MATHEMATICS

ON THE ALGEBRAIC APPROXIMATION OF FUNCTIONS. III

BY

JOHN COATES

(Communicated by Prof. J. POPKEN at the meeting of January 29, 1966)

VII.

19. We next study the properties of function vectors which are normal at *every* system, or in the notation of Mahler, are *perfect*. Perfectness is thus a global property. We shall now establish a number of global properties of such perfect function vectors.

Firstly, there is a global analogue of the First Uniqueness Theorem.

Third Uniqueness Theorem. (Mahler, (8)). *If the function vector* $\mathbf{f}$ *is perfect, and*

$$\mathfrak{a}_k(\varrho_1\varrho_2\cdots\varrho_m)\ ,\ \mathfrak{w}_{jk}(\varrho_1\varrho_2\cdots\varrho_m) \qquad (j,k=1,2,\ldots,m)$$

are a non-trivial system of German polynomials and its remainders belonging to the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, *then*

1. $\overline{|\mathfrak{a}_k(\varrho_1\varrho_2\cdots\varrho_m)|} = \sigma-\varrho_k \qquad (k=1,2,\ldots,m);$

2. *every system of German polynomials belonging to the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$ *is a constant multiple of the system*

$$\mathfrak{a}_k(\varrho_1\varrho_2\cdots\varrho_m) \qquad (k=1,2,\ldots,m);$$

3. *at least one of the remainders*

$$\mathfrak{w}_{jk}(\varrho_1\varrho_2\cdots\varrho_m) \qquad (j,k=1,2,\ldots,m)$$

has order equal to $\sigma+1$.

Proof. Firstly, we prove part 1. Suppose that, on the contrary, there exists an integer l, with $1\leqslant l\leqslant m$, such that

$$\overline{|\mathfrak{a}_l(\varrho_1\varrho_2\cdots\varrho_m)|} < \sigma-\varrho_l.$$

Then the polynomials

$$\mathscr{C}_h = \sum_{k=1}^{m} A_{hk}(\varrho_1+\delta_{l1}\,\varrho_2+\delta_{l2}\cdots\varrho_m+\delta_{lm})\ \mathfrak{a}_k(\varrho_1\varrho_2\cdots\varrho_m) \qquad (h=1,2,\ldots,m)$$

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has order equal to $\sigma+1$.

Proof. Firstly, we prove part 1. Suppose that, on the contrary, there exists an integer l, with $1 \leqslant l \leqslant m$, such that

$$\overline{|\mathfrak{a}_l(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \sigma - \varrho_l .$$

Then the polynomials

$$\mathscr{C}_h = \sum_{k=1}^{m} A_{hk}(\varrho_1 + \delta_{l1} \varrho_2 + \delta_{l2} \cdots \varrho_m + \delta_{lm})\, \mathfrak{a}_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (h=1, 2, \ldots, m)$$

ON THE ALGEBRAIC APPROXIMATION OF FUNCTIONS. II

BY

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IV.

10. We now introduce the *local* property of *normality* at one system $\varrho_1, \varrho_2, \ldots, \varrho_m$.

Definition: *The function vector* $\boldsymbol{f}$ *is said to be normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$ *if*

1. *the function vector* $\boldsymbol{f}$ *vanishes at none of the primes in* Π; *and*
2. *for each suffix* $h = 1, 2, \ldots, m$, *there exists a system of Latin polynomials*

$$\bar{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

such that

$$\overline{|\bar{a}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \varrho_h.$$

For the rest of this part, let the function vector $\boldsymbol{f}$ be normal at the fixed, but arbitrary, system $\varrho_1, \varrho_2, \ldots, \varrho_m$.

To avoid having unwieldy constants in our formulae, it is convenient to introduce a slight change in the notation of the last part. Put

$$A_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \frac{\bar{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)}{\bar{\alpha}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)} \qquad (h, k = 1, 2, \ldots, m),$$

$$R_h(\varrho_1 \varrho_2 \cdots \varrho_m) = \frac{\bar{r}_h(\varrho_1 \varrho_2 \cdots \varrho_m)}{\bar{\alpha}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)} \qquad (h = 1, 2, \ldots, m),$$

so that

$$R_h(\varrho_1 \varrho_2 \cdots \varrho_m) = \sum_{k=1}^{m} A_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, f_k \qquad (h = 1, 2, \ldots, m),$$

where the constant $\bar{\alpha}_{hh}(\varrho_1\, \varrho_2 \ldots \varrho_m)$ is the coefficient of ψ_{ϱ_h} in the interpolation series for $\bar{a}_{hh}(\varrho_1\, \varrho_2 \ldots \varrho_m)$. By this definition, the coefficient of ψ_{ϱ_h} in the interpolation series for $A_{hh}(\varrho_1\, \varrho_2 \ldots \varrho_m)$ is equal to 1. Let $A(\varrho_1\, \varrho_2 \ldots \varrho_m)$ be the $m \times m$ matrix

$$A(\varrho_1 \varrho_2 \cdots \varrho_m) = A_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)_{h,k=1,2,\ldots,m},$$

and let $D(\varrho_1\, \varrho_2 \ldots \varrho_m)$ be the determinant of this matrix. The degree of this determinant is equal to σ, and thus, by the result of § 7, the value of the determinant is

$$D(\varrho_1 \varrho_2 \cdots \varrho_m) = \alpha \psi_\sigma, \text{ with } \alpha \neq 0 \in F.$$

But, expanding the determinant, we see that the coefficient of ψ_σ in its interpolation series is equal to 1, and therefore, more exactly,

$$D(\varrho_1 \varrho_2 \cdots \varrho_m) = \psi_\sigma.$$

First Uniqueness Theorem. *If the function vector* $\mathbf{f}$ *is normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$, *and, if for each suffix* $h = 1, 2, \ldots, m$,

$$\mathfrak{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \;,\; \mathfrak{w}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j, k = 1, 2, \ldots, m)$$

are any non-trivial system of German polynomials, and its remainders, belonging to the system

$$\varrho_1 - \delta_{h1}, \; \varrho_2 - \delta_{h2}, \; \ldots, \; \varrho_m - \delta_{hm},$$

then, for $h = 1, 2, \ldots, m$,

1. $\overline{|\mathfrak{a}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \sigma - \varrho_h$;

2. *every system of German polynomials belonging to the system* $\varrho_1 - \delta_{h1}$, $\varrho_2 - \delta_{h2}, \ldots, \varrho_m - \delta_{hm}$ *is a constant multiple of the system*

$$\mathfrak{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m);$$

3. *at least one of the remainders*

$$R_h(\varrho_1 \varrho_2 \cdots \varrho_m) \;,\; \mathfrak{w}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j, k = 1, 2, \ldots, m)$$

has order equal to σ.

Proof. Firstly, we prove part 1. Suppose that, on the contrary, there exists an integer l, with $1 \leqslant l \leqslant m$, such that

$$\overline{|\mathfrak{a}_{ll}(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \sigma - \varrho_l.$$

The polynomials

$$\mathscr{C}_j = \sum_{k=1}^{m} A_{jk}(\varrho_1 \varrho_2 \cdots \varrho_m) \, \mathfrak{a}_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j = 1, 2, \ldots, m)$$

are then expressions of the form $e\begin{pmatrix} r_1 \, r_2 \ldots r_m \, s \\ \mathfrak{w}_1 \, \mathfrak{w}_2 \ldots \mathfrak{w}_m \, \mathfrak{s} \end{pmatrix}$, with parameter values

$$r_k = \varrho_k + \delta_{jk} \;,\; \mathfrak{w}_k = \varrho_k,$$
$$s = \sigma + 1 \quad,\; \mathfrak{s} = \sigma - 1.$$

From these values, (D) and (0) give the estimates

$$\overline{|\mathscr{C}_j|} \leqslant \max_{k=1,\ldots,m} \{\varrho_k + \delta_{jk} - 1) + (\sigma - 1 - \varrho_k)\} \leqslant \sigma - 1,$$

$$\underline{|\mathscr{C}_j|} \geqslant \min\{\sigma, \sigma\} = \sigma.$$

Hence all the polynomials $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_m$ must be zero. Thus we have m homogeneous linear equations, with non-zero determinant $D(\varrho_1 \varrho_2 \cdots \varrho_m)$, for the non-trivial system of polynomials

$$\mathfrak{a}_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\ldots,m).$$

But this is impossible, whence the assertion.

Secondly, we show that part 1 implies part 2. For any integer h, with $1 \leqslant h \leqslant m$, let

$$\mathfrak{a}^*_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ \mathfrak{w}^*_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j,k=1,2,\ldots,m)$$

be a non-trivial system of German polynomials and its remainders, belonging to the system $\varrho_1-\delta_{h1}, \varrho_2-\delta_{h2}, \ldots, \varrho_m-\delta_{hm}$. Then we can choose a constant β such that

$$\overline{|\mathfrak{a}^*_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m) - \beta \mathfrak{a}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \sigma - \varrho_h.$$

Then the new system of polynomials

$$\mathfrak{a}^{**}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \mathfrak{a}^*_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) - \beta \mathfrak{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\ldots,m)$$

and its remainders satisfy the inequalities

$$\overline{|\mathfrak{a}^{**}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)|} \leqslant \sigma - \varrho_k - 1 \qquad (k=1,2,\ldots,m),$$

$$\underline{|\mathfrak{w}^{**}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m)|} \geqslant \sigma \qquad (j,k=1,2,\ldots,m).$$

But this is impossible, by part 1, unless the new system is trivial, and so the assertion follows.

Finally, we prove part 3. Suppose that, on the contrary, there exists an integer l, with $1 \leqslant l \leqslant m$, such that all the remainder series

$$R_l(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ \mathfrak{w}_{ljk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j,k=1,2,\ldots,m)$$

have order greater than σ. Then the polynomial

$$\mathscr{C}_l = \sum_{k=1}^{m} A_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, \mathfrak{a}_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m)$$

is an expression of the form $e\begin{pmatrix} r_1\, r_2 \ldots r_m\, s \\ \mathfrak{w}_1\, \mathfrak{w}_2 \ldots \mathfrak{w}_u\, \mathfrak{s} \end{pmatrix}$, which, by (D) and (0), is easily seen to satisfy

$$\overline{|\mathscr{C}_l|} = \sigma,$$

$$\underline{|\mathscr{C}_l|} \geqslant \min\,\{\sigma+1, \sigma+1\} = \sigma+1.$$

But this is impossible, whence the assertion. This completes the proof.

11. We next prove an analogous theorem for the Latin polynomials. Put

$$\mathfrak{A}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \frac{\mathfrak{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)}{\beta_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)} \qquad (h, k = 1, 2, \ldots, m),$$

$$\mathfrak{R}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \frac{\mathfrak{w}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m)}{\beta_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)} \qquad (h, j, k = 1, 2, \ldots, m),$$

so that

$$\mathfrak{R}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \mathfrak{A}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) f_j - \mathfrak{A}_{hj}(\varrho_1 \varrho_2 \cdots \varrho_m) f_k \qquad (h, j, k = 1, 2, \ldots, m),$$

where the constant $\beta_{hh}(\varrho_1 \varrho_2 \ldots \varrho_m)$ is the coefficient of $\psi_{\sigma-\varrho_h}$ in the interpolation series for $\mathfrak{a}_{hh}(\varrho_1 \varrho_2 \ldots \varrho_m)$. The oecfficient of $\psi_{\sigma-\varrho_h}$ in the interpolation series for $\mathfrak{A}_{hh}(\varrho_1 \varrho_2 \ldots \varrho_m)$ is therefore 1, and thus, by the First Uniqueness Theorem, the polynomial systems and their remainders

$$\mathfrak{A}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \;,\; \mathfrak{R}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (h, j, k = 1, 2, \ldots, m)$$

are uniquely determined. Let $\mathfrak{A}(\varrho_1 \varrho_2 \ldots \varrho_m)$ be the $m \times m$ matrix

$$\mathfrak{A}(\varrho_1 \varrho_2 \cdots \varrho_m) = (\mathfrak{A}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m))_{h,k=1,2,\ldots,m},$$

and let $\mathfrak{D}(\varrho_1 \varrho_2 \ldots \varrho_m)$ be the determinant of this matrix. The degree of this determinant is equal to $(m-1)\sigma$, and thus, by the results of § 8, the value of the determinant is

$$\mathfrak{D}(\varrho_1 \varrho_2 \cdots \varrho_m) = \beta \psi_\sigma^{m-1}, \text{ with } \beta \neq 0 \in F.$$

But, expanding the determinant, we see that the coefficient of ψ_σ^{m-1} in its expansion is equal to 1, and therefore

$$\mathfrak{D}(\varrho_1 \varrho_2 \cdots \varrho_m) = \psi_\varrho^{m-1}.$$

Second Uniqueness Theorem. *If the function vector* $\mathbf{f}$ *is normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$, *and if for each suffix* $h = 1, 2, \ldots, m$,

$$a_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \;,\; r_h(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

are any non-trivial system of Latin polynomials, and its remainder, belonging to the system

$$\varrho_1 + \delta_{h1},\; \varrho_2 + \delta_{h2},\; \ldots,\; \varrho_m + \delta_{hm},$$

then, for $h = 1, 2, \ldots, m$,

1. $\overline{|a_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \varrho_h$;

2. *every system of Latin polynomials belonging to the system* $\varrho_1 + \delta_{h1}$, $\varrho_2 + \delta_{h2}, \ldots, \varrho_m + \delta_{hm}$ *is a constant multiple of the system*

$$a_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m);$$

3. *at least one of the remainders*

$$r_h(\varrho_1 \varrho_2 \cdots \varrho_m),\; \mathfrak{R}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j, k = 1, 2, \ldots, m)$$

has order equal to σ.

Proof. The proof is completely analogous to the proof of the First Uniqueness Theorem. Firstly ,we prove part 1. Suppose, on the contrary, that there exists an integer l, with $1 \leqslant l \leqslant m$, such that

$$\overline{|a_{ll}(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \varrho_l.$$

Then the polynomials

$$\mathscr{E}_j = \sum_{k=1}^{m} \mathfrak{A}_{jk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, a_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j = 1, 2, \ldots, m)$$

are expressions of the forme $e\begin{pmatrix} r_1\, r_2 \ldots r_m\, s \\ \mathfrak{w}_1\, \mathfrak{w}_2 \ldots \mathfrak{w}_m\, \mathfrak{s} \end{pmatrix}$, with parameter values

$$r_k = \varrho_k \quad , \ \mathfrak{w}_k = \varrho_k - \delta_{jk},$$

$$s = \sigma + 1 \ , \quad \mathfrak{s} = \sigma - 1.$$

From these values, (D) and (0) give the estimates

$$\overline{|\mathscr{E}_j|} \leqslant \max_{k=1,\ldots,m} \{(\varrho_k - 1) + (\sigma - 1 - \varrho_k + \delta_{jk})\} \leqslant \sigma - 1,$$

$$\underline{|\mathscr{E}_j|} \geqslant \min\, \{\sigma, \sigma\} = \sigma.$$

Hence all the polynomials $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_m$ are zero. Thus we have m homogeneous linear equations, with non-zero determinant $\mathfrak{D}(\varrho_1, \varrho_2 \ldots \varrho_m)$, for the non-trivial system of polynomials

$$a_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m).$$

But this is impossible, whence the assertion.

Secondly, we show that part 1 implies part 2. For any integer h, with $1 \leqslant h \leqslant m$, let

$$a^*_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ r^*_h(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

be a non-trivial system of Latin polynomials, and its remainder, belonging to the system $\varrho_1 + \delta_{h1}, \varrho_2 + \delta_{h2}, \ldots, \varrho_m + \delta_{hm}$. Then we can choose a constant α so that

$$\overline{|a^*_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m) - \alpha a_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \varrho_h.$$

Thus the new system of polynomials

$$a^{**}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) = a^*_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) - \alpha a_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m),$$

and its remainder satisfy the inequalities

$$\overline{|a^{**}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)|} \leqslant \varrho_k - 1 \qquad (k = 1, 2, \ldots, m),$$

$$\underline{|r^{**}_h(\varrho_1 \varrho_2 \cdots \varrho_m)|} \geqslant \sigma.$$

But this is impossible, by part 1, unless the new system is trivial, and the assertion follows.

Finally, we prove part 3. Suppose that, on the contrary, there exists an integer l, with $1 \leqslant l \leqslant m$, such that all the remainders

$$r_l(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ \mathfrak{R}_{ljk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j, k = 1, 2, \ldots, m)$$

have order greater than σ. Then the polynomial

$$\mathscr{E}_l = \sum_{k=1}^{m} \mathfrak{A}_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, a_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m)$$

is an expression of the forme $e\begin{pmatrix} r_1\, r_2 \ldots r_m\, s \\ \mathfrak{w}_1\, \mathfrak{w}_2 \ldots \mathfrak{w}_m\, \mathfrak{s} \end{pmatrix}$, which, by (D) and (0), is easily seen to satisfy

$$\overline{|\mathscr{E}_l|} = \sigma,$$

$$\underline{|\mathscr{E}_l|} \geqslant \min \{\sigma+1, \sigma+1\} = \sigma+1.$$

But this is impossible, whence the assertion. This completes the proof.

12. The original definition of normality was given in terms of the Latin polynomials. However, we could equally well have defined normality in terms of the German polynomials, as is shown by the following criterion.

Criterion 1. *The function vector* $\boldsymbol{f}$ *is normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$ *if and only if*

1. *the function vector* $\boldsymbol{f}$ *vanishes at none of the primes in* Π; *and*

2. *for each suffix* $h = 1, 2, \ldots, m$, *there exists a system of German polynomials*

$$\bar{\mathfrak{a}}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

such that

$$\overline{|\bar{\mathfrak{a}}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \sigma - \varrho_h.$$

Proof. The necessity is an immediate consequence of the First Uniqueness Theorem. The sufficiency follows by noting that, if the conditions (1) and (2) hold, then, by repeating the argument in § 11, the Second Uniqueness Theorem can be proved independently of the First Uniqueness Theorem. In particular, this would prove that the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1, \varrho_2, \ldots, \varrho_m$.

It is now worthwhile to review the basic facts on normality, which we have proven so far in this part. Essentially, we have shown that, given the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, if either of the determinants

$$d(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ \mathfrak{d}(\varrho_1 \varrho_2 \cdots \varrho_m)$$

is non-zero, then the approximation is locally unique in the following sense. Firstly, the Latin and German matrices

$$A(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ \mathfrak{A}(\varrho_1 \varrho_2 \cdots \varrho_m)$$

are uniquely determined, and these matrices have non-zero determinants

$$D(\varrho_1\varrho_2\cdots\varrho_m)=\psi_\sigma\;,\;\mathfrak{D}(\varrho_1\varrho_2\cdots\varrho_m)=\psi_\sigma^{m-1},$$

respectively. Secondly, for $h=1, 2, \ldots, m$, the Latin and German remainders

$$R_h(\varrho_1\varrho_2\cdots\varrho_m)\;,\;\mathfrak{R}_{hjk}(\varrho_1\varrho_2\cdots\varrho_m)\qquad (j,k=1,2,\ldots,m)$$

are also uniquely determined, and at least one of them has order equal to σ.

13. In the theory given so far there has always been a complete symmetry between the Latin and German polynomials. However, we now give a criterion for normality in terms of the Latin polynomials, where there does not appear to be an analogous criterion in terms of the German polynomials.

Criterion 2. *The function vector* $\boldsymbol{f}$ *is normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$ *if and only if*

1. *the function vector* $\boldsymbol{f}$ *vanishes at none of the primes in* Π;

2. *there exists no non-trivial system of Latin polynomials, which, together with its remainder, satisfies the inequalities*

$$\underline{|a_k(\varrho_1\varrho_2\cdots\varrho_m)|}\leqslant\varrho_k-1\qquad (k=1,2,\ldots,m),$$

$$\underline{|\;r(\varrho_1\varrho_2\cdots\varrho_m)|}>\sigma-1.$$

Proof. The necessity is an immediate consequence of the Second Uniqueness Theorem. Conversely, the sufficiency is obvious.

Criterion 2 implies the following local uniqueness property of the approximation.

Corollary. *If the function vector* $\boldsymbol{f}$ *is normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$, *then the Latin polynomial system belonging to the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$ *is uniquely determined except for a constant factor.*

Proof. Let

$$a_k(\varrho_1\varrho_2\cdots\varrho_m)\;,\;a_k^*(\varrho_1\varrho_2\cdots\varrho_m)\qquad (k=1,2,\ldots,m)$$

be any two systems of Latin polynomials belonging to the system $\varrho_1, \varrho_2, \ldots, \varrho_m$ at which the function vector $\boldsymbol{f}$ is supposed normal. Then we can choose a constant α such that their respective remainders satisfy

$$\underline{|r(\varrho_1\varrho_2\cdots\varrho_m)-\alpha r^*(\varrho_1\varrho_2\cdots\varrho_m)|}>\sigma-1.$$

But

$$a_k(\varrho_1\varrho_2\cdots\varrho_m)-\alpha a_k^*(\varrho_1\varrho_2\cdots\varrho_m)\qquad (k=1,2,\ldots,m)$$

is a system of Latin polynomials belonging to the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, and therefore, by Criterion 2, it must be trivial. This completes the proof.

V.

14. We next prove a remarkable theorem which asserts that the *local* property of normality implies certain *global* properties of the approximation.

We begin by introducing the notion of a *normality zigzag*. An infinite set of systems

$$\Sigma = \{(\varrho_1^{(n)}, \varrho_2^{(n)}, \ldots, \varrho_m^{(n)}) \mid n = 0, 1, \ldots\}$$

is said to be a *normality zigzag* of the function vector $\boldsymbol{f}$ if

(1) the function vector f is normal at every system in Σ;

(2) $\varrho_1^{(0)} = 0, \varrho_2^{(0)} = 0, \ldots, \varrho_m^{(0)} = 0$;

(3) for all non-negative integers n, there exists an integer h_n, with $1 \leqslant h_n \leqslant m$, such that

$$\varrho_1^{(n+1)} = \varrho_1^{(n)} + \delta_{h_n 1}\ , \ \varrho_2^{(n+1)} = \varrho_2^{(n)} + \delta_{h_n 2}, \ldots,\ \varrho_m^{(n+1)} = \varrho_m^{(n)} = \delta_{h_m m}.$$

We note that every function vector, which vanishes at none of the primes in Π, is normal at the system 0, 0, …, 0. As before, we write systems in Σ without brackets around them when there is no danger of confusion.

Normality Zigzag Theorem. *The function vector $\boldsymbol{f}$ is normal at the system $\varrho_1, \varrho_2, \ldots, \varrho_m$ if and only if this system belongs to at least one normality zigzag of the function vector.*

Proof. The sufficiency is obvious. Conversely, let us suppose that the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1, \varrho_2, \ldots, \varrho_m$. We shall construct a normality zigzag

$$\Sigma = \{(\varrho_1^{(n)}, \varrho_2^{(n)}, \ldots, \varrho_m^{(n)}) \mid n = 0, 1, \ldots\}, \text{ with } \varrho_1^{(\sigma)} = \varrho_1, \varrho_2^{(\sigma)} = \varrho_2, \ldots, \varrho_m^{(\sigma)} = \varrho_m,$$

to which the system $\varrho_1, \varrho_2, \ldots, \varrho_m$ belongs. This construction will use all the facts which we have so far proven on normality. The proof is divided into two parts, the descent and the ascent.

Firstly, we construct the systems

(4) $$\varrho_1^{(n)}, \varrho_2^{(n)}, \ldots, \varrho_m^{(n)} \text{ where } n = 0, 1, \ldots, \sigma.$$

We can suppose that the system $\varrho_1, \varrho_2, \ldots, \varrho_m$ is non-trivial, otherwise there is nothing to prove. If

$$a_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

is a non-trivial system of Latin polynomials belonging to the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, then there exists an integer l, with $1 \leqslant l \leqslant m$, such that ϱ_l is positive and

(5) $$\overline{|a_l(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \varrho_l - 1.$$

For suppose that, on the contrary

$$\overline{|a_k(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \varrho_k - 1 \qquad (k = 1, 2, \ldots, m).$$

Then the new system of Latin polynomials

$$a_k^*(\varrho_1 \varrho_2 \cdots \varrho_m) = p_\sigma a_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\ldots,m)$$

is non-trivial, and, together with its remainder, satisfies the inequalities

$$\overline{|a_k^*(\varrho_1 \varrho_2 \cdots \varrho_m)|} \leqslant \varrho_k - 1 \qquad (k=1,2,\ldots,m),$$
$$\underline{|r^*(\varrho_1 \varrho_2 \cdots \varrho_m)|} > \sigma - 1.$$

But, since the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, this is impossible by Criterion 2, whence the assertion (5). Further, by the Corollary to Criterion 2, we conclude that every non-trivial system of Latin polynomials belonging to the system $\varrho_1, \varrho_2, \ldots, \varrho_m$ satisfies (5). I assert that we can take

$$\varrho_1^{(\sigma-1)} = \varrho_1 - \delta_{l1}\ ,\ \varrho_2^{(\sigma-1)} = \varrho_2 - \delta_{l2}, \ldots,\ \varrho_m^{(\sigma-1)} = \varrho_m - \delta_{lm}.$$

To prove this, it suffices to show that the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1 - \delta_{l1}, \varrho_2 - \delta_{l2}, \ldots, \varrho_m - \delta_{m_{lm}}$. But this follows immediately from (5), since (5) implies that there exists no non-trivial system of Latin polynomials satisfying the inequalities

$$\overline{|a_k(\varrho_1 - \delta_{l1}\, \varrho_2 - \delta_{l2} \cdots \varrho_m - \delta_{lm})|} \leqslant \varrho_k - \delta_{lk} - 1 \qquad (k=1,2,\ldots,m),$$
$$\underline{|\ r(\varrho_1 - \delta_{l1}\, \varrho_2 - \delta_{l2} \cdots \varrho_m - \delta_{lm})|} > \sigma - 2.$$

If the system

$$\varrho_1 - \delta_{l1}\ ,\ \varrho_2 - \delta_{l2}, \ldots,\ \varrho_m - \varrho_{lm}$$

is non-trivial, we can repeat this procedure, until, after σ steps, we obtain the trivial system

$$0\ ,\ 0, \ldots,\ 0.$$

The function vector $\boldsymbol{f}$ is then normal at all systems so constructed, and this therefore gives the systems (4).

Secondly, we construct the systems

$$\varrho_1^{(n)}, \varrho_2^{(n)}, \ldots, \varrho_m^{(n)} \text{ where } n = \sigma+1, \sigma+2, \ldots. \tag{6}$$

If

$$\mathfrak{a}_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\ldots,m)$$

is a non-trivial system of German polynomials belonging to the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, then there exists, by the First Uniqueness Theorem, an integer j, with $1 \leqslant j \leqslant m$, such that

$$\overline{|\mathfrak{a}_j(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \sigma - \varrho_j. \tag{7}$$

I assert that the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1 + \delta_{j1}$, $\varrho_2 + \delta_{j2}, \ldots, \varrho_m + \delta_{jm}$, so that we can take

$$\varrho_1^{(\sigma+1)} = \varrho_1 + \delta_{j1}\ ,\ \varrho_2^{(\sigma+1)} = \varrho_2 + \delta_{j2}, \ldots,\ \varrho_m^{(\sigma+1)} = \varrho_m + \delta_{jm}.$$

For suppose that, on the contrary,

$$\overline{|R_j(\varrho_1 \varrho_2 \cdots \varrho_m)|} > \sigma.$$

Then the polynomial

$$\mathscr{C}_j = \sum_{k=1}^{m} A_{jk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, \mathfrak{a}_k(\varrho_1 \varrho_2 \cdots \varrho_m)$$

is an expression of the forme $e\begin{pmatrix} r_1\, r_2 \ldots r_m\, s \\ \mathfrak{w}_1\, \mathfrak{w}_2 \ldots \mathfrak{w}_m\, \mathfrak{s} \end{pmatrix}$, with parameter values

$$r_k = \varrho_k + \delta_{jk} \;,\; \mathfrak{w}_k = \varrho_k,$$

$$s = \sigma + 2 \quad , \quad \mathfrak{s} = \sigma.$$

(D) and (0) therefore give the estimates

$$\overline{|\mathscr{C}_j|} \leqslant \max_{k=1,\ldots,m} \{(\varrho_k + \delta_{jk} - 1) + (\sigma - \varrho_k)\} \leqslant \sigma,$$

$$\underline{|\mathscr{C}_j|} \geqslant \min\, \{\sigma + 1, \sigma + 1\} = \sigma + 1.$$

However, by equation (7), it is clear that in fact

$$\overline{|\mathscr{C}_j|} = \sigma.$$

But this is impossible, whence the assertion.

We can repeat this procedure for the system

$$\varrho_1 + \delta_{j1} \;,\; \varrho_2 + \delta_{j2}, \ldots, \varrho_m + \delta_{jm}$$

and continue in this manner indefinitely. The function vector $\boldsymbol{f}$ is then normal at all systems so constructed, and this therefore gives the systems (6).

On taking together the systems in (4) and (6), we obtain the required normality zigzag. This completes the proof.

Since every function vector, which vanishes at none of the primes in Π, is trivially normal at the system $0, 0, \ldots, 0$, the Normality Zigzag Theorem has the following immediate corollary.

Crollary. *Every function vector* $\boldsymbol{f}$, *which vanishes at none of the primes in* Π, *is normal at infinitely many systems* $\varrho_1, \varrho_2, \ldots, \varrho_m$.

However, this result is, as one would expect, very weak, and it is trivial when all the primes in Π are equal.

15. The function vector f has therefore a set of normality zigzags

$$\{\Sigma_1, \Sigma_2, \ldots\}$$

such that every system $\varrho_1, \varrho_2, \ldots, \varrho_m$, at which $\boldsymbol{f}$ is normal, belongs to at least one of these normality zigzags. A fundamental problem of this theory can now be formulated as follows.

Problem 1. *Given a set of systems $\varrho_1, \varrho_2, \ldots, \varrho_m$, determine conditions on the function vector $\boldsymbol{f}$, which imply that every system in this set belongs to at least one of the normality zigzags*

$$\Sigma_1, \Sigma_2, \ldots \text{ of } \boldsymbol{f}.$$

The most important case of this problem arises when the set given consists of all systems $\varrho_1, \varrho_2, \ldots, \varrho_m$. We shall study the properties of function vectors satisfying this stronger condition later.

VI.

16. We now show that there exist simple relations linking the Latin and German matrices belonging to two different systems, $\varrho_1, \varrho_2, \ldots, \varrho_m$ and $\varrho_1', \varrho_2', \ldots, \varrho_m'$, at which the function vector $\boldsymbol{f}$ is normal.

Given that the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, then the Latin and German matrices

$$A(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ \mathfrak{A}(\varrho_1 \varrho_2 \cdots \varrho_m)$$

are non-singular and uniquely determined. The inverses of each of these matrices can easily be determined. For, by § 8, m^2 equations hold

$$\sum_{k=1}^{m} A_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, \mathfrak{A}_{jk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \delta_{hj} \varepsilon_j \psi_\sigma, \text{ with } \varepsilon_h \in F, \qquad (h, j = 1, 2, \ldots, m).$$

The constants $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ are all non-zero, since the degree of the left hand side is equal to σ whenever $h = j$. However, the coefficient of ψ_σ in the interpolation for the left hand side is equal to 1 whenever $h = j$, and so each of the constants $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ is equal to 1. Hence the m^2 equations are

$$\sum_{k=1}^{m} A_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, \mathfrak{A}_{jk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \delta_{hj} \psi_\sigma \qquad (h, j = 1, 2, \ldots, m)$$

and so are equivalent to the single matrix equation

$$A(\varrho_1 \varrho_2 \cdots \varrho_m)\, \mathfrak{A}'(\varrho_1 \varrho_2 \cdots \varrho_m) = \psi_\sigma I$$

where I denotes the $m \times m$ unit matrix. This equation implies that

$$A(\varrho_1 \varrho_2 \cdots \varrho_m)^{-1} = \frac{1}{\psi_\sigma}\, \mathfrak{A}'(\varrho_1 \varrho_2 \cdots \varrho_m),$$

$$\mathfrak{A}(\varrho_1 \varrho_2 \cdots \varrho_m)^{-1} = \frac{1}{\psi_\sigma}\, A'(\varrho_1 \varrho_2 \cdots \varrho_m).$$

From these formulae, the elements of $A(\varrho_1 \varrho_2 \cdots \varrho_m)^{-1}$ and $\mathfrak{A}(\varrho_1 \varrho_2 \cdots \varrho_m)^{-1}$ lie in the quotient field of ω.

17. Assume now that the function vector $\boldsymbol{f}$ is also normal at the system $\varrho_1', \varrho_2', \ldots, \varrho_m'$, with sum σ'. Let, say, $\sigma' \geqslant \sigma$. Naturally the systems $\varrho_1, \varrho_2, \ldots, \varrho_m$ and $\varrho_1', \varrho_2', \ldots, \varrho_m'$ are not necessarily in the same normality zigzag of $\boldsymbol{f}$.

Define the matrices

$$P\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} = A(\varrho_1'\varrho_2'\cdots\varrho_m')\,A(\varrho_1\varrho_2\cdots\varrho_m)^{-1} =$$

$$= \frac{1}{\psi_\sigma}\,A(\varrho_1'\varrho_2'\cdots\varrho_m')\,\mathfrak{A}'(\varrho_1\varrho_2\cdots\varrho_m),$$

$$\mathfrak{P}\begin{pmatrix}\varrho_1'\varrho_1'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} = \mathfrak{A}(\varrho_1'\varrho_2'\cdots\varrho_m')\,\mathfrak{A}(\varrho_1\varrho_2\cdots\varrho_m)^{-1} =$$

$$= \frac{1}{\psi_\sigma}\,\mathfrak{A}(\varrho_1'\varrho_2'\cdots\varrho_m')\,A'(\varrho_1\varrho_2\cdots\varrho_m),$$

so that

$$A(\varrho_1'\varrho_2'\cdots\varrho_m') = P\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}A(\varrho_1\varrho_2\cdots\varrho_m),$$

$$\mathfrak{A}(\varrho_1'\varrho_2'\cdots\varrho_m') = \mathfrak{P}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\mathfrak{A}(\varrho_1\varrho_2\cdots\varrho_m).$$

We call $P\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}$ and $\mathfrak{P}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}$ the Latin and German *transformation matrices*, respectively. The elements of these transformation matrices are given explicitly by the equations

$$P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} = \frac{1}{\psi_\sigma}\sum_{k=1}^{m} A_{hk}(\varrho_1'\varrho_2'\cdots\varrho_m')\,\mathfrak{A}_{jk}(\varrho_1\varrho_2\cdots\varrho_m) \qquad (j, h = 1, 2, \ldots, m),$$

$$\mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} = \frac{1}{\psi_\sigma}\sum_{k=1}^{m} \mathfrak{A}_{hk}(\varrho_1'\varrho_2'\cdots\varrho_m')\,A_{jk}(\varrho_1\varrho_2\cdots\varrho_m) \qquad (j, h = 1, 2, \ldots, m).$$

From these formulae, the elements of these transformation matrices lie in the quotient field of ω, and their denominators are factors of ψ_σ. In fact, we shall deduce from $\sigma' \geqslant \sigma$ that their elements are polynomials.

The polynomials

$$\psi_\sigma P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}, \quad \psi_\sigma \mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} \qquad (h, j = 1, 2. \ldots, m)$$

are expressions of the form $e\begin{pmatrix}r_1 r_2 \ldots r_m s\\ \mathfrak{w}_1 \mathfrak{w}_2 \ldots \mathfrak{w}_m \mathfrak{s}\end{pmatrix}$ with parameter values

$$r_k = \varrho_k' + \delta_{hk}\ ,\ \mathfrak{w}_k = \varrho_k - \delta_{jk}\ ;\ r_k = \varrho_k + \delta_{jk}\ ,\ \mathfrak{w}_k = \varrho_k' - \delta_{hk},$$

$$s = \sigma' + 1 \quad , \quad \mathfrak{s} = \sigma - 1 \quad ; \quad s = \sigma + 1 \quad , \quad \mathfrak{s} = \sigma' - 1,$$

and thus (D) and (0) give the estimates

$$\overline{\left|\psi_\sigma P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \leqslant \max_{k=1,\ldots,m} \{(\varrho_k'+\delta_{hk}-1)+(\sigma-1-\varrho_k+\delta_{jk})\} = $$
$$= \max_{k=1,\ldots,m} \{\sigma+\varrho_k'-\varrho_k+\delta_{hk}+\delta_{jk}-2\},$$

$$\underline{\left|\psi_\sigma P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \geqslant \min\{\sigma',\sigma\}=\sigma.$$

$$\overline{\left|\psi_\sigma \mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \leqslant \max_{k=1,\ldots,m} \{(\varrho_k+\delta_{jk}-1)+(\sigma'-1-\varrho_k'+\delta_{hk})\} = $$
$$= \max_{k=1,\ldots,m} \{\sigma'+\varrho_k-\varrho_k'+\delta_{jk}+\delta_{hk}-2\},$$

$$\underline{\left|\psi_\sigma \mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \geqslant \min\{\sigma',\sigma\}=\sigma.$$

Hence all of the polynomials

$$\psi_\sigma P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}, \quad \psi_\sigma \mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} \qquad (h,j=1,2,\ldots,m)$$

have orders at least equal to σ, proving our assertion that all the elements

$$P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}, \quad \mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} \qquad (h,j=1,2,\ldots,m)$$

are polynomials, of degrees satisfying the inequalities

$$\overline{\left|P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \leqslant \max_{k=1,\ldots,m} \{\varrho_k'-\varrho_k+\delta_{jk}+\delta_{hk}-2\} \qquad (h,j=1,2,\ldots,m),$$

$$\overline{\left|\mathfrak{P}_{kj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \leqslant \max_{k=1,\ldots,m} \{\sigma'-\sigma+\varrho_k-\varrho_k'+\delta_{jk}+\delta_{hk}-2\} \qquad (h,j=1,2,\ldots,m),$$

From the properties of the Latin and German matrices, we also deduce that

$$\left|P\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right| = \frac{\psi_{\sigma'}}{\psi_\sigma},$$

$$\left|\mathfrak{P}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right| = \left(\frac{\psi_{\sigma'}}{\psi_\sigma}\right)^{m-1},$$

$$P\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} \mathfrak{P}'\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} = \frac{\psi_{\sigma'}}{\psi_\sigma} I.$$

18. One can easily obtain explicit expressions for the Latin and German transformation matrices if we suppose that the two systems $\varrho_1, \varrho_2, \ldots, \varrho_m$

and $\varrho_1', \varrho_2', \ldots, \varrho_m'$ are related as follows:

either

$$\varrho_1' = \varrho_1 + 1 \;\;, \; \varrho_2' = \varrho_2 + 1, \ldots, \; \varrho_m' = \varrho_m + 1,$$

or

$$\varrho_1' = \varrho_1 + \delta_{h1} \;\;, \; \varrho_2' = \varrho_2 + \delta_{h2}, \ldots, \; \varrho_m' = \varrho_m + \delta_{hm},$$

or

$$\varrho_1' = \varrho_1 + \delta_{h1} - \delta_{k1} \;\;, \; \varrho_2' = \varrho_2 = \delta_{h2} - \delta_{k2}, \ldots, \; \delta_m' = \varrho_m + \delta_{hm} - \delta_{km}.$$

However, we omit these expressions.

The following problem concerning these transformation matrices was proposed to me by Mahler.

Problem 2. *Given a sequence of systems*

$$\varrho_1^{(n)}, \varrho_2^{(n)}, \ldots, \varrho_m^{(n)} \qquad\qquad (n = 0, 1, 2, \ldots),$$

and a sequence of Latin transformation matrices

$$P\begin{pmatrix} \varrho_1^{(n+1)} & \varrho_2^{(n+1)} & \ldots & \varrho_m^{(n+1)} \\ \varrho_1^{(n)} & \varrho_2^{(n)} & \ldots & \varrho_m^{(n)} \end{pmatrix} \qquad\qquad (n = 0, 1, 2, \ldots)$$

or a sequence of German transformation matrices

$$\mathfrak{P}\begin{pmatrix} \varrho_1^{(n+1)} & \varrho_2^{(n+1)} & \ldots & \varrho_m^{(n+1)} \\ \varrho_1^{(n)} & \varrho_2^{(n)} & \ldots & \varrho_m^{(n)} \end{pmatrix} \qquad\qquad (n = 0, 1, 2, \ldots)$$

does there exist a function vector to which these transformation matrices belong?

The case of particular interest is when

$$\varrho_1^{(n)} = \varrho_2^{(n)} = \ldots = \varrho_m^{(n)} = n \qquad\qquad (n = 0, 1, 2, \ldots).$$

(To be continued)

are expressions of the form $e\begin{pmatrix} r_1 r_2 \dots r_m s \\ \mathfrak{w}_1 \mathfrak{w}_2 \dots \mathfrak{w}_m \mathfrak{s} \end{pmatrix}$, with parameter values

$$r_k = \varrho_k + \delta_{hk} + \delta_{lk} \ , \ \mathfrak{w}_k = \varrho_k + \delta_{lk},$$
$$s = \sigma + 2 \qquad , \ \mathfrak{s} = \sigma.$$

(D) and (O) therefore give the estimates

$$\overline{|\mathscr{C}_h|} \leqslant \max_{k=1,\dots,m} \{(\varrho_k + \delta_{hk} + \delta_{lk} - 1) + (\sigma - \varrho_k - \delta_{lk})\} \leqslant \sigma,$$
$$\underline{|\mathscr{C}_h|} \geqslant \{\sigma+1, \sigma+1\} = \sigma+1,$$

so that all of the polynomials $\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_m$ are equal to zero. We have then m homogeneous linear equations, with non-zero determinant $D(\varrho_1 + \delta_{l1}\ \varrho_2 + \delta_{l2} \dots \varrho_m + \delta_{lm})$, for the non-trivial system of polynomials

$$\mathfrak{a}_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \dots, m).$$

But this is impossible, whence the assertion.

Secondly, by an argument similar to that in the proof of part 2 of the First Uniqueness Theorem, part 2 of the present one follows from part 1.

Finally, we prove part 3. Suppose that, on the contrary,

$$\underline{|\mathfrak{w}_{jk}(\varrho_1 \varrho_2 \cdots \varrho_m)|} > \sigma + 1 \qquad (j, k = 1, 2, \dots, m).$$

Then

$$\mathfrak{a}_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \dots, m)$$

would be a non-trivial system of German polynomials belonging to the system $\varrho_1 + 1, \varrho_2, \dots, \varrho_m$, contrary to part 1 of the theorem. This completes the proof.

20. Secondly, there is also a global analogue of the Second Uniqueness Theorem. Before we state this analogue, it is convenient to make the following change in convention. From now on we shall assume that the degree of the zero polynomial is any negative integer or $-\infty$.

Fourth Uniqueness Theorem. (Mahler, [8]). *If the function vector* $\boldsymbol{f}$ *is perfect, and*

$$a_k(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ r(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \dots, m)$$

are a non-trivial system of Latin polynomials and its remainder belonging to the system $\varrho_1, \varrho_2, \dots, \varrho_m$, *then*

1. $\overline{|a_k(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \varrho_k - 1 \qquad (k = 1, 2, \dots, m)$;

2. *every system of Latin polynomials belonging to the system* $\varrho_1, \varrho_2, \dots, \varrho_m$ *is a constant multiple of the system*

$$a_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \dots, m);$$

3. *the remainder* $r(\varrho_1\ \varrho_2 \dots \varrho_m)$ *has order equal to* $\sigma - 1$.

Proof. We first prove part 1. Suppose that, on the contrary, there exist an integer l, with $1 \leqslant l \leqslant m$, such that ϱ_l is non-zero and

$$\overline{|a_l(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \varrho_l - 1.$$

But then the system of Latin polynomials

$$a_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

contradicts the normality of $\mathbf{f}$ at the system $\varrho_1 - \delta_{l1}, \varrho_2 - \delta_{l2}, \ldots, \varrho_m - \delta_{lm}$, whence the assertion.

Part 2 follows from part 1 as in the proof of the Second Uniqueness Theorem.

Part 3 follows immediately from Criterion 2. This completes the proof.

The Fourth Uniqueness Theorem implies the following corollary.

Corollary. *If the function vector $\mathbf{f}$ is perfect, then it is linearly independent over the quotient field of ω.*

In the light of this corollary, the Normality Zigzag Theorem is even more surprising, since it holds for essentially arbitrary function vectors $\mathbf{f}$.

21. Finally, we give three additional criteria for the function vector $\mathbf{f}$ to be perfect. These three criteria are global analogues of the definition of normality, Criterion 1 and Criterion 2, respectively.

Criterion 3. (Mahler, [8]). *The function vector $\mathbf{f}$ is perfect if and only if*

1. *the function fector f vanishes at none of the primes in Π;*

2. *for every system $\varrho_1, \varrho_2, \ldots, \varrho_m$, there exists a system of Latin polynomials*

$$\bar{a}_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

such that

$$\overline{|\bar{a}_k(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \varrho_k - 1 \qquad (k = 1, 2, \ldots, m).$$

Proof. The sufficiency is obvious. The necessity follows from the Fourth Uniqueness Theorem.

Criterion 4. (Mahler, [8]). *The function vector $\mathbf{f}$ is perfect if and only if*

1. *the function vector vanishes at none of the primes in Π;*

2. *for every system $\varrho_1, \varrho_2, \ldots, \varrho_m$, there exists a system of German polynomials*

$$\bar{\mathfrak{a}}_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

such that

$$\overline{|\bar{\mathfrak{a}}_k(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \sigma - \varrho_k \qquad (k = 1, 2, \ldots, m).$$

Proof. The sufficiency follows from Criterion 1. The necessity follows from the Third Uniqueness Theorem.

Criterion 5. (Mahler, [8]). *The function vector* $\mathbf{f}$ *is perfect if and only if, for all systems* $\varrho_1, \varrho_2, \ldots, \varrho_m$, *there exists no non-trivial system of Latin polynomials, which, together with its remainder, satisfies the inequalities*

$$\overline{|a_k(\varrho_1 \varrho_2 \cdots \varrho_m)|} \leqslant \varrho_k - 1 \qquad (k = 1, 2, \ldots, m),$$

$$\underline{|\ r(\varrho_1 \varrho_2 \cdots \varrho_m)|} > \sigma - 1.$$

Proof. The necessity follows from the Fourth Uniqueness Theorem. To prove the sufficiency it suffices, by Criterion 2, to prove that $\mathbf{f}$ vanishes at none of the primes in Π. If, on the contrary, $\mathbf{f}$ vanishes at the prime $p_\lambda \in \Pi$, then no remainder could have order equal to $\lambda - 1$, contrary to hypothesis.

VIII.

22. To conclude this paper, we apply the general theory to concrete rings and function vectors.

Let F be a field, of arbitrary characteristic, and $F[z]$ the ring of polynomials in the indeterminate z with coefficients in F. If we define the valuation $\overline{|\ \ |}$ on $F[z]$ to be the degree of a polynomial, it is clear that $F[z]$ satisfies the conditions for the ring ω. Furthermore, F becomes the field of constants in § 1, which was also called F. In fact, ω will always be isomorphic to such a polynomial ring $F[z]$. Henceforth we take $\omega = F[z]$.

The sequence of primes Π will now be of the form

$$\Pi\colon\ z - z_1,\ z - z_2,\ z - z_3,\ \ldots,$$

where

$$z_1,\ z_2,\ z_3,\ \ldots$$

is an arbitrary infinite sequence of equal or distinct elements of F.

We next give two examples of weld rings of the sequence of primes Π.

Firstly, if all the primes in Π are equal, say to $z - \zeta$, where ζ is some fixed element of F, then the integral domain $F(z - \zeta)$ of all formal power series in $z - \zeta$ with coefficients in F is a weld ring of Π.

If the primes in Π are not all equal, then the ring of all formal power series will no longer be a weld ring of Π, because the expansion constants are not unique. For example, if we take $\omega = F[z]$, $\Omega = F(z)$, and

$$\Pi\colon\ z - 1,\ z,\ z,\ \ldots,$$

then

$$0 = 0 + (z-1)\,0 = 1 + (z-1)\,(1 + z + z^2 + \ldots),$$

so that the expansion constants for 0 are not unique.

However, if the field F has characteristic zero, and is complete under a valuation, we can obtain the following important examples of a weld ring. A function $f(z)\colon F \to F$ is said to be *analytic* at $\zeta \in F$ if it has a power series development

$$f(z) = f(\zeta) + f'(\zeta)\,(z - \zeta) + \frac{f''(\zeta)}{2!}\,(z - \zeta)^2 + \ldots$$

which is convergent, with respect to the complete valuation on ζ, in some neighbourhood of ζ in the topology on F determined by the complete valuation. This function is then said to be analytic on a subset of F if it is analytic at every point of this subset. Let

$$z_1,\ z_2,\ z_3,\ \dots$$

be an arbitrary infinite sequence of equal or distinct elements of some connected open subset of F. Then any ring of functions, which are analytic on this connected open subset, is a weld ring of

$$\Pi\colon\ z-z_1,\ z-z_2,\ z-z_3,\ \dots .$$

For example, such weld rings exist if $F=C$, the field of complex numbers, or $F=P$, the field of p-adic numbers.

These are the only examples of weld rings which I know at the moment. However, I feel that there are probably further examples of such rings.

23. We now give some examples of function vectors in these particular rings.

For the following three examples we let F be any field of characteristic zero, and we take

$$\Pi_0\colon\ z, z, z, \dots,$$

so that we can choose $\Omega=F(z)$. We define the formal power series e^{ω}, $(1-z)^{\omega}$, and $\log(1-z)$ in the usual manner.

Example 1. If $\omega_1, \omega_2, \dots, \omega_m$ are arbitrary distinct elements of F, then the exponential function vector

$$e^{\omega_1 z},\ e^{\omega_2 z},\ \dots,\ e^{\omega_m z}$$

is perfect with respect to Π_0, (2), (3), (4), (5).

Example 2. If $\omega_1, \omega_2, \dots, \omega_m$ are arbitrary elements of F such that, for $j\neq k$, $\omega_j-\omega_k$ is not an integral multiple of the unit element of F, then the binomial function vector

$$(1-z)^{\omega_1},\ (1-z)^{\omega_2},\ \dots,\ (1-z)^{\omega_m}$$

is perfect with respect to Π_0, (2), (6).

These function vectors, and an example to be given shortly, are the only examples I know of perfect function vectors.

Example 3. The logarithmic function vector

$$\log^{m-1}(1-z),\ \log^{m-2}(1-z),\ \dots,\ 1$$

is normal with respect to Π_0 at every system $\varrho_1, \varrho_2, \dots, \varrho_m$ such that $\varrho_1<\varrho_2<\dots<\varrho_m$, (2), (9).

It is not known whether the logarithmic function vector is perfect. These three examples are discussed in the references noted with each.

24. We conclude this paper by proving new results on the exponential function vector. Let $F=C$ be the field of complex numbers, and take $\omega=C[z]$. Let Π_1 be

$$\Pi_1:\ z,\ z-1,\ z-2,\ \ldots .$$

Then the ring of entire analytic functions is a weld ring of Π_1. From now on, log α will always denote the principal value of the logarithmic function, and we define $\alpha^z=e^{z\log\alpha}$. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are distinct complex numbers, none of which is zero, we shall prove that the power function vector

$$\alpha_1^z,\ \alpha_2^z,\ \ldots,\ \alpha_m^z$$

is perfect with respect to Π_1, and we shall also construct the Latin and German polynomial systems associated with this function vector.

We use some simple facts from the calculus of finite differences, which can be found in books on the subject (1), (10). It will be convenient to express for which system of complex of complex numbers $\beta_1, \beta_2, \ldots, \beta_\mu$ the Latin and German polynomials are formed. Therefore, if μ is any integer, with $1\leqslant\mu\leqslant m$, and $\beta_1, \beta_2, \ldots, \beta_\mu$ are distinct non-zero complex numbers, let

$$a_k\left(z\left|\begin{matrix}\beta_1\beta_2\ldots\beta_\mu\\ \varrho_1\varrho_2\cdots\varrho_\mu\end{matrix}\right.\right),\ r\left(z\left|\begin{matrix}\beta_1\beta_2\ldots\beta_\mu\\ \varrho_1\varrho_2\cdots\varrho_\mu\end{matrix}\right.\right)=\sum_{k=1}^{\mu}a_k\left(z\left|\begin{matrix}\beta_1\beta_2\ldots\beta_\mu\\ \varrho_1\varrho_2\cdots\varrho_\mu\end{matrix}\right.\right)\beta_k^z\qquad(k=1,2,\ldots,\mu),$$

$$\mathfrak{a}_k\left(z\left|\begin{matrix}\beta_1\beta_2\ldots\beta_\mu\\ \varrho_1\varrho_2\cdots\varrho_\mu\end{matrix}\right.\right),\ \mathfrak{w}_{jk}\left(z\left|\begin{matrix}\beta_1\beta_2\ldots\beta_\mu\\ \varrho_1\varrho_2\cdots\varrho_\mu\end{matrix}\right.\right)=$$

$$=\mathfrak{a}_k\left(z\left|\begin{matrix}\beta_1\beta_2\ldots\beta_\mu\\ \varrho_1\varrho_2\cdots\varrho_\mu\end{matrix}\right.\right)\beta_j^z-\mathfrak{a}_j\left(z\left|\begin{matrix}\beta_1\beta_2\ldots\beta_\mu\\ \varrho_1\varrho_2\cdots\varrho_\mu\end{matrix}\right.\right)\beta_k^z\qquad(j,k=1,2,\ldots,\mu)$$

be a non-trivial system of Latin polynomials and its remainder, and a non-trivial system of German polynomials and its remainders, respectively, belonging to the system of non-negative integers $\varrho_1, \varrho_2, \ldots, \varrho_\mu$. Here the trivial case $\mu=1$ has been included for convenience.

Let Δ and δ be the difference and translation operators

$$\Delta f(z)=f(z+1)-f(z)\ ,\ \mathfrak{d}f(z)=f(z+1),$$

respectively, so that, if $a(z)$ is a polynomial, and $\alpha\neq 0, 1$,

$$\Delta^\varrho\alpha^{-z}a(z)=\alpha^{-z}a^*(z),\ \text{where}\ a^*(z)=\alpha^{-\varrho}(\mathfrak{d}-\alpha)^\varrho a(z).$$

From the latter identity, it follows that $a^*(z)$ is a polynomial of the same degree as $a(z)$. If $b(\delta)$ is a polynomial in the operator δ such that $b(1)\neq 0$, the operator $b(\delta)^{-1}$ is defined for any polynomial $a(z)$ by

$$b(\mathfrak{d})^{-1}a(z)=\{b(1+\Delta)\}^{-1}a(z)=\sum_{k=0}^{\infty}\beta_k\Delta^k a(z)$$

where $\sum_{k=0}^{\infty} \beta_k \Delta^k$ is the Taylor series for $\{b(1+\Delta)\}^{-1}$. The sum need not be extended past the degree of $a(z)$. Evidently

$$b(\mathfrak{d})^{-1} b(\mathfrak{d})\, a(z) = a(z).$$

Next, if z is any non-negative integer, and $f(z)$ is a function defined for such z, write

$$Jf(z) = \sum_{t=0}^{z-1} f(t).$$

The operator J has then the properties

$$\Delta J f(z) = f(z) \quad , \quad J\Delta f(z) = f(z) - f(0).$$

By definition, not all of the polynomials

$$a_k\left(z \left| \begin{matrix} \alpha_1\, \alpha_2 \ldots \alpha_m \\ \varrho_1\, \varrho_2 \cdots \varrho_m \end{matrix}\right.\right) \qquad (k = 1, 2, \ldots, m)$$

are identically zero; so let us suppose say, that $a_l\left(z \left| \begin{matrix} \alpha_1\, \alpha_2 \ldots \alpha_m \\ \varrho_1\, \varrho_2 \cdots \varrho_m \end{matrix}\right.\right)$ does not vanish identically. For $h \neq l$,

$$\alpha_h^{\varrho_h} \Delta^{\varrho_h} \alpha_h^{-z}\, r\left(z \left| \begin{matrix} \alpha_1\, \alpha_2 \ldots \alpha_m \\ \varrho_1\, \varrho_2 \cdots \varrho_m \end{matrix}\right.\right) =$$

$$= \sum_{k=1}^{m} a_k^*(z) \left(\frac{\alpha_k}{\alpha_h}\right)^z, \text{ where } a_k^*(z) = (\alpha_k \delta - \alpha_h)^{\varrho_h} a_k\left(z \left| \begin{matrix} \alpha_1\, \alpha_2 \ldots \alpha_m \\ \varrho_1\, \varrho_2 \cdots \varrho_m \end{matrix}\right.\right),$$

and thus

$$\overline{|a_k^*(z)|} = \overline{\left| a_k\left(z \left| \begin{matrix} \alpha_1\, \alpha_2 \ldots \alpha_m \\ \varrho_1\, \varrho_2 \cdots \varrho_m \end{matrix}\right.\right)\right|},$$

$$\underline{\left|\alpha_h^{\varrho_h} \Delta^{\varrho_h} \alpha_h^{-z}\, r\left(z \left| \begin{matrix} \alpha_1\, \alpha_2 \ldots \alpha_m \\ \varrho_1\, \varrho_2 \cdots \varrho_m \end{matrix}\right.\right)\right|} \geqslant \underline{\left| r\left(z \left| \begin{matrix} \alpha_1\, \alpha_2 \ldots \alpha_m \\ \varrho_1\, \varrho_2 \cdots \varrho_m \end{matrix}\right.\right)\right|} - \varrho_h.$$

In particular, it follows that $a_l^*(z)$ is not identically zero.

It is now easy to prove that the function vector $\alpha_1^z, \alpha_2^z, \ldots, \alpha_m^z$ is perfect with respect to the sequence of primes Π_1. For suppose that, on the contrary,

$$\underline{\left| r\left(z \left| \begin{matrix} \alpha_1\, \alpha_2 \ldots \alpha_m \\ \varrho_1\, \varrho_2 \cdots \varrho_m \end{matrix}\right.\right)\right|} = \sigma - 1 + \nu,$$

where ν is some *positive* integer. Then, repeating the previous argument $m-1$ times, we would obtain a function $r\left(z \left| \begin{matrix} \frac{\alpha_l}{\alpha_h} \\ \varrho_l \end{matrix}\right.\right) = a_l\left(z \left| \begin{matrix} \frac{\alpha_l}{\alpha_h} \\ \varrho_l \end{matrix}\right.\right) \left(\frac{\alpha_l}{\alpha_h}\right)^z$

satisfying

$$\overline{\left| a_l\left(z \left| \begin{matrix} \frac{\alpha_l}{\alpha_h} \\ \varrho_l \end{matrix}\right.\right)\right|} = \overline{\left| a_l\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \cdots \varrho_m \end{matrix}\right.\right)\right|} \leqslant \varrho_l - 1, \quad \underline{\left| r\left(z \left| \begin{matrix} \frac{\alpha_l}{\alpha_h} \\ \alpha_l \end{matrix}\right.\right)\right|} \geqslant \varrho_l - 1 + \nu.$$

Hence, in particular, $a_l\left(z \left| \begin{matrix} \frac{\alpha_l}{\alpha_h} \\ \varrho_l \end{matrix}\right.\right)$ is not identically zero. But since $\nu > 0$, and the function $\left(\frac{\alpha_l}{\alpha_h}\right)^z$ never vanishes, this is impossible, whence the assertion. Thus we have proven the following.

Theorem. *If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are distinct non-zero complex numbers, then the power function vector $\alpha_1^z, \alpha_2^z, \ldots, \alpha_m^z$ is perfect with respect to the sequence of primes*

$$\Pi_1\colon\ z,\ z-1,\ z-2, \ldots$$

The complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ are said to be *multiplicatively independent* if there exists no relation of the form $\alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\ldots\alpha_m^{\lambda_m} = 1$, where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are arbitrary integers, not all zero. The above theorem then has the immediate corollary.

Corollary. *If the complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ are multiplicatively independent, then the functions*

$$z,\ \alpha_1^z,\ \alpha_2^z, \ldots, \alpha_m^z$$

are algebraically independent over the field of complex numbers.

25. Next, we construct the Latin polynomial system and its remainder for $\alpha_1^z, \alpha_2^z, \ldots, \alpha_m^z$. Let us normalize constants and suppose that

$$r\left(\sigma - 1 \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \cdots \varrho_m \end{matrix}\right.\right) = 1.$$

Then, from the general theory of perfect function vectors,

$$a_k\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \cdots \varrho_m \end{matrix}\right.\right) \text{ and } r\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \cdots \varrho_m \end{matrix}\right.\right)$$

are uniquely determined, and hence, in particular $r\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \cdots \varrho_m \end{matrix}\right.\right)$ is symmetric in the pairs $(\alpha_1, \varrho_1), (\alpha_2, \varrho_2), \ldots, (\alpha_m, \varrho_m)$. Further

$$r\left(z \left| \begin{matrix} \alpha_1 \ldots \alpha_{h-1}\,\alpha_{h+1} \ldots \alpha_m \\ \varrho_1 \cdots \varrho_{h-1}\,\varrho_{h+1} \cdots \varrho_m \end{matrix}\right.\right) = \alpha_h^z\,\alpha_h^{\varrho_h}\,\Delta^{\varrho_h}\,\alpha_h^{-z}\,r\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \cdots \varrho_m \end{matrix}\right.\right)$$

$$a_k\left(z \left| \begin{matrix} \alpha_1 \ldots \alpha_{h-1}\,\alpha_{h+1} \ldots \alpha_m \\ \varrho_1 \cdots \varrho_{h-1}\,\varrho_{h+1} \cdots \varrho_m \end{matrix}\right.\right) = (\alpha_k\delta - \alpha_h)^{\varrho_h}\,a_k\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \cdots \varrho_m \end{matrix}\right.\right) \quad (k = 1, 2, \ldots, m, \ [k \neq h]),$$

and a simple computation shows again that

$$r\left(\sigma-1-\varrho_h\left|\begin{matrix}\alpha_1\cdots\alpha_{h-1}\,\alpha_{h+1}\cdots\alpha_m\\ \varrho_1\cdots\varrho_{h-1}\,\varrho_{h+1}\cdots\varrho_m\end{matrix}\right.\right)=1.$$

We can then repeat this argument with $m-1$ instead of m, and so on. The Latin polynomial systems so constructed are uniquely determined, and their remainders are symmetric in the appropiate pairs.

It follows that necessarily

$$r\left(z\left|\begin{matrix}\alpha_k\\ \varrho_k\end{matrix}\right.\right)=\alpha^{-(\varrho_k-1)}\binom{z}{\varrho_k-1}\alpha_k^z \qquad (k=1,2,\ldots,m),$$

and thus

$$\left\{\prod_{\substack{h=1\\ h\neq k}}^{m}(\alpha_k\delta-\alpha_h)^{\varrho_h}\right\}a_k\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\cdots\alpha_m\\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)=a_k\left(z\left|\begin{matrix}\alpha_k\\ \varrho_k\end{matrix}\right.\right)=\alpha_k^{-(\varrho_k-1)}\binom{z}{\varrho_k-1}$$
$$(k=1,2,\ldots,m).$$

The Latin polynomial system is therefore given explicitly by the formulae

$$a_k\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\cdots\alpha_m\\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)=\alpha_k^{-(\varrho_k-1)}\left\{\prod_{\substack{h=1\\ h\neq k}}^{m}(\alpha_k\delta-\alpha_h)^{-\varrho_h}\right\}\binom{z}{\varrho_k-1} \qquad (k=1,2,\ldots,m).$$

These expressions can be written as contour integrals in the following manner. The function $\prod\limits_{\substack{h=1\\ h\neq k}}^{m}(\alpha_k\mathfrak{z}-\alpha_h)^{-\varrho_h}$ can be expanded as a power series

$$\prod_{\substack{h=1\\ h\neq k}}^{m}(\alpha_k\mathfrak{z}-\alpha_h)^{-\varrho_k}=\sum_{l=0}^{\infty}\nu_l^{(k)}(\mathfrak{z}-1)^l \qquad (k=1,2,\ldots,m),$$

where, by Cauchy's Integral Formula,

$$\nu_l^{(k)}=\frac{1}{2\pi i}\int_{C_1}\prod_{\substack{h=1\\ h\neq k}}^{m}(\alpha_k\mathfrak{z}-\alpha_h)^{-\varrho_h}\frac{d\mathfrak{z}}{(\mathfrak{z}-1)^{l+1}} \qquad (k=1,2,\ldots,m),$$

C_1 being a positively oriented circle with centre $\mathfrak{z}=1$ and sufficiently small radius. Now

$$a_k\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\cdots\alpha_m\\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)=\alpha_k^{-(\varrho_k-1)}\sum_{l=0}^{\varrho_k-1}\nu_l^{(k)}\binom{z}{\varrho_k-l-1} \qquad (k=1,2,\ldots,m),$$

whence

$$a_k\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\cdots\alpha_m\\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)=\frac{\alpha_k^{-(\varrho_k-1)}}{2\pi i}\int_{C_1}\frac{\mathfrak{z}^z-\sum\limits_{l=\varrho_k}^{\infty}\binom{z}{l}(z-1)^l\,dz}{\prod\limits_{\substack{h=1\\ h\neq k}}^{m}(\alpha_k\mathfrak{z}-\alpha_h)^{\varrho_h}\cdot(z-1)^{\varrho_k}} \qquad (k=1,2,\ldots,m)$$

$$=\frac{\alpha_k}{2\pi i}\int_{C_1}\frac{\mathfrak{z}^z\,d\mathfrak{z}}{\prod\limits_{h=1}^{m}(\alpha_k\mathfrak{z}-\alpha_h)^{\varrho_h}} \qquad (k=1,2,\ldots,m),$$

because the function

$$\prod_{h=1}^{m}(\alpha_k \mathfrak{z}-\alpha_h)^{-\varrho_h}\sum_{l=\varrho_k}^{\infty}\binom{z}{l}(\mathfrak{z}-1)^l \qquad (k=1,2,\dots,m)$$

is analytic at the point $\mathfrak{z}=1$. For $k=1, 2, \dots, m$, let C_k be a positively oriented circle with centre $\mathfrak{z}=1$ and sufficiently small radius, and let C_∞ be any positively oriented closed contour, which contains all the points $\alpha_1, \alpha_2, \dots, \alpha_m$ in its interior, and which does not cross the non-positive real axis. Then, by Cauchy's Residue Theorem,

$$\alpha_k^z a_k\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\dots\alpha_m\\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_{C_k}\frac{z^z\,d\mathfrak{z}}{\prod_{h=1}^{m}(\mathfrak{z}-\alpha_h)^{\varrho_h}} \qquad (k=1,2,\dots,m),$$

$$r\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\dots\alpha_m\\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_{C_\infty}\frac{\mathfrak{z}^z\,d\mathfrak{z}}{\prod_{h=1}^{m}(\mathfrak{z}-\alpha_h)^{\varrho_h}}.$$

Evidently $r\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\dots\alpha_m\\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)$ is a solution of the difference equation

$$\prod_{h=1}^{m}(\delta-\alpha_h)^{\varrho_h}\,y(z)=0,$$

which, in addition, satisfies the initial conditions

$$y(0)=0,\ \Delta y(0)=0,\ \dots,\ \Delta^{\sigma-1}y(0)=1.$$

There is a simple expression for the remainder $r\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\dots\alpha_m\\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)$ when z ranges over the non-negative integers. For such values of z,

$$r\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\dots\alpha_m\\ \\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)=\alpha_m^{-\varrho_m}\alpha_m^z J^{\varrho_m} r\left(z\left|\begin{matrix}\frac{\alpha_1}{\alpha_m}\ \frac{\alpha_2}{\alpha_m}\cdots\frac{\alpha_{m-1}}{\alpha_m}\\ \\ \varrho_1\ \ \varrho_2\ \cdots\ \varrho_{m-1}\end{matrix}\right.\right),$$

so that

$$r\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\dots\alpha_m\\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)=(\alpha_m^{-\varrho_m}\alpha_m^z J^{\varrho_m})\dots\left(\alpha_2^{-\varrho_2}\left(\frac{\alpha_1}{\alpha_3}\right)^z J^{\varrho_2}\right)\left(\alpha_1^{-(\varrho_1-1)}\alpha_1^z\binom{z}{\varrho_1-1}\right).$$

Now, by a well known result, $J^\varrho f(z)=\sum_{\tau=0}^{z-1}\binom{z-t-1}{\varrho-1}f(t)$ and therefore

$$r\left(z\left|\begin{matrix}\alpha_1\,\alpha_2\dots\alpha_m\\ \varrho_1\,\varrho_2\cdots\varrho_m\end{matrix}\right.\right)=\alpha_m^{-\varrho_m}\dots\alpha_2^{-\varrho_2}\alpha_1^{-(\varrho_1-1)}\sum_{t_{m_1}=0}^{z}\cdots\sum_{t_1=0}^{t_2-1}\binom{z-t_{m-1}-1}{\varrho_m-1}\cdots$$

$$\cdots\binom{t_1}{\varrho_1-1}\times\alpha_m^{z-t_{m-1}}\dots\alpha_1^{t_1}.$$

26. Finally, we construct the German polynomial system and its remainders for $\alpha_1^z, \alpha_2^z, \ldots, \alpha_m^z$. Put

$$F(\mathfrak{z}) = \prod_{l=1}^{m} (\mathfrak{z}-\alpha_l)^{\varrho_l},$$

$$\mathcal{J}(z|\mathfrak{z}) = \sum_{\lambda=0}^{\infty} \frac{\mathfrak{z}^{\lambda} \dfrac{d^{\lambda}}{d\mathfrak{z}^{\lambda}} F(\mathfrak{z})}{\psi_{\lambda+1}(z)}.$$

where, as before, $\psi_\lambda(z) = z(z-1) \ldots (z-\lambda+1)$.

Then

$$\int F(\mathfrak{z})\, \mathfrak{z}^{-z-1} d\mathfrak{z} = -\mathfrak{z}^{-z} \mathcal{J}(z|\mathfrak{z}),$$

and the German polynomial system and its remainders are given explicitly by the formulae

$$\mathfrak{a}_k\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \ldots \varrho_m \end{matrix} \right.\right) = \psi_{\sigma+1}(z)\, \mathcal{J}(z|\alpha_k) \qquad (k=1,2,\ldots,m),$$

$$\mathfrak{w}_{jk}\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \ldots \varrho_m \end{matrix} \right.\right) = \psi_{\sigma+1}(z) \int_{\alpha_k}^{\alpha_j} F(\mathfrak{z})\, \mathfrak{z}^{-z-1} d\mathfrak{z}, \qquad (j,k=1,2,\ldots,m),$$

provided the integral is defined along a contour which does not intersect the non-positive real axis. Further if $\mathscr{R}(z) < 0$, the expressions for the German polynomial system can be written as integrals

$$\mathfrak{a}_k\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \ldots \varrho_m \end{matrix} \right.\right) = \psi_{\sigma+1}(z) \int_1^0 \prod_{l=1}^{m} (\alpha_k \mathfrak{z} - \alpha_l)^{\varrho_l}\, \mathfrak{z}^{-z-1} d\mathfrak{z} \text{ if } \mathscr{R}(z) < 0 \qquad (k=1,2,\ldots,m).$$

Using all these explicit expressions, one can easily determine explicitly the matrices

$$A\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \ldots \varrho_m \end{matrix} \right.\right), \quad \mathfrak{A}\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \ldots \varrho_m \end{matrix} \right.\right).$$

These matrices have the following properties

$$\left| A\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \ldots \varrho_m \end{matrix} \right.\right) \right| = \psi_\sigma(z), \quad \left| \mathfrak{A}\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \ldots \varrho_m \end{matrix} \right.\right) \right| = \psi_\sigma^{m-1}(z),$$

$$A\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \ldots \varrho_m \end{matrix} \right.\right) \mathfrak{A}'\left(z \left| \begin{matrix} \alpha_1\,\alpha_2 \ldots \alpha_m \\ \varrho_1\,\varrho_2 \ldots \varrho_m \end{matrix} \right.\right) = \psi_\sigma(z)\, I.$$

If $\mathscr{R}(z) < 0$, this matrix equation leads to a matrix generalization of the Beta function. For, in terms of the explicit expressions for the Latin and German polynomials, this equation is

$$\left(\varrho_h!\,\alpha_h^{\varrho_h}\prod_{\substack{l=1\\ l\neq 1}}^{m}(\alpha_h-\alpha_l)^{\varrho_l}\cdot\frac{\alpha_k}{2\pi i}\int_{C_1}\frac{\mathfrak{z}^z\,d\mathfrak{z}}{\prod_{l=1}^{m}(\alpha_k\mathfrak{z}-\alpha_l)^{\varrho_l+\delta_{hl}}}\right)_{hk}\cdot$$

$$\cdot\left(\frac{\alpha_h^{-(\varrho_h-1)}}{(\varrho_h-1)!}\prod_{\substack{l=1\\ l\neq h}}^{m}(\alpha_h-\alpha_l)^{-\varrho_l}\int_1^0\prod_{l=1}^{m}(\alpha_k\mathfrak{z}-\alpha_l)^{\varrho_l-\delta_{hl}}\mathfrak{z}^{-z-1}\,d\mathfrak{z}\right)'_{hk}=I.$$

In the special case $m=1$ this functional equation reduces to

$$\frac{1}{2\pi i}\int_{C_1}\frac{\mathfrak{z}^z\,d\mathfrak{z}}{(\mathfrak{z}-1)^{\varrho+1}}\cdot\int_1^0(\mathfrak{z}-1)^{\varrho-1}\mathfrak{z}^{-z-1}\,d\mathfrak{z}=\frac{1}{\varrho},$$

which is a limiting case of the functional equation

$$B(x+1,-y)\,B(-x,y)=\frac{\pi}{y}\,\frac{\sin\pi(y-x)}{\sin\pi x\cdot\sin\pi y}$$

for the Beta function. Thus we can regard both the matrices

$$A\left(z\middle|\begin{matrix}\alpha_1\,\alpha_2\ldots\alpha_m\\ \varrho_1\,\varrho_2\,\cdots\,\varrho_m\end{matrix}\right),\quad \mathfrak{A}\left(z\middle|\begin{matrix}\alpha_1\,\alpha_2\ldots\alpha_m\\ \varrho_1\,\varrho_2\,\cdots\,\varrho_m\end{matrix}\right)$$

as being generalizations of the Beta function. These matrices have other properties in common with the Beta function. For example, by computing the transformation matrix, one can easily show that in the special case $m=1$, the matrix equation

$$A\left(z\middle|\begin{matrix}\alpha_1 & \alpha_2 & \ldots\alpha_m\\ \varrho+1 & \varrho+1 & \ldots\varrho+1\end{matrix}\right)=P\left(z\middle|\begin{matrix}\varrho+1 & \varrho+1 & \ldots\varrho+1\\ \varrho & \varrho & \cdots\varrho\end{matrix}\right)A\left(z\middle|\begin{matrix}\alpha_1\,\alpha_2\ldots\alpha_m\\ \varrho\;\varrho\;\cdots\varrho\end{matrix}\right)$$

is a limiting case of the functional equation

$$B(x,y)=\frac{x+y}{y}\,B(x,y+1)$$

for the Beta function.

As we saw in § 23, the power function vector $\alpha_1^z, \alpha_2^z, \ldots, \alpha_m^z$ is also perfect with respect to the sequence of primes

$$\Pi_0\colon\ z,\ z,\ \ldots.$$

This result leads to a matrix generalization of the Gamma function (5), just as our results led to a matrix generalization of the Beta function.

This suggests that the Latin and German polynomial systems associated with Π_0 and Π_1 are related. The following important question also arises.

Problem 3. *Determine other sequences of primes with respect to which the power function vector* $\alpha_1^z, \alpha_2^z, \ldots, \alpha_m^z$ *is perfect.*

Further results in this direction will undoubtedly be of importance in problems of diophantine approximations.

Note Added in Proof

Using the results of part IV of these papers, I have recently obtained a very general theorem on the perfectness of any system of analytic functions which are linearly independent over the field of rational functions. Unfortunately, the proof does not allow one to construct the Latin and German polynomial systems. Details will appear in a further paper in this series.

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