Abstract

  The textbook treatment for the valuation of warrants takes as a state variable the value of the firm and shows that the value of a warrant is equal to the value of a call option on the equity of the firm multiplied by a dilution factor. This approach however applies only for the not so realistic case where the firm issues only a single warrant. By a single warrant we mean n warrants with a single exercise price and time to maturity. What happens however in the "real world" where corporations issue multiple warrants and executive stock options? What happens for example when firms issue warrants of different exercise prices, different maturities and different dilution factors? Valuing each type of warrants independently of the others is clearly inappropriate and will result in mispricing. In this paper we derive distribution-free (and distribution-specific) formulae for firms that issue warrants with different maturities, different strike prices, and different dilution factors and for firms that issue warrants of the same maturity but different strike prices (and different dilution factors). The distinction we make between warrants and executive stock options is simply a matter of whether the contract is traded or not. We use the term warrant to cover both cases.

ON THE VALUATION OF WARRANTS AND EXECUTIVE STOCK OPTIONS: PRICING FORMULAE FOR FIRMS WITH MULTIPLE WARRANTS/EXECUTIVE OPTIONS

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1. INTRODUCTION

Warrants, executive/employee stock options, and exchange traded call options are all examples of contingent claims. Conceptually they are all rather similar instruments. They do after all provide the holder with a call option on the underlying.[1] However despite their superficial similarity, warrants and executive options differ in a number of important ways from vanilla exchange traded options. These differences, initially between warrants and standard call options and in the very recent years between executive stock options (ESOs) and exchange traded options, have inspired a great deal of academic and practical interest that has spanned the subject not only from a purely "asset pricing/valuation" perspective or from a purely "corporate finance/agency theory" perspective but has also resulted in an amalgamation of the two in the sense that valuation of warrants and ESOs has occasionally not been free of considerations such as corporate policies, managerial incentives, stockholders attitudes etc. Furthermore, as interest and more importantly issuance of these corporate instruments, particularly ESOs, has surged over the years, and all this amidst widespread demand for more financial transparency in corporate accounts, assessing correctly their true cost/value to stockholders/stakeholders has never been more imperative. We first briefly outline a few, from a long list of features, that distinguish warrants and ESOs from standard exchange traded options and can potentially complicate

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[1] According to Hull (2000): *Warrants* are call options that often come into existence as a result of a bond issuance. They are added to the bond issue to make it more attractive to investors. Typically, a warrant lasts for a number of years. Once they have been created, they sometimes trade separately from the bonds to which they were originally attached. *Executive stock options* are call options issued to executives to motivate them to act in the best interests of the company's shareholders. They are usually at-the-money when they are first issued. After a period of time they become vested and can be exercised. Unlike warrants and exchange-traded stock options they cannot be sold. They often last as long as 10 or 15 years.

their valuation. We then concentrate in one particular feature, namely the capital structure or dilution effect of warrants and ESOs, which provides the benchmark for this paper.

To start with, warrants often contain special provisions that allow the issuing company the right to call the warrants; see Schultz (1993), as well as the possibility to alter their maturity and/or exercise price; see Longstaff (1990), Howe and Wei (1993). The same also applies to ESOs where resetting the terms of already issued options that have gone "underwater" (i.e. (deep)-out-of-the-money) is a common phenomenon amongst young, developing corporations. This usually involves specifying a new strike price which at the time of resetting is often set equal to the current stock price. Extending the maturity of the options or canceling an executive's existing options and granting him or her new options with a lower strike price, although not as frequent, are also in the agenda. Among others, Brenner, Sundaram, and Yermack (2000), Carter and Lynch (2001) examine the "repricing" of ESOs and allocate it to poor firm-specific performance. The empirical evidence in Chance, Kumar, and Todd (2000) suggest that ESOs are usually repriced when the stock declines by about 25%. Acharya, John, and Sundaram (2000) investigate the optimality of resetting ESOs and find that a corporate policy that allows/accommodates for some resetting is almost always optimal. The default in this study will be that all the aforementioned special features, which we hereby refer to as special exercise provisions, are exogenous. However, if required, it is also possible to endogenize them in the valuation. Brenner, Sundaram and Yermack (2000) note for example that an ESO that will be repriced, i.e. whose strike price will change to a lower strike price the first time the stock price falls below a pre-specified barrier, can be valued as a portfolio of a down-and-out call with the initial strike price and a down-and-in call with the new lower strike price.

Moving on now from the special exercise provisions of the stake-issuers to value enhancing exercise strategies of the stake-holders, Emanuel (1983), Constantinides (1984), and Spatt and Sterbenz (1988) have examined exercise strategies by warrant-holders to increase the value of their warrants (e.g. sequential exercise – early exercise) and the actions of the warrants-issuers to neutralize this (e.g. reinvestment policies). Along the same lines, managerial incentives, arising from the issuance of ESOs, to increase the stock price, increase risk, reduce dividend yield (see Johnson and Tian (2000), Carpenter (2000)), and incentives in timing the exercises of ESOs; see Carpenter and Remmers (2001), are also fruitful areas for research. Again, for the purposes of this paper, we treat value-enhancing (or sometimes value-reducing) managerial strategies as exogenous factors.

Turning to maturity issues, both warrants and ESOs have much longer maturities than exchange traded options (compare for example 3-5 years and 10-15 years respectively with up to 1 year for standard options). Lauterbach and Schultz (1990) and Rubinstein (1995) are good references in identifying some of the difficulties associated with applying standard option models to long-term options. The results presented in this study are distribution free and thus applicable to a wide range of option pricing models, including

models such as the Constant Elasticity of Variance (CEV) of Cox and Ross (1976) or Merton's (1976) Jump-diffusion, which might be more suitable for valuing long-term options.

From a financial economics point of view, the theoretically most interesting feature of warrants and executive stock options as opposed to standard calls is their capital structure effect on the issuing firm. When warrants and executive options are exercised, new shares are issued and any exercise proceeds are paid into the firm. As a result, the interests of the firm's existing shareholders are diluted, including possible dilution in wealth, voting rights and the underlying stock price. The source of this dilution is the game that is played between the stakeholders and existing shareholders. A way to deal with the so-called dilution effect, originally suggested by Black and Scholes (1973), popularized by Galai and Schneller (1978) and also discussed in Lauterbach and Schultz (1990), Schulz and Trautmann (1994) is to value warrants as call options on the value of the firm (not the stock price) multiplied by a dilution factor. This framework has subsequently become the standard treatment for warrant valuation and appears in almost every textbook on derivative securities.

Over the past decade, empirical literature on warrant pricing (see for example Bensoussan, Crouhy, and Galai (1992), Veld (1994), Sidenius (1996)) has suggested that there is no need to follow the textbook treatment to value warrants (i.e. the value of the firm approach). Instead based on their empirical results and simulations these studies recommend "option-like" warrant valuation. This simply means valuing the warrant as if it was identical to a call option, without involving in the valuation process the (typically unobservable) value of the firm and ignoring any dilution. Surprisingly such an approximation works very well for a large number of cases. Cox and Rubinstein (1985) and Ingersoll (1987) recognize that when dilution is sufficiently small then it may be ignored. Schulz and Trautmann (1994) and Darsinos and Satchell (2002) have illustrated that "option-like" valuation will indeed price "correctly" and efficiently in-the-money, near-the-money and long maturity out-of-the-money warrants (and ESOs). However, they also find that for deep-out-of-the-money and near maturity out-of-the-money warrants in general, this approximation will significantly overprice warrants with the overpricing an increasing function of the dilution factor[2].

The purpose of this paper is to extend the theoretical framework for warrant and ESO valuation so that it correctly accounts for the dilution effect in the presence of multiple warrant (or ESO) issuances. Both the existing theoretical framework and the "option-like" valuation approximation outlined above are only appropriate for the case when firms issue just a single warrant. By a single warrant we mean n identical

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[2] Regarding the dilution factor, there appear to be no specific limits placed upon the total number of options granted. Each bunch of share options to be awarded is put before the shareholders in the general meeting for their approval. Over time, the total granted can rise to a considerable percentage, often well in excess of 10% of the issued capital (particularly so for high-tech firms).

outstanding warrants with a single exercise price and time to maturity. For the more realistic case where firms issue multiple warrants with different exercise prices and different maturities, application of the current textbook approach will result in overpricing. Indeed, the existing framework does not take into account the interdependence between warrants and the potential dilution effect of each issuance on the remaining warrants. The formulae derived in this paper incorporate the potential dilution effect of all the outstanding warrants on each particular issuance thus adjusting downwards the "true" value/cost of these instruments. Furthermore, just because "option-like" valuation is often a good approximation technique for the single warrant/ESO case does not *a priori* imply (and rightly so) that it will prove useful for approximating the value of multiple warrant issuances. By deriving the theoretical formulae for the valuation of multiple issuances, we provide the correct benchmark against which to check the quality of this approximation. This line of enquiry is particularly useful since corporations in their annual reports, according to the FASB mandates, often calculate the value/cost of their ESOs using "option-like" valuation. We believe that an extension of the existing framework is mandated since rarely in practice do companies issue only a single Executive Stock Option or warrant. For example, Marquardt (1999) examines 58 Fortune 100 firms over a 21 year period and finds an average of 17 contracts per firm. Furthermore, a casual glance at the accounts of any company that grants options to its executives/employees reveals a variety of maturities and exercise prices.

The distinction we make in this paper between warrants and executive share options is simply a matter of whether the contract is traded or not. Moreover we take the term executive stock options to contain the popular subset of employee share option schemes which have been successfully introduced amongst corporations during the past few years. Generally, we shall use the term warrant to cover all cases (i.e. warrants, executive options and employee options).[3] Issues of non-transferability, delayed vesting and suboptimal exercise policies of Executive Stock Options (ESOs) are not discussed in this paper. For these (and a few other) differences between exchange-traded options and ESOs we refer the reader to Rubinstein (1995). Here it suffices to say that the reduction in value of an executive option due to the non-transferability constraint is usually handled my multiplying the value that would otherwise be obtained (i.e. the value of a traded warrant) by one minus the probability that the executive/employee will leave the firm before exercise is possible. The default in this study will be that the probability of "premature leave" is zero (it is straightforward however to modify this assumption).

Regarding now the delayed vesting constraint or exercise policy of ESOs in general, it is indeed the case that most option plans do not permit employees to exercise

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[3] It is worth mentioning here that warrants are usually issued deep-out-of-the-money while ESO's are typically issued at-the-money (see for example Noreen and Wolfson (1981)). Hall and Murphy (2000) investigate the economic rationale behind this approach and find that by setting the exercise price at- or near-the grant-date market price, corporations maximize the incentives of their employees and executives.

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their granted options until after a predefined period of time has elapsed (for example, an executive option has typically a maturity of 10 years; however through delayed vesting, exercise is usually not permitted for a period after grant, typically 3 years). In other words ESOs are neither European nor American. In its Exposure Draft, "Accounting for Stock-based Compensation," FASB allows valuing ESO's as European but requires using the so called "expected life of the option" instead of the actual time to expiration. Carpenter (1998) examines the exercise policies of managers which can be executive-specific when the individual preferences, endowments, hedging and transferability constraints come into play. She shows that early exercise of an ESO is not consistent with exercise patterns observed in the data. Executives hold options long enough and deep enough into the money before exercising to capture a significant amount of their potential value. Detemple and Sundaresan (1999) find that ESOs may be exercised prematurely even when the underlying asset pays no dividends. In any case, as indicated by Rubinstein (1995) an ESO has two values. The one is highly personal depending on the preferences and financial circumstances of each employee. The other, which is the one that should be used by the corporation in its external financial statements, is the value of the option according to the effect its existence has on the value of the underlying stock. This is the notion of value that we use in this paper. As a final note, ESOs are particularly popular amongst high-tech firms. Such corporations promise rapid growth but also pay little or no dividends. (see amongst many others Microsoft Corporation, Oracle, Cisco Systems, AoL Time-Warner, etc). This has the additional advantage that American-type ESOs can, for the purposes of valuation, be considered as European.

The plan for the paper is as follows. In Section 2 we outline the textbook treatment for warrant valuation. We show that the price of a warrant can be evaluated as a call option on the equity value of the firm multiplied by a dilution factor equal to $1/(1+\lambda)$. (λ represents the ratio of the total number of new shares issued upon exercise of the warrants to the total number of existing shares). However we claim that this framework applies only for firms with a single warrant/ESO issuance, which in practice is rarely the case. In Section 3 we extend the theoretical framework to accommodate for multiple warrants. In particular, in section 3.1 we derive pricing formulae for firms that issue warrants of different maturities, different strike prices (and different dilution factors), while in section 3.2 we provide formulae for firms with warrants of common maturity but different exercise prices (and different dilution factors). The derived formulae are distribution free and are thus applicable to a wide range of process assumptions for the state variable. In Section 4 we make the derived formulae operational under the very widely-used assumption of Geometric Brownian Motion for the value process. Section 5 contains an empirical illustration where we value the outstanding ESOs of Cisco Systems. We find that application of the single warrant valuation technique in the multiple setting of Cisco Systems results in significant overpricing. On the other hand, Black-Scholes "option-like" valuation results in underpricing. This re-enforces existing concerns on the optimality of

using standard, unmodified option pricing models to account for the cost of stock-based compensation. Section 6 concludes.

2. A FRAMEWORK FOR WARRANT PRICING

2.1 The Textbook Treatment for the Valuation of Warrants: Firms with a Single Warrant Issuance

Valuation of warrants dates as back as the valuation of traded options. In fact, according to Black (1979), the celebrated option valuation model of Black and Scholes (1973) was originally developed for the pricing of warrants. However, warrants and executive stock options are written by companies on their own stock. When they are exercised, the company issues more shares and sells them to the stakeholders for the strike price. This subsequently dilutes the equity of existing shareholders. The textbook treatment for warrant valuation thus mandates that the warrants should be regarded as call options on a share of the total equity value of the firm, where equity is defined as the sum of the value of its shares and the value of its warrants.[4] We therefore start with the following simplifying assumption:

ASSUMPTION 1: *The warrant-issuing firm is an all equity firm with no outstanding debt.*

This then implies that stocks and warrants are the only sources of financing that the company is using. Hence, at current time $t = 0$, the company has a total equity value V_0 of:

$$V_0 = NS_0 + nW(S_0) \tag{1}$$

where

N: The number of outstanding shares.

S_0: The current price per share.

n: The number of outstanding warrants

$W(S_0)$: The current price of a warrant on a share.

Similarly the value of the firm per share (total equity per share) $v_0 = V_0 / N$ is:

$$v_0 = S_0 + \lambda W(S_0) \tag{2}$$

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[4] Alternatively, valuation of warrants can also be performed by pricing them as call options on the stock price of the warrant-issuing firm. This of course requires knowledge of the distribution of the stock price of the warrant-issuing firm, which will be different both from the distribution of the value of the firm and from the distribution of the stock price of a non-warrant issuing firm (see Darsinos and Satchell (2002) for more details).

where $\lambda = n/N$ is the dilution factor. More generally v_t, V_t, and S_t will be the values of the above entities at time t.

Suppose now that each warrant entitles the holder to purchase one share of the firm at time T for a strike price of K per share.[5]

- CASE 1: *The warrants are exercised at maturity.*

If the outstanding warrants are exercised at time T, the company receives a cash inflow of nK and the total equity value increases to $V_T + nK$. This value is then distributed among $N + n$ shares so that the price per share immediately after exercise becomes

$$S_T = (V_T + nK)/(N + n)$$

Hence, if the warrants are exercised, the payoff to each warrant holder is

$$W(S_T) = \frac{V_T + nK}{N + n} - K = \frac{N}{N + n}(\frac{V_T}{N} - K) = \frac{1}{1 + \lambda}(\frac{V_T}{N} - K)$$
$$= \frac{1}{1 + \lambda}(v_T - K)$$

Of course the warrants will be exercised only if $v_T > K$.

- CASE 2 (Trivial): *The warrants are not exercised at maturity.*

If the warrants are not exercised then no dilution of equity takes place. At maturity the price per share will be

$$S_T = V_T$$

and the payoff to the warrant holders will be zero

$$W(S_T) = 0$$

The warrants will not be exercised when $v_T \le K$.

We have therefore illustrated that the payoff to a warrant holder at maturity will be

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[5] We assume that each warrant entitles the holder to purchase one share of the firm. It is straightforward to modify this assumption and assume that each warrant entitles the holder to purchase p numbers of shares. For simplicity the default in this study will be $p = 1$.

$$W(S_T) = \frac{1}{1+\lambda}(v_T - K)^+ \tag{3}$$

But this is the payoff of $1/(1+\lambda)$ call options on the total equity value of the firm per share v. Hence the value of a warrant at current time $t = 0$, $W(S_0)$, can be obtained as:

$$W(S_0) = \exp(-r\tau) \times \left[\int_{-\infty}^{K} 0 f_{RN}(v_T / v_0) + \frac{1}{1+\lambda} \int_{K}^{\infty} (v_T - K) f_{RN}(v_T \setminus v_0) \right] \tag{4}$$

where $f_{RN}(v_T / v_0)$ is the conditional risk-neutral distribution of the value of the firm per share. Writing the above in symbolic notation, suppressing any subscripts, we have

$$W(S) \equiv \frac{1}{1+\lambda} C(v, T, K, \sigma_v, r) \tag{5}$$

where $C(\ )$ denotes a call option (pricing function), T is the time to maturity of the warrant, σ_v and r represent the equity volatility and risk-free rate respectively.[6] This then implies (see also equation (2)) that the following representation for the stock price of a warrant-issuing firm must hold:

$$S = v - \frac{\lambda}{1+\lambda} C(v, T, K, \sigma_v, r) \tag{6}$$

In other words, the stock price of a warrant/executive option issuing firm is given as a linear combination of the equity value process and an option on the equity value process.

Theoretically, as first indicated by Galai and Schneller (1978), warrant valuation can also be performed using the stock price distribution of the warrant-issuing firm. Symbolically this means valuing warrants as:

$$W(S) \equiv D(S, T, K, \sigma_S, r) \tag{7}$$

where
$D(\ )$ represents a call option valuation function.
σ_S denotes the stock return volatility.

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[6] For clarity we note here that when we refer to equity volatility or stock volatility we don't really mean the absolute volatility of the equity or the stock but rather the volatility of the rate of return on the equity or stock respectively.

Valuing warrants using equation (5) or equation (7) will produce identical results. The two equations represent after all equivalent approaches to warrant valuation. Note however that $D(\ )$ is different from $C(\ )$. The two call option pricing functions are different in the sense that $C(\ )$ is derived based on the process followed by the value of the firm, while $D(\ )$ depends on the process of the stock price of the warrant-issuing firm. Darsinos and Satchell (2002) derive the distribution of the stock price of a warrant-issuing firm and show that it is markedly different from the distribution of the stock price of a non-warrant issuing firm (particularly so for high to moderate dilution factors). However they also find that despite of the fact that the risk-neutral distributions of a warrant-issuing firm and a non-warrant-issuing firm are different, valuation by taking expectations of the discounted payoff of the warrants over the two different risk-neutral distributions produces warrant prices very close to each other for a large number of cases even when the log-stock price distribution of the warrant-issuing firm exhibits marked skewness and kurtosis. The significance of this result is that for a large number of cases warrants can be valued using the following "option-like" valuation approximation[7]:

$$W(S) \equiv C(S,T,K,\sigma_S,r) \tag{8}$$

where

$$C(S,T,K,\sigma_S,r) \approx D(S,T,K,\sigma_S,r) = \frac{1}{1+\lambda} C(v,T,K,\sigma_v,r)$$

Exceptions occur for deep-out-of-the-money and close to maturity out-of-the-money warrants in general. In such cases, the "option-like" valuation approximation of equation (8) will overprice warrants, with the pricing error increasing the higher the dilution factor.

In any case, the framework presented thus far applies only for the not so realistic case where the firm under consideration issues only a single warrant. What happens however in the "real" world where corporations issue multiple warrants and ESOs? Do the same formulae apply for firms that issue warrants and ESOs with multiple exercise prices and maturities? We provide the answer to this question in the following section.

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[7] To avoid any misunderstanding, we note here that the nature of the so-called "option-like" valuation approximation is that it takes the distribution of the stock price of a non-warrant-issuing firm as an approximation to the distribution of the stock price of a warrant-issuing firm. Furthermore to make things more specific, the pricing function $C(\ )$ is usually taken to be the Black-Scholes model. However if $C(\ )$ is Black-Scholes, then, in the presence of warrants, $D(\ )$ will not be Black-Scholes. The nature of the approximation is thus that $D(\ )$ is approximated by the Black-Scholes model.

3. EXTENDING THE FRAMEWORK FOR WARRANT PRICING:
FIRMS WITH MULTIPLE WARRANTS

Most, if not all companies, issue more than one type of warrants/executive stock options. By this we mean that companies issue q types ($i = 1,2...,q$) of warrants each with its own exercise price K_i and time to maturity T_i. Moreover for each type of warrants there is a number n_i of outstanding warrants. Under such a specification, the company's current value of the firm per share is given by:

$$v = S + \sum_{i=1}^{q} \lambda_i W_i(S) \tag{9}$$

where
$\lambda_i = n_i / N$ is the dilution factor of each type of warrants.
$W_i(S)$: The price of a type-i warrant on a share at current time $t=0$.

This then implies the following representation for the stock price of the warrant-issuing firm:

$$S = v - \sum_{i=1}^{q} \lambda_i W_i(S) \tag{10}$$

However this time we cannot write the multiple-warrant case as an analogue of the single warrant case. In other words the following relationship does *not* hold:

$$S = v - \sum_{i=1}^{q} \frac{\lambda_i}{1 + \lambda_i} C(v, T_i, K_i, \sigma_V, r_i) \tag{11}$$

That is, we *cannot* price each type of warrants independently of the others as

$$W_i(S) = \frac{1}{1 + \lambda_i} C(v, T_i, K_i, \sigma_V, r_i) \tag{12}$$

for $i = 1, 2, ..., q$. Valuing each type of warrants as a single issuance using equation (12) will result in overpricing. We need a valuation formula that takes into account the interdependence between warrants and the potential dilution effect of each issuance on the value of the other warrants. To simplify exposition we will consider two subcases. In section 3.1 we first consider the case where the firm issues warrants of different maturities and different strike prices while in section 3.2 we consider the case where the corporation issues warrants of common maturity but with multiple exercise prices.

3.1 Firms with warrants of different maturity and different strike price

To start with, assume for simplicity a firm with two warrants of different maturities T_1 and T_2, and different strike prices K_1 and K_2.[8] Without loss of generality take $T_1 < T_2$. Then the current value of the firm per share is given by:

$$v_0 = S_0 + \lambda_1 W_1(S_0) + \lambda_2 W_2(S_0)$$

where $\lambda_1 = n_1/N$ and $\lambda_2 = n_2/N$ are the dilution factors.

- CASE 1: *The first-type warrants are exercised at maturity T_1.*

If the first-type warrants are exercised at time T_1 the company receives a cash inflow of $n_1 K_1$ and the total equity value increases to $V_{T_1} + n_1 K_1$. This value is then distributed among $N + n_1$ shares so that the price per share immediately after exercise becomes

$$S_{T_1} = (V_{T_1} + n_1 K_1)/(N + n_1)$$

Hence, if the first-type warrants are exercised, the payoff to their holders (at maturity) is:

$$W_1(S_{T_1}) = \frac{V_{T_1} + n_1 K_1}{N + n_1} - K_1 = \frac{N}{N + n_1}\left(\frac{V_{T_1}}{N} - K_1\right) = \frac{1}{1 + \lambda_1}\left(\frac{V_{T_1}}{N} - K_1\right) \tag{13}$$
$$= \frac{1}{1 + \lambda_1}(v_{T_1} - K_1)$$

By a similar argument, *provided that the first-type warrants have been exercised* at time T_1, we have that if at time T_2 the second-type warrants are exercised the company will receive another cash inflow of $n_2 K_2$. The price per share immediately after exercise now becomes:

$$S_{T_2} = (V_{T_2} + n_2 K_2)/(N + n_1 + n_2)$$

Hence, if the second-type warrants are exercised, the payoff to the warrant holders is

$$W_2(S_{T_2}) = \frac{V_{T_2} + n_2 K_2}{N + n_1 + n_2} - K_2 = \frac{N}{N + n_1 + n_2}\left(\frac{V_{T_2}}{N} - (K_2 + \frac{n_1}{N}K_2)\right) \tag{14}$$

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[8] Note here that the strike prices can also be the same.

$$= \frac{1}{1 + \lambda_1 + \lambda_2}(v_{T_2} - (K_2 + \lambda_1 K_2))$$

The payoffs in equations (13) and (14) will be valid, if and only if, the first-type warrants have been exercised at time T_1. We express the probability that the above payoffs are valid as $\Pr ob(v_{T_1} > K_1)$.

- CASE 2: *The first-type warrants are not exercised at time T_1.*

Then of course at maturity:

$$W_1(S_{T_1}) = 0 \tag{15}$$

Valuation of the second-type warrants now reduces to valuing a single warrant issuance. We can therefore show that the payoff of the T_2-maturity warrants will be:

$$W_2(S_{T_2}) = \frac{1}{1 + \lambda_2}(v_{T_2} - K_2)^+ \tag{16}$$

These payoff structures will hold if and only if the first-type warrants have not been exercised at time T_1. The probability of this happening is naturally $1 - \Pr ob(v_{T_1} > K_1)$.

We can now derive pricing formulae for the two warrants. For the T_1-maturity warrants this is straightforward. We just have (using equations (13) and (15)) that their current value is:

$$W_1(S_0) = \exp(-rT_1) \times \frac{1}{1 + \lambda_1} \int_{K1}^{\infty}(v_{T_1} - K_1)f_{RN}(v_{T_1} \backslash v_0)dv$$

Or in symbolic notation

$$W_1(S_0) = \frac{1}{1 + \lambda_1}C(v, T_1, K_1, \sigma_v, r) \tag{17}$$

When valuing the T_2-maturity warrants we must take into account the potential dilution effect of the T_1-maturity warrants. We therefore have (using equations (14) and (16)):

$$W_2(S_0) = (1 - \Pr ob(v_{T_1} > K_1)) \times \exp(-rT_2) \times \frac{1}{1 + \lambda_2} \int_{K_2}^{\infty}(v_{T_2} - K_2)f_{RN}(v_{T_2} \backslash v_0)dv$$

$$+ \Pr ob(v_{T_1} > K_1) \times \exp(-rT_2) \times \frac{1}{1 + \lambda_1 + \lambda_2} \int_{K_2 + \lambda_1 K_2}^{\infty}[v_{T_2} - (K_2 + \lambda_1 K_2)]f_{RN}(v_{T_2} \backslash v_0)dv$$

Or in symbolic notation:

$$W_2(S_0) = (1 - \mathrm{Pr}\,ob(v_{T_1} > K_1)) \times \frac{1}{1 + \lambda_2} C(v, T_2, K_2, \sigma_v, r) \tag{18}$$

$$+ \mathrm{Pr}\,ob(v_{T_1} > K_1) \times \frac{1}{1 + \lambda_1 + \lambda_2} C(v, T_2, K_2 + \lambda_1 K_2, \sigma_v, r)$$

We also get the following representation for the stock price of our warrant issuing-firm:

$$S = v - \frac{\lambda_1}{1 + \lambda_1} C(v, K_1, T_1, \sigma_V, r)$$

$$- (1 - \mathrm{Pr}\,ob(v_{T_1} > K_1)) \times \frac{\lambda_2}{1 + \lambda_2} C(v, T_2, K_2, \sigma_V, r) \tag{19}$$

$$- \mathrm{Pr}\,ob(v_{T_1} > K_1) \times \frac{\lambda_2}{1 + \lambda_1 + \lambda_2} C(v_0, T_2, K_2 + \lambda_1 K_2, \sigma_V, r)$$

Expressions for the case of q different warrants can also be derived in a similar manner. In fact the above formulae are valid for a firm with q-types of warrants with maturities $T_1 < T_2 < T_3 < ... < T_n < ... < T_q$. To derive the value of the warrants with maturity T_3 we need to calculate 2^{3-1} probabilities (or more precisely products of probabilities). The valuation formula for the T_3-*maturity* warrants will thus consist of 4 terms and is given by:

$$W_3(S_0) = (1 - \mathrm{Pr}\,ob(v_{T_1} > K_1)) \times (1 - \mathrm{Pr}\,ob(v_{T_2} > K_2) \times \frac{1}{1 + \lambda_3} C(v, T_3, K_3, \sigma_v, r)$$

$$+ (1 - \mathrm{Pr}\,ob(v_{T_1} > K_1)) \times \mathrm{Pr}\,ob(v_{T_2} > K_2) \times \frac{1}{1 + \lambda_2 + \lambda_3} C(v, T_3, K_3 + \lambda_2 K_3, \sigma_v, r)$$

$$+ \mathrm{Pr}\,ob(v_{T_1} > K_1) \times (1 - \mathrm{Pr}\,ob(v_{T_2} > K_2)) \times \frac{1}{1 + \lambda_1 + \lambda_3} C(v, T_3, K_3 + \lambda_1 K_3, \sigma_v, r)$$

$$+ \mathrm{Pr}\,ob(v_{T_1} > K_1) \times \mathrm{Pr}\,ob(v_{T_2} > K_2) \times \frac{1}{1 + \lambda_1 + \lambda_2 + \lambda_3} C(v, T_3, K_3 + \lambda_1 K_3 + \lambda_2 K_3, \sigma_v, r)$$

In general to find the valuation formula for the T_n-*maturity* warrants, for any n, we define

$$\begin{aligned} \delta_i &= 1 \text{ if } v_{T_i} > K_i \\ \delta_i &= 0 \text{ if } v_{T_i} \le K_i \end{aligned} \qquad i = 1, 2, ..., n-1$$

Then there are 2^{n-1} possible values of $(\delta_1,...,\delta_{n-1})$. Let S be the set of all such values. Also let

$$p_i = \Pr ob(v_{T_i} > K_i).$$

Hence the formula will consist of 2^{n-1} terms and is given by:

$$W_n(S_0) = \sum_S \left(\left(\prod_{i=1}^{n-1} p_i^{\delta_i} \times (1-p_i)^{1-\delta_i} \right) \times \frac{C(v,T_n,K_n + \sum_{i=1}^{n-1} \delta_i \lambda_i K_n, \sigma_v, r)}{1 + \sum_{i=1}^{n-1} \delta_i \lambda_i + \lambda_n} \right) \quad (20)$$

This is the valuation formula for a firm with warrants of different maturity (and different or common strike). We make practical use of this formula later, in our empirical application of section 5 where we value the multiple ESOs of Cisco Systems. We now turn to deriving the formula for a firm with warrants of common maturity but different strike prices.

3.2. Firms with warrants of common maturity but multiple exercise prices

Again for ease of exposition consider first a firm with 2 warrants with exercise prices K_1 and K_2, and common time to maturity T. Without loss of generality we assume that $K_1 < K_2$.

- CASE 1: *Both warrants are exercised at maturity T.*

If both warrants are exercised at maturity the company receives a cash inflow of $n_1 K_1 + n_2 K_2$ and the total equity value increases to $V_T + n_1 K_1 + n_2 K_2$. This value is then distributed among $N + n_1 + n_2$ shares so that the price per share immediately after exercise becomes

$$S_T = \frac{V_T + n_1 K_1 + n_2 K_2}{N + n_1 + n_2}$$

Hence if the warrants are exercised the payoff to their holders is:

$$W_1(S_T) = \frac{V_T + n_1 K_1 + n_2 K_2}{N + n_1 + n_2} - K_1 = \frac{V_T + n_2 K_2 - NK_1 - n_2 K_1}{N + n_1 + n_2} \quad (21)$$

$$= \frac{1}{1 + \lambda_1 + \lambda_2}(\frac{V_T}{N} + \lambda_2(K_2 - K_1) - K_1)$$

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$$= \frac{1}{1 + \lambda_1 + \lambda_2}(v_T - (K_1 - \lambda_2(K_2 - K_1)))$$

Similarly

$$W_2(S_T) = \frac{V_T + n_1 K_1 + n_2 K_2}{N + n_1 + n_2} - K_2 = \frac{1}{1 + \lambda_1 + \lambda_2}(\frac{V_T}{N} + \lambda_1(K_1 - K_2) - K_2) \qquad (22)$$

$$= \frac{1}{1 + \lambda_1 + \lambda_2}(v_T - (K_2 - \lambda_1(K_1 - K_2)))$$

Of course both warrants will be exercised if and only if $v_T > K_1 - \lambda_2(K_2 - K_1)$ and $v_T > K_2 - \lambda_1(K_1 - K_2)$. Since we have $K_1 < K_2$, this then implies that both warrants will be exercised if and only if

$$v_T > K_2 - \lambda_1(K_1 - K_2)$$

- ▪ CASE 2: *Only one warrant is exercised at maturity*

If only the warrant with the lowest strike price is exercised then using a similar procedure as above, we can show that the payoff to the warrant holders is:

$$W_1(S_T) = \frac{V_T + n_1 K_1}{N + n_1} - K_1 = \frac{1}{1 + \lambda_1}(\frac{V_T}{N} - K_1) = \frac{1}{1 + \lambda_1}(v_T - K_1) \qquad (23)$$

and

$$W_2(S_T) = 0 \qquad (24)$$

The warrant with the lowest exercise price will be exercised if and only if $v_T > K_1$.

- ▪ CASE 3 *(Trivial): None of the warrants are exercised*

The final case (which is redundant) is when none of the warrants are exercised. Then the payoff to their holders is zero. This will be the case if $v_T \leq K_1$.

We can now derive pricing formulae for the two warrants. Let us first obtain a valuation formula for $W_1(S_0)$. The first warrant has two different nonzero payoffs depending on whether one or both warrants are exercised:

$$W_1(S_0) = \exp(-rT) \times [\frac{1}{1 + \lambda_1} \int_{K_1}^{K_2 - \lambda_1(K_1 - K_2)} (v_T - K_1) f_{RN}(v_T / v_0) dv \qquad (25)$$

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$$+ \frac{1}{1 + \lambda_1 + \lambda_2} \int_{K_2 - \lambda_1(K_1 - K_2)}^{\infty} (v_T - (K_1 - \lambda_2(K_2 - K_1))) f_{RN}(v_T \setminus v_0) dv]$$

This then implies that

(26)

$$W_1(S_0) = \frac{1}{1 + \lambda_1} C(v, T, K_1, \sigma_V, r) - \frac{\lambda_2}{(1 + \lambda_1 + \lambda_2)(1 + \lambda_1)} C(v, T, K_2 - \lambda_1(K_1 - K_2), \sigma_V, r)$$

The calculations required to arrive to equation (26) (from equation (25)) are exhibited in the Appendix. To obtain a valuation formula for $W_2(S_0)$, we note that the second-type warrants have a non-negative payoff only when both warrants are exercised. Hence it follows immediately that:

$$W_2(S_0) = \exp(-rT) \times \frac{1}{1 + \lambda_1 + \lambda_2} \int_{K_2 - \lambda_1(K_1 - K_2)}^{\infty} [v_T - (K_2 - \lambda_1(K_1 - K_2)] f_{RN}(v_T \setminus v_0) dv$$

or in symbolic notation

$$W_2(S_0) = \frac{1}{1 + \lambda_1 + \lambda_2} C(v, T, K_2 - \lambda_1(K_1 - K_2), \sigma_v, r) \qquad (27)$$

It is now straightforward to try and generalize the formulae. For example, for a firm with 3 warrants with exercise prices $K_1 < K_2 < K_3$ and common time to maturity T, using the same methodology as above, we get:

$$W_1(S_0) = \frac{1}{1 + \lambda_1} C(v, T, K_1, \sigma_V, r) \qquad (28)$$

$$- \frac{\lambda_2}{(1 + \lambda_1 + \lambda_2)(1 + \lambda_1)} C(v, T, K_2 - \lambda_1(K_1 - K_2), \sigma_V, r)$$

$$- \frac{\lambda_3}{(1 + \lambda_1 + \lambda_2 + \lambda_3)(1 + \lambda_1 + \lambda_2)} C(v, T, K_3 - \lambda_1(K_1 - K_3) - \lambda_2(K_2 - K_3), \sigma_V, r)$$

$$W_2(S_0) = \frac{1}{1 + \lambda_1 + \lambda_2} C(v, T, K_2 - \lambda_1(K_1 - K_2), \sigma_V, r) \qquad (29)$$

$$- \frac{\lambda_3}{(1 + \lambda_1 + \lambda_2 + \lambda_3)(1 + \lambda_1 + \lambda_2)} C(v, T, K_3 - \lambda_1(K_1 - K_3) - \lambda_2(K_2 - K_3), \sigma_V, r)$$

$$W_3(S_0) = \frac{1}{1 + \lambda_1 + \lambda_2 + \lambda_3} C(v, T, K_3 - \lambda_1(K_1 - K_3) - \lambda_2(K_2 - K_3), \sigma_V, r) \qquad (30)$$

Generally for q-types of warrants with common maturity T and multiple strike prices $K_1 < K_2 < K_3 < ... < K_n < ... < K_q$ we have that for any n:

$$W_n(S_0) = \frac{1}{1 + \sum_{i=1}^{n} \lambda_i} C(v, T, K_n - \sum_{i=1}^{n} \lambda_i (K_i - K_n), \sigma_v, r) \tag{31}$$

$$- \sum_{i=n+1}^{q} [\frac{\lambda_i}{(1 + \sum_{j=1}^{i} \lambda_j)(1 + \sum_{j=1}^{i-1} \lambda_j)} C(v, T, K_i - \sum_{j=1}^{i} \lambda_j (K_j - K_i), \sigma_v, r)]$$

where we use the convention that $\sum_{i=n+1}^{q} (\ ) = 0$ when $n+1 > q$.

4. SPECIFYING A PROCESS FOR THE STATE VARIABLE

So far our analysis has been distribution free and the results presented above are valid for a range of process assumptions for the value of the firm. However to make the formulae operational we must now specify the law that governs the evolution of v. In other words, we must assume a process for the state variable v, so that the call option pricing function $C(\ )$ obtains a specific functional form and can therefore be evaluated. The literature on warrant pricing has mainly concentrated on three processes for the value of the firm. Apart from Black and Scholes's Geometric Brownian Motion which assumes constant volatility, Cox and Ross's (1976) Constant Elasticity of Variance (CEV) model (see Noreen and Wolfson (1981), Lauterbach and Schultz (1990), Schulz and Trautmann (1989)) and Merton's (1976) Jump Diffusion model (see Kremer and Roenfeldt (1993)) have also been used to model the value process. The results presented above will be valid under any specification. Of course the functional form of $C(\ )$ will depend on the particular process chosen. It is not the aim of this paper to go into a detailed review of the literature on the most appropriate value process for warrant pricing. An exhaustive study that reviews the empirical research under alternative stochastic processes has already been conducted by Veld (1994). Here it suffices to quote one of the conclusions of Veld: "There is no conclusive evidence to replace (dividend corrected) models in which a constant volatility is assumed (i.e. Black-Scholes like models) by more complicated models such as the Jump Diffusion or the CEV model." For operational convenience and since it is the absolute standard in the industry we therefore assume hereafter that the value process is a lognormal diffusion:

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ASSUMPTION 2: *The total equity value of the firm per share v is governed by Geometric Brownian motion:*

$$dv = \mu_v v dt + \sigma_v v dB$$

where

μ_v : The expected rate of return on the value of the firm's assets

σ_v : The standard deviation of the rate on the value of the firm's assets.

B : A standard Brownian motion.

Assumption 2 implies that in a risk-neutral world:

$$v_T = v_0 \exp\left((r - \frac{\sigma_v^2}{2})T + \sigma_v \sqrt{T} Z \right) \tag{32}$$

where

$\sqrt{T} Z = B_T - B_0$; i.e. Z is a standardized normal variable.

As a byproduct of assumption 2, we can now calculate explicitly, using equation (32), the probability that $\Pr ob(v_{T_i} > K_i)$. (Remember that this probability appears in the pricing equation for multiple warrants with different maturities; i.e. equation (20)). We therefore have that

$$p_i = \Pr ob(v_{T_i} > K_i) = \Pr ob(v_0 \exp\left((r - \frac{\sigma_v^2}{2})T_i + \sigma_v \sqrt{T_i} Z \right) > K_i)$$

$$= \Pr ob(Z > \frac{\ln(\frac{K_i}{v_0}) - (r - \frac{\sigma_v^2}{2})T_i}{\sigma_v \sqrt{T_i}})$$

$$= \Pr ob(Z > -\frac{\ln(\frac{v_0}{K_i}) + (r - \frac{\sigma_v^2}{2})T_i}{\sigma_v \sqrt{T_i}})$$

$$= \Phi(\frac{\ln(\frac{v_0}{K_i}) + (r - \frac{\sigma_v^2}{2})T_i}{\sigma_v \sqrt{T_i}})$$

where $\Phi(\ )$ denotes the standard normal cumulative distribution function (remember that Z is a standard normal random variable). Furthermore the pricing function

$$C(v, T_n, K_n + \sum_{i=1}^{n-1} \delta_i \lambda_i K_n, \sigma_v, r)$$

that also appears in equation (20) takes the familiar form

$$C^{BS}(v, T_n, K_n + \sum_{i=1}^{n-1} \delta_i \lambda_i K_n, \sigma_v, r)$$

where $C^{BS}(\ )$ denotes the Black-Scholes option price for a European call.

We have now a fully operational warrant pricing formula and apart from v and σ_v the remaining arguments of the formula can be observed in the market or extracted from the annual accounts of each firm. However it is also possible to estimate v and σ_v using observed stock market data for S and σ_S. To do that we just need to solve a system of two nonlinear simultaneous equations. Remember that from equation (10) we have the following representation for the stock price of a warrant/ESO issuing firm:

$$S = v - \sum_{i=1}^{q} \lambda_i W_i(S) \tag{33}$$

where we now know that each $W_i(S)$ ($i = 1, 2, ... n, ..., q$) is evaluated using the formula of equation (20). It is also well known (see for example Schulz and Trautmann (1994), Darsinos and Satchell (2002)) that

$$\sigma_S = \sigma_v \times (\frac{\partial S}{\partial v} \frac{v}{S})$$

which in our case becomes

$$\sigma_S = (1 - \sum_{i=1}^{q} \lambda_i \frac{\partial W_i(S)}{\partial v}) \sigma_v \frac{v}{S} \tag{34}$$

Equations (33) and (34) can be solved simultaneously for the two unknown arguments v and σ_v.

Turning now to the formula for q-types of warrants with common maturity T and multiple strike prices $K_1 < K_2 < K_3 < ... < K_n < ... < K_q$ we have that under Black-Scholes assumptions for any n:

$$W_n(S_0) = \frac{1}{1 + \sum_{i=1}^{n} \lambda_i} C^{BS}(v, T, K_n - \sum_{i=1}^{n} \lambda_i (K_i - K_n), \sigma_v, r) \tag{35}$$

$$- \sum_{i=n+1}^{q} [\frac{\lambda_i}{(1 + \sum_{j=1}^{i} \lambda_j)(1 + \sum_{j=1}^{i-1} \lambda_j)} C^{BS}(v, T, K_i - \sum_{j=1}^{i} \lambda_j (K_j - K_i), \sigma_v, r)]$$

This follows straightforwardly from equation (31). In this case as well the latent parameters v and σ_v can be estimated using the numerical procedure outlined above where of course this time we substitute equation (35) for the value of each $W_i(S)$ $(i = 1,2,...n,...,q)$

5. AN EMPIRICAL APPLICATION

As an empirical application of the theoretical framework presented thus far we now turn to valuing the Executive Stock Options of Cisco Systems. Cisco Systems is one of the leading technology companies and has stock options issued to about half its employees (announced January 2002)[9]. At July 28, 2001 the company's ESOs were classified into 5 major categories according to strike and maturity, and their aggregate dilution potential was approximately 15%. Table 1 exhibits all the relevant information as extracted from the Company's 2001 annual report.

TABLE 1
Cisco Systems: Information on Outstanding Executive Stock Options at July 28, 2001

Number of ESO Issuances: $n =$	Maturity (Years)	Exercise Price per Share ($)	Number Outstanding (millions)
1	$T_1 = 4.45$	$K_1 = 4.97$	226 ($\lambda_1 = 3.1\%$)
2	$T_2 = 6.64$	$K_2 = 13.98$	255 ($\lambda_2 = 3.5\%$)
3	$T_3 = 7.52$	$K_3 = 37.45$	338 ($\lambda_3 = 4.6\%$)
4	$T_4 = 7.54$	$K_4 = 55.85$	214 ($\lambda_4 = 2.9\%$)
5	$T_5 = 7.77$	$K_5 = 69.35$	27 ($\lambda_5 = 0.4\%$)
Total	$T = 6.66$	$K = 29.41$	1,060 ($\lambda = 14.5\%$)

(a) The last row in the table as represented by "Total" is an attempt to treat the multiple ESO issuances as a single "representative" issuance with maturity and exercise price given by a weighted average of the individual strikes and maturities respectively. Specifically the single "representative" maturity is given by
$T = (226/1060)*4.45+(255/1060)*6.64+(338/1060)*7.52+(214/1060)*7.54+(27/1060)*7.77=6.66$
and the single "representative" strike by
$K = (226/1060)*4.97+(255/1060)*13.98+(338/1060)*37.45+(214/1060)*55.85+(27/1060)*69.35=29.41$.
(b) At July 28, 2001, there were 7,324 million ordinary shares in issue (required to calculate the dilution factors).

Additional information extracted from stock market data and the firm's accounts include that, at July 28, 2001 the share price of Cisco Systems was $S = \$19.06$, the expected future dividend was $d = 0\%$, the risk-free interest rate was $r = 5.4\%$ and the expected future stock return volatility was $\sigma_S = 34.8\%$. Cisco uses projected volatility rates, which are based upon historical volatility rates trended into future years. We wish to value the company's

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[9] See *Business 2.0*, The Hidden Cost of Stock Options, David Futrelle, February 01, 2002.

ESOs using: (i) the Multiple Warrant Valuation Framework (Multiple-WVF) introduced in this paper, (ii) the standard Single Warrant Valuation Framework (Single-WVF), and (iii) the Black-Scholes Warrant Valuation Framework (BS-WVF).

(i) The Multiple-WVF: Reproducing a programmable analogue of equation (20), (combined with assumption 2 of section 4) we have that the valuation formula for a firm with q Executive Stock Options of maturities $T_1 < ... < T_q$ and of strikes $K_1,...,K_q$ is given, for any $n = 1,2,...,q$, by

$$W_n(S_0) = \sum_{j=1}^{2^{n-1}} \left[\left(\prod_{i=1}^{n-1} p_i^{\delta_{i,j}} \times (1-p_i)^{1-\delta_{i,j}} \right) \times \frac{C^{BS}(v,T_n,K_n + \sum_{i=1}^{n-1} \delta_{i,j}\lambda_i K_n, \sigma_v, r)}{1 + \sum_{i=1}^{n-1} \delta_{i,j}\lambda_i + \lambda_n} \right]$$

where $p_i = \Phi(\dfrac{\ln(\dfrac{v}{K_i}) + (r - \dfrac{\sigma_v^2}{2})T_i}{\sigma_v \sqrt{T_i}})$ and $\delta_{i,j} = 1 - Rounddownward(\mathrm{mod}[\dfrac{j-1}{2^{(n-1)-i}},2])$ [10].

To make the formula operational we need to obtain estimates for the value of the firm v and volatility σ_v. To do that we follow the procedure outlined above in section 4, utilizing the fact that we have information on S and σ_S (i.e. we solve numerically the simultaneous equations (33) and (34) using a Newton-Raphson iterative procedure). We obtain the following values: $v = \$20.32$ and $\sigma_v = 36.2\%$. It's worth emphasizing that to obtain these estimates we substituted the Multiple-WVF formula for each $W_i(S)$ ($i = 1,2,...n,...,q$) in equations (33) and (34).

(ii) The Single-WVF: The Single-WVF constitutes the standard textbook treatment for the valuation of warrants. In a multiple warrant setting, it takes the following form: for a firm with q Executive Stock Options of maturities $T_1 < ... < T_q$ and of strikes $K_1,...,K_q$, we have that for any $n = 1,2,...,q$

$$W_n(S) = \frac{1}{1 + \lambda_n} C^{BS}(v,T_n,K_n,\sigma_v,r)$$

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[10] mod[x, z] represents modulo division. This returns the value that is the remainder of the *integer* division of x by z.

where the latent parameters v and σ_v are inferred once again by solving simultaneously equations (33) and (34). This time however we substitute the Single-WVF formula for each $W_i(S)$ $(i=1,2,...n,...,q)$. In other words we solve the following system of equations

$$S = v - \sum_{i=1}^{q} \frac{\lambda_i}{1+\lambda_i} C^{BS}(v,T_i,K_i,\sigma_V,r_i)$$

$$\sigma_S = (1 - \sum_{i=1}^{q} \frac{\lambda_i}{1+\lambda_i} \frac{\partial C^{BS}(v,T_i,K_i,\sigma_v,r_i)}{\partial v})\sigma_v \frac{v}{S}$$

We thus obtain: $v = \$20.39$ and $\sigma_v = 36.2\%$.

(iii) The BS-WVF: The Black-Scholes Warrant Valuation Framework (or "option-like" warrant valuation framework) is popular amongst corporations since it is consistent with the FASB mandates for accounting for the value/cost of stock-based compensation. It involves valuing ESOs and warrants as standard call options. Hence the formula for a firm with q Executive Stock Options of maturities $T_1 < ... < T_q$ and of strikes $K_1,...,K_q$ is simply given, for any $n=1,2,...,q$, by

$$W_n(S) = C^{BS}(S,T_n,K_n,\sigma_S,r)$$

In this case all the parameters involved in the valuation are extractable from the annual accounts and the financial markets. Cisco states in its annual report that the fair value of each option grant is estimated on the date of grant using the Black-Scholes option pricing model. However it acknowledges that because employee stock options have characteristics significantly different from those of traded options, and because changes in the subjective input assumptions can materially affect the fair value estimate, the existing models do not necessarily provide a reliable single measure of the fair value of the Company's options.

The Multiple-WVF provides the benchmark against which to check the accuracy of the Single-WVF and the BS-WVF. The latter two, are of course approximations to the "true" value/cost of ESOs in the presence of multiple-issuances. In Table 2, we report the values of the company's ESOs using all three valuation frameworks. We then evaluate the quality of the Single-WVF and the BS-WVF approximations by reporting the percentage mispricing error arising from their application. This is calculated as [(Single-WVF – Multiple-WVF) / Multiple-WVF] and [(BS-WVF - Multiple-WVF) / Multiple-WVF] respectively.

TABLE 2

(i) Valuation of Cisco's ESOs under the Multiple-, Single-, and BS- Warrant Valuation Frameworks and
(ii)Percentage Mispricing Error arising from the use of the Single-WVF and the BS-WVF approximations.

	(i) $ Value / Cost			(ii) % Mispricing Error	
$n =$	Multiple-WVF	Single-WVF	BS-WVF	of Single-WVF	of BS-WVF
1	$W_1(S) = 15.95$	$W_1(S) = 16.02$	$W_1(S) = 15.18$	0.4%	-4.8%
2	$W_2(S) = 11.17$	$W_2(S) = 11.72$	$W_2(S) = 10.82$	4.9%	-3.2%
3	$W_3(S) = 5.62$	$W_3(S) = 6.21$	$W_3(S) = 5.42$	10.6%	-3.6%
4	$W_4(S) = 3.64$	$W_4(S) = 4.17$	$W_4(S) = 3.41$	14.6%	-6.1%
5	$W_5(S) = 2.95$	$W_5(S) = 3.43$	$W_5(S) = 2.68$	16.5%	-9.1%
Total	$W(S) = 6.12$	$W(S) = 6.12$	$W(S) = 6.15$	N/A	0.5%

(a) The last row in the table as represented by "Total" gives the $ Value/Cost per ESO when one attempts to treat the multiple ESOs as a single issuance. It corresponds to the parameter values given in the last row of table 1.

It is clear from the table that use of the textbook, Single-Warrant Valuation Framework will result in significant overpricing with the percentage mispricing error becoming progressively higher as the number of intervening ESO issuances increases. Indeed the more ESO issuances there are before each particular issuance can be exercised, the larger the failure to account for the likelihood of dilution of the firm's equity prior to exercise of the issuance under consideration. This unavoidably leads to overpricing. In aggregate now, if the firm erroneously used the Single-WVF (instead of the Multiple-WVF) to account for the total cost/value of its ESOs it would overestimate their "true" cost/value by approximately $483.6 million.[11] Turning now to the Black-Scholes Warrant Valuation Framework we observe that its application will also misprice the firm's ESOs, but this time in the opposite direction (i.e. this time we observe underpricing of Cisco's ESOs). Despite of the fact that the BS-WVF has proved to be an accurate approximation in single-warrant settings, its performance in a multiple setting is, unfortunately, not nearly as good (in the case of Cisco, pricing errors ranged from 3.2% to 9.1%). If it were to be used, the BS-WVF approximation would underestimate the total cost/value of the firm's options by approximately $385.5 million.

In the last rows of tables 1 and 2 we have attempted to treat the multiple ESO issuances of Cisco Systems as a single "representative" ESO issuance (with strike and maturity given by a weighted average of the individual strikes and maturities respectively) and value it accordingly. In this case the Multiple-WVF and the Single-WVF coincide and the values obtained for v and σ_v are $19.94 and 36.3% respectively. The theoretical value

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[11] To arrive at this result simply just subtract the total cost of Cisco's options arising from the use of the Multiple-WVF (i.e. (15.95*226 + 11.17*255 + 5.62*338 + 3.64*214 + 2.95*27) = $9,210.5 million) from the total cost arising from the use of the Single-WVF (i.e. (16.02*226 + 11.72*255 + 6.21*338 + 4.17*214 + 3.43*27) = $9,694.1 million).

of this "representative" ESO is then found to be $6.12. Consistent with what has been said so far, the BS-WVF approximation in this single-warrant setting performs very well, giving a value of $6.15 per outstanding ESO, a mispricing error of only 0.5%. Attempting however to assess the total cost/value of the firm's ESO using this single "representative" ESO setting is definitely inappropriate. The total cost/value of the company's ESOs under this approach amounts to approximately $6.487 billion, an understatement of a staggering $2.723 billion when compared to their theoretical cost of $9.21 billion arising from the use of the Multiple Warrant Valuation Framework. (see also footnote 11). Indeed the message going through from this last exercise is that it is not generally a good idea to approximate multiple ESO issuances by a representative single issuance, value it accordingly, and subsequently assess the total cost/value of the company's options under such a specification.

6. CONCLUSION

Over the years, interest and more importantly issuance of executive stock options as a form of stock based compensation has surged. As a result it is now a common phenomenon for corporations to issue a large number of Executive Stock Options schemes, each with its own exercise price and maturity. When it comes to valuing corporate instruments such as executive options and warrants one of the difficulties that immediately arises is their capital structure effect on the issuing firm. In single-issuance settings this has never really been a problem since not only there is a well developed theory to account for the so-called dilution effect but also it has been demonstrated by various studies that "option-like" valuation results in nearly identical prices for these instruments as the textbook, value of the firm, approach. However, an investigation on the applicability and suitability of the current valuation methods in multiple-issuances settings, despite their increasing popularity, has not so far been performed. In this paper we have extended the theoretical framework of Galai and Schneller (1978) to account for multiple warrant and ESO issuances and derived distribution free formulae for firms with warrants and ESOs of several maturities and strikes. We have illustrated that, when applied in multiple settings, the textbook, single issuance approach can result in significant overpricing. Likewise we have also shown that "option-like" valuation is not that good an approximation in multiple settings and will typically result in underpricing. This re-enforces existing concerns on the optimality of using standard, unmodified option pricing models to account for the cost of stock-based compensation. Finally, for the purposes of assessing the total cost of a firm's options, the practice of treating multiple issuances as a single representative issuance (with strike and maturity given by a weighted average of the individual strikes and maturities respectively) is potentially very dangerous and will produce highly misleading values. In the example considered in the text, this leads to an "underpricing" of approximately 30%;

thus underestimating the impact of ESOs on the firm's value. Likewise, with average percentage gains or losses of the order of 10% of capital, application of the single warrant valuation framework or of the Black-Scholes option-like framework can have significant effects on different groups of shareholders, be they employees or shareholders. The multiple warrant valuation framework introduced in this paper should be used instead.

7. APPENDIX

Starting from equation (25) we need to arrive to equation (26). We first reproduce equation (25):

$$W_1(S_0) = \exp(-rT) \times [\frac{1}{1+\lambda_1} \int_{K_1}^{K_2 - \lambda_1(K_1 - K_2)} (v_T - K_1) f_{RN}(v_T / v_0) dv \qquad (25)$$

$$+ \frac{1}{1+\lambda_1+\lambda_2} \int_{K_2-\lambda_1(K_1-K_2)}^{\infty} (v_T - (K_1 - \lambda_2(K_2 - K_1))) f_{RN}(v_T \setminus v_0) dv]$$

$$= \exp(-rT) \times [\frac{1}{1+\lambda_1}\left(\int_0^{K_2-\lambda_1(K_1-K_2)} (v_T - K_1) f_{RN}(v_T / v_0) dv - \int_0^{K_1} (v_T - K_1) f_{RN}(v_T / v_0) dv \right)$$

$$+ \frac{1}{1+\lambda_1+\lambda_2}(\int_{K_2-\lambda_1(K_1-K_2)}^{\infty} (v_T - (K_2 - \lambda_1(K_1 - K_2))) f_{RN}(v_T \setminus v_0) dv$$

$$+ \int_{K_2-\lambda_1(K_1-K_2)}^{\infty} ((K_2 - \lambda_1(K_1 - K_2)) - (K_1 - \lambda_2(K_2 - K_1))) f_{RN}(v_T \setminus v_0) dv)]$$

$$= \exp(-rT) \times [\frac{1}{1+\lambda_1}(- \int_0^{K_2-\lambda_1(K_1-K_2)} ((K_2 - \lambda_1(K_1 - K_2)) - v_T + K_1 - (K_2 - \lambda_1(K_1 - K_2))) f_{RN}(v_T / v_0) dv$$

$$+ \int_0^{K_1} (K_1 - v_T) f_{RN}(v_T / v_0) dv)$$

$$+ \frac{1}{1+\lambda_1+\lambda_2}(\int_{K_2-\lambda_1(K_1-K_2)}^{\infty} (v_T - (K_2 - \lambda_1(K_1 - K_2))) f_{RN}(v_T \setminus v_0) dv$$

$$+ \int_{K_2-\lambda_1(K_1-K_2)}^{\infty} ((K_2 - \lambda_1(K_1 - K_2)) - (K_1 - \lambda_2(K_2 - K_1))) f_{RN}(v_T \setminus v_0) dv)]$$

$$= -\frac{1}{1+\lambda_1} P(v, T, K_2 - \lambda_1(K_1 - K_2), \sigma_v, r) + \exp(-rT)(K_2 - K_1) \Pr ob[v_T < K_2 - \lambda_1(K_1 - K_2)]$$

$$+\frac{1}{1+\lambda_1}P(v,T,K_1,\sigma_v,r)$$

$$+\frac{1}{1+\lambda_1+\lambda_2}C(v,T,K_2-\lambda_1(K_1-K_2),\sigma_v,r)$$

$$+\exp(-rT)(K_2-K_1)\Pr ob[v_T>K_2-\lambda_1(K_1-K_2)]$$

where $C(\ )$ and $P(\ )$ denote call and put option (pricing functions) respectively. Hence we have:

$$W_1(S_0)=\exp(-rT)(K_2-K_1)+\frac{1}{1+\lambda_1+\lambda_2}C(v,T,K_2-\lambda_1(K_1-K_2),\sigma_v,r)$$

$$+\frac{1}{1+\lambda_1}P(v,T,K_1,\sigma_v,r)$$

$$-\frac{1}{1+\lambda_1}P(v,T,K_2-\lambda_1(K_1-K_2),\sigma_v,r)$$

Now using Put-Call parity the above expression simplifies to:

$$W_1(S_0)=\exp(-rT)(K_2-K_1)+\frac{1}{1+\lambda_1+\lambda_2}C(v,T,K_2-\lambda_1(K_1-K_2),\sigma_v,r)$$

$$+\frac{1}{1+\lambda_1}(C(v,T,K_1,\sigma_v,r)+K_1\exp(-rT)-v)$$

$$-\frac{1}{1+\lambda_1}(C(v,T,K_2-\lambda_1(K_1-K_2),\sigma_v,r)+(K_2-\lambda_1(K_1-K_2))\exp(-rT)-v)$$

Some straightforward algebra and we get equation (26):

$$W_1(S_0)=\frac{1}{1+\lambda_1}C(v,T,K_1,\sigma_V,r)-\frac{\lambda_2}{(1+\lambda_1+\lambda_2)(1+\lambda_1)}C(v,T,K_2-\lambda_1(K_1-K_2),\sigma_V,r)$$

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