



Mapping class groups of highly connected $(4k + 2)$ -manifolds

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Abstract

We compute the mapping class group of the manifolds $\sharp^g(S^{2k+1} \times S^{2k+1})$ for $k > 0$ in terms of the automorphism group of the middle homology and the group of homotopy $(4k + 3)$ -spheres. We furthermore identify its Torelli subgroup, determine the abelianisations, and relate our results to the group of homotopy equivalences of these manifolds.

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The classical mapping class group Γ_g of a genus g surface naturally generalises to all even dimensions $2n$ as the group of isotopy classes

$$\Gamma_g^n = \pi_0 \text{Diff}^+(W_g)$$

of orientation-preserving diffeomorphisms of the g -fold connected sum $W_g = \sharp^g(S^n \times S^n)$. Its action on the middle cohomology $H(g) := H^n(W_g; \mathbf{Z}) \cong \mathbf{Z}^{2g}$ provides a homomorphism $\Gamma_g^n \rightarrow \text{GL}_{2g}(\mathbf{Z})$ whose image is the symplectic group $\text{Sp}_{2g}(\mathbf{Z})$ in the surface case $2n = 2$, and a certain arithmetic subgroup $G_g \subset \text{Sp}_{2g}(\mathbf{Z})$ or $G_g \subset$

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$O_{g,g}(\mathbf{Z})$ in general, the description of which we shall recall later. The kernel $T_g^n \subset \Gamma_g^n$ of the resulting extension

$$0 \longrightarrow T_g^n \longrightarrow \Gamma_g^n \longrightarrow G_g \longrightarrow 0 \tag{1}$$

is known as the *Torelli group*—the subgroup of isotopy classes acting trivially on homology. In contrast to the surface case, the Torelli group in high dimensions $2n \geq 6$ is comparatively manageable: there is an extension

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow T_g^n \longrightarrow H(g) \otimes S\pi_n\mathrm{SO}(n) \longrightarrow 0 \tag{2}$$

due to Kreck [35], which relates T_g^n to the finite abelian group of homotopy spheres Θ_{2n+1} and the image of the stabilisation map $S: \pi_n\mathrm{SO}(n) \rightarrow \pi_n\mathrm{SO}(n+1)$ whose isomorphism class is shown in Table 1.

The description of Γ_g^n up to these two extension problems has found a variety of applications [2,5–7,18,23,29,33,38,39], especially in relation to the study of moduli spaces of manifolds [22]. The remaining extensions (1) and (2) have been studied more closely for particular values of g and n [15,19,21,36,37,48] but are generally not well-understood (see e.g. [15, p.1189], [21, p.873], [2, p.425]). In the present work, we resolve the remaining ambiguity for $n \geq 3$ odd, resulting in a complete description of the mapping class group Γ_g^n and the Torelli group T_g^n in terms of the arithmetic group G_g and the group of homotopy spheres Θ_{2n+1} .

To explain our results, note that (1) and (2) induce further extensions

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow \Gamma_g^n \longrightarrow \Gamma_g^n / \Theta_{2n+1} \longrightarrow 0 \tag{3}$$

$$0 \longrightarrow H(g) \otimes S\pi_n\mathrm{SO}(n) \longrightarrow \Gamma_g^n / \Theta_{2n+1} \longrightarrow G_g \longrightarrow 0, \tag{4}$$

which express Γ_g^n in terms of G_g and Θ_{2n+1} up to two extension problems that are similar to (1) and (2), but are more convenient to analyse as both of their kernels are abelian. We resolve these two extension problems completely, beginning with an algebraic description of the second one in Sect. 2, which enables us in particular to decide when it splits.

Theorem A *For $n \geq 3$ odd and $g \geq 1$, the extension*

$$0 \longrightarrow H(g) \otimes S\pi_n\mathrm{SO}(n) \longrightarrow \Gamma_g^n / \Theta_{2n+1} \longrightarrow G_g \longrightarrow 0$$

splits for $n \neq 3, 7$. For $n = 3, 7$, it splits if and only if $g = 1$.

Unlike (4), the extension (3) is central and thus classified by a class in $H^2(\Gamma_g^n / \Theta_{2n+1}; \Theta_{2n+1})$, which our main result, Theorem B below, identifies in terms of three cohomology classes

- (i) $\frac{\mathrm{sgn}}{8} \in H^2(\Gamma_g^n / \Theta_{2n+1}; \mathbf{Z})$ for $n \neq 3, 7$ odd,
- (ii) $\frac{\chi^2}{2} \in H^2(\Gamma_g^n / \Theta_{2n+1}; \mathbf{Z})$ for $n \equiv 3 \pmod{4}$ and $n \neq 3, 7$,
- (iii) $\frac{\chi^2 - \mathrm{sgn}}{8} \in H^2(\Gamma_g^n / \Theta_{2n+1}; \mathbf{Z})$ for $n = 3, 7$,

Table 1 $S\pi_n\text{SO}(n)$ for $n \geq 3$, except that $S\pi_6\text{SO}(6) = 0$

$n \pmod{8}$	0	1	2	3	4	5	6	7
$S\pi_n\text{SO}(n)$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	\mathbf{Z}	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	\mathbf{Z}

which can be expressed algebraically, using our description of Γ_g^n/Θ_{2n+1} mentioned above. Referring to Sect. 3 for the precise definition of these classes, we encourage the reader to think of them geometrically in terms of their pullbacks along the composition

$$\text{BDiff}^+(W_g) \longrightarrow \text{B}\Gamma_g^n \longrightarrow \text{B}(\Gamma_g^n/\Theta_{2n+1})$$

induced by taking path components and quotients: the pullback of the first class, which is closely related to *Meyer’s signature cocycle* [43], evaluates a class $[\pi] \in \text{H}_2(\text{BDiff}^+(W_g); \mathbf{Z})$ represented by an oriented smooth fibre bundle $\pi : E \rightarrow S$ over a closed oriented surface S with fibre W_g to an eighth of the signature $\text{sgn}(E)$ of its total space, the pullback of the second one assigns such a bundle the Pontryagin number $p_{(n+1)/4}^2(E)$ up to a fixed constant, and the pullback of the third class evaluates $[\pi]$ to a certain linear combination of $\text{sgn}(E)$ and $p_{(n+1)/4}^2(E)$. In addition to these three classes, our identification of the extension (3) for $n \geq 3$ odd involves two particular homotopy spheres: the first one, $\Sigma_P \in \Theta_{2n+1}$, is the *Milnor sphere*—the boundary of the E_8 -plumbing [8, Sect. V], and the second one, $\Sigma_Q \in \Theta_{2n+1}$, arises as the boundary of the plumbing of two copies of a linear D^{n+1} -bundles over S^{n+1} classified by a generator of $S\pi_n\text{SO}(n)$. We write $(-)\cdot\Sigma : \text{H}^2(\Gamma_g^n/\Theta_{2n+1}; \mathbf{Z}) \rightarrow \text{H}^2(\Gamma_g^n/\Theta_{2n+1}; \Theta_{2n+1})$ for the change of coefficients induced by $\Sigma \in \Theta_{2n+1}$.

Theorem B *For $n \geq 3$ odd and $g \geq 1$, the central extension*

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow \Gamma_g^n \longrightarrow \Gamma_g^n/\Theta_{2n+1} \longrightarrow 0$$

is classified by

- (i) $\frac{\text{sgn}}{8} \cdot \Sigma_P \in \text{H}^2(\Gamma_g^n/\Theta_{2n+1}; \Theta_{2n+1})$ for $n \equiv 1 \pmod{4}$,
- (ii) $\frac{\text{sgn}}{8} \cdot \Sigma_P + \frac{\chi^2}{2} \cdot \Sigma_Q \in \text{H}^2(\Gamma_g^n/\Theta_{2n+1}; \Theta_{2n+1})$ for $n \equiv 3 \pmod{4}$ if $n \neq 3, 7$,
- (iii) $\frac{\chi^2 - \text{sgn}}{8} \cdot \Sigma_Q \in \text{H}^2(\Gamma_g^n/\Theta_{2n+1}; \Theta_{2n+1})$ for $n = 3, 7$.

Moreover, this extension splits if and only if $n \equiv 1 \pmod{4}$ and $g = 1$.

The extension (2) describing the Torelli group T_g^n is the pullback of the extension determined in Theorem B along the map $H(g) \otimes S\pi_n\text{SO}(n) \rightarrow \Gamma_g^n/\Theta_{2n+1}$, so the combination of the previous result with our identification of Γ_g^n/Θ_{2n+1} provides an algebraic description of both Γ_g^n and T_g^n in terms of G_g and Θ_{2n+1} . We derive several consequences from this, beginning with deciding when the more commonly considered extensions (1) and (2) split.

Corollary C *Let $n \geq 3$ odd and $g \geq 1$.*

(i) *The extension*

$$0 \longrightarrow \mathbb{T}_g^n \longrightarrow \Gamma_g^n \longrightarrow G_g \longrightarrow 0$$

does not split for $g \geq 2$, but admits a splitting for $g = 1$ and $n \neq 3, 7$.

(ii) *The extension*

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow \mathbb{T}_g^n \longrightarrow H(g) \otimes S\pi_n \text{SO}(n) \longrightarrow 0$$

does not split for $n \equiv 3 \pmod{4}$, but splits G_g -equivariantly for $n \equiv 1 \pmod{4}$.

Abelian quotients

The second application of our description of the groups Γ_g^n and \mathbb{T}_g^n is a computation of their abelianisations.

Although Theorem A exhibits the extension (4) as nontrivial in some cases, its abelianisation turns out to split (see Corollary 2.4), so there exists a splitting

$$\mathbb{H}_1(\Gamma_g^n / \Theta_{2n+1}) \cong \mathbb{H}_1(G_g) \oplus (H(g) \otimes S\pi_n \text{SO}(n))_{G_g},$$

which participates in the following identification of the abelianisation of Γ_g^n and \mathbb{T}_g^n .

Corollary D *Let $g \geq 1$ and $n \geq 3$ odd.*

(i) *The extension (3) induces a split short exact sequence*

$$0 \longrightarrow \Theta_{2n+1}/K_g \longrightarrow \mathbb{H}_1(\Gamma_g^n) \xrightarrow{p_*} \mathbb{H}_1(G_g) \oplus (H(g) \otimes S\pi_n \text{SO}(n))_{G_g} \longrightarrow 0,$$

where $K_g = \langle \Sigma_P, \Sigma_Q \rangle$ for $g \geq 2$ and $K_g = \langle \Sigma_Q \rangle$ for $g = 1$.

(ii) *The extension (4) induces a split short exact sequence of G_g -modules*

$$0 \longrightarrow \Theta_{2n+1}/\langle \Sigma_Q \rangle \longrightarrow \mathbb{H}_1(\mathbb{T}_g^n) \xrightarrow{\rho_*} H(g) \otimes S\pi_n \text{SO}(n) \longrightarrow 0.$$

In particular, the commutator subgroup of \mathbb{T}_g^n is generated by Σ_Q .

These splittings of $\mathbb{H}_1(\Gamma_g^n)$ and $\mathbb{H}_1(\mathbb{T}_g^n)$ are constructed abstractly, but can often be made more concrete by means of a refinement of the mapping torus as a map

$$t: \Gamma_g^n \longrightarrow \Omega_{2n+1}^{\tau_{>n}}$$

to the bordism group of closed $(2n+1)$ -manifolds M equipped with a lift of their stable normal bundle $M \rightarrow \text{BO}$ to the n -connected cover $\tau_{>n}\text{BO} \rightarrow \text{BO}$. To state the resulting more explicit description of the abelianisations of Γ_g^n and \mathbb{T}_g^n , we write σ'_n for the minimal positive signature of a closed smooth n -connected $(2n+2)$ -dimensional manifold. For $n \neq 1, 3, 7$ odd, the intersection form of such a manifold is unimodular and even, so σ'_n is divisible by 8.

Corollary E *Let $n \geq 3$ odd and $g \geq 1$.*

(i) *The morphism*

$$t_* \oplus p_* : H_1(\Gamma_g^n) \longrightarrow \Omega_{2n+1}^{\tau > n} \oplus H_1(G_g) \oplus (H(g) \otimes S\pi_n \text{SO}(n))_{G_g}$$

is an isomorphism for $g \geq 2$, and for $g = 1$ if $n = 3, 7$. For $g = 1$ and $n \neq 3, 7$, it is surjective, has kernel of order $\sigma'_n/8$ generated by Σ_P , and splits for $n \equiv 1 \pmod{4}$.

(ii) *The morphism*

$$t_* \oplus \rho_* : H_1(T_g^n) \longrightarrow \Omega_{2n+1}^{\tau > n} \oplus (H(g) \otimes S\pi_n \text{SO}(n))$$

is an isomorphism for $n = 3, 7$. For $n \neq 3, 7$, it is surjective, has kernel of order $\sigma'_n/8$ generated by Σ_P , and splits G_g -equivariantly for $n \equiv 1 \pmod{4}$.

Remark (i) In Theorem G below, we determine the abelianisation of $\Gamma_{g,1}^n$ and $T_{g,1}^n$ for $n \geq 4$ even in which case the morphisms $t_* \oplus p_*$ and $t_* \oplus \rho_*$ are isomorphisms for all $g \geq 1$.

- (ii) As shown in [34, Prop. 2.15], the minimal signature σ'_n is nontrivial for n odd, grows very quickly with n , and can be expressed in terms of Bernoulli numbers.
- (iii) For some values of g and n , Theorem E leaves open whether $t_* \oplus p_*$ and $t_* \oplus \rho_*$ split. The morphisms p_* and ρ_* always split by Theorem D, and in Sect. 4.1 we relate the question of whether there exist compatible splittings of t_* to a known open problem in the theory of highly connected manifold, showing in particular that such splittings do exist when assuming a conjecture of Galatius and Randal-Williams.
- (iv) Work of Thurston [52] shows that the component $\text{Diff}_0(W_g) \subset \text{Diff}^+(W_g)$ of the identity diffeomorphism is perfect as a discrete group, so the abelianisation of the full diffeomorphism group $\text{Diff}^+(W_g)$ considered as a discrete group agrees with $H_1(\Gamma_g^n)$.

In view of Theorem E, it is of interest to determine the bordism groups $\Omega_{2n+1}^{\tau > n}$, the abelianisation $H_1(G_g)$, and the coinvariants $(H(g) \otimes S\pi_n \text{SO}(n))_{G_g}$. The computation

$$(H(g) \otimes S\pi_n \text{SO}(n))_{G_g} \cong \begin{cases} 0 & g \geq 2 \text{ or } n = 3, 6, 7 \text{ or } n \equiv 5 \pmod{8} \\ \mathbf{Z}/2^2 & g = 1 \text{ and } n \equiv 0 \pmod{8} \\ \mathbf{Z}/2 & \text{otherwise} \end{cases} \quad (5)$$

is straightforward (see Lemma A.2 and Table 1). The abelianisation of G_g is known and summarised in Table 2 (see Lemma A.1). Finally, the bordism groups $\Omega_{2n+1}^{\tau > n}$ are closely connected to the stable homotopy groups of spheres: the canonical map

$$\pi_{2n+1}^s \cong \Omega_{2n+1}^{\text{fr}} \longrightarrow \Omega_{2n+1}^{\tau > n}$$

factors through the cokernel of the J -homomorphism and work of Schultz and Wall [50,55] implies that the induced morphism is often an isomorphism (see Corollary 3.6).

Table 2 The abelianisation of G_g for n odd

$H_1(G_g)$	$g = 1$	$g = 2$	$g \geq 3$
$n = 1, 3, 7$	$\mathbf{Z}/12$	$\mathbf{Z}/2$	0
$n \neq 1, 3, 7$ odd	$\mathbf{Z}/4 \oplus \mathbf{Z}$	$\mathbf{Z}/4 \oplus \mathbf{Z}/2$	$\mathbf{Z}/4$

Table 3 Some abelianisations of T_g^n and Γ_g^n

	$n = 3$	$n = 5$	$n = 7$	$n = 9$
$H_1(T_g^n)$				
$g = 0$	$\mathbf{Z}/28$	$\mathbf{Z}/992$	$\mathbf{Z}/2 \oplus \mathbf{Z}/8128$	$\mathbf{Z}/2 \oplus \mathbf{Z}/261632$
$g \geq 1$	\mathbf{Z}^{2g}	$\mathbf{Z}/992$	$\mathbf{Z}/2 \oplus \mathbf{Z}^{2g}$	$\mathbf{Z}/2 \oplus (\mathbf{Z}/2)^{2g} \oplus \mathbf{Z}/261632$
$H_1(\Gamma_g^n)$				
$g = 0$	$\mathbf{Z}/28$	$\mathbf{Z}/992$	$\mathbf{Z}/2 \oplus \mathbf{Z}/8128$	$\mathbf{Z}/2 \oplus \mathbf{Z}/261632$
$g = 1$	$\mathbf{Z}/12$	$\mathbf{Z}/4 \oplus \mathbf{Z} \oplus \mathbf{Z}/992$	$\mathbf{Z}/12 \oplus \mathbf{Z}/2$	$\mathbf{Z}/4 \oplus \mathbf{Z} \oplus (\mathbf{Z}/2)^2 \oplus \mathbf{Z}/261632$
$g = 2$	$\mathbf{Z}/2$	$\mathbf{Z}/4 \oplus \mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2 \oplus \mathbf{Z}/4$
$g \geq 3$	0	$\mathbf{Z}/4$	$\mathbf{Z}/2$	$\mathbf{Z}/2 \oplus \mathbf{Z}/4$

Theorem (Schultz, Wall). *For n odd, the natural morphism $\text{coker}(J)_{2n+1} \rightarrow \Omega_{2n+1}^{\tau_{>n}}$ is surjective with cyclic kernel. It is an isomorphism for $n \equiv 1 \pmod{4}$ and for $n = 3, 7$.*

Combined with Theorem E, this reduces the computation of the abelianisation of Γ_g^n and T_g^n in many cases to determining the cokernel of the J -homomorphism—a well-studied problem in stable homotopy theory. Table 3 shows the resulting calculation of the abelianisations of the groups Γ_g^n and T_g^n for the first few values of n .

Remark After the completion of this work, Burkund–Hahn–Senger [11] and Burkund–Senger [12] showed that for n odd, the homotopy sphere $\Sigma_Q \in \Theta_{2n+1}$ bounds a parallelisable manifold if and only if $n \neq 11$. This implies in particular that aside from the case $n = 11$

- (i) the canonical map $\text{coker}(J)_{2n+1} \rightarrow \Omega_{2n+1}^{\tau_{>n}}$ is an isomorphism, which extends the theorem attributed to Schultz and Wall above,
- (ii) the conjecture of Galatius–Randal-Williams mentioned in the third part of the previous remark holds, and
- (iii) the minimal signature σ'_n appearing in Theorem E is computable from [34, Prop. 2.15].

Homotopy equivalences

As an additional application of our results, we briefly discuss the group $\pi_0 \text{hAut}^+(W_g)$ of homotopy classes of orientation-preserving homotopy equivalences.

The natural map $\Gamma_g^n \rightarrow \pi_0 \text{hAut}^+(W_g)$ can be seen to factor through the quotient $\Gamma_g^n / \Theta_{2n+1}$ and to induce a commutative diagram of the form

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H(g) \otimes S\pi_n \text{SO}(n) & \longrightarrow & \Gamma_g^n / \Theta_{2n+1} & \longrightarrow & G_g \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & H(g) \otimes S\pi_{2n} S^n & \longrightarrow & \pi_0 \text{hAut}^+(W_g) & \longrightarrow & G_g \longrightarrow 0,
 \end{array} \tag{6}$$

which exhibits the lower row—an extension describing $\pi_0 \text{hAut}^+(W_g)$ due to Baues [1]—as the extension pushout of the extension (4) along the left vertical morphism, which is induced by the restriction $J : S\pi_n \text{SO}(n) \rightarrow S\pi_{2n} S^n$ of the unstable J -homomorphism, where $S\pi_{2n} S^n$ is the image of the suspension map $S : \pi_{2n} S^n \rightarrow \pi_{2n+1} S^{n+1}$. By Theorem A, the upper row splits in most cases and thus induces a compatible splitting of Baues’ extension. In the cases in which the upper row does not split, we show that Baues’ extension cannot split either.

Corollary F *Let $n \geq 3$ odd and $g \geq 1$.*

- (i) *For $n \neq 3, 7$, the two extensions in (6) admit compatible splittings. For $n = 3, 7$, either of the extensions splits if and only if $g = 1$.*
- (ii) *The induced morphism $H_1(\pi_0 \text{hAut}^+(W_g)) \rightarrow H_1(G_g)$ is an isomorphism for $g \geq 2$, and a split epimorphism with kernel the coinvariants $(H(g) \otimes S\pi_{2n} S^n)_{G_g}$, which vanish for $g \geq 2$ or $n = 3, 7$, and are isomorphic to the group $S\pi_{2n} S^n / (2 \cdot S\pi_{2n} S^n)$ otherwise.*

The groups $\Gamma_{g,1}^n$ for n even

Some parts in our analysis of $\Gamma_{g,1}^n$ go through when $n \geq 4$ is even as well, but a few key steps do not and would require new arguments. For instance, a different approach to the extension problem (4) would be necessary, as well as an extension of Theorem 3.12 to incorporate the Arf invariant. The abelianisation of the groups $\Gamma_{g,1}^n$ and $T_{g,1}^n$, however, can be determined without fully solving the extensions (3) and (4) if n is even. It turns out that in this case, the morphisms considered in Theorem E are isomorphisms for all $g \geq 1$, which we shall prove as part of Sect. 4.2.

Theorem G *For $n \geq 4$ even and $g \geq 1$, the morphisms*

$$\begin{aligned}
 t_* \oplus p_* : H_1(\Gamma_{g,1}^n) &\longrightarrow \Omega_{2n+1}^{\tau > n} \oplus H_1(G_g) \oplus (H(g) \otimes S\pi_n \text{SO}(n))_{G_g} \\
 t_* \oplus \rho_* : H_1(T_{g,1}^n) &\longrightarrow \Omega_{2n+1}^{\tau > n} \oplus (H(g) \otimes S\pi_n \text{SO}(n))
 \end{aligned}$$

are isomorphisms for $g \geq 1$. The group of coinvariants $(H(g) \otimes S\pi_n \text{SO}(n))_{G_g}$ is described in (5).

Other highly connected manifolds

Instead of restricting to W_g , one could consider any $(n-1)$ -connected almost parallelisable manifold M of dimension $2n \geq 6$. Baues' and Kreck's work [1,35] applies in this generality, so there are analogues of the sequences (1)–(6) describing $\pi_0 \text{Diff}^+(M)$ and $\pi_0 \text{hAut}^+(M)$. However, for n odd—the case of our main interest—Wall's classification of highly connected manifolds [55] implies that any such manifold is diffeomorphic to a connected sum $W_g \sharp \Sigma$ with an exotic sphere $\Sigma \in \Theta_{2n}$, aside from those of Kervaire invariant 1, which only exist in dimensions 6, 14, 30, 62, and possibly 126 by work of Hill–Hopkins–Ravenel [25]. The mapping class group $\pi_0 \text{Diff}^+(W_g \sharp \Sigma)$ for $\Sigma \in \Theta_{2n}$ and n odd in turn is understood in terms of Γ_g^n : Kreck's work [35, Lem. 3, Thm 3] shows that the former is a quotient of the latter by a known element $\Sigma' \in \Theta_{2n+1}$ of order at most 2, which is trivial if and only if $\eta \cdot [\Sigma] \in \text{coker}(J)_{2n+1}$ vanishes.

Previous results

The extensions (1) and (2) and their variants (3) and (4) have been studied by various authors before, and some special cases of our results were already known:

- (i) As an application of their programme on moduli spaces of manifolds, Galatius–Randal-Williams [21] determined the abelianisation of Γ_g^n for $g \geq 5$ and used this to determine the extension (3) for $n \equiv 5 \pmod{8}$ up to automorphisms of Θ_{2n+1} as long $g \geq 5$. Our work recovers and extends their results, also applies to low genera $g < 5$, and does not rely on their work on moduli spaces of manifolds.
- (ii) Theorems A and F for $n = 3, 7$ reprove results due to Crowley [15].
- (iii) Baues [1, Thm 8.14, Thm 10.3] showed that the lower extension in (6) splits for $n \neq 3, 7$ odd, which we recover as part of the first part of Corollary F.
- (iv) The case $(g, n) = (1, 3)$ of Theorem A and Theorem C (ii) can be deduced from work of Krylov [37] and Fried [19], who also showed that the extension of Theorem C (i) does not split in this case. Krylov [36, Thms 2.1, 3.2, 3.3] moreover established the case $n \equiv 5 \pmod{8}$ of Theorem C (i) for $g = 1$. For $n \neq 3, 7$, he also proved the case $n \equiv 3 \pmod{4}$ of Theorem A and the case $n \equiv 3 \pmod{4}$ of Theorem C (ii) for $g = 1$.

Further applications

Our main result Theorem B has been used in [32] in conjunction with Galatius–Randal-Williams' work on moduli spaces of manifolds [22] to compute the second stable homology of the *theta-subgroup* of $\text{Sp}_{2g}(\mathbf{Z})$ (see Sect. 1.2), or equivalently, the second quadratic symplectic algebraic K -theory group of the integers $\text{KSp}_2^q(\mathbf{Z})$.

Outline

Section 1 serves to recall foundational material on diffeomorphism groups and their classifying spaces, as well as to introduce different variants of the extensions (1) and (2) and to establish some of their basic properties. In Sect. 2, we study the action of Γ_g^n on the set of stable framings of W_g to identify the extension (3) and prove Theorem A. Section 3 aims at the proof of our main result Theorem B, which requires some preparation. We recall the relation between relative Pontryagin classes and obstruction theory in Sect. 3.1, discuss aspects of Wall’s classification of highly connected manifolds in Sect. 3.2, relate this class of manifolds to $W_{g,1}$ -bundles over surfaces with certain boundary conditions in Sect. 3.3 (which incidentally is the key geometric insight to prove Theorem B), construct the cohomology classes appearing in the statement of Theorem B in Sects. 3.4 and 3.5, and finish with the proof of Theorem B in Sect. 3.6. In Sect. 4, we analyse the extensions (1) and (2) and compute the abelianisation of Γ_g^n and T_g^n , proving Corollaries C–E and Theorem G. Section 5 briefly discusses the group of homotopy equivalences and proves Corollary F. In the appendix, we compute various low-degree (co)homology groups of the symplectic group $\mathrm{Sp}_{2g}(\mathbf{Z})$ and its arithmetic subgroup $G_g \subset \mathrm{Sp}_{2g}(\mathbf{Z})$.

1 Variations on two extensions of Kreck

1.1 Different flavours of diffeomorphisms

Throughout this work, we write

$$W_g = \sharp^g(S^n \times S^n) \quad \text{and} \quad W_{g,1} = \sharp^g(S^n \times S^n) \setminus \mathrm{int}(D^{2n})$$

for the g -fold connected sum of $S^n \times S^n$, including $W_0 = S^{2n}$, and the manifold obtained from W_g removing the interior of an embedded disc $D^{2n} \subset W_g$. Occasionally, we view the manifold $W_{g,1}$ alternatively as the iterated boundary connected sum $W_{g,1} = \natural^g W_{1,1}$ of $W_{1,1} = S^n \times S^n \setminus \mathrm{int}(D^{2n})$. We call g the *genus* of W_g or $W_{g,1}$ and denote by $\mathrm{Diff}^+(W_g)$ and $\mathrm{Diff}^+(W_{g,1})$ the groups of orientation-preserving diffeomorphisms, not necessarily fixing the boundary in the case of $W_{g,1}$. We shall also consider the subgroups

$$\mathrm{Diff}_\partial(W_{g,1}) \subset \mathrm{Diff}_{\partial/2}(W_{g,1}) \subset \mathrm{Diff}^+(W_{g,1})$$

of diffeomorphisms required to fix a neighbourhood of the boundary $\partial W_{g,1} \cong S^{2n-1}$ or a neighbourhood of a chosen disc $D^{2n-1} \subset \partial W_{g,1}$ in the boundary, respectively. All groups of diffeomorphisms are equipped with the smooth topology so that

- (i) $\mathrm{BDiff}^+(W_{g,1})$ and $\mathrm{BDiff}^+(W_g)$ classify smooth oriented $W_{g,1}$ -bundles or W_g -bundles, respectively,
- (ii) $\mathrm{BDiff}_\partial(W_{g,1})$ classifies $(W_{g,1}, S^{2n-1})$ -bundles, i.e. smooth $W_{g,1}$ -bundles with a trivialisation of their S^{2n-1} -bundle of boundaries, and
- (iii) $\mathrm{BDiff}_{\partial/2}(W_{g,1})$ classifies $(W_{g,1}, D^{2n-1})$ -bundles, that is, smooth $W_{g,1}$ -bundles with a trivialised D^{2n-1} -subbundle of its S^{2n-1} -bundle of boundaries.

Taking path components, we obtain various groups of isotopy classes

$$\Gamma_g^n = \pi_0 \text{Diff}^+(W_g), \quad \Gamma_{g,1/2}^n = \pi_0 \text{Diff}_{\partial/2}(W_g), \quad \text{and} \quad \Gamma_{g,1}^n = \pi_0 \text{Diff}_{\partial}(W_{g,1}).$$

Extending diffeomorphisms by the identity provides a map $\text{Diff}_{\partial}(W_{g,1}) \rightarrow \text{Diff}^+(W_g)$, which induces an isomorphism on path components by work of Kreck as long as $n \geq 3$.

Lemma 1.1 (Kreck) *The canonical map $\Gamma_{g,1}^n \rightarrow \Gamma_g^n$ is an isomorphism for $n \geq 3$.*

Proof Taking the differential at the centre of the disc induces a fibration $\text{Diff}^+(W_g) \rightarrow \text{Fr}^+(W_g)$ to the oriented frame bundle of W_g . Its fibre is the subgroup of diffeomorphisms that fix a point and its tangent space, so it is equivalent to the subgroup of diffeomorphisms fixing a small disc around that point, which is in turn equivalent to $\text{Diff}_{\partial}(W_{g,1})$. We thus have fibration sequences of the form

$$\text{Diff}_{\partial}(W_{g,1}) \longrightarrow \text{Diff}^+(W_{g,1}) \longrightarrow \text{Fr}^+(W_g) \quad \text{and} \quad \text{SO}(2n) \longrightarrow \text{Fr}^+(W_g) \longrightarrow W_g$$

whose long exact sequences show that the morphism in question is surjective and also that its kernel is generated by a single isotopy class given by “twisting” a collar $[0, 1] \times S^{2n-1} \subset W_{g,1}$ using a smooth based loop in $\text{SO}(2n)$ that represents a generator of $\pi_1 \text{SO}(2n) \cong \mathbf{Z}/2$ (see [35, p. 647]). It follows from [35, Lem. 3 b), Lem. 4] that this isotopy class is trivial as W_g bounds the, handlebody $\natural^g(D^{n+1} \times S^n)$, which is parallelisable. \square

For the purpose of studying the mapping class group Γ_g^n , we can thus equally well work with $\text{Diff}_{\partial}(W_{g,1})$ instead of $\text{Diff}^+(W_g)$, which is advantageous since there is a stabilisation map $\text{Diff}_{\partial/2}(W_{g,1}) \rightarrow \text{Diff}_{\partial/2}(W_{g+1,1})$ by extending diffeomorphisms over an additional boundary connected summand via the identity, which restricts to a map $\text{Diff}_{\partial}(W_{g,1}) \rightarrow \text{Diff}_{\partial}(W_{g+1,1})$ and thus induces stabilisation maps of the form

$$\Gamma_{g,1/2}^n \longrightarrow \Gamma_{g,1/2}^n \quad \text{and} \quad \Gamma_{g,1}^n \longrightarrow \Gamma_{g+1,1}^n, \tag{1.1}$$

that allow us to compare mapping class groups of different genera. The group $\Gamma_{0,1}$ has a convenient alternative description: gluing two closed d -discs along their boundaries via a diffeomorphism supported in a disc $D^d \subset \partial D^{d+1}$ gives a morphism $\pi_0 \text{Diff}_{\partial}(D^d) \rightarrow \Theta_{d+1}$ to Kervaire–Milnor’s [30] finite abelian group Θ_d of oriented homotopy d -spheres up to h -cobordism. By work of Cerf [13], this is an isomorphism for $d \geq 5$, so we identify these two groups henceforth. Iterating the stabilisation map yields a sequence of maps

$$\Theta_{2n+1} = \pi_0 \text{Diff}_{\partial}(D^{2n}) = \Gamma_{0,1}^n \longrightarrow \Gamma_{g,1}^n \longrightarrow \Gamma_{g,1/2}^n. \tag{1.2}$$

Lemma 1.2 *For $n \geq 3$, the image of Θ_{2n+1} in $\Gamma_{g,1}^n$ is central and becomes trivial in $\Gamma_{g,1/2}^n$. The induced morphism*

$$\left(\Gamma_{g,1}^n / (\text{im}(\Theta_{2n+1} \rightarrow \Gamma_{g,1}^n)) \right) \longrightarrow \Gamma_{g,1/2}^n$$

is an isomorphism.

Proof Every diffeomorphism of $W_{g,1}$ supported in a disc is isotopic to one that is supported in an arbitrary small neighbourhood of the boundary and thus commutes with any diffeomorphism in $\text{Diff}_\partial(W_{g,1})$ up to isotopy, which shows the first part of the claim. For the others, we consider the sequence of topological groups

$$\text{Diff}_\partial(W_{g,1}) \longrightarrow \text{Diff}_{\partial/2}(W_{g,1}) \longrightarrow \text{Diff}_\partial(D^{2n-1}) \tag{1.3}$$

induced by restricting diffeomorphisms in $\text{Diff}_{\partial/2}(W_{g,1})$ to the moving part of the boundary. This is a fibration sequence by the parametrised isotopy extension theorem. Mapping this sequence for $g = 0$ into (1.3) via the iterated stabilisation map, we see that the looped map $\Omega\text{Diff}_\partial(D^{2n-1}) \rightarrow \text{Diff}_\partial(W_{g,1})$ induced by the fibration sequence (1.3) factors as the composition

$$\Omega\text{Diff}_\partial(D^{2n-1}) \longrightarrow \text{Diff}_\partial(D^{2n}) \longrightarrow \text{Diff}_\partial(W_{g,1})$$

of the map defining the Gromoll filtration with the iterated stabilisation map. Since the first map in this factorisation is surjective on path components by Cerf’s work [13], the claim will follow from the long exact sequence on homotopy groups of (1.3) once we show that the map $\Gamma_{g,1/2}^n \rightarrow \pi_0\text{Diff}_\partial(D^{2n-1}) = \Theta_{2n}$ has trivial image. Using that any orientation preserving diffeomorphism fixes any chosen oriented codimension 0 disc up isotopy by the isotopy extension theorem, one sees that this image agrees with the inertia group of W_g , which vanishes by work of Kosinski and Wall [31,54]. \square

1.2 Wall’s quadratic form

We recall Wall’s quadratic form associated to an $(n - 1)$ -connected $2n$ -manifold [55], specialised to the case of our interest—the iterated connected sums $W_g = \sharp^g(S^n \times S^n)$ in dimension $2n \geq 6$.

The intersection form $\lambda: H(g) \otimes H(g) \rightarrow \mathbf{Z}$ on the middle cohomology $H(g) := H^n(W_g; \mathbf{Z})$ is a nondegenerate $(-1)^n$ -symmetric bilinear form. We use Poincaré duality to identify $H(g)$ with $\pi_n(W_g; \mathbf{Z}) \cong H_n(W_g)$ and a result of Haefliger [24] to represent classes in $\pi_n(W_g)$ by embedded spheres $e: S^n \hookrightarrow W_g$, unique up to isotopy as long as $n \geq 4$. As W_g is stably parallelisable, the normal bundle of such e is stably trivial and hence gives a class $q([e])$ in

$$\ker(\pi_n(\text{BSO}(n)) \rightarrow \pi_n(\text{BSO}(n + 1))) \cong \mathbf{Z}/\Lambda_n = \begin{cases} \mathbf{Z} & \text{if } n \text{ even} \\ \mathbf{Z}/2 & \text{if } n \text{ odd, } n \neq 3, 7, \\ 0 & \text{if } n = 3, 7. \end{cases}$$

where Λ_n is the image of the usual map $\pi_n(\text{SO}(n + 1)) \rightarrow \pi_n(S^n) \cong \mathbf{Z}$ (see e.g. [41, §1.B]).¹ This defines a function $q: H(g) \rightarrow \mathbf{Z}/\Lambda_n$, which Wall [55] showed to satisfy the following two properties.

- (i) $q(k \cdot [e]) = k^2 \cdot q([e])$ and

¹ Note that \mathbf{Z}/Λ_n is the trivial in the case $n = 3$ where Haefliger’s result does not apply.

(ii) $q([e] + [f]) = q([e]) + q([f]) + \lambda([e], [f]) \pmod{\Lambda_n}$.

Note that for n even, (i) and (ii) imply $q([e]) = \frac{1}{2}\lambda([e], [e]) \in \mathbf{Z}$, so q can in this case be recovered from λ . The triple $(H(g), \lambda, q)$ is the *quadratic form* associated to W_g . The decomposition $W_g = \sharp^g(S^n \times S^n)$ into connected summands induces a basis $(e_1, \dots, e_g, f_1, \dots, f_g)$ of $H(g) \cong \mathbf{Z}^{2g}$ with respect to which q and λ have the form

$$q: \begin{matrix} \mathbf{Z}^{2g} & \longrightarrow & \mathbf{Z}/\Lambda_n \\ (x_1, \dots, x_g, y_1, \dots, y_g) & \longmapsto & \sum_{i=1}^g x_i y_i \end{matrix} \quad \text{and} \quad J_{g,(-1)^n} = \begin{pmatrix} 0 & I \\ (-1)^n I & 0 \end{pmatrix},$$

so the automorphism group of the quadratic form can be identified as

$$G_g := \text{Aut}(H(g), \lambda, q) \cong \begin{cases} \text{O}_{g,g}(\mathbf{Z}) & n \text{ even} \\ \text{Sp}_{2g}^q(\mathbf{Z}) & n \text{ odd, } n \neq 3, 7 \\ \text{Sp}_{2g}(\mathbf{Z}) & n = 3, 7, \end{cases}$$

where

$$\begin{aligned} \text{O}_{g,g}(\mathbf{Z}) &= \{A \in \mathbf{Z}^{2g \times 2g} \mid A^T J_{g,1} A = J_{g,1}\}, \\ \text{Sp}_{2g}(\mathbf{Z}) &= \{A \in \mathbf{Z}^{2g \times 2g} \mid A^T J_{g,-1} A = J_{g,-1}\}, \\ \text{Sp}_{2g}^q(\mathbf{Z}) &= \{A \in \text{Sp}_{2g}(\mathbf{Z}) \mid qA = q\}. \end{aligned}$$

In the theory of theta-functions, the finite index subgroup $\text{Sp}_{2g}^q(\mathbf{Z}) \leq \text{Sp}_{2g}(\mathbf{Z})$ is known as the *theta group*; it is the stabiliser of the standard theta-characteristic with respect to the transitive $\text{Sp}_{2g}(\mathbf{Z})$ -action on the set of even characteristics (see e.g. [58]). Using this description, it is straightforward to compute its index in $\text{Sp}_{2g}(\mathbf{Z})$ to be $2^{2g-1} + 2^{g-1}$.

1.3 Kreck’s extensions

To recall Kreck’s extensions [35, Prop.3] describing $\Gamma_{g,1}^n$ for $n \geq 3$, note that an orientation-preserving diffeomorphism of W_g induces an automorphism of the quadratic form $(H(g), \lambda, q)$. This provides a morphism $\Gamma_{g,1}^n \rightarrow G_g$, which Kreck proved to be surjective using work of Wall [56].² This explains the first extension

$$0 \longrightarrow \text{T}_{g,1}^n \longrightarrow \Gamma_{g,1}^n \longrightarrow G_g \longrightarrow 0. \tag{1.4}$$

The second extension describes the Torelli subgroup $\text{T}_{g,1}^n \subset \Gamma_{g,1}^n$ and has the form

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow \text{T}_{g,1}^n \xrightarrow{\rho} H(g) \otimes S\pi_n \text{SO}(n) \longrightarrow 0, \tag{1.5}$$

where $S\pi_n \text{SO}(n)$ denotes the image of the morphism $S: \pi_n \text{SO}(n) \rightarrow \pi_n \text{SO}(n+1)$ induced by the usual inclusion $\text{SO}(n) \subset \text{SO}(n+1)$. The isomorphism type of this

² Kreck phrases his results in terms of pseudo-isotopy instead of isotopy. By Cerf’s “pseudo-isotopy implies isotopy” [13], this does not make a difference as long as $n \geq 3$.

image can be extracted from work of Kervaire [28] to be as shown in Table 1. As a diffeomorphism supported in a disc acts trivially on cohomology, the morphism $\Theta_{2n+1} \rightarrow \Gamma_{g,1}^n$ in (1.2) has image in $\Gamma_{g,1}^n$, which explains first map in the extension (1.5). To define the second one, we canonically identify $H(g) \otimes S\pi_n\text{SO}(n)$ with $\text{Hom}(H_n(W_g; \mathbf{Z}), S\pi_n\text{SO}(n))$ and note that for a given isotopy class $[\phi] \in \mathbb{T}_g^n$ and a class $[e] \in H_n(W_g; \mathbf{Z})$ represented by an embedded sphere $e: S^n \hookrightarrow W_g$, the embedding $\phi \circ e$ is isotopic to e , so we can assume that ϕ fixes e pointwise by the isotopy extension theorem. The derivative of ϕ thus induces an automorphism of the once stabilised normal bundle $\vartheta(e) \oplus \varepsilon$, which after choosing a trivialisation $\vartheta(e) \oplus \varepsilon \cong \varepsilon^{n+1}$ defines the image $\rho([\phi])([e]) \in \pi_n\text{SO}(n + 1)$ of $[e]$ under the morphism $\rho([\phi]) \in \text{Hom}(H_n(W_g; \mathbf{Z}), S\pi_n\text{SO}(n))$, noting that $\rho([\phi])([e])$ is independent of all choices and actually lies in the subgroup $S\pi_n\text{SO}(n) \subset \pi_n\text{SO}(n + 1)$ (see [35, Lem. 1]).

Instead of the extensions (1.4) and (1.5), we shall mostly be concerned with two closely related variants which we describe now. By Kreck’s result, the morphism $\Theta_{2n+1} \rightarrow \Gamma_{g,1}^n$ is injective, so gives rise to an extension $0 \rightarrow \Theta_{2n+1} \rightarrow \Gamma_{g,1}^n \rightarrow \Gamma_{g,1}^n / \Theta_{2n+1} \rightarrow 0$, which combined with the canonical identification $\Gamma_{g,1}^n / \Theta_{2n+1} \cong \Gamma_{g,1/2}^n$ of Lemma 1.2 has the form

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow \Gamma_{g,1}^n \longrightarrow \Gamma_{g,1/2}^n \longrightarrow 0 \tag{1.6}$$

and agrees with the extension induced by taking path components of the chain of inclusions $\text{Diff}_\partial(D^{2n}) \subset \text{Diff}_\partial(W_{g,1}) \subset \text{Diff}_{\partial/2}(W_{g,1})$. The action of $\Gamma_{g,1/2}^n$ on $H(g)$ preserves the quadratic form as $\Gamma_{g,1}^n \rightarrow \Gamma_{g,1/2}^n$ is surjective by Lemma 1.2, so this action yields an extension

$$0 \longrightarrow H(g) \otimes S\pi_n\text{SO}(n) \longrightarrow \Gamma_{g,1/2}^n \xrightarrow{p} G_g \longrightarrow 0, \tag{1.7}$$

which, via the isomorphism $\mathbb{T}_{g,1}^n / \Theta_{2n+1} \cong H(g) \otimes S\pi_n\text{SO}(n)$ induced by (1.5), corresponds to the quotient of the extension (1.4) by Θ_{2n+1} , using $\Gamma_{g,1}^n / \Theta_{2n+1} \cong \Gamma_{g,1/2}^n$ once more.

Lemma 1.3 *The action of G_g on $H(g) \otimes S\pi_n\text{SO}(n) \cong \text{Hom}(H_n(W_g; \mathbf{Z}), S\pi_n\text{SO}(n))$ induced by the extension (1.7) is through the standard action of G_g on $H_n(W_g; \mathbf{Z})$.*

Proof In view of the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{T}_{g,1}^n & \longrightarrow & \Gamma_{g,1}^n & \longrightarrow & G_g \longrightarrow 0 \\ & & \downarrow \rho & & \downarrow & & \parallel \\ 0 & \longrightarrow & H(g) \otimes S\pi_n\text{SO}(n) & \longrightarrow & \Gamma_{g,1/2}^n & \xrightarrow{p} & G_g \longrightarrow 0, \end{array}$$

it suffices to establish the identity $\rho(\phi \circ \psi \circ \phi^{-1}) = p(\phi) \cdot \rho(\psi)$ for all $\phi \in \mathbb{T}_{g,1}^n$ and $\psi \in \mathbb{T}_{g,1}^n$. Unwrapping the definition of ρ , the image of $p(\phi) \cdot \rho(\psi)$ on a homology class in $H_n(W_{g,1}; \mathbf{Z})$ is given by the automorphism

$$\varepsilon^{n+1} \xrightarrow{F^{-1}} \vartheta(\phi^{-1} \circ e) \oplus \varepsilon \xrightarrow{d(\psi)} \vartheta(\phi^{-1} \circ e) \oplus \varepsilon \xrightarrow{F} \varepsilon^{n+1},$$

where e is an embedded sphere which represents the homology class and is pointwise fixed by $\phi \circ \psi \circ \phi^{-1}$ and F is any choice of framing of $\vartheta(\phi^{-1} \circ e) \oplus \varepsilon$. For the particular choice of framing

$$\vartheta(e) \oplus \varepsilon \xrightarrow{d(\phi^{-1}) \oplus \varepsilon} \vartheta(\phi^{-1} \circ e) \oplus \varepsilon \xrightarrow{F} \varepsilon^{n+1}$$

to compute the image of $[e] \in H_n(W_{g,1}; \mathbf{Z})$ under $\rho(\phi \circ \psi \circ \phi^{-1})$, the claimed identity is a consequence of the chain rule for the differential. \square

1.4 Stabilisation

Iterating the stabilisation map (1.1) induces a morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Theta_{2n+1} & \longrightarrow & \Gamma_{h,1}^n & \longrightarrow & \Gamma_{h,1/2}^n \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Theta_{2n+1} & \longrightarrow & \Gamma_{g,1}^n & \longrightarrow & \Gamma_{g,1/2}^n \longrightarrow 0, \end{array}$$

of group extensions for $h \leq g$, which exhibits the upper row as the pullback of the lower row, so the extension (1.6) for a fixed genus g determines those for all $h \leq g$. The situation for the extension (1.7) is similar: writing $W_{g,1} \cong W_{h,1} \natural W_{g-h,1}$, we obtain a decomposition $H(g) \cong H(h) \oplus H(g-h)$, which yields a stabilisation map

$$s := (-) \oplus \text{id}_{H(g-h)}: G_h \longrightarrow G_g$$

and two morphisms of G_h -modules, an inclusion $H(h) \rightarrow s^*H(g)$ and a projection $s^*H(g) \rightarrow H(h)$. These morphisms express the extension (1.7) for genus $h \leq g$ as being obtained from that for genus g by pulling back along $s: G_h \rightarrow G_g$ followed by forming the extension pushout along $s^*H(g) \rightarrow H(h)$. They also induce a morphism of extensions of the form

$$\begin{array}{ccccccc} 0 & \longrightarrow & H(h) \otimes S\pi_n \text{SO}(n) & \longrightarrow & \Gamma_{h,1/2}^n & \longrightarrow & G_h \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H(g) \otimes S\pi_n \text{SO}(n) & \longrightarrow & \Gamma_{g,1/2}^n & \longrightarrow & G_g \longrightarrow 0. \end{array}$$

2 The action on the set of stable framings and Theorem A

This section serves to resolve the extension problem

$$0 \longrightarrow H(g) \otimes S\pi_n \text{SO}(n) \longrightarrow \Gamma_{g,1/2}^n \longrightarrow G_g \longrightarrow 0, \tag{2.1}$$

described in the previous section. Our approach is in parts inspired by work of Crowley [15], who identified this extension in the case $n = 3, 7$.

The group $\Gamma_{g,1/2}^n$ acts on the set of equivalence classes of stable framings

$$F: TW_{g,1} \oplus \varepsilon^k \cong \varepsilon^{2n+k} \quad \text{for } k \gg 0$$

that extend the standard stable framing on TD^{2n-1} , by pulling back stable framings along the derivative. As the equivalence classes of such framings naturally form a torsor for the group of pointed homotopy classes $[W_{g,1}, \text{SO}]_*$, the action of $\Gamma_{g,1/2}^n$ on a fixed choice of stable framing F as above yields a function

$$s_F: \pi_0 \text{Diff}_{\partial/2}(W_{g,1}) \longrightarrow [W_{g,1}, \text{SO}]_* \cong \text{Hom}(H_n(W_g; \mathbf{Z}), \pi_n \text{SO}) \cong H(g) \otimes \pi_n \text{SO},$$

where the first isomorphism is induced by $\pi_n(-)$ and the Hurewicz isomorphism, and the second one by the universal coefficient theorem. This function is a 1-cocycle (or *crossed homomorphism*) for the canonical action of $\Gamma_{g,1/2}^n$ on $H(g) \otimes \pi_n \text{SO}$ (cf. [15, Prop. 3.1]) and as this action factors through the map $p: \Gamma_{g,1/2}^n \rightarrow G_g$, we obtain a morphism of the form

$$(s_F, p): \Gamma_{g,1/2}^n \rightarrow (H(g) \otimes \pi_n \text{SO}) \rtimes G_g,$$

which is independent of F up to conjugation in the target by a straightforward check. This induces a morphism from the extension (2.1) to the trivial extension of G_g by the G_g -module $H(g) \otimes \pi_n \text{SO}$,

$$\begin{array}{ccccccc} 0 \rightarrow H(g) \otimes S\pi_n \text{SO}(n) & \longrightarrow & \Gamma_{g,1/2}^n & \xrightarrow{p} & G_g & \rightarrow & 0 \\ & & \downarrow & & \parallel & & \\ & & \downarrow & & \downarrow & & \\ 0 \longrightarrow H(g) \otimes \pi_n \text{SO} & \longrightarrow & (H(g) \otimes \pi_n \text{SO}) \rtimes G_g & \rightarrow & G_g & \rightarrow & 0. \end{array} \tag{2.2}$$

The left vertical map is induced by the natural map $S\pi_n \text{SO}(n) \rightarrow \pi_n \text{SO}$ originating from the inclusion $\text{SO}(n) \subset \text{SO}$ and is an isomorphism for $n \neq 1, 3, 7$ odd as a consequence of the following lemma whose proof is standard (see e.g. [41, §1B]).

Lemma 2.1 *For n odd, the morphism $S\pi_n \text{SO}(n) \rightarrow \pi_n \text{SO}$ induced by the inclusion $\text{SO}(n) \subset \text{SO}$ is an isomorphism for $n \neq 1, 3, 7$ odd. For $n = 1, 3, 7$, it is injective with cokernel $\mathbf{Z}/2$.*

As a result, the diagram (2.2) induces a splitting of (2.1) for $n \neq 3, 7$ odd, since all vertical maps are isomorphisms. This proves the cases $n \neq 3, 7$ of the following reformulation of Theorem A (see Sect. 1.3). We postpone the proof of the cases $n = 3, 7$ to Sect. 3.6.

Theorem 2.2 *For $n \geq 3$ odd, the extension*

$$0 \longrightarrow H(g) \otimes S\pi_n \text{SO}(n) \longrightarrow \Gamma_{g,1/2}^n \xrightarrow{p} G_g \longrightarrow 0$$

splits for $n \neq 3, 7$. For $n = 3, 7$, it splits if and only if $g = 1$.

Even though the extension does not split for $n = 3, 7$, the morphism (2.2) is still injective by Lemma 2.1 and thus expresses the extension in question as a subextension of the trivial extension of G_g by the G_g -module $H(g) \otimes \pi_n \text{SO}$. Crowley [15, Cor. 3.5] gave an algebraic description of this subextension and concluded that it splits if and only if $g = 1$. We proceed differently and prove this fact in Sect. 3.3 directly, which can in turn be used to determine the extension in the following way: by the discussion in Sect. 1.4, it is sufficient to determine its extension class in $H^2(G_g; H(g) \otimes S\pi_n \text{SO}(n))$ for $g \gg 0$. For $n = 3, 7$, we have $G_g \cong \text{Sp}_{2g}(\mathbf{Z})$ with its usual action on $H(g) \otimes S\pi_n \text{SO}(n) \cong \mathbf{Z}^{2g}$. Using work of Djament [16, Thm 1], one can compute $H^2(\text{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}^{2g}) \cong \mathbf{Z}/2$ for $g \gg 0$, so there is only one nontrivial extension of G_g by $H(g) \otimes S\pi_n \text{SO}(n)$, which must be the one in consideration because of the second part of Theorem 2.2. Note that this line of argument gives a geometric proof for the following useful fact on the twisted cohomology of $\text{Sp}_{2g}(\mathbf{Z})$ as a byproduct, which can also be derived algebraically (see for instance [15, Sect. 2]).

Corollary 2.3 *The pullback of the unique nontrivial class in $H^2(\text{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}^{2g})$ for $g \gg 0$ to $H^2(\text{Sp}_{2h}(\mathbf{Z}); \mathbf{Z}^{2h})$ for $h \leq g$ is trivial if and only if $h = 1$.*

We close this section by relating the abelianisation of $\Gamma_{g,1/2}^n$ to that of G_g . The latter is content of Lemma A.1.

Corollary 2.4 *For $n \geq 3$ odd, the morphism*

$$H_1(\Gamma_{g,1/2}^n) \longrightarrow H_1(G_g)$$

is split surjective and has the coinvariants $(H(g) \otimes S\pi_n \text{SO}(n))_{G_g}$ as its kernel, which vanish for $g \geq 2$. For $g = 1$ it vanishes if and only if $n \equiv 5 \pmod{8}$ or $n = 3, 7$, and has order 2 otherwise.

Proof The claim regarding the coinvariants follows from Lemma A.2 and Table 1. Since they vanish for $g \geq 2$, the remaining statement follows from the exact sequence

$$H_2(\Gamma_{g,1/2}^n) \rightarrow H_2(G_g) \rightarrow (H(g) \otimes S\pi_n \text{SO}(n))_{G_g} \rightarrow H_1(\Gamma_{g,1/2}^n) \rightarrow H_1(G_g) \rightarrow 0$$

induced by (2.1), combined with the fact that this extension splits for $g = 1$ by Theorem 2.2. □

3 Signatures, obstructions, and Theorem B

By Lemma 1.2, the extension

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow \Gamma_{g,1}^n \longrightarrow \Gamma_{g,1/2}^n \longrightarrow 0$$

discussed in Sect. 1.3 is central and is as such classified by a class in $H^2(\Gamma_{g,1/2}^n; \Theta_{2n+1})$ with $\Gamma_{g,1/2}^n$ acting trivially on Θ_{2n+1} . In this section, we identify this extension class in

terms of the algebraic description of $\Gamma_{g,1/2}^n$ provided in the previous section, leading to a proof of our main result Theorem B. Our approach is partially based on ideas of Galatius–Randal-Williams [21, Sect. 7], who determined the extension for $n \equiv 5 \pmod{8}$ and $g \geq 5$ up to automorphisms of Θ_{2n+1} .

We begin with an elementary recollection on the relation between Pontryagin classes and obstructions to extending trivialisations of vector bundles, mainly to fix notation.

3.1 Obstructions and Pontryagin classes

Let $k \geq 1$ and $\xi: X \rightarrow \tau_{>4k-1}\mathbf{BO}$ be a map to the $(4k - 1)$ st connected cover of \mathbf{BO} with a lift $\bar{\xi}: A \rightarrow \tau_{>4k}\mathbf{BO}$ over a subspace $A \subset X$ along the canonical map $\tau_{>4k}\mathbf{BO} \rightarrow \tau_{>4k-1}\mathbf{BO}$. Such data has a relative Pontryagin class $p_k(\xi, \bar{\xi}) \in H^{4k}(X, A; \mathbf{Z})$ given as the pullback along the map $(\xi, \bar{\xi}): (X, A) \rightarrow (\tau_{>4k-1}\mathbf{BO}, \tau_{>4k}\mathbf{BO})$ of the unique lift to $H^{4k}(\tau_{>4k-1}\mathbf{BO}, \tau_{>4k}\mathbf{BO}; \mathbf{Z})$ of the pullback $p_k \in H^{4k}(\tau_{>4k-1}\mathbf{BO}; \mathbf{Z})$ of the usual Pontryagin class $p_k \in H^{4k}(\mathbf{BSO}; \mathbf{Z})$. The class $p_k(\xi, \bar{\xi})$ is related to the primary obstruction $\chi(\xi, \bar{\xi}) \in H^{4k}(X, A; \pi_{4k-1}\mathbf{SO})$ to solving the lifting problem

$$\begin{array}{ccc} A & \xrightarrow{\bar{\xi}} & \tau_{>4k}\mathbf{BO} \\ \downarrow & \nearrow \text{---} & \downarrow \\ X & \xrightarrow{\xi} & \tau_{>4k-1}\mathbf{BO} \end{array}$$

by the equality

$$p_k(\xi, \bar{\xi}) = \pm a_k(2k - 1)! \cdot \chi(\xi, \bar{\xi}),$$

up to the choice of a generator $\pi_{4k-1}\mathbf{SO} \cong \mathbf{Z}$ (cf. [44, Lem. 2]). We suppress the lift $\bar{\xi}$ from the notation whenever there is no source of confusing. For us, $X = M$ will usually be a compact oriented $8k$ -manifold and $A = \partial M$ its boundary, in which case we can evaluate $\chi^2(\xi, \bar{\xi}) \in H^{8k}(M, \partial M; \mathbf{Z})$ against the relative fundamental class $[M, \partial M]$ to obtain a number $\chi^2(\xi, \bar{\xi}) \in \mathbf{Z}$. The following two sources of manifolds are relevant for us.

Example 3.1 Fix an integer $n \equiv 3 \pmod{4}$.

- (i) For a compact oriented n -connected $(2n + 2)$ -manifold whose boundary is a homotopy sphere, there is a (up to homotopy) unique lift $M \rightarrow \tau_{>n}\mathbf{BO}$ of the stable oriented normal bundle. On the boundary ∂M , this lifts uniquely further to $\tau_{>n+1}\mathbf{BO}$, so we obtain a canonical class $\chi(M) \in H^{n+1}(M, \partial M; \mathbf{Z})$ and a characteristic number $\chi^2(M) \in \mathbf{Z}$.
- (ii) Consider a $(W_{g,1}, D^{2n-1})$ -bundle $\pi: E \rightarrow B$, i.e. a smooth $W_{g,1}$ -bundle with a trivialisised D^{2n-1} -subbundle of its $\partial W_{g,1}$ -bundle $\partial\pi: \partial E \rightarrow B$ of boundaries. The standard framing of D^{2n-1} induces a trivialisisation of stable vertical tangent bundle $T_\pi E: E \rightarrow \mathbf{BSO}$ over the subbundle $B \times D^{2n-1} \subset \partial E$, which extends uniquely to a $\tau_{>n+1}\mathbf{BO}$ -structure on $T_\pi E|_{\partial E}$ by obstruction theory. Using that W_g is n -parallelisable, another application of obstruction theory shows that the

induced $\tau_{>n}$ BO-structure on $T_\pi E|_{\partial E}$ extends uniquely to a $\tau_{>n}$ BO-structure on $T_\pi E$, so the above discussion provides a class $\chi(T_\pi E) \in H^{n+1}(E, \partial E; \mathbf{Z})$, and, assuming B is an oriented closed surface, a number $\chi^2(T_\pi E) \in \mathbf{Z}$.

3.2 Highly connected almost closed manifolds

As a consequence of Theorem 3.12, we shall see that $(W_{g,1}, D^{2n-1})$ -bundles over surfaces are closely connected to n -connected almost closed $(2n + 2)$ -manifolds. These manifolds were classified by Wall [55], which we now recall for $n \geq 3$ in a form tailored to later applications, partly following [34, Sect. 2].

A compact manifold M is *almost closed* if its boundary is a homotopy sphere. We write $A_d^{\tau_{>n}}$ for the abelian group of almost closed oriented n -connected d -manifolds up to oriented n -connected bordism restricting to an h -cobordism on the boundary. Recall that $\Omega_d^{\tau_{>n}}$ denotes the bordism group of closed d -manifolds M equipped with a $\tau_{>n}$ BO-structure on their stable normal bundle $M \rightarrow \mathbf{BO}$, i.e. a lift $M \rightarrow \tau_{>n}\mathbf{BO}$ to the n -connected cover. By classical surgery, the group $\Omega_d^{\tau_{>n}}$ is canonically isomorphic to the bordism group of closed oriented n -connected d -manifolds up to n -connected bordism as long as $d \geq 2n + 1$, so we will use both descriptions interchangeably. There is an exact sequence

$$\Theta_{2n+2} \longrightarrow \Omega_{2n+2}^{\tau_{>n}} \longrightarrow A_{2n+2}^{\tau_{>n}} \xrightarrow{\partial} \Theta_{2n+1} \longrightarrow \Omega_{2n+1}^{\tau_{>n}} \longrightarrow 0 \quad (3.1)$$

due to Wall [57, p. 293] in which the two outer morphisms are the obvious ones, noting that homotopy d -spheres n -connected for $n < d$. The morphisms $\Omega_{2n+2}^{\tau_{>n}} \rightarrow A_{2n+2}^{\tau_{>n}}$ and $\partial: A_{2n+2}^{\tau_{>n}} \rightarrow \Theta_{2n+1}$ are given by cutting out an embedded disc and by assigning to an almost closed manifold its boundary, respectively. By surgery theory, the subgroup

$$\mathbf{b}A_{2n+2} := \text{im}(A_{2n+2}^{\tau_{>n}} \xrightarrow{\partial} \Theta_{2n+1})$$

of homotopy $(2n + 1)$ -spheres bounding n -connected manifolds contains the cyclic subgroup $\mathbf{bP}_{2n+2} \subset \Theta_{2n+2}$ of homotopy $(2n + 1)$ -spheres bounding parallelisable manifolds, so the right end of (3.1) receives canonical a map from Kervaire–Milnor’s exact sequence [30],

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{bP}_{2n+2} & \longrightarrow & \Theta_{2n+1} & \longrightarrow & \text{coker}(J)_{2n+1} \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & \mathbf{b}A_{2n+2} & \longrightarrow & \Theta_{2n+1} & \longrightarrow & \Omega_{2n+1}^{\tau_{>n}} \longrightarrow 0, \end{array} \quad (3.2)$$

which in particular induces a morphism $\text{coker}(J)_{2n+1} \rightarrow \Omega_{2n+1}^{\tau_{>n}}$, concretely given by representing a class in $\text{coker}(J)_{2n+1}$ by a stably framed manifold and restricting its stable framing to a $\tau_{>n}$ BO-structure.

3.2.1 Wall's classification

For our purposes, Wall's computation [55,57] of $A_{2n+2}^{\tau_{>n}}$ is for $n \geq 3$ odd is most conveniently stated in terms of two particular almost closed n -connected $(2n + 2)$ -manifolds, namely

- (i) *Milnor's E_8 -plumbing P* , arising from plumbing together 8 copies of the disc bundle of the tangent bundle of the standard $(n + 1)$ -sphere such that the intersection form of P agrees with the E_8 -form (see e.g.[8, Ch. V.2]), and
- (ii) the manifold Q , obtained from plumbing together two copies of a linear D^{n+1} -bundle over the $(n + 1)$ -sphere representing a generator of $S\pi_n\text{SO}(n)$.

The following can be derived from Wall's work (see e.g. [34, Thm 2.1]).

Theorem 3.2 (Wall) *For $n \geq 3$ odd, the bordism group $A_{2n+2}^{\tau_{>n}}$ satisfies*

$$A_{2n+2}^{\tau_{>n}} \cong \begin{cases} \mathbf{Z} \oplus \mathbf{Z}/2 & \text{if } n \equiv 1 \pmod{8} \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } n \equiv 3 \pmod{4} \\ \mathbf{Z} & \text{if } n \equiv 5 \pmod{8}. \end{cases}$$

The first summand is generated by P in all cases but $n = 3, 7$ where it is generated by \mathbf{HP}^2 and \mathbf{OP}^2 . The second summand for $n \equiv 5 \pmod{8}$ is generated by Q .

From a consultation of Table 1, one sees that the group $S\pi_n\text{SO}(n)$ vanishes for $n \equiv 5 \pmod{8}$, so $Q \in A_{2n+2}^{\tau_{>n}}$ is trivial in this case, which shows that the subgroup bA_{2n+2} is for all $n \geq 3$ odd generated by the boundaries

$$\Sigma_P := \partial P \in \text{bA}_{2n+2} \quad \text{and} \quad \Sigma_Q := \partial Q \in \text{bA}_{2n+3}.$$

In the cases $n \equiv 1 \pmod{8}$ in which Q defines a $\mathbf{Z}/2$ -summand, its boundary Σ_Q is trivial by a result of Schultz [50, Cor. 3.2, Thm 3.4 iii)]. For $n \equiv 3 \pmod{4}$ on the other hand, it is nontrivial by a calculation of Kosinski [31, p. 238–239].

Theorem 3.3 (Kosinski, Schultz) *The homotopy sphere $\Sigma_Q \in \Theta_{2n+1}$ is trivial for $n \equiv 1 \pmod{4}$ and nontrivial for $n \equiv 3 \pmod{4}$*

In the exceptional dimensions $n = 3, 7$, the homotopy sphere Σ_Q agrees with the inverse of the Milnor sphere Σ_P , as explained in [34, Cor. 2.8 ii)].

Lemma 3.4 $\Sigma_Q = -\Sigma_P$ for $n = 3, 7$.

For $n \geq 3$ odd, the Milnor sphere $\Sigma_P \in \Theta_{2n+1}$ is well-known to be nontrivial and to generate the cyclic subgroup $\text{bP}_{2n+2} \subset \Theta_{2n+1}$ whose order can be expressed in terms of numerators of divided Bernoulli numbers (see e.g. [41, Lem. 3.5 (2), Cor. 3.20]), so Theorem 3.3 has the following corollary.

Corollary 3.5 *For $n \geq 3$ odd, the subgroup bA_{2n+1} is nontrivial. It is generated by Σ_P for $n \equiv 1 \pmod{4}$, by Σ_Q for $n = 3, 7$, and by Σ_P and Σ_Q for $n \equiv 3 \pmod{4}$.*

Combining the previous results with the diagram (3.2), we obtain the following result, which we already mentioned in the introduction.

Corollary 3.6 *The natural morphism $\text{coker}(J)_{2n+1} \rightarrow \Omega_{2n+2}^{\tau_{>n}}$ is an isomorphism for $n \equiv 1 \pmod{4}$ and for $n = 3, 7$. For $n \equiv 3 \pmod{4}$, it is an epimorphism whose kernel is generated by the class $[\Sigma_Q] \in \text{coker}(J)_{2n+1}$.*

$[\Sigma_Q] \in \text{coker}(J)_{2n+1}$ is conjecturally trivial [21, Conj. A] for all n odd. Until recently (see Remark 3.8), this was only known for $n = 3, 7$ and $n \equiv 1 \pmod{4}$ by the results above.

Conjecture 3.7 (Galatius–Randal-Williams) $[\Sigma_Q] = 0$ for all $n \geq 3$ odd.

Remark 3.8 As mentioned in the introduction, after the completion of this work, Burklund–Hahn–Senger [11] and Burklund–Senger [12] showed that $[\Sigma_Q]$ vanishes in $\text{coker}(J)_{2n+1}$ for n odd if and only if $n \neq 11$, confirming Conjecture 3.7 for $n \neq 11$ and disproving it for $n = 11$. This has as a consequence that, for $n \neq 11$ odd, the subgroup bA_{2n+1} is generated by Σ_P even for $n \equiv 3 \pmod{4}$ and that the morphism $\text{coker}(J)_{2n+1} \rightarrow \Omega_{2n+2}^{\tau_{>n}}$ discussed in Corollary 3.6 is an isomorphism.

3.2.2 Invariants

It follows from Theorems 3.2 and 3.3 that the boundary of an n -connected almost closed $(2n + 2)$ -manifold M is determined by at most two integral bordism invariants of M . Concretely, we consider the signature $\text{sgn}: A_{2n+2}^{\tau_{>n}} \rightarrow \mathbf{Z}$ and for $n \equiv 3 \pmod{4}$ the characteristic number $\chi^2: A_{2n+2}^{\tau_{>n}} \rightarrow \mathbf{Z}$, explained in Example 3.1. As discussed for example in [34, Sect. 2.1], these functionals evaluate to

$$\text{sgn}(P) = 8 \quad \text{sgn}(Q) = 0 \quad \chi^2(P) = 0 \quad \chi^2(Q) = \begin{cases} 8 & \text{for } n = 3, 7 \\ 2 & \text{otherwise} \end{cases}, \quad (3.3)$$

and on the closed manifolds \mathbf{HP}^2 and \mathbf{OP}^2 to

$$\text{sgn}(\mathbf{HP}^2) = \text{sgn}(\mathbf{OP}^2) = 1 \quad \chi^2(\mathbf{HP}^2) = \chi^2(\mathbf{OP}^2) = 1, \quad (3.4)$$

which results in the following formula for boundary spheres of highly connected manifolds when combined with the discussion above.

Proposition 3.9 *For $n \geq 3$ odd, the boundary $\partial M \in \Theta_{2n+1}$ of an almost closed oriented n -connected $(2n + 2)$ -manifold M satisfies*

$$\partial M = \begin{cases} \text{sgn}(M)/8 \cdot \Sigma_P & \text{if } n \equiv 1 \pmod{4} \\ \text{sgn}(M)/8 \cdot \Sigma_P + \chi^2(M)/2 \cdot \Sigma_Q & \text{if } n \equiv 3 \pmod{4} \text{ and } n \neq 3, 7 \\ (\chi^2(M) - \text{sgn}(M))/8 \cdot \Sigma_Q & \text{if } n = 3, 7. \end{cases}$$

3.2.3 The minimal signature

As in the introduction, we denote by σ'_n the minimal positive signature of a smooth closed n -connected $(2n + 2)$ -manifold. This satisfies $\sigma'_n = 1$ for $n = 1, 3, 7$ as witnessed by $\mathbf{C}P^2$, $\mathbf{H}P^2$, and $\mathbf{O}P^2$, and in all other cases, it can be expressed in terms of the subgroup $\mathbf{b}A_{2n+2} \subset \Theta_{2n+1}$ as follows.

Lemma 3.10 *For $n \geq 3$ odd, the quotient $\mathbf{b}A_{2n+2}/\langle \Sigma_Q \rangle$ is a cyclic group generated by the class of Σ_P . It is trivial if $n = 3, 7$ and of order $\sigma'_n/8$ otherwise.*

Proof For $n = 3, 7$, the claim is a consequence of Theorem 3.2 and Lemma 3.4. In the case $n \neq 3, 7$, it follows from taking vertical cokernels in the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \langle \text{ord}(\Sigma_Q) \cdot Q \rangle & \longrightarrow & \Omega_{2n+2}^{\tau > n} / \Theta_{2n+1} & \xrightarrow{\text{sgn}} & \sigma'_n \cdot \mathbf{Z} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \langle Q \rangle & \longrightarrow & A_{2n+2}^{\tau > n} & \xrightarrow{\text{sgn}} & 8 \cdot \mathbf{Z} \longrightarrow 0
 \end{array}$$

with exact rows, obtained from a combination of Theorem 3.2 with (3.1) and (3.3). \square

Remark 3.11 In [34, Prop. 2.15], the minimal signature σ'_n was expressed in terms of Bernoulli numbers and the order of $[\Sigma_Q] \in \text{coker}(J) \in \Theta_{2n+1}$, from which one can conclude that for $n \neq 1, 3, 7$, the signature of such manifolds is divisible by 2^{n+3} if $(n + 1)/2$ is odd and by $2^{n-2\nu_2(n+1)}$ otherwise, where $\nu_2(-)$ denotes the 2-adic valuation (see [34, Cor. 2.18]).

3.3 Bundles over surfaces and almost closed manifolds

In order to identify the cohomology class in $H^2(\Gamma_{g,1/2}^n; \Theta_{2n+1})$ that classifies the central extension

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow \Gamma_{g,1}^n \longrightarrow \Gamma_{g,1/2}^n \longrightarrow 0, \tag{3.5}$$

we first determine how it evaluates against homology classes, i.e. identify its image

$$d_2: H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \longrightarrow \Theta_{2n+1}$$

under the map h participating in the universal coefficient theorem

$$\begin{aligned}
 0 &\rightarrow \text{Ext}(H_1(\Gamma_{g,1/2}^n; \mathbf{Z}), \Theta_{2n+1}) \rightarrow H^2(\Gamma_{g,1/2}^n; \Theta_{2n+1}) \\
 &\xrightarrow{h} \text{Hom}(H_2(\Gamma_{g,1/2}^n; \mathbf{Z}); \Theta_{2n+1}) \rightarrow 0,
 \end{aligned}$$

followed by resolving the remaining ambiguity originating from the Ext-term. Indicated by our choice of notation, the morphism d_2 can be viewed alternatively as the first possibly nontrivial differential in the E_2 -page of the Serre spectral sequence of

the extension (3.5) (cf. [26, Thm 4]). Before identifying this differential, we remind the reader of two standard facts we shall make frequent use of.

- (i) The canonical map of spectra $\mathbf{MSO} \rightarrow \mathbf{HZ}$ is 4-connected, so pushing forward fundamental classes induces an isomorphism $\Omega_*^{\mathbf{SO}}(X) \rightarrow H_*(X; \mathbf{Z})$ for $* \leq 3$ and any space X , and
- (ii) the 1-truncation of a connected space X (in particular the natural map $BG \rightarrow B\pi_0 G$ for a topological group G) induces a surjection $H_2(X; \mathbf{Z}) \rightarrow H_2(K(\pi_1 X, 1); \mathbf{Z})$, whose kernel agrees with the image of the Hurewicz homomorphism $\pi_2 X \rightarrow H_2(X; \mathbf{Z})$.

The key geometric ingredient to identify the differential d_2 is the following result.

Theorem 3.12 *Let $n \geq 3$ be odd and $\pi: E \rightarrow S$ a $(W_{g,1}, D^{2n-1})$ -bundle over an oriented closed surface S . There exists a class $E' \in A_{2n+2}^{\tau_{>n}}$ such that*

- (i) *its boundary $\partial E' \in \Theta_{2n+1}$ is the image of the class $[\pi] \in H^2(\text{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z})$ under the composition*

$$H_2(\text{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \xrightarrow{d_2} \Theta_{2n+1},$$

- (ii) *it satisfies $\text{sgn}(E) = \text{sgn}(E')$, and $\chi^2(T_\pi E) = \chi^2(E')$ if $n \equiv 3 \pmod{4}$.*

Proof By the isotopy extension theorem, the restriction map to the moving part of the boundary $\text{Diff}_{\partial/2}(W_{g,1}) \rightarrow \text{Diff}_\partial(D^{2n-1})$ is a fibration. As its image is contained in the component of the identity (see the proof of Lemma 1.2), this fibration induces the upper row in a map of fibrations

$$\begin{CD} \text{Diff}_\partial^{\text{id}}(D^{2n-1}) @>>> \text{BDiff}_\partial(W_{g,1}) @>>> \text{BDiff}_{\partial/2}(W_{g,1}) \\ @VVV @VVV @VVV \\ \text{B}\Theta_{2n+1} @>>> \text{B}\Gamma_{g,1}^n @>>> \text{B}\Gamma_{g,1/2}^n \end{CD} \tag{3.6}$$

whose bottom row is induced by the extension (3.5). The two right vertical maps are induced by taking components and the left vertical map is the induced map on homotopy fibres. The latter agrees with the delooping of the Gromoll map $\Omega\text{Diff}_\partial^{\text{id}}(D^{2n-1}) \rightarrow \text{Diff}_\partial(D^{2n})$ followed by taking components, which one checks by looping the fibre sequences and using that

$$\Omega\text{Diff}_\partial^{\text{id}}(D^{2n-1}) \longrightarrow \text{Diff}_\partial(W_{g,1})$$

is given by “twisting” a collar $[0, 1] \times S^{2n-1} \subset W_{g,1}$, meaning that it sends a smooth loop $\gamma \in \Omega\text{Diff}_\partial^{\text{id}}(D^{2n-1}) \subset \Omega\text{Diff}_\partial^{\text{id}}(S^{2n-1})$ to the diffeomorphism that is the identity outside the collar and is given by $(t, x) \mapsto (t, \gamma(t) \cdot x)$ on the collar. Now consider

the commutative square

$$\begin{array}{ccc}
 H_2(\text{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) & \longrightarrow & H_2(\text{BDiff}_{\partial}^{\text{id}}(D^{2n-1}); \mathbf{Z}) \\
 \downarrow & & \downarrow \\
 H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) & \longrightarrow & \Theta_{2n+1}
 \end{array} \tag{3.7}$$

obtained from delooping (3.6) once to the right and using $H_2(B^2\Theta_{2n+1}) \cong \Theta_{2n+1}$. By transgression, the bottom arrow agrees with the differential d_2 in the statement. Combining this with the Hurewicz theorem, the square (3.7) gives a factorisation

$$H_2(\text{BDiff}_{\partial/2}(W_{g,1})) \rightarrow H_2(\text{BDiff}_{\partial}^{\text{id}}(D^{2n-1})) \cong \pi_2(\text{BDiff}_{\partial}^{\text{id}}(D^{2n-1})) \rightarrow \Theta_{2n+1}$$

of the map in the first part of the statement, which thus has the following geometric description: a smooth $(W_{g,1}, D^{2n-1})$ -bundle $\pi : E \rightarrow S$ represents a class $[\pi] \in H^2(\text{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z})$ and its image under the first map in the composition is the class $[\pi_+] \in H_2(\text{BDiff}_{\partial}^{\text{id}}(D^{2n-1}); \mathbf{Z})$ of its $(\partial W_{g,1}, D^{2n-1})$ -bundle $\pi_+ : E_+ \rightarrow S$ of boundaries, which in turn maps under the inverse of the Hurewicz homomorphism to a $(\partial W_{g,1}, D^{2n-1})$ -bundle $\pi_- : E_- \rightarrow S^2$ over the 2-sphere that is bordant, as a bundle, to π_- . That is, there exists a $(\partial W_{g,1}, D^{2n-1})$ -bundle $\bar{\pi} : \bar{E} \rightarrow K$ over an oriented bordism K between S and S^2 that restricts to π_+ over S and to π_- over S^2 . We claim that the image of the $(\partial W_{g,1}, D^{2n-1})$ -bundle π_- under the final map in the composition is the homotopy sphere $\Sigma_{\pi} \in \Theta_{2n+1}$ obtained by doing surgery on the total space E_- along the trivialised subbundle $D^{2n-1} \times S^2 \subset E_-$. This is most easily seen by thinking of a class in $\pi_k \text{BDiff}_{\partial}(D^d)$ as a smooth bundle $D^d \rightarrow P \rightarrow D^k$ together with a trivialisation $\varphi : D^d \times \partial D^k \cong P|_{\partial D^k}$ and a trivialised ∂D^k -subbundle $\psi : \partial D^d \times D^k \hookrightarrow P$ such that φ and ψ agree on $\partial D^d \times \partial D^k$. From this point of view, the morphism $\pi_k \text{BDiff}_{\partial}(D^d) \rightarrow \pi_{k-1} \text{BDiff}_{\partial}(D^d \times D^1) \cong \pi_{k-1} \text{BDiff}_{\partial}(D^{d+1})$ induced by the Gromoll map is given by sending such a bundle $p : P \rightarrow D^k \cong D^{k-1} \times D^1$ to the $(D^d \times D^1)$ -bundle $(\text{pr}_1 \circ p) : P \rightarrow D^{k-1}$, and the isomorphism $\pi_1 \text{BDiff}_{\partial}(D^d) \cong \Theta_{d+1}$ is given by assigning to a disc bundle $D^d \rightarrow P \rightarrow D^1$ the manifold $P \cup_{\partial D^d \times D^1 \cup D^d \times \partial D^1} D^d \times D^1$. Decomposing the sphere into half-discs $S^n = D_+^n \cup D_-^n$, we see from this description that the composition $\pi_k \text{BDiff}_{\partial}(D^d) \rightarrow \Theta_{d+k}$ of the iterated Gromoll map with the isomorphism $\pi_1 \text{BDiff}_{\partial}(D^{d+k-1}) \cong \Theta_{d+k}$ maps a class represented by an S^d -bundle $S^d \rightarrow Q \rightarrow S^k$ with a trivialisation $\varphi : D_+^k \times S^d \cong Q|_{D_+^k}$ and a trivialised subbundle $\psi : S^k \times D_+^d \hookrightarrow Q$ that agree on $D_+^k \times D_+^d$ to the homotopy sphere

$$\begin{aligned}
 & \left(Q \setminus \text{int}(\varphi(D_+^k \times S^d) \cup \psi(S^k \times D_+^d)) \right) \cup (D^k \times D^d) \\
 & \cong \left(Q \setminus \text{int}(\psi(S^k \times D_+^d)) \right) \cup_{S^k \times \partial D^d} (D^{k+1} \times \partial D^d)
 \end{aligned}$$

obtained by doing surgery along the trivialised D^d -subbundle, where $D^k \times D^d$ is glued to $Q \setminus \text{int}(\varphi(D_+^k \times S^d) \cup_{\partial(D^k \times S^d)} \psi(S^k \times D_+^d))$ along the embedding

$$\begin{aligned} \partial(D^k \times D^d) &= \partial D^k \times D^d \cup D^k \times \partial D^d \\ &\xrightarrow{\varphi|_{\partial D_+^k \times D_+^d} \cup \psi|_{D_+^k \times \partial D_+^d}} \partial \left(Q \setminus \text{int}(\varphi(D_+^k \times S^d) \cup \psi(S^k \times D_+^d)) \right). \end{aligned}$$

This in particular implies the claim we made above in the case $k = 2$ and $d = 2n - 1$.

As a consequence of this description of the morphism in consideration, the image $\Sigma_\pi \in \Theta_{2n+1}$ of the class $[\pi]$ comes equipped with a nullbordism, namely $N := E \cup_{E_+} \bar{E} \cup_{E_-} W$, where W is the trace of the performed surgery. Omitting the trivialised D^{2n-1} -subbundles, the situation can be summarised schematically as follows

$$\begin{array}{ccccccccccc} W_{g,1} & & \partial W_{g,1} & & \partial W_{g,1} & & \partial W_{g,1} & & & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\ E & \supset & E_+ & \subset & \bar{E} & \supset & E_- & \subset & W & \supset & \Sigma_\pi \\ \downarrow \pi & & \downarrow \pi_+ & & \downarrow \bar{\pi} & & \downarrow \pi_- & & & & \\ S & = & S & \subset & K & \supset & S^2. & & & & \end{array}$$

A choice of a stable framing of K induces stable framings on S and S^2 and thus a stable isomorphism $TE \cong T_\pi E \oplus \pi^*TS \cong_s T_\pi E$ using which the canonical $\tau_{>n}$ BO-structure on $T_\pi E$ and the $\tau_{>n+1}$ BO-one on $T_\pi E|_{E_+}$ (see Example 3.1) induce a $\tau_{>n}$ BO-structure on TE and a $\tau_{>n+1}$ BO-structure on $TE|_{E_+} \cong_s TE_+$. With these choices, we have $\chi(T_\pi E) = \chi(TE, TE_+)$. By construction, the restriction of this $\tau_{>n+1}$ BO-structure to $TE_+|_{S \times D^{2n-1}} \cong_s TS$ agrees with the $\tau_{>n+1}$ BO-structure on TS obtained from the stable framing of K , so we obtain a $\tau_{>n+1}$ BO-structure on $T\bar{E}|_{E_+ \cup K \times D^{2n-1}}$, which by obstruction theory extends to one on $T(\bar{E} \cup_{E_-} W)$: the relative Serre spectral sequence shows that $H^*(\bar{E}, E_+ \cup K \times D^{2n-1})$ vanishes for $* \leq 2n - 2$ and thus that $H^{i+1}(\bar{E} \cup_{E_-} W, E_+ \cup K \times D^{2n-1}; \pi_i(\tau_{\leq n}SO)) \cong H^{i+1}(\bar{E} \cup_{E_-} W, \bar{E}; \pi_i(\tau_{\leq n}SO)) \cong H^{i+1}(W, E_-; \pi_i(\tau_{\leq n}SO)) \cong H^{i+1}(D^3, S^2; \pi_i(\tau_{\leq n}SO)) = 0$ for $i \leq 2n - 3$, using $W \simeq E_- \cup_{S^2} D^3$ and $\pi_2SO = 0$. The restriction of this $\tau_{>n+1}$ BO-structure on $T\bar{E}|_{E_+ \cup K \times D^{2n-1}}$ to a $\tau_{>n}$ BO and the canonical $\tau_{>n}$ BO-structure on $TE \cong T_\pi E$ (see Example 3.1) assemble to a $\tau_{>n}$ BO-structure on N . By construction, the canonical restriction map (using excision)

$$H^*(E, E_+; \mathbf{Z}) \cong H^*(N, \bar{E} \cup_{E_-} W; \mathbf{Z}) \longrightarrow H^*(N, \Sigma_\pi; \mathbf{Z})$$

sends $\chi(T_\pi E) = \chi(TE, TE_+)$ to $\chi(TN, T\Sigma_\pi)$, so we conclude $\chi^2(T_\pi E) = \chi^2(TN, T\Sigma_\pi)$. To finish the proof, note that the $\tau_{>n}$ BO-structure on TN allows us to do surgery away from the boundary on N to obtain an n -connected manifold E' , which gives a class in $A_{2n+2}^{\tau_{>n}}$ as aimed for: $\partial E' = \Sigma_\pi$ holds by construction, $\chi^2(E') = \chi^2(TN, T\Sigma_\pi) = \chi^2(T_\pi E)$ by the bordism invariance of Pontryagin numbers (see Example 3.1), and $\text{sgn}(E') = \text{sgn}(N) = \text{sgn}(E)$ by the additivity and bordism invariance of the signature. \square

Combining the previous result with Proposition 3.9, we conclude that

$$H_2(\text{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \xrightarrow{d_2} \Theta_{2n+1}$$

sends a homology class $[\pi]$ represented by a bundle $\pi : E \rightarrow S$ to a certain linear combination of Σ_P and Σ_Q whose coefficients involve the invariants $\text{sgn}(E)$ and $\chi^2(T_\pi E)$. In the following two subsections, we shall see that these functionals

$$\text{sgn} : H_2(\text{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow \mathbf{Z} \quad \text{and} \quad \chi^2 : H_2(\text{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow \mathbf{Z}$$

factor through the composition

$$H_2(\text{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \xrightarrow{(SF,p)^*} H_2((H(g) \otimes \pi_n \text{SO}) \rtimes G_g; \mathbf{Z})$$

and have a more algebraic description in terms of $H(g) \otimes \pi_n \text{SO} \rtimes G_g$. This uses

$$(SF, p) : \Gamma_{g,1/2}^n \longrightarrow (H(g) \otimes \pi_n \text{SO}) \rtimes G_g,$$

induced by acting on a stable framing of $W_{g,1}$ (agreeing with the usual framing on D^{2n-1}) as explained in Sect. 2.

3.4 Signatures of bundles of symplectic lattices

The standard action of the symplectic group $\text{Sp}_{2g}(\mathbf{Z})$ on \mathbf{Z}^{2g} gives rise to a local system $\mathcal{H}(g)$ over $\text{BSp}_{2g}(\mathbf{Z})$ and the usual symplectic form on \mathbf{Z}^{2g} gives a morphism $\lambda : \mathcal{H}(g) \otimes \mathcal{H}(g) \rightarrow \mathbf{Z}$ of local systems to the constant system. To an oriented closed surface S with a map $f : S \rightarrow \text{BSp}_{2g}(\mathbf{Z})$, we can associate a bilinear form

$$\langle -, - \rangle_f : H^1(S; f^* \mathcal{H}(g)) \otimes H^1(S; f^* \mathcal{H}(g)) \rightarrow \mathbf{Z},$$

defined as the composition

$$H^1(S; f^* \mathcal{H}(g)) \otimes H^1(S; f^* \mathcal{H}(g)) \xrightarrow{\sim} H^2(S; f^* \mathcal{H}(g) \otimes \mathcal{H}(g)) \xrightarrow{\lambda} H^2(S; \mathbf{Z}) \xrightarrow{[S]} \mathbf{Z}.$$

As both the cup-product and the symplectic pairing λ are antisymmetric, the form $\langle -, - \rangle_f$ is symmetric. The usual argument for the bordism invariance of the signature shows that its signature $\text{sgn}(\langle -, - \rangle_f)$ depends only on the bordism class $[f] \in \Omega_2^{\text{SO}}(\text{BSp}_{2g}(\mathbf{Z})) \cong H_2(\text{Sp}_{2g}(\mathbf{Z}); \mathbf{Z})$ and thus induces a morphism

$$\text{sgn} : H_2(\text{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \longrightarrow \mathbf{Z}, \tag{3.8}$$

which is compatible with the usual inclusion $\text{Sp}_{2g}(\mathbf{Z}) \subset \text{Sp}_{2g+2}(\mathbf{Z})$.

Remark 3.13 As $\mathrm{Sp}_{2g}(\mathbf{Z})$ is perfect for $g \geq 3$ (see Lemma A.1), the morphism (3.8) determines a unique cohomology class $\mathrm{sgn} \in H^2(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z})$. There is a well-known purely algebraically defined cocycle representative of this class due to Meyer [43], known as the *Meyer cocycle*.

The morphism (3.8) measures signatures of total spaces of smooth bundles over surfaces (even of fibrations of Poincaré complexes). More precisely, for a compact oriented $(4k+2)$ -manifold M , the action of its group of diffeomorphisms on the middle cohomology induces a morphism $\mathrm{Diff}^+(M) \rightarrow \mathrm{Sp}_{2g}(\mathbf{Z})$ for $2g = \mathrm{rk}(H^{2k+1}(M))$ and the resulting composition

$$H_2(\mathrm{BDiff}^+(M); \mathbf{Z}) \longrightarrow H_2(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\mathrm{sgn}} \mathbf{Z}$$

can be shown to map a homology class represented by a smooth bundle over a surface to the signature of its total space. This fact can either be proved along the lines of [14] or extracted from [42] and it has in particular the following consequence.

Lemma 3.14 *For n odd, the composition*

$$H_2(\mathrm{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow H_2(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\mathrm{sgn}} \mathbf{Z}$$

sends the class of an $(W_{g,1}, D^{2n-1})$ -bundle $\pi : E \rightarrow S$ to the signature $\mathrm{sgn}(E)$ of its total space.

We proceed by computing the image of the signature morphism (3.8) and of its pullback to the theta-subgroup $\mathrm{Sp}_{2g}^q(\mathbf{Z}) \subset \mathrm{Sp}_{2g}(\mathbf{Z})$ as defined in Sect. 1.2.

Lemma 3.15 *The signature morphism satisfies*

$$\begin{aligned} \mathrm{im} \left(H_2(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\mathrm{sgn}} \mathbf{Z} \right) &= \begin{cases} 0 & \text{if } g = 1 \\ 4 \cdot \mathbf{Z} & \text{if } g \geq 2 \end{cases} \\ \mathrm{im} \left(H_2(\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\mathrm{sgn}} \mathbf{Z} \right) &= \begin{cases} 0 & \text{if } g = 1 \\ 8 \cdot \mathbf{Z} & \text{if } g \geq 2. \end{cases} \end{aligned}$$

Proof The signatures realised by classes in $H_2(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z})$ are well-known (see e.g. [3, Lem 6.5, Thm 6.6 (vi)]). To prove that the signature of classes in $H_2(\mathrm{Sp}_{2g}^q(\mathbf{Z}))$ is divisible by 8, recall from Sects. 1.2 and 1.3 that for n odd the morphism $\mathrm{Diff}_{\partial/2}(W_{g,1}) \rightarrow \mathrm{Sp}_{2g}(\mathbf{Z})$ lands in the subgroup $\mathrm{Sp}_{2g}^q(\mathbf{Z}) \subset \mathrm{Sp}_{2g}(\mathbf{Z})$ as long as $n \neq 1, 3, 7$, so we have a composition

$$H_2(\mathrm{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \longrightarrow H_2(\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\mathrm{sgn}} \mathbf{Z},$$

which maps the class of a bundle $\pi : E \rightarrow S$ by Lemma 3.14 to $\mathrm{sgn}(E)$. The latter agrees by Theorem 3.12 with the signature of an almost closed n -connected

$(2n + 2)$ -manifold, so it is divisible by 8 as the intersection form of such manifolds is unimodular and even (see e.g. [55]). This proves the claimed divisibility, since the first two morphisms in the composition are surjective, the first one because of the second remainder at the beginning of Sect. 3.3 and the second one by Corollary 2.4. As the signature morphism vanishes on $H_2(\mathrm{Sp}_2(\mathbf{Z}); \mathbf{Z})$ by the first part, it certainly vanishes on $H_{2g}(\mathrm{Sp}_2^g(\mathbf{Z}); \mathbf{Z})$. Consequently, by the compatibility of the signature with the inclusion $\mathrm{Sp}_{2g}(\mathbf{Z}) \subset \mathrm{Sp}_{2g+2}(\mathbf{Z})$, the remaining claim follows from constructing a class in $H_2(\mathrm{Sp}_{2g}^g(\mathbf{Z}); \mathbf{Z})$ of signature 8 for $g = 2$. Using $H_2(\mathrm{Sp}_4(\mathbf{Z}); \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}/2$ (see e.g. [3, Lem. A.1(iii)]) and the first part of the claim, the existence of such a class is equivalent to the image of $H_2(\mathrm{Sp}_4^g(\mathbf{Z}); \mathbf{Z})$ in the torsion free quotient $H_2(\mathrm{Sp}_4(\mathbf{Z}); \mathbf{Z})_{\mathrm{free}} \cong \mathbf{Z}$ containing 2. That it contains 10 is ensured by transfer, since the index of $\mathrm{Sp}_4^g(\mathbf{Z}) \subset \mathrm{Sp}_4(\mathbf{Z})$ is 10 (see Sect. 1.2). As $H_1(\mathrm{Sp}_4(\mathbf{Z}); \mathbf{Z})$ and $H_1(\mathrm{Sp}_4^g(\mathbf{Z}); \mathbf{Z})$ are 2-torsion by Lemma A.1, it therefore suffices to show that $H_2(\mathrm{Sp}_4^g(\mathbf{Z}); \mathbf{F}_5) \rightarrow H_2(\mathrm{Sp}_4(\mathbf{Z}); \mathbf{F}_5)$ is nontrivial, for which we consider the level 2 congruence subgroup $\mathrm{Sp}_4(\mathbf{Z}, 2) \subset \mathrm{Sp}_4(\mathbf{Z})$, i.e. the kernel of the reduction map $\mathrm{Sp}_4(\mathbf{Z}) \rightarrow \mathrm{Sp}_4(\mathbf{Z}/2)$, which is surjective (see e.g. [45, Thm 1] for an elementary proof). From the explicit description of $\mathrm{Sp}_{2g}^g(\mathbf{Z})$ presented in Sect. 1.2, one sees that it contains the congruence subgroup $\mathrm{Sp}_4(\mathbf{Z}, 2)$. As a result, it is enough to prove that $H_2(\mathrm{Sp}_4(\mathbf{Z}, 2); \mathbf{F}_5) \rightarrow H_2(\mathrm{Sp}_4(\mathbf{Z}); \mathbf{F}_5)$ is nontrivial, which follows from an application of the Serre spectral sequence of the extension

$$0 \longrightarrow \mathrm{Sp}_4(\mathbf{Z}, 2) \longrightarrow \mathrm{Sp}_4(\mathbf{Z}) \longrightarrow \mathrm{Sp}_4(\mathbf{Z}/2) \longrightarrow 0,$$

using that $H_1(\mathrm{Sp}_4(\mathbf{Z}, 2); \mathbf{F}_5)$ vanishes by a result of Sato [49, Cor. 10.2] and that the groups $H_*(\mathrm{Sp}_4(\mathbf{Z}/2); \mathbf{F}_5)$ are trivial in low degrees, because of the exceptional isomorphism between $\mathrm{Sp}_4(\mathbf{Z}/2)$ and the symmetric group in 6 letters as explained for instance in [46, p. 37]. □

- Remark 3.16** (i) There are at least two other proofs for the divisibility of the signature of classes in $H_2(\mathrm{Sp}_{2g}^g(\mathbf{Z}); \mathbf{Z})$ by 8. One can be extracted from the proof of [21, Lem. 7.5 i)] and another one is given in [4, Thm 12.1]. The proof in [21] shows actually something stronger, namely that the form $\langle -, - \rangle_f$ associated to a class $[f] \in H_2(\mathrm{Sp}_{2g}^g(\mathbf{Z}); \mathbf{Z})$ is always even. We shall give a different proof of this fact as part of the second part of Lemma 3.19 below.
- (ii) For $g \geq 4$, the existence of a class in $H_2(\mathrm{Sp}_{2g}^g(\mathbf{Z}); \mathbf{Z})$ of signature 8 was shown as part of the proof of [21, Thm 7.7], using that the image of

$$H_2(\mathrm{Sp}_{2g}(\mathbf{Z}, 2); \mathbf{Z}) \longrightarrow H_2(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \cong \mathbf{Z}$$

for $g \geq 4$ is known to be $2 \cdot \mathbf{Z}$ by a result of Putman [47, Thm F]. However, this argument breaks for small values of g , in which case the image of $\mathrm{sgn}: H_2(\mathrm{Sp}_{2g}^g(\mathbf{Z}); \mathbf{Z}) \rightarrow \mathbf{Z}$ was not known before, at least to the knowledge of the author.

By Lemma 3.15, the signatures of classes in $H_2(\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$ are divisible by 8, so we obtain a morphism of the form

$$\mathrm{sgn}/8: H_2(\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z}) \longrightarrow \mathbf{Z}.$$

To lift this morphism to a cohomology class in $H^2(\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$, we consider

$$a: \mathbf{Z}/4 \longrightarrow \mathrm{Sp}_2^q(\mathbf{Z}) \subset \mathrm{Sp}_{2g}^q(\mathbf{Z}), \tag{3.9}$$

induced by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Sp}_2^q(\mathbf{Z})$. By Lemma A.1, this is an isomorphism on abelianisations for $g \geq 3$ and thus induces a splitting of the universal coefficient theorem

$$a^* \oplus h: H^2(\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\cong} H^2(\mathbf{Z}/4; \mathbf{Z}) \oplus \mathrm{Hom}(H_2(\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z}), \mathbf{Z}). \tag{3.10}$$

This splitting is compatible with the inclusion $\mathrm{Sp}_{2g}^q(\mathbf{Z}) \subset \mathrm{Sp}_{2g+2}^q(\mathbf{Z})$, so we can define a lift of the divided signature $\mathrm{sgn}/8$ to a class $H^2(\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$ as follows.

Definition 3.17 (i) Define the class

$$\frac{\mathrm{sgn}}{8} \in H^2(\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$$

for $g \gg 0$ via the splitting (3.10) by declaring its image in the first summand to be trivial and to be $\mathrm{sgn}/8$ in the second. For small g , the class $\frac{\mathrm{sgn}}{8} \in H^2(\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$ is defined as the pullback of $\frac{\mathrm{sgn}}{8} \in H^2(\mathrm{Sp}_{2g+2h}^q(\mathbf{Z}); \mathbf{Z})$ for $h \gg 0$.

(ii) Define the class

$$\frac{\mathrm{sgn}}{8} \in H^2(\Gamma_{g,1/2}^n; \mathbf{Z}) \quad \text{for } n \neq 1, 3, 7 \text{ odd}$$

as the pullback of the same-named class along the map $\Gamma_{g,1/2}^n \rightarrow G_g \cong \mathrm{Sp}_{2g}^q(\mathbf{Z})$ induced by the action on the middle cohomology.

3.5 Framing obstructions

To describe the invariant

$$\chi^2: H_2(\mathrm{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow \mathbf{Z}$$

explained in Sect. 3.1 more algebraically, note that a map $f: S \rightarrow \mathrm{B}(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}))$ from an oriented closed connected surface S to $\mathrm{B}(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}))$ induces a 1-cocycle

$$\pi_1(S; *) \longrightarrow \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}) \longrightarrow \mathbf{Z}^{2g}$$

and hence a class $[f] \in H^1(S; f^*\mathcal{H}(g))$. The composition $S \rightarrow B(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})) \rightarrow \mathrm{BSp}_{2g}(\mathbf{Z})$ defines a bilinear form $\langle -, - \rangle_f$ on $H^1(S; f^*\mathcal{H}(g))$ as explained in Sect. 3.4, and hence a number $\langle [f], [f] \rangle_f \in \mathbf{Z}$. Varying f , this gives a morphism

$$\chi^2 : H_2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \longrightarrow \mathbf{Z},$$

which is compatible with natural inclusion $\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}) \subset \mathbf{Z}^{2g+2} \rtimes \mathrm{Sp}_{2g+2}(\mathbf{Z})$ and takes for $n \equiv 3 \pmod{4}$ part in a composition

$$H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \xrightarrow{(SF,P)} H_2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\chi^2} \mathbf{Z},$$

where the first morphism is induced by acting on a stable framing F of $W_{g,1}$ as in Sect. 2. A priori, this requires three choices: a stable framing, a generator $\pi_n \mathrm{SO} \cong \mathbf{Z}$, and a symplectic basis $H(g) \cong \mathbf{Z}^{2g}$. However, the composition turns out to not be affected by these choices and the following proposition shows that it is related to the invariant of $(W_{g,1}, D^{2n-1})$ -bundles explained in Example 3.1.

Proposition 3.18 *For $n \equiv 3 \pmod{4}$, the composition*

$$H_2(\mathrm{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \xrightarrow{(SF,P)} H_2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\chi^2} \mathbf{Z}$$

sends the class of a $(W_{g,1}, D^{2n-1})$ -bundle $\pi : E \rightarrow S$ to $\chi^2(T_\pi E)$.

Proof The relative Serre spectral sequence of

$$(W_{g,1}, \partial W_{g,1}) \longrightarrow (E, \partial E) \xrightarrow{\pi} S$$

induces canonical isomorphisms

$$H^{2n+2}(E, \partial E; \mathbf{Z}) \cong H^2(S; \mathbf{Z}) \quad \text{and} \quad H^{n+1}(E, \partial E; \mathbf{Z}) \cong H^1(S; f^*\mathcal{H}(g)),$$

where f denotes the composition

$$S \longrightarrow \mathrm{BDiff}_{\partial/2}(W_{g,1}) \longrightarrow \mathrm{B}\Gamma_{g,1/2}^n \xrightarrow{P} \mathrm{BSp}_{2g}(\mathbf{Z}).$$

By the compatibility of the Serre spectral sequence with the cup-product and after identifying $H^1(S; f^*\mathcal{H}(g))$ with $H^1(\pi_1(S; *); H(g) \otimes \mathrm{Sp}_n \mathrm{SO}(n))$, it suffices to show that the second isomorphism sends $\chi(T_\pi E) \in H^{n+1}(E, \partial E; \mathbf{Z})$ up to signs to the class represented by the cocycle

$$\pi_1(S; *) \longrightarrow \Gamma_{g,1/2}^n \xrightarrow{SF} [W_{g,1}, \mathrm{SO}]_* \cong H(g) \otimes \pi_n \mathrm{SO}, \tag{3.11}$$

involving the choice of stable framing $F : TW_{g,1} \oplus \varepsilon^k \cong \varepsilon^{2n+k}$ as in Sect. 2. As a first step, we describe this isomorphism more explicitly:

Note that $H^{n+1}(E, \partial E; \pi_n \text{SO}) \cong H^{n+1}(E; \pi_n \text{SO})$ as $H^*(\partial E; \pi_n \text{SO})$ is trivial for $* = n, n+1$. Unwinding the construction of the Serre spectral sequence using a skeletal filtration of S , one sees that after fixing an identification $W_{g,1} \cong \pi^{-1}(*) \subset E$, the image of a class $x \in H^{n+1}(E, \pi_n \text{SO})$ under the isomorphism in question is represented by the cocycle $\pi_1(S; *) \rightarrow H(g) \otimes \pi_n \text{SO}$ which maps a loop $\omega: ([0, 1], \{0, 1\}) \rightarrow (S, *)$ to the class obtained from a choice of lift $\tilde{x} \in H^{n+1}(E, \pi^{-1}(*); \pi_n \text{SO})$ by pulling it back along

$$(W_{g,1} \times [0, 1], W_{g,1} \times \{0, 1\}) \longrightarrow (\omega^* E, W_{g,1} \times \{0, 1\}) \longrightarrow (E, \pi^{-1}(*)),$$

where the second morphism is induced by pulling back the bundle $\pi: E \rightarrow S$ along ω and the first morphism is the unique (up to homotopy) trivialisation $\omega^* E \cong [0, 1] \times W_{g,1}$ relative to $W_{g,1} \times \{0\}$ of the pullback bundle over $[0, 1]$; here we used the canonical isomorphism $H^{n+1}(W_{g,1} \times [0, 1], W_{g,1} \times \{0, 1\}; \pi_n \text{SO}) \cong H(g) \otimes \pi_n \text{SO}$.

Recall from Example 3.1 that the class $\chi(T_\pi E) \in H^{n+1}(E; \pi_n \text{SO})$ is the primary obstruction to extending the canonical $\tau_{>n}$ BO-structure on $T_\pi E$ to a $\tau_{>n+1}$ BO-structure. The choice of framing F induces such an extension on the fibre $W_{g,1} \cong \pi^{-1}(*) \subset E$ and thus induces a lift of $\chi(T_\pi E)$ to a relative class in $H^{n+1}(E, \pi^{-1}(*); \pi_n \text{SO})$. This uses that the $\tau_{>n}$ BO structure on $W_{g,1}$ induced by the framing agrees with the restriction of the $\tau_{>n}$ BO-structure on $T_\pi E$ to $\pi^{-1}(*) \cong W_{g,1}$ by obstruction theory. Using the above description, we see that the image of $\chi(T_\pi E)$ under the isomorphism in question is represented by the cocycle $\pi_1(S; *) \rightarrow H(g) \otimes \pi_n \text{SO}$ that sends a loop ω to the primary obstruction in $H^n(W_{g,1}; \pi_n \text{SO}) \cong H(g) \otimes \pi_n \text{SO}$ to solving the lifting problem

$$\begin{array}{ccccc} W_{g,1} \times \{0, 1\} & \longrightarrow & \pi^{-1}(*) & \longrightarrow & \tau_{>n+1} \text{BSO} \\ \downarrow \subset & & \downarrow & \dashrightarrow & \downarrow \\ W_{g,1} \times [0, 1] & \longrightarrow & E & \xrightarrow{T_\pi E} & \text{BSO}, \end{array} \tag{3.12}$$

which agrees with the corresponding obstruction when replacing BSO by $\tau_{>n}$ BSO. Here the left square in the diagram is given by the trivialisation explained earlier. There is a useful alternative description of this obstruction class: relative to the subspace $W_{g,1} \times \{0\} \subset W_{g,1} \times [0, 1]$ there is a unique (up to homotopy) lift in (3.12), so the obstruction to finding a lift relative to the subspace $W_{g,1} \times \{0, 1\}$ can be seen as an element in the group of path-components of the fibre of the principal fibration $\text{Map}_*(W_{g,1}, \tau_{>n+1} \text{BSO}) \rightarrow \text{Map}_*(W_{g,1}, \text{BSO})$, which is exactly $[W_{g,1}, \tau_{\leq n} \text{SO}]_* \cong H(g) \otimes \pi_n \text{SO}$.

To see that the cocycle we just described agrees with (3.11), note that the function $s_F: \Gamma_{g,1/2}^n \rightarrow H(g) \otimes \pi_n \text{SO} \cong [W_{g,1}, \tau_{\leq n} \text{SO}]_*$ induced by acting on the stable framing F arises as the connecting map $\pi_1(\text{BDiff}_{\partial/2}^{\tau_{>n+1}}(W_{g,1}); [F]) \rightarrow \pi_0 \text{Map}_*(W_{g,1}, \tau_{\leq n} \text{SO})$ of the fibration

$$\text{Map}_*(W_{g,1}, \tau_{\leq n} \text{SO}) \longrightarrow \text{BDiff}_{\partial/2}^{\tau_{>n+1}}(W_{g,1}) \longrightarrow \text{BDiff}_{\partial/2}(W_{g,1}),$$

where $\text{BDiff}_{\partial/2}^{\tau_{>n+1}}(W_{g,1})$ is the space that classifies $(W_{g,1}, D^{2n-1})$ -bundles with a $\tau_{>n+1}\text{BO}$ -structure on the vertical tangent bundle extending the given $\tau_{>n+1}\text{BO}$ -structure on the restriction to the trivial D^{2n-1} -subbundle induced by the standard framing of D^{2n-1} ; here we identified the space of $\tau_{>n+1}\text{BO}$ -structures of $W_{g,1}$ relative to D^{2n-1} with the mapping space $\text{Map}_*(W_{g,1}, \tau_{\leq n}\text{SO})$ by using the choice of stable framing F , which also induces the basepoint $[F] \in \text{BDiff}_{\partial/2}^{\tau_{>n+1}}(W_{g,1})$. This shows that the value of (3.11) on a loop $[\omega] \in \pi_1(S; *)$ is given by the component in $[W_{g,1}, \tau_{\leq n}\text{SO}]_* \cong H(g) \otimes \pi_n\text{SO}$ obtained by evaluating a choice of path-lift

$$\begin{array}{ccccc} \{0\} & \longrightarrow & * & \xrightarrow{F} & \text{BDiff}_{\partial/2}^{\tau_{>n+1}}(W_{g,1}) \\ \downarrow \subset & & \downarrow & \dashrightarrow & \downarrow \\ [0, 1] & \xrightarrow{\omega} & S & \longrightarrow & \text{BDiff}_{\partial/2}(W_{g,1}). \end{array}$$

at the end point. Such a path-lift precisely classifies a lift as in (3.12) relative to the subspace $W_{g,1} \times \{0\} \subset W_{g,1} \times [0, 1]$, so the claim follows from the second description of the obstruction to solving the lifting problem (3.12) mentioned above. \square

Lemma 3.19 *Let $g \geq 1$.*

(i) *The image of the composition induced by inclusion and χ^2*

$$H_2(\mathbf{Z}^{2g}; \mathbf{Z}) \longrightarrow H_2(\mathbf{Z}^{2g} \times \text{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z}) \longrightarrow H_2(\mathbf{Z}^{2g} \times \text{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\chi^2} \mathbf{Z}$$

contains $2 \cdot \mathbf{Z}$ and agrees with $2 \cdot \mathbf{Z}$ for $g = 1$.

(ii) *We have*

$$\begin{aligned} \text{im} \left(H_2(\mathbf{Z}^{2g} \times \text{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z}) \xrightarrow{(\text{sgn}, \chi^2)} \mathbf{Z} \oplus \mathbf{Z} \right) &= \begin{cases} \langle (0, 2) \rangle & \text{if } g = 1 \\ \langle (8, 0), (0, 2) \rangle & \text{if } g \geq 2. \end{cases} \\ \text{im} \left(H_2(\mathbf{Z}^{2g} \times \text{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \xrightarrow{(\text{sgn}, \chi^2)} \mathbf{Z} \oplus \mathbf{Z} \right) &= \begin{cases} \langle (0, 2) \rangle & \text{if } g = 1 \\ \langle (4, 0), (0, 1) \rangle & \text{if } g \geq 2. \end{cases} \end{aligned}$$

(iii) *For $n = 3, 7$, we have*

$$\text{im} \left(H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \xrightarrow{(s_F, p)^*} H_2(\mathbf{Z}^{2g} \times \text{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\chi^2 - \text{sgn}} \mathbf{Z} \right) = 8 \cdot \mathbf{Z}.$$

Proof By the compatibility of χ^2 with the stabilisation maps, it suffices to show the first part for $g = 1$, which follows from checking that the image of a generator in $H_2(\mathbf{Z}^2; \mathbf{Z}) \cong \mathbf{Z}$ is mapped to ± 2 under χ^2 by chasing through the definition. As the signature morphism pulls back from $\text{Sp}_{2g}(\mathbf{Z})$, the second part follows from the first part and Lemma 3.15 by showing that the image of χ^2 is for $\text{Sp}_{2g}^q(\mathbf{Z})$ always divisible by $2 \cdot \mathbf{Z}$ and for $\text{Sp}_{2g}(\mathbf{Z})$ divisible by 2 if and only if $g = 1$. In the case of

$\mathrm{Sp}_{2g}^q(\mathbf{Z})$, this can be shown “geometrically” as in the proof of Lemma 3.15: choose $n \equiv 3 \pmod{4}$, $n \neq 3, 7$ and consider the composition

$$\mathrm{H}_2(\mathrm{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow \mathrm{H}_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \xrightarrow{(s_F, p)} \mathrm{H}_2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z}) \xrightarrow{\chi^2} \mathbf{Z}.$$

The first morphism is surjective by the second reminder at the beginning of Sect. 3.3 and the second morphism is an isomorphism as a result of Sect. 2, so it suffices to show that the composition is divisible by 2, which in turn follows from a combination of Lemma 3.18, Theorem 3.12 and the fact that in these dimensions, an n -connected almost closed $(2n + 2)$ -manifold E' satisfies $\chi^2(E') \in 2 \cdot \mathbf{Z}$ by a combination of Theorem 3.2 and (3.3). For $\mathrm{Sp}_{2g}(\mathbf{Z})$, we argue as follows: in the case $g = 1$, we show that the morphism $\mathrm{H}_2(\mathbf{Z}^2; \mathbf{Z}) \rightarrow \mathrm{H}_2(\mathbf{Z}^2 \rtimes \mathrm{Sp}_2(\mathbf{Z}); \mathbf{Z})$ is surjective, which will exhibit the claimed divisibility as a consequence of (i). It follows from Lemma A.3 that the group $\mathrm{H}_1(\mathrm{Sp}_2(\mathbf{Z}); \mathbf{Z}^2)$ vanishes and as $\mathrm{Sp}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})$, we also have $\mathrm{H}_2(\mathrm{Sp}_2(\mathbf{Z}); \mathbf{Z}) = 0$,³ so an application of the Serre spectral sequence to $\mathbf{Z}^2 \rtimes \mathrm{Sp}_2(\mathbf{Z})$ shows the claimed surjectivity. This leaves us with proving that χ^2 is not divisible by 2 for $g \geq 2$ for which we use that there is class $[f : \pi_1 S \rightarrow \mathrm{Sp}_{2g}(\mathbf{Z})] \in \mathrm{H}_2(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z})$ of signature 4 by Lemma 3.15, so the form $\langle -, - \rangle_f$ cannot be even and hence there is a 1-cocycle $g : \pi_1 S \rightarrow \mathbf{Z}^{2g}$ for which $\langle [g], [g] \rangle_f$ is odd, which means that the image $\chi^2([g, f])$ of the class $[(g, f)] \in \mathrm{H}_2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z})$ induced by the morphism $(g, f) : \pi_1 S \rightarrow \mathbf{Z}^2 \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})$ is odd. For the last part, note that the argument we gave for the divisibility in the second part for $\mathrm{Sp}_{2g}^q(\mathbf{Z})$ shows for $n = 3, 7$ that the image of the composition in (iii) is contained in $8 \cdot \mathbf{Z}$, since $\chi^2(E') - \mathrm{sgn}(E')$ is divisible by 8 if $n = 3, 7$. Hence, to finish the proof, it suffices to establish the existence of a class in $\mathrm{H}_2(\Gamma_{g,1/2}^n; \mathbf{Z})$ for $n = 3, 7$ on which the composition evaluates to 8. To this end, we consider the square

$$\begin{array}{ccc} \mathrm{H}_2(\mathbf{Z}^{2g} \otimes S\pi_n \mathrm{SO}(n); \mathbf{Z}) & \longrightarrow & \mathrm{H}_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \\ \downarrow & & \downarrow (s_F, p) \\ \mathrm{H}_2(\mathbf{Z}^{2g} \otimes \pi_n \mathrm{SO}; \mathbf{Z}) & \longrightarrow & \mathrm{H}_2((\mathbf{Z}^{2g} \otimes \pi_n \mathrm{SO}) \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}), \end{array}$$

induced by the embedding (s_F, p) of the extension describing $\Gamma_{g,1/2}^n$ into the trivial extension of $\mathrm{Sp}_{2g}(\mathbf{Z})$ by $\mathbf{Z}^{2g} \otimes \pi_n \mathrm{SO}$ (see Sect. 2). By the first part, there is a class $[f] \in \mathrm{H}_2(\mathbf{Z}^{2g} \otimes \pi_n \mathrm{SO}; \mathbf{Z})$ with $\chi^2([f]) = 2$ and trivial signature, since the signature morphism pulls back from $\mathrm{Sp}_{2g}(\mathbf{Z})$. As a result of Lemma 2.1, the cokernel of the left vertical map in the square is 4-torsion if $n = 3, 7$, so $4 \cdot [f]$ lifts to $\mathrm{H}_2(\mathbf{Z}^{2g} \otimes S\pi_n \mathrm{SO}(n); \mathbf{Z})$ and provides a class as desired. \square

³ This can be seen for instance from the Meyer–Vietoris sequence of the well-known decomposition $\mathrm{SL}_2(\mathbf{Z}) \cong \mathbf{Z}/4 *_{\mathbf{Z}/2} \mathbf{Z}/6$, where $\mathbf{Z}/4$ is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{Z}/6$ by $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, and $\mathbf{Z}/2$ by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Similar to the construction of $\frac{\text{sgn}}{8} \in H^2(\text{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$, we would like to lift the morphism resulting from the second part of Lemma 3.19

$$\chi^2/2: H_2(\mathbf{Z}^{2g} \rtimes \text{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z}) \rightarrow \mathbf{Z}$$

to a class $\frac{\chi^2}{2} \in H^2(\mathbf{Z}^{2g} \rtimes \text{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$. To this end, observe that $\text{Sp}_{2g}^q(\mathbf{Z}) \subset \mathbf{Z}^{2g} \rtimes \text{Sp}_{2g}^q(\mathbf{Z})$ induces an isomorphism on abelianisations for $g \geq 2$ since the coinvariants $(\mathbf{Z}^{2g})_{\text{Sp}_{2g}^q(\mathbf{Z})}$ vanish in this range by Lemma A.2. The morphism

$$a: \mathbf{Z}/4 \longrightarrow \text{Sp}_2^q(\mathbf{Z}) \subset \mathbf{Z}^{2g} \rtimes \text{Sp}_{2g}^q(\mathbf{Z})$$

considered in Sect. 3.4 thus induces a splitting

$$a^* \oplus h: H^2(\mathbf{Z}^{2g} \rtimes \text{Sp}_{2g}^q(\mathbf{Z})) \xrightarrow{\cong} H^2(\mathbf{Z}/4) \oplus \text{Hom}(H_2(\mathbf{Z}^{2g} \rtimes \text{Sp}_{2g}^q(\mathbf{Z})), \mathbf{Z}), \quad (3.13)$$

for $g \geq 3$, analogous to the splitting (3.10). As before, this splitting is compatible with the inclusions $\mathbf{Z}^{2g} \rtimes \text{Sp}_{2g}^q(\mathbf{Z}) \subset \mathbf{Z}^{2g+2} \rtimes \text{Sp}_{2g+2}^q(\mathbf{Z})$, so the following is valid.

Definition 3.20 (i) Define the class

$$\frac{\chi^2}{2} \in H^2(\mathbf{Z}^{2g} \rtimes \text{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$$

for $g \gg 0$ via the splitting (3.13) by declaring its image in the first summand to be trivial and to be $\chi^2/2$ in the second. For small g , the class $\frac{\chi^2}{2}$ is defined as the pullback of the class for $g \gg 0$.

(ii) Define the class

$$\frac{\chi^2}{2} \in H^2(\Gamma_{g,1}^n; \mathbf{Z}) \quad \text{for } n \neq 1, 3, 7 \text{ and } n \equiv 3 \pmod{4}$$

as the pullback of the same-named class along the map

$$\Gamma_{g,1/2}^n \xrightarrow{(SF,P)} (H(g) \otimes S\pi_n \text{SO}(n)) \rtimes G \cong \mathbf{Z}^{2g} \rtimes \text{Sp}_{2g}^q(\mathbf{Z}).$$

(iii) Define the class

$$\frac{\chi^2 - \text{sgn}}{8} \in H^2(\Gamma_{g,1/2}^n; \mathbf{Z}) \cong \text{Hom}(H_2(\Gamma_{g,1/2}^n; \mathbf{Z}), \mathbf{Z}) \quad \text{for } n = 3, 7$$

for $g \gg 0$ as image of $(\chi^2 - \text{sgn})/8$ ensured by Lemma 3.19, and for small g as the pullback of $\frac{\chi^2 - \text{sgn}}{8} \in H^2(\Gamma_{g+h,1/2}; \mathbf{Z})$ for $h \gg 0$.

The isomorphism $H^2(\Gamma_{g,1/2}^n; \mathbf{Z}) \cong \text{Hom}(H_2(\Gamma_{g,1/2}^n; \mathbf{Z}), \mathbf{Z})$ for $n = 3, 7$ in the previous definition is assured by Corollary 2.4 and Lemma A.1, as the abelianisation $H_1(\Gamma_{g,1/2}^n; \mathbf{Z})$ vanishes for $g \gg 0$.

We finish this subsection with an auxiliary lemma convenient for later purposes.

Lemma 3.21 For $g = 1$, the class $\frac{\text{sgn}}{8} \in H^2(\text{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$ and the pullback of the class $\frac{\chi^2}{2} \in H^2(\mathbf{Z}^{2g} \times \text{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$ to $H^2(\text{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$ are trivial.

Proof Both classes evaluate trivially on $H_2(\text{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z})$. For the first class, this is a consequence of Lemma 3.15, for the second this holds by construction. Moreover, both classes pull back trivially to $H^2(\mathbf{Z}/4; \mathbf{Z})$ along the morphism $a: \mathbf{Z}/4 \rightarrow \text{Sp}_{2g}^q(\mathbf{Z})$ by definition. Although this morphism does not induce an isomorphism on abelianisations for $g = 1$, it still induces one on their torsion subgroups by Lemma A.1 and this is sufficient to deduce the assertion from the universal coefficient theorem. \square

3.6 The proof of Theorem B

We are ready to prove our main result Theorem B, which we state equivalently in terms of the central extension

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow \Gamma_g^n \longrightarrow \Gamma_{g,1/2}^n \longrightarrow 0,$$

explained in Sect. 1.3. Our description of its extension class in $H^2(\Gamma_{g,1/2}^n; \Theta_{2n+1})$ involves the cohomology classes in $H^2(\Gamma_{g,1/2}^n; \mathbf{Z})$ of Definitions 3.17 and 3.20, and the two homotopy spheres Σ_P and Σ_Q in the subgroup $\text{bA}_{2n+2} \subset \Theta_{2n+1}$ examined in Sect. 3.2.1. We write $(-)\cdot \Sigma \in H^2(\Gamma_{g,1/2}^n; \mathbf{Z}) \rightarrow H^2(\Gamma_{g,1/2}^n; \Theta_{2n+1})$ for the change of coefficients induced by $\Sigma \in \Theta_{2n+1}$.

Theorem 3.22 For $n \geq 3$ odd, the extension

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow \Gamma_{g,1}^n \longrightarrow \Gamma_{g,1/2}^n \longrightarrow 0$$

is classified by the class

$$\begin{cases} \frac{\text{sgn}}{8} \cdot \Sigma_P & \text{if } n \equiv 1 \pmod{4} \\ \frac{\text{sgn}}{8} \cdot \Sigma_P + \frac{\chi^2}{2} \cdot \Sigma_Q & \text{if } n \equiv 3 \pmod{4}, n \neq 3, 7 \\ \frac{\chi^2 - \text{sgn}}{8} \cdot \Sigma_Q & \text{if } n = 3, 7. \end{cases}$$

and its induced differential $d_2: H_2(\Gamma_{g,1/2}^n) \rightarrow \Theta_{2n+1}$ has image

$$\text{im} \left(H_2(\Gamma_{g,1/2}^n) \xrightarrow{d_2} \Theta_{2n+1} \right) = \begin{cases} \text{bA}_{2n+2} & \text{if } g \geq 2 \\ \langle \Sigma_Q \rangle & \text{if } g = 1. \end{cases}$$

Moreover, the extension splits if and only if $g = 1$ and $n \equiv 1 \pmod{4}$.

Proof As all cohomology classes involved are compatible with the stabilisation map $\Gamma_{g,1/2}^n \rightarrow \Gamma_{g+1,1/2}^n$, it is sufficient to show the first part of the claim for $g \gg 0$ (see Sect. 1.4). We assume $n \neq 3, 7$ first. Identifying $\Gamma_{g,1/2}^n$ with $(H(g) \otimes \pi_n \text{SO}) \rtimes G_g$ via the isomorphism (s_F, p) of Sect. 2, the morphism $a: \mathbf{Z}/4 \rightarrow (H(g) \otimes \pi_n \text{SO}) \rtimes G_g$

of the previous section induces a morphism between the sequences of the universal coefficient theorem

$$\begin{array}{ccccccc}
 0 & \rightarrow & \text{Ext}(H_1(\Gamma_{g,1/2}^n), \Theta_{2n+1}) & \rightarrow & H^2(\Gamma_{g,1/2}^n; \Theta_{2n+1}) & \rightarrow & \text{Hom}(H_2(\Gamma_{g,1/2}^n), \Theta_{2n+1}) \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Ext}(\mathbf{Z}/4, \Theta_{2n+1}) & \longrightarrow & H^2(\mathbf{Z}/4; \Theta_{2n+1}) & \longrightarrow & \text{Hom}(H_2(\mathbf{Z}/4), \Theta_{2n+1}) \rightarrow 0.
 \end{array}$$

By the exactness of the rows and the vanishing of $H_2(\mathbf{Z}/4; \mathbf{Z})$, it is sufficient to show that the extension class in consideration agrees with the classes in the statement when mapped to $\text{Hom}(H_2(\Gamma_{g,1/2}^n; \mathbf{Z}), \Theta_{2n+1})$ and $H^2(\mathbf{Z}/4, \Theta_{2n+1})$. Regarding the images in the Hom-term, it is enough to identify them after precomposition with the epimorphism

$$H_2(\text{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \longrightarrow H_2(\Gamma_{g,1/2}^n; \mathbf{Z}),$$

so from the construction of $\frac{\text{sgn}}{8}$ and $\frac{\chi^2}{2}$ together with Lemmas 3.15 and 3.19, we see that it suffices to show that $H_2(\text{BDiff}_{\partial/2}(W_{g,1}); \mathbf{Z}) \rightarrow \Theta_{2n+1}$ induced by the extension class maps the class of a bundle $\pi : E \rightarrow S$ to $\text{sgn}(E)/8 \cdot \Sigma_P$ if $n \equiv 1 \pmod{4}$ and to $\text{sgn}(E)/8 \cdot \Sigma_P + \chi^2(E)/2 \cdot \Sigma_Q$ otherwise, which is a consequence of Theorem 3.12 combined with Proposition 3.9. By construction, the classes $\frac{\text{sgn}}{8} \cdot \Sigma_P$ and $\frac{\chi^2}{2} \cdot \Sigma_Q$ vanish in $H^2(\mathbf{Z}/4; \Theta_{2n+1})$, so the claim for $n \neq 3, 7$ follows from showing that the extension class is trivial in $H^2(\mathbf{Z}/4; \Theta_{2n+1})$, i.e. that the pullback of the extension to $\mathbf{Z}/4$ splits, which is in turn equivalent to the existence of a lift

$$\begin{array}{ccc}
 & & \Gamma_{g,1}^n \\
 & \nearrow \text{dashed} & \downarrow (s_F, p) \\
 \mathbf{Z}/4 & \longrightarrow & G_g \subset (H(g) \otimes \pi_n \text{SO}) \times G_g.
 \end{array}$$

Using the standard embedding $W_1 = S^n \times S^n \subset \mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$, we consider the diffeomorphism

$$\begin{aligned}
 S^n \times S^n &\longrightarrow S^n \times S^n \\
 (x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) &\longmapsto (-y_1, \dots, y_{n+1}, x_1, \dots, x_{n+1}),
 \end{aligned}$$

which is of order 4, maps to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{Sp}_2^q(\mathbf{Z})$, and has constant differential, so it vanishes in $H(g) \otimes \pi_n \text{SO}$. As the natural map $\Gamma_{1,1}^n \rightarrow \Gamma_1^n$ is an isomorphism by Lemma 1.1, this diffeomorphism induces a lift as required for $g = 1$, which in turn provides a lift for all $g \geq 1$ via the stabilisation map $\Gamma_{1,1}^n \rightarrow \Gamma_{g,1}^n$. For $n = 3, 7$, the abelianisation of $\Gamma_{g,1/2}^n$ vanishes for $g \gg 0$ due to Corollary 2.4, so it suffices to identify the extension class with the classes in the statement in $\text{Hom}(H_2(\Gamma_{g,1/2}^n; \mathbf{Z}), \Theta_{2n+1})$ which follows as in the case $n \neq 3, 7$.

Lemma 3.19 (iii) implies that the image of d_2 for $n = 3, 7$ is generated by Σ_Q for all $g \geq 1$. For $n \neq 3, 7$, the map $\Gamma_{g,1/2}^n \rightarrow (H(g) \otimes \pi_n \text{SO}) \times G_g$ is an isomorphism (see Sect. 2), so Lemma 3.19 tells us that the image of d_2 for $n \equiv 3 \pmod{4}$ is generated by Σ_P and Σ_Q if $g \geq 2$ and by Σ_Q if $g = 1$. For $n \equiv 1 \pmod{4}$, it follows from Lemma 3.15 that the image of d_2 is generated by Σ_P if $g \geq 2$ and that it is trivial for $g = 1$. In sum, this implies the second part of the claim by Corollary 3.5, and also that the differential d_2 does not vanish for $g \geq 2$, so the extension is nontrivial in these cases. For $n \equiv 3 \pmod{4}$, the homotopy sphere Σ_Q is nontrivial by Theorem 3.3, so d_2 does not vanish for $g = 1$ either. Finally, in the case $n \equiv 1 \pmod{4}$, the extension is classified by $\frac{\text{sgn}}{8} \cdot \Sigma_P$, which is trivial for $g = 1$ by Lemma 3.21, so the extension splits and the proof is finished. \square

It is time to make good for the missing part of the proof of Theorem 2.2.

Proof of Theorem 2.2 for $n = 3, 7$. We have $G_g = \text{Sp}_{2g}(\mathbf{Z})$, so the case $g = 1$ follows from the fact that $H^2(\text{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}^{2g} \otimes \pi_n \text{SO}(n))$ vanishes by Lemma A.3. To prove the case $g \geq 2$, note that a hypothetical splitting $s: \text{Sp}_{2g}(\mathbf{Z}) \rightarrow \Gamma_{g,1/2}^n$ of the upper row of the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbf{Z}^{2g} \otimes S\pi_n \text{SO}(n) & \longrightarrow & \Gamma_{g,1/2}^n & \longrightarrow & \text{Sp}_{2g}(\mathbf{Z}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow (s_F, p) & & \parallel \\
 0 & \longrightarrow & \mathbf{Z}^{2g} \otimes \pi_n \text{SO} & \longrightarrow & (\mathbf{Z}^{2g} \otimes \pi_n \text{SO}) \times \text{Sp}_{2g}(\mathbf{Z}) & \longrightarrow & \text{Sp}_{2g}(\mathbf{Z}) \longrightarrow 0
 \end{array}$$

induces a splitting $(s_F, p) \circ s$ of the lower row, which agrees with the canonical splitting of the lower row up to conjugation with $\mathbf{Z}^{2g} \otimes \pi_n \text{SO}$, because such splittings up to conjugation are a torsor for $H^1(\text{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}^{2g} \otimes \pi_n \text{SO})$ which vanishes by Lemma A.3. Lemma 3.15 on the other hand ensures that there is a class $[f] \in H_2(\text{Sp}_{2g}(\mathbf{Z}); \mathbf{Z})$ with signature 4, so $s_*[f] \in H_2(\Gamma_{g,1/2}^n; \mathbf{Z})$ satisfies $\text{sgn}(s_*[f]) = 4$ and $\chi^2(s_*[f]) = 0$, which contradicts Lemma 3.19 (iii). \square

4 Kreck’s extensions and their abelian quotients

The inclusion $\mathbb{T}_{g,1}^n \subset \Gamma_{g,1}^n$ of the Torelli group extends to a pullback diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Theta_{2n+1} & \longrightarrow & \mathbb{T}_{g,1}^n & \longrightarrow & H(g) \otimes S\pi_n \text{SO}(n) \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Theta_{2n+1} & \longrightarrow & \Gamma_{g,1}^n & \longrightarrow & \Gamma_{g,1/2}^n \longrightarrow 0
 \end{array} \tag{4.1}$$

of extensions whose bottom row we identified in Theorem 3.22. We now apply this to obtain information about the top row, and moreover to compute the abelianisations of $\Gamma_{g,1}^n$ and $\mathbb{T}_{g,1}^n$ in terms of the homotopy sphere $\Sigma_Q \in \Theta_{2n+2}$, the subgroup $\text{bA}_{2n+2} \subset \Theta_{2n+2}$ of Sect. 3.2, and the abelianisation of $\Gamma_{g,1/2}^n$ as computed in Corollary 2.4.

Theorem 4.1 *Let $n \geq 3$ be odd and $g \geq 1$.*

(i) The kernel K_g of the morphism $\Theta_{2n+1} \rightarrow H_1(\Gamma_{g,1}^n)$ is generated by Σ_Q for $g = 1$ and agrees with the subgroup \mathfrak{bA}_{2n+2} for $g \geq 2$. The induced extension

$$0 \longrightarrow \Theta_{2n+1}/K_g \longrightarrow H_1(\Gamma_{g,1}^n) \longrightarrow H_1(\Gamma_{g,1/2}^n) \longrightarrow 0$$

splits.

(ii) The extension

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow T_{g,1}^n \longrightarrow H(g) \otimes S\pi_n \text{SO}(n) \longrightarrow 0$$

is nontrivial for $n \equiv 3 \pmod{4}$ and splits G_g -equivariantly for $n \equiv 1 \pmod{4}$. The image of its differential $d_2: H_2(H(g) \otimes S\pi_n \text{SO}(n); \mathbf{Z}) \rightarrow \Theta_{2n+1}$ is generated by Σ_Q .

(iii) The commutator subgroup of $T_{g,1}^n$ is generated by Σ_Q and the resulting extension

$$0 \longrightarrow \Theta_{2n+1}/\Sigma_Q \longrightarrow H_1(T_{g,1}^n) \longrightarrow H(g) \otimes S\pi_n \text{SO}(n) \longrightarrow 0$$

splits G_g -equivariantly.

Proof By the naturality of the Serre spectral sequence, the morphism of extension (4.1) induces a ladder of exact sequences

$$\begin{array}{ccccccc} H_2(H(g) \otimes S\pi_n \text{SO}(n)) & \xrightarrow{d_2^T} & \Theta_{2n+1} & \longrightarrow & H_1(T_{g,1}^n) & \longrightarrow & H(g) \otimes S\pi_n \text{SO}(n) \longrightarrow 0 \\ & \downarrow & & & \parallel & & \downarrow \\ H_2(\Gamma_{g,1/2}^n) & \xrightarrow{d_2^T} & \Theta_{2n+1} & \longrightarrow & H_1(\Gamma_{g,1}^n) & \longrightarrow & H_1(\Gamma_{g,1/2}^n) \longrightarrow 0 \end{array}$$

from which we see that the kernel K_g in question agrees with the image of the differential d_2^T , which we described in Theorem 3.22. By the universal coefficient theorem, the pushforward of the extension class of the bottom row of (4.1) along the quotient map $\Theta_{2n+1} \rightarrow \Theta_{2n+1}/\text{im}(d_2^T)$ is classified by a class in $\text{Ext}(H_1(\Gamma_{g,1/2}^n), \Theta_{2n+1}/\text{im}(d_2^T))$, which also describes the exact sequence in (i). By a combination of Theorem 3.22 and Corollary 3.5, this class is trivial for $n = 3, 7$, and for $n \neq 3, 7$ as long as $g \geq 2$. For $g = 1$ and $n \neq 3, 7$, this class agrees with the image of $\frac{\text{sgn}}{8} \cdot \Sigma_P$ in $H^2(\Gamma_{g,1}^n; \Theta_{2n+1}/\text{im}(d_2^T))$ and therefore vanishes as $\frac{\text{sgn}}{8} \in H^2(G_g; \mathbf{Z})$ is trivial in view of Lemma 3.21.

The extension class of the bottom row of (4.1), determined in Theorem 3.22, pulls back to the extension class of the top one. Since $\frac{\text{sgn}}{8} \in H^2(\Gamma_{g,1/2}^n; \mathbf{Z})$ is pulled back from G_g by construction, it is trivial in $H(g) \otimes S\pi_n \text{SO}(n)$, so the extension in (ii) is trivial for $n \equiv 1 \pmod{4}$, classified by $\frac{\chi^2}{2} \cdot \Sigma_Q$ for $n \equiv 3 \pmod{4}$ if $n \neq 3, 7$, and by $\frac{\chi^2 - \text{sgn}}{8} \cdot \Sigma_Q$ if $n = 3, 7$. From this, the claimed image of d_2^T follows from Lemma 3.19 and its proof. For $n \equiv 3 \pmod{4}$, the homotopy sphere Σ_Q is nontrivial, so the extension does not split. This shows the second part of the statement, except for the claim regarding the equivariance, which will follow from the third part, since Σ_Q is trivial for $n \equiv 1 \pmod{4}$ by Theorem 3.3.

The diagram above shows that the commutator subgroup of $T_{g,1}^n$ agrees with the image of d_2^T , which we already showed to be generated by Σ_Q . To construct a G_g -equivariant splitting as claimed, note that the quotient of the lower row of (4.1) by Σ_Q pulls back from G_g by Theorem 3.22 because $\frac{\text{sgn}}{8} \in H^2(\Gamma_{g,1}^n; \Theta_{2n+1})$ has this property. Consequently, there is a central extension $0 \rightarrow \Theta_{2n+1}/\Sigma_Q \rightarrow E \rightarrow G_g \rightarrow 0$ fitting into a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Theta_{2n+1}/\Sigma_Q & \longrightarrow & T_{g,1}/\Sigma_Q & \longrightarrow & H(g) \otimes S\pi_n\text{SO}(n) \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Theta_{2n+1}/\Sigma_Q & \longrightarrow & \Gamma_{g,1}^n/\Sigma_Q & \longrightarrow & \Gamma_{g,1/2}^n \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow^p \\
 0 & \longrightarrow & \Theta_{2n+1}/\Sigma_Q & \longrightarrow & E & \longrightarrow & G_g \longrightarrow 0
 \end{array}$$

whose middle vertical composition induces a splitting as claimed, using that the right column in the diagram is exact. \square

The previous theorem, together with Lemma 1.1, Corollary 2.4, and Theorem 3.2 has Theorem C (ii) and Theorem D as a consequence. It also implies the following.

Corollary 4.2 *For $n \geq 3$ odd and $g \geq 1$, $T_{g,1}^n$ is abelian if and only if $n \equiv 1 \pmod{4}$.*

4.1 A geometric splitting

The splittings of the abelianisations of $\Gamma_{g,1}^n$ and $T_{g,1}^n$ provided by Theorem 4.1 are of a rather abstract nature. Aiming towards splitting these sequences more geometrically, we consider the following construction.

A diffeomorphism $\phi \in \text{Diff}_\partial(W_{g,1})$ fixes a neighbourhood of the boundary pointwise, so its mapping torus T_ϕ comes equipped with a canonical germ of a collar of its boundary $S^1 \times \partial W_{g,1} \subset T_\phi$ using which we obtain a closed oriented $(n-1)$ -connected $(2n+1)$ -manifold \tilde{T}_ϕ by gluing in $D^2 \times S^{2n-1}$. By obstruction theory and the fact that W_g is n -parallelisable, the stable normal bundle $\tilde{T}_\phi \rightarrow \text{BO}$ has a unique lift to $\tau_{>n}\text{BO} \rightarrow \text{BO}$ compatible with the lift on $D^2 \times S^{2n-1}$ induced by its standard stable framing. This gives rise to a morphism

$$t: \Gamma_{g,1}^n \longrightarrow \Omega_{2n+1}^{\tau_{>n}}$$

which is compatible with the stabilisation map $s: \Gamma_{g,1}^n \rightarrow \Gamma_{g,1}^n$ since

$$ts(\phi) = [T_\phi \# (S^1 \times S^n \times S^n) \cup_{S^1 \times S^{2n-1}} D^2 \times S^{2n-1}] = t(\phi) + [S^1 \times S^n \times S^n] = t(\phi),$$

where $S^1 \times S^n \times S^n$ carries the $\tau_{>n}\text{BO}$ -structure induced by the standard stable framing, which bounds. Using this, it is straight-forward to see that the composition

$$\Theta_{2n+1} = \Gamma_{0,1}^n \longrightarrow \Gamma_{g,1}^n \xrightarrow{t} \Omega_{2n+1}^{\tau_{>n}}$$

of the iterated stabilisation map with t agrees with the canonical epimorphism appearing in Wall’s exact sequence (3.1), so its kernel agrees with the subgroup bA_{2n+2} . Together with the first part of Theorem 4.1, we conclude that the dashed arrow in the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Theta_{2n+1}/K_g & \longrightarrow & H_1(\Gamma_{g,1}^n) & \xrightarrow{p_*} & H_1(\Gamma_{g,1/2}^n) \longrightarrow 0 \\
 & & & & \downarrow t_* & & \\
 & & & \dashrightarrow & \Omega_{2n+1}^{\tau > n} & &
 \end{array}$$

is an isomorphism if and only if $K_g = \text{bA}_{2n+2}$ which is the case for $g \geq 2$, and for $g = 1$ as long as $n = 3, 7$ since Σ_Q generates bA_{2n+2} for $n = 3, 7$ by Corollary 3.5. Consequently, in these cases, the morphism

$$t_* \oplus p_* : H_1(\Gamma_{g,1}^n) \longrightarrow \Omega_{2n+1}^{\tau > n} \oplus H_1(\Gamma_{g,1/2}^n) \tag{4.2}$$

is an isomorphism, whereas its kernel for $g = 1$ coincides with the quotient $\text{bA}_{2n+2}/\Sigma_Q$, which is nontrivial as long as $n \neq 3, 7$, generated by Σ_P , and whose order can be interpreted in terms of signatures (see Lemma 3.10). Since p_* splits by Theorem 4.1 and the natural map $\text{coker}(J)_{2n+1}/[\Sigma_Q] \rightarrow \Omega_{2n+1}^{\tau > n}$ is an isomorphism by Corollary 3.6, the morphism (4.2) splits for $g = 1$ if and only if the natural map

$$\Theta_{2n+1}/\Sigma_Q \longrightarrow \text{coker}(J)_{2n+1}/[\Sigma_Q] \tag{4.3}$$

does. Brumfiel [10, Thm 1.3] has shown that this morphism always splits before taking quotients, so the map (4.3) (and hence also (4.2)) in particular splits whenever $[\Sigma_Q] \in \text{coker}(J)_{2n+1}$ is trivial, which is conjecturally always the case (see Conjecture 3.7) and known in many cases as a result of Theorem 3.3 (see also Remark 3.8).

The situation for $H_1(\mathbb{T}_{g,1}^n)$ is similar. By Theorem 4.1, the morphism ρ_* in

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Theta_{2n+1}/\Sigma_Q & \longrightarrow & H_1(\mathbb{T}_{g,1}^n) & \xrightarrow{\rho_*} & H(g) \otimes S\pi_n\text{SO}(n) \longrightarrow 0 \\
 & & & & \downarrow t_* & & \\
 & & & \dashrightarrow & \Omega_{2n+1}^{\tau > n} & &
 \end{array}$$

splits G_g -equivariantly and since the morphism t_* is defined on $H_1(\Gamma_{g,1}^n)$, its restriction to $H_1(\mathbb{T}_{g,1}^n)$ is G_g -equivariant when equipping $\Omega_{2n+1}^{\tau > n}$ with the trivial action. By an analogous discussion to the one above, the kernel of the resulting morphism of G_g -modules

$$t_* \oplus \rho_* : H_1(\mathbb{T}_{g,1}^n) \longrightarrow \Omega_{2n+1}^{\tau > n} \oplus H(g) \otimes S\pi_n\text{SO}(n) \tag{4.4}$$

is trivial for $n = 3, 7$ and given by the quotient $\text{bA}_{2n+2}/\Sigma_Q$ for $n \neq 3, 7$. Moreover, this morphism splits if and only if it splits G_g -equivariantly,⁴ which is precisely the

⁴ If (s_t, s_ρ) is a non-equivariant splitting of $t_* \oplus \rho_*$, then $(s_t, (\text{id} - s_t t_*)s')$ is an equivariant splitting of $t_* \oplus \rho_*$, where s' an equivariant splitting of ρ_* ensured by Theorem 4.1. This uses that s_t is already equivariant since $\rho_* s_t = 0$, so the image of s_t is contained in the image of $\Theta_{2n+1}/\Sigma_Q \rightarrow H_1(\mathbb{T}_{g,1}^n)$ which is fixed by the action.

case if the natural map (4.3) admits a splitting. We summarise this discussion in the following corollary, which implies Theorem E when combined with Lemma 3.10.

Corollary 4.3 *Let $g \geq 1$ and $n \geq 3$ odd.*

(i) *The morphism*

$$t_* \oplus p_* : H_1(\Gamma_{g,1}^n) \longrightarrow \Omega_{2n+1}^{\tau_{>n}} \oplus H_1(\Gamma_{g,1/2}^n)$$

is an isomorphism for $g \geq 2$. For $g = 1$, it is an epimorphism and its kernel is given by $\mathfrak{bA}_{2n+2}/\Sigma_Q$.

(ii) *The morphism*

$$t_* \oplus \rho_* : H_1(\mathbb{T}_{g,1}^n) \longrightarrow \Omega_{2n+1}^{\tau_{>n}} \oplus H(g) \otimes S\pi_n\mathrm{SO}(n)$$

is an epimorphism and has kernel $\mathfrak{bA}_{2n+2}/\Sigma_Q$.

(iii) *The morphism $t_* \oplus p_*$ splits for $g = 1$ if and only if*

$$\Theta_{2n+1}/\Sigma_Q \longrightarrow \mathrm{coker}(J)_{2n+1}/[\Sigma_Q]$$

splits, which is the case for $n \equiv 1 \pmod{4}$. The same holds for $t_ \oplus \rho_*$ for all $g \geq 2$.*

4.2 Abelianising $\Gamma_{g,1}^n$ and $\mathbb{T}_{g,1}^n$ for n even

For $n \geq 4$ even, the abelianisations of $\Gamma_{g,1}^n$ and $\mathbb{T}_{g,1}^n$ can be computed without fully determining the extensions

$$\begin{aligned} 0 &\longrightarrow \Theta_{2n+1} \longrightarrow \Gamma_{g,1}^n \longrightarrow \Gamma_{g,1/2}^n \longrightarrow 0 \\ 0 &\longrightarrow H(g) \otimes S\pi_n\mathrm{SO}(n) \longrightarrow \Gamma_{g,1/2}^n \longrightarrow G_g \longrightarrow 0. \end{aligned}$$

Indeed, arguing similarly as in the proof of Corollary 2.4, the second extension provides an isomorphism $H_1(\Gamma_{g,1/2}^n) \cong H_1(G_g) \oplus (H(g) \otimes S\pi_n\mathrm{SO}(n))_{G_g}$, using that the coinvariants vanish also for $G_g = \mathrm{O}_{g,g}(\mathbf{Z})$ as long as $g \geq 2$ by Lemma A.2 and that the extension splits for $g = 1$, which is straightforward to check by noting that it is easy to lift elements of

$$G_1 = \langle \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \rangle \cong \mathbf{Z}/2 \oplus \mathbf{Z}/2$$

to $\Gamma_{1,1}^n \cong \Gamma_n^1 \cong \pi_0\mathrm{Diff}^+(S^n \times S^n)$ (so in particular to $\Gamma_{1,1/2}^n$) using the flip of the factors and a diffeomorphism of S^n of degree -1 .⁵ In contrast to the case $n \geq 3$, the resulting analogues of the morphisms (4.2) and (4.4) for $n \geq 4$ even are isomorphisms for all $g \geq 1$; this is Theorem G.

⁵ There are several mistakes in the literature related to the fact that $G_1 = \mathrm{O}_{1,1}(\mathbf{Z})$ is isomorphic to $(\mathbf{Z}/2)^2$ and not $\mathbf{Z}/4$: in [35, p. 645], it should be $\tilde{\pi}_0\mathrm{Diff}(S^2 \times S^2) \cong (\mathbf{Z}/2)^2$, in [49, Thm 1] it should be $\tilde{\pi}_0\mathrm{Diff}(S^p \times S^q)/\tilde{\pi}_0\mathrm{SDiff}(S^p \times S^q) \cong (\mathbf{Z}/2)^2$ for $p = q$ even, and finally in the proof of [36, Thm 2.6] it should be $\mathrm{Aut}(H_k(S^k \times S^k)) \cong (\mathbf{Z}/2)^2$ for k even.

Proof of Theorem G Wall’s exact sequence (3.1) is also valid for $n \geq 4$ even, so implies similarly to the case n odd that both morphisms in question are surjective and that their kernels agree with the quotients of \mathfrak{bA}_{2n+2} by the images of the differentials

$$H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \xrightarrow{d_2} \Theta_{2n+1} \quad \text{and} \quad H_2(H(g) \otimes S\pi_n\text{SO}(n); \mathbf{Z}) \xrightarrow{d_2} \Theta_{2n+1}$$

induced by the extensions (1.5) and (1.6), so we have to show that these images agree with \mathfrak{bA}_{2n+2} . By comparing the extensions for different g via the stabilisation map (see Sect. 1.4) and noting that the second differential factors through the first, we see that it suffices to show this for the second differential in the case $g = 1$. Plumbing disc bundles over S^{n+1} defines a pairing of the form

$$S\pi_n\text{SO}(n) \otimes S\pi_n\text{SO}(n) \rightarrow A_{2n+2}^{\tau_{>n}}$$

which can be seen to be surjective for $n \geq 4$ even, unlike in the case $n \geq 3$ odd [57, p. 295]. The composition of the pairing with the boundary map $\partial: A_{2n+2}^{\tau_{>n}} \rightarrow \Theta_{2n+1}$ with image \mathfrak{bA}_{2n+2} is usually called the *Milnor pairing*. Using the surjectivity, it is enough to show that the image of the second differential contains the image of the Milnor pairing. To do so, we rewrite the extension inducing the second differential via the canonical isomorphism $T_{1,1}^n \cong T_1^n = \ker(\pi_0\text{Diff}^+(S^n \times S^n) \rightarrow G_1)$ resulting from Lemma 1.1 as

$$0 \longrightarrow \Theta_{2n+1} \xrightarrow{l} T_1^n \xrightarrow{\rho} S\pi_n\text{SO}(n)^{\oplus 2} \longrightarrow 0. \tag{4.5}$$

and constructs in an explicit cocycle $f: S\pi_n\text{SO}(n)^{\oplus 2} \times S\pi_n\text{SO}(n)^{\oplus 2} \rightarrow \Theta_{2n+1}$ classifying this extension as follows: define morphisms

$$t_1: S\pi_n\text{SO}(n) \rightarrow T_1^n \quad \text{and} \quad t_2: S\pi_n\text{SO}(n) \rightarrow T_1^n$$

by assigning to a class $\eta \in S\pi_n\text{SO}(n)$ represented by $\eta: (S^n, D_+^n) \rightarrow (\text{SO}(n), \text{id}) \subset (\text{SO}(n+1), \text{id})$ the diffeomorphism $t_1(\eta)(x, y) = (x, \eta(x) \cdot y)$ and $t_2(\eta)(x, y) = (\eta(y) \cdot x, y)$, where $D_+^n \subset S^n$ is the upper half-disc. Choosing the centre $* \in D_+^n \subset S^n$ as the base point, the diffeomorphisms $t_1(\eta)$ and $t_2(\eta)$ fix both spheres $* \times S^n, S^n \times * \subset S^n \times S^n$, one because η is a based map and the other because it factors through the stabilisation map. From the description of ρ in Sect. 1.3, we see that $\rho t_1(\eta) = (\eta, 0)$ and $\rho t_2(\eta) = (0, \eta)$, so the function $S\pi_n\text{SO}(n)^{\oplus 2} \rightarrow T_1^n$ mapping (η, ξ) to $t_1(\eta) \circ t_2(\xi)$ is a set-theoretical section to ρ , which implies that the function defined by

$$f: S\pi_n\text{SO}(n)^{\oplus 2} \times S\pi_n\text{SO}(n)^{\oplus 2} \longrightarrow \Theta_{2n+1} \\ ((\eta_1, \xi_1), (\eta_2, \xi_2)) \longmapsto t_1(\eta_1)t_2(\xi_1)t_1(\eta_2)t_2(\xi_2)(t_1(\eta_1 + \eta_2)t_2(\xi_1 + \xi_2))^{-1}$$

defines a 2-cocycle that represents the extension class of (4.5) in $H^2(S\pi_n\text{SO}(n)^{\oplus 2}; \Theta_{2n+1})$ (see e.g. [9, Ch. IV.3]); here we identified Θ_{2n+1} with its image in T_1^n . The differential is the image of this class under the map $H^2(S\pi_n\text{SO}(n)^{\oplus 2}; \Theta_{2n+1}) \rightarrow \text{Hom}(H_2(S\pi_n\text{SO}(n)^{\oplus 2}); \Theta_{2n+1})$ participating in the universal coefficient theorem (see Sect. 3.3), so with respect to the canonical isomorphism $\Lambda^2(S\pi_n\text{SO}(n)^{\oplus 2}) \cong$

$H_2(S\pi_n\text{SO}(n)^{\oplus 2})$ (see e.g. [9, Thm V.6.4 (iii)], the differential takes the form (see e.g. [9, Ex. IV.4.8 (c), Ex. V.6.5])

$$\begin{aligned} \Lambda^2(S\pi_n\text{SO}(n)^{\oplus 2}) &\longrightarrow \Theta_{2n+1} \\ (\eta_1, \xi_1) \wedge (\eta_2, \xi_2) &\longmapsto f((\eta_1, \xi_1), (\eta_2, \xi_2)) - f((\eta_2, \xi_2), (\eta_1, \xi_1)). \end{aligned}$$

The precomposition of this differential with the map $S\pi_n\text{SO}(n) \otimes S\pi_n\text{SO}(n) \rightarrow \Lambda^2(S\pi_n\text{SO}(n)^{\oplus 2})$ mapping $\eta \otimes \xi$ to $(\eta, 0) \wedge (0, \xi)$ thus agrees with the pairing

$$S\pi_n\text{SO}(n) \otimes S\pi_n\text{SO}(n) \rightarrow \Theta_{2n+1}$$

that sends $\eta \otimes \xi$ to the commutator $t_1(\eta)t_2(\xi)t_1(\eta)^{-1}t_2(\xi)^{-1}$. By the discussion in [40, p. 834], this coincides with the Milnor pairing, so its image is bA_{2n+2} by the discussion above, which concludes the claim. \square

4.3 Splitting the homology action

We conclude our study of Kreck’s extensions and its abelianisations with the following result, which proves the remaining first part of Theorem C. The reader shall be reminded once more of Lemma 1.1, saying that the natural map $\Gamma_{g,1}^n \rightarrow \Gamma_g^n$ is an isomorphism for $n \geq 3$.

Theorem 4.4 *For $n \geq 3$ odd, the extension*

$$0 \longrightarrow T_{g,1}^n \longrightarrow \Gamma_{g,1}^n \longrightarrow G_g \longrightarrow 0$$

does not split for $g \geq 2$, but admits a splitting for $g = 1$ and $n \neq 3, 7$.

Proof The case $n = 3, 7$ and $g \geq 2$ is clear, since Theorem A shows that under this assumption even the quotient of the extension by the subgroup $\Theta_{2n+1} \subset T_{g,1}^n$ does not split. To deal with the other cases, we consider the morphism of extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Theta_{2n+1} & \longrightarrow & \Gamma_{g,1}^n & \longrightarrow & \Gamma_{g,1/2}^n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T_{g,1}^n & \longrightarrow & \Gamma_{g,1}^n & \longrightarrow & G_g \longrightarrow 0. \end{array}$$

By the naturality of the Serre spectral sequence, we have a commutative square

$$\begin{array}{ccc} H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) & \xrightarrow{d_2} & \Theta_{2n+1} \\ \downarrow & & \downarrow \\ H_2(G_g; \mathbf{Z}) & \xrightarrow{d_2} & H_1(T_{g,1}^n; \mathbf{Z})_{G_g} \end{array}$$

whose right vertical map has kernel generated by Σ_Q by the computation of $H_1(T_{g,1}^n; \mathbf{Z})$ as a G_g -module in Theorem 4.1. Therefore, to finish the proof of the

first claim, it suffices to show that the differential $d_2: H_2(\Gamma_{g,1/2}^n; \mathbf{Z}) \rightarrow \Theta_{2n+1}/\Sigma_Q$ is nontrivial for $g \geq 2$ and $n \neq 3, 7$, which follows from Theorem 3.22 together with the fact that $\text{bA}_{2n+2}/\Sigma_Q$ is nontrivial in these cases by Lemma 3.10 and Remark 3.11. Turning towards the second claim, we assume $n \neq 3, 7$ and recall that the isomorphism $(s_F, p): \Gamma_{g,1/2}^n \rightarrow (H(g) \otimes \pi_n \text{SO}) \times G_g$ induces a splitting of the right vertical map $\Gamma_{g,1/2}^n \rightarrow G_g$ in the above diagram (see Sect. 2), so the claim follows from showing that the pullback of $\Gamma_{g,1}^n \rightarrow \Gamma_{g,1}^n$ along $G_g \subset (H(g) \otimes \pi_n \text{SO}) \times G_g \cong \Gamma_{g,1}^n$ splits for $g = 1$, which is a consequence of Theorem B and Lemma 3.21. \square

Remark 4.5 Theorem 4.4 leaves open whether $\Gamma_{g,1}^n \rightarrow G_g$ admits a splitting for $g = 1$ in dimensions $n = 3, 7$. Krylov [37, Thm 2.1] and Fried [19, Sect. 2] showed that this can not be the case for $n = 3$ and we expect the same to hold for $n = 7$.

5 Homotopy equivalences

Our final result Corollary F is concerned with the morphism of extensions

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H(g) \otimes S\pi_n \text{SO}(n) & \longrightarrow & \Gamma_g^n / \Theta_{2n+1} & \longrightarrow & G_g \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & H(g) \otimes S\pi_{2n} S^n & \longrightarrow & \pi_0 \text{hAut}^+(W_g) & \longrightarrow & G_g \longrightarrow 0,
 \end{array} \tag{5.1}$$

underlying work of Baues [1, Thm 10.3], relating the mapping class group Γ_g^n to the group $\pi_0 \text{hAut}^+(W_g)$ of homotopy classes of orientation preserving homotopy equivalences. The left vertical morphism is induced by the restriction of the unstable J homomorphism $J: \pi_n \text{SO}(n + 1) \rightarrow \pi_{2n+1} S^{n+1}$ to the image of the stabilisation $S: \pi_n \text{SO}(n) \rightarrow \pi_n \text{SO}(n + 1)$ in the source and to the image of the suspension map $S: \pi_{2n} S^n \rightarrow \pi_{2n+1} S^{n+1}$ in the target, justified by the fact that the square

$$\begin{array}{ccc}
 \pi_n \text{SO}(n) & \xrightarrow{S} & \pi_n \text{SO}(n + 1) \\
 \downarrow J & & \downarrow J \\
 \pi_{2n} S^n & \xrightarrow{S} & \pi_{2n+1} S^{n+1}
 \end{array}$$

commutes up to sign (see [53, Cor. 11.2]).

Proof of Corollary F By Theorem A, the upper row of (5.1) splits for $n \neq 1, 3, 7$ odd, so the first part of (i) is immediate. The second part follows from Theorem A as well if we show that the existence of a splitting of the lower row for $n = 3, 7$ is equivalent to one of the upper row. To this end, note that for $n = 3, 7$, we have isomorphisms $S\pi_n \text{SO}(n) \cong \mathbf{Z}$ and $S\pi_{2n} S^n = \text{Tor}(\pi_{2n+1} S^{n+1}) \cong \mathbf{Z}/d_n$ for $d_3 = 12$ and $d_7 = 120$ with respect to which the J -homomorphism $J: S\pi_n \text{SO}(n) \rightarrow S\pi_n S^n$ is given by reduction by d_n (see e.g. [53, Ch. XIV]). By Lemma A.3, the group $H^2(G_g; H(g))$ is annihilated by 2, so in particular by d_n . Using this, the first claim follows from the

long exact sequence on cohomology induced by the exact sequence

$$0 \rightarrow H(g) \otimes S\pi_n \text{SO}(n) \xrightarrow{d_n \cdot (-)} H(g) \otimes S\pi_n \text{SO}(n) \xrightarrow{J_*} H(g) \otimes S\pi_{2n} S^n \rightarrow 0$$

of G_g -modules, since the extension class of the lower row is obtained from that of the upper one by the change of coefficients $J_*: H(g) \otimes S\pi_n \text{SO}(n) \rightarrow H(g) \otimes S\pi_{2n} S^n$.

To prove the second part, we consider the exact sequence

$$(H(g) \otimes S\pi_{2n} S^n)_{G_g} \rightarrow H_1(\pi_0 \text{hAut}^+(W_g)) \rightarrow H_1(G_g) \rightarrow 0$$

induced by the Serre spectral sequence of the lower row of (5.1). The left morphism is split injective as long as the extension splits, which, together with the first part and a consultation of Lemma A.2, exhibits $H_1(\pi_0 \text{hAut}^+(W_g))$ to be as asserted. \square

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Appendix A: Low-degree cohomology of symplectic groups

This appendix contains various results on the low-degree (co)homology of the integral symplectic group $\text{Sp}_{2g}(\mathbf{Z})$ and its theta subgroup $\text{Sp}_{2g}^q(\mathbf{Z})$ (see Sect. 1.2).

Lemma A.1 (i) *The abelianisations of $\text{Sp}_{2g}(\mathbf{Z})$ and $\text{Sp}_{2g}^q(\mathbf{Z})$ satisfy*

$$H_1(\text{Sp}_{2g}(\mathbf{Z})) \cong \begin{cases} \mathbf{Z}/12 & \text{if } g = 1 \\ \mathbf{Z}/2 & \text{if } g = 2 \\ 0 & \text{if } g \geq 3 \end{cases} \quad \text{and} \quad H_1(\text{Sp}_{2g}^q(\mathbf{Z})) \cong \begin{cases} \mathbf{Z}/4 \oplus \mathbf{Z} & \text{if } g = 1 \\ \mathbf{Z}/4 \oplus \mathbf{Z}/2 & \text{if } g = 2 \\ \mathbf{Z}/4 & \text{if } g \geq 3. \end{cases}$$

- (ii) *The element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \text{Sp}_2(\mathbf{Z}) \subset \text{Sp}_{2g}(\mathbf{Z})$ generates $H_1(\text{Sp}_{2g}(\mathbf{Z}))$ for $g = 1, 2$.*
- (iii) *The element $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{Sp}_2^q(\mathbf{Z}) \subset \text{Sp}_{2g}^q(\mathbf{Z})$ generates the $\mathbf{Z}/4$ -summand in the group $H_1(\text{Sp}_{2g}^q(\mathbf{Z}))$ for all $g \geq 1$.*
- (iv) *The \mathbf{Z} -summand in $H_1(\text{Sp}_2^q(\mathbf{Z}))$ is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in \text{Sp}_2^q(\mathbf{Z})$ and the $\mathbf{Z}/2$ -summand in $H_1(\text{Sp}_4^q(\mathbf{Z}))$ is generated by $\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \in \text{Sp}_4^q(\mathbf{Z})$, where $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

(v) *The stabilisation map*

$$H_1(\mathrm{Sp}_{2g}(\mathbf{Z})) \longrightarrow H_1(\mathrm{Sp}_{2g+2}(\mathbf{Z}))$$

is surjective for all $g \geq 1$ and the stabilisation map

$$H_1(\mathrm{Sp}_{2g}^q(\mathbf{Z})) \longrightarrow H_1(\mathrm{Sp}_{2g+2}^q(\mathbf{Z}))$$

is surjective for $g \geq 2$, but has cokernel $\mathbf{Z}/2$ for $g = 1$.

Proof The fact that the abelianisation of $\mathrm{Sp}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and of order 12 is well-known, and so is the isomorphism type of $H_1(\mathrm{Sp}_{2g}(\mathbf{Z}))$ for $g \geq 2$ (see e.g. [3, Lem. A.1 (ii)]). The remaining claims regarding $\mathrm{Sp}_{2g}(\mathbf{Z})$ follows from showing that $H_1(\mathrm{Sp}_2(\mathbf{Z})) \rightarrow H_1(\mathrm{Sp}_4(\mathbf{Z}))$ is nontrivial, which can for instance be extracted from the proof of [20, Thm 2.1]: in their notation $\Gamma_1 = \mathrm{Sp}_4(\mathbf{Z})$ and the map $i_\infty: \mathrm{SL}_2(\mathbf{Z}) \rightarrow \Gamma_1$ identifies with the stabilisation $\mathrm{Sp}_2(\mathbf{Z}) \rightarrow \mathrm{Sp}_4(\mathbf{Z})$. The isomorphism type of $H_1(\mathrm{Sp}_{2g}^q(\mathbf{Z}))$ for $g \geq 2$ is determined in [17, Thm 2] and for $g = 1$ in [58, Thm 1]. The first claim of (ii) follows from the main formula of [27] for $g \geq 3$ and from [58, Cor. 2] for $g = 1, 2$, which also gives the claimed generator of the \mathbf{Z} -summand in $H_1(\mathrm{Sp}_2^q(\mathbf{Z}))$. The proof of [17, Thm 2] provides the asserted generator of the $\mathbf{Z}/2$ -summand in $H_1(\mathrm{Sp}_4^q(\mathbf{Z}))$. The final claim follows from the first four items once we show that the image of $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ in $H_1(\mathrm{Sp}_4^q(\mathbf{Z}))$ vanishes, which is another consequence of the formulas in [58, Cor. 2]. \square

Lemma A.2 *The (co)invariants of the standard actions of $\mathrm{Sp}_{2g}^q(\mathbf{Z})$ and $\mathrm{O}_{g,g}(\mathbf{Z})$ on $\mathbf{Z}^{2g} \otimes A$ for an abelian group A satisfy*

$$\begin{aligned} (\mathbf{Z}^{2g} \otimes A)_{\mathrm{Sp}_{2g}^q(\mathbf{Z})} &\cong (\mathbf{Z}^{2g} \otimes A)_{\mathrm{O}_{g,g}(\mathbf{Z})} \cong \begin{cases} A/2 & \text{if } g = 1 \\ 0 & \text{if } g \geq 2 \end{cases} \text{ and} \\ (\mathbf{Z}^{2g} \otimes A)^{\mathrm{Sp}_{2g}^q(\mathbf{Z})} &\cong (\mathbf{Z}^{2g} \otimes A)^{\mathrm{O}_{g,g}(\mathbf{Z})} \cong \begin{cases} \mathrm{Hom}(\mathbf{Z}/2, A) & \text{if } g = 1 \\ 0 & \text{if } g \geq 2 \end{cases}. \end{aligned}$$

The same applies to the action of $\mathrm{Sp}_{2g}(\mathbf{Z})$, except that the (co)invariants also vanish for $g = 1$.

Proof We prove the claim for $\mathrm{Sp}_{2g}^{(q)}(\mathbf{Z})$ first. The self-duality of \mathbf{Z}^{2g} induced by the symplectic form implies

$$(\mathbf{Z}^{2g} \otimes A)^{\mathrm{Sp}_{2g}^{(q)}(\mathbf{Z})} \cong \mathrm{Hom}(\mathbf{Z}^{2g}, A)^{\mathrm{Sp}_{2g}^{(q)}(\mathbf{Z})} \cong \mathrm{Hom}((\mathbf{Z}^{2g})_{\mathrm{Sp}_{2g}^{(q)}(\mathbf{Z})}, A),$$

so the computation of the invariants is a consequence of that of the coinvariants. To settle the case $g \geq 2$ for both $\mathrm{Sp}_{2g}^q(\mathbf{Z})$ and $\mathrm{Sp}_{2g}(\mathbf{Z})$, it thus suffices to prove that the coinvariants $(\mathbf{Z}^{2g} \otimes A)_{\mathrm{Sp}_{2g}^q(\mathbf{Z})}$ with respect to the smaller group vanish. We consider

multiplication $g \cdot (-) : M \rightarrow M$ by an element $g \in G$ of a discrete group G on a G -module M is equivariant with respect to the conjugation $c_g : G \rightarrow G$ by g and induces the identity on $H^*(G; M)$ by the usual argument. If $g \in G$ is central, conjugation by g is trivial, so $H^*(G; M)$ is annihilated by the action of $(1 - g)$ on M .

Since $H^*(\mathrm{Sp}_{2g}^{(q)}(\mathbf{Z}); \mathbf{Z}^{2g})$ is 2-torsion, the self-duality of \mathbf{Z}^{2g} combined with the universal coefficient theorem for nontrivial coefficients (see e.g. [51, p. 283]) implies

$$\mathrm{Ext}_{\mathbf{Z}}^1(H_{i-1}(\mathrm{Sp}_{2g}^{(q)}(\mathbf{Z}); \mathbf{Z}^{2g}), \mathbf{Z}) \cong H^i(\mathrm{Sp}_{2g}^{(q)}(\mathbf{Z}); \mathbf{Z}^{2g}),$$

so the computation of $H^*(\mathrm{Sp}_{2g}^{(q)}(\mathbf{Z}); \mathbf{Z}^{2g})$ for $* \leq 1$ is a consequence of Lemma A.2.

From the Lyndon–Hochschild–Serre spectral sequence for the extension

$$0 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Sp}_2^{(q)}(\mathbf{Z}) \longrightarrow \mathrm{PSP}_2^{(q)}(\mathbf{Z}) \longrightarrow 0$$

with coefficients in \mathbf{Z}^2 , we see that

$$H^2(\mathrm{Sp}_2^{(q)}(\mathbf{Z}); \mathbf{Z}^2) \cong H^1(\mathrm{PSP}_2^{(q)}(\mathbf{Z}); \mathbf{F}_2^2)$$

and the right hand side can easily be computed to be as claimed by applying the Mayer–Vietoris sequence to the presentation

$$\mathrm{PSP}_2(\mathbf{Z}) = \langle S, T \mid S^2 = 1, T^3 = 1 \rangle \quad \text{and} \quad \mathrm{PSP}_2^q(\mathbf{Z}) = \langle S, R \mid S^2 = 1 \rangle$$

for

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad R = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},$$

which is well-known for $\mathrm{PSP}_2(\mathbf{Z}) \cong \mathrm{PSL}_2(\mathbf{Z})$ and appears for instance in [58, p. 385] for $\mathrm{PSP}_2^q(\mathbf{Z})$. □

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