ON THE ALGEBRAIC APPROXIMATION OF FUNCTIONS. II

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IV.

10. We now introduce the *local* property of *normality* at one system $\varrho_1, \varrho_2, \ldots, \varrho_m$.

Definition: *The function vector* $\boldsymbol{f}$ *is said to be normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$ *if*

1. *the function vector* $\boldsymbol{f}$ *vanishes at none of the primes in* Π; *and*
2. *for each suffix* $h = 1, 2, \ldots, m$, *there exists a system of Latin polynomials*

$$\bar{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

such that

$$\overline{|\bar{a}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \varrho_h.$$

For the rest of this part, let the function vector $\boldsymbol{f}$ be normal at the fixed, but arbitrary, system $\varrho_1, \varrho_2, \ldots, \varrho_m$.

To avoid having unwieldy constants in our formulae, it is convenient to introduce a slight change in the notation of the last part. Put

$$A_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \frac{\bar{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)}{\bar{\alpha}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)} \qquad (h, k = 1, 2, \ldots, m),$$

$$R_h(\varrho_1 \varrho_2 \cdots \varrho_m) = \frac{\bar{r}_h(\varrho_1 \varrho_2 \cdots \varrho_m)}{\bar{\alpha}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)} \qquad (h = 1, 2, \ldots, m),$$

so that

$$R_h(\varrho_1 \varrho_2 \cdots \varrho_m) = \sum_{k=1}^{m} A_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, f_k \qquad (h = 1, 2, \ldots, m),$$

where the constant $\bar{\alpha}_{hh}(\varrho_1 \varrho_2 \ldots \varrho_m)$ is the coefficient of ψ_{ϱ_h} in the interpolation series for $\bar{a}_{hh}(\varrho_1 \varrho_2 \ldots \varrho_m)$. By this definition, the coefficient of ψ_{ϱ_h} in the interpolation series for $A_{hh}(\varrho_1 \varrho_2 \ldots \varrho_m)$ is equal to 1. Let $A(\varrho_1 \varrho_2 \ldots \varrho_m)$ be the $m \times m$ matrix

$$A(\varrho_1 \varrho_2 \cdots \varrho_m) = A_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)_{h,k=1,2,\ldots,m},$$

and let $D(\varrho_1 \varrho_2 \ldots \varrho_m)$ be the determinant of this matrix. The degree of this determinant is equal to σ, and thus, by the result of § 7, the value of the determinant is

$$D(\varrho_1 \varrho_2 \cdots \varrho_m) = \alpha \psi_\sigma, \text{ with } \alpha \neq 0 \in F.$$

But, expanding the determinant, we see that the coefficient of ψ_σ in its interpolation series is equal to 1, and therefore, more exactly,

$$D(\varrho_1 \varrho_2 \cdots \varrho_m) = \psi_\sigma.$$

First Uniqueness Theorem. *If the function vector* $\mathbf{f}$ *is normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$, *and, if for each suffix* $h = 1, 2, \ldots, m$,

$$\mathfrak{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ \mathfrak{w}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j, k = 1, 2, \ldots, m)$$

are any non-trivial system of German polynomials, and its remainders, belonging to the system

$$\varrho_1 - \delta_{h1}, \ \varrho_2 - \delta_{h2}, \ \ldots, \ \varrho_m - \delta_{hm},$$

then, for $h = 1, 2, \ldots, m$,

1. $\overline{|\mathfrak{a}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \sigma - \varrho_h$;

2. *every system of German polynomials belonging to the system* $\varrho_1 - \delta_{h1}$, $\varrho_2 - \delta_{h2}, \ldots, \varrho_m - \delta_{hm}$ *is a constant multiple of the system*

$$\mathfrak{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m);$$

3. *at least one of the remainders*

$$R_h(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ \mathfrak{w}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j, k = 1, 2, \ldots, m)$$

has order equal to σ.

Proof. Firstly, we prove part 1. Suppose that, on the contrary, there exists an integer l, with $1 \leqslant l \leqslant m$, such that

$$\overline{|\mathfrak{a}_{ll}(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \sigma - \varrho_l.$$

The polynomials

$$\mathscr{C}_j = \sum_{k=1}^{m} A_{jk}(\varrho_1 \varrho_2 \cdots \varrho_m) \ \mathfrak{a}_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j = 1, 2, \ldots, m)$$

are then expressions of the form $e\begin{pmatrix} r_1 \, r_2 \ldots r_m \, s \\ \mathfrak{w}_1 \, \mathfrak{w}_2 \ldots \mathfrak{w}_m \, \mathfrak{s} \end{pmatrix}$, with parameter values

$$r_k = \varrho_k + \delta_{jk} \ , \ \mathfrak{w}_k = \varrho_k,$$
$$s = \sigma + 1 \quad , \ \mathfrak{s} = \sigma - 1.$$

From these values, (D) and (0) give the estimates

$$\overline{|\mathscr{C}_j|} \leqslant \max_{k=1,\ldots,m} \{\varrho_k + \delta_{jk} - 1) + (\sigma - 1 - \varrho_k)\} \leqslant \sigma - 1,$$

$$\underline{|\mathscr{C}_j|} \geqslant \min \{\sigma, \sigma\} = \sigma.$$

Hence all the polynomials $\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_m$ must be zero. Thus we have m homogeneous linear equations, with non-zero determinant $D(\varrho_1 \varrho_2 \dots \varrho_m)$, for the non-trivial system of polynomials

$$\mathfrak{a}_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\dots,m).$$

But this is impossible, whence the assertion.

Secondly, we show that part 1 implies part 2. For any integer h, with $1 \leqslant h \leqslant m$, let

$$\mathfrak{a}^*_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \;,\; \mathfrak{w}^*_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j,k=1,2,\dots,m)$$

be a non-trivial system of German polynomials and its remainders, belonging to the system $\varrho_1 - \delta_{h1}, \varrho_2 - \delta_{h2}, \dots, \varrho_m - \delta_{hm}$. Then we can choose a constant β such that

$$\overline{|\mathfrak{a}^*_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m) - \beta \mathfrak{a}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \sigma - \varrho_h.$$

Then the new system of polynomials

$$\mathfrak{a}^{**}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \mathfrak{a}^*_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) - \beta \mathfrak{a}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\dots,m)$$

and its remainders satisfy the inequalities

$$\overline{|\mathfrak{a}^{**}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)|} \leqslant \sigma - \varrho_k - 1 \qquad (k=1,2,\dots,m),$$

$$\underline{|\mathfrak{w}^{**}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m)|} \geqslant \sigma \qquad (j,k=1,2,\dots,m).$$

But this is impossible, by part 1, unless the new system is trivial, and so the assertion follows.

Finally, we prove part 3. Suppose that, on the contrary, there exists an integer l, with $1 \leqslant l \leqslant m$, such that all the remainder series

$$R_l(\varrho_1 \varrho_2 \cdots \varrho_m) \;,\; \mathfrak{w}_{ljk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j,k=1,2,\dots,m)$$

have order greater than σ. Then the polynomial

$$\mathscr{C}_l = \sum_{k=1}^{m} A_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, \mathfrak{a}_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m)$$

is an expression of the form $e\begin{pmatrix} r_1 r_2 \dots r_m s \\ \mathfrak{w}_1 \mathfrak{w}_2 \dots \mathfrak{w}_u \mathfrak{s} \end{pmatrix}$, which, by (D) and (0), is easily seen to satisfy

$$\overline{|\mathscr{C}_l|} = \sigma,$$

$$\underline{|\mathscr{C}_l|} \geqslant \min\{\sigma+1, \sigma+1\} = \sigma+1.$$

But this is impossible, whence the assertion. This completes the proof.

11. We next prove an analogous theorem for the Latin polynomials. Put

$$\mathfrak{A}_{hk}(\varrho_1\varrho_2\cdots\varrho_m)=\frac{\mathfrak{a}_{hk}(\varrho_1\varrho_2\cdots\varrho_m)}{\beta_{hh}(\varrho_1\varrho_2\cdots\varrho_m)}\qquad(h,k=1,2,\dots,m),$$

$$\mathfrak{R}_{hjk}(\varrho_1\varrho_2\cdots\varrho_m)=\frac{\mathfrak{w}_{hjk}(\varrho_1\varrho_2\cdots\varrho_m)}{\beta_{hh}(\varrho_1\varrho_2\cdots\varrho_m)}\qquad(h,j,k=1,2,\dots,m),$$

so that

$$\mathfrak{R}_{hjk}(\varrho_1\varrho_2\cdots\varrho_m)=\mathfrak{A}_{hk}(\varrho_1\varrho_2\cdots\varrho_m)f_j-\mathfrak{A}_{hj}(\varrho_1\varrho_2\cdots\varrho_m)f_k\qquad(h,j,k=1,2,\dots,m),$$

where the constant $\beta_{hh}(\varrho_1\varrho_2\dots\varrho_m)$ is the coefficient of $\psi_{\sigma-\varrho_h}$ in the interpolation series for $\mathfrak{a}_{hh}(\varrho_1\varrho_2\dots\varrho_m)$. The oecfficient of $\psi_{\sigma-\varrho_h}$ in the interpolation series for $\mathfrak{A}_{hh}(\varrho_1\varrho_2\dots\varrho_m)$ is therefore 1, and thus, by the First Uniqueness Theorem, the polynomial systems and their remainders

$$\mathfrak{A}_{hk}(\varrho_1\varrho_2\cdots\varrho_m)\ ,\ \mathfrak{R}_{hjk}(\varrho_1\varrho_2\cdots\varrho_m)\qquad(h,j,k=1,2,\dots,m)$$

are uniquely determined. Let $\mathfrak{A}(\varrho_1\varrho_2\dots\varrho_m)$ be the $m\times m$ matrix

$$\mathfrak{A}(\varrho_1\varrho_2\cdots\varrho_m)=(\mathfrak{A}_{hk}(\varrho_1\varrho_2\cdots\varrho_m))_{h,k=1,2,\dots,m},$$

and let $\mathfrak{D}(\varrho_1\varrho_2\dots\varrho_m)$ be the determinant of this matrix. The degree of this determinant is equal to $(m-1)\sigma$, and thus, by the results of § 8, the value of the determinant is

$$\mathfrak{D}(\varrho_1\varrho_2\cdots\varrho_m)=\beta\psi_\sigma^{m-1},\ \text{with}\ \beta\neq 0\in F.$$

But, expanding the determinant, we see that the coefficient of ψ_σ^{m-1} in its expansion is equal to 1, and therefore

$$\mathfrak{D}(\varrho_1\varrho_2\cdots\varrho_m)=\psi_\varrho^{m-1}.$$

Second Uniqueness Theorem. *If the function vector* $\mathbf{f}$ *is normal at the system* $\varrho_1,\varrho_2,\dots,\varrho_m$, *and if for each suffix* $h=1,2,\dots,m$,

$$a_{hk}(\varrho_1\varrho_2\cdots\varrho_m)\ ,\ r_h(\varrho_1\varrho_2\cdots\varrho_m)\qquad(k=1,2,\dots,m)$$

are any non-trivial system of Latin polynomials, and its remainder, belonging to the system

$$\varrho_1+\delta_{h1},\ \varrho_2+\delta_{h2},\ \dots,\ \varrho_m+\delta_{hm},$$

then, for $h=1,2,\dots,m$,

1. $\overline{|a_{hh}(\varrho_1\varrho_2\cdots\varrho_m)|}=\varrho_h$;

2. *every system of Latin polynomials belonging to the system* $\varrho_1+\delta_{h1}$, $\varrho_2+\delta_{h2},\dots,\varrho_m+\delta_{hm}$ *is a constant multiple of the system*

$$a_{hk}(\varrho_1\varrho_2\cdots\varrho_m)\qquad(k=1,2,\dots,m);$$

3. *at least one of the remainders*

$$r_h(\varrho_1\varrho_2\cdots\varrho_m),\ \mathfrak{R}_{hjk}(\varrho_1\varrho_2\cdots\varrho_m)\qquad(j,k=1,2,\dots,m)$$

has order equal to σ.

Proof. The proof is completely analogous to the proof of the First Uniqueness Theorem. Firstly ,we prove part 1. Suppose, on the contrary, that there exists an integer l, with $1 \leqslant l \leqslant m$, such that

$$\overline{|a_{ll}(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \varrho_l.$$

Then the polynomials

$$\mathscr{E}_j = \sum_{k=1}^{m} \mathfrak{A}_{jk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, a_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j=1,2,\ldots,m)$$

are expressions of the forme $e\begin{pmatrix} r_1\, r_2 \ldots r_m\, s \\ \mathfrak{w}_1\, \mathfrak{w}_2 \ldots \mathfrak{w}_m\, \mathfrak{s} \end{pmatrix}$, with parameter values

$$r_k = \varrho_k \quad , \ \mathfrak{w}_k = \varrho_k - \delta_{jk},$$

$$s = \sigma + 1 \ , \quad \mathfrak{s} = \sigma - 1.$$

From these values, (D) and (0) give the estimates

$$\overline{|\mathscr{E}_j|} \leqslant \max_{k=1,\ldots,m} \{(\varrho_k - 1) + (\sigma - 1 - \varrho_k + \delta_{jk})\} \leqslant \sigma - 1,$$

$$\underline{|\mathscr{E}_j|} \geqslant \min \{\sigma, \sigma\} = \sigma.$$

Hence all the polynomials $\mathscr{E}_1, \mathscr{E}_2, \ldots, \mathscr{E}_m$ are zero. Thus we have m homogeneous linear equations, with non-zero determinant $\mathfrak{D}(\varrho_1, \varrho_2 \ldots \varrho_m)$, for the non-trivial system of polynomials

$$a_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\ldots,m).$$

But this is impossible, whence the assertion.

Secondly, we show that part 1 implies part 2. For any integer h, with $1 \leqslant h \leqslant m$, let

$$a^*_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ r^*_h(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\ldots,m)$$

be a non-trivial system of Latin polynomials, and its remainder, belonging to the system $\varrho_1 + \delta_{h1}, \varrho_2 + \delta_{h2}, \ldots, \varrho_m + \delta_{hm}$. Then we can choose a constant α so that

$$\overline{|a^*_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m) - \alpha a_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \varrho_h.$$

Thus the new system of polynomials

$$a^{**}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) = a^*_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) - \alpha a_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\ldots,m),$$

and its remainder satisfy the inequalities

$$\overline{|a^{**}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)|} \leqslant \varrho_k - 1 \qquad (k=1,2,\ldots,m),$$

$$\underline{|r^{**}_h(\varrho_1 \varrho_2 \cdots \varrho_m)|} \geqslant \sigma.$$

But this is impossible, by part 1, unless the new system is trivial, and the assertion follows.

Finally, we prove part 3. Suppose that, on the contrary, there exists an integer l, with $1 \leqslant l \leqslant m$, such that all the remainders

$$r_l(\varrho_1 \varrho_2 \cdots \varrho_m) \;\; , \;\; \mathfrak{R}_{ljk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j, k = 1, 2, \ldots, m)$$

have order greater than σ. Then the polynomial

$$\mathscr{E}_l = \sum_{k=1}^{m} \mathfrak{A}_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, a_{lk}(\varrho_1 \varrho_2 \cdots \varrho_m)$$

is an expression of the forme $e\begin{pmatrix} r_1 r_2 \ldots r_m s \\ \mathfrak{w}_1 \mathfrak{w}_2 \ldots \mathfrak{w}_m \mathfrak{s} \end{pmatrix}$, which, by (D) and (0), is easily seen to satisfy

$$\overline{|\mathscr{E}_l|} = \sigma,$$

$$\underline{|\mathscr{E}_l|} \geqslant \min \{\sigma+1, \sigma+1\} = \sigma+1.$$

But this is impossible, whence the assertion. This completes the proof.

12. The original definition of normality was given in terms of the Latin polynomials. However, we could equally well have defined normality in terms of the German polynomials, as is shown by the following criterion.

Criterion 1. *The function vector* $\boldsymbol{f}$ *is normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$ *if and only if*

1. *the function vector* $\boldsymbol{f}$ *vanishes at none of the primes in* Π; *and*

2. *for each suffix* $h = 1, 2, \ldots, m$, *there exists a system of German polynomials*

$$\bar{\mathfrak{a}}_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

such that

$$\overline{|\bar{\mathfrak{a}}_{hh}(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \sigma - \varrho_h.$$

Proof. The necessity is an immediate consequence of the First Uniqueness Theorem. The sufficiency follows by noting that, if the conditions (1) and (2) hold, then, by repeating the argument in § 11, the Second Uniqueness Theorem can be proved independently of the First Uniqueness Theorem. In particular, this would prove that the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1, \varrho_2, \ldots, \varrho_m$.

It is now worthwhile to review the basic facts on normality, which we have proven so far in this part. Essentially, we have shown that, given the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, if either of the determinants

$$d(\varrho_1 \varrho_2 \cdots \varrho_m) \;\; , \;\; \mathfrak{d}(\varrho_1 \varrho_2 \cdots \varrho_m)$$

is non-zero, then the approximation is locally unique in the following sense. Firstly, the Latin and German matrices

$$A(\varrho_1 \varrho_2 \cdots \varrho_m) \;\; , \;\; \mathfrak{A}(\varrho_1 \varrho_2 \cdots \varrho_m)$$

are uniquely determined, and these matrices have non-zero determinants

$$D(\varrho_1 \varrho_2 \cdots \varrho_m) = \psi_\sigma \ , \ \mathfrak{D}(\varrho_1 \varrho_2 \cdots \varrho_m) = \psi_\sigma^{m-1},$$

respectively. Secondly, for $h=1, 2, \ldots, m$, the Latin and German remainders

$$R_h(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ \mathfrak{R}_{hjk}(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (j, k = 1, 2, \ldots, m)$$

are also uniquely determined, and at least one of them has order equal to σ.

13. In the theory given so far there has always been a complete symmetry between the Latin and German polynomials. However, we now give a criterion for normality in terms of the Latin polynomials, where there does not appear to be an analogous criterion in terms of the German polynomials.

Criterion 2. *The function vector* $\boldsymbol{f}$ *is normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$ *if and only if*

1. *the function vector* $\boldsymbol{f}$ *vanishes at none of the primes in* Π;

2. *there exists no non-trivial system of Latin polynomials, which, together with its remainder, satisfies the inequalities*

$$\overline{|a_k(\varrho_1 \varrho_2 \cdots \varrho_m)|} \leqslant \varrho_k - 1 \qquad (k = 1, 2, \ldots, m),$$

$$\underline{|\ r(\varrho_1 \varrho_2 \cdots \varrho_m)|} > \sigma - 1.$$

Proof. The necessity is an immediate consequence of the Second Uniqueness Theorem. Conversely, the sufficiency is obvious.

Criterion 2 implies the following local uniqueness property of the approximation.

Corollary. *If the function vector* $\boldsymbol{f}$ *is normal at the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$, *then the Latin polynomial system belonging to the system* $\varrho_1, \varrho_2, \ldots, \varrho_m$ *is uniquely determined except for a constant factor.*

Proof. Let

$$a_k(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ a_k^*(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

be any two systems of Latin polynomials belonging to the system $\varrho_1, \varrho_2, \ldots, \varrho_m$ at which the function vector $\boldsymbol{f}$ is supposed normal. Then we can choose a constant α such that their respective remainders satisfy

$$\underline{|r(\varrho_1 \varrho_2 \cdots \varrho_m) - \alpha r^*(\varrho_1 \varrho_2 \cdots \varrho_m)|} > \sigma - 1.$$

But

$$a_k(\varrho_1 \varrho_2 \cdots \varrho_m) - \alpha a_k^*(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \ldots, m)$$

is a system of Latin polynomials belonging to the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, and therefore, by Criterion 2, it must be trivial. This completes the proof.

V.

14. We next prove a remarkable theorem which asserts that the *local* property of normality implies certain *global* properties of the approximation.

We begin by introducing the notion of a *normality zigzag*. An infinite set of systems

$$\Sigma = \{(\varrho_1^{(n)}, \varrho_2^{(n)}, \dots, \varrho_m^{(n)}) \mid n = 0, 1, \dots\}$$

is said to be a *normality zigzag* of the function vector $\mathbf{f}$ if

(1) the function vector f is normal at every system in Σ;

(2) $\varrho_1^{(0)} = 0, \varrho_2^{(0)} = 0, \dots, \varrho_m^{(0)} = 0$;

(3) for all non-negative integers n, there exists an integer h_n, with $1 \leqslant h_n \leqslant m$, such that

$$\varrho_1^{(n+1)} = \varrho_1^{(n)} + \delta_{h_n 1}\ ,\ \varrho_2^{(n+1)} = \varrho_2^{(n)} + \delta_{h_n 2}, \dots,\ \varrho_m^{(n+1)} = \varrho_m^{(n)} = \delta_{h_m m}.$$

We note that every function vector, which vanishes at none of the primes in Π, is normal at the system 0, 0, …, 0. As before, we write systems in Σ without brackets around them when there is no danger of confusion.

Normality Zigzag Theorem. *The function vector* $\mathbf{f}$ *is normal at the system* $\varrho_1, \varrho_2, \dots, \varrho_m$ *if and only if this system belongs to at least one normality zigzag of the function vector.*

Proof. The sufficiency is obvious. Conversely, let us suppose that the function vector $\mathbf{f}$ is normal at the system $\varrho_1, \varrho_2, \dots, \varrho_m$. We shall construct a normality zigzag

$$\Sigma = \{(\varrho_1^{(n)}, \varrho_2^{(n)}, \dots, \varrho_m^{(n)}) \mid n = 0, 1, \dots\}, \text{ with } \varrho_1^{(\sigma)} = \varrho_1, \varrho_2^{(\sigma)} = \varrho_2, \dots, \varrho_m^{(\sigma)} = \varrho_m,$$

to which the system $\varrho_1, \varrho_2, \dots, \varrho_m$ belongs. This construction will use all the facts which we have so far proven on normality. The proof is divided into two parts, the descent and the ascent.

Firstly, we construct the systems

(4) $$\varrho_1^{(n)}, \varrho_2^{(n)}, \dots, \varrho_m^{(n)} \text{ where } n = 0, 1, \dots, \sigma.$$

We can suppose that the system $\varrho_1, \varrho_2, \dots, \varrho_m$ is non-trivial, otherwise there is nothing to prove. If

$$a_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k = 1, 2, \dots, m)$$

is a non-trivial system of Latin polynomials belonging to the system $\varrho_1, \varrho_2, \dots, \varrho_m$, then there exists an integer l, with $1 \leqslant l \leqslant m$, such that ϱ_l is positive and

(5) $$\overline{|a_l(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \varrho_l - 1.$$

For suppose that, on the contrary

$$\overline{|a_k(\varrho_1 \varrho_2 \cdots \varrho_m)|} < \varrho_k - 1 \qquad (k = 1, 2, \dots, m).$$

Then the new system of Latin polynomials

$$a_k^*(\varrho_1 \varrho_2 \cdots \varrho_m) = p_\sigma a_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\ldots,m)$$

is non-trivial, and, together with its remainder, satisfies the inequalities

$$\overline{|a_k^*(\varrho_1 \varrho_2 \cdots \varrho_m)|} \leqslant \varrho_k - 1 \qquad (k=1,2,\ldots,m),$$
$$\underline{|r^*(\varrho_1 \varrho_2 \cdots \varrho_m)|} > \sigma - 1.$$

But, since the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, this is impossible by Criterion 2, whence the assertion (5). Further, by the Corollary to Criterion 2, we conclude that every non-trivial system of Latin polynomials belonging to the system $\varrho_1, \varrho_2, \ldots, \varrho_m$ satisfies (5). I assert that we can take

$$\varrho_1^{(\sigma-1)} = \varrho_1 - \delta_{l1} \ , \ \varrho_2^{(\sigma-1)} = \varrho_2 - \delta_{l2}, \ldots, \ \varrho_m^{(\sigma-1)} = \varrho_m - \delta_{lm}.$$

To prove this, it suffices to show that the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1 - \delta_{l1}, \varrho_2 - \delta_{l2}, \ldots, \varrho_m - \delta_{m_{lm}}$. But this follows immediately from (5), since (5) implies that there exists no non-trivial system of Latin polynomials satisfying the inequalities

$$\overline{|a_k(\varrho_1 - \delta_{l1}\, \varrho_2 - \delta_{l2} \cdots \varrho_m - \delta_{lm})|} \leqslant \varrho_k - \delta_{lk} - 1 \qquad (k=1,2,\ldots,m),$$
$$\underline{|\ r(\varrho_1 - \delta_{l1}\, \varrho_2 - \delta_{l2} \cdots \varrho_m - \delta_{lm})|} > \sigma - 2.$$

If the system

$$\varrho_1 - \delta_{l1} \ , \ \varrho_2 - \delta_{l2}, \ldots, \ \varrho_m - \varrho_{lm}$$

is non-trivial, we can repeat this procedure, until, after σ steps, we obtain the trivial system

$$0 \ , \ 0, \ldots, \ 0.$$

The function vector $\boldsymbol{f}$ is then normal at all systems so constructed, and this therefore gives the systems (4).

Secondly, we construct the systems

(6) $$\varrho_1^{(n)}, \varrho_2^{(n)}, \ldots, \varrho_m^{(n)} \text{ where } n = \sigma+1, \sigma+2, \ldots.$$

If

$$\mathfrak{a}_k(\varrho_1 \varrho_2 \cdots \varrho_m) \qquad (k=1,2,\ldots,m)$$

is a non-trivial system of German polynomials belonging to the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, then there exists, by the First Uniqueness Theorem, an integer j, with $1 \leqslant j \leqslant m$, such that

(7) $$\overline{|\mathfrak{a}_j(\varrho_1 \varrho_2 \cdots \varrho_m)|} = \sigma - \varrho_j.$$

I assert that the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1 + \delta_{j1}$, $\varrho_2 + \delta_{j2}, \ldots, \varrho_m + \delta_{jm}$, so that we can take

$$\varrho_1^{(\sigma+1)} = \varrho_1 + \delta_{j1} \ , \ \varrho_2^{(\sigma+1)} = \varrho_2 + \delta_{j2}, \ldots, \ \varrho_m^{(\sigma+1)} = \varrho_m + \delta_{jm}.$$

For suppose that, on the contrary,

$$\overline{|R_j(\varrho_1\varrho_2\cdots\varrho_m)|} > \sigma.$$

Then the polynomial

$$\mathscr{C}_j = \sum_{k=1}^{m} A_{jk}(\varrho_1\varrho_2\cdots\varrho_m)\,\mathfrak{a}_k(\varrho_1\varrho_2\cdots\varrho_m)$$

is an expression of the forme $e\begin{pmatrix} r_1\, r_2 \ldots r_m\, s \\ \mathfrak{w}_1\, \mathfrak{w}_2 \ldots \mathfrak{w}_m\, \mathfrak{s}\end{pmatrix}$, with parameter values

$$r_k = \varrho_k + \delta_{jk}\ ,\ \mathfrak{w}_k = \varrho_k,$$

$$s = \sigma + 2\quad ,\quad \mathfrak{s} = \sigma.$$

(D) and (0) therefore give the estimates

$$\overline{|\mathscr{C}_j|} \leqslant \max_{k=1,\ldots,m} \{(\varrho_k + \delta_{jk} - 1) + (\sigma - \varrho_k)\} \leqslant \sigma,$$

$$\underline{|\mathscr{C}_j|} \geqslant \min\,\{\sigma+1, \sigma+1\} = \sigma + 1.$$

However, by equation (7), it is clear that in fact

$$\overline{|\mathscr{C}_j|} = \sigma.$$

But this is impossible, whence the assertion.

We can repeat this procedure for the system

$$\varrho_1 + \delta_{j1}\ ,\ \varrho_2 + \delta_{j2}, \ldots,\ \varrho_m + \delta_{jm}$$

and continue in this manner indefinitely. The function vector $\boldsymbol{f}$ is then normal at all systems so constructed, and this therefore gives the systems (6).

On taking together the systems in (4) and (6), we obtain the required normality zigzag. This completes the proof.

Since every function vector, which vanishes at none of the primes in Π, is trivially normal at the system $0, 0, \ldots, 0$, the Normality Zigzag Theorem has the following immediate corollary.

Crollary. *Every function vector* $\boldsymbol{f}$, *which vanishes at none of the primes in* Π, *is normal at infinitely many systems* $\varrho_1, \varrho_2, \ldots, \varrho_m$.

However, this result is, as one would expect, very weak, and it is trivial when all the primes in Π are equal.

15. The function vector f has therefore a set of normality zigzags

$$\{\Sigma_1, \Sigma_2, \ldots\}$$

such that every system $\varrho_1, \varrho_2, \ldots, \varrho_m$, at which $\boldsymbol{f}$ is normal, belongs to at least one of these normality zigzags. A fundamental problem of this theory can now be formulated as follows.

Problem 1. *Given a set of systems* $\varrho_1, \varrho_2, \ldots, \varrho_m$, *determine conditions on the function vector* $\boldsymbol{f}$, *which imply that every system in this set belongs to at least one of the normality zigzags*

$$\Sigma_1, \Sigma_2, \ldots \text{ of } \boldsymbol{f}.$$

The most important case of this problem arises when the set given consists of all systems $\varrho_1, \varrho_2, \ldots, \varrho_m$. We shall study the properties of function vectors satisfying this stronger condition later.

VI.

16. We now show that there exist simple relations linking the Latin and German matrices belonging to two different systems, $\varrho_1, \varrho_2, \ldots, \varrho_m$ and $\varrho_1', \varrho_2', \ldots, \varrho_m'$, at which the function vector $\boldsymbol{f}$ is normal.

Given that the function vector $\boldsymbol{f}$ is normal at the system $\varrho_1, \varrho_2, \ldots, \varrho_m$, then the Latin and German matrices

$$A(\varrho_1 \varrho_2 \cdots \varrho_m) \ , \ \mathfrak{A}(\varrho_1 \varrho_2 \cdots \varrho_m)$$

are non-singular and uniquely determined. The inverses of each of these matrices can easily be determined. For, by § 8, m^2 equations hold

$$\sum_{k=1}^{m} A_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, \mathfrak{A}_{jk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \delta_{hj} \varepsilon_j \psi_\sigma, \text{ with } \varepsilon_h \in F, \qquad (h, j = 1, 2, \ldots, m).$$

The constants $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ are all non-zero, since the degree of the left hand side is equal to σ whenever $h = j$. However, the coefficient of ψ_σ in the interpolation for the left hand side is equal to 1 whenever $h = j$, and so each of the constants $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ is equal to 1. Hence the m^2 equations are

$$\sum_{k=1}^{m} A_{hk}(\varrho_1 \varrho_2 \cdots \varrho_m)\, \mathfrak{A}_{jk}(\varrho_1 \varrho_2 \cdots \varrho_m) = \delta_{hj} \psi_\sigma \qquad (h, j = 1, 2, \ldots, m)$$

and so are equivalent to the single matrix equation

$$A(\varrho_1 \varrho_2 \cdots \varrho_m)\, \mathfrak{A}'(\varrho_1 \varrho_2 \cdots \varrho_m) = \psi_\sigma I$$

where I denotes the $m \times m$ unit matrix. This equation implies that

$$A(\varrho_1 \varrho_2 \cdots \varrho_m)^{-1} = \frac{1}{\psi_\sigma} \mathfrak{A}'(\varrho_1 \varrho_2 \cdots \varrho_m),$$

$$\mathfrak{A}(\varrho_1 \varrho_2 \cdots \varrho_m)^{-1} = \frac{1}{\psi_\sigma} A'(\varrho_1 \varrho_2 \cdots \varrho_m).$$

From these formulae, the elements of $A(\varrho_1 \varrho_2 \cdots \varrho_m)^{-1}$ and $\mathfrak{A}(\varrho_1 \varrho_2 \cdots \varrho_m)^{-1}$ lie in the quotient field of ω.

17. Assume now that the function vector $\boldsymbol{f}$ is also normal at the system $\varrho_1', \varrho_2', \ldots, \varrho_m'$, with sum σ'. Let, say, $\sigma' \geqslant \sigma$. Naturally the systems $\varrho_1, \varrho_2, \ldots, \varrho_m$ and $\varrho_1', \varrho_2', \ldots, \varrho_m'$ are not necessarily in the same normality zigzag of $\boldsymbol{f}$.

Define the matrices

$$P\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} = A(\varrho_1'\varrho_2'\cdots\varrho_m')\,A(\varrho_1\varrho_2\cdots\varrho_m)^{-1} =$$

$$= \frac{1}{\psi_\sigma}\,A(\varrho_1'\varrho_2'\cdots\varrho_m')\,\mathfrak{A}'(\varrho_1\varrho_2\cdots\varrho_m),$$

$$\mathfrak{P}\begin{pmatrix}\varrho_1'\varrho_1'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} = \mathfrak{A}(\varrho_1'\varrho_2'\cdots\varrho_m')\,\mathfrak{A}(\varrho_1\varrho_2\cdots\varrho_m)^{-1} =$$

$$= \frac{1}{\psi_\sigma}\,\mathfrak{A}(\varrho_1'\varrho_2'\cdots\varrho_m')\,A'(\varrho_1\varrho_2\cdots\varrho_m),$$

so that

$$A(\varrho_1'\varrho_2'\cdots\varrho_m') = P\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} A(\varrho_1\varrho_2\cdots\varrho_m),$$

$$\mathfrak{A}(\varrho_1'\varrho_2'\cdots\varrho_m') = \mathfrak{P}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} \mathfrak{A}(\varrho_1\varrho_2\cdots\varrho_m).$$

We call $P\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}$ and $\mathfrak{P}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}$ the Latin and German *transformation matrices*, respectively. The elements of these transformation matrices are given explicitly by the equations

$$P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} = \frac{1}{\psi_\sigma}\sum_{k=1}^{m} A_{hk}(\varrho_1'\varrho_2'\cdots\varrho_m')\,\mathfrak{A}_{jk}(\varrho_1\varrho_2\cdots\varrho_m) \qquad (j,h=1,2,\ldots,m),$$

$$\mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} = \frac{1}{\psi_\sigma}\sum_{k=1}^{m} \mathfrak{A}_{hk}(\varrho_1'\varrho_2'\cdots\varrho_m')\,A_{jk}(\varrho_1\varrho_2\cdots\varrho_m) \qquad (j,h=1,2,\ldots,m).$$

From these formulae, the elements of these transformation matrices lie in the quotient field of ω, and their denominators are factors of ψ_σ. In fact, we shall deduce from $\sigma' \geqslant \sigma$ that their elements are polynomials.

The polynomials

$$\psi_\sigma P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}, \quad \psi_\sigma \mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} \qquad (h,j=1,2.\ldots,m)$$

are expressions of the form $e\begin{pmatrix}r_1\,r_2\ldots r_m\,s\\ \mathfrak{w}_1\,\mathfrak{w}_2\ldots\mathfrak{w}_m\,\mathfrak{s}\end{pmatrix}$ with parameter values

$$r_k = \varrho_k' + \delta_{hk}\;,\;\; \mathfrak{w}_k = \varrho_k - \delta_{jk}\;\;;\;\; r_k = \varrho_k + \delta_{jk}\;,\;\; \mathfrak{w}_k = \varrho_k' - \delta_{hk},$$

$$s = \sigma' + 1 \quad,\quad \mathfrak{s} = \sigma - 1 \quad;\quad s = \sigma + 1 \quad,\quad \mathfrak{s} = \sigma' - 1,$$

and thus (D) and (0) give the estimates

$$\overline{\left|\psi_\sigma P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \leqslant \max_{k=1,\dots,m}\{(\varrho_k'+\delta_{hk}-1)+(\sigma-1-\varrho_k+\delta_{jk})\} = \\ = \max_{k=1,\dots,m}\{\sigma+\varrho_k'-\varrho_k+\delta_{hk}+\delta_{jk}-2\},$$

$$\underline{\left|\psi_\sigma P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \geqslant \min\{\sigma',\sigma\}=\sigma.$$

$$\overline{\left|\psi_\sigma \mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \leqslant \max_{k=1,\dots,m}\{(\varrho_k+\delta_{jk}-1)+(\sigma'-1-\varrho_k'+\delta_{hk})\} = \\ = \max_{k=1,\dots,m}\{\sigma'+\varrho_k-\varrho_k'+\delta_{jk}+\delta_{hk}-2\},$$

$$\underline{\left|\psi_\sigma \mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \geqslant \min\{\sigma',\sigma\}=\sigma.$$

Hence all of the polynomials

$$\psi_\sigma P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}, \quad \psi_\sigma \mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} \qquad (h,j=1,2,\dots,m)$$

have orders at least equal to σ, proving our assertion that all the elements

$$P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}, \qquad \mathfrak{P}_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} \qquad (h,j=1,2,\dots,m)$$

are polynomials, of degrees satisfying the inequalities

$$\overline{\left|P_{hj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \leqslant \max_{k=1,\dots,m}\{\varrho_k'-\varrho_k+\delta_{jk}+\delta_{hk}-2\} \qquad (h,j=1,2,\dots,m),$$

$$\overline{\left|\mathfrak{P}_{kj}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right|} \leqslant \max_{k=1,\dots,m}\{\sigma'-\sigma+\varrho_k-\varrho_k'+\delta_{jk}+\delta_{hk}-2\} \qquad (h,j=1,2,\dots,m),$$

From the properties of the Latin and German matrices, we also deduce that

$$\left|P\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right| = \frac{\psi_{\sigma'}}{\psi_\sigma},$$

$$\left|\mathfrak{P}\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix}\right| = \left(\frac{\psi_{\sigma'}}{\psi_\sigma}\right)^{m-1},$$

$$P\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} \mathfrak{P}'\begin{pmatrix}\varrho_1'\varrho_2'\cdots\varrho_m'\\ \varrho_1\varrho_2\cdots\varrho_m\end{pmatrix} = \frac{\psi_{\sigma'}}{\psi_\sigma} I.$$

18. One can easily obtain explicit expressions for the Latin and German transformation matrices if we suppose that the two systems $\varrho_1, \varrho_2, \dots, \varrho_m$

and $\varrho_1', \varrho_2', \ldots, \varrho_m'$ are related as follows:

either

$$\varrho_1' = \varrho_1 + 1 \ , \ \varrho_2' = \varrho_2 + 1, \ldots, \ \varrho_m' = \varrho_m + 1,$$

or

$$\varrho_1' = \varrho_1 + \delta_{h1} \ , \ \varrho_2' = \varrho_2 + \delta_{h2}, \ldots, \ \varrho_m' = \varrho_m + \delta_{hm},$$

or

$$\varrho_1' = \varrho_1 + \delta_{h1} - \delta_{k1} \ , \ \varrho_2' = \varrho_2 = \delta_{h2} - \delta_{k2}, \ldots, \ \delta_m' = \varrho_m + \delta_{hm} - \delta_{km}.$$

However, we omit these expressions.

The following problem concerning these transformation matrices was proposed to me by Mahler.

Problem 2. *Given a sequence of systems*

$$\varrho_1^{(n)}, \varrho_2^{(n)}, \ldots, \varrho_m^{(n)} \qquad (n = 0, 1, 2, \ldots),$$

and a sequence of Latin transformation matrices

$$P\begin{pmatrix} \varrho_1^{(n+1)} & \varrho_2^{(n+1)} & \ldots & \varrho_m^{(n+1)} \\ \varrho_1^{(n)} & \varrho_2^{(n)} & \ldots & \varrho_m^{(n)} \end{pmatrix} \qquad (n = 0, 1, 2, \ldots)$$

or a sequence of German transformation matrices

$$\mathfrak{P}\begin{pmatrix} \varrho_1^{(n+1)} & \varrho_2^{(n+1)} & \ldots & \varrho_m^{(n+1)} \\ \varrho_1^{(n)} & \varrho_2^{(n)} & \ldots & \varrho_m^{(n)} \end{pmatrix} \qquad (n = 0, 1, 2, \ldots)$$

does there exist a function vector to which these transformation matrices belong?

The case of particular interest is when

$$\varrho_1^{(n)} = \varrho_2^{(n)} = \ldots = \varrho_m^{(n)} = n \qquad (n = 0, 1, 2, \ldots).$$

(To be continued)