Asymptotically cylindrical Calabi–Yau and special Lagrangian geometry

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This dissertation is submitted on 13 April 2017 for the degree of Doctor of Philosophy
Abstract

We study asymptotically cylindrical Calabi–Yau manifolds and their asymptotically cylindrical special Lagrangian submanifolds. As a prototype problem, we also consider an extension of Hodge theory to general asymptotically cylindrical manifolds.

For our study of asymptotically cylindrical Calabi–Yau manifolds, we restrict to complex dimension three. We regard a Calabi–Yau structure as a pair of closed forms $(\Omega, \omega)$; the assumption that the structure is asymptotically cylindrical gives an asymptotic condition on $(\Omega, \omega)$. Regarding the Riemannian products of Calabi-Yau threefolds with $S^1$ as $G_2$ manifolds, we show that the asymptotically cylindrical deformations of a Calabi–Yau structure (with possibly varying asymptotic limit) are unobstructed. Locally, the spaces of deformations are given by appropriate spaces of harmonic forms. We then show that we can glue asymptotically cylindrical Calabi–Yau manifolds, and that if we do so the “gluing map” of moduli spaces is essentially a local diffeomorphism. In particular, it is an open mapping.

In the case of asymptotically cylindrical special Lagrangian submanifolds, we no longer explicitly restrict to dimension three; we assume only that we have a gluing theorem for Calabi–Yau manifolds of the kind obtained in dimension three. McLean and others have constructed deformation spaces of special Lagrangian submanifolds; we show that gluing of asymptotically cylindrical special Lagrangian submanifolds is again unobstructed. As in the Calabi–Yau case, we can define a “gluing map” and this map is a local diffeomorphism of moduli spaces.

In both cases, the local diffeomorphism property gives a “local Mayer–Vietoris principle” for deformations. In the special Lagrangian case, the linearisation of the “ungluing” map so defined is just the map of harmonic forms induced by Hodge theory from the natural map of de Rham cohomology; in the Calabi–Yau case it is only slightly more involved.
Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.

It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.

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April 2017
Acknowledgements

First of all, I am deeply indebted to my supervisor, Alexei Kovalev, without whose patient advice and guidance this thesis would not have been written. Equally, I am grateful for the corrections and suggestions for improvement provided by Mihalis Dafermos and Dominic Joyce; these have noticeably improved the final version. I would particularly like to thank Professor Joyce for his comments on Theorem 3.5.14.

I am also grateful to the Engineering and Physical Sciences Research Council, both for my own financial support, and for the block grant for the Cambridge Centre for Analysis, which was a very enjoyable doctoral training experience.

Finally, I must thank all those, both individuals and institutions, who have contributed to keeping me close to the rails and work on this thesis proceeding in a forward direction at positive rate. Any attempt to enumerate the individuals would necessarily be incomplete and probably suffer from a bias towards the nearer past, so I shall not try: most of them know who they are. On the institutional level, I would particularly like to thank King’s College for eight and a half years of study in a spectacular environment, and the 28th Cambridge Scout Group for only slightly less time demonstrating what can be done if you forget that it’s impossible.
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Chapter 1

Introduction

Calabi–Yau manifolds, that is manifolds with Riemannian holonomy contained in $SU(n)$, or equivalently Ricci-flat Kähler manifolds with a nonvanishing holomorphic volume form, are so called after Yau’s 1978 proof [76] of the Calabi conjecture constructed large numbers of compact examples. They form one of the best-understood kinds of manifold with special holonomy, and also form an important class of examples of Ricci-flat manifolds. Furthermore, they are conjectured to be useful in physics: in certain forms of supersymmetric string theory, spacetime is conjectured to take the form $M \times K$ where $K$ is compact, Ricci-flat and Kähler. See [10].

In 1982, Harvey and Lawson introduced [31] the idea of a special Lagrangian submanifold of a Calabi–Yau manifold as an example of their notion of a calibrated submanifold, which extended the complex submanifolds of a Kähler manifold. As calibrated submanifolds, special Lagrangians are minimal submanifolds; in fact, they are volume-minimising in their homology class.

On physical grounds, it is conjectured that there is a duality between Calabi–Yau threefolds known as “mirror symmetry” (see for example the review of Gross [29]). One of the formulations of this given in [29] is that there should be an isomorphism between the special Lagrangian submanifolds of a Calabi–Yau manifold equipped with a flat $U(1)$ bundle and the complex submanifolds of its mirror pair equipped with a holomorphic line bundle: this gives another reason to study special Lagrangian submanifolds. Specifically, a conjecture of Strominger, Yau and Zaslow [72] says that a Calabi–Yau manifold and its mirror pair both admit special Lagrangian torus fibrations and the generic tori are dual (in the sense that one is an appropriately generalised first cohomology group of the other).

We study the interaction of Calabi–Yau manifolds and special Lagrangian submanifolds with the notion of an (exponentially) asymptotically cylindrical manifold found in for instance Lockhart [55]. We study their deformations, the gluings of matching manifolds and submanifolds, and the resulting gluing map between deformation spaces.

It turns out that the deformations of Calabi–Yau structures and special Lagrangian submanifolds are locally parametrised by the kernel of the Laplace–Beltrami operator. Hence, our
study of asymptotically cylindrical Calabi–Yau structures and special Lagrangian submanifolds requires as a preliminary a corresponding study of the Laplace–Beltrami operator on asymptotically cylindrical manifolds.

In each of these three cases, our objective is a theorem on the gluing map between deformation spaces. To establish such a theorem, we of course need to construct both a gluing map and deformation spaces. Gluing constructions have a distinguished history. The idea is that to construct the solution to a differential equation on a generalised connected sum $M^T$ of $M_1$ and $M_2$, we first find solutions on $M_1$ and $M_2$ with appropriate boundary conditions, then cut these off to find an approximate solution on $M^T$. Then, by applying perturbative analysis, we can argue that there is a solution close to this approximate solution.

An important early example was the construction of self-dual conformal structures on four-manifolds, initiated by Floer [24]; another example is given by Kovalev–Singer [53], who used this approach to find anti-self-dual metrics on 4-dimensional manifolds. In turn, such constructions lead to questions about which deformations of the glued structure arise from compatible deformations of the structures being glued and which, if any, arise from choices in the gluing (for instance, in the cutoffs and connected sums).

For instance, after showing that anti-self-dual connections on four-manifolds could be glued, Donaldson and Kronheimer showed [20, Theorem 7.2.63] showed that every deformation of an anti-self-dual connection on a four-manifold obtained by gluing can be obtained by either deformation of the connections being glued, variation of the length of the neck in gluing, or a change in the identification of the pieces, in an essentially unique way.

In some cases, particularly for the Laplace–Beltrami operator and the deformation spaces of special Lagrangians, this work has already been done (or is trivial), and we merely review it. For instance, since the Laplace–Beltrami operator is linear, the “deformations of a harmonic form” are isomorphic to the whole space of harmonic forms. In the compact case, these are then given by the classical Hodge Decomposition Theorem, which states that on a compact oriented Riemannian manifold without boundary, the space of harmonic forms is finite-dimensional and that any $k$-form $\alpha$ can be decomposed orthogonally as $h + \Delta \beta$, where $\Delta$ is the Laplace–Beltrami operator and $h$ and $\beta$ are $k$-forms with $\Delta h = 0$. It then follows that there is an isomorphism between harmonic forms and de Rham cohomology, enabling us to frame topological questions in analytic terms and vice versa. In particular, after extending this to asymptotically cylindrical manifolds, we may then study the gluing construction using topology, in which the questions essentially reduce to the Mayer–Vietoris sequence. Melrose [61] has identified harmonic representations for the cohomology of a $b$-manifold, which is close in spirit to our asymptotically cylindrical manifolds. A closer discussion of these questions on asymptotically cylindrical manifolds, which actually proves slightly more than we do, has been given by Nordström [64].

For the deformations of Calabi–Yau manifolds, there have been a number of previous works in the asymptotically cylindrical case. Most of these studies have used the notion of a logarith-
mic deformation of a complex structure. By a result of Haskins–Hein–Nordström [32, Theorem C], any asymptotically cylindrical Calabi–Yau manifold of dimension greater than 2 is given by removing a divisor from a suitable orbifold. Kawamata [47] studied deformations of complex manifolds compactifiable in a similar sense; he called those corresponding to a fixed compactification logarithmic deformations.

Kovalev [51] studied deformations of Ricci-flat Kähler metrics. Given an asymptotically cylindrical Calabi–Yau manifold, he showed that we can define an orbifold of Ricci-flat metrics around its metric, and that locally all such metrics with the same limit are Kähler for some logarithmic deformation of the complex structure. However, [51] did not consider how the complex structure varied in general: in particular, it did not consider variations of complex structure not leading to a change in the metric.

More recently, Conlon, Mazzeo and Rochon [15] combined [32, Theorem C] with [47] to show (their Theorem C) that the complex deformation theory of asymptotically cylindrical Calabi–Yau manifolds is unobstructed. Conversely to Kovalev, they did not explicitly consider deformations of the Kähler form, but observed (Lemma 9.1) that if the first Betti number of the compactifying orbifold is zero, then Kähler classes remain Kähler under such deformations of complex structure. Combining this with Theorem 4.1.14 below, which says that for any Kähler class we can find an asymptotically cylindrical Calabi–Yau metric, we essentially expect that every Kähler class on the orbifold gives a deformation of the metric corresponding to this complex deformation.

To obtain a result comparable to Theorem 4.2.39, which deals with deformations of both the complex structure and the metric, we would also have to combine this with [51] so that we can vary the Kähler class without the complex structure and both the Kähler class and the complex structure simultaneously. We would also need to extend [15, Lemma 9.1] to the case where the first Betti number of the compactifying orbifold is nonzero.

Instead, we will restrict to threefolds and use $G_2$ manifolds, that is those seven-dimensional manifolds with holonomy contained in the exceptional Lie group $G_2$. Since the subgroup of $G_2$ fixing a nonzero vector is isomorphic to $SU(3)$, the Riemannian product of a Calabi–Yau threefold and a general one-manifold is a $G_2$ manifold. We will use this correspondence to obtain results for Calabi–Yau threefolds from corresponding results for $G_2$ manifolds: because a small perturbation of a $G_2$ structures as a 3-form is again a $G_2$ structure, the $G_2$ analysis is often easier than trying to work with Calabi–Yau structures directly.

For instance, the $G_2$ moduli space has already been studied in the asymptotically cylindrical case. The result we use is Theorem 4.2.15: in the compact case it is due to Bryant and Harvey, but the first published proof was provided by Joyce; in adapting it for the results we need, we follow the simplified proof of Hitchin [35] and its elaboration for the asymptotically cylindrical case by Nordström [65]. Ebin [21] also constructed a moduli space of Riemannian metrics, and some ideas from his paper have become standard.
As for gluing, we know that Calabi–Yau manifolds can be glued by deformation in the sense of complex algebraic geometry. Using the smoothing results of Friedman [25] and Kawamata–Namikawa [48], Lee [54] showed that we can deform the singular space given by identifying the manifolds “at infinity” to give a smooth manifold. This work leaves open the question of what the topology of the deformed space is and so whether it can be regarded as a gluing given by cutting off the structures to their asymptotically cylindrical limits, joining them to form a manifold with a long neck, and perturbing to maintain a Calabi–Yau structure. In a similar vein, it remains open whether all the different possible ways of deforming the singular space yield structures equivalent up to diffeomorphism, and so it’s not clear that using these deformations gives a well-defined map on moduli spaces. We therefore will not consider this approach.

Instead, we again restrict to threefolds and use a gluing result on $G_2$ structures. The idea is that we can easily construct a $G_2$ structure with torsion, and that this torsion is small and can be removed by a perturbation argument; directly constructing a Calabi–Yau structure with torsion would be rather harder. Removing small torsion was previously studied by Joyce [40] with a detailed result designed for use desingularising conical singularities; in the asymptotically cylindrical gluing setting, a more direct result was obtained by Kovalev [50].

As for our result on the gluing map of deformation spaces, Nordström [66] proved the analogous $G_2$ result: that every deformation of a $G_2$ structure obtained by gluing two matching asymptotically cylindrical $G_2$ structures is obtained, uniquely, by one of perturbing the glued structures, perturbing the identification of the two manifolds, or perturbing the length of the neck.

The relation between Calabi–Yau threefolds and $G_2$ manifolds was used by Chan [11] in the context of desingularising conical Calabi–Yau orbifolds. He showed that a “nearly Calabi–Yau structure” on a real six-manifold $M$ can be deformed to a Calabi–Yau structure by passing through a $G_2$ structure on $M \times S^1$. The idea was also used by Doi and Yotsutani [18] for gluing asymptotically cylindrical $SU(3)$ structures, in the simply connected case. In distinction to Chan’s approach of gluing $SU(3)$ structures approximately first and then passing to $G_2$, they crossed with $S^1$ first and then glued the resulting $G_2$ structures using Kovalev’s result. Our analysis is similar but more explicit, and there is no need for the assumption of simple connectedness.

Finally, this relation is also used in physics, for instance, by de la Ossa, Larfors, and Svanes [16]. By putting a $G_2$ structure on the product of their $SU(3)$ manifold and a half-line, they constrain the evolution of an $SU(3)$ structure, with torsion, (according to various physical hypotheses), and so study paths in the “moduli space” of $SU(3)$ structures with torsion. They showed for instance that if the $G_2$ structure does not satisfy specific torsion conditions, then various components of the torsion of $SU(3)$ structures may become nonzero along these paths even if they were zero to start with.

We concentrate on the case of Calabi–Yau threefolds because then we only have to introduce
one extra dimension to reach seven dimensional $G_2$ manifolds. Similar arguments should work for lower dimension. Taking the Riemannian product with a complex torus would enable us to construct a Calabi–Yau threefold from a Calabi–Yau surface, though it would probably be necessary to consider the space of Calabi–Yau structures on such a torus. Reducing Calabi–Yau curves to Calabi–Yau surfaces should also be possible in the same way, though since we need the case of the torus care would be required to avoid circular reasoning.

We now turn to special Lagrangians. The deformation theory of calibrated submanifolds in general and special Lagrangian submanifolds in particular was initiated by McLean [60]. One of the key ideas in the special Lagrangian case is the use of the Kähler form to translate normal vector fields on $L$ into one-forms on $L$.

If $M$ is an asymptotically cylindrical Calabi–Yau manifold, and $L$ is an asymptotically cylindrical special Lagrangian submanifold, then $L$ with the induced metric is itself asymptotically cylindrical. Hence, deformation results follow by combining the asymptotically cylindrical Laplace–Beltrami theory of Lockhart [55] with the McLean results. This was carried out by Salur and Todd [70]. A slightly different deformation result for minimal Lagrangian submanifolds with boundary constrained to lie in an appropriate symplectic submanifold has also been obtained by Butscher [8].

Joyce has written a series of papers [41–45] on special Lagrangians with conical singularities in a fixed compact (generalised) Calabi–Yau manifold. For instance, he provides desingularisations of such singularities by gluing in a sufficiently small asymptotically conical submanifold of $\mathbb{C}^n$. This gluing argument uses the Lagrangian Neighbourhood Theorem to ensure that all submanifolds we deal with are Lagrangian. However, once we have a compact Lagrangian submanifold that is close to special Lagrangian he gives a general result on perturbing it to become special Lagrangian [44, Theorem 5.3]. Pacini extended this result to the case of asymptotically conical Lagrangian submanifolds in $\mathbb{C}^n$ given by patching together special Lagrangians with conical singularities and asymptotically conical ends (see [67, Theorem 6.3]).

In the asymptotically cylindrical setting, where there is a change of Calabi–Yau structure arising from the Calabi–Yau gluing, it is much harder to remain Lagrangian and so the analysis is rather different. There are also other gluing constructions of submanifolds satisfying appropriate partial differential equations: for instance, Lotay has considered similar desingularisation problems for coassociative submanifolds of $G_2$ manifolds (see for example [57]) and Butscher constructs contact stationary Legendrian submanifolds by a gluing method in [9].

1.1 Outline

We now describe the structure of the thesis and the main results.

Chapter 2 consists mostly of general review material. We first set up the notions of an asymptotically cylindrical manifold and make various adapted definitions. Since it is required
for our asymptotically cylindrical Hodge theory in chapter 3, we briefly discuss the cohomology of such a manifold. We then describe Banach spaces of forms in section 2.2, which will be needed throughout.

In chapter 3, we consider the Laplace–Beltrami operator. We state existence results for the “Poisson’s equation” given by this operator on both compact and asymptotically cylindrical manifolds. In the compact case this is just the Hodge decomposition theorem, and in the asymptotically cylindrical case we use Lockhart [55] on asymptotically cylindrical manifolds. In section 3.4, we apply a result of Maz'ja–Plamenevskiĭ [59] to the structure of harmonic forms on asymptotically cylindrical manifolds. We then summarise, following Melrose [61], an asymptotically cylindrical version of Hodge theory. In section 3.5 we turn to gluing. In subsection 3.5.1 we give some basic definitions of patching of manifolds and forms, pass to a gluing map of harmonic forms, and then give some results that follow from the Mayer–Vietoris sequence. In particular, we prove Theorem 3.5.10, which says that our gluing map is an inverse to the restriction map of the Mayer–Vietoris sequence, and infer a complement for its image from exactness. We then establish, in subsection 3.5.2, a vital estimate (Theorem 3.5.14) saying that we can find a lower bound for the Laplace–Beltrami operator on the gluings of asymptotically cylindrical manifolds shrinking only polynomially in the length of the cylindrical neck we allow. We use this estimate in subsection 3.5.3 to establish that the direct sum obtained from Mayer–Vietoris approaches orthogonality.

In chapter 4, we study Calabi–Yau threefolds, which we define as threefolds equipped with a suitable pair of closed differential forms \((\Omega, \omega)\). Our first main result is Theorem 4.1.26, that Calabi–Yau structures on a six-manifold \(M\) correspond to \(S^1\)-invariant \(G_2\) structures on \(M \times S^1\) in a natural way. We then turn to moduli spaces. We show first that there is a moduli space of torsion-free \(S^1\)-invariant \(G_2\) structures which is locally diffeomorphic to the moduli space of torsion-free \(G_2\) structures (Theorem 4.2.29). We then use the correspondence between \(S^1\)-invariant \(G_2\) structures and Calabi–Yau structures to identify a family of subspaces of this moduli space as the moduli space and Calabi–Yau structures; we show all of these subspaces are diffeomorphic, and infer that the moduli space of \(S^1\)-invariant \(G_2\) structures is the product of the moduli space of Calabi–Yau structures with an open subset of a suitable cohomology space (Theorem 4.2.39). We then turn to gluing; we show, by passing to a \(G_2\) structure, gluing the \(G_2\) structures, and returning to a Calabi–Yau structure, that Calabi–Yau structures can be glued in Theorem 4.3.3, and that this gluing defines, if we allow variation of neck length, a diffeomorphism of moduli spaces, in Theorem 4.3.46.

In chapter 5, we study special Lagrangian \(n\)-folds of Calabi–Yau manifolds, that is submanifolds where \(\text{Re} \Omega|_L = \text{vol}_L\) or, essentially equivalently, \(\text{Im} \Omega|_L = 0 = \omega|_L\). As a preliminary, we describe asymptotically cylindrical submanifolds and discuss the restriction map of forms in section 5.1, proving that the restriction map of forms can be bounded in terms of the second fundamental form in Theorem 5.1.3. In this section, we also briefly describe patching methods
for asymptotically cylindrical submanifolds extending the material of subsection 3.5.1.

We give our definitions and some examples and then review (and correct slightly, altering the dimension of the asymptotically cylindrical deformation space in Theorem 5.2.13) the deformation theory of McLean and Salur–Todd in subsection 5.2. We then turn to gluing. We assume that there is a Calabi–Yau $n$-fold gluing result of a kind that follows from chapter 4 in the $n = 3$ case, and show that we can combine our patching with a perturbation argument inspired by the deformation theory to give a gluing theorem for asymptotically cylindrical special Lagrangian submanifolds in Theorem 5.4.10. This generalises slightly to a theorem of the form “nearly special Lagrangian submanifolds can be made special Lagrangian”, for a suitable sense of nearly special Lagrangian, which we state as Theorem 5.4.17.

In section 5.5, we give a careful discussion of how to identify normal vector fields on nearby pairs of submanifolds and how these give derivatives for various natural maps on the “manifold of submanifolds”. For instance, we show that given two curves of submanifolds $L_s$ and $L'_s$, the curve of normal vector fields such that $\exp_{v_s}(L_s) = L'_s$ is smooth, and give an expression for the tangent to $v_s$ at zero in terms of the tangents to the curves $L_s$ and $L'_s$ (for this expression, see Proposition 5.5.16). This material is inspired by Palais [68] and Hamilton [30], but we could not find it in the literature.

In section 5.7, we prove our final theorem, that (for fixed neck-length), the gluing map of special Lagrangians defines a local diffeomorphism of moduli spaces as Theorem 5.7.13. We do this by explicitly computing the derivative of this gluing map using the material of section 5.5 and then showing that it is close to the gluing map of harmonic forms we constructed in chapter 3.

In chapter 6, we suggest some further questions provided by the material of chapters 4 and 5.
Chapter 2

General preliminaries

In this chapter, we set up definitions and notations that will be needed for the rest of the thesis. In section 2.1, we define asymptotically cylindrical manifolds and asymptotically translation invariant objects on them. In section 2.2, we define various spaces of forms that will be essential for analysis, and state some relevant basic analytic results that have applications throughout.

Throughout the thesis, the manifold $M$ will be taken to be oriented and Riemannian. When giving analytic results in chapters 2 and 3, we will write $R \Omega^k(M)$ for the space of differential $k$-forms on $M$ of a given regularity $R$. For example, $C^\infty \Omega^k(M)$ will be the smooth $k$-forms, and other spaces such as $e^{-\epsilon t} \Omega^k(M)$ will be introduced in due course. We will also need sections of more general bundles; for example, we will need to consider covariant derivatives of $k$-forms, which are sections of $T^*M \otimes \wedge^k T^*M$. Given such a bundle $E$, we shall write the space of sections of it as $S(E)$, so that $\Omega^k(M) = S(\wedge^k T^*M)$. As in the case of forms, we shall prefix $S(E)$ with regularity notations such as $C^\infty S(E)$ or $L^2 S(E)$. If no regularity notation is present, then we are working with the space of smooth forms or sections.

2.1 Asymptotically cylindrical manifolds

We begin by defining the notion of an asymptotically cylindrical manifold. Here we follow the tradition of Lockhart–McOwen [56], Maz’ja–Plamenevskii [59], and to some extent Kovalev–Singer [53].

We begin by defining the relevant topological notion.

**Definition 2.1.1.** A smooth manifold $M$ is said to have *ends* if it can be decomposed into a compact manifold $M^{\text{cpt}}$ with compact (oriented) boundary $N$, and the product manifold $N \times [0, \infty)$, where these two parts are identified by the obvious identification between $N = \partial M^{\text{cpt}}$ and $N \times \{0\}$. The number of ends is the number of connected components of $N$, so if $M$ has no ends $M$ is compact.

Note that $M \setminus M^{\text{cpt}}$ is not canonically identified with $N \times [0, \infty)$. Nevertheless, we shall...
generally assume that a manifold with ends is equipped with a choice of identification of \( M \setminus M^{cpt} \) with \( N \times [0, \infty) \). The main exception is in subsection 4.1.2.

In particular, this means that we have a well-defined coordinate function \( t \) on \( M \setminus M^{cpt} \). Furthermore, this can be extended to a non-negative global smooth function \( t \) on \( M \). We take this to be the coordinate on \( N \times [1, \infty) \), zero on \( M^{cpt} \) and between zero and one on \( N \times [0, 1] \). Given such a function we can find cutoff functions enabling us to define forms on \( M \) in terms of forms on \( N \) that might not extend over \( M^{cpt} \). Let \( \psi : \mathbb{R} \to \mathbb{R} \) be a smooth function with

\[
\psi(t) = \begin{cases} 
1, & \text{for } t \geq 2, \\
0, & \text{for } t \leq 1.
\end{cases}
\] (2.1.1)

Then let \( \psi_T(t) = \psi(t - T + 2) \) so that \( \psi_T(t) = 1 \) for \( t \geq T \) and \( \psi_T(t) = 0 \) for \( t \leq T - 1 \). We shall use \( \varphi_T \) for \( 1 - \psi_T \). We reserve \( \phi \) for a \( G_2 \) structure in chapter 4; we shall also use \( \phi \) for bump and cutoff functions of other kinds when no confusion will result. Using the global function \( t \) on any manifold \( M \) with ends, \( \psi_T \) and \( \varphi_T \) define smooth functions \( \psi_T \) and \( \varphi_T \) on \( M \).

Given a manifold \( M \) with ends, we write \( H^k_{\text{rel}}(M) \) for its de Rham cohomology; we write \( H^k_{\text{abs}}(M) \) for its compactly supported de Rham cohomology. In chapters 4 and 5, we shall only very rarely use \( H^k_{\text{rel}}(M) \), and so we shall just write the de Rham cohomology as \( H^k(M) \) there.

For this chapter and chapter 3, we write \( H^k_{\text{abs}}(M) \) and \( H^k_{\text{rel}}(M) \) explicitly.

The relative cohomology \( H^k_{\text{rel}}(M) \) is the relative cohomology of the pair \( (M, \mathbb{R}^n) \times N \).

Hence, we have the following

**Theorem 2.1.2.** There is an exact sequence

\[
\cdots \to H^k_{\text{rel}}(M) \to H^k_{\text{abs}}(M) \to H^k(N) \to H^{k+1}_{\text{rel}}(M) \to \cdots,
\] (2.1.2)

where the maps are given by inclusion, pullback to a cross-section, and \([\alpha] \mapsto [\psi'(t)dt \wedge \alpha] \), for \( \psi \) a cutoff function as in (2.1.1).

We omit the proof of this, which can be found in general in Bott–Tu [7, p.78].

We briefly note some properties of these spaces \( H^k_{\text{rel}}(M) \). First, by excision (see for example [33, p.201]), we obtain that \( H^k_{\text{rel}}(M) \) is isomorphic to the cohomology of the quotient space \( M/(N \times (0, \infty)) \) for \( k > 0 \) given by identifying all of the ends together to a single point. For \( k = 0 \), we obtain that \( H^0_{\text{rel}}(M) \) has dimension one less than \( H^0(M/(N \times (0, \infty))) \). Secondly, we note that Poincaré duality (for example [33, Theorem 3.35] or [7, p.44]) implies that \( H^k_{\text{rel}}(M) \cong (H^{n-k}_{\text{abs}}(M))^* \) for each \( k \), at least if these spaces are finite-dimensional: they will be in the case of a manifold with ends, and in particular for a compact manifold.

We now introduce a metric, and so pass from a manifold with ends to an asymptotically cylindrical manifold.

**Definition 2.1.3.** Suppose \((M, g)\) is an oriented Riemannian manifold with ends. \((M, g)\) is said
to be *asymptotically cylindrical with rate* $\delta > 0$ if there is a $g_N$ and constants $C_r$ such that

$$|\nabla^r(g|_{N\times[0,\infty)} - g_N - dt^2)| < C_re^{-\delta t}$$

(2.1.3)

for every $r = 0, 1, \ldots$, where $t$ is the global smooth function on $M$, $\nabla$ is the Levi-Civita connection induced by $g$, and $|\cdot|$ is the metric induced by $g$ on the appropriate space of tensors. We write $\tilde{g}$ for the cylindrical metric $g_N + dt^2$; $(M, g)$ is said to be (eventually) cylindrical if $g = \tilde{g}$ for $t$ large enough.

$(M, g)$ is said to be *asymptotically cylindrical* if it is asymptotically cylindrical with rate $\delta$ for some $\delta > 0$.

**Remarks.** Clearly there are other metrics, for example $2dt^2 + g_N$, that could be referred to as cylindrical; we will ignore them as only arising by reparametrising the end in $t$. Moreover, this reparametrisation of the end is tantamount to changing the identification of $M \setminus M^\text{cpt}$ with $N \times [0, \infty)$ referred to under Definition 2.1.1; hence, Definition 2.1.3 depends on exactly which identification we fixed.

Analogously, we may define asymptotically translation invariant sections of vector bundles.

**Definition 2.1.4.** Suppose that $M$ is an asymptotically cylindrical manifold. Given a bundle $E$ associated to the tangent bundle over $M$, a section $\tilde{\alpha}$ of $E|_N$ extends to a section of $E|_{N\times[0,\infty)}$ by extending parallel in $t$ using the Levi-Civita connection. A section $\alpha$ of $E$ is then said to be *asymptotically translation-invariant (with rate $0 < \delta' < \delta$)* if there is a section $\tilde{\alpha}$ of $E|_N$ and (2.1.3) holds (with $\delta'$) for $|\nabla^r(\alpha - \tilde{\alpha})|$ for $t > T$. In general, given a section $\alpha$ of such a bundle, $\tilde{\alpha}$ will be its limit in this sense. Note that $dt^2 + g|_N$ is not asymptotically translation invariant for any rate greater than $\delta$, which is why we restrict to $\delta' < \delta$.

**Remark.** It is easy to see that if $M$ is a fixed manifold with ends, hence with fixed identification, $(M, g_1)$ is asymptotically cylindrical with rate $\delta$ and $\alpha$ is asymptotically translation invariant with rate $\delta' < \delta$ and $g_2$ is another asymptotically cylindrical Riemannian metric on $M$ with rate $\epsilon > \delta'$, then $\alpha$ is also asymptotically translation invariant on $(M, g_2)$ with rate at least $\delta'$. Consequently, we may refer to asymptotically translation invariant sections without specifying an asymptotically cylindrical metric.

To simplify statements, fields on asymptotically cylindrical manifolds will often be implicitly assumed to be asymptotically translation invariant in chapters 4 and 5; for the purposes of this chapter and chapter 3, we will generally make this assumption explicitly when we use it, as fields with controlled growth will also be important.

### 2.2 Function spaces

We would like to consider our partial differential equations as maps between Banach spaces.

There are two natural families of Banach spaces of functions to which we may extend, the
Sobolev and Hölder spaces. We define these in this section. In chapter 3, we principally consider Sobolev spaces, though we explain how our results transfer to the Hölder case. This is because Sobolev spaces are better adapted to various analytic problems. In chapters 4 and 5, we work mostly with Hölder spaces as these are conceptually simpler and the analysis is not complicated enough to require Sobolev spaces.

In both cases, we shall use the notion of a jet bundle to simplify the definition. We take the basic existence results from Palais [68]; a standard reference is Saunders [71].

**Proposition 2.2.1** ([68, ch. 2]). Let \( M \) be a smooth manifold and let \( E \) be a vector bundle over \( M \). There exists a (unique up to isomorphism) vector bundle \( J^s(E) \), called the jet bundle, such that at every point \( p \in M \), the fibre \( J^s(E)_p \) is given by the quotient of smooth sections of \( E \) by the sections vanishing to order \( s \) at \( p \), and whenever \( \sigma \) is a smooth section of \( E \), we can write \( j^k(\sigma) \) a smooth section of \( J^s(E) \) given by the equivalence class of \( \sigma \) at every point. We call the operation of getting \( j^s(\sigma) \) from \( \sigma \) jet prolongation.

Given a metric on \( M \) and connection on a bundle \( E \), we can find a metric and connection on these jet bundles. We use the following lemma which we take from [37, Lemma 2.1].

**Lemma 2.2.2.** Let \( M \) be a Riemannian manifold, and let \( E \) be a vector bundle. Suppose we have a linear connection on \( E \). Then using the Levi-Civita connection and the Leibniz rule we have induced connections on the tensor products \( \otimes^k T^* M \otimes E \). Hence, we may define a bundle map

\[
\begin{align*}
J^sM & \longrightarrow \bigoplus_{t=0}^s S^t(T^* M) \otimes E, \\
[\sigma] & \longmapsto (\sigma_p, \text{Sym}(\nabla\sigma)_p, \ldots, \text{Sym}(\nabla^s\sigma)_p).
\end{align*}
\]

(2.2.1)

where \( S^t \) is the symmetric product and \( \text{Sym} \) are the symmetrisation operators. (2.2.1) is a vector bundle isomorphism.

A metric and connection on \( J^sE \) follow immediately.

Typically, we will take \( E \) to be a vector bundle associated to the tangent bundle of a Riemannian manifold, and hence can use the Levi-Civita connection associated to the metric on \( M \); but see subsection 5.1.2 below, in particular Remark 5.1.10, for an example of another connection we will use briefly.

We note also that if \( F \) is a smooth fibre bundle, Palais [68, ch. 15] shows that we can also define a jet bundle \( J^s(F) \). Moreover, this construction is functorial, so that if \( F \) is a subbundle of a bundle associated to the tangent bundle, \( J^s(F) \) is a smooth subbundle of \( J^s(E) \). This means we have a metric and connection on \( J^s(F) \) also.

Now we make our first definition of a norm on differential forms.

**Definition 2.2.3.** For each \( p \in (1, \infty) \) and integer \( s \geq 0 \), we define for \( \alpha \in C^\infty \Omega^k(M) \)

\[
\|\alpha\|_{W^s_p\Omega^k(M)} = \| j^s\alpha \|_p = \sum_{t=0}^s \|\text{Sym}(\nabla^t\alpha)\|_p,
\]

(2.2.2)
provided that this is finite, where $\| \cdot \|_p$ is the Lebesgue $p$-norm.

The space $W^p_0(M)$ is then defined as the completion of (the appropriate subspace of) $C^\infty_0(M)$ under this norm. We will write $L^p$ for $W^p_0$, and we may omit $\Omega_k(M)$ in the case of functions.

It is worth considering other potential ways of constructing the spaces $W^p_0(M)$. For example, we could define them as the subspace of currents (the dual of compactly supported smooth forms) such that these derivatives, interpreted distributionally, can all be represented by measurable forms with finite $L^p$ norm, or we could define them as in Definition 2.2.3 but by taking the completion of the compactly supported forms. Clearly the definition we have given lies between these two. According to Eichhorn in [22, p.11], none of these need be the same in general.

However, if $s = 1$ and the manifold $M$ is complete, then all three of these coincide. This can be interpreted as saying that smooth forms (with compact support) are dense in the appropriate subspace of distributions. A proof of this fact can be found in Eichhorn [22, Proposition 1.4]; see also [4, Theorem 2.6], which does not consider the corresponding space of currents. The case of higher $s$ requires a steadily increasing set of assumptions ($s = 2$ requires additionally that the curvature is bounded; $s = 3$ requires this and that its derivatives are bounded, and so on.) We also require some control on the injectivity radius. For details, see Eichhorn [22, Proposition 1.6] or Aubin [3, Théorème 1]; again, Aubin does not consider the current case.

We now define the other natural choice of spaces, the Hölder spaces. These are less analytically useful than the Sobolev spaces, but have the advantage that global geometry is not involved in their definition, which makes using them on manifolds by arguing locally rather easier than in the Sobolev case.

**Definition 2.2.4.** Let $(M, g)$ be a Riemannian manifold. Let $s$ be a non-negative integer, and $\mu \in [0, 1)$. Let $E$ be a bundle associated to the tangent bundle, so that the Riemannian metric and $\nabla$ define a metric and connection on the bundle $E$. Given a section $\sigma$ of $E$, define

$$
\| \sigma \|_{C^{0,\mu}S(E)} = \sup_{x \in M} \left( g(\sigma_x, \sigma_x) \right)^{\frac{1}{2}} + \sup_{x, y \in M, d(x, y) < \delta} \frac{|\sigma_x - \sigma_y|}{d(x, y)^\mu},
$$

(2.2.3)

where $\delta$ is the injectivity radius and $|\sigma_x - \sigma_y|$ is given by parallel transporting $\sigma_x$ with $\nabla$ from the fibre $E_x$ to the fibre $E_y$ along the geodesic from $x$ to $y$ and then measuring the difference. Note that since parallel transport is an isometry, this is symmetric in $x$ and $y$.

Since $J^s(E)$ is also a bundle associated to the tangent bundle, we then define

$$
\| \sigma \|_{C^{s,\mu}S(E)} = \| j^s \sigma \|_{C^{0,\mu}S(J^sE)},
$$

(2.2.4)

where $j^s \sigma$ is the jet prolongation from Proposition 2.2.1.

If $\mu = 0$, we just write $C^s$. 

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This definition is taken from Joyce [40, p.5]. As in the Sobolev case, if $M$ is not compact, smooth forms need not have finite $C^{s,\mu}$ norm for any $s$ and $\mu$. As we will mostly be working with forms on compact manifolds and asymptotically translation invariant forms on asymptotically cylindrical manifolds, this will not be a major issue.

The following proposition explains why $C^s$ norms are convenient to work with.

**Proposition 2.2.5.** Let $f : E \to F$ be a smooth bundle map between fibre bundles over a manifold $M$ that are subbundles of vector bundles associated to the tangent bundle. For each open subset $U$ in $M$ with compact closure, open subset $V$ of $E|_U$ with compact closure in $E$, constant $K$, and positive integer $k$, we have a constant $C_{s,K,V}$ such that whenever the sections $\sigma_1$ and $\sigma_2$ of $E|_U$ lie in $V$, and have derivatives up to order $s$ bounded by $K$,

$$\| f(\sigma_1)|_U - f(\sigma_2)|_U \|_{C^s} \leq C_{s,K,V} \| \sigma_1|_U - \sigma_2|_U \|_{C^s}.$$  \hfill (2.2.5)

**Proof.** By passing to the jet bundles $J^s(E)$ and $J^s(F)$ using that their construction is functorial (see the discussion after Lemma 2.2.2), we may suppose that $s = 0$. The constraint that $\sigma_i$ lie in $V$ and have derivatives bounded by $K$ implies that the jets lie in a compact subset of the jet bundles. The result is then immediate from continuity. \hfill \Box

**Remark.** In the Sobolev space case, the argument of Proposition 2.2.5 still applies but evidently cannot so easily localised. Since we will use Proposition 2.2.5 to get a local constant and then argue by other methods than this constant can be bounded uniformly, it is easier to use $C^s$ spaces than having to work with a partition of unity.

On an asymptotically cylindrical manifold, we define weighted Sobolev spaces following Lockhart [55] and Lockhart & McOwen [56]. We take, as usual, the completion of the smooth forms under a certain norm.

**Definition 2.2.6.** Suppose that $\delta \in \mathbb{R}$. $\delta t$ is a function defined canonically only on the ends, so we introduce, for $\alpha \in C^\infty_\Omega^k(M)$, the norm

$$\| \alpha \|_{e^{\delta t}W^p_\Omega^k(M)} = \| \psi e^{-\delta t} \alpha + (1 - \psi) \alpha \|_{W^p_\Omega^k(M)},$$  \hfill (2.2.6)

and we define the space $e^{\delta t}W^p_\Omega^k(M)$ as the completion of $C^\infty_\Omega^k(M)$ under this norm (meaning the completion of those smooth forms for which this is finite).

Furthermore, as we will be interested in smooth forms (that is, those with all derivatives), set

$$e^{\delta t}W^p_\infty^k(M) = \bigcap_{s \in \mathbb{N}} e^{\delta t}W^p_s\Omega^k(M).$$  \hfill (2.2.7)

(We can fix some embedding of all of these into $e^{\delta t}L^p$, so this intersection does make sense.)
Remark. We have fixed a smooth extension of $t$ over $M$ and so we could equivalently set $e^{\delta t}W^p_s$ to be the currents arising as images of this multiplication on a subset of currents. This would give an isomorphic Banach space, with a different norm.

We can in the same way extend the Hölder norms of Definition 2.2.4 to form the $e^{\delta t}C^{s,\mu}$ norms. Note that it follows from a weighted version of the Sobolev embedding theorem that any form $\alpha$ in $e^{\delta t}W^p_\infty\Omega^k(M)$ has all of its $e^{\delta t}C^{s,\mu}$ norms finite, so that $\alpha$ and all its derivatives have bounded growth. We also use these norms for vector fields on manifolds, and in particular normal vector fields in chapter 5.

Considering $\delta < 0$, these weighted norms define norms on exponentially decaying forms. We will require for later work a $C^\infty$ topology on asymptotically translation invariant forms with a given rate $\delta$. Specifically, we define a $C^\infty_\delta$ norm on a subset of asymptotically translation invariant sections $\alpha$ of a bundle $E$ associated to the tangent bundle by

$$\|\alpha\|_{\mathcal{C}^\infty_\delta(E)} = \|(1 - \psi)\alpha + \psi e^{\delta t}(\alpha - \tilde{\alpha})\|_{\mathcal{C}^\infty S(E)} + \|\tilde{\alpha}\|_{\mathcal{C}^\infty_\delta S|_{\mathcal{N}}}(2.2.8)$$

The topology induced by (2.2.8) is called the extended weighted topology with weight $\delta$, as in the case of $e^{\delta t}C^{s,\mu}$. Note that if, for a given $\alpha$, there is some fixed $\delta$ so that $\|\alpha\|_{\mathcal{C}^\infty_\delta(E)}$ is finite for all $s$, then $\alpha$ must be asymptotically translation invariant with any rate greater than $\delta$. Conversely, if $\alpha$ is asymptotically translation invariant, with rate greater than $\delta$, then all these norms are finite.

In the same way we have a $C^{s,\mu}_\delta$ topology, and by taking the inverse limit we also have a $C^\infty_\delta$ topology. See also the discussion of Sobolev extended weighted topologies in, for example, [51, section 3] and [2, p.58ff.].

These weighted and extended weighted topologies (as opposed to the norms) depend only on the decay rate of the metric, since by compactness of $\mathcal{N}$ and $\mathcal{M}^{\text{cpt}}$ all metrics with the same decay rate are Lipschitz equivalent.

We have thus far only used the asymptotically cylindrical end to find a suitable family of weights. We will occasionally need to work in local coordinates to appeal to analytic results in Euclidean spaces, and so we state the following lemma saying how $e^{\delta t}W^2_s\Omega^k$ can be identified in local coordinates.

**Lemma 2.2.8.** Suppose $M$ is asymptotically cylindrical, and $u$ is a form on $M$. Then $u \in e^{\delta t}W^2_s\Omega^k(M)$ if and only if both

i) for all $\phi$ compactly supported in a single chart on $M$, the vector valued function $\phi u$ on an open subset of $\mathbb{R}^n$ is in $W^2_s$ (with respect to the appropriate Euclidean metric)

ii) for all $\tilde{\phi}$ compactly supported in a single chart on the cross-section $\mathcal{N}$, the vector valued function $e^{-\delta t}\tilde{\phi}u$ on $(1, \infty) \times \mathcal{U}$ for an open subset $\mathcal{U}$ of $\mathbb{R}^{n-1}$ is in $W^2_s$. 

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Chapter 3

Harmonic forms

In this chapter, we discuss the first and simplest of the three partial differential equations we shall consider: the Laplace-Beltrami operator, or Laplacian. Much of this is review material for which we only give statements and references. The most interesting parts are the review of asymptotically cylindrical Hodge theory leading to Theorem 3.4.5 and our discussion of the perturbative gluing method applied to harmonic forms contained in section 3.5: in particular, we prove for later use in chapter 5 that on one-forms with the cross-section of $M$ connected, this defines an isomorphism between appropriate spaces of harmonic forms in Corollary 3.5.12, and extend this to say that the isomorphism can be bounded below independently of a natural gluing parameter in Proposition 3.5.19.

3.1 Definition and local analysis

We make the following

**Definition 3.1.1.** Let $\delta$ be the map on $k$-forms given by $\delta \alpha = (-1)^{(k+1)+1} \ast d \ast \alpha$, where $\ast$ is the Hodge star. Then let the Laplace-Beltrami operator $\Delta$ be given by $d \delta + \delta d$. Finally, let

$$\mathcal{H}^k(M) = \{ \alpha \in C^\infty \Omega^k(M) : \Delta \alpha = 0 \}$$

be the vector space of *harmonic* $k$-forms.

Note that $\ast d \ast$ reduces the degree of forms. Thus it is a good candidate to be the (formal) adjoint of exterior differentiation. This is true, up to a change of sign, on compact manifolds without boundary and is a straightforward application of Stokes’ theorem.

**Proposition 3.1.2.** The formal adjoint of $d : C^\infty \Omega^k(M) \rightarrow C^\infty \Omega^{k+1}(M)$ is given by $d^* = \delta$. That is, $(\alpha, \delta \beta) = (d \alpha, \beta)$ for any smooth $k$-form $\alpha$ and smooth $(k+1)$-form $\beta$.

**Remark.** Strictly, to define an adjoint of an operator, the operator must map between Banach spaces and the adjoint is defined between the dual spaces; thus, to discuss the adjoint of $d$ we
would need to discuss which Banach space we embed $C^\infty_\Omega^k(M)$ into and then find the dual space and identify the domain of the adjoint operator. Here, the adjoint is purely formal; this is a map from smooth forms to smooth forms that obeys the right expression on inner products.

This implies that $\Delta$ is formally self-adjoint.

Given a linear differential operator such as $\Delta$, if we restrict to a chart of our manifold $M$ (on which, in particular, we can trivialise the bundle of forms) and work entirely in this chart, we obtain some other linear differential operator on an open subset $U$ of Euclidean space $\mathbb{R}^n$. Specifically, suppose that $\mathcal{L}$ is a linear differential operator of order $m$ from a bundle of rank $k$ to a bundle of rank $l$. (Note that for $\Delta$, evidently, $k = l$, and $m = 2$.) Such a differential operator may in a suitable chart be written as a sum over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i$ positive integers, as

$$\mathcal{L}u = \sum_{\alpha: \sum \alpha_i \leq m} A_\alpha(t_1, \ldots, t_n) D\alpha u,$$

(3.1.2)

where $A_\alpha(t_1, \ldots, t_n)$ are $l \times k$ matrices, and $D\alpha$ is the partial derivative differentiating $\alpha_i$ times in the $t_i$ coordinate. We can thus define, locally, the principal symbol, as the matrix-valued function

$$(x, \xi) \mapsto \sum_{\alpha: \sum \alpha_i = m} A_\alpha(t) \xi^\alpha$$

(3.1.3)

as the dummy variable $\xi$ varies through $\mathbb{R}^n$. We can then make

**Definition 3.1.3.** A linear differential operator is **elliptic** if its principal symbol (3.1.3) is an invertible square matrix whenever $\xi \neq 0$.

Evidently, this is only possible if $k = l$; if not, then we could ask that (3.1.3) be of full rank. The operator is then said to be overdetermined or underdetermined elliptic according to whether $k < l$ or $k > l$ respectively.

Note that these definitions are not dependent on coordinate system. In particular, if $k = l$, it is equivalent that if

$$\mathcal{L}(\phi^m \alpha)(p) = 0,$$

(3.1.4)

for any smooth form $\alpha$ and smooth function $\phi$ with $\phi(p) = 0$ and $d\phi(p) \neq 0$, then $\alpha(p) = 0$ (see for instance [75, Definition 6.28]). In [75, p.250], Warner proves

**Proposition 3.1.4.** The Laplace-Beltrami operator is elliptic.

**Remark.** We could also define the Sobolev spaces of Definition 2.2.3, at least for $s$ even, by using the Laplacian rather than the covariant derivative and taking the norm $\|(1 + \Delta)^{s/2} \alpha\|_p$. This will not, however, define the same space in general; for a counterexample due to Dodziuk and Min-Oo, see [17, p.67].

We will be interested in the form of the Laplacian on an asymptotically cylindrical manifold. Elementary calculation shows that
Proposition 3.1.5. Let \((M, g)\) be a cylindrical manifold \(M = \mathbb{R} \times N\). Let \(\eta\) be a smooth form on \(N\): we take it to have degree \(k\) or \(k - 1\) to make the form we consider on the left hand side of the equation below a \(k\)-form. Let \(f\) be a function of \(t\) only. Then, sufficiently far along this end we have

\[
\Delta(f\eta) = -f''\eta + f\bar{\Delta}\eta, \quad \Delta(f dt \wedge \eta) = -f''dt \wedge \eta + f dt \wedge \bar{\Delta}\eta,
\]

(3.1.5)

where \(\bar{\Delta}\) refers to the Laplacian of \(g_N\) on \(N\).

Generalising, we may now refer to \(\bar{\Delta}(dt \wedge \theta) = dt \wedge \bar{\Delta}\theta\). Moreover, by choosing a basis for \(\wedge^k T^* N\), we may differentiate forms on \(M\) in \(t\). As \(\wedge^k T^* N\) is independent of \(t\), the integral and differential are independent of the choice of basis, and the fundamental theorem of calculus applies. We may then say that \(\Delta = -\frac{\partial^2}{\partial t^2} + \bar{\Delta}\).

We also briefly observe that for asymptotically cylindrical metrics, \(\Delta\) decays to \(\bar{\Delta} = -\frac{\partial^2}{\partial t^2} + \bar{\Delta}\).

Proposition 3.1.6. Suppose that \((M, g)\) is asymptotically cylindrical, and that \(g\) is decaying at rate \(e^{-\delta t}\), for some \(\delta\), to \(\tilde{g} = dt \otimes dt + g_N\). Write \(\bar{\Delta}\) for the Laplace-Beltrami operator associated to \(\tilde{g}\). We write \(O(\beta)\) for some quantity \(\gamma\) with \(|\nabla^r \gamma| \leq C_r(|\nabla^r \beta| + |\nabla^{r+1} \beta| + |\nabla^{r+2} \beta|)\) for every \(r\) for some constants \(C_r\), where \(\nabla\) is the covariant derivative associated to the metric \(g\). Then for any \(k\)-form \(\alpha\), we have \(\Delta\alpha - \bar{\Delta}\alpha = O(e^{-\delta t} \alpha)\), where the implied constants are independent of \(\alpha\).

We shall use results analogous to Proposition 3.1.6 throughout: generally, we shall just say “because the operator depends continuously on the metric”.

3.2 Regularity theory

It is well known that solutions to elliptic equations have the greatest possible regularity. This implies that harmonic forms are smooth. We make some precise statements that are useful for the asymptotically cylindrical case. Unless our local coordinates form an isometry, the Laplace-Beltrami operator need not be given by the Laplacian in local coordinates, and so we state these local results in terms of general elliptic differential operators. For asymptotically cylindrical manifolds, we can consider things in local coordinates using Lemma 2.2.8.

We assert the following standard result for elliptic regularity on the interior of bounded domains of Euclidean space.

Theorem 3.2.1. Suppose \(U\) is a bounded domain in Euclidean space, that \(u \in W^2_1(U)\) (meaning potentially a vector-valued function), and that \(L\) is a second-order strictly elliptic (meaning that the relevant symbol is bounded away from zero) differential operator, its coefficients being bounded with their derivatives. Suppose further that \(Lu = v \in W^2_0(U)\), in a suitable weak
sense. Then, for any $\phi$ smooth and compactly supported in $U$ we have $\phi u \in W^{2,s+2}_s(U)$, with the estimate

$$\| \phi u \|_{W^{2,s+2}_s(U)} \leq C(\| u \|_{W^2_s(U)} + \| v \|_{W^2_s(U)}),$$

(3.2.1)

where the constant depends on $\phi$ and $U$ (meaning the interplay of the geometry of the support of $\phi$ and this domain), $s$ and the bounds and the lower bound on the ellipticity of $L$ and its derivatives.

This is proved in Gilbarg and Trudinger [27, Theorem 8.10]. If $L$ is sufficiently smooth, then we can pass to the space of currents and give similar results for $u$ of lower regularity a priori. For instance, this is the approach taken in Warner [75, ch. 6].

This immediately implies, of course, that all harmonic forms are smooth by working in local coordinates. In the asymptotically cylindrical case, we use Lemma 2.2.8 and its proof to obtain

**Theorem 3.2.2.** Suppose $M$ is asymptotically cylindrical, $u \in e^{\delta t}W^2_1\Omega^k(M)$, and that $v = \Delta u \in e^{\delta t}W^2_s\Omega^k(M)$ (where $\Delta$ is taken in a weak sense). Then $u \in e^{\delta t}W^2_{s+2}\Omega^k(M)$, and we have an estimate

$$\| u \|_{e^{\delta t}W^2_{s+2}\Omega^k(M)} \leq C(\| u \|_{e^{\delta t}W^2_s\Omega^k(M)} + \| v \|_{e^{\delta t}W^2_s\Omega^k(M)}),$$

(3.2.2)

for some constant $C$ independent of $u$.

The main analytic difficulty of this proof is to construct the partition of unity over the end enabling us to work with compact support and so apply Lemma 2.2.8. It is clear that $\Delta$ becomes a well-defined second-order operator on Euclidean space that we can apply Theorem 3.2.1 to. Compare also Lockhart [55, Proposition 5.3], which says that harmonic forms are smooth using a similar estimate.

We may, of course, have more regularity on $u$ a priori; as the higher Sobolev norms certainly bound the $L^2$ norm the same result holds. From this, and a very similar argument, we obtain the following

**Corollary 3.2.3.** Any harmonic form in $e^{\delta t}W^2_r\Omega^k(M)$ for some $r$ is in $e^{\delta t}W^2_s\Omega^k(M)$ for every integer $s$. In particular, it is smooth and has bounded growth with all its derivatives.

We also briefly give a Hölder version of Theorem 3.2.2. We begin by stating the following analogue of Theorem 3.2.1 which follows straightforwardly from [27, Theorem 6.6 and Problem 6.1]: we have to introduce $\phi$, but $\phi u$ must also satisfy an elliptic differential equation with right hand side depending on $v$ and $\phi u$ can certainly be bounded appropriately by $u$.

**Theorem 3.2.4.** Suppose $U$ is a bounded subset of Euclidean space, and that $u$ is a smooth vector-valued function on $U$. Suppose that $L$ is a second-order strictly elliptic (meaning that the relevant symbol is bounded away from zero) differential operator, its coefficients being bounded...
with their derivatives up to the boundary of $\mathcal{U}$. Suppose $L u = v$, and that $\phi$ is compactly supported in $\mathcal{U}$. Then

$$
\| \phi u \|_{C^{s, \mu}(\mathcal{U})} \leq C (\| v \|_{C^{s-2, \mu}(\mathcal{U})} + \| u \|_{C^{0}(\mathcal{U})}),
$$

for $k$ sufficiently large, where the constant $C$ depends on $\phi$ and the bounds on the operators, and the norms on the right hand side are up to the boundary of $\mathcal{U}$.

This is known as the Schauder estimate. It is essential that $\mu \neq 0$.

It is also possible to get estimates replacing the term $\| u \|_{C^{0}(\mathcal{U})}$ on the right hand side with a weaker norm such as $\| u \|_{L^2(\mathcal{U})}$. Intuitively, this is because this term is only required to control the kernel of the operator $L$, which Theorem 3.2.2 implies is smooth provided that it is in an appropriate $L^2$ space. To do this, we use estimates of the form

**Proposition 3.2.5.** Suppose $\mathcal{U}$ is a bounded subset of Euclidean space. For every $\epsilon > 0$, there exists a constant $K_\epsilon$ such that

$$
\| u \|_{C^{0}(\mathcal{U})} \leq \epsilon \| u \|_{C^1(\mathcal{U})} + K_\epsilon \| u \|_{L^2(\mathcal{U})},
$$

where these norms are up to boundary of $\mathcal{U}$.

This can be proved by the same method as the similar argument given by Jost [39, Lemma 11.1.2]. (Note that this section was substantially revised from the first edition of the book; a more direct argument can be found there).

It is not trivial to combine Proposition 3.2.5 with Theorem 3.2.4. This is essentially because what we would like to do is choose $\epsilon$ small enough that $C \epsilon < 1$, so that we can bound the second term of (3.2.3) using (3.2.4) and then absorb the $C^1$ term on the left hand side. However, we cannot control $u$ by $\phi u$, as $\phi$ is compactly supported in $\mathcal{U}$, and thus this doesn’t quite work. The desired similar estimate to (3.2.3) with an $L^2$ control (replacing $\phi u$ with $u$ on a compact subdomain $\mathcal{V}$ of $\mathcal{U}$ and restricting to the cases where these are balls) can nevertheless be achieved by examining carefully how the constants of (3.2.3) and (3.2.4) depend on the radii of these balls. See for instance [39, proof of Theorem 11.1.2]. When we work with compact manifolds, however, these problems do not arise.

Using the density of smooth functions a result more closely analogous to Theorem 3.2.1 follows, and again by using Lemma 2.2.8 we can infer a global weighted version corresponding to Theorem 3.2.2.

**Theorem 3.2.6.** Suppose $M$ is asymptotically cylindrical, $u \in e^{\delta t} C^{2, \mu}\Omega^k(M)$, and that $v = \Delta u \in e^{\delta t} C^{s, \mu}\Omega^k(M)$. Then $u \in e^{\delta t} C^{s+2, \mu}\Omega^k(M)$, and we have an estimate

$$
\| u \|_{e^{\delta t} C^{s+2, \mu}\Omega^k(M)} \leq C(\| u \|_{e^{\delta t} C^{0}\Omega^k(M)} + \| v \|_{e^{\delta t} C^{s, \mu}\Omega^k(M)}),
$$

(3.2.5)
for some constant $C$ independent of $u$.

## 3.3 Existence theory

In this section, we state some basic existence results for the equation $\Delta u = v$ with $u$ and $v$ in suitable spaces of forms on compact manifolds and asymptotically cylindrical manifolds. On compact manifolds, the result is the classical Hodge decomposition theorem. We state the following variant.

**Theorem 3.3.1 (Hodge Decomposition).** Suppose $M$ is compact without boundary. The space $\mathcal{H}^k(M)$ of harmonic forms on $M$ is finite-dimensional and consists of smooth closed and co-closed forms. Given $v \in C^\infty \Omega^k(M)$, we can write $v = h + \Delta u$ for $h \in \mathcal{H}^k(M)$ and $u \in C^\infty \Omega^k(M)$. Moreover, harmonic forms, exact forms, and coexact forms are all mutually orthogonal and so this yields an orthogonal decomposition of smooth forms. If we write this decomposition as $v = h + dw_1 + \delta w_2$, then we may choose $w_1$ and $w_2$ such that

$$\|w_1\|_{W^{s+1}_{2,1}\Omega^{k-1}(M)} \leq C\|v\|_{W^{2s}_{2,1}\Omega^k(M)},$$

for each $s$, and similarly for $w_2$.

For a proof, see Warner [75, Theorem 6.8]. The fact that all harmonic forms are smooth and $u$ is smooth rests on the elliptic regularity results of section 3.2.

This yields our first and simplest theorem on the representation of de Rham cohomology. An asymptotically cylindrical variant of Corollary 3.3.2 is one of the main goals of this chapter.

**Corollary 3.3.2.** Suppose that $M$ is compact and without boundary. Then there is an isomorphism between harmonic forms $\mathcal{H}^k(M)$ and de Rham cohomology $H^k(M)$.

We can obtain corresponding results for Hölder spaces. Similar arguments as in the proof of Theorem 3.3.1 prove estimates corresponding to (3.3.1) for these spaces. That is,

$$\|w_1\|_{C^{s+1,\mu}\Omega^{k-1}(M)} \leq C\|v\|_{C^{s,\mu}\Omega^k(M)}.$$  

(3.3.2)

Because of the condition $\mu \neq 0$ in the Schauder estimate of Theorem 3.2.4, $\mu$ cannot be zero here.

In the asymptotically cylindrical case, the required result is slightly less standard. We state the following, which basically follows from the work of Lockhart [55, particularly Theorem 5.2 and Corollary 6.2] and the elliptic regularity results in the previous section.

**Theorem 3.3.3.** Let $M$ be asymptotically cylindrical and $\epsilon > 0$ sufficiently small so no nonzero eigenvalue of the Laplacian $\Delta$ on $(N, g_N)$ is less than $\epsilon^2$ (by self-adjointness nonzero eigenvalues are positive). The set of harmonic forms in $e^{\pm \epsilon t} W^2_\infty \Omega^k(M)$ is finite-dimensional, and given
u \in e^{\varepsilon t}W^2_{\infty} \Omega^k(M), we can write
\[ u = \Delta v + u', \tag{3.3.3} \]
where $u' \in e^{-\varepsilon t}W^2_{\infty} \Omega^k(M)$ is harmonic and $v \in e^{\varepsilon t}W^2_{\infty} \Omega^k(M)$.

\textbf{Remark.} This is weaker than Theorem 3.3.1. We do not have orthogonality of harmonic forms and the image in general, and nor do we have that harmonic forms are closed and coclosed. Both of these are essentially because integration by parts arguments do not necessarily hold. Later (Proposition 3.4.2) we shall see that provided a harmonic form does not grow too fast, it is indeed closed and coclosed.

Theorem 3.3.3 also extends to the case of Hölder spaces.

### 3.4 Hodge theory

We now extend Corollary 3.3.2 to asymptotically cylindrical manifolds. We find spaces of harmonic forms isomorphic to both $H^k_{\text{abs}}(M)$ and $H^k_{\text{rel}}(M)$. We shall recall that any harmonic form growing sufficiently slowly approaches, towards infinity, one of the form $\eta + dt \wedge \theta + \tilde{\eta} + t\tilde{\theta}$, where $\eta, \tilde{\eta}, \theta$ and $\tilde{\theta}$ are forms on the cross-section, and we shall mainly consider those approaching $\eta$ or $dt \wedge \theta$. In Theorem 3.4.4, we state the relationship between possible $\eta$ and $\theta$: if both are $k$-forms, then they are orthogonal, and all harmonic forms on $N$ arise as sum of such forms. We then prove that the required isomorphisms exist and give a topological interpretation of Theorem 2.1.2 corresponding to that of Melrose [61, Proposition 6.18] in Theorem 3.4.5. See also an earlier reinterpretation of this result by Nordström in [65, subsection 5.2] and [64, subsection 2.3.4]; details for much of this section can also be found there.

We begin by considering the limits of harmonic forms. We observe the following result on the structure of such limits from work of Maz’ja–Plamenevskiĭ. This follows by combining [59, Theorem 8.2], applied to the Laplacian corresponding to the cylindrical metric, with a result on the structure of harmonic forms on cylinders that we can view as a special case of [59, Theorem 4.2].

\textbf{Proposition 3.4.1.} Suppose $M$ is asymptotically cylindrical with rate $\varepsilon$. Let $\delta > \hat{\delta};$ suppose that $u \in e^{\delta t}W^2_{\infty} \Omega^k(M)$ has $\Delta u = 0$ far enough along, say for $t > T$, that neither $\delta$ nor $\hat{\delta}$ is a square root of an eigenvalue of $\Delta$ acting on $k$ or $(k-1)$-forms on $(N, g_N)$, and that $\lambda_1, \ldots, \lambda_m$ are the square roots of eigenvalues between $\delta$ and $\hat{\delta}$. Then there exist $p_i$, linear polynomials in $t$ and smooth in the cross-section $N$, such that
\[ u = \varphi \left( \sum_{i=1}^m e^{\lambda_i t} p_i \right) + u', \quad p_i = \eta_i + dt \wedge \theta_i + t\tilde{\eta}_i + tdt \wedge \tilde{\theta}_i. \tag{3.4.1} \]

In these expressions, $u' \in e^{\max\{\delta, \delta-\epsilon\}}W^2_{\infty} \Omega^k(M)$, $\eta_i$ and $\tilde{\eta}_i$ are $k$-forms on $N$, and $\theta_i$ and $\tilde{\theta}_i$ are $(k-1)$-forms on $N$ such that $\eta_i, \tilde{\eta}_i, \theta_i$ and $\tilde{\theta}_i$ are eigenforms for $\Delta$ on $N$ with eigenvalues
In particular, if the eigenvalues of the Laplacian on $N$ on $k$-forms and $(k-1)$-forms do not coincide, each $\lambda_i$ only yields two terms in the $p_i$.

In particular, since the 0 eigenvalue is isolated, we may apply Proposition 3.4.1 with $\hat{\delta} = -\delta$ where $\delta$ is sufficiently small that both there are no eigenvalues except 0 in $(-\delta, \delta)$ and $\delta$ is smaller than twice the decay rate of the metric; then $\eta_i$ and $\theta_i$ are harmonic, and $u'$ is in a $e^{-\delta t}$-weighted space. This shows, explicitly, that a harmonic form in $e^{\delta t}W^2_{\infty}\Omega^k(M)$ is of the form

$$\psi(\eta + dt \wedge \theta + t\tilde{\eta} + tdt \wedge \tilde{\theta}) + u', \quad (3.4.2)$$

for $u'$ exponentially decaying.

If we then restrict to the asymptotically translation invariant case, we obtain the following integration by parts result.

**Proposition 3.4.2.** If $M$ is asymptotically cylindrical, and $u$ is harmonic and of the form $\psi(\eta + dt \wedge \theta) + u'$ with $u'$ decaying exponentially as $t \to \infty$, then it is closed and coclosed.

This follows immediately from the fact that $du$ and $\delta u$ are exponentially decaying.

We may now define the subspaces of the set $\mathcal{H}^k(M)$ of harmonic forms on $M$ that will correspond to $H^k_{\text{abs}}(M)$ and $H^k_{\text{rel}}(M)$.

**Definition 3.4.3.** Let

$$H^k_{\text{abs}} = \{ u \in \mathcal{H}^k(M) : u = \psi \eta + u' \}, \quad H^k_{\text{rel}} = \{ u \in \mathcal{H}^k(M) : u = \psi (dt \wedge \theta) + u' \}, \quad (3.4.3)$$

where $\eta \in \mathcal{H}^k(N)$, $\theta \in \mathcal{H}^{k-1}(N)$, and $u' \in e^{-\epsilon t}W^2_{\infty}\Omega^k(M)$.

We have the following decomposition of $\mathcal{H}^k(N)$.

**Theorem 3.4.4.** For each $k$, $\mathcal{H}^k(N)$ is the orthogonal direct sum of those harmonic $k$-forms that arise in $H^k_{\text{abs}}$, which we will denote by $A^k$, and those that arise in $H^k_{\text{rel}}$, which we denote by $R^k$.

Moreover, for each $\alpha \in \mathcal{H}^k(N)$ there exists a canonical choice of representative in $H^k_{\text{abs}}$ or $H^k_{\text{rel}}$ respectively. Specifically, if $\alpha$ corresponds to class in $H^k_{\text{rel}}$ we can find a unique exact harmonic form with limit $dt \wedge \alpha$ (which is the differential of a harmonic form); similarly, if $\alpha$ corresponds to a class in $H^k_{\text{abs}}$ we can find a unique coexact harmonic form with limit $\alpha$. This exact or coexact form is equivalently determined by being orthogonal to decaying harmonic forms. The differential and codifferential of a harmonic form in $e^{\delta t}W^2_{\infty}\Omega^k(M)$ are such exact and coexact harmonic forms.

Theorem 3.4.4 can be proved by the same methods [61, Lemma 6.15 and Proposition 6.16] as Melrose used for his setting. We note an example application. Since $M$ is oriented and Riemannian there is always a volume form, which must be harmonic. It follows that $\text{vol}_N \in R^{n-1}$,
and thus that there is no harmonic form of the form $\psi \text{vol}_N + u'$. To take a specific example, consider a capped half-cylinder with end $(0, \infty) \times S^1$; this says that there is no harmonic covector field approaching $d\theta$. On the other hand, on the whole cylinder $\mathbb{R} \times S^1$ there clearly is such a field — but there the volume form more accurately corresponds to the pair $(d\theta, -d\theta)$ because of the orientation, and there is no harmonic covector field that reverses its direction.

Theorem 3.4.4 is essentially the proof of exactness at $\mathcal{H}^k(N)$ in our interpretation of Theorem 2.1.2 in terms of harmonic forms, which we now give; this of course includes the isomorphisms between $H^k_{\text{abs}}$ and $\mathcal{H}^k_{\text{abs}}$ and $H^k_{\text{rel}}$ and $\mathcal{H}^k_{\text{rel}}$. This is a reinterpretation of Proposition 6.18 of Melrose [61] in our slightly more general setting.

**Theorem 3.4.5.** Suppose that $M$ has asymptotically cylindrical ends. Then we can represent the exact sequence of Theorem 2.1.2 with harmonic forms in a natural way: there exists a commutative diagram

$$
\cdots \xrightarrow{f} H^k(N) \xrightarrow{g} \mathcal{H}^{k+1}_{\text{rel}}(M) \xrightarrow{h} H^k_{\text{rel}}(M) \xrightarrow{\pi_{\text{rel}}} H^k_{\text{rel}}(N) \xrightarrow{\text{bd}} H^k_{\text{rel}}(M) \xrightarrow{\pi_{\text{rel}}} H^k_{\text{rel}}(N) \xrightarrow{\text{bd}} \cdots
$$

(3.4.4)

where the arrows in the top row are the maps of Theorem 2.1.2, all the vertical maps are isomorphisms, the bottom row is exact, and $f$, $g$, and $h$ are defined by

$$
f : \alpha \mapsto \text{bd}\pi_{\text{rel}}\alpha, \quad g : dv + u' \mapsto u', \quad h : \psi\eta + u' \mapsto \eta, \quad (3.4.5)
$$

where we need to define $\text{bd}\pi_{\text{rel}}$: the other two maps are defined by the uniqueness of these decompositions (uniqueness of the exact representative in Theorem 3.4.4 and uniqueness of limit).

For the first of these, $\pi_{\text{rel}}$ is simply the orthogonal projection $\mathcal{H}^k(N) \to \mathcal{R}^k$. bd is defined by taking $\text{bd}\alpha$ to be the unique exact harmonic form with limit $dt \wedge \alpha$.

To prove this, we will need some way of obtaining relative cohomology, as thus far we have not used compactly supported forms. We thus require a technical lemma. Clearly, the forms in $e^{-ct}W_2^\infty \Omega^k(M)$ form a complex under exterior differentiation, so we can obtain another cohomology from them. The lemma says that this cohomology is $H^k_{\text{rel}}(M)$.

**Lemma 3.4.6.** The natural inclusion of compactly supported $k$-forms into $e^{\delta t}W_2^\infty \Omega^k(M)$ induces an isomorphism on cohomology, for $\delta < 0$.

This can be proved identically to the version of Melrose [61, Proposition 6.13].

The following lemma is related to Lemma 3.4.6. We give a proof, as this lemma will be used throughout the thesis.
Lemma 3.4.7. Suppose that \( u \) is an exponentially decaying (with all derivatives) closed form on the end \( N \times (0, \infty) \) of an asymptotically cylindrical manifold \( M \) with cross-section \( N \). Then there is a form \( v \) on \( N \times (0, \infty) \) such that \( dv = u|_{N \times (0, \infty)} \).

Proof. Suppose that
\[
u = u_1 + dt \wedge u_2,
\]
where \( u_1 \) and \( u_2 \) may depend on \( t \) but have no \( dt \) part. They are also exponentially decaying, because the pointwise ratio \( \frac{u_1, dt \wedge u_2}{|u_1| |dt \wedge u_2|} \) converges exponentially to zero (as a consequence of the convergence of the metric) and so we have for large enough \( t \) that \( |u|^2 \geq \frac{1}{2} (|u_1|^2 + |dt \wedge u_2|^2) \). Thus we may consider the integral
\[
v = \int_t^\infty -u_2 \, dt'.
\]
The integral converges, as \( u_2 \) decays exponentially, to define a \((k-1)\)-form. Moreover,
\[
dv = dt \wedge u_2 + \int_t^\infty -d_N u_2 \, dt',
\]
where \( d_N \) is the exterior differential on the cross-section. Since \( u \) is closed, \( d_N u_2 = \frac{\partial u_1}{\partial t} \) and so we have the result.

Remark. Using Lemma 3.4.6, we can also give a topological proof of Lemma 3.4.7. An exponentially decaying form \( u \) represents a class in \( H^k_{rel}( (0, \infty) \times N_i \) relative to \( (3, \infty) \times N_i \). But by excision as described after Theorem 2.1.2, this is the cohomology of the space given by identifying \( (3, \infty) \times N_i \) to a point (except in degree zero, which we may exclude as no constant decays exponentially); as this is contractible, it follows that \( H^k_{rel}( (0, \infty) \times N ) \) is trivial. Thus \( u \) defines the trivial class in it, and hence is the differential of something decaying exponentially.

We will also need to be slightly more careful with \( H^k_{abs}(M) \). We are now considering spaces of the form \( e^{\delta t} W^2_{\infty} \Omega^k(M) \) and for each \( \delta \) there are smooth forms growing fast enough not to be in this space. The corresponding lemma is

Lemma 3.4.8. The natural inclusion of \( e^{\delta t} W^2_{\infty} \Omega^k(M) \) into the smooth \( k \)-forms \( C^\infty \Omega^k(M) \) induces an isomorphism on cohomology, for \( \delta > 0 \).

This is the other half of Melrose’s corresponding result [61, Proposition 6.13]; again, the proof goes through as in that case as the slightly different analytic setup is not relevant here.

The case of \( \delta = 0 \) is much nastier and in general this cohomology is infinite-dimensional.

Proof of Theorem 3.4.5. It is obvious from Theorem 3.4.4 that the bottom row is exact. It remains to prove the vertical maps are isomorphisms and the diagram commutes. The case of forms on \( N \) is done by Corollary 3.3.2.
We consider first $H^k_{\text{abs}}$ and $H^k_{\text{rel}}$. Suppose $u \in H^k_{\text{abs}}$; then $u$ is an asymptotically translation invariant harmonic form so closed by Proposition 3.4.2. Thus an element of $H^k_{\text{rel}}$ represents a class in $H^k_{\text{abs}}$. To define an inverse to this map, we want to write a closed form as a harmonic form of the required kind plus an exact form, and we appeal to the existence result for $\Delta$ on asymptotically cylindrical manifolds stated as Theorem 3.3.3. Given a cohomology class represented by the $k$-form $v$, by Lemma 3.4.8 we may suppose that $v \in e^{\epsilon t} W^2_{\infty} \Omega^k(M)$. By Proposition 3.3.3, we thus have

$$v = \Delta u + h,$$

(3.4.9)

for some $u \in e^{\epsilon t} W^2_{\infty} \Omega^k(M)$ and $h \in e^{-\epsilon t} W^2_{\infty} \Omega^k(M)$ harmonic so in particular closed and coclosed. We consider $w = du$. It follows as $v$ is closed that $w$ is harmonic. By Theorem 3.4.4, $h + \delta w = h + \delta du$ is in $H^k_{\text{abs}}$. We now check that this map is inverse to the map $H^k_{\text{rel}} \to H^k_{\text{abs}}$, as the representative of this cohomology class, assuming for the moment it is in $H^k_{\text{abs}}$. As the difference $v - h - \delta w = d\delta u$ is exact, the harmonic representative of a cohomology class does indeed lie in that cohomology class, and we have the identity for one composition. On the other hand, if $v$ is itself harmonic and of the form $\psi \eta + u'$, it is coclosed by Proposition 3.4.2; hence $v - h - \delta w$ is exact, coclosed, and asymptotically translation invariant so we may integrate by parts to infer it is zero, meaning the other composition is also the identity.

We represent $H^k_{\text{rel}}$ by the cohomology of the complex of decaying forms, as justified by Lemma 3.4.6. Given $dv + u' = \psi(dt \wedge \theta) + u'' \in H^k_{\text{rel}}$, we map it to the class of the clearly decaying form $u' + \psi' dt \wedge \theta$. We shall appeal to the five lemma to prove that this map is an isomorphism; we already know that the rows of the diagram are exact and that the remaining vertical maps are isomorphisms, so it remains to prove the diagram commutes, as this implies that this is an isomorphism by the five lemma. We deal with each square in turn.

Firstly, given $\alpha \in \mathcal{H}^k(N)$, write $\alpha = \eta + \theta$ for $\eta \in \mathcal{A}^k$ and $\theta \in \mathcal{R}^k$. We have to show that as classes of relative cohomology $[\psi' dt \wedge \alpha] = [\psi' dt \wedge \theta]$, i.e. that $\psi' dt \wedge \eta$ is the differential of a decaying form. But $\eta \in \mathcal{A}^k$ and so there is a harmonic, thus closed, $u = \psi \eta + u'$ with $u$ decaying; then $d(-u') = \psi' dt \wedge \eta$, as desired.

Secondly, given $dv + u' = \psi(dt \wedge \theta) + u'' \in H^k_{\text{rel}}$, we have to show $[u' + \psi' dt \wedge \theta] = [u']$ as classes of absolute cohomology, but as $\psi' dt \wedge \theta = d(\psi \theta)$ this is obvious.

Thirdly and finally, we have to show that $\eta$ is the harmonic representative of the cohomology class given by pulling back $\psi \eta + u'$ to a cross-section. We apply Lemma 3.4.7. By that lemma, $u'|_{N \times (0, \infty)}$ is an exact form on $N \times (0, \infty)$. Consequently, the pullback of $\psi \eta + u'$ to any cross-section represents the same cohomology class as the pullback of $\psi \eta$. By taking a cross-section far enough along this is $[\eta]$, and thus $\eta$ is its harmonic representative.

Thus the diagram does commute, as desired, and so also the map $H^k_{\text{rel}} \to H^k_{\text{rel}}$ is an isomorphism. □
3.5 Gluing of harmonic forms

In this section, we apply the perturbative gluing strategy described in chapter 1 to the Laplace–Beltrami operator. We first make preliminary definitions of approximate gluing or patching for manifolds and forms, and in Definition 3.5.9 we pass to a gluing map of harmonic forms. We then use the Mayer–Vietoris sequence to show that the restriction map is inverse to this in Theorem 3.5.10 and give some corollaries. In subsection 3.5.2, we give a definition of patching for metrics and then show in Theorem 3.5.14 that with this metric (or something close to it) and for \( \alpha \) orthogonal to harmonics, \( \|d\alpha\| + \|d^*\alpha\| \geq CT^l\|\alpha\| \) for appropriate constants \( C \) and \( l \); this is very similar to bounding the first eigenvalue below (if we stated it in terms of \( L^2 \) norms this would be basically equivalent). This enables us to estimate how close things are to harmonic by estimating their Laplacian. In subsection 3.5.3, we do this, and show (Proposition 3.5.18) that the image of the gluing map becomes orthogonal to the space of harmonic forms we had identified as a complement as \( T \to \infty \). This material of this section will be useful for the corresponding gluing maps in chapter 5, for which it forms a model example.

3.5.1 Definitions and cohomological results

First, we need to patch together manifolds with ends; in order to do this, we need to know which manifolds we can patch. We make

**Definition 3.5.1.** Suppose that \( M_1 \) and \( M_2 \) are manifolds with ends, with corresponding cross-sections \( N_1 \) and \( N_2 \). \( M_1 \) and \( M_2 \) are said to match if we have an orientation-reversing diffeomorphism \( F : N_1 \to N_2 \). Such an \( F \) induces a further orientation-preserving map

\[
F : N_1 \times (0, 1) \to N_2 \times (0, 1),
\]

\[
(n, t) \mapsto (F(n), 1 - t). \tag{3.5.1}
\]

Given a pair of matching manifolds with ends \((M_1, M_2)\), we can thus make

**Definition 3.5.2.** Suppose \( M_1 \) and \( M_2 \) are matching manifolds with ends as in Definition 3.5.1. Fix \( T > 1 \), a “gluing parameter”. In practice \( T \) will be taken large enough to provide various analytic estimates. Let

\[
M_1 \supset M_1^{\text{tr}(T+1)} := M_1^{\text{cpt}} \cup N_1 \times (0, T + 1), \tag{3.5.2}
\]

and define \( M_2^{\text{tr}(T+1)} \) similarly. As in Definition 3.5.1, \( F \) defines an orientation-preserving diffeomorphism between \( N_1 \times (T, T + 1) \) and \( N_2 \times (T, T + 1) \). Then we consider

\[
M^T = \frac{M_1^{\text{tr}(T+1)} \cup M_2^{\text{tr}(T+1)}}{F}, \tag{3.5.3}
\]
the identification of these two manifolds by $F$. $M^T$ is a closed and oriented manifold.

Note that a subset of $M^T$ is parametrised by $(-T - \frac{1}{2}, T + \frac{1}{2}) \times N$, with $(-\frac{1}{2}, \frac{1}{2}) \times N$ corresponding to the identification region; we shall call this subset the neck of $M^T$. We shall use the subsets $M^T_{i'}$ for varying $T'$. Evidently, $M^T_{i'}$ is always a subset of $M_i$ and defines a subset of $M^T$ if $T' \leq T + 1$.

We need to observe how $M^T$ depends on our parameters $F$ and $T$. Beginning with $F$, we note that if $F_1$ and $F_2$ are isotopic then the manifolds $M^T$ constructed using them are diffeomorphic; however, there is no natural isotopy class for $F$. Thus our definition depends on the isotopy class of $F$, which is essentially an arbitrary choice: we shall ignore this choice for the time being, though it will re-emerge later in Propositions 4.3.24 and 4.3.33. We may suppress $F$ and just write that the two cross-sections $N_1$ and $N_2$ are equal, therefore.

For the parameter $T$, we note that all of the manifolds $M^T$ are diffeomorphic (by compression and extension of the neck). $T$ dependence will be important later in Definition 3.5.5, and it is for this reason that we make the parameter $T$ clear in the notation $M^T$.

Applying the Mayer–Vietoris sequence (see Bott & Tu [7, p.23 and p.26]) to the subsets $M^T_{i'(T+1)}$ of $M^T$, whose intersection $(-\frac{1}{2}, \frac{1}{2}) \times N$ is homotopy equivalent to $N$, we obtain

**Theorem 3.5.3.** There exists an exact sequence

$$\cdots \to H^k(M^T) \to H^k_{\text{abs}}(M_1) \oplus H^k_{\text{abs}}(M_2) \to H^k(N) \to H^{k+1}(M^T) \to \cdots,$$

(3.5.4)

where the maps between spaces of the same degree are given by restriction and pullbacks under appropriate smooth maps giving diffeomorphisms or homotopy equivalences. The coboundary map is given by $\partial(\alpha) = [d\phi \wedge \alpha]$ where $\phi$ is some function on $M^T$ with $\phi(t) = 1$ for $t > \frac{1}{2}$ and $\phi(t) = 0$ for $t < -\frac{1}{2}$ (with respect to the parametrisation of the neck).

We now turn to approximately gluing differential forms so that we can apply the gluing strategy described in the introduction to the Laplace–Beltrami operator. This is done by combining Theorem 3.4.5 on representing cohomology with harmonic forms with the Mayer–Vietoris sequence of Theorem 3.5.3. First we need to define approximate gluing of differential forms, and we begin with a matching condition. Note that here there is no need for us to have a metric on $M^T$, though we do need asymptotically cylindrical metrics on $M_1$ and $M_2$ to define asymptotically translation invariant forms.

**Definition 3.5.4.** Let $M_1$ and $M_2$ be matching manifolds with ends as in Definition 3.5.1, and let $q_1$ and $q_2$ be asymptotically cylindrical metrics on them. Suppose that $\alpha_1$ and $\alpha_2$ are asymptotically translation invariant $p$-forms on $M_1$ and $M_2$ respectively. The diffeomorphism $F$ (extended as in Definition 3.5.1) induces a pullback map $F^*$ from the limiting bundle $\bigwedge^p T^* M_2|_{N_2}$ to $\bigwedge^p T^* M_1|_{N_1}$. $\alpha_1$ and $\alpha_2$ are said to match if $F^* \hat{\alpha}_2 = \hat{\alpha}_1$. 

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We can now immediately approximately glue matching forms by cutting them off to the common limit $\tilde{\alpha}$. In practice, we shall not use this definition. Instead we shall assume our forms are closed and maintain closedness, which will be very useful for instance for Theorem 3.5.10. We use Lemma 3.4.7 to make

**Definition 3.5.5.** Suppose that $M_1$ and $M_2$ are matching manifolds with ends in the sense of Definition 3.5.1, with identification $F$, and with asymptotically cylindrical metrics $g_1$ and $g_2$. Let $T$ and $M^T$ be as in Definition 3.5.2. Suppose that $\alpha_1$ and $\alpha_2$ are a pair of matching differential forms in the sense of Definition 3.5.4, and that $\alpha_1$ and $\alpha_2$ are both closed.

Because we have convergence with all derivatives, the limits $\tilde{\alpha}_i$ are closed on $N_i$, and hence, treated as constants, on the end of $M_i$. By Lemma 3.4.7 we then have that $\alpha_i - \tilde{\alpha}_i$ is exact on the end, and so can be written as $d\eta_i$ there. Let $\psi_T$ be as in equation (2.1.1) and define

$$\hat{\alpha}_i = \alpha_i - d(\psi_T \eta_i)$$

(3.5.5)
on the end, and $\alpha_i$ off the end. On the overlap of $M_1^{tr(T+1)}$ and $M_2^{tr(T+1)}$, $\hat{\alpha}_i = \tilde{\alpha}_i$, and so the two forms are identified by $F$. Thus they define a form $\gamma_T(\alpha_1, \alpha_2)$ on $M^T$, and this form is closed because the $\hat{\alpha}_i$ are closed.

**Remark 3.5.6.** Note that if we vary the cutoff function, the corresponding $\tilde{\alpha}_i$ differ by a compactly supported form near $T$ that is the differential of a compactly supported form. Hence, two gluings with different cutoff function define the same cohomology class $[\gamma_T(\alpha_1, \alpha_2)]$, and since $\alpha_i$ decays exponentially, two gluings with different cutoff function must also be exponentially close together (with respect to any norm).

Note that $T$ dependence is important here. Whilst there are natural diffeomorphisms between $M^T$ and $M^{T+1}$, it is not clear how these diffeomorphisms would interact with $\gamma_T(\alpha_1, \alpha_2)$ and $\gamma_{T+1}(\alpha_1, \alpha_2)$. We could choose things carefully and apply appropriate diffeomorphisms to effectively vary $T$ for forms without actually varying $T$ for the manifold, but to do so would make the notation more complex for no practical gain.

We now specialise to harmonic forms. We suppose that $M_1$ and $M_2$ are asymptotically cylindrical, and $M^T$ has some metric. We will need to know that harmonic forms match if and only if their corresponding cohomology classes match. Hence, we make the following definition of matching asymptotically cylindrical manifolds.

**Definition 3.5.7.** Let $M_1$ and $M_2$ be matching manifolds with ends as in Definition 3.5.1, and let $g_1$ and $g_2$ be asymptotically cylindrical metrics on $M_1$ and $M_2$ respectively. Since $g_i$ is asymptotically cylindrical, there are metrics $g_{N_i}$ on $N_i$ such that $g_i$ decays exponentially to $g_{N_i} + dt \otimes dt$. $g_1$ and $g_2$ are said to match if $F^* g_{N_2} = g_{N_1}$.

We shall assume that $M_1$ and $M_2$ have matching metrics, and we now start our gluing analysis by considering what a pair of matching harmonic forms $\alpha_1$ on $M_1$ and $\alpha_2$ on $M_2$ looks
like. By assumption these must be asymptotically translation invariant, and so using Proposition 3.4.1 we can split them up as

$$\alpha_i = \psi(\eta_i + dt \wedge \theta_i) + \alpha'_i.$$  

(3.5.6)

The matching condition of Definition 3.5.4 then becomes $F^*\eta_2 = \eta_1$ and $F^*\theta_2 = -\theta_1$.

We shall further restrict to only gluing forms in $\mathcal{H}_{\text{abs}}^k$, i.e. assuming $\theta_i = 0$. Complementary analysis applies in the case of $\mathcal{H}_{\text{rel}}^k$: for details, see Nordström [66, Theorem 3.1]. Then a gluing map is a map from the subset $\mathcal{D}$ of $\mathcal{H}_{\text{abs}}^k(M_1) \oplus \mathcal{H}_{\text{abs}}^k(M_2)$ of matching forms to $\mathcal{H}^k(M^T)$. But the Mayer-Vietoris sequence gives a map

$$r : \mathcal{H}^k(M^T) \to \mathcal{H}_{\text{abs}}^k(M_1) \oplus \mathcal{H}_{\text{abs}}^k(M_2),$$  

(3.5.7)

so it is of interest whether this is related to our gluing maps.

We begin with a simple proposition on $\mathcal{D}$.

**Proposition 3.5.8.**

$$\mathcal{D} = \text{im } r.$$  

(3.5.8)

*Proof.* By definition $\mathcal{D}$ is the subspace of pairs of harmonic forms in $\mathcal{H}_{\text{abs}}^k(M_1) \oplus \mathcal{H}_{\text{abs}}^k(M_2)$ with $F^*\eta_2 = \eta_1$. Since the restriction map $\mathcal{H}^k(M_1) \to \mathcal{H}^k(N_1)$ is given by the operation $\psi \eta + u' \mapsto \eta$, $\mathcal{D}$ is the kernel of the map $\mathcal{H}_{\text{abs}}^k(M_1) \oplus \mathcal{H}_{\text{abs}}^k(M_2) \to \mathcal{H}^k(N)$ in the Mayer-Vietoris sequence. The result follows immediately by exactness.

Thus, for any gluing map of harmonic forms $g$, it makes sense to consider the compositions $rg : \mathcal{D} \to \mathcal{D}$ and $gr : \mathcal{H}^k(M^T) \to \mathcal{H}^k(M^T)$. If $H^{k-1}(M^T)$ is nontrivial, then $r$ has nontrivial kernel, so $gr$ is certainly not the identity: but it might be the identity on a suitable complement.

Since, by the Hodge decomposition theorem (Theorem 3.3.1), we know that there is a well-defined projection to harmonic forms on the compact Riemannian manifold $M^T$, to define such a gluing map it suffices to take a gluing map of a subspace of matching translation invariant forms containing the harmonic forms and then apply this projection.

**Definition 3.5.9.** Suppose that $M_1$ and $M_2$ are a pair of asymptotically cylindrical manifolds matching in the sense of Definition 3.5.1, with metrics matching in the sense of Definition 3.5.7, and that we equip $M^T$ with some Riemannian metric.

Suppose that $\alpha_1$ and $\alpha_2$ are matching bounded harmonic forms. By Proposition 3.4.2 $\alpha_1$ and $\alpha_2$ are closed, and so we can form the closed form $\gamma_T(\alpha_1, \alpha_2)$ using Definition 3.5.5. Using the Hodge decomposition theorem we may take the harmonic part of $\gamma_T(\alpha_1, \alpha_2)$ to form the harmonic form $\Gamma_T(\alpha_1, \alpha_2)$.

As $\gamma_T(\alpha_1, \alpha_2)$ is closed, $\Gamma_T(\alpha_1, \alpha_2) - \gamma_T(\alpha_1, \alpha_2)$ is exact by the Hodge Decomposition Theorem. This enables a topological proof of
Theorem 3.5.10. Let $M_1$ and $M_2$ be a matching pair of asymptotically cylindrical manifolds as in Definition 3.5.7. Equip the glued manifold $M^T$ with any Riemannian metric. Let $r$ be the restriction map $\mathcal{H}^k(M) \to \mathcal{D}$ given by the Mayer–Vietoris sequence and Theorem 3.4.5 and $\Gamma_T$ be as in Definition 3.5.9. $r\Gamma_T$ is the identity map on $\mathcal{D}$.

Proof. We consider $\mathcal{D}$ as a subspace of $H^k_{\text{abs}}(M_1) \oplus H^k_{\text{abs}}(M_2)$ and prove this in terms of the maps $\Gamma_T : \mathcal{D} \to H^k(M^T)$ and $r : H^k(M^T) \to \mathcal{D}$. As $\Gamma_T(\alpha_1, \alpha_2)$ is in the same cohomology class as $\gamma_T(\alpha_1, \alpha_2)$ it then suffices to prove that $r\gamma_T$ is the identity map. Directly applying $\gamma_T$ and $r$, we obtain the forms $\hat{\alpha}_i|_{M^\text{tr}(T+1)}$ (from Definition 3.5.5) on $M^\text{tr}(T+1)$. We want to show that under the relevant diffeomorphism between $M_i$ and $M^\text{tr}(T+1)$, these give the same cohomology classes as $\alpha_i$ on $M_i$. However, the relevant diffeomorphism has the inclusion of $M^\text{tr}(T+1)$ into $M_i$ as a homotopy inverse; hence $\hat{\alpha}_i|_{M^\text{tr}(T+1)}$ defines the same cohomology class as $\hat{\alpha}_i$ on $M_i$. It is thus sufficient that $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are in the same cohomology classes as $\alpha_1$ and $\alpha_2$, which is true by construction.

We now obtain by straightforward linear algebra

Corollary 3.5.11. $\Gamma_T r$ is the identity $\im \Gamma_T \to \im \Gamma_T$; we thus have a direct sum decomposition $\mathcal{H}^k(M^T) = \im \Gamma_T \oplus \ker r$.

In particular, in the case of one-forms where $N$ is connected we obtain

Corollary 3.5.12. If $M_1$, $M_2$ and $M^T$ are as in Theorem 3.5.10 and the cross-section $N$ of $M_1$ and $M_2$ is connected, the gluing map $\Gamma_T$ applied to one-forms is an isomorphism.

Proof. We know that $\Gamma_T$ is injective, so we just have to show that it is surjective; by Corollary 3.5.11 this is equivalent to showing that $\ker r$ is the zero space. But on one-forms the relevant part of the Mayer–Vietoris sequence becomes

$$
\cdots \to H^0(M_1) \oplus H^0(M_2) \to H^0(N) \to H^1(M) \to H^1(M_1) \oplus H^1(M_2) \to \cdots. \tag{3.5.9}
$$

If $N$ is connected, the map $H^0(M_1) \oplus H^0(M_2) \to H^0(N)$ is surjective. By exactness, the map $H^0(N) \to H^1(M)$ is the zero map, and so by exactness again $H^1(M) \to H^1(M_1) \oplus H^1(M_2)$ is injective. But this is $r$, so $\ker r$ is zero as required.

3.5.2 The Laplacian on a glued manifold

We now turn to results that depend on the metric on $M^T$. We define a patching of matching asymptotically cylindrical metrics and then prove a lower bound result for the Laplace–Beltrami operator, which is Theorem 3.5.14.

There is of course a natural metric on $M^T$ if asymptotically cylindrical metrics match, defined as follows.
Definition 3.5.13. Suppose that $g_1$ and $g_2$ are matching metrics on the matching pair of manifolds $(M_1, M_2)$, with limits $\tilde{g}_i = dt \otimes dt + g_{N_i}$.

Form the metrics on $M_1$ and $M_2$ defined by $\hat{g}_i = g_i + \psi_T(\tilde{g}_i - g_i)$, where as in (2.1.1) the smooth function $\psi_T$ is zero for $t < T - 1$ and one for $t \geq T$. These metrics can be restricted to $M_1^{T(T+1)}$ and $M_2^{T(T+1)}$ and then included in the obvious way into $M^T$, where they agree in the subset $(t, y) \in (-\frac{1}{2}, \frac{1}{2}) \times N$ which we identify. Thus we can combine them to form the metric $g$ on $M^T$.

Since this is the natural metric on $M^T$, we do not necessarily give it an indication of the value of $T$. Occasionally, however, it will be necessary to do so, and in these cases we write the glued metric as $g^T$.

Before continuing with our study of glued harmonic forms, we find a bound for the inverse of the Laplacian on $(M^T, g^T)$ which we will use in our further study. We will prove

Theorem 3.5.14. Let $M^T$ be a compact manifold obtained by approximately gluing two asymptotically cylindrical manifolds with ends $M_1$ and $M_2$ as in Definition 3.5.2. Suppose that it has a Riemannian metric $g$, which is within a uniform bound of the Riemannian metric $g^T$ given by gluing the asymptotically cylindrical metrics $g_1$ and $g_2$ as in Definition 3.5.13.

Let $k$ be a positive integer and $\mu \in (0, 1)$. Then if $\alpha$ is orthogonal to harmonic forms on $M^T$, then

$$\|d\alpha\|_{C^{k,\mu}} + \|d^*\alpha\|_{C^{k,\mu}} \geq C T^l \|\alpha\|_{C^{k+1,\mu}},$$

(3.5.10)

for some positive constants $C$ and $l$, which may both depend on the geometry of $M_1$ and $M_2$; $C$ may also depend on the regularity $k + \mu$.

This is fundamentally the composition of Chanillo–Treves’ result [12, Theorem 1.1] which we state as Theorem 3.5.16 below with the Schauder estimate of Theorem 3.2.4 and a suitable $L^2$ estimate. The method of Chanillo–Treves rests on the notion of an admissible covering of a Riemannian manifold.

Definition 3.5.15. Let $(M, g)$ be a Riemannian manifold. A covering of $M$ by open sets $U_1, \ldots, U_N$ is admissible if there are two positive numbers $A$ and $r$ (with $r$ substantially less than both 1 and the injectivity radius of $M$) such that

i) For each $i$, there exists $p_i \in U_i$ such that $U_i$ is the image under the exponential map of a ball of radius $r$ about 0 in $T_{p_i}M$.

ii) The exponential map is a diffeomorphism on the ball of radius $4^n r$ about 0 in $T_{p_i}M$.

iii) The derivative of the exponential map throughout the balls of radius $4^n r$ is bounded above by $A$ and below by $\frac{1}{A}$.

Then Chanillo and Treves prove
Theorem 3.5.16 ([12, Theorem 1.1]). Suppose there exists an admissible covering \( \{ U_1, \ldots, U_N \} \) of \( M \) with parameters \( N, r \) and \( A \). Then there is a constant \( C > 0 \) depending on the dimension \( n \) and \( A \) such that for every \( d \)-exact \( p \)-form \( \alpha \), we can find a \((p-1)\)-form \( \beta \) such that \( \partial \beta = \alpha \) and

\[
\| \beta \|_{L^2} \leq C r N^{4p} \| \alpha \|_{L^2}. \tag{3.5.11}
\]

In particular, (3.5.11) is true for the coexact \( \beta \) satisfying \( \partial \beta = \alpha \), as this is \( L^2 \)-minimal.

Since the Hodge star is an isometry, we obtain a corresponding result for \( \partial^* \), which is [12, Theorem 1.2]. It follows immediately that we obtain

\[
\| \alpha \|_{L^2} \leq C N^l (\| \partial \alpha \|_{L^2} + \| \partial^* \alpha \|_{L^2}), \tag{3.5.12}
\]

for some positive \( l \) and \( C \), whenever \( \alpha \) is orthoharmonic, since we certainly have that if \( \alpha = \beta + \gamma \) with \( \beta \) and \( \gamma \) the coexact and exact parts respectively, then \( \| \alpha \|_{L^2} \leq \| \beta \|_{L^2} + \| \gamma \|_{L^2} \).

We may now complete the proof of Theorem 3.5.14.

Proof of Theorem 3.5.14. Since \( M^T \) has a metric that is close to the glued metric, and using compactness of the cross-section, we may choose an admissible covering in the sense of Definition 3.5.15 with \( r \) and \( A \) depending uniformly on \( T \) and \( N \) growing linearly with \( T \). Moreover we may suppose that the exponential map is bounded with all derivatives uniformly on these open sets.

We now apply standard elliptic regularity theory to \( \partial + \partial^* \). As a map from \( k \)-forms to \((k+1)\)-forms and \((k-1)\)-forms, \( \partial + \partial^* \) is generally not an elliptic operator in the sense of section 3.2, because it does not map between bundles of the same dimension. Hence, we work with the composition with its adjoint, which is precisely the Laplace–Beltrami operator \( \Delta \). We note the interior Schauder estimate, Theorem 3.2.4, applies to \( \Delta \) in any set of coordinates.

In particular, therefore, for a sufficiently small ball \( U \) in our admissible covering, we have

\[
\| u \|_{C^{k,\mu}(U)} \leq C (\| \Delta u \|_{C^{k-2,\mu}(M)} + \| u \|_{C^0(M)}), \tag{3.5.13}
\]

with \( C \) depending only on the bound on the exponential map and its derivatives.

On the other hand, we have

\[
\| \Delta u \|_{C^{k,\mu}(M)} \leq \| \partial^* du \|_{C^{k-2,\mu}(M)} + \| dd^* u \|_{C^{k,\mu}(M)} \leq C (\| du \|_{C^{k-1,\mu}(M)} + \| d^* u \|_{C^{k-1,\mu}(M)}), \tag{3.5.14}
\]

for some constant \( C \) independent of the metric, as we can express \( du \) as a sum of appropriate covariant derivatives (e.g. [26, Lemma 2.61(i)]), and the Hodge star is an isometry.

If we have \( N \) balls in the admissible cover, combining (3.5.13) and (3.5.14) on each using a
partition of unity gives the estimate
\[ \| u \|_{C^{k,\mu}(M)} \leq CN(\| du \|_{C^{k-1,\mu}(M)} + \| d^* u \|_{C^{k-1,\mu}(M)} + \| u \|_{C^0(M)}). \]  
(3.5.15)

Suppose \( C \) is at least the required constant in (3.5.12) too, and at least one. Now by Proposition 3.2.5 and using the uniform equivalence of norms arising from the estimate on derivatives of exponential maps, there exists a constant \( K \), which we may assume at least one, such that for a sufficiently small ball \( U \) in the covering we have
\[ \| u \|_{C^0(U)} \leq \frac{1}{2CN} \| u \|_{C^1(M)} + K \| u \|_{L^2(M)} . \]  
(3.5.16)

Note that \( K \) can be chosen independent of \( U \) because the derivatives are controlled uniformly and the balls can all be chosen to be the same size. Hence, taking a maximum, we in particular have
\[ \| u \|_{C^0(M)} \leq \frac{1}{2CN} \| u \|_{C^1(M)} + K \| u \|_{L^2(M)} . \]  
(3.5.17)

Combining this with (3.5.12), we obtain
\[ \| u \|_{C^0(M)} \leq 2KCN^{l+1}(\| du \|_{C^{l-1,\mu}(M)} + \| d^* u \|_{C^{l-1,\mu}(M)} + \| du \|_{L^2(M)} + \| d^* u \|_{L^2(M)}). \]  
(3.5.18)

We can then combine this with (3.5.15) to get
\[ \| u \|_{C^{k,\mu}(M)} \leq 2KCN^{l+1}(\| du \|_{C^{k-1,\mu}(M)} + \| d^* u \|_{C^{k-1,\mu}(M)} + \| du \|_{L^2(M)} + \| d^* u \|_{L^2(M)}). \]  
(3.5.19)

As the volume is bounded linearly in \( N \), and \( N \) is bounded linearly in \( T \), this immediately yields the required estimate.

Finally we note that we have the following

**Corollary 3.5.17.** Let \( M^T \) be a compact manifold obtained by gluing two asymptotically cylindrical manifolds with ends \( M_1 \) and \( M_2 \) as in Definition 3.5.2. Suppose that it has a Riemannian metric \( g \), which is within a uniform bound of the Riemannian metric \( g^T \) given by gluing the asymptotically cylindrical metrics \( g_1 \) and \( g_2 \) as in Definition 3.5.13.

Let \( k \) be a positive integer and \( \mu \in (0,1) \). Then for every form \( \alpha \), there exist constants \( C \) and \( l \) such that
\[ \| hpt \alpha \|_{C^{k,\mu}} \leq CT^l \| \alpha \|_{C^{k,\mu}} . \]  
(3.5.20)

These constants may depend on the geometry of \( M_1 \) and \( M_2 \), and \( C \) may also depend on the regularity \( k + \mu \).

**Proof.** By the triangle inequality, it suffices to prove that there exist \( C' \) and \( l' \) such that \( \beta = \alpha - hpt \alpha \) satisfies
\[ \| \beta \|_{C^{k,\mu}} \leq C'T^{l'} \| \alpha \|_{C^{k,\mu}} . \]  
(3.5.21)
Theorem 3.5.14 provides such a bound given a bound
\[\|d\alpha\|_{C^{k-1,\mu}} + \|d^*\alpha\|_{C^{k-1,\mu}} \leq C''T'\|\alpha\|_{C^{k,\mu}},\] (3.5.22)
and such a bound is immediate (indeed \(l'' = 0\)).

**Remarks.** As in [12, Theorem 1.3], Theorem 3.5.16 can be extended to the Laplacian, using that
\[\|\Delta \alpha\|_{L^2}^2 = \|dd^*\alpha\|_{L^2}^2 + \|d^*d\alpha\|_{L^2}^2,\] (3.5.23)
and hence to give an \(L^2\) lower bound for \(\Delta \alpha\) it is enough to lower bound \(d\alpha\) and \(d^*\alpha\) separately.

The remainder of the proof of Theorem 3.5.14 is independent of the particular (overdetermined) elliptic operator required and indeed is simpler for an elliptic operator such as \(\Delta\). Hence, the corresponding result holds.

\(d + d^*: \Omega^{p-1} \oplus \Omega^{p+1} \to \Omega^p\), as well as having problems with combining the exact and coexact parts, is underdetermined elliptic and therefore elliptic regularity does not hold for it. Hence, we cannot get a theorem analogous to Theorem 3.5.14 by these methods in that case.

### 3.5.3 Metric-dependent results

We now turn to metric-dependent results on the gluing of harmonic forms: that is, we suppose \(M^T\) has the glued metric of Definition 3.5.13 or something close to it.

We have that the kernel of \(r\) is the image of the coboundary map \(\partial\) in the Mayer-Vietoris sequence, so can in principle be identified. However, this does not enable us to identify the range of \(\Gamma_T\) defined in Definition 3.5.9, as we know nothing a priori about how the direct sum decomposition of Corollary 3.5.11 arises.

Taking the glued metric \(g^T\) on \(M^T\), we can consider a limit as \(T \to \infty\). We shall show that in this limit the decomposition becomes orthogonal in an appropriate sense: thus if we consider a sequence of manifolds with necks of increasing length, we can hope that the term obtained by projecting onto the orthocomplement of the image of the coboundary map approaches something obtained by gluing. The result essentially relies on the fact that \(\Gamma_T\) and \(\gamma_T\) should be close.

We state

**Proposition 3.5.18.** Suppose that for every \(\alpha_1, \alpha_2\) we have
\[\|\Gamma_T(\alpha_1, \alpha_2) - \gamma_T(\alpha_1, \alpha_2)\|_{L^2\Omega^k(M^T)} \leq Ae^{-\epsilon T},\] (3.5.24)
where \(\gamma_T\) is as in Definition 3.5.5 and \(\Gamma_T\), as in Definition 3.5.9, is the orthogonal projection of the form constructed by \(\gamma_T\) to harmonic forms, for some \(A\) depending on the norms of \(\alpha_1\) and \(\alpha_2\) and some \(\epsilon\). Observe that \(H_{\text{abs}}^k(M) \oplus H_{\text{abs}}^k(N) \oplus H^{k-1}(N)\) is finite-dimensional. If we take
(α₁, α₂, αₕ) to be a triple in this space and consider

\[ \sup_{\|\alpha_1\| \leq 1, \|\alpha_2\| \leq 1, \|\alpha_ₜ\| \leq 1} \{ \langle \gamma_T(\alpha_1, \alpha_₂), \partial \alpha_ₜ \rangle \} \quad (3.5.25) \]

where these norms are the \( L^2 \) norms on the relevant space of forms, this is certainly finite, by linearity. Moreover, the set over which we take the supremum is independent of \( T \). The supremum (3.5.25) converges to 0 as \( T \to \infty \), proving that the two sets “approach \( L^2 \)-orthogonality”.

**Proof.** We have that \( \partial \alpha_ₜ \) is the harmonic representative of the cohomology class \( [d\phi \wedge \alpha_ₜ] \) where \( \phi \) is one for \( t > \frac{1}{2} \) and zero for \( t < -\frac{1}{2} \).

Thus, for any \( \alpha_1, \alpha_2 \) and \( \alpha_ₜ \),

\[ \langle \gamma_T(\alpha_1, \alpha_₂), \partial \alpha_ₜ \rangle = \langle \gamma_T(\alpha_1, \alpha_₂), d\phi \wedge \alpha_ₜ \rangle, \quad (3.5.26) \]

since the difference of \( \partial \omega \) and \( d\phi \wedge \omega \) is exact and so orthogonal to the harmonic form \( \gamma_T(\alpha_1, \alpha_₂) \).

But also, by direct computation using the fact that \( \Gamma(\alpha_1, \alpha_₂) \) is simply given by \( \eta \) in this region, we also have that \( \langle \Gamma(\alpha_1, \alpha_₂), d\phi \wedge \alpha_ₜ \rangle = 0 \). Therefore,

\[ \langle \gamma(\alpha_1, \alpha_₂), \partial \alpha_ₜ \rangle = \langle \gamma(\alpha_1, \alpha_₂) - \Gamma(\alpha_1, \alpha_₂), d\phi \wedge \alpha_ₜ \rangle \]

\[ \leq \| \gamma(\alpha_1, \alpha_₂) - \Gamma(\alpha_1, \alpha_₂) \| \| d\phi \wedge \alpha_ₜ \|. \quad (3.5.27) \]

By our choice of \( \phi \) this second term is independent of \( T \); by the hypothesis on the decay of the first term, the result follows.

It only remains to prove this hypothesis, that is to show that the part of \( \gamma_T(\alpha_1, \alpha_₂) \) orthogonal to harmonic forms decays exponentially in \( T \). We will use Theorem 3.5.14. This says, more or less, that the glued metric on \( M^T \) has a Laplacian that can be inverted with a constant only depending polynomially on \( T \). Hence, if \( \Delta \gamma_T(\alpha_1, \alpha_₂) \) decays exponentially, then so too must the part orthogonal to harmonic forms. It suffices by symmetry to consider \( \Delta \hat{\alpha}_₁ \). We know that this is zero for \( t < T - 1 \), as then \( \hat{\alpha}_₁ \) coincides with \( \alpha_₁ \) and the metric \( g \) on \( M^T \) coincides with \( g₁ \).

It remains to consider the neck. Clearly, \( \psi_T \) and its derivatives are bounded above independent of \( T \); this implies that the glued metric \( g^T \) of Definition 3.5.13 and \( \hat{\alpha}_₁ \) are exponentially close to \( \tilde{g} \) and \( \hat{\alpha} \) here. We know that \( \hat{\alpha} \) is \( \tilde{g} \)-harmonic, and so we see that \( \Delta \hat{\alpha}_₁ \) is exponentially small, as required.

We could now use similar arguments to show that any “gluing map of harmonic forms” (not necessarily using the special properties of closed forms and Lemma 3.4.7) is close to \( \Gamma_T \) and hence argue that every such gluing map is injective, and has image complementary to \( \text{im} \partial \), and also if we have family in \( T \) converging to \( \Gamma_T \) its image again approaches orthogonality.

Finally, we return to the case of one-forms, where we know that \( \Gamma_T \) is an isomorphism if the cross-section is connected by Corollary 3.5.12. We can also prove that \( \Gamma_T \) can be bounded
below independently of $T$.

**Proposition 3.5.19.** Let $M_1$ and $M_2$ be matching asymptotically cylindrical Riemannian manifolds. Suppose that the metric on $M^T$ is exponentially close in $T$ to that given by Definition 3.5.13. The gluing map $\Gamma_T$ of harmonic one-forms of Definition 3.5.9 is an isomorphism, if $N$ is connected. Moreover, we have an estimate

$$\|\Gamma_T(\alpha_1, \alpha_2)\|_{C^k} \geq C(\|\alpha_1\|_{C^k} + \|\alpha_2\|_{C^k}),$$

(3.5.28)

where $C$ is independent of $T$ and the norms on the right hand side are the extended weighted norms of Definition 2.2.7.

**Proof.** That this gluing map is an isomorphism follows from Corollary 3.5.12, so we only have to show the lower bound (3.5.28).

We show first that (3.5.28) holds when we replace $\Gamma_T$ with the approximate gluing map $\gamma_T$ of Definition 3.5.5. Since $\gamma_T(\alpha_1, \alpha_2)$ is just given by identifying the cutoffs $\hat{\alpha}_1$ and $\hat{\alpha}_2$ it suffices to show that there exists $C$ independent of $T$ such that

$$\|\hat{\alpha}_i\|_{C^k(M)} \geq C\|\alpha_i\|_{C^k(M)}.$$  

(3.5.29)

We argue using the restriction to $\hat{\alpha}_i$ to the neighbourhood $M_i^{tr^2}$ of $M_i^{cpt}$ and the limit $\tilde{\alpha}_i$ of $\hat{\alpha}_i$. If $T > 3$, then $\hat{\alpha}_i|_{M_i^{tr^2}} = \alpha_i|_{M_i^{tr^2}}$; for any $T$, $\tilde{\alpha}_i$ is again the limit of $\hat{\alpha}_i$. We thus obtain for every $T > 3$

$$\|\alpha_i|_{M_i^{tr^2}}\|_{C^k(M_i^{tr^2})} + \|\hat{\alpha}_i\|_{C^k(N)} \leq 2\|\hat{\alpha}_i\|_{C^k(M)}.$$  

(3.5.30)

Hence, it suffices to prove that there is a constant $C$ such that

$$\|\alpha_i\|_{C^k(M)} \leq C(\|\alpha_i|_{M_i^{tr^2}}\|_{C^k(M_i^{tr^2})} + \|\hat{\alpha}_i\|_{C^k(N)}).$$  

(3.5.31)

Since the space of harmonic one-forms is finite-dimensional and the map $\alpha_i \mapsto (\alpha_i|_{M_i^{tr^2}}, \hat{\alpha}_i)$ is linear, such a constant must exist provided that this map is injective. Suppose then that $\alpha_i|_{M_i^{tr^2}} = 0$ and $\hat{\alpha}_i = 0$. Examining the proof of Lemma 3.4.7, we see that, working over $(0, \infty) \times N$, $\alpha_i = dt$ for some decaying $\eta$ and moreover $\eta$ is translation invariant and closed on $(0, 2) \times N$, so defines a closed translation invariant form on the whole end. Subtracting this translation invariant form from $\eta$ and extending over $M^{cpt}$ by zero, we find that $\alpha_i$ is the differential of an asymptotically translation invariant form on $M$. In particular, $\alpha_i$ must define the trivial class in $H^1_{abs}(M_i)$. Using that $N$ is connected, we see from Theorem 3.4.5 that $\alpha_i$ must itself be zero (as $H^1_{rel}(M) \rightarrow H^1_{abs}(M)$ is injective), giving injectivity. This proves that (3.5.28) holds when we replace $\Gamma_T$ by $\gamma_T$.

Now $\Delta \gamma_T(\alpha_1, \alpha_2)$ is exponentially small as in the proof of Proposition 3.5.18 and using that the metrics are exponentially close, so by Theorem 3.5.14 so is the orthoharmonic part of
$\gamma_T(\alpha_1, \alpha_2)$. Hence $\Gamma_T(\alpha_1, \alpha_2)$ is exponentially close to $\gamma_T(\alpha_1, \alpha_2)$ and the result follows. □
Chapter 4

Calabi-Yau manifolds

We now turn to the first significant part of original work in the thesis, a study of asymptotically cylindrical Calabi–Yau threefolds using torsion-free \(G_2\) structures. In section 4.1, we recall the definitions of a Calabi–Yau and \(G_2\) structure and show that a suitably symmetric torsion-free \(G_2\) structure corresponds to a Calabi–Yau structure, leading to the complete description of the correspondence in Theorem 4.1.26. As a digression, in subsection 4.1.2 we show that Calabi–Yau structures exist on a large class of manifolds, so that the results of this chapter have a reasonable number of applications. We also give a slight extension of the uniqueness result (Proposition 4.1.15) corresponding to the existence result in the asymptotically cylindrical case; this extension will be useful in chapter 5.

In section 4.2, we turn to moduli spaces. We define a moduli space of \(S^1\)-invariant \(G_2\) structures in Definition 4.2.11, and show that the arguments of section 4.1 extend to give a bijection (Proposition 4.2.14) between it and the product of the moduli space of Calabi–Yau structures and an open subset of \(H^1(M \times S^1)\). By analysing the arguments of Hitchin [35] and Nordström [65] showing that the \(G_2\) moduli space is smooth, we then show that this moduli space is smooth in Theorem 4.2.29; in fact, it is locally diffeomorphic to an open subset of the full \(G_2\) moduli space. Using these together, we then argue that the \(SU(3)\) moduli space can be assumed to be smooth and its smooth product with the open subset of \(H^1(M \times S^1)\) is the \(S^1\)-invariant \(G_2\) moduli space in Theorem 4.2.39.

Finally, we deal with gluing in section 4.3. Firstly, we show (Theorem 4.3.3) that there is (indeed, are potentially multiple) sensible gluing maps defined on matching \(SU(3)\) structures. In subsection 4.3.3 we define moduli spaces of gluing data and show in Theorem 4.3.35 that the \(S^1\)-invariant \(G_2\) version decomposes into the product of the Calabi–Yau version and a cohomology space; we then define a gluing map between Calabi–Yau moduli spaces in Definition 4.3.42, and show it is well-defined in Proposition 4.3.44. Finally, we show in Theorem 4.3.46 that this gluing map is a local diffeomorphism.
4.1 Definitions and preliminary statements

$G$-structures for appropriate Lie groups $G \subseteq GL(n, \mathbb{R})$ and the relations between them implied by inclusions of different Lie groups are the fundamental arguments of our analysis. We will follow the definitions and in large part the approach of [34, section 2] when dealing with $SU(n)$ in subsection 4.1.1, and [40, section 10.1] when dealing with $G_2$ in subsection 4.1.3. Between these, in subsection 4.1.2, we follow Haskins–Hein–Nordström [32] in checking that asymptotically cylindrical Calabi–Yau manifolds do indeed exist, and improve their uniqueness result slightly. We then pass, in subsection 4.1.4, to the relationship between $SU(3)$ and $G_2$ structures, essentially following Chan [11].

In general, a $G$-structure on the smooth manifold $M$ is informally a smooth collection of inclusions

$$G \hookrightarrow GL_\mathbb{R}(T_p M)$$

as $p$ varies in $M$. More formally, it is a principal subbundle of the frame bundle with fibre $G$.

Rather than approaching these structures in this sense, using principal bundles, we will define the structures we want as stabilisers of finite sets of differential forms. This is not quite the same thing, because we can get the same principal bundle from different sets of differential forms, for example simply by rescaling. However, the set of forms induces a unique inclusion of $G$ (fixing the action of $GL(n, \mathbb{R})$ on the tangent space), and so we shall abuse notation and henceforth refer to such a set of forms as a structure.

4.1.1 $SU(n)$ structures

We begin with $SU(n)$ structures.

**Definition 4.1.1.** Let $M$ be a $2n$-dimensional manifold. An $SU(n)$ structure on $M$ is (induced by) a pair $(\Omega, \omega)$ where $\Omega$ is a smooth complex $n$-form on $M$ and $\omega$ is a smooth real 2-form on $M$ such that at every point $p$ of $M$:

i) $\Omega_p = \beta_1 \wedge \cdots \wedge \beta_n$ for some $\beta_i \in T^*_p M \otimes \mathbb{C}$,

ii) $\Omega_p \wedge \bar{\Omega}_p \neq 0$,

iii) $\Omega_p \wedge \bar{\Omega}_p = \frac{(-2)^n n^2}{n!} \omega^n_p$,

iv) $\omega_p \wedge \Omega_p = 0$,

v) $\omega_p(v, J_p v) > 0$ for every $v \in T_p M$ where $J_p$ is as in the following proposition.

We shall call $(\Omega_p, \omega_p)$ satisfying (i)-(v) above an $SU(n)$ structure on the vector space $T_p M$. 

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The standard definition is actually that an $SU(n)$ structure is a principal $SU(n)$-subbundle of the frame bundle of $M$ (for instance, see [40, section 2.6]). We shall show that a pair of forms as in Definition 4.1.1 determines such a subbundle by taking the stabiliser at each point. We begin by noting that the stabiliser of $(\Omega_p, \omega_p)$ is isomorphic to $SU(n)$, that is that the pair induces an (almost) complex structure and a hermitian metric on each vector space $T_p M$.

**Proposition 4.1.2** ([34, section 2]). Suppose that $M$ is a $2n$-dimensional manifold and $(\Omega, \omega)$ is an $SU(n)$ structure on it. For each $p \in M$, there is a unique complex structure $J_p$ on $T_p M$ with respect to which $\Omega_p$ is an $(n, 0)$-form and, with respect to $J$, $\omega_p$ is the fundamental form of an hermitian metric $g_p$ on $T_p M$.

Even then, Definition 4.1.1 is not quite the definition found in Hitchin [34]. There, it is required that on every neighbourhood we can decompose as a wedge of smooth $\beta_i$. We use a pointwise definition for two reasons. Firstly, it is more natural to expect a smooth structure to be a structure on each point varying smoothly than to require additional conditions. Secondly, with the usual definition, local decomposability would have to be checked when we obtain $SU(3)$ structures from $G_2$ structures, as a smooth local decomposability condition is not required of $G_2$ structures in Definition 4.1.17.

Fundamentally, Definition 4.1.1 says that an $SU(n)$ structure is a pair of forms with certain properties at every point. Therefore, it is a section of a subbundle of the bundle of forms. We make

**Definition 4.1.3.** Let $SU_n(M)$ be the subset of the bundle $(\wedge^n T^* M \otimes_{\mathbb{R}} \mathbb{C}) \oplus \wedge^2 T^* M$

$$\bigcup_{p \in M} \{ (\Omega_p, \omega_p) \in (\wedge^n T^*_p M \otimes_{\mathbb{R}} \mathbb{C}) \oplus \wedge^2 T^*_p M : (\Omega_p, \omega_p) \text{ is an } SU(n) \text{ structure on } T_p M \}. \quad (4.1.2)$$

We have

**Lemma 4.1.4.** $SU_n(M)$ is a smooth subbundle of $(\wedge^n T^* M \otimes_{\mathbb{R}} \mathbb{C}) \oplus \wedge^2 T^* M$.

**Proof.** We work with the notion of an $SU(n)$ structure on the vector space $\mathbb{R}^{2n}$, i.e. a pair of forms satisfying (i)-(v) of Definition 4.1.1. (i)-(v) are preserved by the action of the Lie group $GL(2n, \mathbb{R})$ on forms; thus, the set of $SU(n)$ structures on $\mathbb{R}^{2n}$ is a representation $\rho$ of $GL(2n, \mathbb{R})$, a subrepresentation of the standard representation $\rho_s$ on $(\wedge^n (\mathbb{R}^{2n})^* \otimes_{\mathbb{R}} \mathbb{C}) \oplus \wedge^2 (\mathbb{R}^{2n})^*$.

Moreover, it follows from the proof of Proposition 4.1.2 that this action is transitive; the proof gives us a basis in which $\Omega$ is $(e_1 + ie_2) \wedge \cdots \wedge (e_{2n-1} + ie_{2n})$ and $\omega$ is $e_1 \wedge e_2 + \cdots + e_{2n-1} \wedge e_{2n}$. Hence, the set of $SU(n)$ structures is diffeomorphic to the homogeneous space $GL(2n, \mathbb{R})_{SU(n)}$. Moreover, the inclusion map of $SU(n)$ structures into forms can just be given by this $GL(2n, \mathbb{R})$ action, and so this diffeomorphism yields a smooth inclusion $GL(2n, \mathbb{R})_{SU(n)} \to (\wedge^n (\mathbb{R}^{2n})^* \otimes_{\mathbb{R}} \mathbb{C}) \oplus \wedge^2 (\mathbb{R}^{2n})^*$. 

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We may now construct the associated bundle of the frame bundle of $M$ with respect to the representation $\rho$. As $\rho$ is a subrepresentation of $\rho_s$, its associated bundle is a subbundle of the associated bundle of $\rho_s$: but the associated bundle of $\rho_s$ is the vector bundle $(\bigwedge^n T^*M \otimes \mathbb{C}) \oplus \bigwedge^2 T^*M$. The associated bundle of $\rho$ is precisely $SU_n(M)$. 

We can now infer that given an $SU(n)$ structure $(\Omega, \omega)$ the almost complex structure $J$ and hermitian metric $g$ are depend smoothly on $(\Omega, \omega)$ in the following sense.

**Proposition 4.1.5.** The construction of Proposition 4.1.2 passes to a smooth bundle map

$$SU_n(M) \rightarrow TM \otimes T^*M \oplus S^2T^*M,$$

with values in the pairs $(J_p, g_p)$ with $J_p$ a complex structure on $T_pM$ and $g_p$ a metric on $T_pM$.

**Proof.** Since the $SU(n)$ structures on $\mathbb{R}^{2n}$ are a $GL(2n, \mathbb{R})$-homogeneous space, and the map from $GL(2n, \mathbb{R})$ to $TM \otimes T^*M \oplus S^2T^*M$ is clearly smooth, the construction of Proposition 4.1.2 defines a smooth map from the $SU(n)$ structures on $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n} \otimes (\mathbb{R}^{2n})^\ast \oplus S^2(\mathbb{R}^{2n})^\ast$. Moreover, it is clear that this map is $GL(2n, \mathbb{R})$-equivariant. By constructing the associated bundles corresponding to the standard representations of $GL(2n, \mathbb{R})$ on this space, we find the required smooth map of bundles. 

This immediately implies

**Corollary 4.1.6.** Suppose that $M$ is a $2n$-dimensional manifold and $(\Omega, \omega)$ is an $SU(n)$ structure on it. Then there is a unique almost complex structure $J$ on $M$ with respect to which $\Omega$ is an $(n, 0)$-form and, with respect to $J$, $\omega$ is the fundamental form of an hermitian metric $g$.

It is easy to see also that the stabiliser of an $SU(n)$ structure $(\Omega_p, \omega_p)$ does indeed define a subgroup $SU(n)$ of $GL(T_pM)$, and so that the stabiliser of $(\Omega, \omega)$ constructs a principal $SU(n)$-subbundle of the frame bundle, just as in [40, Example 2.6.2].

We can extend Definition 4.1.1 to submanifolds as follows.

**Definition 4.1.7.** Suppose that $L$ is an $n$-dimensional submanifold of a $2n$-dimensional manifold $M$. Let $SU_n(M)$ be the bundle of $SU(n)$ structures on $M$ given by Definition 4.1.3, and consider the pullback bundle $SU_n(M)|_L$. A section of this bundle will be called an $SU(n)$ structure around $L$: concretely, such a section is a section $(\Omega, \omega)$ of

$$\bigwedge^n T^*M \oplus \bigwedge^2 T^*M|_L,$$

such that at every point $p$ of $L$ $(\Omega(p), \omega(p))$ is an $SU(n)$ structure on the vector space $T_pM$.

Of course, an $SU(n)$ structure on $M$ restricts to an $SU(n)$ structure around $L$, and an $SU(n)$ structure on a tubular neighbourhood of $L$ would too. We make Definition 4.1.7 as
it is conceptually simpler to work only on $L$: compare in particular Proposition 5.4.16. This definition will primarily be useful in chapter 5: in the present chapter, we do not work with submanifolds.

In the case of an $SU(n)$ structure on a manifold $M$, we would like slightly more than the almost complex structure and hermitian metric obtained from Corollary 4.1.6. We would like $I$ to be a complex structure and $\omega$ a Kähler form. In the principal bundle setting, the condition is the structure being “torsion-free”: the existence of a torsion-free connection on $TM$ that is compatible with our choice of $G \subset GL_\mathbb{R}(T_pM)$ in the sense that parallel transports around loops at $p$ are contained in $G$. In our case, this means we expect to be able to parallel transport $I$ and $\Omega$ to give a global complex structure and volume form, and $\omega$ to give a parallel form and hence a Kähler metric.

To avoid having to think about parallel transport, we would like to express these integrability conditions using the forms. To achieve this, we add further conditions, making

**Definition 4.1.8.** Suppose that $(\Omega, \omega)$ is an $SU(n)$ structure. It is said to be *torsion-free*, or a Calabi–Yau structure, if $d\Omega$ and $d\omega$ are both zero.

Note that Definition 4.1.8 is equivalent to the vanishing of the intrinsic torsion of the principal $SU(n)$-subbundle (see [40, Proposition 2.6.5], and note that any torsion-free connection compatible with the principal $SU(n)$-subbundle is also compatible with the larger principal $SO(2n)$-subbundle, and so is the Levi-Civita connection for the induced metric).

Now, we indeed have

**Proposition 4.1.9.** Suppose that $(\Omega, \omega)$ is a torsion-free $SU(n)$ structure. The induced almost complex structure $I$ is a complex structure, and the hermitian metric induced from $\omega$ is Kähler and Ricci-flat.

We omit the proof, which is standard. See [34] and [36, Corollary 4.B.23].

**Remarks.** We make two remarks. Firstly, in the three-complex-dimensional case, the smoothness of the complex structure $J$ can also be shown directly from the smoothness of the 3-form, adopting the definition in [35, equation (9)] of the complex structure in terms of certain quantities derived from the form, which are clearly smooth.

Secondly, the scaling condition iii) is not used in the proof of Corollary 4.1.6: it is used for Proposition 4.1.9. In fact, in [41–45], Joyce calls a structure in which iii) may fail an almost Calabi–Yau structure.

We now have to combine Definition 4.1.1 with Definition 2.1.3 to define an asymptotically cylindrical $SU(n)$ structure, a restriction of the class of all $SU(n)$ structures on a manifold with an end.

**Definition 4.1.10.** An $SU(n)$ structure on a manifold $M$ with an end is said to be *asymptotically cylindrical* if the induced metric $g$ is asymptotically cylindrical and, with respect to $g$, $\Omega$ and $\omega$ are asymptotically translation invariant.
Note that if \((\Omega, \omega)\) are asymptotically translation invariant forms for some cylindrical metric, it need not be the case that \(\frac{\partial}{\partial t}\) is orthogonal to \(N\). Conversely, any almost complex structure admitting a non vanishing \((n, 0)\) form and such that a cylindrical metric is hermitian can be combined with that cylindrical metric to yield an \(SU(n)\) structure inducing a cylindrical metric: in particular, this almost complex structure need not be asymptotically translation invariant, so that the fact that the induced metric \(g\) is asymptotically cylindrical does not imply that \(\Omega\) and \(\omega\) are asymptotically translation invariant. That is, the two conditions of Definition 4.1.10 are independent of each other.

On the other hand, if the structure is torsion-free so that it defines a complex structure and Kähler metric, the complex structure is parallel with respect to the induced metric. In particular, therefore, it is translation invariant with respect to this metric (in a suitably generalised sense) and it is easy to deduce that \(\Omega\) must also be translation invariant. Hence, if we restricted to torsion-free structures, it would suffice to assume that the induced metric were asymptotically translation invariant.

**Remark 4.1.11.** In their study of asymptotically cylindrical Ricci-flat Kähler manifold, Haskins–Hein–Nordström [32] make this torsion-freeness assumption. They show that then simply-connected irreducible asymptotically cylindrical Calabi-Yau manifolds have \(N = (S^1 \times X)/\langle \Phi \rangle\) for a certain isometry \(\Phi\) of finite order (which may be greater than one: see [32, Example 1.4]). Our results in this chapter therefore apply to this setting. On the other hand, in chapter 5, we will assume (Hypothesis 5.2.5) that \(N = S^1 \times X\).

**Remark 4.1.12.** If we have a torsion-free asymptotically cylindrical \(SU(n)\) structure on \(M\), then it induces a Ricci-flat metric. If the cross-section \(N\) of \(M\) has disconnected cross-section, it admits a line between different components of the cross-section; by the Cheeger–Gromoll splitting theorem [14] it is then a product cylinder \(N \times \mathbb{R}\), and so not especially interesting. We may thus assume \(N\) is connected wherever required. In this chapter, this is only explicitly required in Lemma 4.2.32, but that lemma is often used after its proof; in chapter 5, however, we will often assume the cross-section of a submanifold is connected.

We will also find it useful to know that diffeomorphisms of a Calabi–Yau manifold isotopic to the identity are isometries if and only if they are automorphisms of the underlying Calabi–Yau structures. The infinitesimal version of this (that Killing fields are holomorphic vector fields and vice versa) is a special case of [49, Theorem III.5.2].

**Lemma 4.1.13.** Suppose \(M\) is compact or has an end, that \((\Omega, \omega)\) is an (asymptotically cylindrical) torsion-free \(SU(n)\) structure, and that \(\Phi \in \text{Diff}_0(M)\). Then \(\Phi^* g_{(\Omega, \omega)} = g_{\Omega, \omega}\), i.e. \(\Phi\) is an isometry, if and only if \(\Phi^* \Omega = \Omega\) and \(\Phi^* \omega = \omega\), i.e. \(\Phi\) is an automorphism of the \(SU(n)\) structure.

**Proof.** If \(\Phi\) is an automorphism, it is clearly an isometry, because the metric is obtained from \((\Omega, \omega)\) in a natural fashion, so commuting with the pullback. Conversely, if \(\Phi\) is an isometry, it
pulls back $\Omega$ and $\omega$ to parallel forms with respect to the induced metric. If $M$ is compact, it also preserves the cohomology classes $[\Omega]$ and $[\omega]$; since an exact parallel form is zero, it preserves $\Omega$ and $\omega$ and so is an automorphism. If $M$ has an end, then the limit cohomology classes are also preserved, and the differences $\Phi^*\Omega - \Omega$ and $\Phi^*\omega - \omega$ are exponentially decaying parallel forms trivial in cohomology, and hence zero. □

4.1.2 Existence of torsion-free $SU(n)$ structures

We have a concept of an asymptotically cylindrical $SU(3)$ structure. These are the objects of our study: to ensure the rest of the discussion is not vacuous, we had better check that such structures exist. This is well-known: we briefly discuss its origin and then quote a result, the remainder of the work for which is close analysis. We then discuss a little of this analysis to show that one of the choices involved in this construction does not lead to a different $SU(3)$ structure. This will be helpful in constructing suitable submanifolds in subsection 5.2.2 below.

We begin with the compact case. The Calabi conjecture, proved by Yau [76], says that a compact Kähler manifold whose canonical bundle has trivial first Chern class can be equipped with a Ricci-flat Kähler metric in the same Kähler class. This follows by applying the continuity method to solve a complex Monge-Ampere equation, as the Ricci curvature is roughly given by $\partial\bar{\partial}$ of the logarithm of the determinant of the metric. The difficulty in the proof is the necessary a priori estimates, which carefully use $L^p$ spaces for increasing values of $p$.

In particular, if the canonical bundle of a compact Kähler manifold is holomorphically trivial, then so is the anticanonical bundle and we may find a holomorphic, hence closed, form $\Omega$ as in Definition 4.1.1. Then the first Chern class of the canonical bundle must also be trivial, and so by the Calabi conjecture, we may find a Ricci-flat Kähler metric with Kähler form $\omega$. We find by an argument analogous to Proposition 4.1.9, the other direction of [36, Corollary 4.B.23] that $\omega^n$ must be proportional to $\Omega \wedge \bar{\Omega}$, and so by rescaling $\Omega$ we find a torsion-free $SU(n)$ structure on the compact manifold.

The standard example is then to take a quintic curve in $\mathbb{C}P^4$; by the adjunction formula this has trivial canonical bundle and it is clearly Kähler. Of course, there are many more.

The asymptotically cylindrical case is somewhat harder. To follow the compact case, we first need some notion of a manifold with trivial canonical bundle admitting asymptotically cylindrical Kähler metrics. We begin by thinking about noncompact complex manifolds in general. Having trivial canonical bundle just means that they have nowhere zero holomorphic sections of the canonical bundle, so we may take the complement of an canonical divisor in any compact Kähler manifold. Note that it does not necessarily follow from this that the manifold has a Ricci-flat metric, since the proof of the Calabi conjecture essentially uses compactness.

More importantly, at this point we don’t even have an end and so “asymptotically cylindrical” is meaningless. We look for an end by taking a tubular neighbourhood of the anticanonical divisor; if the normal bundle is topologically trivial, this is diffeomorphic to the product of the
divisor and \( \{ z : 0 < |z| < 1 \} \) (although to get a well-behaved complex structure we will also need holomorphic triviality of this bundle). Now, taking a canonical divisor, on this tubular neighbourhood we expect, if things behave nicely, the holomorphic volume form to be given by

\[
z dz \wedge \Omega_D,
\]

where \( \Omega_D \) is some holomorphic volume form on the divisor \( D \). We identify \( z = e^{-t+i\theta} \) so that the divisor \( z = 0 \) corresponds to \( t = \infty \), and this yields

\[
e^{2(-t+i\theta)}(-dt + id\theta) \wedge \Omega_D.
\]

This decays rapidly, and is difficult to deal with in an asymptotically cylindrical framework (because its limit is zero). However, if the canonical bundle is trivial, the anticanonical bundle is also, and under the same identification we have

\[
\frac{dz}{z} \wedge \Omega_D = (-dt + id\theta) \wedge \Omega_D,
\]

which is rather easier to deal with. Hence we consider the same analysis as above applied to an anticanonical divisor.

In general, we assume that the normal bundle is holomorphically trivial, which means that the arguments above still work at least in the limit, because we have the expected behaviour for the holomorphic coordinate \( z \). Still more work is required to get a metric that is cylindrical on this and then to prove a variant of the Calabi conjecture applying to this setting, which essentially corresponds to guaranteeing well-posedness by imposing boundary conditions. The following theorem is taken from Haskins–Hein–Nordström [32], and shows that we can find a suitable \( \omega \).

**Theorem 4.1.14** ([32], Theorems D and E, weakened). Let \( \tilde{M} \) be a smooth compact Kähler manifold of complex dimension \( n \geq 2 \), and \( D \) be a smooth anticanonical divisor in it which has holomorphically trivial normal bundle. Then for each Kähler class on \( \tilde{M} \), \( M = \tilde{M} \setminus D \) admits an asymptotically cylindrical Ricci-flat Kähler metric \( g \), where asymptotically cylindrical means with respect to a diffeomorphism around \( D \)

\[
U \setminus D \to D \times \{ 0 < |z| < 1 \} \cong D \times S^1 \times (0, \infty),
\]

constructed using the trivial normal bundle of \( D \) (with certain special properties, see below) and by writing \( z = e^{-t+i\theta} \).

The Kähler form is in the cohomology class given by restricting the original Kähler class to \( M \), and its limit is the product of \( d\theta^2 \) on \( S^1 \) with the Calabi–Yau metric on \( D \) in the Kähler class given by restricting the original Kähler class to \( D \). Moreover, the metric is unique subject to the
diffeomorphism (4.1.8) of the tubular neighbourhood to $D \times S^1 \times (0, \infty)$ and these properties.

The fundamental idea follows from work of Tian and Yau [74] under slightly more restrictive assumptions, as Kovalev [50] explains when constructing building blocks for the gluing construction of $G_2$ manifolds. The extensions of Haskins, Hein, and Nordström [32] are that we get representatives for each Kähler class, we can include the case where $D$ is an orbifold (though we have not included this in the statement of Theorem 4.1.14), and we can include the case where $\bar{M}$ is not fibred over $\mathbb{C}P^1$ (and so we don’t have to have $b^1(D) = 0$). The uniqueness statement seems to first appear as Proposition 3.11 of [50]; the idea is essentially the same as the compact case.

We then still need such a $\bar{M}$. There are many possible options, generally taking some kind of weakened Fano manifold as $\bar{M}$. The simplest example, however, is to take $\bar{M}$ as complex projective space $\mathbb{C}P^{n+1}$. The anticanonical divisor $D$ is then the solution to a homogeneous polynomial of degree $n + 1$, which, provided none of its zeros are also zeros of its first partial derivatives, is smooth by the implicit function theorem. $D$ might nevertheless have nontrivial self-intersection, in the sense that small perturbations of $D$ could meet $D$ in a non-empty (complex) curve: this would imply the normal bundle of $D$ is nontrivial (as if it were trivial we could obviously find a perturbation not meeting $D$). We thus blow up the points of self-intersection as in Kovalev [50], and as in that paper the resulting submanifold has trivial normal bundle.

We can in fact improve the uniqueness result of Theorem 4.1.14 slightly, and this will be useful for constructing certain submanifolds in subsection 5.2.2 below. (4.1.8) requires a diffeomorphism $\Phi$ between $\Delta \times D$ and an appropriate neighbourhood of $D$ in $\bar{M}$. There may not be such a diffeomorphism that is biholomorphic, in general. Haskins–Hein–Nordström prove, however, that provided the diffeomorphism satisfies [32, Observation A.2], Theorem 4.1.14 follows. We shall show that the constructed metric is in fact independent of the choice of diffeomorphism here. By the uniqueness part of Theorem 4.1.14, it suffices to show that if $\Phi_1$ and $\Phi_2$ are two such diffeomorphisms, the metrics we construct with each are asymptotically cylindrical with respect to the structure given by the other, since we are by hypothesis using the same Kähler class.

**Proposition 4.1.15.** Let $\Phi_1$ and $\Phi_2$ be diffeomorphisms from $\Delta \times D$ to their images in $\bar{M}$, satisfying

$$\Phi_i(0, x) = x \text{ for all } x \in D, \Phi_i^* J - J_0 \text{ along } \{0\} \times D, \Phi_i^* J - J_0 = 0 \text{ on all of } T\Delta, \tag{4.1.9}$$

where $J$ and $J_0$ are the complex structures on $\bar{M}$ and $\Delta \times D$ respectively. (These are the conditions in [32, Observations A.1 and A.2]). Given a fixed Kähler class on $\bar{M}$, let $g_1$ and $g_2$ be the Calabi–Yau metrics on $\bar{M} \setminus D$ constructed by Theorem 4.1.14 with these diffeomorphisms so that it is clear that $g_i$ is asymptotically cylindrical with respect to the diffeomorphism $\Psi_i$ given by composing $\Phi_i$ with $z = e^{-t+i\theta}$. Then $g_1$ is also asymptotically cylindrical with respect
to $\Psi_2$, and vice versa; the limits are the same.

In particular, by [32, Theorem E], $g_1 = g_2$.

Proof. The construction in [32, subsection 4.2], is that if $\Phi_i$ is such a diffeomorphism then an asymptotically cylindrical metric near $D$ can be chosen by taking near $D$

$$\frac{i}{2} \frac{dz_i \wedge d\bar{z}_i}{|z_i|^2} + \omega,$$

(4.1.10)

where $\omega$ is the Kähler form on $D$ corresponding to the Calabi-Yau metric in the restriction of the Kähler class and $z_i$ is the function $\tilde{M} \to \Delta$ given by composing $\Phi_i^{-1}$ with the projection. Because we are using the same Kähler class in both cases, $\omega$ does not depend on $i$.

Much of the work in [32, subsection 4.2] is simply in showing that this can be cut off under the differentials without affecting the asymptotics. For instance, on $\Delta \times D$ (4.1.10) can be expressed as $\frac{i}{2} \partial \bar{\partial} (\log |z|)^2 + \omega$; the conditions in and arising from (4.1.9) are primarily to ensure that this still has the same asymptotic when $\partial$ and $\bar{\partial}$ are given by the complex structure on $\tilde{M}$.

Using the same ideas, we shall show that

$$\frac{dz_1 \wedge d\bar{z}_1}{|z_1|^2} = \frac{dz_2 \wedge d\bar{z}_2}{|z_2|^2} + \text{decaying terms},$$

(4.1.11)

the decaying terms having $z_2$ or $d\bar{z}_2$ factors, which imply by simple calculation that the metric induced from $\Psi_1$ is asymptotically cylindrical with respect to the structure of $\Psi_2$, as these become $O(e^{-t})$. Moreover, this implies $g_1$ has the same limit as $g_2$.

To do this, we shall primarily work locally in $D$. We know by [32, proof of Observation A.3] that for any local holomorphic defining function $w$ for $D$ in $\tilde{M}$, we may write locally

$$f_1 z_1 + z_1^2 h_1 = w = f_2 z_2 + z_2^2 h_2,$$

(4.1.12)

where $f_1$ and $f_2$ are nonzero holomorphic functions on an open subset of $D$, and $h_1$ and $h_2$ are smooth functions on a neighbourhood of this open subset in $\tilde{M}$. Hence,

$$z_1 = \frac{z_2 f_2 + z_2 h_2}{f_1 + z_1 h_1};$$

(4.1.13)

this fraction is a smooth function, which we shall call $h$ without subscript. Hence, we obtain

$$z_1 = \frac{f_2}{f_1} z_2 + z_2 \frac{h_2}{f_1} - z_2^2 h h_1^2.$$

(4.1.14)

In other words, there is a smooth function $\tilde{h}$ such that

$$z_1 = \frac{f_2}{f_1} z_2 + z_2^2 \tilde{h}.$$

(4.1.15)
Expanding out $\frac{dz_1}{z_1}$ using the Leibniz rule, we see that
\[
\frac{dz_1}{z_1} = \frac{dz_2}{z_2} + f_1 \left( \frac{f_2}{f_1} \right) + \text{decaying terms.}
\] (4.1.16)

It remains to show that $\frac{f_1}{f_2}$ is a constant; then the middle term vanishes and (4.1.11) holds.

Since $f_1$ and $f_2$ are nonzero holomorphic functions, $\frac{f_1}{f_2}$ is holomorphic. We claim that this holomorphic function is independent of the local defining function $w$. Then, by taking a cover of $D$, $\frac{f_1}{f_2}$ extends to a holomorphic function on the compact complex manifold $D$ and so is a constant.

To do this, we note that $f_i = dw\left( \frac{\partial}{\partial z_i} \right)$. Let $w'$ be another local holomorphic defining function, and let $p \in D$ be in the relevant open subset. $(dw)_p$ and $(dw')_p$ both lie in $(T^{1,0})_p \widetilde{M}$, and moreover as $w$ and $w'$ are defining functions they both vanish on $T^1_{\tilde{M}}$. It follows that $(dw)_p = a(dw')_p$ for some $a$, and $a$ cancels in the quotient $\frac{f_1}{f_2}$. As described above, the result then follows.

\[\square\]

In particular, we may obtain the following

**Corollary 4.1.16.** Suppose that $\tilde{M}_1$ and $\tilde{M}_2$ are smooth compact Kähler manifolds of complex dimension $n \geq 2$. Suppose $D_i$ is a smooth anticanonical divisor in $\tilde{M}_i$, for each $i$, with holomorphically trivial normal bundle. Suppose that $f : \tilde{M}_1 \rightarrow \tilde{M}_2$ is a biholomorphism with $f(D_1) = D_2$. Suppose $[\tilde{\omega}_2]$ is a Kähler class on $\tilde{M}_2$ so that $[\tilde{\omega}_1] = f^*[\tilde{\omega}_2]$ is a Kähler class on $\tilde{M}_1$. Theorem 4.1.14 constructs asymptotically cylindrical metrics on $\tilde{M}_1 \setminus D_1$ and $\tilde{M}_2 \setminus D_2$. The map $f$ induces an isometry between these.

**Remark.** By taking $\tilde{M}_1 = \tilde{M}_2$ and $f = \text{id}$, this includes Proposition 4.1.15.

**Proof.** We apply Proposition 4.1.15. Let $\Phi_2$ be a diffeomorphism for $\tilde{M}_2$ satisfying (4.1.9). Since $f$ is a biholomorphism, the composition $f^{-1}\Phi_2$ is satisfies (4.1.9) for $\tilde{M}_1$.

Now $f^*g_2$ is a Calabi–Yau metric on $\tilde{M}_1 \setminus D_1$, constructed from $[\tilde{\omega}_1] = f^*[\tilde{\omega}_2]$ and using the diffeomorphism $f^{-1}\Psi_2$. On the other hand $g_1$ was constructed from $[\tilde{\omega}_1]$ using the diffeomorphism $\Psi_1$. As both of these satisfy (4.1.9), each is asymptotically cylindrical with respect to the other; they have the same limit, as they were constructed with the same Kähler class, and they lie in the same Kähler class for the same reason. Hence $g_1 = f^*g_2$ as required. \[\square\]

### 4.1.3 $G_2$ structures

As a technical tool, we will also use $G_2$ structures on 7-dimensional manifolds. We take a slightly different approach to these, as although we could in principle give properties analogous to those in Definition 4.1.1 defining a $G_2$ structure, this would be far less natural than it is in the $SU(n)$ case.
To motivate our definition, we note that in the $SU(n)$ case, if we restrict to a point $p$, so that we have forms $(\Omega_p, \omega_p)$ on a vector space $T_p M$, arguments similar to those proving Proposition 4.1.9 show that there is a basis $e_1, \ldots, e_{2n}$ of $T_p^* M$ such that

\[(\Omega_p, \omega_p) = ((e_1 + ie_2) \wedge \cdots \wedge (e_{2n-1} + ie_{2n}), e_1 \wedge e_2 + \cdots e_{2n-1} \wedge e_{2n}). \tag{4.1.17}\]

That is, an $SU(n)$ structure at the point $p$ can be defined by a such a basis. In the $G_2$ case, we adopt this viewpoint for the definition.

**Definition 4.1.17.** A $G_2$ structure on a seven-dimensional manifold $M$ is a smooth three-form such that at every point $p$ there is a basis $e_1, \ldots, e_7$ of $T_p^* M$ such that

\[
\phi_p = e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_4 \wedge e_5 + e_1 \wedge e_6 \wedge e_7 + e_2 \wedge e_4 \wedge e_6 - e_2 \wedge e_5 \wedge e_7 - e_3 \wedge e_4 \wedge e_7 - e_3 \wedge e_5 \wedge e_6. \tag{4.1.18}\]

The set of $G_2$ structures is the orbit of any one $G_2$ structure under $GL(V^*)$. Since the stabiliser of $\phi$ under the action is $G_2$, we can see that these form an open subset of $\Lambda^3 V^*$ and so of $\Omega^3(M)$ (for instance, with respect to the uniform topology if $M$ is compact). Every $G_2$ structure induces a metric, by taking the corresponding basis to be orthonormal. Thus, $\phi$ induces a 4-form $*\phi \phi$. We use this 4-form to define torsion-freeness of a $G_2$ structure.

**Definition 4.1.18.** A $G_2$ structure $\phi$ on $M$ is torsion-free if the forms $\phi$ and $*\phi \phi$ are both closed.

As in the $SU(n)$ case, this is equivalent to torsion-freeness of the principal $G_2$-subbundle given by the stabiliser. Torsion-freeness would more naturally be described by $\phi$ being parallel with respect to the metric it induces; that Definition 4.1.18 implies that is a result of Fernández and Gray [23], the converse being obvious as parallel forms are certainly closed. Further as in the $SU(n)$ case, the conditions of Definition 4.1.18 give conditions on the induced metric: here all it usefully implies is that the induced metric is Ricci-flat.

Also as in the $SU(n)$ case, we require a notion of asymptotically cylindrical $G_2$ structure. As in Definition 4.1.10, we make

**Definition 4.1.19.** A $G_2$ structure on a manifold $M$ with an end is said to be asymptotically cylindrical if the induced metric is asymptotically cylindrical and, with respect to this metric, $\phi$ is asymptotically translation invariant.

The fact that torsion-free $G_2$ structures are parallel yields the analogue of Lemma 4.1.13 in the compact and asymptotically cylindrical $G_2$ settings, with the same proof; the fact they induce Ricci-flat metrics gives the analogue of Remark 4.1.12 in the asymptotically cylindrical $G_2$ setting.
4.1.4 \( SU(3) \) structures and \( G_2 \) structures

We now proceed to the relationship between \( SU(3) \) and \( G_2 \) structures induced by the inclusion \( SU(3) \subset G_2 \). This subsection closely follows ideas in Chan’s analysis of the three-dimensional conical gluing problem, [11]. Only the details of the asymptotically cylindrical case are original. The beginnings of these ideas can be found in [40, Proposition 11.1.2], which corresponds to the easier directions of Propositions 4.1.22 and 4.1.24: that if we have a Calabi–Yau structure \((\Omega, \omega)\) on a six-manifold \(M\) (without any global conditions), we can induce a torsion-free \( G_2 \) structure on \( M \times \mathbb{R} \) by \( \phi = \text{Re} \Omega + d\theta \wedge \omega \), with corresponding metric \( g_M + d\theta^2 \). Our setup, following Chan, is more general. Because in gluing we introduce a perturbation to the \( G_2 \) structure will remain in the proper subspace \( \{ \text{Re} \Omega + d\theta \wedge \omega : (\Omega, \omega) \text{ an } SU(3) \text{ structure} \} \). For instance, and more geometrically, it is not clear that the perturbed \( G_2 \) structure will either have \( \frac{\partial}{\partial \theta} \) orthogonal to \( M \) or have \( \frac{\partial}{\partial \theta} \) of length one. We introduce \( z \), therefore, as a generalisation of \( d\theta \): a 1-form with nonzero coefficient of \( d\theta \). In practice, we shall soon assume it has positive coefficient of \( d\theta \), and by the end of this subsection we will assume that \( z = Ld\theta + v \) for a constant \( L \) and a closed 1-form \( v \) on \( M \). In particular, if \( b^1(M) = 0 \), we could reduce further when discussing moduli spaces, and we may as well assume in section 4.2 and the relevant parts of section 4.3 that \( z = Ld\theta \).

However, we work in full generality to start with. For simplicity, we begin with the vector space case, where \( z \) is just a covector.

**Proposition 4.1.20.** Suppose given a six-dimensional vector space \( V \), and suppose that \( z \in (V \oplus \mathbb{R})^* \cong V^* \oplus \mathbb{R} \) is complementary to \( V^* \). Suppose further that \((\Omega, \omega)\) is an \( SU(3) \) structure on \( V \) (that is, is a pair of forms with the appropriate conditions) with associated metric \( g_{\Omega, \omega} \). Then the three-form

\[
\phi = \text{Re} \Omega + z \wedge \omega
\]  

(4.1.19)

is a \( G_2 \) structure on \( V \oplus \mathbb{R} \) with associated metric \( z \otimes z + g_{\Omega, \omega} \). Moreover, given a \( G_2 \) structure \( \phi \) on \( V \oplus \mathbb{R} \), there exist exactly two possible triples \((z, \Omega, \omega)\) and \((z', \Omega', \omega')\), with \((\Omega, \omega)\) and \((\Omega', \omega')\) \( SU(3) \) structures on \( V \), such that \( \phi \) was obtained from this triple as in (4.1.19). They satisfy \( z' = -z \), \( \Omega' = \bar{\Omega} \), and \( \omega' = -\omega \).

**Remark.** Note that if we chose a different six-dimensional subspace of \( V \oplus \mathbb{R} \) we would get a different decomposition of the form (4.1.19).

**Proof.** First choose a basis \( e_2, e_3, \ldots, e_7 \) of \( V^* \) so that \((\Omega, \omega)\) is the standard \( SU(3) \) structure:

\[
\Omega = (e_2 + ie_3) \wedge (e_4 + ie_5) \wedge (e_6 + ie_7), \quad \omega = e_2 \wedge e_3 + e_4 \wedge e_5 + e_6 \wedge e_7.
\]  

(4.1.20)

Then \( e_1 = z, e_2, \ldots, e_7 \) is a basis of \((V \oplus \mathbb{R})^* \) and \( \text{Re} \Omega + z \wedge \omega \) is the \( G_2 \) structure given by (4.1.18) with respect to this basis; hence, this construction always yields a \( G_2 \) structure. The
metric follows by this construction: the dual basis vector to \( z \) is orthogonal to \( V \) and length 1, and \( V \) has its original metric.

Conversely, suppose given a \( G_2 \) structure \( \phi \) on \( V \oplus \mathbb{R} \). We want to choose a basis so that \( \phi \) is the \( G_2 \)-structure given by (4.1.18) and \( V^* \) is spanned by \( e_2, \ldots, e_7 \). First choose any basis such that \( \phi \) is given by (4.1.18). We need a change of basis for which \( \phi \) is still given by (4.1.18) (i.e. an element of \( G_2 \)) mapping \( \text{span}\{e_2, \ldots, e_7\} \) to \( V^* \). As \( G_2 \subset SO(7) \), this is equivalent to taking \( e_1 \) to \( V^* \)'s unit normal, and this is possible because \( G_2 \) is transitive on \( S^6 \). In this basis, we then have the structure of the previous paragraph.

To show that any \( \phi \) is given precisely by these two triples, we begin by noting that if \( \text{Re} \Omega + z \wedge \omega \) is a \( G_2 \)-structure then \( z \) and \( V^* \) are orthogonal with respect to the induced metric and \( z \) has length 1. Therefore, \( z \) is a unit normal vector to \( V^* \) with respect to the metric of \( \phi \) and is determined up to sign. Fix a possible \( z \), and then consider \( SU(3) \) structures \( (\Omega, \omega) \) on \( V \) such that \( \phi = z \wedge \omega + \text{Re} \Omega \); the fact that \( z \) is complementary means \( \phi \) and \( z \) uniquely determine \( \omega \) and so \( \text{Re} \Omega \). It is easy to check that given any such \( z, \omega \) and \( \Omega \),

\[
* \phi = \frac{1}{2} \omega \wedge z + z \wedge \text{Im} \Omega,
\]

and so as \( \phi \) uniquely determines \( * \phi \) it also uniquely determines \( \text{Im} \Omega \), using \( z \) again.

If we reverse the sign of \( z \), this merely reverses the signs of \( \text{Im} \Omega \) and \( \omega \), and so gives the second triple.

At this point, we have two options. We can either hold on to this 2:1 correspondence throughout, or we can make a uniform choice. We will do the latter, partly for notational simplicity and partly to guarantee that the set of \( z \)'s is connected (a similar result will be technically useful later.) Since \( (\Omega, \omega) \leftrightarrow (\bar{\Omega}, -\omega) \) is an isomorphism of \( SU(3) \) structures, it has no serious effect on the results.

We shall express this choice as an orientation on \( \mathbb{R} \) (and later the corresponding manifold \( S^1 \)); this fixes the sign of \( z \) by demanding that its relevant component should be positive with respect to a standard form \( d\theta \) on \( \mathbb{R} \); equivalently, this is defining which orientation \( V \) has as a subspace of \( V \oplus \mathbb{R} \).

It is clear that the correspondence of Proposition 4.1.20 extends to global structures in exactly the same way, although we may not get a product. We require the notion of a structure being \( S^1 \)-invariant to ensure we get a product.

**Definition 4.1.21.** Let \( M \) be a six-dimensional manifold. Consider the product \( M \times S^1 \), and let \( \frac{\partial}{\partial \theta} \) be the vector field corresponding to a global function \( \theta \) giving a coordinate on the circle. The diffeomorphism \( \Theta \) is given by the flow of \( \frac{\partial}{\partial \theta} \) for some time (corresponding to a rotation of the circle). A differential form \( \alpha \) on \( M \times S^1 \) is said to be \( S^1 \)-invariant if its Lie derivative in the \( \frac{\partial}{\partial \theta} \) direction is zero, or equivalently it is preserved by pullback by \( \Theta \). Any other tensor is said to be \( S^1 \)-invariant if the same conditions hold (since \( \Theta \) is a diffeomorphism we can
consider pushforward by its inverse). A map of tensors is $S^1$-equivariant if it commutes with the appropriate pullback and pushforward maps induced by $\Theta$.

We may now state

**Proposition 4.1.22.** Let $M$ be a six-dimensional manifold admitting an $SU(3)$ structure. Let $z$ be an $S^1$-invariant (i.e. invariant by rotation) covector field on $M \times S^1$ that is always complementary to the subbundle $T^*M$ and has positive orientation with respect to the circle, i.e. \( \int_{(p) \times S^1} z > 0 \) everywhere. Then the construction of Proposition 4.1.20 yields a $G_2$-structure on $M \times S^1$.

Conversely, if $M \times S^1$ admits a $G_2$ structure, the structure is constructed as in Proposition 4.1.20 from some unique section of the bundle $\pi^* \mathfrak{su}_n(M)$, where $\pi$ is the projection $M \times S^1 \rightarrow M$ (that is, a structure on $T_pM$ at each point $(p, \theta)$, but potentially varying with $\theta$) and some unique complementary covector field on $M \times S^1$ with suitable orientation at each point. In particular, if the $G_2$ structure is $S^1$-invariant, then both of these sections are $S^1$-invariant, so reduce to an $SU(3)$ structure on $M$ and an $S^1$-invariant complementary covector field with suitable orientation as in the previous paragraph.

Moreover, the maps $(z, \Omega, \omega) \mapsto \phi$ and $\phi \mapsto (z, \Omega, \omega)$ are smooth maps of Fréchet spaces.

**Proof.** Proposition 4.1.20 proves the first two paragraphs pointwise; we have to show that the resulting sections are smooth, and the final paragraph. We prove the final paragraph in proving smoothness in the first and second. For the first, if $z, \omega$ and $\Omega$ are smooth sections of the relevant bundles then so is $\phi = z \wedge \omega + \text{Re} \, \Omega$, and as this is multilinear it is clearly a smooth function of $(z, \Omega, \omega)$.

For the second paragraph, we show directly that $z, \Omega,$ and $\omega$ are smooth functions of $\phi$, and so in particular are themselves smooth if $\phi$ is. We begin by showing that $z$ is smooth. We observe that $z = (\frac{\partial}{\partial \theta})^\flat$ at all points of $M \times S^1$, where $\theta$ is a positively oriented coordinate (meaning $d\theta$ is positive with respect to our choice of orientation).

Indeed, if $u \in T^*_pM \subset T^*_p(M \times S^1)$ we have

\[
\left\langle u, \left(\frac{\partial}{\partial \theta}\right)^\flat \right\rangle = u \left(\frac{\partial}{\partial \theta}\right) = 0,
\]

so $(\frac{\partial}{\partial \theta})^\flat$ is indeed orthogonal to $T^*_pM$, and $\left\langle \frac{\partial}{\partial \theta} \right\rangle$ is clearly unit. Since $(\frac{\partial}{\partial \theta})^\flat (\frac{\partial}{\partial \theta}) = |\frac{\partial}{\partial \theta}|^2 > 0$, the orientation is positive, and so $z$ is indeed $\left(\frac{\partial}{\partial \theta}\right)^\flat$.

Note that although $\frac{\partial}{\partial \theta}$ is independent of the coordinates on $M$, the metric need not be, so $z$ depends on our position on $M$. We need to consider $(\frac{\partial}{\partial \theta})^\flat$ with respect to the metric $g$, which is $t \frac{\partial}{\partial \theta} g$. The interior product is linear and continuous; the map $g \mapsto \left\| \frac{\partial}{\partial \theta} \right\|$ is clearly smooth, as the square root of the smooth function $g \mapsto g(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta})$. It is therefore enough, to prove $z$ is smooth,
to prove that
\[ \phi \mapsto g_\phi \tag{4.1.23} \]
is smooth. This essentially follows by the computation in Hitchin [35]: both
\[ B_\phi : (u, v) \mapsto \iota_u \phi \wedge \iota_v \phi \wedge \phi \tag{4.1.24} \]
and \( K_\phi \), the reinterpretation of the bilinear form \( B_\phi \) as an endomorphism, are smooth functions of \( \phi \). Similarly, so are the determinant of \( K_\phi \) and
\[ g_\phi = (\det K_\phi)^{-\frac{1}{2}} B_\phi \tag{4.1.25} \]
, using compactness and asymptotic cylindricality to ensure that \( \det K_\phi \) is bounded away from zero.

Then the division of \( \phi \) and \( *\phi \) into “the \( z \) part” and “the other part” is smooth, because it is linear and continuous; it follows that \( \Omega \) and \( \omega \) are smooth. \( \Box \)

We also have to check that the correspondence of Propositions 4.1.20 and 4.1.22 respects the notions of asymptotically cylindrical structure that we have defined.

**Proposition 4.1.23.** Suppose that \( M \) is a six-dimensional smooth manifold with an end. Let \((\Omega, \omega)\) be an \( SU(3) \) structure on \( M \) and \( z \) an \( S^1 \)-invariant complementary covector field in the sense of Proposition 4.1.22 (appropriately oriented). Suppose that \( \phi \) is the corresponding \( S^1 \)-invariant \( G_2 \) structure on \( M \times S^1 \). Then \( \phi \) is asymptotically cylindrical if and only if \((\Omega, \omega)\) is asymptotically cylindrical, \( z \) is asymptotically translation-invariant, and \( z(\frac{\partial}{\partial t}) \to 0 \) exponentially uniformly on \( N \).

**Proof.** It is clear that if \( \Omega, \omega, \) and \( z \) are asymptotically translation invariant, then so too is \( \phi \). By Proposition 4.1.20, the induced asymptotically translation invariant metric has limit \( \tilde{g}_{\Omega, \omega} + \tilde{z} \otimes \tilde{z} = g_N + dt \otimes dt + \tilde{z} \otimes \tilde{z} \) (again we use \( \tilde{\cdot} \) to denote limit). We thus have to show that \( \tilde{z} \otimes \tilde{z} \) can be taken as a form on only \( N \times S^1 \). As \( z(\frac{\partial}{\partial t}) \to 0 \), \( \tilde{z} \) has no \( dt \) component and this is indeed the case.

For the converse, we observe that both the asymptotically cylindrical \( G_2 \)-structure \( \phi \) and its limit \( \tilde{\phi} \) must split in the usual way and therefore we have
\[ \phi = \text{Re } \Omega + z \wedge \omega \to \text{Re } \tilde{\Omega} + \tilde{z} \wedge \tilde{\omega} = \tilde{\phi}, \tag{4.1.26} \]
with respect to the metric induced by either. Since \( \phi \to \tilde{\phi} \), exponentially with all derivatives, and the map \( \phi \mapsto (z, \Omega, \omega) \) is a continuous map of Fréchet spaces with implicit constants (in continuity arguments) bounded since \( \phi \) is asymptotically translation invariant, we must have \( z \to \tilde{z} \) and so on too. So \( z, \Omega, \) and \( \omega \) are asymptotically translation invariant.
Furthermore,
\[ z \left( \frac{\partial}{\partial t} \right) = g \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial t} \right) / \left\| \frac{\partial}{\partial \theta} \right\|, \] 
(4.1.27)
and since \( \phi \) is asymptotically cylindrical the right hand side of (4.1.27) tends to zero uniformly in \( N \) and exponentially in \( t \). Thus \( \tilde{z} \otimes \tilde{z} \) has no \( dt \otimes dt \) component, and as in the first paragraph \((\Omega, \omega)\) must be asymptotically cylindrical.

We now have to consider torsion-freeness. We need to find a condition on \( z \) which combined with \((\Omega, \omega)\) being torsion-free implies the \( G_2 \) structure \( \text{Re} \Omega + z \wedge \omega \) is torsion-free. Because torsion-freeness, as a differential equation, is a global condition, we restrict to \( M \) compact or asymptotically cylindrical.

**Proposition 4.1.24.** Suppose that \( M \) is a compact six-dimensional smooth manifold. Let \((\Omega, \omega)\) be an SU(3) structure on \( M \) and \( z \) be an \( S^1 \)-invariant covector field on the product \( M \times S^1 \) complementary to \( T^*M \) with positive orientation. Suppose that \( \phi \) is the corresponding \( S^1 \)-invariant \( G_2 \)-structure on \( M \times S^1 \). Then \( \phi \) is torsion-free if and only if \( z \) is closed and \((\Omega, \omega)\) is torsion-free.

The same holds if \((\Omega, \omega)\) is an asymptotically cylindrical SU(3) structure and \( z \) an asymptotically translation invariant complementary covector field, with \( \tilde{z} \) having no \( dt \) component, so that \( \phi \) is an asymptotically cylindrical \( G_2 \) structure.

**Proof.** Firstly, given a torsion-free SU(3) structure and \( z \) closed, we have
\[ \phi = z \wedge \omega + \text{Re} \Omega, \quad \ast \phi = \frac{1}{2} \omega \wedge \omega + z \wedge \text{Im} \Omega. \] 
(4.1.28)
Since \( d \) is a real operator, \( \tilde{\Omega} \) is closed, and so both of these are closed.

Conversely, if \( \phi \) is a torsion-free \( S^1 \)-invariant \( G_2 \)-structure on \( M \times S^1 \), we begin by considering the covector field \( z \). Let \( \frac{\partial}{\partial \theta} \) be the vector field on \( S^1 \) induced by a standard coordinate \( \theta \) (positively orientated and inducing the rotation we use for “\( S^1 \)-invariance”).

Since \( \phi \) is \( S^1 \)-invariant, \( \frac{\partial}{\partial \theta} \) is a Killing field. A Bochner argument (see, for instance, [6, Theorem 1.84]) shows that Killing fields on compact Ricci-flat manifolds are parallel, and we know that the metric associated to the torsion-free \( G_2 \)-structure \( \phi \) is Ricci-flat. Consequently, if \( M \) is compact, \( \frac{\partial}{\partial \theta} \) is parallel. We want to show that \( \frac{\partial}{\partial \theta} \) is also parallel in the asymptotically cylindrical case. We note that \( \tilde{\phi} \) defines a translation-invariant \( G_2 \)-structure on the limit \( N \times \mathbb{R} \times S^1 \). \( \frac{\partial}{\partial \theta} \) is a translation-invariant Killing vector field on \( N \times \mathbb{R} \times S^1 \), and thus we may imagine we work on \( N \times S^1 \times S^1 \) to deduce that \( \frac{\partial}{\partial \theta} \) is parallel with respect to the limit metric. Now we know that \( \nabla \frac{\partial}{\partial \theta} \) decays, we may do the integration by parts required by the Bochner argument and deduce that \( \frac{\partial}{\partial \theta} \) is parallel.

Thus also \( \| \frac{\partial}{\partial \theta} \| \) is constant, so \( \left( \frac{\partial}{\partial \theta} \right)^\flat \| \frac{\partial}{\partial \theta} \| \) is parallel and in particular closed.

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Then the torsion-freeness of $\phi$ yields from the formulae for $\phi$ and $*\phi \phi$ in terms of $\omega$ and $\Omega$

\[0 = z \wedge d\omega + d \text{Re} \Omega, \quad 0 = \omega \wedge d\omega + z \wedge d \text{Im} \Omega.\] (4.1.29)

Since $d\omega$, $d \text{Re} \Omega$ and $d \text{Im} \Omega$ are all forms on $M$, it follows since $z$ is complementary to the subbundle of such forms that they are all zero, and so $(\Omega, \omega)$ is torsion-free. \qed

We now introduce some terminology to simplify the rest of the chapter.

**Definition 4.1.25.** Let $M$ be a six-dimensional manifold. A closed $S^1$-invariant covector field $z = Ld\theta + v$ for which $L > 0$ is called a *twisting*. If $M$ is an asymptotically cylindrical manifold, we require also that the limit $\tilde{z}$ satisfies $\tilde{z}(\frac{\partial}{\partial t}) = 0$.

Combining all of this subsection, and restricting to the torsion-free case, we have the critical relationship between Calabi–Yau structures and $G_2$ structures.

**Theorem 4.1.26.** If $M$ is a compact six-manifold, there is a homeomorphism

\[\{\text{torsion-free } S^1\text{-invariant } G_2 \text{ structures on } M \times S^1\} \leftrightarrow \{\text{torsion-free } SU(3) \text{ structures on } M\} \times \mathbb{R}_{>0} \times \{\text{closed } 1\text{-forms on } M\}.\]

If $M$ is a six-manifold with an end, there is a homeomorphism

\[\{\text{torsion-free asymptotically cylindrical } S^1\text{-invariant } G_2 \text{ structures on } M \times S^1\} \leftrightarrow \{\text{torsion-free asymptotically cylindrical } SU(3) \text{ structures on } M\} \times \mathbb{R}_{>0} \times \left\{\text{closed asymptotically translation invariant } 1\text{-forms } v \text{ on } M \text{ with } \tilde{v}\left(\frac{\partial}{\partial t}\right) = 0\right\}.\]

In both cases, this homeomorphism is defined by the map

\[(\Omega, \omega, L, v) \leftrightarrow \text{Re} \Omega + (Ld\theta + v) \wedge \omega.\] (4.1.30)

**Remark.** The whole subsection applies equally if the one-dimensional factor is a line instead of a circle, because by invariance we can join the ends to form a circle. Thus the same argument applies on the end of an asymptotically cylindrical $G_2$ manifold, for instance, but asymptotic cylindricality means that the closed 1-form $z$ must be $dt$.

For $SU(2) \subset SU(3)$ we may argue similarly, again requiring that our two extra vectors $u$ and $v$ be closed and considering $T^2$-invariant $SU(3)$ structures. The desired holomorphic covector is, up to scaling, $\left(\frac{\partial}{\partial z}\right)^\flat$ for a suitable holomorphic coordinate $z$ on the torus (in the same way as insisting that $\theta$ be a suitable coordinate on the circle). $\frac{\partial}{\partial z}$ is Killing and so parallel as before, and so its real and imaginary parts are, so a similar result holds.
4.2 Moduli spaces

For the rest of the chapter, all $SU(3)$ and $G_2$ structures will be torsion-free unless stated otherwise. We now want to push the relationship between $SU(3)$ structures and $G_2$ structures discussed in subsection 4.1.4 and culminating there in Theorem 4.1.26 slightly further. In this section, we define moduli spaces and prove Theorem 4.2.39 on how the $SU(3)$ moduli space relates to the $G_2$ moduli space. The section falls into three parts. In subsection 4.2.1, we set up a moduli space of $S^1$-invariant torsion-free $G_2$ structures (Definition 4.2.11). We choose this $S^1$-invariant $G_2$ moduli space so that we have a bijection between it and the product of the Calabi–Yau moduli space with the “moduli space” $Z$ of potential twistings $z$, using the relationship between Calabi–Yau structures and $G_2$ structures. We then have to use this bijection to show the Calabi–Yau moduli space is a manifold. In subsection 4.2.2, we prove that the $S^1$-invariant $G_2$ moduli space is locally homeomorphic to the moduli space of $G_2$ structures and so a manifold (Theorem 4.2.29), by closely following the proof that the $G_2$ moduli space itself is a manifold. We also give an idea for an alternative proof of Theorem 4.2.29 and a discussion of where it runs into difficulty. We then discuss the space $Z$ of classes of twistings in subsection 4.2.3, identifying it as an open subset of a vector space. Finally, in subsection 4.2.4, we return to the relationship between Calabi–Yau structures and $S^1$-invariant $G_2$ structures. We show that the projection map given by Proposition 4.2.14 from the $S^1$-invariant $G_2$ moduli space to the “moduli space” $Z$ of potential twistings $z$ is a smooth surjective submersion, so that each $SU(3)$ moduli-space fibre is a smooth manifold. To prove Theorem 4.2.39, it only then remains to show that these manifolds are all diffeomorphic and that the product structure of Proposition 4.2.14 is compatible with the manifold structures.

First of all, we must give the standard definition of a moduli space. Consider the spaces of all Calabi–Yau or $G_2$ structures. These are typically infinite-dimensional spaces, and have natural symmetries by pullback by diffeomorphisms of the manifold $M$. We therefore take a quotient to ignore these symmetries: we shall see that this quotient reduces the moduli space to a finite number of dimensions.

Concretely, suppose $M$ is a (smooth) compact 6-manifold. We henceforth restrict attention to 6-manifolds which admit Calabi–Yau structures to avoid having to consider the possibility of empty moduli spaces. On manifolds with ends we will further restrict to manifolds for which these structures can be chosen asymptotically cylindrical. Quotienting by the pullback action of the identity component of the diffeomorphism group, we make

**Definition 4.2.1.** If $M$ is a compact 6-manifold,

$$
\mathcal{M}_{SU(3)}(M) = \frac{\{\text{Calabi–Yau structures on } M\}}{\text{Diff}_0 \text{ equivalence}}.
$$

(4.2.1)

To make a a similar definition in the asymptotically cylindrical case, we need to restrict to
a subset of the diffeomorphisms, as general diffeomorphisms obviously do not have to preserve
the cylindrical asymptotic, and so do not act by pullback on asymptotically cylindrical metrics.
Following Nordström [65], we make

**Definition 4.2.2** ([65, Definition 2.19]). A diffeomorphism \( \Phi \) of the asymptotically cylindri-
cal manifold \((M, g)\) is *asymptotically cylindrical* if there is a diffeomorphism \( \tilde{\Phi} \) of \( N \) and a
parameter \( L \in \mathbb{R} \) such that

\[
\Phi(n, t) \to (\tilde{\Phi}(n), t + L)
\]

exponentially, meaning that on restriction to \( N \times (T, \infty) \) for some large \( T \), we have \( \Phi =
\exp V \circ (\tilde{\Phi}(n), t + L) \) for some vector field \( V \) on \( M \) decaying exponentially with all derivatives.

It is then easy to see that the pullback by an asymptotically cylindrical diffeomorphism of
an asymptotically cylindrical metric is asymptotically cylindrical. Note that in the same way
as for the topology on fields, whether a diffeomorphism \( \Phi \) of \( M \) is asymptotically cylindrical
does not depend essentially on the asymptotically cylindrical metric \( g \) (compare Proposition
6.22 of [65]).

The asymptotically cylindrical diffeomorphisms form a group. As in the compact case,
we would like to restrict to the identity component on this group; to do this, we will need a
topology. We extend the extended weighted topologies of Definition 2.2.7 to asymptotically
cylindrical diffeomorphisms found in Definition 4.2.2. This extension essentially defines the
topology used, although not in so many words, by Kovalev [51, p.148] in his choice of norm on
the generating vector fields, and Nordström implicitly (some such assumption is required to get
the bottom of [65, p.336]); we have stated it explicitly to avoid confusion about what “isotopic
to the identity” means.

**Definition 4.2.3.** Let \((M, g)\) be an asymptotically cylindrical manifold. Fix \( \delta > 0 \), smaller
than the decay rate of \( g \), and consider the subset of asymptotically cylindrical diffeomorphisms
that decay at rate at least \( \delta \) with respect to this metric, \( \text{Diff}^\delta_\delta \). We define a topology on \( \text{Diff}^\delta_\delta \) as
follows.

We give neighbourhoods of the identity and extend by multiplication to give a topology
on the whole group. Let \( \Phi \) be an asymptotically cylindrical diffeomorphism which we are
considering for inclusion in this neighbourhood. By definition, \( \Phi = \exp V \circ (\tilde{\Phi}(n), t + L) \)
far enough along the end. If \( (\tilde{\Phi}, L) \) is not close to the identity, then we do not include \( \Phi \) in
the neighbourhood. Consequently, we may suppose that \( \tilde{\Phi} = \exp W \), where this exponential
map is taken with respect to \( \tilde{g} \), so that \( \Phi = \exp_\tilde{g} V \circ \exp_\tilde{g} (W + L\partial/\partial t) \). Then \( V + W + L\partial/\partial t \)
is an asymptotically translation invariant vector field. That is, we have identified a subset of
\( \text{Diff}^\delta_\delta \) containing the identity all of whose elements define an asymptotically translation invariant
vector field, and where the identity defines the zero vector field.

To define our neighbourhoods of the identity, we then take the neighbourhoods of zero with
respect to the extended weighted topologies described in Definition 2.2.7 on the corresponding
asymptotically translation invariant vector fields.

We may now define a topology on the set \( \text{Diff}^g \) of all asymptotically cylindrical diffeomorphisms.

**Definition 4.2.4.** If \((M, g)\) is an asymptotically cylindrical manifold, \( U \subset \text{Diff}^g \) is open if and only if \( U \cap \text{Diff}^g_\delta \) is open in the topology of Definition 4.2.3 for every \( \delta > 0 \).

This topology is also independent of \( g \). Given two asymptotically cylindrical metrics \( g \) and \( g' \), metric-independence of the notion of asymptotically cylindrical diffeomorphism used the result that for every \( \delta \) there exists \( \delta' \) with \( \text{Diff}^g_\delta \subset \text{Diff}^{g'}_{\bar{\delta}} \); it is equally straightforward to see that this inclusion is continuous with respect to these topologies.

For concreteness, we now make

**Definition 4.2.5.** The asymptotically cylindrical diffeomorphism \( \Phi \) is **asymptotically cylindrically isotopic to the identity** if it lies in the identity component \( \text{Diff}_0 \) of \( \text{Diff} \). For simplicity, if \( M \) is an asymptotically cylindrical manifold, then if we say “\( \Phi \) is a diffeomorphism of \( M \) isotopic to the identity”, we shall mean that \( \Phi \) is an asymptotically cylindrical diffeomorphism asymptotically cylindrically isotopic to the identity.

Furthermore, each potential limit \((\tilde{\Phi}, L)\) of an asymptotically cylindrical diffeomorphism defines a closed subspace, as the map to the limit is continuous; we shall say that diffeomorphisms are **isotopic with fixed limit** if they can be joined by a continuous path in such a subspace (in particular, of course, this implies that they have the same limits).

As the map \( \Phi \mapsto \tilde{\Phi} \) is continuous, this definition also automatically gives us a well-defined map from a quotient by \( \text{Diff}_0 \) on \( M \) to a quotient by \( \text{Diff}_0 \) on \( N \). It is interesting to note that we also have a stronger result. If an asymptotically cylindrical diffeomorphism is connected to the identity by a continuous path of asymptotically cylindrical diffeomorphisms, with no control on their rates, there is a possibly different path with some control: specifically, there is a least \( \delta \) or slowest rate of convergence, along the new path. We do not claim that \( \delta \) is the rate of the final diffeomorphism: \( \delta \) might be even less, but is strictly positive. This result might potentially be useful if it were necessary to estimate the behaviour towards the limit of the \( M \) along such a path of diffeomorphisms.

We may now make

**Definition 4.2.6.** If \( M \) is a 6-manifold with an end,

\[
\mathcal{M}_{\text{SU}(3)}(M) = \left\{ \text{asymptotically cylindrical Calabi–Yau structures on } M \right\} / \text{Diff}_0 \text{ equivalence.}
\]

(4.2.3)

There is also a natural action by the rescaling \((\Omega, \omega) \mapsto (a^{3/2} \Omega, a \omega)\) for a fixed constant \( a \).

We will not quotient by this, as it makes the setup of the moduli spaces slightly more complex:
for details of the results we would get, and an example of the resulting complexity, see Remark 4.2.31 below.

In the $G_2$ case, similarly, we restrict to 7-manifolds that admit (asymptotically cylindrical) torsion-free $G_2$ structures. We correspondingly make

**Definition 4.2.7.** If $M$ is a 6-manifold, either compact or with an end,

$$
\mathcal{M}_{G_2}(M \times S^1) = \{(\text{asymptotically cylindrical) torsion-free } G_2 \text{ structures on } M \times S^1\} / \text{Diff}_0^{\text{equivalence}}.
$$

Note that $M \times S^1$ has an end if and only if $M$ does.

### 4.2.1 Setup of the $S^1$-invariant $G_2$ moduli space

By Theorem 4.1.26, we have in both the compact and asymptotically cylindrical cases a bijection roughly given by

$$
\{\text{Calabi–Yau structures}\} \oplus \mathbb{R}_{>0} \oplus \{\text{closed 1-forms}\} \leftrightarrow \{S^1\text{-invariant torsion-free } G_2\text{-structures}\}.
$$

We want to show that this bijection induces a local bijection of moduli spaces, which will be a homeomorphism. Note that since we have not proved that the asymptotically cylindrical $SU(3)$ moduli space is a manifold, we cannot yet ask for a diffeomorphism, and in fact we will use this homeomorphism to define a smooth manifold structure on the $SU(3)$ moduli space. In fact, at the moment we don’t even have a good notion of the moduli space of $S^1$-invariant $G_2$ structures: we could guess at a definition, but it would then not be at all clear that it is smooth or that we can locally take its topology as the $C^\infty$ topology on a subset of structures.

In this subsection, therefore, we make definitions to set up the moduli space of $S^1$-invariant $G_2$ structures. We need to restrict the diffeomorphisms involved in the quotient, to preserve $S^1$-invariance; evidently, we should choose the set of diffeomorphisms we use so that the map induced from Theorem 4.1.26 is well-defined and we would like it to be a bijection. The point of this subsection is to choose our definitions so that we have a bijection.

In the same way, to pass to a moduli space version of (4.2.4), we have to take a quotient of the set of twistings. We have a choice of what to quotient the twistings by, and again, we will choose our quotient in the hope of getting a bijection.

We shall use the following set of diffeomorphisms

**Definition 4.2.8.** Suppose that $M$ is a compact or asymptotically cylindrical manifold with an $S^1$-invariant Ricci-flat metric $g$. Let the space $\text{Diff}_0^{S^1}$ be the identity path-component of

$$
\left\{ \Phi \in \text{Diff}_0 : \Phi_\ast \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} \right\}.
$$

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In Proposition 4.2.22, we shall show that $\text{Diff}^{S^1}_0$ is equal to the identity path-component of diffeomorphisms satisfying the weaker condition that $\Phi \frac{\partial}{\partial \theta}$ is a Killing field for the metric $g$; note that since $g$ is $S^1$-invariant, $\frac{\partial}{\partial \theta}$ is certainly a Killing field for it.

The first diffeomorphisms to note that we have in $\text{Diff}^{S^1}_0 (M \times S^1)$ are those induced from diffeomorphisms of the six-dimensional manifold $M$. That is, we begin with the entirely elementary

Lemma 4.2.9. If $\Phi$ is a diffeomorphism of $M^6$ isotopic to the identity (recall that we are omitting the words “asymptotically cylindrical” for brevity) then the diffeomorphism $\hat{\Phi} : (x, \theta) \mapsto (\Phi(x), \theta)$ of $M^6 \times S^1$ lies in $\text{Diff}^{S^1}_0 (M \times S^1)$. If $(\Omega, \omega)$ is a (asymptotically cylindrical) Calabi–Yau structure and $Ld\theta + v$ a twisting, the torsion-free $G_2$ structures $\Phi^* (\text{Re} \Omega) + (Ld\theta + \Phi^* v) \wedge \Phi^*(\omega)$ and $\text{Re} \Omega + (Ld\theta + v) \wedge \omega$ are identified by $\hat{\Phi}$.

Proof. Given any diffeomorphism $\Psi$ of $M$, $\tilde{\Psi} : (x, \theta) \mapsto (\Psi(x), \theta)$ gives a diffeomorphism of $M \times S^1$, and $\tilde{\Psi} \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta}$. We have a curve of diffeomorphisms $\Phi_s$ defining the isotopy of $\Phi$ to the identity; the curve $\hat{\Phi}_s$ shows that $\hat{\Phi}$ is in the same path-component as the identity in the space (4.2.5). Consequently, $\hat{\Phi} \in \text{Diff}^{S^1}_0(M \times S^1)$. The final sentence is obvious.

We would like $\Phi^* (\text{Re} \Omega) + (Ld\theta + v) \wedge \Phi^*(\omega)$ and $\text{Re} \Omega + (Ld\theta + v) \wedge \omega$ to be identified in our $S^1$-invariant $G_2$ moduli space, as they correspond to the same element of the $SU(3)$ moduli space with the same twisting. Lemma 4.2.9 says that it is sufficient to choose some more diffeomorphisms so that $\text{Re} \Omega + (Ld\theta + \Phi^* v) \wedge \omega$ and $\text{Re} \Omega + (Ld\theta + v) \wedge \omega$ are identified. More concretely, we shall identify $S^1$-invariant $G_2$ structures where the twisting differs by $df$ for some (asymptotically translation invariant) $f$; $v - \Phi^* v$ is exact and it’s clear that the resulting $f$ can be chosen to be asymptotically translation invariant if necessary, by its explicit form as the integral of an asymptotically translation invariant integrand.

Lemma 4.2.10. If $(\Omega, \omega)$ is a Calabi–Yau structure on $M$, $L d \theta + v$ is a twisting, and $f$ is a bounded function on $M$, there is a diffeomorphism $\Phi \in \text{Diff}^{S^1}_0 (M \times S^1)$ pulling back the $S^1$-invariant $G_2$ structure $\text{Re} \Omega + (L d \theta + v) \wedge \omega$ to $\text{Re} \Omega + (L d \theta + v + df) \wedge \omega$.

Proof. Consider the curve of diffeomorphisms

$$\Phi_s : (x, \theta) \mapsto (x, \theta + \frac{s f(x)}{L}).$$

Each $\Phi_s$ is clearly smooth and smoothly invertible by $(x, \theta) \mapsto (x, \theta - \frac{2 f(x)}{L})$. Moreover, it is easy to see that $\Phi_s \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta}$ for all $s$: hence $\Phi = \Phi_1 \in \text{Diff}^{S^1}_0 (M \times S^1)$.

As all the $M^6$ coordinates are left unchanged, and the structure is invariant by $S^1$ (and so
Re Ω, ω and v are), Φ acts as the identity on them. However, by definition

\[ \Phi^*(d\theta) = d(\theta \circ \Phi) = d\left( \theta + \frac{f(x)}{L} \right) = d\theta + \frac{df}{L}. \]

We obtain

\[ \Phi^*(Re \Omega + (Ld\theta + v) \land \omega) = Re \Omega + (Ld\theta + v + df) \land \omega, \]

as claimed.

Lemmas 4.2.9 and 4.2.10 show that if we quotient by \( \text{Diff}_{S1}^0(M \times S^1) \) we have a well-defined map from the Calabi–Yau moduli space. We shall thus make

**Definition 4.2.11.**

\[ \mathcal{M}^S_{G_2}(M \times S^1) := \frac{S^1\text{-invariant torsion-free } G_2 \text{ structures}}{\text{Diff}_{S1}^0(M \times S^1)}. \]

We shall call \( \mathcal{M}^S_{G_2} \) the \( S^1 \)-invariant \( G_2 \) moduli space.

It remains to choose the "moduli space" \( Z \) of twistings \( z \). Given Definition 4.2.11, Lemma 4.2.10 implies that we have to quotient by the differentials of (asymptotically translation invariant) functions in order to make the induced map an injection. Consequently, we make

**Definition 4.2.12.** Let the set of twisting-classes \( Z \) be the quotient of the twistings of Definition 4.1.25

\[ \{ \text{closed } S^1\text{-invariant } 1\text{-forms } z = Ld\theta + v : L > 0 \text{ (with } \tilde{z}(\frac{\partial}{\partial t}) = 0) \text{ on } M \times S^1 \}, \]

\[ \{ \text{differentials of } S^1\text{-invariant (asymptotically translation invariant) functions} \}. \]

\[ Z \]

has a relatively simple description, which is discussed in subsection 4.2.3 leading to Lemma 4.2.32 below.

To justify making these definitions, we need to check that we do indeed have a well-defined bijection

\[ \mathcal{M}_{SU(3)} \times Z \to \mathcal{M}^S_{G_2}, \]

induced from Theorem 4.1.26. Well-definition follows from the two preceding lemmas; injectivity is

**Lemma 4.2.13.** Suppose that \( \Phi \in \text{Diff}_{S1}^0(M \times S^1) \) and that \( (\Omega, \omega) \) is a Calabi–Yau structure on \( M \) with \( z = Ld\theta + v \) a closed complementary covector field with positive orientation (i.e. \( L > 0 \)). Then, choosing a point on \( S^1 \) and so an identification of \( S^1 \) with \( \mathbb{R}/\mathbb{Z} \), \( \Phi \) is of the form \( (p, \theta) \mapsto (f(p), \theta + g(p)) \) for some smooth functions \( f \) and \( g \). Consequently, there exist
Φ₁ ∈ Diff₀(M) and Φ₂ the time-1 flow of \( f \frac{∂}{∂θ} \) (for \( f \) asymptotically translation invariant, if necessary) such that

\[
\Phi^*Ω = \Phi_2^*\Phi_1^*Ω, \quad \Phi^*ω = \Phi_2^*\Phi_1^*ω, \quad \Phi^*z = \Phi_2^*\Phi_1^*z.
\] (4.2.12)

**Remark.** In Lemma 4.2.13, we do not claim that \( \Phi = \Phi_1\Phi_2 \), though this will of course be true up to an isometry.

**Proof.** We have chosen an origin, so write \( \Phi(p, 0) = (f(p), g(p)) \) for smooth functions \( f \) and \( g \). Now suppose \( (p, θ') ∈ S^1 \), and consider the curve \( (p, sθ') \) between \( (p, 0) \) and \( (p, θ) \) in \( M × S^1 \). At all points of this curve its derivative is \( θ' \frac{∂}{∂θ} \). Consequently, the derivative of its image is \( θ' \frac{∂}{∂θ} \) and so its image is \( (f(p), g(p) + sθ') \). Hence \( \Phi(p, θ') = (f(p), g(p) + θ') \), as required. Note that \( f \) is hence a diffeomorphism.

Now let \( Φ_1 \) be the extension of the map \( f \) as in Lemma 4.2.9. It is then clear that we have \( Φ_1^*Ω = Φ^*Ω \) and \( Φ_1^*ω = Φ^*ω \).

The \( θ \) component is preserved by \( Φ_1 \), and so \( Φ_1^*(tdθ + v) = tdθ + Φ^*v \). By Stokes’ theorem, \( Φ^*v - Φ_1^*v \) is exact, and in the asymptotically cylindrical case is the differential of an asymptotically translation invariant function. Thus there exists such a \( Φ_2 \), by Lemma 4.2.10.

**Proposition 4.2.14.** The map induced by the bijection of Theorem 4.1.26 is a well-defined homeomorphism \( M_{SU(3)} × Z → M_{G_2}^1(M × S^1) \). (4.2.13)

The fact that (4.2.13) is a homeomorphism follows immediately as the maps between structures are continuous since they are so pointwise and we just take the quotient topology. However, we know very little about what the topology on the right hand side looks like.

### 4.2.2 Smoothness of the \( S^1 \)-invariant \( G_2 \) moduli space

In order to use the bijection of Proposition 4.2.14 to show the moduli space of Calabi–Yau structures \( M_{SU(3)} \) is a manifold, we first have to show that \( M_{G_2}^1 \) is a manifold. The objective of this subsection is to prove Theorem 4.2.29, which says that \( M_{G_2}^1 \) is a manifold and in fact is locally diffeomorphic to \( M_{G_2} \). The idea of the proof is that the constructions of the \( G_2 \) moduli space due to Hitchin [35] and Nordström [65] work by, given a \( G_2 \) structure, constructing a geometrically natural premoduli space of structures and arguing that this premoduli space is locally homeomorphic to the moduli space. Since the construction is geometrically natural, if the \( G_2 \) structure concerned is \( S^1 \)-invariant, it turns out that the premoduli space also consists of \( S^1 \)-invariant structures, and the result then follows in the same way.

In the \( G_2 \) case, the standard result is

**Theorem 4.2.15** ([35] for (a), [65, Proposition 6.18] for (b), cf. [21, Theorem 7.1] for (c)(i), (c)(ii) by combining the proofs of (a) and (b) with (c)(i)). Let \( M \) be a six-dimensional manifold.
a) If $M$ is compact, the moduli space of torsion-free $G_2$ structures on $M \times S^1$ is a smooth manifold. It is locally diffeomorphic to $H^3(M \times S^1) \cong H^3(M) \oplus H^2(M)$ by the well-defined map that takes a representative of a moduli class to its cohomology class.

b) If $M$ has an end, the moduli space of asymptotically cylindrical torsion-free $G_2$ structures on $M \times S^1$ is a smooth manifold. It is locally diffeomorphic to $H_3(M \times S^1) \cong H_3(M) \oplus H_2(N) \oplus H_1(N)$, by the well-defined map taking a representative $\phi$ of a moduli class with limit $\tilde{\phi} = \tilde{\phi}_1 + dt \wedge \tilde{\phi}_2$ to the pair $([\phi], [\tilde{\phi}_2])$. This submanifold is wholly determined by the requirement that there be a Calabi–Yau structure $(\Omega, \omega)$ on $N \times S^1$ with $(\text{Re} \, \Omega, \omega) = (\tilde{\phi}_1, \tilde{\phi}_2)$, which follows from subsection 4.1.4.

c) In either case, given a (asymptotically cylindrical) torsion-free $G_2$ structure $\phi$ on $M \times S^1$, we may find a set $U$ of such structures containing $\phi$ and with the following properties.

i) All the structures in the chart have the same group $\text{Aut}$ of (asymptotically cylindrical) automorphisms isotopic to the identity.

ii) There are neighbourhoods $D$ of $[\text{id}] \in \text{Diff}_0 \text{Aut}$ and $V$ of $\phi$ in the set of all (asymptotically cylindrical) torsion-free $G_2$ structures such that the pullback map defines a bijection between the product $D \times U$ and $V$ (in fact, a homeomorphism).

The set $U$ is called a slice neighbourhood.

We will now show that the proof (and hence the statement) of Theorem 4.2.15 survives our passing to the $S^1$-invariant setting: that is, we will prove Theorem 4.2.29, which says that $M_{G_2}^{S^1}$ is a manifold locally diffeomorphic to $M_{G_2}$. Essentially, the proof is that the whole proof of Theorem 4.2.15 depends on geometrically natural, and so $S^1$-invariant, operators, and thus can be taken $S^1$-invariant; we get that both moduli spaces are locally diffeomorphic to each other essentially because they arise from the same application of the implicit function theorem.

We concentrate on the proofs of (a) and (b); (c)(i) follows in exactly the same way as in the general case, and then (c)(ii) also follows. We begin with (a), as it is simpler. From the work of Hitchin, we extract Propositions 4.2.17 and 4.2.18 which together prove (a). Of course, since we rely on the implicit function theorem, these should properly be stated in terms of suitable Banach spaces. However, the choice of Banach spaces is straightforward and of no relevance to the introduction of $S^1$-invariance; consequently we omit it.

We first need a slice for the $\text{Diff}_0$ action at some $G_2$ structure $\phi$, that is, essentially, a local cross-section of the quotient. To choose a slice, we need the following standard fact about two-forms on a manifold with a torsion-free $G_2$ structure.

**Lemma 4.2.16** (cf. [69, Lemma 11.4]). *Let the seven-manifold $X$ admit a $G_2$ structure $\phi$. We then have an isomorphism of bundles*

$$\bigwedge^2 T^*(X) = \bigwedge^2_7 \oplus \bigwedge^2_{14}, \quad (4.2.14)$$
where $\bigwedge^2_7$ is a rank-seven bundle given by contractions of $\phi$ with tangent vectors, $\bigwedge^2_{14}$ is a rank-fourteen bundle, and the fibres of these sub-bundles are orthogonal with respect to the inner product induced on two-forms by $\phi$.

We apply Lemma 4.2.16 in the case where $X = M \times S^1$. It is virtually sufficient to prove the following proposition. Note that here we make no torsion-freeness assumption on the three-forms other than $\phi$.

**Proposition 4.2.17** (cf. [35, bottom of p.23]). Let $\phi$ be a torsion-free $G_2$ structure on the compact seven-manifold $M \times S^1$. Let

$$E = \{ \alpha \in \Omega^3(M \times S^1) : d\alpha = 0, d^*\alpha \in \Omega^2_{14} \},$$

(4.2.15)

where $\Omega^2_{14}$ is the subspace of 2-forms which at every point are in the subbundle $\bigwedge^2_{14}$.

Then $E$ is $L^2$-orthogonal to the space of Lie derivatives $\mathcal{L}_X \phi$ of $\phi$, with respect to the metric at $\phi$, and the sum of these spaces is the set of all closed three-forms. (We choose the Banach spaces so that the projection onto $E$ is continuous). Consequently, locally $E$ is transverse to the orbits of the identity component of the diffeomorphism group.

Now we have to pass to the torsion-free $G_2$ structures in the slice $E$.

**Proposition 4.2.18** ([35, p. 35]). Let $\phi_0$ be a torsion-free $G_2$ structure on $M \times S^1$ as in the previous proposition, and let $E$ be as stated there. Define a map $F$ from a neighbourhood of $\phi_0 \in E$ to exact forms by $F(\phi) = P(\ast_0 \ast \phi)$ where $\ast_0$ is the Hodge star induced by $\phi_0$, $\ast$ is the Hodge star induced by $\phi$, and $P$ is the orthogonal projection onto exact forms induced by $\phi_0$.

If $F(\phi) = 0$ for $\phi$ sufficiently close to $\phi_0$, then $\phi$ is itself a torsion-free $G_2$ structure.

The derivative $DF$ has kernel consisting of harmonic forms and is surjective to the exact forms, proving (a) of Theorem 4.2.15.

We have to transfer Propositions 4.2.17 and 4.2.18 to the $S^1$-invariant setting. We begin with the slice.

We first need to show that $\text{Diff}_{S^1}^0$ has a well-defined tangent space so that we can still work with the space of Lie derivatives. In other words, we must prove

**Proposition 4.2.19.** $\text{Diff}_{S^1}^0$ is a Fréchet manifold. Its tangent space is given by the $S^1$-invariant vector fields.

**Proof.** We know that the identity component of all diffeomorphisms is a Fréchet manifold, in fact a Fréchet Lie group; see, for example, [62, p. 1041]. Strictly this result is for the compact case, but fundamentally the same argument extends to asymptotically cylindrical manifolds.

Since $\text{Diff}_{S^1}^0$ is a subgroup, it suffices to show it is locally a submanifold around the identity. A neighbourhood of the identity in $\text{Diff}_0$ is given, for any smooth Riemannian metric $g$, by the
diffeomorphisms given by the Riemannian exponential map applied to sufficiently small vector fields, and using these vector fields defines the smooth structure.

From Lemma 4.2.13 and a trivial calculation, we know that $\text{Diff}_0^{S^1}$ are the diffeomorphisms of the form $\Phi(p, \theta) = (f(p), g(p) + \theta)$; equivalently, they are precisely the diffeomorphisms that commute with the rotations $\Theta$. We choose $g$ to be $S^1$-invariant, for simplicity, and we must show that the vector fields such that $\exp_v$ commutes with $\Theta$ are a submanifold with the specified tangent space. In fact we shall show that they are precisely the $S^1$-invariant vector fields small enough to be in the neighbourhood; the intersection of a vector subspace with the neighbourhood is clearly a submanifold.

Suppose given a sufficiently small vector field $v$. We have to compare $\Theta \exp_v(p, \theta)$ and $\exp_v \Theta(p, \theta)$, for some $(p, \theta) \in M \times S^1$. Suppose that $\Theta$ is rotation by $\theta'$. $\Theta \exp_v(p, \theta)$ is the endpoint of the image under $\Theta$ of the geodesic with initial velocity $v(p, \theta)$; since the metric is $S^1$-invariant, $\Theta$ is an isometry, and so $\Theta \exp_v(p, \theta)$ is the endpoint of the geodesic with initial velocity $(\Theta_\ast v)(p, \theta + \theta')$. On the other hand, $\exp_v \Theta(p, \theta)$ is the endpoint of the geodesic with initial velocity $v(p, \theta + \theta')$. By assumption, $v$ and hence $\Theta_\ast v$ are sufficiently small that the exponential map is injective, and consequently we have $\exp_v \Theta = \Theta \exp_v$ if and only if $\Theta_\ast v = v$.

This argument also extends to the Banach manifolds given by considering reduced regularities. Note that these are groups, though they are not Banach Lie groups. Similarly, it extends to asymptotically cylindrical diffeomorphisms with appropriately restricted limits, and so on.

**Remark.** In the Banach case, we could avoid using the group structure on $\text{Diff}_0^{S^1}$ as follows. We have established that $\Phi \in \text{Diff}_0^{S^1}(M \times S^1)$ is of the form $(p, \theta) \mapsto (f(p), \theta + g(p))$, and vice versa. Consequently, it is uniquely determined by the smooth map $p \mapsto (f(p), g(p))$. It is clear that the space of $C^k$ maps from $M$ to $M \times S^1$ is a Banach manifold, since it is the $C^k$ sections of the fibre bundle $M \times M \times S^1 \to M$; see Palais [68]. We want to show that the map $C^k(M, M \times S^1) \to C^k(M \times S^1, M \times S^1)$ is smooth (and an immersion). The critical idea is to notice, again essentially from Palais [68], the following facts, for any finite-dimensional manifolds $M, N, K$:

i) The map $C^k(M \times K, N) \to C^k(M \times K, N \times K)$ given by mapping $f$ to $(f, \pi_K)$, where $\pi_K : M \times K \to K$ is the projection, is a smooth map.

ii) The map $C^k(M, N) \to C^k(M, K)$ given by composition with a fixed smooth map is smooth.

iii) The map $C^k(M, N) \to C^k(K, N)$ given by composition with a fixed smooth map is smooth.

The first of these follows by the Whitney-sum observation (additivity of $C^k$) on vector bundles. The second and third are more or less immediate from the standard hypotheses in Palais.
Consequently, given \( h = (f, g) \) in \( C^k(M, M \times S^1) \), the map \( h \mapsto (h, \pi_{S^1}) \) in \( C^k(M \times S^1, M \times S^1 \times S^1) \) is smooth, using the first and third. Composing with the fixed smooth map given by identity in the \( M \) component and addition in the \( S^1 \) component gives precisely the final result. Hence the whole map is smooth. It is obvious that this map is an immersion.

That is, the tangent space to the orbit of a \( G_2 \) structure \( \phi \) under Diff\( S^1_0 \) is given by the Lie derivatives \( L_X \phi \) with \( X S^1 \)-invariant. We shall prove that for \( \phi S^1 \)-invariant, this is equivalent to the \( S^1 \)-invariant Lie derivatives \( L_X \phi \). We begin with the simplest case: an \( S^1 \)-invariant \( G_2 \) structure on a compact manifold.

**Lemma 4.2.20.** Let \( M \) be a compact six-dimensional manifold and let \( \phi \) be an \( S^1 \)-invariant torsion-free \( G_2 \) structure on \( M \times S^1 \). Suppose that \( X \) is a vector field on \( M \times S^1 \) and \( L_X \phi \) is \( S^1 \)-invariant. Then \( X \) is \( S^1 \)-invariant.

**Proof.** We begin by showing that any Killing field \( X \) is \( S^1 \)-invariant. As \( M \) is Ricci-flat, a Killing field is parallel; hence \( L_{\partial/\partial \theta} X = [X, \partial/\partial \theta] = 0 \) and this is equivalent to \( S^1 \)-invariance.

Now suppose that \( L_X \phi \) is \( S^1 \)-invariant, but not necessarily zero. Then we have \( L_{\partial/\partial \theta} L_X \phi = 0 = L_X L_{\partial/\partial \theta} \phi \), and so \( [X, \partial/\partial \theta] \phi = 0 \). By the previous paragraph we find that \( [X, \partial/\partial \theta] \) is \( S^1 \)-invariant.

We may now work locally on \( M \). Pick some open subset of \( M \) on which we have coordinates \( x_1, \ldots, x_n \), and on this subset write

\[
X = a_0 \frac{\partial}{\partial \theta} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}. \tag{4.2.16}
\]

We see by elementary computation that

\[
0 = [[X, \partial/\partial \theta], \partial/\partial \theta] = \frac{\partial^2 a_0}{\partial \theta^2} \frac{\partial}{\partial \theta} + \sum_{i=1}^n \frac{\partial^2 a_i}{\partial \theta^2} \frac{\partial}{\partial x_i}. \tag{4.2.17}
\]

It follows that each of the \( a_i \) (possibly zero) is of the form \( A_i \theta + B_i \), where \( A_i \) and \( B_i \) are functions on \( M \). But as there is no globally defined function \( \theta \), \( A_i \) must be identically zero. It follows that \( X \) is independent of \( \theta \), as required. \( \square \)

In the asymptotically cylindrical case, we will need a couple of statements very similar to Lemma 4.2.20.

**Lemma 4.2.21.**

i) Let \( N \) be a compact five-dimensional manifold and let \((\Omega, \omega)\) be an \( S^1 \)-invariant Calabi-Yau structure on \( N \times S^1 \). Suppose that \( X \) is a vector field on \( N \times S^1 \) and \( L_X \Re \Omega \) is \( S^1 \)-invariant. Then \( X \) is \( S^1 \)-invariant, and in particular any Killing field is \( S^1 \)-invariant.

ii) Let \( M \) be a six-dimensional manifold with an end and let \( \phi \) be an asymptotically cylindrical torsion-free \( G_2 \) structure on \( M \times S^1 \). Suppose that \( X \) is an asymptotically translation
invariant vector field on $M \times S^1$ such that $\mathcal{L}_X \phi$ is also $S^1$-invariant. Then $X$ is $S^1$-invariant; in particular, again, any Killing field is $S^1$-invariant.

Proof. (ii) is essentially identical. The only part of Lemma 4.2.20 that required compactness of $M$ was showing that Killing fields are parallel, and this was briefly explained in Proposition 4.1.24.

(i) is slightly more involved, as it is not immediately clear that $\mathcal{L}_X \text{Re } \Omega = 0$ implies that $X$ is a Killing field and so parallel. However, it follows from Hitchin [35], as follows.

Hitchin proves that $\text{Im } \Omega$ can be determined at each point from $\text{Re } \Omega$. Moreover, the map $\text{Re } \Omega \mapsto \text{Im } \Omega$ is smooth. Hence, if $\mathcal{L}_X \text{Re } \Omega = 0$, we have a curve of Calabi-Yau structures corresponding to a curve of diffeomorphisms; at each point $\text{Im } \Omega$ depends smoothly on $\text{Re } \Omega$, and so $\mathcal{L}_X \text{Im } \Omega$ depends linearly on $\mathcal{L}_X \text{Re } \Omega$, and must in turn be zero. Hence, if $\mathcal{L}_X \text{Re } \Omega = 0$, then $\mathcal{L}_X \Omega = 0$, i.e. $X$ is holomorphic. But then it follows by the argument from Kobayashi [49, Theorem III.5.2] mentioned before Lemma 4.1.13 that $\mathcal{L}_X [\frac{\partial}{\partial \theta}] \omega = 0$. Then $X$ is Killing, and we can apply the previous argument.

A similar argument proves a non-infinitesimal version. It does not quite imply Lemma 4.2.20 as we would need to integrate up: we would need to show that a vector field $X$ with $\mathcal{L}_X \phi S^1$-invariant induces a curve $\Phi_s$ of diffeomorphisms so that $\Phi_s^* \phi$ is $S^1$-invariant for all $s$ sufficiently small.

Proposition 4.2.22. Let $M \times S^1$ have an $S^1$-invariant Ricci-flat metric $g$. The space $\text{Diff}_0^{S^1}$ defined in Definition 4.2.8 is also the identity path-component of the set

$$\{ \Phi \in \text{Diff}_0(M \times S^1) : \Phi_s \frac{\partial}{\partial \theta} \text{ is Killing} \}. \quad (4.2.18)$$

Proof. Because $\frac{\partial}{\partial \theta}$ is certainly a Killing field for the metric $g$, it is clear that (4.2.5) is a subspace of (4.2.18), and consequently $\text{Diff}_0^{S^1}(M \times S^1)$ is contained in the identity path-component of (4.2.18). It suffices to show that (4.2.5) is open and closed in (4.2.18), for then connectedness of the identity component of (4.2.18) implies it is all of $\text{Diff}_0^{S^1}(M \times S^1)$.

Closedness follows immediately from continuity of the pushforward.

For openness, we suppose that $\Phi_0$ is in $\text{Diff}_0^{S^1}$; we need to show that there is an open neighbourhood $U$ of $\Phi_0$ in $\text{Diff}_0$ such that if $\Phi \in U$ and $\Phi_s \frac{\partial}{\partial \theta}$ is Killing, then $\Phi_s \frac{\partial}{\partial \theta}$ is $\frac{\partial}{\partial \theta}$. We work around some $p \in M$. We note first that by the argument of Lemma 4.2.13, $\Phi_0$ carries $\{p\} \times S^1$ onto $\{q\} \times S^1$ for some point $q$ of $M$ depending on $p$. Let $V$ be a small chart around $p$ and let $W$ be a small chart around $q$ such that $\Phi_0(V \times S^1) \subset W \times S^1$. Choose a smaller neighbourhood $V'$ whose closure is compact and contained in $V$; then for $\Phi$ sufficiently close to $\Phi_0$, $\Phi(V' \times S^1) \subset W \times S^1$.

We may consequently analyse $\Phi$ in terms of the coordinates on $V \times S^1$ and $W \times S^1$; that is,
we shall write

$$\Phi(x_1, \ldots, x_n, \theta) = (y_1(x_1, \ldots, x_n, \theta), \ldots, y_n(x_1, \ldots, x_n, \theta), \theta'(x_1, \ldots, x_n, \theta)). \quad (4.2.19)$$

Now $\Phi_\ast \frac{\partial}{\partial \theta}$ is Killing, and so parallel as in the proof of Proposition 4.1.24, hence $S^1$-invariant. It follows that for each $i$, $\frac{\partial^2 y_i}{\partial \theta^2} = 0$. We deduce that for fixed $x_1, \ldots, x_n$, $y_i = A_i \theta + B_i$, where $A_i$ and $B_i$ depend on $x_1, \ldots, x_n$. It follows immediately that $A_i = 0$, as otherwise we do not have a well-defined map from the circle. Hence, we find that

$$\Phi_\ast \frac{\partial}{\partial \theta} = A \frac{\partial}{\partial \theta}, \quad (4.2.20)$$

for some function $A(x_1, \ldots, x_n, \theta)$. Taking the second derivatives of $\theta'$ in the same way, $A$ must be independent of $\theta$; since we still need a well-defined map from the circle, we also know that $A \in \mathbb{Z}$ for all points of $M$. Since $\Phi$ is close to $\Phi_0$ in the $C^1$ topology and $\Phi_0, \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta}$ we obtain $A = 1$ identically, as required.

**Remark.** It follows immediately from Proposition 4.2.22 that the identity component of the isometry group of a manifold $M \times S^1$ with an $S^1$-invariant Ricci-flat metric $g$ is contained in $\text{Diff}_0^S$. Hence, when we quotient by the automorphism group as in Theorem 4.2.15(c)(ii), it doesn’t matter whether we work with $\text{Diff}_0$ or $\text{Diff}_{S^1}^S$ as in a local neighbourhood of the origin the subgroups of isometries are the same. In practice, of course, the fact that Killing fields are $S^1$-invariant would also prove that the subgroups of isometries are the same locally around the identity.

We can now show that slices such as that in Proposition 4.2.17 restrict to slices in the $S^1$-invariant setting. For the asymptotically cylindrical case, we will need analogous results to Proposition 4.2.17 for Calabi-Yau structures as well, and so Proposition 4.2.23 contains three cases corresponding to Lemma 4.2.20 and the two cases of Lemma 4.2.21.

**Proposition 4.2.23.** Suppose that $W \times S^1$ is a six- or seven-dimensional manifold, where $W$ is one of a compact or asymptotically cylindrical six-manifold, or a compact five-manifold, and $W \times S^1$ has either an $S^1$-invariant torsion-free $G_2$ structure $\phi$ or an $S^1$-invariant Calabi-Yau structure $(\Omega, \omega)$ respectively. Let $\alpha$ be $\phi$ or $\text{Re} \, \Omega$ respectively. Suppose that

$$Y = E \oplus \{L_X \alpha : X \text{ vector field}\} \quad (4.2.21)$$

is an orthogonal splitting for some vector spaces of three-forms $E$ and $Y$.

Let $E'$ and $Y'$ be the intersection of $E$ and $Y$ respectively with the set of $S^1$-invariant three-forms. Then we have

$$Y' = E' \oplus \{L_X \phi : X \text{ $S^1$-invariant vector field}\}, \quad (4.2.22)$$

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again as an orthogonal splitting.

Proof. Lemmas 4.2.20 and 4.2.21 say that the space of Lie derivatives on the right hand side of (4.2.22) is precisely
\[ \{ \mathcal{L}_X \alpha : X \text{ vector field} \} \cap \{ S^1\text{-invariant forms} \}, \] (4.2.23)
and thus we just need to show that the orthogonal splitting is preserved when we intersect throughout with \( S^1\)-invariant forms. The two components obviously remain orthogonal, so it suffices to show that the projection to the Lie derivative part is \( S^1\)-equivariant, i.e. commutes with the pullback by any rotation \( \Theta \). We observe that \( \Theta^* \mathcal{L}_X \alpha = \mathcal{L}_{\Theta^{-1} X} \Theta^* \alpha \) for all \( X \).

Now given a 3-form \( \beta \) its Lie derivative part is the unique 3-form \( \gamma \) given by a Lie derivative of \( \alpha \) such that
\[ \langle \beta - \gamma, \mathcal{L}_X \alpha \rangle = 0 \quad \text{for all vector fields } X. \] (4.2.24)
As \( \Theta \) is an isometry, we have for any \( X \)
\[ \langle \Theta^* \beta - \Theta^* \gamma, \mathcal{L}_X \alpha \rangle = \langle \beta - \gamma, (\Theta^{-1})^* \mathcal{L}_X \alpha \rangle = \langle \beta - \gamma, \mathcal{L}_{\Theta^{-1} X} \alpha \rangle = 0, \] (4.2.25)
and from above \( \Theta^* \gamma \) is a Lie derivative of \( \Theta^* \alpha = \alpha \), and thus the Lie derivative part of \( \Theta^* \beta \) is \( \Theta^* \gamma \). In particular, if \( \beta \) is \( S^1\)-invariant its Lie derivative part is \( S^1\)-invariant. Thus its \( E \) part is too, and lies in \( E' \). We obtain the orthogonal splitting
\[ Y' = E' \oplus \{ \mathcal{L}_X \phi : [X, \frac{\partial}{\partial \theta}] = 0 \}. \] (4.2.26)
\[ \square \]
Applying Proposition 4.2.23 to Proposition 4.2.17, we obtain

**Corollary 4.2.24.** Let \( \phi \) be an \( S^1\)-invariant \( G_2 \) structure on \( M \times S^1 \). With respect to \( \phi \), let
\[ E' = \{ \alpha \in \Omega^3(M \times S^1) : \alpha \text{ } S^1\text{-invariant, } \text{d}\alpha = 0, \text{d}^* \alpha \in \Omega^2_{14} \}, \] (4.2.27)
where \( \Omega^2_{14} \) is the subbundle of 2-forms which at every point are in the subbundle corresponding to the 14-dimensional subrepresentation of \( G_2 \) on the space of alternating two-forms. Then \( E' \) is \( L^2\)-orthogonal to the space of Lie derivatives \( \mathcal{L}_X \phi \) of \( \phi \) for \( X \) \( S^1\)-invariant, with respect to the metric at \( \phi \), and the sum of these spaces is the set of all closed \( S^1\)-invariant three-forms. Consequently, locally \( E \) is transverse to the orbits of \( \text{Diff}_0 \).

We now proceed to the \( S^1\)-invariant version of the implicit function theorem argument for Proposition 4.2.18. We have to show that when we pass to the \( S^1\)-invariant setting the derivative \( DF \) is still surjective and has the same kernel. Again, we state and prove a more general version that will be used for the asymptotically cylindrical case. We first prove some easy lemmas
saying that harmonic forms are $S^1$-invariant. The proof is very similar to the proof of Lemma 4.1.13.

**Lemma 4.2.25.** Suppose $M \times S^1$ is a compact manifold with an $S^1$-invariant Riemannian metric, and $\alpha$ a harmonic form on it. Then $\alpha$ is $S^1$-invariant.

**Proof.** Consider a rotation $\Theta$. As $\Theta$ is isotopic to the identity, $[\Theta^\ast \alpha] = [\alpha]$. As the metric is $S^1$-invariant, $\Theta$ is also an isometry and so $\Theta^\ast \alpha$ is also a harmonic form. Thus, by Hodge decomposition, $\Theta^\ast \alpha = \alpha$, i.e. $\alpha$ is $S^1$-invariant. $\square$

**Lemma 4.2.26.** Suppose $M \times S^1$ is a manifold with an end $N \times S^1 \times (0, \infty)$, equipped with an asymptotically cylindrical $S^1$-invariant Riemannian metric, and $\alpha$ an asymptotically translation invariant harmonic form on it. Then $\alpha$ is $S^1$-invariant.

**Proof.** We apply asymptotically cylindrical Hodge theory from chapter 3 and the argument in Lemma 4.2.25. By Theorem 3.4.4 combined with Theorem 3.4.5, $\alpha$ is the sum of a decaying harmonic form, an exact harmonic form, and a coexact harmonic form. We first show that the exact harmonic form $\beta$, say, is $S^1$-invariant.

By the uniqueness of the exact representative in Theorem 3.4.4, the map from the exact harmonic form to its limit is injective. By Lemma 4.2.25, its limit is $S^1$-invariant. Thus, the exact harmonic form $\Theta^\ast \beta$ has the same limit: so $\Theta^\ast \beta = \beta$, as required. By the same argument the coexact harmonic form is also $S^1$-invariant. Taking the difference, we now have to show that the decaying harmonic form, $\gamma$ say, is $S^1$-invariant. Again, $\Theta^\ast \gamma$ is a decaying harmonic form, and it follows from Theorem 3.4.5 that the map from such forms to cohomology is injective. Thus $\Theta^\ast \gamma = \gamma$ exactly as in Lemma 4.2.25. $\square$

We now prove our general $S^1$-invariance proposition which we will apply to Proposition 4.2.18.

**Proposition 4.2.27.** Let $M$ be a compact or asymptotically cylindrical manifold. Let $F : X \rightarrow Y$ be a smooth $S^1$-equivariant nonlinear map between Banach spaces of (asymptotically translation invariant) differential forms on $M \times S^1$ (with $S^1$-invariant norms). Suppose that $DF$ is surjective and its kernel consists of harmonic forms. Suppose further that in the compact case $X$ is continuously contained in the space of $L^2$ forms and in the asymptotically cylindrical case we can find an $S^1$-invariant complement $W$ to the kernel of $DF$ in $X$. Then the restriction

$$F^{S^1} : X' := X \cap \{ S^1\text{-invariant forms} \} \rightarrow Y \cap \{ S^1\text{-invariant forms} \} =: Y'$$

(4.2.28)

is a well-defined map, with $DF^{S^1}$ surjective and the same kernel as $DF$.

**Proof.** Since $F$ is $S^1$-equivariant, the image of an $S^1$-invariant form under it is an $S^1$-invariant form. Consequently, $F^{S^1}$ is a well-defined map.
We now consider the derivative. By hypothesis, its kernel consists of (asymptotically translation invariant) harmonic forms; by Lemmas 4.2.25 and 4.2.26 we know that these are $S^1$-invariant, and consequently we know that the kernel of $DF^{S^1}$ agrees with the kernel of $DF$. For surjectivity, note first that $DF^{S^1}$ must also be $S^1$-equivariant.

Suppose we have a form $\alpha \in F(X')$, and let $\dot{\alpha}$ be a tangent to $Y'$ at $\alpha$, so we have to show that $\dot{\alpha}$ is in the image of $DF^{S^1}$. Since $DF$ is surjective, we can find $\dot{\beta}$ with $DF(\dot{\beta}) = \dot{\alpha}$. There is always an $S^1$-invariant complement $W$ to the finite-dimensional kernel of $DF$; in the asymptotically cylindrical case, this is assumed, and in the compact case we may let $W$ be the $L^2$-orthogonal complement and note that since the metric is $S^1$-invariant, $W$ is also $S^1$-invariant. We suppose that $\dot{\beta}$ lies in $W$.

Now, given a rotation $\Theta$, since $DF^{S^1}$ is $S^1$-equivariant and $\dot{\alpha}$ is $S^1$-invariant, $\Theta^*\dot{\beta}$ also maps to $\dot{\alpha}$. Since $W$ is $S^1$-invariant, $\Theta^*\dot{\beta}$ is also in $W$. Consequently the difference $\dot{\beta} - \Theta^*\dot{\beta}$ is in $W$ and maps to zero; hence it is zero, and $\dot{\beta}$ is $S^1$-invariant. Hence, $\dot{\alpha} = DF^{S^1}(\dot{\beta})$, and $DF^{S^1}$ is surjective.

Applying Proposition 4.2.27 to Proposition 4.2.18 gives

**Corollary 4.2.28.** Let $\phi_0$ be a torsion-free $S^1$-invariant $G_2$ structure on $M \times S^1$ and let $E$ and $F$ be as in Proposition 4.2.18. When we restrict all the spaces to be $S^1$-invariant (passing to $E'$ and so forth), the kernel of $DF$ is unchanged and it remains surjective.

**Proof.** We apply Proposition 4.2.27. We have to check that $F$ is $S^1$-equivariant and that its kernel consists of harmonic forms. The fact the kernel consists of harmonic forms is stated in Proposition 4.2.18. For $S^1$-equivariance, we recall that $F(\phi) = P(\ast_0 \ast \phi)$ where $\ast_0$ and $\ast$ are the Hodge stars induced by $\phi_0$ and $\phi$ and $P$ is the orthogonal projection onto exact forms induced by $\phi_0$. Since $\phi_0$ is $S^1$-invariant, $\ast_0$ and $P$ are $S^1$-equivariant ($P$ was proved to be so as part of Proposition 4.2.27). On the other hand, it is clear that the map $\phi \mapsto \ast \phi$ is $S^1$-equivariant. Consequently, $F$ is $S^1$-equivariant. By Proposition 4.2.27, the result follows.

Corollaries 4.2.24 and 4.2.28 essentially prove the compact case of Theorem 4.2.29.

The asymptotically cylindrical case is similar but more involved. Propositions 4.2.23 and 4.2.27 will suffice to prove everything, but we have to apply them to the proof of smoothness of the asymptotically cylindrical $G_2$ moduli space in Nordström [65] which is substantially more complicated than Hitchin’s proof for the compact case. We consequently only summarise this case.

We begin by considering the limit as in [65, section 4]. The limit is a torsion-free Calabi–Yau structure on the cross-section $N \times S^1$, so first of all requires us to define an $S^1$-invariant Calabi–Yau moduli space on $N \times S^1$. We could define such a moduli space using the compact case arguments in the present chapter, passing to $T^2$-invariant $G_2$ structures. Setting up a moduli space of $T^2$-invariant $G_2$ structures and repeating all the compact $S^1$-invariant arguments above for $T^2$-invariance would require significantly more effort, and so we just follow [65, section 4].
As in the compact $G_2$ case, the proof is fundamentally that we first choose a slice and then consider the torsion-freeness map from that slice. In our case, we work with a base Calabi–Yau structure $(\Omega, \omega)$ which is $S^1$-invariant. The slice is defined in [65, Proposition 4.7], by identifying an orthogonal complement to the Lie derivatives of $\text{Re} \Omega$ (essentially symmetrically to that in Proposition 4.2.17). It is then not necessary to consider the Lie derivatives of $\omega$, as each diffeomorphism can be identified by its $\text{Re} \Omega$ part. The Calabi–Yau case of Proposition 4.2.23 says that when all forms in these splittings are taken $S^1$-invariant, the reduced orthogonal complement is still an orthogonal complement.

The map defining torsion-freeness is $F$ in [65, Definition 4.12], viz.

$$F(\beta, \gamma) = (P_1(*\hat{\beta}), P_2(\beta \wedge \gamma), \frac{1}{4} \beta \wedge \hat{\beta} - \frac{1}{6} \gamma^3),$$  

(4.2.29)

where, for $(\beta, \gamma)$ close to $(\text{Re} \Omega, \omega)$ $\hat{\beta}$ is the imaginary part of the unique decomposable complex $3$-form of which $\beta$ is the real part, $P_1$ is the orthogonal projection to those three-forms in the slice which are orthogonal to harmonic forms, and $P_2$ is an orthogonal projection on closed five-forms induced by the Calabi–Yau structure (to the harmonic forms and exterior derivatives of $(3, 1) + (1, 3)$ forms).

Wedge products are clearly $S^1$-equivariant. The map $\beta \mapsto \hat{\beta}$ is $S^1$-equivariant by uniqueness of the decomposable complex $3$-form, so to show that $F$ is $S^1$-equivariant we only need to show that the two projections are. However, the Calabi–Yau structure with respect to which these splittings are taken is $S^1$-invariant, so the subspaces that these splittings project to and their complements are $S^1$-invariant. It follows as in Proposition 4.2.23 that the projection maps are $S^1$-equivariant.

[65, Proposition 4.14] says that $DF$ is surjective and [65, Proposition 4.15] says that the kernel of $DF$ consists of harmonic forms. By Proposition 4.2.27 it follows that the kernel is the same and the derivative remains surjective when we pass to the $S^1$-invariant case.

This proves that the $S^1$-invariant moduli space of Calabi–Yau structures on $N \times S^1$ is a smooth manifold locally diffeomorphic to that for all Calabi–Yau structures on $N \times S^1$.

Only a subspace of these Calabi–Yau structures might arise as limits of a torsion-free $G_2$ structure: we have to check that this subspace is still a manifold. In the non-$S^1$-invariant case, this is [65, Proposition 6.2]. Again, the proof is that structures arising as limits correspond to the kernel of two nonlinear maps (taken consecutively). Note that the tangent space to the Calabi–Yau moduli space already consists of harmonic forms, so this hypothesis of Proposition 4.2.27 does not need checking. The nonlinear maps concerned are composites of the wedge product, orthogonal projections to $S^1$-invariant subspaces determined by the base Calabi–Yau structure, and the orthogonal projection to the complement of such an $S^1$-invariant subspace with respect to the metric induced by the Calabi–Yau structure we consider. These are clearly all $S^1$-equivariant, and so Proposition 4.2.27 shows that the derivatives remain surjective, so that the subspace of the $S^1$-invariant moduli space corresponding to limits of $S^1$-invariant torsion-
free $G_2$ structures is indeed a submanifold.

We now must pass to the full asymptotically cylindrical setting. We must, here, restrict to asymptotically cylindrical structures and diffeomorphisms with a fixed decay rate $\delta > 0$ to define our Banach spaces, as in [65].

The slice we take in the full asymptotically cylindrical setting is given in [65, Proposition 6.11]. We restrict to the subspace of asymptotically translation invariant closed 3-forms with suitable limits, and in particular to vector fields (defining diffeomorphisms) whose limits are Killing fields for the limit structure. We then take $E$ to be the subspace satisfying $d^*\alpha \in \Omega^2_{14}$ as in Proposition 4.2.17. Then we again have that $E$ and the space of Lie derivatives are orthogonal complements. By Proposition 4.2.23, it follows that this orthogonal splitting is preserved when we pass to the $S^1$-invariant setting.

The final map we consider is $F$ of [65, Definition 6.13]: $F(\phi) = P(\star_0 \ast \phi)$ exactly as in Proposition 4.2.18; by [65, Proposition 6.15] the kernel of $F$ is precisely the torsion-free $G_2$ structures. Exactly as in Corollary 4.2.28, $F$ is $S^1$-equivariant. [65, Proposition 6.17] says that the kernel of the derivative, when we restrict to the slice, consists exactly of harmonic forms, and the derivative is surjective. Moreover, the kernel is all the harmonic forms with suitable limits, and by construction every nonzero limit arises as a limit of a harmonic form, and decaying forms orthogonal to harmonic forms form an $S^1$-invariant complement for the kernel. Consequently by Proposition 4.2.27 the derivative has the same kernel and is surjective in the $S^1$-invariant setting too.

We have now essentially proved

**Theorem 4.2.29.** In both the compact and asymptotically cylindrical cases the moduli space of $S^1$-invariant $G_2$ structures $\mathcal{M}_{S^1 G_2}$ of Definition 4.2.11 is a manifold and is locally diffeomorphic to the moduli space of $G_2$ structures $\mathcal{M}_{G_2}$ of Definition 4.2.7.

The remaining parts of this proof are passing from the reduced-regularity and fixed-decay-rate Banach spaces to smooth and all asymptotically cylindrical structures: $S^1$-invariance is irrelevant to these, which thus follow exactly as in [35] and [65].

Note that the map $\mathcal{M}_{S^1 G_2} \to \mathcal{M}_{G_2}$ is not known to be an inclusion map as it need not be globally injective. More evidently, it may not be globally surjective. However, if the fundamental group $\pi_1(M)$ is finite, then $H^1(M \times S^1)$ is one-dimensional, and so if $M \times S^1$ has a torsion-free $G_2$ structure it has a one-dimensional vector space of parallel Killing fields. On the other hand, the identity component of the isometry group of $M \times S^1$ is a compact Lie group (see for instance [51, Lemma 3.6] for the asymptotically cylindrical case). Hence, this identity component is a one-dimensional compact group, hence is the circle. That is, we always have a circle action on $M \times S^1$. As the Killing field is parallel so cannot vanish, this action is free. This shows that any torsion-free $G_2$ structure on $M \times S^1$ with $\pi_1(M)$ finite is “$S^1$-invariant” for some notion of $S^1$-invariance.

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Remark. It is tempting to try to prove Theorem 4.2.29 more directly. We outline this “more direct” proof in the compact case, and explain where it runs into difficulty.

Every $S^1$-invariant $G_2$ structure is a $G_2$ structure, and every $S^1$-invariant diffeomorphism is a diffeomorphism. Consequently, there is a natural map $\mathcal{M}^{S_1}_{G_2} \rightarrow \mathcal{M}_{G_2}$; this map is well-defined and continuous. If we could show directly that this map was a local homeomorphism, we could induce this to induce a manifold structure on $\mathcal{M}^{S_1}_{G_2}$ that is a local diffeomorphism.

Locally, we have a slice neighbourhood of $G_2$ structures around an $S^1$-invariant $G_2$ structure $\phi_0$; that is, we have a neighbourhood $U$ of the class of $\phi_0$ in $\mathcal{M}_{G_2}$. The inverse image $V$ of this under $\mathcal{M}^{S_1}_{G_2} \rightarrow \mathcal{M}_{G_2}$ is a neighbourhood of the class of $\phi_0$ in $\mathcal{M}^{S_1}_{G_2}$.

If we could show that $V \rightarrow U$ is a homeomorphism, then $\mathcal{M}^{S_1}_{G_2} \rightarrow \mathcal{M}_{G_2}$ is a local homeomorphism and we are done. It is fairly easy to see that this is surjective: from Theorem 4.2.15 the slice neighbourhood for $\mathcal{M}_{G_2}$ around an $S^1$-invariant $G_2$ structure $\phi_0$ must consist of $S^1$-invariant $G_2$ structures, because we know its representatives are preserved by the isometry $\Theta$ of $\phi_0$. Hence, each class in $U$ has an $S^1$-invariant representative, and these representatives define classes in $V$: hence $V \rightarrow U$ is surjective.

Injectivity is somewhat more complicated. We have to check that if $\phi_1$ and $\phi_2$ are $S^1$-invariant representatives of the same class in $U$, they define the same class in $V$. That is, given $\phi_1$ a slice representative and $\Phi \in \text{Diff}_0$, we have to show $\phi_2 = \Phi^* \phi_1$ is also $\Psi^* \phi_1$ for some $\Psi \in \text{Diff}^{S^1}_0$.

If we have a curve $\phi_t$ of $S^1$-invariant $G_2$ structures from $\phi_1$ to $\phi_2$ then, at least after passing to a suitable $C^{k,\alpha}$ space, we know by Theorem 4.2.15 that for all $t$, $\phi_t$ is the pullback of an element in the slice and can thus be written as $\Phi^*_t \phi_t$ with $\phi_t$ always in the slice. (In particular, $\hat{\phi}_2 = \hat{\phi}_1 = \hat{\phi}_1$). We know also, as $\hat{\phi}_t$ is in the slice, that $\hat{\phi}_t$ is $S^1$-invariant. Theorem 4.2.15 says that isometries of $\hat{\phi}_t$ are isometries of $\hat{\phi}_0$, and it follows that $\Phi^*_t \phi_0$ is also $S^1$-invariant, by considering the isometry $\Phi_t \circ \Theta \circ \Phi_t^{-1}$ of $\phi_t$. It then follows from Proposition 4.2.22 that $\Phi_t$ is $S^1$-invariant for all $t$; in particular, $\Phi_2$ is $S^1$-invariant, and this proves that the map $V \rightarrow U$ is injective.

We only require such curves locally, because $V$ is a neighbourhood. Thus this proof reduces to

Claim 4.2.30. The $S^1$-invariant torsion-free $G_2$ structures form a locally path-connected subset of torsion-free $G_2$ structures.

Without torsion-freeness this claim is evident because of the openness of $G_2$ structures. The natural thing to do is to take a path of torsion-free $G_2$ structures and average out the rotation, but we cannot take averages because the map defining the torsion is non-linear. Removing the remaining torsion would therefore require some analysis: it should be possible, but is unlikely to be easier than the arguments we have outlined in this subsection.

To complete this extended remark, we will observe where Claim 4.2.30 comes in our original proof, or, essentially equivalently, how we have local path-connectedness for all torsion-free
$G_2$ structures (before passing to $S^1$-invariant ones). The idea is that we only use the whole set of torsion-free $G_2$ structures as the product of diffeomorphisms and the slice. First, we show that the set of torsion-free $G_2$ structures is locally this product, essentially by using the implicit function theorem on $(\Phi, \phi) \mapsto \Phi^* \phi$; then we use the implicit function theorem again to determine that the torsion-free $G_2$ structures in what’s left are also a manifold, and so locally path-connected. Since we already know that the diffeomorphisms are a manifold, so locally path-connected, we know that the set of torsion-free $G_2$ structures is a product of locally path-connected spaces and so locally path-connected. It would perhaps be possible to combine these applications of the implicit function theorem and show Claim 4.2.30 directly, by showing that $S^1$-invariant torsion-free $G_2$ structures are themselves a manifold; but the whole point of this “more direct argument” is to avoid these two applications of the implicit function theorem (which correspond, for instance, to Corollaries 4.2.24 and 4.2.28).

Before turning to the components $\mathcal{M}_{SU(3)}$ and $Z$ of $\mathcal{M}^{S^1}_{G_2}$, we deal with the question posed after Definition 4.2.6, of what would happen if we defined our moduli spaces to also quotient by the rescaling action. We make the following

**Remark 4.2.31.** Suppose for simplicity that $M$ is compact; similar arguments will apply in the asymptotically cylindrical case. We know that there is a natural rescaling action on both $SU(3)$ structures and $G_2$ structures. The action induced on $S^1$-invariant $G_2$ structures by rescaling of $SU(3)$ structures is not just rescaling: it maps the $G_2$ structures $a^2 \Re \Omega + az \wedge \omega$ to $\Re \Omega + z \wedge \omega$. Consequently, if we quotient by rescaling of $SU(3)$ structures, we have to quotient by this partial rescaling of $S^1$-invariant $G_2$ structures, otherwise Proposition 4.2.14 no longer holds. If we also quotient by rescaling the $G_2$ structures, we find in particular that $\Re \Omega + z \wedge \omega$ is identified with $a^2 \Re \Omega + a^{\frac{3}{2}} z \wedge \omega$ and hence with $\Re \Omega + a^{\frac{3}{2}} z \wedge \omega$; that is, we are also quotienting by rescaling of $Z$. The natural slice to take for $Z$ is to recall that an element $z$ of $Z$ is of the form $L[d\theta] + [v]$ for some $[v] \in H^1(M)$, and merely insist that $L = 1$. An easy calculation shows that if $L = 1$ and $(\Omega, \omega)$ induces a metric of volume one, then so too does $\Re \Omega + z \wedge \omega$.

Consequently, we could quotient by these and establish the following analogue of Proposition 4.2.14:

$$\{[\Omega, \omega] \in \mathcal{M}_{SU(3)} : \text{Vol}([\Omega, \omega]) = 1\} \times H^1(M) = \{[\phi = \Re \Omega + z \wedge \omega] \in \mathcal{M}^{S^1}_{G_2} : \text{Vol}([\phi]) = \text{Vol}([\Omega, \omega]) = 1\}. \tag{4.2.30}$$

Of course, the analogue of Theorem 4.2.29 in this case is completely false: even if we quotient by rescaling of $G_2$ structures, so that we work with $\{[\phi] \in \mathcal{M}_{G_2} : \text{Vol}([\phi]) = 1\}$, $\{[\phi = \Re \Omega + z \wedge \omega] \in \mathcal{M}^{S^1}_{G_2} : \text{Vol}([\phi]) = \text{Vol}([\Omega, \omega]) = 1\}$ must be a proper subspace. Consequently, to prove that this space is smooth we would essentially have to proceed by the same argument as in this subsection and then continue. It is in this sense that we claimed after Definition 4.2.6 that quotienting by the rescaling action added additional complexity for no practical gain.

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4.2.3 The space of twisting classes $Z$

We now know that $\mathcal{M}_{G_2}^{S^1} = Z \times \mathcal{M}_{SU(3)}$ is a manifold; we want to know that both factors are manifolds and if we take the product manifold structure on the right hand side this identification is a diffeomorphism. The quotient $Z$ of Definition 4.2.12 is easier, so we will start there. $Z$ is clearly an open subset of a vector space; the purpose of this subsection is to obtain a description of this vector space intrinsic to $M$. We use the Hodge theory described in chapter 3.

This is the only place where we explicitly need Remark 4.1.12 and its $G_2$ analogue: that all interesting Ricci-flat asymptotically cylindrical manifolds have connected cross-section $N$.

Lemma 4.2.32. Let $M \times S^1$ be compact or asymptotically cylindrical. In the case that $M \times S^1$ is asymptotically cylindrical, suppose further that the cross-section $N$ of $M$ is connected. Then the quotient

$$ \frac{\text{closed $S^1$-invariant covector fields $z$ (with $z(\frac{\partial}{\partial t}) \to 0$ exponentially)}}{\text{differentials of $S^1$-invariant (asymptotically translation invariant) functions}} $$

is isomorphic to $H^1(M \times S^1) = H^1(M) \times \mathbb{R}$; in particular, if we restrict to $Z$, where the $[d\theta]$ component must be positive, we get the open subset $Z = H^1(M) \times \mathbb{R}_{>0}$.

**Proof.** It is clear that $Z$ corresponds to cohomology classes containing a positive $[d\theta]$ component, so it is enough to show that (4.2.31) is isomorphic to $H^1(M \times S^1)$. In the compact case, (4.2.31) reduces to

$$ \frac{\text{closed $S^1$-invariant covector fields $z$}}{\text{differentials of $S^1$-invariant functions}}. $$

(4.2.32)

Since a closed $S^1$-invariant covector field is of the form $v + cd\theta$ where $v$ is a closed 1-form on $M$ and $c$ is a constant, and an $S^1$-invariant function is just a function on $M$, it is easy to see that (4.2.32) is indeed isomorphic to $H^1(M) \oplus \mathbb{R} = H^1(M \times S^1)$.

We clearly have a map from closed $S^1$-invariant covector fields $z$ on $M \times S^1$, with $z(\frac{\partial}{\partial t}) \to 0$ exponentially, to $H^1(M \times S^1)$. We have to show that the induced map from (4.2.31) to $H^1(M \times S^1)$ is a well-defined bijection. Since the quotient in (4.2.31) is by exact forms, the induced map is clearly well-defined. To show that it is injective, we note first that as $N \times S^1$ is connected the long exact sequence of Theorem 2.1.2 reduces to

$$ 0 \to H^1_{\text{rel}}(M \times S^1) \to H^1_{\text{abs}}(M \times S^1) \to H^1(N \times S^1) \to \cdots. $$

(4.2.33)

We have to show that if $v$ is $S^1$-invariant with $v(\frac{\partial}{\partial t}) \to 0$ exponentially and $v$ represents the zero cohomology class of $H^1_{\text{abs}}(M \times S^1)$, then $v$ is the differential of an asymptotically translation invariant $S^1$-invariant function. Since $\bar{v}(\frac{\partial}{\partial t}) = 0$, $\bar{v}$ is a form on $N \times S^1$, and its cohomology class is just the image of $[v]$ under the restriction map; hence, $[\bar{v}] = 0$. By the compact case, $\bar{v} = dg$ for some $S^1$-invariant function $g$. Then $v - d(\psi g)$, where as in equation (2.1.1) $\psi$ is one for $t$ large and zero for $t$ small, has zero limit, and so by Lemma 3.4.6 represents
a class in \( H^1_{\text{rel}}(M \times S^1) \). On the other hand, the image of this class in \( H^1_{\text{abs}}(M \times S^1) \) must be trivial, and so the class in \( H^1_{\text{rel}} \) must be trivial by (4.2.33). Applying Lemma 3.4.6 again, we have \( v - d(\psi g) = dh \) for some decaying function \( h \); since \( v - d(\psi g) \) is \( S^1 \)-invariant, we may assume that \( h \) is. Then we have \( v = d(\psi g + h) \); \( \psi g + h \) is an asymptotically translation invariant \( S^1 \)-invariant function, as required.

To show surjectivity, suppose that \([v]\) is a cohomology class. By Corollary 3.5.12, we can find a harmonic representative \( v \) for this class, and \( v((\partial/\partial t) \to 0 \) exponentially. By Lemma 4.2.26, \( v \) is \( S^1 \)-invariant and hence the map is clearly surjective.

4.2.4 Smoothness of the \( SU(3) \) moduli space

We now turn to the \( M_{SU(3)} \) factor. Since we have it as a subspace of \( \mathcal{M}_{G_2}^{S^1} \) by Proposition 4.2.14, and we understand the structure of \( \mathcal{M}_{G_2}^{S^1} \) by Theorem 4.2.29, we have a reasonable knowledge of its structure as a topological space. It remains to understand the \( \mathcal{M}_{SU(3)} \) factor as a smooth manifold. We use the projection \( \pi_Z \) from \( \mathcal{M}_{G_2}^{S^1} \) to \( Z \). We will show that \( \pi_Z \) is a submersion. Its fibres are precisely \( \mathcal{M}_{SU(3)} \)'s, and so the implicit function theorem will give a family of manifold structure on \( \mathcal{M}_{SU(3)} \). We then have to check that the manifold structure is independent of which fibre we take, and that consequently we indeed have a smooth product.

Firstly, we now know that \( \mathcal{M}_{SU(3)}(M) \) is locally homeomorphic to a subset of cohomology.

**Proposition 4.2.33.** In the compact case,

\[
[\Omega, \omega] \mapsto ([\text{Re } \Omega], [\omega]) \in H^3(M) \oplus H^2(M)
\]

is a local homeomorphism to its image. In the asymptotically cylindrical case,

\[
[\Omega, \omega] \mapsto ([\text{Re } \Omega], [\omega], [\text{Re } \hat{\Omega}_2], [\hat{\omega}_2]) \in H^3(M) \oplus H^2(M) \oplus H^2(N) \oplus H^1(N)
\]

is a local homeomorphism to its image, where \( \text{Re } \hat{\Omega}_2 \) and \( \hat{\omega}_2 \) are, as in Theorem 4.2.15, the appropriate components of \( \text{Re } \hat{\Omega} = \text{Re } \hat{\Omega}_1 + dt \wedge \text{Re } \hat{\Omega}_2 \) and \( \hat{\omega} = \hat{\omega}_1 + dt \wedge \hat{\omega}_2 \).

**Proof.** We first apply Theorem 4.2.29, which says that locally \( \mathcal{M}_{G_2}^{S^1} \) is homeomorphic to \( \mathcal{M}_{G_2} \) and hence to cohomology. Then, in the compact case, the result follows by combining Proposition 4.2.14 with the Künneth theorem. Specifically, given a point \([\Omega, \omega]\), take a neighbourhood of \([\text{Re } \Omega + d\theta \land \omega] \) in \( \mathcal{M}_{G_2}^{S^1} \) that is homeomorphic to a neighbourhood in \( \mathcal{M}_{G_2}(M \times S^1) \) and so to a neighbourhood in \( H^3(M \times S^1) \). Then the map (4.2.34) is given by the composition

\[
\mathcal{M}_{SU(3)} \longrightarrow \mathcal{M}_{G_2}^{S^1} \longrightarrow H^3(M \times S^1) \longrightarrow H^3(M) \oplus H^2(M)
\]

\[
[\Omega', \omega'] \mapsto [\text{Re } \Omega' + d\theta \land \omega'] \mapsto [\text{Re } \Omega' + d\theta \land \omega'] \mapsto ([\text{Re } \Omega'], [\omega']).
\]
Consequently, (4.2.34) is continuous because every individual step is. The inverse can be written in exactly the same way and so is also continuous (the last map being the projection $\mathcal{M}^{S^1}_{G_2} \to \mathcal{M}_{SU(3)}$).

In the asymptotically cylindrical case, the only difficulty is that we have to use the Künneth theorem on the cross-section as well. The map from $[\Omega', \omega']$ to $[\phi' = \text{Re} \Omega' + d\theta \wedge \omega']$ is a local homeomorphism to its image exactly as in the compact case. We already know that (with the notation of Theorem 4.2.15) the map

$$[\phi'] \mapsto ([\phi'], [\tilde{\phi}']) \in H^3(M \times S^1) \oplus H^2(N \times S^1)$$

(4.2.36)

is a local homeomorphism to its image; finally, the map from $H^3(M \times S^1) \oplus H^2(N \times S^1)$ to $H^3(M) \oplus H^2(M) \oplus H^2(N) \oplus H^1(N)$ is again continuous in both directions, using the Künneth theorem for both $H^3(M)$ and $H^2(N)$. \hfill \Box

Note that we know nothing about the image of the maps (4.2.34) and (4.2.35) as yet: we certainly do not know that they are submanifolds, which would require us to prove in turn that the image is transverse to the comparable subset given by perturbing $z$, and so Proposition 4.2.33 doesn’t immediately give us a manifold structure on $\mathcal{M}_{SU(3)}$.

We now seek to show that the topological product $\mathcal{M}^{S^1}_{G_2}(M \times S^1) \cong \mathcal{M}_{SU(3)}(M) \times Z$ can actually be taken smooth; in particular, that $\mathcal{M}_{SU(3)}(M)$ is a manifold. We will make use of the projection map

$$\pi_Z : \mathcal{M}^{S^1}_{G_2}(M \times S^1) \to Z.$$ (4.2.37)

We will show first that $\pi_Z$ is smooth, then that it is a surjective submersion; the implicit function theorem then implies that the fibres, which are clearly $\mathcal{M}_{SU(3)}$’s, have manifold structures.

For smoothness, we work locally, and so may assume we have a subset of torsion-free $G_2$ structures (open in some suitable slice). We already know from Proposition 4.1.22 that the map taking a $G_2$ structure to the twisting $z$ is smooth (as a map of Fréchet spaces). Since the map from $z$ to its cohomology class $[z]$ is linear and continuous, it is evidently smooth.

Thus we have a smooth map between finite-dimensional manifolds

$$\pi_Z : \mathcal{M}^{S^1}_{G_2} \to Z.$$ (4.2.38)

To show that $\pi_Z$ is a surjective submersion we will use the following elementary

**Lemma 4.2.34.** Suppose that $(\Omega, \omega)$ is an (asymptotically cylindrical) Calabi–Yau structure on $M$ and that $z(s)$ is a smooth curve of closed 1-forms with $[z(s)] \in Z$ for all $s$. Then the curve

$$[\text{Re} \Omega + z(s) \wedge \omega] \in \mathcal{M}^{S^1}_{G_2}(M \times S^1)$$

(4.2.39)

is smooth.
Proof. It is obvious that \( \text{Re } \Omega + z(s) \wedge \omega \) is a smooth curve of closed three-forms. The map to \( \mathcal{M}^3_{G_2} \) is given locally by taking certain cohomology classes, by Theorem 4.2.15. This map is linear and continuous and so is smooth to cohomology classes; hence it is smooth to the image of \( \mathcal{M}^3_{G_2} \) in cohomology. (In the compact case, of course, this image is just an open subset). \( \square \)

Lemma 4.2.34 yields

**Proposition 4.2.35.** The map \( \pi_Z \) of (4.2.37) is a surjective submersion.

**Proof.** Surjectivity is already done, since \( \pi_Z \) is the projection from a product.

To prove \( \pi_Z \) is a submersion is easy using Lemma 4.2.34. Given a tangent vector \([y] \in H^1(M \times S^1) \) at \([z] \in Z\), we have to show that for every \([\phi] \in \pi_Z^{-1}([z])\) there is a tangent vector at \([\phi] \) that maps to \([y]\) under \( D\pi_Z \). By Proposition 4.2.14, we know any such \([\phi] = \text{Re } \Omega + z(\omega)\) for some Calabi–Yau structure \((\Omega, \omega)\) and representative \(z\). Pick some representative \(y\) for the tangent. \( z + sy \) is a smooth curve and, by openness of \( Z \), \([z + sy] \in Z\) for \( s \) small enough. By Lemma 4.2.34, therefore, \( \gamma(s) = \text{Re } \Omega + (z + sy) \wedge \omega \) is a curve in \( \mathcal{M}^3_{G_2} \) through \([\phi]\). We consider its tangent at \([\phi] = \gamma(0)\).

\[
D\pi_Z \left( \frac{d}{ds} \bigg|_0 \gamma \right) = \frac{d}{ds} \bigg|_0 (\pi \circ \gamma) = \frac{d}{ds} \bigg|_0 ([s \mapsto z + sy]) = [y], \quad (4.2.40)
\]

and so we have a submersion. \( \square \)

In particular, the implicit function theorem now proves that every \( \mathcal{M}_{SU(3)} \) fibre has a smooth structure, possibly different for each fibre. As we already have a topological product, we next have to show that all these smooth structures are the same and that the map to the fibre implied by the product is smooth; it is then straightforward to show that we obtain a smooth product.

Recall that \( Z \) is \( \mathbb{R}_{>0} \times H^1(M) \), the quotient of \( S^1 \)-invariant one-forms with suitable boundary conditions and orientation by suitable exact one-forms.

**Proposition 4.2.36.** Suppose \([z_1]\) and \([z_2]\) are classes of \( Z \). The map \( \pi_Z^{-1}([z_1]) \to \pi_Z^{-1}([z_2]) \) given using the product structure of Proposition 4.2.14 by projection to \( \mathcal{M}_{SU(3)} \) and inclusion is a diffeomorphism when these fibres are equipped with their submanifold smooth structures.

**Proof.** We first prove the case where \( M \) is compact. Fix a class \([\phi_1] = \text{Re } \Omega + z_1(\omega)\) in \( \pi_Z^{-1}([z_1])\) and its image \([\phi_2] = \text{Re } \Omega + z_2(\omega)\) in \( \pi_Z^{-1}([z_2])\). Locally around these two, we know by Theorems 4.2.29 and 4.2.15 that \( \mathcal{M}^3_{G_2} \) is locally diffeomorphic to \( H^3(M \times S^1) \). Consequently, the fibres \( \pi_Z^{-1}([z_1]) \) and \( \pi_Z^{-1}([z_2]) \) are locally submanifolds of \( H^3(M \times S^1) \). The map in the statement defines a map between those submanifolds, which we want to show is smooth. It suffices to show that there is a smooth map on \( H^3(M \times S^1) \) which agrees with this map on \( \pi_Z^{-1}[z_1] \).

Suppose that \([z_1] = L_1[d\theta] + [v_1]\) and \([z_2] = L_2[d\theta] + [v_2]\). By the Künneth theorem, we know that \( H^3(M \times S^1) \cong H^3(M) \oplus H^2(M) \). We define a map on \( H^3(M \times S^1) \cong H^3(M) \oplus H^2(M) \)
by
\[ H^3(M) \oplus H^2(M) \ni ([\alpha], [\beta]) \mapsto ([\alpha] + \frac{[v_2] - [v_1]}{L_1} \wedge [\beta], \frac{L_2}{L_1}[\beta]). \tag{4.2.41} \]

This is linear and so certainly smooth. Suppose now that \((\alpha, \beta) \in \pi_Z^{-1}([z])\) is close to \([\phi_1]\). Then \(\alpha + d\theta \wedge \beta = [\text{Re } \Omega' + (L_1d\theta + v_1) \wedge \omega']\). It follows that \([\alpha] = [\text{Re } \Omega' + v_1 \wedge \omega']\) and \([\beta] = [L_1\omega']\); hence the image of this map is \([\text{Re } \Omega' + v_2 \wedge \omega'], [L_2\omega']\) = \([\text{Re } \Omega' + (L_2d\theta + v_2) \wedge \omega']\). This is precisely the image under the map in the statement, and this proves the result in the compact case.

The asymptotically cylindrical case is very similar: the additional linear map \(H^2(N) \oplus H^1(N) \ni ([\alpha], [\beta]) \mapsto ([\alpha], \frac{L_2}{L_1}[\beta])\) for the limit factor behaves identically, and the fact that \(M_{G_2}^{S^1}\) is only diffeomorphic to a submanifold of \(H^3(M \times S^1) \oplus H^2(N \times S^1)\) does not affect the argument.

We can be more concrete about what the smooth structure on the fibre \(\pi_Z^{-1}([z])\) is. To set up our moduli space of gluing data in Proposition 4.3.33, we will need to know that the moduli spaces have coordinates corresponding to a suitable set of structures (essentially slice coordinates as in the \(G_2\) case). We could in principle have introduced these on each of the fibres and checked that the corresponding smooth structures coincided, but the argument via cohomology is easier.

**Proposition 4.2.37.** Suppose that \(\Omega, \omega \in M_{SU(3)}\), and that \((\Omega, \omega)\) is a Calabi–Yau structure representing it. Then there exists a subset \(U\) of Calabi–Yau structures containing \((\Omega, \omega)\) such that \(U\) is diffeomorphic to a neighbourhood of \([\Omega, \omega] \in M_{SU(3)}\), and such that the group of automorphisms \((\Omega', \omega')\) isotopic to the identity is independent of \((\Omega', \omega') \in U\).

**Proof.** First we write \(\phi = \text{Re } \Omega + d\theta \wedge \omega; \phi\) is an \(S^1\)-invariant \(G_2\) structure. Consequently, by Theorems 4.2.29 and 4.2.15, there exists a chart \(V\) for \(M_{G_2}^{S^1}\) diffeomorphic to a set of torsion-free \(S^1\)-invariant \(G_2\) structures and such that the group of automorphisms of \(\phi'\) isotopic to the identity is independent of \(\phi' \in V\).

By Theorem 4.1.26, such torsion-free \(S^1\)-invariant \(G_2\) structures \(\phi'\) are given by a twisting \(z'\) and a Calabi–Yau structure \((\Omega', \omega')\). Let \(U\) be the set of Calabi–Yau structures
\[ U = \{ (\Omega', \omega') : \exists z' \in [d\theta] \text{s.t. } \text{Re } \Omega' + z' \theta \wedge \omega' \in V \}. \tag{4.2.42} \]

\(U\) is precisely the set \(\pi_Z^{-1}([d\theta])\) expressed in the local coordinates provided by \(V\); hence \(U\) defines a chart for \(M_{SU(3)}\) containing \((\Omega, \omega)\), as needed.

It remains to check that the automorphisms isotopic to the identity don’t vary with the Calabi–Yau structure in \(U\). We apply the ideas of subsection 4.2.1. Suppose that \((\Omega', \omega')\) and \((\Omega'', \omega'')\) are structures in \(U'\), and \(\Phi\) is an automorphism of \((\Omega', \omega')\) isotopic to the identity. There are \(z', z'' \in [d\theta]\) such that \(\text{Re } \Omega' + z' \wedge \omega', \text{Re } \Omega'' + z'' \wedge \omega'' \in V\); since \(\Phi * z' - z'\) is exact, by Lemma 4.2.10 we may find a diffeomorphism \(\Psi\) corresponding to a time-1 flow in the \(\frac{\partial}{\partial t}\)
direction such that $\Phi \circ \Psi$ is an automorphism of the $G_2$ structure $\text{Re} \; \Omega' + z' \wedge \omega'$ (clearly isotopic to the identity). Consequently it is an automorphism of $\text{Re} \; \Omega'' + z'' \wedge \omega''$, and so its $M$ part in the sense of Lemma 4.2.13 is an automorphism of $(\Omega'', \omega'')$. It is easy to see that this $M$ part is precisely $\Phi$, which proves the result.

Now we have a fixed smooth structure on $\mathcal{M}_{SU(3)}$ we can prove

**Proposition 4.2.38.** The projection map $\mathcal{M}^{S^1}_{G_2} \to \mathcal{M}_{SU(3)}$ is smooth.

**Proof.** We now know that cohomology classes provide local coordinates for both $\mathcal{M}^{S^1}_{G_2}$ and its submanifold $\mathcal{M}_{SU(3)} = \pi^{-1}_Z([d\theta])$, and therefore it is enough to show the smoothness of the projection map at the level of cohomology classes ($H^3(M)$ in the compact case, and $H^3(\mathbb{R}^3) \oplus H^3(N)$ as in Theorem 4.2.15 in the asymptotically cylindrical case). We take a neighbourhood $U = \mathcal{M}' \times Z'$ with $\mathcal{M}'$ and $Z'$ are both charts, by the fact that $\mathcal{M}^{S^1}_{G_2}$ is a topological product.

Smoothness at the level of cohomology, however, is obvious. For compact manifolds, the map is

$$[\phi'] = [\text{Re} \; \Omega' + z' \wedge \omega'] \mapsto ([\text{Re} \; \Omega' + d\theta \wedge \omega']); \quad (4.2.43)$$

that is, it is the addition of $[d\theta - z'] \wedge [\omega']$. We can work on a slice neighbourhood, so $\phi'$ is smooth, and then $[z']$ and $[\omega']$ are smooth by Proposition 4.1.22. In the asymptotically cylindrical case, we note that $z'(\frac{\partial}{\partial t})$ and $z(\frac{\partial}{\partial t})$ are zero by the boundary conditions of Definition 4.1.25; thus the map on the $H^2(\mathbb{R}^3)$ term is the identity, and certainly smooth.

Together these yield the culmination of our work on deformations.

**Theorem 4.2.39.** Let $\mathcal{M}_{SU(3)}(M)$ be the Calabi–Yau moduli space of Definition 4.2.6, $\mathcal{M}^{S^1}_{G_2}$ be the $S^1$-invariant $G_2$ moduli space of Definition 4.2.11, and let $Z$ be the quotient of Definition 4.2.12. By Theorem 4.2.29, $\mathcal{M}^{S^1}_{G_2}$ is a smooth manifold, and by Lemma 4.2.32 $Z$ is an open subset of $H^1(M \times S^1)$. There exists a smooth structure on $\mathcal{M}_{SU(3)}(M)$ such that

$$\mathcal{M}^{S^1}_{G_2} = Z \times \mathcal{M}_{SU(3)}(M) \quad (4.2.44)$$

is a smooth product.

**Proof.** We already have a single smooth structure on the fibre $\mathcal{M}_{SU(3)}$ such that both projections of the product $\mathcal{M}^{S^1}_{G_2} = \mathcal{M}_{SU(3)} \times Z$ are smooth. We have to show that the combination of these two has an isomorphism for its derivative.

We work at $[\phi] \in \pi^{-1}_Z([z])$. We know by Proposition 4.2.35 that $D\pi_Z : T\mathcal{M}^{S^1}_{G_2} \to TZ$ is surjective and by construction its kernel is $T\pi^{-1}_Z(z)$. On the other hand, $\pi_{\mathcal{M}_{SU(3)}} : \pi^{-1}_Z([z]) \to \mathcal{M}_{SU(3)}$ is essentially the identity, and so $D\pi_{\mathcal{M}_{SU(3)}} : T\pi^{-1}_Z([z]) \to T\mathcal{M}_{SU(3)}$ is also the identity. It follows immediately that $D\pi_Z \oplus D\pi_{\mathcal{M}_{SU(3)}} : T\mathcal{M}^{S^1}_{G_2} \to TZ \oplus T\mathcal{M}_{SU(3)}$ is an isomorphism, and so that we have a local diffeomorphism for the smooth product structure. Hence, as smoothness is a local property, we have a global smooth product.
We obtain the following

**Corollary 4.2.40.** If $M$ is a compact six-manifold admitting a torsion-free $SU(3)$ structure, then the dimension of the moduli space of such structures is $b^3(M) + b^2(M) - b^1(M) - 1$. If $M$ is a (connected) asymptotically cylindrical six-manifold admitting a torsion-free asymptotically cylindrical $SU(3)$ structure, then the dimension of the moduli space of such structures is $b^4(M) + b^3(M) + \frac{1}{2}b^3(N) + \frac{1}{2}b^2(N) - 2b^1(M) - 3$.

**Proof.** It follows from Theorem 4.2.39 that in both cases the dimension of the moduli space is $\dim \mathcal{M}_{G_2} - \dim Z$. We have immediately from Lemma 4.2.32 that $\dim Z = b^1(M) + 1$; on the other hand, from Theorem 4.2.29 we know that $\mathcal{M}_{G_2}$ has the same dimension as $\mathcal{M}_{G_2}(M \times S^1)$. In the compact case, this is $b^3(M \times S^1) = b^3(M) + b^2(M)$; in the asymptotically cylindrical case from [65, Lemma 6.29] we obtain

$$\dim M_{SU(3)}(M) = (b^4(M \times S^1) + \frac{1}{2}b^3(N \times S^1) - b^1(M \times S^1) - 1) - b^1(M) - 1$$

$$= b^4(M) + b^3(M) + \frac{1}{2}b^3(N) + \frac{1}{2}b^2(N) - 2b^1(M) - 3. \quad (4.2.45)$$

4.3 Gluing asymptotically cylindrical Calabi–Yau manifolds

We now turn to questions of gluing. The idea of this section is again to apply the relationship between Calabi–Yau and torsion-free $G_2$ structures to obtain a Calabi–Yau version of results that are already known in the $G_2$ case. The final objective is to prove Theorem 4.3.46, which states that the gluing map on Calabi–Yau structures induced from the gluing map on $G_2$ structures defines a local diffeomorphism from a moduli space of gluing data to the moduli space of Calabi–Yau structures.

We must first show that we can induce a gluing map of Calabi–Yau structures from the gluing map of $G_2$ structures. We do this in subsection 4.3.1: we analyse the proof of the gluing result for $G_2$ structures found in Kovalev [50, section 5] to prove that asymptotically cylindrical $S^1$-invariant $G_2$ structures can be glued to form an $S^1$-invariant $G_2$ structure, and then Theorem 4.1.26 gives us a large family of gluing maps (Theorem 4.3.3). We then proceed to the analysis of gluing between moduli spaces, following [66]. In subsection 4.3.2, we show that this family of gluing maps defines a unique map to the moduli space $\mathcal{M}_{SU(3)}$ of Definition 4.2.1. In subsection 4.3.3 we set up the moduli space of gluing data. Chiefly we follow [66], but in a few places this paper was abbreviated from Nordström’s thesis [64] and we need the full version. As this moduli space is induced from the moduli spaces on the asymptotically cylindrical ends, it is perhaps not surprising that a result analogous to Theorem 4.2.39 remains true: this result is Theorem 4.3.35 below. In subsection 4.3.4, we then restrict to the data that may be glued, and define the gluing map on the moduli space of gluing data. Finally, in subsection 4.3.5, we analyse the gluing map of $G_2$ structures in terms of this product structure on the moduli space.

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of gluing data, and identify what deformations in each component correspond to. This analysis enables us to prove Theorem 4.3.46, by saying that the deformations of $G_\mathbb{2}$ gluing data corresponding to deformations of the Calabi–Yau gluing data give Calabi–Yau deformations, but that the deformations corresponding to the twistings do not affect the final Calabi–Yau structure.

### 4.3.1 Gluing of structures

In this subsection, we define what we mean by “gluing” and show that this operation can be carried out for Calabi–Yau structures. We use our notion of gluing of closed forms from Definition 3.5.5 and briefly review the perturbation argument for $G_\mathbb{2}$ gluing. We then show in Theorem 4.3.3 that this argument passes to the $S^1$-invariant case, using uniqueness, and so defines a collection of gluing maps for Calabi–Yau structures.

Suppose that $(M_1, M_2)$ is a pair of matching six-manifolds in the sense of Definition 3.5.1. Suppose that the $M_i$ have a torsion-free asymptotically cylindrical Calabi–Yau structure. Such structures are defined by asymptotically cylindrical closed forms, so are said to match precisely if they match as forms in the sense of Definition 3.5.4. Likewise twistings (as in Definition 4.1.25) are said to match if the forms match. We seek to find a torsion-free Calabi–Yau structure on the manifold $M_T$ given by Definition 3.5.2.

We note further that since the cross-section of $M_i \times S^1$ is $N_i \times S^1$ and we may extend $F$ by

$$F : N_1 \times S^1 \rightarrow N_2 \times S^1,$$

$$ (n, \theta) \rightarrow (F(n), \theta, 1 - t),$$

(4.3.1) $M_1 \times S^1$ and $M_2 \times S^1$ are also matching manifolds, and moreover the result of gluing them as in Definition 3.5.2 is $M_T \times S^1$.

Given a pair of matching torsion-free $G_\mathbb{2}$ structures on matching seven-manifolds $M_1$ and $M_2$, we can glue them using Definition 3.5.5 to get a (not necessarily torsion-free) $G_\mathbb{2}$-structure $\phi^T$ on $M_T$. By construction, $d\phi^T = 0$ and $d*_{\phi^T} \phi^T$ can be bounded with all derivatives by bounds decaying exponentially in $T$, and we seek to perturb $\phi^T$ to find a torsion-free $G_\mathbb{2}$-structure. Proposition 4.3.1 and Theorem 4.3.2 below carry out this perturbation. The proposition, which provides the setup, is essentially due to Joyce and the theorem is summarised from Kovalev [50, section 5], though the same result can be obtained by using the work of Joyce. The second paragraph of the theorem is easy to establish from the proof, using lower semicontinuity of the first eigenvalue of the Laplacian in the metric (e.g. [65, Lemma 5.5]). In our statements, we restrict to the case where the seven-manifold is of the form $M \times S^1$, for clarity: the statements are in fact true for general seven-manifolds, with the same proofs.

**Proposition 4.3.1 ([40, Theorem 10.3.7]).** Let $M \times S^1$ be a compact seven-dimensional Riemannian manifold whose metric is defined by a closed, but not necessarily torsion-free, $G_\mathbb{2}$-structure
where the remainder term \( R \) satisfies \(|R(d\eta) - R(d\xi)| \leq \epsilon |d\eta - d\xi| (|\eta| + |\xi|)\) for some constant \( \epsilon \).

Then \( \phi + d\eta \) is a torsion-free \( G_2 \)-structure.

**Theorem 4.3.2.** Let \( M^T \times S^1 \) be a compact seven-manifold as in Definition 3.5.2, with \( \phi^T \) given by gluing asymptotically cylindrical \( G_2 \) structures as in Definition 3.5.5. We may choose \( \tilde{\phi}^T \) by using the approximate gluing (as in Definition 3.5.5) of the closed forms \( \ast \phi_1 \) and \( \ast \phi_2 \) as our closed approximation to \( \ast \phi \). Then for \( T > T_0 \) sufficiently large we may find a small 2-form \( \eta \) solving (4.3.2). \( d\eta \) is unique of its size given \( \tilde{\phi}^T \).

Moreover, \( \phi^T + d\eta \) can be chosen to be continuous in the structures \( \phi_1 \) and \( \phi_2 \) with respect to a suitable extended weighted \( C^\infty_\delta \) topology defined after Definition 2.2.7 (that is, with a suitable choice of \( \delta \) so that \( \phi_1 \) and \( \phi_2 \) lie in the appropriate space), and \( T_0 \) can be chosen to be upper semi-continuous in these structures.

A straightforward extension of Theorem 4.3.2 yields the theorem that Calabi–Yau structures can be glued.

**Theorem 4.3.3.** Suppose \( M_1 \) and \( M_2 \) are asymptotically cylindrical Calabi–Yau threefolds. Let \( (\Omega_1, \omega_1) \) and \( (\Omega_2, \omega_2) \) be Calabi–Yau structures on \( M_1 \) and \( M_2 \) matching in the sense of Definition 3.5.4, and let \( (z_1, z_2) \) be a pair of twistings matching in the same sense. \( \phi_1 = \text{Re} \Omega_1 + z_1 \wedge \omega_1 \) and \( \phi_2 = \text{Re} \Omega_2 + z_2 \wedge \omega_2 \) define are \( S^1 \)-invariant torsion-free \( G_2 \) structures, matching in the same sense. Write \( \phi^T \) for the approximate gluing of these torsion-free \( G_2 \) structures given by Definition 3.5.5. There exists \( T_0 > 0 \) such that, for all \( T > T_0 \), \( \phi^T \) can be perturbed to give an \( S^1 \)-invariant torsion-free \( G_2 \) structure on \( M^T \times S^1 \). In particular, we get a Calabi–Yau structure \((\Omega^T, \omega^T)\) on \( M^T \).

For each choice of matching twistings, this procedure gives a well-defined and continuous map \( (\Omega_1, \omega_1, \Omega_2, \omega_2) \mapsto (\Omega^T, \omega^T) \) of Calabi–Yau structures.

**Proof.** By Propositions 4.1.22 and 4.1.24, \( \text{Re} \Omega_1 + z_1 \wedge \omega_1 \) and \( \text{Re} \Omega_2 + z_2 \wedge \omega_2 \) are indeed \( S^1 \)-invariant torsion-free \( G_2 \) structures on \( M_i \times S^1 \); they obviously match, since taking the real part and the wedge product commute with pullback. The approximate gluing procedure of Definition 3.5.5 is clearly invariant under the rotation, and so our approximate gluing \( \phi^T \) is \( S^1 \)-invariant. We know by Theorem 4.3.2 that for \( T > T_0 \) sufficiently large we can perturb \( \phi^T \) to a torsion-free \( G_2 \) structure: we need to check that that structure is \( S^1 \)-invariant.
To follow the theorem, we need \( \hat{\phi}^T \) to be \( S^1 \)-invariant. If a \( G_2 \) structure \( \phi \) is \( S^1 \)-invariant, then \(*_\phi \phi\) is also \( S^1 \)-invariant, because pullback by the isometric rotation \( \Theta \) commutes with the Hodge star. Thus, the approximation to \(*_\phi \phi\) given by gluing \(*_{\phi_1} \phi_1\) and \(*_{\phi_2} \phi_2\) is \( S^1 \)-invariant and hence so is \( \hat{\phi}^T \).

We may now check that the solution \( d\eta \), where \( \eta \) solves (4.3.2), is \( S^1 \)-invariant. Since \( \phi^T \), and so the metric being used, and \( \hat{\phi}^T \) are both \( S^1 \)-invariant, the operator defining (4.3.2) commutes with \( \Theta \), hence if \( \eta \) satisfies (4.3.2), so too does \( \Theta^* \eta \). The uniqueness statement then implies that \( \Theta^* d\eta = d\Theta^* \eta = d\eta \), i.e. that \( d\eta \) is \( S^1 \)-invariant.

Using Propositions 4.1.22 and 4.1.24 again, \( \phi^T + d\eta \) then yields our Calabi–Yau structure \((\Omega^T, \omega^T)\) on \( M^T \).

The claim of continuity on structures is immediate from the claim in Theorem 4.3.2.

At this point, we have recovered in more generality the result of Doi–Yotsutani [18]. Their argument proceeds as in the first paragraph of the proof of Theorem 4.3.3, except with slightly more assumptions, to obtain a torsion-free \( G_2 \) structure on \( M^T \times S^1 \). They then argue as follows.

**Lemma 4.3.4** ([18, Lemma 3.14]). Suppose \( M^T \) is a simply connected compact manifold and \( M^T \times S^1 \) admits a torsion-free \( G_2 \) structure. Then \( M^T \) admits a Ricci-flat Kähler metric.

**Sketch of original proof.** Consider the universal cover \( M^T \times \mathbb{R} \) of \( M^T \times S^1 \). \( M^T \times \mathbb{R} \) also admits a torsion-free \( G_2 \) structure, and so a Ricci-flat metric, and by the Cheeger-Gromoll splitting theorem [14] the metric on \( M^T \times \mathbb{R} \) is given by a Riemannian product \( N \times \mathbb{R} \). The metric induced on \( N \) is Ricci-flat Kähler, by holonomy considerations. By classification theory for compact simply connected spin \( 6 \)-manifolds, \( M^T \) and \( N \) are diffeomorphic; hence \( M^T \) admits a Ricci-flat Kähler metric.

Our work makes this argument much more concrete, as well as generalising to the not necessarily simply connected case. We shall assume that the torsion-free \( G_2 \) structure on \( M^T \times S^1 \) is \( S^1 \)-invariant; by the proof of Theorem 4.3.3, this assumption requires no further hypotheses on the structures to be glued.

**Concrete variant of proof of Lemma 4.3.4 if \( G_2 \) structure is \( S^1 \)-invariant.** Using \( S^1 \)-invariance, we write the torsion-free \( G_2 \) structure on \( M^T \times S^1 \) as \( \text{Re} \Omega + (Ld\theta + v) \wedge \omega \), where \( z = Ld\theta + v \) is a twisting.

We describe the Riemannian universal cover of \( M^T \times S^1 \) with the \( G_2 \) structure \( \text{Re} \Omega + (Ld\theta + v) \wedge \omega \). We need to equip the universal cover \( M^T \times \mathbb{R} \) with a torsion-free \( G_2 \) structure, and we take the torsion-free \( G_2 \) structure \( \text{Re} \Omega + (Ld\theta) \wedge \omega \), where \( \theta \) is the coordinate along \( \mathbb{R} \).

We now need to define a projection \( \pi : M^T \times \mathbb{R} \to M^T \times S^1 \) such that

\[
\pi^* (\text{Re} \Omega + (Ld\theta + v) \wedge \omega) = \text{Re} \Omega + Ld\theta \wedge \omega. \tag{4.3.3}
\]
Since $M^T$ is simply connected, $b^1(M^T) = 0$ and so we may write $v = df$ for some function $f$ on $M^T$. Define $\pi$ by $(x, \theta) \mapsto (x, [\theta - \frac{f(x)}{L}])$; it is easy to see that $\pi$ satisfies (4.3.3) and so is a Riemannian covering of $M^T \times S^1$ by $M^T \times \mathbb{R}$; since $M^T$ is simply connected, it is the universal cover. Note that the torsion-free $G_2$ structure $\text{Re } \Omega + (Ld\theta) \wedge \omega$ is the product structure that can be obtained by using Cheeger-Gromoll, which shows immediately that $N = M^T$.

We thus get the Calabi–Yau structure $(\Omega, \omega)$ on $M^T$, and in particular a Ricci-flat Kähler metric. 

The major gain from our concrete approach is that we get an explicit identification of the factor $N$ in the universal cover and $M^T$ (the identity), and an explicit universal cover; in particular, there is a sensible relation between the Calabi–Yau structures we glue and the resulting Calabi–Yau structure. This will be vital in ensuring that Hypothesis 5.3.1 holds in the case $n = 3$, which underlies our special Lagrangian results in chapter 5. For instance, see Proposition 4.3.7 below.

### 4.3.2 Gluing to $\mathcal{M}_{SU(3)}$

The next thing to try to do is show that the gluing map of Calabi–Yau structures defined by Theorem 4.3.3 is independent of the twistings $z_1$ and $z_2$. Unfortunately, this is not obviously true in general, since we only know that the perturbation we make in Theorem 4.3.2 to maintain torsion-freeness is small: we do not know how it is related to the inclusion of Calabi–Yau structures. However, we can show that the gluing map is indeed independent of $z_1$ and $z_2$ as a map to the moduli space $\mathcal{M}_{SU(3)}(M^T)$ defined in Definition 4.2.1: that is, the Calabi–Yau structure may depend on the twistings, but different twistings result in Calabi–Yau structures that are at worst pullbacks of each other.

We use cohomology. We know that the perturbation made in Theorem 4.3.2 is an exact form, so does not change the cohomology class of the $G_2$ structure. This cohomology class can be decomposed, for instance as in the proof of Proposition 4.2.38, into cohomology classes corresponding to the twisting and the Calabi–Yau structure. If it were the case that the cohomology classes corresponding to the Calabi–Yau structure did not change under this perturbation, then since they originally are just given by gluing the Calabi–Yau structures using Definition 3.5.5 (which we will prove momentarily), they would be independent of the twistings used. It would then follow from Proposition 4.2.33 that the $\mathcal{M}_{SU(3)}$ class is also independent of the twistings used.

Unfortunately, it is not quite true that the cohomology classes corresponding to the Calabi–Yau structure do not change under the perturbation: whilst it is the case for $[\text{Re } \Omega]$, it is possible we might have to rescale $[\omega]$. In this subsection, we adjust the previous paragraph to provide a correct argument proving the result in a similar way.
We begin with the following simple lemma saying that in cohomology the wedge product of glued forms is the gluing of the wedge products.

**Lemma 4.3.5.** Suppose that \( \alpha_1, \beta_1 \) and \( \alpha_2, \beta_2 \) are two pairs of closed matching asymptotically translation invariant forms on \( M_1 \) and \( M_2 \), so that \( \alpha_1 \wedge \beta_1 \) and \( \alpha_2 \wedge \beta_2 \) is also a pair of closed matching asymptotically translation invariant forms. The approximate gluing of Definition 3.5.5 gives well-defined cohomology classes \( [\gamma_T(\alpha_1, \alpha_2)], [\gamma_T(\beta_1, \beta_2)], \) and \( [\gamma_T(\alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2)] \). Then

\[
[\gamma_T(\alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2)] = [\gamma_T(\alpha_1, \alpha_2)] \wedge [\gamma_T(\beta_1, \beta_2)].
\]

(4.3.4)

**Proof.** It is sufficient to show that on the subsets \( M^{\text{tr}(T+1)}_i \) of \( M_i \) used in the approximate gluing, \( \alpha_i \wedge \beta_i \) and the wedge product \( \hat{\alpha}_i \wedge \hat{\beta}_i \) differ by \( d\gamma_i \) where \( \gamma_i \) is supported away from the identified region of \( M^{\text{tr}(T+1)}_i \), for then when we identify we get

\[
\gamma_T(\alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2) - \gamma_T(\alpha_1, \alpha_2) \wedge \gamma_T(\beta_1, \beta_2) = d\gamma_1 + d\gamma_2,
\]

(4.3.5)

with \( \gamma_1 \) and \( \gamma_2 \) having disjoint support and the result follows.

We will therefore drop the subscripts. As in Definition 3.5.5, divide \( \alpha \) and \( \beta \), on the end, into a limit part and an exact part

\[
\alpha = \check{\alpha} + d\alpha', \quad \beta = \check{\beta} + d\beta'.
\]

(4.3.6)

We then get

\[
\alpha \wedge \beta = \check{\alpha} \wedge \check{\beta} + d(\alpha' \wedge \check{\beta} + (-1)^{\text{deg} \check{\alpha}} \check{\alpha} \wedge \beta' + \alpha' \wedge d\beta').
\]

(4.3.7)

As after (2.1.1), let \( \varphi_T \) be zero for \( t > T \) and 1 for \( t \leq T - 1 \). The cutoffs \( \check{\alpha} \) and \( \check{\beta} \) in Definition 3.5.5 are \( \check{\alpha} + d(\varphi_T \alpha') \) and \( \check{\beta} + d(\varphi_T \beta') \) and it follows that the wedge product of the cutoffs is given exactly by replacing \( \alpha' \) and \( \beta' \) with \( \varphi_T \alpha' \) and \( \varphi_T \beta' \) in (4.3.7). Similarly the cutoff of the wedge product is given by introducing a \( \varphi_T \) into the exterior derivative in (4.3.7). On taking the difference, all terms but the last then cancel, to give

\[
d(\varphi_T \alpha' \wedge d\beta' - \varphi_T \alpha' \wedge d(\varphi_T \beta')) = d(\varphi_T \alpha' \wedge d\beta' - \varphi_T \alpha' \wedge d\beta' - \varphi_T \alpha' \wedge d\varphi_T \wedge \beta').
\]

(4.3.8)

Since \( \varphi_T - \varphi_T^2 = \varphi_T(1 - \varphi_T) \) and \( \varphi_T d\varphi_T \) are both supported in \( (T - 1, T) \), (4.3.8) is supported in \( (T - 1, T) \); that is, away from the identified region. Thus we have the claimed result. \( \square \)

Combining Lemma 4.3.5 with standard results on the cohomology ring of a compact Kähler manifold, we obtain our result on how the cohomology classes \( [\text{Re} \Omega] \) and \( [\omega] \) differ from the approximate gluings of the \( \text{Re} \Omega_i \) and \( \omega_i \).

**Proposition 4.3.6.** Suppose that \( \text{Re} \Omega_1 + (Ld\theta + v_1) \wedge \omega_1 \) and \( \text{Re} \Omega_2 + (Ld\theta + v_2) \wedge \omega_2 \) are matching torsion-free asymptotically cylindrical \( S^1 \)-invariant \( G_2 \) structures obtained from
torsion-free asymptotically cylindrical \((\Omega_i, \omega_i)\) and twistings \(L_i d\theta + v_i\) by Propositions 4.1.22 and 4.1.24. (Note that \(L_1 = L_2\) since these two \(G_2\) structures match.)

Suppose that for some \(T\) these glue as in Theorem 4.3.3 to the \(S^1\)-invariant torsion-free \(G_2\) structure \(\phi\) and we have \(\phi = \text{Re}\ \Omega + (L'd\theta + v) \wedge \omega\) for a Calabi–Yau structure \((\Omega, \omega)\) and twisting \(L'd\theta + v\). Then there exists \(c > 0\) such that

\[
L' = cL, \quad [\text{Re}\ \Omega] = [\gamma_T(\text{Re}\ \Omega_1, \text{Re}\ \Omega_2)],
\]

\[
[w] = \frac{1}{c} [\gamma_T(\omega_1, \omega_2)], \quad [v] = c[\gamma_T(v_1, v_2)].
\] (4.3.9)

**Proof.** Removing the torsion does not affect the cohomology class of \(\phi\), so we have

\[
[\gamma_T(\text{Re}\ \Omega_1 + (Ld\theta + v_1) \wedge \omega_1, \text{Re}\ \Omega_2 + (Ld\theta + v_2) \wedge \omega_2)] = [\text{Re}\ \Omega + (L'd\theta + v) \wedge \omega]. \quad (4.3.10)
\]

Using Lemma 4.3.5 and the obvious linearity of \(\gamma_T\), we obtain

\[
[\gamma_T(\text{Re}\ \Omega_1, \text{Re}\ \Omega_2)] + [Ld\theta + \gamma_T(v_1, v_2) \wedge \gamma_T(\omega_1, \omega_2)] = [\text{Re}\ \Omega] + [L'd\theta + v] \wedge [\omega]. \quad (4.3.11)
\]

Since \(\gamma_T\) can be defined on any pair of matching asymptotically cylindrical manifolds and commutes with matching maps of such pairs (provided the cutoff functions are chosen appropriately), and we can choose inclusions so that \((M_1, M_2)\hookrightarrow (M_1 \times S^1, M_2 \times S^1)\) is such a pair, \(\gamma_T(\text{Re}\ \Omega_1, \text{Re}\ \Omega_2)\), \(\gamma_T(v_1, v_2)\), and \(\gamma_T(\omega_1, \omega_2)\) are in the subset \(H^*(M^T)\) of \(H^*(M^T \times S^1)\) (corresponding to having no \(d\theta\) terms). Evidently, \([\text{Re}\ \Omega], [v],\) and \([\omega]\) also lie in the subset \(H^*(M^T)\).

By the Künneth theorem, therefore, we have

\[
L[\gamma_T(\omega_1, \omega_2)] = L'[\omega]. \quad (4.3.12)
\]

Recalling that \(L\) and \(L'\) are both positive, set \(c = \frac{L}{L'}\). Hence \([\omega] = \frac{1}{c} [\gamma_T(\omega_1, \omega_2)].\)

Taking the other component from the Künneth isomorphism, and writing \([\gamma_T(\omega_1, \omega_2)]\) as \(c[\omega]\), we have

\[
[\gamma_T(\text{Re}\ \Omega_1, \text{Re}\ \Omega_2)] + [\gamma_T(v_1, v_2)] \wedge c[\omega] = [\text{Re}\ \Omega] + [v] \wedge [\omega]. \quad (4.3.13)
\]

Now \([\omega]\) is a Kähler class on \(M^T\) and we have

\[
[\omega] \wedge [\text{Re}\ \Omega] = 0 = [\gamma_T(\text{Re}\ \Omega_1 \wedge \omega_1, \text{Re}\ \Omega_2 \wedge \omega_2)] = [\gamma_T(\text{Re}\ \Omega_1, \text{Re}\ \Omega_2)] \wedge [\omega], \quad (4.3.14)
\]

again using Lemma 4.3.5. Since \(c \neq 0\), (4.3.14) means that \([\text{Re}\ \Omega]\) and \([\gamma_T(\text{Re}\ \Omega_1, \text{Re}\ \Omega_2)]\) are classes of primitive 3-cohomology. The remaining two equations now follow from the Lefschetz decomposition as the primitive 3-cohomology and 1-cohomology components of (4.3.13). □
Proposition 4.3.6 will also be useful in chapter 5, where it will be used to ensure the gluing of special Lagrangians is not topologically obstructed from being perturbed to a special Lagrangian. Similarly, we will need the following observation.

**Proposition 4.3.7.** Let $M_1$ and $M_2$ be a matching pair of six-manifolds with $(\Omega_1, \omega_1)$ and $(\Omega_2, \omega_2)$ matching Calabi–Yau structures on them. Fix a matching pair of twistings $z_1$ and $z_2$, and let $(\Omega^T, \omega^T)$ be the Calabi–Yau structure obtained on $M^T$ by Theorem 4.3.3. Then there exists $\epsilon$ and a sequence of constants $C_k$ such that for all $k$

$$\|\gamma_T(\Omega_1, \Omega_2) - \Omega^T\|_{C^k} + \|\gamma_T(\omega_1, \omega_2) - \omega^T\|_{C^k} \leq C_k e^{-\epsilon T}. \quad (4.3.15)$$

**Remark 4.3.8.** Estimates of the form (4.3.15) will be used throughout our gluing analyses. If a family of objects on the glued manifolds $M^T$ or the glued submanifolds $L^T$ in the next chapter satisfies an estimate of the form (4.3.15) (for all $k$) with respect to the metrics on $M^T$ (or $L^T$), we shall say that the objects decay exponentially in $T$. For another example, see Lemma 5.1.15 below.

**Proof.** As in Theorem 4.3.3, let $\phi_1 = \text{Re} \Omega_1 + z_1 \wedge \omega_1$ and $\phi_2 = \text{Re} \Omega_2 + z_2 \wedge \omega_2$. Let $\phi^T$ be the torsion-free $S^1$-invariant $G_2$ structure constructed by Theorem 4.3.3, and we have $\phi^T = \text{Re} \Omega^T + z^T \wedge \omega^T$. It follows from the proof of Theorem 4.3.2 (compare [50, Theorem 5.34], and note that we may use stronger Banach space norms in the proof) that $\phi^T - \gamma_T(\phi_1, \phi_2)$ decays exponentially in $T$. On the other hand, the analysis of Lemma 4.3.5 implies that $\gamma_T(\phi_1, \phi_2) - (\gamma_T(\text{Re} \Omega_1, \text{Re} \Omega_2) + \gamma_T(z_1, z_2) \wedge \gamma_T(\omega_1, \omega_2))$ decays exponentially in $T$, so that $\text{Re} \Omega^T + z^T \wedge \omega^T - (\gamma_T(\text{Re} \Omega_1, \text{Re} \Omega_2) + \gamma_T(z_1, z_2) \wedge \gamma_T(\omega_1, \omega_2))$ decays exponentially in $T$.

Since $\gamma_T(\text{Re} \Omega_1, \text{Re} \Omega_2) + \gamma_T(z_1, z_2) \wedge \gamma_T(\omega_1, \omega_2)$ always lies in a compact subset and its derivatives are bounded, the result follows from Proposition 2.2.5.

Proposition 4.3.6 implies that the naive argument that changing the twisting doesn’t change the cohomology classes of $[\text{Re} \Omega]$ and $[\omega]$ (since the cohomology is “just that given by gluing $\text{Re} \Omega_i$ and $\omega_i$”) and so locally doesn’t change the $\mathcal{M}_{SU(3)}$ class fails. To save it, we need to make sure the constant $c$ of Proposition 4.3.6 doesn’t change as we change the twistings, which we do shortly in Lemma 4.3.9.

An interesting question would be whether this is just a consequence of the proof, or a real restriction. If we regard $c$ instead as a possible rescaling of the holomorphic volume form $\Omega$, it is not obvious that the scaling of the holomorphic volume form should be preserved by gluing: so its being a real restriction is possible. We could ask whether the scaling of the holomorphic volume form is uniquely determined by the cohomology. It is clear that if $\Omega$ is known, $c$ is determined for each $\omega$ by condition iii) of Definition 4.1.1

$$\Omega \wedge \bar{\Omega} = \frac{(-2)^n n^2}{n!} \omega^n, \quad (4.3.16)$$

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and that if $[\Omega]$ is known, exactly the same applies for each $[\omega]$. We know that $\Im \Omega$ is determined by $\Re \Omega$ from [35]. Unfortunately, it is less clear that $[\Im \Omega]$ is determined by $[\Re \Omega]$. Using torsion-freeness we can essentially say that this holds locally.

**Lemma 4.3.9.** Let $(\Omega, \omega)$ be a Calabi–Yau structure. There is an open neighbourhood $U$ of $(\Omega, \omega)$ in Calabi–Yau structures, and $\epsilon > 0$, such that if $(\Omega_1, \omega_1)$ and $(\Omega_2, \omega_2)$ both lie in $U$, with $[\Re \Omega_1] = [\Re \Omega_2]$ and $[\omega_1] = C[\omega_2]$ for $|C| < \epsilon$, then $C = 1$.

**Proof.** We work locally around the class of $\mathcal{M}_{G_2}^{S^1}$ corresponding to $\Re \Omega + d\theta \wedge \omega$. We know that there is an open subset $V$ of $\mathcal{M}_{G_2}^{S^1}$ around this class which is homeomorphic to an open subset of $H^3(M \times S^1)$ as in Theorem 4.2.15 and Theorem 4.2.29. By reducing the open set if necessary, we may assume using Proposition 4.2.14 that $V$ is a product of open sets $U'$ and $W$ in $\mathcal{M}_{SU(3)}$ and $Z$ respectively. Let $U$ be the set of structures whose moduli class is in $U'$. $W$ is an open set containing $[d\theta]$ so it contains an interval of the line $\mathbb{R}[d\theta]$; choose $\epsilon < \frac{1}{2}$ so that it contains $(1 - 2\epsilon, 1 + 2\epsilon)[d\theta]$.

Now suppose given two structures $(\Omega_i, \omega_i)$ as in the statement. Clearly, the $S^1$-invariant torsion-free $G_2$ class $[\Re \Omega_1 + d\theta \wedge \omega_1]$ lies in $V$, and since $|\frac{1}{C} - 1| < \frac{1}{1-\epsilon} < 2\epsilon$, the same is true of the class $[\Re \Omega_2 + \frac{1}{C}d\theta \wedge \omega_2]$. Moreover, we have the equality

$$[\Re \Omega_1 + d\theta \wedge \omega_1] = [\Re \Omega_2 + \frac{1}{C}d\theta \wedge \omega_2]$$

in $H^3$. Since $V$ is homeomorphic to its image in $H^3(M \times S^1)$, we also have equality in $\mathcal{M}_{G_2}^{S^1}$. Applying Proposition 4.2.14 again, we find that $C = 1$. \hfill $\Box$

We now use Lemma 4.3.9 to prove our foreshadowed well-definition result.

**Proposition 4.3.10.** Let $(\Omega_1, \omega_1)$ and $(\Omega_2, \omega_2)$ be matching Calabi–Yau structures; let $z_1$ and $z_2$ be two matching twistings, and $z_1'$ and $z_2'$ be another matching pair of twistings. Then, for neck-length parameter $T$ sufficiently large (depending on a curve of $G_2$ structures that will appear in the proof), the $\mathcal{M}_{SU(3)}$ parts of the results of gluing the $G_2$ structures $\Re \Omega_i + z_i \wedge \omega_i$ and $\Re \Omega_i + z_i' \wedge \omega_i$ are equal, and the $Z$ parts are given by the same multiple of the approximate gluing, though the approximate gluing may be different in the two cases.

**Proof.** The set of matching pairs of twistings is precisely $\mathbb{R}_{>0}$ times the vector space of matching closed 1-forms (with appropriate limits) on $M_1$ and $M_2$. Therefore it is path-connected, and so there exists a path in it $(z_1(s), z_2(s))$ with $z_i(0) = z_i$ and $z_i(1) = z_i'$.

For $T$ sufficiently large (by semi-continuity of a minimal $T$ from Theorem 4.3.2 and compactness of $[0, 1]$), the resulting pairs of matching torsion-free $S^1$-invariant $G_2$-structures

$$\left(\Re \Omega_1 + z_1(s) \wedge \omega_1, \Re \Omega_2 + z_2(s) \wedge \omega_2\right)$$

(4.3.18)

can be glued to give torsion-free $S^1$-invariant $G_2$ structures $\phi(s)$; $\phi(s)$ is a continuous curve,
again by Theorem 4.3.2. Using the proof of Proposition 4.1.22, we can split these up as

$$\phi(s) = \text{Re} \Omega(s) + z(s) \wedge \omega(s),$$

(4.3.19)

for continuous curves of Calabi–Yau structures and twistings. In the notation of Proposition 4.3.6, it follows that $L'$ is continuous and so $c$ is (because these are determined from $z(s)$); hence we know that the cohomology classes satisfy

$$[\text{Re} \Omega(s)] = [\gamma_T (\text{Re} \Omega_1, \text{Re} \Omega_2)],$$

(4.3.20)

$$[\omega(s)] = \frac{1}{c(s)} [\gamma_T (\omega_1, \omega_2)],$$

(4.3.21)

for a continuous positive function $c(s)$. In particular, we see that $(\Omega(s), \omega(s))$ is a continuous curve of Calabi–Yau structures with $[\text{Re} \Omega(s)]$ fixed and $[\omega(s)]$ only varying in a line. It follows from Lemma 4.3.9 that $c(s)$ is locally constant, and hence it is constant; thus, $[\text{Re} \Omega(s)]$ and $[\omega(s)]$ are both fixed, and so so is the moduli class of $(\Omega(s), \omega(s))$, which proves the first claim.

The second claim follows since we also have $[z(s)] = c(s)[\gamma_T (z_1(s), z_2(s))]$ from Proposition 4.3.6.\hfill \Box

Proposition 4.3.10 essentially says that the images under gluing of a pair of pairs of $G_2$ structures differing just by varying the twistings themselves just differ by varying the twisting and potentially diffeomorphism.

In subsection 4.3.5, we shall analyse the gluing map between moduli spaces for $G_2$ closely to work out how the gluing map between Calabi–Yau moduli spaces behaves. Proposition 4.3.10 will be used to say that a variation corresponding to a twisting glues to a variation corresponding a twisting: we would like to know what happens when we vary the Calabi–Yau structure or the gluing parameter $T$.

In varying the Calabi–Yau structure, there are two complications over varying the twisting. Firstly, it is not at all clear that $\mathcal{M}_{SU(3)}$ is connected, so we will need to assume the existence of the curve used in the proof of Proposition 4.3.10. In any case, the factor $c$ may vary. Thus our result is noticeably weaker.

**Proposition 4.3.11.** Suppose that $z_1$ and $z_2$ are a pair of matching twistings. Let $(\Omega_1, \omega_1)$ and $(\Omega_2, \omega_2)$ be a pair of matching Calabi–Yau structures, and let $(\Omega'_1, \omega'_1)$ and $(\Omega'_2, \omega'_2)$ be another such pair. Suppose that there exists a continuous curve through matching pairs of Calabi–Yau structures joining these pairs. Then, for neck-length parameter sufficiently large (depending on this curve), the $Z$ parts of the results of gluing the $G_2$ structures $\text{Re} \Omega_i + z_i \wedge \omega_i$ and $\text{Re} \Omega'_i + z_i \wedge \omega'_i$ are proportional.

**Proof.** By choosing $T$ large, as in the proof of Proposition 4.3.10, we get a continuous curve of glued structures; write them as $\Omega(s) + z(s) \wedge \omega(s)$. By Proposition 4.3.6, we know that $[z(s)] = c(s)[\gamma_T (z_1, z_2)]$. Hence $[z(0)] = \frac{c(0)}{c(1)}[z(1)]$.\hfill \Box

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The only remaining question is the effect of varying the neck-length parameter \( T \): again we don’t get a great deal, but we can obtain the following

**Proposition 4.3.12.** Suppose \((\Omega_i, \omega_i)\) are matching Calabi–Yau structures and \( z_i \) are matching twistings. Let \( T \) and \( T' \) be a pair of positive reals exceeding the minimal gluing parameter \( T_0 \) for the associated matching \( G_2 \) structures \( \text{Re} \Omega_i + z_i \wedge \omega_i \). Then as in Proposition 4.3.11 the \( Z \) parts of the glued structures are proportional.

**Proof.** Choose a curve \( T(s) \) from \( T \) to \( T' \), always greater than \( T_0 \). As in Propositions 4.3.10 and 4.3.11, write the curve of glued structures as \( \Omega(s) + z(s) \wedge \omega(s) \) and use Proposition 4.3.6 to get \([z(s)] = c(s)[\gamma_{T(s)}(z_1, z_2)]\).

The statement follows as in Proposition 4.3.11 if \([\gamma_{T(s)}(z_1, z_2)]\) is independent of \( s \). Because the common limit of \( z_1 \) and \( z_2 \) has no \( dt \) term, the natural diffeomorphism pulls back the gluing with a large \( T \) to the smaller \( T \) with only a compactly supported error (see [66, Proposition 3.2]), and so \([\gamma_{T(s)}(z_1, z_2)]\) is indeed independent of \( s \). □

### 4.3.3 Moduli spaces of gluing data

By combining Theorem 4.3.3 with Proposition 4.3.10, we have thus shown that there is a single well-defined gluing map from matching pairs of Calabi–Yau structures to the moduli space \( \mathcal{M}_{\text{SU}(3)}(M^T) \). Next, we must define a moduli space of gluing data and show that this gluing map induces a well-defined map between these moduli spaces. To define the space of gluing data, we basically follow the ideas and notation for the \( G_2 \) case in [66]. Here, Nordström restricts to the special case in which the first Betti number of the glued manifold is zero for simplicity, though the result is true in general. In our case, \( b^1(M^T \times S^1) \) is clearly nonzero, and though we could similarly argue for the special case when \( b^1(M^T) = 0 \), we will follow the full generality analysis provided by Nordström in [64, subsection 6.3.2] in the relevant place.

In this subsection, we define a quotient which we expect to define a sensible space of gluing data, and show that this quotient is a manifold. The idea here, which is used in [66], is to use a sequence of larger and larger spaces, and show each in turn is a manifold. The smallest is the space \( \mathcal{B} \) of “matching moduli classes”; the second is the space \( \tilde{\mathcal{G}} \) of “moduli classes of matching pairs”, and finally we end up with the space \( \tilde{\mathcal{G}} \) of “moduli classes of matching pairs and gluing parameters” which we require.

We first review the definitions and results in the \( G_2 \) case, following Nordström [66]: these pass to the \( S^1 \)-invariant \( G_2 \) case with very little additional work. We show that \( \tilde{\mathcal{G}}_{S^1 \mathcal{G}_2} \) is a principal \( \mathbb{R} \)-bundle over \( \mathcal{G}_{S^1 \mathcal{G}_2} \), and that \( \mathcal{G}^{S^1}_{G_2} \) is a bundle over \( \mathcal{B}^{S^1}_{G_2} \), which defines its coordinates (Proposition 4.3.24). We provide some detail of the proof that \( \mathcal{G}^{S^1}_{G_2} \) is a bundle over \( \mathcal{B}^{S^1}_{G_2} \), as this material is not available in [66] and is not fully given even in [64].

We then pass simultaneously to the analogous spaces for Calabi–Yau structures and the relationship between the \( G_2 \) and Calabi–Yau cases. We show that the analogous spaces for
Calabi–Yau structures are smooth and the inclusion maps from the Calabi–Yau versions to the $G_2$ versions are smooth. For smoothness of the spaces, we use both similar methods to the $G_2$ case and what we already known about the relationship between $G_2$ and Calabi–Yau structures. The final theorem of this subsection (Theorem 4.3.35) is a gluing-data version of Theorem 4.2.39, saying that the space of “$S^1$-invariant $G_2$ gluing data” is a product of the space of “Calabi–Yau gluing data” with a suitable space of twistings in the sense of Definition 4.1.25.

We now begin by summarising the $G_2$ case, with the minor changes required to make the results $S^3$-invariant. We will make some of the definitions in greater generality, however, as otherwise we would have to make exactly parallel definitions in the Calabi–Yau case.

To define a moduli space of gluing data, we need to define an action of a matching pair on a triple of gluing data. The type of the structure is irrelevant here. Consequently, we shall write $υ$ for the structures, which we shall use generally in this subsection when giving an argument that applies in both cases.

**Definition 4.3.13** ([66, Definition 2.3]). Suppose $M_1$, $M_2$, and $F$ are as in Definition 3.5.1. Suppose that $Φ_1$ and $Φ_2$ are asymptotically cylindrical diffeomorphisms of $M_1$ and $M_2$ with limits $(\tilde{Φ}_1, L_1)$. They are said to match if $F^{-1}\tilde{Φ}_2 F = \tilde{Φ}_1$ as a map $N_1 \to N_1$. Note that we do not require $L_1 = L_2$.

A matching pair $(Φ_1, Φ_2)$ is isotopic to the identity as a matching pair if $Φ_1$ and $Φ_2$ are both asymptotically cylindrically isotopic to the identity as in Definition 4.2.5 and we may choose the isotopies $Φ_{1,s}$ and $Φ_{2,s}$ such that the pair of diffeomorphisms $(Φ_{1,s}, Φ_{2,s})$ matches for all $s$. As in Definition 4.2.5, we shall omit ‘as a matching pair’ and simply speak of a pair of diffeomorphisms being isotopic to the identity.

To define a moduli space of gluing data, we need to define an action of a matching pair on a triple of gluing data. The type of the structure is irrelevant here. Consequently, we shall write $υ$ for the structures, which we shall use generally in this subsection when giving an argument that applies in both cases.

**Definition 4.3.14** (cf. [66, Definition 2.3]). Suppose that $(Φ_1, Φ_2)$ is a matching pair of diffeomorphisms with limits $(\tilde{Φ}_1, L_1)$. We define an action on triples $(υ_1, υ_2, T)$ where $υ_1$ and $υ_2$ are (asymptotically cylindrically Calabi–Yau or, potentially $S^1$-invariant, $G_2$) structures on $M_1$ and $M_2$ respectively and $T$ is a real number, which will eventually be the gluing parameter, by

$$(Φ_1, Φ_2)(υ_1, υ_2, T) = (Φ_1^*υ_1, Φ_2^*υ_2, T - \frac{1}{2}(L_1 + L_2)).$$  \hspace{1cm} (4.3.22)$$

(4.3.22) is clearly an action on triples of structures. The first thing to show is that (4.3.22) preserves the subspace of triples where the structures match as in Definition 3.5.4, but this is obvious from the definition of matching diffeomorphisms. Therefore we make
Definition 4.3.15. Let
\[ \tilde{G}^{S^1}_{G_2} = \frac{\text{matching pairs of torsion-free } S^1\text{-invariant } G_2 \text{ structures and parameters } T}{\text{matching pairs of } S^1\text{-invariant diffeomorphisms isotopic to the identity}}. \] (4.3.23)

$S^1$-invariant diffeomorphisms are as defined in Definition 4.2.8, for consistency of our moduli spaces.

We require the pairs to be isotopic to the identity as matching pairs so that they can be “glued” to form diffeomorphisms isotopic to the identity in Theorem 4.3.38, and hence give a well-defined gluing map from a subset of $\tilde{G}^{S^1}_{G_2}$ to the space $\mathcal{M}^{S^1}_{G_2}$ of Definition 4.2.11. We note that the orbit of a matching pair of $G_2$ structures under matching pairs of diffeomorphisms isotopic to the identity is closed: by Theorem 4.2.15, the orbit of a $G_2$ structure under diffeomorphisms isotopic to the identity is closed (and in fact the diffeomorphisms converge to a diffeomorphism giving the new point of the orbit), so we only have to check that a pair of diffeomorphisms being a matching pair isotopic to the identity is a closed condition, which is obvious. Hence, $\tilde{G}^{S^1}_{G_2}$ is Hausdorff. The same applies for Calabi–Yau structures, combining Theorem 4.2.15 with Theorem 4.2.39 and using the closedness of exact forms.

We consider two additional spaces of gluing data, both of which are smaller than $\tilde{G}^{S^1}_{G_2}$, from which we can construct $\tilde{G}^{S^1}_{G_2}$, and hence infer that it is a manifold. First of all, we define the smallest possible space of gluing data: the subspace of the product of the moduli spaces on each part corresponding to matching moduli classes.

Definition 4.3.16. Let $\mathcal{B}^{S^1}_{G_2}$ be the space of matching pairs in the $S^1$-invariant $G_2$ moduli spaces on $M_1$ and $M_2$, that is:
\[ ([\phi_1], [\phi_2]) \in \mathcal{M}^{S^1}_{G_2}(M_1) \times \mathcal{M}^{S^1}_{G_2}(M_2) \] (4.3.24)
such that there exist representatives $\phi_1$ and $\phi_2$ matching in the sense of Definition 3.5.4.

The final space is the space of pairs of matching classes quotiented by matching diffeomorphisms, defined as for $\tilde{G}$ but forgetting the parameter $T$. Because we have to deal with two different kinds of structures the notation is already quite involved, so we give it a specific name. The action by matching pairs of diffeomorphisms is just that restricted from Definition 4.3.14.

Definition 4.3.17. Let $\hat{G}^{S^1}_{G_2}$ be the space of matching pairs of torsion-free $S^1$-invariant $G_2$ structures and parameters $T$, defined as:
\[ \hat{G}^{S^1}_{G_2} = \frac{\text{matching pairs of torsion-free } S^1\text{-invariant } G_2 \text{ structures and parameters } T}{\text{matching pairs of } S^1\text{-invariant diffeomorphisms isotopic to the identity}}. \] (4.3.25)

The notation we are adopting is rather different from that of [66]. In that, our $\mathcal{B}$ is called $\mathcal{M}_y$, following a general principle of using $y$ subscripts to denote matching objects. Similarly, the set of matching diffeomorphisms is $D_y$, and then our space $\hat{G}$ is just denoted $(X_y \times \mathbb{R})/D_y$ and our space $\hat{G}$ is denoted $X_y/D_y$, or $B$. It is possible that our using $\mathcal{B}$ for a different space may...
cause confusion; as in [66], the reason for this notation is that it is the base space of a suitable bundle.

It is clear that $\hat{G}^{S_1}_{G_2}$ is a principal $\mathbb{R}$-bundle over $G^{S_1}_{G_2}$ Therefore, by taking the natural smooth structure on such a bundle, it is enough to show that $G^{S_1}_{G_2}$ is smooth.

The argument is essentially that $\hat{G}^{S_1}_{G_2}$ is a manifold because it is a covering space of $B^{S_1}_{G_2}$. First, therefore, we have to check that $B^{S_1}_{G_2}$ is a manifold. We have

**Proposition 4.3.18** (cf. [66, Proposition 4.3]). $B^{S_1}_{G_2}$ is a smooth manifold. Moreover, around the classes of any matching pair of structures there exist charts for $B^{S_1}_{G_2}$ consisting of matching pairs of structures.

Strictly, of course, Nordström’s argument is for the case of $B_{G_2}$, defined as in Definition 4.3.16 but removing the constraints on $S^1$-invariance; but we have shown that locally $M^{S_1}_{G_2}$ is an open subset of $M_{G_2}$, and therefore locally $B^{S_1}_{G_2}$ is an open subset of $B_{G_2}$. As open subsets of a submanifold are submanifolds, the $S^1$-invariant result is immediate. The existence of such charts is not part of [66, Proposition 4.3] but is clear from its proof.

We now proceed to show $\hat{G}^{S_1}_{G_2}$ is a manifold. We give details for this proof, as it is not found in full generality in [66] but only in [64]; even there, not all the details are given. The factor by which $\hat{G}^{S_1}_{G_2}$ is bigger than $B^{S_1}_{G_2}$ appears in [64, p.140] (except of course for not requiring $S^1$-invariance), though we use a somewhat different setup. As it will reappear in the Calabi–Yau case, in Proposition 4.3.33, we define it separately. We would like to define it in general, as it only really depends on the Riemannian metric, but to prove its properties we need slice results analogous to those of Theorem 4.2.15 and Proposition 4.3.23. These results could be obtained more generally from Ebin [21], but we do not need them, and so we make a more restricted definition.

We first need to weaken the notion of isotopic with fixed limit from Definition 4.2.5. We will be interested in isotopies $\Phi_s$ such that for some fixed diffeomorphism $\Psi$, $(\Psi, \Phi_s)$ is an isotopy of matching pairs in the sense of Definition 4.3.13, and thus we make

**Definition 4.3.19.** Suppose that $\Phi$ and $\Psi$ are asymptotically cylindrical diffeomorphisms of an asymptotically cylindrical manifold $M$, isotopic in the sense of Definition 4.2.5. An isotopy is a curve $\Phi_s$ of asymptotically cylindrical diffeomorphisms; taking limits, an isotopy gives us a curve $(\Phi_s, L_s)$ of diffeomorphisms of $N$ and real numbers. $\Phi$ and $\Psi$ are *isotopic with fixed Diff($N$) limit* if there is an isotopy such that $\Phi_s$ is independent of $s$.

To have fixed Diff($N$) limit is clearly weaker than having fixed limit in the sense of Definition 4.2.5, and is precisely what is needed to obtain an isotopy of matching pairs.

**Definition 4.3.20.** Suppose that $(M_1, g_1)$ and $(M_2, g_2)$ are matching asymptotically cylindrical Ricci-flat manifolds, with metrics induced from torsion-free $G_2$ or Calabi–Yau structures.
Suppose that the cross-section is $N$ and $g_1$ and $g_2$ induce the metric $\tilde{g}$ on it, suppressing the diffeomorphism $F$ of Definitions 3.5.1 and 3.5.4. Consider the set of diffeomorphisms

\[
\left\{ \text{diffeomorphisms of } N \times [0,3] \text{ of the form } (n,t) \mapsto (f_t(n),t) \text{ where } f_t = \text{id} \text{ for } t \in [0,1], f_t = f_2 \text{ for } t \in [2,3], \text{ and } f_2 \text{ is an isometry of } \tilde{g} \right\}.
\]

(4.3.26)

Let $\tilde{A}(\tilde{g})$ be the quotient of this set by isotopies preserving the diffeomorphism on $N \times ([0,1] \cup [2,3])$.

It is clear that a class of $\tilde{A}(\tilde{g})$ induces diffeomorphisms on $M_1$ and $M_2$, and that taking a different representative gives diffeomorphisms that are isotopic with fixed limit in the sense of Definition 4.2.5. Consider the subgroup of $\tilde{A}(\tilde{g})$ consisting of classes whose induced diffeomorphisms on $M_1$ are isotopic with fixed $\text{Diff}(N)$ limit to isometries of $M_1$, and the analogous subgroup for $M_2$. Consider the subgroup $G$ generated by these two subgroups, and let $A(g_1, g_2)$ be the quotient $\tilde{A}(\tilde{g}) G$.

By a minor abuse of notation, when the metric is induced from Calabi–Yau structures we shall write $\tilde{A}(\tilde{\Omega}, \tilde{\omega})$ and $A(\Omega_1, \omega_1, \Omega_2, \omega_2)$.

By a slightly larger abuse of notation, when the metric is induced from a $S^1$-invariant torsion-free $G_2$ structures, we will write $\tilde{A}(\tilde{\phi})$ and $A(\phi_1, \phi_2)$ to be the sets given by requiring all the diffeomorphisms above to be $S^1$-invariant in the sense of Definition 4.2.8.

Elements of $\tilde{A}(\tilde{g})$ give an “action” on pairs of structures, thus:

**Definition 4.3.21.** Suppose that $g_1$ and $g_2$ are Ricci-flat metrics on $M_1$ and $M_2$ induced by matching Calabi–Yau or torsion-free $G_2$ structures. Let $\tilde{A}(\tilde{g})$ be as in Definition 4.3.20. Suppose that $g_1'$ and $g_2'$ are two other Ricci-flat metrics induced by matching structures. Define an action of $\tilde{A}(\tilde{g})$ on the set of such metrics by

\[
[\Phi](g_1, g_2) = (g_1, \Phi^* g_2).
\]

(4.3.27)

Note that (4.3.27) is not well-defined; however, it is well-defined up to pullback by a matching pair of diffeomorphisms isotopic to the identity, and in practice we will only be using (4.3.27) as a map to spaces (such as $\hat{G}_{G_2}^{S^1}$) where we have quotiented by pullback by matching pairs isotopic to the identity.

We hope that $A(\phi_1, \phi_2)$ will define the other smooth component in a local statement “$\hat{G}_{G_2}^{S^1} = A(\phi_1, \phi_2) \times B_{G_2}^{S^1}$”. We thus need to know that $A(\phi_1, \phi_2)$ is smooth.

**Proposition 4.3.22.** With the notation of Definition 4.3.20, $\tilde{A}(\tilde{g})$ is a finite-dimensional abelian Lie group. Its tangent space at the class of the identity is the space of Killing fields on $N$. A diffeomorphism defining a class of $\tilde{A}(\tilde{g})$ defines a class of $G$ precisely if its image under the map of Definition 4.3.21 is in the orbit of $(g_1, g_2)$ by matching pairs isotopic to the identity and so $G$ is a closed subgroup, with tangent spaces the sums of the subspaces of Killing fields on $N$ that have...
extensions to Killing fields on $M_1$ and $M_2$. Hence $A(g_1, g_2)$ is also a finite-dimensional manifold, and the map of Definition 4.3.21 passes to a well-defined map of $A(g_1, g_2)$ on structures up to pullback by matching pairs of diffeomorphisms isotopic to the identity.

Locally around the class of the identity, $A(\phi_1, \phi_2)$ as defined is equal to $A(g_1, g_2)$ for the induced metrics.

**Proof.** To show $\tilde{A}(\tilde{g})$ is a finite-dimensional Lie group, we note that it is equivalent, by careful use of bump functions, to the space of curves in $\text{Diff}_0(N)$ from the identity to isometries of $N$, modulo homotopy with fixed end points. The corresponding group in [64, Definition 6.3.6] is in fact defined as this space of curves modulo homotopy. As in [64], standard arguments show that its identity component (in the sense of isotopy through diffeomorphisms in the subset of (4.3.26)) is the universal cover of the identity component of the isometry group of $N$. The identity component of the isometry group is a compact Lie group, with Lie algebra given by the Killing fields. Hence, the space of Killing fields is the universal cover, and by considering each component in turn we consequently have that the whole of $\tilde{A}(\tilde{g})$ is a finite-dimensional Lie group, with tangent space at the class of the identity given by the Killing fields on $N$.

We now have to show that a diffeomorphism defining a class of $\tilde{A}(\tilde{g})$ defines a class of $G$ if and only if its image under the map of Definition 4.3.21 is in the orbit of the matching pair of structures inducing $(g_1, g_2)$ by matching pairs of diffeomorphisms isotopic to the identity. Since this orbit is closed by the remark after Definition 4.3.15, and we can find local continuous maps giving representatives from $\tilde{A}(\tilde{g})$ to diffeomorphisms since $\tilde{A}(\tilde{g})$ is finite-dimensional, we may then deduce that $G$ is a closed subgroup and the rest of the first paragraph follows. We note from Lemma 4.1.13 and and the remark after Definition 4.1.19 that, within $\text{Diff}_0$, an isometry of a metric induced by a Calabi–Yau or torsion-free $G_2$ structure is an automorphism of that structure. We shall, as in Definition 4.3.14, write $(\upsilon_1, \upsilon_2)$ for the matching pair of structures, which is either $(\phi_1, \phi_2)$ or $(\Omega_1, \omega_1, \Omega_2, \omega_2)$.

Note that if we take a different diffeomorphism representing the same class of $\tilde{A}(\tilde{g})$, the two diffeomorphisms are isotopic, by an isotopy preserving limits. Since such an isotopy changes the result by a matching pair of diffeomorphisms isotopic to the the identity, using this different representative does not affect whether the image by the map of Definition 4.3.21 lies in the orbit, so the argument will be independent of the representative we choose.

Firstly, if the diffeomorphism $\Phi$ representing a class of $\tilde{A}(\tilde{g})$ is isotopic, preserving $\text{Diff}(N)$ limit, to an automorphism $\Psi$ of $\upsilon_2$, then we have that $\Psi^{-1}\Phi$ is isotopic, preserving $\text{Diff}(N)$ limit, to the identity. Consequently the matching pair $(id, \Psi^{-1}\Phi)$ is isotopic to the identity, and hence $(\upsilon_1, \Phi^*\upsilon_2) = (\upsilon_1, \Phi^*(\Psi^{-1})^*\upsilon_2)$ lies in the orbit by matching pairs isotopic to the identity. Consequently, representatives of the part of $G$ corresponding to $M_2$ map into the orbit under the map of Definition 4.3.21.

Secondly, if the diffeomorphism $\Phi$ representing a class of $\tilde{A}(\tilde{g})$ is isotopic, preserving $\text{Diff}(N)$ limits, to an automorphism $\Psi$ of $\upsilon_1$, then since $\Phi$ is isotopic to the identity the matching
pair \((\Phi, \Phi)\) is isotopic to the identity, and so \((\Psi, \Phi)\) is isotopic to the identity. Consequently, \((v_1, \Phi^*v_2) = (\Psi^*v_1, \Phi^*v_2)\) lies in the orbit.

We have now shown that every diffeomorphism representing a class of \(G\) maps into the orbit under the map of Definition 4.3.21. It remains to show that if \(\Phi\) represents a class of \(\tilde{\mathcal{A}}(\tilde{g})\) and \((v_1, \Phi^*v_2)\) is in the orbit, then \(\Phi\) represents a class of \(G\). We suppose that \((v_1, \Phi^*v_2) = (\Psi_1^*v_1, \Psi_2^*v_2)\) for some matching pair \((\Psi_1, \Psi_2)\) isotopic to the identity. In particular, the \(\text{Diff}(N)\) part of their common limit is isotopic to the identity. This isotopy is a curve as at the start of the current proof, so defines a diffeomorphism \(\tilde{\Psi}\) giving a class of \(\tilde{\mathcal{A}}(\tilde{g})\). We will show that \(\tilde{\Psi}\) is isotopic with fixed \(\text{Diff}(N)\) limit to both \(\Psi_1\) and \(\Psi_2\). That is, \(\tilde{\Psi}\) is isotopic with fixed \(\text{Diff}(N)\) limit to the automorphism \(\Psi_1\) of \(v_1\) and also \(\Phi\tilde{\Psi}^{-1}\) is isotopic with fixed \(\text{Diff}(N)\) limit to the automorphism \(\Phi\Psi_2^{-1}\) of \(v_2\). That is, the classes defined by \(\tilde{\Psi}\) and \(\Phi\tilde{\Psi}^{-1}\) are in \(G\); it follows that the class defined by \(\Phi\) lies in \(G\).

We note that the extensions of \(\tilde{\Psi}\) to \(M_1\) and \(M_2\) are also isotopic to the identity and that moreover these isotopies can be chosen to match the original isotopies of \(\Psi_1\) and \(\Psi_2\) (which, since they match each other, have the same isotopy at the limit). Inverting one of these isotopies, we find an isotopy with fixed \(\text{Diff}(N)\) limit between \(\text{id}\) and \(\tilde{\Psi}^{-1}\Psi_i\) for either \(i\); composing with \(\tilde{\Psi}\) gives the result.

For the tangent space, we use an infinitesimal version of the previous argument: it would be possible, but more complicated, to extract tangent spaces from our description of \(G\). Suppose a Killing field \(X\) maps, under the derivative of the map of Definition 4.3.21, into the tangent space of the orbit extending \(X\) to \(M_2\). That is, we have \((v_1, \mathcal{L}_Xv_2) = (\mathcal{L}_{Y_1}v_1, \mathcal{L}_{Y_2}v_2)\) for some matching pair of vector fields \(Y_1\) and \(Y_2\). It follows that \(X - Y_2\) is a Killing field for \(g_2\) and that \(Y_1\) is a Killing field for \(g_1\); hence, since \(Y_1\) and \(Y_2\) match, \(X\) is the sum of Killing fields extending to Killing fields for \(g_1\) and \(g_2\). The converse follows by reversing the argument.

To prove the claim for \(A(\phi_1, \phi_2)\) we just observe that locally around the class of the identity these manifolds are given by their tangent spaces, and all Killing fields on \(S^1\)-invariant Ricci-flat manifolds are \(S^1\)-invariant. \(\square\)

Remark. In fact, we can say slightly more: the dimension of \(A(g_1, g_2)\) may be established using the Mayer-Vietoris theorem, see for instance [66, Proof of Proposition 4.2], which shows that it is zero under the condition \(b^1(M^T) = 0\) (of course, in our case we are working with \(b^1(M^T \times S^1) > 0\)).

Before working further with these ideas, we note that we will need to extract elements of \(A(g_1, g_2)\) from pairs of matching structures. Consequently, we need to take slightly more care with the slice arguments proving Theorem 4.2.15, to make sure we can determine diffeomorphisms from structures. Specifically, we require the following result, claimed without proof by Nordström [64, second sentence of p.141].

**Proposition 4.3.23.** Suppose one of the following holds.

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i) Let $N$ be a compact five-dimensional manifold. Let $U$ be a sufficiently small neighbourhood in a subspace around the translation-invariant $S^1$-invariant $G_2$ structure $\phi_0$ on $N \times S^1 \times \mathbb{R}$ such that $U$ is transverse to the orbit of the identity component $\text{Diff}^1_{S^1}(N \times S^1)$ defined in Definition 4.2.8.

ii) Let $M$ be a six-dimensional manifold with an end. Let $U$ be a sufficiently small neighbourhood in a subspace around the asymptotically cylindrical $S^1$-invariant $G_2$ structure $\phi_0$ on $M \times S^1$, consisting of structures whose limits are torsion-free $S^1$-invariant $G_2$ structures with the same automorphism groups as the limit of $\phi_0$, and such that $U$ is transverse to the orbit of the identity component of diffeomorphisms in $\text{Diff}^1_{S^1}(M \times S^1)$ that have limits automorphisms of the limit structure $\tilde{\phi}_0$.

Then, in both cases, on the image \( \left\{ \frac{\text{Diff}^1_{S^1}}{\text{Aut}(\phi_0) \cap \text{Diff}^1_{S^1}} \right\}^* U \) of the pullback map, the map from a smooth structure to the class of diffeomorphisms required is continuous and smooth.

**Proof.** It suffices to prove that the analogous map to $\frac{\text{Diff}_0}{\text{Aut}(\phi_0) \cap \text{Diff}_0}$ is smooth and continuous, as $\frac{\text{Diff}^1_{S^1}}{\text{Aut}(\phi_0) \cap \text{Diff}^1_{S^1}}$ is a submanifold of it, by Proposition 4.2.19 and using Proposition 4.2.22 to see that the quotient remains locally the same.

We prove (i); (ii) is entirely analogous. Both are straightforward applications of the inverse function theorem. Note that the forms in $U$ are not constrained to be torsion-free $G_2$ except in their limits, in case (ii).

Consider the pullback map from $C^{2,\alpha}$ diffeomorphisms and $C^{1,\alpha}$ 3-forms in $U$ to $C^{1,\alpha}$ 3-forms. By combining Baier’s result on the smoothness of pullbacks in the diffeomorphism [5, Theorem 2.2.15] with linearity in the form, the pullback map is smooth. The derivative is an isomorphism, since $U$ is transverse to the derivative orbits and we have removed the automorphisms, and so by the inverse function theorem there is a small neighbourhood of $\phi_0$ in $U$, and a small neighbourhood of the identity, which we call $D$, on which the inverse is continuous and smooth. When we restrict to smooth diffeomorphisms, the pullback map must remain continuous and smooth (as a smooth map to a submanifold).

We now just have to globalise in diffeomorphisms. Given some diffeomorphism $\Phi$, consider the subset $D\Phi$ and the slice neighbourhood $U$. The image of $D\Phi \times U$ under the pullback map is just the pullback by $\Phi$ of the image of $D \times U$. Consequently, the map from a point of the image $\phi$ to the diffeomorphism class is given by composing the inverse of $\Phi$ with the diffeomorphism class required for $\Phi^{-1} \ast \phi$, which depends smoothly on $\phi$ by the previous paragraph. Since composition and pullback by fixed smooth maps are smooth, it follows that the composition depends smoothly on $\phi$.

**Remark.** The slice $U$ exists by the proof of Theorem 4.2.15. The required transition to the $S^1$-invariant setting is carried out beginning on page 67. The slice required for case (i) occurs in...
the paragraph immediately before (4.2.29) and the slice required for case (ii) is the penultimate paragraph before Theorem 4.2.29.

We may now use $A(\phi_1, \phi_2)$ to find charts for $\hat{G}_{G_2}^{S^1}$.

**Proposition 4.3.24** (cf. [64, p.141]). Suppose that $[\phi_1, \phi_2] \in \hat{G}_{G_2}^{S^1}$. We have a chart $U = U_1 \times U_2$ around $([\phi_1], [\phi_2]) \in \mathcal{B}_{G_2}^{S^1} \subset \mathcal{M}_{G_2}^{S^1} \times \mathcal{M}_{G_2}^{S^1}$ consisting of matching pairs of structures, such that all these pairs and their limits have the same identity components of their isometry groups, and that for each $\phi'_i \in U_i$, for $\Phi^*\phi'_i$ sufficiently close to $\phi_i$ there is a continuous map $\Phi^*\phi'_i \mapsto [\Phi] \in \frac{\text{Diff}_{S^1}^{G_2}}{\text{Aut} \circ \text{Diff}_{S^1}^{G_2}}$

Then an open subset of $A(\phi_1, \phi_2) \times U$ is homeomorphic to an open neighbourhood of $[\phi_1, \phi_2]$ in $\hat{G}_{G_2}^{S^1}$, by the map from $A(\phi_1, \phi_2) \times U$ to $\hat{G}_{G_2}^{S^1}$

$$([\Phi], (\phi'_1, \phi'_2)) \mapsto [\phi'_1, \Phi^*\phi'_2],$$

(4.3.28)

where we take the extension of $\Phi$ to a diffeomorphism of $M_2$.

**Proof.** The set $U$ exists by Proposition 4.3.18 and the properties required are just properties on $M_1$ and $M_2$ so hold by the slice theorem Theorem 4.2.15. The existence of the required continuous map is given by Proposition 4.3.23.

We first show that (4.3.28) gives a well-defined element of $\hat{G}_{G_2}^{S^1}$. Since $(\phi'_1, \phi'_2)$ are the representatives of the point of $U$ in the chart, they match, and have the same identity components of their automorphism groups as $(\phi_1, \phi_2)$, as do their limits. It follows immediately from Proposition 4.3.22 that (4.3.28) is a well-defined map, since we have quotiented by the stabiliser.

Now we show injectivity. If we have $[\phi'_1, \Phi^*\phi'_2] = [\phi''_1, \Psi^*\phi''_2]$ in $\hat{G}$, in particular these define the same class in $\mathcal{B}$. Thus so do $(\phi'_1, \phi'_2)$ and $(\phi''_1, \phi''_2)$, and by hypothesis both of these pairs lie in $U$. Since $U$ is a slice neighbourhood, it follows that $\phi'_1 = \phi''_1$ and $\phi'_2 = \phi''_2$. It remains to show that if $[\phi'_1, \Phi^*\phi'_2] = [\phi'_1, \Psi^*\phi'_2]$, then $[\Phi] = [\Psi]$ in $A(\phi_1, \phi_2)$. Again, as in Proposition 4.3.22 we have shown that we have quotiented by the stabiliser, we indeed have $[\Phi] = [\Psi]$.

It is clear that (4.3.28) is continuous, so it only remains to show that it maps to an open subset and its inverse there is continuous. We shall construct the open set and the inverse on it simultaneously, taking a sequence of smaller open sets as required. First of all, the projection $\hat{G}_{G_2}^{S^1} \to \mathcal{B}_{G_2}^{S^1}$ is continuous, and so the preimage of $U$ is open. This preimage is our first open set $V_1$. We also have a natural map from an open subset of $\hat{G}_{G_2}^{S^1}$ contained in $V_1$ to $A(\phi_1, \phi_2)$, as follows. Suppose given $[\phi''_1, \phi''_2] \in V_1$, which projects to $([\phi'_1], [\phi'_2]) \in U$. By definition, there then exist slice structures $\phi'_1$ and $\phi'_2$ and asymptotically cylindrical diffeomorphisms $\Phi_1$ and $\Phi_2$ such that $\phi''_1 = \Phi_1^*\phi_i$. By construction, $\phi'_1$ and $\phi'_2$ match, but $\Phi_1$ and $\Phi_2$ need not; note that $\Phi_1$ and $\Phi_2$ are only defined up to isometries, but changing them by an isometry will have no effect on the final class of $A(\phi_1, \phi_2)$. Since $\Phi_1$ is asymptotically cylindrically asymptotic to the identity, its limit is isotopic to the identity, and hence the $\text{Diff}(N)$ part is. The isotopy from the identity to the $\text{Diff}(N)$ part of its limit defines, as in the proof of Proposition 4.3.22,
a diffeomorphism \( \Psi_1 \) representing a class of \( \tilde{A}(\tilde{\phi}) \), such that \( (\Phi_1, \Psi_1) \) is isotopic to the identity as a matching pair. On the other hand, \( \Phi_2 \) is also asymptotically cylindrically asymptotic to the identity, so we have a diffeomorphism \( \Psi_2 \) such that \( (\Psi_2, \Phi_2) \) is isotopic to the identity as a matching pair. Let \( \Phi' = \Psi_2^{-1}\Psi_1^{-1} \); the diffeomorphism \( \Phi' \) defines a class of \( A(\phi_1, \phi_2) \).

On a suitably small open set \( V_2 \), \( \Phi_1 \) and \( \Phi_2 \) depend continuously on \( \phi_1' \) and \( \phi_2' \), by the hypothesis. Consequently, since the isotopy can clearly be chosen continuously in the diffeomorphism, so do \( \Psi_1 \) and \( \Psi_2 \). Since inversion is continuous, the diffeomorphism \( \Phi' = \Psi_2^{-1}\Psi_1^{-1} \) also depends continuously on \( \phi_1'' \) and \( \phi_2'' \), and so in an even smaller open subset \( V_3 \) \( \Phi' \) defines a class of \( A(\phi_1, \phi_2) \) depending continuously on \( \phi_1'' \) and \( \phi_2'' \).

We have now constructed an open subset \( V_3 \) of \( \mathcal{G}^S_{G_2} \) and a map to \( U \times A(\phi_1, \phi_2) \) which we hope to be the inverse. It is clearly continuous, by construction. We have to check that it is an inverse, that is that \([\phi_1', (\Psi_2\Psi_1^{-1})^\ast\phi_2'] = [\phi_1'', \phi_2'']\). We know that the pairs \((\Phi_1^{-1}, \Psi_1^{-1})\) and \((\text{id}, \Phi_2^{-1}\Psi_2)\) are both isotopic to the identity as matching pairs. Consequently, we have

\[
[\phi_1'', \phi_2''] = [\Phi^\ast\phi_1', \Phi_2^\ast\phi_2'] = [\Phi_1^\ast\phi_1', \Psi_2^\ast\phi_2'] = [\phi_1', (\Psi_2\Psi_1^{-1})^\ast\phi_2'].
\] (4.3.29)

We now want to show that the set of local homeomorphisms obtained in Proposition 4.3.24 yields a manifold structure on \( \mathcal{G}^S_{G_2} \). We have to show that these charts form an atlas, i.e.

**Proposition 4.3.25.** Suppose given two open subsets of \( \mathcal{G}^S_{G_2} \) as in Proposition 4.3.24, so homeomorphic by the map in the proposition to the product of an open subset of \( U \times A(\phi_1, \phi_2) \) and \( U' \times A(\chi_1, \chi_2) \) respectively. Suppose that these subsets intersect; on the intersection, the transition map

\[
([\phi_1'], [\phi_2'], [\Phi]) \mapsto [\phi_1', \Phi^\ast\phi_2'] = [\chi_1', \Phi^\ast\chi_2'] \mapsto ([\chi_1'], [\chi_2'], [\Phi'])
\] (4.3.30)

is smooth, where \( \chi_1' \) and \( \chi_2' \) are the representatives of the classes \([\phi_1']\) and \([\phi_2']\) in the other chart, and \([\Phi']\) is the relevant class of \( A(\chi_1, \chi_2) \).

**Proof.** The map \(([\phi_1', \phi_2']) \mapsto ([\chi_1'], [\chi_2'])\) is the identity in \( \mathcal{B}^S_{G_2} \), and so is smooth. Consequently also the maps to the slice representatives \( \chi_1' \) and \( \chi_2' \) are smooth. For the map to \( \Phi' \), we note that the structures \( \phi_1' \) and \( \Phi^\ast\phi_2' \) depend smoothly on \([\phi_1'], [\phi_2']\) and \([\Phi]\). It is obvious that \([\phi_1']\) and \([\phi_2']\) depend smoothly on these classes, by linearity; for \([\Phi]\), we first note that since the components \( \tilde{A}(\tilde{\phi}) \) are identified with the finite-dimensional space of Killing fields, we may choose a representative \( \Phi \) for \([\Phi]\) smoothly, and then the pullback is smooth by [5, Theorem 2.2.15].

Consequently it suffices to show the map in Proposition 4.3.24 determining \([\Phi']\) from the structures \( \phi_1', \chi_1', \Phi^\ast\phi_2' \) and \( \chi_2' \) is smooth. Exactly the same argument works, using the smoothness result of Proposition 4.3.23. \(\square\)
We now proceed to the Calabi-Yau case. The spaces are set up as in the $S^1$-invariant $G_2$ case, but to show they are manifolds requires some further work. We begin with the definitions. We first make, exactly as in Definition 4.3.15,

**Definition 4.3.26.** Let

$$\tilde{G}_{SU(3)} = \frac{\text{matching pairs of Calabi-Yau structures and parameters } T}{\text{pairs of diffeomorphisms isotopic to the identity as matching pairs}},$$

where structures match if they match in the sense of Definition 3.5.4, isotopy as matching pairs is as in Definition 4.3.13, and the action is given by Definition 4.3.14.

We want to show that $\tilde{G}_{SU(3)}$ is smooth and that we can include it into $\tilde{G}_{G_2}^{S^1}$ as a smooth submanifold. It follows from our earlier analysis (in subsection 4.1.4) that to define such a map we need a matching pair of twistings $(z_1, z_2)$, with the usual boundary condition $\bar{z}_i(\frac{\partial}{\partial t}) = 0$. Such a pair of twistings immediately gives a map from matching pairs of Calabi-Yau structures to matching pairs of $S^1$-invariant $G_2$ structures, using Theorem 4.1.26. This map from pairs of Calabi-Yau structures to pairs of $G_2$ structures does not immediately induce a map $\tilde{G}_{SU(3)} \to \tilde{G}_{G_2}^{S^1}$; it may not be well-defined between these spaces, as if the triples $(\Omega_1, \omega_1, \Omega_2, \omega_2, T)$ and $(\Omega_1', \omega_1', \Omega_2', \omega_2', T')$ are identified by the isotopic-to-the-identity matching pair $(\Phi_1, \Phi_2)$ then the extension $(\hat{\Phi}_1, \hat{\Phi}_2)$ given as in Lemma 4.2.9 need not identify $(\text{Re } \Omega_1 + z_1 \wedge \omega_1, \text{Re } \Omega_2 + z_2 \wedge \omega_2, T)$ with $(\text{Re } \Omega_1' + z_1' \wedge \omega_1', \text{Re } \Omega_2' + z_2' \wedge \omega_2, T')$. We will have to take a quotient of twistings as in Proposition 4.2.14. We thus make, using notation inspired by Definition 4.3.16,

**Definition 4.3.27.** Let $M_1$ and $M_2$ be as in Definition 3.5.1, and let $B_Z$ be the space of matching pairs of twisting classes; that is, $([z_1], [z_2])$ in $Z(M_1) \times Z(M_2)$ such that there are representatives $z_1$ and $z_2$ matching in the sense of Definition 3.5.4. Here $Z(M_i)$ is the open subset of $H^1(M_i \times S^1)$ of Lemma 4.2.32.

**Remark.** We know that a twisting is of the form $Ld\theta + v$ for $v$ a 1-form on $M$. Thus if two twistings $L_i d\theta + v_i$ match, we have $L_1 = L_2$ and $v_1$ matches with $v_2$ (we used the first of these in Proposition 4.3.6). Thus $B_Z$ is the product of $\mathbb{R}_{>0}$ with the set of matching pairs $([v_1], [v_2])$. Now the forms $v_i$ have no $dt$ component in the limit and so $[\tilde{v}_1 - \tilde{v}_2]$ is a well-defined element of $H^1(N \times S^1)$, and it is easy to see using Mayer-Vietoris that a pair $([v_1], [v_2])$ equivalently matches if and only if $[\tilde{v}_1 - \tilde{v}_2]$ is zero. Thus the set of matching pairs is a vector space, and $B_Z$ is the product of $\mathbb{R}_{>0}$ with a vector space. It follows that $B_Z$ is a manifold, and in particular path-connected. Analogously to our use of path-connectedness of the space of twistings in Proposition 4.3.10, we will use path-connectedness of $B_Z$ in Proposition 4.3.32 below to show that which element of $B_Z$ we use is not very important.

These are indeed the classes we need to use to define our inclusion maps.
**Definition 4.3.28.** Suppose that \(([z_1], [z_2]) \in \mathcal{B}_Z\). Define the map

\[ 	au_{[z_1],[z_2]} : \tilde{G}_{SU(3)} \to \tilde{G}_{G_2}^{z_1} \]

by taking a pair of matching representatives \((z_1, z_2)\) and then mapping

\[ (\Omega_1, \omega_1, \Omega_2, \omega_2, T) \mapsto [\Re \Omega_1 + z_1 \wedge \omega_1, \Re \Omega_2 + z_2 \wedge \omega_2, T], \]

for any representative quintuple \((\Omega_1, \omega_1, \Omega_2, \omega_2, T)\).

**Proposition 4.3.29.** The map \(\tau_{[z_1],[z_2]} : \tilde{G}_{SU(3)} \to \tilde{G}_{G_2}^{z_1}\) of Definition 4.3.28 is well-defined and injective.

**Proof.** First suppose that \((\Omega_1, \omega_1, \Omega_2, \omega_2, T)\) and \((\Omega'_1, \omega'_1, \Omega'_2, \omega'_2, T')\) are two quintuples representing the same class of \(\mathcal{M}_{SU(3)}\). Suppose that we use the matching pairs \((z_1, z_2)\) and \((z'_1, z'_2)\), both of which represent \(([z_1], [z_2])\), to define the corresponding \(S^1\)-invariant \(G_2\) structures. There is a matching pair of diffeomorphisms \((\Phi_1, \Phi_2)\) isotopic to the identity in the sense of Definition 4.3.13 such that \(\Omega_1 = \Phi_1^* \Omega'_1\) and so on as in Definition 4.3.14. It is clear from the definitions that extending these diffeomorphisms to \(M_4 \times S^1\) as in Lemma 4.2.9 gives a matching pair, still isotopic to the identity in the sense of Definition 4.3.13, that acts on \((\Re \Omega'_1 + z'_1 \wedge \omega'_1, \Re \Omega'_2 + z'_2 \wedge \omega'_2, T')\) to give \((\Re \Omega_1 + \Phi_1^* (z'_1) \wedge \omega_1, \Re \Omega_2 + \Phi_2^* (z'_2) \wedge \omega_2, T)\).

To prove the result, it thus suffices to show that \((\Re \Omega_1 + \Phi_1^* (z'_1) \wedge \omega_1, \Re \Omega_2 + \Phi_2^* (z'_2) \wedge \omega_2, T)\) and \((\Re \Omega_1 + z_1 \wedge \omega_1, \Re \Omega_2 + z_2 \wedge \omega_2, T)\) represent the same class of \(G_{G_2}^{S^1}\). Since \((\Phi_1^* z'_1, \Phi_2^* z'_2)\) is another matching pair of representatives for \(([z_1], [z_2])\), by relabelling it suffices to prove the special case in which \((\Omega_1, \omega_1, \Omega_2, \omega_2, T) = (\Omega'_1, \omega'_1, \Omega'_2, \omega'_2, T')\).

Since \(z_i - z'_i\) are exact and asymptotically translation invariant, as in the proof of Lemma 4.2.32 they are \(df_i\), for some asymptotically translation invariant \(f_i\). Hence, there are asymptotically cylindrical diffeomorphisms \(\Phi_i \in \text{Diff}^{S^1}_0\) identifying \(\Re \Omega_i + z_i \wedge \omega_i\) and \(\Re \Omega_i + z'_i \wedge \omega_i\) on \(M_4 \times S^1\) by Lemma 4.2.10. We have to check that \((\Phi_1, \Phi_2)\) can be chosen to be isotopic to the identity as a matching pair and that the common limit is of the form \((\tilde{\Phi}_i, 0)\), i.e. has no translation component, so that \(T\) is unaffected. By the proof of Lemma 4.2.10, \(\Phi_i\) is the time-1 flow of \(f_i \frac{\partial}{\partial \theta}\), so its limit certainly has no translation component (which would correspond to a flow by \(\frac{\partial}{\partial \theta}\)). It then only remains to show that \(f_1\) and \(f_2\) can be chosen to match, as then the flow defines a matching isotopy. However, the proof of Lemma 4.2.32 also yields that the limits of the \(f_i\) only depend on the limits of the differences \(z'_i - z_i\); hence, that \(f_1\) and \(f_2\) match follows, if we make appropriate choices, from the fact that these differences match.

For injectivity, we apply the proof of Lemma 4.2.13. If

\[ [\Re \Omega_1 + z_1 \wedge \omega_1, \Re \Omega_2 + z_2 \wedge \omega_2, T] = [\Re \Omega'_1 + z_1 \wedge \omega'_1, \Re \Omega'_2 + z_2 \wedge \omega'_2, T'], \]

then there is a pair of \(S^1\)-invariant diffeomorphisms isotopic to the identity as a matching pair.
pulling back $\text{Re } \Omega_i + z_i \wedge \omega_i$ to $\text{Re } \Omega'_i + z_i \wedge \omega'_i$. Taking the $M_i$ parts of these diffeomorphisms as in Lemma 4.2.13 gives diffeomorphisms of $M_i$ pulling back $\Omega_i$ to $\Omega'_i$ and $\omega_i$ to $\omega'_i$. Evidently these diffeomorphisms are also isotopic to the identity as a matching pair, and as the $M$ part must include the translation part of the limit, the action on $T$ also gives $T'$. Hence

$$[\Omega_1, \omega_1, \Omega_2, \omega_2, T] = [\Omega'_1, \omega'_1, \Omega'_2, \omega'_2, T'];$$

(4.3.35)

i.e. $\iota_{[z_1],[z_2]}$ is injective.

To show that $\tilde{G}_{SU(3)}$ is smooth, and the maps $\iota_{[z_1],[z_2]}$ are smooth inclusions, we introduce smaller moduli spaces exactly analogous to those in the $S^1$-invariant $G_2$ case. We begin with the space corresponding to $B_{SU(3)}^{S^1}$ defined in Definition 4.3.16.

**Definition 4.3.30.** Let $B_{SU(3)}$ be the space of matching pairs in the $SU(3)$ moduli spaces on $M_1$ and $M_2$, that is:

$$([\Omega_1, \omega_1], [\Omega_2, \omega_2]) \in \mathcal{M}_{SU(3)}(M_1) \times \mathcal{M}_{SU(3)}(M_2)$$

(4.3.36)

such that there exist representatives $(\Omega_1, \omega_1)$ and $(\Omega_2, \omega_2)$ matching in the sense of Definition 3.5.4.

We also need the space corresponding to $\tilde{G}_{G_2}^{S^1}$ of Definition 4.3.17.

**Definition 4.3.31.** Let

$$\tilde{G}_{SU(3)} = \frac{\text{matching pairs of Calabi–Yau structures}}{\text{pairs of diffeomorphisms isotopic to the identity as matching pairs}}.$$  

(4.3.37)

To show that $\tilde{G}_{SU(3)}$ is a manifold, we argue, just as in the $S^1$-invariant $G_2$ case, that $B_{SU(3)}$ and $\tilde{G}_{SU(3)}$ are manifolds. We also have to show that the inclusion maps corresponding to $\iota_{[z_1],[z_2]}$ are all smooth.

We begin with $B_{SU(3)}$. Using Theorem 4.2.39, and essentially arguing as in the proof of that theorem that $B_{SU(3)}$ is the fibre of a smooth submersion to $B_{G_2}^{S^1} \to B_Z$, we prove that $B_{SU(3)}$ is a manifold. Proposition 4.3.32 is the Calabi–Yau analogue of Proposition 4.3.18.

**Proposition 4.3.32.** $B_{SU(3)}$ is a smooth manifold. Moreover, there exist charts consisting of matching pairs of structures around every point, with their groups of automorphisms isotopic to the identity independent of the point in the slice.

We have a diffeomorphism $B_{SU(3)} \times B_Z \to B_{G_2}^{S^1}$. In particular, the inclusion map $B_{SU(3)} \to B_{G_2}^{S^1}$ given by any pair of matching cohomology classes $([z_1],[z_2])$ is a well-defined smooth immersion.
Proof. We have shown in Theorem 4.2.39 that
\[
\mathcal{M}^{S^1}_{G_2}(M_1 \times S^1) \times \mathcal{M}^{S^1}_{G_2}(M_2 \times S^1) \\
= \mathcal{M}_{SU(3)}(M_1) \times Z(M_1) \times \mathcal{M}_{SU(3)}(M_2) \times Z(M_2). 
\] (4.3.38)

That is, given a pair of $S^1$-invariant $G_2$ moduli classes $[\varphi_1]$ and $[\varphi_2]$ we can express them in terms of Calabi–Yau structures by pairs $([\Omega_1, \omega_1], [z_1])$ and $([\Omega_2, \omega_2], [z_2])$. If moreover $([\varphi_1], [\varphi_2]) \in B^{S^1}_{G_2}$, then there exist representatives $\phi_1$ and $\phi_2$ that match. By applying uniqueness in Proposition 4.1.22 to the limits, it follows immediately that the corresponding representatives $z_i, \omega_i$, and $\Omega_i$ all match.

Conversely, given a matching pair of Calabi–Yau classes and a matching pair of twisting classes, taking matching representatives for these pairs and then combining them as in Proposition 4.1.22 gives a matching pair of $S^1$-invariant $G_2$ structures and hence of classes.

It follows therefore that the submanifold $B^{S^1}_{G_2}$ can be expressed in terms of the product structure of (4.3.38) as
\[
B^{S^1}_{G_2} = B_{SU(3)} \times B_Z. 
\] (4.3.39)

We proceed exactly as in the case of proving that $\mathcal{M}_{SU(3)}$ is a manifold by showing that $B_{SU(3)}$ is the fibre of a surjective submersion. By the remark after Definition 4.3.27, $B_Z$ is the product of $\mathbb{R}_{>0}$ with a vector space, and so a manifold. An obvious smooth path of structures and hence of classes (since a path of structures defines a path of cohomology classes) yields that the map $B^{S^1}_{G_2} \to B_Z$ is a submersion; it follows that we have a collection of manifold structures on $B_{SU(3)}$ by the implicit function theorem.

The natural inclusion map from $B_{SU(3)}$ to $\mathcal{M}_{SU(3)} \times \mathcal{M}_{SU(3)}$ is a smooth immersion from the smooth structure given on $B_{SU(3)}$ by the implicit function theorem, because it is the composition
\[
B_{SU(3)} \hookrightarrow B^{S^1}_{G_2} \hookrightarrow \mathcal{M}^{S^1}_{G_2} \times \mathcal{M}^{S^1}_{G_2} \twoheadrightarrow \mathcal{M}_{SU(3)} \times \mathcal{M}_{SU(3)}.
\] (4.3.40)

It follows that the smooth structure on $B_{SU(3)}$ is independent of the point of $B_Z$.

We already know that $B^{S^1}_{G_2}$ is a product as a topological space, as a subspace of the topological product (4.3.38). It remains to check that the bijective homeomorphism $B_{SU(3)} \times B_Z \to B^{S^1}_{G_2}$ is a diffeomorphism. We may choose coordinates as in the statement by applying Proposition 4.2.37. Now the maps from an $S^1$-invariant $G_2$ structure to its $Z$ and Calabi–Yau parts are smooth, and combining these we see that the map $B^{S^1}_{G_2} \to B_{SU(3)} \times B_Z$ is smooth; similarly given a Calabi–Yau structure and a twisting the map to an $S^1$-invariant $G_2$ structure is smooth, so the map $B_{SU(3)} \times B_Z \to B^{S^1}_{G_2}$ is smooth, and hence is indeed a diffeomorphism.

We now turn to the smoothness of $\hat{G}_{SU(3)}$. The coordinate charts are set up as in Proposition 4.3.24.
**Proposition 4.3.33.** Suppose that \([\Omega_1, \omega_1, \Omega_2, \omega_2] \in \hat{\mathcal{G}}_{SU(3)}\). We have a chart \(U\) for \(\mathcal{B}_{SU(3)}\) around \([\Omega_1, \omega_1, \Omega_2, \omega_2]\) consisting of matching pairs of structures, such that all these pairs and their limits have the same groups of isometries isotopic to the identity, and that for each \((\Omega, \omega)\) on \(M_1\) or \(M_2\), for \((\Phi^* \Omega, \Phi^* \omega)\) sufficiently close to \((\Omega_i, \omega_i)\) there is a continuous map \((\Phi^* \Omega_i, \Phi^* \omega_i) \mapsto \Phi\) (which is not necessarily unique).

Then an open subset of \(A(\Omega_1, \omega_1, \Omega_2, \omega_2) \times U\) is homeomorphic to an open neighbourhood of \([\Omega_1, \omega_1, \Omega_2, \omega_2]\) in \(\hat{\mathcal{G}}_{SU(3)}\), by the map from \(A(\Omega_1, \omega_1, \Omega_2, \omega_2) \times U\) to \(\hat{\mathcal{G}}_{G_2}^{S^1}\)

\[
([\Phi], (\Omega_1', \omega_1', \Omega_2', \omega_2')) \mapsto [\Omega_1', \omega_1', \Phi^* \Omega_2', \Phi^* \Omega_2'],
\]

where \(A(\Omega_1, \omega_1, \Omega_2, \omega_2)\) is as in Definition 4.3.20, and its action is as described in Definition 4.3.21.

**Proof.** Once the first paragraph of the proposition is established, the rest follows by exactly the same methods as in Proposition 4.3.24.

To establish the first paragraph, we note that we have charts with the required property on isometries by Proposition 4.3.32, and a continuous map giving diffeomorphisms between Calabi–Yau structures is the following composition of which every step is continuous. Given a pair of structures \((\Omega, \omega)\) and \((\Omega', \omega')\) close by and representing the same moduli class, we have that the continuous images \(\text{Re} \Omega + d\theta \wedge \omega\) and \(\text{Re} \Omega' + d\theta \wedge \omega'\) are also close by and represent the same moduli class. In turn, therefore, we have a continuously dependent \(S^1\)-invariant \(\Phi\) pulling back the first to the second, by the result of Proposition 4.3.24. We know that \((\Omega, \omega)\) and \((\Omega', \omega')\) represent the same class of \(\mathcal{M}_{SU(3)}\), so there is a diffeomorphism \(\Phi'\) pulling back one to the other. Composing the extension of \(\Phi'\) using Lemma 4.2.9 with \(\Phi^{-1}\) clearly gives an automorphism of the \(G_2\) structure, and hence an isometry of the product metric. Therefore restricting the composition to \(M\) as in Lemma 4.2.13 is also an isometry, and it follows by Lemma 4.1.13 that it is an automorphism of the Calabi–Yau structure. In particular, we see that \(\Phi^* \Omega = \Omega'\) and \(\Phi^* \omega = \omega'\). The map of diffeomorphisms given by restricting \(\Phi\) to \(M\) is continuous, as we see in Lemma 4.2.13 that restricting an \(S^1\)-invariant diffeomorphism to \(M\) is essentially composition with an inclusion and a projection: that is, the final step is continuous, as required.

We now have to show that \(\hat{\mathcal{G}}_{SU(3)}\) is smooth and the inclusion maps \(\iota_{[z_1],[z_2]} : \hat{\mathcal{G}}_{SU(3)} \to \hat{\mathcal{G}}_{G_2}^{S^1}\) induced as in Definition 4.3.28 are smooth. These are closely related:

**Proposition 4.3.34.** Let \((\Omega_1, \omega_1, \Omega_2, \omega_2)\) define a class of \(\hat{\mathcal{G}}_{SU(3)}\) and let \((z_1, z_2)\) define a class of \(\mathcal{B}_Z\). Let \((\phi_1 = \text{Re} \Omega_1 + z_1 \wedge \omega_1, \phi_2 = \text{Re} \Omega_2 + z_2 \wedge \omega_2)\) define the class \(\iota_{[z_1],[z_2]}([\Omega_1, \omega_1, \Omega_2, \omega_2]) \in \hat{\mathcal{G}}_{G_2}^{S^1}\). The manifolds \(A(\Omega_1, \omega_1, \Omega_2, \omega_2)\) and \(A(\phi_1, \phi_2)\) of Definition 4.3.20 can be naturally identified in a neighbourhood of the class of the identity. The inclusion map \(\iota_{[z_1],[z_2]}\) can thus be
examined locally in terms of the local homeomorphisms of Propositions 4.3.33 and 4.3.24 as

\[ A(\Omega_1, \omega_1, \Omega_2, \omega_2) \times B_{SU(3)} \rightarrow \hat{G}_{SU(3)} \rightarrow \hat{G}^S_{G_2} \rightarrow A(\phi_1, \phi_2) \times B^S_{G_2}. \] (4.3.42)

It is smooth. Moreover, it is an immersion, so the coordinate charts defined on \( \hat{G}_{SU(3)} \) in Proposition 4.3.33 form an atlas.

**Proof.** Locally around the identity, both \( A(\phi_1, \phi_2) \) and \( A(\Omega_1, \omega_1, \Omega_2, \omega_2) \) are manifolds. Consequently, we may work with the tangent spaces at the identity. These are quotients as identified in Proposition 4.3.22: the quotient of Killing fields on the cross-section by those that extend to Killing fields on the asymptotically cylindrical pieces.

It is clear that a Killing field on \( N \) extends to a Killing field on \( N \times S^1 \). Conversely, given a Killing field on \( N \times S^1 \), it is (since parallel and so \( S^1 \)-invariant) \( X + c \frac{\partial}{\partial \theta} \), with \( X \) a Killing field on \( N \) and \( c \) a constant, which defines a map from Killing fields on \( N \times S^1 \) to Killing fields on \( N \). Hence we have maps between \( \hat{A}(\hat{\phi}) \) and \( \hat{A}(\hat{\Omega}, \hat{\omega}) \). We have to show first that the maps induced on the quotients \( A(\phi_1, \phi_2) \) and \( A(\Omega_1, \omega_1, \Omega_2, \omega_2) \) by these are well-defined. If a Killing field on \( N \) extends to a Killing field on \( M_1 \), say, then clearly the corresponding Killing field on \( N \times S^1 \) extends as a Killing field to \( M_1 \times S^1 \). Conversely, if a Killing field \( X + c \frac{\partial}{\partial \theta} \) on \( N \times S^1 \) extends to a Killing field on \( M_1 \times S^1 \), then since \( c \frac{\partial}{\partial \theta} \) is itself a Killing field on \( M_1 \), \( X \) must also so extend. Thus these maps are well-defined. That the maps are inverse to each other also follows easily from the fact that \( c \frac{\partial}{\partial \theta} \) extends to an \( S^1 \)-invariant Killing field of \( M_1 \). Thus, in a sufficiently small subset of the identity, \( A(\Omega_1, \omega_1, \Omega_2, \omega_2) \) and \( A(\phi_1, \phi_2) \) are naturally identified.

For the second claim, for notational simplicity setting \( \Psi = \Phi^{-1} \), in these coordinates \( \iota_{[z_1],[z_2]} \) becomes

\[
(\Omega_1, \omega_1, [\Omega_2, \omega_2], [\Phi]) \mapsto ([\Omega_1, \omega_1], (\Phi^* \Omega_2, \Phi^* \omega_2)) \quad (4.3.43)
\]

\[
\mapsto [\text{Re } \Omega_1 + z_1 \wedge \omega_1, \Phi^* (\text{Re } \Omega_2 + \Psi^* z_2 \wedge \omega_2)] \quad (4.3.44)
\]

\[
\mapsto ([\text{Re } \Omega_1 + z_1 \wedge \omega_1], [\text{Re } \Omega_2 + \Psi^* z_2 \wedge \omega_2], [\Phi]). \quad (4.3.45)
\]

The map from \( A(\Omega_1, \omega_1, \Omega_2, \omega_2) \) to \( A(\phi_1, \phi_2) \) is clearly the identity under the identification of the previous paragraph and so smooth, so it is sufficient to check that the map to the \( B^S_{G_2} \) component is smooth. We show that the \( B^S_{G_2} \) component is independent of \( \Phi \). Then the \( B^S_{G_2} \) component is just \( ([\text{Re } \Omega_1 + z_1 \wedge \omega_1], [\text{Re } \Omega_2 + z_2 \wedge \omega_2]) \), which depends smoothly on the class \( ([\Omega_1, \omega_1], [\Omega_2, \omega_2]) \) of \( B_{SU(3)} \) by Proposition 4.3.32.

So, it is enough to show that as moduli classes we have \( [\text{Re } \Omega_2 + z_2 \wedge \omega_2] = [\text{Re } \Omega_2 + \Psi^* z_2 \wedge \omega_2] \). On an appropriate neighbourhood of the class of the identity in \( A(\Omega_1, \omega_1, \Omega_2, \omega_2) \), possibly reducing the size of the charts, it is sufficient to show the equalities of cohomology classes

\[
[\text{Re } \Omega_2 + z_2 \wedge \omega_2] = [\text{Re } \Omega_2 + \Psi^* z_2 \wedge \omega_2], \quad [\text{Re } \hat{\Omega}_{2,2} + \hat{z}_2 \wedge \hat{\omega}_{2,2}] = [\text{Re } \hat{\Omega}_{2,2} + \hat{\Psi}^* \hat{z}_2 \wedge \hat{\omega}_{2,2}]. \quad (4.3.46)
\]
where the additional subscript 2 in the second equation denotes the relevant components of \( \tilde{\Omega}_2 = \tilde{\Omega}_{2,1} + dt \wedge \tilde{\Omega}_{2,2} \) and \( \tilde{\omega}_2 = \tilde{\omega}_{2,1} + dt \wedge \tilde{\omega}_{2,2} \) as in Theorem 4.2.15. Using that theorem, we know that structures that are sufficiently close and have these cohomology classes the same define the same moduli classes. But \( \Phi \) is isotopic to the identity, and so so is \( \Psi \), and \( \omega_2 \) and \( \omega_{2,2} \) are closed (since \( \omega_2 \) is parallel): it follows that the cohomology classes are the same.

\( \iota_{[z_1],[z_2]} \) is now obviously an immersion, because the identity is and the inclusion of \( \mathcal{B}_{SU(3)} \) into \( \mathcal{B}_{G_2}^{S^1} \) is (by Proposition 4.3.32 again). Since the manifold structure on \( \mathcal{G}_{G_2}^{S^1} \) is defined in Definitions 4.3.15 and 4.3.26 are manifolds. With \( \mathcal{B}_Z \) as defined in Definition 4.3.27, we have a diffeomorphism

\[
\mathcal{G}_{SU(3)} \times \mathcal{B}_Z \to \mathcal{G}_{G_2}^{S^1},
\]

(4.3.47)

induced from that of Theorem 4.2.39.

**Proof.** Using the maps \( \iota_{[z_1],[z_2]} \) of Definition 4.3.28, we have a map \( \mathcal{G}_{SU(3)} \times \mathcal{B}_Z \to \mathcal{G}_{G_2}^{S^1} \)

\[
([\Omega_1, \omega_1, \Omega_2, \omega_2, T], ([z_1], [z_2])) \mapsto \iota_{[z_1],[z_2]}([\Omega_1, \omega_1, \Omega_2, \omega_2, T]).
\]

(4.3.48)

(4.3.48) is smooth because in local coordinates, by Proposition 4.3.34, it reduces to the corresponding map \( \mathcal{B}_{SU(3)} \times \mathcal{B}_Z \to \mathcal{B}_{G_2}^{S^1} \), the identity on the \( T \) component, and the identity \( A(\Omega_1, \omega_1, \Omega_2, \omega_2) \to A(\phi_1, \phi_2) \). Using that the map on \( \mathcal{B} \) spaces is a diffeomorphism, it follows that (4.3.48) is a smooth local diffeomorphism.

It is clearly a surjection, as any representative \((\phi_1, \phi_2, T)\) of a class of \( \mathcal{G}_{G_2}^{S^1} \) can be written as \((\text{Re } \Omega_1 + z_1 \wedge \omega_1, \text{Re } \Omega_2 + z_2 \wedge \omega_2, T)\) for some matching pair of Calabi–Yau structures and matching pair of twistings as in Proposition 4.3.32. It is an injection because if

\[
[\text{Re } \Omega_1 + z_1 \wedge \omega_1, \text{Re } \Omega_2 + z_2 \wedge \omega_2, T] = [\text{Re } \Omega'_1 + z'_1 \wedge \omega'_1, \text{Re } \Omega'_2 + z'_2 \wedge \omega'_2, T'],
\]

(4.3.49)

then there are asymptotically cylindrical diffeomorphisms relating these \( S^1 \)-invariant \( G_2 \) structures, and in particular we see that \([z_1] = [z'_1]\) as in Lemma 4.2.13. Injectivity then follows by injectivity in Proposition 4.3.29.
Thus (4.3.47) is a global diffeomorphism, as claimed.

4.3.4 Restricting to data that can be glued

In this subsection, we define the subspaces of the quotient $\tilde{\mathcal{G}}_{SU(3)}^{S^1}$ and $\tilde{\mathcal{G}}_{G_2}$ that actually glue and the corresponding gluing maps (Definitions 4.3.37 and 4.3.42). We have to define the gluing on Calabi–Yau structures by our inclusions $\tilde{\mathcal{G}}_{SU(3)} \to \tilde{\mathcal{G}}_{G_2}^{S^1}$, as there is more than one such inclusion, there is more than one possible such gluing map. Consequently, we must prove that the Calabi–Yau gluing map is independent of the inclusion we consider: this is the final result of this subsection, Proposition 4.3.44, and essentially follows by combining Proposition 4.3.10 with the fact that the gluing map is well-defined on the $G_2$ moduli space (Theorem 4.3.38).

We begin by defining $\tilde{\mathcal{G}}_{G_2}^{S^1} \subset \tilde{\mathcal{G}}_{G_2}$: our definition, Definition 4.3.36, is adapted from Nordström [66, Definition 2.4]. From Theorem 4.3.2, we know that any pair of $S^1$-invariant $G_2$ structures glues for gluing parameter $T > T_0$ for some $T_0$, and $T_0$ is upper semi-continuous in the pair of structures. Thus, if we defined $\tilde{\mathcal{G}}_{G_2}^{S^1} \subset \tilde{\mathcal{G}}_{G_2}$ to be the subset of gluing classes with representatives $(\phi_1, \phi_2, T)$ such that $\phi_1$ and $\phi_2$ glue with parameter $T$, $\tilde{\mathcal{G}}_{G_2}^{S^1}$ would be an open subset and so a manifold.

However, [66, Proposition 4.4] says that the derivative of the gluing map between moduli spaces of $G_2$ structures (and hence of $S^1$-invariant $G_2$ structures) is an isomorphism for $T > T'_0$ for some, possibly larger, $T'_0$. The proof of [66, Theorem 3.1] enables us to infer that $T'_0$ is also upper semi-continuous in the structures: in order to prove Theorem 4.3.46, that the gluing map between moduli spaces of Calabi–Yau structures is a local diffeomorphism, we would like to have $T > T'_0$ as well. Therefore we make

**Definition 4.3.36** (cf. [66, Definition 2.4]). Let $\tilde{\mathcal{G}}_{G_2}^{S^1} \subset \tilde{\mathcal{G}}_{G_2}$ be the subset of $G_2$ gluing data classes that have a representative $(\phi_1, \phi_2, T)$ with $T$ large enough that $\phi_1$ and $\phi_2$ can be glued with parameter $T$ in the sense of Theorem 4.3.2 and the derivative of the gluing map is an isomorphism at the triple $(\phi_1, \phi_2, T)$.

We see that $\tilde{\mathcal{G}}_{G_2}^{S^1}$ is an open subset of $\tilde{\mathcal{G}}_{G_2}$. We may then define a gluing map from $\tilde{\mathcal{G}}_{G_2}^{S^1}$ in the obvious way. Note that as $T$ is now varying, we cannot sensibly use $M^T$ for the glued manifold as in Theorem 4.3.2. We shall call it $M$.

**Definition 4.3.37.** The gluing map $\tilde{\mathcal{G}}_{G_2}^{S^1}$ to $\mathcal{M}_{G_2}^{S^1}(M \times S^1)$ is defined as follows. Given a class in $\tilde{\mathcal{G}}_{G_2}^{S^1}$, by definition it admits a representative $(\phi_1, \phi_2, T)$ that glues in the sense of Theorem 4.3.2 (and by the proof of Theorem 4.3.3 the resulting structure is $S^1$-invariant). Take the class of the result in $\mathcal{M}_{G_2}^{S^1}(M \times S^1)$.

As there is likely to be more than one such representative, the gluing map of Definition 4.3.37 may in principle be ill-defined. However, $\tilde{\mathcal{G}}_{G_2}^{S^1}$ was properly chosen, as we have
Theorem 4.3.38 (cf. [66, Proposition 4.1]). The map $G_{G_2}^{S^1} \to \mathcal{M}_{G_2}^{S^1}$ given by Definition 4.3.37 is well-defined.

Of course, Nordström proved that the corresponding map $G_{G_2} \to \mathcal{M}_{G_2}$, with $G_{G_2}$ defined analogously to $G_{G_2}^{S^1}$, was well-defined: since both $G_{G_2}$ and $\mathcal{M}_{G_2}$ are locally diffeomorphic to the corresponding $S^1$-invariant spaces and the map is defined identically, the $S^1$-invariant result immediately follows.

The most natural definition of $G_{SU(3)}$ would be to take those classes of Calabi–Yau gluing data that have representatives $(\Omega_1, \omega_1, \Omega_2, \omega_2, T)$ that glue using the inclusions $\iota_{[z_1], [z_2]}$ of Definition 4.3.28. However, it is possible that the required $T$ may depend on $z_i$; it is possible that it might not, but to establish independence of $T$ from $z$ would require close analysis of the role which $z$ plays in the gluing problem, which would be tantamount to analysing Calabi–Yau gluing directly. We will therefore have to work initially with subsets depending on the class $([z_1], [z_2]) \in B_Z$, but we will then take the union to define our space $\mathcal{G}_{SU(3)}$, and check that the gluing map is still well-defined. First, we make

Definition 4.3.39. Let

$$G_{SU(3),([z_1],[z_2])} = \iota^{-1}_{[z_1],[z_2]}(G_{G_2}^{S^1}) = \{[\Omega_1, \omega_1, \Omega_2, \omega_2, T] : [\text{Re } \Omega_1 + z_1 \wedge \omega_1, \text{Re } \Omega_2 + z_2 \wedge \omega_2, T] \in G_{G_2}^{S^1}\}. \quad (4.3.50)$$

$G_{SU(3),([z_1],[z_2])}$ is the inverse image of an open subset under a continuous map so open. Note that, for any choice of $([z_1], [z_2])$, every class of $\hat{G}_{SU(3)}$ is included in $G_{SU(3),([z_1],[z_2])}$ for sufficiently large $T$, because every pair of matching $S^1$-invariant $G_2$-structures glues and the derivative is an isomorphism for $T$ sufficiently large.

We now define a family of gluing maps:

Definition 4.3.40. Define the gluing map on the space of gluing data $G_{SU(3),([z_1],[z_2])}$ given by Definition 4.3.39 by the composition

$$G_{SU(3),([z_1],[z_2])} \to G_{G_2}^{S^1} \to \mathcal{M}_{G_2}^{S^1} \to \mathcal{M}_{SU(3)}, \quad (4.3.51)$$

where the first map is the inclusion $\iota_{[z_1],[z_2]}$, the second map is the gluing map of Definition 4.3.37, and the third map is the appropriate projection of Theorem 4.2.39.

Rather than a family of spaces of gluing data and corresponding gluing maps, we would like a single space with a single gluing map. We make

Definition 4.3.41. Let

$$G_{SU(3)} = \bigcup_{([z_1],[z_2]) \in B_Z} G_{SU(3),([z_1],[z_2])}. \quad (4.3.52)$$

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\[ G_{SU(3)} \] is also an open subset of \( \tilde{G}_{SU(3)} \), and so a manifold. We can define a gluing map on it in the natural way.

**Definition 4.3.42.** Define the **gluing map** \( G_{SU(3)} \rightarrow M_{SU(3)} \) by taking the map of Definition 4.3.40 on each of the open subsets \( G_{SU(3),([z_1],[z_2])] \).

The gluing map of Definition 4.3.42 is not a priori well-defined. If we are given a class \([\Omega_1,\omega_1,\Omega_2,\omega_2, T]\) of \( G_{SU(3)} \), there may be a pair of pairs \(([z_1],[z_2])\) and \(([z_1'],[z_2'])\) such that \( \xi_{[z_1],[z_2]}([\Omega_1,\omega_1,\Omega_2,\omega_2, T]) \) and \( \xi_{[z_1'],[z_2']}([\Omega_1,\omega_1,\Omega_2,\omega_2, T]) \) both lie in \( G_{SU(3)}^1 \). We have to show that under the two maps of Definition 4.3.40 \([\Omega_1,\omega_1,\Omega_2,\omega_2, T]\) has the same image.

We have already proved Proposition 4.3.10, which says that the \( M_{SU(3)} \) components of the gluing of the two pairs \((\Omega_1 + z_1 \land \omega_1, \Omega_2 + z_2 \land \omega_2, T + S)\) and \((\Omega_1 + z_1' \land \omega_1, \Omega_2 + z_2' \land \omega_2, T + S)\) are equal for \( S \) large enough. We now have two distinct classes in \( G_{SU(3)}^1 \) corresponding to our two inclusions. For each of these classes, there exist representatives that glue, but we know nothing about how the representatives for the different classes are related. However, for \( S \) large enough, we know that the explicit representatives \((\Omega_1 + z_1 \land \omega_1, \Omega_2 + z_2 \land \omega_2, T + S)\) and \((\Omega_1 + z_1' \land \omega_1, \Omega_2 + z_2' \land \omega_2, T + S)\) glue, and we may apply Proposition 4.3.10 to deduce that these have the same \( M_{SU(3)} \) component. We will show that the equality of the \( M_{SU(3)} \) component is independent of increasing the gluing parameter, so that although \((\Omega_1 + z_1 \land \omega_1, \Omega_2 + z_2 \land \omega_2, T + S)\) and \((\Omega_1 + z_1' \land \omega_1, \Omega_2 + z_2' \land \omega_2, T + S)\) may not glue, the result of gluing the classes \([\Omega_1 + z_1 \land \omega_1, \Omega_2 + z_2 \land \omega_2, T]\) and \([\Omega_1 + z_1' \land \omega_1, \Omega_2 + z_2' \land \omega_2, T]\) must also have the same \( M_{SU(3)} \) component.

**Lemma 4.3.43.** Suppose that \((\Omega_1,\omega_1, T)\) and \((\Omega_1',\omega_1', T')\) are representatives of the same class in \( \tilde{G}_{SU(3)} \) and for matching pairs of twistings \( z_i \) and \( z_i' \), possibly defining different twisting classes, the resulting representatives \((\text{Re } \Omega_1 + z_i \land \omega_1, T)\) and \((\text{Re } \Omega_1' + z_i' \land \omega_1', T')\) for the corresponding classes of \( \tilde{G}_{SU(3)}^1 \) glue and they continue to glue if the parameters \( T \) and \( T' \) are increased. Suppose that there exists \( S > 0 \) such that the results of gluing \((\text{Re } \Omega_1 + z_i \land \omega_1, T + S)\) and \((\text{Re } \Omega_1' + z_i' \land \omega_1', T' + S)\) have the same \( M_{SU(3)} \) component. Then so do the results of gluing \((\text{Re } \Omega_1 + z_i \land \omega_1, T)\) and \((\text{Re } \Omega_1' + z_i' \land \omega_1', T')\).

**Proof.** We have two curves in \( G_{SU(3)}^1 \), defined on \([0, S]\) by gluing the curves \( s \mapsto (\text{Re } \Omega_1 + z_i \land \omega_1, \text{Re } \Omega_2 + z_2 \land \omega_2, T + s) \) and \( s \mapsto s(\text{Re } \Omega_1' + z_i' \land \omega_1', \text{Re } \Omega_2' + z_2' \land \omega_2', T' + s) \). We consider the projection of these to \( M_{SU(3)} \), and call them \(((\Omega(s),\omega(s)))\) and \(((\Omega'(s),\omega'(s)))\). By Proposition 4.2.33, \( M_{SU(3)} \) is locally represented by the cohomology of \( \text{Re } \Omega \) and \( \omega \), and so these curves are determined by their values at a point and the corresponding curves of cohomology classes. Now these cohomology classes are

\[
[\gamma_{T+s}(\text{Re } \Omega_1, \text{Re } \Omega_2)] \quad \text{and} \quad c(s)[\gamma_{T+s}(\omega_1, \omega_2)],
\]

and the same with primes, where \( \gamma_T \) is the gluing map of Definition 3.5.5, and the functions \( c \) and \( c' \) are as in Proposition 4.3.6.
By assumption, \([\Omega(S), \omega(S)] = [\Omega'(S), \omega'(S)]\). In particular, we have the same for the cohomology classes:

\[
[\gamma_{T+S}(\Re \Omega_1, \Re \Omega_2)] = [\gamma_{T+S}(\Re \Omega'_1, \Re \Omega'_2)] , \quad (4.3.54)
\]
\[
c(s)[\gamma_{T+S}(\omega_1, \omega_2)] = c'(s)[\gamma_{T+S}(\omega'_1, \omega'_2)] . \quad (4.3.55)
\]

As \(\Omega_i, \omega_i\) and \(\Omega'_i, \omega'_i\) define the same Calabi–Yau class, they have agreeing cohomology and agreeing limit cohomology, it follows that \([\gamma_{T+S}(\Re \Omega_1, \Re \Omega_2)] = [\gamma_{T+S}(\Re \Omega'_1, \Re \Omega'_2)]\) for all \(s\) (because they agree at \(s = S\) and the change as we reduce \(s\) are given by the Mayer-Vietoris sequence and the limit cohomology; see, for example, [66, Proposition 3.2]).

The Kähler parts are complicated slightly by the functions \(c\) and \(c'\). Under the restriction map to the cohomology of \(M_1\), \([\gamma_{T+S}(\omega_1, \omega_2)]\) and \([\gamma_{T+S}(\omega'_1, \omega'_2)]\) give \([\omega_1]\) and \([\omega'_1]\), respectively, which are equal by assumption and nonzero by non-degeneracy of the Kähler form. Thus from (4.3.55) we see that

\[
[\gamma_{T+S}(\omega_1, \omega_2)] = [\gamma_{T+S}(\omega'_1, \omega'_2)] , \quad c(s) = c'(s) . \quad (4.3.56)
\]

The same argument as in the previous paragraph then shows the equality of the cohomology classes over the whole curve. Now Lemma 4.3.9 shows that there exists \(\epsilon\) such that \(c(s) = c'(s)\) for \(s > S - \epsilon\). Since the cohomology represents \(\mathcal{M}_{SU(3)}\) locally homeomorphically we get that the curves in \(\mathcal{M}_{SU(3)}\) agree for \(s > S - \epsilon\); by continuity they also agree at \(s = S - \epsilon\). Generalising the above argument, and using the connectedness of \([0, S]\), it follows that the curves in \(\mathcal{M}_{SU(3)}\) agree at \(s = 0\).

As envisaged, Lemma 4.3.43 enables us to prove a far stronger well-definition result than Proposition 4.3.10.

**Proposition 4.3.44.** The gluing map \(G_{SU(3)} \to \mathcal{M}_{SU(3)}\) of Definition 4.3.42 is well-defined.

**Proof.** Suppose that \([\Omega_1, \omega_1, \Omega_2, \omega_2, T]\) is a class of \(G_{SU(3)}\) and there exist two twisting class pairs \([z_1, z_2]\) and \([z'_1, z'_2]\) such that the classes \([\Re \Omega_i + z_i \land \omega_i, T]\) and \([\Re \Omega'_i + z'_i \land \omega_i, T]\) both lie in \(G_{G_2}^{S^1}\). That is, there are representatives

\[
(\Re \Omega_1 + z_1 \land \omega_1, \Re \Omega_2 + z_2 \land \omega_2, T) \text{ and } (\Re \Omega'_1 + z'_1 \land \omega'_1, \Re \Omega'_2 + z'_2 \land \omega'_2, T'), \quad (4.3.57)
\]

both of which glue and continue to glue if \(T\) and \(T'\) are increased.

Now take \((\Phi_1, \Phi_2)\) to be a matching pair of diffeomorphisms of \(M_1\) and \(M_2\) isotopic to the identity pulling back \((\Omega'_i, \omega'_i, T')\) to \((\Omega_i, \omega_i, T)\), which represents the same \(G_{SU(3)}\) class, by construction. Then there exists \(S\) sufficiently large so that

\[
(\Re \Omega_1 + \Phi^*_1 z'_1 \land \omega_1, \Re \Omega_2 + \Phi^*_2 z'_2 \land \omega_2, T + S) \quad (4.3.58)
\]

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also glues. Since the action of \((\Phi_1, \Phi_2)\) is affine on the gluing parameter, gluing (4.3.58) defines the same class of \(\mathcal{M}^{S_1}_{G_2}\) as gluing
\[
(\text{Re } \Omega_1' + z_1' \land \omega_1', \text{Re } \Omega_2' + z_2' \land \omega_2', T' + S).
\] (4.3.59)

By Theorem 4.3.38, the results of gluing these two are thus the same.

Also, however, by Proposition 4.3.10, and possibly increasing \(S\) some more, the result of gluing (4.3.58) has the same \(\mathcal{M}_{SU(3)}\) component as the result of gluing
\[
(\text{Re } \Omega_1 + z_1 \land \omega_1, \text{Re } \Omega_2 + z_2 \land \omega_2, T + S).
\] (4.3.60)

Thus we have that the \(\mathcal{M}_{SU(3)}\) component giving by gluing (4.3.60) is the same as for gluing (4.3.59). By Lemma 4.3.43, we can then reduce to \(S\) to zero, which proves the proposition.

**Remark.** The proof of Proposition 4.3.44 is closely allied to Nordström’s proof of the \(G_2\) theorem, Theorem 4.3.38, which again works by taking a curve for far larger gluing parameter and arguing on cohomology, and then increasing the gluing parameter in a controlled fashion. It would not be that hard to combine the two proofs (essentially proving Proposition 4.3.10 in a representation-independent way).

### 4.3.5 The main theorem

We now turn to the fact that the gluing map constructs an open subset of the moduli space \(\mathcal{M}_{SU(3)}\). This is Theorem 4.3.46 below. The idea is that by the previous work (Propositions 4.3.24 and 4.3.33 and the fact that the spaces \(\tilde{G}\) are principal \(\mathbb{R}\)-bundles over the spaces \(\hat{G}\)) we locally have
\[
\mathcal{G}_{SU(3)} = A(\Omega_1, \omega_1, \Omega_2, \omega_2) \times \mathbb{R}_{>0} \times B_{SU(3)}
\subset A(\text{Re } \Omega_1 + z_1 \land \omega_1, \text{Re } \Omega_2 + z_2 \land \omega_2) \times \mathbb{R}_{>0} \times B^{S_1}_{G_2} = \mathcal{G}^{S_1}_{G_2},
\] (4.3.61)

and we have by Theorem 4.2.39 that
\[
\mathcal{M}^{S_1}_{G_2} = \mathcal{M}_{SU(3)} \times Z,
\] (4.3.62)

and by Nordström’s work [66] we understand the gluing map defined (by Definition 4.3.37 below) on \(\mathcal{G}^{S_1}_{G_2}\) and its derivative, which is an isomorphism. We thus just have to consider what happens in terms of the splittings in (4.3.61) and (4.3.62). We consider the \(B_Z\) part, \(B_{SU(3)}\) part, the \(A\) part, and the \(\mathbb{R}_{>0}\) part in turn, showing that variations in \(B_Z\) lead to variations in the \(Z\) component of \(\mathcal{M}_{SU(3)} \times Z\) but variations in the other three parts lead to variations in \(\mathcal{M}_{SU(3)}\) and perhaps rescaling of the \(Z\) component. It then follows easily that the composition defined
by Definition 4.3.42 also has derivative an isomorphism, and Theorem 4.3.46 then follows from
the inverse function theorem.

Formally, we should first note Nordström’s result for the $G_2$ case, which says ([66, Theorem 2.3]; see also our Definition 4.3.36) that the map

$$G_{G_2} \to M_{G_2}$$

(4.3.63)
defined analogously to Definition 4.3.37 is a local diffeomorphism. As we know that $G_{G_2}$ and $M_{G_2}$ are locally diffeomorphic to $G^{S^1}_{G_2}$ and $M^{S^1}_{G_2}$ respectively, and the gluing map commutes with these local diffeomorphisms as it maps $S^1$-invariant structures to $S^1$-invariant structures (proof of Theorem 4.3.3), Nordström’s result implies

**Theorem 4.3.45.** The gluing map

$$G^{S^1}_{G_2} \to M^{S^1}_{G_2}$$

(4.3.64)
defined in Definition 4.3.37 has derivative an isomorphism, and is thus a local diffeomorphism.

We want to use the same ideas to show

**Theorem 4.3.46.** The map

$$G_{SU(3)} \to G^{S^1}_{G_2} \to M^{S^1}_{G_2} \to M_{SU(3)}$$

(4.3.65)
defined by Definition 4.3.42 is also a local diffeomorphism.

As we are only interested in a local result, we may work on an open subset $G_{SU(3),([z_1],[z_2])}$ as in Definition 4.3.39. We apply the inverse mapping theorem, as Nordström did to prove the corresponding result for $G_2$. He showed (in the proof of [66, Proposition 4.4]) that the map between harmonic forms given by the derivative is essentially the gluing map given by Definition 3.5.5, and that whilst this gluing map isn’t an isomorphism, it is injective, and a complement of its image can be obtained by varying $T$. We have to show the derivative remains an isomorphism when we pre- and post-compose the inclusion by $t_{[z_1],[z_2]}$ and the projection $M^{S^1}_{G_2} \to M_{SU(3)}$ of Theorem 4.2.39.

We need to consider the tangent spaces of $G_{SU(3),([z_1],[z_2])}$, $G^{S^1}_{G_2}$, $M^{S^1}_{G_2}$ and $M_{SU(3)}$ and how they are related to each other. By the work in subsection 4.3.3, we essentially have the following

**Proposition 4.3.47.** The tangent space to $G_{SU(3),([z_1],[z_2])}$ is the direct sum $T_{SU(3)} \oplus TA \oplus T\mathbb{R}$, where $T_{SU(3)}$ is the inclusion of the tangent space to $B_{SU(3)}$ and $A = A(\Omega_1, \omega_1, \Omega_2, \omega_2)$ is defined in Definition 4.3.20. The tangent space to $G^{S^1}_{G_2}$ is the direct sum $T_{SU(3)} \oplus T_Z \oplus TA \oplus T\mathbb{R}$, where $T_{SU(3)}$ is as before, $T_Z$ is the inclusion of the tangent space to $B_Z$, and $A = A(\phi_1, \phi_2)$, and the inclusion of $TG_{SU(3)}$ is given by the inclusion of the appropriate components (recalling that the two groups $A$ have the same tangent spaces by Proposition 4.3.34).
The tangent space to $\mathcal{M}^{S^1}_{G_2}$ is the direct sum of the tangent space to $\mathcal{M}_{SU(3)}$ and the tangent space to $Z$ at the corresponding points, and we write these as $T_{SU(3)}$ and $T_Z$.

**Proof.** Locally, $\mathcal{G}_{SU(3),[z_1],[z_2]}$ is diffeomorphic to $\tilde{\mathcal{G}}_{SU(3)}$ as it is an open subset. But $\tilde{\mathcal{G}}_{SU(3)}$ is locally diffeomorphic to $\mathcal{G}_{SU(3)} \times \mathbb{R}$, as a principal bundle, which in turn is locally diffeomorphic to $\mathcal{B}_{SU(3)} \times \mathcal{G}$, as a principal bundle, which in turn is locally diffeomorphic to $\mathcal{B}_{SU(3)} \times \mathbb{R}$ by Theorem 4.3.35. For $\mathcal{G}^{S^1}_{G_2}$, we again work locally and note that Theorem 4.3.35 says that $\tilde{\mathcal{G}}^{S^1}_{G_2}$ is the product of $\tilde{\mathcal{G}}_{SU(3)}$ with $B_Z$. The inclusion is the identity for $TA \oplus T\mathbb{R}$ because Theorem 4.3.35 preserves these components. For the inclusion $\mathcal{B}_{SU(3)} \rightarrow \mathcal{B}^{S^1}_{G_2}$, the fact the inclusion is the identity on the $\mathcal{B}_{SU(3)}$ component follows from the definition of $\mathcal{B}_{SU(3)}$ as a submanifold in Proposition 4.3.32, and the corresponding product structure.

The compact case is even easier. 

We begin by considering Proposition 4.3.10, which almost immediately implies

**Proposition 4.3.48.** Suppose that $[\phi_1, \phi_2, T] = [\text{Re } \Omega_1 + z_1 \wedge \omega_1, \text{Re } \Omega_2 + z_2 \wedge \omega_2, T] \in \mathcal{G}^{S^1}_{G_2}$. Suppose that $([z'_1], [z'_2]) \in T\mathcal{B}_{G_2}$, so that $[z'_1 \wedge \omega_1, z'_2 \wedge \omega_2, 0] \in T_Z \subset T\mathcal{G}^{S^1}_{G_2}$. Applying the derivative of the gluing map of Definition 4.3.37 takes $[z'_1 \wedge \omega_1, z'_2 \wedge \omega_2, 0]$ into the $T_Z$ component of $\mathcal{M}^{S^1}_{G_2}$

**Proof.** Suppose that we have chosen our representatives such that the triple $(\text{Re } \Omega_1 + z_1 \wedge \omega_1, \text{Re } \Omega_2 + z_2 \wedge \omega_2, T)$ glues. Choose a curve of matching pairs of twistings $s_i(s)$ with $s_i(0) = z_i$ and $s_i' = z_i'$, for some representative. Upper semi-continuity of the minimal parameter $T_0$ implies that $(\text{Re } \Omega_1 + z_1(s) \wedge \omega_1, \text{Re } \Omega_2 + z_2(s) \wedge \omega_2, T)$ will also glue for $s$ small. Note that these triples define a curve through $[\text{Re } \Omega_1 + z_1 \wedge \omega_1, \text{Re } \Omega_2 + z_2 \wedge \omega_2, T]$ in the same way as Lemma 4.2.34, with its tangent exactly $[z'_1 \wedge \omega_1, z'_2 \wedge \omega_2, 0]$. Proposition 4.3.10 says that the curve in $\mathcal{M}^{S^1}_{G_2}$ constructed by gluing $(\text{Re } \Omega_1 + z_1(s) \wedge \omega_1, \text{Re } \Omega_2 + z_2(s) \wedge \omega_2, T)$ has fixed $\mathcal{M}_{SU(3)}$ component, and so its tangent lies in $T_Z$.

Using the analogous proposition for Calabi–Yau structures (Proposition 4.3.11), we obtain

**Proposition 4.3.49.** Suppose that $[\Omega, \omega, T] \in \mathcal{G}_{SU(3),([z_1],[z_2])}$, and that $[\Omega', \omega', 0]$ is a tangent vector in $T\mathcal{G}_{SU(3)}$. Pick representatives of $\Omega$ and $\omega$, and let $z_1, z_2$ be a pair of matching twistings such that the triple $(\text{Re } \Omega_1 + z_1 \wedge \omega_1, \text{Re } \Omega_2 + z_2 \wedge \omega_2, T)$ is gluable. Then the image of the tangent $[\text{Re } \Omega'_1 + z_1 \wedge \omega'_1, 0] \in T_{SU(3)} \subset T\mathcal{G}^{S^1}_{G_2}$ under the derivative of the gluing map lies in $T_{SU(3)} \oplus \mathbb{R}[z] \subset T\mathcal{M}^{S^1}_{G_2}$, where $[z]$ is the twisting class given by applying Definition 3.5.5 to $z_1$ and $z_2$.

**Proof.** Pick some forms $\Omega'_1$ and $\omega'_1$ representing $[\Omega', \omega', 0]$. Take a curve of matching Calabi–Yau structures $(\Omega(s),\omega(s))$ such that $\Omega_i(0) = \Omega_i$, $\omega_i(0) = \omega_i$, $\Omega'_i = \Omega'_i$, $\omega'_i$. By upper semi-continuity of the minimal parameter $T_0$, $(\text{Re } \Omega_1(s) + z_1 \wedge \omega_1(s), \text{Re } \Omega_2(s) + z_2 \wedge \omega_2(s), T)$ is gluable for $s$ sufficiently small.
We know by Proposition 4.3.11 that the image of \((\text{Re } \Omega_1(s) + z_1 \wedge \omega_1(s), \text{Re } \Omega_2(s) + z_2 \wedge \omega_2(s), T)\) in \(\mathcal{M}^{S^1}_{G_2}\) is of the form \([\text{Re } \Omega(s) + c(s)z \wedge \omega(s)]\) for a curve of Calabi–Yau structures \((\Omega(s), \omega(s))\); the result clearly follows by differentiating in \(s\).

\[\square\]

In the same way, we have a similar result for the automorphism component \(TA\).

**Proposition 4.50.** As in Proposition 4.49, let \([\Omega_1, \omega_1, T] \in \mathcal{G}_{SU(3),([z_1],[z_2])}\). Choose representatives \(\Omega_1\) and \(\omega_1\) and let \(z_1, z_2\) be a pair of matching twistings such that the triple \((\phi_1 = \text{Re } \Omega_1 + z_1 \wedge \omega_1, \phi_2 = \text{Re } \Omega_2 + z_2 \wedge \omega_2, T)\) is gluable. Consider \(X \in TA(\Omega_1, \omega_1, \Omega_2, \omega_2) = TA(\phi_1, \phi_2) \subset T\mathcal{G}^{S^1}_{G_2}\). Its image under the derivative of the gluing map lies in \(T_{SU(3)} \oplus \mathbb{R}[z]\).

**Proof.** Let \(\Phi_s\) be the curve of diffeomorphisms of \(M_2\) generated by curve of Killing fields \(sX\) as in Proposition 4.3.22. By upper semi-continuity of the minimal \(T_0\), \((\Omega_1, \omega_1, \Phi_s^*\Omega_2, \Phi_s^*\omega_2, T)\) is gluable for \(s\) sufficiently small. Exactly as in Proposition 4.49, by Proposition 4.3.11, the image of \((\Omega_1, \omega_1, \Phi_s^*\Omega_2, \Phi_s^*\omega_2, T)\) in \(\mathcal{M}^{S^1}_{G_2}\) is of the form \([\text{Re } \Omega(s) + c(s)z \wedge \omega(s)]\); the result follows.

\[\square\]

Finally, we have to consider the effect of varying the neck length \(T\). Exactly the same argument as in Propositions 4.3.48–4.3.50 shows that it follows from Proposition 4.3.12 that \(T\mathbb{R}\) maps into \(T_{SU(3)} \oplus \mathbb{R}[z]\).

In sum, therefore, we have shown that \(T_Z\) maps into \(T_Z\) but that \(T\mathcal{G}_{SU(3),([z_1],[z_2])}\) maps into \(T\mathcal{M}_{SU(3)} \oplus \mathbb{R}[z]\). The proof of Theorem 4.3.46 is now straightforward linear algebra, as follows.

**Proof of Theorem 4.3.46.** We consider the derivative of the gluing map

\[\mathcal{G}_{SU(3)} \rightarrow \mathcal{M}_{SU(3)},\]  

(4.3.66)

around any given point. Because \(\mathcal{G}_{SU(3)}\) is given by a union of open sets, we may suppose that this point lies in \(\mathcal{G}_{SU(3),([z_1],[z_2])}\), and then by Definition 4.3.40 the gluing map is locally given by the composition

\[\mathcal{G}_{SU(3),([z_1],[z_2])} \rightarrow \mathcal{G}^{S^1}_{G_2} \rightarrow \mathcal{M}^{S^1}_{G_2} \rightarrow \mathcal{M}_{SU(3)}.\]  

(4.3.67)

Its derivative is therefore given in terms of the decomposition in Proposition 4.3.47 by

\[T_{SU(3)} \oplus TA \oplus T\mathbb{R} \hookrightarrow T_{SU(3)} \oplus T_Z \oplus TA \oplus T\mathbb{R} \xrightarrow{\gamma} T_{SU(3)} \oplus T_Z \rightarrow T_{SU(3)},\]  

(4.3.68)

where the middle map \(\gamma\) is the derivative of the gluing map on \(S^1\) invariant moduli spaces and is therefore an isomorphism by Theorem 4.3.45.

Now we have that the components in the tangent space of \(\mathcal{G}^{S^1}_{G_2}\) corresponding to the tangent space of \(\mathcal{G}_{SU(3)}\) are mapped into \(T_{SU(3)} \oplus \mathbb{R}[z]\) (by Propositions 4.3.49, 4.3.50, and the following comment) and the additional component \(T_Z\) is mapped into \(T_Z\) (by Proposition 4.3.48). Moreover, we know that \(\mathbb{R}[z]\) is mapped to by \(T_Z\), by taking the obvious curve of twistings.
that is, by taking \([z_1 \wedge \omega_1, z_2 \wedge \omega_2, 0] \in T_Z\). It follows that the composition is injective, because if a vector in \(T_{SU(3)}\) maps to zero under the whole map, then under the \(G_2\) gluing it must map to \((0, c[z])\). But \((0, c[z])\) is also the image of something in \(T_Z\), which proves that the \(G_2\) gluing map is not injective, a contradiction. The composition is also surjective, simply because anything in \(T_{SU(3)}\) is mapped to by some tangent vector in \(T\tilde{G}_2\), and if we ignore the \(T_Z\) component we still get the same \(T_{SU(3)}\) component under the composition. Hence the composition of derivatives is an isomorphism and the gluing map of Calabi–Yau structures is a local diffeomorphism.

Remark. We could also have used a simplified version of [66, Proposition 3.1] to show that Definition 3.5.5 defines an isomorphism between the two \(T_Z\) components, rather than analysing the remaining components separately.

Finally, we make a remark on the possibility of complex gluing parameter \(T\), which is natural when we are gluing complex manifolds.

Remark. By the Haskins–Hein–Nordström structure theory ([32]), any asymptotically cylindrical Calabi–Yau has cross-section a finite quotient of \(S^1 \times X\) for some \(X\), and so has a rotation map on its asymptotically cylindrical end; we could say that taking gluing parameter \(T + iS\) corresponds to a rotation of one asymptotically cylindrical end.

Evidently, if we permitted complex gluing parameter, the gluing map would still cover an open subset. Local injectivity depends on how variation of \(S\) interacts with the proof of Theorem 4.3.46, in particular how the automorphism of rotation on the end behaves as a class in \(A(\Omega_1, \omega_1, \Omega_2, \omega_2)\). If it defines the trivial class, then the moduli class we obtain on gluing is independent of \(S\). If it defines a nontrivial class, then we lose local injectivity, as changing \(S\) is equivalent to a change in \(T\tilde{G}_{SU(3)}\). Note, however, that considering this question is less natural in this case, as there need not be a natural (almost) complex structure on the moduli spaces \(M_{SU(3)}\). Indeed, comparing Corollary 4.2.40 with the examples in [18], we find that there exist examples where the moduli space is odd-dimensional.
Chapter 5

Special Lagrangian submanifolds

Calabi–Yau manifolds have various families of distinguished submanifolds. The most obvious family is the complex submanifolds, since Calabi–Yau manifolds are complex. There are other families that are less well understood. We are concerned here with special Lagrangian submanifolds, first introduced by Harvey and Lawson in [31].

In this chapter, we prove three theorems of a similar nature to those in the previous chapter, though by different methods. We also give a corrected, if slightly restricted, version of a known theorem (Theorem 5.2.13 on the asymptotically cylindrical deformations with moving boundary), and give some examples of asymptotically cylindrical special Lagrangian submanifolds in subsection 5.2.2. In Hypothesis 5.3.1, we assume a result on the gluing of Calabi–Yau manifolds that generalises to higher $n$ the one obtained in the previous chapter, and then prove in Theorem 5.4.10 that asymptotically cylindrical special Lagrangian submanifolds can be glued. This follows by considering the same argument as in the deformation case. The main difficulty is to find a bound on the inverse of the linearisation, and this is done by showing that the inverse of the linearisation depends continuously on the Calabi–Yau structure and so by comparing with the original structures, we can assume we are just working with $d + d^*$ as in the deformation case. Extending this slightly, by showing we can find an auxiliary $SU(n)$ structure around any given nearly special Lagrangian submanifold in the sense of Definition 4.1.7, leads to Theorem 5.4.17, that any nearly special Lagrangian submanifold can be perturbed to a special Lagrangian.

Finally, we show in Theorem 5.7.13 that the gluing map of special Lagrangians so defined is a local diffeomorphism of moduli spaces, at least if the special Lagrangians have connected cross-section. Intuitively, this is obvious, because the gluing map is just “patch and become special Lagrangian” so its derivative should just be “patch and become harmonic”, and Corollary 3.5.12 from chapter 3 says that this is an isomorphism. The proof is technical, as we need to understand exactly how all the one-forms and normal vector fields concerned interrelate: most of the theory in section 5.5 is developed for this purpose.
5.1 Generalities on submanifolds

In this section, we make some definitions and prove some results on general submanifolds $L$ of Riemannian manifolds $M$; some of this material will be needed for our review of basic special Lagrangian theory in section 5.2. In subsection 5.1.1, we define norms on objects on such a submanifold. In subsection 5.1.2, we prove an important result on restriction, or how these norms interact with the norms on objects on the ambient manifold $M$. Specifically, we prove Theorem 5.1.3, that (locally) if the second fundamental form of $L$ in $M$ is bounded in $C^{k-1}$, then the restriction maps are $C^k$ bounded. We then proceed in subsection 5.1.3 to define asymptotically cylindrical submanifolds (Definition 5.1.11) and give their basic properties. Finally, we describe an approximate gluing or patching map of asymptotically cylindrical submanifolds in Definition 5.1.14, and study how this map interacts with the restriction maps of forms.

5.1.1 Norms

We begin by defining norms on forms on, and normal vector fields to, a submanifold $L$ of the Riemannian manifold $(M,g)$. Finally, we define $C^k$ norms on the second fundamental form, which will be required for Theorem 5.1.3.

If $L$ is such a submanifold, $(L,g|_L)$ is itself Riemannian: write $\nabla^L$ for the induced Levi-Civita connection. It follows that we have natural $C^{k,\mu}$ norms on sections of bundles associated to the tangent bundle $TL$ just as in Definition 2.2.4. We will also need norms on sections of the normal bundle $\nu_L$. The metric $g$ defines a metric on this bundle: in order to define a $C^k$ norm on its sections, we need a connection on $\nu_L$.

In order to do this, we use

**Lemma 5.1.1** (Gauss and Weingarten formulae; e.g. [6, Theorem 1.72]). Let $L$ be an immersed submanifold of the Riemannian manifold $(\mathbb{R}^n, g)$, where $g$ need not necessarily be Euclidean. Let $X$ and $Y$ be vector fields on $L$, and $\xi$ a normal field to $L$. Let $\nabla$ be the Levi-Civita connection of $g$. Then, suppressing the identification of $X$ and $Y$ as sections of $T\mathbb{R}^n|_L$ by $\iota_*$, and writing $\pi^0_1$ and $\pi^1_1$ for the orthogonal projections from $T\mathbb{R}^n|_L$ to the subbundles $TL$ and $\nu_L$ respectively, we have

\[
\begin{align*}
\pi^0_1(\nabla_X Y) &= \nabla^L_X Y, & \pi^1_1(\nabla_X Y) &= \Pi(X, Y), \\
\pi^0_1(\nabla_X \xi) &= S(Y, \xi), & \pi^1_1(\nabla_X \xi) &= D_X \xi.
\end{align*}
\]

(5.1.1)

where $\nabla^L$ is the Levi-Civita connection corresponding to the restricted metric on $L$, $D$ is a connection on the bundle $\nu_L$, and $\Pi$ and $S$ are tensors $\Pi : TL \oplus TL \to \nu_L$, $S : TL \oplus \nu_L \to TL$.

In fact, $\Pi$ and $S$ provide the same information, since for all such $X, Y, \xi$

\[
g(-S(X, \xi), Y) = g(\Pi(X, Y), \xi).
\]

(5.1.2)

$D$ of (5.1.1) is then a connection on $\nu_L$. We again call this connection $\nabla^L$. Combining it
with the Levi-Civita connection $\nabla^L$ on $T^* L$ induces connections on $\otimes^l T^* L \otimes \nu_L$ by the Leibniz rule, enabling us to make

**Definition 5.1.2.** Let $(M, g)$ be a Riemannian manifold and $L$ a submanifold with tangent bundle $TL$ and normal bundle $\nu_L$. Let $\nabla^L$ be the Levi-Civita connection induced on $TL$, the connection $D$ on $\nu_L$ given by (5.1.1), and the induced connections on tensor products of these bundles. Given a section $s$ of $\nu_L$, define

$$\|s\|_{C^k} = \sum_{l=0}^k g((\nabla^L)^l s, (\nabla^L)^l s).$$

(5.1.3)

Finally, consider the section $\Pi$ of $T^* L \otimes T^* L \otimes \nu_L$ defined by (5.1.1). Again, we may define a $C^k$ norm on this using the connections $\nabla^L$.

### 5.1.2 The restriction map

Suppose that $(M, g)$ is a Riemannian manifold, and $L \subset M$ is a submanifold. Suppose $\alpha$ is a differential form on $M$ with $\|\alpha\|_{C^k}$ small. We would like to know that $\|\alpha|_{L}\|_{C^k}$ is also small. In this subsection, we prove that this holds locally given a local bound on the second fundamental form. We shall globalise it, for instance, in Proposition 5.1.12 to an asymptotically cylindrical application.

More precisely, we prove

**Theorem 5.1.3.** Let $L$ be an immersed submanifold of $(\mathbb{R}^n, g)$; note that the metric $g$ need not be Euclidean. If the second fundamental form $\Pi$ has finite $C^{k-1}(L)$ norm in the sense described immediately after Definition 5.1.2, then the restriction map

$$C^k \Omega^p(\mathbb{R}^n) \to C^k \Omega^p(L)$$

(5.1.4)

is bounded for all $p$; moreover, its bound depends only on $p$ and the $C^{k-1}(L)$ bound on $\Pi$.

A $p$-form on $\mathbb{R}^n$ is a section of $\bigwedge^p T^* \mathbb{R}^n$. We begin by considering the structure of this bundle.

**Lemma 5.1.4.** Let $\iota : L \to (\mathbb{R}^n, g)$ be an immersion. Using the musical isomorphisms $TL \cong T^* L$ and $T\mathbb{R}^n \cong T^* \mathbb{R}^n$ induced from $g|_L$ and $g$, we may define an inclusion $\iota_*$ of $T^* L$ as a subbundle of $T^* \mathbb{R}^n$. Its orthogonal complement is the annihilator of the subbundle $TL$ of $T\mathbb{R}^n$ under the natural pairing. We write this orthogonal complement $\nu^*_L$, so that $T^* \mathbb{R}^n = T^* L \oplus \nu^*_L$.

In particular, we see that for any $p$,

$$\bigwedge^p T^* \mathbb{R}^n = \bigoplus_{q=0}^p \bigwedge^{p-q} T^* L \wedge \bigwedge^q \nu^*_L,$$

(5.1.5)
and that this is an orthogonal splitting. Let the corresponding projection maps be \( \{ \pi_q^p \}_{q=0}^{p} \), \( \pi_0^p \) defines a map \( \bigwedge^n T^*\mathbb{R}^n \to \bigwedge^p T^* L \); this map is precisely the pullback map \( \iota^* \).

**Remark.** The notation for our projection maps is deliberately similar to that used in Lemma 5.1.1, but \( \pi_q^p \) are defined on the dual spaces.

**Proof.** We define the inclusion map \( \iota_* : T^* L \to T^* \mathbb{R}^n \) by \( \iota_*(\alpha) = (\iota_* \alpha^*)^\flat \). As the musical isomorphisms are isometric we have that the orthogonal complement of \( T^* L \) is

\[
\{ \beta \in T^* \mathbb{R}^n : g(\beta, (\iota_* \alpha^*)^\flat) = 0 \quad \text{for all} \quad \alpha \in T^* L \} = \{ \beta \in T^* \mathbb{R}^n : g(\beta^\sharp, (\iota_* v)) = 0 \quad \text{for all} \quad v \in TL \} = \{ \beta \in T^* \mathbb{R}^n : \beta(\iota_* v) = 0 \quad \text{for all} \quad v \in TL \};
\]

that is, the annihilator of \( TL \). The orthogonal splitting (5.1.5) follows immediately from the splitting \( T^* \mathbb{R}^n = T^* L \oplus \nu_L^* \).

It remains to show that \( \pi_0^p = \iota^* \). Since both are linear maps it suffices to check equality on a decomposable form \( \alpha_1 \wedge \cdots \wedge \alpha_p \). By definition, we then have

\[
\iota^*(\alpha_1 \wedge \cdots \wedge \alpha_p) = \iota^*(\alpha_1) \wedge \cdots \wedge \iota^*(\alpha_p), \quad \pi_0^p(\alpha_1 \wedge \cdots \wedge \alpha_p) = \pi_0^1(\alpha_1) \wedge \cdots \wedge \pi_0^1(\alpha_p) \quad \text{(5.1.7)}
\]

where \( \pi_0^1 \) is the orthogonal projection map \( \pi_0^1 : T^* \mathbb{R}^n \to T^* L \). Hence it suffices to prove that \( \pi_0^1 = \iota^* \).

That is, we have to check that for \( \alpha \in T^* \mathbb{R}^n \), \( \iota_* \iota^* \alpha = \pi_0^1 \alpha \). Clearly \( \iota_* \iota^* \alpha \in T^* L \), so we have to check that \( \alpha - \iota_* \iota^* \alpha \in \nu_L^* \). For any \( v \in TL \), we have

\[
\iota_* \iota^* \alpha(\iota_* v) = g(\iota_* (\iota^* \alpha)^\sharp, \iota_* v) = g(\iota^* \alpha^\sharp, v) = \iota^* \alpha(v) = \alpha(\iota_* v), \quad \text{(5.1.8)}
\]

using that the musical isomorphism is an isometry and the metric on \( L \) is the pullback of \( g \) on \( \mathbb{R}^n \). Hence \( \alpha - \iota_* \iota^* \alpha \) is in the annihilator of \( TL \), which is \( \nu_L^* \). This proves \( \pi_0^1 = \iota^* \). \( \square \)

As orthogonal projections decrease norm, Lemma 5.1.4 implies the case \( k = 0 \) of Theorem 5.1.3. For the case of \( k > 0 \), we extend Lemma 5.1.1. We first make a notational change. For a tangent field \( Z \) on \( L \), we shall write \( \Pi_Z Y \) for \( \Pi(Z, Y) \) and \( S_Z \xi \) for \( S(Z, \xi) \). This is because \( \Pi \) and \( S \) arise as differentials, and it is easiest to regard \( \Pi_Z \) and \( S_Z \) as a family of operators depending on \( Z \). We will then suppress \( Z \): all the estimates below, of course, depend on appropriate norms of \( Z \), but as to compute we may choose some local trivialisation of \( TL \), this is not important.

We begin by noting that \( \Pi \) and \( S \) are essentially the same. In particular, bounds in terms of one can be easily recast in terms of the other, since we have

**Lemma 5.1.5.** The map \( \Pi \leftrightarrow S \) implied by (5.1.2) is a \( C^k(L) \) isometry for every \( k \), in the sense of the norm described on \( \Pi \) after Definition 5.1.2, and the corresponding definition for \( S \).
**Proof.** This is essentially because this map is a tensor product of musical isomorphisms. By induction, for any vector fields $Z_1, \ldots, Z_k$ on $L$, it is easy to see using the Leibniz rule that

$$g(- (\nabla^L_{Z_1} \cdots \nabla^L_{Z_k} S)(X, \xi), Y) = g( (\nabla^L_{Z_1} \cdots \nabla^L_{Z_k} \Pi)(X, Y), \xi), \quad (5.1.9)$$

so it suffices to prove that if two tensors are related by (5.1.2), then they have equal size in $C^0$. This follows if $e_1, \ldots, e_m$ is an orthonormal basis of $T_p L$ and $e_{m+1}, \ldots, e_n$ is an orthonormal basis of $(\nu_L)_p$, then

$$g(S, S) = \sum_{i,j=1}^m \sum_{k=m+1}^n |g(S(e_k, e_i), e_j)|^2$$

$$= \sum_{i,j=1}^m \sum_{k=m+1}^n |g(\Pi(e_i, e_j), e_k)|^2 = g(\Pi, \Pi). \quad \square$$

We now extend $S_Z, \Pi_Z$ and the connection $\nabla^L_Z$ to $p$-forms, as follows.

**Definition 5.1.6.** Let $\alpha = \lambda + \nu$ be a section of $T^*\mathbb{R}^n|_L$, where $\lambda = \pi^L_0 \alpha$ and $\nu = \pi^L_1 \alpha$. Define

$$\nabla^L \alpha = (\nabla^L \lambda)^b, \quad \Pi \alpha = (\Pi(\alpha^b))^b, \quad S \alpha = (S(\alpha^b))^b. \quad (5.1.11)$$

Now let

$$\alpha = \lambda_1 \wedge \cdots \wedge \lambda_q \wedge \nu_{q+1} \wedge \cdots \wedge \nu_p$$

be a section of the subbundle $\bigwedge^q T^* M \wedge \bigwedge^{p-q} \nu_M$ of $\bigwedge^p T^* \mathbb{R}^n|_M$, and define

$$\nabla^L \alpha = (\nabla^L \lambda_1) \wedge \cdots \wedge \lambda_q \wedge \nu_{q+1} \wedge \cdots \wedge \nu_p + \lambda_1 \wedge \cdots \wedge (\nabla^L \lambda_q) \wedge \nu_{q+1} \wedge \cdots \wedge \nu_p$$

$$+ \lambda_1 \wedge \cdots \wedge \lambda_q \wedge \nabla^L(\nu_{q+1}) \wedge \cdots \wedge \nu_p + \lambda_1 \wedge \cdots \wedge \lambda_q \wedge \nu_{q+1} \wedge \cdots \wedge \nabla^L(\nu_p), \quad (5.1.13)$$

$$\Pi \alpha = \Pi(\lambda_1) \wedge \cdots \wedge \lambda_q \wedge \nu_{q+1} \wedge \cdots \wedge \nu_p + \lambda_1 \wedge \cdots \wedge \Pi(\lambda_q) \wedge \nu_{q+1} \wedge \cdots \wedge \nu_p, \quad (5.1.14)$$

$$S \alpha = \lambda_1 \wedge \cdots \wedge \lambda_q \wedge S(\nu_{q+1}) \wedge \cdots \wedge \nu_p + \lambda_1 \wedge \cdots \wedge \lambda_q \wedge \nu_{q+1} \wedge \cdots \wedge S(\nu_p). \quad (5.1.15)$$

By linearity, we can extend these maps to all sections of the subbundles $\bigwedge^q T^* M \wedge \bigwedge^{p-q} \nu_M$. In turn, for a general section of $\bigwedge^p T^* \mathbb{R}^n|_M$, define

$$\nabla^L \alpha = \sum_{q=0}^p \nabla^L \pi_q \alpha, \quad \Pi \alpha = \sum_{q=0}^p \pi_q \alpha, \quad S \alpha = \sum_{q=0}^p S \pi_q \alpha. \quad (5.1.16)$$

**Proposition 5.1.7.** For all $p$ and $q$ and every $Z$, $\nabla^L_Z$ is a well-defined connection on $\bigwedge^q T^* M \wedge \bigwedge^p \nu_M$.\]
\[ \wedge^{p-q}\nu_M^\ast \text{ and } \Pi_Z \text{ and } S_Z \text{ are well-defined tensorial maps} \]

\[
\Pi_Z : \bigwedge^q T^* M \wedge \bigwedge^{p-q}\nu_M^\ast \rightarrow \bigwedge^{q-1} T^* M \wedge \bigwedge^{p-q+1}\nu_M^\ast. \tag{5.1.17}
\]

\[
S_Z : \bigwedge^q T^* M \wedge \bigwedge^{p-q}\nu_M^\ast \rightarrow \bigwedge^{q+1} T^* M \wedge \bigwedge^{p-q-1}\nu_M^\ast. \tag{5.1.18}
\]

**Proof.** Since the musical isomorphisms are certainly tensorial, it is clear that \( \nabla_Z^p \), \( \Pi_Z \) and \( S_Z \) are well-defined connections and tensors when \( p = 1 \). It is clear since the musical isomorphism maps \( TM \) to \( T^* M \) and \( \nu_M \) to \( \nu_M^\ast \) that they map between the required spaces.

We now pass to the case \( p > 1 \). To check that \( \nabla_M, \Pi, \text{ and } S \) are well-defined, we have to check that if a section \( \alpha_1 \) of \( \bigwedge^q T^* L \wedge \bigwedge^{p-q}\nu_L^\ast \) of the form (5.1.12) can be written as the sum of two sections \( \alpha_2 \) and \( \alpha_3 \) also of the form (5.1.12), then \( \nabla^L, \Pi \) and \( S \) are linear on this triple.

The condition that \( \alpha_1 \) is of the form (5.1.12) guarantees \( \alpha_2 \) and \( \alpha_3 \) are very similar. At every \( p \in L \), we can choose a basis \( \lambda_1^{(2)}, \ldots, \lambda_p^{(2)}, \lambda_1^{(3)}, \ldots, \lambda_k^{(3)}, \lambda_1^{(1)}, \ldots, \lambda_\lambda^{(1)} \) of \( T_p^\ast L \) and a basis \( \nu_1^{(2)}, \ldots, \nu_{p-q}^{(2)}, \nu_1^{(3)}, \ldots, \nu_{k'}^{(3)}, \nu_1^{(1)}, \ldots, \nu_{\lambda'}^{(1)} \) of \( (\nu_L^\ast)_p \) such that

\[
\alpha_2 = \lambda_1^{(2)} \wedge \cdots \wedge \lambda_q^{(2)} \wedge \nu_1^{(2)} \wedge \cdots \wedge \nu_{p-q}^{(2)}, \tag{5.1.19}
\]

\[
\alpha_3 = \lambda_1^{(2)} \wedge \cdots \wedge \lambda_{q-k}^{(2)} \wedge \lambda_1^{(3)} \wedge \cdots \wedge \lambda_{k'}^{(3)} \wedge \nu_1^{(2)} \wedge \cdots \wedge \nu_{p-q-k'} \wedge \nu_1^{(3)} \wedge \cdots \wedge \nu_{k'}^{(3)}. \tag{5.1.20}
\]

For notational simplicity, we write

\[
\zeta_2 = \lambda_1^{(2)} \wedge \cdots \wedge \lambda_q^{(2)}, \quad \eta_2 = \nu_1^{(2)} \wedge \cdots \wedge \nu_{p-q}^{(2)}, \tag{5.1.21}
\]

\[
\zeta_3 = \lambda_1^{(2)} \wedge \cdots \wedge \lambda_{q-k}^{(2)} \wedge \lambda_1^{(3)} \wedge \cdots \wedge \lambda_{k'}^{(3)}, \quad \eta_3 = \nu_1^{(2)} \wedge \cdots \wedge \nu_{p-q-k'} \wedge \nu_1^{(3)} \wedge \cdots \wedge \nu_{k'}^{(3)}. \tag{5.1.22}
\]

so that \( \alpha_2 = \zeta_2 \wedge \eta_2 \) and \( \alpha_3 = \zeta_3 \wedge \eta_3 \).

Now \( \alpha_1 = \zeta_2 \wedge \eta_2 + \zeta_3 \wedge \eta_3 \) is by hypothesis decomposable. Therefore, the vector space \( V \) of 1-forms \( \beta \) such that \( \beta \wedge \alpha_1 = 0 \) is at least \( p \)-dimensional. If \( \beta \wedge \alpha_1 = 0 \), then \( \pi_{q}^{p+1}(\beta \wedge \alpha_1) = (\pi_{q}^{1}\beta) \wedge \alpha_1 \) and \( \pi_{q+1}^{p+1}(\beta \wedge \alpha_1) = (\pi_{q}^{1}\beta) \wedge \alpha_1 = 0 \). Hence, we can split \( V \) into \( V \cap T_p^\ast L \) and \( V \cap (\nu_L^\ast)_p \). So consider \( \beta \in V \cap T_p^\ast L \). We have

\[
0 = \beta \wedge \zeta_2 \wedge \eta_2 + \beta \wedge \zeta_3 \wedge \eta_3. \tag{5.1.23}
\]

Either \( k' \neq 0 \), so that \( \eta_2 \) and \( \eta_3 \) are linearly independent, and their wedges with forms in \( \bigwedge^{q+1} T_p^\ast L \) are, so that \( \beta \wedge \zeta_2 = 0 = \beta \wedge \zeta_3 \), or \( k' = 0 \) so that \( \eta_2 = \eta_3 \) and \( \beta \wedge \zeta_2 = -\beta \wedge \zeta_3 \).

Clearly \( \text{span}\{\lambda_1^{(2)}, \ldots, \lambda_{q-k}^{(2)}\} \) is contained in \( V \cap T_p^\ast L \). For the remainder, suppose that

\[
\beta = \sum_{i=q-k+1}^{q} a_i\lambda_i^{(2)} + \sum_{i=1}^{k} b_i\lambda_i^{(3)} + \sum_{i=1}^{\lambda} c_i\lambda_i^{(1)}. \tag{5.1.24}
\]

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Then we have
\[
\beta \wedge \zeta_2 = \sum_{i=1}^{k} b_i \lambda_i^{(3)} \wedge \lambda_1^{(2)} \wedge \cdots \wedge \lambda_q^{(2)} + \sum_{i=1}^{N} c_i \lambda_i^{(1)} \wedge \lambda_1^{(2)} \wedge \cdots \wedge \lambda_q^{(2)}, \tag{5.1.25}
\]
and
\[
\beta \wedge \zeta_3 = \sum_{i=q-k+1}^{q} a_i \lambda_i^{(2)} \wedge \lambda_1^{(2)} \wedge \cdots \wedge \lambda_{q-k}^{(2)} \wedge \lambda_1^{(3)} \wedge \cdots \wedge \lambda_k^{(3)} + \sum_{i=1}^{N} c_i \lambda_i^{(1)} \wedge \lambda_1^{(2)} \wedge \cdots \wedge \lambda_{q-k}^{(2)} \wedge \lambda_1^{(3)} \wedge \cdots \wedge \lambda_k^{(3)}. \tag{5.1.26}
\]

It is clear that all of the terms in each of (5.1.25) and (5.1.26) are linearly independent. Hence \( \beta \wedge \zeta_2 = 0 = \beta \wedge \zeta_3 \) if and only if all the \( a_i, b_i \) and \( c_i \) are zero. It follows that if \( k' \neq 0 \), then \( V \cap T^*_pL = \text{span} \{ \lambda_1^{(2)}, \ldots, \lambda_{q-k}^{(2)} \} \), and so has dimension \( q - k \).

If \( k > 1 \), we also have that all the terms in (5.1.25) and (5.1.26) are linearly independent, considered together, and so \( \beta \wedge \zeta_2 = -\beta \wedge \zeta_3 \) if and only if all the \( a_i, b_i \) and \( c_i \) are zero. Hence again if \( k > 1 \), \( V \cap T^*_pL \) has dimension \( q - k \).

That is, we have shown that \( V \cap T^*_pL \) has dimension \( q - k \) unless \( k' = 0 \) and \( k \leq 1 \). A similar argument shows that \( V \cap (\nu^*_L)_p \) has dimension \( p - q - k' \) unless \( k = 0 \) and \( k' \leq 1 \). Hence, \( V \) has dimension less than \( p \) unless \( k + k' \leq 1 \). After relabelling our bases, this gives two possible cases for this triple:

\[
\alpha_1 = (\lambda_1 + \lambda'_1) \wedge \cdots \wedge \lambda_q \wedge \nu_{q+1} \wedge \cdots \wedge \nu_p, \tag{5.1.27}
\]
or
\[
\alpha_1 = \lambda_1 \wedge \cdots \wedge \lambda_q \wedge (\nu_{q+1} + \nu'_{q+1}) \wedge \cdots \wedge \nu_p. \tag{5.1.28}
\]

By choosing trivialisations, we can assume we have this decomposition locally. It then follows immediately that (5.1.13), (5.1.14) and (5.1.15) are linear on the triple as required.

We also have to check that the answer doesn’t depend on the stated decomposition, i.e. that if locally

\[
\alpha = \lambda_1 \wedge \cdots \wedge \lambda_q \wedge \nu_{q+1} \wedge \cdots \wedge \nu_p \tag{5.1.29}
\]
\[
= \lambda'_1 \wedge \cdots \wedge \lambda'_q \wedge \nu'_{q+1} \wedge \cdots \wedge \nu'_p, \tag{5.1.30}
\]
then applying (5.1.13), (5.1.14) and (5.1.15) to these two decompositions gives the same result. We begin by noting that since \( \nabla^L \) is a connection and \( \Pi \) and \( S \) are tensors on \( T^*L \) and \( \nu^*_L \), we may assume that \( \lambda_2, \ldots, \lambda_q, \nu_{q+1}, \nu_p \) and \( \lambda'_2, \ldots, \lambda'_q, \nu'_{q+1}, \ldots, \nu'_p \) are unit, since we may factorise out functions. Having factorised out functions, we have that (5.1.29) and (5.1.30) must differ by terms that can be written to involve the wedges of 1-forms with themselves. It is clear that such
terms vanish under the three maps (5.1.13), (5.1.14) and (5.1.15), and so using linearity we see that (5.1.29) and (5.1.30) have the same image.

The final sum over possible values of \( q \) is direct, so that the maps of (5.1.16) are certainly well-defined and linear of the required kind. \( \square \)

We now use the operators of (5.1.16) to generalise Lemma 5.1.1 to differential forms.

**Proposition 5.1.8.** Let \( \alpha \) be a section of \( \bigwedge^p T^*R^n|_L \). Then

\[
\nabla \alpha = \nabla^L \alpha + S \alpha + \Pi \alpha. \tag{5.1.31}
\]

In particular, we have for every \( q \)

\[
\pi^p_q \nabla^p_q \alpha = \nabla^L \pi^p_q \alpha = \pi^p_q \nabla^L \alpha, \tag{5.1.32}
\]

\[
\pi^p_{q+1} \nabla^p_q \alpha = \Pi \pi^p_q \alpha = \pi^p_{q+1} \Pi \alpha, \quad \pi^p_{q-1} \nabla^p_q \alpha = S \pi^p_q \alpha = \pi^p_{q-1} S \alpha. \tag{5.1.33}
\]

**Proof.** By linearity, we may assume that there exists \( q \) such that \( \pi^p_q \alpha = \alpha \), and that \( \alpha \) is decomposable. That is, locally,

\[
\alpha = \lambda_1 \wedge \cdots \wedge \lambda_{p-q} \wedge \nu_{p-q+1} \wedge \cdots \wedge \nu_p, \tag{5.1.34}
\]

where \( \lambda_i \) are local sections of \( T^*L \) and \( \nu_i \) are local sections of \( \nu^*_L \).

Then, by the Leibniz rule, we see that

\[
\nabla_X \alpha = \sum_{j=1}^{p-1} \lambda_1 \wedge \cdots \wedge \lambda_{j-1} \wedge \nabla_X \lambda_j \wedge \lambda_{j+1} \cdots \wedge \lambda_{p-q} \wedge \nu_{p-q+1} \wedge \cdots \wedge \nu_p
\]

\[
+ \sum_{j=p-q+1}^p \lambda_1 \wedge \cdots \wedge \lambda_{p-q} \wedge \nu_{p-q+1} \wedge \cdots \wedge \nu_{j-1} \wedge \nabla_X \nu_j \wedge \nu_{j+1} \wedge \cdots \wedge \nu_p. \tag{5.1.35}
\]

Now since the musical isomorphism on \( \mathbb{R}^n \) maps \( T^*L \) to \( TL \) and \( \nu^*_L \) to \( \nu_L \), by the definition of \( T^*L \) and \( \nu^*_L \) as subbundles, it follows from the definition (5.1.11) that \( \nabla_X \lambda_i = \nabla^L_X \lambda_i + \Pi \nabla^L \lambda_i \) and \( \nabla_X \nu_i = \nabla^L_X \nu_i + S \nabla^L \nu_i \). Following through Definition 5.1.6, we obtain the result. \( \square \)

We need to know that the extended map \( S \) of (5.1.16) is bounded in terms of our second fundamental form bound. We shall show the same for \( \Pi \) as this is slightly simpler.

**Lemma 5.1.9.** Suppose that the second fundamental form \( \Pi \) on \( L \), as in Lemma 5.1.1, has a bound \( \|\Pi\|_{C^k} \leq A \). Then the extensions \( \Pi \) and \( S \) of (5.1.16) have bounds \( \|\Pi\|_{C^k} \leq CA \) and \( \|S\|_{C^k} \leq CA \) where \( C \) is independent of the second fundamental form \( \Pi \).

**Proof.** We begin by noting that the dual operator \( \Pi : T^*L \to \nu^*_L \) on \( T^*_L \) must have the same \( C^k \) bound as \( \Pi : TL \to \nu_L \). The dual operator is obtained from the original operator by composition
with musical isomorphisms on \((\mathbb{R}^n, g)\), so this is immediate if the musical isomorphisms of \((\mathbb{R}^n, g)\) define \(C^k\) isometries between \(T^*L\) and \(TL\) and \(\nu^*_L\) and \(\nu_L\). It is clear by the definition of \(T^*L\) and \(\nu^*_L\) that the musical isomorphisms restrict to isomorphisms between these bundles, and it remains to show that the musical isomorphisms commute with the connections \(\nabla^L\) on these bundles. This is immediate as \(\nabla^L\) is also defined using musical isomorphisms.

The extension of \(\Pi\) in (5.1.16) is essentially given by \(\Pi \wedge \text{id} \wedge \cdots \wedge \text{id} + \text{id} \wedge \Pi \wedge \text{id} \wedge \cdots \wedge \text{id} + \cdots + \text{id} \wedge \cdots \wedge \Pi\), and some natural isomorphism (that’s evidently an isometry) between products of endomorphisms and endomorphisms of products. It is clear that this satisfies the appropriate bound.

The same applies for \(S\), and we apply Lemma 5.1.5 to note that on \(L\) we also have \(\|S\|_{C^k} \leq A\) for \(S\) as in Lemma 5.1.1.

We can now give

**Proof of Theorem 5.1.3.** For \(k = 0\), as \(\pi^p_0\) is an orthogonal projection, and \(\iota_*\) is an isometry, the result is immediate. For larger \(k\), we need to find a good expression for successive derivatives on \(L\) of \(\pi^p_0\alpha\); that is, we are interested in \(\nabla^L_{X_1} \cdots \nabla^L_{X_k} \pi^p_0\alpha\).

By Proposition 5.1.7, we have that

\[
\pi^p_0(\nabla_X - S_X)\alpha = \pi^p_0(\nabla^L_X + \Pi_X)\alpha = \pi^p_0\nabla^L_X\alpha = \nabla^L_X\pi^p_0\alpha.
\]  

(5.1.36)

By induction, it follows that

\[
\nabla^L_{X_1} \cdots \nabla^L_{X_k} \pi^p_0\alpha = \pi^p_0(\nabla_{X_1} - S_{X_1}) \cdots (\nabla_{X_k} - S_{X_k})\alpha.
\]  

(5.1.37)

We now only need to bound \((\nabla_{X_1} - S_{X_1}) \cdots (\nabla_{X_k} - S_{X_k})\alpha\) in \(C^k\), perhaps depending on \(C^{k-1}\) bounds on the terms \(S\); this is immediate using the Leibniz rule and Lemma 5.1.9.

We will eventually apply Theorem 5.1.3 both to asymptotically cylindrical manifolds and to the glued manifolds \(M^T\) of Definition 3.5.2 with the glued metric of Definition 3.5.13.

**Remark 5.1.10.** We could also define norms on forms on \(L\) by using the ambient connection in the isomorphism of Lemma 2.2.2. The same arguments prove that the \(C^k\) norm defined in this way is Lipschitz equivalent to the standard \(C^k\) norm, and the Lipschitz constant depends only on the \(C^{k-1}\) norm of the second fundamental form of \(L\) in \(M\).

**5.1.3 Asymptotically cylindrical submanifolds and their patching**

In this subsection, we describe what it will mean for a submanifold to be asymptotically cylindrical (Definition 5.1.11), following Joyce–Salur [46, Definition 2.9] and Salur–Todd [70, Definition 2.6]. We show that an asymptotically cylindrical submanifold of an asymptotically cylindrical manifold is itself an asymptotically cylindrical manifold (globalising a variant of
Theorem 5.1.3) when equipped with the restricted metric. We then extend the patching or approximate gluing of subsection 3.5.1 and give such a patching of matching submanifolds, and show how this relates to the other patchings in subsection 3.5.1. For instance, we prove in Lemma 5.1.15 that in cohomology, the patching map of forms commutes with the restriction map to a submanifold. Finally, we give another globalisation of Theorem 5.1.3: we show that the second fundamental form of a patched submanifold can be bounded independently of $T$ and hence so can the restriction map in Proposition 5.1.17.

We make the following definition of an asymptotically cylindrical submanifold.

**Definition 5.1.11.** Let $(M, g)$ be an asymptotically cylindrical Riemannian manifold, with cross-section $N$. Let $L$ be a submanifold of $M$. $L$ is called asymptotically cylindrical (with cross-section $K$) if there exists $R \in \mathbb{R}$ and an exponentially decaying normal vector field $v$ on $K \times (R, \infty)$ (in the sense that its inclusion and derivatives are exponentially decaying in the metric on $M$) such that $L$ is the union of a compact part with boundary $\exp_v(K \times \{R\})$ and $\exp_v(K \times (R, \infty))$. Note that $\exp_v(K \times \{R\})$ need not be contained in $\{t = R\}$, as the metric on $M$ is not necessarily cylindrical.

If $v \equiv 0$ far enough along $K \times (R, \infty)$, we shall call $L$ cylindrical.

For notational clarity, we shall write the coordinate on $(R, \infty)$ as $r$. Evidently, if $\iota : L \rightarrow M$ is the inclusion of a cylindrical submanifold, then $r = t \circ \iota$ at least far enough along the end of $L$, but in the asymptotically cylindrical case this is not quite the case and so we use a different letter. We note also that $K$ need not be connected.

It is interesting to compare Definition 5.1.11 with Definition 4.2.2 of an asymptotically cylindrical diffeomorphism. Given a pair of manifolds $L$ and $M$ with ends, we can define a cylindrical inclusion of the end of $L$ into the end of $M$ as an inclusion $(k, r) \mapsto (\tilde{\iota}(k), r + R)$ for some fixed $R$ and inclusion $\tilde{\iota}$ of $K$ into $N$. Then Definition 5.1.11 is just saying that the inclusion of the ends is close to a cylindrical inclusion, in the same way as Definition 4.2.2 says that $\Phi$ decays to a cylindrical diffeomorphism.

We note the following proposition, which is intuitively obvious.

**Proposition 5.1.12.** Suppose that $(M, g)$ is an asymptotically cylindrical manifold with cross-section $N$, and that $L$ is an asymptotically cylindrical submanifold in the sense of Definition 5.1.11. Then $L$, with its restricted metric, is itself an asymptotically cylindrical manifold.

**Remark.** This proposition is used without proof, as an obvious result, by Joyce–Salur [46] and Salur–Todd [70].

**Proof.** We can work on an end $r > R$ of $L$ for $R$ sufficiently large. Suppose that $K \times (R, \infty)$ is a cylindrical submanifold, and that $L = \exp_v(K \times (R, \infty))$. Extend $v$ to a vector field on a tubular neighbourhood of $K \times (R, \infty)$, for instance by extending parallel in the normal directions so that $v$ is still exponentially decaying as $r = t \circ \iota \to 0$. Then $\exp_v$ defines a diffeomorphism between
tubular neighbourhoods, and by definition \((\exp_v^* g)|_{K \times (R, \infty)} = g|_{\exp_v(K \times (R, \infty))}\). Thus, provided that \(\exp_v^* g\) is an asymptotically cylindrical metric, it suffices to assume that \(L\) is cylindrical. Since \(v\) itself decays exponentially, \(\exp_v\) decays exponentially to the identity, which in turn implies \(\exp_v^* g\) is an asymptotically cylindrical metric. So we can indeed assume that \(L\) is cylindrical.

Now, it is clear that the pullback of a cylindrical metric to a cylindrical submanifold is cylindrical. Let \(\tilde{g}\) be the cylindrical metric on \(M\). Then \(g - \tilde{g}\) is exponentially decaying, so we just have to show that \((g - \tilde{g})|_L\) is exponentially decaying. Provided the second fundamental form of \(L\) in \(M\) is bounded, it follows from a similar argument to Theorem 5.1.3 that the restriction is again exponentially decaying, and this proves the result.

But II is obviously a continuous map in the inclusion and the metric, and both of these are continuous functions tending to a limit as \(t \to \infty\). Hence II must itself be continuous and tend to a limit as \(t \to \infty\); since this limit (the second fundamental form of the cylindrical inclusion) is finite, it follows that II is bounded as required.

Of course, a similar argument shows that the induced metric on the bundle \(TM|_L\) is asymptotically translation invariant: it is easy to see, then, that we may equivalently ask that \(v\) in Definition 5.1.11 decays exponentially with respect to the induced metric on \(TM|_{K \times (R, \infty)}\).

We may then extend Definition 5.1.2 to define weighted and extended weighted spaces of normal vector fields on asymptotically cylindrical submanifolds just as in section 2.2.

We now extend the notion of approximate gluing to submanifolds. We have to begin by defining the pairs \((L_1, L_2)\) of submanifolds that we expect to glue.

**Definition 5.1.13.** Let \(M_1\) and \(M_2\) be matching asymptotically cylindrical manifolds in the sense of Definition 3.5.1, so that we have orientation-reversing \(F : N_1 \to N_2\). Let \(L_1\) and \(L_2\) be asymptotically cylindrical submanifolds of \(M_1\) and \(M_2\); by definition, they have limits \(K_1 \times (R_1, \infty)\) and \(K_2 \times (R_2, \infty)\). Say that they match if \(F(K_1) = K_2\).

A matching pair of submanifolds is approximately glued as follows.

**Definition 5.1.14.** Let \(M_1\) and \(M_2\) be a matching pair of asymptotically cylindrical manifolds and let \(L_1\) and \(L_2\) be a matching pair of asymptotically cylindrical submanifolds in the sense of Definition 5.1.13.

Then by definition

\[ L_i = L_i^{\text{cpt}} \cup (\exp_v((R_i, \infty) \times K_i)). \]  

(5.1.38)

Consider the cutoff function \(\phi_T\). Note that \(\phi_T \circ \iota\) is a well-defined function on \(K \times (R, \infty)\) with \(\phi_T \circ \iota(k, r) \equiv 1\) for \(r < T - 1\) as \(r = t \circ \iota\) for the cylindrical inclusion \(\iota\). Hence for \(T > R_i + 1\)

\[ \tilde{L}_i = L_i^{\text{cpt}} \cup (\exp_{(\phi_T \circ \iota)}((R_i, \infty) \times K_i)) \]  

(5.1.39)

is a submanifold.
We want to be able to identify \( \hat{L}_1 \) and \( \hat{L}_2 \) over the identified regions \((T, T + 1) \times N_i \) of \( M_i \); that is, we want to show that \( \hat{L}_i \cap (N_i \times (T, T + 1)) \) are identified by \( F \). To do this, we need to understand the function \( t \circ \tilde{i} \) on \( \hat{L}_i \). For \( r > T \), we have \( \varphi_T \tilde{i}(k, r) = 0 \), so that \( \tilde{i}(k, r) = \tilde{i}(k, r) \), and so \( t \circ \tilde{i} = r \). In particular, \( t \circ \tilde{i}(k, T) = T \) for all \( k \in K_i \).

Furthermore, note that by choosing \( T \) sufficiently large \( \hat{i}_* \frac{\partial}{\partial r} \) and \( \hat{i}_* \frac{\partial}{\partial \tau} \) can be chosen as close as we like. Since \( \hat{i}_* \frac{\partial}{\partial r} = \frac{\partial}{\partial r} \), it follows that \( dt(\hat{i}_* \frac{\partial}{\partial r}) > 0 \) throughout \( \hat{L}_i \). In particular, \( t \circ \tilde{i}(k, r) \) is always a strictly increasing function of \( r \) for fixed \( k \). It then follows that for \( r < T, t \circ \tilde{i}(k, r) < T \), and for \( r > T + 1 \), \( t \circ \tilde{i}(k, r) > T + 1 \), so that \( \hat{L}_i \cap (N \times (T, T + 1)) = K_i \times (T, T + 1) \). Since \( L_i \) match, these are identified by \( F \) and we can form the gluing of \( \hat{L}_1 \) and \( \hat{L}_2 \), a submanifold \( L^T \) of \( M^T \).

We may also consider \( L_1, L_2 \) and \( L^T \) constructed in Definition 5.1.14 as ambient manifolds. It is then clear that \( L_1 \) and \( L_2 \) match in the sense of Definition 3.5.1, just using the restriction of \( F \), and that \( L^T \) is the gluing of \( L_1 \) and \( L_2 \) as ambient manifolds in the sense of Definition 3.5.2, though not necessarily with parameter \( T \). Hence, given matching pairs of metrics and closed forms on \( L_1 \) and \( L_2 \), Definitions 3.5.5 and 3.5.13 give metrics and forms on \( L^T \). Now suppose that a matching pair of closed forms on \( L_1 \) and \( L_2 \) is induced from a matching pair of closed forms on \( M_1 \) and \( M_2 \), i.e. the pair of forms is \( \alpha_1 |_{L_1} \) and \( \alpha_2 |_{L_2} \). Then both \( \alpha = \gamma_T(\alpha_1, \alpha_2) \big|_{L^T} \) and \( \alpha' = \gamma_T(\alpha_1 |_{L_1}, \alpha_2 |_{L_2}) \) define forms on \( L^T \). These forms need not be equal in general. Note that by Remark 3.5.6, equality would require an appropriate choice of cutoff functions for the two parts.

We give an example to show that equality need not occur even for carefully chosen cutoff functions before turning to the positive results given by Lemma 5.1.15. \( \gamma_T(\alpha_1 |_{L_1}, \alpha_2 |_{L_2}) \) is given by identifying the forms \( \alpha_1 |_{L_i} \) on \( L_i \), and \( \gamma_T(\alpha_1, \alpha_2) \big|_{L^T} \) is given by identifying the forms \( \hat{\alpha}_i \big|_{L_i} \) on the diffeomorphic manifolds \( \hat{L}_i \), where \( \hat{\alpha}_i \) represents a cutoff object. Hence, it suffices to prove that \( \hat{\alpha}_i |_{L_i} \) is not equal to \( \alpha_i |_{L_i} \).

We may thus work with only one asymptotically cylindrical manifold. For simplicity we shall take \( M \) two-dimensional, and \( L \) one-dimensional. \( L \) has more than one end, but we shall take a form \( \alpha \) supported on one end only.

Let \( M \) have cross-section \( S^1 \), so that it has end \((0, \infty) \times S^1 \), and let the end we consider of \( L \) be

\[
\{(r, e^{-r}) : r > 0\}. \tag{5.1.40}
\]

Then the corresponding end of \( \hat{L} \) is

\[
\hat{L} = \{(r, \varphi_T(r)e^{-r}) : r > 0\}. \tag{5.1.41}
\]

Take \( \alpha \) to be \( d(\psi \cos \theta) \) where the cutoff function \( \psi \) is as in (2.1.1): \( \psi \equiv 1 \) for \( t > 2 \), \( \psi \equiv 0 \) for \( t < 1 \). Hence, for \( T \) large enough, \( \hat{\alpha} = \alpha \). Hence, we see that

\[
\hat{\alpha} |_L = d(\psi(r) \cos(\varphi_T(r)e^{-r})). \tag{5.1.42}
\]

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On the other hand, we have
\[
\alpha|_L = d(\psi(r) \cos(e^{-r})),
\] (5.1.43)
and so
\[
\hat{\alpha}|_L = d(\psi(r)\varphi^L_T(r) \cos(e^{-r})),
\] (5.1.44)
where \(\varphi^L\) is some other cutoff function \(\varphi^L\) with the same properties as \(\varphi\): again, Remark 3.5.6 means that, if true, equality is only true for the right choice of cutoff functions.

Then we obtain on the end the equality
\[
d(\psi(r) \cos(\varphi_T(r)e^{-r})) = d(\psi(r)\varphi^L_T(r) \cos(e^{-r})),
\] (5.1.45)
and so that there exists a constant \(C\) such that
\[
\psi(r)(\cos(\varphi_T(r)e^{-r}) - \psi(r)\varphi'_T(r) \cos(e^{-r})) = C
\] (5.1.46)
throughout the end. Considering \(r\) very close to zero, we see that \(C = 0\). On the other hand, for \(r\) very large, \(\psi(r) = 1\) but \(0 = \varphi_T(r) = \varphi^L_T(r)\), and so (5.1.46) yields \(\cos(0) - 0 = 0\): which is false.

We turn now to the weaker result that can be achieved. We shall assume that the ends of \(M_i\) and \(L_i\) have the same parametrisation, so that \(\gamma_T\) is the gluing map on \(L_1\) and \(L_2\) corresponding to their gluing as submanifolds with gluing parameter \(T\) and similarly for the cutoff functions; by reparametrising \(M_i\) this may assumed without loss of generality.

**Lemma 5.1.15.** Let \(M_1\) and \(M_2\) be a matching pair of asymptotically cylindrical manifolds, and let \(L_1\) and \(L_2\) be matching asymptotically cylindrical submanifolds. Let \(L^T\) be the glued submanifold of Definition 5.1.14, and \(\alpha_1\), and \(\alpha_2\) be a matching pair of asymptotically translation invariant closed forms on \(M_1\) and \(M_2\). Then as cohomology classes
\[
[\gamma_T(\alpha_1|_{L_1}, \alpha_2|_{L_2})] = [\gamma_T(\alpha_1, \alpha_2)]|_{L^T},
\] (5.1.47)
and there exist constants \(C_k\) and \(\epsilon\) such that for all \(k\):
\[
\|\gamma_T(\alpha_1|_{L_1}, \alpha_2|_{L_2}) - \gamma_T(\alpha_1, \alpha_2))|_{L^T}\| \leq C_k e^{-\epsilon T}.
\] (5.1.48)

**Proof.** We show first that (5.1.47) and (5.1.48) hold if \(L_1 = \hat{L}_1\) and \(L_2 = \hat{L}_2\) are cylindrical far enough along the end. In fact, we first show that if we choose the cutoff functions \(\varphi^L_T = \varphi_i \circ \iota\), then \(\gamma_T(\alpha_1|_{L_1}, \alpha_2|_{L_2}) = \gamma_T(\alpha_1, \alpha_2)|_{L^T}\) in this case.

As in the negative discussion, it suffices to show that \(\alpha_i|_{L_i} = \hat{\alpha}_i|_{L_i}\). We assume that \(\alpha_i = \beta_i + d\zeta_i\), where \(\beta_i\) is translation-invariant and \(\zeta_i\) is decaying. Then \(\alpha_i|_{L_i} = \beta_i|_{L_i} + d(\zeta_i|_{L_i})\).
Since $L_i$ is cylindrical, $\beta_i|_{L_i}$ is also translation invariant, and so $\hat{\alpha}_i|_{\hat{L}_i} = \beta_i|_{L_i} + d(\varphi_T^*\zeta_i|_{L_i})$. On the other hand, $\hat{\alpha}_i = \beta_i + d(\varphi_T^*\zeta_i)$, so that $\hat{\alpha}_i|_{\hat{L}_i} = \beta_i|_{L_i} + d((\varphi_T \circ i)\zeta_i)$. Hence, indeed, these are equal if $\varphi_T^* = \varphi_i \circ i$.

The general result follows if we vary the cutoff function by Remark 3.5.6.

Now suppose that $L_i$ is only asymptotically cylindrical. By the cylindrical case, it suffices to show that the relationships of (5.1.47) and (5.1.48) hold between $\gamma_T(\alpha_1|_{L_1}, \alpha_2|_{L_2})$ and $\gamma_T(\alpha_1|_{L_1}, \alpha_2|_{L_2})$. That is, we want to show that $\gamma(\alpha_1|_{L_1} - \alpha_1|_{L_1}, \alpha_2|_{L_2} - \alpha_2|_{L_2})$ is exponentially small and exact. In turn, this follows if $\alpha_i|_{L_i} - \alpha_i|_{\hat{L}_i} = d\beta_i$ for a decaying form $\beta_i$ on $L_i$: Then $\gamma(\alpha_1|_{L_1} - \alpha_1|_{L_1}, \alpha_2|_{L_2} - \alpha_2|_{L_2}) = d(\varphi_1\beta_1 + \varphi_2\beta_2)$ and as in Remark 3.5.6 this is exponentially small in $T$ and exact.

Note that $\alpha_i|_{L_i} - \alpha_i|_{\hat{L}_i}$ is supported on the end of $L_i$, and so if it is decaying and exact it is $d\beta_i$ for $\beta_i$ decaying. Moreover, on its support, we have

$$\alpha_i|_{L_i} - \alpha_i|_{\hat{L}_i} = v^*\exp^v\alpha_i - \exp^v\alpha_i,$$

where $v$ is the normal vector field defining the asymptotically cylindrical submanifold $L_i$ and $i$ is the inclusion of $K \times (R_1, \infty)$. Since $v$ is exponentially decaying so that $\exp_v$ is exponentially close in $T$ to $\exp_{\varphi_T^*v}$ and clearly $\exp_v$ is homotopic to $\exp_{\varphi_T^*v}$, this shows that $\alpha_i|_{L_i} - \alpha_i|_{\hat{L}_i}$ is decaying and exact.

A slightly generalised version of the argument of Lemma 5.1.15 using Definition 3.5.13 rather than Definition 3.5.5 proves that

**Proposition 5.1.16.** Let $M^T$ and $L^T$ be as in Definition 5.1.14. Consider the metric given on $L^T$ by direct gluing of the metrics on $L_1$ and $L_2$ by cutting them off and identifying them over an appropriate region; consider also the metric given on $L^T$ by restricting the metric on $M^T$ defined similarly to $L^T$. The difference of these two metrics decays exponentially in $T$ to zero with all derivatives, with respect to either of them.

We also note at this point that the second fundamental form of $L^T$ in $M^T$ can be bounded independently of $T$.

**Proposition 5.1.17.** Let $M^T$ and $L^T$ be as in Definition 5.1.14. Then the second fundamental form of $L^T$ in $M^T$ can be bounded in $C^k$ uniformly in $T$ for every $k$. In particular, by Theorem 5.1.3, the restriction maps of $C^k$ $p$-forms can be bounded uniformly in $T$.

**Proof.** We divide $M^T$ into the three parts $M_1^{tr(T-2)}$, $M_2^{tr(T-2)}$, and $(-3, 3) \times N$. First consider $M_1^{tr(T-2)}$. Provided $T$ is sufficiently large, on this region, $\gamma^T = g_1$ and $L^T = L_1$; note that since we don’t know the image of $\{T-1\} \times K$ under the appropriate decaying normal vector field lies in $\{T-1\} \times N$, it isn’t clear that the cutoff $\hat{L}_1$ agrees with $L_1$ up to $T-1$, which is why we use $T-2$ here. Hence the fundamental form is the same as that of $L_1$ in $(M_1, g_1)$. As in
the proof of Proposition 5.1.12 the second fundamental form of \( L_i \) in \( M_i \) is bounded, and so the second fundamental form of \( L^T \) in \( M^T \) is bounded on this subset independently of \( T \). Similarly, on \( M_2^{T_{1+T-2}} \), the second fundamental form is the same as that for \( L_2 \) in \( (M_2, g_2) \).

Now consider \((-3, 3) \times N\). Here \( g^T \) is exponentially close in \( T \) to the cylindrical metric \( \tilde{g} = g_N + dt \otimes dt \) and \( L^T \) is exponentially close in \( T \) to \((-3, 3) \times K\). Hence the second fundamental form is exponentially close to the second fundamental form of \( K \) in \( N \), which again is independent of \( T \).

It follows that throughout \( M^T \) the second fundamental form can be bounded independently of \( T \).

### 5.2 Definitions, examples and deformations

In this section, we first define special Lagrangian submanifolds and make some elementary remarks on the structure of the ends of asymptotically cylindrical such submanifolds. Then, in subsection 5.2.2, we show that there are asymptotically cylindrical special Lagrangian submanifolds in some of the asymptotically cylindrical Calabi–Yau manifolds we found in subsection 4.1.2. Finally, in subsection 5.2.3 we summarise the deformation theory due to McLean [60] in the compact case and extended by Salur and Todd [70] to the asymptotically cylindrical case. We give fuller details in the asymptotically cylindrical case, particularly for the result (Theorem 5.2.13) that the deformations with moving boundary form a manifold with specified dimension, as the argument in [70] is somewhat unclear and the dimension found in [70, Theorem 1.2] is not quite correct.

#### 5.2.1 Definitions

We first define special Lagrangian submanifolds and discuss asymptotically cylindrical special Lagrangian submanifolds.

**Definition 5.2.1.** Let \( M \) be a Calabi–Yau \( n \)-fold and \( L \) an oriented \( n \)-submanifold. \( L \) is special Lagrangian if and only if it is calibrated with respect to \( \Re \Omega \); that is, \( \Re \Omega |_L = \text{vol}_L \).

The analysis becomes easier if we can remove explicit dependence on the volume form; thus, we use the following equivalent definition

**Lemma 5.2.2 ([31, Corollary III.1.11]).** \( L \) is special Lagrangian, for some choice of orientation, if and only if \( \omega |_L = 0 \) and \( \Im \Omega |_L = 0 \).

A simple extension of the proof of Lemma 5.2.2, essentially just considering each line of the proof in turn and passing from equality to continuity gives
Proposition 5.2.3. Suppose that $L$ is an $n$-submanifold of a $2n$-manifold $M$ and $(\Omega_i, \omega_i)$ is a sequence of (torsion-free) Calabi–Yau structures on $M$. If

$$\text{Re} \Omega_i|_L - \text{vol}_{i,L} \to 0,$$  

(5.2.1)

in the sense that the $C^k$ norms, say, given by $g_i|_L$ converge to 0 then in the same sense we have

$$\text{Im} \Omega_i|_L \to 0, \quad \omega|_L \to 0.$$  

(5.2.2)

That is, combining with the reverse implication, the conditions of Definition 5.2.1 and Lemma 5.2.2 extend to define the same notion of a “nearly special Lagrangian” submanifold. We shall almost exclusively use the extension of Lemma 5.2.2; the only exception is in Lemma 5.4.15, where we need to worry about orientation.

Harvey and Lawson proved [31, Theorem III.2.7] that any special Lagrangian $L$ which is locally of the form \( \{(x_1, \ldots, x_n, \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}) \} \subset \mathbb{R}^n \oplus i\mathbb{R}^n = \mathbb{C}^n \) for a real-valued $C^2$ function $F$ on a domain of $\mathbb{R}^n$ is smooth, in fact real analytic, because it is a minimal submanifold. They use the regularity theory of Morrey for second-order elliptic systems (such as the minimal surface equation), which in turn uses integral equations for difference quotients to establish its regularity results. On the other hand, there are special Lagrangian submanifolds even of $\mathbb{R}^{2n}$ with singularities (cones, for instance). Harvey and Lawson’s first such example is $|z|^2 - |w|^2 = 0 = \text{Re}(zw)$, but they give others, in particular that the canonical embedding of the normal bundle of a minimal surface in $\mathbb{R}^n$ into $\mathbb{R}^{2n}$ is a special Lagrangian for the Calabi–Yau structure $(e^{i\theta} \Omega_0, \omega_0)$ on $\mathbb{R}^{2n}$, for some $\theta$ which may not be zero.

We now turn to the idea of an asymptotically cylindrical special Lagrangian submanifold of an asymptotically cylindrical Calabi–Yau. For the purposes of this chapter, we restrict the cross-section $N$ of a manifold $M$ with an end and the definition of asymptotically cylindrical Calabi–Yau from Definition 4.1.10 so that we have more structure on the cylindrical end. This restriction follows Salur and Todd [70], as we will use the deformation theory of that paper.

We make

Definition 5.2.4. $\mathbb{R} \times N = \mathbb{R} \times S^1 \times X$ is said to be a Calabi–Yau cylinder if it is a Calabi–Yau manifold and its Calabi–Yau structure is given by

$$\Omega = (dt + id\theta) \wedge \Omega_{xs}, \quad \omega = dt \wedge d\theta + \omega_{xs},$$  

(5.2.3)

where $(t, \theta)$ are standard coordinates on $\mathbb{R} \times S^1$ and $(\Omega_{xs}, \omega_{xs})$ is a Calabi–Yau structure on $X$.

By assuming that the limit of an asymptotically cylindrical Calabi–Yau manifold must be a Calabi–Yau cylinder in the sense of Definition 5.2.4, we obtain the following
Hypothesis 5.2.5. For the purposes of this chapter, any asymptotically cylindrical Calabi–Yau manifold as defined in Definition 4.1.10 has \( N = S^1 \times X \), there is a Calabi–Yau structure \( (\Omega_{xs}, \omega_{xs}) \) on \( X \), and the limit Calabi–Yau structure \( (\tilde{\Omega}, \tilde{\omega}) \) is given by (5.2.3).

There exist asymptotically cylindrical Calabi–Yau manifolds that do not satisfy Hypothesis 5.2.5; see Remark 4.1.11 for some details of the relatively restricted problems that can happen if \( n > 3 \) and the manifold is simply connected. If \( n = 2 \), or for instance there is a torus factor that can be split so that a factor can be taken with \( n = 2 \), then there are further examples of worse limit behaviour.

We now make the obvious definition that an asymptotically cylindrical special Lagrangian submanifold of an asymptotically cylindrical Calabi–Yau manifold is a special Lagrangian submanifold in the sense of Definition 5.2.1 which is asymptotically cylindrical in the sense of Definition 5.1.11. Note that although Remark 4.1.12 requires that the end \( N \times (0, \infty) = X \times S^1 \times (0, \infty) \) of the Calabi–Yau manifold \( M \) be connected, asymptotically cylindrical special Lagrangians can have multiple ends contained in this one end.

Using Hypothesis 5.2.5, we obtain the following result on the structure of the cross-section \( K \) of \( L \).

**Proposition 5.2.6.** Let \( M \) be an asymptotically cylindrical Calabi–Yau manifold with cross-section \( N = X \times S^1 \) as in Hypothesis 5.2.5, and \( L \) be an asymptotically cylindrical special Lagrangian submanifold with cross-section \( K \), so that \( L = \exp_v(K \times (R, \infty)) \) far enough along the end, where \( v \) decays exponentially with all derivatives. For each connected component \( K' \) of \( K \), there is a special Lagrangian \( Y \) in \( X \), and a point \( p \in S^1 \), such that \( K' = Y \times \{ p \} \subset X \times S^1 = N \).

**Proof.** We know that \( \text{Im} \Omega|_L = 0 = \omega|_L \). By continuity of the restriction map, it follows that the cylindrical limit submanifold is special Lagrangian with respect to the limit structure. That is,

\[
(dt \wedge \text{Im} \Omega_{xs} + d\theta \wedge \text{Re} \Omega_{xs})|_{K \times \mathbb{R}} = 0 = (dt \wedge d\theta + \omega_{xs})|_{K \times \mathbb{R}}. \tag{5.2.4}
\]

Now we know that the inclusion map of \( K \times \mathbb{R} \) into \( N \times \mathbb{R} \) is given by \( \iota(k, r) = (\tilde{\iota}(k), r) \) for some \( \tilde{\iota} : K \to N \). Hence, \( d\iota|_{K \times \mathbb{R}} = dr \) but we may write the restrictions \( \text{Re} \Omega_{xs}|_{K \times \mathbb{R}} \) and so forth as restrictions to \( K \). We find

\[
dr \wedge (\text{Im} \Omega_{xs}|_{K}) + d\theta|_{K} \wedge \text{Re} \Omega_{xs}|_{K} = 0 = dr \wedge d\theta|_{K} + \omega_{xs}|_{K}. \tag{5.2.5}
\]

Since \( dr \) is linearly independent of forms on \( K \), we can then reduce (5.2.5) to

\[
d\theta|_{K} \wedge \text{Re} \Omega_{xs}|_{K} = 0, \quad \text{Im} \Omega|_{xs}|_{K} = 0, \quad \omega|_{xs}|_{K} = 0, \quad d\theta|_{K} = 0. \tag{5.2.6}
\]

Note the first of these equations is automatic as an \( n \)-form on the \((n-1)\)-dimensional manifold \( K \) must be zero. For each component \( K' \), it follows from \( d\theta|_{K} = 0 \) that \( \theta \) can only take one
value on $K'$. Hence, $K'$ is certainly of the form $\{p\} \times Y$. It follows from the first two equations of (5.2.6) that $Y$ is special Lagrangian in $X$.

Remark. Definition 5.1.11 and Proposition 5.2.6 together form a version of the definition of asymptotically cylindrical special Lagrangian found in [70, Definition 2.6]. Our definition, which appears simpler, makes the identifications between the Calabi–Yau cylinder and the end of the asymptotically cylindrical Calabi–Yau manifold tacitly and implicitly; Salur and Todd give a more concrete explanation. Note that these identifications include the choice of identification for a manifold with ends discussed after Definition 2.1.1.

### 5.2.2 Examples of asymptotically cylindrical special Lagrangians

As in subsection 4.1.2, in this subsection we will explain why our objects of study do indeed exist. We will provide a fairly general method for a subset of the asymptotically cylindrical Calabi–Yau manifolds of subsection 4.1.2 in Proposition 5.2.10; we will then specialise and give a concrete example.

To do this, we will use antiholomorphic involutive isometries, as mentioned, for instance, by Joyce–Salur [46, top of p.1118].

**Definition 5.2.7.** Let $M$ be a Kähler manifold with complex structure $J$ and Kähler form $\omega$. A diffeomorphism $\sigma$ of $M$ is called an *antiholomorphic involutive isometry* if

$$\sigma^2 = \text{id}, \quad \sigma^* J = -J, \quad \sigma^* \omega = -\omega,$$

(5.2.7)

with pullback of the complex structure meaning the appropriate combination of pullback and pushforward.

We will mostly be interested, as usual, in Calabi–Yau manifolds. We will restrict slightly in that case.

**Definition 5.2.8.** Let $M$ be a Calabi–Yau manifold with Calabi–Yau structure $(\Omega, \omega)$. The diffeomorphism $\sigma$ is called a *antiholomorphic involutive isometry with fixed phase* or a *real structure* if

$$\sigma^2 = \text{id}, \quad \sigma^* \Omega = \bar{\Omega}, \quad \sigma^* \omega = -\omega.$$

(5.2.8)

The term “real structure” is taken from [40, p.304].

$\Omega$ determines the complex structure $J$ (Lemma 4.1.2); hence an antiholomorphic involutive isometry with fixed phase is indeed an antiholomorphic involutive isometry. On the other hand, similar arguments show that $J$ determines $\Omega$ up to scale, the absolute value of this scale is fixed by $\omega$, and the scale must be constant by torsion-freeness; hence, $\omega$ and $J$ fix $\Omega$ up to phase. This proves that if $\sigma$ is an antiholomorphic involutive isometry of a Calabi–Yau manifold, $\sigma^* \Omega = e^{i\alpha} \bar{\Omega}$ for some $\alpha$. Hence, $\sigma^* e^{-\frac{10}{\tau}} \Omega = e^{\frac{10}{\tau}} \bar{\Omega}$ so that $\sigma$ is an antiholomorphic involutive
isometry with fixed phase for \((e^{-\frac{i}{2} \Omega}, \omega)\). Because of this, we may blur the distinction between the two.

We have

**Lemma 5.2.9.** The fixed point set of an antiholomorphic involutive isometry with fixed phase is a special Lagrangian submanifold.

An antiholomorphic involutive isometry is an isometry, and so its fixed point set is a manifold. See, for example, Kobayashi [49, Theorem 5.1]. This further proves that its tangent space at the fixed point \(p\) of \(\sigma\) is precisely the subspace of \(T_p M\) fixed by \(\sigma_*\). For every \(v \in T_p M\), \(\sigma_*J_p(v) = -J_p(\sigma_*v)\), so that \(J_p\) interchanges the two eigenspaces of this involution; hence both are \(n\)-dimensional. Now also as \(\sigma_*\) fixes the tangent space, we must have \(\omega_p(v, w) = 0\) for any \(v\) and \(w\) in \(T_p L\), and similarly for \(\text{Im} \Omega\).

Note that the argument above implies that the fixed point set of a general antiholomorphic involutive isometry is minimal Lagrangian.

If \(M\) is an asymptotically cylindrical Calabi–Yau, we want to find an antiholomorphic involutive isometry \(\sigma\) (with fixed phase) such that this special Lagrangian is asymptotically cylindrical. To do this, we induce \(\sigma\) from our construction of \(M\). Specifically, we have

**Proposition 5.2.10.** As in Theorem 4.1.14, let \(\tilde{M}\) be a smooth compact Kähler manifold of complex dimension \(n \geq 2\) and \(D\) a smooth anticanonical divisor with holomorphically trivial normal bundle. Let \(\sigma\) be an antiholomorphic involutive isometry of \(\tilde{M}\). Suppose moreover that \(\sigma\) acts on \(D\), and \(D\) contains some fixed points of \(\sigma\).

By Theorem 4.1.14, \(M = \tilde{M} \setminus D\) admits an asymptotically cylindrical Ricci-flat Kähler metric, in the restriction of this Kähler class. Moreover, the restriction of \(\sigma\) to \(M\) defines an antiholomorphic involutive isometry of \(M\) which is asymptotically cylindrical in the sense of Definition 4.2.2, and its fixed point set is an asymptotically cylindrical submanifold in the sense of Definition 5.1.11.

**Proof.** Since \(\sigma\) acts on \(D\), it also acts on \(M\). Hence the restriction of \(\sigma\) to \(M\) is well-defined and squares to the identity.

Now the complex structure on \(M\) is just the restriction of the complex structure on \(\tilde{M}\). Hence, we immediately have \(\sigma^* J = -J\). It remains to show that \(\sigma\) induces an isometry. To do this, we apply Corollary 4.1.16. \(\sigma\) is precisely a biholomorphism between \(\tilde{M}\) and its complex conjugate, preserving the anticanonical divisor. Hence, Corollary 4.1.16 proves that it induces an isometry in this case.

Now [65, Proposition 6.22] implies that \(\sigma\) is an asymptotically cylindrical diffeomorphism, as an isometry of asymptotically cylindrical metrics. It only remains to show that the fixed point set, which we know by Lemma 5.2.9 is a submanifold, is in fact asymptotically cylindrical. To do this, we use that \(\sigma\) is an involution again.
Because $\sigma$ is asymptotically cylindrical, there exist a diffeomorphism $\tilde{\sigma}$ of $N$, a real number $l$, and a decaying normal vector field $v$ such that $\sigma(t, x) = \exp_v((t + l, \tilde{\sigma}(x)))$. Since $\sigma$ is an involution, we must have $l = 0$. By abuse of notation, we shall also write $\tilde{\sigma}$ for the induced diffeomorphism on $(T, \infty) \times N$.

We now show that the fixed points of $\sigma$, far enough along the end, are the image of the fixed points of $\tilde{\sigma}$ under $\exp_{\tilde{\sigma}}$. The fixed points of $\tilde{\sigma}$ are certainly a cylindrical submanifold. Since $v$ is exponentially decaying, and the map from a vector field along a submanifold to the normal vector field with the same image preserves exponential decay (this follows by similar arguments to Proposition 5.5.21 below), this shows that the fixed points of $\sigma$ are an asymptotically cylindrical submanifold, as required. Given $p$ with $\tilde{\sigma}(p) = p$, we must have $\sigma(p) = \exp_v(p)$. For $t$ large enough, uniformly in the cross-section, there is a unique minimising geodesic from $p$ to $\exp_v(p)$. Since $\sigma$ is an involution, $\sigma(\exp_v(p)) = p$, and so this geodesic is reversed by the isometry $\sigma$. Hence its midpoint $\exp_{\tilde{\sigma}}(p)$ is fixed by $\sigma$. This shows that the fixed points of $\sigma$ contain this image (at least far enough along the end); the reverse argument is only slightly more complicated.

This shows that the fixed points of $\sigma$ are an asymptotically cylindrical submanifold with the same limit as the fixed points of $\tilde{\sigma}$ (in other words, the limit is the fixed points of $\tilde{\sigma}$ on $N$). It remains to prove that $\tilde{\sigma}$ fixes some points of $N = D \times S^1$, otherwise all we have done is exhibited a closed special Lagrangian submanifold. We shall consider $\tilde{\sigma}(x, \theta)$. By [65, Proposition 6.22], this can be given by the limit approached by the image under $\sigma$ of the unique geodesic half-line approaching $(x, \theta)$. Any curve approaching $(x, \theta)$ must, passing to a curve in $\tilde{M}$, approach $(x, 0)$. Hence the image approaches $(\sigma|_D(x), 0)$. This shows that $\tilde{\sigma}(x, \theta) = (\sigma|_D(x), \theta')$ for some $\theta'$. Now also we know by continuity that $\tilde{\sigma}$ is antiholomorphic on the Calabi–Yau cylinder $\mathbb{R} \times S^1 \times D$. Since we clearly have $\tilde{\sigma} \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta}$, we must have $\tilde{\sigma} \frac{\partial}{\partial \theta} = -\frac{\partial}{\partial \theta}$. Hence also the isometry $\tilde{\sigma}$ preserves the orthogonal complement $TD$ of $\text{span}\{\frac{\partial}{\partial \theta}\}$. It follows that $\theta'$ depends only on $\theta$. Moreover, the isometry thus given on $S^1$ is a reflection. Hence it has two fixed points. Thus for each fixed point of $\sigma|_D$ we have two fixed points of $\tilde{\sigma}$ on $N$. Since by hypothesis $\sigma|_D$ has fixed points, $\tilde{\sigma}$ has fixed points, and the resulting submanifold is indeed asymptotically cylindrical.

Remark. It is not hard to find non-involutive asymptotically cylindrical diffeomorphisms whose fixed point set is not an asymptotically cylindrical submanifold. For instance, working locally, one can obtain the cone $\{(x, t) : |x| < e^{-t}\}$ in $(0, 1) \times (0, \infty)$, and easily extend this to $S^1 \times (0, \infty)$ (perhaps introducing some more fixed points).

Hence, if we choose the appropriate phase for the holomorphic volume form, we get an asymptotically cylindrical special Lagrangian submanifold.

In particular, we shall consider the simplest examples of such $\tilde{M}$, that is the blowups of Fano manifolds in the self-intersection of their canonical divisor, as described in the final paragraph of subsection 4.1.2. There is no need to worry about metrics here, and so we restrict to
antiholomorphic involutions, that is involutions \( \sigma \) with \( \sigma^* J = -J \).

**Proposition 5.2.11** (cf. Kovalev [52, top of p.19]). Suppose that \( \tilde{M} \) is a Fano manifold with \( D \) a smooth anticanonical divisor and \( \sigma \) is an antiholomorphic involution which acts on \( D \). Suppose further that we may find a submanifold representing the self-intersection of \( D \) on which \( \sigma \) acts. Then if we blow up this self-intersection, the resulting manifold admits an antiholomorphic involution acting on the anticanonical divisor given by the proper transform of \( D \).

**Proof.** That the resulting complex manifold admits an antiholomorphic involutive isometry essentially follows from the universal property of blowup. We begin by noting that we may blow up \( \tilde{M} \) in the self-intersection of \( D \) to obtain the projection \( \pi : \tilde{\tilde{M}} \to \tilde{M} \). Then if we multiply the complex structures on \( \tilde{\tilde{M}} \) and \( \tilde{M} \) by \(-1\), \( \pi \) remains a biholomorphism away from \( D \) and continues to be given there by the projection of the projectivisation of the normal bundle. Hence, by uniqueness of blowups, this is essentially the blow up of \( \tilde{M} \) with the complex structure multiplied by \(-1\).

Now \( \sigma \pi \) defines a map from \( \tilde{\tilde{M}} \) to \( \tilde{M} \) and this is an antiholomorphic map. Since \( \sigma \) acts on the self-intersection, the inverse image of the self-intersection is the exceptional divisor, and elsewhere it is locally biholomorphic. Hence, by the universal property of blow-up (essentially the uniqueness result [28, p.604] in this case) it passes through the blow-up of \( \tilde{M} \) with its conjugate structure: but we know that that is just the original blow-up, and so the factorisation gives us an antiholomorphic involutive isometry, as required.

It remains to show that this factorisation acts on the proper transform of \( D \). But if a point lies in the proper transform, it is the limit of a sequence in the preimage of \( D \) and the complement of the self-intersection; this is preserved by \( \sigma \), and so the image must also be such a limit, as required. \( \square \)

We can then choose any Kähler class \( [\omega] \) on the blowup and consider \( [\omega] - \sigma^* [\omega] \) to get an antiholomorphic involutive isometry.

The only explicit example we discussed in subsection 4.1.2 was to take \( \tilde{M} = \mathbb{C} P^n \) and \( D = \{ p(x_1, x_2, x_3, x_4, \ldots x_{n+1}) = 0 \} \) for a homogeneous polynomial \( p \) of degree \( n + 1 \) that is always a submersion at its zeros. If \( p \) has real coefficients, \( D \) is preserved by the involution of \( \mathbb{C} P^n \) given by complex conjugation of \( \mathbb{C}^{n+1} \). Since we may perturb \( p \) to get a transverse submanifold without losing the real coefficients, we may suppose \( \sigma \) acts on the self-intersection, and so apply Proposition 5.2.11 to find an antiholomorphic involution of the blowup. Choosing an appropriate Kähler class, applying Proposition 5.2.10, and then choosing the appropriate phase for the holomorphic volume form, we find an asymptotically cylindrical Calabi–Yau manifold admitting a special Lagrangian submanifold.

We know that each end of this special Lagrangian has limit \( \{ p \} \times Y \), for some special Lagrangian \( Y \) in the proper transform of \( D \). It is clear from the construction that \( Y \) is a component of the fixed points of the induced map on the blowup restricted to this proper transform. This
set of fixed points is identified with the fixed points of \( \sigma|_D \), which are given by the product of the solutions to \( \{ p(x_1, x_2, x_3, x_4, \ldots, x_{n+1}) = 0 \} \) in the real projective space \( \mathbb{R}P^n \) with \( \mathbb{R}P^1 \) (as we may change the argument). As a basic example, therefore, taking \( n = 3 \) and the divisor \( x_1^4 + x_2^3 + x_3^2 + x_4 = 0 \), this method does not yield an asymptotically cylindrical special Lagrangian. However, \( x_1^4 + x_2^3 + x_3^2 - x_4 \) will; so will other similar examples. Moreover, each component of this set corresponds to two ends of the special Lagrangian, again by the construction. Thus a special Lagrangian constructed by this method always has an even number of ends; since the fixed points of \( \sigma|_D \) may not be connected, it follows that there are examples with more than two. Thus, the deformation results in the next subsection do not apply without further work, and as it relies on this nor will our Theorem 5.7.13 on the gluing of deformed special Lagrangians.

Similar arguments will apply more generally to Fano manifolds constructed as complete intersections; we should be able to choose the required polynomials to be real inductively, including the anticanonical divisor and its perturbation.

### 5.2.3 Deformation theory

In this subsection, we give a summarised version of McLean’s deformation result [60] and the extension to asymptotically cylindrical special Lagrangians given by Salur and Todd. We shall outline the proof in the compact case as our gluing result will rest on applying the same idea to perturb a nearly special Lagrangian submanifold to a special Lagrangian submanifold; in the asymptotically cylindrical case, we shall give some details to improve the results and provide clearer proofs. In particular, note that Theorem 5.2.13 is not quite in agreement with [70, Theorem 1.2].

We begin with the compact case. Let \( M \) be a Riemannian manifold. If \( L \) and \( L' \) are closed submanifolds of \( M \) with \( L \) and \( L' \) close in a \( C^1 \) sense, then there is a normal vector field \( v \) to \( L \) such that \( L' \) is the image of \( v \) under the Riemannian exponential map; in this case, we write \( L' = \exp_v(L) \). This is because \( L \) and \( L' \) are the images of similar inclusions \( \iota \) and \( \iota' \) of the same ambient manifold. Hence, \( L' \) must lie in a tubular neighbourhood of \( L \), and \( L \) and \( L' \) should have similar tangent spaces, so that the projection of the tubular neighbourhood will be a local diffeomorphism on restriction to \( L' \). It is clear that this map must then be a global diffeomorphism. Conversely, of course, a \( C^1 \)-small normal vector field \( v \) defines a \( C^1 \)-close submanifold \( L' = \exp_v(L) \).

Moreover, the regularities of \( L' \) and \( v \) are the same. To see this, we just note that in the coordinates on this tubular neighbourhood the inclusion of \( L \) as \( L' \) is \( p \mapsto v(p) \in \nu_L \). The identification of the tubular neighbourhood with \( \nu_L \) is smooth, and this immediately shows that \( v \) is \( C^{k,\mu} \) if and only if \( L' \) is. In particular smooth submanifolds are given by smooth normal vector fields.

Before turning to the deformation problem itself, we note that the homogeneous equations
\[ \text{Im} \Omega|_L = 0 \text{ and } \omega|_L = 0 \text{ of Lemma 5.2.2 are only equivalent to the original definition (Definition 5.2.1) Re } \Omega|_L = \text{vol}_L \text{ up to orientation. However, if } L' \text{ is close to the special Lagrangian } L \text{ so that } \exp_v(L) = L', \text{ there is a natural choice of orientation on } L' \text{ so that } \exp_v \text{ is orientation-preserving. Hence, if } \text{Im} \Omega|_{L'} = 0 \text{ and } \omega|_{L'} = 0, \text{ so that } \text{Re } \Omega|_{L'} = \pm \text{vol}_{L'}, \text{ we may assume that } \text{Re } \Omega|_{L'} = +\text{vol}_{L'}. \text{ Hence, Definition 5.2.1 and Lemma 5.2.2 may still be treated as equivalent.} \\

The point of the discussion above is that to work with submanifolds it suffices to work with normal vector fields. Hence, to understand which submanifolds } L' \text{ close to } L \text{ are special Lagrangian, we want to understand the zero set of the nonlinear map} \\
\[ F : v \mapsto (\exp_v^* \text{Im } \Omega|_L, \exp_v^* \omega|_L). \] (5.2.9) \\

We will apply the implicit function theorem to } F \text{ and thus we have to show} \\
i) \text{ What the derivative } DF \text{ of } F \text{ is.} \\
i) \text{ That on restriction of } F \text{ to appropriate Banach spaces of vector fields and forms } DF \text{ is surjective and therefore the zero set is a submanifold.} \\
ii) \text{ That the zero set obtained by this method is independent of the Banach space we have used.} \\
i) \text{ and ii) are covered in detail by McLean [60, Theorem 3-6]. A calculation using Cartan’s magic formula shows that the linearisation } DF \text{ is} \\
\[ DF(u) = (d(\iota_u \omega|_L), d^* (\iota_u \omega|_L)), \] (5.2.10) \\
recalling that since } L \text{ is Lagrangian, } u \mapsto \iota_u \omega|_L \text{ is an isomorphism between the normal and cotangent bundles of } L. \text{ We will generalise this isomorphism for our purposes in Lemma 5.4.3 below.} \\
ii) \text{ is the technical part. We need to use some Banach space of forms, say } C^{1,\mu}. \text{ It is easy to see that } F \text{ maps only to exact forms, and so we consider } F \text{ as a map from } C^{1,\mu} \text{ normal vector fields to } C^{0,\mu} \text{ forms.} \\
By Baier [5, Theorem 2.2.15], we know that } F \text{ is then a smooth map of Banach spaces. } DF \text{ then becomes (5.2.10) between } C^{1,\mu} \text{ normal vector fields and exact } C^{0,\mu} \text{ forms; since } u \mapsto \iota_u \omega|_L \text{ is an isomorphism between } C^{1,\mu} \text{ normal vector fields and } C^{1,\mu} \text{ forms, and } d + d^* \text{ is surjective, it follows that (5.2.10) is surjective.} \\
Finally, for iii), the choice of Banach space is a choice of regularity for our solutions of } F. \text{ But by the analysis of the beginning of 5.2.3, these vector fields have the same regularity as the submanifolds concerned. Thus iii) reduces to the well-understood elliptic regularity of calibrated (and hence minimal) submanifolds, and is essentially already done.} \\
We also review an asymptotically cylindrical extension of this theory. This is due to Salur and Todd [70]. We recall by Proposition 5.2.6, Definition 5.1.11 of an asymptotically cylindrical
special Lagrangian \( L \) required that \( L \) decay to a special Lagrangian \( Y \times \{ p \} \times (R, \infty) \) in the cylinder \( X \times S^1 \times (R, \infty) \). The question of what asymptotically cylindrical submanifolds \( L' \) close to \( L \) are also special Lagrangian therefore falls into two parts: firstly, what happens when we fix \( Y \) and \( p \), and secondly what happens when we vary them.

We begin by discussing normal vector fields corresponding to asymptotically cylindrical deformations of an asymptotically cylindrical manifolds. We would like to say that these are asymptotically translation invariant in some sense. We will give a definition of these much later in Definition 5.5.17.

In particular, if the asymptotically cylindrical deformations have the same limit, we expect to get exponentially decaying normal vector fields. It is clear that by adding an exponentially decaying normal vector field on \( K \times (R, \infty) \) to the vector field \( v \) of Definition 5.1.11 also gives an asymptotically cylindrical submanifold. The setup of the tubular neighbourhood theorem used by Salur and Todd [70, p. 110-1] precisely uses this: they define the map \( \Xi \) to be applying the exponential map to this sum, and then choose an isomorphism \( \zeta \) between the normal bundles of (the end of) \( L \) and \( K \times (R, \infty) \); this isomorphism is just a pushforward and so it is reasonably obvious that it preserves exponentially decaying normal vector fields. The converse result follows in the same way.

For the asymptotically translation invariant case Salur and Todd use the rigidity of cylindrical special Lagrangians to reduce the problem (and hence to avoid needing a general definition of asymptotically translation invariant normal vector fields). They say that a cylindrical special Lagrangian deformation of the limit corresponds to a translation-invariant one-form using the isomorphism between one-forms and normal vector fields on a special Lagrangian on \( K \times (R, \infty) \), and hence gives a well-defined one-form on the end of \( L \), and then they combine these one-forms with exponentially decaying one-forms.

We are proceeding more directly: we have \( L \) a well-defined submanifold of \( M \), and consider the exponential map. It is simplest to suppose that we take the exponential map corresponding to a cylindrical metric on \( M \); our choice of exponential map does not affect the final result. We have to show that there is a uniform lower bound on the injectivity radius of \( M \) with the cylindrical metric to ensure our tubular neighbourhood is going to contain all reasonably nearby submanifolds. But this is clear, as with the cylindrical metric the injectivity radius only depends on the geometry of the cylinder and of the compact complement \( M^{cpt} \). As before, we find that \( C^1 \)-uniformly close \( C^{k, \mu} \) submanifolds correspond to \( C^1 \)-uniformly-small \( C^{k, \mu} \) normal vector fields. We now have to pass to the subset corresponding to asymptotically cylindrical submanifolds.

Again using the isomorphism between one-forms and normal vectors, we can assume that the Riemannian exponential map is defined on \( T^* L \). If a one-form on an asymptotically cylindrical submanifold is asymptotically translation invariant, then its image under the Riemannian exponential map is also an asymptotically cylindrical submanifold. This result is still needed
by Salur and Todd, in the specific cylindrical case. This is much less obvious a priori than the exponentially decaying case: whilst the image certainly decays in a $C^0$ sense to the corresponding cylindrical manifold, it is not clear why the associated normal vector field must decay with all its derivatives.

If we suppose that the Riemannian metric we use on $M$ is actually cylindrical, this follows because combining one-forms is a smooth operation. Specifically, by the definition of “asymptotically cylindrical submanifold”, we have to show that $\exp_\beta \exp_\alpha(K \times (R, \infty)) = \exp_\gamma(K' \times (R, \infty))$ is asymptotically cylindrical, that is that we can find $K'$ and $\gamma$ so that $\gamma$ is exponentially decaying. $K'$ can be obtained from the limit, and then pointwise, $\gamma$ can be chosen to depend smoothly on $\alpha$ and $\beta$, with a well-defined limit in this map as $t \to \infty$. Indeed, since the metric is cylindrical, we do not need to worry about it in this limit: for everything else, we have smooth dependence and so exponential convergence to this limit. Conversely, given two asymptotically cylindrical submanifolds with limits $\exp_\alpha(K_1 \times (R, \infty))$ and $\exp_\beta(K_2 \times (R, \infty))$ close enough that one is the image under the Riemannian exponential map of a one-form $\gamma$, $\gamma$ must have a well-defined limit (corresponding to the normal vector field between $K_1$ and $K_2$) and again $\gamma$ at each point depends smoothly on $\alpha$ and $\beta$ and continuously on $t$ with a well-defined limit, so decays exponentially to the limit $\tilde{\gamma}$. For further details of arguments like this, see section 5.5 below, particularly Proposition 5.5.21.

That is, a sensible definition of asymptotically translation invariant normal vector field is just “a normal vector field whose corresponding one-form is asymptotically translation invariant”. We shall show this is equivalent to the general definition we shall give in Definition 5.5.17 in Proposition 5.5.33 later.

In particular, an exponentially decaying normal vector field gives an asymptotically cylindrical submanifold with limit the image under the zero vector field of the limit of $L$, i.e. with the same limit. Conversely, given a close asymptotically cylindrical submanifold with the same limit, we can find such an asymptotically translation invariant normal vector field and since $K_1 = K_2$ this normal vector field must decay exponentially.

We begin with the case in which we fix the limit $\tilde{K} = \{p\} \times Y$, that is the case of exponentially decaying normal vector fields. We obtain the following variant of [70, Theorem 1.1]. The improvement we make is that we do not explicitly require a decay rate: we say that the space of deformations is independent of the decay rate provided it is sufficiently small.

**Theorem 5.2.12.** Let $M$ be an asymptotically cylindrical Calabi–Yau manifold, with cross-section $N = X \times S^1$, where $X$ is a compact connected Calabi–Yau manifold, and let $L$ be an asymptotically cylindrical special Lagrangian submanifold of $M$ with cross-section $K = Y \times \{p\}$ where $Y$ is a special Lagrangian submanifold of $X$.

The space of special Lagrangian deformations of $L$ that are also asymptotically cylindrical with limit $L \times \{p\} \times (T', \infty)$ and sufficiently small decay rate is a manifold, with tangent space at $L$ given by the (exponentially) decaying harmonic normal vector fields.
Remark. Note that Salur–Todd only proved this result in the three-dimensional case, but the proof obviously generalises.

Sketch proof. For each rate \( \delta \), the existence of a smooth moduli space of deformations of rate at least \( \delta \) that is a manifold follows exactly as in [70], by applying the implicit function theorem to (5.2.9) on suitably weighted Banach spaces.

We thus only need to show that this moduli space can be taken independent of \( \delta \). It is clear that if we consider things decaying slower, we get the same solutions and maybe some more: we have to show that as we consider decay rate getting faster, the space of deformations does not shrink to \( \{ L \} \), which would correspond to all the deformations decaying slower than \( L \).

Suppose \( L' \) is any such deformation. Then, since we have a manifold, there is a curve of special Lagrangians \( L_s \) with \( L_0 = L \) and \( L_1 = L' \). The tangents to this curve are exponentially decaying harmonic normal vector fields on \( L_s \). Since all the submanifolds \( L_s \) have the same limit, it follows that there exists \( \delta > 0 \) (depending only on \( L \)) such that all the normal vector fields decay at least at rate \( \delta \) (by Proposition 3.4.1). Hence, so does the whole curve \( L_s \) and in particular \( L' \).

Note that it may also be possible to prove independence of \( \delta \) using a (boundary) elliptic regularity approach: to prove a priori that all special Lagrangians with this cross-section decay at least at a certain rate, as with the harmonic forms (Proposition 3.4.1). For an example of this in a rather different setting, see Kovalev–Singer [53, Proposition 5.13].

We now pass to the case where the limit \( K = Y \times \{ p \} \) may change, that is the asymptotically translation invariant normal vector fields. We will sketch how to prove the following variant of [70, Theorem 1.2].

**Theorem 5.2.13.** Let \( M \) be an asymptotically cylindrical Calabi–Yau manifold, with cross-section \( X \times S^1 \), where \( X \) is a compact connected Calabi–Yau manifold, and let \( L \) be an asymptotically cylindrical special Lagrangian in \( M \) with cross-section asymptotic to \( Y \times \{ p \} \) where \( Y \) is a special Lagrangian submanifold of \( X \). Suppose \( Y \) is connected.

The space of special Lagrangian deformations of \( L \) that are also asymptotically cylindrical with sufficiently small decay rate is a manifold with tangent space at \( L \) given by the harmonic normal vector fields on \( L \) such that \( \iota \omega|_L \) is in the space \( H^{1, \text{abs}}(L) \) of Definition 3.4.3.

The most important difference between this and [70, Theorem 1.2] is that there it is claimed that the tangent space has slightly greater dimension. Salur and Todd work more generally with \( Y \) potentially disconnected and claim that we can find a deformation curve with limit corresponding to the normal vector field limits \( \frac{\partial}{\partial \theta} \) (for each end), and this is not the case in general; in this connected case, where there is only one end, this does not occur at all, and in the disconnected case a different argument would be required for these limits. We also give a slightly clearer explanation of which limits that do correspond to deformation curves in...
Proposition 5.2.15, and we will need this argument later in Proposition 5.7.11 where we will apply it to show that the space of matching special Lagrangian deformations is a submanifold. Hence we give some details.

Suppose that \( L' \) is such a deformation and the corresponding asymptotically translation invariant normal vector field \( \nu \) has limit \( \tilde{\nu} \). Then in the cylindrical Calabi–Yau manifold \( \mathbb{R} \times S^1 \times X \), if we extend \( \tilde{\nu} \) to be translation invariant then \( \exp_{\tilde{\nu}}(\mathbb{R} \times Y \times \{p\}) \) must also be a translation invariant special Lagrangian. Our first problem is thus to identify the possible limits \( \tilde{\nu} \).

Again, we apply (5.2.9) and (5.2.10); since the only noncompactness is \( \mathbb{R} \), and we assume translation-invariance, McLean’s result applies and locally the \( \tilde{\nu} \) such that \( \exp_{\tilde{\nu}}(\mathbb{R} \times Y \times \{p\}) \) is special Lagrangian are diffeomorphic to translation-invariant harmonic normal vector fields.

Now \( \hat{M} = X \times S^1 \times \mathbb{R} \) is an orthogonal product, by the definition of a Calabi–Yau cylinder. \( \frac{\partial}{\partial t} \) is a tangent vector to the special Lagrangian, and so normal vector fields to \( \tilde{L} = Y \times \{p\} \times \mathbb{R} \) must be of the form \( u + f \frac{\partial}{\partial \theta} \) where \( u \) is a normal vector field on \( Y \) and \( f \) a function. Since we are looking for translation-invariant deformations, both \( u \) and \( f \) are independent of \( t \). To identify the harmonic normal vector fields, we note that \( \iota_{u+f} \tilde{\omega} = \iota_u \omega_{\mathbb{S}^1} + f dt \), and so that \( \iota_{u+f} \frac{\partial}{\partial \theta} \tilde{\omega} \big|_L = \iota_u \omega_{\mathbb{S}^1} |_Y + f dr \). Direct computation gives

\[
\Delta(\iota_{u+f} \frac{\partial}{\partial \theta} \tilde{\omega} |_L) = \Delta_Y(\iota_u \omega_{\mathbb{S}^1} |_Y) + (\Delta_Y f) dr. \tag{5.2.11}
\]

If \( u + f \frac{\partial}{\partial \theta} \) is a harmonic normal vector fields, then since harmonic functions are constant on components, \( f \) must be constant on the components of \( Y \). This establishes

**Lemma 5.2.14.** Translation-invariant harmonic normal vector fields on the end are precisely of the form \( u + C \frac{\partial}{\partial \theta} \), where \( u \) is a harmonic normal vector field on \( Y \) (that is, an infinitesimal deformation of \( Y \) in \( X \)), and \( C \) is constant on the components of \( Y \).

In particular, McLean’s analysis gives translation invariant special Lagrangian submanifolds corresponding to these normal vector fields. Moreover, a dimension-counting argument shows that these are all the normal vector fields giving cylindrical special Lagrangians (since this is essentially equivalent to asking for special Lagrangian deformations on \( X \times S^1 \times S^1 \)). This thus completes our analysis of the possible limits \( \tilde{\nu} \).

We further have to consider which of these deformations arises as the limit of an asymptotically cylindrical deformation.

**Proposition 5.2.15.** Let \( L_0 \) be an asymptotically cylindrical special Lagrangian submanifold of the asymptotically cylindrical Calabi–Yau manifold \( M \), with cross-sections \( K_0 = \{p\} \times Y \) and \( N = S^1 \times X \) respectively. We know that the set of cylindrical deformations of the special Lagrangian \( \mathbb{R} \times \{p\} \times Y \) in \( \mathbb{R} \times S^1 \times X \) is a manifold. Suppose further that \( Y \), and so \( K \), is connected.

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The set of limits of asymptotically cylindrical deformations of \( L \) is a submanifold \( K \) of this submanifold of cylindrical deformations. Its tangent space is precisely those harmonic normal vector fields on \( K \) that arise as limits of harmonic normal vector fields on \( L \).

**Proof.** Let \( K_s \) be a deformation of the limit \( K \), and extend it to \( L_s \). We have to show that we can choose \( L_s \) to be special Lagrangian. This can be done if and only if the exact and exponentially decaying forms \( \text{Im} \Omega|_{L_s} \) and \( \omega|_{L_s} \) are the differentials of decaying forms for any such extension: see [70, p.121].

We know that \( \text{Im} \Omega|_{L_s} \) and \( \omega|_{L_s} \) are decaying closed forms, so they define classes of the relative cohomology groups \( H^n_{\text{rel}}(L) \) and \( H^2_{\text{rel}}(L) \) (cf. Lemma 3.4.6). By Lemma 3.4.6, to ask for these forms to be the differential of a decaying form is to ask for these classes to be the trivial classes.

As in [70], we know that \( \omega \) and \( \text{Im} \Omega \) restrict to exact forms on a tubular neighbourhood of \( L_0 \) (since they restrict to zero on \( L_0 \)). Consequently we may find \( \tau_1 \) and \( \tau_2 \) asymptotically translation invariant with limits \( \tilde{\tau}_1 \) and \( \tilde{\tau}_2 \) such that \( \tau_1|_{L_0} = 0 \) and \( d\tau_1 = \omega, d\tau_2 = \text{Im} \Omega \) on a tubular neighbourhood of \( L_0 \).

We note that we have the map \( \partial : H^p(K) \to H^p_{\text{rel}}(L) \) of Theorem 2.1.2, treating \( L \) as a manifold with ends. Examining the definition of \( \partial \), we see that \( [\omega|_{L_s}] \in H^2_{\text{rel}}(L) \) is given by \( \partial([\tilde{\tau}_1|_K,]) \) and \( [\text{Im} \Omega|_{L_s}] \in H^2_{\text{rel}}(L) \) is \( \partial([\tilde{\tau}_2|_K,]) \).

That is, we need to show that the space of deformations \( K_s \) such that \( ([\tilde{\tau}_1|_K], [\tilde{\tau}_2|_K]) \) are both in \( \ker \partial \) is a submanifold with the desired tangent space.

We begin by considering deformations \( K_s \) that arise from definitions \( Y_s \) of the special Lagrangian \( Y_0 \) in \( X \); that is, that arise from the \( u \) component of \( u + C \partial \theta \). Such deformations \( K_s \) are homotopic to \( K \) in the submanifold \( \{p\} \times X \) of \( N = S^1 \times X \). We know that \( \text{Im} \Omega = d\theta \wedge \text{Re} \Omega_{\text{us}} + dt \wedge \text{Im} \Omega_{\text{us}} \) restricts to zero on \( \{p\} \times X \), so that \( \tilde{\tau}_2 \) is closed on this submanifold. Hence, for any such \( K_s \), we have \( \tilde{\tau}_2|_K = 0 \). In particular, \( \partial([\tilde{\tau}_2|_K]) = 0 \). Hence we only have to show that the deformations satisfying \( \partial([\tilde{\tau}_1|_K]) = 0 \) form a manifold with the desired properties. We consequently consider the nonlinear map

\[
K_s \mapsto \partial([\tilde{\tau}_1|_K]).
\]

This linearises using Cartan’s magic formula and the assumption \( \tilde{\tau}_1|_{K_0} = 0 \) to give

\[
v \mapsto \partial([t_v \tilde{\omega}|_{K_0}]).
\]

To prove that the kernel of (5.2.12) is a submanifold, we only have to prove that its linearisation (5.2.13) is surjective to the image of \( \partial \) in \( H^2_{\text{rel}}(L) \). Note that since by assumption \( v \) is a tangent to a curve of special Lagrangian deformations, \( t_v \tilde{\omega}|_{K_0} \) is a harmonic one-form, and so uniquely defines a cohomology class in \( H^1(K) \). By the Künneth formula, \( H^1(K) \cong H^1(Y) \). Now every class in \( H^1(Y) \) is represented by some harmonic one-form, and hence as \( t_v \tilde{\omega}|_{K_0} \) for some such
normal vector field $v$ to $Y$. Hence, (5.2.13) is a surjection onto the image of $\partial$. Its kernel is, by Theorem 3.4.5, precisely the harmonic one-forms on $K$ that arise as limits of harmonic forms.

It follows that we have a submanifold of potential deformations $Y_s$ that extend to asymptotically cylindrical special Lagrangian deformations of $L_s$.

It remains to consider the effect of introducing the normal vector fields $C\frac{\partial}{\partial \theta}$, which correspond to the perturbed limits $\{p^t\} \times Y$. If these limits define the limits of a curve of special Lagrangians $L_s$, then by linearising this curve we would find an asymptotically translation invariant harmonic normal vector field $v$ with limit $C\frac{\partial}{\partial \theta}$: this follows simply by the linearisation (5.2.10). Consequently, $\iota_v \omega|_L$ would be a harmonic one-form on $L_0$ with limit $Cdr$. This is impossible by Theorem 3.4.4: 1 is the limit of the harmonic form 1, so there cannot be a harmonic form with limit $dr$. The same applies when combining a curve varying $p$ with any other curve $Y_s$. Consequently, the only special Lagrangian deformations of the limit that may extend to special Lagrangian deformations of $L$ are those arising from special Lagrangian deformations of $Y_s$, as described above. \hfill\Box

**Remark.** Proposition 5.2.15 is essentially claimed in [70, Proposition 6.4], and we have followed the ideas of their proof. Their proof itself is somewhat unclear and possibly erroneous: we know that the $\tau_i$ decay on $L$, but Salur–Todd seem in that proposition to claim they have zero limit on $L_s$ too, so we have expanded on it. On [70, p. 123], Salur and Todd further claim that the normal vector fields $C\frac{\partial}{\partial \theta}$ give further perturbations $L_s$ of $L_0$, which is unfortunately untrue also.

Thus the (nonlinear) space of acceptable limits is a manifold, with tangent space at zero the harmonic normal vector fields $v$ to $Y$ such that $\iota_v \omega|_K$ arises as the limit of a harmonic form on $L_0$.

To obtain our final result, we take a space of vector fields $v$ defined as “decaying vector fields” $\oplus$ “acceptable limits”. In order to do this, we have to take some choice of vector field for each acceptable limit $\tilde{v}$. Whilst there is no canonical way to do this, if we assume that they are eventually constant any differences can be absorbed into the decaying vector fields. The image of such a vector field under the map $v \mapsto (\exp^* \Omega, \exp^* \omega)$ is obviously exponentially decaying, and is still exact by homotopy, and so we may still consider a right hand side of decaying exact forms; furthermore, by the above, the right hand side can be considered as differentials of decaying forms. We thus work on the (nonlinear) subspace of all normal vector fields corresponding to asymptotically cylindrical submanifolds that decay to a limit in $K$; the tangent space to this is normal vector fields decaying to harmonic normal vector fields that are limits of harmonic normal vector fields on $L$, and so the linearisation comes down to the effect of $(d, d\ast)$ from forms that decay to harmonic limits that are the limits of harmonic forms to the differentials of decaying forms. This is surjective exactly as in the decaying case, and evidently its kernel is precisely bounded harmonic forms. We get dimension of the moduli space the dimension of the bounded harmonic 1-forms, which is $b^1(L)$. 

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This completes the proof of Theorem 5.2.13. We thus see that the following lemma was not explicitly necessary; we restate it partly to give a more precise statement, though the proof is identical, but principally as it might be helpful for other boundary conditions, which we will discuss in chapter 6.

**Lemma 5.2.16** ([70, Lemma 6.2]). Let \( L = Y \times \{ p \} \times [0, 1] \) be special Lagrangian in \( M = X \times S^1 \times [0, 1] \). Let \( L' \) be a special Lagrangian homologous to \( L \). Further suppose that \( L' \) is transverse to \( X \times S^1 \times \{ t \} = N \times \{ t \} \) for all \( t \); note that this transversality does not follow from the homology assumption, though it follows if \( L' \) is a small perturbation of \( L \). Then \( L' \) is again of the form \( Y' \times \{ p' \} \times [0, 1] \).

This again proves that all the deformations of \( \tilde{L} = Y \times \{ p \} \times \mathbb{R} \) are \( \tilde{L}' = Y' \times \{ p' \} \times \mathbb{R} \), and so infinitesimally are given by \( u + C \frac{\partial}{\partial \theta} \) as in Lemma 5.2.14.

### 5.3 Constructing an approximate special Lagrangian

In this section, we apply the approximate gluing definitions given in subsection 5.1.3 to the asymptotically cylindrical special Lagrangian submanifolds of section 5.2, to construct a submanifold \( L^T \). We will primarily be showing that \( L^T \) is nearly special Lagrangian. We first discuss the hypothesis we shall assume for Calabi–Yau gluing and how it can be obtained from chapter 4 in the case of \( n = 3 \); then we deduce in Proposition 5.3.5 that \( L^T \) is nearly special Lagrangian.

We shall work with a pair of matching asymptotically cylindrical Calabi–Yau manifolds, and we shall assume the following gluing result for Calabi–Yau manifolds.

**Hypothesis 5.3.1** ( Ambient Gluing). Let \( M_1 \) and \( M_2 \) be a matching pair of asymptotically cylindrical Calabi–Yau manifolds. There exist \( T_0 > 0 \), a constant \( \epsilon \) and a sequence of constants \( C_k \) such that for gluing parameter \( T > T_0 \) there exists a Calabi–Yau structure \( (\Omega^T, \omega^T) \) on the glued manifold \( M^T \) constructed in Definition 3.5.2 with

\[
\|\Omega^T - \gamma_T(\Omega_1, \Omega_2)\|_{C^k} + \|\omega^T - \gamma_T(\omega_1, \omega_2)\|_{C^k} \leq C_k e^{-cT},
\]

\[
[\Im \Omega^T] = [\gamma_T(\Im \Omega_1, \Im \Omega_2)], \quad [\omega^T] = c[\gamma_T(\omega_1, \omega_2)],
\]

for some \( c > 0 \), where \( \gamma_T \) is the patching map of closed forms defined in Definition 3.5.5.

This hypothesis says that we can perturb the approximate gluing \( (\gamma_T(\Omega_1, \Omega_2), \gamma_T(\omega_1, \omega_2)) \) to get a Calabi–Yau structure. (5.3.1) says that the perturbation from our approximate gluing to the glued structure is small, and (5.3.2) says that these perturbations are basically by exact forms. Perturbing by exact forms is the obvious thing to try, as we want to get closed forms as perturbations of closed forms. In chapter 4 we didn’t quite obtain that. This may not be
necessary for the result, but would require extra arguments to remove (see the discussion before Lemma 4.3.9). As it does no harm here, we admit such rescaling.

Hypothesis 5.3.1 holds if \( n = 3 \). This follows from Theorem 4.3.3, combined with Proposition 4.3.6 and Proposition 4.3.7. Note that Proposition 4.3.6 gave (5.3.2) for \( \Re \Omega \), rather than \( \Im \Omega \) as here: but this is fine as \((i \Omega, \omega)\) is a Calabi–Yau structure whenever \((\Omega, \omega)\) is.

If \( n = 4 \), a gluing result for \( SU(4) \) structures is known. For instance, Doi–Yotsutani have followed their work [18] which we discussed around Lemma 4.3.4 with an extension to the case of \( n = 4 \) by passing through \( \text{Spin}(7) \): see [19]. It is not immediate from this work that we can achieve (5.3.1) and (5.3.2). Indeed, the analogue of (5.3.2) for the \( \text{Spin}(7) \) structures is very unlikely to be true, as (as with \( SU(4) \) structures) applying \( \gamma_T \) to \( \text{Spin}(7) \) structures yields a four-form that is very unlikely to define a \( \text{Spin}(7) \) structure. We would have to show that there was an exact perturbation that was a \( \text{Spin}(7) \) structure, rather than merely any perturbation as Doi–Yotsutani use (by taking the nearest \( \text{Spin}(7) \) structure pointwise and then perturbing as in [40, Theorem 13.6.1]). Even then, the \( SU(4) \) structure induced from a \( \text{Spin}(7) \) structure is not unique, and further work would be required to get (5.3.2).

For higher \( n \) still, it is not immediately obvious that the complex structure parts of asymptotically cylindrical Calabi–Yau structures can be glued easily (because the set of decomposable complex \( n \)-forms is highly nonlinear).

Hence, it seems unlikely that Hypothesis 5.3.1 (in particular (5.3.2)) holds in the case of \( n > 3 \). In Proposition 5.3.5 below, we will obtain the consequences of Hypothesis 5.3.1 for our approximately glued submanifold; we will then discuss what weaker conditions than Hypothesis 5.3.1 might yield these consequences.

In Joyce’s work on the desingularisation of cones (e.g. [44]), (5.3.2) is not difficult to obtain. The gluing is carried out in such a way that \( \omega|_L = 0 \). For \([\Im \Omega|_L] \), we observe that \( H^n(L) \) is one-dimensional so it suffices to consider the pairing with the homology class of \([L]\): but the desingularised submanifold \( L \) is homologous to the original special Lagrangian and so this pairing is zero.

Hypothesis 5.3.1 has the following consequence, of which we give details as another application of the tripartite splitting argument of Proposition 5.1.17.

**Proposition 5.3.2.** Let \( M_1 \) and \( M_2 \) be a matching pair of asymptotically cylindrical Calabi–Yau manifolds. Let \( g^T \) be the gluing of the asymptotically cylindrical metrics \( g_1 \) and \( g_2 \) on \( M^T \) and \( g(\Omega^T, \omega^T) \) be the metric given by the Calabi–Yau structures in Hypothesis 5.3.1. Then there exist constants \( C_k \) and \( \epsilon \) such that

\[
\|g^T - g(\Omega^T, \omega^T)\|_{C^k} \leq C_k e^{-\epsilon T},
\]

where these norms are taken with either metric.

**Proof.** For each \( T \) we divide \( M^T \) into the three parts \( M_1^0(T^{-1}), M_2^0(T^{-1}), \) and \((-2, 2) \times N \) as
Hypothesis 5.3.1 holds. Then with respect to the metric induced from $(\Omega_1, \omega_2)$ agrees with $\Omega_1$ and $\gamma_T(\omega_1, \omega_2)$ agrees with $\omega_1$, so that by Hypothesis 5.3.1, $(\Omega^T, \omega^T) - (\Omega_1, \omega_1)$ is exponentially small in $T$ in the sense of (5.3.1). Now by Proposition 4.1.5 the map $(\Omega_p, \omega_p) \mapsto g_p$ is a smooth bundle map. As in Proposition 2.2.5, it induces smooth maps between jet bundles and hence $g(\Omega^T, \omega^T) - g(\Omega_1, \omega_1) = g(\Omega^T, \omega^T) - g_1$ must also be exponentially small in $T$, that is that on $M_1^{tr(T-1)}$ we can find $C_k$ and $\epsilon$ so that (5.3.3) holds. $M_2^{tr(T-1)}$ is identical.

For $(-2, 2) \times N$, since $(\Omega_1, \omega_1)$ decay exponentially to the cylindrical Calabi–Yau structure $(\hat{\Omega}, \hat{\omega})$ we know that $\gamma_T(\Omega_1, \Omega_2)$ and $\gamma_T(\omega_1, \omega_2)$ are exponentially close in $T$ to $\hat{\Omega}$ and $\hat{\omega}$ respectively. Hence, by Hypothesis 5.3.1 we know that $(\Omega^T, \omega^T) - (\hat{\Omega}, \hat{\omega})$ is exponentially small in $T$. As in the previous paragraph, we infer that $g(\Omega^T, \omega^T) - g(\hat{\Omega}, \hat{\omega})$ is exponentially small in $T$. On the other hand since $g_1$ and $g_2$ decay exponentially to the cylindrical metric $\tilde{g} = g(\hat{\Omega}, \hat{\omega})$, we also have that $g^T$ is exponentially close to $\tilde{g}$, ie that there is some other $C_k$ and $\epsilon$ so that (5.3.3) holds on $(-2, 2) \times N$.

Taking the larger of the two $C_k$’s for each $k$ and the smaller of the two $\epsilon$s we obtain (5.3.3) throughout $M^T$. ☐

Combining Proposition 5.3.2 with Proposition 5.1.16, we obtain

**Corollary 5.3.3.** Let $M_1$ and $M_2$ be a matching pair of asymptotically cylindrical Calabi–Yau manifolds, and let $L_1$ and $L_2$ be a matching pair of asymptotically cylindrical submanifolds. Let $M^T$ and $L^T$ be the glued manifold and submanifold as in Definition 5.1.14 and suppose Hypothesis 5.3.1 holds. Then we have two metrics on $L^T$. Firstly, we have a metric obtained by direct gluing of the metrics on $L_1$ and $L_2$ as in Definition 3.5.13. Secondly, we have the metric given on $L^T$ by restricting the metric on $M^T$ induced by $(\Omega^T, \omega^T)$. The difference of these two metrics decays exponentially in $T$ to zero with all derivatives, with respect to either of them.

Similarly, combining Proposition 5.3.2 with Proposition 5.1.17, we obtain

**Corollary 5.3.4.** Let $M_1$ and $M_2$ be a matching pair of asymptotically cylindrical Calabi–Yau manifolds, and let $L_1$ and $L_2$ be a matching pair of asymptotically cylindrical submanifolds. Let $M^T$ and $L^T$ be the glued manifold and submanifold as in Definition 5.1.14 and suppose Hypothesis 5.3.1 holds. Then with respect to the metric induced from $(\Omega^T, \omega^T)$, the second fundamental form of $L^T$ in $M^T$ is bounded in $C_k$ for all $k$ uniformly in $T$. In particular, the restriction maps of $C_k$ $p$-forms are bounded independently of $T$.

We now proceed to show that if $L_1$ and $L_2$ are a matching pair of asymptotically cylindrical special Lagrangians and Hypothesis 5.3.1 holds, then $L^T$ is nearly special Lagrangian in $M^T$.

**Proposition 5.3.5.** Suppose that $(M_1, M_2)$ is a pair of matching asymptotically cylindrical Calabi–Yau manifolds, and $L_1$ and $L_2$ are asymptotically cylindrical special Lagrangian submanifolds matching in the sense of Definition 5.1.13. Let $L^T$ be the glued submanifold constructed by Definition 5.1.14 and suppose that Hypothesis 5.3.1 holds. Then

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i) \( [\text{Im } \Omega^T|_{LT}] = 0 = [\omega^T|_{LT}] \),

ii) There exist constants \( C_k \) and \( \epsilon \) such that

\[
\| \text{Im } \Omega^T|_{LT} \|_{C^k} + \| \omega^T|_{LT} \|_{C^k} \leq C_k e^{-\epsilon T}.
\] (5.3.4)

**Proof.** For i), by Hypothesis 5.3.1 and the cohomology part of Lemma 5.1.15, we have

\[
[\text{Im } \Omega^T|_{LT}] = [\gamma_T(\text{Im } \Omega_1, \text{Im } \Omega_2)|_{LT}] = [\gamma_T(\text{Im } \Omega_1|_{L_1}, \text{Im } \Omega_2|_{L_2})] = 0
\] (5.3.5)

and similarly for \( \omega \).

For ii), we note by the norm part of Lemma 5.1.15 and the fact that \( L_1 \) and \( L_2 \) are special Lagrangian an estimate of the form (5.3.4) holds for \( \| \gamma_T(\text{Im } \Omega_1, \text{Im } \Omega_2)|_{LT} \|_{C^k} \) and \( \| (\omega^T - \gamma(\omega_1, \omega_2))|_{LT} \| \). This follows from the norm part (5.3.1) of Hypothesis 5.3.1 and Corollary 5.3.4.

We now return to the question of what we may expect when \( n > 3 \). We certainly need to suppose that our Calabi–Yau structures can be glued, and it seems likely that a perturbative argument ought to work. Hence, suppose that we have Calabi–Yau structures \((\Omega^T, \omega^T)\) and that (5.3.4) holds; the question is how we may obtain something resembling \([\text{Im } \Omega^T|_{LT}] = 0 = [\omega^T|_{LT}]\).

If we have a Kähler class \([\omega^T_0]\) whose restriction to \( L^T \) is zero, then, rescaling \( \Omega^T \) if necessary, by the Calabi conjecture we may find \( \omega^T_1 \) in this class such that \((\Omega^T, \omega^T_1)\) is a Calabi–Yau structure and \([\omega^T_1|_{LT}] = 0\).

In particular, if the holonomy of \((\Omega^T, \omega^T)\) is exactly \( SU(n) \), then we know that there are no parallel \((2, 0) + (0, 2)\) forms and hence by a Bochner argument no harmonic such forms; see, for instance, [40, p.125]. With \( J \) the complex structure corresponding to \((\Omega^T, \omega^T)\) we thus know by Hodge theory that \( J\gamma_T(\omega_1, \omega_2) \) lies in the same cohomology class as \( \gamma_T(\omega_1, \omega_2) \), since they have the same harmonic representative (note that \( J \) commutes with exterior differentiation as it is parallel). Then \( \omega^T_0 = \frac{1}{2}(J\gamma_T(\omega_1, \omega_2) + \gamma_T(\omega_1, \omega_2)) \) is a closed \((1, 1)\) form close to \( \gamma_T(\omega_1, \omega_2) \), and in particular defines a Kähler metric. \( \omega^T_0 \) is in the cohomology class \([\gamma_T(\omega_1, \omega_2)]\), and so restricts to an exact form on \( L^T \) just as before. We can then find a Ricci-flat metric in this Kähler class as above.

We always have holonomy \( SU(n) \) if \( n \) is odd and \( M^T \) is simply connected; see [40, Proposition 6.2.3]. If \( n \) is even, nevertheless the asymptotically cylindrical Calabi–Yau manifolds \( M_1 \) and \( M_2 \) have holonomy exactly \( SU(n) \) if they are simply connected and irreducible (see [32, Theorem B]). It would remain to prove that the holonomy will not reduce during this gluing. This is natural, and might be achievable via an argument related to section 3.5. We want to
show the non-existence of a harmonic $(2, 0) + (0, 2)$ form, and we know that no such forms exist on the asymptotically cylindrical components, so this comes down to asking how the gluing of harmonic forms from that chapter interacts with bidegree. Alternatively, we could attempt to formalise the argument that gluing should increase the set of “interesting” loops for holonomy purposes and so should not reduce the holonomy group.

It remains to deal with the holomorphic volume form; we suppose $\omega^T_0$ has been found as above in general. Now, $[\text{Re } \Omega^T]|_{LT}$ and $[\text{Im } \Omega^T]|_{LT}$ lie in the one-dimensional space $H^n(L^T)$. Hence, there exists $\alpha$ such that $[\text{Im}(e^{i\alpha} \Omega^T)]|_{LT}$ is zero. Thus $(e^{i\alpha} \Omega^T, \omega^T_1)$ is a Calabi-Yau structure for which Proposition 5.3.5 (i) holds.

It remains to verify (ii) for this structure, that is it remains to show that $\omega^T_1$ and $\alpha$ are close to $\omega^T$ and $0$ respectively. The former requires us to know how we’ve chosen the Kähler class, and so we cannot deal with here in general. In the specific case of a manifold with holonomy $SU(n)$ above, we note that $\omega^T_0$ is close to $\omega^T$, with small Ricci form. In particular, this implies that the rescaling of $\Omega^T$ required is small. We wish to know that the Ricci-flat Kähler form $\omega^T_1$ is close to $\omega^T_0$. This follows by the apriori estimates for the Calabi conjecture; for instance, see [40, Theorems C1 and C2, p. 101]. As for $\alpha$, it suffices to observe that $\text{Re } \Omega^T|_{LT}$ is close to $\text{vol}_{LT}$ and $\text{Im } \Omega^T|_{LT}$ is close to zero. The latter follows just as in Proposition 5.3.5, and the former can be proved by similar methods, using the similarity of the metrics to ensure similarity of the volume forms.

In particular, assuming the details above can be filled in, the argument we give should extend to a gluing map of minimal Lagrangian submanifolds (that is special Lagrangians with any phase, where the phase may change during the gluing) provided that the holonomy of $M^T$ is exactly $SU(n)$. Alternatively, we can glue special Lagrangians provided our $\omega^T_1$ can be chosen in general: but it may be necessary to choose this form depending on the submanifolds we are interested in.

### 5.4 The SLing map: existence and well-definition

Suppose that $M_1$ and $M_2$ are a pair of matching asymptotically cylindrical Calabi–Yau manifolds, and $L_1$ and $L_2$ are a pair of matching asymptotically cylindrical special Lagrangian submanifolds and that Hypothesis 5.3.1 applies. Then we know from Proposition 5.3.5 that the submanifold $L^T$ constructed by gluing $L_1$ and $L_2$ as in Definition 5.1.14 is close to special Lagrangian. We now have to show that $L^T$ can be perturbed into a special Lagrangian submanifold. More generally, we seek to show that any such nearly special Lagrangian submanifold $L$ can be perturbed into a special Lagrangian submanifold, and to define a uniform “SLing map” giving this perturbation. This is carried out in this section. We first state Condition 5.4.1 on a submanifold $L$ of a Calabi–Yau manifold $M$ and show that an inverse function (or contraction mapping) argument means that whenever Condition 5.4.1 applies then $L$ can be perturbed
into a special Lagrangian. One of the most important conditions is then dealt with in Lemma 5.4.3. This will be absolutely essential for all our later work, and extends the isomorphism between one-forms and normal vector fields used by McLean to the nearly special Lagrangian case. We then show that the “remainder term” can be sensibly bounded, in Proposition 5.4.9, and complete the proof that $L^T$ satisfies Condition 5.4.1 and so can be perturbed into a special Lagrangian in Theorem 5.4.10. We then give a definition (Definition 5.4.11) of an “SLing map” applicable whenever Condition 5.4.1 holds. In Theorem 5.4.17, we show that Condition 5.4.1 holds whenever $\text{Im } \Omega|_L$ and $\omega|_L$ are sufficiently small and exact, with sufficient smallness depending on $L$, so that any nearly special Lagrangian submanifold can be perturbed to a special Lagrangian.

### 5.4.1 General setup

In this first subsection, we describe the inverse function (or contraction mapping) argument that we will use and identify the hypotheses required as Condition 5.4.1. We shall then prove our first results towards obtaining them, Lemma 5.4.3 and Proposition 5.4.4. Lemma 5.4.3 says firstly that if $L^T$ is nearly special Lagrangian then the map $v \mapsto \iota_v \omega|_L$ gives an isomorphism between normal vector fields and one-forms, generalising the special Lagrangian case, and secondly gives (not particularly explicit) bounds on the $C^{k,\mu}$ norms of this in both directions. Proposition 5.4.4 says that the linearisation of the map to which we apply the inverse function theorem does not change very much if we change the $SU(n)$ structure by a small amount.

For the inverse function argument, we rely on essentially the same idea as in the deformation theory of subsection 5.2.3. That is, we consider the nonlinear map of (5.2.9)

$$F : \{\text{normal vector fields on } L^T\} \rightarrow \{n\text{-forms and } 2\text{-forms on } L\},$$

$$v \mapsto ((\exp^*_v \text{ Im } \Omega^T)|_{L^T}, (\exp^*_v \omega^T)|_{L^T}). \tag{5.4.1}$$

Since $(\Omega^T, \omega^T)$ is a torsion-free Calabi–Yau structure, this linearises as in subsection 5.2.3 using Cartan’s magic formula to give

$$D_0F : v \mapsto (d(\iota_v \text{ Im } \Omega^T)|_{L^T}, d(\iota_v \omega^T)|_{L^T}). \tag{5.4.2}$$

However, since $L^T$ is not special Lagrangian, we do not have that $\iota_v \omega^T$ has no normal component, and we certainly do not have the equality $\iota_v \text{ Im } \Omega^T|_{L^T} = * (\iota_v \omega^T|_{L^T})$ of [60, equation (3.7)].

If we suppose that these do hold, and further that the inverse of the linearisation $D_0F$ can be controlled appropriately, then an inverse function argument would prove that if $\text{Im } \Omega^T|_{L^T}$ and $\omega^T|_{L^T}$ are sufficiently small then $L^T$ can be perturbed to special Lagrangian. This is moreover the case if these equalities only nearly hold (indeed, since the second is only used for bounding
the linearisation we not need to assume it separately), and thus we formalise this as

**Condition 5.4.1** (SLing conditions). Let $M$ be a Calabi–Yau manifold. The closed submanifold $L$ satisfies the SLing conditions if for some $k$ and $\mu$

i) The map $u \mapsto t_u\omega|_L$ is an isomorphism between $C^{k+1,\mu}$ normal vector fields to $L$ and $C^{k+1,\mu}$ 1-forms on $L$ and there exists $C_1 \geq 1$ such that

$$\frac{1}{C_1} \|u\|_{C^{k+1,\mu}} \leq \|t_u\omega|_{L^T}\| \leq C_1 \|u\|_{C^{k+1,\mu}}. \quad (5.4.3)$$

ii) The linear map $u \mapsto (dt_u\omega|_L, dt_u \Im \Omega|_L)$ is an isomorphism from $C^{k+1,\mu}$ normal vector fields $u$ such that $t_u\omega|_L$ is $L^2$-orthogonal to harmonic forms (these normal vector fields will later be called orthoharmonic: see the discussion after Definition 5.6.1) onto exact $C^{k,\mu}$ 2- and $n$-forms, and there exists $C_2 \geq 1$ such that for any such $u$,

$$\|u\|_{C^{k+1,\mu}} \leq C_2 (\|dt_u\omega|_L\|_{C^{k,\mu}} + \|dt_u \Im \Omega|_L\|_{C^{k,\mu}}). \quad (5.4.4)$$

iii) There exists $r > 0$ such that whenever $\|u\|_{C^{k+1,\mu}} < r$ and $\|v\|_{C^{k+1,\mu}} < r$, the remainder term satisfies the following bound for some constant $C_3$ depending on $r$

$$\| (\exp_u^* \Im \Omega|_L - \exp_v^* \Im \Omega|_L - dt_u \Im \Omega|_L + dt_v \Im \Omega|_L, \exp_u^* \omega|_L - \exp_v^* \omega|_L - dt_u \omega|_L + dt_v \omega|_L)\|_{C^{k,\mu}} \leq C_3 \|u - v\|_{C^{k+1,\mu}} (\|u\|_{C^{k+1,\mu}} + \|v\|_{C^{k+1,\mu}}). \quad (5.4.5)$$

iv) $\| (\Im \Omega|_L, \omega|_L)\|_{C^{k,\mu}} \leq \min\{\frac{1}{8C_1C_2C_3}, \frac{r}{2C_1C_2}\}$, and we have the cohomology conditions $[\Im \Omega|_L] = 0$ and $[\omega|_L] = 0$.

If $L$ satisfies Condition 5.4.1, we can perturb it.

**Proposition 5.4.2.** Suppose that $L$ is a closed submanifold of the Calabi–Yau manifold $M$ and that $L$ satisfies Condition 5.4.1. Then there exists a normal vector field $v$ to $L$ such that $\exp_v(L)$ is special Lagrangian. We have $\|v\|_{C^{k+1,\mu}} \leq 2C_1C_2 (\|\Im \Omega|_L, \omega|_L\|_{C^{k,\mu}}$; note that by Condition 5.4.1 (iv), this is always less than $\min\{\frac{1}{4C_1C_2}, r\}$.

**Proof.** We apply the implicit function theorem, carefully following its proof to get this quantified statement. Define a map $F$ from $C^{k+1,\mu}$ normal vector fields on $L$ to the direct sum

$$\{\text{exact } C^{k,\mu} \ n\text{-forms}\} \oplus \{\text{exact } C^{k,\mu} \ 2\text{-forms}\} \oplus H^1(L) \quad (5.4.6)$$

by

$$F : v \mapsto (\Im \Omega|_{\exp_v(L)}, \omega|_{\exp_v(L)}, [\hpt t_v\omega|_L]), \quad (5.4.7)$$

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where hpt represents the harmonic part of a form (using Theorem 3.3.1). Note that Condition 5.4.1 (iv) implies that $\text{Im } \Omega|_{\exp_v(L)}$ and $\omega|_{\exp_v(L)}$ are indeed exact. The linearisation of (5.4.7) at 0 is

$$D_0F : v' \mapsto (d(t_{v'} \text{Im } \Omega|_L), d(t_{v'} \omega|_L), [\text{hpt } t_{v'} \omega|_L]),$$

(5.4.8)
since hpt is clearly linear.

By Condition 5.4.1(i), the space of $C^{k+1,\mu}$ normal vector fields is isomorphic to the space of $C^{k+1,\mu}$ one-forms. Hence, we may decompose it as the direct sum $\mathcal{H} \oplus O$ where $\mathcal{H}$ is the space of normal vector fields corresponding to harmonic one-forms and $O$ is the space of normal vector fields corresponding to one-forms orthogonal to harmonics; again, compare the discussion after Definition 5.6.1, where these are simply described as harmonic and orthoharmonic normal vector fields. We have

$$D_0F|_{\mathcal{H}} : v' \mapsto (0, 0, [t_{v'}\omega]).$$

(5.4.9)

This is evidently an isomorphism, and by Condition 5.4.1(i), its inverse is bounded by $C_1$. On the other hand,

$$D_0F|_{O} : v' \mapsto (d(t_{v'} \text{Im } \Omega|_L), d(t_{v'} \omega|_L), 0).$$

(5.4.10)

Condition 5.4.1(ii) says precisely that this is an isomorphism with inverse bounded by $C_2$. Since $C_1$ and $C_2$ are both at least one, and so both less than or equal to $C_1C_2$, applying the triangle inequality it follows that $D_0F$ is an isomorphism with inverse bounded by $C_1C_2$.

We now proceed to the contraction argument required for the proof of the inverse function theorem. We shall look for $F^{-1}(0, 0, 0)$, which is certainly special Lagrangian if it exists. We note that

$$F(v) = (\text{Im } \Omega|_L, \omega|_L, 0) + D_0F(v) + R(v),$$

(5.4.11)

where the remainder term $R$, by Condition 5.4.1 (iii) and linearity of the third term of $F$, satisfies

$$\|R(u) - R(v)\|_{C^{k,\mu}} \leq \|u - v\|_{C^{k+1,\mu}}(\|u\|_{C^{k+1,\mu}} + \|v\|_{C^{k+1,\mu}})$$

provided $\|u\|_{C^{k+1,\mu}} < r$ and $\|v\|_{C^{k+1,\mu}} < r$. So we seek a normal vector field $v$ satisfying

$$v = (D_0F)^{-1}((\text{Im } \Omega|_L, \omega|_L, 0) + R(v)).$$

(5.4.12)

That is, we want a fixed point of the map

$$v \mapsto (D_0F)^{-1}((\text{Im } \Omega|_L, \omega|_L, 0) + R(v)).$$

(5.4.13)

We thus need to show that (5.4.13) is a contraction. By the estimates on $R$ and $(D_0F)^{-1}$ found
above, we know that

\[(D_0F)^{-1}((\text{Im }\Omega|_L,\omega|_L,0) + R(v)) - (D_0F)^{-1}((\text{Im }\Omega|_L,\omega|_L,0) + R(u))\|_{C^{k+1,\mu}}
=\|(D_0F)^{-1}(R(u) - R(v))\|_{C^{k+1,\mu}}
\leq C_1C_2C_3\|u - v\|_{C^{k+1,\mu}}\] (5.4.14)

provided that \(\|u\|_{C^{k+1,\mu}} < r\) and \(\|v\|_{C^{k+1,\mu}} < r\). In particular (5.4.13) is a contraction with Lipschitz constant \(\frac{1}{2}\) on the set \(\{\|v\|_{C^{k+1,\mu}} < \min\{\frac{1}{4C_1C_2C_3}, r\}\}\). We iterate (5.4.13) starting at \(v_0 = 0\), and show inductively that Condition 5.4.1 (iv) implies that \(\|v_n\| < \min\{\frac{1}{4C_1C_2C_3}, r\}\) for all \(n\). In particular, we shall show

\[\|v_n\|_{C^{k+1,\mu}} \leq 2(1 - 2^{-n})\|v_1\| \leq (1 - 2^{-n}) \min\left\{\frac{1}{4C_1C_2C_3}, r\right\}.\] (5.4.15)

Certainly \(v_1 = (D_0F)^{-1}(\text{Im }\Omega|_L,\omega|_L,0)\), and Condition 5.4.1 (iv) implies that this is bounded by \(\min\{\frac{1}{4C_1C_2C_3}, \frac{1}{2}\}\), as required. Given (5.4.15) for \(n\), it is easy to deduce (5.4.15) for \(n + 1\) and hence, by induction, (5.4.15) holds for all \(n\). Moreover, \(v_n\) converges in the Banach space of \(C^{k+1,\mu}\) normal vector fields to a normal vector field \(v\) with \(F(v) = 0\). That is, in particular, \(\exp_v(L)\) is special Lagrangian.

To complete the proof, we take the limit in (5.4.15) to get

\[\|v\|_{C^{k+1,\mu}} \leq 2\|v_1\| \leq 2C_1C_2\|\text{Im }\Omega|_L,\omega|_L\|.\] (5.4.16)

It therefore suffices to show that if \(M_1\) and \(M_2\) are a matching pair of Calabi–Yau manifolds and \(L_1\) and \(L_2\) a matching pair of special Lagrangian submanifolds and Hypothesis 5.3.1 holds, then the submanifold \(L^T\) of \(M^T\) constructed as in Definition 5.1.14 satisfies Condition 5.4.1.

Parts (iii) and (iv) of Condition 5.4.1 are relatively straightforward; we need to discuss parts (i) and (ii). We begin with (i). We know from McLean [60, Remark 3-4] that \(v \mapsto \iota_v\omega|_L\) defines an isomorphism between normal vectors to a special Lagrangian submanifold, and cotangent vectors to this submanifold. Moreover, this map is an isometry, so that if \(L\) is special Lagrangian (i) holds with \(C_1 = 1\). We extend this to the case where \((\Omega, \omega)\) is only defined locally and \(L\) is close to special Lagrangian.

**Lemma 5.4.3.** Suppose that \((\Omega, \omega)\) is an \(SU(n)\) structure around \(L\) (as in Definition 4.1.7) and \(p\) is a point of \(L\) such that \(|(\omega|_L)_p| < 1\) (i.e., \(|\omega(u, v)| < |u||v|\) for all \(u, v \in T_pL\)). Then the complex structure \(J_p : T_pM \rightarrow T_pM\) does not take any tangent vector to another tangent vector, and at \(p\) the map \(u \mapsto \iota_u\omega|_L\) from normal vectors to tangent covectors is an isomorphism; in particular \(J_p\) takes no normal vector to a normal vector.

Now suppose the uniform norm \(\|\omega|_L\|_{C^0} < 1\), so that the preceding paragraph holds at all points. Then \(v \mapsto \iota_v\omega|_L\) defines a map from smooth normal vector fields on \(L\) to smooth 1-forms
on \( L \). Provided that \( \| \omega \|_{C^{k,\mu}} L \) is sufficiently small compared to the \( C^{k,\mu} \) norm of \( J \) and the \( C^k \) norm of the second fundamental form of \( L \) in \( M \), we have bounds

\[
e c \| v \|_{C^{k,\mu}} L \leq \| \iota_v \omega \|_{C^{k,\mu}} L \leq C \| v \|_{C^{k,\mu}}, \tag{5.4.17}
\]

where the constants \( c \) and \( C \) depend on the \( C^{k,\mu} \) norms of \( \omega \mid L \) and the induced almost complex structure \( J \) and the \( C^k \) norm of the second fundamental form of \( L \) in \( M \).

**Proof.** For the first part, we work at \( p \in L \). Suppose that \( v \in T_p L \subset T_p M \) such that \( Jv \in T_p L \). \( J_p \) is an isometry and so

\[
|v|^2 = g(v, v) = \omega(v, J_p v) < |v||Jv| = |v|^2. \tag{5.4.18}
\]

This is a contradiction, proving the first claim.

It follows that the map from tangent vectors to normal vectors given by taking the normal component of \( J_p v \) is an isomorphism as it is an injective linear map between spaces of the same dimension. That is, given a normal vector \( v \) we can find a tangent vector \( u \) such that \( v \) is the normal component of \( Ju \); i.e. we can find a pair \( u \) and \( u' \) of tangent vectors such that \( v = J_p u + u' \).

To show \( u \mapsto \iota_u \omega \mid L \) is an isomorphism, we show that it too is injective. We note that \( \iota_u \omega \) is \( (J_p u)' \). The musical isomorphism takes the tangent and normal parts of \( T_p M \) to the tangent and normal parts of \( T^*_p M \) respectively, and we know that the restriction map is just taking the normal part. Hence, if \( u \mapsto \iota_u \omega \mid L \) is not injective there exists \( u \) such that \( J_p u \) is normal. As above, we can find tangent vectors \( v \) and \( v' \) such that \( u = J_p v + v' \), and hence \( J_p v = v' - u \). Applying \( J_p \) to this and rearranging,

\[
J_p v' = -v - J_p u. \tag{5.4.19}
\]

Since \( v \) and \( v' \) are tangential, \( u \) and \( J_p u \) are normal and nonzero, and \( J_p \) is an isometry, we obtain

\[
|v| < | -v - J_p u | = |J_p v'| = |v'| < |u + v'| = |J_p v| = |v|, \tag{5.4.20}
\]

which is evidently a contradiction. This completes the first part of the proof.

For the second part, we shall work with the norms on sections of \( TM \mid L \) induced by the ambient connection; the same arguments as in Remark 5.1.10 (analogous to Theorem 5.1.3) say that the restrictions of these norms to normal vector fields are Lipschitz equivalent to the restricted norms of Definition 5.1.2, with Lipschitz constant depending on the \( C^k \) norm of the second fundamental form.

As in subsection 5.1.2, we introduce the notations \( \pi^1_1 \) and \( \pi^1_0 \) for the normal and tangent parts of vector fields and one-forms; by construction these maps commute with the musical isomorphism (compare Proposition 5.1.7). Since the musical isomorphism is a \( C^k \) isometry for
every \( k \) (when we use the ambient connection), to estimate \( \|\iota_u \omega\|_{L^{k,\mu}} \) it suffices to estimate \( \|\pi_1^1 J u\|_{C^{k,\mu}} \). Again, the proof of Theorem 5.1.3 shows that

\[
\|\pi_1^1 \alpha\|_{C^{k,\mu}} \leq C' \|\alpha\|_{C^{k,\mu}},
\]  

(5.4.21)

where \( C' \) depends on the \( C^k \) norm of the second fundamental form. The upper bound in (5.4.17) follows immediately; it remains to show the lower bound.

To do this, we use a similar idea to the first part. Given a normal vector field \( v \), we can find local tangent vector fields \( u \) and \( u' \) such that

\[
v = Ju + u'.
\]  

(5.4.22)

Now we note that \( u' \) is small: if \( w \) is also a local tangent vector field, taking the inner product of \( w \) with (5.4.22) yields

\[
0 = \omega(u, w) + g(u', w).
\]  

(5.4.23)

Since \( u \) and \( w \) are both tangential, \( \|\omega(u, w)\|_{C^{k,\mu}} \) can be bounded by \( \|\omega\|_L \|u\|_{C^{k,\mu}} \|w\|_{C^{k,\mu}} \). Hence, since the \( C^{k,\mu} \) norm of a covector field can be identified as its operator norm as a map from \( C^{k,\mu} \) vector fields to \( C^{k,\mu} \) functions, we have that \( \|u'\|_{C^{k,\mu}} \) can be bounded by \( \|u\|_{C^{k,\mu}} \|\omega\|_L \|w\|_{C^{k,\mu}} \). In particular, we find

\[
\|u\|_{C^{k,\mu}} \leq C_1 \|Ju\|_{C^{k,\mu}} \leq C_1 \|Jv\|_{C^{k,\mu}} + C_1 \|u'\|_{C^{k,\mu}} \leq C_1 \|v\|_{C^{k,\mu}} + C_1 \|u\|_{C^{k,\mu}} \|\omega\|_L \|w\|_{C^{k,\mu}},
\]  

(5.4.24)

where \( C_1 \) depends on the \( C^{k,\mu} \) norm of \( J \), and hence that if \( C_1 \|\omega\|_L \|w\|_{C^{k,\mu}} \) is sufficiently small

\[
\|u\|_{C^{k,\mu}} \leq \frac{C_1}{1 - C_1 \|\omega\|_L \|w\|_{C^{k,\mu}}} \|v\|_{C^{k,\mu}}.
\]  

(5.4.25)

Now we apply \( J \) to (5.4.22), obtaining

\[
Jv = -u + Ju'.
\]  

(5.4.26)

Consequently, the normal part of \( Jv \) is the normal part of \( Ju' \) and we have the estimate

\[
\|\pi_1^1 Jv\|_{C^{k,\mu}} \leq C_2 \|Ju'\|_{C^{k,\mu}} \leq C_3 \|u'\|_{C^{k,\mu}} \\
\leq C_3 \|u\|_{C^{k,\mu}} \|\omega\|_L \|w\|_{C^{k,\mu}} \leq \frac{C_1 C_3 \|\omega\|_L \|w\|_{C^{k,\mu}} \|v\|_{C^{k,\mu}}}{1 - C_1 \|\omega\|_L \|w\|_{C^{k,\mu}}},
\]  

(5.4.27)

where \( C_2 \) depends on the \( C^k \) norm of the second fundamental form and \( C_3 \) depends on the \( C^k \) norm of the second fundamental form and the \( C^{k,\mu} \) norm of \( J \). If \( \frac{C_1 C_3 \|\omega\|_L \|w\|_{C^{k,\mu}}}{1 - C_1 \|\omega\|_L \|w\|_{C^{k,\mu}}} \) is sufficiently
small we then have the desired lower bound
\[
\|\pi_0^*Jv\|_{C^{k,\mu}} \geq \left(1 - \frac{C_1C_3\|\omega|_L\|_{C^{k,\mu}}}{1 - C_1\|\omega|_L\|_{C^{k,\mu}}}\right)\|v\|_{C^{k,\mu}}.
\] (5.4.28) \hfill \Box

We now turn to part (ii) of Condition 5.4.1. To understand this, we will show that \(v \mapsto \iota_v \Im \Omega^T|_L\) and \(v \mapsto *\iota_v \omega^T|_L\) are similar.

We will work locally, and take a different \(SU(n)\) structure \((\Omega', \omega')\) around an open subset \(U\) of \(L^T\), again in the sense of Definition 4.1.7, so that \(U\) is special Lagrangian with respect to \((\Omega', \omega')\), and then we know from [60, equation (3.7)] that \(v \mapsto \iota_v \Im \Omega'|_U = v \mapsto *\iota_v \omega'|_U\).

Then we have to show that \(v \mapsto \iota_v \Im \Omega^T|_U\) is close to \(v \mapsto \iota_v \Im \Omega'|_U\) and similarly for \(v \mapsto *\iota_v \omega^T|_U\). That these follow provided that \((\Omega', \omega')\) is close enough to \((\Omega^T, \omega^T)\) is the content of the following.

**Proposition 5.4.4.** Suppose that \(M\) is a \(2n\)-dimensional manifold, and \(L\) is an \(n\)-dimensional submanifold. Let \(SU_n(M)\) be the bundle of \(SU(n)\) structures over \(M\) from Definition 4.1.3.

Suppose that \((\Omega_1, \omega_1)\) and \((\Omega_2, \omega_2)\) are two sections of \(SU_n(M)|_L\). Assume further that \((\Omega_1, \omega_1)\) is the restriction to \(L\) of a Calabi–Yau structure on \(M\), so that there is a well-defined metric \(g\) on \(M\), giving a well-defined \(C^k\) norm on \(TM|_L\), and suppose that the second fundamental form with respect to \(g\) of \(L\) in \(M\) is bounded in \(C^{k-1}\) by \(R\).

Then there exist \(C\) and \(\delta_0\) depending on \(R\) such that if
\[
\|\Omega_1 - \Omega_2\|_{C^k} + \|\omega_1 - \omega_2\|_{C^k} < \delta < \delta_0,
\] (5.4.29)
then
\[
\|\left((u \mapsto *\iota_u \omega_1|_L) - (u \mapsto *\iota_u \omega_2|_L)\right)\|_{C^k} + \|\left((u \mapsto \iota_u \Im \Omega_1|_L) - (u \mapsto \iota_u \Im \Omega_2|_L)\right)\|_{C^k} < C\delta,
\] (5.4.30)
where these norms are the induced norms on the bundle \(\nu^*_L \otimes \wedge^{n-1} T^* L\).

**Proof.** Since the map \((\Omega, \omega) \mapsto g\), contraction, and the map \(g \mapsto *g\) are smooth bundle maps,
\[
(\Omega, \omega) \mapsto (u \mapsto *\iota_u \omega|_L - u \mapsto \iota_u \Im \Omega|_L)
\] (5.4.31)
is a smooth bundle map. We may then apply Proposition 2.2.5 using the ambient Levi-Civita connection. Since \((\Omega_1, \omega_1)\) is parallel and \(\delta_0\) is small, this proposition applies provided \((\Omega_1, \omega_1)\) and \((\Omega_2, \omega_2)\) lie in a compact subset of \(SU_n(M)\). By trivialising so that \((\Omega_1, \omega_1)\) is the standard structure, we may assume this locally. It follows just as in Proposition 2.2.5 by extending to a smooth, hence Lipschitz continuous, map of jet bundles that (5.4.30) holds when the \(C^k\) norms are those induced using the ambient connection.

But by the proof of Theorem 5.1.3 (see Remark 5.1.10), the \(C^k\) norms induced using the ambient connection are Lipschitz equivalent with a constant depending on the \(C^{k-1}\) norm of
the second fundamental form to the \( C^k \) norms induced using the restricted connection, which is bounded by \( R \). (5.4.30) with the \( C^k \) norms induced using the restricted connection follows.

**Remark.** Moreover, the same proof shows that if \((\Omega_1, \omega_1)\) is not parallel but has bounded derivatives the result is still true, but we may have to reduce \( \delta_0 \) further.

### 5.4.2 The gluing theorem

In this subsection we combine Theorem 3.5.14 with a special case of Proposition 5.4.4 to prove Theorem 5.4.10, that the approximate special Lagrangian constructed in section 5.3 can be perturbed to be special Lagrangian. As a preliminary, we will consider the nonlinearity and show that Condition 5.4.1 (iii) holds with a constant independent of \( T \).

The idea of this proof is obvious. The map \( u \mapsto (\exp^* \Omega|_L - d\iota_u \Omega|_L, \exp^* \omega|_L - d\iota_u \omega|_L) \), at any \( p \in L \), only depends on the geometry of \( M \) around \( p \). Unfortunately, it also depends on the derivatives of \( u \). But it is easy to see it only depends on the first derivative and so gives a well-defined map from the first jet space (fibre of the jet bundle) \((J^1(\nu_L))_{p} \) to \( \wedge^n T^*_p L \oplus \wedge^2 T^*_p L \); this pointwise map then only depends on the geometry locally around \( p \). Since the local geometry of the glued Calabi–Yau manifolds \( M_T \) is reasonably independent of \( T \), any bounds we can give on the map should also be independent of \( T \).

However, the submanifold \( L \) and hence \( \nu_L \) also depend on \( T \), and so to prove this we cannot use \( J^1(\nu_L) \). For the same reason we need to be careful with \((\Omega, \omega)\) as well; in this case, we need to know their values at \( p \) and at \( \exp_p(u) \). As we then take the exterior derivative, we also need the first jet at \( p \). However, the final restriction to \( L \) of a form on \( TM|_L \) we already know by Theorem 5.1.3 depends on the second fundamental form, and by Corollary 5.3.4 this is bounded independently of \( T \). Hence, it suffices to consider

**Definition 5.4.5.** Let \( \mathcal{F} \) be the subset of \((J^1 TM \oplus J^1 \wedge^n T^* M \oplus J^1 \wedge^2 T^* M) \times (\wedge^n T^* M \oplus \wedge^2 T^* M)\) consisting of pairs \(((u, \nabla u), (\alpha_0, \nabla \alpha_0), (\beta_0, \nabla \beta_0)), (\alpha, \beta)\) such that

\[
\pi_{\wedge^n T^* M \oplus \wedge^2 T^* M}(\alpha, \beta) = \exp(u).
\]

It is easy to see

**Lemma 5.4.6.** The intersection of \( \mathcal{F} \) with the set of elements with \((u, \nabla u)\) small is a submanifold and is a smooth bundle over \( M \).

**Proof.** For simplicity, put \( B_1 = J^1 TM \oplus J^1 \wedge^n T^* M \oplus J^1 \wedge^2 T^* M \) and \( B_2 = \wedge^n T^* M \oplus \wedge^2 T^* M \), with \( \pi_1 \) and \( \pi_2 \) the corresponding projections. Then \( \mathcal{F} \) is the preimage in \( B_1 \times B_2 \) of the diagonal submanifold of \( M \times M \) under

\[
(b_1, b_2) = (((u, \nabla u), (\alpha_0, \nabla \alpha_0), (\beta_0, \nabla \beta_0)), (\alpha, \beta)) \mapsto (\exp_{\pi_1(b_1)}(u), \pi_2(b_2)).
\]
Since each component of (5.4.33) is certainly a submersion at \((u, \nabla u) = 0\), \(\mathcal{F}\) is a submanifold around these points.

We may now make

**Definition 5.4.7.** Let \(G\) be the bundle map from \(\mathcal{F}\) to \(\bigwedge^n T^*_p M \oplus \bigwedge^2 T^*_p M\) given by

\[
(((u, \nabla u), (\alpha_0, \nabla \alpha_0), (\beta_0, \nabla \beta_0)), (\alpha, \beta)) \mapsto (\exp_p^* \alpha - dt_u \alpha_0, \exp_p^* \beta - dt_u \beta_0).
\]

(5.4.34)

Note that isometrically embedding an incomplete manifold \(M\) into \(M'\) will enlarge \(\mathcal{F}\), as more points \(\exp_u(p)\) will be in \(M'\). The extended map \(G\) will of course depend on the metric on \(M' \setminus M\). This idea will be needed below to deal with behaviour on the neck of a glued manifold.

We may then prove

**Proposition 5.4.8.** The map \(G\) is smooth. Moreover, it depends continuously on the metric on \(M\) in the sense that if \(g_s\) is a finite-dimensional family of metrics, then \(G\) depends continuously on \(s\). Moreover, we have that for every \(\epsilon, \delta_0\) and \(p\) there exists \(\delta\) depending on \(p\), \(\epsilon\) and \(\delta_0\) such that if \(\pi(u) = p\), \(|s - s'| < \delta\), and \(|(u, \nabla u)| < \delta_0\)

\[
|G(s, (((u, \nabla u), (\alpha_0, \nabla \alpha_0), (\beta_0, \nabla \beta_0)), (\alpha, \beta)))
- G(s', (((u, \nabla u), (\alpha_0, \nabla \alpha_0), (\beta_0, \nabla \beta_0)), (\alpha, \beta)))|| < \epsilon|(u, \nabla u)||(\alpha, \beta)|.
\]

(5.4.35)

In particular, if there is a metric \(g_\infty\) such that \(g_s \to g_\infty\) as \(s \to \infty\), then \(G\) converges to the corresponding \(G_\infty\) as \(s \to \infty\), and we have that for every \(\epsilon, \delta_0\) and \(p\) there exists \(K\) such that if \(\pi(u) = p\), \(s > K\) and \(|(u, \nabla u)| < \delta_0\) (5.4.35) holds.

**Proof.** The contraction and exterior derivative terms are obviously smooth, and are independent of the metric. The difficulty is the terms \(\exp_p^* \alpha\) and \(\exp_p^* \beta\). To define these, we imagine \(u\) is defined locally around a point \(p\), and so defines a local diffeomorphism. We shall show below that then the value of \(\exp_p^* \alpha\) at \(p\) only depends on the first jet at \(p\).

We begin by showing that the pushforward map \((\exp_p)_*\) from \(T_p M\) to \(T_{\exp_p(u)} M\) has these properties (viewed as a map on \(J^1(TM) \oplus TM\)); then we may dualise. Given \(((u, \nabla u), v)\) at \(p \in M\), by hypothesis we have a unique geodesic \(\gamma\) with initial velocity \(u\). Construct the Jacobi field \(X\) along \(\gamma\) with \(X_0 = v\) and \((\nabla_{\gamma'} X)_0 = \nabla_v u\). The pushforward \((\exp_p)_* v\) is precisely the final value of this Jacobi field, as we shall now show.

Let \(\gamma_s\) be a curve in \(L\) whose tangent at \(p\) is \(v\). The geodesics with initial velocity \(u_{\gamma(s)}\) form a smooth variation through geodesics and their final positions define the smooth curve \(\exp u \circ \gamma\) in \(\exp_p(L)\). Thus the final value of the Jacobi field \(X\) corresponding to this variation through geodesics is precisely \((\exp_p)_* v\). But \(X_0 = v\) and \((\nabla_{\gamma'} X)_0 = \nabla_v u\), since by commutativity of partial derivatives (or torsion-freeness of the Levi-Civita connection) \((\nabla_{\gamma'} X)_0\) is equivalent to the derivative in \(s\) of the initial velocities.
To prove that pushforward has the desired properties, therefore, we just have to show that the solution to the Jacobi equation concerned does. It suffices to work locally and in coordinates. Locally, we are just solving a system of ordinary differential equations. The smoothness of the map then reduces to the fact that the solution to a system of ordinary differential equations depends smoothly on the initial conditions, which is well-known; see [1, chapter 2, Corollaries 9 and 10]. Similarly, continuity in $s$ for a smooth family of metrics $g_s$ is simply continuity in the finite-dimensional parameter $s$, and again this is well-known. The analogue to (5.4.35) follows by noting that the map giving the derivative of pushforward in $u$ is also continuous in $s$ and that the pushforward is independent of the metric if $(u, \nabla u) = 0$.

We now dualise, to show that the pullback map from the relevant components of $\mathcal{F}$ to $T^*M$ is smooth and has the same continuity estimate; the final result is then immediate.

To do this, choose a smooth local trivialisation $\{e_i\}$ of $TM$ around $p$ and let $\{\epsilon_i\}$ be its dual trivialisation. Then the dual map is given by

$$\alpha \mapsto \sum_i \alpha(P(e_i))\epsilon_i,$$

(5.4.36)

with $P$ being the pushforward. It is easy to see that (5.4.36) is independent of the trivialisation, and so defines a smooth pullback map $\exp^*_p$ which is continuous in the metric. The estimate (5.4.35) follows immediately from the pushforward case.

Remark. Similarly, in [44, Definition 5.5], Joyce prepares for analysis of the nonlinear term by passing to work with finite-dimensional spaces. He goes on to give a more direct analysis of this term than we will; we will only show that we can bound things uniformly in $T$.

We now argue directly using this.

**Proposition 5.4.9.** Let $M_1$ and $M_2$ be a matching pair of asymptotically cylindrical Calabi–Yau manifolds and let $L_1$ and $L_2$ be a matching pair of asymptotically cylindrical special Lagrangian submanifolds. Let $L^T$ be the approximate gluing of $L_1$ and $L_2$ defined in Definition 5.1.14. Suppose Hypothesis 5.3.1 applies, so that $M^T$ is Calabi–Yau with a suitable Calabi–Yau structure $(\Omega^T, \omega^T)$. Then for any $k$ and $\mu$, for $T$ sufficiently large, Condition 5.4.1(iii) holds with constants $C_3$ and $r$ independent of $T$.

**Proof.** We may work locally around each point $p$ of $L^T$, and look for a pointwise estimate,

$$|G(((u, \nabla u), (\Im \Omega_p, \nabla \Im \Omega_p), (\omega_p, \nabla \omega_p)), (\Im \Omega^\text{exp}_p(p), \omega^\text{exp}_p(p)))|
\geq |G(((v, \nabla v), (\Im \Omega_p, \nabla \Im \Omega_p), (\omega_p, \nabla \omega_p)), (\Im \Omega^\text{exp}_p(p), \omega^\text{exp}_p(p)))| + 2C((|u, \nabla u|) + |(v, \nabla v)|)(u, \nabla u - \nabla v)|.$$  

(5.4.37)

As $G$ has zero linearisation around $u = 0$, and all of these maps are smooth in $p$ (including that the sections $\Omega$ and $\omega$ are), some such local constant can be found for each $p$ when we...
restrict to \((u, \nabla u)\) in a ball around zero of radius \(r_p\). By differentiating the smooth map \(G\) a similar estimate holds for the derivatives; this is analogous to Proposition 2.2.5. Since \(M^T\) is compact, we can find \(C\) and \(r_p\) independently of \(p\) for each \(T\); it remains to prove that they are independent of \(T\), as the difference between this and the remainder term we wish to bound is just the restriction to \(L^T\), which we know is uniformly bounded in \(T\).

We appeal to the continuity of \(G\) and hence of these constants in the metric, using (5.4.35) to infer that the derivatives are also continuous. We will note when we do this that \((\Omega, \omega)\) also converge. The most obvious space of parameters to consider is just \(T\) (large enough). Unfortunately, as \(T \to \infty\) the metric on \(M^T\) becomes increasingly singular and so there is no \(g_\infty\) with which we may compare. Hence we need to be slightly more careful, and we work locally.

We begin by choosing \(T'_0 > T_0\), that is large enough that the Calabi–Yau structures can be appropriately glued, and considering \(G\) on \(M^T_1 \subset M^T\). \(M^T_1\) is incomplete, so we further consider it as a subspace of \(M^T_1 \subset M^T\) as indicated above to make sure \(\mathcal{F}\) is large enough; \(G\) is then well-defined on \(M^T_1\) for \((u, \nabla u)\) small enough, uniformly in \(t\). Now by Hypothesis 5.3.1, the Calabi–Yau structure \((\Omega^T, \omega)^{M^T_1}\) converges to \((\Omega_1, \omega_1)^{M^T_1}\) as \(T \to \infty\); hence, by Proposition 4.1.5, the metric converges to the metric \(g_1\) induced by \((\Omega_1, \omega_1)\). \(C\) and \(r_p\) can now be bounded independently of \(T\) on \(M^T_1\), by first finding \(C\) and \(r_p\) for \(g_1\) and using smoothness in the forms and the continuity part of Proposition 5.4.8. Similar arguments apply for \(M^T_2\).

If \(T < T'_0 - \frac{1}{2}\), then \(M^T_1\) and \(M^T_2\) intersect and so we have covered all of \(M^T\). Otherwise, it remains to consider the subset \((-T - 1 + T'_0, T + 1 - T'_0) \times N\) of the neck of \(M^T\). Since we want to work on a fixed manifold, we shall consider \((-1, 1) \times N\), and include this as the subset \((t - 1, t + 1) \times N\) for \(|t| < T - T'_0\). Again, the Calabi–Yau structures \((\Omega^T, \omega)^{(t-1,t+1)\times N}\) define incomplete metrics, and so to make \(\mathcal{F}\) large enough we extend to the corresponding \((t - 2, t + 2) \times N\), and then \(G\) applied to sufficiently small tangent vectors over \((-1, 1) \times N\) depends continuously on the metric on \((-2, 2) \times N\).

Now \((\Omega^T, \omega)^{(t-1,t+1)\times N}\) gives a family of Calabi–Yau structures on \((-2, 2) \times N\). This family can be parametrised by the pair \((t, T)\) in the set \(\{T \geq T'_0 - \frac{1}{2}, |t + 1| < T - T'_0\}\). Clearly, for each \(t\) as \(T\) approaches infinity, we approach the cylindrical metric \(\tilde{g}\) on \((-2, 2) \times N\) and the Calabi–Yau structure approaches the cylindrical Calabi–Yau structure. Thus by choosing \(T'_0\) large enough we may consider the \(g^{(t-1,t+1)\times N}\) as perturbations of \(\tilde{g}\) and the forms as perturbations of the cylindrical forms. Hence, we can find \(C\) and \(r_p\) independent of both \(t\) and \(T\).

This provides the required constants uniformly in \(T\). \(\square\)

Extending Proposition 5.4.9 to more general settings is rather harder, as the above argument essentially uses gluing. See the discussion after our generalisation, Remark 5.4.14. We may now prove
Theorem 5.4.10. Let $M_1$ and $M_2$ be a matching pair of asymptotically cylindrical Calabi–Yau manifolds and let $L_1$ and $L_2$ be a matching pair of asymptotically cylindrical special Lagrangian submanifolds. Let $L^T$ be the approximate gluing of $L_1$ and $L_2$ defined in Definition 5.1.14. Suppose Hypothesis 5.3.1 applies, so that $M^T$ is Calabi–Yau with a suitable Calabi–Yau structure $(\Omega^T, \omega^T)$. Then for any $k$ and $\mu$, for $T$ sufficiently large, Condition 5.4.1 holds for the submanifold $L^T$ of $M^T$ so there is a normal vector field $v$ such that $\exp_v(L^T)$ is a special Lagrangian submanifold for $(\Omega^T, \omega^T)$. $v$ is smooth and decays exponentially with $T$ (in the usual sense as in Remark 4.3.8).

Proof. It follows from Lemma 5.4.3 combined with Proposition 5.3.5 that Condition 5.4.1(i) holds at least for $T$ sufficiently large and that $C_1$ can be taken uniform in $T$; Proposition 5.4.9 says that Condition 5.4.1(iii) holds for $T$ sufficiently large with $C_3$ and $r$ independent of $T$. It remains to show that Condition 5.4.1(ii) and (iv) hold. We shall first show that (ii) holds and that $C_2$ grows at most polynomially in $T$. Proposition 5.3.5 will then precisely give (iv). That is, we will have proved that Condition 5.4.1 holds for $L^T$, so by Proposition 5.4.2 there is such a $v$ perturbing $L^T$ to a special Lagrangian. The norm of $v$ is controlled again by Proposition 5.4.2: since $C_1$, $C_2$ and $C_3$ grow at most polynomially, and Proposition 5.3.5 says that $\| \text{Im} \Omega^T_{LT} \| + \| \omega^T_{LT} \|$ decays exponentially in $T$, this norm decays exponentially in $T$.

To prove Condition 5.4.1(ii), we consider the two linear maps

$$v \mapsto (d\iota_v \omega^T_{LT}, d\iota_v \text{Im} \Omega^T_{LT}) \quad (5.4.38)$$

and

$$v \mapsto (d\iota_v \omega^T_{LT}, d^* \iota_v \omega^T_{LT}). \quad (5.4.39)$$

Combining Corollary 5.3.3 with Theorem 3.5.14 and Condition 5.4.1(i), (5.4.39) is an isomorphism with a lower bound of the form

$$\|u\|_{C^{k+1,\mu}} \leq C T^r (\|d\iota_v \omega^T_{LT}\|_{C^{k,\mu}} + \|d^* \iota_v \omega^T_{LT}\|_{C^{k,\mu}}). \quad (5.4.40)$$

We shall apply Proposition 5.4.4 to show that the difference of (5.4.38) and (5.4.39) is exponentially small in $T$. It will follow by openness of isomorphisms of Banach spaces that (5.4.38) also has a lower bound of the form (5.4.40), which is Condition 5.4.1(ii) with $C_2$ growing at most polynomially in $T$.

(5.4.38) and (5.4.39) are the composites with the exterior derivative of

$$v \mapsto (\iota_v \omega^T_{LT}, \iota_v \text{Im} \Omega^T_{LT}) \quad (5.4.41)$$

and

$$v \mapsto (\iota_v \omega^T_{LT}, * \iota_v \omega^T_{LT}). \quad (5.4.42)$$
To show that (5.4.38) and (5.4.39) are exponentially close as maps from $C^{k+1,\mu}$ normal vector fields to $C^{k,\mu}$ forms, therefore, it suffices to show that (5.4.41) and (5.4.42) are exponentially close as maps from $C^{k+1,\mu}$ normal vector fields to $C^{k+1,\mu}$ forms, and this is an application of Proposition 5.4.4.

Proposition 5.4.4 is essentially a local result. We have to show that around every point $p$ of $L^T$ we can find an open neighbourhood $U \cap L^T$ of $p$ and an $SU(n)$ structure $(\Omega_U, \omega_U)$ around $U \cap L^T$, in the sense of Definition 4.1.7, with respect to which $U \cap L^T$ is special Lagrangian and so that $U \cap \Lambda$ with the constants $A$ by Corollary 5.3.4, we may assume $T$ would have shown that in particular that the difference of these tensors is bounded by in $T$, so that Condition 5.4.1(ii) holds with exponentially decaying, we would obtain that (5.4.41) and (5.4.42) are exponentially close, and $A$ by at most $U$ is why we have to take $3$.

Specifically, let $U_1 = M_1^{T(T-2)} \subset M^T$, $U_2 = M_2^{T(T-2)} \subset M^T$, and $U_3$ be the subset $(-3, 3) \times N$ of the neck.

We shall now find $(\Omega_U, \omega_U)$. In order to do this, we will work with three open subsets $U_1$, $U_2$ and $U_3$ forming an open cover of $M^T$, so that $\{U_i \cap L^T\}$ form an open cover of $L^T$. If $A(U_1, T)$, $A(U_2, T)$ and $A(U_3, T)$ all decay exponentially in $T$, so does $\max\{A(U_i, T)\}$, and so we have an upper bound $A(U, T)$ independent of $U$ and decaying exponentially in $T$.

Specifically, let $U_1 = M_1^{T(T-2)} \subset M^T$, $U_2 = M_2^{T(T-2)} \subset M^T$, and $U_3$ be the subset $(-3, 3) \times N$ of the neck.

We first deal with $U_i$ for $i = 1, 2$. Here, we have by construction that

$$
\gamma_T(\Omega_1, \Omega_2) = \Omega_i, \quad \gamma_T(\omega_1, \omega_2) = \omega_i, \quad (5.4.44)
$$

and for $T$ sufficiently large $L^T \cap U_i = L_i \cap U_i$. As in Proposition 5.1.17, to ensure this is why we have to take $U_i = M_i^{T(T-2)}$. Consequently, $(\Omega_i, \omega_i)$ defines an $SU(n)$ structure around $U_i \cap L^T$ for which $U_i \cap L^T$ is special Lagrangian. By Hypothesis 5.3.1, $(\Omega_i, \omega_i) = (\gamma_T(\Omega_1, \Omega_2), \gamma_T(\omega_1, \omega_2))$ is exponentially close to $(\Omega_T, \omega_T)$ on $U_i \cap L^T$, so we are done.

It remains to find an $SU(n)$ structure around $U_3 \cap L^T$ with the desired properties. We know that $U_3 \cap L^T$ is the intersection with $U_3$ of the image under an exponentially small normal vector
field of \( K \times (T - 4, T + 4) \) with respect to the asymptotically cylindrical metrics. Note that where these metrics are not defined or do not agree, the vector field is zero, so this is a well-defined notion. Since the glued metric is exponentially close to the asymptotically cylindrical metrics, this is still the case for the glued metric. That is, there is some normal vector field \( v_T \), so that \( L^T \cap U_3 = \exp_{v_T} (K \times (T - 4, T + 4)) \cap U_3 \) and \( v_T \) decays exponentially in \( T \).

Note that \( K \times (T - 4, T + 4) \) is special Lagrangian with respect to the cylindrical \( SU(n) \) structure \((\hat{\Omega}, \hat{\omega})\). By construction, we have that \((\gamma_T(\Omega_1, \Omega_2), \gamma_T(\omega_1, \omega_2))\) is exponentially close to \((\hat{\Omega}, \hat{\omega})\) on \( U_3 \) and by Hypothesis 5.3.1 again we consequently have that \((\hat{\Omega}, \hat{\omega})\) is exponentially close to \((\Omega^T, \omega^T)\) on \( U_3 \).

We may extend \( v_T \) to a tubular neighbourhood \( V_1 \) of \( K \times (T - 4, T + 4) \), containing \( N \times (T - 4, T + 4) \cap L^T \), which \( \exp_{v_T} \) maps locally diffeomorphically to a tubular neighbourhood \( V_2 \) of \( L^T \cap U_3 \). We may choose this extension, which we shall also call \( v_T \), to also decay exponentially in \( T \). Let \( F \) be the inverse of the diffeomorphism \( \exp_{v_T} : V_1 \to V_2 \). Let our \( SU(n) \) structure \((\Omega_{U_3}, \omega_{U_3})\) around \( L^T \cap U_3 \) be \( F^*(\hat{\Omega}, \hat{\omega}) \) (which we think of as defined on \( V_1 \) for simplicity). Since \( v_T \) is exponentially small, and \( \hat{\Omega} \) and \( \hat{\omega} \) are bounded with their derivatives independently of \( T \),

\[
(\Omega_{U_3}, \omega_{U_3}) - (\hat{\Omega}, \hat{\omega}) \tag{5.4.45}
\]

is exponentially small, and so too is

\[
(\Omega_{U_3}, \omega_{U_3}) - (\Omega^T, \omega^T) \tag{5.4.46}
\]
on \( V_1 \cap V_2 \). In particular, (5.4.46) is exponentially small on \( L^T \cap U_3 \). However, it is easy to see

\[
F^*((dt + id\theta) \wedge \Omega_{xs}, dt \wedge d\theta + \omega_{xs})|_{\exp_{v_T}(K \times (T - 4, T + 4))} = ((dt + id\theta) \wedge \Omega_{xs}, dt \wedge d\theta + \omega_{xs})|_{(K \times (T - 4, T + 4))} = (\text{vol}, 0). \tag{5.4.47}
\]

That is, \( L^T \cap U_3 \) is special Lagrangian with respect to \((\Omega_{U_3}, \omega_{U_3})\). That is, \((\Omega_{U_3}, \omega_{U_3})\) has the desired properties. This completes the proof of Condition 5.4.1 (ii).

\( v \) is smooth since the special Lagrangian is smooth and it is a normal vector field between smooth submanifolds; see the argument at the beginning of subsection 5.2.3. For the exponential decay of all the \( C^{k,\mu} \) norms at a fixed rate, we note that how large \( T \) needs to be in this argument depends on \( k \) and \( \mu \). However, for all \( k \) and \( \mu \) we have \( \|v\|_{C^{k+1,\mu}} \) decaying exponentially in \( T \) for \( T \) large enough; moreover, with the same rate (which is just the rate of exponential decay in Proposition 5.3.5). Since \( v \) is smooth, it follows that we can extend this to smaller \( T \).

\[\square\]

This is the gluing theorem for special Lagrangians. We note that it follows from the proof of Proposition 5.4.2 that there is more than one possible special Lagrangian perturbation of \( L^T \), and the family of perturbations corresponds to the harmonic normal vector fields on \( L^T \). In par-
ticular, if we vary the harmonic normal vector field we can construct a whole local deformation space to our glued special Lagrangian. We chose the normal vector field we did as this gives the smallest estimate on it.

Remark. We may compare this with the similar analysis in Joyce [44] and Pacini [67]. Both of these construct a submanifold that is close to special Lagrangian and then argue that it may be deformed. However, in both those cases, careful application of the Lagrangian neighbourhood theorem is used to ensure that the initial submanifold corresponding to \( L^T \) is itself Lagrangian. If we defined \( L^T \) with similar care we could presumably obtain that \( \gamma_T(\omega_1, \omega_2)|_{L^T} = 0 \), but as we have had to introduce a perturbation to \( \omega \) to obtain a Calabi–Yau structure, we cannot obtain that \( \omega^T|_{L^T} = 0 \).

If \( L^T \) is Lagrangian, then we may again apply the Lagrangian neighbourhood theorem to infer that the one-forms corresponding to Lagrangian deformations of it under the specialised isomorphism of Condition 5.4.1(i) are closed. The assumption that they are orthoharmonic thus becomes that they are exact, and this rather simplifies the analysis.

In section 5.7, we will analyse a gluing map to show that it defines a local diffeomorphism on deformations of special Lagrangians. This means we need one single gluing map defined on nearly special Lagrangian submanifolds \( L \), and we need to make a uniform choice of this harmonic normal field. Thus we make the following definition

**Definition 5.4.11.** The map \( \text{SLing} \) is defined from submanifolds of \( M^T \) satisfying Condition 5.4.1 to special Lagrangian submanifolds of \( M^T \) by, given a submanifold \( L \), finding a normal vector field \( v \) to \( L \) as in Proposition 5.4.2 and letting \( \text{SLing}(L) = \exp_v(L) \).

Theorem 5.4.10 can then be interpreted as saying that the domain of \( \text{SLing} \) contains all approximate gluings of asymptotically cylindrical special Lagrangians for sufficiently large \( T \).

### 5.4.3 Generalisations

In this subsection, we prove that the domain of \( \text{SLing} \), or equivalently the set of submanifolds \( L \) satisfying Condition 5.4.1, is larger than merely the patchings of section 5.3. We first prove that this is an open subset of submanifolds; then, we prove in Theorem 5.4.17 that any “nearly special Lagrangian” in the sense that \( \omega|_L \) and \( \text{Im} \Omega|_L \) are exact and sufficiently small (unfortunately depending on \( L \)) can also be perturbed to be special Lagrangian.

We begin with the openness result.

**Proposition 5.4.12.** Condition 5.4.1 is open in submanifolds with the \( C^\infty \) topology, so that the domain of \( \text{SLing} \) is an open set.

**Proof.** Let Condition 5.4.1 hold for \( L_0 \) with \( k \) and \( \mu \). We will show that if a normal vector field \( v \) to \( L_0 \) is sufficiently small in \( C^{k+1,\mu} \) then Condition 5.4.1 also holds for \( \exp_v(L_0) \). We
certainly have that the maps
\[ u \mapsto \iota_u \omega \big|_{\exp(v)(L)}, \quad u \mapsto (d\iota_u \omega \big|_{\exp(v)(L)}, d\iota_u \text{Im } \Omega \big|_{\exp(v)(L)}) \tag{5.4.48} \]
from \(C^{k,\mu}\) normal vector fields to \(C^{k,\mu}\) forms depend continuously on \(v\). This is because pull-back by a \(C^{k+1,\mu}\) diffeomorphism is a continuous map of \(C^{k,\mu}\) forms; for instance, this follows from Baier's result [5, Theorem 2.2.15], which says it is a smooth map.

It follows since isomorphisms form an open subset of linear maps between Banach spaces that Condition 5.4.1(i) and (ii) are open conditions. Moreover, the norms of these maps must also be continuous, and since inversion of linear maps is a continuous map on this open subset, the implied constants are continuous.

That (iii) is open follows for the same reason: the map has changed only slightly and still has zero constant and linear part, so there must still be a quadratic bound with the implied constants continuous.

Finally, it is clear that \([\text{Im } \Omega \big|_{\exp(v)(L_0)}] = [\text{Im } \Omega \big|_{L_0}] = 0\) and similarly for \(\omega\). Continuity of the implied constants shows that (iv) is open: the norm \(\|([\text{Im } \Omega \big|_{\exp(v)(L_0)}, \omega \big|_{\exp(v)(L_0)})]\|_{C^{k,\mu}}\) is continuous in \(v\), and so the condition that it is less than \(\min\{\frac{1}{8C_1C_2C_3}, \frac{r}{2C_1C_2}\}\) must be open.

We now turn to a more direct generalisation.

Examining the proof of Theorem 5.4.10 shows that we used that \(L^T\) was the gluing of a pair of asymptotically cylindrical special Lagrangians in three places: firstly, to exhibit nearby local \(SU(n)\) structures with respect to which the submanifold is special Lagrangian (proof of Theorem 5.4.10); secondly, to show that the bound on the restriction map didn’t grow as \(T\) did (Corollary 5.3.4); and thirdly, to provide a lower bound on the Laplacian (Theorem 3.5.14). A lower bound on the Laplacian over a family of manifolds is difficult to obtain in general: however, if we work only with one submanifold \(L\) then the restriction map is automatically bounded and the Laplacian is automatically bounded below. This means that the proof of Theorem 5.4.10 gives the following statement.

**Proposition 5.4.13.** Suppose \(M\) is a \(2n\)-dimensional Calabi–Yau manifold with Calabi–Yau structure \((\Omega, \omega)\) and that \(L\) is a closed submanifold. Let \(k\) be a positive integer, and \(A\) be a positive constant. Suppose that around each point \(p\) of \(L\) there is a local \(SU(n)\) structure \((\Omega'_p, \omega'_p)\) around a neighbourhood of \(p\) in \(L\), in the sense of Definition 4.1.7, with
\[
\|\Omega'_p - \Omega\|_{C^{k+2}} + \|\omega'_p - \omega\|_{C^{k+2}} \leq A, \tag{5.4.49}
\]
and so that a neighbourhood of \(p\) in \(L\) is a special Lagrangian with respect to \((\Omega'_p, \omega'_p)\).

If \(A\) is sufficiently small, then for any \(\mu \in (0, 1)\) Condition 5.4.1 (ii) holds for \(k\) and \(\mu\). We note that (iii) holds automatically with some constants \(C_3\) and \(r\). Hence, if \(\|\text{Im } \Omega \big|_L\|_{C^{k+1}} + \|\omega \big|_L\|_{C^{k+1}}\) is also sufficiently small depending on \(A\), the second fundamental form of \(L\) in \(M\),
the inverse Laplacian bound, \( r \), and \( C_3 \), and also \( |\text{Im} \Omega|_L = 0 = |\omega|_L \) then (iv) holds and \( \text{SLing}(L) \) exists.

**Remark 5.4.14.** When \( k = 0 \) and restricting to the Lagrangian case, Joyce [44, Proposition 5.8] gave a rather more direct estimate of the constant corresponding to \( C_3 \) in (iii). Fundamentally, that argument shows that for any given \( r \), estimating \( C_3 \) rests on estimating the derivatives of the nonlinear map \( v \to (\exp^*_v \text{Im} \Omega|_L, \exp^*_v \omega|_L) \); that is, by the chain rule, linearity of restriction to \( L \), and Theorem 5.1.3, it rests on the derivatives of \( v \to (\exp^*_v \text{Im} \Omega|_L, \exp^*_v \omega|_L) \) and the second fundamental form. Since \( \text{Im} \Omega \) and \( \omega \) are parallel and of fixed size, estimating these derivatives depends entirely on estimating the pullback map \( J^1(\nu_L) \to \bigwedge^n T^*M \oplus \bigwedge^2 T^*M|_L \), and its derivatives; \( r \) controls exactly how large a ball in \( J^1(\nu_L) \) we admit. Based on the Rauch comparison result (Proposition 5.5.1) compared with the identification of pushforward with Jacobi fields in Proposition 5.4.8, it seems plausible that this should only depend on the curvature of \( M \) and its derivatives. In any case, it should be possible to choose \( r \) and then estimate these derivatives independently of \( L \), by extending to work with the corresponding map \( J^1(TM) \to \bigwedge^n T^*M \oplus \bigwedge^2 T^*M \).

This estimate on the derivatives of pullback corresponds roughly to (iii) of [44, Theorem 5.3], which is an estimate on certain adapted derivatives of \( \text{Im} \Omega \) considered as a form on \( T^*L \), under the identification given by the Lagrangian neighbourhood theorem (that is, on the pullback of \( \text{Im} \Omega \) under an appropriate diffeomorphism). In [44, Proposition 5.8], this yields the required derivatives, because the pushforward just reduces to an algebraic map under this identification.

We shall now show that the existence of \((\Omega_p', \omega_p')\) follows from smallness of \( |\text{Im} \Omega|_L \) and \( |\omega|_L \). Combining this with Proposition 5.4.13 leads to Theorem 5.4.17.

We begin with a pointwise construction.

**Lemma 5.4.15.** Let \( V \) be a \( 2n \)-dimensional vector space, and \( L \) an \( n \)-dimensional subspace. For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \((\Omega, \omega)\) is an \( SU(n) \) structure on \( V \) and \( |\text{Im} \Omega|_L| + |\omega|_L| + |\text{Re} \Omega|_L - \text{vol}_L| < \delta \) with respect to the metric induced by \((\Omega, \omega)\), then there is an \( SU(n) \) structure \((\Omega', \omega')\) such that \( L \) is special Lagrangian with respect to \((\Omega', \omega')\) and \(|\Omega' - \Omega| + |\omega' - \omega| < \epsilon\).

**Proof.** We are given \( \epsilon > 0 \). Assume without loss of generality that \((\Omega, \omega)\) is the standard \( SU(n) \) structure \( \Omega = (e_1 + ie_2) \wedge \cdots \wedge (e_{2n-1} + ie_{2n}), \omega = e_1 \wedge e_2 + \cdots + e_{2n-1} \wedge e_{2n} \). Fix \( \delta > 0 \); we shall choose it to be sufficiently small. We will look for \((\Omega', \omega') = (\epsilon \Omega, \omega')\) so that we can avoid the decomposability condition on \( \Omega' \).

We begin by finding \( \omega' \). Let \( e_1, \ldots, e_n \) be an orthonormal basis for \( L \). For \( \delta \) sufficiently small \( e_1, Je_1, \ldots, e_n, Je_n \) is a basis for \( V \), since if we have a dependence relation it may be written as

\[
\sum_{i=1}^{n} a_i Je_i \in \text{span}\{e_1, \ldots, e_n\}. \quad (5.4.50)
\]
This says that \( \sum_{i=1}^{n} a_i Je_i \) lies in \( L \), but applying \( J \) to the left hand side also evidently gives a vector in \( L \), and this is a contradiction to Lemma 5.4.3 if \( \delta \leq 1 \). Let \( \xi_1, -J\xi_1, \ldots, \xi_n, -J\xi_n \) be the dual basis to \( e_1, Je_1, \ldots, e_n, Je_n \). Note that \( J\xi_i \) is indeed the negative of the dual basis vector to \( Je_i \) since

\[
J\xi_i(\xi_j) = -\delta_{ij}, \quad J\xi_i(e_j) = \xi_i(\xi_j) = 0.
\]  

(5.4.51)

We now note that \( \omega|_L = 0 \) if and only if \( \omega(e_i, e_j) = 0 \) for all \( i \) and \( j \), so we set

\[
\omega' = \omega - \sum_{i,j} \omega(e_i, e_j) (\xi_i \wedge \xi_j + J\xi_i \wedge J\xi_j).
\]  

(5.4.52)

We then have

\[
J\omega' = J\omega - \sum_{i,j} \omega(e_i, e_j) (J\xi_i \wedge J\xi_j + (-\xi_i) \wedge (-\xi_j)) = \omega',
\]  

(5.4.53)

and so \( \omega' \) is a \((1,1)\)-form as required, and \( \omega' \wedge \Omega = 0 \). Using (5.4.51), it follows that \( \omega'(e_i, e_j) = 0 \) for all \( i \) and \( j \). That is, \( \Omega \wedge \omega' = 0 \), and \( \omega'|_L = 0 \). We note that

\[
|\omega - \omega'| \leq \sum_{i,j} |\omega(e_i, e_j) (\xi_i \wedge \xi_j + J\xi_i \wedge J\xi_j) |
\leq 2|\omega|_L \sum_{i,j} |e_i||e_j||\xi_i||\xi_j|.
\]  

(5.4.54)

since \( J \) is an isometry. Certainly \( |e_i| = 1 \) by construction. We have

\[
|\xi_i| = 1 / \inf_{e' \in \text{span}\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n\}} |e_i + Je + e'| .
\]  

(5.4.55)

Now using that \( J \) is an isometry,

\[
|e_i + Je + e'|^2 = 1 + |e|^2 + |e'|^2 - \omega(e_i + e', e) \geq 1 + |e|^2 + |e'|^2 - |\omega|_L |e|(1 + |e'|).
\]  

(5.4.56)

Applying elementary calculus to find the minimum of this right hand side, we obtain

\[
|e_i + Je + e'|^2 \geq \frac{16 - 12|\omega|_L^2 + 2|\omega|_L^4}{(4 - |\omega|_L^2)^2}.
\]  

(5.4.57)

Hence,

\[
|\xi_i| < \frac{4 - \delta^2}{\sqrt{16 - 12\delta^2 + 2\delta^4}},
\]  

(5.4.58)

provided \( \delta < \sqrt{2} \).
Thus,
\[ |\omega - \omega'| \leq 2n^2 \frac{\delta(4 - \delta^2)}{\sqrt{16 - 12\delta^2 + 2\delta^4}}. \]  
(5.4.59)

Clearly this is less than \( \epsilon \) if \( \delta \) is sufficiently small.

We must now choose \( c \) so that \((c\Omega, \omega')\) is an \( SU(n) \) structure. Comparing with Definition 4.1.1, the remaining hypotheses are (iii), that the resulting \( g' \) is positive definite, and (iv), the scaling condition
\[ |c|^2 \Omega \wedge \bar{\Omega} = \frac{(-2)^n q^2}{n!} (\omega')^n. \]  
(5.4.60)

Positive definiteness is immediate as we chose \( \omega' \) so that \( e_1, Je_1, \ldots, Je_n \) are orthonormal with respect to \( g' \).

We now find \( |c| \) to ensure (5.4.60). We note that
\[ (\omega')^n - \omega^n = \sum_{i=1}^{n} \binom{n}{i} \omega^{n-i}(\omega' - \omega)^i. \]  
(5.4.61)

Hence, we may make \((\omega')^n - \omega^n\) as small as we like by choosing \( \delta \) small enough. In particular, we may choose \( \delta \) small enough that \(|(\omega')^n - \omega^n| < |\omega^n|\). Then since \( \Lambda^{2n} V^* \) is a one-dimensional vector space, we see that
\[ (\omega')^n = \left(1 \pm \frac{|\omega^n - (\omega')^n|}{|\omega^n|}\right) \omega^n. \]  
(5.4.62)

We can find \( C > 0 \) so that
\[ C^2 = \left(1 \pm \frac{|\omega^n - (\omega')^n|}{|\omega^n|}\right). \]  
(5.4.63)

Thus, if we set \( |c| = C \), then \((c\Omega, \omega')\) is a \( SU(n) \) structure on \( V \).

We now choose the argument of \( c \) so that \( L \) is a special Lagrangian subspace with respect to \((c\Omega, \omega')\). With respect to \((C\Omega, \omega')\) we certainly have that \( L \) is a Lagrangian subspace, so as in [31, Corollary III.1.11], there exists some \( \theta \) such that \( \text{Re}(Ce^{i\theta}\Omega)|_L = \text{vol}|_L \). Explicitly, we know that \( \text{Re} \Omega|_L = a\text{vol}_L \) and \( \text{Im} \Omega|_L = b\text{vol}_L \), with \( a^2 + b^2 = 1 \), so that there is \(-\theta \) such that \( a = \cos \theta \) and \( b = \sin \theta \). It follows that setting \( c = Ce^{i\theta} \) and thus \( \Omega' = Ce^{i\theta}\Omega \) means that the \( SU(n) \) structure \((\Omega', \omega')\) has \( L \) as a special Lagrangian.

To complete the proof, we need to estimate
\[ |\Omega' - \Omega| = |Ce^{i\theta} - 1| (\sqrt{2})^n. \]  
(5.4.64)

That is, we need to show that \(|Ce^{i\theta} - 1| \leq |C - 1| + |e^{i\theta} - 1|\) can be made as small as we like by choosing \( \delta \) sufficiently small.
We begin by using (5.4.63), which implies

$$|1 - C^2| = \frac{|\omega^n - (\omega')^n|}{|\omega^n|}.$$  \hspace{1cm} (5.4.65)\]

By (5.4.61), we know that this can chosen as small as we like by choosing \( \delta \) small enough.

Finally, we note that to estimate \( |e^{i\theta} - 1| \) it suffices by standard trigonometric identities to estimate \( |\cos \theta - 1| \). But \( \cos \theta \text{vol}_L = \text{Re}(C \Omega)|_L \), so

$$|\cos \theta - 1| = |C \text{Re} \Omega|_L - \text{vol}_L| \leq |C| \text{Re} \Omega|_L - \text{vol}_L| + |C - 1| \leq |C|\delta + |C - 1|,$$ \hspace{1cm} (5.4.66)

by hypothesis. But we already know that \( |C - 1| \) can be made as small as we like and so (5.4.66) can be made as small as we like by choosing \( \delta \) small enough, as required. \( \square \)

We now pass to constructing a local \( SU(n) \) structure around \( L \), as in Definition 4.1.7, that is a local section of the bundle \( SU_n(M)|_L \) of Definition 4.1.3. We can in fact do this construction globally on \( L \).

**Proposition 5.4.16.** Let \( M \) be a \( 2n \)-dimensional manifold, and \( L \) an \( n \)-dimensional submanifold. For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that if \( (\Omega, \omega) \) is a torsion-free \( SU(n) \) structure on \( M \) and \( \| \text{Im} \Omega|_L \|_{C^k} + \| \omega|_L \|_{C^k} + \| \text{Re} \Omega|_L - \text{vol}_L \|_{C^k} < \delta \), then there is an \( SU(n) \) structure \( (\Omega', \omega') \) around \( L \) such that \( L \) is special Lagrangian with respect to \( (\Omega', \omega') \) and \( \| \Omega' - \Omega \|_{C^k} + \| \omega' - \omega \|_{C^k} < \epsilon \).

**Proof.** Fix \( \epsilon > 0 \). By Lemma 5.4.15, there exists a \( \delta_0 \) such that at every point \( p \) of \( M \), if \( (\Omega_p, \omega_p) \) is an \( SU(n) \) structure on \( T_pM \) with \( \| \text{Im} \Omega_p(T_pM) \|_L + \| \omega_p(T_pM) \|_L + \| \text{Re} \Omega_p(T_pM) - \text{vol}_{T_pM} \|_L < \delta \) then there exists \( (\Omega'_p, \omega'_p) \) with the desired properties. In particular, if \( \| \text{Im} \Omega_L \|_L + \| \omega_L \|_L + \| \text{Re} \Omega_L - \text{vol}_L \|_L < \delta_0 \), we can find global forms \( \Omega' \) and \( \omega' \), with \( \| \Omega' - \Omega \|_{L^\infty} + \| \omega' - \omega \|_{L^\infty} < \epsilon \) and \( L \) special Lagrangian with respect to \( (\Omega', \omega') \).

We now show that \( \Omega' \) and \( \omega' \) are smooth, and in particular continuous, so that the \( L^\infty \) norms in the previous paragraph are just \( C^0 \) norms. Examining the proof of Lemma 5.4.15 shows that \((\tilde{\Omega}, \tilde{\omega})\) is constructed by a smooth bundle map

$$SU_n(M)|_L \oplus \bigwedge^n T^* L \otimes \mathbb{C} \oplus \bigwedge^2 T^* L \to SU_n(M)|_L,$$ \hspace{1cm} (5.4.67)

so \((\Omega', \omega')\) is certainly smooth. Of course, \((\Omega, \omega, \Omega|_L, \omega|_L)\) is not a general section of the bundle \( SU_n(M)|_L \oplus \bigwedge^n T^* L \otimes \mathbb{C} \oplus \bigwedge^2 T^* L \). We may then use Proposition 2.2.5 as in the proof of Proposition 5.4.4. Because \((\Omega, \omega)\) is assumed to be torsion-free and \( \Omega|_L \) and \( \omega|_L \) are assumed to be small in \( C^k \), we have the required bounded derivatives; by using the standard structure as in Proposition 5.4.4, we may assume we are in a compact subset of the bundle. The argument of Proposition 2.2.5 then proves that by choosing \( \delta \) small so that the \( k \)-jets of \((\Omega, \omega, \Omega|_L, \omega|_L)\)
are very close to the $k$-jets of $(\Omega, \omega, 0, 0)$, the $k$-jets of $(\Omega', \omega')$ and $(\Omega, \omega)$ are within $\epsilon$ of each other.

Consequently, we find the following theorem saying that all “nearly special Lagrangian” submanifolds can be perturbed to special Lagrangian. It is perhaps not the best possible result, since there is a requirement that $\|\Omega|_L\|$ and $\|\omega|_L\|$ be small with respect to a constant that depends in ways we do not describe on $(\Omega, \omega)$ and $L$.

**Theorem 5.4.17.** Suppose $M$ is a $2n$-dimensional Calabi–Yau manifold with Calabi–Yau structure $(\Omega, \omega)$ and that $L$ is a closed submanifold. Suppose that for some $k$, $\|\text{Im} \Omega|_L\|_{C^{k+2}} + \|\omega|_L\|_{C^{k+2}}$ is sufficiently small, in terms of a constant depending on the second fundamental form of $L$ in $M$, a lower bound on $d + d^*$ on $L$ in the sense of Theorem 3.5.14, and the constants $r$ and $C_3$ of Condition 5.4.1(iii) (with appropriate regularities). Suppose further that $[\Omega|_L] = 0$ and $[\omega|_L] = 0$. Then Condition 5.4.1 holds (for any $\mu$) and we can thus perturb $L$ to a special Lagrangian submanifold.

**Proof.** We apply Proposition 5.4.13. Comparing the hypotheses we just have to show that we can find the local $SU(n)$ structures around $L$ $(\Omega'_p, \omega'_p)$ which are close in $C^{k+2}$ to $(\Omega, \omega)$ and have $L$ as a special Lagrangian. Proposition 5.4.16 precisely gives us such structures (indeed, a global structure) provided that $\|\text{Im} \Omega|_L\|_{C^{k+2}} + \|\omega|_L\|_{C^{k+2}}$ is sufficiently small.

**Remark.** As mentioned in Remark 5.4.14, a bound on $C_3$, for any given $r$, only depends on the derivatives of pullback as a map from sufficiently small (depending on $r$) normal vector fields to forms on $TM|_L$ and the second fundamental form. Consequently, we expect that it only depends on the curvature of $M$, its derivatives, and the second fundamental form, and at any rate depends on the second fundamental form and something bounded independently of $L$. Hence Theorem 5.4.17 could be regarded as a rather less precise but somewhat extended version of Joyce’s result [44, Theorem 5.3] which says that Lagrangian submanifolds that are sufficiently close to special Lagrangian can be perturbed to be special Lagrangian.

It may also be possible to prove such an extension by arguing directly that a submanifold that is close to Lagrangian with exact restriction of $\omega$ can be perturbed to be Lagrangian, using some appropriate extension of the Lagrangian neighbourhood theorem. The proof of the Lagrangian neighbourhood theorem for a Lagrangian $L$ begins by constructing a diffeomorphism between $T^*L$ and an open subset of $M$, such that the inclusion of the Lagrangian $L$ corresponds to the inclusion of the zero section and the symplectic structures agree over $L$. Then Moser’s trick is used to adjust the diffeomorphism away from $L$ so that the symplectic structures agree. The construction of this preliminary diffeomorphism relies on the Lagrangian condition, and so would need to be altered non-trivially to extend to our case.
5.5 Identifying normal vector fields on nearby submanifolds

In this section, we study the interactions of normal vector fields and families of submanifolds in a manifold. Often, we will silently assume these are closed, but as our arguments will mostly be local they will readily generalise; in some cases, we will use these on asymptotically cylindrically manifolds. The general question we are mainly interested in can be stated as follows. Suppose that $L_s$ and $L'_s$ are two nearby curves of closed submanifolds. Then there are normal vector fields $w_s$ to each $L_s$ such that $L'_s = \exp_{w_s}(L_s)$, there is a curve of normal vector fields to $L_0$ giving the curve $L_s$, and these is a curve of normal vector fields to $L'_0$ giving the curve $L'_s$. We are interested, for later use, in how these normal vector fields depend on each other. We will mostly show that the curves of vector fields we need exist and are smooth, and identify the derivatives of these maps. (For instance, for the example above see, essentially, Proposition 5.5.16). It is necessary to work directly, rather than just taking a base submanifold and using one tubular neighbourhood and regarding all changes of submanifold and normal vector field in the coordinates given by this tubular neighbourhood, as in Definition 5.4.11 we defined $SL_{\text{In}}$ in terms of normal vector fields on the manifold which we are perturbing to special Lagrangian.

Using these results, we will also define a patching map of asymptotically translation invariant normal vector fields like those in subsections 3.5.1 and 5.1.3 and show (Proposition 5.5.24) that it is close to the derivative of the patching map of submanifolds in Definition 5.1.14. We work in two phases. Firstly, we will work with submanifolds of closed Riemannian manifolds in general. This is because these results could be used for other questions where analysis of finite-dimensional families of submanifolds and maps between them would be useful, and it is clearer to prove the results in general first. Secondly, we shall use Lemma 5.4.3 to translate these questions on normal vector fields into questions on one-forms for nearly special Lagrangian submanifolds of Calabi–Yau manifolds.

5.5.1 General Riemannian manifolds

In this subsection, we work with submanifolds that need not be (close to) special Lagrangian. The main results useful for later work are Proposition 5.5.16, which identifies the derivative of the map from a pair of submanifolds to a normal vector field between them, and Proposition 5.5.9 which is required to show that this map is smooth and will be used later to give analytic controls on this derivative (this result is structurally the same as Proposition 5.4.8 that we needed in the previous section). We then return to the question of asymptotically translation invariant normal vector fields. We give a definition of such fields in Definition 5.5.17 and prove some basic properties of these and how they relate to the identifications we needed to give Proposition 5.5.16. We then give a patching map for normal vector fields (Definition 5.5.23) using the identification maps we needed to give Proposition 5.5.16. Finally, we show that this patching map is close to the derivative of the patching map of asymptotically cylindrically submanifolds in
Proposition 5.5.24.

As at the beginning of subsection 5.2.3, suppose that $L$ is a submanifold of the Riemannian manifold $(M,g)$ and that $L' = \exp_v(L)$ is a small deformation of it. We will need some way of identifying normal vector fields on $L$ with normal vector fields on $L'$. We will define two such identification maps. We begin by noting the following estimate, which follows by combining the Rauch comparison theorem [13, Theorem 1.28] with the corresponding result of Berger [13, Theorem 1.29] using the triangle inequality, and which is of vital importance when transporting vectors using Jacobi fields.

**Proposition 5.5.1** (corollary of “Rauch Comparison Theorem”). Let $(M, g)$ be a Riemannian manifold with sectional curvature bounded below and above by the constants $-c$ and $C$ respectively. Let $\gamma$ be a geodesic in $M$ with initial velocity $v_0$. Let $X$ be a Jacobi field along $\gamma$ with $|X_0| = A$ and $|\nabla_{\frac{\partial}{\partial t}} X_0| = B$. Then for $|v_0| < \frac{\sqrt{c}}{2}$ we have estimates

$$A \cos \left( |v_0| \sqrt{C} \right) - \frac{B}{\sqrt{c}} \sinh \left( |v_0| \sqrt{c} \right) \leq |X_t| \leq A \cosh \left( |v_0| \sqrt{c} \right) + \frac{B}{\sqrt{c}} \sinh \left( |v_0| \sqrt{c} \right). \quad (5.5.1)$$

Moreover, we have the following similar result which we take from Jost [38], applying a rescaling to the geodesic.

**Proposition 5.5.2** ([38, Theorem 4.5.3]). Let $(M, g)$ be a Riemannian manifold with sectional curvature bounded below and above by the constants $-C$ and $C$, for some fixed $C > 0$. Let $\gamma$ be a geodesic in $M$ with initial velocity $v_0$. Let $X$ be a Jacobi field along $\gamma$ with $|X_0| = A$ and $|\nabla_{\frac{\partial}{\partial t}} X_0| = B$. Suppose further that $X_0$ and $|\nabla_{\frac{\partial}{\partial t}} X_0|$ are linearly dependent. Let $P_t$ be parallel transport from $\gamma(0)$ to $\gamma(t)$ along $\gamma$. Then for each $t$,

$$|X_t - P_t(X_0 + t(\nabla_{\frac{\partial}{\partial t}} X_0))| \leq A(\cosh(\sqrt{C}|v_0|t) - 1) + B\left( \frac{1}{\sqrt{C}|v_0|} \sinh(\sqrt{C}|v_0|t) - t \right). \quad (5.5.2)$$

To ensure we have linear dependence, we split $X$ into two Jacobi fields using linearity as for Proposition 5.5.1. We then obtain from the triangle inequality and by calculation

**Corollary 5.5.3.** Let $(M, g)$ be a Riemannian manifold with sectional curvature bounded below and above by the constants $-C$ and $C$, for some fixed $C > 0$. Let $\gamma$ be a geodesic in $M$ with initial velocity $v_0$. Let $X$ be a Jacobi field along $\gamma$ with $|X_0| = A$ and $|\nabla_{\frac{\partial}{\partial t}} X_0| = B$. Let $P_t$ be parallel transport from $\gamma(0)$ to $\gamma(t)$ along $\gamma$. Then for each $t$,

$$|X_t - P_t(X_0 + t(\nabla_{\frac{\partial}{\partial t}} X_0))| \leq A(\cosh(\sqrt{C}|v_0|t) - 1) + B\left( \frac{1}{\sqrt{C}|v_0|} \sinh(\sqrt{C}|v_0|t) - t \right). \quad (5.5.3)$$

We now begin our work by defining the identification maps required pointwise. We will do this with pointwise representatives of $L$ and $L'$; it would be possible to work with any point...
in a fixed $L$, but in due course (for example in Proposition 5.5.10) we will need to consider $L$
 varying.

To do this, recall

**Definition 5.5.4.** Let $M$ be a manifold. Then for each $k$ we have a fibre bundle $\text{Gr}_k(TM)$, the
bundle of $k$-planes in $TM$. For instance, this is the subbundle of unit decomposable $k$-forms.

At a point, we can represent $L$ locally by its tangent space. This is precisely a point of
$\text{Gr}_k(TM)$ where $k$ is the dimension of $L$. To determine $L'$, we use the normal vector field $v$
such that $L' = \exp_v(L)$. The value of $v$ at a point of $L$ just determines a point of $L'$; we will need
the tangent space to $L'$ also. We will use the jet bundles of Proposition 2.2.1; specifically, we
will use the first jet of $v$. This is comparable to the use of the first jet bundle for $u$ in Definition
5.4.7.

As we will mostly be working with $n$-dimensional special Lagrangian submanifolds of
2$n$-dimensional Calabi–Yau manifolds, we will restrict to the case of $n$-dimensional submanifold.
Consequently, we will define our transfer maps on the Whitney sum $TM \oplus J^1(TM) \oplus
\text{Gr}_n(TM)$. Note that the $J^1(TM)$ term will be the 1-jet at a point of a small deformation and
so can be expected to be small.

**Definition 5.5.5.** Let $M$ be a $2n$-dimensional Riemannian manifold. Suppose that

$$(u, (v, \nabla v), \ell) \in TM \oplus J^1(TM) \oplus \text{Gr}_n(TM),$$  

(5.5.4)

with $\pi(u, (v, \nabla v), \ell) = p$ and $(v, \nabla v)$ small. We note that $v$, being small, defines a geodesic $\gamma$
in $M$.

Choose a basis for the subspace $\ell$ of $T_pM, e_1, \ldots, e_n$. For each $e_i$ construct the Jacobi field
$X^{(i)}$ along $\gamma$ with $X^{(i)}_0 = e_i$ and $(\nabla_{\frac{d}{dt}} X^{(i)})_0 = \nabla_{e_i} v$. The final values of the $X^{(i)}$
form a subset of $T_{\exp_p(v)}M$. If this subset is not linearly independent, then it follows by linearity of the
Jacobi equation that we could choose $e_1$ so that the final value of $X^{(1)}$ is zero; but for $v$ and $\nabla v$
sufficiently small this is impossible by Proposition 5.5.1. Hence, this set is linearly independent
and forms a basis for an $n$-dimensional subspace $\ell'$ of $T_{\exp_p(v)}M$.

Now find Jacobi fields $X_1$ and $X_2$ along $\gamma$ with initial conditions $(X_1)_0 = 0 = (\nabla_{\frac{d}{dt}} X_2)_0$
and $(\nabla_{\frac{d}{dt}} X_1)_0 = u = (X_2)_0$ and suppose that $X_i$ has final value $w_i \in T_{\exp_p(v)}M$. We then take
$T_i(u, (v, \nabla v), \ell)$ to be the part of $w_i$ orthogonal to the subspace $\ell'$.

$T_i$ is only defined on an open subset of $TM \oplus J^1(TM) \oplus \text{Gr}_n(TM)$ that is not merely $(v, \nabla v)$
small. We shall thus sometimes extend to $M'$ into which we can embed $M$ isometrically, as for
$G$ in Definition 5.4.7.

We now consider what the $T_i$ look like for fixed submanifolds.
**Definition 5.5.6.** Let $M$ be a Riemannian manifold, and let $L$ and $L'$ be submanifolds of $M$. Suppose that there is a normal vector field $v$ on $L$ such that $\exp_p(L) = L'$. Suppose that $\|v\|_{C^1}$ is sufficiently small.

We evidently have an inclusion $\nu_L \to TM$. That is, given a normal vector $u$ at a point $p \in L$ we have the triple $(u, (v, \nabla v), T_pL) \in TM \oplus J^1(TM|_L) \oplus \text{Gr}_n(TM)$. We may extend $(v, \nabla v)$ to $J^1(TM)|_L$ in some way, and then we may consider $T_1(u, (v, \nabla v), T_pL)$, with $T_1$ as in Definition 5.5.5 and $i = 1, 2$. This defines two bundle maps $\nu_L \to TM|_{L'}$. Examining Definition 5.5.5 shows that these bundle maps are independent of our extension of $(v, \nabla v)$.

Definition 5.5.6 can be simplified.

**Proposition 5.5.7.** As in Definition 5.5.6, let $L$ and $L'$ be submanifolds of the Riemannian manifold $M$. Suppose that there is a normal vector field $v$ on $L$ such that $\exp_p(L) = L'$, and $v$ is sufficiently small in a $C^1$ sense. Suppose $u \in (\nu_L)_p$ and let $\gamma$ be the geodesic in $M$ with initial velocity $v_p$. Let $X_1$ be the Jacobi field along $\gamma$ with initial conditions $(X_1)_0 = 0$ and $(\nabla_\gamma X_1)_0 = u$; similarly, let $X_2$ be the Jacobi field along $\gamma$ with $(X_2)_0 = u$ and $(\nabla_\gamma X_2)_0 = 0$. Let $w_1$ and $w_2$ be the final values of $X_1$ and $X_2$ respectively. Then $T_1(u)$ is the normal part of $w_1$. It follows that $T_1$ and $T_2$ define bundle maps from $\nu_L$ to $\nu_{L'}$.

**Proof.** The Jacobi fields $X_1$ and $X_2$ are constructed exactly as in Definition 5.5.5. We have to show that taking the normal part is precisely the orthogonal projection onto the complement of $\ell'$ described there. That is, we have to show that the space $\ell'$ of Definition 5.5.5 is $T_{\exp_p(v)}L'$.

We recall from the proof of Proposition 5.4.8 that the final value of the Jacobi field $X$ with $X_0 = e$ and $(\nabla_\gamma X)_0 = \nabla_e v$ is a tangent vector to $L'$. Comparing with Definition 5.5.5, this immediately shows that $\ell' \subset T_{\exp_p(v)}L'$. As both of these are $n$-dimensional, however, we have equality as required. 

Note that since the Jacobi equation is linear, we have that for fixed $M, L, L'$ and $v$ any normal vector field obtained by a Jacobi transport followed by taking the normal part is a combination of $T_1$ and $T_2$; we use this in Proposition 5.5.15 below.

We now show that the bundle maps $T_1$ and $T_2$ are bundle isomorphisms.

**Proposition 5.5.8.** The bundle maps $T_1$ and $T_2$ of Definition 5.5.6 define bundle isomorphisms $\nu_L \to \nu_{L'}$ if $\|v\|_{C^1}$ is sufficiently small.

**Proof.** By definition $T_1 : \nu_L \to \nu_{L'}$ are bundle maps over $\exp_p : L \to L'$. Both $\nu_L$ and $\nu_{L'}$ have rank $n$, so it suffices to prove that there is no normal vector $u \in (\nu_L)_p$ such that $T_1(u) = 0$.

Let $\gamma$ be the geodesic with initial velocity $v_p$. We have, as in the proof of Proposition 5.5.7, that every tangent vector at $\exp_p(p)$ is the final value of a Jacobi field $X$ along $\gamma$, with initial conditions $X_0 = e$ and $(\nabla_\gamma X)_0 = \nabla_e v$, for some $e \in T_pL$. 

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Therefore, we must show that if we have \( e \in T_p L \) and \( u \in (v_L)_p \), and Jacobi fields \( X^0 \) and \( X^1 \) along \( \gamma \) with \( (X^0)_0 = e \) and \( (\nabla_{\gamma}^T X^0)_0 = \nabla_e v \), and either \( (X^1)_0 = u \) and \( (\nabla_{\gamma}^T X^1)_0 = 0 \), or vice versa, then the final values of \( X^0 \) and \( X^1 \) differ.

To do this, we will apply Corollary 5.5.3. Let \( P_t \) be the parallel transport along \( \gamma \) and then let
\[
Y_t^0 = P_t(e + t\nabla_e v), \quad Y_t^1 = P_t(u) \quad \text{or} \quad Y_t^1 = P_t(tu),
\]
where \( Y^1 \) depends on the initial conditions of \( X^1 \). Note that \( Y_t^1 = P_t(u) \) in both cases. By Corollary 5.5.3, we have
\[
|X^0_t - Y^0_t| \leq |e|(\cosh(\sqrt{C}|v_p|) - 1) + |\nabla_e v| \left( \frac{\sinh(\sqrt{C}|v_p|)}{\sqrt{C}|v_p|} - 1 \right), \quad (5.5.6)
\]
\[
|X^1_t - Y^1_t| \leq |u| \max\{\cosh(\sqrt{C}|v_p|) - 1, \frac{\sinh(\sqrt{C}|v_p|)}{\sqrt{C}|v_p|} - 1\}. \quad (5.5.7)
\]

Hence, by choosing \( ||v||_{C^1} \) sufficiently small so that both \( v_p \) and \( (\nabla v)_p \) are small, we may suppose
\[
|X^0_t - Y^0_t| \leq \frac{1}{7}|e|, \quad |X^1_t - Y^1_t| \leq \frac{1}{7}|u|. \quad (5.5.8)
\]

Now \( u \) is normal and \( e \) tangential so the inner product between \( u \) and \( v \) is zero. Since parallel transport preserves inner product, we find \( \langle Y^0_t, Y^1_t \rangle = \langle \nabla_e v, u \rangle \). It follows that \( ||Y^0_t, Y^1_t|| \leq |(\nabla v)_p||e||u| \). Again, by choosing \( ||v||_{C^1} \) small, we may suppose this inner product is smaller in absolute value than \( \frac{1}{7}|e||u| \). Finally, since parallel transport preserves the metric, \( |Y^0_t| = |e + \nabla_e v| \), and hence again for \( ||v||_{C^1} \) small, we have
\[
\frac{6}{7}|e| \leq |Y^0_t| \leq \frac{8}{7}|e|. \quad (5.5.9)
\]

Evidently, \( |Y^1_t| = |v| \), and combining these with (5.5.8) yields
\[
\frac{5}{7}|e| \leq |X^0_t| \leq \frac{9}{7}|e|, \quad \frac{6}{7}|e| \leq |X^1_t| \leq \frac{8}{7}|e|. \quad (5.5.10)
\]

We then obtain by combining these with our inner product estimate
\[
|\langle X^0_t, X^1_t \rangle| \leq \frac{1}{7}|e||u| + \frac{9}{7}|e|\frac{1}{7}|u| + \frac{8}{7}|e|\frac{1}{7}|u| + \frac{1}{7}|u|\frac{1}{7}|e| \leq \frac{25}{14}||X^0_t||X^1_t|| = \frac{5}{6}||X^0_t||X^1_t||. \quad (5.5.11)
\]

Hence, \( X^0_t \) and \( X^1_t \) cannot be equal for \( ||v||_{C^1} \) sufficiently small. \( \square \)

Hence, the \( T_i \) define isomorphisms from normal vector fields on \( L \) to normal vector fields on
\( L' \). Note that their inverses need not be the \( T_i \). Note also that the construction of \( \ell' \) in Definition 5.5.5 precisely gives a further isomorphism

\[
T_pM \cong T_{\exp_p(v)}M, \tag{5.5.12}
\]

with \( \ell' \) being the image of \( \ell \). Passing to a map of tangent vectors as in Definition 5.5.6, this isomorphism corresponds to the pushforward of tangent vectors by the diffeomorphism \( \exp_v \) from \( L \) to \( L' \). We have shown as part of Proposition 5.4.8 that it is smooth and depends continuously on the metric as a map of triples \((u, (v, \nabla v), \ell)\); in fact, of course, this map is essentially independent of the space \( \ell \).

We now turn to the case where \( L \) and \( L' \) vary. We shall assume they vary within finite-dimensional families \( U \). This is so that we have a straightforward inverse function theorem (which is not true in the general Fréchet space case that would arise if we worked with all smooth submanifolds: see Hamilton [30]) but do not have to worry about the details of precisely which regularity spaces we are using (which would be complicated, as we will frequently use composition of maps, which is not even continuously differentiable as a map giving \( C^k \) maps from pairs of \( C^k \) maps, for instance).

We seek to show, for instance, that the transfer map \( T_i \) can be said to define a smooth map from normal vector fields on \( L_0 \) to normal vector fields on \( L_\alpha \) in our family \( U \). As with the definition, we begin with a pointwise result.

**Proposition 5.5.9.** Let \( M \) be a Riemannian manifold. Consider the maps

\[
T_i : TM \oplus J^1(TM) \oplus \text{Gr}_n(TM) \to TM \tag{5.5.13}
\]

defined as in Definition 5.5.5. These maps are smooth. They depend continuously on the metric on \( M \) in the sense described in Proposition 5.4.8, and (5.4.35) becomes

\[
|T_i(s, u, (v, \nabla v), \ell) - T_i(s', u, (v, \nabla v), \ell)| < \epsilon |(u, (v, \nabla v))|. \tag{5.5.14}
\]

Moreover, if we extend these maps by the identity to

\[
T_i : TM \oplus J^1(TM) \oplus \text{Gr}_n(TM) \to TM \oplus J^1(TM) \oplus \text{Gr}_n(TM), \tag{5.5.15}
\]

they are injective immersions. That is, their images are submanifolds and the inverse maps defined on these images are smooth, and depend continuously on the metric in the same sense.

**Proof.** Smoothness and continuity follow exactly as in the proof of Proposition 5.4.8. That is, locally, we are just solving some systems of ordinary differential equations and then taking some inner products. The solution to a system of ordinary differential equations depends smoothly on the initial conditions and continuously on a parameter, as in that proposition. The estimate
(5.5.14) follows as there: the map giving the derivative of $T_i$ in $(v, \nabla v)$ is also continuous in $s$ and $T_i(s, u, 0, 0, \ell) = T_i(s', u, 0, 0, \ell)$ for every $\ell$ and $u$: integrating up yields (5.5.14). We note that $\text{Gr}_n(TM)$ is compact over each point $p$, and so there is no need for an estimate of the size of $\ell$.

Since $T_i$ forms a bundle isomorphism if we fix the $J^1(TM) \oplus \text{Gr}_n(TM)$ component, the extended map (5.5.15) is certainly injective. The derivative of the identity is the identity, and $T_i$ is linear in the first component. It follows that to prove (5.5.15) is an immersion we only have to show that $T_i$ is injective, which we have already shown in Proposition 5.5.8.

We may now prove

**Proposition 5.5.10.** Let $L_s$ and $L'_s$ be finite-dimensional smooth compact families of submanifolds of $M$, with $L_0$ and $L'_0$ close enough that there exists a normal vector field $v(0,0)$ to $L_0$ whose image under the Riemannian exponential map is $L'_0$.

Then for $s$ and $s'$ small enough and $v(0,0)$ small enough, there exists a normal vector field to $L_s$ whose image under the Riemannian exponential map is $L'_s$. This normal vector field is the image under $T_2$ of a normal vector field $v(s,s')$ on $L_0$. The map $(s,s') \mapsto v(s,s')$ is a smooth map.

**Proof.** Note that we assume we have chosen our parametrisation of the families $L_s$ and $L'_s$, so that these families are diffeomorphic to $U$ and $U'$: hence, $v(s,s')$ depending smoothly on $(s,s')$ is equivalent to its depending smoothly on $(L_s, L'_s)$. The existence of the normal vector fields is immediate. The difficulty is to prove that $v(s,s')$ depends smoothly on $(s,s')$. We know that $\exp$ is a smooth map $TM \to M$. Write $u_s$ for the normal vector field to $L_0$ whose image is $L_s$.

Now consider the map

$$
\begin{array}{ccc}
\nu_L \times U & \rightarrow & M \times U, \\
(v,s) & \mapsto & (\exp(T_2(v, j^1(u_s)_{\pi(v)}, T_{\pi(v)}L_0)), s).
\end{array}
$$

where $j^1$ is the jet prolongation map of Proposition 2.2.1, and we choose some smooth extension for the normal components of the derivative. (5.5.16) is smooth: the evaluation maps of $u_s$ and its derivative are linear and continuous, $T_2$ is smooth by Proposition 5.5.9, and $\exp$ is certainly smooth. Moreover, it is clear that its derivative at $(0_p,s)$ always contains an identity component in $s'$, so is an isomorphism if and only if the derivative $T_{0_p} \nu_L \to T_{\exp(p,s)}M$ is an isomorphism.

We show that this is the case if $s$ is sufficiently small. $T_{0_p} \nu_L$ can be decomposed as the direct sum $T_{0_p}(\nu_L)_p \oplus T_pL$; clearly, $T_{0_p}(\nu_L)_p \cong (\nu_L)_p$. If we consider $v' \in T_pL$, we can choose some curve $\gamma$ in $L$ through $p$ with tangent $v'$ and then consider the corresponding curve $0_\gamma$ in $\nu_L$. The image of this curve under (5.5.16) is precisely $\exp_{u_s} \circ \gamma$, so for $u_s$ sufficiently small, $T_pL \subset T_{0_\gamma} \nu_L$ is mapped isomorphically to $T_{\exp(p,s)}L \subset T_{\exp(p,s)}M$. If $v' \in T_{0_p}(\nu_L)_p$, we get the curve $sv'$. The image of this curve is $\exp(T_2(sv'))$; since the derivative of $\exp$ at

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zero is the identity, the tangent at zero is just $T_2(v')$ and hence $T_{0p}((\nu_L)_p) \subset T_{0p}\nu_L$ is mapped isomorphically to $(\nu_L)_{exp_p(u_s)} \subset exp_p(u_s)M$, by Proposition 5.5.8.

Hence, for $s$ sufficiently small, the derivative $T_{0p}\nu_L \rightarrow T_{exp_p(u_s)}M$ is an isomorphism, and so (5.5.16) is a local diffeomorphism. Since $L_0$ is compact, we can find a tubular neighbourhood $T$ of $L_0$ on which the inverse map

$$T \times U \rightarrow \nu_{L_0} \times U$$

is well-defined. If $v(0,0)$ is sufficiently small, then $L_0'$ lies in $T$, and hence so does $L_{s'}'$ for $s'$ sufficiently small. Clearly, $L_{s'}'$ can be essentially equivalently viewed as a smooth family of inclusions of $L$ into $M$. Composing this family of inclusions with (5.5.17) yields the result, since composition of smooth maps is smooth.

Remark. The same holds if $L_s$ and $L_{s'}'$ are asymptotically cylindrical submanifolds with decay rate uniformly bounded from below. The argument to construct a tubular neighbourhood on which (5.5.17) is smooth goes through just as before, noting that the metric converges to a limit, so (5.5.16) does, essentially by the same argument as in Proposition 5.5.9. We have to check that composition behaves in this case: but this works exactly as in the compact case for the same reasons (we can find partial derivatives in both $s$ and $s'$ and they all commute, and the tubular neighbourhoods can still be chosen uniformly).

We now explain the purpose of defining $T_1$ and $T_2$. We begin with $T_1$.

**Proposition 5.5.11.** Let $M$, $L$, $L'$, $v$ and $T_1$ be as in Definition 5.5.6. Let $v_s$ be a smooth curve of normal vector fields on $L$ with $v_0 = v$. Then $exp_{v_s}(L)$ defines a smooth curve of submanifolds, and when $s = 0$ it passes through $exp_v(L) = L'$. Therefore, there exists a smooth curve $w_s$ of normal vector fields to $L'$ such that

$$exp_{v_s}(L) = exp_{w_s}(L').$$

(5.5.18)

We have $w' = T_1v'$ where $w'$ and $v'$ are the derivatives at 0 of these curves of vector fields.

**Proof.** For smoothness of the curve $w_s$, we apply Proposition 5.5.10 with $L_s = L'$ for all $s$, and $L_{s'}' = exp_{v_s}(L)$. Then $w_s$ is precisely the normal vector field to $L'$ given by Proposition 5.5.10 and therefore depends smoothly on $s$. We now need to show that the derivative of this curve at zero is given by $T_1(v')$.

We begin by noting that we can find sections $x_s$ of $TM|_{L'}$, such that for all $p \in L_0$ we have $exp_{v_s}(p) = exp_{x_s}exp_{v_0}(p)$. We can evaluate $x'$, as follows. By construction, and the fact the derivative of $exp$ at 0 is the identity, $x'_{exp_v}(p)$ is the derivative of the curve $exp_p(v_s)$ at $s = 0$. For each $s$, $exp_p(v_s)$ is given by the final position of a geodesic; since $v_s(p)$ is smooth in $s$, we have a smooth variation through geodesics. $x'$ is then the final value of the corresponding Jacobi field
Along the geodesic $\gamma$ corresponding to $v_0(p)$. Since all the geodesics start at the same point $p$, we certainly have $X_0 = 0$. For $(\nabla_\gamma X)_0$, we note that this as usual is equal to the derivative in $s$ of the initial velocities, hence $v'$. That is, $x'$ is given by the final value of the Jacobi field along $\gamma$ with initial conditions $X_0 = 0$, $(\nabla_\gamma X)_0 = v'$. Note that this is just evaluating the derivative of the Riemannian exponential map: for an alternative proof, see [26, Corollary 3.46].

We now have to move from $x'$ to $w'$. To determine $w_s$ from $x_s$, we compose $\exp_{x_s}$ with the projection $\pi$ of a tubular neighbourhood map around $L'$, invert the resulting diffeomorphism of $L'$, and then compose this inverted diffeomorphism with $\exp_{x_s}$. To evaluate the derivative of this is fairly straightforward. The derivative of the curve $\pi \circ \exp_{x_s}$ of diffeomorphisms of $L$ is just the tangential part of $x'$ (see [30, Example 4.4.5]). On inversion around the identity $\pi \circ \exp_{x_0}$, we get the negative of this tangential part. Recomposing this with $\exp_{u_s}$ gives addition of the derivatives, and so

$$w' = \text{npt } x' = T_1 v'. \tag{5.5.19}$$

$T_2$ is not quite the most obvious other way of constructing a curve of submanifolds through $L'$. We introduce the following notation.

**Definition 5.5.12.** Let $M$ be a Riemannian manifold and let $L$ be a submanifold. Let $u$ be a normal vector field on $L$, so that $\exp_{su}(L)$ forms a curve of submanifolds for $u$ sufficiently small. Let $v$ be a normal vector to $L$, with $\pi(v) = p \in L$. Let $\gamma$ be the geodesic with initial velocity $u_p$, so that $\gamma(s) \in \exp_{su}(L)$ for each $s$. Let $J$ be the Jacobi field along $\gamma$ with initial conditions $J_0 = v$ and $(\nabla_\gamma J)_0 = 0$. Then $J_s \in TM|_{\exp_{su}(L)}$ for each $s$, so we can consider its normal part. This gives another curve of tangent vectors $N_s$ with $\pi(N_s) = \gamma(s)$. We may consider the derivative $N'$ of $N_s$ at $s = 0$. We define

$$\text{npt}'_u v = N'. \tag{5.5.20}$$

We can be slightly more explicit about $\text{npt}'_u v$.

**Proposition 5.5.13.** Let $M$, $L$, $u$ and $v$ be as in Definition 5.5.12. We can choose sections $e_i$ of $TM|_\gamma$ so that $\{e_i(s)\}$ forms an orthonormal basis of $T_{\gamma(s)} \exp_{su}(L)$ for each $s$. In particular, we have $e_i = e_i(0)$ an orthonormal basis for $T_p L$ and $e'_i$ their derivatives at zero. In terms of these sections we can write

$$\text{npt}'_u v = \sum_{i=1}^n -g'(e_i, v)e_i - g(e'_i, v)e_i. \tag{5.5.21}$$

**Proof.** To find sections $e_i$, we can just construct Jacobi fields for pushforward as in the proof of Proposition 5.4.8 to construct a basis $T_{\gamma(s)} \exp_{su}(L)$, noting these depend smoothly on $s$, and
then apply the Gram–Schmidt process. With $J$ and $N$ as in Definition 5.5.12, we then have

$$N_s = J_s - \sum g(e_i(s), J_s)e_i.$$  \hfill (5.5.22)

Differentiating this, and using that $v$ is normal so orthogonal to the $e_i$, we get (5.5.21). \hfill \Box

It is hard to explicitly determine the $e'_i$, so we shall not attempt to understand the right hand side of (5.5.21) further, and rely on the fact that it will be small if $v$ is. Now $\text{nt}'_u v$ is not necessarily a normal vector on $L$, but we can extend the map $T_1$ of Definition 5.5.6 to sections of $TM|_L$ (the tangent fields being close to annihilated by the final orthogonal projection). We can thus make

**Definition 5.5.14.** Let $M$ be a Riemannian manifold and let $L$ be a submanifold. Let $u$ and $v$ be normal vector fields on $L$, so that $\text{nt}'_u v$ is a section of $TM|_L$. Then let $T_3u$ be $T_1(\text{nt}'_u v)$ where we extend the map $T_1$ of Definition 5.5.6 constructing a normal vector field on $\text{exp}_v(L)$.

Note that $T_1u$ and $T_2u$ are linear maps of $u$, but $T_3u$ is not. Moreover, we can write $T_3u$ also as a map induced from a map on a bundle as in Definition 5.5.5, but we would have to define it on $J^1(TM) \oplus J^1(TM) \oplus \text{Gr}_n(TM)$, as the condition in Definition 5.5.12 that we take the normal part of $J_s$ on $\text{exp}_{su}(L)$ requires the first jet of $u$ to determine the tangent space $T_{\text{exp}_{su}(p)}\text{exp}_{su}(L)$. We will not give details of these arguments.

We can now explain the relevance of $T_2$.

**Proposition 5.5.15.** Let $M$, $L$, $L'$, $v$ and $T_2$ be as in Definition 5.5.6. Let $u_s$ be a smooth curve of normal vector fields on $L$ with $u_0 = 0$ and derivative $u'$; let $T_3$ be as in Definition 5.5.14. Let $T_{2,s}$ be the transfer maps from $L$ to $\text{exp}_{u_s}(L)$ given by taking the appropriate map $T_2$ from Definition 5.5.6. Then $\text{exp}_{T_2,s(v)}\text{exp}_{u_s}(L)$ defines a smooth curve of submanifolds, and when $s = 0$ it passes through $\text{exp}_v(L) = L'$. Therefore, there exists a smooth curve $w_s$ of normal vector fields to $L'$ such that

$$\text{exp}_{T_2,s(v)}\text{exp}_{u_s}(L) = \text{exp}_{w_s}(L').$$ \hfill (5.5.23)

We have $w' = T_2u' + T_3u'$.

**Proof.** To see that $w_s$ is a smooth curve of normal vector fields, we show that $\text{exp}_{T_2,s(v)}\text{exp}_{u_s}(L)$ is a smooth curve of submanifolds; then we may apply the argument at the beginning of Proposition 5.5.11. Since, as at the end of Proposition 5.5.11, by composition with the projection of a tubular neighbourhood, inversion, and re-composition, a smooth curve of immersions $L \hookrightarrow M$ induces a smooth curve of submanifolds, we have to show that if $\iota$ is the inclusion of $L$ then $\text{exp}_{T_2,s(v)}\exp_{u_s} \iota$ is a smooth curve of maps; clearly as $s = 0$ gives $\iota$ it is then locally a smooth curve of immersions.
This follows straightforwardly from Proposition 5.5.9. Consider the map

\[
\begin{align*}
J^1(u_L) & \longrightarrow M, \\
(u, \nabla u) & \longmapsto \exp(T_2(v_{\pi(u)}), (u, \nabla u), T_{\pi(u)}L)).
\end{align*}
\]  

(5.5.24)

where \( T_2 \) is as in Definition 5.5.5. This is a smooth map; composing it with the curve \( j^1(u_s) \) of first jets of \( u_s \) gives the curve of immersions required. Hence this curve is smooth and so is \( w_s \).

To find \( w' \), as in Proposition 5.5.11, we first construct for each \( p \in L \) a family of tangent vectors \( x_s \) to \( \exp_p(v) \) such that

\[
\exp_{T_2(s)} \exp_{u_s}(p) = \exp_x \exp_v(p).
\]  

(5.5.25)

Again as in Proposition 5.5.11, we identify \( \exp_{T_2(s)} \exp_{u_s}(p) \) as the curve of final positions of a variation through geodesics, so that \( x'(p) \) is the final value of the Jacobi field \( X \) corresponding to this variation. These geodesics have initial positions \( \exp_{u_s}(p) \) and initial velocities \( T_{2,s}(v_p) \). Differentiating in \( s \), we see that \( X_0 = u_p' \).

Now also \( x_s \) at \( \exp_p(v_p) \) is given by composing \( \exp_{\exp_p(v)}^{-1} \) with (5.5.25). Hence, it depends smoothly on the curve of 1-jets of \( u_s \) at \( p \). In particular, \( x' \) at \( \exp_p(v_p) \) depends smoothly on the 1-jets of \( u_0 = 0 \) and of \( u' \). These 1-jets remain the same if we replace \( (u_s) \) by \( (sv') \), and so to find \( x' \) we may suppose that \( u_s = sv' \).

With this curve, we are in the situation of Definition 5.5.12. We let \( \gamma(s) \) be the geodesic with initial velocity \( u_s' \), and let \( J_s \) be the Jacobi field along \( \gamma \) with initial conditions \( J_0 = v_p \) and \( (\nabla_{\frac{d}{ds}} J)_0 = 0 \). Then \( T_{2,s}(v_p) \) is just the normal part of \( J_s \). Hence, the derivative in \( s \) of \( T_{2,s}(v_p) \) is just \( \nabla_{u_s'} v \). This is equivalently \( (\nabla_{\frac{d}{ds}} X)_0 \).

That \( w' \) is given by taking the normal part of \( x' \) follows exactly as in Proposition 5.5.11. Hence \( w' \) is given by taking the normal part of the final value of Jacobi fields, and so is a linear combination of \( T_1 \) and \( T_2 \): examining the initial conditions precisely gives \( w' = T_2u' + T_3u' \). \( \square \)

Putting Propositions 5.5.11 and 5.5.15 together, we obtain

**Proposition 5.5.16.** Let \( M, L, L' \), and \( v \) be as in Definition 5.5.6. Suppose that \( u_s \) is a curve of normal vector fields to \( L_0 \) and \( w_s \) is a curve of normal vector fields to \( L' \). For each \( s \), we can find a normal vector field \( v_s \) to \( \exp_{u_s}(L_0) \) so that

\[
\exp_{v_s} \exp_{u_s}(L_0) = \exp_{w_s} \exp_{v_0}(L_0).
\]  

(5.5.26)

The transfer map \( T_2 \) of Definition 5.5.6 defines an isomorphism between the normal bundles to \( L_0 \) and \( \exp_{u_s}(L_0) \). Hence, we may identify \( v_s \) with a normal vector field on \( L \), and so \( (v_s) \) is a
smooth curve of such normal vector fields. We have

$$w' = T_1v' + T_2u' + T_3u', \quad (5.5.27)$$

and so the derivative of the map $\mathbf{(u_s, w_s)} \mapsto v_s$ is given by

$$v' = T^{-1}_1(T_2u' + T_3u' - w'). \quad (5.5.28)$$

Proof. That $v_s$ is smooth follows immediately from Proposition 5.5.10.

A similar argument generalising the smooth map (5.5.24) used in Proposition 5.5.15 can be used to show that $w_s$ is a smooth map of $u_s$ and $v_s$. As before, we define a curve of diffeomorphisms between $\nu_{L_0}$ and a tubular neighbourhood of $L_0$, and then $w_s$ is just the composition of these diffeomorphisms with the sections $v_s$ of $\nu_{L_0}$.

This enables us to identify the derivative map giving $w'$ as a linear map of $u'$ and $v'$. The case $u_s \equiv 0$ gives $w' = T_1(v')$ by Proposition 5.5.11; the case $v_s \equiv v$ gives $w' = T_2u' + T_3u'$ by Proposition 5.5.15. By linearity of the derivative, we have (5.5.27). (5.5.28) follows immediately.

We must also consider asymptotically translation invariant normal vector fields on asymptotically cylindrical submanifolds. First of all, we need to ask how we can identify such a normal vector field. We briefly discussed this question for the asymptotically cylindrical deformation theory of Theorems 5.2.12 and 5.2.13. In that case, we explained briefly that we could take $v$ asymptotically translation invariant if and only if $v_0 \omega|_L$ is. In general, it is easy to define exponentially decaying such fields, as the submanifolds $L$ are themselves asymptotically cylindrical and so we have exponential weights. To define asymptotically translation invariant vector fields, we use that each asymptotically cylindrical submanifold has a corresponding cylindrical submanifold.

Definition 5.5.17. Let $L$ be an asymptotically cylindrical submanifold of the asymptotically cylindrical manifold $M$. Let $K \times (R, \infty)$ be the cylindrical end, so that the end of $L$ is $\exp_v(K \times (R, \infty))$ for $v$ decaying.

Translation gives an action on $TM|_{K \times (R, \infty)}$, and consequently a notion of translation invariant vector field. Since we always have a notion of exponentially decaying vector fields, we obtain a notion of asymptotically translation invariant vector fields on $K \times (R, \infty)$.

We may extend $v$ to obtain an asymptotically cylindrical diffeomorphism with limit the identity (as in Definition 4.2.2) of tubular neighbourhoods of $K \times (R, \infty)$ and the end of $L$. Pushforward by this diffeomorphism induces a map from vector fields along $K \times (R, \infty)$ to vector fields along the end of $L$; as $v$ and its first derivative are exponentially decaying, it follows as in Definition 5.5.5 (that is, by using the Rauch comparison estimate Proposition 5.5.1) that this defines an isomorphism far enough along the end.

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We then say that a vector field along $L$ is asymptotically translation invariant precisely if it is the image of an asymptotically translation invariant vector field under this pushforward. An asymptotically translation invariant normal vector field is simply asymptotically translation invariant and normal. We say an asymptotically translation invariant vector field’s limit is the translation invariant vector field along $K \times (R, \infty)$ given by the limit of the vector field on $K \times (R, \infty)$ of which it is a pushforward.

Remark. Note that even on $K \times (R, \infty)$ there need not be any translation invariant normal vector fields, because the metric is not translation invariant.

In Proposition 5.5.33 below, we shall show that this definition is equivalent to the method used for deformation theory in the special Lagrangian (and nearly special Lagrangian) case. The purpose of using pushforward for this transfer and not restricting to normal vector fields in the definition is that it makes the following result trivial.

**Lemma 5.5.18.** Suppose that $L_1$ and $L_2$ are asymptotically cylindrical submanifolds with the same limit, so that there is an exponentially decaying normal vector field $w$ to $L_1$ such that $\exp_w(L_1) = L_2$. Extend $w$ to define an asymptotically cylindrical diffeomorphism of tubular neighbourhoods, with limit the identity. A vector field along $L_1$ is asymptotically translation invariant if and only if its image under the pushforward by this diffeomorphism is. Moreover, the vector field and its pushforward have the same limit.

**Proof.** We restrict to the ends of $L_1$, $L_2$ and the corresponding cylindrical end $\tilde{L}$. As the compact parts are irrelevant for this discussion, we will just write these ends as $L_1$ and $L_2$. We have $L_1 = \exp_{v_1}(\tilde{L})$, $L_2 = \exp_{v_2}(\tilde{L})$; again, make some extensions $v_1$ and $v_2$ to define asymptotically cylindrical diffeomorphisms with limit the identity between tubular neighbourhoods. Suppose $u$ is an asymptotically translation invariant vector field on $L_1$; that is, it is the pushforward by $\exp_{v_1}$ of an asymptotically translation invariant vector field on $\tilde{L}$. We want to show that the pushforward by $\exp_w$ of $u$ is a translation invariant vector field on $L$ with the same limit; that is, we want to show that the pushforward by the composition $\exp_w \exp_{v_1}$ of an asymptotically translation invariant vector field is the pushforward by $\exp_{v_2}$ of an asymptotically translation invariant vector field with the same limit. Equivalently, it suffices to show that $\exp_{v_2}^{-1} \exp_w \exp_{v_1}$ (which is essentially a diffeomorphism of the end $\tilde{L}$) preserves asymptotically translation invariant vector fields and their limits. But for asymptotic translation invariance, we just note that this is an asymptotically cylindrical diffeomorphism, and so pullback induces a corresponding asymptotically cylindrical metric on the tubular neighbourhood of $\tilde{L}$, and the question just becomes the independence of asymptotic translation invariance on metric. This is immediate as usual. As the diffeomorphism has limit the identity, it preserves the limits of the vector fields.

The reverse implication is equally obvious. □

This implies in particular that Definition 5.5.17 is independent of the extension of $v$ used.
We will now explain some consequences of Definition 5.5.17, particularly with respect to the transfer maps $T_1$, $T_2$ and $T_3$.

Before doing so, we make some more definitions. To show results for the inverses of $T_1$ and $T_2$, we need to make sure that these are well-defined pointwise, and so we have to restrict to bundles where we assume the vectors are normal. We thus make the following

**Definition 5.5.19.** Let $M$ be a Riemannian manifold. Let $N$ be the subbundle of $TM \oplus J^1(TM) \oplus \text{Gr}_n(TM)$ (over $M$) consisting of those $(u, (v, \nabla v), \ell)$ such that $u$ is normal to the subspace $\ell$. Let $N'$ be the subspace

$$\{(u, (v, \nabla v), \ell) \in TM \times (J^1(TM) \oplus \text{Gr}_n(TM)) : \pi_{TM}(u) = \exp_{\pi_{J^1(TM) \oplus \text{Gr}_n(TM)}((v, \nabla v), \ell)}(v) \text{ and } u \text{ is normal to } \ell'\}, \tag{5.5.29}$$

where $\ell'$ is the subspace constructed in Definition 5.5.5.

**Proposition 5.5.20.** $N$ is a smooth subbundle and $N'$ is a smooth submanifold locally around the points $(u, (0, 0), \ell)$, with either projection map defining a bundle structure.

**Proof.** We begin with $N$. We can define a smooth map $TM \oplus \text{Gr}_n(TM) \rightarrow TM$ by, given $(u, \ell) \in T_pM \oplus \text{Gr}_n(T_pM)$, taking the component of $u$ that lies in $\ell$. This map is smooth because locally we can define smooth maps from $\text{Gr}_n(TM)$ to orthonormal bases, and then this is just taking an appropriate set of inner products. $N$ is then the kernel of the map on $TM \oplus J^1(TM) \oplus \text{Gr}_n(TM)$ given by ignoring the $J^1(TM)$ component and applying this map. This is evidently a submersion of bundles, and so the kernel is indeed a subbundle.

$N'$ is similar once we know that the subset

$$\{(u, (v, \nabla v), \ell) \in TM \times (J^1(TM) \oplus \text{Gr}_n(TM)) : \pi_{TM}(u) = \exp_{\pi_{J^1(TM) \oplus \text{Gr}_n(TM)}((v, \nabla v), \ell)}(v)\} \tag{5.5.30}$$

is a submanifold around the points $(u, (0, 0), \ell)$. This follows just as in the proof of Lemma 5.4.6.

We may now pass to our results on the actions of the $T_i$ on asymptotically translation invariant normal vector fields.

**Proposition 5.5.21.** Let $L$ be an asymptotically cylindrical submanifold of $M$; let $L'$ be another asymptotically cylindrical submanifold with the same limit. Let $u$ be an asymptotically translation invariant normal vector field on $L$. Then

i) $\|u\|_{C^k}$ is finite for every $k$.

ii) $T_1u$ and $T_2u$ are asymptotically translation invariant normal vector fields on $L'$ with the same limit as $u$. Similarly, if $u'$ is an asymptotically translation invariant normal vector field to $L'$, $T_1^{-1}u'$, $T_2^{-1}u'$ are asymptotically translation invariant normal vector fields on $L$.
with the same limits as $u'$. Finally, $T_3u$ is an exponentially decaying normal vector field on $L'$.

iii) If the ambient metric is cylindrical as in subsection 5.2.3, asymptotically translation invariant normal vector fields correspond to nearby asymptotically cylindrical submanifolds, with the limits also corresponding.

Proof. We begin with (i). By definition, away from a compact part, $u$ is the image under push-forward of an asymptotically translation invariant vector field $w$ on $K \times (R, \infty)$. $w$ is clearly bounded in $C^k$ for every $k$, at least for $R$ large enough, as the metric converges to the cylindrical metric. We have to show that pushforward preserves this, and this follows easily from the smoothness and continuity result obtained as part of Proposition 5.4.8. For instance, $|u_{\exp, (p)}|$ is precisely given by applying the pointwise pushforward map with $w_p$ and $(v_p, (\nabla v)_p)$; this depends smoothly on $p$ and converges to a limit as we approach the end, using the continuity in the metric. This proves the $C^0$ bound; the $C^k$ bound is similar by using the derivatives.

The major part of this proposition is (ii). We shall prove the result for $T_1$ and $T_2$ and their inverses; $T_3$ is similar. As usual we shall argue using the ideas of Proposition 5.5.9. It follows from Lemma 5.5.18 that the result is true just for applying pushforward, so it suffices to check two things. Firstly, we shall show that these four maps are close to pushforward in the sense that their differences decay exponentially. Secondly, we shall show that exponentially decaying vector fields are asymptotically translation invariant with zero limit. This will show that the image under these maps is the sum of two asymptotically translation invariant vector fields, and so is asymptotically translation invariant; since it is clear that the limits defined in Definition 5.5.17 are additive, it also shows the limits are preserved.

We begin by working with $T$ which is either $T_1$ or $T_2$. This is defined pointwise by a map on $TM \oplus J^1(TM) \oplus Gr_n(TM)$. By Proposition 5.5.9, this map is smooth and depends continuously on the metric in the sense of that proposition. Similarly, pushforward, which we shall denote $P$, is defined pointwise by a map on $TM \oplus J^1(TM)$, which by the appropriate part of Proposition 5.4.8 is smooth and depends continuously on the metric. We shall restrict to the submanifold $N$ of $TM \oplus J^1(TM) \oplus Gr_n(TM)$ defined in Definition 5.5.19. For any such $u$ and $\ell$, we immediately have

$$|T(u, 0, \ell) - P(u, 0)| = 0.$$  \hspace{1cm} (5.5.31)

Since these maps are smooth, it follows that for each $p$ and $\rho$ we have a constant $C$ with

$$|T(u, (v, \nabla v), \ell) - P(u, (v, \nabla v))| \leq C|(v, \nabla v)||u|,$$  \hspace{1cm} (5.5.32)

as a pointwise estimate, for $|v, \nabla v| < \rho$. We may fix $\rho$ small once for all; since $v$ decays with all its derivatives, we will end up in this case far enough along the end. Note that since the Grassmannian of $n$-planes in $T_pM$ is compact, it is immediate that we can choose the constant
independently of ℓ. Since we may differentiate and obtain the same results for jets, we similarly have a local estimate

\[ \|T(u, (v, \nabla v), \ell) - P(u, (v, \nabla v))\|_{C^k} \leq C_k \|v\|_{C^{k+1}} \|u\|_{C^k}. \]  

(5.5.33)

Again, we get this strictly by choosing \( \rho > 0 \) and then for \( \|v\|_{C^{k+1}} < \rho \), with \( C_k \) depending on \( \rho \), but just as above may choose \( \rho \) once for all. Just as in Proposition 5.4.9, we now only have to show that \( C_k \) may be chosen uniformly as \( v \) is exponentially decaying and \( u \) is bounded by (i). But, using that we have (5.5.14) and its analogue for pushforward, the derivatives are continuous in the metric. \( L \) itself can be regarded as a finite-dimensional parameter space of metrics, and as \( \rho \) heads to the end of \( L \), the metric converges to a limit \( \tilde{g} \). Hence, the whole map depends continuously on the point of \( L \), and we may choose a uniform bound.

This shows that the image of a normal vector field by \( T \) and by pushforward differ by an exponentially decaying vector field.

As for the inverses, we know from the inverse part of Proposition 5.5.9 and the analogous result, proved identically, for the inverse of pushforward that these satisfy the same properties on \( N' \): the result is then proved in exactly the same way.

It now only remains to show that these exponentially decaying vector fields are asymptotically translation invariant with zero limit, that is they are the pushforwards of exponentially decaying vector fields on \( \tilde{L} \). To do this, we again apply the inverse of pushforward. Just as above, we know that we can take an appropriate smooth submanifold of \( TM \times J^1(TM) \), and define a map that is smooth and continuous in the metric. Note that as here we need to ensure the vector field to \( L \) is normal, this submanifold is not \( N' \). It follows in the same way that the image of a pair of exponentially decaying objects is exponentially decaying, since the push-forward of zero is always zero and we converge to a constant metric. The converse result can be proved the same way: an asymptotically translation invariant vector field with zero limit is exponentially decaying.

As for (iii), the relationship between asymptotically translation invariant normal vector fields and asymptotically cylindrical manifolds holds just as sketched in subsection 5.2.3; passing to pointwise operators on appropriate bundles as in this section enables us to formalise that argument. Note that the condition that the limit was zero for (ii) meant we did not have to assume cylindricality of the ambient metric, whereas that will be necessary in this case.

Remarks. We make three remarks. Firstly, it seems very plausible that if \( M, L \) and \( L' \) are asymptotically cylindrical with \( L' = \exp_v(L) \), but \( L' \) and \( L \) have different limits, so that \( v \) is only asymptotically translation invariant, and \( v \) is small in the extended weighted topologies defined analogously to those in section 2.2, then \( T_i \) again maps asymptotically translation invariant normal vector fields to asymptotically translation invariant normal vector fields, and so on. (Similarly, we expect (iv) to hold even for an asymptotically cylindrical \( M \).) To see the
difficulty that arises in the proof, consider the case where everything except the metric on \( M \) is translation invariant. That is, consider \( L = K \times (R, \infty) \) and \( L' = \exp_v(K \times (R, \infty)) \) with \( v \) translation invariant either in the natural sense on a cylindrical submanifold or in the sense that the corresponding one-form is translation invariant. To show that \( T_i u \) is asymptotically translation invariant, we would have to quantify the dependence of \( T_i u \) on the metric at each point of \( L \). This is difficult using the standard results on ordinary differential equations we used in the proof of Proposition 5.5.9, because to get a quantified dependence we would need to explicitly use the metric (rather than using a parameter space) and for each point of \( L \), \( T_i u \) depends on the metric along a whole curve, and consequently we do not have a parameter in a finite-dimensional space. It seems likely, however, that some suitable result will be true.

Secondly, we could try to replace Definition 5.5.17. If \( M \) is asymptotically cylindrical, then we can equip \( TM \) with a metric so that it is itself asymptotically cylindrical. We may then consider asymptotically cylindrical submanifolds of \( TM \): it seems plausible that these should consist of pairs of an asymptotically cylindrical submanifold \( L \) and an asymptotically translation invariant section of \( TM|_L \).

Finally, we note more generally from the proof of Proposition 5.5.21 that for \( L' = \exp_v(L) \), the difference of the vector fields \( T_i u \) and \( (\exp_v)_s u \) on \( L' \) can be bounded by \( u \) on \( L \) and a bound on \( v \). This idea will be useful later, although we will not use this statement of it directly.

Using Proposition 5.5.21, we will now define a patching of normal vector fields. We first define what it will mean for asymptotically translation invariant normal vector fields to match.

**Definition 5.5.22.** Let \( M_1 \) and \( M_2 \) be a matching pair of asymptotically cylindrical manifolds and let \( L_1 \) and \( L_2 \) be a matching pair of asymptotically cylindrical submanifolds as in Definition 5.1.13. Suppose that \( v_1 \) and \( v_2 \) are normal vector fields on \( L_1 \) and \( L_2 \) respectively, with limits \( \tilde{v}_1 \) and \( \tilde{v}_2 \) on \( K_1 \) and \( K_2 \) respectively. We say that \( v_1 \) and \( v_2 \) match if \( F_* \tilde{v}_1 = \tilde{v}_2 \).

Defining a patching is not totally straightforward. Recall from Definition 5.1.14 that to glue asymptotically cylindrical submanifolds \( L_1 \) and \( L_2 \), we cut them off to form \( \hat{L}_1 \) and \( \hat{L}_2 \) and then identify. Consequently, to define a normal vector field on \( L^T \), we will need normal vector fields on \( \hat{L}_1 \) and \( \hat{L}_2 \), and so to define a gluing map of normal vector fields, we will need to transfer normal vector fields from \( L_i \) to \( \hat{L}_i \). We do this with the maps \( T_1 \) and \( T_2 \) of Definition 5.5.6, and also using \( T_3 \) of Definition 5.5.14.

To make proving Proposition 5.5.24, that the gluing of normal vector fields is nearly the derivative of the gluing of submanifolds, as easy as possible, we make a somewhat complicated definition of the gluing of normal vector fields.

**Definition 5.5.23.** Suppose that \( M_1 \) and \( M_2 \) are matching asymptotically cylindrical Riemannian manifolds, \( L_1 \) and \( L_2 \) are matching asymptotically cylindrical submanifolds, and \( u_1 \) and \( u_2 \) are matching normal vector fields on \( L_1 \) and \( L_2 \).
Let $K_1 \times (R_1, \infty)$, $v_1$, $K_2 \times (R_2, \infty)$ and $v_2$ be as in Definition 5.1.11, and let $\hat{L}_1$ and $\hat{L}_2$ be as in Definition 5.1.14. Let $T_1$ and $T_2$ be the transfer maps of Definition 5.5.6 with $L = K_i \times (R_i, \infty)$ and $L' = \exp_{v_i}((R_i, \infty) \times K_i)$; let $\hat{T}_1$ and $\hat{T}_2$ be the corresponding transfer maps with $L = K_i \times (R_i, \infty)$ and $L' = \exp_{\varphi_{v_i}}((R_i, \infty) \times K_i)$. Let $\hat{u}$ be the limit $\hat{u}_1 = \hat{u}_2$, and let $\hat{T}_2$ be the limit map of $T_2$ (that therefore takes normal vector fields on $K_i$ to normal vector fields on $\exp_2(K_i)$).

For each $i$, $\hat{T}_2^{-1}\hat{u}$ extends to a translation invariant section of $TM|_{K_i \times (R, \infty)}$, and we may take the normal part of this to form a normal vector field $w_i$ on $K_i \times (R, \infty)$. Now consider the normal vector field $u_i - T_2w_i - T_3w_i$ on $\exp_{v_i}((R_i, \infty) \times K_i)$. By Proposition 5.5.21, this is asymptotically translation invariant with zero limit, so is exponentially decaying. Let

$$x_i = T_1^{-1}(u_i - T_2w_i - T_3w_i).$$

(5.5.34)

$x_i$ is an exponentially decaying normal vector field on $K_i \times (R, \infty)$ by Proposition 5.5.21 again. Then $\hat{T}_1x_i + \hat{T}_2w_i + \hat{T}_3w_i$ are asymptotically translation invariant normal vector fields on $\exp_{\varphi_{v_i}}((R_i, \infty) \times K_i)$.

Now let

$$\hat{u}_i = \hat{T}_1(\varphi_Tx_i) + \hat{T}_2w_i + \hat{T}_3w_i.$$  

(5.5.35)

Since $T > R - 2$, on $K_i \times (R, R + \epsilon)$ for sufficiently small $\epsilon$ we have $v_i = \varphi_{v_i}$. Calculation then shows that $\hat{T}_1\varphi_Tx_i + \hat{T}_2w_i + \hat{T}_3w_i = u_i$ on $\exp_{\varphi_{v_i}}((R, R + \epsilon) \times K_i) = \exp_{\varphi_{v_i}}((R, R + \epsilon) \times K_i)$. Hence (5.5.35) can be extended to give an asymptotically translation invariant normal vector field on the submanifold $\hat{L}_i$.

Furthermore, on $\exp_{v_i}((T, T + 1) \times K_i)$ we have $\hat{u}_i = \hat{T}_2w_i$. This is then independent of $i$ as $w_i$ is chosen to be translation invariant.

Hence when we identify the $\hat{L}_i$ to give a submanifold $L^T$ of $M^T$, we may also identify $\hat{u}_1$ and $\hat{u}_2$ to give a section of $TM^T|_{L^T}$. Since the metric on $M^T$ is not just an identification of $g_1$ and $g_2$, this section need not be normal: so, finally, take its normal part.

With this definition, we can prove

**Proposition 5.5.24.** Let $\mathcal{U}$ parametrise a finite-dimensional submanifold of matching pairs $(L_1, L_2)$. Then the restriction to $\mathcal{U}$ of the gluing map of submanifolds defined in Definition 5.1.14 is a smooth map to its image in the space of deformations of $L^T$. Its derivative is exponentially close (in $T$) to the gluing map of normal vector fields described in Definition 5.5.23.

**Proof.** First we show that the gluing map described in Definition 5.1.14 is smooth. The identification map is obviously smooth, so we just have to show that the cutoff map $L_i \rightarrow \hat{L}_i$ is smooth. Hence we work in a single asymptotically cylindrical manifold $M$, and with a family $L_s$ of asymptotically cylindrical submanifolds with cross-sections $K_s$. Given $L_s$, $\hat{L}_s$ is given by cutting off the normal vector field $v_s$ to $K_s \times (R, \infty)$ that gives the end of $L_s$. $v_s$ depends smoothly
on \( L_s \) and \( K_s \) by the remark after Proposition 5.5.10, and \( K_s \) certainly depends smoothly on \( L_s \). The required cutoff function is \( \varphi_s = \varphi_T \circ \exp_{w_s} \circ t \) where \( t \) is the inclusion of \( K_0 \times (R, \infty) \) and \( w_s \) is the normal vector field to \( K_0 \times (R, \infty) \) giving \( K_s \times (R, \infty) \). Again using the remark after Proposition 5.5.10, \( w_s \) depends smoothly on \( K_s \). Hence the normal vector fields \( \varphi_s v_s \) on \( K_s \times (R, \infty) \) depends smoothly on \( L_s \). By composition it follows that \( \hat{L}_s \) depends smoothly on \( s \).

To evaluate the derivative, we again may just work with \( L_i \mapsto \hat{L}_i \) as the derivative of the identification map is obviously again the identification map. By using the transfer map \( T_{2,s} \) as in Proposition 5.5.15 to consider the family \( v_s \) of normal vector fields in the previous paragraph on the same space, we can write

\[
L_s = \exp_{T_{2,s}(v_s)}(K_s \times (R, \infty)) = \exp_{T_{2,s}(v_s)}(K_0 \times (R, \infty)), \tag{5.5.36}
\]

\[
\hat{L}_s = \exp_{T_{2,s}(\varphi_s v_s)}(K_0 \times (R, \infty)) \tag{5.5.37}
\]

where \( v_s \) is a decaying normal vector field on \( K_0 \times (R, \infty) \) and \( w_s \) gives \( K_s \times (R, \infty) \) from \( K_0 \times (R, \infty) \). We note that as \( M \) is not cylindrical, \( w_s \) need not be translation invariant, but nevertheless \( w_s \) can be uniquely determined by its limit \( \hat{w}_s \). Since \( \mathcal{U} \) is finite-dimensional, around each point we also know that the \( v_s \) must decay at some uniform rate. That is, the derivatives \( v' \) and \( w' \) in \( s \) decay and are uniquely determined by their limit respectively.

We now let \( T_1, T_2, T_3, \hat{T}_1, \hat{T}_2 \) and \( \hat{T}_3 \) be as in Definition 5.5.23. We use these maps to analyse the derivative. It follows from Proposition 5.5.16 that the normal vector field to \( L_0 \) giving the tangent to the curve \( L_s \) is \( w' = T_2 w' + T_1 v' + T_3 w' \); similarly, the normal vector field to \( \hat{L}_0 \) giving the tangent to the curve \( \hat{L}_s \) is \( \hat{T}_2 w' + \hat{T}_1 (\varphi_s v_s)' + \hat{T}_3 w' \). Applying the product rule, and working at each point \( p \in K_0 \times (R, \infty) \), we have that this is \( \hat{T}_2 w' + \hat{T}_1 (\varphi_0 v') + \nabla_{w'} \varphi_T + \hat{T}_3 w' \). Note that \( \varphi_T \) is defined on \( M \), so the normal derivative \( \nabla_{w'} \varphi \) makes sense.

The derivative is consequently the map

\[
T_2 w' + T_1 v' + T_3 w' \mapsto \hat{T}_2 w' + \hat{T}_1 (\varphi_0 v') + \nabla_{w'} \varphi_T + \hat{T}_3 w'. \tag{5.5.38}
\]

In order to obtain a complete description of this map we have to explain how to obtain \( w' \) and \( v' \) from \( T_2 w' + T_1 v' + T_3 w' \). Since \( v_0 \) is exponentially decaying, it follows by (ii) of Proposition 5.5.21 that \( T_1 v' \) is asymptotically translation invariant, with the same limit as \( v' \), but since \( v' \) is exponentially decaying \( T_1 v' \) is too. Similarly it follows that \( T_3 w' \) is exponentially decaying.

On the other hand, since \( w' \) is uniquely determined by its limit and the limit of \( T_2 w' \) is the action of the limit map \( \hat{T}_2 \) on the limit \( \hat{w}' \), \( T_2 w' \) is also uniquely determined by its limit. The limit of \( T_2 w' \) is the limit of \( T_2 w' + T_1 v' + T_3 w' \), hence the map of (5.5.38) is well-defined.

Note that at a point with \( r < T - 1 \), we have that \( T_1 = \hat{T}_1, T_2 = \hat{T}_2, T_3 = \hat{T}_3, \varphi_T \equiv 1 \), and \( \nabla \varphi_T = 0 \). Hence (5.5.38) becomes the identity map at these points, and so it can be extended to a map from normal vector fields on \( L_0 \) to normal vector fields on \( \hat{L}_0 \).
Rewriting (5.5.38) in terms of finding $w'$, then $v'$ and then evaluating the right hand side we see that the difference between (5.5.38) and the direct gluing map of Definition 5.5.23 is that in that case we just apply $\varphi_0$ to $v'$ (which is $x_i$ in the notation of Definition 5.5.23) rather than $\varphi + \nabla_u \varphi$. Thus the difference is $\hat{T}_1(\nabla_{w'} \varphi_T v')$. Since $v'$ is exponentially decaying, by Proposition 5.5.21 so is $\hat{T}_1 v'$. But also $\hat{T}_1$ is given by a bundle map, so $\hat{T}_1(\nabla_{w'} \varphi_T v')$ is just $\hat{T}_1 v'$ multiplied by the composition of $\nabla_{w'} \varphi_T$ with the diffeomorphism from $L_0$ to $K_0 \times (R, \infty)$; it is then clear that the whole difference is exponentially small in $T$. 

\[ \square \]

### 5.5.2 Nearly special Lagrangian submanifolds

In this subsection, we specialise to the case where our submanifolds are close to special Lagrangian. By Lemma 5.4.3, on such submanifolds normal vector fields can be identified with one-forms. This means that we can write the maps $T_1$ and $T_2$ of Definition 5.5.5 and hence of Definition 5.5.6, and the map $T_3$ of Definition 5.5.14, in terms of one-forms: we carry this out carefully. We then return again to asymptotically translation invariant normal vector fields. We shall show that a normal vector field being asymptotically translation invariant is equivalent to the corresponding one-form being so in Proposition 5.5.33. Finally, we show that the patching map of Definition 5.5.23 is then close to the patching map of one-forms with respect to the identification.

To do this, we first need to introduce the requirement that the submanifold be “nearly special Lagrangian” to the bundle $TM \oplus J^1(TM) \oplus Gr_n(TM)$ used in Definition 5.5.5 and to the corresponding “target bundle” in which the image of the extended map (5.5.15) lies. We make the following

**Definition 5.5.25.** Let $M$ be a Calabi–Yau manifold. Let $O$ be the subbundle of $Gr_n(TM)$ consisting of those subspaces $\ell$ for which $|\omega|_\ell < 1$. Let $O'$ be the subbundle of $J^1(TM) \oplus O$ such that also the subspace $\ell'$ constructed in Definition 5.5.5 satisfies $|\omega|_{\ell'} < 1$.

Since the map from $\ell$ to $|\omega|_\ell$ is smooth and so continuous, and the map from $((v, \nabla v), \ell)$ to $\ell'$ is also smooth, $O$ and $O'$ are open subbundles.

Recall the spaces $N$ and $N'$ from Definition 5.5.19. $N$ is the subbundle of $TM \oplus J^1(TM) \oplus Gr_n(TM)$ consisting of $(u, (v, \nabla v), \ell)$ with $u$ normal to $\ell$. $N'$ is the corresponding final space: the subspace of $TM \times (J^1(TM) \oplus Gr_n(TM))$ given by $(u, (v, \nabla v), \ell)$ with $u$ a tangent vector at $\exp_p(v)$ (if $v$ and $\ell$ are at $p$) and $u$ normal to the subspace $\ell'$ of $T_{\exp_p(v)} M$ given by pushforward of $\ell$. We showed in Proposition 5.5.20 that these were manifolds. To pass to one-forms, we will also need the corresponding subspace of $T^* M \oplus J^1(TM) \oplus Gr_n(TM)$. We make

**Definition 5.5.26.** Let $M$ be a Calabi–Yau manifold. Let $F$ be the subbundle of $T^* M \oplus J^1(TM) \oplus Gr_n(TM)$ consisting of $(\alpha, (v, \nabla v), \ell)$ with $\alpha^2$ in $\ell$.

The intersection of $N$, $N'$ and $F$ with $O'$ in their $Gr_n(TM)$ component are then also open subbundles. We may make
Definition 5.5.27. Let $I : N \cap O' \to F \cap O'$ be given by taking $(u, (v, \nabla v), \ell)$, contracting $u$ with $\omega$, and then taking the orthogonal projection to the tangential part.

Let $I' : N' \cap O' \to F \cap O'$ be given similarly by contracting $u$ with $\omega$ at $\exp_v(p)$, pulling back the resulting covector using $(v, \nabla v)$ by applying the dual of the map (5.5.12), and the taking the orthogonal projection to the tangential part.

$I$ and $I'$ are pointwise representatives of the map $v \mapsto \iota_v \omega|_L$; we use them to show that this map “depends smoothly on $L$”.

We have

Proposition 5.5.28. $I$ and $I'$ are smooth maps, depending continuously on the metric as in Proposition 5.5.9, and locally around any $(u, 0, 0, \ell)$ are diffeomorphisms.

Proof. The contraction maps and orthogonal projections are clearly smooth and continuous in the metric; hence $I$ is. It remains to deal with the pullback to show that $I'$ is smooth and continuous in the metric. This follows exactly as in the proof of Proposition 5.4.8. 

This enables us to make our definition of what $T_1$ and $T_2$ look like in terms of one-forms.

Definition 5.5.29. Let $M$ be a Calabi–Yau manifold and consider $\mathcal{F} \cap O'$. Define maps $\tau_1$ and $\tau_2$ on this space by $\tau_i = I' \circ T_i \circ I^{-1}$, with $T_i$ the extended map from Proposition 5.5.9. These maps are also local diffeomorphisms around any $(\alpha, 0, 0, \ell)$. As this is linear in the first component, we may equivalently regard $\tau_i$ as a map $J^1(TM) \oplus \text{Gr}_n(TM) \to T^*M \otimes TM$. Similarly define a map $\tau_3$ on a suitable bundle by composition of the pointwise version of the map $T_3$ of Definition 5.5.14 with suitably extended versions of $I'$ and $I$: note from the discussion after that definition that we need the first jet of $\alpha$ to do this. This is thus more complicated, and throughout this discussion we will omit the details.

Generalising Definition 5.5.6, we have

Definition 5.5.30. Let $M$ be a Calabi–Yau manifold and $L$ a submanifold such that $\|\omega|_L\|_{C^0} < 1$. Given a sufficiently small normal vector field $v$ on $L$, we can construct $L' = \exp_v(L)$ and again have $\|\omega|_{L'}\|_{C^0} < 1$. For $i = 1, 2$, we can then define a map $\tau_i$ from one-forms on $L$ to one-forms on $L'$ by taking $\alpha$ to $\tau_i(\alpha, (v, \nabla v), T_pL)$ where $\nabla v$ is extended somehow as in Definition 5.5.6. Similarly, we can define a map $\tau_3$ by taking some extension of $\alpha$; the map will not depend on the extension.

We then have that these maps of one-forms correspond to the maps $T_i$ of normal vector fields.
Proposition 5.5.31. Let $M$, $L$ and $L'$ be as in Definition 5.5.30, and $i \in \{1, 2\}$. Then the following diagram commutes:

\[
\begin{array}{ccc}
\text{normal vector fields on } L & \xleftarrow{\tau_i \omega|_L} & \text{one-forms on } L \\
\downarrow T_i \text{ (Def. 5.5.6)} & & \downarrow \tau_i \text{ (Def. 5.5.30)} \\
\text{normal vector fields on } L' & \xleftarrow{\tau_i' \omega|_{L'}} & \text{one-forms on } L = L' \\
\end{array}
\]

(5.5.39)

where the horizontal maps are determined by Lemma 5.4.3. For example, given a one-form $\alpha$ on $L$, we have the one-form $\tau_i \alpha$ on $L$. We find another one-form on $L$ by taking a normal vector field $u$ on $L$ such that $\nu_u \omega|_L = \alpha$, applying $T_i$ to $u$ to get a normal vector field on $L'$, and then finding the corresponding one-form $\nu_{T_i u} \omega|_{L'}$, and these two one-forms are equal. Similarly, given a one-form on $L = L'$, the one-forms on $L$ constructed by $\tau_i^{-1}$ and $T_i^{-1}$ are the same. Finally, the corresponding diagram with $T_3$ and $\tau_3$ also commutes.

Proof. For $T_1$ and $T_2$, it suffices to show that $I$ and $I'$ as in Definition 5.5.27 define maps between normal vector fields and sections corresponding to the isomorphism of Lemma 5.4.3. That is, we need to show that if $u$ is a normal vector field on $L$, then the map given by $I(u, (v_p, (\nabla v)_p), T_p L)$ is well-defined and gives the one-form $\nu_u \omega|_L$ and that if $u$ is a normal vector field on $L'$, then $I'(u_{\exp_p (v_p)}, (v_p, (\nabla v)_p), T_p L)$ is well-defined and gives $\nu_{u} \omega|_{L'}$. This is essentially immediate from the definition.

As before, we omit the details for the correspondence of $T_3$ and $\tau_3$. \(\square\)

Furthermore, we have the following regularity result for the $\tau_i$ of Definition 5.5.29 corresponding to Proposition 5.5.9.

Proposition 5.5.32. For $i = 1, 2, 3$, $\tau_i$ are smooth maps, and depend continuously on the Calabi–Yau structure in the same sense as $T_i$ depends continuously on the metric (described in Proposition 5.5.9); in particular, we have the estimate corresponding to (5.5.14).

Proof. For $i = 1, 2$, since $T_i$ is smooth by Proposition 5.5.9 and $I'$ and $I^{-1}$ are smooth by Proposition 5.5.28, it is immediate that $\tau_i$ are smooth maps. It remains to show that they have the appropriate continuity property. Since composition is a continuous operation, it suffices to verify that $I'$, $I^{-1}$ and $T_i$ have the appropriate continuity property. Since the metric depends continuously on the Calabi–Yau structure, as in Proposition 4.1.5, $T_i$ certainly does. For $I'$ and $I^{-1}$, we have dependence on $\omega$ at points and continuous dependence on the metric: again, each of these is continuous in the Calabi–Yau structure.

The case of $\tau_3$ is similar; again, we omit the details. \(\square\)

We now prove that asymptotically translation invariant one-forms correspond to asymptotically translation invariant normal vector fields. That is, Definition 5.5.17 is equivalent to the one-forms we used in subsection 5.2.3.
Proposition 5.5.33. Let $M$ be an asymptotically cylindrical Calabi–Yau manifold, and let $L$ be an asymptotically cylindrical submanifold so that $\omega|_{\mathring{L}}$ is small with all derivatives, so that the estimate (5.4.17) of Lemma 5.4.3 applies in every $C^k$ space. Then a normal vector field $v$ on $L$ is asymptotically translation invariant in the sense of Definition 5.5.17 if and only if the one-form $\iota_v \omega|_{\mathring{L}}$ is asymptotically translation invariant.

Proof. As usual, we are only interested in the end of $L$, and this is asymptotic to a cylindrical submanifold $\mathring{L}$. There is an exponentially decaying normal vector field $v$ on $\mathring{L}$ whose image under the exponential map is $L$; this extends to a diffeomorphism $\exp_v$ between tubular neighbourhoods decaying to the identity, and Definition 5.5.17 says a vector field $u$ along $L$ is asymptotically translation invariant if and only if it is $\left(\exp_v\right)_* w$ for some asymptotically translation invariant vector field $w$ along $\mathring{L}$.

It suffices to suppose that $L = \mathring{L}$, as follows. With the notation as above,

$$\iota_u \omega|_{\mathring{L}} = \iota_{\left(\exp_v\right)_* w} \exp_v \omega|_{\exp_v(\mathring{L})} = \iota_v \exp_v^* \omega|_{\mathring{L}}.$$  \hfill (5.5.40)

Moreover, if $u$ is normal, $w$ is normal with respect to the metric $\exp_v^* g$ and vice versa. Hence, the result is true for $L$ if and only if it is true for $\mathring{L}$ with the metric $\exp_v^* g$ and the 2-form $\exp_v^* \omega$; these correspond to the asymptotically cylindrical Calabi–Yau structure $(\exp_v^* \Omega, \exp_v^* \omega)$, and as $v$ is exponentially decaying and $\omega$ asymptotically translation invariant $\exp_v^* \omega$ also restricts to a small form (essentially as in Proposition 5.5.21), and so this reduces the problem to $\mathring{L}$.

If $L$ is cylindrical, we just note that $\omega$ determines an asymptotically translation invariant section of $\bigwedge^2 T^* M|_L$. The projection map $T^* M|_L \to T^* L$ is also asymptotically translation invariant; indeed, as $L$ is cylindrical it is translation invariant, and so we obtain an asymptotically translation invariant section of $T^* L \otimes (T^* M|_L)$. Hence, if $v$ is an asymptotically translation invariant normal vector field, the corresponding one-form is asymptotically translation invariant. Moreover, we know that if we restrict this section to $\nu_L$, it defines an asymptotically translation invariant section of $T^* L \otimes (\nu_L)^*$ (this is essentially composition with the asymptotically translation invariant inclusion of $(\nu_L)^*$ into $T^* M|_L$). By Lemma 5.4.3, this section consists of isomorphisms, so we can invert it; it is easy to see that the inverse section is again asymptotically translation invariant. This completes the cylindrical case, and hence the proof. \hfill $\square$

Remarks. Note that it now follows from Proposition 5.5.21 that $\tau_1, \tau_1^{-1}, \tau_2$ and $\tau_2^{-1}$ preserve asymptotically translation invariant one-forms and $\tau_3$ maps them to decaying one-forms (noting that the limits correspond using $\mathring{u} \mapsto \iota_{\mathring{u}} \mathring{\omega}|_{\mathring{L}}$, which also follows from the proof of Proposition 5.5.33). These can alternatively be proved directly using similar methods to Proposition 5.5.21, replacing the pushforward map with the identity map of $T^*_p L$.

Moreover, since $\exp_v^* \omega$ is close to $\omega$, the argument of Proposition 5.5.33 combined with the third of the Remarks after Proposition 5.5.21 implies that $\tau_1$ and so forth are close to the identity if the normal vector field $v$ between submanifolds is small. This can also be proved
more simply directly replacing pushforward with the identity; see Proposition 5.7.4 below for a particular case.

We now turn back to gluing normal vector fields. If $L_1$ and $L_2$ are a matching pair of special Lagrangians, and Hypothesis 5.3.1 holds, then by Proposition 5.3.5 and Lemma 5.4.3 $v \mapsto \iota_v \omega|_{L^T}$ gives an isomorphism between normal vector fields and one-forms on $L^T$. As $L_1$ and $L_2$ are themselves special Lagrangian we also have such an isomorphism on $L_1$ and $L_2$. With respect to these isomorphisms, we have

**Proposition 5.5.34.** Let $M_1$, $M_2$, $L_1$, $L_2$ and $L^T$ be as in Proposition 5.3.5. Suppose $T$ is sufficiently large that $v \mapsto \iota_v \omega|_{L^T}$ is an isomorphism. Then we can induce a gluing map of normal vector fields from the gluing map of one-forms. This map differs from the gluing map of Definition 5.5.23 by a linear map decaying exponentially in $T$ in the sense of Remark 4.3.8.

**Proof.** Let $\alpha_1$ and $\alpha_2$ be matching asymptotically translation invariant one-forms with common limit $\hat{\alpha}$. We shall show that the difference of the one-form given by gluing these using Definition 5.5.23 and $\gamma_T(\alpha_1, \alpha_2)$ is exponentially decaying; this proves the result as it follows from Proposition 5.3.5 and Lemma 5.4.3 that the isomorphism between normal vector fields and one-forms on $L^T$ is bounded uniformly in $T$.

Let $u_i$ be the normal vector field corresponding to $\alpha_i$ using the Calabi–Yau structure $(\Omega_i, \omega_i)$, and let $u$ be the gluings of $u_1$ and $u_2$ using Definition 5.5.23.

We prove this in two stages. Firstly, we prove that on each cutoff cylindrical submanifold $\hat{L}_i$, with cutoff normal vector field $\hat{u}_i$ as in Definition 5.5.23, and cutoff one-form $\hat{\alpha}_i$, $\iota_{\hat{u}_i} \omega_i|_{\hat{L}_i} - \hat{\alpha}_i$ is exponentially small in $T$. We then note that $\iota_{\hat{u}_i} \omega_i|_{\hat{L}_i} - \iota_{\hat{w}_i} \omega_i|_{\hat{L}_i}$ is exponentially small in $T$. This proves that $\iota_u (\gamma_T(\omega_1, \omega_2)|_{L^T} - \gamma_T(\alpha_1, \alpha_2)$ is exponentially small in $T$. Secondly, we use Hypothesis 5.3.1 to show that $\iota_u (\gamma_T(\omega_1, \omega_2))|_{L^T}$ is exponentially small in $T$.

To do the first of these we simply replace $T_1$, $T_2$, $T_3$, $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$ used in Definition 5.5.23 with the corresponding maps $\tau_1, \tau_2, \tau_3, \hat{\tau}_1, \hat{\tau}_2$ and $\hat{\tau}_3$ of one-forms as in Definition 5.5.29: Proposition 5.5.31 says that these are just the one-form versions of the $T_i$ and $\hat{T}_i$.

By construction, $\hat{\alpha}_i$ and $\iota_{\hat{u}_i} \omega_i|_{\hat{L}_i}$ are both equal to $\alpha$, for $t < T - \frac{3}{2}$, say. (As the cutoff in the normal vector field case is taken on $K \times (R, \infty)$, we have to allow a slight change of parameter $t$ since this may not be preserved by the map from $K \times (R, \infty)$ to $L_i$.) Hence, it suffices to consider the difference where $t > T - \frac{3}{2}$, and so it suffices to prove that $\tau_1$, $\tau_2$, $\hat{\tau}_1$ and $\hat{\tau}_2$ is exponentially close in $T$ to the identity, and $\tau_3$ and $\hat{\tau}_3$ are exponentially small in $T$. This follows by the reasoning described in the second remark before this proposition, as we now are working with $v$ which we may assume exponentially small in $T$.

It now only remains to show that $\iota_u (\gamma_T(\omega_1, \omega_2))|_{L^T}$ is exponentially small in $T$. However, Hypothesis 5.3.1 says that $\gamma_T(\omega_1, \omega_2) - \omega^T$ is exponentially small in $T$; $u$ is uniformly bounded in $T$ since $\hat{u}_1$ and $\hat{u}_2$ are, so $\iota_u (\gamma_T(\omega_1, \omega_2))$ is exponentially small in $T$, and Corollary 5.3.4 then implies that the restriction is.

□
5.6 The Laplacian on normal vector fields

In this section, we use Lemma 5.4.3 similarly to subsection 5.5.2 to define a Laplacian on the normal vector fields on a submanifold that is close to special Lagrangian. This consequently defines a notion of the harmonic part of a normal vector field. We then show in Proposition 5.6.5 that, with an appropriate identification of normal vector fields from section 5.5, this harmonic part depends smoothly on the submanifold concerned and we identify the derivative of the map \( v \mapsto \text{hpt} \, v \) in Proposition 5.6.6, in terms of the derivative of the Laplacian induced by the identifications.

We make

**Definition 5.6.1.** Suppose that \( L \subset M \) is a submanifold and \( (\Omega, \omega) \) is an SU(\( n \)) structure around it in the sense of Definition 4.1.7. This induces a Riemannian metric on \( L \) and consequently, as in chapter 3, a Laplace–Beltrami operator on the differential forms on \( L \). Suppose that Lemma 5.4.3 holds for the \( SU(n) \) structure \( (\Omega, \omega) \) around \( L \), that is

\[
v \mapsto \iota_v \omega|_Lan

is an isomorphism between normal vectors and one-forms. Then (5.6.1) induces linear differential operators \( \Delta \) and \( d + d^* \) on normal vector fields.

Since by the estimates of (5.4.17) we have an isomorphism between differential one-forms and normal vector fields of given regularity, all the properties of \( d + d^* \) and \( \Delta \) carry over. In particular

i) The kernel of \( \Delta \) is a finite-dimensional space of normal vector fields forming the kernel of \( \Delta \); if \( \Delta v = 0 \), \( v \) will be called harmonic.

ii) If \( L \) is compact the kernels of \( \Delta \) and \( d + d^* \) coincide.

iii) If \( L \) is compact, for any normal vector field \( v \) we may write \( v = u + \Delta w \) with \( u \) harmonic.

We shall then write \( u = \text{hpt}(v) \).

Since the image of \( \Delta \) on 1-forms is \( L^2 \)-orthogonal to the harmonic forms, we shall call a normal vector field in the image of \( \Delta \) orthoharmonic. Note that as (5.6.1) need not be an isometry if \( L \) is not special Lagrangian, harmonic normal vector fields and orthoharmonic normal vector fields need not be \( L^2 \)-orthogonal.

We now note that \( \text{hpt} \) and a left inverse \( \Delta^{-1} \), considered as maps of forms, depend smoothly on the metric \( g_s \). As in, for instance, Proposition 5.5.10, we shall choose a finite-dimensional family of metrics parametrised by \( \mathcal{U} \).
Proposition 5.6.2. Let $M$ be a compact manifold and $g_s$ be a finite-dimensional smooth family of smooth metrics on it parametrised by $s \in U \subset \mathbb{R}^m$, with a base point $0 \in U$. Then the map

$$hpt : U \times \Omega^p(M) \rightarrow \Omega^p(M)$$

is smooth, perhaps after shrinking the neighbourhood $U$ of $g_0$.

Proof. Since $M$ is a compact manifold, we know by Corollary 3.3.2 that for any Riemannian metric $g$ the space of $g$-harmonic $p$-forms on $M$ bijects with the de Rham cohomology space $H^p(M)$. $H^p(M)$ is independent of $g$, so we first ask if the $g_s$-harmonic representative of a given class $[\alpha]$ of $H^p(M)$ depends smoothly on $s$. Because this question can be answered using only closed forms, it is slightly simpler that approaching the question of the harmonic part of a given form directly.

By Corollary 3.3.2 for the metric $g_0$, we can find a $g_0$-harmonic $\alpha$ representing $[\alpha]$. $\alpha$ is smooth by elliptic regularity. We pick an integer $k$ and $\mu \in (0, 1)$ and apply the inverse function theorem to the following map

$$F : U \times \{\text{exact } C^{k+1,\mu}(p-1)\text{-forms}\} \rightarrow U \times \{\text{exact } C^{k,\mu}(n-p+1)\text{-forms}\},$$

$$(s, \beta) \rightarrow (s, d \ast g_s(\alpha + \beta)).$$

(5.6.3)

Since the Hodge star depends smoothly on the metric, $F$ is a smooth map. We want to show that at $(0, 0)$ its derivative is an isomorphism.

To prove this, since the identity from $T_0U$ to $T_0U$ is obviously an isomorphism, it suffices to show that

$$\beta \mapsto d \ast g_0 \beta$$

is an isomorphism from exact $C^{k+1,\mu}$ forms to exact $C^{k,\mu}$ forms. Since $g_0$ is smooth, we know that $\ast g_0$ is an isomorphism from exact $C^{k+1,\mu}$ forms to coexact $C^{k+1,\mu}$ forms. We further know by standard Hodge theory that $d$ is an isomorphism from coexact $C^{k+1,\mu}$ forms to exact $C^{k,\mu}$ forms; hence (5.6.4) and so $DF_{(0,0)}$ are isomorphisms.

Let $F^{-1}(s, 0) = (s, \beta_s)$. $\beta_s$ is closed and $d \ast g_s(\alpha + \beta_s) = 0$. Since $\alpha$ is also closed, $\alpha + \beta_s$ is closed and $g_s$-coclosed: hence it is $g_s$-harmonic. Hence $\beta_s$ is smooth, and so our choice of $k$ and $\mu$ was unimportant.

Moreover, since $\beta_s$ is exact, $[\alpha + \beta_s] = [\alpha]$. That is, we have shown that for $s$ close to 0, the harmonic representative $\alpha + \beta_s$ of the cohomology class $[\alpha]$ depends smoothly on $s$.

Now for any $\beta$, $hpt(s, \beta)$ is given by the $L^2$-orthogonal projection of $\beta$ onto the harmonic $p$-forms with respect to $g_s$, so it suffices to show that we may find an orthonormal basis $\alpha_i(s)$
for the harmonic $p$-forms depending smoothly on $s$, and then write

$$hpt(s, \beta) = \sum_{i=1}^{b_p(M)} \left< \alpha_i(s), \beta \right>_g \alpha_i(s). \tag{5.6.5}$$

To do this, choose a basis $[\alpha_1], \ldots, [\alpha_{b_p(M)}]$ of $H^p(M)$. On some neighbourhood of $0 \in \mathcal{U}$ the $g_s$-harmonic representatives of $[\alpha_i]$ depends smoothly on $s$ for all $i$. Also, the $g_s$-harmonic representatives form a basis for the $g_s$-harmonic forms, by Hodge decomposition again. By applying the Gram–Schmidt process, we can find an orthonormal basis $\alpha_i(s)$ for the $g_s$-harmonic forms depending smoothly on $s$, and then (5.6.5) is smooth and we are done.

We can use a similar argument to prove that we can find a left inverse to the Laplacian smoothly in $s$.

**Proposition 5.6.3.** Let $M$ be a compact manifold and $g_s$ be a finite-dimensional smooth family of smooth metrics on it parametrised by $s \in \mathcal{U} \subset \mathbb{R}^m$, with a base point $0 \in \mathcal{U}$. Then the map

$$\Delta^{-1} : \mathcal{U} \times \Omega^p(M) \to \{g_0\text{-orthoharmonic } p\text{-forms}\} \tag{5.6.6}$$

such that $\Delta_{g_s} \Delta^{-1}(s, \alpha) = \alpha - hpt_{g_s} \alpha$ is well-defined and smooth, perhaps after shrinking the neighbourhood $\mathcal{U}$ of $g_0$.

**Proof.** Again, we use an inverse function theorem argument. Pick an integer $k$ and $\mu \in (0, 1)$, and consider the map $F$ defined by

$$\mathcal{U} \times \{C^{k+2, \mu} g_0\text{-orthoharmonic } p\text{-forms}\} \times H^p(M) \to \mathcal{U} \times C^{k, \mu} \Omega^p(M), \quad (s, \alpha, [\beta]) \mapsto (s, \Delta_{g_s} \alpha + hpt_{g_s} \beta), \tag{5.6.7}$$

where we let $\beta$ be some closed representative of $[\beta]$ (clearly, $hpt_{g_s} \beta$ depends only on $[\beta]$). By Proposition 5.6.2, $F$ is smooth. As in that proposition, to show that the derivative at $(0, \alpha, [\beta])$ is an isomorphism we can ignore the term in $s$, and it suffices to prove that

$$(\alpha', [\beta']) \mapsto \Delta_{g_0} \alpha' + hpt_{g_0} \beta' \tag{5.6.8}$$

is an isomorphism. This is again immediate by Theorem 3.3.1. Consequently on a small neighbourhood of $(0, \alpha, [\beta])$ $F$ is invertible. $\Delta^{-1}(s, \alpha)$ is precisely the second component of $F^{-1}(s, \alpha)$, so it exists, is unique, and depends smoothly on $s$ for $s$ close enough to $0$.

If $\alpha$ is smooth, elliptic regularity (Theorem 3.2.2) shows that the second component of $F^{-1}(s, \alpha)$ is smooth.

Propositions 5.6.2 and 5.6.3 pass immediately to smoothness of the corresponding operators.
on normal vector fields on a compact nearly special Lagrangian submanifold $L$ of a Calabi–Yau manifold $M$. We begin by defining these operators.

**Definition 5.6.4.** Let $M$ be a Calabi–Yau manifold and let $L_s$ be a finite-dimensional smooth family of smooth submanifolds parametrised by $s \in \mathcal{U} \subset \mathbb{R}^m$, with a base point $0 \in \mathcal{U}$. Suppose that for all $s \in \mathcal{U}$, $v \mapsto \iota_v \omega|_{L_s}$ is an isomorphism of bundles. Let $T_s$ be the family of transfer maps from $L_0$ to $L_s$, given by $T_2$ from Definition 5.5.6, so that $T_s = T_{2,s}$ in Proposition 5.5.15.

Define maps

$$hpt : \mathcal{U} \times \{\text{normal vector fields on } L_0\} \to \{\text{normal vector fields on } L_0\} \quad (5.6.9)$$

and

$$\Delta^{-1} : \mathcal{U} \times \{\text{normal vector fields on } L_0\} \to \{\text{orthoharmonic normal vector fields on } L_0\}, \quad (5.6.10)$$

as follows.

$hpt(s, v)$ is given by taking $T_s(v)$ a normal vector field on $L_s$, taking the harmonic part $u$ using the correspondence between 1-forms and normal vector fields, and then setting $hpt(s, v) = T_s^{-1}(u)$.

$\Delta^{-1}(s, v)$ is given by taking the normal vector field $T_s(v)$ on $L_s$ and then finding an orthoharmonic normal vector field $u$ on $L_0$ such that $\Delta T_s u = v - hpt(s, v)$.

**Proposition 5.6.5.** Let $M$, $\{L_s\}$, $hpt$ and $\Delta^{-1}$ be as in Definition 5.6.4. Then the maps $hpt$ and $\Delta^{-1}$ are well-defined and smooth, perhaps after reducing the neighbourhood $\mathcal{U}$ of $L_0$.

**Proof.** Composing $T_s$ with the isomorphisms $v \mapsto \iota_v \omega|_{L_s}$ yields

$$\mathcal{U} \times \{\text{normal vector fields on } L_0\} \to \Omega^1(L), \quad (s, v) \mapsto (s, \iota_{T_s(v)} \omega|_{L_s}). \quad (5.6.11)$$

It is enough to show that (5.6.11) is a smooth map with smooth inverse, as the results will then follow by composition.

For each $s$, (5.6.11) is just $\tau_2$ from Definition 5.5.29 composed with the map between normal vector fields and one-forms on $L_0$. It follows since the map between normal vector fields and one-forms is invertible by Lemma 5.4.3, that it suffices to prove the corresponding result for the map $\tau_2$ of one-forms. We showed in Proposition 5.5.31 that $\tau_2$ was invertible. It remains to show that $\tau_2$ depends smoothly on $s$ and its inverse is smooth in $s$.

That $\tau_2$ is smooth in $s$ is immediate from Proposition 5.5.32, since the pair $(v_s, \nabla v_s)$ depends smoothly on $s$. Combining the argument of Proposition 5.5.31 with the identity map in $s$ again shows that (5.6.11) is a local diffeomorphism and so has inverse smooth in $s$. 

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Composing (5.6.11) and its inverse with the smooth map \( \text{hpt} \) of Proposition 5.6.2 gives the map \( \text{hpt} \) of Definition 5.6.4. Consequently, this is a smooth map. Composing (5.6.11) and its inverse with the smooth map \( \Delta^{-1} \) of Proposition 5.6.3 gives the map \( \Delta^{-1} \) of Definition 5.6.4, and so this is also a well-defined smooth map.

Indeed, we can evaluate the derivative of \( \text{hpt} \):

**Proposition 5.6.6.** Suppose that \( M \) is a Calabi–Yau manifold and \( L_s \) is a finite-dimensional smooth family of smooth submanifolds as in Definition 5.6.4. Let \( T_s \) and \( \text{hpt} \) be as in Definition 5.6.4. Let \( v_s \) be a smooth curve of normal vector fields to \( L_0 \). Then \( \text{hpt}_s v_s \) is again a smooth curve of normal vector fields to \( L_0 \), and its derivative satisfies

\[
\begin{align*}
\text{hpt}_0 \left( \frac{d}{ds} \bigg|_{s=0} \text{hpt}_s v_s \right) &= \text{hpt}_0 (v' - \Delta' u), \\
\Delta_0 \left( \frac{d}{ds} \bigg|_{s=0} \text{hpt}_s v_s \right) &= -\Delta' \text{hpt}_0 v_0.
\end{align*}
\]

where \( u \) satisfies \( \Delta_0 u = v - \text{hpt}_0 v \), and \( \Delta' \) is the derivative of the induced operator \( \Delta_s \) in \( s \) (where \( \Delta_s \) is again induced by the transfer operator \( T_s \)).

**Proof.** That \( \text{hpt}_s(v_s) \) is a smooth curve follows immediately from Proposition 5.6.5.

To identify the derivative, we observe that we must have

\[
\Delta_s \text{hpt}_s v_s \equiv 0 \quad v_s - \text{hpt}_s v_s = \Delta_s u_s,
\]

for some smooth curve of normal vector fields \( u_s \). Note that this curve exists and is smooth again by Proposition 5.6.5 (and \( u_s \) is chosen to be orthoharmonic on \( L_0 \)).

We differentiate (5.6.14) to obtain

\[
\begin{align*}
\Delta'(\text{hpt}_0 v_0) + \Delta_0 \left( \frac{d}{ds} \bigg|_{s=0} \text{hpt}_s v_s \right) &= 0, \\
v' - \frac{d}{ds} \bigg|_{s=0} \text{hpt}_s v_s &= \Delta_0 u' + \Delta' u_0.
\end{align*}
\]

Applying \( \text{hpt}_0 \) to (5.6.16) gives

\[
\text{hpt}_0 \left( \frac{d}{ds} \bigg|_{s=0} \text{hpt}_s v_s \right) = \text{hpt}_0 (v' - \Delta' u). 
\]

**5.7 Gluing as a local diffeomorphism**

We will now show that the gluing map of special Lagrangians given by combining the SLing map of Definition 5.4.11 with the approximate gluing map of Definition 5.1.14 is a local diffeomorphism of moduli spaces for \( T \) sufficiently large, if the cross-section is connected. This is
Theorem 5.7.13. We will show that the derivative of this map is an isomorphism. We begin in Proposition 5.7.2 by identifying the derivative $D_L \text{SLing}$ of the SLing map under a hypothesis that the transfer maps of section 5.5 behave well with respect to the harmonic normal vector fields, which is essentially a condition that the base point $L$ is close enough to $\text{SLing}(L)$. We then show that in the gluing case, provided $T$ is sufficiently large, this hypothesis is satisfied and when viewed as a map of one-forms under the isomorphism of Lemma 5.4.3, the derivative of SLing is close to taking the harmonic part of the one-form on the special Lagrangian submanifold given by gluing our asymptotically cylindrical pair (Proposition 5.7.10). Finally, we show in Proposition 5.7.11 that the space of matching special Lagrangian submanifolds around any pair is a manifold, so that our derivative for the approximate gluing map in Proposition 5.5.24 applies, and prove using the linear harmonic theory in Proposition 3.5.19 that the composition of the derivatives is an isomorphism.

Since the spaces of special Lagrangian deformations are all finite-dimensional, we will work with finite-dimensional manifolds of smooth submanifolds; equivalently, given a submanifold $L_0$, we will work with finite-dimensional submanifolds of the space of normal vector fields on $L_0$. As explained before Proposition 5.5.9, this also simplifies the analysis.

We start by considering SLing on such a finite-dimensional manifold $\mathcal{U}$ of perturbations $L_s$ of $L_0$. Recall that $\text{SLing}(L_s)$ is defined as the special Lagrangian $L'_s$ given by $\exp_s(L_s)$ where $\text{hpt}(v) = 0$ in the sense of the discussion after Definition 5.6.1.

We expect that on $\mathcal{U}$ SLing will map into a neighbourhood of the special Lagrangian deformations of $\text{SLing}(L_0)$. For notational simplicity, let $L'_0 = \text{SLing}(L_0)$ and let $\mathcal{U}'$ be a neighbourhood of $L'_0$ in the space of special Lagrangians.

More formally, we let $\mathcal{U}$ be an open set in a finite-dimensional space of perturbations of the submanifold $L_0$ and $\mathcal{U}'$ an open set of the special Lagrangian deformations of $L'_0 = \text{SLing}(L_0)$. We write

$$F : \mathcal{U} \times \mathcal{U}' \longrightarrow \mathcal{U} \times H^1(L),$$

$$(L_s, L'_s') \mapsto (L, \alpha), \quad (5.7.1)$$

where if $v$ is the normal vector field on $L_s$ giving $L'_s$, $\alpha$ is the cohomology class of the harmonic part of $\iota_v \omega|_{L_s}$. Note that since $L$ is close to $L_0$ and $L'$ is close to $\text{SLing}(L_0)$, $F$ is a well-defined map if $\mathcal{U}$ and $\mathcal{U}'$ are small enough. We have

$$F(L_s, \text{SLing}(L_s)) = (L_s, 0), \quad (5.7.2)$$

for all $L_s \in \mathcal{U}$. We shall show that $F$ is smooth and establish its derivative at $(L_0, \text{SLing}(L_0))$, $D_{(L_0, \text{SLing}(L_0))}F$, show that $D_{(L_0, \text{SLing}(L_0))}F$ is an isomorphism, and consequently invert it to find $D_{L_0}\text{SLing}$.

**Proposition 5.7.1.** Let $M$ be a Calabi–Yau manifold, and let $L_0$ be a submanifold of $M$ in the domain of the SLing map of Definition 5.4.11. Let $\mathcal{U}$, $L'_0$ and $\mathcal{U}'$ be as described above. Let $v_0$
be the orthoharmonic normal vector field on $L_0$ such that $\exp_{v_0}(L_0) = \text{SLing}(L_0)$. Then the map $F$ of (5.7.1) is smooth. We have

$$D_{(L_0, L_0')} F : T_{L_0} U \oplus T_{L_0'} U' \rightarrow T_{L_0} U \oplus H^1(L),$$

$$(v, u) \mapsto (v, [\text{hpt} (T_1^{-1}(T_2v + T_3v - u) - \Delta'x) \omega|_{L_0}]),$$

where $T_1$ and $T_2$ are as in Definition 5.5.6 (for the transfer from $L_0$ to $L_0'$ via $v_0$), $T_3$ is similarly as in Definition 5.5.14, $\Delta'$ is the derivative defined in Proposition 5.6.6, and $x$ is (any) normal vector field to $L_0$ with $\Delta x = v_0$.

**Proof.** We consider $F$ as a composition of two maps. Firstly we find a normal vector field $w(s, s')$ on $L_0$ such that $T_{2,s}(w(s, s'))$ is the normal vector field on $L_s$ giving $L'_s$, where $T_{2,s}$ is taken between $L_0$ and $L_s$. Then we take the cohomology class of the harmonic part of the one-form $\omega|_{L_s}$ with respect to the metric induced on $L_s$.

By Proposition 5.5.10, $w(s, s')$ depends smoothly on $(s, s')$. By Proposition 5.6.5 (and composing a map $I'$ induced from $I'$ of Definition 5.5.27 to translate normal vector fields into one-forms) we know that $\text{hpt}$ is smooth in precisely the sense required. It follows that $F$ is smooth.

We now find the derivative $D_{(L_0, L_0')} F$. We first recall from Proposition 5.5.16 that the derivative of the map $(s, s') \mapsto w(s, s')$ is given by

$$(v, u) \mapsto T_1^{-1}(T_2v + T_3v - u),$$

where $v$ is a normal vector field to $L_0$ and $u$ is a normal vector field to $L'_0$. Given curves $L_s$ and $L'_s$ with normal vector fields $v$ and $u$, let $w' = T_1^{-1}(T_2v + T_3v - u)$ be the derivative of the corresponding curve $w(s)$ of normal vector fields on $L_0$.

We computed in Proposition 5.6.6 that the derivative of a curve $\text{hpt}_s T_2(w(s))$ has harmonic part $\text{hpt}(w' - \Delta'x)$, where $\Delta x$ gives the orthoharmonic part of $w(0)$, and Laplacian $-\Delta' \text{hpt} w(0)$. In this case, $w(0) = v_0$ and consequently it is orthoharmonic. Consequently the derivative of the curve $\text{hpt}_s T_2(w(s))$ is harmonic, and so it is

$$\text{hpt}_0(w' - \Delta'x) = \text{hpt}_0(T_1^{-1}(T_2v + T_3v - u) - \Delta'x),$$

where $\Delta x = v_0$.

From Proposition 5.7.1 we obtain

**Proposition 5.7.2.** Let $M$, $L_0$, $U$, $L_0'$, $U'$, and $v_0$ be as in Proposition 5.7.1. Suppose that for every nonzero harmonic normal vector field $u$ on $L_0'$, $\text{hpt} T_1^{-1}(u)$ is a nonzero harmonic normal vector field on $L_0$. Then $\text{SLing}$, restricted to a sufficiently small open subset of $U$, is a smooth
map. Its derivative is given by mapping a normal vector field \( v \) to the unique harmonic normal vector field \( u \) to \( L'_0 \) such that

\[
\text{hpt } T_1^{-1}u = \text{hpt}(T_1^{-1}(T_2v + T_3v) - \Delta'v),
\]

where \( T_1, T_2 \) and \( T_3 \) are as in Proposition 5.7.1, \( \Delta' \) is as in Proposition 5.6.6 and \( \Delta x = v_0 \).

**Remark.** Note that if \( L_0 \) is special Lagrangian, so \( L'_0 = L_0 \) and \( v_0 = 0 \) (and we may take \( x = 0 \)), \( T_1 \) and \( T_2 \) are the identity and \( T_3 \) is zero, so (5.7.6) reduces to \( u = \text{hpt } v \).

**Proof.** We apply the inverse function theorem to the map \( F \) of (5.7.1) at \( (L_0, L'_0) \). We first show that \( D_{(L_0, L'_0)}F \) is an isomorphism. We know that \( D_{(L_0, L'_0)}F \) is a linear map between spaces of the same finite dimension so it suffices to prove that it is injective. That is, we suppose that \( v \in T_{L_0}U \) and \( u \in T_{L'_0}U' \) satisfy \( D_{(L_0, L'_0)}F(v, u) = 0 \). Applying (5.7.3), \( v = 0 \) automatically and

\[
0 = \xi_{\text{hpt}}(T_1^{-1}(T_2v+T_3v-u) + \Delta'x)\xi|_{L_0} = \xi_{\text{hpt}}T_1^{-1}u\xi|_{L_0}.
\]

The hypothesis then implies that \( u \) must be zero, so that \( (v, u) = 0 \) and \( D_{(L_0, L'_0)}F \) is injective.

By the inverse function theorem, therefore, when we restrict to a small neighbourhood of \( (L_0, L'_0) \) in \( U \times U' \), \( F \) becomes a diffeomorphism. \( \text{SLing}(L_s) \) is precisely given by the harmonic normal vector field \( u \) to \( L'_0 \) so that \( DF_{(L_0, L'_0)}(v, u) = 0 \). Rearranging to obtain (5.7.6) is straightforward.

We now return to the gluing setup and begin translating Proposition 5.7.2 into one-forms so that we can appeal to the gluing results on harmonic forms given in section 3.5. We set up notation for this analysis as follows: unfortunately, the notation is rather involved.

**Convention 5.7.3.** Let \((M_1, M_2)\) be a matching pair of asymptotically cylindrical Calabi–Yau manifolds and let \((L_1, L_2)\) be a matching pair of asymptotically cylindrical special Lagrangian submanifolds as in Definition 5.1.13. Suppose that Hypothesis 5.3.1 holds so that \((\Omega^T, \omega^T)\) is a Calabi–Yau structure on \( M^T \). Let \( L_0(T) \) be the family of glued submanifolds of \( M^T \) given by approximately gluing as in Definition 5.1.14. Suppose \( T_0 \) is sufficiently large that for \( T > T_0 \) sufficiently large Condition 5.4.1 applies with \( k \) and \( \mu \), and let \( L'_0(T) \) be the family of special Lagrangian submanifolds for \((\Omega^T, \omega^T)\) given by perturbing \( L_0(T) \) as in Theorem 5.4.10. Let \( v_0(T) \) be the normal vector field to \( L_0(T) \) giving \( L'_0(T) \), and let \( x^T \) be a normal vector field on \( L_0(T) \) with \( \Delta x^T = v_0(T) \). Finally we note from Theorem 5.4.10 that there exists a fixed \( \epsilon \) and constants \( C_{k, \mu} \) such that \( \|v_0(T)\|_{C^{k+1, \mu}} \leq C_{k, \mu}e^{-\epsilon T} \) for every \( k \) and \( \mu \).

**Remark.** Note that we are making absolutely no assumption or claim on the regularity of the families \( L_0(T) \) and \( L'_0(T) \) as families of submanifolds of the fixed smooth manifold underlying \( M^T \).
We seek to understand $D_{L_0(T)}SL^i$ for sufficiently large $T$. As $L_0(T)$ and $L_0'(T)$ are close to special Lagrangian and special Lagrangian, respectively, we can identify normal vector fields on them with one-forms. We then obtain that $D_{L_0(T)}SL^i$, if defined, is the map from a one-form $\alpha$ on $L_0(T)$ to the unique harmonic one-form $\beta$ on $L_0'(T)$ such that

$$\text{lpt } \tau_1^{-1}\beta = \text{lpt}(\tau_1^{-1}(\tau_2 \alpha + \tau_3 \alpha) + \Delta'_\alpha x^T),$$

(5.7.8)

where $\Delta'_\alpha x^T$ is given by finding the normal vector field $v$ on $L_0$ corresponding to $\alpha$, constructing the transfer operators $T_{2,s}$ from $L_0$ to $\exp_{\alpha}(L_0)$, constructing the curve of normal vector fields $T_{2,s}^1 \Delta T_{2,s} x^T$, and then taking the derivative of this curve in $s$ at zero, and $\tau_1$, $\tau_2$ and $\tau_3$ are as in Definition 5.5.30.

We first show that for $T$ large enough, $\beta$ satisfying (5.7.8) is defined, by showing that the $\tau_i$ behave well as $T$ gets large.

**Proposition 5.7.4.** Let $M_1$, $M_2$, $L_1$, $L_2$, $L_0(T)$, $L_0'(T)$ and $\epsilon$ be as in Convention 5.7.3. Consider the maps $\tau_1$, $\tau_2$ and $\tau_3$ defined in Definition 5.5.30 from one-forms on $L_0(T)$ to one-forms on $L_0'(T)$; these maps also depend on $T$. There exists a sequence of constants $C_k$ such that for every $k$

$$\|\alpha - \tau_1 \alpha\|_{C^k} + \|\alpha - \tau_2 \alpha\|_{C^k} + \|\tau_3 \alpha\|_{C^k} \leq C_k e^{-\epsilon T} \|\alpha\|_{C^k}.$$

(5.7.9)

**Proof.** Fix $k$; the proof is analogous to that of Proposition 5.4.9 and Proposition 5.5.21. We know by Proposition 5.5.32 that the $\tau_i$ defined in Definition 5.5.29 are smooth maps with $\tau_i(\alpha, (0, 0), \ell) = \alpha$ for $i = 1, 2$ and $\tau_3(\alpha, (0, 0), \ell) = 0$. Using Proposition 2.2.5 and linearity, fixing some $r > 0$ and supposing $\|v_0(T)\|_{C^{k+1}} < r$, we obtain locally around $p \in M^T$

$$\|\alpha - \tau_1 \alpha\|_{C^k} + \|\alpha - \tau_2 \alpha\|_{C^k} + \|\tau_3 \alpha\|_{C^k} \leq C_k \|v_0(T)\|_{C^{k+1}} \|\alpha\|_{C^k},$$

(5.7.10)

for some constant $C_k$ depending on the geometry of $M^T$ near $p$ and $r$.

As explained in Convention 5.7.3, $\|v_0(T)\|_{C^{k+1}}$ decays exponentially in $T$, so that it suffices to show that $C_k$ and $r$ can be bounded uniformly in $p$ and $T$. This, however, follows just as in Proposition 5.4.9. \qed

**Remark.** It should be possible in principle to quantify the constant $C_k$, but this would require very careful analysis of possible growth rates in families of Jacobi fields to extend the corollary of the Rauch comparison theorem stated as Proposition 5.5.1 to the $C^k$ case. We do not need to do this, since we are only working with glued submanifolds and so can argue more directly, but it would perhaps be useful in a more general setting.

The other major ingredient in (5.7.8) (as well as the $\tau_i$) is the Laplacian. We have the following
Proposition 5.7.5. Let $M_1$, $M_2$, $L_1$, $L_2$, $M^T$, $L_0(T)$, $L_0'(T)$, $v_0(T)$ and $\epsilon$ be as in Convention 5.7.3. We have two metrics $g(\Omega^T, \omega^T)|_{L_0(T)}$ and $g(\Omega^T, \omega^T)|_{L_0'(T)}$ on the submanifold $L_0(T)$ of $M^T$. There exist constants $C_k$ such that for all $k$

$$\|g(\Omega^T, \omega^T)|_{L_0(T)} - g(\Omega^T, \omega^T)|_{L_0'(T)}\| \leq C_k e^{-\epsilon T}. \tag{5.7.11}$$

Secondly, these two metrics induce two Laplacians $\Delta_{L_0(T)}$ and $\Delta_{L_0'(T)}$ on forms on these submanifolds. There exist constants $C_{k,\mu}'$ such that for any integer $k$ and $\mu \in (0,1)$ and for any form $\alpha$

$$\|\Delta_{L_0(T)}\alpha - \Delta_{L_0'(T)}\alpha\|_{C^k,\mu} \leq C_{k,\mu}' e^{-\epsilon T}\|\alpha\|_{C^{k+2,\mu}}, \tag{5.7.12}$$

and the same estimate for $d^*$. Thirdly, possibly increasing $C_{k,\mu}'$, there also exists $r$ such that if $\alpha$ is orthoharmonic with respect to the metric on $L_0(T)$ or $L_0'(T)$,

$$\|d\alpha\|_{C^k,\mu} + \|d^*\alpha\|_{C^k,\mu} \geq C_{k,\mu}' T^r \|\alpha\|_{C^{k+1,\mu}}, \tag{5.7.13}$$

where the Laplacian is that on $L_0(T)$ or $L_0'(T)$ respectively.

Proof. That $g(\Omega^T, \omega^T)|_{L_0(T)} - g(\Omega^T, \omega^T)|_{L_0'(T)}$ decays exponentially in $T$ with all derivatives follows by an argument similar to that of Proposition 5.7.4; (5.7.12) and its $d^*$ version then follow from smoothness of the Laplacian and Hodge star written in local coordinates as functions of the metric.

As for (5.7.13), for $L_0(T)$ this follows in the proof of Theorem 5.4.10 by using Corollary 5.3.3 and Theorem 3.5.14. For $L_0'(T)$, we just use (5.7.11) and the same argument. 

Proposition 5.7.5 has the following two corollaries. Firstly, it implies that the harmonic parts of a form taken on $L_0(T)$ and $L_0'(T)$ are similar.

Corollary 5.7.6. Let $M_1$, $M_2$, $L_1$, $L_2$, $M^T$, $L_0(T)$, $L_0'(T)$, $v_0(T)$ and $\epsilon$ be as in Convention 5.7.3. Given a form $\alpha$ on $L_0(T) = L_0'(T)$, write $hpt_{L_0(T)}$ and $hpt_{L_0'(T)}$ for its harmonic part with respect to the two metrics of Proposition 5.7.5. Then for every $k$ and $\mu$ there exist constants $C_{k,\mu}$ such that for every form $\alpha$,

$$\|hpt_{L_0(T)}\alpha - hpt_{L_0'(T)}\alpha\|_{C^k,\mu} \leq C_{k,\mu} e^{-\epsilon T}\|\alpha\|_{C^k,\mu}. \tag{5.7.14}$$

Furthermore, we have for some constants $r$ and $C_{k,\mu}$

$$\|hpt_{L_0'(T)}\alpha\|_{C^k,\mu} \leq C_{k,\mu} T^r \|\alpha\|_{C^k,\mu}. \tag{5.7.15}$$

Proof. We suppress the dependence of $L_0$ and $L_0'$ on $T$, for notational simplicity. As in showing that $hpt$ is smooth in Proposition 5.6.2, we begin by demonstrating the result for harmonic forms and working with a basis. Let $\alpha_1, \ldots, \alpha_{b_p(L_0)}$ be an orthonormal basis of harmonic $p$-forms for
the metric on $L_0$. By the $d^*$ version of (5.7.12), $d^*_{L_0} \alpha_i$ decays exponentially in $T$ for each $i$. Since we have the lower bound (5.7.13), it follows that the $L_0'$-orthoharmonic part of $\alpha_i$ also decays exponentially in $T$. Applying the Gram–Schmidt process, we find, at least for $T$ large enough, an orthonormal basis $\alpha_i'$ of harmonic $p$-forms on $L_0'$ with $\alpha_i' - \alpha_i$ decaying exponentially in $T$.

Then $\text{hpt}_{L_0(T)}$ is simply given by taking $L^2$-inner products with $\alpha_i$, and $\text{hpt}_{L_0'(T)}$ is given by taking $L^2$-inner products with $\alpha_i'$; hence, as the difference of metrics also converges exponentially, we see that these maps converge exponentially.

We already know a bound of the form (5.7.15) holds on $\text{hpt}_{L_0(T)}$ by using Corollary 3.5.17 and Corollary 5.3.3; the bound on $L_0'(T)$ follows either in the same way using (5.7.11) or by applying the previous result.

Secondly, combining Proposition 5.7.5 with Proposition 5.7.4 implies that the transfer maps induce isomorphisms between harmonic normal vector fields.

**Corollary 5.7.7.** Let $M_1$, $M_2$, $L_1$, $L_2$, $M^T$, $L_0(T)$, $L_0'(T)$ and $\nu_0(T)$ be as in Convention 5.7.3. Let $T_i$ be either $T_1$ or $T_2$ from Definition 5.5.6. For $T$ sufficiently large, $\text{hpt}_{L_0(T)} \circ T_i$ defines an isomorphism between harmonic normal vector fields. So does $\text{hpt}_{L_0'(T)} \circ T_i^{-1}$.

**Proof.** That $\text{hpt}_{L_0(T)} \circ T_i$ is an isomorphism is equivalent, by the definition of hpt on normal vector fields, to the statement that $\text{hpt}_{L_0(T)} \circ T_i$ is an isomorphism. We shall show that $\text{hpt}_{L_0'(T)} \circ T_i$ is injective on one-forms: since $L_0(T)$ and $L_0'(T)$ have the same first Betti number and consequently equal-dimensional spaces of harmonic one-forms, it will follow that $\text{hpt}_{L_0'(T)} \circ T_i$ is an isomorphism.

Pick some $k$ and $\mu$, and suppose that $\alpha$ is a harmonic one-form on $L_0(T)$ with $\|\alpha\|_{C^{k,\mu}} = 1$. By Proposition 5.7.4, we see that $\|\tau_i \alpha - \alpha\|_{C^{k,\mu}}$ is bounded by a constant decaying exponentially in $T$. By the bound (5.7.15), we obtain that $\|\text{hpt}_{L_0(T)} \tau_i \alpha - \text{hpt}_{L_0'(T)} \alpha\|_{C^{k,\mu}}$ is similarly bounded. By Corollary 5.7.6, we also have that $\|\text{hpt}_{L_0'(T)} \alpha - \alpha\|_{C^{k,\mu}}$ can also be bounded by a constant decaying exponentially in $T$. Hence we obtain that $\|\text{hpt}_{L_0'(T)} \tau_i \alpha - \alpha\|_{C^{k,\mu}}$ decays exponentially in $T$, and it follows that $\text{hpt}_{L_0'(T)} \tau_i \alpha$ cannot be zero.

That $\text{hpt} \circ T_i^{-1}$ is an isomorphism follows in exactly the same way.

In particular, for $T$ sufficiently large and $\mathcal{U}$ sufficiently small, combining Corollary 5.7.7 with Proposition 5.7.2 shows that SLing is a smooth map on $\mathcal{U}$ with derivative at $L_0(T)$ given by Proposition 5.7.2.

**Remark.** Slightly restricted versions of Propositions 5.7.4 and 5.7.5 also hold in a more general setting. If $M$ is a Calabi–Yau manifold and $L_0$ is a closed nearly special Lagrangian submanifold satisfying Condition 5.4.1, then SLing($L_0$) exists. If SLing($L_0$) is close enough to $L_0$ in $C^k$, then working with a suitably low regularity in the propositions shows again that $\text{hpt}_{L_0} \tau_i^{-1}$ is an isomorphism. Then, as in the gluing case, Proposition 5.7.2 applies and we have that
SLing is again a smooth map with derivative given by (5.7.6) on a small enough space $U$ of perturbations of $L_0$.

We now know that $D_{L_0(T)}SLing$ exists and in terms of one-forms is given by (5.7.8). It remains to show that this map is close to taking the harmonic part $\text{hpt}_{L_0(T)}$. Proposition 5.7.4 controls the $\tau_i$; it remains to deal with the term $\Delta'_{\alpha}x^T$. We have

**Proposition 5.7.8.** Let $M_1$, $M_2$, $L_1$, $L_2$, $M^T$, $L_0(T)$, $L'_0(T)$, $v_0(T)$ and $x^T$ be as in Convention 5.7.3. Consider the map of one-forms

$$\alpha \mapsto \Delta'_{\alpha}x^T,$$

defined after (5.7.8). This is a linear map of $\alpha$ and for every $k$ there is $C_k$, independent of $T$, such that

$$\|\Delta'_{\alpha}x^T\|_{C^k} \leq C_k \|\alpha\|_{C^{k+1}} \|x^T\|_{C^{k+2}}.$$  

(5.7.17)

**Proof.** $\Delta'_{\alpha}x^T$ is given by finding the normal vector field $v$ on $L_0$ corresponding to $\alpha$, constructing the transfer operators $T_{2,s}$ from $L_0$ to $\exp_{sv}(L_0)$, constructing the curve of normal vector fields $T_{2,s}^{-1}\Delta T_{2,s}x^T$, and then taking the derivative of this curve in $s$ at zero. That is, it is the derivative at zero of the map $\Delta_{\alpha}x^T$ given by finding the normal vector field $v$ corresponding to $\alpha$, the corresponding transfer operator $T_2$, and then evaluating $T_2^{-1}\Delta T_2x^T$. Since this is a smooth map by the argument of Proposition 5.6.5, it follows immediately that the derivative is well-defined and in particular linear.

The proof of the bound is similar to Proposition 5.7.4, though is more involved. We argue that the map $\Delta_{\alpha}x^T$ can be determined locally, as for the $\tau_i$, by the first jet of $\alpha$, the second jet of $x^T$, and the tangent space to $L$ at the point, and is continuous in the Calabi–Yau structure in the same sense as Proposition 5.5.32. Hence, the derivative $\Delta'_{\alpha}x^T$ can also be determined by the first jet of the derivative, the second jet of $x^T$, and the tangent space and is continuous in the Calabi–Yau structure in the sense of (5.5.14). But then the bound follows by exactly the same argument as in Proposition 5.7.4: the argument of Proposition 2.2.5 gives it locally, and the uniform constant follows by comparing with $(\Omega_1, \omega_1)$, $(\Omega_2, \omega_2)$ and the cylindrical Calabi–Yau structure $(\tilde{\Omega}, \tilde{\omega})$.

We obtain

**Corollary 5.7.9.** In the setup of Convention 5.7.3, for each $k$ there exists $C_k$ so that

$$\|\Delta'_{\alpha}x^T\|_{C^k} \leq C_k e^{-\epsilon T} \|\alpha\|_{C^{k+1}}.$$  

(5.7.18)

**Proof.** Given (5.7.17), it suffices to prove $x^T$ is exponentially decaying in $T$. We are working on $L^0(T)$ so this follows in exactly the same way as (5.7.13) from Lemma 5.4.3, Corollary 5.3.3 and the lower bound on $\Delta$ (rather than $d + d^*$) which we noted at the end of subsection 3.5.2.

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We can now turn to our main result identifying $D_{L_0(T)\text{SLing}}$ with $\text{hpt}_{L_0(T)}$ on one-forms.

**Proposition 5.7.10.** Let $M_1, M_2, L_1, L_2, M^T, L_0(T), L_0'(T)$ and $\epsilon$ be as in Convention 5.7.3. Given a one-form $\alpha$ on $L_0(T)$ we can construct a harmonic one-form with respect to the metric on $L_0'(T)$ by applying either the harmonic projection $\text{hpt}_{L_0(T)}$ or $D_{L_0(T)\text{SLing}}$. We have constants $C_k$ such that

$$
\| (\text{hpt}_{L_0'(T)} - D_{L_0(T)\text{SLing}}) \alpha \|_{C^k} \leq C_k \epsilon^{-\frac{3}{2}} \| \alpha \|_{C^{k+1}}. 
$$

(5.7.19)

**Proof.** We recall from (5.7.8) that $D_{L_0(T)\text{SLing}}(\alpha)$ is the unique $L_0'(T)$-harmonic $\beta$ such that

$$
\text{hpt}_{L_0(T)} \tau_1^{-1} \beta = \text{hpt}_{L_0(T)} (\tau_1^{-1} (\tau_2 \alpha + \tau_3 \alpha) + \Delta' \alpha x^T).
$$

(5.7.20)

We first show that $\tau_1^{-1} (\tau_2 \alpha + \tau_3 \alpha) + \Delta' \alpha x^T$ is exponentially close to $\alpha$ in $T$. We have

$$
\| \tau_1^{-1} (\tau_2 \alpha + \tau_3 \alpha) + \Delta' \alpha x^T - \alpha \|_{C^k} \leq \| \tau_1^{-1} (\tau_2 \alpha + \tau_3 \alpha) - (\tau_2 \alpha + \tau_3 \alpha) \|_{C^k}
+ \| \tau_2 \alpha - \alpha \|_{C^k} + \| \tau_3 \alpha \|_{C^k} + \| \Delta' \alpha x^T \|_{C^k}.
$$

(5.7.21)

Applying Proposition 5.7.4 and Corollary 5.7.9, we have that there exist $C_k$ such that

$$
\| \tau_1^{-1} (\tau_2 \alpha + \tau_3 \alpha) + \Delta' \alpha x^T - \alpha \|_{C^k} \leq C_k \epsilon^{-\frac{3}{2}} (\| \tau_2 \alpha + \tau_3 \alpha \|_{C^k} + 2 \| \alpha \|_{C^k} + \| \alpha \|_{C^{k+1}}).
$$

(5.7.22)

Then applying Proposition 5.7.4 again to $\| \tau_2 \alpha + \tau_3 \alpha \|_{C^k} \leq \| \alpha \|_{C^k} + \| \tau_2 \alpha - \alpha \|_{C^k} + \| \tau_3 \alpha \|_{C^k}$ we obtain possibly different $C_k$ such that

$$
\| \tau_1^{-1} (\tau_2 \alpha + \tau_3 \alpha) + \Delta' \alpha x^T - \alpha \|_{C^k} \leq C_k \epsilon^{-\frac{3}{2}} \| \alpha \|_{C^{k+1}}.
$$

(5.7.23)

By applying $\text{hpt}_{L_0(T)}$, which is bounded at most polynomially in $T$ by Proposition 5.7.5, and increasing $C_k$ some more, it follows that

$$
\| \text{hpt}_{L_0(T)} \tau_1^{-1} D_{L_0(T)\text{SLing}}(\alpha) - \text{hpt}_{L_0(T)} \alpha \| \leq C_k \epsilon^{-\frac{3}{2}} \| \alpha \|_{C^{k+1}}.
$$

(5.7.24)

Applying Proposition 5.7.4 to the difference $\tau_1^{-1} D_{L_0(T)\text{SLing}}(\alpha) - D_{L_0(T)\text{SLing}}(\alpha)$, and combining the resulting estimate with (5.7.24) and the polynomial bound on $\text{hpt}_{L_0(T)}$ again we find another $C_k$ such that

$$
\| \text{hpt}_{L_0(T)} D_{L_0(T)\text{SLing}}(\alpha) - \text{hpt}_{L_0(T)} \alpha \| \leq C_k \epsilon^{-\frac{3}{2}} (\| \alpha \|_{C^{k+1}} + \| D_{L_0(T)\text{SLing}}(\alpha) \|_{C^k}).
$$

(5.7.25)

But then by Corollary 5.7.6 and the fact that $D_{L_0(T)\text{SLing}}(\alpha)$ is $L_0'(T)$-harmonic, we obtain for yet another $C_k$

$$
\| D_{L_0(T)\text{SLing}}(\alpha) - \text{hpt}_{L_0(T)} \alpha \|_{C^k} \leq C_k \epsilon^{-\frac{3}{2}} (\| \alpha \|_{C^{k+1}} + \| D_{L_0(T)\text{SLing}}(\alpha) \|_{C^k}).
$$

(5.7.26)

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We can then simply use the triangle inequality

\[ \|D_L(T)SLing(\alpha)\|_{C^k} \leq \|hpt_{L_0(T)}\alpha\|_{C^k} + \|D_L(T)SLing(\alpha) - hpt_{L_0(T)}\alpha\|_{C^k} \]  

(5.7.27)

to bound the right hand side of (5.7.26). The \( \|D_L(T)SLing(\alpha) - hpt_{L_0(T)}\alpha\|_{C^k} \) term can be absorbed on the left hand side for \( T \) large enough; the \( \|hpt_{L_0(T)}\alpha\|_{C^k} \) term is polynomially bounded by \( \|\alpha\|_{C^{k+1}} \) using Corollary 5.7.6 again. Thus we get the required estimate. \( \square \)

We now turn to the approximate gluing map. We know from Proposition 5.5.24 and 5.5.34 that on any manifold of matching pairs this is smooth and its derivative is exponentially close in \( T \) to the approximate gluing map of one-forms under the identification.

From the asymptotically cylindrical deformation Theorem 5.2.13, we infer that the set of matching special Lagrangians is a finite-dimensional manifold.

**Proposition 5.7.11.** Let \( M_1 \) and \( M_2 \) be a matching pair of asymptotically cylindrical Calabi–Yau manifolds, and let \( L_1 \) and \( L_2 \) be a matching pair of asymptotically cylindrical special Lagrangian submanifolds as in Definition 5.1.13. Assume that \( K_1 \) and hence \( K_2 \) are connected. We have a set of pairs \((L_1', L_2')\) where \( L_1' \) is a special Lagrangian deformation of \( L_1 \) and \( L_2' \) is a special Lagrangian deformation of \( L_2 \). The subset of \((L_1', L_2')\) such that \( L_1' \) and \( L_2' \) also match is a manifold.

**Proof.** We know that the set of deformations of \( L_i \) is given by a finite-dimensional submanifold of the set of asymptotically translation invariant normal vector fields on \( L_i \). If \( L_i' = \exp_{v_i}(L_i) \), then \( L_1' \) and \( L_2' \) match if and only if \( v_1 \) and \( v_2 \) do, since the limits are \( K_1' = \exp_{\tilde{v}_i}(K_i) \) and the metrics agree since the \( M_i \) match.

The map from \( L_1' \) to \( K_1' \) is evidently smooth. By Proposition 5.2.15, which follows [70, Proposition 6.4], \( K_1' \) lies in a submanifold \( K_i \) of possible limits. The derivative of \( L_1' \mapsto K_1' \) is the map \( v_i \mapsto \tilde{v}_i \) given by taking the limit of a harmonic normal vector field such that \( \tau_i \omega|_{L_i} \) is an absolute harmonic one-form. That is, the derivative is the limit map from absolute harmonic normal vector fields to the limits of these, and hence is surjective.

For notational simplicity, write \( K_1 = K_2 \): that is, use \( F \) to identify the two cross-sections.

Note that \( K_1 \) and \( K_2 \) are submanifolds of the manifold of all translation invariant special Lagrangian deformations \( K' \) of \( K \times \mathbb{R} \) in the cylinder \( N \times \mathbb{R} \), as in Proposition 5.2.15. Let \( \tilde{\partial}_i : H^1(K) \to H^2_{\text{reg}}(L_i) \) be the map of the exact sequence given by Theorem 2.1.2 for the asymptotically cylindrical manifold \( L_i \). Let \( \tilde{\tau}_i \) be the limit of an asymptotically translation invariant form \( \tau_i \) on a tubular neighbourhood of \( L_i \) with \( \tau_i|_{L_i} = 0 \) and \( d\tau_i = \omega \) on this tubular neighbourhood. Note that both \( \tau_1 \) and \( \tau_2 \) are examples of the form which we called \( \tau_1 \) in Proposition 5.2.15, with \( \tau_1 \) on \( L_1 \) and \( \tau_2 \) on \( L_2 \). By Proposition 5.2.15, \( K_i \) is the kernel of the map on special Lagrangian deformations \( K'_i \) of \( K = \{p\} \times Y_i \)

\[ K'_i \mapsto \tilde{\partial}_i([\tilde{\tau}_i|_{K'_i}]). \]  

(5.7.28)
To show that the joint kernel of the two maps of (5.7.28) is a manifold, we want to show that the two maps are as similar as possible.

We note that $d\tilde{\tau}_1 - d\tilde{\tau}_2 = \tilde{\omega} - \tilde{\omega} = 0$. Consequently, for any deformation $K_s$ of $K_0$, we have that \[ [(\tilde{\tau}_1 - \tilde{\tau}_2)|_{K_s}] = [(\tilde{\tau}_1 - \tilde{\tau}_2)|_{K_0}] = 0. \]

Since we shall only need these restricted cohomology classes, we shall write $\tilde{\tau} = \tilde{\tau}_1$.

We know, as a consequence of the calculation of its linearisation in Proposition 5.2.15, that the map from $K_s$ to $[\tilde{\tau}|_{K_s}]$ is a submersion into $H^1(K)$. Consequently, as $\ker \partial_1 \cap \ker \partial_2$ is a vector subspace of $H^1(K)$, we have that $K_1 \cap K_2$ is a submanifold of the manifold of all special Lagrangian deformations of $K$.

Consequently, the space $(K_1 \cap K_2) \times (K_1 \cap K_2)$ is a submanifold of $K_1 \times K_2$. The desired submanifold of matching deformations of $(L_1, L_2)$ is precisely the inverse image of this submanifold under the submersion $(L'_1, L'_2) \mapsto (K'_1, K'_2)$.

Consequently, we have

**Proposition 5.7.12.** If we restrict the gluing map of submanifolds given by Definition 5.1.14 to pairs of matching asymptotically cylindrical special Lagrangian submanifolds, for $T$ large enough, the approximate gluing map is a smooth immersion and so maps to a finite-dimensional submanifold of the deformations of $L^T$.

**Proof.** Propositions 5.5.24 and 5.5.34 show that the derivative of this gluing map is close to the approximate gluing map of one-forms for $T$ large enough. By the argument of Proposition 3.5.19 the approximate gluing map of one-forms is injective and bounded below independently of $T$ when restricted to the matching harmonic forms, and so this derivative must also be injective.

We may now combine Proposition 5.7.10 with Proposition 3.5.19 (on the harmonic gluing map) to prove that the gluing map of special Lagrangians is a local diffeomorphism of moduli spaces.

**Theorem 5.7.13.** Let $M_1$ and $M_2$ be a matching pair of asymptotically cylindrical Calabi–Yau manifolds, and suppose that Hypothesis 5.3.1 holds so $M_1$ and $M_2$ can be glued to give a Calabi–Yau manifold $M^T$. Let $L_1$ and $L_2$ be a matching pair of asymptotically cylindrical special Lagrangians in $M_1$ and $M_2$, with $K_1$ (and hence $K_2$) connected. By Theorem 5.4.10, there exists $T_0 > 0$ such that $L_1$ and $L_2$ can be glued to a form a special Lagrangian in $M^T$ for all $T > T_0$. Moreover, this applies for any sufficiently small deformation of $L_1$ and $L_2$ as a matching pair, and hence we obtain a gluing map from the deformation space of matching pairs of submanifolds in Proposition 5.7.11 to the space of deformations of the gluing of $L_1$ and $L_2$. This map is a local diffeomorphism for $T$ sufficiently large.

**Proof.** This gluing map is the composition of the approximate gluing map of Definition 5.1.14 with the SLing map of Definition 5.4.11. We shall show that the derivative of this gluing map,
regarded as a map of one-forms, is exponentially close to the gluing map $\Gamma_T$ of harmonic one-forms in Definition 3.5.9; this derivative is the composition of the derivatives of the approximate gluing and SLing. For the proof we use the notation of Convention 5.7.3.

As in the previous proposition, by Proposition 5.5.24, the derivative of the approximate gluing map is exponentially close to the approximate gluing map of normal vector fields given in Definition 5.5.23, and by Proposition 5.5.34 this is exponentially close to the approximate gluing map of one-forms of Definition 3.5.5.

By Proposition 5.7.10, $D_{L_{0}(T)} SLing$ is exponentially close to $hpt_{SLing}(L_{0}(T))$. In particular, it is bounded polynomially in $T$, by Corollary 5.7.6.

By composition (using that the harmonic part map is polynomially bounded) we find that the difference between the derivative of the gluing map of special Lagrangians and the gluing map $\Gamma_T$ of harmonic forms with respect to the metric $g(\Omega^T, \omega^T)|_{L_{0}^{T}(T)}$ is exponentially small in $T$. Combining Corollary 5.3.3 with Proposition 5.7.5, this metric is exponentially close in $T$ to that given by Definition 3.5.13, so, since by Proposition 3.5.19 $\Gamma_T$ is bounded below uniformly in $T$ and is an isomorphism, it follows that the derivative of the gluing map of special Lagrangians is an isomorphism for $T$ sufficiently large. By the inverse function theorem, the result follows. □
Chapter 6

Concluding remarks

We collect some further questions that may be of interest extending the material of chapters 4 and 5. We treat these questions in the order they arise in the text.

The most obvious question arising from chapter 4 is to what extent the results of that chapter can be extended to higher $n$. We discuss, as an example, the result that we may glue Calabi–Yau structures such that Hypothesis 5.3.1, which we need for our special Lagrangian results, holds. The most interesting condition in the hypothesis is the cohomology condition. There is no obvious reason why for higher $n$ we would need to admit a constant of proportionality $c$ in (5.3.2), that is, rescale the Kähler class, rather than just take a pair of exact forms to alter $\Omega$ and $\omega$. Hence, we may think of the question of whether $c = 1$ discussed before Lemma 4.3.9 as a preliminary question, which would be useful to say what kind of result might be expected for higher $n$. In the same way, we discussed what is known about the $n = 4$ case after Hypothesis 5.3.1; the questions there would also provide some insight.

For $n > 4$, arguments via other exceptional holonomy spaces are no longer available and other methods would be necessary, possibly using algebraic geometry methods. It would thus be interesting to work out how the material of chapter 4 corresponds to the algebraic geometry theory of Calabi–Yau manifolds, and for instance how the deformation theory corresponds to the deformation theory of asymptotically cylindrical Calabi–Yau manifolds using logarithmic deformations in [15].

On the special Lagrangians of chapter 5, we can see more avenues of further research. Furthermore, they are slightly more straightforward than the general questions indicated above. First of all, it should be straightforward to extend the deformation results and Theorem 5.7.13 to special Lagrangians with disconnected cross-section, and so include the examples of subsection 5.2.2. We could then consider extending the results to special Lagrangian submanifolds of asymptotically cylindrical Calabi–Yau manifolds with different asymptotic conditions.

The first alternative asymptotic we might consider is a helical special Lagrangian in the Calabi–Yau cylinder of Definition 5.2.4. These are of the form $H \times Y$ where $H$ is a helix in $\mathbb{R} \times S^1$. We note that $H = \{(r \cos \alpha, r \sin \alpha)\}$ is a special Lagrangian in $\mathbb{R} \times S^1$ for the Calabi–
Yau structure \((e^{-ia}(dt + id\theta), dt \wedge d\theta)\) on \(\mathbb{R} \times S^1\). It follows by straightforward calculation that if \(Y\) is special Lagrangian for the Calabi–Yau structure \((e^{ia}\Omega_{xs}, \omega_{xs})\) on \(X\), then \(H \times Y\) is special Lagrangian for the Calabi–Yau structure \((e^{-ia}(dt + id\theta)) \wedge e^{ia}\Omega_{xs}, dt \wedge d\theta + \Omega_{xs}\) = \((\tilde{\Omega}, \tilde{\omega})\).

The limiting cases of \(H \times Y\) are \(\alpha = 0\), the cylindrical case, and \(\alpha = \pi/2\), the case of a special Lagrangian \(\{p\} \times S^1 \times Y\). Except when \(\alpha = \pi/2\), an asymptotically helical submanifold is also asymptotically cylindrical considered intrinsically, and we can expect almost all of the chapter to generalise readily. One of the more difficult points would be that a priori given an asymptotically helical special Lagrangian, its asymptotically helical special Lagrangian deformations may need not be given by asymptotically translation invariant normal vector fields. A generalisation of Lemma 5.2.16 (which says that all special Lagrangian deformations of cylindrical special Lagrangians in a cylindrical Calabi–Yau are cylindrical) to infer that special Lagrangian deformations of helical special Lagrangians are again helical, with the same value of \(\alpha\), could perhaps be used to obtain this.

We could then generalise further to a definition corresponding to the asymptotically periodic ends of Taubes [73]. The Laplace-Beltrami estimate of Theorem 3.5.14 and the second fundamental form estimate established in Corollary 5.3.4 should go through as in the asymptotically cylindrical case, and so we should obtain Condition 5.4.1 easily. The deformation theory, however, would be more complicated, but the linear theory corresponding to chapter 3 has been treated by Taubes and further more recently by Mrowka, Ruberman and Saveliev [63]. Thus, though some more work will be needed, a deformation result is not implausible.

Finally, we could also investigate what additional conditions on the special Lagrangians \(L_1\) and \(L_2\) give Condition 5.4.1 mostly easily. To obtain (i), by examining the proof of Lemma 5.4.3, it seems likely that an assumption that the section fundamental forms of \(L_i\) in \(M_i\) are bounded with all derivatives should be sufficient; the Laplace-Beltrami estimate would be more difficult.

Secondly, we can ask further questions on the space of special Lagrangians in the glued Calabi–Yau \(M^T\). Of course, the additional gluing results described above would construct additional special Lagrangians: we now describe some more general questions. A sensible initial question is how the special Lagrangians in \(M^T\) depend on the parameter \(T\). For instance, given a special Lagrangian \(L\) in \(M^T\), does it yield a special Lagrangian in \(M^{T'}\)?

For \(T'\) close to \(T\), given suitable topological assumptions, this comes down to the extended deformation theorem given by Marshall [58, Theorem 3.2.9]. These topological assumptions should not hold in the case of submanifolds with helical necks which we suggested above, which is natural, as helical ends should only be gluable at discrete positions.

If \(T'\) is far from \(T\), we would have to show that the estimates required for the small deformation remain stable as \(T\) moves, which would require careful analysis. It seems possible that given \(M^T\) with a cylindrical neck, an argument of the form of the proof of Theorem 3.5.14 can be run depending only on say the volume and second fundamental form of the submanifold \(L\),
giving the Laplace-Beltrami estimate we would require. This would mean that to control the estimates we would only have to control the second fundamental form.
Bibliography


