Abstract

This paper sets out to provide a risk-management tool (namely the distribution of the stock price of a warrant-issuing firm) and at the same time resolves an outstanding issue between the theory and the empirical evidence of the warrant pricing literature. In their seminal article on warrant pricing, Galai and Schneller (1978) make the following statement: "…if the distribution of the firm's liquidation value is lognormal, the value of its share price is *not* lognormally distributed". On the other hand recent empirical studies suggest that assuming lognormality for the stock price distribution of a warrant-issuing firm gives a very good approximation for the value of a warrant (this is the so-called "option-like" warrant valuation approximation). In this paper we derive the "theoretical" distribution of the stock price for a warrant-issuing firm and show that dilution is reflected and incorporated in the underlying stock price prior to expiration. We also show that despite of the fact that the (risk-neutral) distribution of a warrant-issuing firm and a non-warrant issuing firm is different, valuation by taking expectations of the discounted payoff of the warrant over the two different risk-neutral distributions produces warrant prices very close to each other for a large number of cases even when the log-stock price distribution of the warrant-issuing firm exhibits marked skewness and kurtosis. Exceptions occur for *deep-out-of-the-money* and *close to maturity out-of-the-money* warrants in general. In such cases the "option-like" approximation will significantly overprice warrants. The distinction we make in this paper between warrants and executive stock options is simply a matter of whether the contract is traded or not. We use the term warrant to cover both cases.

THE IMPLIED DISTRIBUTION FOR STOCKS OF COMPANIES WITH WARRANTS AND/OR EXECUTIVE STOCK OPTIONS

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May 2002

1. INTRODUCTION

A natural application for option pricing models is the valuation of warrants. However, unlike call options, warrants and executive stock options are written by companies on their own stock and form part of their equity. When exercised, they increase the number of outstanding shares in the firm and thus have a dilution effect.[1] The textbook treatment of warrant valuation therefore mandates that instead of taking the stock price of the firm as the underlying asset, one should use the total equity value of the firm as the state variable[2]. To illustrate; in the classical Black-Scholes framework for option valuation it is assumed that the stock price process is governed by a Geometric Brownian Motion (or lognormal diffusion). In a similar spirit, a Black-Scholes framework for warrant valuation assumes that Geometric Brownian Motion is the process governing the total equity value of the firm (instead of the stock process; we illustrate why this is the case in section 2 below). But then if lognormal diffusion is the process governing the equity value of the firm what is the process governing its stock price? And even more importantly, what is the distribution of the stock price of this warrant-issuing firm? Answering this question is the purpose of this paper.

The first insight is given by Galai and Schneller (1978): "…if the distribution of the firm's liquidation value (total equity) is lognormal, the value of its share price is not…".[3] Knowledge of the distribution of the stock price of a warrant issuing firm is very important for a number of reasons such as risk management purposes (e.g. Value-at-Risk calculations), credit management purposes (e.g. estimating the probability of default of a firm)[4], not to mention the fact that as noted by many authors (Galai and Schneller (1978), Galai (1989), Sidenius

[1] Other minor differences between call options and warrants could regard adjustment for dividends or the possibility that expiration dates of certain warrants may be changed.
[2] In other words a warrant is to be regarded as a call option on a share of the total equity of the firm, where equity is defined as the sum of the value of its shares and the value of its warrants (originally suggested by Black and Scholes (1973)).
[3] The same observation is made by Sidenius (1996).
[4] In this case the debt of the firm should be also incorporated in its capital structure.

(1996)) the value of a warrant is equal to the value of a call option on the *stock of the warrant-issuing firm*. In fact this lack of knowledge of the distribution of the stock price was one of the motivations that led Galai and Schneller to price a warrant by reference to the value of a call option on the stock of an identical firm without warrants, adjusted by a dilution factor. Or putting this last statement in another way where no reference to "identical" firms is made, the state variable is nothing else than the total equity value of the warrant-issuing firm, which for simplicity and tractability can be assumed to be lognormally distributed (see for example Schulz and Trautmann (1994)).

Over the past decade, empirical literature on warrant pricing (see for example Bensoussan, Crouhy, and Galai (1992), Schulz and Trautmann (1994), Sidenius (1996)) has suggested that there is no need to follow the textbook treatment to value warrants (i.e. the equity value approach). Instead, based on their empirical results and simulations these studies recommend "option-like" warrant valuation. This simply means valuing the warrant as if it was identical to a call option, without involving in the valuation process the (typically unobservable) value of the firm and ignoring any dilution. Surprisingly, such an *approximation* works very well for a large number of cases (see Section 2.2 for more details). Indeed Cox and Rubinstein (1985) and Ingersoll (1987) have long ago recognised that when dilution is sufficiently small then it may me ignored. Moreover, in harmony with the findings of Schultz and Trautmann (1994), we illustrate in this paper that dilution can often also be ignored when the dilution potential is high. Note however that this result is not universal and we present a few cases where the practice of "option-like" valuation will lead to overpricing. Regarding now the size of the dilution factor, there appear to be no specific limits placed upon the total number of options granted by corporations. In fact, over time, the total granted can rise to a considerable percentage, often well in excess of 10% (or 15%) of the issued capital.

In relation to the above paragraphs, it is important to stress here that warrant valuation theory does *not* state that "option-like" valuation is wrong. On the contrary it states that such an approach is equivalent to the equity value approach. However in order for this to be exact, one should use the process of the stock price of *the warrant-issuing firm*. This process is not the same as, for example, the process of the stock price of an otherwise identical *non-warrant-issuing-firm*. Thus the nature of the so-called "option-like" warrant valuation approximation is that it takes the distribution of the stock price of a *non-warrant-issuing-firm* as an approximation to the distribution of the stock price of a *warrant-issuing firm*.

Moreover, the processes followed by the stock price and the total equity of the warrant-issuing firm cannot, in general, be of the same form. Indeed as put by Sidenius (1996), once the total equity process is given, the stock process is completely fixed and vice-versa and one cannot independently specify the form of both processes. In particular, to assume that they are both Geometric Brownian Motion is inconsistent, except in special cases (e.g. no outstanding warrants). In this paper we show that the initial assumption that the equity value of the firm is lognormal leads to a process for the stock price with stochastic drift and variance parameters. Moreover the volatility of the stock process depends on the "moneyness" and time to maturity of the warrants and exhibits a "smile or smirk" similar to the one observed in options markets. This agrees with Schulz and Trautmann (1994) where they suggest that even if the volatility

of the equity value is constant the stock volatility will be non-stationary. Note however an exception: For the special case where the warrant is at-the-money, as is sometime the case with executive stock options,[5] an interesting result lurks. We show that both the equity value and the stock price are driven by Geometric Brownian Motions and are thus lognormally distributed. Although convenient, the result is nonetheless quite restrictive since it is only locally (instantaneously) applicable or requires the warrant to be permanently at-the-money. Departures from that assumption imply that the lognormal distribution will be just an approximation for the true distribution of the stock price and in fact an increasingly bad one as the warrant moves progressively away from the money. We therefore go on next to derive the "true" distribution for the stock price of a warrant-issuing firm. This distribution must already reflect the potential dilution of equity thus making the warrant identical to a call option on the stock price of the warrant-issuing firm.

To sum up, we are faced with two phenomena. On the one hand we have Galai and Schneler (1978) indicating that the distribution of the stock price of a warrant issuing firm is not lognormal. On the other hand we have empirical studies suggesting that assuming lognormality for the stock price distribution gives a very good approximation for the value of a warrant. Surely something interesting is going on and requires investigation. This is the gap in the literature that this paper sets out to fill.

The structure of the paper is as follows. In section 2 we set up a framework for warrant pricing. We start with a discussion of the theoretical approaches that exist for valuing warrants. In particular, we show that the value of a warrant can be written either as an option on a share on the total equity value of the firm or as an option on its stock price and we elaborate on some interesting insights that arise from this relationship. We also present some approximations that have been suggested for the valuation of warrants. We then go on to derive the process that governs the stock price of a warrant-issuing firm (based on the fact that we have previously assumed that the value process is a Geometric Brownian Motion). Following is a subsection that illustrates how the unobserved value and volatility, (and drift) of an executive stock option issuing firm can be inferred by solving a system of simultaneous equations. We end the section with a note on the non-tradability of the value of an executive-stock-option-issuing firm and the implications of this on no-arbitrage pricing and hedging arguments. In section 3 we go on to derive the "true" distribution of the stock price for our warrant-issuing firm. We do this not by solving the stochastic process that governs the firm's stock price, but by exploiting the monotonicity of the stock price with the equity value of the firm and applying a non-linear transformation. In section 4 we relax an assumption that we made in section 3. The assumption was that the volatility of the equity value of the firm is observable/inferable. We relax this assumption by utilising a Bayesian approach where the diffusion parameters for the value process are treated as random variables and can therefore be integrated out or eliminated via non-linear transformations. Section 5 illustrates empirically

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[5] See for example Noreen and Wolfson (1981) where they state that executive stock options are typically issued with the exercise price equal to the stock price. (p. 385, Footnote 2). For the economic rationale behind this practice see Hall and Murphy (2000) where the authors show that by setting the exercise price at or near the grant-date market price corporations maximise the incentives of their employees and executives.

what has been said in the previous sections. Concluding remarks and issues for further research follow in section 6.

2.   A FRAMEWORK FOR WARRANT PRICING

2.1 The Textbook Treatment for the Valuation of Warrants

Warrants and executive stock options are written by companies on their own stock. When they are exercised, the company issues more stock and sells them to the option holder for the strike price. This subsequently dilutes the equity of existing shareholders. A way out of this, illustrated below and originally suggested by Black and Scholes (1973) and also discussed in Lauterbach and Schultz (1990), Schulz and Trautmann (1994) is to regard a warrant as a call option on a share of the total equity of the firm where equity is defined as the sum of the value of its shares and the value of its warrants. In this case the underlying state variable is the total equity value of the firm and in the absence of arbitrage, the value of a warrant is shown to be equal to the value of a call option on the equity of the firm multiplied by a dilution factor equal to $1/(1+\lambda)$ [6]. One difficulty that arises with this approach is that the value and volatility of the firm's equity is not directly observable. It is possible however, as we illustrate later, to infer these values using numerical routines.

An alternative way is to price a warrant as a call option on the stock of the warrant-issuing firm. This of course requires knowledge of the distribution of the stock-price of the warrant-issuing firm. In this case the potential dilution should already be reflected in the stock price and thus there is no need for a specific dilution adjustment. Hence the warrant is identical to a call option on the stock price of the firm. Although this approach has never been used in empirical work for valuing warrants (due to the lack of knowledge of the stock price distribution) it raises important issues in option pricing and suggests that using for example the Black-Scholes model to price exchange-traded stock options on firms that happen to issue warrants may result in mispricing since the wrong process (i.e. the lognormal) is assumed for the state variable (i.e. the stock price). Or in other words, two stock options, one on a firm with warrants and the other on an identical firm without warrants should normally sell for different prices since their price processes will typically be different.[7] Looking at the bigger picture, this implies that when pricing a stock option, the capital structure of the firm should be modelled and reflected in the stock price process. We leave an investigation on the impact of the capital structure of a firm on its stock price distribution for a forthcoming paper.

The distinction we make in this paper between warrants and executive share options is simply a matter of whether the contract is traded or not. Moreover we take the term executive stock options to contain the popular subset of employee share option schemes which have been successfully introduced amongst corporations during the past few years. Issues of non-

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[6] λ represents the ratio of the total number of new shares issued upon exercise of the warrants to the total number of existing shares.
[7] Note though that it is possible for different processes for the state variable to have the same distribution.

transferability, delayed vesting and exercise policies of Executive Stock Options (ESO's) are not discussed in this paper. For these (and a few other) differences between exchange-traded options and ESO's we refer the reader to Rubinstein (1995). Here it suffices to say that the reduction in value of an executive option due to the non-transferability constraint is usually handled my multiplying the value that would otherwise be obtained (i.e. the value of a traded warrant) by one minus the probability that the executive/employee will leave the firm before exercise is possible. The default in this study will be that the probability of "premature leave" is zero (it is straightforward however to modify this assumption).

Regarding now the delayed vesting constraint or exercise policy of ESO's in general, it is indeed the case that most option plans do not permit employees to exercise their granted options until after a predefined period of time has elapsed (for example, an executive option has typically a maturity of 10 years; however through delayed vesting, exercise is usually not permitted for a period after grant, typically 3 years). In other words ESO's are neither European nor American. In its Exposure Draft, "Accounting for Stock-based Compensation," FASB allows valuing ESO's as European but requires using the so called "expected life of the option" instead of the actual time to expiration. Carpenter (1998) defines the cost of an ESO as the market value of the option to an unrestricted outside investor and examines the exercise policies of managers who are subject to transferability and hedging constraints. She shows that early exercise of an ESO is not consistent with exercise patterns observed in the data. Executives hold options long enough and deep enough into the money before exercising to capture a significant amount of their potential value. In another note now, ESO's are particularly popular amongst high-tech firms. Such corporations promise rapid growth but also pay little or no dividends. (see amongst many others Microsoft Corporation, Oracle, Cisco Systems, AoL Time-Warner, etc). This has the advantage that American ESO's can be valued as European. Generally, in this study we shall use the term warrant to cover all cases (i.e. warrants, executive options and employee options). We start first with a simplifying assumption:

ASSUMPTION 1: *The warrant-issuing firm is an all equity firm with no outstanding debt.* [8]

Assumption 1 implies that stocks and warrants are the only sources of financing that the company is using. Hence, the company has a current total equity value *V* of:

$$V = NS + nW(S) \tag{2.1.1}$$

where

N: The number of outstanding shares.

S: The price per share at time *t*.

n: The number of outstanding warrants

$W(S)$: The price of a warrant on a share at time *t*. (Occasionally we will just write *W* to

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[8] We make this assumption for simplicity. Note however that this approach can be extended to the more realistic case where the firm also has debt in its capital structure.

denote the value of the warrant).

Similarly the value of the firm per share (total equity per share) $v = V/N$ is:

$$v = S + \lambda W(S) \tag{2.1.2}$$

where $\lambda = n/N$ is the dilution factor.

Suppose now that each warrant entitles the holder to purchase one share of the firm at time T for a strike price of K per share.[9] If the warrants are exercised the company receives a cash inflow of nK and the total equity value increases to $V_T + nK$. This value is then distributed among $N + n$ shares so that the price per share immediately after exercise becomes

$$S_T = (V_T + nK)/(N + n) \tag{2.1.3}$$

Hence, if the warrant is exercised, the payoff to the warrant holder is

$$\frac{V_T + nK}{N + n} - K = \frac{N}{N + n}\left(\frac{V_T}{N} - K\right) = \frac{1}{1 + \lambda}\left(\frac{V_T}{N} - K\right) = \frac{1}{1 + \lambda}(v_T - K) \tag{2.1.4}$$

Of course the warrant will be exercised only if $v_T > K$. Hence the payoff to the warrant holder at maturity is

$$\frac{1}{1 + \lambda}(v_T - K)^+ \tag{2.1.5}$$

We have therefore shown that the value of the warrant $W(S)$ is equal to the value of $1/(1 + \lambda)$ call options on a share on the total equity value of the firm v:

$$W(S) = \frac{1}{1 + \lambda}C(v) \equiv \frac{1}{1 + \lambda}C(v, \tau, K, \sigma_v, r) \tag{2.1.6}$$

where $C(...)$ denotes a call option valuation function, τ, σ_v, K, and r represent the (non-stochastic) time to maturity, equity volatility[10], strike price, and risk-free rate respectively.

LEMMA 1: *As noted in Galai and Schneller* (1978) *the value of a warrant is also equal to the value of a call option on the stock of the warrant-issuing firm. The exercise value of such a call at maturity is*

$$(S - K)^+ = \frac{1}{1 + \lambda}(v_T - K)^+ \tag{2.1.7}$$

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[9] We make this assumption for simplicity. It is straightforward to amend it and assume that each warrant entitles the holder to purchase p numbers of shares.

[10] For clarity we should note here that when we refer to equity volatility or stock volatility we don't really mean the absolute volatility of the equity or the stock but rather the volatility of the rate of return on the equity or stock respectively.

which is the same as the maturity value of the warrant.

However in this case the valuation procedure requires knowledge of the distribution of the stock price S of the warrant-issuing firm. Moreover since the process for the stock price is not, in general, lognormally distributed the Black-Scholes valuation formula will no longer be valid. Assuming no arbitrage, the value of the warrant W is given in this case by

$$W(S) = e^{-r\tau} E^Q((S_T - K)^+ \setminus \Im_t) = e^{-r\tau} \int_K^\infty (S_T - K)^+ f_{RN}(S_T \setminus S_t) dS_T \qquad (2.1.8)$$

or

$$W(S) = D(S) \equiv D(S, \tau, K, \sigma_S, r) \qquad (2.1.9)$$

where

$D(...)$ denotes a call option valuation function, not necessarily the Black-Scholes.

σ_S denotes the stock return volatility.

Q is the risk-neutral (martingale) measure for the discounted stock price (if it exists; we show later that it does).

$\Im_t$ is the filtration of the stock price at time t.

$f_{RN}(S_T \setminus S_t)$ is the risk-neutral conditional distribution of the stock price S_T given the security price at time t.

Note that we have not yet made any distributional (or process) assumptions for *v* or for *S*. We have shown however that the value of a warrant *W* can be written either as an option on a share on the value of the firm *v*:

$$W(S) = \frac{1}{1+\lambda} C(v)$$

or as an option on its stock price *S*:

$$W(S) = D(S)$$

It is worth noting that $C(...)$ and $D(...)$ represent two different option pricing functions: $C(...)$ is derived based on the process followed by *v*, while $D(...)$ depends on the process followed by *S*. Of course as already mentioned and as it will be shown below once one of the processes is specified, the other will be derived as a by-product and thus be completely fixed as well.

2.2. Some Suggested Approximations for the Valuation of Warrants

To value warrants via the formula $W(S) = (1/(1+\lambda))C(v, \tau, K, \sigma_v, r)$ one needs to estimate the value *v* and volatility σ_V of the firm. For the case of traded warrants this is a straightforward task. For the case of executive stock options however this can be tricky, since *v* is unobservable, and requires numerical inference. (We discuss this issue in Section 2.6

below). Similarly valuing warrants via the formula $W(S) = D(S)$ requires knowledge of the process or the distribution of the stock price. The latter has not so far been available.

As a result, academics and practitioners have proposed approximation techniques, not all necessarily correct. For example, as noted by Galai (1989) and Crouhy and Galai (1991) a common mistake that is often made in practice is that warrant prices are calculated as

$$W(S) = \frac{1}{1+\lambda} C(S, \tau, K, \sigma_S, r) \tag{2.2.1}$$

This procedure, which is based upon a misinterpretation of the Galai and Schneller (1978) result, is incorrect and will lead to (downward) biased warrant values. Galai and Schneller (1978) suggested that a warrant can be priced by reference to the value of a call option on the stock of an "identical" firm without warrants, adjusted by a dilution factor. Since however "identical" firms hardly ever exist, the correct interpretation is that (in the absence of "identical" firms) the state variable should not be the stock price but the total equity value of the warrant-issuing firm.

An alternative approximation that seems to work quite well is what has come to be known as "option-like" warrant valuation: A number of empirical studies (see for example Bensoussan, Crouhy and Galai (1992), Schulz and Trautmann (1994), Sidenius (1996)) have suggested that there is no need to follow the textbook treatment to value warrants. Instead, based on their empirical results (and simulations) they suggest that warrants could just as well be valued as

$$W(S) = C(S, \tau, K, \sigma_S, r) \tag{2.2.2}$$

For example Schulz and Trautmann (1989) conclude: "To obtain warrant values with acceptable accuracy, adjustments to the Black/Scholes formula are not needed except perhaps for deep-out-of-the-money warrants." (In their study C(…) is assumed to be the Black-Scholes option pricing formula). Along the same lines is the conclusion of Sidenius (1996): "… the warrant prices obtained using the stock price method are identical to the results arising from the textbook method."

Although "option-like" warrant valuation is only an approximation and has no theoretical foundation it appears to work quite well. According to Veld (1994):

"This result can be explained by looking at the modifications of the precise warrant valuation model in relation to the original Black-Scholes formula:
1) The stock price S is replaced by the equity value per share of common stock v.
2) The standard deviation of the returns on common stock (σ_S) is replaced by the standard deviation on the firm's equity (σ_V)
3) The entire formula is multiplied by the dilution factor.
The fact that only marginal differences exist can be attributed to the fact that the effects from replacing S by v (modification 1) and σ_S by σ_V (modification 2) are outweighed by the multiplication by the dilution factor (modification 3)."

But is there a theoretical justification to support this approximation? The theoretical prerequisite for "option-like" warrant valuation to work well is not hard to infer. Simply the underlying (risk-neutral) distributions behind the option pricing functions $C(S, \tau, K, \sigma_S, r)$ and $D(S, \tau, K, \sigma_S, r)$ (i.e. behind the processes of the stock price of a non-warrant-issuing and a warrant-issuing firm respectively) should behave in such a way so that taking expectations of the discounted payoff of the warrant (over the two risk-neutral distributions) should produce near identical values. Indeed, later in the paper (Section 5) we will illustrate that despite of the fact that the risk-neutral distributions of a warrant-issuing firm and a non-warrant issuing firm are different, valuation by taking expectations of the discounted payoff over the two different distributions produces warrant prices very close to each other for a large number of cases. Note however that we will also provide counter-examples as a word of caution, where "option-like" approximation might not always work well and result in "poor" warrant prices and large mispricing errors.

2.3 Insights arising from the Textbook Treatment Valuation of Warrants

THEOREM 1: *Assuming no arbitrage, the following relationship between the two option pricing functions C(…) and D(…) must hold:*

$$C(v) = (1 + \lambda)D\left(v - \frac{\lambda}{1+\lambda}C(v)\right) \qquad (2.3.1)$$

or rearranging (2.3.1) *and writing it in terms of S:*

$$D(S) = \frac{1}{1+\lambda}C(S + \lambda D(S)) \qquad (2.3.2)$$

This relationship is independent of distributional assumptions.

PROOF: Equating equations (2.1.6) and (2.1.9) we get:

$$D(S) = \frac{1}{1+\lambda}C(v) \qquad (2.3.3)$$

or

$$C(v) = (1 + \lambda)D(S) \qquad (2.3.4)$$

Now equations (2.1.2) and (2.1.6) imply:

$$v = S + \frac{\lambda}{1+\lambda}C(v) \qquad (2.3.5)$$

Similarly equations (2.1.2) and (2.1.9) imply:

$$v = S + \lambda D(S) \qquad (2.3.6)$$

Then (solving (2.3.5) and (2.3.6) in terms of S)

$$S = v - \lambda D(S) = v - \frac{\lambda}{1+\lambda} C(v) \qquad (2.3.7)$$

Substituting (2.3.7) in (2.3.4) we get (2.3.1). Similarly substituting (2.3.6.) in (2.3.3) we get (2.3.2).

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EXAMPLE 1: *Consider an option pricing formula linear in (a share on) the equity value of the firm v; i.e. $C(v) = a + bv$. Then from equation (2.3.2)*

$$D(S) = \frac{1}{1+\lambda} \big(a + b(S + \lambda D(S)) \big)$$

Solving for $D(S)$ we get:

$$D(S) = \frac{a}{1+\lambda - \lambda b} + \frac{b}{1+\lambda - \lambda b} S \;\; \rightarrow \;\; D(S) = \frac{1}{1+\lambda - \lambda b} C(S)$$

Hence $D(S)$ is also linear in S. Similarly if we start from an option pricing formula linear in S; i.e. $D(S) = a + bS$, using equation (2.3.1) we get

$$C(v) = (1+\lambda)\left(a + b\left(v - \frac{\lambda}{1+\lambda} C(v) \right) \right)$$

Solving for $C(v)$ we get

$$C(v) = \frac{(1+\lambda)a}{1+\lambda b} + \frac{(1+\lambda)b}{1+\lambda b} v \;\; \rightarrow \;\; C(v) = \frac{1+\lambda}{1+\lambda b} D(v)$$

Hence $C(v)$ is also linear in v.

This example might seem out of place at this point, but it isn't. For example as is well known the Black-Scholes formula for an at-the-money option is a linear function of the underlying (for more details see Remark 1 below). Moreover the value of an option on a stock that follows a binomial process is a linear function of the stock price (i.e. the binomial model).

For the general case, (i.e. a non-linear (differentiable) option pricing formula), given $C(v)$ or $D(S)$, one can use Taylor series expansions to find an option pricing function for $D(S)$ or $C(v)$ respectively, as a power series (see for example Butler and Schachter (1986) and Knight and Satchell (1997)[11]). Assume for example that we know $C(v)$. Then provided that the regularity conditions are satisfied we can write $C(v)$ as a convergent Taylor series; i.e. $C(v) = \sum \beta_i (v - v')^i$, where v' is the point of expansion (assume for simplicity it is zero). Now the unknown $D(S)$ can also be written as a polynomial function, i.e. $D(S) = \sum d_j (S - S')^j$, where S' is the point of expansion (assume for simplicity it is zero)

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[11] In both these cases, expansions are written in terms of the standard deviation.

and the d_j's are the unknown coefficients of $D(S)$ that we need to calculate. Then from equation (2.3.2) we have:

$$D(S) = \frac{1}{1+\lambda} \sum_i \beta_i \left(S + \lambda D(S)\right)^i$$

$$= \frac{1}{1+\lambda} \sum_i \beta_i \binom{k}{i} S^{k-i} (\lambda D(S))^i$$

Hence

$$\sum d_j S^j = \frac{1}{1+\lambda} \sum_i \beta_i \binom{k}{i} S^{k-i} (\lambda \sum_j d_j S^j)^i$$

Equating coefficients we obtain $D(S)$ as a (convergent) Taylor series in S.

2.4 Specifying a Process for the State Variable

It is now time to become more specific. To value the option whose payoff is given in equation (2.1.5) (i.e. whose payoff is: $[1/(1+\lambda)](v_T - K)^+$) one must assume a process for v. The literature on warrant pricing has mainly concentrated on three processes for the value of the firm. Appart from Black and Scholes's Geometric Brownian Motion which assumes constant volatility, Cox and Ross's (1976) Constant Elasticity of Variance (CEV) model (see Noreen and Wolfson (1981), Lauterbach and Schultz (1990), Schulz and Trautmann (1989)) and Merton's (1976) Jump diffusion model (see Kremer and Roenfeldt (1993)) have also been used to model the value process. In this paper we will not go into a detailed review of the literature on warrant pricing. An exhaustive study that reviews the empirical research under alternative processes has already been conducted by Veld (1994). Here it suffices to quote one of the conclusions of Veld: "There is no conclusive evidence to replace (dividend corrected) models in which a constant volatility is assumed (i.e. Black/Scholes (1973) like models) by more complicated models such as the Jump Diffusion or the CEV model."[12]

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[12] In the Constant Elasticity of Variance model (CEV), the assumption of a constant volatility is replaced by the assumption of a constant elasticity of variance. In this model it is generally assumed that the elasticity factor is defined in a way that the volatility decreases as the stock price increases. Special cases of the CEV model are the Square Root model, which assumes that the volatility is inversely related to the square root of the stock price and the Absolute model, which assumes that the volatility is inversely proportional to the stock price.

The Jump Diffusion model, developed by Merton (1976), also drops the assumption of constant volatility. The model assumes a two part stochastic process generating stock returns: (a) small continuous price movements generated by the same process as assumed by Black and Scholes and (b) large infrequent jumps generated by a Poisson process.

Of course when these models are used for warrant valuation the state variable is the equity value of the firm and not the stock price.

ASSUMPTION 2: *The (unobservable) share on the total equity value v of the firm follows a Geometric Brownian Motion:*

$$dv = \mu_V v dt + \sigma_V v dZ \tag{2.4.1}$$

where

μ_V : The expected rate of return on the value of the firm's assets.

σ_V : The standard deviation of the rate of return on the value of the firm's assets.

Z: A standard Brownian motion.

Assumption (2) and equations (2.1.5) - (2.1.6) then imply that:

$$W(S) \equiv W(v, \sigma_V) = \frac{1}{1+\lambda} C^{BS}(v, \tau, K, \sigma_V, r) \tag{2.4.2}$$

where $C^{BS}(...)$ denotes the Black-Scholes option price and v a share on the total equity value of the firm. Specifically:

$$C^{BS}(v, \tau, K, \sigma_V, r) = v\Phi(d_1) - Ke^{-r\tau}\Phi(d_2) \tag{2.4.3}$$

where

$$d_1 = \frac{\left[\ln(\frac{v}{Ke^{-r\tau}}) + \frac{\sigma_V^2 \tau}{2}\right]}{\sigma_V \sqrt{\tau}} \quad \text{and} \quad d_2 = d_1 - \sigma_V \sqrt{\tau} \tag{2.4.4}$$

Also

$\tau = T - t$: The time to maturity of the outstanding warrants

K: The exercise price of the warrants

r: The risk-free rate of interest.

$\Phi(...)$: The standard normal cumulative distribution function.

REMARK 1: *If the warrant is currently at-the-money (i.e. $K = ve^{r\tau}$), as is often the case with executive stock options (see Noreen and Wolfson (1981), then some considerable simplifications occur in the valuation formula: Equation (2.4.4) reduces to*

$$d_1 = \frac{\sigma_V \sqrt{\tau}}{2}$$

and the Black-Scholes valuation formula (2.4.3) simplifies to:

$$C_{ATM}^{BS}(v, \tau, K, \sigma_V, r) = v(2\Phi(\frac{\sigma_V \sqrt{\tau}}{2}) - 1).$$

NOTE 1: *Remark* (1) *can be interpreted as a special case where the exercise price is "stochastic" so that the relationship $K = ve^{r\tau}$ holds currently. In other words the result is only locally (instantaneously) applicable or alternatively applicable at discrete random times.*

2.5 The Process Governing the Stock Price of the Warrant-Issuing Firm

In the previous section we assumed that Geometric Brownian Motion is the process governing the equity value of the firm v. What is then the process governing the stock price S? Solving equation (2.1.2) for S we get:

$$S = v - \lambda W(S) \tag{2.5.1}$$

Also from equation (2.4.2) we have

$$W(S) = \frac{1}{1+\lambda} C^{BS}(v, \tau, K, \sigma_V, r)$$

Then the following representation for the stock price S must hold:

$$S = v - \frac{\lambda}{1+\lambda} C^{BS}(v, \tau, K, \sigma_V, r) \tag{2.5.2}$$

Now since

$$\frac{\partial S}{\partial v} = 1 - \frac{\lambda}{1+\lambda} \frac{\partial C^{BS}}{\partial v} = 1 - \frac{\lambda}{1+\lambda} \Phi(d_1) > 0,$$

there is a unique one-to-one relationship between S and v and therefore equation (2.5.2) can be inverted for v via a straightforward numerical method (e.g. Newton-Raphson). We write

$$v = \Psi(S) \tag{2.5.3}$$

where $\Psi(\ )$ is the inverse function of the relationship given in (2.5.2). In general $\Psi(\ )$ is not known explicitly. However, when the warrant is at-the-money the above equation can be written in analytic form:

$$v = \Psi(S) = \frac{(1+\lambda)S}{1 + 2\lambda - 2\lambda\Phi(\sigma_V \sqrt{\tau}/2)}$$

LEMMA 2: *Given that the equity value of the firm v follows a Geometric Brownian Motion, then the process for the stock price S is given by*

$$dS = \left((1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\mu_V \Psi(S) + \frac{\lambda}{1+\lambda} rKe^{-r\tau}\Phi(d_2) \right) dt + \left((1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\sigma_V \Psi(S) \right) dZ \tag{2.5.4}$$

where
$$d_1 = \frac{\left[\ln(\frac{\Psi(S)}{Ke^{-r\tau}}) + \frac{\sigma_V^2 \tau}{2}\right]}{\sigma_V \sqrt{\tau}}$$
and
$$d_2 = d_1 - \sigma_V \sqrt{\tau}$$

$\phi = \Phi'(...)$ *denotes the standard normal probability density function.*

PROOF: See Section 1 in the Appendix. □

COROLLARY 1: *Suppose we have an at-the-money warrant (i.e. $v = Ke^{-r\tau}$).[13] Then given that the equity value of the firm follows a Geometric Brownian Motion, the process for the stock price is also governed by a Geometric Brownian Motion:*

$$dS = \mu_V S dt + \sigma_V S dZ$$

PROOF: See Section 1 in the Appendix. □

The process given in Lemma (2), represents the process for the stock price. It is a process with stochastic drift and variance parameters. Note however that although the parameters of the process are stochastic they should only exhibit the stochastic behaviour allowed by the functional dependence on the stock price S (and time). In other words the underlying process for the stock price is of the form: $dS = \mu(S,t)dt + \sigma(S,t)dZ$. This equation describes the most general set up that goes beyond the case of a purely deterministic (time-dependent) volatility and drift, but still allows risk-neutral valuation without introducing other hedging instruments apart from the underlying itself. Thus the process given in Lemma (2) belongs to the popular class of stochastic volatility models called "restricted stochastic volatility models" (see for example Rebonato (1999)) or "level-dependent volatility models". These models have of course the advantage that they preserve market completeness. Hence a risk-neutral measure for the discounted stock price exists and as stated in Lemma (1) the value of the warrant will be given by:

$$W(S) = e^{-r\tau} E^Q((S_T - K)^+ \setminus \mathfrak{I}_t) = e^{-r\tau} \int_K^\infty (S_T - K)^+ f_{RN}(S_T \setminus S_t) dS_T \qquad (2.5.5)$$

From this it is obvious that knowledge of the risk-neutral density $f_{RN}(S_T \setminus S_t)$ is sufficient to value the warrant. To obtain the risk-neutral density one needs first to apply the Cameron-Martin-Girsanov change of measure to make the discounted stock price a martingale (see Duffie (1996), Baxter and Rennie (1997), Bjork (1998)).

One can in theory obtain the transition probabilities for the stock price by solving the stochastic differential equation (SDE) of Lemma (2) (the risk-neutral transition probabilities can similarly be obtained by solving the stochastic differential equation arising under the risk-neutral measure that makes S a martingale). This is however too hard a task. Alternatively one

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[13] This corollary is of theoretical interest only and should be interpreted in conjunction with Note 1 in the previous page.

can make do without solving the SDE. Instead shifting from it to a second order partial differential equation (PDE) is also possible. Indeed under some regularity conditions for the stock price's drift and diffusion coefficients, the transition probabilities can be obtained as fundamental solutions of Kolmogorov's backward equation or indeed Kolmogorov's forward (Fokker-Planck) equation.[14] Still however, the backward and forward equations have been explicitly solved only in a few simple cases (e.g. Black-Scholes, Ornstein-Uhlenbeck, or Cox and Ross's Constant Elasticity of Variance processes). These have usually been found by taking the Fourier and Laplace transformation of the transition probabilities. In general, one must rely on numerical methods such as solving numerically the PDE (i.e. the Kolmogorov's equations), or performing Monte Carlo integration of the SDE (i.e. equation (2.5.4)).

Finally a number of approximations have also been suggested. For example Jarrow and Rudd (1982) have proposed replacing the unknown density of the SDE by an approximate density, typically adding free skewness and kurtosis to the lognormal density, so as to allow for departures from the Black-Scholes formula. However, the disadvantage of this approach is that the approximate density ignores the underlying process of the state variable. More recently Ait-Sahalia (1999) proposed a method for obtaining a closed form approximation for the transition density. In particular he proposes transforming (standardising) the SDE of say equation (2.5.4) to an SDE with unit diffusion (say $dP = \mu_p(P,t)dt + dZ$). The point of making the transformation from S to P is that it is possible to construct an expansion for the transition density of P. Ait-Sahalia then suggests approximating the transition density of P by using a Hermite polynomial expansion around a Normal density function. Once this is done one obtains an approximation to the transition density of S as a non-linear transformation of that of P.

What we intend to do in Section 3 to derive the distribution of the stock price of a warrant-issuing firm can be viewed as similar in spirit to Ait-Sahalia (1999). However in our case we take advantage of the relationship between the stock price of a firm and its equity value per share so that our "standardised" SDE is the equity value process. By assumption this is Geometric Brownian Motion (thus there is no need to transform to unit diffusion) and therefore the transition densities of our standardised SDE are available in the well known lognormal form. Note however that although our analysis assumes lognormality for the value process, it is obvious that other cases can also be discussed.

2.6 Inferring the Value, Volatility, and Drift of the Firm

In Section (2.4), equation (2.4.2) we have shown that the value of a warrant can be calculated as:

$$W(S) \equiv \frac{1}{1+\lambda} C^{BS}(v, \tau, K, \sigma_V, r)$$

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[14] For further details and for the regularity conditions see for example Arnold (1974).

Let us briefly illustrate how v and σ_V can be inferred so that they can be used in the valuation formula. We start with equation (2.1.2) (i.e. $v = S + \lambda W(S)$). Clearly, if the warrants are traded (i.e. $W(S)$ is observed) and one believes that the traded price conveys information one can infer v (and σ_V) directly from the above equation. However if the warrants are not traded (e.g. the case of executive share options) or one wants to use the "theoretical" value of the firm to find the "theoretical value" of the warrant the following procedure is employed:

From equation (2.5.2) above we have that the following representation for the stock price S must hold:

$$S = v - \frac{\lambda}{1+\lambda} C^{BS}(v, \tau, K, \sigma_V, r) \qquad (2.6.1)$$

The paper by Schulz and Trautmann (1994) presents a methodology for arriving at both the unobserved equity value v and volatility σ_V of a warrant-issuing firm. The authors use equation (2.6.1) together with the fact that the elasticity of a stock gives the percentage change in the stock's value for a percentage change in the firm's equity value; i.e.

$$\sigma_S = \sigma_V \cdot \varepsilon_{S,V} \qquad (2.6.2)$$

where σ_S is the observable/estimable stock volatility and $\varepsilon_{S,V}$ represents the stock elasticity with respect to the value of the firm. (Relationship (2.6.2) is a standard result in option pricing theory where the stock's elasticity, $\varepsilon_{S,V}$, gives the percentage change in the stock's value for a percentage change in the firm's value; see Jarrow and Rudd (1983), p.110). They then approximate the elasticity by

$$\varepsilon_{S,V} \equiv \frac{\partial S}{\partial v} \frac{v}{S} = (1 - \lambda \frac{\partial W}{\partial v}) \frac{v}{S}.$$

Where $\frac{\partial W}{\partial v} = \frac{1}{1+\lambda} \Phi(d_1)$. Hence they obtain a system of two equations

$$S = v - \frac{\lambda}{1+\lambda} C^{BS}(v, \tau, K, \sigma_V, r)$$

$$\sigma_S = (1 - \frac{\lambda}{1+\lambda} \Phi(d_1)) \frac{v}{S} \sigma_V \qquad (2.6.3)$$

which can then be solved simultaneously for the two unknown arguments v and σ_V. We are able to independently verify/derive equation (2.6.3) from our results of the previous section. Remember that we have shown that the process for the stock price is of the form: $dS = \mu(S,t)dt + \sigma(S,t)dZ$. Then the process for the return of the stock price will be given by:

$$\frac{dS}{S} = \frac{\mu(S,t)}{S} dt + \frac{\sigma(S,t)}{S} dZ$$

where $\sigma_S = \dfrac{\sigma(S,t)}{S}$ is the stock return volatility. In particular from equation (2.5.4) we have that:

$$dS = \left(\overbrace{(1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\mu_V \Psi(S) + \frac{\lambda}{1+\lambda}rKe^{-r\tau}\Phi(d_2)}^{\mu(S,t)} \right)dt + \left(\overbrace{(1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\sigma_V \Psi(S)}^{\sigma(S,t)} \right)dZ$$

Hence

$$\sigma_S = \frac{\sigma(S,t)}{S} = (1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\frac{\Psi(S)}{S}\sigma_V = (1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\frac{v}{S}\sigma_V$$

But this is identical to equation (2.6.3) above.

It is worth repeating here that such a specification for the stock price process and subsequently volatility implies that the stock return volatility of a warrant-issuing firm will be stochastic and will depend (among other things) on the level of the stock price S and on time. Or put a bit differently the stock return volatility will depend on the "moneyness" of the warrants and on their time to maturity. In fact our derived process for the stock price implies/produces a volatility "smile or smirk" similar to the one observed in options markets. Thus, just incorporating warrants/executive options on the stock price process of the firms that happen to issue them, could help reduce some of the observed biases between theoretical and observed options prices. For a detailed discussion on the non-stationarity and general behaviour of stock volatility we refer the reader to Schulz and Trautmann (1994).

Finally it is worth noting that after having obtained values for v and σ_V it is straightforward to also obtain the growth rate (drift) of the firm μ_V. We know that

$$\mu_S = \frac{\mu(S,t)}{S} = \frac{(1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\mu_V \Psi(S) + \frac{\lambda}{1+\lambda}rKe^{-r\tau}\Phi(d_2)}{S}$$

Hence solving for the drift of the firm we get:

$$\mu_V = \frac{S\mu_S - \frac{\lambda}{1+\lambda}rKe^{-r\tau}\Phi(d_2)}{(1 - \frac{\lambda}{1+\lambda}\Phi(d_1))v} .$$

2.7 The Non-Tradability of the Value of the Firm

For the general case where the warrants of the firm are not traded (i.e. the popular case of executive share options) the value of the firm v is clearly a non-tradable quantity. Does this mean that parts of our analysis that have been based on no-arbitrage arguments are invalid? Clearly not. We illustrate below why this is the case (see also the seminal papers of Harrison and Kreps (1979) and Harrison and Pliska (1981)):

We have that the non-tradable v is modelled with the stochastic differential: $dv = \mu_V v \, dt + \sigma_V v \, dZ$. Although v is non-tradable, a deterministic function of v, $S = f(v)$ is tradable[15]. In particular

$$S = v - \frac{\lambda}{1+\lambda} C^{BS}(v, \tau, K, \sigma_V, r)$$

$$= v - \frac{\lambda}{1+\lambda}\left(v\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)\right)$$

In Lemma (2) above we have shown using Ito's formula that S has differential increment:

$$dS = \overbrace{\left((1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\mu_V v + \frac{\lambda}{1+\lambda}rKe^{-r\tau}\Phi(d_2)\right)}^{\mu(S,t)} dt + \overbrace{\left((1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\sigma_V v\right)}^{\sigma(S,t)} dZ$$

As S is tradable we can write down its market price of risk θ. Assuming that the discount rate is constant at r we have that

$$\theta = \frac{\mu(S,t) - rS}{\sigma(S,t)}$$

$$\Rightarrow \quad \theta = \frac{(1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\mu_V v + \frac{\lambda}{1+\lambda}rKe^{-r\tau}\Phi(d_2) - r\left(v - \frac{\lambda}{1+\lambda}\left(v\Phi(d_1) - Ke^{-r\tau}\Phi(d_2)\right)\right)}{(1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\sigma_V v}$$

$$\Rightarrow \quad \theta = \frac{(1 - \frac{\lambda}{1+\lambda}\Phi(d_1))v(\mu_V - r)}{(1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\sigma_V v}$$

$$\Rightarrow \quad \theta = \frac{\mu_V - r}{\sigma_V}$$

This is the market price of risk θ for the stock price of a warrant-issuing firm. Note that θ is a constant and therefore the Cameron-Martin-Girsanov boundedness condition $E_P[\exp(\frac{1}{2}\int_0^T \theta_t^2 dt)] < \infty$ is automatically satisfied. This is sufficient for the existence of the risk-neutral measure that makes the discounted equity value a martingale (the risk-neutral measure is the measure under which the discounted stock price is a martingale (i.e. a tradable)). In other words what we are saying is that using the martingale representation theorem we can construct v out of S and a riskless cash bond. Indeed the market price of risk is simply another way of writing the change from nature's measure P to the risk neutral measure Q. The behaviour of S under the risk neutral measure Q is

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[15] We also acknowledge the dependence of S on time. For clearness of the argument however we ignore that dependence for the time being.

$$dS = \left((1 - \frac{\lambda}{1+\lambda} \Phi(d_1)) \sigma_V v \right) (dZ + \frac{\mu_V - r}{\sigma_V} dt)$$

$$- (1 - \frac{\lambda}{1+\lambda} \Phi(d_1)) v(\mu_V - r) dt + \left((1 - \frac{\lambda}{1+\lambda} \Phi(d_1)) \mu_V v + \frac{\lambda}{1+\lambda} rKe^{-r\tau} \Phi(d_2) \right) dt$$

$$dS = \left((1 - \frac{\lambda}{1+\lambda} \Phi(d_1)) \sigma_V v \right) d\tilde{Z} + rv(1 - \frac{\lambda}{1+\lambda} \Phi(d_1)) dt + \frac{\lambda}{1+\lambda} rKe^{-r\tau} \Phi(d_2) dt$$

($\tilde{Z}$ is the Brownian motion under Q)

$$dS = \left((1 - \frac{\lambda}{1+\lambda} \Phi(d_1)) \sigma_V v \right) d\tilde{Z} + r(v - \frac{\lambda}{1+\lambda} (v\Phi(d_1) + Ke^{-r\tau} \Phi(d_2))) dt$$

(but now $S = v - \frac{\lambda}{1+\lambda} C^{BS}(v, \tau, K, \sigma_V, r)$)

$$dS = rSdt + \left((1 - \frac{\lambda}{1+\lambda} \Phi(d_1)) \sigma_V \Psi(S) \right) d\tilde{Z}$$

which of course makes the behaviour of the discounted stock price a martingale as required.

Since θ is the change of measure from P to Q we can also write the behaviour of v under the risk-neutral measure:

$$dv = \sigma_V v(dZ + \frac{\mu_V - r}{\sigma_V} dt) - v(\mu_V - r) dt + \mu_V dt$$

$$\Rightarrow \qquad dv = rvdt + \sigma_V v d\tilde{Z}$$

Thus if we have claims on v, they can be priced via the normal expectation route, using this risk neutral SDE for v.

3. THE DISTRIBUTION OF THE STOCK PRICE FOR A WARRANT-ISSUING FIRM

In this section we derive the "theoretical" distribution for the stock price of a warrant-issuing firm when the underlying distribution for the total equity value of the firm is assumed to be lognormal. We do this by exploiting the monotonicity of the stock price with respect to the equity value of the firm and applying a non-linear transformation:

From Assumption 2 we have that: $dv = \mu_V vdt + \sigma_V vdZ$. From this it follows that a share on the value of the firm v is lognormally distributed. Its probability density function is then given by:

$$f(v \setminus v_0, \mu_V, \sigma_V) = \frac{1}{v\sqrt{2\pi t}\sigma_V} \exp(-\frac{(\ln(v/v_0) - (\mu_V - \sigma_V^2/2)t)^2}{2\sigma_V^2 t}) \tag{3.1}$$

where v_0 represents the value of the firm per share at time 0.[16]

Remember now from equation (2.5.2) that the following representation for the stock price must hold: $S = v - (\lambda/(1+\lambda))C^{BS}(v,\tau,K,\sigma_V,r)$. Since there exists a one-to-one relationship between S and v (see the arguments presented for the derivation of equation (2.5.3)) we can solve the above equation for v, thus obtaining v as a function of S; i.e. $v = \Psi(S)$.

If we now start from the distribution of the equity value, i.e. $f(v \setminus v_0, \mu_V, \sigma_V)$, and consider the transformation $v = \Psi(S)$ we obtain

$$f(S \setminus S_0, \mu_V, \sigma_V) = f(v \equiv \Psi(S) \setminus S_0, \mu_V, \sigma_V)\frac{1}{\partial S/\partial v} \qquad (3.2)$$

$$= \frac{(1+\lambda)\exp(-\left((\ln(\frac{\Psi(S)}{\Psi(S_0)})-(\mu_V-\sigma_V^2/2)t)^2 \right)/2\sigma_V^2 t)}{(1+\lambda-\lambda\Phi(d_1))\Psi(S)\sqrt{2\pi t}\,\sigma_V}$$

where $d_1 = \dfrac{\left[\ln(\dfrac{\Psi(S)}{Ke^{-r(\tau-t)}}) + \dfrac{\sigma_V^2(\tau-t)}{2}\right]}{\sigma_V\sqrt{\tau-t}}$

This is the "theoretical" distribution for the stock price of a warrant-issuing firm. It reflects the potential for dilution from the outstanding warrants and although the parameters μ_v, σ_v are usually not directly observable, they can be inferred using numerical routines. To infer the drift and volatility of the firm we will use the procedure of Section (2.6).

PROPOSITION 1: *The risk-neutral distribution for the stock price exists, and can be shown to be equal to:*

$$f_{RN}(S \setminus S_0, \sigma_V) = \frac{(1+\lambda)\exp(-[\ln(\frac{\Psi(S)}{\Psi(S_0)})-(r-\sigma_V^2/2)t]^2/2\sigma_V^2 t)}{(1+\lambda-\lambda\Phi(d_1))\Psi(S)\sqrt{2\pi t}\,\sigma_V}$$

In particular for the case when $t = \tau$, the risk-neutral distribution has the following simple form:

$$f_{RN}(S \setminus v_0 \equiv \Psi(S_0), \sigma_V) = \frac{1}{\sqrt{2\pi\tau}\sigma_V}\frac{1+\lambda}{(1+\lambda)S-\lambda K}\exp(-(\ln(\frac{(1+\lambda)S-\lambda K}{v_0})-(r-\frac{\sigma_V^2}{2})\tau)^2/2\sigma_V^2\tau)$$

This distribution can be used to value warrants via equation

$$W(S) = e^{-r\tau}E^Q((S_T-K)^+ \setminus \Im_t) = e^{-r\tau}\int_K^\infty (S_T-K)^+ f_{RN}(S_T \setminus S_t)dS_T$$

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[16] Note here that we will use $f(\ldots)$ to denote probability density functions generally and not one specific probability density. The argument of $f(\ldots)$ as well as the context in which it is used will identify the particular probability density being considered.

PROOF: See Section 2 in the Appendix. $\qquad\square$

THEOREM 2: As the time to maturity of the warrants tends to infinity, the stock price of the warrant-issuing firm converges in distribution to the lognormal distribution:

$$\lim_{\tau\to\infty} f(S) \to \Lambda(\ln S_0 + (\mu_v - \frac{\sigma_V^2}{2})t, \sigma_V^2 t)$$

or

$$\lim_{\tau\to\infty} f(S) \to \Lambda(\ln S_0 + ((1+\lambda)\mu_S - \frac{\sigma_S^2}{2})t, \sigma_S^2 t)$$

$\Lambda(\mu, \sigma^2)$ denotes the lognormal distribution with parameters μ and σ^2.

PROOF. By assumption we have that: $v \sim \Lambda(\ln v_0 + (\mu_v - \frac{\sigma_V^2}{2})t, \sigma_V^2 t)$. We have also shown that the following representation for the stock price must hold: $S = v - (\lambda/(1+\lambda))C^{BS}(v, \tau, K, \sigma_V, r)$.

But $\lim_{\tau\to\infty} C^{BS}(v, \tau, K, \sigma_V, r) = v$. Hence at the limit $S = \frac{v}{1+\lambda}$. The result now follows from the property of the lognormal distribution that if $X \sim \Lambda(\mu, \sigma^2)$ and b and c are constants, where $c > 0$ (say $c = e^a$), then cX^b is $\Lambda(a + b\mu, b^2\sigma^2)$. In our case we have that $S = \frac{v}{1+\lambda}$ and $v \sim \Lambda(\ln v_0 + (\mu_v - \frac{\sigma_V^2}{2})t, \sigma_V^2 t)$. Hence $c = 1/(1+\lambda)$ and $b = 1$. Therefore it must be the case that:

$$\lim_{\tau\to\infty} f(S) \to \Lambda(\ln(\frac{1}{1+\lambda}) + \ln v_0 + (\mu_v - \frac{\sigma_V^2}{2})t, \sigma_V^2 t)$$

However at the limit we also have that $v_0 = \Psi(S_0) = (1+\lambda)S_0$. Hence

$$\lim_{\tau\to\infty} f(S) \to \Lambda(\ln S_0 + (\mu_v - \frac{\sigma_V^2}{2})t, \sigma_V^2 t)$$

Finally it is not hard to show that $\lim_{\tau\to\infty} \sigma_V = \sigma_S$ (use equation (2.6.3) combined with the fact that $v = S(1+\lambda)$) and $\lim_{\tau\to\infty} \mu_V = (1+\lambda)\mu_S$ (follows directly from $v = S(1+\lambda)$). Note of course that the result also follows from our derived distribution of equation (3.2). Just evaluate $\lim_{\tau\to\infty} f(S \setminus S_0, \mu_V, \sigma_V)$ to confirm the result. $\qquad\square$

The importance of theorem (2) is that, ceteris paribus, firms that offer long-horizon executive share options are less likely to suffer initially from "non-normal" distributions than those with short horizon options.

Now according to corollary (1), (and example (1)), we must have that the distribution for the stock price of an at-the-money warrant-issuing firm must also be lognormal. Let us confirm that this is indeed the case:

If the warrant is at the money (i.e. $K = ve^{r\tau}$) we have:

$$S = v - \frac{\lambda}{1+\lambda} C_{ATM}^{BS}(v, \tau, K, \sigma_V, r)$$

$$= v(1 - \frac{\lambda(2\Phi(\sigma_V \sqrt{\tau}/2) - 1)}{(1+\lambda)})$$

$$= v(\frac{1 + 2\lambda - 2\lambda\Phi(\sigma_V \sqrt{\tau}/2)}{1+\lambda})$$

$$\Rightarrow \quad v = \Psi(S) = \frac{(1+\lambda)S}{1 + 2\lambda - 2\lambda\Phi(\sigma_V \sqrt{\tau}/2)}$$

Also
$$\frac{\partial v}{\partial S} = \frac{(1+\lambda)}{1 + 2\lambda - 2\lambda\Phi(\sigma_V \sqrt{\tau}/2)}$$

Then

$$f_{ATM}(S \backslash S_0, \mu_V, \sigma_V) = f(v \equiv \frac{(1+\lambda)S}{1 + 2\lambda - 2\lambda\Phi(\sigma_V \sqrt{\tau}/2)} \backslash v_0 \equiv \Psi(S_0), \mu_V, \sigma_V) \frac{\partial v}{\partial S}$$

$$= \frac{1}{S\sqrt{2\pi t}\sigma_V} \exp(-\frac{(\ln(\frac{(1+\lambda)S}{(1+2\lambda-2\lambda\Phi(\sigma_V \sqrt{\tau}/2))v_0})) - (\mu_V - \sigma_V^2/2)t)^2}{2\sigma_V^2 t})$$

$$= \frac{1}{S\sqrt{2\pi t}\sigma_V} \exp(-\frac{(\ln(S/S_0) - (\mu_V - \sigma_V^2/2)t)^2}{2\sigma_V^2 t})$$

Hence it is indeed the case that $S \sim \Lambda(\ln S_0 + (\mu_v - \frac{\sigma_V^2}{2})t, \sigma_V^2 t)$.

3.1 The distribution of the warrant value W

Let us now obtain an expression for the distribution of the warrant value W. We start from $f(v \backslash v_0, \mu_V, \sigma_V)$ and remembering that

$$W = \frac{1}{1+\lambda} C^{BS}(v, \tau, K, \sigma_V, r) \Leftrightarrow (1+\lambda)W = C^{BS}(v, \tau, K, \sigma_V, r)$$

we consider the following transformation:

$$v = C_v^{BS^{-1}}((1+\lambda)W, \tau, K, \sigma_V, r) \equiv C_v^{BS^{-1}}((1+\lambda)W, \sigma_V) \tag{3.1.1}$$

where $\frac{\partial W}{\partial v} = \frac{\Delta}{1+\lambda} = \frac{\Phi(d_1)}{1+\lambda}$. Note that $C_v^{BS^{-1}}( )$ denotes the inverse of the Black-Scholes option pricing formula with respect to the value of the firm v. (In general $C_v^{BS^{-1}}( )$ is not known explicitly. An exception occurs when the warrant is at-the-money). Then

$$f(W \backslash v_0, \mu_V, \sigma_V) = f\left(v \equiv C_v^{BS^{-1}}((1+\lambda)W, \sigma_V) \backslash v_0, \mu_V, \sigma_V\right) \frac{1}{\partial W/\partial v}$$

$$= \frac{(1+\lambda)\exp\left(-\frac{(\ln(C_v^{BS^{-1}}((1+\lambda)W, \sigma_V)/v_0) - (\mu_V - \sigma_V^2/2)t)^2}{2\sigma_V^2 t}\right)}{C_v^{BS^{-1}}((1+\lambda)W, \sigma_V)\Phi(d_1')\sqrt{2\pi t}\sigma_V}$$

where $d_1' = \dfrac{\left[\ln\left(\dfrac{C_v^{BS^{-1}}\left((1+\lambda)W, \sigma_V\right)}{Ke^{-r\tau}}\right) + \dfrac{\sigma_V^2 \tau}{2}\right]}{\sigma_V \sqrt{\tau}}$

Again, if the warrant is at-the-money, it is not hard to show that the distribution of the warrant value W will also be lognormal with

$$W \sim \Lambda(\ln W_0 + (\mu_v - \frac{\sigma_V^2}{2})t, \sigma_V^2 t)$$

Note that so far we have conditioned on σ_V and μ_V. However these parameters are typically unobservable and there are arguments that the numerical routines presented in section (2.6) are not adequate to correctly infer the "true" values of these parameters. In the following section we present a methodology where we can relax the assumption that the diffusion parameters for the equity value process should be observable/estimable.

4. A BAYESIAN FRAMEWORK

We now develop a Bayesian framework where we relax the assumption that σ_V and μ_V are observable. Employing a Bayesian approach allows us to account for the unobservability of the drift and diffusion parameters since these parameters are treated as random variables which can eventually be integrated out or eliminated via a non-linear transformation. Let us first identify prior distributions for the drift and the volatility parameters of the unobservable value process of the firm. In identifying these distributions we follow standard Bayesian methodology as presented in Raiffa and Schlaifer (1961), Zellner (1971) and more recently in Bauwens, Lubrano, and Richard (1999) and Darsinos and Satchell (2001):

ASSUMPTION 3. *When the variance σ_v^2 of an independent normal process is assumed known but the mean μ_v is a random variable, the most convenient distribution for μ_V (the natural conjugate of the likelihood of the sample) is the normal distribution. The conditional distribution of the expected rate of return μ_V is therefore given by:*

$$f(\mu_V \setminus \sigma_V, m) = \frac{\sqrt{t}}{\sqrt{2\pi}\sigma_V} \exp\left(-\frac{t(\mu_V - m)^2}{2\sigma_V^2}\right)$$

where m is a hyperparameter.

ASSUMPTION 4. *We can assign an Inverted-Gamma-1 distribution with hyperparameters* λ, θ *as the prior distribution for* σ_V^2. *Its prior probability density function is then given by:*

$$f(\sigma_V^2 \setminus \eta, \theta) = \frac{\eta^\theta}{\Gamma(\theta)} \frac{1}{(\sigma^2)^{\theta+1}} \exp\left(-\frac{\eta}{\sigma^2}\right)$$

or

$$f(\sigma_V \setminus \eta, \theta) = 2\sigma \frac{\eta^\theta}{\Gamma(\theta)} \frac{1}{(\sigma^2)^{\theta+1}} \exp\left(-\frac{\eta}{\sigma^2}\right)$$

Although m, η, θ are hyperparameters which are also unobservable, it is much easier to obtain estimates for these from analysts forecasts (from long-run values of company data, growth targets set by each company, etc) than obtaining directly estimates for the drift and diffusion parameters of the value process. Note here that when the distributions are conditional on any prior parameters (i.e. η, θ and m) and/or on v_0 we will not explicitly condition on them and we shall refer to them as unconditional. (e.g. we will write $f(\sigma_V \setminus \eta, \theta) \equiv f(\sigma_V)$).

4.1. Derivation of the Unconditional Distribution for the Stock Price of a Warrant-Issuing Firm

Let us first obtain the joint density of drift and volatility. Directly from Assumptions (3) and (4) we get:

$$f(\mu_V, \sigma_V) = f(\sigma_V) f(\mu_V \setminus \sigma_V)$$

Now multiplying the above equation with equation (3.1) we get:

$$f(v, \mu_V, \sigma_V) = f(\mu_V, \sigma_V) f(v \setminus \mu_V, \sigma_V)$$

Finally
$$f(v, \sigma_V) = \int f(v, \sigma_V, \mu_V) \partial \mu_V$$

Darsinos and Satchell (2001) have shown that:

$$f(v, \sigma_V) = \frac{1}{\sqrt{\pi t} v} \frac{\eta^\theta}{\Gamma(\theta)} \frac{1}{\sigma_V^{2(\eta+1)}} \exp\left(-\frac{\eta}{\sigma_V^2}\right) \exp\left(-\frac{1}{4\sigma_V^2 t}\left[\ln\left(\frac{v}{v_0}\right) - (m_V - \frac{1}{2}\sigma_V^2)t\right]^2\right)$$

Now start from $f(v, \sigma_V)$ and consider the transformation:

$$v = v$$

$$\sigma_V = C_\sigma^{BS^{-1}}(v, \tau, K, (1+\lambda)W, r) \equiv C_\sigma^{BS^{-1}}(v, (1+\lambda)W)$$

where $C_\sigma^{BS^{-1}}(\ )$ denotes the inverse function of the Black-Scholes option price with respect to equity volatility. (This is the so-called implied volatility of the option price). In general $C_\sigma^{BS^{-1}}(\ )$ is not known explicitly. However, when the warrant is at-the-money the above equation can be written as

$$\sigma_V = C_\sigma^{BS^{-1}}(v,(1+\lambda)W) = \frac{2}{\sqrt{\tau}}\Phi^{-1}\left(\frac{1}{2}(\frac{(1+\lambda)W}{v}+1)\right).$$

The Jakobian of the transformation is given by:

$$\frac{1}{J} = \begin{vmatrix} \partial v/\partial v & \partial v/\partial \sigma_V \\ \partial W/\partial v & \partial W/\partial \sigma_V \end{vmatrix} = \frac{Vega}{1+\lambda} \qquad \text{where} \qquad Vega = \phi\left(\frac{\ln(\frac{v}{Ke^{-r\tau}})+\frac{\sigma_V^2\tau}{2}}{\sigma_V\sqrt{\tau}}\right)v\sqrt{\tau}$$

We therefore get

$$f(v,W) = f\left(v,\sigma_V \equiv C_\sigma^{BS^{-1}}(v,(1+\lambda)W)\right)|J|$$

$$= \frac{1+\lambda}{Vega'\sqrt{\pi}v}\frac{\eta^\theta}{\Gamma(\theta)}\frac{1}{C_\sigma^{BS^{-1}}(v,(1+\lambda)W)^{2(\eta+1)}}\exp(-\frac{\eta}{C_\sigma^{BS^{-1}}(v,(1+\lambda)W)^2})$$

$$\times\exp\left(-\frac{1}{4C_\sigma^{BS^{-1}}(v,(1+\lambda)W)^2 t}[\ln(\frac{v}{v_0})-(m_V-\frac{1}{2}C_\sigma^{BS^{-1}}(v,(1+\lambda)W)^2)t]^2\right)$$

where $Vega' = \phi\left(\frac{\ln(\frac{v}{Ke^{-r\tau}})+\frac{C^{BS^{-1}}(v,(1+\lambda)W)^2\tau}{2}}{C^{BS^{-1}}(v,(1+\lambda)W)\sqrt{\tau}}\right)v\sqrt{\tau}.$

Now, from equation (2.1.2) we have that $S=v-\lambda W$. If we next consider the transformation:

$$\begin{aligned} v &= S+\lambda W \\ W &= W \end{aligned} \qquad \text{where} \qquad J = \begin{vmatrix} \frac{\partial v}{\partial S}=1 & \frac{\partial v}{\partial W}=\lambda \\ \frac{\partial W}{\partial S}=0 & \frac{\partial W}{\partial W}=1 \end{vmatrix} = 1$$

we get

$$f(S,W) = f\left(v=S+\lambda W,W\right)|J| =$$

$$= \frac{1+\lambda}{Vega\sqrt{\pi}(S+\lambda W)}\frac{\eta^\theta}{\Gamma(\theta)}\frac{1}{C_\sigma^{BS^{-1}}(S+\lambda W,(1+\lambda)W)^{2(\eta+1)}}\exp(-\frac{\eta}{C_\sigma^{BS^{-1}}(S+\lambda W,(1+\lambda)W)^2})$$

$$\times\exp\left(-\frac{1}{4C_\sigma^{BS^{-1}}(S+\lambda W,(1+\lambda)W)^2 t}[\ln(\frac{S+\lambda W}{v_0})-(m_V-\frac{1}{2}C_\sigma^{BS^{-1}}(S+\lambda W,(1+\lambda)W)^2)t]^2\right)$$

Finally

$$f(S) = \int f(S,W)\,dW$$

$$
= \int \left\{
\begin{array}{l}
\dfrac{1+\lambda}{Vega\sqrt{\pi t}\,(S+\lambda W)}\,\dfrac{\eta^{\theta}}{\Gamma(\theta)}\,\dfrac{1}{C_{\sigma}^{BS^{-1}}(S+\lambda W,(1+\lambda)W)^{2(\eta+1)}}\,\exp\!\left(-\dfrac{\eta}{C_{\sigma}^{BS^{-1}}(S+\lambda W,(1+\lambda)W)^{2}}\right) \\[2ex]
\times\exp\!\left(-\dfrac{1}{4C_{\sigma}^{BS^{-1}}(S+\lambda W,(1+\lambda)W)^{2}t}[\ln(\dfrac{S+\lambda W)}{v_0}) - (m_V - \dfrac{1}{2}C_{\sigma}^{BS^{-1}}(S+\lambda W,(1+\lambda)W)^{2})t]^2\right)
\end{array}
\right\} \partial W
$$

4.2 Derivation of the Distribution for the Value of a Warrant W

To derive the distribution of the value of the warrant W there are a number of alternatives. In section (4.1) above we derive both $f(S,W)$ and $f(v,W)$. It is therefore obvious that $f(W) = \int f(S,W)\,dW$ or $f(W) = \int f(v,W)\,dW$. However for speed and simplicity in the numerical calculations that need to be performed we recommend the following:

Remembering that $W(S) \equiv W(v,\sigma_V) \equiv \dfrac{1}{1+\lambda}C^{BS}(v,\tau,K,\sigma_V,r) \equiv \dfrac{1}{1+\lambda}C^{BS}(v,\sigma_V)$, start from $f(v,\sigma_V)$ and consider the transformation:

$$
\begin{aligned}
v &= C_v^{BS^{-1}}\big((1+\lambda)W,\sigma_V\big) \\
\sigma_V &= \sigma_V
\end{aligned}
\qquad \text{where} \qquad
\frac{1}{J} =
\begin{vmatrix}
\dfrac{\partial W}{\partial v} & \dfrac{\partial W}{\partial \sigma_V} \\[2ex]
\dfrac{\partial \sigma_V}{\partial v} & \dfrac{\partial \sigma_V}{\partial \sigma_V}
\end{vmatrix}
=
\begin{vmatrix}
\dfrac{\Phi(d_1)}{1+\lambda} & \dfrac{Vega}{1+\lambda} \\[2ex]
0 & 1
\end{vmatrix}
= \frac{\Phi(d_1)}{1+\lambda}.
$$

Hence

$$f(W,\sigma_V) = f\big(v = C_v^{BS^{-1}}\big((1+\lambda)W,\sigma_V\big),\sigma_V\big)|J| =$$

$$
\frac{1+\lambda}{\sqrt{\pi t}\,C_v^{BS^{-1}}\big((1+\lambda)W,\sigma_V\big)\Phi(d_1')}\,\frac{\eta^{\theta}}{\Gamma(\theta)}\,\frac{\exp(-\eta/\sigma_V^2)}{\sigma_V^{2(\eta+1)}}\exp\!\left(-\frac{1}{4\sigma_V^2 t}[\ln(\frac{C_v^{BS^{-1}}\big((1+\lambda)W,\sigma_V\big)}{v_0}) - (m_V - \frac{1}{2}\sigma_V^2)t]^2\right)
$$

where $d_1' = \dfrac{\left[\ln(\dfrac{C_v^{BS^{-1}}\big((1+\lambda)W,\sigma_V\big)}{Ke^{-r\tau}}) + \dfrac{\sigma_V^2 \tau}{2}\right]}{\sigma_V\sqrt{\tau}}$

Finally $f(W) = \int f(W,\sigma_V)\,\partial\sigma_V$.

4.3 Derivation of the conditional distribution of the stock price

To derive the conditional (on the warrant value) distribution of the stock price we need one final calculation:

$$f(S \setminus W) = \frac{f(S, W)}{f(W)}$$

where the numerator is derived in section (4.1) and the denominator in section (4.2). Of course the cost of removing the dependence on the latent parameters by employing the Bayesian approach is that most calculations must be performed numerically based on the quasi-analytic expressions provided above.

5. EMPIRICAL BEHAVIOUR OF THE STOCK PRICE DISTRIBUTION

5.1 Comparing the Stock Price Distribution of a Warrant-Issuing and a Non-Warrant-Issuing Firm: An Illustration for Risk Management

In Section 3, equation (3.2), we have derived the theoretical distribution of a warrant-issuing firm. We reproduce it here for reference:

$$f(S_t \setminus S_0, \mu_V, \sigma_V) = \frac{(1+\lambda)\exp\left(-\left((\ln(\frac{\Psi(S_t)}{\Psi(S_0)}) - (\mu_V - \sigma_V^2/2)t)^2\right)/2\sigma_V^2 t\right)}{(1+\lambda-\lambda\Phi(d_1))\Psi(S_t)\sqrt{2\pi t}\sigma_V}$$ (5.1.1)

where $d_1 = \dfrac{\left[\ln(\dfrac{\Psi(S_t)}{Ke^{-r(\tau-t)}}) + \dfrac{\sigma_V^2(\tau-t)}{2}\right]}{\sigma_V\sqrt{\tau-t}}$

We also have by assumption, that the stock price of a non-warrant-issuing firm is lognormally distributed:

$$f(S_t \setminus S_0, \mu_S, \sigma_S) = \frac{1}{S_t\sqrt{2\pi t}\sigma_S}\exp\left(-\frac{(\ln(S_t/S_0) - (\mu_S - \sigma_S^2/2)t)^2}{2\sigma_S^2 t}\right)$$ (5.1.2)

In Table 1 below we compare the two distributions by means of reporting summary statistics (i.e. mean, standard deviation, skewness, and kurtosis) for the daily, weekly, and monthly stock price distributions of a non-warrant-issuing firm (i.e. a lognormal distribution) and of warrant-issuing firms with different dilution factors $\lambda = 5\%$, 50%, and 100% and different degrees of "moneyness" for their warrants. The calculations are based on parameters values: current stock price: $S_0 = 100$, stock return volatility: $\sigma_S = 25\%$, time to maturity of warrants: $\tau = 2$, years risk-free rate: $r = 5\%$, drift rate: $\mu_S = 5\%$, dividend yield: $d = 0$. The equity volatility σ_V and equity drift μ_V of the warrant-issuing firms are calculated using the procedure outlined in Section (2.6).

We are particularly interested to observe how the skewness and kurtosis values of the stock price distributions of the warrant-issuing firms deviate from the respective benchmark values of skewness and kurtosis of the lognormal distribution. We define the so-called "percentage deviation" from the benchmark skewness and kurtosis values of the lognormal distribution as:

$$Percentage\ Deviation = \frac{Warrant\text{-}issuing\ firm's\ Skewnes\ orKurtosiss\ value\ -\ Lognormal\ Skewness\ or\ Kurtosis\ value}{Lognormal\ Skewness\ or\ Kurtosis\ value}$$

Our results are exhibited in Table 1.

Summary statistics for the daily, weekly, and monthly stock price distributions of:

1. A firm with no warrants (i.e. the underlying stock price distribution is lognormal)
2. A warrant/executive option issuing firm with dilution factor $\lambda = 5\%$ and different degrees of moneyness.
3. A warrant/executive option issuing firm with dilution factor $\lambda = 50\%$ and different degrees of moneyness.
4. A warrant/executive option issuing firm with dilution factor $\lambda = 100\%$ and different degrees of moneyness.

The numbers in parentheses denote the percentage deviation of the skewness and kurtosis values of the warrant-issuing-firm's stock price distribution from the respective benchmark values of the lognormal distribution.

Stock Price Distribution:	Mean	Std. Dev.	Skewness	(% Dev.)	E. Kurtosis	(% Dev.)
Daily						
Lognormal $\lambda = 0$	100.02	1.575	0.0473		0.0040	
In-the-money warrants						
$\lambda = 5\%$, $K = 80$	100.02	1.576	0.0472	(-0.2%)	0.0041	(2.5%)
$\lambda = 50\%$, $K = 80$	100.02	1.576	0.0474	(0.2%)	0.0048	(20.0%)
$\lambda = 100\%$, $K = 80$	100.01	1.576	0.0493	(4.2%)	0.0057	(42.5%)
Near-the-money warrants						
$\lambda = 5\%$, $K = 100$	100.02	1.575	0.0461	(-2.5%)	0.0038	(-5.0%)
$\lambda = 50\%$, $K = 100$	100.02	1.575	0.0384	(-18.8%)	0.0032	(-20.0%)
$\lambda = 100\%$, $K = 100$	100.00	1.578	0.0338	(-28.5%)	0.0031	(-22.5%)
Out-of-the-money warrants						
$\lambda = 5\%$, $K = 120$	100.02	1.575	0.0455	(-3.8%)	0.0037	(-7.5%)
$\lambda = 50\%$, $K = 120$	100.02	1.575	0.0334	(-29.4%)	0.0019	(-52.5%)
$\lambda = 100\%$, $K = 120$	100.01	1.575	0.0244	(-48.4%)	0.0011	(-72.5%)
Weekly						
Lognormal $\lambda = 0$	100.10	3.526	0.1057		0.0199	
In-the-money warrants						
$\lambda = 5\%$, $K = 80$	100.10	3.528	0.1055	(-0.2%)	0.0203	(2.0%)
$\lambda = 50\%$, $K = 80$	100.10	3.527	0.1060	(0.3%)	0.0240	(20.6%)
$\lambda = 100\%$, $K = 80$	100.09	3.527	0.1104	(4.5%)	0.0286	(43.7%)
Near-the-money warrants						
$\lambda = 5\%$, $K = 100$	100.10	3.525	0.1030	(-2.6%)	0.0192	(-3.5%)
$\lambda = 50\%$, $K = 100$	100.10	3.525	0.0860	(-18.6%)	0.0161	(-19.1%)
$\lambda = 100\%$, $K = 100$	100.08	3.531	0.0756	(-28.5%)	0.0155	(-22.1%)
Out-of-the-money warrants						
$\lambda = 5\%$, $K = 120$	100.10	3.530	0.1019	(-3.6%)	0.0184	(-7.5%)
$\lambda = 50\%$, $K = 120$	100.09	3.525	0.0750	(-29.0%)	0.0098	(-50.8%)
$\lambda = 100\%$, $K = 120$	100.09	3.525	0.0547	(-48.3%)	0.0050	(-74.9%)
Monthly						
Lognormal $\lambda = 0$	100.40	7.080	0.2119		0.0799	
In-the-money warrants						
$\lambda = 5\%$, $K = 80$	100.40	7.084	0.2115	(-0.2%)	0.0820	(2.6%)
$\lambda = 50\%$, $K = 80$	100.40	7.081	0.2134	(0.7%)	0.0973	(21.8%)
$\lambda = 100\%$, $K = 80$	100.39	7.083	0.2225	(5.0%)	0.1164	(45.2%)
Near-the-money warrants						
$\lambda = 5\%$, $K = 100$	100.40	7.078	0.2065	(-2.6%)	0.0773	(-3.3%)
$\lambda = 50\%$, $K = 100$	100.40	7.076	0.1725	(-18.6%)	0.0651	(-18.5%)
$\lambda = 100\%$, $K = 100$	100.38	7.086	0.1520	(-28.3%)	0.0632	(-20.9%)
Out-of-the-money warrants						
$\lambda = 5\%$, $K = 120$	100.40	7.080	0.2040	(-3.7%)	0.0740	(-7.4%)
$\lambda = 50\%$, $K = 120$	100.40	7.075	0.1511	(-28.7%)	0.0394	(-50.7%)
$\lambda = 100\%$, $K = 120$	100.39	7.073	0.1095	(-48.3%)	0.0210	(-73.7%)

It is clear from table 1 that the existence of warrants in the capital structure of a firm does affect its stock price distribution, particularly so for firms with moderate to high dilution factors. In particular, regarding the mean and standard deviation of the stock price, there is no significant difference between warrant-issuing firms (of any dilution and moneyness) and non-warrant-issuing firms. However when it comes to the skewness and kurtosis values, we observe a clear pattern. Firms with *in-the-money warrants* tend to have similar (or slightly higher) values of skewness and significantly higher values of excess kurtosis when compared with the lognormal distribution. Firms with *near-the-money warrants* tend to have lower values of skewness and excess kurtosis than their respective non-warrant-issuing counterparts. Finally firms with *out-of-the-money warrants* tend to have significantly lower values of skewness and excess kurtosis when compared with non-warrant-issuing (lognormal) firms. Needless to say, these effects are magnified for all cases, the higher the dilution factor.

An intuitive interpretation of these results is that firms with out-of-the-money warrants tend to be riskier than firms with in the money warrants. Indeed our results suggest that the log-price (and return) of a firm with out-of-the-money warrants will have a longer and thinner left tail than the normal distribution while a firm with in-the-money warrants a shorter and fatter tail. (To obtain the log price or log return distribution of a warrant-issuing firm one need only take the $\log S_t$ or $\log S_t / S_{t-1}$ transform of the distribution of equation (5.1.1)). For Value-at-Risk calculations this is crucial. For example if a normal approximation is used for the log-price (and return) distributions of all firms we will tend to calculate a VaR that is too low for firms with out-of-the money (and near-the-money) warrants and too high for firms with in the money warrants.

Indeed, practitioners often interpret a firm with in-the-money executive stock options as a good indication that the firm has been meeting its growth targets and *ceteris paribus* is expected to do so in the future. On the other hand, the presence of out-of-the money executive options can be interpreted as a signal of underperformance or financial distress.

Typically warrants are issued out-of-the-money and executive stock options are issued near-the-money. Both instruments are issued with very long maturities (for example executive stock options last for as long as $10 - 15$ years). We do not present distributional results for firms with very long maturity warrants/executive options since based on Section 3, Theorem 2, we have that the distribution of the stock price of these firms converges to the lognormal as the time to maturity increases indefinitely. Instead, we model the stock price distribution of these firms when their warrants are within a few years to maturity.

5.2 Comparing the Risk-Neutral Stock Price Distribution of a Warrant-Issuing and a Non-Warrant-Issuing Firm: An Application to Executive Option/Warrant Valuation

In Section 3, Proposition 2, we have derived the risk-neutral distribution of a warrant-issuing firm. We reproduce it here for reference:

$$(5.2.1)$$

$$f_{RN}(S_\tau \backslash v_0 \equiv \Psi(S_0), \sigma_V) = \frac{1}{\sqrt{2\pi\tau}\sigma_V} \frac{1+\lambda}{(1+\lambda)S_\tau - \lambda K} \exp(-(\ln(\frac{(1+\lambda)S_\tau - \lambda K}{v_0}) - (r - \frac{\sigma_V^2}{2})\tau)^2 / 2\sigma_V^2\tau)$$

This distribution can be used to value warrants via equation

$$W(S) \equiv D(S) = e^{-r\tau}E^Q((S_\tau - K)^+ \backslash \Im_t) = e^{-r\tau}\int (S_\tau - K)^+ f_{RN}(S_\tau \backslash S_0)dS_\tau \qquad (5.2.2)$$

To date, this approach has never been used to value warrants due to the lack of knowledge of $f_{RN}(S_\tau \backslash S_0)$. Valuation of warrants has been of course carried out using the standard value of the firm approach:

$$W(S) \equiv C(v) = \frac{1}{1+\lambda} C^{BS}(v, \tau, K, \sigma_V, r) \qquad (5.2.3)$$

Needless to say that equations (5.2.2) and (5.2.3) produce identical warrant values. They are after all equivalent approaches to warrant valuation.

Now "option-like" warrant valuation suggests that instead of using the distribution of equation (5.2.1) one can obtain a very good approximation for the value of the warrant by using a Black-Scholes framework and valuing the warrant as if it was identical to a call option on the stock of the firm. (This basically means using the lognormal risk-neutral distribution in equation (5.2.2) above or equivalently using $W(S) = C^{BS}(S, \tau, K, \sigma_S, r)$). The lognormal risk neutral distribution (i.e. the risk-neutral distribution of a non-warrant-issuing firm) is given by:

$$f_{RN}(S \backslash S_0, \sigma_S) = \frac{1}{S\sqrt{2\pi\tau}\sigma_S} \exp(-\frac{(\ln(S/S_0) - (r - \sigma_S^2/2)\tau)^2}{2\sigma_S^2\tau}) \qquad (5.2.4)$$

In Table 2 below we compare the two risk-neutral distributions by means of reporting summary statistics (i.e. mean, standard deviation, skewness, and kurtosis) for the stock price distributions of a non-warrant-issuing firm (i.e. a lognormal distribution) and of warrant-issuing firms with different dilution factors $\lambda = 5\%$, 50%, and 100%, different degrees of "moneyness" for their warrants, and different maturities ($\tau = 2$ years and $\tau = 3$ months).

TABLE 2

Summary statistics for the risk-neutral distributions of the stock price of:

1. A firm with no warrants (i.e. the underlying risk-neutral distribution is lognormal)
2. A warrant-issuing firm with dilution factor $\lambda = 5\%$, different degrees of moneyness and different maturities
3. A warrant-issuing firm with dilution factor $\lambda = 50\%$, different degrees of moneyness and different maturities
4. A warrant-issuing firm with dilution factor $\lambda = 100\%$, different degrees of moneyness and different maturities

Stock Price Distribution:	Mean	Std. Deviation	Skewness	Excess Kurtosis
Lognormal $\lambda = 0$ ($\tau = 2$ years)	110.52	40.33	1.14	2.41
In-the-money warrants:				
$\lambda = 5\%$, $K = 80$	110.67	40.16	1.18	2.58
$\lambda = 50\%$, $K = 80$	111.59	39.08	1.48	4.12
$\lambda = 100\%$, $K = 80$	112.09	38.70	1.77	6.01
Near-the-money warrants:				
$\lambda = 5\%$, $K = 100$	111.00	39.76	1.18	2.55
$\lambda = 50\%$, $K = 100$	113.89	36.30	1.44	3.92
$\lambda = 100\%$, $K = 100$	115.56	34.32	1.71	5.64
Out-of-the-money warrants:				
$\lambda = 5\%$, $K = 120$	111.53	39.37	1.17	2.52
$\lambda = 50\%$, $K = 120$	117.56	33.32	1.37	3.50
$\lambda = 100\%$, $K = 120$	121.00	29.54	1.57	4.66
Deep-out-of-the-money warrants:				
$\lambda = 5\%$, $K = 180$	113.91	38.64	1.15	2.44
$\lambda = 50\%$, $K = 180$	134.19	28.17	1.19	2.64
$\lambda = 100\%$, $K = 180$	145.90	21.77	1.23	2.79
Lognormal $\lambda = 0$ ($\tau = 3$ months)	101.26	12.71	0.38	0.255
Close-to-maturity-out-of-the-money-warrants:				
$\lambda = 5\%$, $K = 120$	102.18	12.16	0.38	0.258
$\lambda = 50\%$, $K = 120$	107.65	8.79	0.39	0.275
$\lambda = 100\%$, $K = 120$	110.81	6.75	0.40	0.287

The calculations are based on parameters values: Current stock price: $S = 100$, Stock return volatility: $\sigma_S = 25\%$, Time to maturity of warrants: $\tau = 2$ years (for the last case of "Close-to-maturity-out-of-the-money warrants" a time to maturity of $\tau = 3$ months was used instead), Risk-free rate: $r = 5\%$, Dividend yield: $d = 0$.
The risk-neutral lognormal distribution is given in equation (5.2.4)
The parameter specific (i.e. specific on dilution, moneyness, time to maturity, etc) risk-neutral distribution for the stock price of a warrant-issuing firm is given in equation (5.2.1).

It is obvious from Table 2 that the risk-neutral distributions of warrant and non-warrant-issuing firms are markedly different. Generally the log-stock price distribution of high dilution warrant-issuing firms exhibits higher mean, lower standard deviation and marked skewness and excess kurtosis when compared to the log-stock price of a non-warrant-issuing firm (i.e. a normal distribution). Thus, the next question that naturally arises is how much this will affect the pricing of warrants. In Table 3 below we report the warrant values obtained using the theoretical approach and the "option-like" approximation approach. We compare these values by what we call the "percentage mispricing error". This is calculated as:

$$\text{Percentage Mispricing Error} = \frac{(\textit{``option-like'' warrant valuation approximation} - \textit{theoretical value of warrant})}{\textit{Theoretical value of warrant}}$$

<u>TABLE 3</u>

Comparison of the theoretical value of a warrant with the "option-like warrant valuation approximation" for a variety of dilution factors and degrees of moneyness.

	Total Equity Value: v	Equity volatility: σ_V	Theoretical Value of Warrant	Option-Like Warrant Valuation Approximation	Percentage Mispricing Error
In-the-money warrants					
$\lambda = 5\%$, $K = 80$	101.53	25.7%	30.54	30.53	-0.03%
$\lambda = 50\%$, $K = 80$	115.26	30.9%	30.53	30.53	0%
$\lambda = 100\%$, $K = 80$	130.46	35.6%	30.46	30.53	0.23%
Near-the-money warrants					
$\lambda = 5\%$, $K = 100$	100.93	25.6%	18.66	18.65	-0.05%
$\lambda = 50\%$, $K = 100$	109.34	30.4%	18.68	18.65	-0.16%
$\lambda = 100\%$, $K = 100$	118.64	34.8%	18.66	18.65	-0.05%
Out-of-the-money warrants					
$\lambda = 5\%$, $K = 120$	100.54	25.5%	10.71	10.73	0.19%
$\lambda = 50\%$, $K = 120$	105.27	29.1%	10.55	10.73	1.71%
$\lambda = 100\%$, $K = 120$	110.39	32.5%	10.39	10.73	3.27%
Deep-out-of-the-money warrants					
$\lambda = 5\%$, $K = 180$	100.08	25.1%	1.670	1.711	2.46%
$\lambda = 50\%$, $K = 180$	100.69	26.0%	1.379	1.711	24.1%
$\lambda = 100\%$, $K = 180$	101.16	26.6%	1.162	1.711	47.2%
Close-to-maturity-out-of-the-money-warrants:					
$\lambda = 5\%$, $K = 120$	100.03	25.1%	0.531	0.546	2.82%
$\lambda = 50\%$, $K = 120$	100.22	25.9%	0.432	0.546	26.4%
$\lambda = 100\%$, $K = 120$	100.36	26.5%	0.360	0.546	51.7%

The calculations are based on parameters values: Current stock price: $S = 100$, Stock return volatility: $\sigma_S = 25\%$, Time to maturity of warrants: $\tau = 2$ years (for the last case of "Close-to-maturity-out-of-the-money warrants" a time to maturity of $\tau = 3$ months was used instead), Risk-free rate: $r = 5\%$, Dividend yield: $d = 0$.

The total equity value and volatility of the firm are calculated by solving numerically the system of simultaneous equations of section (2.6) (i.e. equations (2.6.1) and (2.6.3)).

The theoretical value of the warrant is calculated either by taking the expected value of the discounted payoff of the warrant over the respective risk-neutral distribution for the stock price (see equation (5.2.1); summary statistics for the respective risk-neutral distribution for the stock price for each case are given in table 1) or equivalently using equation (5.2.3).

The "option-like warrant valuation approximation" ignores any dilution and is calculated either by taking the expected value of the discounted payoff of the warrant over the lognormal risk-neutral distribution for the stock price (see equation (5.2.4)) or equivalently using equation (2.2.2). (This approach is of course identical to valuing an ordinary call option on the stock of the firm).

The percentage mispricing error is calculated as:
("option-like" warrant valuation approximation – theoretical value of warrant) / theoretical value of warrant.

The message from table 3 is clear. For *in-* and *near-the-money* warrants option-like valuation is a viable and convenient alternative to the textbook treatment valuation. However it is also clear that option-like valuation will result in a large mispricing error for *deep-out-of-the-money* warrants and for *close-to-maturity-out-of-the-money* warrants in general.[17] For such cases one is advised to follow the "theoretical" approach to warrant valuation. We would now like to draw to the attention of the reader that such cases might not be as infrequent as it might at first appear. Note that this paper has assumed that the warrant-issuing firm under investigation is debt free. Suppose we consider a simple extension were the firm issues both debt and warrants. Suppose for example an "idealised" scenario where the firm has also issued a zero-coupon bond with the same maturity as the warrant. Let the face value of that debt be *F*. In this case we have to take into account the fact that debt is senior to both stock and warrant. Hence at exercise of the warrant, equation (2.1.4) of section (2.1) must be modified by subtracting the face value of the debt from the value of the firm. In this case equation (2.1.4) becomes:

$$\frac{V_T + nK - F}{N + n} - K = \frac{N}{N + n}(\frac{V_T - F}{N} - K) = \frac{1}{1 + \lambda}(v_T - f - K)$$

where $f = \frac{F}{N}$ is the face value of the debt per share. Thus in the presence of debt a warrant with an exercise price of *K*, is valued as a warrant with an exercise price of $f + K$. In other words, if the warrant is at any moment in time at-the-money or out-of-the-money, incorporating a high face value of debt makes it immediately a deep-out-of-the-money warrant.

6. CONCLUDING REMARKS AND ISSUES FOR FURTHER RESEARCH

6.1 Concluding Remarks

In this paper we have derived the stock price distribution of a firm with warrants and/or executive stock options in its capital structure. We have illustrated that the existence of moderate to high dilution warrants in the corporate structure of a firm does make its stock price distribution markedly different from the stock price distribution of say a non-warrant-issuing firm (warrants/executive options with high dilution potential are typically encountered in high-tech firms; see for example amongst many others Microsoft Corporation, Intel Corporation, Oracle, Cisco Systems, AoL Time-Warner, etc). Indeed, for risk-management calculations we argue that the lognormal distribution for the stock price is not appropriate. Our simulations suggest that a normal approximation to the log-stock price (or return) will lead to a Value-at-Risk that is too low for firms with out-of-the-money and near-the-money warrants and too high for firms with in-the- money warrants. Regarding now the valuation of

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[17] Similar findings have also been reported by Schulz and Trautmann (1994).

warrants and executive stock options we find that "option-like" valuation will price "correctly" and efficiently in-the-money, near-the-money and long maturity out-of-the-money warrants. However for deep-out-of-the-money and near-maturity out-of-the-money warrants in general, we find that "option-like" valuation will significantly overprice warrants, with the percentage mispricing error exceeding in some highly diluted cases 50% the theoretical price. This can be significant since (traded) warrants are typically issued very deep-out-of-the-money.

6.2. Issues for Further Research

The way is now open for deriving the distribution of the stock price for a firm with debt (or debt and warrants) in its capital structure. We believe that the implications of this will be significant not only for Value-at-Risk calculations but also for credit management purposes since it will be possible to estimate the probability of default from a stock price distribution that explicitly incorporates esoteric capital structures of firms (i.e. equity, debt, warrants, etc). Moreover using a similar approach with the one employed in section (2.6) it will be possible to infer all the unobservable parameters of the derived distribution (i.e. the drift and volatility of the value of the firm).

Regarding now option valuation in general, we saw that just incorporating warrants in the stock price distribution is a first step to the right direction. We know that the Black-Scholes model overprices deep-out-of-the-money stock options. We illustrated that by incorporating warrants in the stock price distribution some of this bias was eliminated. Remember that a warrant is identical to an ordinary call option when the "correct" distribution is used. Thus the textbook treatment to warrant valuation applies also to option valuation. Indeed we believe that more attention should be paid to valuing options using the "value of the firm" approach. Geske (1979) was the first to develop the compound option model. He showed that the stock in a levered firm can be viewed as a call option on the value of the firm. Thus a call option on the stock can be viewed as a compound option on the value of the firm. The advantage of this approach is that it eliminates a number of Black-Scholes biases since it captures the pattern of implied volatilities observed for equity options. Conceptually the "value of the firm" approach is very appealing since it provides a unified framework for stock option, warrant, and debt valuation.

7. APPENDIX

SECTION 1

Proof of LEMMA 2: From equation (2.5.2) we have the following representation for the stock price: $S = v - qC^{BS}(v,\tau,K,\sigma_V,r)$ where $q = \dfrac{\lambda}{(1+\lambda)}$. Then by Ito

$$dS = \frac{\partial S}{\partial v}dv + \frac{\partial S}{\partial t}dt + \frac{1}{2}\frac{\partial^2 S}{\partial v^2}(dv)^2$$

$$= (1 - q\Delta)(\mu v dt + \sigma v dZ) - q\Theta dt - \frac{q}{2}\Gamma\sigma^2 v^2 dt$$

where

$$\Delta = \frac{\partial C^{BS}}{\partial v} = \Phi(d_1),$$

$$\Gamma = \frac{\partial^2 C^{BS}}{\partial v^2} = \frac{\phi(d_1)}{v\sigma_V\sqrt{\tau}},$$

$$\Theta = \frac{\partial C^{BS}}{\partial t} = -\frac{v\phi(d_1)\sigma_V}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2).$$

Now using equation (2.5.3) we get:

$$dS = (1 - q\Delta)(\mu\Psi(S)dt + \sigma\Psi(S)dZ) - q\Theta dt - \frac{q}{2}\Gamma\sigma^2\Psi(S)^2 dt$$

Substituting for $q, \Delta, \Gamma,$ and Θ, we have:

$$dS = \left((1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\mu_V\Psi(S) - \frac{\lambda}{(1+\lambda)}\left(-\frac{\Psi(S)\phi(d_1)\sigma_V}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2)\right) - \frac{\lambda}{2(1+\lambda)}\left(\frac{\phi(d_1)}{\Psi(S)\sigma_V\sqrt{\tau}}\right)\sigma_V^2\Psi(S)^2 \right)dt$$

$$+ \left((1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\sigma_V\Psi(S) \right)dZ$$

$\Rightarrow$

$$dS = \left(\overbrace{(1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\mu_V\Psi(S) + \frac{\lambda}{1+\lambda}rKe^{-r\tau}\Phi(d_2)}^{\mu(S,t)} \right)dt + \left(\overbrace{(1 - \frac{\lambda}{1+\lambda}\Phi(d_1))\sigma_V\Psi(S)}^{\sigma(S,t)} \right)dZ$$

$\square$

Proof of COLLORARY 1: Let us first start with the general case, (i.e. when the warrant is not necessarily at-the-money). This time the general case is different from the general case of Lemma 2 because *K* is stochastic. Then from equation (2.5.2):

$$S = v - \gamma C^{BS}(v,\tau,K,\sigma_V,r)$$

where
$K = K(v,t)$ (i.e. *K is stochastic*),
$\gamma = \lambda/(1+\lambda)$.

Then applying Ito calculus we get:

$$dS = (\frac{\partial S}{\partial v} + \frac{\partial S}{\partial K}\frac{\partial K}{\partial v})dv + (\frac{\partial S}{\partial t} + \frac{\partial S}{\partial K}\frac{\partial K}{\partial t})dt + \frac{1}{2}(\frac{\partial^2 S}{\partial v^2} + \frac{\partial^2 S}{\partial K^2}\frac{\partial^2 K}{\partial v^2})(dv)^2$$

where

$$\frac{\partial S}{\partial v} = 1 - \gamma\Phi(d_1),$$

$$\frac{\partial S}{\partial K} = -\gamma(-e^{-r\tau}\Phi(d_2)),$$

$$\frac{\partial S}{\partial t} = -\gamma(-\frac{v\phi(d_1)\sigma_V}{2\sqrt{\tau}} - rKe^{-r\tau}\Phi(d_2)),$$

$$\frac{\partial^2 S}{\partial v^2} = -\gamma\frac{\phi(d_1)}{v\sigma_V\sqrt{\tau}}$$

$$\frac{\partial^2 S}{\partial K^2} = -\gamma(e^{-r\tau}\frac{\phi(d_2)}{K\sigma_V\sqrt{\tau}})$$

Hence

$$dS = \left((1 - \gamma\Phi(d_1) + \gamma e^{-r\tau}\Phi(d_2)\frac{\partial K}{\partial v}\right)(\mu_V v dt + \sigma_V v dZ)$$

$$+ (\gamma\frac{v\phi(d_1)\sigma_V}{2\sqrt{\tau}} + \gamma rKe^{-r\tau}\Phi(d_2) + \gamma e^{-r\tau}\Phi(d_2)\frac{\partial K}{\partial t})dt$$

$$+ \frac{1}{2}(-\gamma\frac{\phi(d_1)}{v\sigma_V\sqrt{\tau}} - \gamma e^{-r\tau}\frac{\phi(d_2)}{K\sigma_V\sqrt{\tau}}\frac{\partial^2 K}{\partial v^2})(\sigma_V^2 v^2 dt)$$

Now when the warrant is at-the-money $K = K(v,t) = ve^{r\tau}$. Then

$$dS = ((1 - \gamma\Phi(d_1) + \gamma\Phi(-d_1))(\mu_V v dt + \sigma_V v dZ)$$

$$+ (\gamma\frac{v\phi(d_1)\sigma_V}{2\sqrt{\tau}} + \gamma rv\Phi(-d_1) - r\gamma\Phi(-d_1)v)dt$$

$$+ \frac{1}{2}(-\gamma\frac{\phi(d_1)}{v\sigma_V\sqrt{\tau}})(\sigma_V^2 v^2 dt)$$

i.e.

$$dS = ((1 - \gamma\Phi(d_1) + \gamma\Phi(-d_1))(\mu_V v dt + \sigma_V v dZ)$$

$$= ((1 + \gamma - 2\gamma\Phi(d_1))(\mu_V v dt + \sigma_V v dZ)$$

$$= \mu_V S dt + \sigma_V S dZ$$

Hence the process for the stock price S is also a Geometric Brownian Motion.

$\square$

SECTION 2

PROPOSITION 1: *The risk-neutral distribution for the stock price exists (see section (2.7))
and can be shown to be equal to:*

$$f_{RN}(S \setminus S_0, \sigma_V) = \frac{(1+\lambda)\exp(-[\ln(\frac{\Psi(S)}{\Psi(S_0)}) - (r - \sigma_V^2/2)t]^2 / 2\sigma_V^2 t)}{(1 + \lambda - \lambda\Phi(d_1))\Psi(S)\sqrt{2\pi t}\sigma_V}$$

In particular for the case when $t = \tau$, the risk-neutral distribution has the following simple form:

$$f_{RN}(S \setminus v_0, \sigma_V) = \frac{1}{\sqrt{2\pi\tau}\sigma_V} \frac{1+\lambda}{(1+\lambda)S - \lambda K} \exp(-(\ln(\frac{(1+\lambda)S - \lambda K}{v_0}) - (r - \frac{\sigma_V^2}{2})\tau)^2 / 2\sigma_V^2 \tau)$$

PROOF: In Section (2.7) we have shown that the market price of risk for the stock price of a warrant- issuing firm is $\theta_t = \theta = \frac{\mu_V - r}{\sigma_V}$. We denote market's measure by P and the risk neutral measure by Q. Given P and Q equivalent measures we can define the Radon-Nikodym derivative $\frac{dQ}{dP}$ such that:

$$E_Q[S] = E_P[\frac{dQ}{dP}S]$$

Now since the market price of risk is constant the Cameron-Martin-Girsanov boundedness condition $E_P[\exp(\frac{1}{2}\int_0^T \theta_t^2 dt)] < \infty$ is automatically satisfied. We therefore have that:

$$\frac{dQ}{dP} = \exp\left\{-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_S^2 ds\right\}$$

$$= \exp\left\{-\frac{\mu_V - r}{\sigma_V}W_t - \frac{(\mu_V - r)^2}{2\sigma_V^2}t\right\}$$

Hence

$$E_P[\frac{dQ}{dP}S] = E_p[\exp\left\{-\frac{\mu_V - r}{\sigma_V}W_t - \frac{(\mu_V - r)^2}{2\sigma_V^2}t\right\}S]$$

But we know the distribution of the stock price S under market's measure P. It is given by equation (3.2) in Section 3. Hence

$$E_P[\frac{dQ}{dP}S] = \int_0^t \exp\left\{-\frac{\mu_V - r}{\sigma_V}W_t - \frac{(\mu_V - r)^2}{2\sigma_V^2}t\right\} \frac{(1+\lambda)\exp(-\frac{[\ln(\frac{\Psi(S)}{\Psi(S_0)}) - (\mu_V - \sigma_V^2/2)t]^2}{2\sigma_V^2 t})}{(1 + \lambda - \lambda\Phi(d_1))\Psi(S)\sqrt{2\pi t}\sigma_V} S dS$$

$$= \int_0^t \frac{(1+\lambda)}{(1 + \lambda - \lambda\Phi(d_1))\Psi(S)\sqrt{2\pi t}\sigma_V} \exp\left(-\frac{\left((\ln(\frac{\Psi(S)}{\Psi(S_0)}) - (\mu_V - \frac{\sigma_V^2}{2})t)^2\right) + (\mu_V - r)^2 t^2 + 2(\mu_V - r)t\sigma_V W_t}{2\sigma_V^2 t}\right) S dS$$

Let us now concentrate on

$$\left(\left(\ln\left(\frac{\Psi(S)}{\Psi(S_0)}\right)-\left(\mu_V-\frac{\sigma_V^2}{2}\right)t\right)^2\right)+\left(\mu_V-r\right)^2t^2+2\left(\mu_V-r\right)t\sigma_V W_t \tag{A.1}$$

First though remember that by assumption 2 we have that: $v=v_0\exp\left(\left(\mu_V-\frac{\sigma_V^2}{2}\right)t+\sigma_V W_t\right)$

which implies that $W_t=\dfrac{\ln\left(\frac{v}{v_0}\right)-\left(\mu_V-\frac{\sigma_V^2}{2}\right)t}{\sigma_V}$ or since $v=\Psi(S)$

$$W_t=\frac{\ln\left(\frac{\Psi(S)}{\Psi(S_0)}\right)-\left(\mu_V-\frac{\sigma_V^2}{2}\right)t}{\sigma_V}$$

Substituting into equation (A.1) the above expression for W_t we get:

$$\left(\left(\ln\left(\frac{\Psi(S)}{\Psi(S_0)}\right)-\left(\mu_V-\frac{\sigma_V^2}{2}\right)t\right)^2\right)+\left(\mu_V-r\right)^2t^2+2\left(\mu_V-r\right)t\left(\ln\left(\frac{\Psi(S)}{\Psi(S_0)}\right)-\left(\mu_V-\frac{\sigma_V^2}{2}\right)t\right)$$

But this is a quadratic expansion of the form $(a+b)^2=a^2+b^2+2ab$. Hence

$$\left(\ln\left(\frac{\Psi(S)}{\Psi(S_0)}\right)-\left(\mu_V-\frac{\sigma_V^2}{2}\right)t+\left(\mu_V-r\right)t\right)^2=\left(\ln\left(\frac{\Psi(S)}{\Psi(S_0)}\right)-\left(r-\frac{\sigma_V^2}{2}\right)t\right)^2$$

Substituting into the integral above we have

$$\int_0^t\frac{(1+\lambda)}{(1+\lambda-\lambda\Phi(d_1))\Psi(S)\sqrt{2\pi t}\sigma_V}\exp\left(-\frac{\left(\ln\left(\frac{\Psi(S)}{\Psi(S_0)}\right)-\left(r-\frac{\sigma_V^2}{2}\right)t\right)^2}{2\sigma_V^2 t}\right)SdS$$

But this is exactly

$$\int_0^t f_{RN}(S\setminus S_0,\sigma_V)SdS=E_Q[S]$$

where

$$f_{RN}(S\setminus S_0,\sigma_V)=\frac{(1+\lambda)\exp(-[\ln\left(\frac{\Psi(S)}{\Psi(S_0)}\right)-(r-\sigma_V^2/2)t]^2/2\sigma_V^2 t)}{(1+\lambda-\lambda\Phi(d_1))\Psi(S)\sqrt{2\pi t}\sigma_V}$$

Indeed the risk-neutral distribution of the stock price follows directly by applying the non-linear transformation $v = \Psi(S)$ to the risk-neutral distribution of the equity value. This is the well-known lognormal distribution of Geometric Brownian Motion with μ_v replaced by r.

Now when $t = \tau$, then $t - \tau = 0$. Hence $S = v - \dfrac{\lambda}{1+\lambda} C^{BS}(v, \tau - t, K, \sigma_V, r)$ becomes

$$S = v - \frac{\lambda}{1+\lambda}(v - K)^+$$

→

$$v = \Psi(S) = (1+\lambda)S - \lambda K \qquad \text{or} \qquad v = S.$$

Also

$$\Phi(d_1) = 1.$$

Therefore the risk-neutral distribution takes the simple form:

$$f_{RN}(S \setminus v_0, \sigma_V) = \frac{1}{\sqrt{2\pi\tau}\sigma_V} \frac{1+\lambda}{(1+\lambda)S - \lambda K} \exp(-(\ln(\frac{(1+\lambda)S - \lambda K}{v_0}) - (r - \frac{\sigma_V^2}{2})\tau)^2 / 2\sigma_V^2\tau)$$

and is defined within $(K, +\infty)$.

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