Separability within alternating groups and randomness

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Declaration

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Abstract

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This thesis promotes known residual properties of free groups, surface groups, right angled Coxeter groups and right angled Artin groups to the situation where the quotient is only allowed to be an alternating group. The proofs follow two related threads of ideas.

The first thread leads to ‘alternating’ analogues of extended residual finiteness in surface groups [Sco78], right angled Artin groups and right angled Coxeter groups [Hag08]. Let W be a right-angled Coxeter group corresponding to a finite non-discrete graph $\mathscr{G}$ with at least 3 vertices. Our main theorem says that $\mathscr{G}^c$ is connected if and only if for any infinite index convex-cocompact subgroup H of W and any finite subset $\{\gamma_1, \dots, \gamma_n\} \subset W \setminus H$ there is a surjective homomorphism f from W to a finite alternating group such that $f(\gamma_i) \notin f(H)$. A corollary is that a right-angled Artin group splits as a direct product of cyclic groups and groups with many alternating quotients in the above sense.

Similarly, finitely generated subgroups of closed, orientable, hyperbolic surface groups can be separated from finitely many elements in an alternating quotient, answering positively a conjecture of Wilton [Wil12].

The second thread uses probabilistic methods to provide ‘alternating’ analogues of subgroup conjugacy separability and subgroup into-conjugacy separability in free groups [BG10]. Suppose $H_1, \dots H_k$ are infinite index, finitely generated subgroups of a non-abelian free group F. Then there exists a surjective homomorphism $f : F \longrightarrow A_m$ such that if H_i is not conjugate into H_j, then $f(H_i)$ is not conjugate into $f(H_j)$.

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Introduction

0.1 Finite index subgroups

Suppose you want to promote a locally injective map of spaces $X \longrightarrow Y$ to an injective map by lifting it to some finite cover of Y. This means that you're looking for a certain kind of a finite-index subgroup of $\pi_1(Y)$.

Or assume that you have a presentation of a group and you want to check whether a given element lies outside a specified subgroup. Then you might want to seek a finite quotient, where the image of the element does not belong to the image of the subgroup.

Both of the above are examples of residual problems. As the name suggests they generalise residues in $\mathbb{Z}$. In groups we are normally looking at the residues modulo finite index subgroups. *A residual property* allows us to preserve some property of separation, distinctness or disjointness in finite quotients. For example, in *residually finite* groups distinctness of two elements $g \neq h$ passes to their images under some finite quotient. Residual finiteness is a very common property: Mal'cev proved that all finitely generated linear groups are residually finite [Mal40]. For topological applications we often want to separate more then just the trivial subgroup. A group is *subgroup separable*, if for any finitely generated subgroup and an element not in this subgroup, there exists some finite quotient in which the image of the element does not belong to the image of the subgroup. Hall proved that free groups are subgroup separable [Hal49] and Scott proved it for surface groups [Sco78, Sco85]. An even stronger property was introduced by Bogopolski-Grunewald: in *subgroup conjugacy separable* groups two non-conjugate subgroups are sent to non-conjugate subgroups in some finite quotient, in *subgroup into-conjugate separable* group the same applies but for the property of being conjugate into. They established that the subgroup conjugacy separability and subgroup into-conjugacy separability for free groups [BG10] and later Bogopolski-Bux had done the same for surface groups [BB14].

Finitely generated subgroups are not always the best class of subgroups to consider when studying residual properties. Sometimes, we deal with a space with a geometric structure and then a nicer class of group may arise from that structure. For example in Chapter 2, subgroups of RAAGs we consider are the groups acting cocompactly on convex subspaces. Not every finitely generated subgroup of a RAAG is separable since subgroup separable finitely presented groups have a solvable membership problem for finitely generated subgroups, but $F_2 \times F_2$ contains a finitely generated subgroup whose membership can't be decided [Mih68]. However, Haglund extended Scott's theorem to convex-cocompact subgroups of RACGs and RAAGs [Hag08].

The convex-cocompact subgroups coincide with finitely generated subgroups in free groups and surface groups.

0.2 Residual properties within other groups

If we allow any finite quotient, we are giving up some control. When can we recover the same information from a subclass of the quotients? For example, suppose that G is residually finite. When is the intersection of all kernels of maps onto simple groups a singleton?

A great deal of work has been done studying residually p-groups. For example right angled Artin groups are residually p-groups for any p [DK92].

Another class of quotient that has often been looked at is finite simple groups. These often arise naturally when studying the "congruence topology" on a linear group, induced by reducing modulo a maximal ideal. For instance, Long and Reid proved that every hyperbolic 3-manifold group is residually finite simple [LR98, Theorem 1.2]. It remains a problem of great interest to find congruence covers of hyperbolic 3-manifold groups with special properties; see, for instance [AS19].

Surface groups, right-angled Artin groups and right-angled Coxeter groups admit a version of subgroup separability within alternating groups. This generalizes the ordinary subgroup separability of these groups [Hag08]. I will extend the results about the residual properties within alternating groups. See Table 0.3 for the timeline of theorems on residual properties and their counterparts within alternating groups.

The separability of convex-cocompact subgroups of special groups is inherited from right angled Artin groups. The same does not apply to the 'alternating' analogue, since a subgroup of an alternating group need not be alternating.

The earliest proofs of residual finiteness and residual properties tend to go by an explicit construction of a quotient. This is also the approach I choose to broaden the scope of the theorems. In chapter 2, I will construct quotients almost by hand to demonstrate that the theorems about subgroup separability within alternating groups apply not only to free groups

[Wil12], but also to surface groups, right angled Artin groups and right angled Coxeter groups.

More elegant proofs of residual properties are formulated later. These later proofs often supply a natural class of quotients, which demonstrate a residual property. For example, Stallings reproved residual finiteness of free groups using a topological argument [Sta83]. In the last chapter, I show that free groups admit a version of subgroup conjugacy separability. This will be done by constructing a fairly natural probability distribution on the quotients, such that the probability of demonstrating a residual property within a class of groups can be bounded away from zero.

To do this, we'll need to understand two things. Firstly, we need to know the group type of the quotient. This will allow us to control the quotient group by changing the probability distribution. Secondly, we need to understand the small scale behaviour of a typical quotient. This will give us useful information about the residual properties.

Example 0.2.1. Two elements of S_n generate S_n or A_n with probabilities $3/4 - o(1)$ and $1/4 - o(1)$ respectively [Dix69]. Hence a typical map $F_2 \longrightarrow S_n$ hits S_n or A_n.

Pick a degree-n covering of $S^1 \vee S^1$ uniformly at random. Pick a random vertex in this covering. The probability that a k-neighbourhood of this vertex is a tree goes to 1 as n goes to infinity.

Combining these two observations, given distinct $g, h \in F_2$ we can easily find a map $f : F_2 \longrightarrow A_n$ with $f(g), f(h)$ distinct. Simply take a random map $F_2 \longrightarrow S_n$ and send n to infinity. The image is A_n with probability $\sim 1/4$ and the probability that $f(g) \neq f(h)$ tends to 1 since a typical ball of radius $l(g) + l(h)$ is a tree. Here $l(g)$ is the word length of g.

There are of course much easier ways to show that free groups are residually alternating, but this idea generalises to show better residual properties in chapter 3.

People had asked before in various settings: "What is a typical quotient?" One can take two random elements [Dix69] or even one restricted element and the other at random [Bab89]. The results of Chapter 3 enable us to impose restrictions on both (or all) generators simultaneously.

Random actions of groups have been examined before, for example Puder-Parzanchevski examined the number of points fixed by a subgroup of a free group under a random permutation action [PP15]. Constructing a specific map is normally easier than estimating a probability that the map demonstrates a separability property. This contrasts with the probabilistic approach, where controlling any individual quotient might be tricky or tedious, but the typical behaviour is easy to understand. Probabilistic methods had been used before to prove that for every infinite class $\mathscr{C}$ of simple groups, every non-abelian free group is residually $\mathscr{C}$ [DPSS03, Theorem 3].

0.3 Results

A subgroup is $\mathscr{C}$-separable if we can demonstrate its separability with maps onto groups in the class $\mathscr{C}$.

Definition 0.3.1 ($\mathscr{C}$-separable)**.** Let H be a subgroup of a finitely generated group G, let $\mathscr{C}$ be a class of groups. We say that H is *$\mathscr{C}$-separable* if for any choice of $\{\gamma_1, \dots, \gamma_m\} \subset G \setminus H$ there is a surjective homomorphism f from G to a group in $\mathscr{C}$ such that $f(\gamma_i) \notin f(H)$ for all i.

We often take $\mathscr{C}$ to be the class of alternating groups $\mathscr{A}$ or the class of symmetric groups $\mathscr{S}$.

We can exploit geometry of right angled Coxeter groups to get a version of residual finiteness within alternating groups. This theorem improves Haglund's theorem [Hag08] that convex-cocompact subgroups of RACGs are separable to alternating quotients. This answers affirmatively a conjecture of Wilton from [Wil12] where the result was proved for free groups.

Theorem A (Alternating quotients of RACGs)**.** *Let $\mathscr{G}$ be a non-discrete finite simplicial graph of size at least* 3*. Then all infinite-index convex-cocompact subgroups of the right-angled Coxeter group associated to $\mathscr{G}$ are $\mathscr{A}$-separable and $\mathscr{S}$-separable if and only if $\mathscr{G}^c$ is connected.*

This is the Theorem 2.2.1 below. The idea of the proof is the same as in the case of free groups: build a suitable finite sheeted covering so that the action on the associated finite index subgroups is alternating. However, there are substantial technical difficulties to overcome. To build these covers explicitly requires a refined understanding of the convex core constructed in the Scott/Haglund argument. A corollary gives an analogous result for right angled Artin groups and surface groups.

Suppose G and H are finitely generated subgroups of a free group. The property 'G is not conjugate to H' passes to their images in some finite quotient of the free group. The same is true for 'is not conjugate into' [BG10, Theorems 1.3 and 1.8]. The proof again goes by an explicit controlled construction.

A multi-subgroup version of these properties would start with non-conjugate subgroups, resp. groups, where none of them conjugates into any other of them. There is a common refinement of these two properties (in case of free groups). We can take a finite collection of subgroups $H_1,\dots,H_k$ and look for a quotient map f such that if $f(H_i)$ is conjugate into $f(H_j)$ then H_i is conjugate into H_j. This version is still true for free groups even with only alternating quotients.

Theorem B. *Suppose $H_1,\dots,H_k$ are infinite index, finitely generated subgroups of a non-abelian free group F. Then there exists a surjective homomorphism $f: F \longrightarrow A_m$ such that if H_i is not conjugate into H_j, then $f(H_i)$ is not conjugate into $f(H_j)$.*

This is Theorem 3.6.11. It provides an 'alternating' improvement of the main theorem of [BG10].

Unlike all the previous proofs, this one is probabilistic. Probabilistic methods had been used in permutation groups before. Recall that two random elements of S_n generate all of S_n or all of A_n with large probability [Dix69]. A similar result applies even if only one of the elements is random and the other does not fix many points [Bab89, Theorem 1]. However, to the best of my knowledge, this is the first time randomness has been used to prove a residual property.

I'll illustrate the proof on a simple example. Suppose we want to find a surjective homomorphism $f: F \longrightarrow S_m$ where $f(\langle a\rangle)$ and $f(\langle [a,b]\rangle)$ are not conjugate. Consider a random map $f: F \longrightarrow S_{1000}$ from among those where a fixes $\{1,\dots,100\}$ pointwise. The generator a is going to fix on average 101 points since a acts as a random permutation on $\{101,\dots,1000\}$ and each of these 900 points is fixed with probability $1/900$. The commutator $[a,b]$ is going to fix roughly 1 point on average, intuitively because if $w = ba^{-1}b^{-1}(v)$, then v is fixed only if $a(w) = v$ and the probability of this happening is about $1/1000$ for every v. (This analysis isn't quite accurate as the action of a isn't independent of the action of $ba^{-1}b^{-1}$. I will formalize this in Chapter 2.)

But this means that $f(\langle [a,b]\rangle)$ is sometimes not conjugate into $f(\langle a\rangle)$, since it fixes fewer points. To make this work in general, we need to control the variance and also the fixed points of characteristic subgroups of H_i's. The following table lists references for some separability results and their alternating analogues.

Statement	Original version	'Alternating' analogue
Free groups are residually finite.	Easy	[KM69]
Free groups are subgroup separable.	[Hal49]	[Wil12]
Surface groups are subgroup separable.	[Sco78, Sco85]	Chapter 2 and [Bur19]
Convex-cocompact subgroups of RACGs are separable.	[Hag08]	Chapter 2 and [Bur19]
Convex-cocompact subgroups of RAAGs are separable.	[Hag08]	Chapter 2 and [Bur19]
Convex-cocompact subgroups of special groups are separable.	[Hag08]	Unknown
Free groups are subgroup conjugacy separable.	[BG10]	Chapter 3
Free groups are subgroup into-conjugacy separable.	[BG10]	Chapter 3

Chapter 1

Background Material

The central tenet of geometric group theory says to judge groups by their actions. The following example illustrates the power of this approach.

Example 1.0.1 (Why to think about actions of groups). A group is free if and only if it acts (simplicially) on a simplicial tree without a fixed point. This property passes to subgroups, hence a subgroup of a free group is free.

1.1 Curvature

1.1.1 Negative curvature

Free groups are in some sense an extreme case. They have no relators and their presentation complexes are graphs. If free groups are extreme what objects are most similar to them? Various notions generalising free groups were gradually discovered, most of them come down to some form of negative curvature. Fundamental groups of surfaces are the most straightforward generalisation.

Definition 1.1.1 (Surface groups)**.** A *genus-g surface group* is given by the presentation

$$\langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1][a_2, b_2] \ldots [a_g, b_g] \rangle .$$

If $g > 1$, we call the surface group *hyperbolic*.

The $4g$-gons in the Cayley complex of a genus-g hyperbolic group overlap at most a single edge. This property is shared by small-cancellation groups.

Definition 1.1.2 (Small cancellation group)**.** Suppose $\lambda \geq 0$, $G = \langle S|R \rangle$, where R is a set of freely reduced and cyclically reduced words in X and is closed under taking cyclic permutations and inverses. The group G is $C'(\lambda)$ if whenever u is an initial segment of two distinct relators r_1, r_2, then $|u| < \lambda |r_1|$. If $\lambda \leq 1/6$, we say that the group is a *small cancellation group*.

Remark 1.1.3. There is also a C and a T small cancellation condition, but it is not relevant to us.

Small cancellation groups have a large number of nice properties. For example they have solvable word problem, and a closed curve of length l bounds an area consisting of at most cl for c depending only on the group and not on l. The second property in fact implies the first one. If γ is a contractible simplicial curve, then the cells enclosed by γ are all within distance of $|\gamma|$ of $\gamma(0)$. There are only finitely many such cells since the Cayley complex is locally compact provided $|R|, |S| < \infty$. So to show that the word w represents a non-trivial element, we only need to list all length $|w|$ curves, which enclose at most $c|w|$ cells each within $|w|$ of the basepoint.

Definition 1.1.4 (Area and Dehn function)**.** Suppose $G = \langle S|R \rangle$. Let F_S be a free group on S, and $w \in \langle\langle R \rangle\rangle$. Then the *area of* w in G is the minimal number of conjugates of relators needed to express w as an element of $\langle\langle R \rangle\rangle$.

$$Area(w) = \min\{n | w = \prod_{i=1}^{n} r_i^{g_i}, \text{ where } r_i \in R \text{ and } g_i \in G\}$$

The *Dehn function* is the maximal area of a word of the given length.

$$Dehn(l) := \max_{w:|w|=l} Area(w)$$

The algebraic definition of area above is equivalent to the topological definition, which instead of counting relators counts cells. The linearity of the Dehn function mentioned above is in fact a group property independent of the presentation (although the constants may change).

Lemma 1.1.5. *[Gre60] The Dehn function of a small cancellation group is bounded above by a linear function.*

This reveals a problem with the definition of small cancellation groups. The property of being small cancellation is in fact a property of a presentation and a group, which admits such a presentation, may also admit a presentation which is not small-cancellation.

In 1987, Gromov came up with a definition of hyperbolicity for metric spaces [Gro87]. Many seemingly unrelated definitions lead to an equivalent notion of negative curvature. For example the groups with linear Dehn function are exactly the hyperbolic groups. This suggests that Gromov hyperbolicity is the correct canonical concept. I will only ever look at proper geodesic spaces, so I'll give a definition for that particular case.

Definition 1.1.6 (Thin triangles, hyperbolicity for a proper geodesic spaces)**.** Suppose X is a proper geodesic space and $x,y,z \in X$. A *triangle* $[x,y,z]$ is a union of geodesics $[x,y]$, $[y,z]$ and $[z,x]$ (the notation is a bit ambiguous since a geodesic does not have to be specified by its endpoints). A triangle $[x,y,z]$ is δ*-thin* if each of its sides belongs to the union of δ-neighbourhoods of the remaining two sides.

The space X is *hyperbolic* if there exists δ such that any triangle in X is δ-thin.

A group G is *hyperbolic* if it acts geometrically (i.e. properly and cocompactly) on a hyperbolic space.

The name is inspired by hyperbolic manifolds, which have hyperbolic fundamental groups. The hyperbolic groups admit a large number of strong properties and generalisations. For example relative hyperbolicity ignores non-hyperbolicity contained in certain subspaces, acylindrical hyperbolicity allows the group to act on a hyperbolic space with weaker conditions, hierarchical hyperbolicity allows one to glue products of hyperbolic spaces. However, this is not the type of situation this thesis is about. Instead I'll be looking at non-positive curvature.

1.1.2 Non-positive curvature

Going from negative to non-positive curvature, one encounters $CAT(0)$*-spaces*. Roughly speaking those are simply connected geodesic metric spaces whose triangles are no thicker than the corresponding triangles in the Euclidean plane. The reference for this subsection is Bridson-Haefliger's book Metric Spaces of Non-Positive Curvature [BH13].

Definition 1.1.7 (Comparison triangle, CAT(0) inequality, CAT(0) space, non-positive curvature)**.** Suppose $[x_1,x_2,x_3]$ is a triangle in a simply connected geodesic metric space (X,d_X). The associated *comparison triangle* $[x'_1,x'_2,x'_3]$ is a triangle in $\mathbb{R}^2$ with $d_X(x_i,x_j) = d_{\mathbb{R}^2}(x'_i,x'_j)$ for all i and j. Let f be the map from $[x'_1,x'_2,x'_3]$ to $[x_1,x_2,x_3]$, which maps each geodesic $[x'_i,x'_j]$ isometrically to $[x_i,x_j]$. The triangle $[x_1,x_2,x_3]$ satisfies the $CAT(0)$ *inequality* if f does not increase any distance.

If every triangle satisfies the $CAT(0)$ inequality, we say that X is a $CAT(0)$ *space*. We call a group $CAT(0)$, if it acts properly and cocompactly on a $CAT(0)$ space. We say a metric space is *non-positively curved* if its universal cover equipped with the natural length metric is $CAT(0)$. There is a rich supply of $CAT(0)$ spaces, which come from cube complexes

There are $CAT(\kappa)$ spaces for every $\kappa \in \mathbb{R}$, where the comparison triangle is taken in a simply connected space of a uniform sectional curvature κ and in case $\kappa > 0$, there is an additional condition on the size of the triangles since large triangles in spheres are not convex. Note that rescaling the metric in such a space also rescales κ, so there are essentially just three fundamentally distinct notions - $CAT(\kappa)$ for $\kappa = -1, 0, 1$.

The simplest example of a non-positively curved space which is not negatively curved is a genus-1 surface, i.e. torus.

Example 1.1.8 (Torus). The torus is a quotient of the Euclidean plane by $\mathbb{Z}^2$. The Euclidean plane is clearly $CAT(0)$, since every triangle in $\mathbb{R}^2$ is its own comparison triangle. At first sight, this is very different from being hyperbolic. It is not $C'(1/6)$, since cells overlap at $1/4$ of their length. A curve of length $4l$ may enclose an area l^2 if we just take a square of side length l. The Dehn function is quadratic. However, there are some similarities, for example the bound on the Dehn function again allows us to solve the word problem.

In fact, every $CAT(0)$ space has at most quadratic Dehn function and hence a solvable word problem [BH13, Proposition 1.6,p.442]. Another difference is that hyperbolicity ignores what happens on scale δ, whereas non-positive curvature can be spoiled by an arbitrarily small triangle, that violates the $CAT(0)$ condition.

1.2 Cube complexes

Small cancellation groups provide a rich supply of hyperbolic groups. We only need to check a finite number of local combinatorial conditions (provided the presentation is finite). Now we'd like to get some such similar supply for $CAT(0)$ groups. We will get it by gluing Euclidean cubes together into a cube complex. The non-positive curvature will again follow from a finite number of local combinatorial conditions (provided the cube complex is compact). First, I'll just define what a cube complex is before showing what the condition for the non-positive curvature is. For further details of the definitions from this section, the reader is referred to [HW08].

Definition 1.2.1 (Cube, face)**.** *An n-dimensional cube C* is I^n, where $I = [-1, 1]$. *A face* of a cube is a subset $F = \{\underline{x} : x_i = (-1)^\varepsilon\}$, where $1 \leq i \leq n$, and ε is 0 or 1.

Note that the face above is a codimension-1 subcube. We can build a cube complex identifying these faces similarly to a simplicial complex. I chose to describe these identifications by listing all cubes and all the inclusions of faces between them.

Definition 1.2.2 (Cube complex)**.** Suppose $\mathscr{C}$ is a set of cubes and $\mathscr{F}$ is a set of maps between these cubes, each of which is an inclusion of a face. Suppose that every face of a cube in $\mathscr{C}$ is an image of exactly one inclusion of a face $f \in \mathscr{F}$. Then *the cube complex X associated to* $(\mathscr{C}, \mathscr{F})$ is the topological space

$$X = (\bigsqcup_{C \in \mathscr{C}} C)/\sim$$

where $\sim$ is the smallest equivalence relation containing $x \sim f(x)$ for every $f \in \mathscr{F}$, $x \in Dom(f)$.

This definition uses a large number of cubes (in particular a face of a cube in $\mathscr{C}$ is an image of unique cube in $\mathscr{C}$). I went for this less efficient definition to make some later concepts such as hyperplanes easier to define.

For example a cube complex, which is a 3-cube consists of one cube, six squares, twelve edges and eight vertices and all the face inclusions between them. Each vertex is included in three edges, each edge in two squares and each square in the unique cube. So this cube complex consists of twenty-seven cubes and fifty-seven maps.

We can equip a cube complex with a metric, where we take each cube to be Euclidean of side length 1 and then take the induced length metric.

In analogy with manifolds, we'd like to have immersed codimension-1 subobjects. We can get them by gluing midcubes, where a midcube is a codimension-1 subcube. This is just a higher dimensional analogy of a midpoint.

Definition 1.2.3 (Midcube)**.** *A midcube M* of a cube I^n is a set of the form $\{\underline{x} : x_i = 0\}$ for some $1 \leq i \leq n$.

If $f : C \to C'$ is an inclusion of a face and M is a midcube of C, then $f(M)$ is contained in a unique midcube M' of C'. Moreover $f|_M \colon M \to M'$ is an inclusion of a face. You might notice that midcubes with their face inclusions form a cube complex themselves. Components of this cube complex are called hyperplanes.

Definition 1.2.4 (Hyperplane)**.** Let X be a cube complex associated to $(\mathscr{C}, \mathscr{F})$. Let $\mathscr{M}$ be the set of midcubes of cubes of $\mathscr{C}$. Let $\mathscr{F}'$ be the set of restrictions of maps in $\mathscr{F}$ to midcubes.

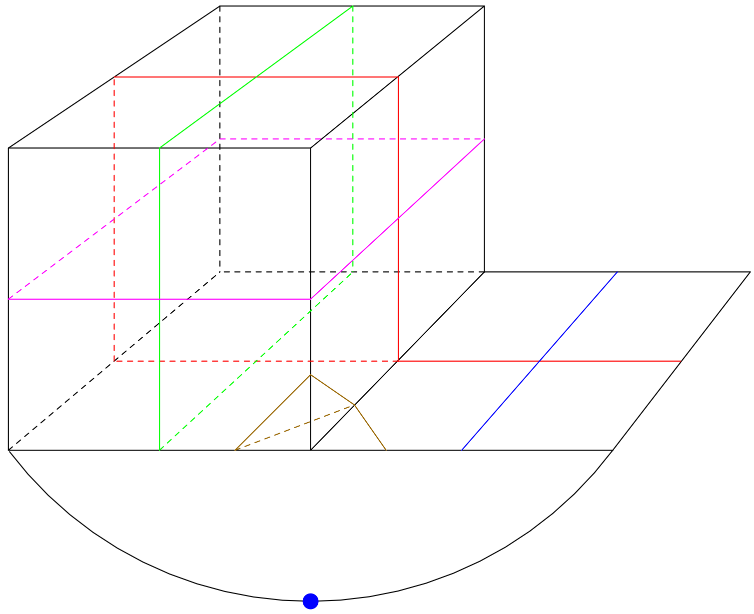

Fig. 1.1 A cube complex formed by a cube, square and an edge with all of its 5 hyperplanes, and a link of one of the vertices.

The pair $(\mathscr{M}, \mathscr{F}')$ satisfies that every face is an image of at most one inclusion of a face, so there is an associated cube complex X'. Moreover, inclusions of midcubes descend to a map $\varphi : X' \to X$. *A hyperplane* H is a connected component of X' together with a map $\varphi|_H$.

Hyperplanes play a fundamental role in the study of cube complexes. The Sageev construction turns information about hyperplane structure into a cube complex [Sag95]. We will discuss this in more detail later. Instead of hyperplanes, one can also talk about walls. A wall is simply a collection of edges intersected by a hyperplane.

Definition 1.2.5 (Elementary parallelism, wall)**.** Suppose X is a cube complex.

Define a relation of *elementary parallelism* on oriented edges of X by $\overrightarrow{e_1} \sim \overrightarrow{e_2}$ if they form opposite edges of a square (pointing in the same direction). Extend this to the smallest equivalence relation. *The wall* $W(\overrightarrow{e})$ is the equivalence class containing $\overrightarrow{e}$. Similarly, we can define an elementary parallelism on unoriented edges and *an unoriented wall* $W(e)$.

We denote by $\overleftarrow{e}$ the edge $\overrightarrow{e}$ with the opposite orientation. There is a bijective correspondence between unoriented walls and hyperplanes, where $W(e)$ corresponds to $H(e)$, a hyperplane which contains the unique midcube of e. We say $H(e)$ is dual to e. By abuse of notation, we sometimes identify $H(e)$ with its image.

In cube complexes, non-positive curvature can be detected by studying the a sphere of small radius around each vertex. See Figure 1.1.

Definition 1.2.6 (Link)**.** Suppose X is a cube complex and $v \in X$ is a vertex. Then *the link of v* is a sphere of small radius around v. It is a simplicial complex, where the simplicial structure comes from the intersections with cubes.

A cube complex is non-positively curved if and only if it is non-positively curved at the vertices in the following sense.

Theorem 1.2.7 (Link condition)**.** *[BH13, 5.20 Theorem] A finite dimensional cube complex is non-positively curved if and only if each link is a flag complex.*

A flag complex is a simplicial complex such that every complete subgraph in the 1-skeleton is a 1-skeleton of some simplex. For example a hollow 3-cube is not non-positively curved as each link is a triangle and there is no 2-simplex such that this triangle is its boundary. On the other hand a standard square tilling of a plane is a non-positively curved complex as each link is a square and the only complete subgraphs are of size 1 or 2.

Haglund and Wise's *A-special complexes*, which are closely related to non-positively curved cube complexes, avoid certain pathological behaviour of hyperplanes [HW08, Definition 3.2].

Definition 1.2.8 (Special cube complex)**.** A cube complex is *special* if the following holds.

1. For all edges $\overrightarrow{e} \notin W(\overleftarrow{e})$. We say the hyperplanes are 2-sided.

2. Whenever $\overrightarrow{e_2} \in W(\overrightarrow{e_1})$, then e_1 and e_2 are not consecutive edges in a square. Equivalently, each hyperplane embeds.

3. Whenever $\overrightarrow{e_2} \in W(\overrightarrow{e_1})$, $\overrightarrow{e_2} \neq \overrightarrow{e_1}$, then the initial point of $\overrightarrow{e_2}$ is not the initial point of $\overrightarrow{e_1}$. We say that no hyperplane *directly self-osculates*.

4. Whenever $\overrightarrow{e_2} \in W(\overrightarrow{e_1})$ and $\overrightarrow{f_2} \in W(\overrightarrow{f_1})$ and e_1 and f_1 form two consecutive edges of a square, if $\overrightarrow{e_2}$ and $\overrightarrow{f_2}$ start at the same vertex, then $\overleftarrow{e_2}$ and $\overrightarrow{f_2}$ are two consecutive edges in some square, and if $\overleftarrow{e_2}$ and $\overrightarrow{f_2}$ start at the same vertex, then $\overrightarrow{e_2}$ and $\overrightarrow{f_2}$ are two consecutive edges in some square. We say that no two hyperplanes *inter-osculate*.

Haglund and Wise have shown that $CAT(0)$ cube complexes are special [HW08, Example 3.3.(3)]. In this thesis, we will only ever use specialness of these complexes.

Every special cube complex is contained in a nonpositively curved cube complex with the same 2-skeleton [HW08, Lemma 3.13]. This nonpositively curved cube complex is also special. A special cube complex is often implicitly replaced with this nonpositively curved cube complex. The hyperplane $H(e)$ separates a $CAT(0)$ cube complex X into two connected components.

It is impossible to talk about special cube complexes without mentioning one of the highest achievements of geometric group theory – the virtually Haken theorem.

Theorem 1.2.9. *[Ago13, Theorem 9.1] Every closed aspherical* 3*-manifold has a finite cover, which contains an embedded* π_1*-injective subsurface.*

The geometrization theorem [Per02, Per03, MT08] reduces this statement to one about the case when the manifold is hyperbolic. In the hyperbolic case, even a stronger statement applies.

Theorem 1.2.10. *[Ago13, Theorem 9.2] Every closed hyperbolic* 3*-manifold has a finite cover, which is a surface bundle over the circle.*

The journey to these theorems is marked by many milestones. Kahn and Markovic had found a 'large number' of surfaces in hyperbolic manifolds [KM12]. Bergeron and Wise had used these surfaces to cubulate these manifolds [BW12]. This in turns implies that they are virtually special [Ago13]. The extensive theory of special cube complexes, particularly the work on hierarchies [HW12, HW15], implies the result.

1.2.1 Right-angled Coxeter and Artin groups

Right-angled Coxeter groups interpolate between an abelian and a non-abelian free product of copies of $\mathbb{Z}/2\mathbb{Z}$. The name stems from the fact that two reflections commute if the planes of reflection are perpendicular. We will formalise this by constructing a space on which this group acts.

Definition 1.2.11 (Right-angled Coxeter group)**.** Given a graph $\mathscr{G}$ with vertex set I, let $S = \{s_i : i \in I\}$. *The right-angled Coxeter group* associated to $\mathscr{G}$ is the group $C(\mathscr{G})$ given by the presentation $\langle S \mid s_i^2 = 1 \text{ for } i \in I, [s_i, s_j] = 1 \text{ for } (i, j) \in E(\mathscr{G})\rangle$.

The right-angled Coxeter group $C(\mathscr{G})$ acts on *the Davis–Moussong complex* $DM(\mathscr{G})$ [HW08]. The Davis–Moussong complex is similar to the Cayley complex, but it does not contain 'duplicate squares' and it contains higher dimensional cubes.

Definition 1.2.12 (Davis–Moussong complex)**.** A right-angled Coxeter group $C(\mathscr{G})$ acts on the Davis-Moussong complex $DM(\mathscr{G})$, which consists of the following:

- $X^0 = C(\mathscr{G})$

- If generators $s_{u_1}, s_{u_2}, \dots, s_{u_n}$ pairwise commute and $g \in C(\mathscr{G})$, then there is a unique n-cube with the vertex set $\{g(\Pi_{j \in P} s_{u_j}) : P \subset \{1, \dots, n\}\}$.[1]

The face inclusion maps come from subset inclusions. The action of the right-angled group on the vertex set is by left multiplication and it extends uniquely to the entire cube complex.

Look at the Figure 2.1. Remove the standard 4-valent tree from the figure. We are left with a tree of squares with each vertex shared by three squares. The link at each vertex is a path of length 3.

Let v_0 be the vertex corresponding to the identity. Denote by e_{s_i} the edge between v_0 and $s_i v_0$. Note that $g s_i g^{-1}$ acts on the left on $DM(\mathscr{G})$ as a reflection in $H(g e_{s_i})$. And as promised the action of s_i and s_j commutes if and only if the fixed hyperplanes are perpendicular. There is also a right action of $C(\mathscr{G})$ on $DM(\mathscr{G})^0$, where s_i sends $g v_0$ to $g s_i v_0$ – the vertex to which g is connected by an edge labelled s_i. This action does not extend to $DM(\mathscr{G})$ unless the Coxeter group is abelian.

More generally, if Γ is a subgroup of $C(\mathscr{G})$, the action of $C(\mathscr{G})$ on the right cosets of Γ can be realised geometrically as an action of $C(\mathscr{G})$ on $\Gamma \backslash DM(\mathscr{G})^0$. This action is given by $(\Gamma h v_0).g = \Gamma h g v_0$. If Γ acts on $DM(\mathscr{G})$ co-compactly, this gives a finite permutation action. We will use this to construct maps from $C(\mathscr{G})$ to S_n.

We want all of this to be some generalisation of free groups. In free groups, we would study finitely generated subgroups. Such subgroups act cocompactly on a convex subspace of the Cayley tree. In cube complexes we will look at subgroups defined by that property.

Definition 1.2.13 (Convex subcomplex, convex-cocompact subgroup)**.** A subcomplex Y of a cube complex X is *(combinatorially geodesically) convex* if any geodesic in $X^{(1)}$ with endpoints in Y is contained in Y.

If G acts on a cube complex X, we say $H < G$ is *convex-cocompact* if there is a non-empty convex subcomplex $Y \subset X$, which is invariant under H and moreover H acts on Y cocompactly. We say, that H acts on X with *core* Y (not to be confused with a normal core of a subgroup).

If X is hyperbolic, this coincides with quasiconvexity [Hag08]. A right angled Coxeter group $C(\Gamma)$ is hyperbolic if and only if Γ contains no induced square [Mou88]. Interpolating between free abelian and free non-abelian groups are right-angled Artin groups. Unlike in Coxeter groups the generators are not involutions.

[1] I could have made the definition more compact at the cost of clarity if I allowed an empty set of commuting generators.

Definition 1.2.14 (Right-angled Artin group)**.** The right-angled Artin group associated to a simplicial graph $\mathscr{G}$ is $A(\mathscr{G}) = \langle g_v : g \in V(\mathscr{G}) \mid g_u g_v = g_v g_u \text{ for } \{u,v\} \in E(\mathscr{G})\rangle$.

The next lemma relates RAAGs and RACGs.

Lemma 1.2.15. *[DJ00] Given a graph $\mathscr{G}$, define a graph $\mathscr{H}$ as follows:*

- $V(\mathscr{H}) = V(\mathscr{G}) \times \{0,1\}$
- *$(u,1)$ and $(v,1)$ are connected by an edge if $\{u,v\}$ is an edge of $\mathscr{G}$. The vertices $(u,0)$ and $(v,1)$ are connected by an edge if u and v are distinct. Similarly, $(u,0)$ and $(v,0)$ are connected by an edge if u and v are distinct.*

The right-angled Artin groups $A(\mathscr{G})$ is a finite-index subgroups of the right-angled Coxeter group $C(\mathscr{H})$ via the inclusion ι extending $g_u \longrightarrow s_{(u,0)}s_{(u,1)}$.

Definition 1.2.16 (Salvetti complex)**.** A right-angled Artin group $A(\mathscr{G})$ acts on Salvetti complex $X = X(\mathscr{G})$, which consists of the following:

- $X^0 = A(\mathscr{G})$
- If generators $g_{u_1}, g_{u_2}, \ldots, g_{u_n}$ pairwise commute and $g \in A(\mathscr{G})$, there is a unique n-cube with the vertex set $\{g(\Pi_{j\in P} g_{u_j}) : P \subset \{1,\ldots,n\}\}$.

The face inclusion maps are given by the inclusions of group elements. The action of the right-angled group on the vertex set is by the left multiplication and it extends uniquely to the entire cube complex.

For the rest of the thesis whenever we talk about the action of a RACG or RAAG on a cube complex, we mean the canonical action on the associated Davis-Moussong Complex or Salvetti complex, respectively.

The Salvetti complex is easily seen to be special. A subgroup of a group acting on a cube complex is *convex-cocompact* if it acts cocompactly on a (combinatorially) convex subcomplex. Note that we're using combinatorial convexity as opposed to the convexity in metric sense. Convex subcomplexes of special cube complexes are special, hence convex-cocompact subgroups of right angled Artin groups are compact special, i.e. they are fundamental groups of compact special cube complexes. Shockingly, the converse is true. Special groups are precisely the convex-cocompact subgroups of right angled Artin groups [HW08].

1.3 Residual properties

Imagine that I have a programme that cannot handle an entire infinite group, but can enumerate finite quotients of this group. How useful is this?

Suppose we'd like to determine whether two elements are equal with the use of the above programme. We will never be able to confirm that they are equal, but we would be able to get a negative answer if there is a finite quotient where the images of the two elements do not coincide. Groups with this property are called residually finite.

Definition 1.3.1 (Residual finiteness)**.** A group G is *residually finite* if for every distinct $g,h \in G$ there exists $f : G \longrightarrow F$ a homomorphism to a finite group with $f(g)$ distinct from $f(h)$.

Can we similarly check subgroup membership, conjugacy of elements or conjugacy of subgroups? These notions correspond respectively to subgroup separability, conjugacy separability and conjugacy separability.

Definition 1.3.2 (Residual properties)**.** A group G is *subgroup separable* if for every $g \in G$ and a finitely generated $H < G$, which does not contain g, there is a homomorphism $f : G \longrightarrow F$ to a finite group with $f(g) \notin f(H)$.

A group G is *conjugacy separable* if for every non-conjugate $g,h \in G$, there is a homomorphism $f : G \longrightarrow F$ to a finite group with $f(g)$ not conjugate to $f(h)$.

A group G is *subgroup conjugacy separable* if for every non-conjugate $H,K < G$, there is a homomorphism $f : G \longrightarrow F$ to a finite group with $f(H)$ not conjugate to $f(K)$ [BG10, Definition 1.2].

A group G is *subgroup into-conjugacy separable* if for every $H,K < G$ with H not conjugate into K, there is a homomorphism $f : G \longrightarrow F$ to a finite group with $f(H)$ not conjugate into $f(K)$ [BG10, Definition 1.6]

Recall from Definition 0.3.1 that G is subgroup separable if and only if every finitely generated subgroup $H < G$ is $\mathscr{C}$-separable, where $\mathscr{C}$ is the class of finite groups.

Clearly this is just a small sample of what can one try to detect in finite quotients. To avoid writing everything multiple times, I will go a bit deeper in just one of the concepts - namely the residual finiteness, since it is the simplest residual property.

Lemma 1.3.3. *Let G be a group. The following are equivalent.*

1. *For any n and distinct elements $g_1, g_2, \ldots, g_n \in G$, there exists a homomorphism $f : G \longrightarrow F$ to a finite group such that $f(g_1), f(g_2), \ldots, f(g_n)$ are distinct.*

2. *G is residually finite (see Definition 1.3.1).*

3. *For any $g \in G$ there exists a homomorphism $f : G \longrightarrow F$ to a finite group such that $f(g) \neq e$.*

Proof. Clearly 1. implies 2. and 2. implies 3.

For 3. implying 1., suppose $g_1, \ldots, g_n \in G$ are distinct. By 3., for every $i \neq j$ there exists a homomorphism $f_{i,j} : G \longrightarrow F_{i,j}$ to a finite group such that $f(g_i g_j^{-1}) \neq e$. Let $f : G \longrightarrow \Pi_{i \neq j} F_{i,j}$ be the product of all $f_{i,j}$. This homomorphism maps g_i's to distinct elements. □

We can also formulate these properties in topological language.

Definition 1.3.4 (Profinite topology)**.** The profinite topology on a group G is the topology generated by finite index normal subgroups of G and their translates.

Lemma 1.3.5. *A group G is residually finite if and only if the profinite topology is Hausdorff.*

Proof. We just need to unpack the definitions.

($\Rightarrow$): Suppose G is residually finite. Given distinct elements $g, h \in G$, there exists a homomorphism $f : G \longrightarrow F$ to a finite group with $f(g) \neq f(h)$. But then $f^{-1}(f(g))$ and $f^{-1}(f(h))$ are disjoint open sets.

($\Leftarrow$): Suppose that the profinite topology is Hausdorff. Given distinct elements $g, h \in G$, let U and V be disjoint open sets containing g and h respectively. Then $U \cap gh^{-1}V$ is an open set and as such contains a non-empty intersection of finitely many cosets of finite index normal subgroups, which contains g. An intersection of finitely many cosets of finite index normal subgroups is itself a coset of a finite index normal subgroup. Say this coset is gK. The quotient map $G \longrightarrow G/K$ sends g and h to different elements. □

The examination of residual properties starts with subgroup separability of free groups [Hal49, Theorem 5.1]. The original proof is algebraic, but there is a simple topological proof. It uses that any locally injective map of compact graphs is a composition of an injective map and a finite degree covering. In other words, any finite graph immersing to another graph is a subspace of some finite index covering space of that graph.

This idea can be traced to Scott's proof of subgroup separability of surface groups [Sco78] (in 1985 a correction came out fixing some errors and filling in details [Sco85]). Stallings uses this tactic to reprove the subgroup separability of free groups [Sta83]. Proving residual properties by promoting precovers to covers becomes known as *Scott's criterion*. In full generality *a precover* is a subspace of a covering space. However, we often require additional properties. For example, if the base space is equipped with a simplicial structure, we might want to require that the map from a precover is a simplicial map. Other times we might want the precovering map to be π_1-injective.

I'll state Scott's criterion more precisely. Suppose X is a connected space, $G = \pi_1(X)$ and element $g \in G$ does not belong to a finitely generated subgroup $H < G$. Let (X_H, x_H) be the based covering space associated to H and $\gamma : S^1 \longrightarrow X$ a path representing g. Suppose (Y, y) is some subspace of (X_H, x_H) with $\pi_1(Y, y) \longrightarrow \pi_1(X_H, x_H)$ a group isomorphism. Suppose $\hat{\gamma}$ is a lift of γ to X_H, which starts at y. Since $g \notin H$, the path $\hat{\gamma}$ isn't a loop. If we can complete $Y \cup Im(\hat{\gamma})$ to a finite index cover, then the group associated to that finite cover contains H, but not g.

There is a close relation between special groups and residual properties. If a fundamental group of a compact connected non-positively curved cube complex is hyperbolic and its convex-cocompact subgroups are separable, then that cube complex is virtually special [HW08, Theorem 8.13].

Finite groups form quite a wild zoo, so I try to restrict the image to a subfamily of groups. Maps to simple groups are of particular interest since taking further quotients gives trivial images and hence maps to simple groups are analogous to primes. Computationally easiest are the maps to alternating groups thanks to Jordan's theorem, which provides criteria for a subgroup of a symmetric group to be alternating.

Definition 1.3.6 (Residually alternating)**.** A group G is residually alternating if for any non-identity element $g \in G$ there exists a surjective homomorphism $f : G \longrightarrow A_n$ onto some alternating group with $f(g) \neq e$.

The history of residual properties of free groups within alternating groups starts with showing that free groups are residually alternating.

Theorem 1.3.7. *[KM69, Theorem 1] Non-abelian free groups are residually alternating.*

Surjectivity is a sensible assumption, since any finite group F is a subgroup of an alternating group - just take the permutation action on itself and embed $S_{|F|}$ in $A_{|F|+2}$.

A sketch of the proof. Any free group is a residually a finitely generated free group, since given any w in the free group, one can project onto the free group generated by the generators, which appear in w. It is enough to show that F_2 is residually alternating, since any finitely generated non-abelian free group is contained in $F_2 := \langle a, b \rangle$ [Pel66].

Let $T = \langle x, y | x^2 \rangle$. Given a word $w \in F_2$, take a map $a \longrightarrow xy^n, b \longrightarrow y$ for n larger than the absolute values of exponents of a and b in w. This is surjective and w does not map to identity, since it is image is a non-trivial word in normal form. So F_2 is residually T. One can show that T is residually alternating by an explicit choice of permutations. □

Recall that in fact non-abelian free groups are residually $\mathscr{C}$ for any infinite set $\mathscr{C}$ of finite simple groups [DPSS03, Theorem 3].

The products of alternating groups aren't alternating, so we can't promote residually alternating to fully residually alternating (separate finitely many elements at the same time) by taking products of maps as in the proof of Lemma 1.3.3. For example $A_3 \times A_3$ is residually alternating (take projections to factors) but not fully residually alternating (enumerate the group).

The next result pushes us much further than fully residually alternating for free groups. Instead, we get an alternating version of full subgroup separability. An infinite index of a subgroup is a necessary condition. Take for example $f : F_2 \longrightarrow \mathbb{Z}_2^2$ given by $a \longrightarrow (1,0)$ and $b \longrightarrow (0,1)$, let $K = \ker(f)$. Then any surjective homomorphism $F_2 \longrightarrow A_m$ maps K onto A_m, since A_m is generated by the squares of its elements and K contains squares of all the elements in F_2. To see a more conceptual argument let $\mathscr{S}$ be a set of some simple groups. If N is proper normal subgroup of a group G and $f : G \twoheadrightarrow S$ where $S \in \mathscr{S}$, then $f(N)$ is e or S. If we're trying to separate N from some $g \in G \setminus N$ in the quotient, then the only useful maps are those with $f(N) = e$, but then f factors through G/N. If G/N has no quotients in $\mathscr{S}$, then N is not $\mathscr{S}$-separable in G.

Similarly, the finite generation is necessary since A_m is generated by its commutators and F_2' contains all commutators and is a proper subgroup of F_2.

Theorem 1.3.8. *[Wil12, Theorem A] Let F be a non-abelian free group. Let H be a finitely generated infinite index subgroup of F. Let $\gamma_1, \dots, \gamma_n \in F \setminus H$. Then there exists a surjective homomorphism $F \longrightarrow A_k$ with $f(\gamma_i) \notin f(H)$ for all i.*

A sketch of proof. Consider the case $F = \langle a, b \rangle$. Let X_H be the cover of presentation complex of F associated to H. Let Y be the subgraph of X_H, which contains all cycles in X_H and all γ_i's to X_H starting at the base point. Since we want to control the quotient, we need to pick a specific way of adding the missing edges. This can be done by making a act with a large orbit and b fix many vertices. □

The first half of the above proof is Scott's criterion applied to free groups. The second half is an explicit construction. The first major new result in this thesis generalizes the construction to right-angled Coxeter groups.

A generalisation of the following theorem will allow us to show that certain groups are alternating with a large probability.

Theorem 1.3.9. *[Dix69] An image of a random homomorphism* $F_2 \longrightarrow S_n$ *is* A_n*, resp.* S_n*, with probabilities which tend to* $1/4$*, resp.* $3/4$*, as n goes to infinity.*

Chapter 2

Separability of RAAGs within alternating groups

2.1 Preliminaries

2.1.1 $\mathscr{A}$-separability

We will establish some properties of $\mathscr{A}$-separability.

Lemma 2.1.1. *Let A and B be non-trivial finitely generated groups. Then $\{e\} < A \times B$ is not $\mathscr{A}$-separable.*

Proof. There are only finitely many surjective homomorphisms from $A \times B$ onto A_2, A_3 and A_4. If $A \times B$ is infinite, then there is a non-identity element g in the kernel of all these maps. Consider elements $(e,b),(a,e)$, where $a \neq e$, $b \neq e$. Suppose $f : A \times B \to A_n$ is a surjective homomorphism, which does not map these elements to e.

By the choice of g, we have $n > 4$. The group $f(A \times e)$ is a normal subgroup of A_n, so it is e or A_n. Similarly for $e \times B$. If both factors mapped onto A_n then for any pair of elements $h_1, h_2 \in A_n$, there is some $a \in A$ and $b \in B$ with $f((a,e)) = h_1$ and $f((e,b)) = h_2$. Therefore $[h_1, h_2] = f([(a,e),(e,b)]) = f(e) = e$ and A_n is commutative, which is a contradiction. We get that at least one of $A \times e$ or $e \times B$ maps to e.

If both A and B are finite and $\{e\} < A \times B$ is $\mathscr{A}$-separable, enumerate $A \times B$ as $\gamma_1, \dots, \gamma_m$. Applying the $\mathscr{A}$-separability condition with respect to this set, we get an isomorphism $f : A \times B \to A_n$. However, A_n is not a direct product, so one of A, B is A_n and the other is trivial. □

This implies that passing to a finite degree extension does not in general preserve $\mathscr{A}$-separability of convex-cocompact subgroups. However passing to a finite-index subgroup does:

Lemma 2.1.2. *Let G be a finitely generated group, let H be a finite-index subgroup of G, and let K be an infinite index subgroup of H. If K is $\mathscr{A}$-separable in G, then it is $\mathscr{A}$-separable in H.*

We need K to be infinite index in H, as otherwise it is possible that $K = N(H)$ in the notation of the proof below. E.g. take $G = A_n$, H a proper subgroup, $K = \{e\}$.

Proof. Suppose $\gamma_1, \dots, \gamma_n \in H \setminus K$.

Let $Core(H) = \bigcap_{g \in G} H^g$ be a normal subgroup contained in H. Then $Core(H)$ is still finite index and let $M = [G : Core(H)]$ be this index. Since G is finitely generated, there are only finitely many surjective homomorphisms $f : G \twoheadrightarrow A_m$ with $m \leq M$. The intersection of preimages of $f(K)$ over such surjective homomorphisms is a finite intersection of finite index subgroups, hence a finite index subgroup. So there exists some $\gamma_0 \in G \setminus K$ such that $f(\gamma_0) \in f(K)$ for all $f : G \twoheadrightarrow A_m$ with $m \leq M$.

As K is $\mathscr{A}$-separable in G, there exists a surjective homomorphism $f : G \twoheadrightarrow A_m$, such that $f(\gamma_i) \notin f(K)$ for all $i \in \{0, \dots, n\}$. By the choice of γ_0 we have $m > M$. But $[A_m : f(Core(H))] \leq M$, so $f(Core(H)) = A_m$. In particular, $f(H) = A_m$ and $f|_H$ is the desired surjective homomorphism. □

2.1.2 Half-spaces

We introduce some important concepts we will need to study cube complexes.

Definition 2.1.3 (Half-space, [Hag08])**.** Suppose X is a cube complex and H is a hyperplane. Let $X \backslash\backslash H$ be the union of cubes disjoint from H. If X is $CAT(0)$, $X \backslash\backslash H$ has two connected components. Call them *half-spaces H^- and H^+*

Definition 2.1.4. If Y is a subspace of a cube complex X, then $N(Y)$ is the union of all cubes intersecting Y. Let $\partial N(Y)$ consist of cubes of $N(Y)$ that do not intersect Y. If Y is a hyperplane and X is a simply connected special cube complex, then $\partial N(Y)$ has two components; call them $\partial N(Y)^+$ and $\partial N(Y)^-$.

Definition 2.1.5 (Convex subcomplex)**.** A subcomplex Y of a cube complex X is *(combinatorially geodesically) convex* if any geodesic in $X^{(1)}$ with endpoints in Y is contained in Y.

The components of the boundary of a hyperplane $\partial N(H)^+$, $\partial N(H)^-$ and half-spaces are combinatorially geodesically convex [Hag08, Lemma 2.10]. Any intersection of half-spaces is convex [Hag08, Corollary 2.16] and a convex subcomplex of a $CAT(0)$ cube complex coincides with the intersection of all half-spaces containing it [Hag08, Proposition 2.17].

Definition 2.1.6 (Bounding hyperplane)**.** A hyperplane *bounds* a convex cube subcomplex $Y \subset X$ if it is dual to an edge with endpoints $v \in Y$ and $v' \notin Y$.

2.1.3 Jordan's Theorem

Definition 2.1.7 (Primitive subgroup)**.** A subgroup $G < S_n$ is called *primitive* if it acts transitively on $\{1,\ldots,n\}$ and it does not preserve any nontrivial partition.

If n is a prime and G is transitive, then the action is primitive.

Our main tool is the following.

Theorem 2.1.8 (Jordan's Theorem)**.** *[DM96, From theorems 3.3A and 3.3D] For each $k > 2$ there exists N such that if $n > N$, $G < S_n$ is a primitive subgroup and there exists $\gamma \in G \setminus \{e\}$, which moves less than k elements, then $G = S_n$ or A_n.*

2.2 The Main Theorem and its consequences

Our main theorem relates the combinatorics of $\mathcal{G}$ to the $\mathscr{A}$-separability of $C(\mathcal{G})$.

Theorem 2.2.1 (Main Theorem)**.** *Let $\mathcal{G}$ be a non-discrete finite simplicial graph of size at least 3. Then all infinite-index convex-cocompact subgroups of the right-angled Coxeter group associated to $\mathcal{G}$ are $\mathscr{A}$-separable and $\mathscr{S}$-separable if and only if $\mathcal{G}^c$ is connected.*

Recall that here convex-cocompact means that it acts cocompactly on a convex subcomplex of the Davis-Moussong complex. A similar result holds for RAAGs.

Corollary 2.2.2. *Let $\mathcal{G}$ be a finite simplicial graph of size at least 2. Then all infinite index convex-cocompact subgroups of the right-angled Artin group associated to $\mathcal{G}$ are $\mathscr{A}$-separable if and only if $\mathcal{G}^c$ is connected.*

Here convex-cocompact means that the subgroup acts cocompactly on a convex subcomplex of the Salvetti complex. There is another action of the Artin group on a cube complex given by embedding the group in right-angled Coxeter group as described in Lemma 1.2.15. We will first show that convex-cocompactness with respect to the Salvetti complex implies convex-cocompactness with respect to the Davis-Moussong complex.

Lemma 2.2.3. *Suppose $\mathscr{G}$ is a simplicial complex, and K a convex-cocompact subgroup of $A(\mathscr{G})$ with respect to the action on $X(\mathscr{G})$. Let $\mathscr{H}$ be as in Lemma 1.2.15 and identify $A(\mathscr{G})$ with a subgroup of $C(\mathscr{H})$ in the same lemma. Then K is convex-cocompact in $C(\mathscr{H})$ with respect to the action on $DM(\mathscr{H})$.*

Proof. Recall that $N(H)$ is the union of all cubes intersecting a hyperplane H. For a hyperplane H in a $CAT(0)$ cube complex X, $N(H) \simeq H \times [0,1]$. We can collapse $N(H)$ onto H. Formally, say $(x,t) \sim (x,t')$ for all $x \in H$ and $t,t' \in [0,1]$. *Collapse of neighbourhood of H* is the quotient map $X \longrightarrow X/\sim$. This is also known as restriction quotient [CS11, HK+18]. We can collapse multiple neighbourhoods simultaneously by quotienting by the smallest equivalence relation, which contains the equivalence relation for each hyperplane.

Let v_0 be a specified vertex in the Davis-Moussong complex, which under the bijection between vertices and group elements corresponds to the identity. Let $f : (DM(\mathscr{H}), v_0) \longrightarrow (Y, y_0)$ be the simultaneous collapse of all hyperplanes labelled by $s_{(v,0)}$ for all $v \in \mathscr{G}$. See Figure 2.1. Here, the base point y_0 is the image of v_0. The equivalence relation commutes with the action of $C(\mathscr{H})$, so there is an induced action of $C(\mathscr{H})$ on Y.

We collapsed all edges with labels from $\mathscr{G} \times \{0\}$ so for all $s_{(v,0)}$ and all $g \in C(\mathscr{H})$, we have $gs_{(v,0)}.y_0 = g.y_0$.

Let $f' : X(\mathscr{G}) \longrightarrow Y$ be defined as follows

- Vertices: Send g to $g.y_0$.
- Edges: Send the edge between g and gg_v to the edge between $g.y_0$ and $gg_v.y_0$. It is indeed an edge as $g.y_0 = gs_{(v,0)}.y_0$ and $gg_v.y_0 = gs_{(v,0)}s_{(v,1)}.y_0$
- Squares: Send the square with vertices g, gg_v, gg_u, gg_ug_v to the square with vertices $g.y_0, gg_v.y_0, gg_u.y_0, gg_ug_v.y_0$.
- Higher dimensions: Extend analogously.

The right-angled Artin group $A(\mathscr{G})$ acts on Y by $g.(h.y_0) = gh.y_0$. The map f' is an $A(\mathscr{G})$-equivariant cube complex isomorphism since $g.f'(h) = g.h.y_0 = gh.y_0 = f'(gh)$.

No two hyperplanes of $C(\mathscr{H})$ labelled $s_{(u,0)}$ and $s_{(v,0)}$ osculate since either the neighbourhoods of the associated hyperplanes do not intersect or u is distinct from v, $(u,0)$ is connected to $(v,0)$ and the associated hyperplanes intersect.

I want to prove that if K acts cocompactly on Z a convex subcomplex of $X(\mathscr{G})$, then it acts cocompactly on $W := f^{-1}f'(Z) \subset DM(\mathscr{H})$. The collapsing map f sends cubes to cubes (of potentially lower dimension), therefore W is a cube complex. To prove cocompactness, it is enough to show that every vertex $y \in Y$ has finitely many vertices in its preimage under f. Suppose x and x' are vertices of $DM(\mathscr{H})$ and that they both map to y. Then there is some sequence $H_1, \ldots, H_k$ of hyperplanes with labels from $\mathscr{G} \times 0$ and vertices $x_1, \ldots, x_{k+1}$ such that $x_1 = x$, $x_{k+1} = x'$ and x_i maps to the same element as x_{i+1} under the collapse of H_i for all i. But then $N(H_i)$ and $N(H_{i+1})$ intersect and as they do not osculate, H_i and H_{i+1} intersect. Since they do not interosculate, x_{i-1}, x_i and x_{i+1} are successive vertices in some square. But now $x_{i+1} \in N(H_{i-1})$ and by induction H_i intersects H_j whenever $i \neq j$. Therefore $H_1, \ldots, H_k$ have distinct labels and $k \leq |\mathscr{G}|$ and the preimage of $y \in Y$ contains at most $2^{|\mathscr{G}|}$ vertices.

It remains to show that W is convex. Let e be an edge in $DM(\mathscr{H})$ with exactly one endpoint in W. The edge e is labelled by some $s_{(v,1)}$ as all edges labelled by $s_{(v,0)}$ either lie entirely in W or have an empty intersection with it. The collapsing map sends parallel edges to parallel edges (unless it sends them both to a vertex) and any sequence of elementary parallelisms in the codomain lifts to the domain, so $f(H(e)) = H(f(e))$. In particular, if $H(e)$ intersects W, then $H(f(e))$ intersects $f'(Z)$ and by the convexity of Z, $f(e)$ lies entirely in $f'(Z)$, which contradicts that e does not lie entirely in W.

So convex-cocompactness with respect to the action on $X(\mathscr{G})$ implies convex-cocompactness with respect to the action on $DM(\mathscr{H})$. □

Proof of Corollary 2.2.2. $\Rightarrow$: If H is a proper component of $\mathscr{G}^c$ and K is the complement of H in $\mathscr{G}^c$, then $A(\mathscr{G}) = A(H^c) \times A(K^c)$ so by Lemma 2.1.1 the trivial subgroup $\{e\}$ is not $\mathscr{A}$-separable in $A(\mathscr{G})$.

$\Leftarrow$: Let $\mathscr{H}$ be as in Lemma 1.2.15.

Suppose U is a proper component of $\mathscr{H}^c$. The vertices $(v,0)$ and $(v,1)$ are not connected by an edge in $\mathscr{H}$, so U^0 is of the form $V \times \{0,1\}$ for some $V \subsetneq \mathscr{G}^0$. But then looking at $V \times \{1\} \subset \mathscr{G} \times \{1\}$ gives that V^0 is a vertex set of a proper component of $\mathscr{G}^c$.

So $\mathscr{G}^c$ being connected implies that $\mathscr{H}^c$ is connected.

By Lemma 2.2.3 K is convex-cocompact in $C(\mathscr{H})$ and hence by Theorem 2.2.1 it is $\mathscr{A}$-separable in $C(\mathscr{H})$. By Lemma 2.1.2 K is also $\mathscr{A}$-separable in $A(\mathscr{G})$. □

Lemma 2.2.4. *[Sco85, Correction to the proof of Theorem 3.1] A closed, orientable, hyperbolic surface group G is a finite index subgroup of $C(C_5)$, where C_5 is a cycle of length* 5. *Moreover, for a suitable embedding $G \hookrightarrow C(C_5)$, all finitely generated subgroups of G are convex-cocompact in $C(C_5)$ with respect to the action on $DM(C_5)$.*

Remark 2.2.5 (Idea of proof). Scott uses a different terminology, so it makes sense to summarise the proof. The natural generators of $C(C_5)$ act on the hyperbolic plane by

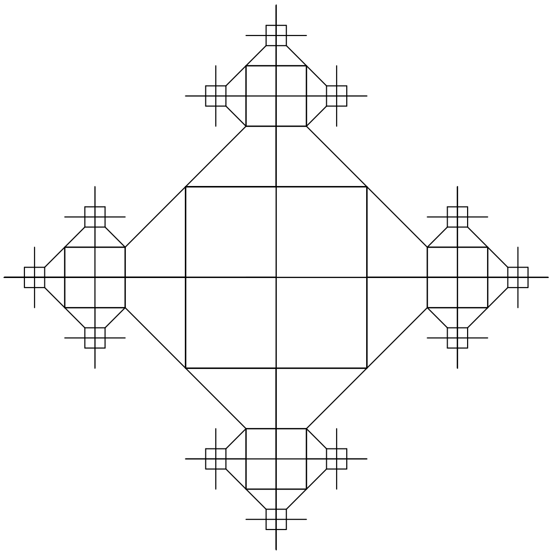

Fig. 2.1 The Salvetti complex for the free group on two generators overlaid with the Davis-Moussong complex for a path of length 3. The Davis-Moussong complex retracts onto the Salvetti complex by the collapse of hyperplanes.

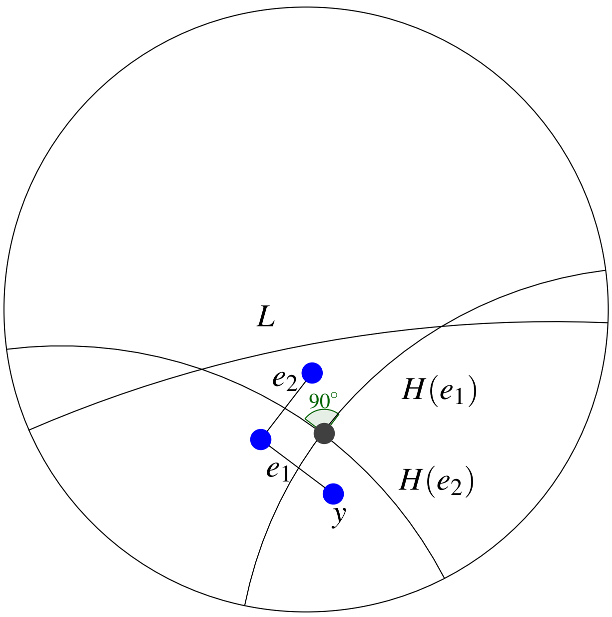

Fig. 2.2 Sketch of proof of Lemma 2.2.4.

reflections in the sides of a right-angled pentagon. Translates of the pentagon give a tiling of the hyperbolic plane. Dual to this cell complex is a square complex $DM(C_5)$. Under this identification, the geodesic lines bounding the pentagons of the tiling become hyperplanes of $DM(C_5)$.

Suppose H is a finitely generated subgroup of the surface group $G = \pi_1(\Sigma)$. Let Σ_H be the covering space associated to H. By Lemma 1.5 in [Sco78], there exists a compact, incompressible subsurface $\Sigma' \subset \Sigma_H$ such that the induced map $\pi_1\Sigma' \longrightarrow \pi_1\Sigma_H$ is surjective. Moreover, by [Sco85, Correction to the proof of Theorem 3.1] we can require Σ' to have a geodesic boundary with respect to a fixed hyperbolic metric on the surface.

Let $\widetilde{\Sigma'}$ be the lift of Σ' to $\mathbb{H}^2 = DM(C_5)$. Let Y be the intersection of all half-spaces containing $\widetilde{\Sigma'}$. Suppose y lies in Y, but not in $N_3(\widetilde{\Sigma'})$ and that e_1, e_2 are the first two edges of the combinatorial geodesic from y to $\widetilde{\Sigma'}$. Since $y \in Y$, both $H(e_1)$ and $H(e_2)$ intersect $\widetilde{\Sigma'}$. Consequently, $H(e_1)$ intersects $H(e_2)$ as $H(e_2)$ does not separate $H(e_1)$ from $\widetilde{\Sigma'}$. Call the intersection y'. The point y is a centre of a pentagon and y' is a vertex of the same pentagon, so the distance between them does not depend on y (for example by specialness of $DM(\mathscr{G})$). See Figure 2.2.

The next part of the proof is illustrated on figure 2.3. The closest boundary component L of $\widetilde{\Sigma'}$ to y is seen from y' at more than the right angle (remember that the hyperplanes are geodesics). But such a point is within distance $\int_{t=0}^{\pi/4} \frac{1}{\cos(t)} dt$ of L. To see this, take L to be the vertical ray through $(0,0)$ in the upper half-plane model. Then the set of points with obtuse subtended angle is contained between rays $y = x$ and $y = -x$. Geodesic between these rays and L is an arc of length

$$\int_{t=0}^{\pi/4} \frac{r\sqrt{\cos^2(t) + \sin^2(t)}}{r\cos(t)} dt = \int_{t=0}^{\pi/4} \frac{1}{\cos(t)} dt$$

Therefore y' (and hence y) is at a uniformly bounded distance from $\widetilde{\Sigma'}$ and the action of H on Y is cocompact.

Corollary 2.2.6. *All finitely generated infinite index subgroups of closed, orientable, hyperbolic surface group G are $\mathscr{A}$-separable in G.*

Proof. By Lemma 2.2.4, finitely generated subgroups of G are convex-cocompact in $C(C_5)$. By the Main Theorem 2.2.1 they are $\mathscr{A}$-separable in $C(C_5)$. By Lemma 2.1.2, they are $\mathscr{A}$-separable in G. □

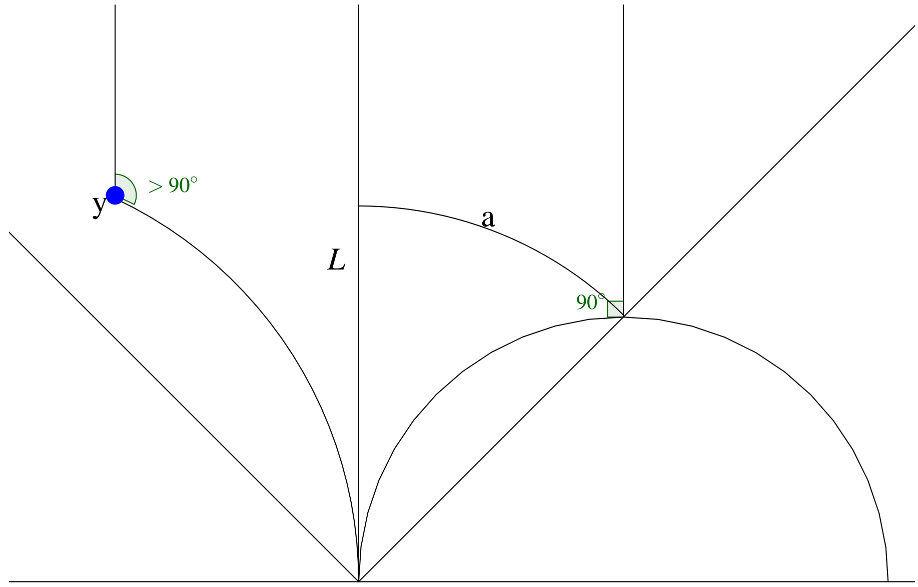

Fig. 2.3 If the angle subtended by L from y is obtuse, then y is uniformly close to L.

2.3 Proof of the Main Theorem

Definition 2.3.1 (Disjoint hyperplanes, bounding hyperplanes, positive half-space)**.** Let X be a cube complex, Y a convex subcomplex. Let $\mathscr{D}(Y)$ be the set of hyperplanes disjoint from Y. Recall from Definition 2.1.6 that a hyperplane bounds Y if it is dual to some e with one endpoint in Y and one not in Y. Let $\mathscr{B}(Y)$ be the set of hyperplanes bounding Y

If $H \in \mathscr{D}(Y)$, denote by H^+ the half-space of $X \backslash\backslash H$ containing Y.

Lemma 2.3.2 (Lemma 13.3 in [HW08])**.** *Any hyperplane H bounding a convex subcomplex Y in a CAT(0) cube complex is disjoint from Y.*

Recall that any intersection of half-spaces is convex and conversely any convex subcomplex is an intersection of the half-spaces containing it. Hence it is equivalent to specify a convex subcomplex or the half-spaces in which it is contained (or the set of disjoint hyperplanes if there can be no confusion about the choice of half-spaces, e.g. if only one choice gives a non-empty intersection).

Definition 2.3.3 (Deletion, vertebra)**.** Suppose G acts on a cube complex X with core Y. Define *deletion* as removing a bounding hyperplane H_0 and all its G-translates from $\mathscr{D}(Y)$. The result of deletion of H_0 is $Y' = \cap_{H \in \mathscr{D}(Y) \setminus G.\{H_0\}} H^+$.

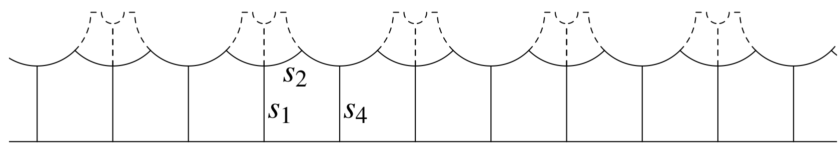

Fig. 2.4 Cocompact subgroup $\langle s_1 s_4 \rangle < C(C_5)$ has a core dual to a row of pentagons. By deletion of the hyperplane labelled s_2, we get a larger core for the same group.

The cube complex $V = H_0^- \cap Y'$ is called a *vertebra*. See Figures 2.4 and 2.5.

A vertebra is an intersection of two combinatorially geodesically convex sets, so it also is combinatorially geodesically convex. In particular, it is connected.

Definition 2.3.4 (Acting without self-intersections)**.** We say G acts *without self-intersections* on a cube complex X, if $N(gH) \cap N(H) \neq \emptyset$ implies $gH = H$ for all hyperplanes H of X and $g \in G$.

Definition 2.3.5 (Special action)**.** An action of G on a cube complex X is *special* if it is without self-intersections and whenever there exists $g \in G$ such that $N(H) \cap N(K) \neq \emptyset$ and $H \cap gK \neq \emptyset$, then $H \cap K \neq \emptyset$.

Lemma 2.3.6. *Suppose that G acts without self-intersections on a locally compact $CAT(0)$ cube complex X with core Y and $H_0 \in \mathscr{B}(Y)$. Then the result Y' of deletion of H_0 is also a core for G. Let $G_{H_0} := \{g \in G | g.H_0 = H_0\}$ be the stabiliser of H_0 in G. If C is a set of orbit representatives for the action of G on the vertices of Y and D is a set of orbit representatives for the action of G_{H_0} on the vertices of the vertebra $V = H_0^- \cap Y'$, then $C' = C \sqcup D$ is a set of orbit representatives for the action of G on the vertices of Y'. Moreover, $Y' \subset N(Y)$.*

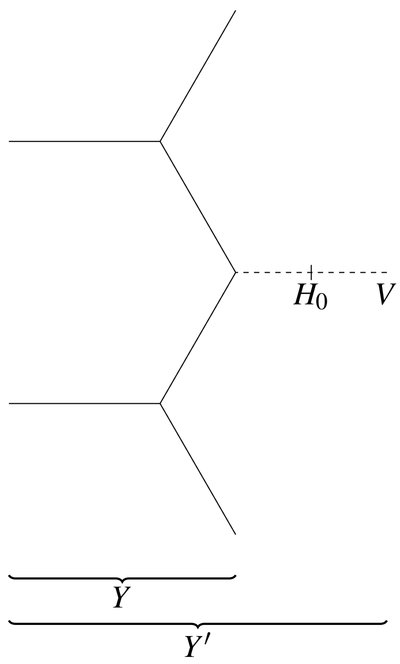

Fig. 2.5 Here X is a 3-regular tree and G is trivial.

Proof. Recall that $CAT(0)$ implies special.

First note that $\mathscr{D}(Y') = \mathscr{D}(Y) \setminus G.\{H_0\}$ by definition and $\mathscr{B}(Y) \setminus G.\{H_0\} \subset \mathscr{B}(Y')$ as a bounding hyperplane Y still bounds Y' unless it is a translate of H_0.

The set of half-spaces containing Y is invariant under G, hence Y' is invariant. The subcomplex Y' is an intersection of half-spaces, hence convex. Suppose $v \in Y' \setminus Y$. Let $v_0, v_1 \ldots, v_k$ be a combinatorial geodesic from v to Y of shortest length with edges $e_1, \ldots, e_k$ and suppose $k > 1$. Let H_i be the hyperplane dual to e_i. Then as $v_{k-1} \notin Y$, we have $H_k \in G.\{H_0\}$. Since G acts on X without self-intersections $H_{k-1} \notin G.\{H_0\}$. And $H_{k-1} \notin \mathscr{D}(Y')$, because $v_0, v_k \in Y'$ and Y' is convex, so $e_{k-1} \in Y'$

Therefore $H_{k-1} \notin \mathscr{D}(Y)$. It must intersect Y, so it is not entirely contained in H_k^- and it intersects H_k. Because the cube complex is special, H_k and H_{k-1} do not interosculate. In particular, there is a square with two consecutive sides e_{k-1} and e_k. Let e'_j be the edge opposite e_j in this square. By Lemma 2.3.2 H_{k-1} does not bound Y and $e'_{k-1} \in Y$. We can now construct a shorter path from v_0 to Y with edges $e_1, \ldots, e_{k-2}, e'_k$. Contradiction.

So $k \leq 1$ and Y' lies in a 1-neighbourhood of Y and therefore the action is cocompact because X is locally compact.

There is a unique edge connecting $v \in Y' \setminus Y$ to Y as any path of length 2 is a geodesic or is contained in some square. In the first case by convexity of Y, we have $v \in Y$. In the second, $H_0 \notin \mathscr{D}(Y)$.

By invariance of Y, the G-translates of V do not intersect Y. Suppose $v \in Y' \setminus Y$. There is a unique hyperplane in $G.\{H_0\}$ dual to an edge e_1, which connects v to Y, say $g.H_0$. Then v belongs to a unique translate of V, namely $g.V$. □

Corollary 2.3.7. *Let $\mathscr{G}$ be a finite simplicial graph. If K is a subgroup of a right-angled Coxeter group $C(\mathscr{G})$ and it acts on the Davis-Moussong complex with core Y, then deletion produces another core.*

Proof. The Davis-Moussong complex $DM(\mathscr{G})$ is a $CAT(0)$ cube complex, hence it is simply connected and special. The action of $C(\mathscr{G})$ on it preserves labels. In this complex any two consecutive edges have distinct labels, so the action is without self-intersections. The restriction to K is also without self-intersections. □

Lemma 2.3.8. *Suppose G acts on a $CAT(0)$ cube complex X with core Y. If $Y' \subset X$ is constructed from Y using a deletion of $H = H(e)$, then each edge in $V = H^- \cap Y'$ is dual to a hyperplane intersecting H.*

Proof. Let e' be an edge in V and H' a hyperplane dual to e'. If $H' \cap H = \emptyset$, H' is contained entirely in H^-. But then H' is disjoint from Y. In particular one of the endpoints of e' is in the opposite half-space of $X \backslash\backslash H'$ to Y.

Since Y' is the intersection of all half-spaces containing Y with the exception of the G-translates of H^+, the hyperplane H' is gH for some $g \in G$.

The subcomplex Y is G-invariant and H bounds Y, hence H' bounds Y. This contradicts $H' \subset H^-$. □

Corollary 2.3.9. *Suppose $G < C(\mathscr{G})$ acts on $DM(\mathscr{G})$ with core Y. If $Y' \subset X$ is constructed from Y using a deletion of $H = H(e)$, then each edge in $V = H^- \cap Y'$ has a label which commutes with the label of e.*

Recall that every edge of $DM(\mathscr{G})$ has some label s associated to it, where s is a basic generator of $C(\mathscr{G})$ such that some conjugate of s swaps the endpoints of the edge. Moreover, edges dual to a hyperplane have the same label, therefore a hyperplane has a well-defined label.

Proof of Corollary 2.3.9. If two hyperplanes in $DM(\mathscr{G})$ intersect, then their labels commute by the definition of squares of $DM(\mathscr{G})$. By Lemma 2.3.8 hyperplanes dual to the edges of V intersect H. Therefore the edges of V have the labels which commute with the label of H.

□

Definition 2.3.10 (Deletion along a path, deletion with labels, tail)**.** Suppose Y is a subcomplex of a $CAT(0)$ complex X and a core for the action of G on X. Suppose $p = e_1e_2 \dots e_n$ is a path in X, which starts in Y. *The deletion of hyperplanes along the path p* is a subcomplex $Y' = \cap H^+$, where H goes over hyperplanes disjoint from Y and from $G.p$.

Suppose additionally that edges of X are labelled in such a way that for every vertex and every label, there is precisely one edge starting at that vertex of the given label. Suppose $v \in Y$, and $s_1, s_2, \dots, s_n$ is a sequence of edge labels, then *the deletion with labels* $s_1, s_2, \dots, s_n$ *at* v is the *deletion of hyperplanes along* p, where p is a path $e_1, e_2, \dots, e_n$ starting at v with e_i labelled s_i.

Suppose Y_n was built from Y_0 using a series of deletion of hyperplanes $H_1, \dots, H_n$. We call $T = Y_n \cap H_1^-$ *a tail.*

Lemma 2.3.11. *Suppose $\mathcal{G}$ is a finite simplicial graph. Suppose $\mathcal{G}^c$ is connected, $|\mathcal{G}| > 1$ and H acts on $DM(\mathcal{G})$ with a core $Y \subsetneq DM(\mathcal{G})$. Then there exists a core Y' which can be obtained from Y by deletion along a path $e_1, e_2 \dots e_n$ with the vertebra $Y' \cap H(e_n)^-$ a single vertex.*

Remark 2.3.12. The hypothesis that $\mathcal{G}^c$ is connected is necessary. Consider the situation when $\mathcal{G}$ is a square. Then $C(\mathcal{G}) = D_\infty \times D_\infty$ and $DM(\mathcal{G})$ is the standard tiling of $\mathbb{R}^2$. Let $H = D_\infty$ be the subgroup generated by two non-commuting generators of $C(\mathcal{G})$. The invariance of the core and cocompactness of the action imply that any core for H is of the form $\mathbb{R} \times [k, l]$ for some $k, l \in \mathbb{Z}$.

Every hyperplane intersecting such a core divides it into two infinite parts.

Proof of Lemma 2.3.11. The proof is depicted in Figure 2.6. Since Y is a proper subcomplex, there exists e_1 such that $H(e_1) = H_1$ bounds Y. Let v_0 be the endpoint of e_1, which lies in Y. Let v_1 be the other endpoint. Say the label of e_1 is s_1. Let Y_1 be a cube complex obtained from Y by deletion of H_1.

Let S_1 be the set of generators labelling the edges of vertebra V_1. Then by Corollary 2.3.9, s_1 commutes with all generators in S_1.

If $e_2 \notin V_1$ is an edge with endpoint v_1, whose label s_2 does not commute with s_1, we can define H_2, Y_2, V_2 and S_2 similarly as before. Just as before the generators of S_2 commute with s_2.

The hyperplanes H_1 and H_2 do not intersect, so $N(H_2) \subset H_1^-$. There is an inclusion of V_2 into V_1 given by sending a vertex of V_2 to the unique vertex of V_1 to which it is connected by an edge labelled s_2. Extending this map to edges and cubes is a label preserving map

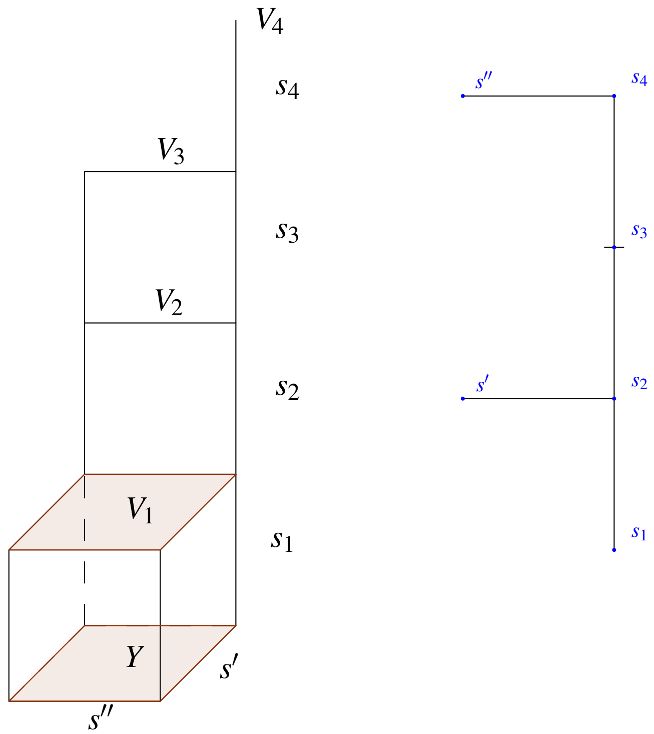

Fig. 2.6 The left figure depicts a gradual removal of hyperplanes labelled s_1 to s_4 in $DM(\mathscr{G})$, where $\mathscr{G}^c$ is depicted on the right. Here we're for simplicity taking $H < C(\mathscr{G})$ to be the trivial group.

between cube complexes V_2 and V_1. It follows that S_2 is a (not necessarily proper) subset of S_1.

We will now show that, by a series of such operations, we can reach a situation where $S_n = \emptyset$. I.e. the vertebra V_n is a single vertex.

Suppose we have already applied deletion i times and S_i is non-empty. We will use a series of deletions to get $S_k \subsetneq S_{k-1} \subset S_{k-2} \subset \ldots \subset S_{i+1} \subset S_i$. By an abuse of notation, we'll identify the vertices of $\mathscr{G}^c$ with the labels and with the generators of the right-angled Coxeter group. (Rather than having a generator s_v for every vertex $v \in V(\mathscr{G})$ and using these as labels.)

Since the group does not split as a product, there exists some $a \in S_i$ and $b \notin S_i$ which do not commute. Since $\mathscr{G}^c$ is connected, there exists a vertex path $s_{i-1}, \ldots, s_k = b$ in $\mathscr{G}^c$ from the vertex s_{i-1}, which is the label of the hyperplane we removed last.

Apply deletion of hyperplanes labelled $s_i, \ldots, s_k$ starting at some vertex of $v \in V_{i-1}$. Note that the jth hyperplane we remove belongs in a subset of $\mathscr{B}(Y_{j-1})$ as $s_i \ldots s_{j-1}v \in V_{j-1}$ and s_j does not commute with s_{j-1}. Moreover, $S_j = \{s \in S_{j-1} : ss_j = s_js\}$. In particular, $S_k \subset S_i$ and a does not belong to S_k as $as_k \neq s_ka$.

Therefore S_k is a proper subset of S_i and we can continue this process until we get an empty S_n. □

Remark 2.3.13. We can even control the label of the hyperplane which was removed last. Indeed, if the last removed hyperplane had label s_i, and b is some other generator, pick a vertex path between s_i and b in $\mathscr{G}^c$. Then remove hyperplanes labelled by vertices on this path, starting at the unique vertex of a vertebra.

By Lemma 2.3.6 there is a set of orbit representatives K for the action of G on Y_n with $T \subset K$.

Haglund shows the following [Hag08, Proof of Theorem A].

Lemma 2.3.14. *Suppose $G < C(\mathscr{G})$ acts on $DM(\mathscr{G})$ with a core Y and with a set of orbit representatives K. Let $\Gamma_0 < C(\mathscr{G})$ be generated by the reflections in the hyperplanes bounding Y. Let $\Gamma_1 = \Gamma_1(Y) = \langle G, \Gamma_0 \rangle$. Then Y is a fundamental domain for the action of Γ_0 on $DM(\mathscr{G})$ and K is a set of orbit representatives for the action of Γ_1 on $DM(\mathscr{G})$.*

Let $C(\mathscr{G})$ act on the right cosets of $\Gamma_1 < C(\mathscr{G})$. We have that $s \in S$ sends $\Gamma_1 g$ to $\Gamma_1 gs = (\Gamma_1 gsg^{-1})g$. But gsg^{-1} is a reflection in the hyperplane $H(ge_s)$. By definition of Γ_0 if $H(ge_s)$ bounds Y, $gsg^{-1} \in \Gamma_0$ and $\Gamma_1 g$ is fixed by s.

Moreover, if $K = \{g_1v_0, \ldots, g_nv_0\}$, then $\{g_1, \ldots, g_n\}$ is a set of right coset representatives for Γ_1.

We will first prove that by a suitable sequence of deletions, we can satisfy the conditions of Jordan's theorem. It follows that we can construct quotients that are either alternating or symmetric.

Definition 2.3.15. If Y is a subset of a cube complex X, then $N_1(Y)$ is a union of closed cubes, which have non-empty intersection with Y. We define inductively $N_i(Y) = N_1(N_{r-1}(Y))$.

If Y is convex, then so is $N_r(Y)$ (as a neighbourhood is obtained by removing bounding hyperplanes and therefore it is an intersection of convex subcomplexes). And if H acts cocompactly on Y, it still acts cocompactly on $N_r(Y)$ assuming that X is locally compact.

Proposition 2.3.16. *Let $C(\mathcal{G})$ be the right-angled Coxeter group associated to $\mathcal{G}$ a finite simplicial graph, $|\mathcal{G}| > 2$, and suppose that $H < C(\mathcal{G})$ acts on the associated Davis-Moussong complex with a proper core Y. Let $\mathcal{C}$ be the class of symmetric and alternating groups. If $\mathcal{G}^c$ is connected, then H is $\mathcal{C}$-separable.*

Proof. As H acts with a proper core, there exists a generator of $C(\mathcal{G})$ not contained in H. Say $s_0 \notin H$.

Suppose $\gamma_1, \ldots, \gamma_n \notin H$.

Fix $v \in Y$. Without loss of generality, we may assume that Y contains $N(v)$ and $\gamma_i v$ for all i (otherwise replace Y with $N_r(Y)$ for a sufficiently large r). Moreover, by Lemma 2.3.11 we may assume that there exists a hyperplane $H_0 \notin \mathcal{D}(Y)$ with $|H_0^- \cap Y| = 1$ and by Remark 2.3.13 we may assume that the label of H_0 is s_0.

As $\mathcal{G}^c$ is connected, there exists a generator s_1 not commuting with s_0. Let v_0 be the unique vertex of $H_0^- \cap Y$. Let e_1 be the edge starting at v_0 with a label s_1. Obtain Y_1 by deleting $H(e_1)$ from the boundary of Y. By Lemma 2.3.6 $Y_1 \subset N(Y)$. If $v_1 \in Y_1 \cap H(e_1)^-$, then there is an edge starting at v_1 with the other endpoint in Y. This edge is labelled s_1 and is dual to $H(e_1)$. Now $N(H(e_1)) \cap Y = v_0$ since otherwise $H(e_1)$ would have to intersect H_0 and s_1 would commute with s_0. Therefore v_1 is uniquely determined as the other endpoint of e_1.

Continue this by taking e_i to be the edge starting at v_{i-1} labelled s_0 for even i and s_1 for odd i and let v_i be the other endpoint of e_i. Let Y_i be Y_{i-1} with $H(e_i)$ deleted from the boundary. Let $Y' = Y_k$ with k to be specified later.

Let Γ_0 be the group generated by the reflections in the hyperplanes bounding Y'. Let $\Gamma_1 = \langle \Gamma_0, H \rangle$. Then $[C(\mathcal{G}) : \Gamma_1] = |H \setminus Y'|$, where $|H \setminus Y'|$ denotes the number of vertices of $H \setminus Y'$. Every successive vertebra consists of a single vertex, so by Lemma 2.3.6 $|H \setminus Y_{i+1}| = |H \setminus Y_i| + 1$. We can choose k to make $|H \setminus Y'|$ a prime. Vertices $V(\Gamma_1 \setminus C(\mathcal{G}))$ is in a natural bijection with $V(H \setminus Y')$ and $V((H \setminus Y')$ is in a natural bijection with $\Gamma_1 \gamma_i$. Since $\gamma_i v \notin H.v$, the coset Γ_1 is different from Γ_1. The group H is a subgroup of Γ_1 so it fixes Γ_1, but γ_i doesn't fix Γ_1 and hence γ_i does not act as an element of H. If f is the homomorphism from $C(\mathscr{G})$ to the symmetric group on the right cosets of Γ_1, then $f(\gamma_i) \notin f(H)$.

Let s_2 be a generator distinct from s_0 and s_1. By the remark after Lemma 2.3.14, we can identify the right cosets of Γ_1 with orbits of Y' under the action of H and we can read off the action from the geometry as follows. Pick $v \in Y$ in an orbit corresponding to $\Gamma_1 g$, let u be a vertex connected to v by an edge labelled s. If $u \notin Y'$, then $\Gamma_1 gs$ is the coset corresponding to $H.u$. Since the tail contains no edge labelled s_2, every coset corresponding to a vertex in the tail is fixed by s_2.

So s_2 moves at most $|H \setminus Y|$ elements. By taking k large enough while $|H \setminus Y'|$ is still a prime, we may ensure that the conditions of Jordan's lemma are satisfied (the primitivity follows from transitivity and a non-existence of non-trivial partition of a prime number of elements into sets of the same size). □

2.4 Changing parity

We shall now prove that we may force the action to be alternating (similarly we can force it to be symmetric). Let $\mathscr{G}$ be a non-discrete finite graph throughout this section.

Definition 2.4.1. Suppose Y is a core for an action of $G < C(\mathscr{G})$ on a $DM(\mathscr{G})$ and suppose s_i is one of the generators of $C(\mathscr{G})$. *The parity of s_i with respect to the core Y* is the parity of s_i acting on the right cosets of $\Gamma_1(Y)$, where $\Gamma_1(Y)$ is the finite index subgroup of $C(\mathscr{G})$ generated by G and the reflections in the hyperplanes bounding Y.

We will modify the construction of the tail in order to make each s_i act as an even permutation (or we will make at least one of s_i act as an odd permutation).

Suppose $g.v_0$ is in the tail. If the edge between $g.v_0$ and $gs.v_0$ is in the tail, then $g.v_0$ and $gs.v_0$ map to distinct vertices in $\Gamma_1 \setminus X$, hence $\Gamma_1 g \neq \Gamma_1 gs$.

If $gs.v_0$ is not in the tail, then the hyperplane dual to this edge bounds Y and the reflection in this hyperplane belongs to Γ_1. Therefore $\Gamma_1 = \Gamma_1 gsg^{-1}$ or equivalently $\Gamma_1 g = \Gamma_1 gs$.

More precisely, suppose H acts with core Y and Y' is the core resulting from deletion of $H_0, \ldots, H_k$, and the label of H_i is s_i. Moreover assume $H_0 \cap Y'$ is a single edge.

Then the parity of s_1 with respect to Y' is the sum of the parity of s_1 with respect to Y and the number of edges labelled s_1 in $H_0^- \cap Y'$. So we can control the parity of s_1 by changing the number of edges with label s_1 in the tail. Suppose that the conditions of Jordan's

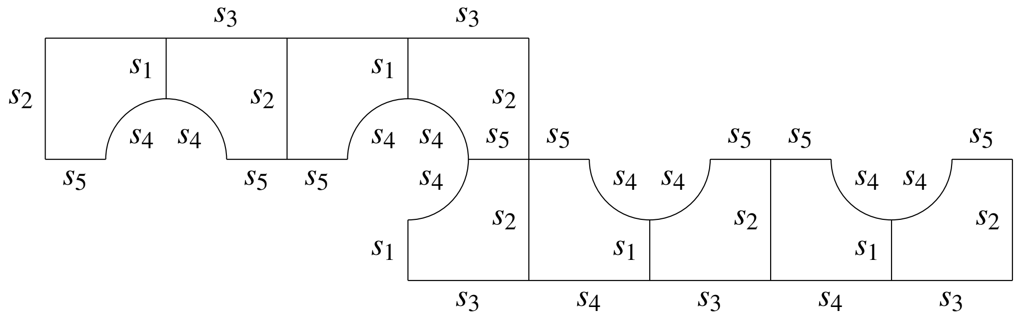

Fig. 2.7 Sketch of the situation in Lemma 2.4.2, where Γ is a cycle of length 5 and $i = 5$. Here we've drawn the hyperplanes. The cube complex would be the dual picture. The lower five squares are the old tail and the upper four squares form the end of the new tail. The figure is a bit deceptive in that the line segments labelled s_5 are not in fact on one line and the line segments labelled s_1 and s_5 don't intersect when extended to lines.

theorem are satisfied with a margin M (i.e. the conditions are satisfied even if s_3 moves $|H \setminus Y| + M$ elements). Taking $M = (|\mathscr{G}| - 2)(2d + 1) + 16$, where d is the diameter of $\mathscr{G}^c$ will be sufficient.

First let us show that we can deal with parity of all generators other than s_1 and s_2.

Lemma 2.4.2. *For any $i \in I \setminus \{1,2\}$, if the tail of Y is a path with labels $s_1, s_2, \ldots, s_1, s_2, s_1$ of length at least $2d_{\mathscr{G}^c}(v_1, v_i) + 1$ starting at vertex V, then there exists a core Y' such that in the associated action the parity of s_i is changed and the parities of no s_j changed for $j \in I \setminus \{1,2,i\}$. Moreover, $|H \setminus Y| = |H \setminus Y'|$ and Y' contains a tail of the same length as Y and the labels of these two paths are the same with the exception of a subpath labelled $s_1, s_2, \ldots, s_1, s_2, s_1$ of length $2d_{\mathscr{G}^c}(v_1, v_i) + 1$.*

Proof. Say $v_1 = v_{i_0}, v_{i_1}, \ldots, v_{i_d} = v_i$ is a path in $\mathscr{G}^c$ of the shortest length. Let Y' be a subcomplex built using deletions of hyperplanes $s_{i_0}, s_{i_1}, \ldots, s_{i_d}, s_{i_{d-1}}, \ldots, s_{i_0}, s_2, s_1, \ldots, s_1$ starting at v.

Compared to Y, the tail of this complex contains two more edges labelled s_{j_i} for $0 < j < d$. It also contains an extra edge labelled $s_{i_d} = s_i$, so the parity of s_i changed and the parity of other generators s_j remains the same for $j \neq 1, 2, i$. □

Now let's change the parity of a generator that appears in the tail.

Lemma 2.4.3. *If the tail of Y contains a path with labels $s_1, s_2, \ldots, s_1, s_2, s_1$ of length at least 7, then there exists a core Y' such that in the associated action only the parity of s_1 changed. Moreover, $|H \setminus Y| = |H \setminus Y'|$ and Y' is built from the same complex as Y using a sequence of*

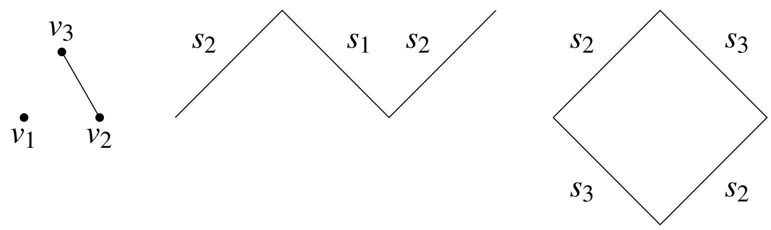

Fig. 2.8 A sketch of the subgraph of $\mathscr{G}$ spanned by v_1, v_2 and v_3, the segment of the old tail and the new square which replaces this segment in the case 1 of the proof of Lemma 2.4.3.

deletions, whose labels agree with that of Y with the exception of 5 *deletions. (We allow a deletion to be replaced by no deletion.)*

Proof.

1. Suppose there exists distinct s_3 and s_4 which commute mutually but neither of which commutes with s_1. Then instead of the deletion of the hyperplanes labelled s_2, s_1, s_2, delete the hyperplanes labelled s_3, s_4. This creates a square. Continue building the tail starting from one of the vertices of the square using the deletions of the hyperplanes with the same labels as before. The new tail contains two fewer s_2 labels, two more of s_3 and two more of s_4 and one fewer s_1 (or the same number of s_2 and two more s_3, if $s_2 = s_4$ etc.). Hence only the parity of s_1 changed.

   To be precise, we need to take the path labelled s_2, s_1, s_2 which is a subpath of a path labelled s_1, s_2, s_1, s_2, s_1 in the tail, as otherwise deleting a hyperplane labelled s_3 could introduce more than just a side of a square. Similarly for the other cases in this proof.

2. Suppose there is some s_3 commuting with s_1, but not s_2. Then instead of the deletion of the hyperplanes labelled s_1, s_2, s_1, s_2, s_1, delete the hyperplanes labelled s_1, s_3 and then delete the hyperplanes labelled s_2 at two of the vertices of the square. This creates a square with two spurs. Continue building the tail starting from the remaining vertex of the square. The new tail contains the same number of s_2 labels, two more of s_3 and one fewer s_1. Hence only the parity of s_1 changed.

3. Lastly, if neither of the above cases holds, then $\mathscr{G}$ consists of v_1, isolated vertices I, vertices S_1 at distance 1 from v_1 and vertices S_2 at distance 2 from v_1. Moreover, there exists a vertex adjacent to v_1 as the graph is non-discrete. Every vertex adjacent to v_1 is adjacent to v_2, so $v_2 \in S_2$.

   The induced graph on vertices of S_2 is discrete because every edge intersects S_1. Take any $u \in S_1$. Consider a path from u to v_1 in $\mathscr{G}^c$. Somewhere along this path we go from a vertex, which is connected to both v_1 and v_2 to a vertex which is connected to neither. Therefore there are s_3 and s_4 such that s_3 commutes with s_1 and s_2 and s_4 does not commute with any of s_1, s_2 and s_3. Now instead of the deletion of the hyperplanes labelled s_1, s_2, s_1, s_2, s_1 delete the hyperplanes labelled s_4, s_1, s_3, s_4. This creates a square with labels s_1, s_3, s_1, s_3. Continue building the tail. We have one fewer s_1, two fewer s_2 and two more of each s_3 and s_4.

Let Y' be the new subcomplex. By construction $|H \setminus Y| = |H \setminus Y'|$ and the sequences of labels of deleted hyperplanes for the two complexes differ at no more than 5 places. □

Using Lemmas 2.4.2 and 2.4.3, we can now modify segments of the tail to make the parity of all elements even (we might need to apply Lemma 2.4.3 twice - once to s_1 and once to s_2). This completes the proof of the main theorem.

Chapter 3

Separability and randomness

3.1 Motivation

Let's say that we want to understand a typical homomorphism between two groups. The simplest domain would be a free group because then the map is specified by its values on generators. The correspondence between the maps and the tuples is bijective, so studying maps from free groups is the same as studying tuples of elements. This also makes any one specific range pretty uninteresting to study. We need a family of groups, ideally one which is easy to describe and work with. In this thesis, I take the symmetric groups.

How large is typically the image of such a map? It is a standard exercise to find a pair of permutations, which generate the entire symmetric group or a pair, which generate the index 2 alternating subgroup. Surprisingly, this is a typical behaviour and as the size of the symmetric group increases, the probability that two random permutations generate the entire symmetric group or its index 2 alternating subgroup tends to 1. In the spirit of the probabilistic method, I will use this to find interesting quotients in situations when an explicit construction might be tedious or even unknown. Sometimes it merely simplifies an argument. We can for example reprove the main theorem from [Wil12] using probabilistic methods.

Corollary 3.1.1 ([Wil12]). *Suppose G is a finitely generated infinite index subgroup of a non-abelian free group F_k and that $g_1, \ldots, g_l \in F_k \setminus G$. Then there exists a surjection $f : F_k \longrightarrow A_n$ onto some alternating group such that $f(g_i) \notin f(G)$ for all i.*

I will indicate how to get this theorem from the results in this chapter.

Proof. Let (X_G, x_G) be the cover of R_k associated to G. Let γ_i be the loop in R_k representing g_i and let $\tilde{\gamma}_i$ be its lift to X_G starting at x_G. Let $Y \subset X_G$ be the union of all loops in X_G and all images of $\tilde{\gamma}_i$.

The graph Y consists of a single component and this component is not a covering of R_k as X_G is a connected infinite degree cover and its finite (non-empty) subgraphs are not coverings. We can apply Theorem 3.2.3 which says that a random group $\Gamma_n(Y)$ with condition Y is S_n or A_n with probabilities which tend to $1-2^{-k}$ and 2^{-k} respectively as n goes to infinity. In particular, probability that the image of G is A_n is eventually positive. The endpoint of $\tilde{\gamma}_i$ isn't x_G, therefore $f(g_i) \notin f(G)$ in any completion of Y and in particular also in those which surject onto an alternating group. □

Another more advanced application is the Theorem B.

Theorem 3.1.2 (Theorem B)**.** *Suppose $H_1,\dots,H_k$ are infinite index, finitely generated subgroups of a non-abelian free group F. Then there exists a surjective homomorphism $f : F \longrightarrow A_m$ such that if H_i is not conjugate into H_j, then $f(H_i)$ is not conjugate into $f(H_j)$.*

The idea of the proof is similar, but we need to be much more careful in the choice of the graph to complete.

3.2 Set-up

The probability that k random elements $a_1,\dots a_k$ of S_n generate S_n (or A_n) tends to $1-2^{-k}$ (or 2^{-k}) as n increases provided $k \geq 2$ [Dix69]. I will generalise this result to the setting with finitely many conditions on $a_1,\dots a_k$. These conditions are given by an immersion of a finite graph into a rose via a correspondence which we now discuss. The basic idea is to start with a graph, which extends to a covering of the presentation complex. We will then look at all the ways it extends to a covering.

We can associate a graph to a k-tuple of elements $a_1,\dots,a_k \in S_n$ as follows. Take n vertices labelled $1,\dots,n$ with i and $a_j(i)$ connected by an oriented edge labelled a_j for all i and j. This graph is a (not necessarily connected) covering of *the rose of k petals R_k*, a graph which has a single vertex and k edges labelled $a_1,\dots,a_k$ respectively. The covering has degree n. This is just the Cayley graph of S_n with respect to $a_1,\dots,a_k$. I'm using the convention that Cayley graph doesn't have to be connected.

This gives a bijective correspondence between the degree n coverings of R_k and k-tuples of elements of S_n. To see the other direction, we need the following definition.

Definition 3.2.1 (Core graph)**.** Given a graph Y, the core of Y, denoted $\mathrm{Core}(Y)$ is the union of all cycles in Y.

Given Γ a subgroup of a free group F_k, let X_Γ be the associated cover of R_k. The *core of* Γ, denoted Core(Γ) is the subspace of Core(X_Γ).

To get a precover from a k-tuple take the core of the covering space associated to the subgroup generated by the elements in the k-tuple.

I will in general look at the coverings where the vertices are not labelled. This means that in fact I'll be using the correspondence between unlabelled degree n coverings and conjugacy classes of k-tuples of elements of S_n. We can use these correspondence to define conditions on a random homomorphism from a finitely generated free group to the symmetric group S_n as follows.

Definition 3.2.2 (Random action)**.** Suppose $G \longrightarrow R_k$ is a label preserving locally injective map of oriented labelled graphs. Such a map is called a *precover* of R_k. Just as a degree n cover corresponds to a permutation $f : [n] \longrightarrow [n]$, a degree n precover corresponds to a partial injective function $f : [n] \nrightarrow [n]$.

Suppose G has at most n vertices. Add vertices to G until there are n vertices in total: let G' be disjoint union of G and a discrete graph with $n - |G|$ vertices.

Let $V_j^{no}(G')$ be the set of vertices of G' without an outgoing edge labelled a_j and $V_j^{ni}(G')$ be the set of vertices without an incoming edge labelled a_j. For all j, choose a bijection f_j between $V_j^{no}(G')$ and $V_j^{ni}(G')$ uniformly at random. Connect v and $f_j(v)$ by an oriented edge labelled a_j.

The resulting graph $\overline{G}$ is *a random degree n completion* of G, the associated homomorphism $\varphi : F_k \longrightarrow S_n$ is *a random homomorphism with condition* G and the associated group $\Gamma_n(G) < S_n$ is *a random group with condition* G. Let's call G a *condition graph*.

We frequently take the condition graph to be a core graph, union of core graphs or some slightly larger superspace of a core graph. A core graph of a finitely generated group is a finite graph, since it is a union of only finitely many cycles. Recall from Theorem 1.3.9 that $\Gamma_n(\emptyset)$ is frequently S_n or A_n. If some component of a graph G is an actual covering of R_2, then $\Gamma_n(G)$ is non-transitive for $n > |G|$. We will prove a converse result:

Theorem 3.2.3 (Main Theorem)**.** *If no component of* G *is a covering of* R_k*, then* $\Gamma_n(G)$ *is* S_n *or* A_n *with probabilities which tend to* $1 - 2^{-k}$ *or* 2^{-k} *respectively as n goes to infinity.*

I will follow the same strategy as Dixon to prove this theorem. In Section 3.3, I shall prove that the random group is transitive. In Section 3.4 we will prove that it is also primitive. In Section 3.5 I will prove that the random group contains a short cycle and that this together with primitivity proves the theorem.

In Section 3.6, I will apply the theorem to show new separability properties of free groups. In particular, infinite index, finitely generated non-conjugate subgroups of a free group map to non-conjugate subgroups of an alternating group under some surjective homomorphism onto an alternating group. This improves Bogopolski-Grunewald's subgroup conjugacy separability [BG10].

3.3 Transitivity

We need to show that a random group is either S_n or A_n. Both A_n and S_n are transitive, so the transitivity is necessary. It also turns out to be one of the conditions used in the converse statement.

Lemma 3.3.1 ([Dix69]). *The group $\Gamma_n(\emptyset)$ is almost always transitive. (i.e. the probability that $\Gamma_n(\emptyset)$ generates a transitive subgroup of S_n tends to 1 as n goes to infinity).*

If a component of G is an actual covering, then no completion is transitive (except for the case when the component is all of G and there are no other vertices). That component will remain a component in any completion. We need to exclude this situation in the generalised version of the theorem.

Lemma 3.3.2. *Assume that no component of G is a covering of R_k. Then $\Gamma_n G$ is almost always transitive and a random completion $\overline{G}$ is almost always connected.*

The idea of the proof is as follows. We're starting from something which intuitively is more connected than a discrete graph. We formalise this intuition by constructing a probability preserving map between random completions of $\emptyset$ and random completions of G, which preserves connectedness. We will do this by replacing components of G with discrete graphs.

Proof. Let $G_1, G_2, \ldots, G_l$ be the connected components of G. Let $E_j(G_i)$ be the set of edges labelled a_j in G_i.

- Case 1: The number of edges $|E_j(G_i)|$ labelled a_j in G_i is the same for all j. Let H_i be the discrete graph with $|V(G_i)|-|E(G_i)|/l$ vertices. Let H be the union of all H_i. Pick a bijection between the "missing edges" at vertices of H_i and the "missing edges" at vertices of G_i - see figure 3.1. This induces a map between random completions. More formally, recall that if G is a graph, then $V_j^{ni}(G)$ and $V_j^{no}(G)$ are the vertices with no incoming and no outgoing edge labelled a_j, respectively. The label a_j appears the same number of times in G_i for all j, so

$$|V_j^{ni}(G_i)| = |V_j^{no}(G_i)| = |V(G_i)| - |E_j(G_i)| = |V(G_i)| - |E(G_i)|/l$$

is independent of j, where $E_j(G_i)$ are the edges of G_i with label a_j. The graph H_i is discrete, so we have $|V_j^{ni}(H_i)| = |V(G_i)| - |E(G_i)|/l$. Pick arbitrary bijections $f_{i,j}^{ni} : V_j^{ni}(H_i) \longrightarrow V_j^{ni}(G_i)$. Let f be a union of these bijections. These maps induce a bijection between the degree n completions of H and degree $n + |E_j(G_i)|$ completions of G as follows. Given a completion $\overline{H}$ of H, consider $(\overline{H} \setminus H) \cup G$. Now connect each open end of an edge in $(\overline{H} \setminus H)$, which was previously attached to $v \in H$ to $f(v)$. This is a completion of G. Call it $f(\overline{H})$ by abuse of notation. This correspondence is bijective as now we could excise G and connect the open ends back to H.

Suppose $f(\overline{H}) = K_1 \sqcup K_2$, where K_i is closed non-empty. For all $v \in H$, the component of G containing $f_{i,j}^{no}(v)$ and $f_{i,j}^{ni}(v)$ does not depend on j. Hence, the closures of $K_i \setminus (G \cap K_i)$ in $\overline{H}$ are two disjoint closed subsets partitioning $\overline{H}$. They are non-epmpty as long as $K \not\subset G$. This is where we use that no component of G is a cover. If $\overline{H}$ is connected, then so is $f(\overline{H})$.

The probability that a random completion of H is connected (hence the associated group is transitive) tends to 1 by Lemma 3.3.1 and therefore the probability that a random completion of G is connected also tends to 1.

- Case 2: Suppose $|E_j(G_i)|$ is not independent of j. We can reduce this situation to case 1, by taking a slightly larger graph G', which satisfies this condition. The key observation will be that most completions of G are also completions of G'.

  If there is some i, j and j' with $|E_j(G_i)| < |E_{j'}(G_i)|$, let v_j be a vertex of G_i with no outgoing edge labelled a_j. Replace G_i by a union of G_i and an a_j-edge starting at v_j and ending at a new leaf. Repeat this process until $|E_j(G_i)|$ becomes independent of j.

  This process terminates since $\sum_i \sum_j \left(\max_{j'} \left(|E_{j'}(G_i)|\right) - |E_j(G_i)|\right)$ is a non-negative integer, which decreases whenever we change the graph. Let G' be the resulting graph.

  The inclusion of G to G' is a π_1-isomorphism on each component and G' contains finitely many more edges than G. If $\overline{G}$ is a random completion of G, then there is a unique map $G' \longrightarrow \overline{G}$ extending the inclusion of G. If this map is injective, then $\overline{G}$ is also a completion of G'. Let's estimate the probability of this event. Build a random completion of G in the same way, we've built G': one edge at a time.

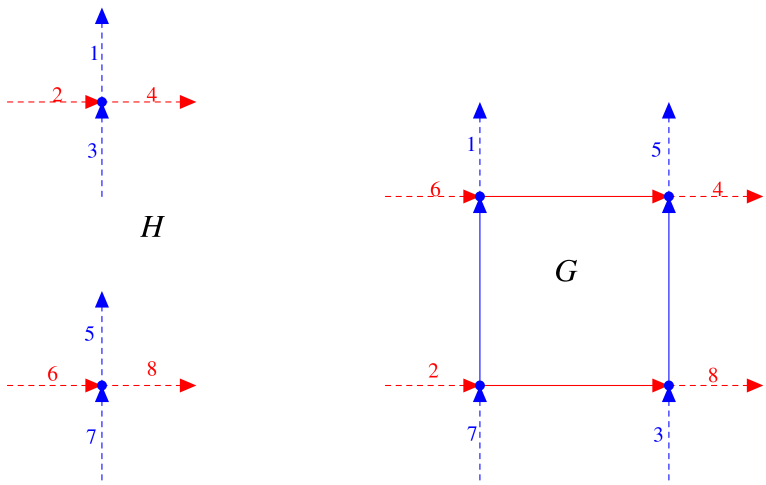

Fig. 3.1 The graph G is the core of $\langle [a,b] \rangle$ and H consists of two vertices. Pick a bijection between the missing edges at H and the missing edges at G. A completion of H corresponds to a completion of G by reconnecting the adjacent edges according to this bijection. If the completion of H is connected, then so is the completion of G.

If first edge e_1 connects to a vertex of G, then the injectivity fails. There are $n - |V(G)|$ vertices not in G. If e_1 connects to one of them, we can continue with the second edge. The second edge e_2 can fail the injectivity in at most $|V(G)|+1$ ways (it might connect back to G or to an endpoint of e_2). It can succeed in at least $n - |V(G)| - 1$ ways. Continue for all new edges. The probability that $G' \longrightarrow \overline{G}$ is injective is at least

$$\frac{n-|V(G)|}{n}\frac{n-|V(G)|-1}{n}\cdots\frac{n-|V(G)|-\Delta}{n}$$

where $\Delta = |E(G')| - |E(G)|$. This quantity goes to 1 as n goes to infinity. This means $G' \longrightarrow \overline{G}$ is almost always injective and a completion of G is almost always a completion of G'. By case 1, a completion of G' is almost always connected, therefore a completion of G is almost always connected. I am implicitly using that the probabilities are compatible in the following sense.

$$\mathbb{P}(\text{ A completion of } G \text{ is } H | H \text{ is a completion of } G') = \mathbb{P}(\text{ A completion of } G' \text{ is } H)$$

This is true, because it does not matter whether we complete G to a completion containing G', or whether we complete G'.

□

3.4 Primitivity

An action of a group Γ on a finite set X is *primitive* if it is transitive and no nontrivial partition of X is preserved by Γ. We have already dealt with the transitivity, so we just need to show non-existence of a preserved partition. Transitivity implies that all sets in the partition have the same size, hence taking n to be a prime (as in chapter 1) ensures primitivity, but we do not need to do that here.

Lemma 3.4.1. *Assume that no component of G is a covering of R_k. Then $\Gamma_n G$ is almost always primitive.*

We use that imprimitive groups are extremely rare.

Proof. By Lemma 2 in [Dix69], the proportion of pairs of elements of S_n, which generate an imprimitive subgroup is at most $n2^{-\frac{n}{4}}$ (and hence this bound also applies to k-tuples).

Let's count what proportion of k-tuples of elements of S_n *respects* G (i.e. how many arise from a completion of G).

Recall that $|E_j(G)|$ is the number of edges in G labelled a_j.

The probability that a random permutation moves vertices according to the edges labelled a_j is

$$\frac{1}{n.(n-1)\dots(n-|E_j(G)|+1)}.$$

If $n > 2|E(G)|$, a random completion respects G with probability at least $(2n)^{-|E(G)|}$. This is only polynomial in n. Even if all k-tuples generating imprimitive subgroups respected G, the proportion of imprimitive random completions of G would be at most

$$\frac{n2^{-\frac{n}{4}}}{(2n)^{-|E(G)|}} = (2n)^{|E(G)|} n2^{-\frac{n}{4}}$$

which goes to zero as n goes to ∞. □

3.5 'Jordan' condition

The final condition (in addition to being primitive) for a subgroup to be A_n or S_n is that it contains a q-cycle for some prime $q \leq n-3$ [Wie14, Theorem 13.9].

Following [Dix69], we define $C_{q,n} \subset S_n$ to consist of those permutations which contain a single cycle of length divisible by q and all the other cycles are of lengths coprime to q.

In particular, if G contains an element of $C_{q,n}$, then it contains a q-cycle. The following lemma is a key step in Dixon's theorem.

Lemma 3.5.1 (Lemma 3 in [Dix69]). *Let $T_n = \bigcup_q C_{q,n}$, where the union is over all primes q such that*

$$(\log n)^2 \leq q \leq n-3\,.$$

Then the proportion u_n of elements of S_n which lie in T_n is at least

$$1 - 4/(3 \log\log n)$$

for all sufficiently large n.

We need to generalise this to the conditional case.

Lemma 3.5.2. *Let G be any graph. Take a random group action with condition G. Almost always some power of a_1 acts as a q-cycle, where $q \leq n-3$ is a prime.*

The generalisation is a bit more complicated. We separate the a_1-edges in the condition graph G into cycles and paths. We will take n very large compared to the size of the cycles. This will allow us to ignore the cycles since they will all be smaller than the prime q. To deal with the paths, one only needs to realise that paths are a typical behaviour. The corresponding walks in the random unconditional completion would almost always be injective, so we can apply the unconditional theorem.

Proof. We are only using one generator, so in this proof we can assume that there is only one generator. The condition graph G consists of loops and paths because no vertex has valency greater than 2. We will deal with both of them separately. The paths do not really cause many issues. As in the proof of transitivity, almost every completion of an empty graph will be also a completion of a union of paths. This will reduce the statement to the unconditional version. To deal with the loops we can use the lower bound of Lemma 3.5.1 and force q to be bigger than the length of all loops. This way a suitable power of a_1 will fix the loops pointwise, and act as a q-cycle on the remaining vertices.

The graph G consists of paths $P_1, \ldots, P_k$ and loops $L_1, \ldots, L_l$. Let v_i be the initial vertex of P_i. Let G' be the union of all the paths P_i.

Let $n' = n - \sum_i |L_i|$. Let D_k be a graph with k vertices and no edges. Pick a bijection f between the vertices of D_k and $\{v_i\}$. Consider the random degree n' completion $\Gamma_{n'}(D_k)$. Then by lemma 3.5.1

$$\mathbb{P}(a_1 \text{ acts as an element of } T_{n'}) \geq 1 - 4/(3 \log\log n')$$

for sufficiently large n'.

There is unique label and orientation preserving map $\overline{f}$ from G' to a completion of a discrete graph, which extends f. If this map $\overline{f}$ is injective, then the completion of D_k is also a completion of G'. I claim that this happens with probability $1-\mathscr{O}(1/n')$. Let's proceed by induction on the sum of length of the paths in G'. If there are no edges, the map $\overline{f}$ is just f and therefore a bijection to its image D_k.

If G' contains an edge, let e be an edge at the end of one of the paths. Let G'' be G' without e and the terminal endpoint $t(e)$, but with the initial endpoint $i(e)$. In other words, G'' is the same graph as G', just with one of the paths shorter by 1. By induction G'' injects with probability $1-\mathscr{O}(1/n')$. Suppose G'' injects. Then the graph G' fails to inject only if $t(e)$ is one of the vertices in D_k. This happens with probability $\frac{k}{n'-|E(G'')|}$ since there are $n'-|E(G'')|-k$ vertices not in the image of G'. Therefore, G' injects with probability $(1-\mathscr{O}((1/n'))\left(1-\frac{k}{n'-|E(G'')|}\right) = (1-\mathscr{O}(1/n'))$.

A random completion of D_k is almost always a random completion of G'. We can restate Lemma 3.5.1 as follows. A random completion of D_k has almost always the property that the induced a_1 belongs to $T_{n'}$. But then the same applies to a random completion of G', because a random completion of D_k is almost always a completion of G'. I.e. some power of a_1 in the random action with condition G' almost always acts as q-cycle, where q is a prime with $(\log n)^2 \leq q \leq n-3$.

Take $n' > \exp(\sqrt{\max|L_i|})$. A random completion of G is just a union of a random completion of G' and the loops L_i. Therefore a_1 almost always acts as a union of an element from $T_{n'}$ and cycles of lengths $|L_i|$. By choice of n', we have $\max|L_i| < q$. Some power of a_1 almost always acts as a union of a q-cycle and cycles shorter than q. Therefore, a higher power of a_1 almost always acts as q-cycle. □

3.6 Subgroup conjugacy separability and randomness

In this section we prove Theorem B. A random action often demonstrates separability properties of a free group. Since the action is often alternating, this demonstrates separability within alternating groups.

Let g and h be two elements of a free group, such that g is not conjugate to either h or h^{-1}. After conjugation, we may assume that g is cyclically reduced and freely reduced. If a homomorphism $f : F_2 \longrightarrow S_n$ is such that $f(g)$ and $f(h)$ have different cycle structures, then g and h remain in different conjugacy classes in the image under f.

A random action with a suitable condition will give different expected numbers of fixed points of g and h and just a small variance. This produces actions, which keep g and h in different conjugacy classes.

Let G be a loop labelled with g. In counting fixed points of g, we need to count how often G lifts to a covering. We can categorise these lifts by their image. I.e. we can count injective lifts of possible images of G.

Definition 3.6.1 (Quotient of a precover)**.** If G is a precover of R_k for some k and K is a graph, then a simplicial surjective locally injective map $f : G \twoheadrightarrow K$ is *a quotient of a precover* G.

Let's say we want to count the number of lifts of a graph H. Then the image of a lift of H is some quotient of H. If we take the union with G, we get some quotient of $G \sqcup H$, where the restriction to G is injective. Counting the lifts of H is therefore the same as counting the injective lifts of those quotients of $G \sqcup H$, where the restriction to G is injective. Let's give this quantity a notation.

Definition 3.6.2. Suppose G and H are precovers, K is a quotient of the precover $G \sqcup H$ and $\overline{G}$ is a completion of G. By $\mu_{K \longrightarrow \overline{G}}$ we denote the number of injective maps from K to $\overline{G}$ such the the composition $G \longrightarrow K \longrightarrow \overline{G}$ is the natural inclusion of G to $\overline{G}$.

Let $\tau_{H \longrightarrow \overline{G}}$ be the total number of maps from H to G.

Note that if $G \longrightarrow K$ is not injective, then $G \longrightarrow K \longrightarrow \overline{G}$ cannot be an inclusion of G and therefore $\mu_{K \longrightarrow \overline{G}} = 0$. Let's express τ using μ.

Lemma 3.6.3. *Suppose that G and H are precovers and $\overline{G}$ is a completion of G. The total number of maps from H to $\overline{G}$ is given by the following.*

$$\tau_{H \longrightarrow \overline{G}} = \sum_{K = (G \sqcup H)/\sim} \mu_{K \longrightarrow \overline{G}}$$

The sum goes over quotients K of the precover $G \sqcup H$.

Proof. Given a map $f : H \longrightarrow \overline{G}$, let $K = G \cup f(H)$. Then K injects to $\overline{G}$ and $G \longrightarrow K \longrightarrow \overline{G}$ is an isomorphism onto $G \subset \overline{G}$.

Conversely, if K is a quotient of $G \sqcup H$ and it injects to $\overline{G}$ and $G \longrightarrow K \longrightarrow G$ is an isomorphism, let f be the map $H \longrightarrow K \longrightarrow G$. □

We will now need to estimate each summand in the previous lemma. If G was empty then the first order estimate would be $n^{\chi(K)}$ [PP15, Theorem 1.8]. To take potentially non-

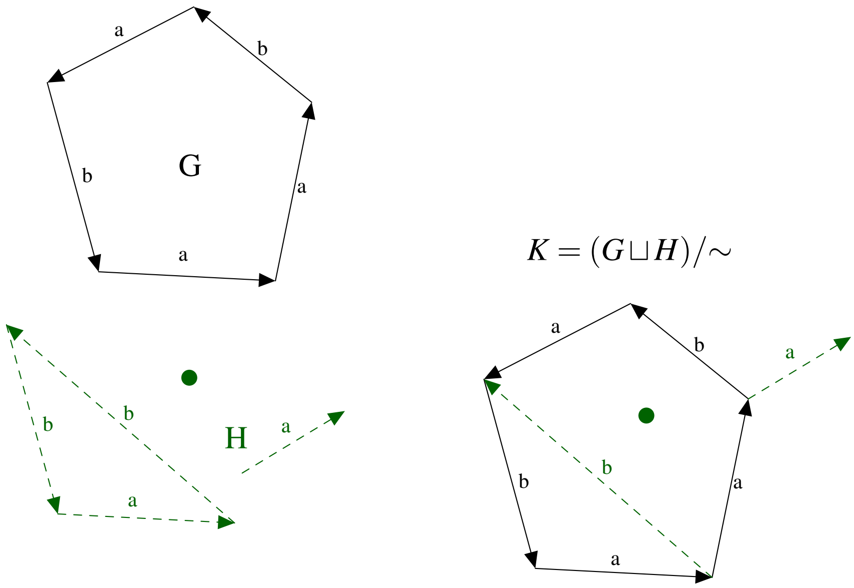

Fig. 3.2 We expect roughly 1 lift of K, since there is about $1/n$ probability that the diagonal b-edge closes up and there are about n possibilities for the location of the isolated vertex. The green a-edge does not contribute anything, because there are roughly n options for its endpoint and each of them appears with probability roughly $1/n$.

empty G into account, I define the relative Euler characteristic be a difference of the Euler characteristics.

Definition 3.6.4 (Relative Euler Characteristic)**.** If K is a quotient of $G \sqcup H$ such that G embeds to K, then *the Euler characteristic of K relative to G* is $\chi_G(K) = \chi(K) - \chi(G)$.

The next lemma gives the expected number of lifts of a quotient of $G \sqcup H$. This quantity makes intuitive sense, since the relative Euler characteristic counts the components of K disjoint from components of G, minus the loops of K, which are not loops of G. See Figure 3.2.

Lemma 3.6.5. *Suppose G and H are precovers and K is a quotient of the precover $G \sqcup H$. Then we can express the expected number of maps from K to the random completion $\overline{G}$ which extend the inclusion of G as follows.*

$$\mathbb{E}(\mu_{K \longrightarrow \overline{G}}) = n^{\chi_G(K)} + \mathscr{O}(n^{\chi_G(K)-1})$$

Here we fix K, G and H and we let $\overline{G}$ be a random degree n completion of G.

Proof. We'll prove this by induction on the number of cells in $K \setminus G$. For the base case of $K = G$, left hand side is 1 and the right hand side is $1 + \mathcal{O}(n^{-1})$.

1. Suppose there exists an edge e of K not contained in G. Let K' be $K \setminus e$. By the induction on the number of cells, we have $\mathbb{E}(\mu_{K' \longrightarrow \overline{G}}) = n^{\chi_G(K')} + \mathcal{O}(n^{\chi_G(K')-1})$.

   There are between n and $n - |E(K')|$ ways for e to lift and only one of them allows K to lift. Hence,
   $$\mathbb{E}(\mu_{K \longrightarrow \overline{G}}) = n^{-1}\mathbb{E}(\mu_{K' \longrightarrow \overline{G}}) + \mathcal{O}(n^{-2})$$

2. If $K \setminus G$ contains no edges, then it is a disjoint union of G and vertices. Suppose $v \in K \setminus G$ is a vertex. Let $K' = K \setminus v$. To lift K, we need to lift K' and specify, where does v go. We always have between n and $n - v(K')$ options for v, so
   $$\mathbb{E}(\mu_{K \longrightarrow \overline{G}}) = n\mathbb{E}(\mu_{K' \longrightarrow \overline{G}}) + \mathcal{O}(1)$$

□

In particular, we can get the highest order term approximation to the total number of expected lifts of H to a completion of G by determining the largest relative Euler characteristic among the quotients of $G \sqcup H$ and the number of quotients, which achieve this minimum.

Definition 3.6.6 (Relative rank, critical graphs and multiplicity)**.** *The relative rank* $r_G(H)$ is $\min \chi_G(K)$, where the minimum goes over quotients of $G \sqcup H$.

We call the quotients which achieve the minimum *critical graphs*. *Relative multiplicity* is the number of critical graphs.

Lemma 3.6.7. *Suppose G and H are precovers, and G' a random completion of G. Then the variance of $\tau_{H \longrightarrow \overline{G}}$ is as follows.*

$$\mathrm{Var}(\tau_{H \longrightarrow \overline{G}}) = \mathbb{E}(\tau_{(H \sqcup H) \longrightarrow \overline{G}}) - \mathbb{E}(\tau_{H \longrightarrow \overline{G}})^2$$

Proof. Write out the expression for the variance.

$$\mathrm{Var}(\tau_{H \longrightarrow G}) = \mathbb{E}(\tau^2_{H \longrightarrow \overline{G}}) - \mathbb{E}(\tau_{H \longrightarrow \overline{G}})^2$$

The expectation of the square $\mathbb{E}(\tau^2_{H \longrightarrow \overline{G}})$ is the same as the expected number of pairs of maps $H \longrightarrow \overline{G}$, which is the same as the number of maps $H \sqcup H \longrightarrow \overline{G}$. □

We will use the Lemmas 3.6.3 and 3.6.7 to count the mean and the variance of the number of the lifts.

Example 3.6.8. Suppose $\gamma_1, \dots, \gamma_k \in F_r$ and $\Gamma_1, \dots, \Gamma_l < F_k$ and each Γ_j has rank at least 2. Suppose that $\gamma_i = u_i^{k_i}$ and that u_i is not a proper power.

Let by abuse of notation γ_i be a core graph of $\langle \gamma_i \rangle$. Let G_j be the core graph of Γ_j. Let graph G be the disjoint union of a_i copies of γ_i and b_j copies of Γ_j.

Now take a random completion of G. We'll count lifts of γ_i and Γ_j. Let's first calculate $\tau_{\gamma_i \longrightarrow \overline{G}}$. For this we'll need to calculate a contribution from each quotient of $G \cup \gamma_i$. The relative rank $r_G(\gamma_i)$ is at most $\chi(\gamma_i) = 0$. It can't be smaller, because then there would need to be a component of a critical graph, which is simply connected. That is not possible, because the quotient map is locally injective and γ_i contains no leaves. When counting the critical graphs, two types arise.

1. The image of γ_i is disjoint from all G (I'm talking about the additionally copy of γ_i, not about one of the copies in G). There are $\sigma(k_i)$ such quotients, where σ counts divisors of an integer.

2. The image of γ_i lies in G. We can express this quantity as a linear function of a_j's and b_j's.
$$\tau_{\gamma_i \longrightarrow G} = \sum_j a_j \tau_{\gamma_i \longrightarrow \gamma_j} + \sum_j b_j \tau_{\gamma_i \longrightarrow G_j}$$

Use Lemma 3.6.5 to get

$$\mathbb{E}(\tau_{\gamma_i \longrightarrow \overline{G}}) = \tau_{\gamma_i \longrightarrow G} + \sigma(k_i) + \mathscr{O}(n^{-1})\,.$$

Let's also compute the variance of $\tau_{\gamma_i \longrightarrow \overline{G}}$. Let $H = \gamma_i \sqcup \gamma_i$. By Lemma 3.6.7,

$$\mathrm{Var}(\tau_{\gamma_i \longrightarrow \overline{G}}) = \mathbb{E}(\tau_{H \longrightarrow \overline{G}}) - \mathbb{E}(\tau_{\gamma_i \longrightarrow \overline{G}})^2\,.$$

We have an estimate for the second term, so let's compute the first one. Again $r_G(H) = 0$. There are four types of quotient contributing to the critical graphs.

1. Image of H are two circles disjoint from G. There are $\sigma(k_i)^2$ such graphs.

2. Both circles of H map to a single circle disjoint from G. There are $D(k_i) = \sum_{d|k_i} d$ such graphs as we need to specify the size of the circle and the distance by which are the images of the two circles shifted.

3. One of the circles maps to G and the other remains disjoint. There are $2\sigma(k_i)\tau_{\gamma_i \longrightarrow G}$ such critical graphs.

4. Both circles map to G. There are $\tau^2_{\gamma_i \longrightarrow G}$ such critical graphs.

Add up all these contributions.

$$\begin{aligned}\mathbb{E}(\tau_{H \longrightarrow \overline{G}}) &= \sigma(k_i)^2 + D(k_i) + 2\sigma(k_i)\tau_{\gamma_i \longrightarrow G} + \tau^2_{\gamma_i \longrightarrow G} + \mathscr{O}(n^{-1}) \\ &= (\sigma(k_i) + \tau_{\gamma_i \longrightarrow G})^2 + D(k_i) + \mathscr{O}(n^{-1})\end{aligned}$$

If we plug it into the expression for variance, most terms cancel out.

$$\begin{aligned}\mathrm{Var}(\tau_{\gamma_i \longrightarrow G}) &= (\sigma(k_i) + \tau_{\gamma_i \longrightarrow G})^2 + D(k_i) + \mathscr{O}(n^{-1}) - (\tau_{\gamma_i \longrightarrow G} + \sigma(k_i) + \mathscr{O}(n^{-1}))^2 \\ &= D(k_i) + \mathscr{O}(n^{-1}).\end{aligned}$$

Let's now compute the number of lifts of G_i. If $b_i \neq 0$, then $\chi_G(G_i) \geq 0$, because we can send G_i to G. Also, $\chi_G(G_i) \leq 0$ since no component of a quotient of $G \sqcup G_i$ is simply connected.

Suppose K is a quotient of $G \sqcup G_i$ such that $G \longrightarrow K$ is an injection. Let L be $q(G_i) \setminus q(G)$, where q is the quotient map. There may be open edges in L, so it is not necessarily a graph. Then $\chi_G(K) = V(L) - E(L)$. If K is a critical graph, then $V(L) = E(L)$. If L is non-empty, it must contain a component L' with $V(L') \geq E(L')$. The component L' is either a tree, a tree minus a leaf, or a rank 1 graph. If a component of L is a genuine graph, then it is also a component of K. Such a component of K is a locally injective quotient of G_i and therefore has rank at least 2. If L' is a tree minus a leaf, then another leaf of L' is a leaf of K. This is impossible since all vertices in $G \sqcup G_i \sqcup G_i$ have valence at least 2 and the quotient map is locally injective. Therefore L is empty and the critical graphs are precisely the quotients arising from the maps from G_i to G. The number of critical graphs is $\tau_{G_i \longrightarrow G}$ and we can use Lemma 3.6.5 to express the expected number of lifts of G_i to a completion of G. Therefore,

$$\mathbb{E}(\tau_{G_i \longrightarrow \overline{G}}) = \tau_{G_i \longrightarrow G} + \mathscr{O}(n^{-1}) = \sum_j b_j \tau_{G_i \longrightarrow G_j} + \mathscr{O}(n^{-1}).$$

Similarly, we can compute the variance using Lemma 3.6.7. We'll need to estimate $\tau_{(G_i \sqcup G_i) \longrightarrow \overline{G}}$. The relative rank of $r_G(G_i \sqcup G_i)$ is at least 0 because we can send both G_i's to a copy of G_i in G. It can't be less, since no component of a quotient of $G \sqcup G_i \sqcup G_i$ is simply connected.

Suppose K is a critical graph and $L = q(G_i \sqcup G_i) \setminus q(G)$ is non-empty. Then there exists a component L' of L, which is either a tree, a tree minus a leaf, or a rank 1 graph. If L' is a tree or a rank 1 graph, then it is a component of a quotient of $G_i \sqcup G_i$. However, the components of quotients of $G_i \sqcup G_i$ have rank at least 2. If L' is a tree minus a vertex, then another leaf of L' is a leaf of K. This is impossible because K is a locally injective quotient of a graph with minimal valence 2. Therefore L is empty, and $G_i \sqcup G_i$ maps to G in any critical quotient.

There are $(\sum_j b_j \tau_{G_i \longrightarrow G_j})^2$ critical graphs, because we need to specify the images of two copies of G_i.

Hence,

$$\mathbb{E}(\tau_{(G_i \sqcup G_i) \longrightarrow \overline{G}}) = (\sum_j b_j \tau_{G_i \longrightarrow G_j})^2 + \mathscr{O}(n^{-1})\,.$$

The leading terms cancel out and we are left with a variance that goes to 0 as n goes to infinity.

$$Var(\tau_{G_i \longrightarrow \overline{G}}) = \mathbb{E}(\tau_{(G_i \sqcup G_i) \longrightarrow \overline{G}}) - \mathbb{E}(\tau_{G_i \longrightarrow \overline{G}})^2 = \mathscr{O}(n^{-1})$$

Eventually, the goal is to separate subgroups using distinct numbers of fixed points. In order to do this, I need the following technical lemmas, which promotes groups commensurable to subgroups to actual subgroups. The first lemma says that a core of a finite index subgroup is a cover of a core.

Lemma 3.6.9. *Suppose $A, B < F_k$ are finitely generated subgroups and A has a finite index in B. Then* $\mathrm{Core}(A)$ *is a degree $[B : A]$ cover of* $\mathrm{Core}(B)$ *and in particular* $\frac{|V(\mathrm{Core}(A))|}{|V(\mathrm{Core}(B))|} = [B : A]$.

Proof. Let X_A and X_B be the covers of R_k associated to A and B. Let $p : X_A \longrightarrow X_B$ be the covering map. Let $d = [B : A]$. Suppose $e \in E(X_A)$ with $p(e) \in \mathrm{Core}(B)$. Then there exists some loop in $\mathrm{Core}(B)$ containing $p(e)$. The d-th power of this loop lifts to a loop in X_A, which contains e, and hence $e \in \mathrm{Core}(A)$. The restriction $p_{\mathrm{Core}(A)}$ is a local homeomorphism which covers $\mathrm{Core}(B)$ evenly and $\mathrm{Core}(A)$ is a cover of $\mathrm{Core}(B)$. □

Lemma 3.6.10. *If H_1, H_2 are finitely generated subgroups of a free group, $G < H_1 \cap H_2$ has finite index in H_1, and all divisors of $[H_1 : G]$ distinct from 1 are larger than $|V(\mathrm{Core}(H_2))|$, then H_1 is a subgroup of H_2.*

Proof. Consider $\mathrm{Core}(H_1 \cap H_2)$. We can get it as a component of the pullback of the maps $\mathrm{Core}(H_i) \longrightarrow X$, where X is the rose R_k. The pullback contains $|V(\mathrm{Core}(H_1)||V(\mathrm{Core}(H_2)|$ vertices, therefore

$$|V(\mathrm{Core}(H_1 \cap H_2)| \leq |V(\mathrm{Core}(H_1)||V(\mathrm{Core}(H_2)|\,.$$

The group G is a finite index subgroup of $H_1 \cap H_2$, which is a finite index subgroup of H_1. By lemma 3.6.9 applied to $H_1 \cap H_2 < H_1$ and to $G < H_1 \cap H_2$, $|V(\text{Core}(H_1))|$ divides $|V(\text{Core}(H_1 \cap H_2)|$, which divides $|V(\text{Core}(G))|$. Then $|V(\text{Core}(H_1 \cap H_2)| = d|V(\text{Core}(H_1)|$, where d divides $\frac{|V(\text{Core}(G))|}{|V(\text{Core}(H_1))|} = [H_1 : G]$. But every nontrivial divisor of $[H_1 : G]$ is larger than $|V(\text{Core}(H_2))|$, so $|V(\text{Core}(H_1 \cap H_2)| = |V(\text{Core}(H_1)|$. Since $\text{Core}(H_1 \cap H_2)$ is a covering of $\text{Core}(H_1)$, the two graphs are in fact equal and $H_1 \cap H_2 = H_1$. □

Finally, we can put everything together in the proof of the following separability property, which can be thought of as an 'alternating' refinement of subgroup into-conjugacy separability. We will do this by using that whenever H_1 is conjugate into H_2, it fixes at least as many elements as H_2. We will also use that the same is true for concrete characteristic subgroups. For example suppose H_1 is not conjugate into H_2 and H_2 is not conjugate into H_1. If H_1 fixes more points than H_2, then $f(H_2)$ is not conjugate into $f(H_1)$. If additionally the intersection of all degree 2 subgroups of H_2 fixes more points than the intersection of all degree 2 subgroups of H_1, then $f(H_1)$ is not conjugate into $f(H_2)$.

Theorem 3.6.11. *Suppose $H_1, H_2, \ldots, H_n < F_r$ are finitely generated subgroups of infinite index. Then there exists a surjective homomorphism $f : F_r \twoheadrightarrow A_m$ such that whenever H_i is not conjugate into H_j, then $f(H_i)$ is not conjugate into $f(H_j)$.*

Proof. Denote the relation of 'is conjugate into' by '$\prec$'. Conjugacy classes of finitely generated subgroups of F_r form a poset with respect to $\prec$ so after reordering and removing duplicates, we may assume that $H_i \prec H_j$ implies $i \leq j$.

Let $p_1, p_2, \ldots, p_n$ be primes larger than $\max_i(V(\text{Core}(H_i))$ with $p_j > p_k^{(k!)^{rkH_k}} V(\text{Core}(H_k))$ whenever $j < k$. Let $G_{i,j}$ be the intersection of all index p_j subgroups of H_i. Let graph G be a union of a_i copies of $\text{Core}(G_{i,i})$, where a_i's are to be specified later. Let $f : F_r \longrightarrow A_m$ be a random map arising from a random completion of G. The group $f(G_{i,j})$ is the intersection of all index p_j subgroups of $f(H_i)$. Indeed, every index p_j subgroup of $f(H_i)$ is an image of an index p_j subgroup of H_i.

If $f(H_i) \prec f(H_j)$, then $fix(f(H_i)) \geq fix(f(H_j))$, but also $f(G_{i,k}) \prec f(G_{j,k})$ and hence $fix(G_{i,k}) \geq fix(G_{j,k})$.

By Example 3.6.8 for every ε there exists $K = K(\varepsilon)$ independent of $a_1, \ldots, a_n$ such that for all sufficiently large m

$$\mathbb{P}(\forall i, j, |fix(G_{i,j}) - \Sigma_k a_k \tau_{\text{Core}(G_{i,j}) \longrightarrow \text{Core}(G_{k,k})}| < K) > 1 - \varepsilon \tag{3.1}$$

In words, the number of fixed points of $G_{i,j}$ belongs with high probability to a specific interval of length $2K$. By controlling the center of the interval, I will ensure that these groups often fix distinct numbers of elements.

If $\tau_{\mathrm{Core}(G_{i,j}) \longrightarrow \mathrm{Core}(G_{k,k})} > 0$, then $G_{i,j} < G_{k,k}^g$ for some g. Both H_i and $G_{k,k}^g$ are subgroups of a free group, and the index of $G_{i,j} < H_i \cap G_{k,k}^g$ in H_i is a power of p_j. The core of $G_{k,k}$ contains at most $p_k^{(k!)^{rkH_k}} V(\mathrm{Core}(H_k))$ vertices. If $j < k$, then $p_j > p_k^{(k!)^{rkH_k}} V(\mathrm{Core}(H_k))$ and by Lemma 3.6.10 $H_i < G_{k,k}^g$. This is a contradiction since the girth of $\mathrm{Core}(H_i)$ is at most $V(\mathrm{Core}(H_i))$ and the girth of $\mathrm{Core}(G_{k,k})$ is at least $p_k > V(\mathrm{Core}(H_i))$.

We also have have $p_j > V(\mathrm{Core}(H_k))$, so Lemma 3.6.10 applied to $H_i, G_{i,j}$ and H_k^g gives that $H_i \prec H_k$.

Let K be such that the probability in Equation 3.1 is at least $p = 1 - 2^{-r-1}$. Let $a_1, \ldots, a_n$ satisfy $a_j > na_{j-1}C + K$, where $C = \max_{i,j,k} \tau_{\mathrm{Core}(G_{i,j}) \longrightarrow \mathrm{Core}(G_{k,k})}$.

All of the following is simultaneously true with probability at least $1 - 2^{-r-1}$. For all j, $fix(G_{j,j}) \geq a_j$. For all i, j, if H_i is not conjugate into H_j, then $\mathrm{fix}(G_{i,j}) \leq \max(0, (j - i)a_{j-1}C + K) < a_j$. Hence $f(H_i)$ is not conjugate into $f(H_j)$.

The probability that the image is A_m tends to 2^{-r} as m goes to infinity (Theorem 3.2.3). In particular, there exists a map f with the described separating properties. □

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