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Class-$\mathcal{K}$ Function Bounds for Positive Definite Functions on Compact Sets

Seungjoon Lee, Jin Gyu Lee, and Hongkeun Kim*

Abstract: This technical note focuses on positive definite functions defined on compact sets which contain the origin in their interior. It provides a class-$\mathcal{K}$ function bounding inequality for the given positive definite function which is satisfied for its entire domain of definition. An application of the result is also given to analyze the stability of perturbed nonlinear systems whose nominal parts contain an asymptotically stable equilibrium point at the origin.

Keywords: Positive definite functions, comparison functions, nonlinear systems, stability analysis

1. INTRODUCTION

Continuous positive definite functions and comparison functions are ubiquitous in control theory because they are deeply related with the stability of dynamical systems. For example, they are used to investigate the Lyapunov stability of nonlinear systems in [1,2], define the input-to-state stability in [3], and characterize the converse Lyapunov theorems in [1,4], to name a few. See [5] for a historical review of comparison functions and some relations with positive definite functions.

As stated above, both positive definite functions and comparison functions are commonly used to study the stability and its several variants, and also simplify the corresponding analyses. Therefore, the relations between them are of particular interest and up to now, various inequalities for them have been established in the literature. In particular, a continuous positive definite function can be globally bounded below by a class-$\mathcal{K}$ function [5,6] and can be bounded above and below by class-$\mathcal{K}$ functions in a neighborhood of the origin [1, Lemma 4.3]. That is, the bounding inequalities hold either locally or globally and to the best of our knowledge, most of existing works belong to one of these two cases.

Motivated by the gap between local and global results, we consider a continuous positive definite function $V : \Omega \to \mathbb{R}$ and investigate the existence of class-$\mathcal{K}$ functions α_1 and α_2 such that the inequality

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{1}$$

is valid for all points x in the compact set Ω, where Ω contains the origin in its interior. Thus, if we are able to find such comparison functions, then the region of validity of (1) is maximized and is the same as the given domain of definition of V. Note in contrast that except for the global case (i.e., $\Omega = \mathbb{R}^n$), [1, Lemma 4.3] guarantees the existence of α_1, α_2 that satisfies (1) only for points in $\bar{B}_r$, where $\bar{B}_r$ is the closed r-ball centered at the origin. A main restriction here is that $\bar{B}_r$ must be contained in Ω. This implies that the region of validity of the inequality can be unacceptably small (i.e., the radius r can be very small) depending on the *shape* of the set Ω and thus, limits the usage of (1).

One of main challenges in finding such comparison functions is that the conventional approaches construct, e.g., a continuous α_1 via a continuous auxiliary function $\psi_1(s) = \inf_{x\in\bar{B}_r\setminus B_s} V(x)$, where B_s is the open s-ball. A natural adaptation of this function to our case may be $\psi_1(s) = \inf_{x\in\Omega\setminus B_s} V(x)$ which however turns out to be discontinuous in general. Even worse, the set of discontinuity of such ψ_1 can be countably infinite. Therefore, a careful construction of α_1 and α_2 would be necessary to ensure their continuity.

Meanwhile, we also provide an application of (1) which addresses the stability of perturbed nonlinear systems. Specifically, we show that if the origin of the nominal part is asymptotically stable, then the solutions of the original perturbed system possess some stability property. In this case, the initial conditions are allowed to be in arbitrary compact subset of the region of attraction of the origin and this is a consequence of (1) which holds for all $x \in \Omega$.

This note is organized as follows. Section 2 includes our main result and its application to the stability of perturbed systems. An example is provided in Section 3 and the conclusion is drawn in Section 4.

Notation: We denote the set of nonnegative real num-

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S. Lee and J. G. Lee are with ASRI, Department of Electrical and Computer Engineering, Seoul National University, Seoul 08826, Korea (e-mails:{seungjoon.lee,ljgman}@cdsl.kr). H. Kim is with School of Mechatronics Engineering, Korea University of Technology and Education, Cheonan 31253, Korea (e-mail: hkkim@koreatech.ac.kr).
* Corresponding author.

bers by $\mathbb{R}_{\geq 0}$. Given a vector $x \in \mathbb{R}^n$ and a nonempty subset Ω of $\mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x and $|x|_\Omega$ denotes the set-distance defined as $|x|_\Omega := \inf_{z\in\Omega} |x-z|$. $\mathcal{C}^0$ and $\mathcal{C}^1$ are the sets of continuous and continuously differentiable functions, respectively. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. We denote it by $\alpha \in \mathcal{K}$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$, denoted as $\beta \in \mathcal{KL}$, if for each fixed $v \geq 0$, $\beta(\cdot, v) \in \mathcal{K}$ and for each fixed $u \geq 0$, $\beta(u,\cdot)$ is decreasing to zero as $v \to +\infty$. Let $V : \Omega \to \mathbb{R}_{\geq 0}$ be a function, where Ω is a subset of $\mathbb{R}^n$ and contains the origin in its interior. Then, V is said to be positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \in \Omega\backslash\{0\}$. Given two sets A and B, $A \subset B$ denotes the proper (or strict) inclusion, whereas $A \subseteq B$ denotes the inclusion with the possibility $A = B$. The boundary of a set A is denoted as ∂A. Set minus is defined as $A \setminus B := \{\xi : \xi \in A, \xi \notin B\}$. The r-ball centered at the origin is denoted as $B_r := \{\xi : |\xi| < r\}$ and $\bar{B}_r$ denotes its closure, i.e., $\bar{B}_r = B_r \cup \partial B_r$. Let $f : X \to Y$ be a function and consider $A \subseteq Y$. Then, the preimage of A under f is defined as $f^{-1}(A) := \{x \in X : f(x) \in A\}$. Half-open intervals are defined as $[a,b) := \{x \in \mathbb{R} : a \leq x < b\}$ and $(a,b] := \{x \in \mathbb{R} : a < x \leq b\}$. Closed interval $[a,b]$ and open interval (a,b) are analogously defined.

2. MAIN RESULT

The main result of this note concerns about positive definite functions defined on compact sets. It provides a construction of class-$\mathcal{K}$ function bounds for a positive definite function $V : \Omega \to \mathbb{R}_{\geq 0}$ which holds over the entire compact domain Ω. Thus, the region of validity of the bounding inequality does not depend on the shape of the set Ω.

Theorem 1: Let $V : \Omega \to \mathbb{R}_{\geq 0}$ be a continuous positive definite function, where $\Omega \subset \mathbb{R}^n$ is a compact set containing the origin in its interior. Then, there exist class-$\mathcal{K}$ functions $\alpha_1, \alpha_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \qquad \forall x \in \Omega. \tag{2}$$

Proof: *(Existence of α_1):* Let $r := \max_{x\in\Omega} |x|$ and $B_s := \{\zeta \in \mathbb{R}^n : |\zeta| < s\}$. Let us also define $\Omega_{1,s} := \Omega \setminus B_s = \Omega \cap \{\zeta \in \mathbb{R}^n : s \leq |\zeta| \leq r\}$ and $\psi_1(s) := \inf_{\chi\in\Omega_{1,s}} V(\chi)$. Note that by construction, $\Omega_{1,s} \subseteq \Omega$ is a nonempty compact set for every $s \in [0,r]$, and $x \in \Omega_{1,|x|}$ for each $x \in \Omega$. Then, $\psi_1 : S \to \mathbb{R}_{\geq 0}$, $S := [0,r]$ is positive definite by the positivity of V, and $V(x) \geq \psi_1(|x|)$ holds for $x \in \Omega$. Moreover, ψ_1 is increasing (but not necessarily strictly increasing) since for $0 \leq s_1 < s_2 \leq r$, $\Omega_{1,s_2} \subset \Omega_{1,s_1}$ and thus, $\psi_1(s_1) \leq \psi_1(s_2)$.

Let $r_{\text{cont}} > 0$ be such that $B_{r_{\text{cont}}} \subset \Omega$. Then, we claim that $\psi_1 : S_{\text{cont}} \to \mathbb{R}$ is continuous, where $S_{\text{cont}} := [0, r_{\text{cont}})$.[1] To prove this, it suffices to show that the preimages of $(-\infty, a)$ and $(b, +\infty)$ under ψ_1 are open in S_{cont} because $(-\infty, a)$ and $(b,+\infty)$ form a subbasis for the topology on $\mathbb{R}$ ([8, p. 103]), where a and b are arbitrary real numbers.

Let us first consider the preimage of $(-\infty, a)$, given by

$$\begin{aligned}\psi_1^{-1}((-\infty,a)) &= \{s \in S_{\text{cont}} : \inf_{\chi\in\Omega_{1,s}} V(\chi) < a\} \\ &= \{s \in S_{\text{cont}} : \exists \zeta \in \Omega_{1,s} \text{ s.t. } V(\zeta) < a\}.\end{aligned}$$

Without loss of generality, $\psi_1^{-1}((-\infty,a))$ is assumed to be nonempty since the empty set is open in S_{cont} vacuously. Let $s \in \psi_1^{-1}((-\infty,a))$ and $\xi \in \Omega_{1,s}$ be such that $V(\xi) < a$. Define $\varepsilon_V := (a - V(\xi))/2 > 0$. Then by the continuity of V, there is $\delta_V > 0$ such that $|\zeta - \xi| < \delta_V$ with $\zeta \in \Omega$ implies $|V(\zeta) - V(\xi)| < \varepsilon_V$. Define

$$\delta := \begin{cases} \delta_0, & \text{if } |\xi| \geq r_{\text{cont}}, \quad (3a) \\ \frac{1}{2}\min(\delta_V, r_{\text{cont}} - |\xi|), & \text{if } |\xi| < r_{\text{cont}}, \quad (3b) \end{cases}$$

where δ_0 is any positive number. We show that the δ-neighborhood of s in S_{cont} is contained in the preimage, i.e., $(s-\delta, s+\delta) \cap S_{\text{cont}} \subseteq \psi_1^{-1}((-\infty,a))$. Let us consider σ satisfying $\sigma \in (s-\delta, s+\delta) \cap S_{\text{cont}}$ and $\sigma > |\xi|$. Note that in this case, $s \leq |\xi| < \sigma < r_{\text{cont}}$ because $\xi \in \Omega_{1,s}$ and $\sigma \in S_{\text{cont}}$. Define

$$\eta := \begin{cases} \delta[1\ 0\ \cdots\ 0]^\top, & \text{if } \xi = 0, \\ \left(1 + \frac{\delta}{|\xi|}\right)\xi, & \text{if } \xi \neq 0. \end{cases}$$

Then, we have $|\eta - \xi| = \delta \leq \delta_V/2 < \delta_V$ by (3b). It also holds that again by (3b),

$$\sigma < s + \delta \leq |\xi| + \delta \leq \frac{1}{2}(|\xi| + r_{\text{cont}}) < r_{\text{cont}}.$$

Since $|\eta| = |\xi| + \delta < r_{\text{cont}}$ from above and $B_{r_{\text{cont}}} \subset \Omega$, the vector η lies in Ω and thus, satisfies that $\eta \in \Omega_{1,\sigma} \cap \{\zeta \in \Omega : |\zeta - \xi| < \delta_V\}$. This implies

$$V(\eta) \leq |V(\eta) - V(\xi)| + V(\xi) < \varepsilon_V + V(\xi) < a,$$

leading to $\sigma \in \psi_1^{-1}((-\infty,a))$. On the other hand, if we consider $\sigma \in (s-\delta, s+\delta) \cap S_{\text{cont}}$ and $\sigma \leq |\xi|$, then $\eta \in \Omega_{1,\sigma}$ and $V(\eta) < a$ hold, where $\eta := \xi$. That is, again $\sigma \in \psi_1^{-1}((-\infty,a))$. Therefore, $(s-\delta, s+\delta) \cap S_{\text{cont}} \subseteq \psi_1^{-1}((-\infty,a))$ and hence, the preimage $\psi_1^{-1}((-\infty,a))$ is open in S_{cont}.

[1] The function ψ_1 can be discontinuous in S. For example, consider the compact set $\Omega = [-1,2]$ and the positive definite function

$$V(x) = \begin{cases} x^2, & -1 \leq x \leq 0, \\ 2x^2, & 0 < x \leq 2. \end{cases}$$

Then, V is of $\mathcal{C}^1$ but ψ_1 is discontinuous at $s = 1$. This is because $\psi_1(s) = s^2$ for $0 \leq s \leq 1$ but $\psi_1(s) = 2s^2$ for $1 < s \leq 2$. In general, the set of points in S at which ψ_1 is discontinuous is at most countable. This follows from the monotonicity of ψ_1 in s [7, Theorem 4.30].

We now turn our attention to the preimage of $(b,+\infty)$ under ψ_1, i.e.,

$$\psi_1^{-1}((b,+\infty)) = \{s \in S_{\text{cont}} : \inf_{\chi\in\Omega_{1,s}} V(\chi) > b\}.$$

Suppose again $\psi_1^{-1}((b,+\infty))$ is nonempty and let $s \in \psi_1^{-1}((b,+\infty))$. Define $\varepsilon_V := (\inf_{\chi\in\Omega_{1,s}} V(\chi) - b)/2 > 0$. Then, by the continuity of V on Ω, for any $\xi \in \Omega_{1,s}$, there is $\delta_{V,\xi} > 0$ such that each vector $\eta \in \Omega \cap B_{\delta_{V,\xi}}(\xi)$ guarantees $|V(\eta) - V(\xi)| < \varepsilon_V$, where $B_{\delta_{V,\xi}}(\xi) := \{\zeta \in \mathbb{R}^n : |\zeta - \xi| < \delta_{V,\xi}\}$. Note that $B_{\delta_{V,\xi}}(\xi)$, $\xi \in \Omega_{1,s}$ cover $\Omega_{1,s}$, i.e, $\Omega_{1,s} \subset \bigcup_{\xi\in\Omega_{1,s}} B_{\delta_{V,\xi}}(\xi)$. Thus by the compactness of $\Omega_{1,s}$, there are finitely many subcovers whose union again contains $\Omega_{1,s}$. Denote such sets as $B_{\delta_i}(\xi_i) := \{\zeta \in \mathbb{R}^n : |\zeta - \xi_i| < \delta_i\}$, $i = 1,\dots,p$. Since $\bigcup_{i=1}^p B_{\delta_i}(\xi_i)$ is open and includes $\Omega_{1,s}$ as a subset, $\Omega_{1,s} \cap \partial \bigcup_{i=1}^p B_{\delta_i}(\xi_i) = \varnothing$. Moreover, because $\partial \bigcup_{i=1}^p B_{\delta_i}(\xi_i)$ is bounded and the boundary of a set is always closed, $\partial \bigcup_{i=1}^p B_{\delta_i}(\xi_i)$ is compact. Therefore, the distance between the two compact sets $\Omega_{1,s}$ and $\partial \bigcup_{i=1}^p B_{\delta_i}(\xi_i)$ is well-defined and positive. Let us denote such distance as δ, i.e.,

$$\delta := \inf_{\zeta\in\partial\bigcup_{i=1}^p B_{\delta_i}(\xi_i),\chi\in\Omega_{1,s}} |\zeta - \chi| > 0.$$

Then, we claim that for each η satisfying $|\eta|_{\Omega_{1,s}} < \delta$, there is an index j such that $\eta \in B_{\delta_j}(\xi_j)$. If not, there is η such that $|\eta|_{\Omega_{1,s}} < \delta$ but $\eta \notin B_{\delta_j}(\xi_j)$ for every $j = 1,\dots,p$. This implies that η is an element of the complement of $\bigcup_{i=1}^p B_{\delta_i}(\xi_i)$ which is a closed set containing $\partial \bigcup_{i=1}^p B_{\delta_i}(\xi_i)$ as its boundary. As a result, $|\eta|_{\Omega_{1,s}} \ge \delta$ by the definition of δ. This contradicts to the assumption that $|\eta|_{\Omega_{1,s}} < \delta$ and thus, proves the claim. Finally, we show $(s-\delta, s+\delta) \cap S_{\text{cont}} \subseteq \psi_1^{-1}((b,+\infty))$. If $\sigma \in [s, s+\delta) \cap S_{\text{cont}}$, then we have $\Omega_{1,\sigma} \subseteq \Omega_{1,s}$, implying that $b < \inf_{\chi\in\Omega_{1,s}} V(\chi) \le \inf_{\chi\in\Omega_{1,\sigma}} V(\chi)$. On the other hand, if $\sigma \in (s-\delta, s) \cap S_{\text{cont}}$, then for every $\eta \in \Omega_{1,\sigma}$, $|\eta|_{\Omega_{1,s}} < \delta$ holds and thus, there is i with $1 \le i \le p$ such that $\eta \in B_{\delta_i}(\xi_i)$. This implies

$$V(\eta) \ge V(\xi_i) - |V(\eta) - V(\xi_i)| > \inf_{\chi\in\Omega_{1,s}} V(\chi) - \varepsilon_V.$$

Note that the last inequality is a consequence of the facts that $\xi_i \in \Omega_{1,s} \subseteq \Omega$, $\eta \in \Omega \cap B_{\delta_i}(\xi_i)$, and V is continuous on Ω. Since the inequality above holds for any $\eta \in \Omega_{1,\sigma}$ and the right-hand side is independent of η, we get

$$\inf_{\eta\in\Omega_{1,\sigma}} V(\eta) \ge \inf_{\chi\in\Omega_{1,s}} V(\chi) - \varepsilon_V > b,$$

yielding that $(s-\delta, s+\delta) \cap S_{\text{cont}} \subseteq \psi_1^{-1}((b,+\infty))$. Therefore, $\psi_1(s)$ is of $\mathcal{C}^0$ on $S_{\text{cont}} = [0, r_{\text{cont}})$.

A function $\alpha_1 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ that satisfies all the required properties is now constructed as follows. Let r_1 be any number such that $0 < r_1 < r_{\text{cont}}$, and define

$$\alpha_1(s) := \begin{cases} \bar{\psi}_1(s)\psi_1(s), & \text{if } 0 \le s \le r_1, \\ \bar{\alpha}_1(s), & \text{if } s > r_1, \end{cases}$$

where $\bar{\psi}_1$ and $\bar{\alpha}_1$ are any $\mathcal{C}^0$ strictly increasing function such that $0 \le \bar{\psi}_1(s) < 1$ for $0 \le s \le r_1$, $\lim_{s\to r_1^+} \bar{\alpha}_1(s) = \bar{\psi}_1(r_1)\psi_1(r_1)$, and $\bar{\alpha}_1(s) \le \psi_1(s)$ for $r_1 < s \le r$. Then, α_1 is strictly increasing and continuous on $\mathbb{R}_{\ge 0}$, and $\alpha_1(0) = 0$, i.e., $\alpha_1 \in \mathcal{K}$. Moreover, it holds that $V(x) \ge \psi_1(|x|) \ge \alpha_1(|x|)$ for all $x \in \Omega$.

(Existence of α_2): Let us define $\Omega_{2,s} := \Omega \cap \bar{B}_s = \Omega \cap \{\zeta \in \mathbb{R}^n : |\zeta| \le s\}$ and $\psi_2(s) := \sup_{\chi\in\Omega_{2,s}} V(\chi)$ for $s \in S$. Then, $\Omega_{2,s} \subseteq \Omega$ is nonempty and compact for each $s \in S$, and $x \in \Omega_{2,|x|}$ for any $x \in \Omega$. In addition, $\psi_2 : S \to \mathbb{R}_{\ge 0}$ is positive definite and $V(x) \le \psi_2(|x|)$ for all $x \in \Omega$. Note that ψ_2 is increasing because $\Omega_{2,s_1} \subset \Omega_{2,s_2}$ whenever $0 \le s_1 < s_2 \le r$. Moreover, it can be verified that ψ_2 is continuous on $S_{\text{cont}} = [0, r_{\text{cont}})$ since for $0 \le s < r_{\text{cont}}$, $\bar{B}_s \subset \Omega$ and hence, $\Omega_{2,s} = \bar{B}_s$.

With a positive constant $r_2 < r_{\text{cont}}$, let us define

$$\alpha_2(s) := \begin{cases} (1 + \bar{\psi}_2(s))\psi_2(s), & \text{if } 0 \le s \le r_2, \\ \bar{\alpha}_2(s), & \text{if } s > r_2, \end{cases}$$

where $\bar{\psi}_2$ and $\bar{\alpha}_2$ are any $\mathcal{C}^0$ strictly increasing function such that $\bar{\psi}_2(s) \ge 0$ for $0 \le s \le r_2$, $\lim_{s\to r_2^+} \bar{\alpha}_2(s) = (1 + \bar{\psi}_2(r_2))\psi_2(r_2)$, and $\bar{\alpha}_2(s) \ge \psi_2(s)$ for $r_2 < s \le r$. Then, α_2 satisfies all the required properties, i.e., $\alpha_2 \in \mathcal{K}$ on $\mathbb{R}_{\ge 0}$ and $V(x) \le \psi_2(|x|) \le \alpha_2(|x|)$ for all $x \in \Omega$. $\square$

Theorem 1 can be applied to various stability problems of nonlinear systems. Among them, we provide a stability result for perturbed nonlinear systems as an application. Consider a nonlinear system given by

$$\dot{x} = f(x) + \varepsilon g(t,x), \qquad x \in \mathbb{R}^n, \tag{4}$$

where $f : D \to \mathbb{R}^n$ is locally Lipschitz, $g : [0,\infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz on $[0,\infty) \times D$ uniformly in t, $g(t,0)$ is bounded for $t \ge 0$, and $D \subseteq \mathbb{R}^n$ is the domain. In addition, $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ is a constant, $\varepsilon_0 > 0$, $f(0) = 0$, and $0 \in D$. In (4), the second term $\varepsilon g(t,x)$ represents perturbations that may result from modeling errors, uncertainties, or disturbances.

Assumption 1: The origin $x = 0$ is an asymptotically stable equilibrium point for the nominal part of (4), i.e.,

$$\dot{x} = f(x), \tag{5}$$

where its region of attraction is given by $R_A \subseteq D$.

Theorem 2: Let Assumption 1 hold. Then, for any compact set $X \subset R_A$, there exists $\varepsilon^\star > 0$ such that for all $|\varepsilon| \le \varepsilon^\star$ and all $x(0) \in X$,

(a) the solution $x(t)$ of (4) satisfies

$$|x(t)| \le \beta(|x(0)|, t) + \gamma(|\varepsilon|), \qquad \forall t \ge 0, \tag{6}$$

where $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$; and

(b) if in addition, $f \in \mathcal{C}^1$, the perturbation is vanishing (i.e., $g(t,0) = 0$ for $t \geq 0$), and $x = 0$ is a *locally* exponentially stable equilibrium point for (5), then $x = 0$ of the perturbed system (4) is locally exponentially stable and $\lim_{t\to+\infty} x(t) = 0$ holds.

Proof: *Proof of (a):* Since the origin of (5) is asymptotically stable by Assumption 1, it follows from [1, Theorem 4.17] that there are positive definite functions $V \in \mathcal{C}^1$ and $W \in \mathcal{C}^0$, both of which are defined for all $x \in R_A$, such that

$$\begin{aligned} &V(x) \to +\infty \text{ as } x \to \partial R_A, \\ &\frac{\partial V}{\partial x} f(x) \leq -W(x), \quad \forall x \in R_A \end{aligned} \tag{7}$$

and every level set of $V(x)$ is a compact subset of R_A. In addition, by (7) and the fact that $X \subset R_A$, there always exists $d > 0$ such that $X \subseteq \Omega := \{x \in \mathbb{R}^n : V(x) \leq d\}$. Note that the level set $\Omega \subset R_A$ is compact.

Let $V(x)$ be a Lyapunov function candidate for the original perturbed system (4). Then, its time derivative along the solution of (4) is given by

$$\dot{V}(x) \leq -W(x) + \varepsilon \frac{\partial V}{\partial x} g(t,x), \quad \forall x \in \Omega,\ t \geq 0. \tag{8}$$

Since $W \in \mathcal{C}^0$ is positive definite and Ω is compact, Theorem 1 guarantees the existence of $\underline{\alpha} \in \mathcal{K}$ such that $W(x) \geq \underline{\alpha}(|x|)$ holds for all $x \in \Omega$. Moreover, a constant $M > 0$ exists such that

$$\left| \frac{\partial V}{\partial x} g(t,x) \right| \leq M, \qquad \forall x \in \Omega,\ t \geq 0,$$

because of the continuity and boundedness of $\partial V/\partial x$ and $g(t,x)$, respectively. These yield that

$$\dot{V}(x) \leq -\underline{\alpha}(|x|) + |\varepsilon| M, \quad \forall x \in \Omega. \tag{9}$$

Meanwhile, since $V \in \mathcal{C}^1$ is positive definite, there also exist $\alpha_1, \alpha_2 \in \mathcal{K}$ by Theorem 1 such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \Omega \tag{10}$$

holds, where $\alpha_1, \alpha_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Thus, (9) becomes

$$\dot{V} \leq -\alpha(V) + |\varepsilon| M, \quad \forall\, V \leq d,$$

where $\alpha := \underline{\alpha} \circ \alpha_2^{-1} \in \mathcal{K}$. Together with Lemma 1 in Appendix, this leads to, for each $|\varepsilon| \leq \varepsilon_1$ and $t \geq 0$,

$$V(x(t)) \leq \bar{\beta}(|V(x(0))|_{[0,\bar{\gamma}(|\varepsilon|M)]}, t) + \bar{\gamma}(|\varepsilon| M), \tag{11}$$

where $\varepsilon_1 > 0$, $\bar{\beta} \in \mathcal{KL}$, and $\bar{\gamma} := \alpha^{-1} \in \mathcal{K}$. Then, by (11) and (10),

$$\begin{aligned} |x(t)| &\leq \alpha_1^{-1}\left(\bar{\beta}(|V(x(0))|, t) + \bar{\gamma}(|\varepsilon| M)\right) \\ &\leq \alpha_1^{-1}\left(2\bar{\beta}(\alpha_2(|x(0)|), t)\right) + \alpha_1^{-1}\left(2\bar{\gamma}(|\varepsilon| M)\right) \end{aligned}$$

is satisfied for all $x(0) \in \Omega$. Note that the last inequality follows from the fact that α_1^{-1} is defined on $\mathbb{R}_{\geq 0}$. Therefore, the inequality (6) holds if we let $\beta(u,v) := \alpha_1^{-1}(2\bar{\beta}(\alpha_2(u), v))$ and $\gamma(u) := \alpha_1^{-1}(2\bar{\gamma}(Mu))$.

Proof of (b): Since the nominal system (5) is exponentially stable, the matrix $A := \frac{\partial f}{\partial x}(0)$ is Hurwitz by [1, Corollary 4.3] and thus, there is a symmetric positive definite matrix $P = P^\top > 0$ satisfying $A^\top P + PA = -I$. Let $V_0(x) := x^\top P x$. Then, its time derivative along the solution of (4) becomes

$$\dot{V}_0(x) = x^\top (A^\top P + PA) x + 2x^\top P(\varepsilon g(t,x) + \tilde{f}(x)),$$

where $\tilde{f}(x) := f(x) - Ax$. Note that by the definition of $\tilde{f}$, there exists $r_{e0} > 0$ such that $\bar{B}_{r_{e0}} \subseteq \Omega$ and $|\tilde{f}(x)| \leq |x|/(8\|P\|)$ holds for all $x \in \bar{B}_{r_{e0}}$, where $\|P\|$ is the induced 2-norm of P. Together with the local Lipschitzness of g, this implies that

$$\begin{aligned} \dot{V}_0(x) &\leq -|x|^2 + L_1|\varepsilon||x|^2 + 2\|P\||x||\tilde{f}(x)| \\ &\leq -(\tfrac{3}{4} - L_1|\varepsilon|)|x|^2, \qquad \forall |x| \leq r_{e0} \end{aligned}$$

for some $L_1 > 0$. If we set $\varepsilon_2 := 1/(4L_1)$, then for all $|\varepsilon| \leq \varepsilon_2$, we have $\dot{V}_0(x) \leq -\frac{1}{2}|x|^2$ for all $x \in \bar{B}_{r_{e0}}$. As a result, if $|x(t_1)| \leq r_e := r_{e0}\sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}}$ for some $t_1 \geq 0$, then $x(t)$ of the perturbed system (4) converges to the origin exponentially fast for all $t \geq t_1$, where $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the smallest and largest eigenvalues of P, respectively.

We now turn our attention to (8) which can be written as $\dot{V}(x) \leq -\underline{\alpha}(|x|) + L_2|\varepsilon||x|$ for all $x \in \Omega$ and for some $L_2 > 0$. Define $r := \max_{x\in\Omega}|x|$ and $\varepsilon_3 := \frac{\underline{\alpha}(r_e)}{2L_2 r}$. Then, for all $x \in \Omega \setminus B_{r_e}$ and $|\varepsilon| \leq \varepsilon_3$,

$$\dot{V}(x) \leq -\frac{1}{2}\underline{\alpha}(|x|) - \frac{1}{2}\underline{\alpha}(r_e) + \varepsilon_3 L_2 r = -\frac{1}{2}\underline{\alpha}(|x|)$$

holds. This implies that for any $x(0) \in \Omega$, the solution of (4) will enter the set $\bar{B}_{r_e}$ within a finite time, say $t = t_1$. Therefore, the result follows if we set $\varepsilon^\star := \min(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$. □

Remark 1: Theorem 2 complements the results of Theorem 4.18 and Lemmas 9.1, 9.2, and 9.3 of [1]. This is because except for the global case, they restrict the initial conditions to be ranged over a ball whose radius might be very small. In contrast to this, Theorem 2 holds for any initial conditions in an arbitrary compact subset of the region of attraction of the nominal system (5). This was possible due to the bounding inequality (2) which is valid for the entire compact set as in Theorem 1.

3. EXAMPLE

Consider the perturbed nonlinear system (4), where $x \in \mathbb{R}$, $f(x) := x(x+1)(x-10)$, and $g(t,x) := \sin(10t)$. Then, its nominal part $\dot{x} = f(x)$ satisfies Assumption 1 and the region of attraction of $x = 0$ is given by $R_A = \{x \in \mathbb{R} : -1 < x < 10\}$. Note that the other two equilibrium points $x = -1$ and $x = 10$ are unstable which can be readily verified via, e.g., the linearization.

Consider the Lyapunov function candidate for the nominal system given by

$$V(x) := \begin{cases} V_n(x), & \text{if } -1 < x \le 0, \\ V_p(x), & \text{if } 0 < x < 10, \end{cases}$$

where

$$V_n(x) := \begin{cases} 33.64 - \frac{6.584}{x+1} + \frac{0.3301}{(x+1)^2}, & \text{if } -1 < x \le -0.9, \\ x^2, & \text{if } -0.9 < x \le 0 \end{cases}$$

and

$$V_p(x) := \begin{cases} 2x^2, & \text{if } 0 < x \le 1, \\ 4x - 2, & \text{if } 1 < x \le 9, \\ 30 + \frac{4}{10-x}, & \text{if } 9 < x < 10. \end{cases}$$

Then, it can be shown that V is $\mathcal{C}^1$ and positive definite, $V(x) \to +\infty$ as $x \to \partial R_A$, and guarantees that $\frac{\partial V}{\partial x} f(x) \le -2x^2$ for each $x \in R_A$.

We now construct $\alpha_1, \alpha_2 \in \mathcal{K}$ such that the inequality (2) holds for $\Omega := [-0.9, 9] \subset R_A$. Note that the approach of [1, Lemma 4.3] is not applicable here because in [1], (2) can be valid only in the region that $|x| < r$ with $r < 1$. Recalling that $\psi_1(s) = \inf_{\chi \in \Omega_{1,s}} V(\chi)$ and $\psi_2(s) = \sup_{\chi \in \Omega_{2,s}} V(\chi)$, where they are given in the proof of Theorem 1, it can be verified that

$$\psi_1(s) = \begin{cases} V_n(-s), & \text{if } 0 \le s \le 0.9, \\ V_p(s), & \text{if } 0.9 < s \le 9, \end{cases}$$

and $\psi_2(s) = \sup_{\chi \in \Omega_{2,s}} V(\chi) = V_p(s)$ for all $s \in [0, 9]$. Note that ψ_1 is discontinuous at $s = 0.9$ and ψ_i, $i = 1, 2$ are already strictly increasing. Based on these functions, we define

$$\alpha_1(s) := \begin{cases} V_n(-s), & \text{if } 0 \le s \le 0.9, \\ \left(1 - \frac{e^{-(s-0.9)}}{2}\right) V_p(s), & \text{if } 0.9 < s \le 9 \end{cases}$$

and $\alpha_2(s) := (1.1 - 0.1e^{-s})\psi_2(s)$. The functions related are plotted in Fig. 1. It shows (2) is indeed valid on $[-0.9, 9]$.

[2]In case of $D = [0, d)$, d can be $+\infty$.

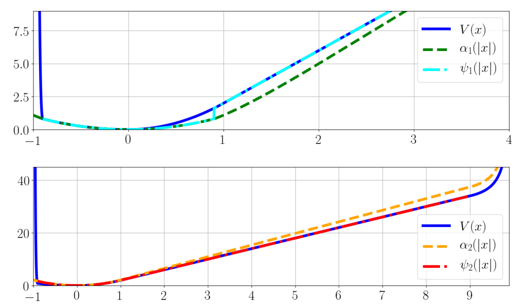

Fig. 1. Plot of $V(x)$, $\psi_i(|x|)$, and $\alpha_i(|x|)$.

Finally, a simulation of the perturbed system (4) is performed and its result is given in Fig. 2. It is seen that the state trajectory is ultimately bounded as expected by Theorem 2. Note that the ultimate bound may be calculated through the proof of Theorem 2 which however may be conservative than the simulation result.

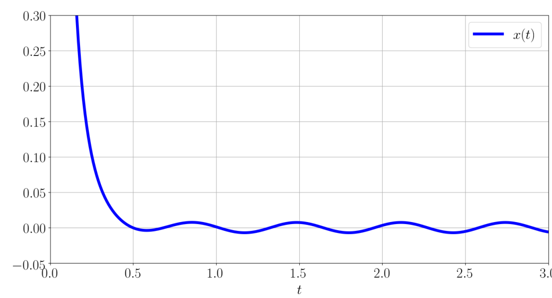

Fig. 2. A simulation result with $x(0) = 9$ and $\varepsilon = 0.1$.

4. CONCLUSION

In this note, class-$\mathcal{K}$ function bounds for positive definite functions have been studied. The emphasis of the result is on the region of validity of the bounding inequality which is the same as the whole compact domain of definition of the given positive definite function. An explicit construction of such class-$\mathcal{K}$ functions is also given, as well as an application of the result to the stability of perturbed systems.

APPENDIX

The lemma below is a generalization of [4, Lemma 4.4] and [1, Lemma 4.4] to where the time derivative of a function is not necessarily strictly negative.

Lemma 1: Let $t_0 \ge 0$ and $y(t)$ be a $\mathcal{C}^1$ function satisfying that $y(t) \ge 0$ for all $t \ge t_0$ and

$$\dot{y} \le -\alpha(y) + \varepsilon M, \quad y \in D, \tag{12}$$

where $\alpha \in \mathcal{K}$, $\varepsilon \in [0,\varepsilon_0]$, $\varepsilon_0 > 0$, and $M \geq 0$. The set D is either $[0,d]$ or $[0,d)$ with $d > 0$.[2] Then, there exist $\varepsilon_1 > 0$ and $\beta \in \mathcal{KL}$ such that for each $\varepsilon \in [0,\varepsilon_1]$,

$$y(t) \leq \beta(|y(t_0)|_{[0,\gamma(\varepsilon M)]}, t-t_0) + \gamma(\varepsilon M), \quad t \geq t_0 \tag{13}$$

holds for all $y(t_0) \in D$, where $\gamma := \alpha^{-1}$. Moreover, $y(t) \in D$ for all $t \geq t_0$.

Proof: First of all, we assume with no loss of generality that α is locally Lipschitz on D. If not, one can always find a locally Lipschitz function $\bar{\alpha} \in \mathcal{K}$ such that $\alpha(y) \geq \bar{\alpha}(y)$ holds, and proceed the proof with the replacement of (12) by $\dot{y} \leq -\bar{\alpha}(y) + \varepsilon M$.

If $M \neq 0$ and $\lim_{y\to d^-} \alpha(y)$ is finite, define $\varepsilon_1 := \min(\varepsilon_0, \lim_{y\to d^-} \alpha(y)/(2M))$. Otherwise, set $\varepsilon_1 := \varepsilon_0$. Then, for all $\varepsilon \in [0,\varepsilon_1]$, $\{0 \leq y \leq \gamma(\varepsilon M)\} \subset \{0 \leq y < d\}$ holds since $\gamma = \alpha^{-1} \in \mathcal{K}$ and $y \leq \gamma(\varepsilon M) \leq \gamma(\varepsilon_1 M) < d$. Moreover, for such ε, the set $\{0 \leq y \leq \gamma(\varepsilon M)\}$ is positively invariant because $\dot{y} \leq -\alpha(y) + \varepsilon M = 0$ whenever $y = \gamma(\varepsilon M)$. In other words, if $y(t_0) \in [0,\gamma(\varepsilon M)]$ and $\varepsilon \in [0,\varepsilon_1]$, we have

$$y(t) \leq \gamma(\varepsilon M) < d, \qquad t \geq t_0. \tag{14}$$

Let us now consider the case that $y(t_0) \in D \setminus [0,\gamma(\varepsilon M)]$. Let q be a number such that $\gamma(\varepsilon M) < q < d$ and define $\phi(y) := \int_q^y \frac{1}{-\alpha(u)+\varepsilon M} du$ for $y \in \mathbb{R}_{\geq 0} \setminus [0,\gamma(\varepsilon M)]$. Then, $\phi \in \mathcal{C}^1$ and is strictly decreasing. Moreover, by the local Lipschitzness of α, it holds that $\lim_{y\to\gamma(\varepsilon M)^+} \phi(y) = +\infty$. Indeed, let $L_\alpha > 0$ be such that $|\alpha(y) - \varepsilon M| \leq L_\alpha |y - \gamma(\varepsilon M)|$ for all $y \in [0,q]$. Then, for $\gamma(\varepsilon M) < y \leq q$,

$$\begin{aligned}\phi(y) &= \int_y^q \frac{1}{\alpha(u) - \varepsilon M} du \geq \int_y^q \frac{1}{L_\alpha (u - \gamma(\varepsilon M))} du \\ &= \{\ln(q - \gamma(\varepsilon M)) - \ln(y - \gamma(\varepsilon M))\}/L_\alpha\end{aligned}$$

is satisfied. Thus, $\phi(y) \to +\infty$ as $y \to \gamma(\varepsilon M)^+$.

Let $w := -\lim_{y\to+\infty} \phi(y) > 0$ (with the possibility that $w = +\infty$). Then, $\phi : \mathbb{R}_{\geq 0} \setminus [0,\gamma(\varepsilon M)] \xrightarrow{\text{onto}} (-w,+\infty)$, $\phi^{-1} : (-w,+\infty) \xrightarrow{\text{onto}} \mathbb{R}_{\geq 0} \setminus [0,\gamma(\varepsilon M)]$, and both of them are strictly decreasing. Let us define

$$\tilde{\beta}(u,v) := \begin{cases} 0, & \text{if } u \in [0,\gamma(\varepsilon M)], \\ \phi^{-1}(\phi(u)+v) - \gamma(\varepsilon M), & \text{otherwise,} \end{cases}$$

and $\beta(u,v) := \tilde{\beta}(u + \gamma(\varepsilon M), v)$. Let $t_1 := \inf\{t \in \mathbb{R} : 0 \leq y(t) \leq \gamma(\varepsilon M)\} = \inf\{t \in \mathbb{R} : |y(t)|_{[0,\gamma(\varepsilon M)]} = 0\}$. Since $y(t_0) \in D \setminus [0,\gamma(\varepsilon M)]$, $0 < t_1 \leq +\infty$ and $y(t) > \gamma(\varepsilon M)$ for $t \in [t_0,t_1)$. Moreover, for such t, $-\alpha(y(t)) + \varepsilon M < 0$ and thus, $y(t)$ is decreasing. Therefore, by (12), we have

$$\int_{t_0}^t \frac{\dot{y}(\tau)}{-\alpha(y(\tau)) + \varepsilon M} d\tau \geq \int_{t_0}^t d\tau = t - t_0.$$

The change of variable, $u = y(\tau)$, leads to the inequality

$$\int_{y(t_0)}^{y(t)} \frac{1}{-\alpha(u) + \varepsilon M} du \geq t - t_0$$

which is equivalent to $\phi(y(t)) \geq \phi(y(t_0)) + t - t_0$. This in turn yields that $y(t) \leq \phi^{-1}(\phi(y(t_0)) + t - t_0)$ and for $t \in [t_0,t_1)$,

$$\begin{aligned}y(t) &\leq \phi^{-1}\left(\phi(|y(t_0)|_{[0,\gamma(\varepsilon M)]} + \gamma(\varepsilon M)) + t - t_0\right) \\ &= \beta(|y(t_0)|_{[0,\gamma(\varepsilon M)]}, t - t_0) + \gamma(\varepsilon M).\end{aligned} \tag{15}$$

Thus, the inequality (13) follows from (14), (15), and the definition of t_1.

It only remains to show that $\beta \in \mathcal{KL}$ and $y(t) \in D$ for $t \geq t_0$. The continuity of β follows from the continuity of ϕ and ϕ^{-1} in their respective domains and $\lim_{u\to 0^+} \beta(u,v) = 0 = \beta(0,v)$. For each fixed v, $\beta(u,v)$ is of class-$\mathcal{K}$ because it is a strictly increasing function of u with its domain of definition being $\mathbb{R}_{\geq 0}$. Note that ϕ^{-1} and ϕ are both strictly decreasing. For each fixed u, $\beta(u,v)$ is decreasing in v and $\lim_{v\to+\infty} \beta(u,v) = 0$ by construction. As a consequence, $\beta \in \mathcal{KL}$. Finally, $y(t) \in D$ since (14) holds if $y(t_0) \in [0,\gamma(\varepsilon M)]$, and whenever $y(t_0) \in D \setminus [0,\gamma(\varepsilon M)]$, we have $y(t) \leq \beta(y(t_0) - \gamma(\varepsilon M), t - t_0) + \gamma(\varepsilon M) \leq \tilde{\beta}(y(t_0), 0) + \gamma(\varepsilon M) = y(t_0)$. □

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where $\alpha \in \mathcal{K}$, $\varepsilon \in [0, \varepsilon_0]$, $\varepsilon_0 > 0$, and $M \geq 0$. The set D is either $[0,d]$ or $[0,d)$ with $d > 0$.[2] Then, there exist $\varepsilon_1 > 0$ and $\beta \in \mathcal{KL}$ such that for each $\varepsilon \in [0, \varepsilon_1]$,

$$y(t) \leq \beta(|y(t_0)|_{[0,\gamma(\varepsilon M)]}, t - t_0) + \gamma(\varepsilon M), \ \ t \geq t_0 \tag{13}$$

holds for all $y(t_0) \in D$, where $\gamma := \alpha^{-1}$. Moreover, $y(t) \in D$ for all $t \geq t_0$.

Proof: First of all, we assume with no loss of generality that α is locally Lipschitz on D. If not, one can always find a locally Lipschitz function $\bar{\alpha} \in \mathcal{K}$ such that $\alpha(y) \geq \bar{\alpha}(y)$ holds, and proceed the proof with the replacement of (12) by $\dot{y} \leq -\bar{\alpha}(y) + \varepsilon M$.

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The change of variable, $u = y(\tau)$, leads to the inequality

$$\int_{y(t_0)}^{y(t)} \frac{1}{-\alpha(u) + \varepsilon M} du \geq t - t_0$$

which is equivalent to $\phi(y(t)) \geq \phi(y(t_0)) + t - t_0$. This in turn yields that $y(t) \leq \phi^{-1}(\phi(y(t_0)) + t - t_0)$ and for $t \in [t_0, t_1)$,

$$\begin{aligned} y(t) &\leq \phi^{-1}\left(\phi(|y(t_0)|_{[0,\gamma(\varepsilon M)]} + \gamma(\varepsilon M)) + t - t_0\right) \\ &= \beta(|y(t_0)|_{[0,\gamma(\varepsilon M)]}, t - t_0) + \gamma(\varepsilon M). \end{aligned} \tag{15}$$

Thus, the inequality (13) follows from (14), (15), and the definition of t_1.

It only remains to show that $\beta \in \mathcal{KL}$ and $y(t) \in D$ for $t \geq t_0$. The continuity of β follows from the continuity of ϕ and ϕ^{-1} in their respective domains and $\lim_{u\to 0^+} \beta(u,v) = 0 = \beta(0,v)$. For each fixed v, $\beta(u,v)$ is of class-$\mathcal{K}$ because it is a strictly increasing function of u with its domain of definition being $\mathbb{R}_{\geq 0}$. Note that ϕ^{-1} and ϕ are both strictly decreasing. For each fixed u, $\beta(u,v)$ is decreasing in v and $\lim_{v\to+\infty} \beta(u,v) = 0$ by construction. As a consequence, $\beta \in \mathcal{KL}$. Finally, $y(t) \in D$ since (14) holds if $y(t_0) \in [0, \gamma(\varepsilon M)]$, and whenever $y(t_0) \in D \setminus [0, \gamma(\varepsilon M)]$, we have $y(t) \leq \beta(y(t_0) - \gamma(\varepsilon M), t - t_0) + \gamma(\varepsilon M) \leq \tilde{\beta}(y(t_0), 0) + \gamma(\varepsilon M) = y(t_0)$. □

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