow \infty$  for  $n$  odd (even). Exceptionally, the sign may be reversed if one or more fields  $\phi_k$  in the solution sequence is the vacuum,  $\phi_k(x) = -1$ .

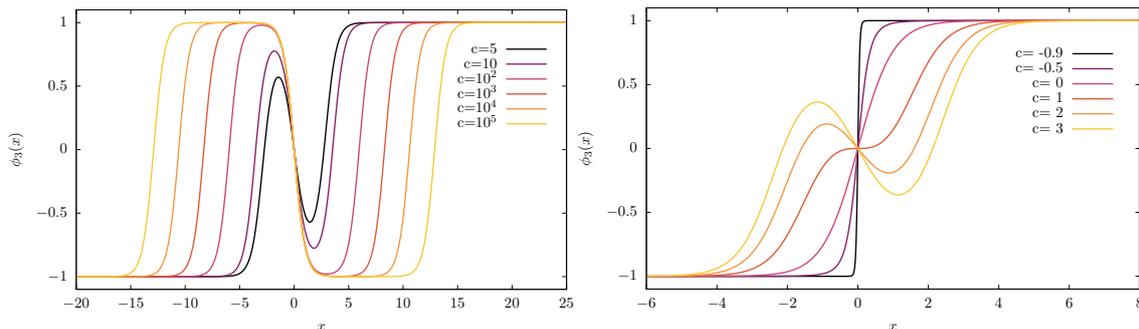
Equation (2.1) for  $\phi_n$  is simply the kink equation (1.5) with impurity  $\phi_{n-1}$ , and as each equation in the sequence is first order, its solution has one constant of integration. Iterating, and allowing these constants to be free, we may interpret  $\phi_n$  as having  $n$  moduli. The arguments in section 1 concerning zeros of  $\phi$  imply that  $\phi_n$  has at most  $n$  zeros, and if it has the maximal number, it is interpreted as a solution with  $n$  kinks and antikinks, whose locations are a choice for the moduli. In this case, the iterated kink equation adds one new kink or antikink to the solution at each step. This is reminiscent of a Bäcklund transformation in sine-Gordon theory, although the details seem quite different.

The first few iterates are field configurations we have previously discussed.  $\phi_1$  obeys the standard kink equation (1.3), having solution  $\phi_1(x) = \tanh(x - a)$  with arbitrary centre  $a$ . Notice that the equation also has the solutions  $\phi_1(x) = -1$  and  $\phi_1(x) = \coth(x - a)$  satisfying the boundary condition, but the latter is excluded because it is singular at  $x = a$ .

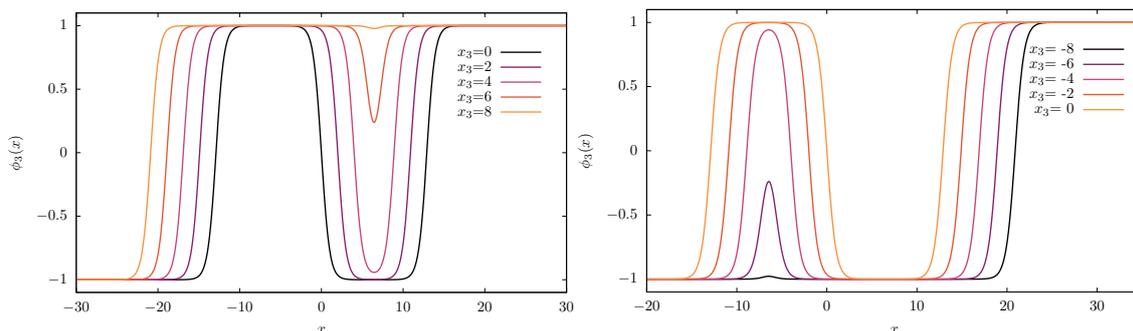
For the second iteration, let us take  $\phi_1$  to be the kink centred at the origin, as the effect of a translation is rather trivial. Equation (2.1) for  $\phi_2$  is the same as equation (1.5) with impurity  $\tanh x$ , having the solutions (1.8) illustrated in figure 1. For all  $x$  and  $c$ ,  $\phi_2(x) < 1$ . As before, the solutions include kink-antikink pairs, and also positive and negative bumps on the background of the vacuum  $\phi_2(x) = -1$ .

Also acceptable at the second iteration is for the impurity to be the vacuum,  $\phi_1(x) = -1$ . This gives solutions for  $\phi_2$  that are either simple kinks or again the vacuum. The family of solutions  $\phi_2$  therefore incorporates all acceptable solutions in the  $\phi_1$  family, including the starting, vacuum solution  $\phi_0$ . An interpretation is that the family of generic  $\phi_2$  kink-antikink solutions is completed by sending the antikink to infinity, and then both the kink and antikink to infinity.

The third iteration is algebraically more complicated. We need to solve eq. (2.1) for  $\phi_3$  with impurity  $\phi_2$  given by eq. (1.8). The explicit solution is given in section 3. The three moduli of the solution are the constant of integration  $x_3$ , the parameter  $c$  in  $\phi_2$ , and the centre of the original kink  $\phi_1$ . Particularly interesting are the solutions with the reflection symmetry  $\phi_3(-x) = -\phi_3(x)$ , which arise when  $\phi_1$  is a kink at the origin,  $\phi_2$  has arbitrary parameter  $c > -1$ , and the constant of integration is chosen to preserve the symmetry. These solutions are shown in figure 2. Use of their 1-dimensional moduli space could resolve some difficulties in modelling kink-antikink-kink dynamics that arose in ref. [14]. Note the appearance of a shape deformation when  $c$  is close to zero, as we anticipated in the approximate solutions (1.13), and the occurrence of kink-antikink-kink



**Figure 2.** Reflection-symmetric solutions  $\phi_3$  for various values of  $c$ .



**Figure 3.** Reflection-asymmetric solutions  $\phi_3$  for  $c = 10^5$ .

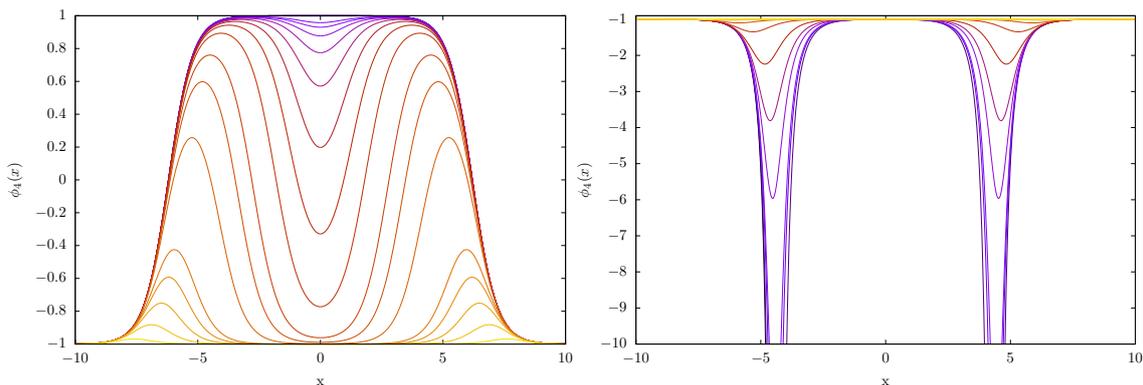
solutions for  $c > 1$ . Figure 3 shows a class of solutions  $\phi_3$  without reflection symmetry, with fixed  $c = 10^5$  and various constants of integration  $x_3$  (see eqs. (3.10) and (3.11)).

We have not systematically attempted a fourth iteration but can make some general observations. A class of solutions  $\phi_4$  consists of kink-antikink-kink-antikink configurations. If these are well separated we can denote their locations, where  $\phi_4(x) = 0$ , by  $a_1, a_2, a_3, a_4$ . The kink-antikink pair at  $a_1$  and  $a_2$  arises from a kink impurity at their midpoint  $\frac{1}{2}(a_1 + a_2)$ . Similarly the antikink-kink pair at  $a_2$  and  $a_3$  arises from an antikink impurity at  $\frac{1}{2}(a_2 + a_3)$ , and so on. So  $\phi_3$  is a solution with kink, antikink and kink locations  $\frac{1}{2}(a_1 + a_2), \frac{1}{2}(a_2 + a_3), \frac{1}{2}(a_3 + a_4)$ . In turn,  $\phi_3$  arises from a kink-antikink solution  $\phi_2$  with kink and antikink locations  $\frac{1}{4}(a_1 + 2a_2 + a_3), \frac{1}{4}(a_2 + 2a_3 + a_4)$ , and finally  $\phi_2$  arises from a single kink impurity  $\phi_1$  centred at  $\frac{1}{8}(a_1 + 3a_2 + 3a_3 + a_4)$ .

Not all solutions  $\phi_4$  are well separated kink-antikink-kink-antikink configurations. Some such solutions, and some alternative types of solution involving bumps, are shown in figure 4. There could also be an interesting class of solutions  $\phi_6$  with two moduli. These would be configurations with a reflection symmetry, where a kink on the left is deformed, and an antikink on the right is similarly deformed. The moduli space could be similar to that proposed in [3] and further discussed in [6, 13].

### 3 Space-deformed kinks

The equation for a kink with impurity (1.5) can be formally integrated [8], and this solution method gives considerable geometrical insight. The method can be applied iteratively to



**Figure 4.** Examples of  $\phi_4$  solutions.

solve the entire set of equations (2.1), but the result involves multiple integrations, and appears algebraically intractable.

Recall that the right hand side of (1.5) vanishes for  $\phi = \pm 1$ , so solutions cannot cross these values. A solution  $\phi(x)$  that approaches  $-1$  as  $x \rightarrow -\infty$  is either (i) trapped between  $-1$  and  $1$ , or (ii) is everywhere less than  $-1$ . We ignore here the vacuum solution  $\phi(x) = -1$ .

Let us first rewrite eq. (1.5) as

$$\frac{d\phi}{1 - \phi^2} = -\chi(x) dx. \tag{3.1}$$

In case (i), the solution is

$$\tanh^{-1} \phi \equiv \frac{1}{2} \log \left( \frac{1 + \phi}{1 - \phi} \right) = - \int^x \chi(x') dx', \tag{3.2}$$

where the lower limit of the integral provides a constant of integration. In case (ii), the solution is

$$\coth^{-1} \phi \equiv \frac{1}{2} \log \left( \frac{\phi + 1}{\phi - 1} \right) = - \int^x \chi(x') dx'. \tag{3.3}$$

(Note that the argument of the logarithm is positive in both cases.) The solutions exploiting the hyperbolic functions are more familiar in the context of kinks, but the logarithmic form can be easier to manipulate algebraically.

The right hand sides of both (3.2) and (3.3) can be expressed as

$$- \int^x \chi(x') dx' = x - a - \int_{-\infty}^x (1 + \chi(x')) dx', \tag{3.4}$$

with  $a$  arbitrary. (Recall that we are assuming that the last integral converges.) We refer to

$$y(x) = x - \int_{-\infty}^x (1 + \chi(x')) dx' \tag{3.5}$$

as the deformed spatial coordinate. The solutions (3.2) and (3.3) are then simply  $\phi(x) = \tanh(y(x) - a)$  and  $\phi(x) = \coth(y(x) - a)$ . These clearly satisfy eq. (3.1), since  $dy = -\chi(x)dx$ .

As  $y$  is finite for all (finite)  $x$ , the solution  $\tanh(y(x) - a)$  is always acceptable. The solution  $\coth(y(x) - a)$  is acceptable only if  $y$  remains less than  $a$  for all  $x$ , otherwise there is a singularity. For example, for  $\chi(x) = \tanh x$ ,  $y(x) = -\log(2 \cosh x)$ , which has a maximal value of  $-\log 2$ . So the  $\coth$  solution is acceptable only for  $a > -\log 2$ . The  $\tanh$  and  $\coth$  solutions together reproduce the solutions (1.8).

The interpretation of the solution  $\tanh(y(x) - a)$  depends on the behaviour of  $y$  as  $x$  increases. If  $\chi$  is everywhere negative, which means that  $\chi \rightarrow -1$  as  $x \rightarrow \infty$ , then  $y$  increases to  $\infty$  monotonically with  $x$ , and the solution is a spatially deformed single kink. If  $\chi < -1$  everywhere, then  $y$  increases more rapidly than  $x$ . The effect is to produce a solution  $\phi(x)$  that is a steepened kink. If  $\chi$  crosses zero at  $x = X$ , then  $\frac{dy}{dx}$  changes sign and part of the profile of  $\phi$  is reflected about  $X$ . Equivalently, there is a spatial fold at  $X$ . If  $\chi$  crosses zero again, there is another reflection, or fold.

When  $\chi \rightarrow -1$  as  $x \rightarrow \infty$ , we can define the overall stretching or compression of the deformed kink,

$$s = \int_{-\infty}^{\infty} (1 + \chi(x')) dx'. \tag{3.6}$$

The asymptotic form of  $\phi(x)$  is  $\tanh(x - a)$  for  $x \ll 0$  and  $\tanh(x - a - s)$  for  $x \gg 0$ . The kink has been stretched by distance  $s$  if  $s > 0$  and compressed by  $|s|$  if  $s < 0$ . Stretching by more than a small distance can introduce kink-antikink pairs.

Analogous to the spatial folding in the relation between  $y$  and  $x$  is to imagine walking the length of a corridor, when it is uncomfortable to walk very slowly, but comfortable to sit for a while. One can walk the length in one go (a kink), and sit the rest of the time, or walk backwards and forwards a few times (kinks and antikinks), sitting less. With more time available one can walk more often backwards and forwards. If the time available is short, one must walk quickly (a steepened kink).

All this analysis applies to the iterated kink equation. Consider a generic sequence of solutions  $\phi_n(x)$ . For  $n$  odd,  $\phi_n$  must be of the  $\tanh$  type, to avoid singularities, but for  $n$  even,  $\phi_n$  can be of  $\tanh$  or  $\coth$  type. For  $n$  odd,  $\phi_n$  is a spatially deformed kink,

$$\phi_n(x) = \tanh \left( x - x_n - \int_{-\infty}^x (1 + \phi_{n-1}(x')) dx' \right), \tag{3.7}$$

whose deformed spatial coordinate is

$$y_n(x) = x - \int_{-\infty}^x (1 + \phi_{n-1}(x')) dx', \tag{3.8}$$

and whose overall stretching/compression is

$$s_n = \int_{-\infty}^{\infty} (1 + \phi_{n-1}(x')) dx'. \tag{3.9}$$

$x_n$  is the arbitrary constant of integration.

An explicit solution for  $\phi_3$  can be found using this approach. Let us assume that  $\phi_1$  is a kink centred at the origin;  $\phi_2$  is then given by eq. (1.8). Using the deformed spatial

coordinate  $y_3$  given by the integral (3.8), we obtain for  $c \geq 0$ ,

$$\phi_3(x) = \tanh \left( x - x_3 - \frac{2c}{\sqrt{c(1+c)}} \tanh^{-1} \left( \sqrt{\frac{c}{1+c}} \tanh x \right) \right), \quad (3.10)$$

and for  $-1 < c \leq 0$ ,

$$\phi_3(x) = \tanh \left( x - x_3 - \frac{2c}{\sqrt{-c(1+c)}} \tanh^{-1} \left( \sqrt{\frac{-c}{1+c}} \tanh x \right) \right). \quad (3.11)$$

The solutions are shown in figure 2 and figure 3. Specifically, in figure 2 we plot  $\phi_3$  for  $x_3 = 0$ . These are the solutions with reflection symmetry. It is clear that the modulus  $c$ , which measures the strength of  $\phi_2$ , controls the emergence of an antikink-kink pair. For large  $c$  such a pair is easily visible in  $\phi_2$ , and the whole solution  $\phi_3$  represents a kink-antikink-kink configuration. When  $c$  approaches zero,  $\phi_2$  tends to the constant  $-1$ , which leads to a single kink for  $\phi_3$ . This single kink solution becomes steeper and steeper as  $c \rightarrow -1$ .

In figure 3 we show the impact of  $x_3$  on  $\phi_3$  for fixed  $c$ . We choose  $c = 10^5$  to better visualise the observed behaviour. Here,  $\phi_2$  represents a well separated kink-antikink pair. For large  $x_3$  the solution describes a single kink monotonically interpolating between the vacua. The impact of  $\phi_2$  is negligible, except on part of the kink tail. When  $x_3$  approaches zero, the single kink interacts strongly with  $\phi_2$  and the kink-antikink pair hidden in  $\phi_2$  has a pronounced effect. Finally, for large negative  $x_3$ , the single kink reappears but on the opposite side of the origin. This variation with  $x_3$  represents a flow on the moduli space, where an incoming kink creates an antikink-kink pair (due to the interaction with  $\phi_2$ ), and later on annihilates this pair leaving an outgoing kink.

#### 4 $\phi^6$ kink as fixed point

The iterated kink equation has a curious fixed point. We find this by setting  $\phi_n = \phi_{n-1}$ . Then eq. (2.1) becomes the  $\phi^6$  kink equation

$$\frac{d\phi}{dx} = -(1 - \phi^2)\phi. \quad (4.1)$$

The generic non-singular solutions, satisfying the boundary condition  $\phi \rightarrow -1$  as  $x \rightarrow -\infty$ , are of the form

$$\phi(x) = - \left( 1 + 2e^{2(x-c)} \right)^{-\frac{1}{2}}, \quad (4.2)$$

with  $c$  arbitrary. These all have the property  $\phi \rightarrow 0$  as  $x \rightarrow \infty$ .

We have not constructed an iterated sequence of solutions  $\phi_n$  with limiting form (4.2). The approach to the limit cannot be uniform in  $x$ .

There is also an interesting 2-cycle of the iteration, a solution of the pair of equations

$$\frac{d\phi}{dx} = -(1 - \phi^2)\psi, \quad (4.3)$$

$$\frac{d\psi}{dx} = -(1 - \psi^2)\phi. \quad (4.4)$$

We assume that  $\phi \rightarrow -1$  and  $\psi \rightarrow -1$  as  $x \rightarrow -\infty$ . Setting  $\psi = \phi\Omega$ , we find that these equations reduce to

$$\frac{d\phi}{dx} = -(1 - \phi^2)\phi\Omega, \tag{4.5}$$

$$\frac{d\Omega}{dx} = -(1 - \Omega^2). \tag{4.6}$$

The equation (4.6) for  $\Omega$  is the usual first order equation for a  $\phi^4$  antikink. It has trivial solutions  $\Omega(x) = \pm 1$ , and non-trivial solutions  $\Omega(x) = -\tanh x$  and  $\Omega(x) = -\coth x$ , or translates of these. If  $\Omega(x) = 1$  then we recover the fixed point solution (4.2), the  $\phi^6$  kink. There is no solution with  $\Omega(x) = -1$  satisfying the boundary conditions. When  $\Omega(x) = -\tanh x$ , then  $\psi(x) = -\phi(x)\tanh x$ , so  $\phi(x) = -\psi(x)\coth x$ . Therefore, multiplication by  $-\tanh x$  and  $-\coth x$  automatically alternate during iteration of the 2-cycle, so we need only consider the case  $\Omega(x) = -\tanh x$ .

The remaining equation (4.5) can be expressed as

$$\frac{d\phi}{(1 - \phi^2)\phi} = \tanh x dx. \tag{4.7}$$

This is the standard equation for a  $\phi^6$  kink, but in terms of a deformed spatial coordinate  $y$ , defined by  $dy = \tanh x dx$ . Integrating, we find that  $\phi(y) = \pm(1 + 2e^{-2(y-c)})^{-\frac{1}{2}}$ , where  $y = \log \cosh x$ . Choosing the appropriate sign, and rearranging, we obtain the solution

$$\phi(x) = -\left(1 + \frac{a}{\cosh^2 x}\right)^{-\frac{1}{2}}, \tag{4.8}$$

with  $a > -1$ , and this is paired in the 2-cycle with

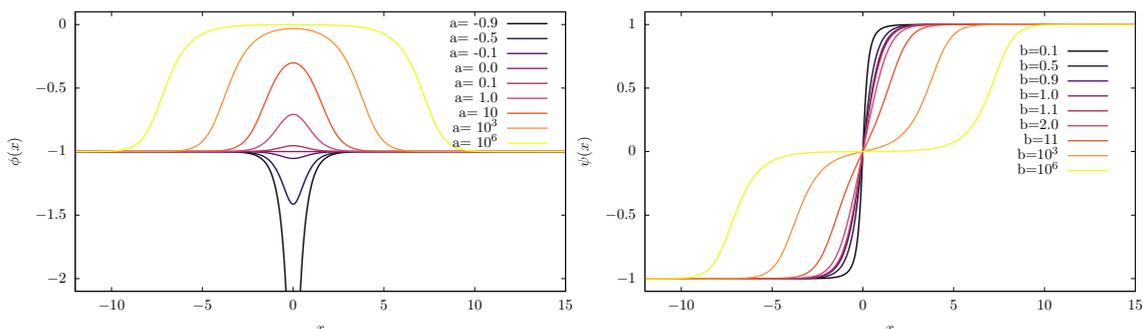
$$\psi(x) = \text{sign}(x) \left(1 + \frac{b}{\sinh^2 x}\right)^{-\frac{1}{2}}, \tag{4.9}$$

where  $b = a+1$ . See figure 5. For large positive  $a$  the field  $\phi$  describes a well separated kink-antikink pair of  $\phi^6$  theory (interpolating between  $-1$  and  $0$ ). When  $a$  decreases the kink and antikink approach each other and finally, for  $a = 0$ , annihilate to the vacuum  $\phi = -1$ . For negative  $a$  the field  $\phi$  forms a negative bump whose strength becomes arbitrarily large as  $a$  approaches  $-1$ . Simultaneously, the field  $\psi$  represents a kink (interpolating between  $-1$  and  $0$ ) and a second kink (interpolating between  $0$  and  $1$ ) of  $\phi^6$  theory. They separate completely as  $a \rightarrow \infty$ . When  $a = 0$  these kinks merge into the kink of  $\phi^4$  theory, and this becomes steeper and steeper as  $a \rightarrow -1$ .

## 5 Energy function for iterated kinks

Here we present an energy function whose stationary points include the solutions of the iterated kink sequence of equations (2.1). Let us start with eq. (1.5) for a kink with given impurity  $\chi$ , rewritten as

$$-\frac{1}{\chi(x)} \frac{d\phi}{dx} = 1 - \phi^2. \tag{5.1}$$



**Figure 5.** 2-cycle solutions  $\phi$  (left) and  $\psi$  (right) for various values of  $a$  and  $b$ .

We shall suppose that the zeros of  $\chi$  (if any) are a discrete set of points, and require that  $\frac{d\phi}{dx}$  is zero at these points. Recall that  $\phi(-\infty) = -1$ , and  $\phi(\infty) = \pm 1$ .

Consider the energy

$$E_\chi = -\frac{1}{2} \int_{-\infty}^{\infty} \left\{ \frac{1}{\chi^2(x)} \left( \frac{d\phi}{dx} \right)^2 + (1 - \phi^2)^2 \right\} \chi(x) dx. \quad (5.2)$$

Formally, this is the standard static energy of  $\phi^4$  theory in terms of the deformed spatial coordinate  $y$ , because  $dy = -\chi(x) dx$ , except that the endpoints of the  $y$ -integration may be non-standard. This energy differs from previous self-dual impurity models which have eq. (1.5) as the corresponding Bogomolny equation [8].

In the usual way, we can complete the square in the integrand, and obtain

$$E_\chi = -\frac{1}{2} \int_{-\infty}^{\infty} \left\{ \frac{1}{\chi(x)} \left( \frac{d\phi}{dx} \right) + (1 - \phi^2) \right\}^2 \chi(x) dx + \left[ \phi(x) - \frac{1}{3} \phi^3(x) \right]_{-\infty}^{\infty}. \quad (5.3)$$

The last term depends only on the field topology — the boundary data of  $\phi$ . The energy  $E_\chi$  is stationary for solutions of eq. (5.1), because a change in  $\phi$  of order  $\varepsilon$  changes  $E_\chi$  at order  $\varepsilon^2$ , though it is not guaranteed to be a minimum unless  $\chi$  is everywhere negative. The energy value is  $E_\chi = \frac{4}{3}$  if  $\phi(\infty) = 1$  and  $E_\chi = 0$  if  $\phi(\infty) = -1$ ; it can be zero because the energy density is negative in any region where  $\chi$  is positive.

It is straightforward to extend the energy function (5.2) to deal with iterated kinks. Define

$$E = -\frac{1}{2} \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \mu_n \left\{ \frac{1}{\phi_{n-1}^2(x)} \left( \frac{d\phi_n}{dx} \right)^2 + (1 - \phi_n^2(x))^2 \right\} \phi_{n-1}(x) dx, \quad (5.4)$$

where  $\mu_n$  are fairly arbitrary positive numbers whose sum is finite. We require that  $\phi_n$  has zero derivative at all locations where  $\phi_{n-1}$  is zero. Completing the square in each term of the sum, we see that  $E$  is stationary when the sequence of iterated kink equations is satisfied.

## 6 Summary

We have introduced a new, iterated equation for kinks in  $\phi^4$  theory. This was motivated by examples of how impurities can affect a kink. Each equation in the iterated sequence

is a first order, static ODE, whose solution includes one new modulus, the constant of integration. In the iterated scheme, the first iteration generates a kink from the vacuum. At the second iteration, this kink is an impurity which acts like a mirror. It generates a kink-antikink configuration, or a positive or negative bump solution around the vacuum, depending on the value of the constant of integration. The third iteration can produce a kink deformed by a variant of the kink's shape mode, and also kink-antikink-kink configurations.

More generally, the  $n$ th iterate generates an  $n$ -dimensional moduli space of solutions which we propose could be useful for modelling the dynamics of  $n$  kinks and antikinks. The bump-like configurations capture the type of fields that occur dynamically when kinks and antikinks annihilate, and that are missed in some existing collective coordinate schemes. It would be interesting to use the standard  $\phi^4$  theory Lagrangian to calculate the metric (equivalently, the kinetic energy for time-varying moduli) and potential energy on these moduli spaces, and to study in detail the classical and quantized dynamics of kinks using these novel collective coordinates.

## A Iterated polynomials

An analogy for the iterated kink equation (2.1) is the iterated equation

$$\frac{du_n}{dx} = -u_{n-1}, \quad n = 0, 1, 2, \dots \tag{A.1}$$

Equation (A.1) is also the linearisation of eq. (2.1) for  $\phi_n \simeq 0$ . We fix  $u_{-1}(x) = 0$ . Then, generically,  $u_0$  is a constant,  $u_1$  is linear in  $x$ ,  $u_2$  is quadratic, and so on.  $u_n(x)$  is a polynomial of degree  $n$ , so it has at most  $n$  real zeros. A zero can be regarded as analogous to the location of a kink or antikink, depending on whether  $\frac{du_n}{dx}$  is positive or negative at the zero. Exceptionally,  $u_n$  is a polynomial of degree  $n - k$  if the first non-zero function in the sequence is  $u_k$ , a non-zero constant.

At each iteration, the number of zeros can increase by at most 1, by Rolle's theorem. If  $u_{n-1}$  has  $N$  zeros, then  $u_n$  has at most  $N + 1$  zeros, or fewer by a multiple of 2. There can never be catching up in the number of zeros.  $u_n$  cannot have  $n$  zeros if any  $u_k$ , for  $k < n$ , has fewer than  $k$  zeros.

The iteration has fixed points satisfying the equation

$$\frac{du}{dx} = -u, \tag{A.2}$$

whose solutions are

$$u(x) = Ae^{-x}. \tag{A.3}$$

As for the iterated kink equation, a fixed point solution  $u(x)$  satisfies different boundary conditions from any of the sequence of solutions  $u_n(x)$ . The fixed point  $u(x) = Ae^{-x}$  is the non-uniform limit of the sequence

$$u_n(x) = A \left( 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots + \frac{(-1)^n}{n!}x^n \right), \tag{A.4}$$

which satisfies eq. (A.1).

A 2-cycle satisfies the equations

$$\frac{du}{dx} = -v, \quad \frac{dv}{dx} = -u. \quad (\text{A.5})$$

Solutions which are not pure exponentials are of the form

$$u(x) = -A \cosh x, \quad v(x) = A \sinh x, \quad (\text{A.6})$$

or translates of this.  $u$  and  $v$  can also be exchanged. The 2-cycle solution (4.8) and (4.9) of the iterated kink equation reduces for large  $a$  and modest  $x$  to the form (A.6), with  $A = \frac{1}{\sqrt{a}}$ . See figure 5.

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