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The unified transform for evolution equations on the half-line with time-periodic boundary conditions*

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Abstract

This paper elaborates on a new approach for solving the generalized Dirichlet-to-Neumann map, in the large time limit, for linear evolution PDEs formulated on the half-line with time-periodic boundary conditions. First, by employing the unified transform (also known as the Fokas method) it can be shown that the solution becomes time-periodic for large t. Second, it is shown that the coefficients of the Fourier series of the unknown boundary values can be determined explicitly in terms of the coefficients of the Fourier series of the given boundary data in a very simple, algebraic way. This approach is illustrated for second-order linear evolution equations and also for linear evolution equations containing spatial derivatives of arbitrary order. The simple and explicit determination of the unknown boundary values is based on the "Q-equation", which for the linearized nonlinear Schrödinger equation is the linear limit of the quadratic Q-equation introduced by Lenells and Fokas [*Proc. R. Soc. A*, 471, 2015]. Regarding the latter equation, it is also shown here that it provides a very simple, algebraic way for rederiving the remarkable results of Boutet de Monvel, Kotlyarov, and Shepelsky [*Int. Math. Res. Not.* issue 3, 2009] for the particular boundary condition of a single exponential.

KEYWORDS
Dirichlet-to-Neumann map, partial differential equations, unified transform

1 | INTRODUCTION

The unified transform, also known as the Fokas method, was introduced by the first author in 1997.[1] It provides a novel approach for solving boundary value problems for linear and integrable nonlinear partial differential equations (PDEs). For integrable PDEs the unified transform is the proper generalization from initial to boundary value problems of the famous inverse scattering transform (IST). Taking into consideration the seminal contributions of Harvey Segur in the development and applications of the IST, we hope that our paper fits well in this issue.

The unified transform has been employed by many authors for the investigation of evolution PDEs in one space variable. For such equations, among the most important results obtained via this method are the following: (i) Linear equations formulated on the half-line or a finite interval have been analyzed by Deconinck, Fokas, Pelloni, and collaborators.[2–17] (ii) Numerical techniques for linear equations are developed in Refs. 18–22. (iii) Novel results in spectral theory are derived in Refs. 23–25. (iv) Evolution PDEs defined on moving domains are analyzed in Refs. 26–28. (v) Unexpected results in the area of null controllability are presented in Ref. 29 (vi) Integrable nonlinear evolution equations have been analyzed by many authors, see, for example, Refs. 30–38. (vii) Nonlinear integrable equations with the so-called *linearizable boundary conditions* are investigated in Refs. 39–43. For this type of special boundary conditions, the unified transform yields a solution, which is as effective as the solution of the corresponding initial value problem on the line obtained via the IST. (viii) It is shown in Refs. 44, 45 that x-periodic initial conditions belong to the linearizable class. Hence, remarkably, such problems can be solved as effectively as the usual initial value problem on the full line. (ix) Himonas, Mantzavinos, and Fokas have initiated a program of study for using the unified transform to derive results regarding the well-posedness of arbitrary nonlinear evolution PDEs.[46–51]

It is well known that for the analysis of elliptic PDEs, one has to face the difficulty of analyzing the Dirichlet-to-Neumann map. Actually, this problem also arises in the analysis of boundary value problems for evolution PDEs. Consider, for example, the case of an evolution equation involving spatial derivatives of order up to two, formulated on the half-line. Let g_0 and g_1 denote the Dirichlet and Neumann boundary values at $x = 0$, i.e.,

$$g_0(t) = u(0, t), \quad g_1(t) = u_x(0, t), \quad t > 0. \tag{1}$$

For a well-posed problem, either g_0 or g_1 or a relationship between them is prescribed as a boundary condition. This corresponds to a Dirichlet or Neumann or Robin boundary value problem, respectively. The generalized Dirichlet-to-Neumann map involves the determination of the unknown boundary value in terms of the given initial and boundary conditions. A vital ingredient of the Fokas method is the so-called *global relation*. For linear evolution equations, this is a linear algebraic equation coupling appropriate integral transforms of all boundary values. By utilizing the global relation it is possible to determine the generalized Dirichlet-to-Neumann map for a linear evolution equation whose highest spatial derivative is of arbitrary order. However, for nonlinear integrable evolution equations the global relation is *nonlinear*. Thus, the generalized Dirichlet-to-Neumann map is characterized, in general, in terms of a *nonlinear* formalism. Despite this serious difficulty, substantial progress has been made via the usage of the global relation: First, it was realized in Ref. 31 that, for linearizable boundary conditions, the above nonlinear step can be bypassed, and that the generalized Dirichlet-to-Neumann map can be solved *explicitly*. Second, for general boundary conditions, the following important results have been obtained: (i) The Dirichlet-to-Neumann map for the nonlinear Schrödinger (NLS) equation can be characterized via two different nonlinear formalisms. The first is based on the analysis of the eigenfunctions of the t-part of the Lax pair evaluated at $x = 0$[33]; see also Refs. 52, 53. The second is based on the Gelfand–Levitan–Marchenko formulation,[34] which extends the pioneering results of Ref. 54. (ii) Boutet de Monvel, Kotlyarov, and Shepelsky obtained a remarkable result, involving the focusing NLS with a Dirichlet boundary condition, which for large t becomes a single exponential:

$$g_0(t) - \alpha e^{i\omega t} \to 0 \text{ sufficiently fast as } t \to \infty, \quad \text{with } \alpha > 0, \quad \omega \in \mathbb{R}; \tag{2}$$

the assumption of α being positive can be taken without loss of generality, because the NLS is invariant under the transformation $u \mapsto ue^{i\theta}$, $\theta \in \mathbb{R}$. It was shown in Ref. 55 that if $\omega \geq \alpha^2$ or $\omega \leq -6\alpha^2$, there exists a solution of the focusing NLS which satisfies (2), and the corresponding Neumann boundary value also asymptotes to a single exponential:

$$g_1(t) \sim \gamma e^{i\omega t}, \quad t \to \infty, \quad \gamma = \pm\alpha\sqrt{\omega - \alpha^2}, \quad \omega \geq \alpha^2, \tag{3a}$$

$$\gamma = i\alpha\sqrt{|\omega| + 2\alpha^2}, \quad \omega \leq -6\alpha^2. \tag{3b}$$

(iii) Rigorous well-posedness results for the Dirichlet-to-Neumann map for decaying Dirichlet data are derived in Ref. 56. (iv) For vanishing initial data, if the Dirichlet boundary condition is a sine wave, i.e., $u(0,t) = \alpha \sin t$, $\alpha \in \mathbb{R}$, then the Neumann boundary function $u_x(0,t)$ for the NLS,[34] and the Neumann boundary functions $u_x(0,t)$ and $u_{xx}(0,t)$ for the modified Korteweg-de Vries equation,[57] can be computed up to and including terms of $O(\alpha^3)$ and, at least up to this order, are asymptotically time-periodic for $t \to \infty$. A similar result has been obtained for the KdV equation,[58] where it is shown that when the Dirichlet condition is a sine wave, then the Neumann boundary values are asymptotically time-periodic at least up to second order in perturbation theory. This perturbative approach introduced in Ref. 34 involves heavy calculations, making it virtually impossible to go further than terms of $O(\alpha^3)$, but gives however a strong indication that (asymptotically) time-periodic Dirichlet boundary conditions lead to asymptotically time-periodic Neumann boundary values.

It is known that, for integrable evolution equations formulated on the full line, the power of the integrability formalism becomes evident in the asymptotic analysis for large t. This is also true for boundary value problems analyzed via the Fokas method. Indeed, for boundary conditions, which *vanish for large* t, the unified transform gives rise to a Riemann–Hilbert formulation with explicit x and t dependence. Hence, it is possible to determine the *structure* of the solution without determining the Dirichlet-to-Neumann map.[59] As a result of the fact that this map remains undetermined, certain constants in the relevant asymptotic formulas remain unknown.

Regarding the case of *t-periodic* boundary conditions, a new approach for the large t asymptotic analysis was introduced in Ref. 60 (see also Ref. 58). This approach is based on the investigation of a certain quadratic equation. We will refer to this equation, which can be obtained directly from the Lax pair evaluated at $x = 0$, as the Q-equation. It was shown in Ref. 60 that for a Dirichlet boundary condition, which asymptotes to a t-periodic function for large t that can be expanded in a power series of ϵ, it is possible to obtain the Neumann boundary value via a series expansion in ϵ, which is explicitly determined in terms of the Dirichlet data.

In the present paper, it is shown that the Q-approach is truly powerful:

(i) For the case of the linear version of the NLS, namely, for the equation

$$iu_t + u_{xx} = 0, \tag{4}$$

it was shown in Ref. 60 that, if the Dirichlet condition is asymptotically t-periodic, then for large t the Neumann boundary value is also t-periodic. Moreover, an explicit formula was obtained in Ref. 60 relating the coefficients of the two Fourier series characterizing the asymptotic Dirichlet and Neumann values. It is shown in Section 2 that the linear version of the Q-formalism provides a much simpler way for computing asymptotically the Dirichlet-to-Neumann map of Equation (4). Similar results are obtained in Section 2 for the heat equation,

$$u_t = u_{xx}, \tag{5}$$

and for the diffusion–convection equation,

$$u_t = u_{xx} + \beta u_x, \quad \beta \geq 0. \tag{6}$$

(ii) For large t, this new approach provides, in a straightforward manner, the asymptotic form of the generalized Dirichlet-to-Neumann map of linear evolution equations containing x-derivatives of arbitrary order, with asymptotically time-periodic boundary data and with a sufficiently decaying initial condition. We illustrate the general approach in Section 3 using the example of the Stokes equation,

$$u_t + u_x + u_{xxx} = 0. \tag{7}$$

The specific implementation of the new method to Equations (4)–(7) makes clear how this method can be easily applied to any linear evolution equation with constant coefficients.

(iii) The determination of the large t-asymptotics of the Dirichlet-to-Neumann map of the diffusion-convection equation (6), which contains the heat equation as a special case for $\beta = 0$, is presented in Section 4 via the unified transform method. The derivation follows similar steps to those used in Ref. 60. However, it presents a slight generalization of the analogous result of Ref. 60, because here we derive the large t asymptotics of $u_x(x,t)$ as opposed to $u_x(0,t)$. Using almost identical steps it is possible to derive an expression for the large t behavior of $u(x,t)$, which shows that $u(x,t)$ becomes t-periodic. By evaluating the expression for $u_x(x,t)$ derived in this way at $x = 0$, we obtain explicitly the relationship between the Fourier coefficients of the Dirichlet and the Neumann values. By comparing this derivation with the very simple procedure of the Q-approach, the advantage of the latter becomes evident. Analogous computations for the Stokes equation (7) can be found in Refs. 58, 61. Similar results can be obtained for other linear equations with spatial derivatives of arbitrary order.

(iv) It is shown in Section 5 that the remarkable results of Ref. 55 follow immediately from the analysis of the Q-equation derived in Ref. 60. Taking into consideration the complexity of the earlier derivations presented in Ref. 55 and in Ref. 60, this result supports further the assertion that the Q-approach provides indeed a direct and quite powerful computational technique.

2 | THE LARGE t BEHAVIOR OF THE DIRICHLET-TO-NEUMANN MAP FOR SECOND-ORDER LINEAR EVOLUTION EQUATIONS WITH t-PERIODIC BOUNDARY CONDITIONS

The derivation of the Q-equation is based on the Lax pair formulation. In turn, this is related to the first step of the Fokas method, namely, rewriting a given linear PDE in a divergence form. This can be achieved in a variety of ways, including the use of the adjoint. Let $u(x,t)$ satisfy the second-order evolution PDE

$$u_t = A_2 u_{xx} + A_1 u_x + A_0 u, \tag{8}$$

where A_0, A_1, and A_2 are complex constants. We assume that $u(x,t)$ is sufficiently smooth and that both $u(x,t)$ and its x-derivatives decay sufficiently fast as $x \to \infty$ for each $t \geq 0$.

The formal adjoint can be obtained from Equation (8) by replacing ${}^{\partial}/_{\partial_t}$ and ${}^{\partial}/_{\partial_x}$ with $-{}^{\partial}/_{\partial_t}$ and $-{}^{\partial}/_{\partial_x}$, respectively. Hence,

$$v_t = -A_2 v_{xx} + A_1 v_x - A_0 v. \tag{9}$$

Multiplying Equations (8) and (9) by v and u, respectively, and then adding the resulting equations, we find

$$(uv)_t = [A_2(v u_x - u v_x) + A_1 uv]_x. \tag{10}$$

Equation (8) admits a solution of the form

$$\mathrm{e}^{ikx-\Omega(k)t}, \text{ where } \Omega(k) = A_2 k^2 - iA_1 k - A_0,\ k \in \mathbb{C}. \tag{11}$$

We assume that $\mathrm{Re}\Omega(k) \geq 0$ for k real, so that the initial value problem is well-posed.[62] We observe that, $\mathrm{e}^{-ikx+\Omega(k)t}$ is a solution of Equation (9). Replacing, in Equation (10), v by this exponential, we find the following one-parameter family of divergence forms:

$$\left(\mathrm{e}^{-ikx+\Omega(k)t} u\right)_t = \left\{\mathrm{e}^{-ikx+\Omega(k)t}[A_2(u_x + iku) + A_1 u]\right\}_x, \quad k \in \mathbb{C}. \tag{12}$$

This equation implies the existence of the function $\mathrm{e}^{-ikx+\Omega(k)t}\mu(k,x,t)$, where

$$\left(\mathrm{e}^{-ikx+\Omega(k)t}\mu\right)_x = \mathrm{e}^{-ikx+\Omega(k)t} u, \tag{13a}$$

$$\left(\mathrm{e}^{-ikx+\Omega(k)t}\mu\right)_t = \mathrm{e}^{-ikx+\Omega(k)t}[A_2(u_x + iku) + A_1 u]. \tag{13b}$$

Simplifying, we obtain the following Lax pair of Equation (8):

$$\mu_x - ik\mu = u, \tag{14a}$$

$$\mu_t + \Omega(k)\mu = A_2(u_x + iku) + A_1 u. \tag{14b}$$

2.1 | The Q-formulation

Suppose that Equation (8) is valid in a given domain D. Equation (12) and Green's Theorem imply that

$$\int_{\partial D} \mathrm{e}^{-ikx+\Omega(k)\tau}\{u\mathrm{d}x + [A_2(u_x + iku) + A_1 u]\mathrm{d}\tau\} = 0, \quad k \in \mathbb{C}, \tag{15}$$

where ∂D denotes the boundary of D.

In the particular case that Equation (8) is formulated on the half-line,

$$D = \{0 < x < \infty, 0 < \tau < t\}, \tag{16}$$

Equation (15) becomes the so-called Global Relation:

$$-\int_0^\infty \mathrm{e}^{-ikx}u(x,t)\mathrm{d}x = -\mathrm{e}^{-\Omega(k)t}\int_0^\infty \mathrm{e}^{-ikx}u(x,0)\mathrm{d}x$$
$$+ A_2\int_0^t \mathrm{e}^{-\Omega(k)(t-s)}g_1(s)\mathrm{d}s + (iA_2k + A_1)\int_0^t \mathrm{e}^{-\Omega(k)(t-s)}g_0(s)\mathrm{d}s. \tag{17}$$

Let $Q(k,t)$ satisfy the t-dependent part of the associated Lax pair evaluated at $x = 0$, i.e.,

$$Q_t(k,t) + \Omega(k)Q(k,t) = A_2 g_1(t) + (iA_2k + A_1)g_0(t), \quad t > 0, \quad k \in \mathbb{C}. \tag{18}$$

Then,

$$Q(k,t) = \mathrm{e}^{-\Omega(k)t}Q(k,0) + A_2\int_0^t \mathrm{e}^{-\Omega(k)(t-s)}g_1(s)\mathrm{d}s + (iA_2k + A_1)\int_0^t \mathrm{e}^{-\Omega(k)(t-s)}g_0(s)\mathrm{d}s. \tag{19}$$

Comparing this equation with the Global Relation (17), it follows that if we choose

$$Q(k,0) = -\int_0^\infty \mathrm{e}^{-ikx}u(x,0)\mathrm{d}x, \quad \mathrm{Im}k \le 0, \tag{20}$$

then

$$-\int_0^\infty \mathrm{e}^{-ikx}u(x,t)\mathrm{d}x = Q(k,t). \tag{21}$$

For the integrals appearing on the right-hand side of Equation (19), we have $t - s \geq 0$. Let us assume that g_0 and g_1 are bounded functions. Thus, as $t \to \infty$, the second and third terms of the right-hand side of Equation (19) are bounded and analytic in the domain

$$\{k \in \mathbb{C} \,:\, \text{Re}\,\Omega(k) > 0\}. \tag{22}$$

Moreover, $Q(k, 0)$ is bounded and analytic for Im$k \leq 0$. Thus, $Q(k, t)$ is well defined in

$$\{k \in \mathbb{C} \,:\, \text{Im}\,k \leq 0, \text{Re}\,\Omega(k) > 0\}. \tag{23}$$

The unified transform yields a representation, which involves the curve ∂D^-, which is the boundary of the domain D^- defined by

$$D^- = \{k \in \mathbb{C} \,:\, \text{Im}\,k < 0, \text{Re}\,\Omega(k) < 0\}. \tag{24}$$

We observe that the domain D^- is the complement (in the lower half of the complex k-plane) of the domain (23) of validity of Q. However, Equation (21) implies that, actually, $Q(k, t)$ has a much larger domain of analyticity. Namely, this equation extends the domain of validity to Im$k \leq 0$ (the entire lower half of the complex k-plane including the real axis), which encompasses $\overline{D^-}$, i.e., D^- and its boundary. Thus, we will require that $Q(k, t)$ is free of singularities in $\overline{D^-}$.

Let us assume that for large t both g_0 and g_1 asymptote sufficiently fast toward smooth time-periodic functions of the general form

$$g_0(t) \sim \sum_{n=-\infty}^{\infty} \alpha_n^{(0)} \mathrm{e}^{in\omega t}, \quad g_1(t) \sim \sum_{n=-\infty}^{\infty} \alpha_n^{(1)} \mathrm{e}^{in\omega t}, \quad t \to \infty, \tag{25}$$

where $\alpha_n^{(0)}, \alpha_n^{(1)} \in \mathbb{C}$, $n \in \mathbb{Z}$ and $\omega > 0$. We seek a solution Q of Equation (18) of the form

$$Q(k, t) \sim \sum_{n=-\infty}^{\infty} q_n(k) \mathrm{e}^{in\omega t}, \quad t \to \infty. \tag{26}$$

Substituting the above asymptotic expressions for g_0, g_1, and Q in Equation (18), we find

$$q_n(k) = \frac{A_2 \alpha_n^{(1)} + (iA_2 k + A_1)\alpha_n^{(0)}}{\Omega(k) + in\omega}, \quad n \in \mathbb{Z}. \tag{27}$$

Remarkably, the condition that $q_n(k)$ does *not* have poles for $k \in \overline{D^-}$, clearly imposes a relationship between $\alpha_n^{(1)}$ and $\alpha_n^{(0)}$. Actually, if the denominator of Equation (27) becomes zero for some k's then, because $in\omega$ is purely imaginary, those k's should lie on the contour defined by Re$\Omega(k) = 0$. Hence, any singularities of $q_n(k)$ in $\overline{D^-}$ occur in fact on its boundary ∂D^-. In other words, the condition that the singularities of $q_n(k)$ on ∂D^- are removable, determines the Dirichlet-to-Neumann map.

Remark 1. The assumption (25), i.e., that (asymptotically) t-periodic Dirichlet boundary conditions lead to asymptotically t-periodic Neumann values, can be shown to hold for appropriate initial-boundary value problems using the Fokas method and the method of steepest descent, see Section 4 for details. In particular, to avoid problems with the steepest descent arguments, we have to make sure that the denominator involved, which for linear evolution PDEs is of the form $\Omega(k) + in\omega$, does not become zero on the steepest descent contour. The corresponding steepest descent contour is the contour that passes through a saddle point of $\Omega(k)$ and on which the imaginary part of $\Omega(k)$ is constant. So, we can a priori consider appropriate t-periodic boundary conditions to avoid problems with the steepest descent arguments.

In what follows we consider three particular examples.

2.2 | The linearized NLS equation

Let $u(x,t)$ solve the linearized version of the NLS, namely,

$$iu_t + u_{xx} = 0. \tag{28}$$

In this case, $A_2 = i$, $A_1 = A_0 = 0$, and we assume that $\alpha_0^{(0)} = 0$ (see the remark at the end of Section 2.1). Thus,

$$\Omega(k) = ik^2, \quad q_n(k) = \frac{\alpha_n^{(1)} + i\alpha_n^{(0)}k}{k^2 + n\omega}, \quad n \in \mathbb{Z}. \tag{29}$$

Denoting $k = k_R + ik_I$ with $k_R, k_I \in \mathbb{R}$, the domain D^- defined by Equation (24) takes the following form:

$$D^- = \{k \in \mathbb{C} : k_I < 0, k_R k_I > 0\}, \tag{30}$$

which is the third quadrant of the complex k-plane. The singularities of $q_n(k)$ occurring on ∂D^- are given by

$$k(n) = \begin{cases} -i\sqrt{n\omega}, & n \geq 0, \\ -\sqrt{-n\omega}, & n < 0. \end{cases} \tag{31}$$

The condition that the above singularities are removable, implies the following relationship between $\alpha_n^{(0)}$ and $\alpha_n^{(1)}$:

$$\alpha_n^{(1)} = -i\alpha_n^{(0)}k(n), \quad n \in \mathbb{Z}, \tag{32}$$

with $k(n)$ defined by Equation (31). This relationship was first given in Ref. 60.

2.3 | The heat equation

Let $u(x,t)$ solve the heat equation, i.e.,

$$u_t = u_{xx}. \tag{33}$$

In this case, $A_2 = 1, A_1 = A_0 = 0$ and we assume that $\alpha_0^{(0)} = 0$. Thus,

$$\Omega(k) = k^2, \quad q_n(k) = \frac{\alpha_n^{(1)} + i\alpha_n^{(0)} k}{k^2 + in\omega}, \quad n \in \mathbb{Z}. \tag{34}$$

The domain D^-, defined by Equation (24), takes the following form (describing a wedge-shaped domain in the lower half of the complex k-plane):

$$D^- = \left\{k \in \mathbb{C} : k_I < 0, \ k_R^2 < k_I^2\right\}, \tag{35}$$

where $k = k_R + ik_I$ with $k_R, k_I \in \mathbb{R}$. The condition that the singularities $k(n)$ of $q_n(k)$ occurring on the boundary ∂D^-

$$k(n) = \begin{cases} \sqrt{\frac{n\omega}{2}}(1 - i), & n \geq 0, \\ \sqrt{\frac{-n\omega}{2}}(-1 - i), & n < 0, \end{cases} \tag{36}$$

are removable, implies the following relationship between $\alpha_n^{(0)}$ and $\alpha_n^{(1)}$:

$$\alpha_n^{(1)} = -i\alpha_n^{(0)} k(n), \quad n \in \mathbb{Z}, \tag{37}$$

with $k(n)$ defined by Equation (36).

2.4 | The diffusion–convection equation

Let $u(x, t)$ solve the diffusion–convection equation:

$$u_t = u_{xx} + \beta u_x, \quad \beta \geq 0. \tag{38}$$

In this case, $A_2 = 1$, $A_1 = \beta$, and $A_0 = 0$ and we assume that $\alpha_0^{(0)} = 0$. Thus,

$$\Omega(k) = k^2 - i\beta k, \quad q_n(k) = \frac{\alpha_n^{(1)} + (ik + \beta)\alpha_n^{(0)}}{k^2 - i\beta k + in\omega}, \quad n \in \mathbb{Z}. \tag{39}$$

The singularities of the denominator lie at

$$\begin{aligned} k_{1,2}(n) &= \frac{i\beta \pm \sqrt{-\beta^2 - 4in\omega}}{2} \\ &= \begin{cases} \frac{1}{2}\left(i\beta \pm \left(-\sqrt{\frac{-\beta^2+\sqrt{\beta^4+16n^2\omega^2}}{2}} + i\sqrt{\frac{\beta^2+\sqrt{\beta^4+16n^2\omega^2}}{2}}\right)\right), & n \geq 0, \\ \frac{1}{2}\left(i\beta \pm \left(\sqrt{\frac{-\beta^2+\sqrt{\beta^4+16n^2\omega^2}}{2}} + i\sqrt{\frac{\beta^2+\sqrt{\beta^4+16n^2\omega^2}}{2}}\right)\right), & n < 0. \end{cases} \end{aligned} \tag{40}$$

For the diffusion–convection equation, the domain D^- takes the form

$$D^- = \left\{k \in \mathbb{C} : k_I < 0,\ k_R^2 - k_I^2 + \beta k_I < 0\right\}, \tag{41}$$

where $k = k_R + ik_I$ with $k_R, k_I \in \mathbb{R}$. It can be readily checked that the singularities that lie on ∂D^- are given by the solutions with the minus sign of Equation (40), which we denote by $k_2(n)$. Hence, the condition that these singularities are removable, implies the following relation between $\alpha_n^{(0)}$ and $\alpha_n^{(1)}$:

$$\alpha_n^{(1)} = -(ik_2(n) + \beta)\alpha_n^{(0)}, \quad n \in \mathbb{Z}, \tag{42}$$

with $k_2(n)$ defined by the solutions with the minus sign given in Equation (40).

3 | THE LARGE t BEHAVIOR OF THE DIRICHLET-TO-NEUMANN MAP FOR AN ARBITRARY LINEAR EVOLUTION EQUATION WITH t-PERIODIC BOUNDARY CONDITIONS

Let $u(x,t)$ satisfy the following general evolution PDE on the half-line, which has spatial derivatives of arbitrary order:

$$u_t + \Omega(-i\partial_x)u = 0, \tag{43}$$

where $\Omega(k)$ is a polynomial of the complex variable k of degree n and $\mathrm{Re}\Omega(k) \geq 0$ for k real. We assume that $u(x,t)$ is sufficiently smooth, up to and including the boundary, and that both $u(x,t)$ and its x-derivatives decay sufficiently fast as $x \to \infty$ for each $t \geq 0$.

A particular solution of Equation (43) is

$$\mathrm{e}^{ikx-\Omega(k)t}, \quad k \in \mathbb{C}. \tag{44}$$

It is shown in Ref. 62 that Equation (43) can be written in the form

$$\left(\mathrm{e}^{-ikx+\Omega(k)t}u(x,t)\right)_t = \left(\mathrm{e}^{-ikx+\Omega(k)t}\sum_{j=0}^{n-1} c_j(k)\partial_x^j u(x,t)\right)_x, \quad k \in \mathbb{C}, \tag{45}$$

where $\{c_j(k)\}_0^{n-1}$ can be calculated explicitly in terms of $\Omega(k)$ as follows:

$$\sum_{j=0}^{n-1} c_j(k)\partial_x^j = i\left.\frac{\Omega(k)-\Omega(l)}{k-l}\right|_{l=-i\partial_x}. \tag{46}$$

Hence, the associated Lax pair is

$$\mu_x - ik\mu = u, \tag{47a}$$

$$\mu_t + \Omega(k)\mu = \sum_{j=0}^{n-1} c_j(k)\partial_x^j u(x,t). \tag{47b}$$

As before, letting $Q(k,t)$ satisfy the t-dependent part of the associated Lax pair evaluated at $x = 0$, we find

$$Q_t(k,t) + \Omega(k)Q(k,t) = \sum_{j=0}^{n-1} c_j(k)g_j(t), \quad t > 0, \quad k \in \mathbb{C}, \tag{48}$$

where the g_j's are defined by $g_j(t) := \partial_x^j u(0,t)$. Assuming the following expansions for large t,

$$g_j(t) \sim \sum_{n=-\infty}^{\infty} \alpha_n^{(j)} e^{in\omega t}, \quad j = 0, 1, \ldots, n-1, \quad t \to \infty, \tag{49}$$

and substituting the above expansions, along with Equation (26), into Equation (48), we find

$$q_n(k) = \frac{\sum_{j=0}^{n-1} c_j(k)\alpha_n^{(j)}}{\Omega(k) + in\omega}, \quad n \in \mathbb{Z}. \tag{50}$$

In the same way as in Section 2, we can show that the singularities of $q_n(k)$ on ∂D^- have to be removable. Hence, this condition determines the Dirichlet-to-Neumann correspondence also for the general linear evolution PDE (43). We illustrate the general approach using the example of the Stokes equation.

3.1 | The Stokes equation

Let $u(x,t)$ solve the Stokes equation, i.e.,

$$u_t + u_x + u_{xxx} = 0. \tag{51}$$

In this case, $\Omega(k) = -ik^3 + ik$ and we assume that $n\omega \neq \frac{2}{3\sqrt{3}}$, $n \in \mathbb{Z}$ (see the remark at the end of Section 2.1). Substituting this in Equation (46), we find

$$\sum_{j=0}^{2} c_j(k)\partial_x^j = -\partial_x^2 - ik\partial_x + k^2 - 1, \tag{52}$$

i.e., $c_2(k) = -1$, $c_1(k) = -ik$, and $c_0(k) = k^2 - 1$. So $q_n(k)$ becomes

$$q_n(k) = \frac{-\alpha_n^{(2)} - ik\alpha_n^{(1)} + (k^2-1)\alpha_n^{(0)}}{i(-k^3 + k + n\omega)}, \quad n \in \mathbb{Z}. \tag{53}$$

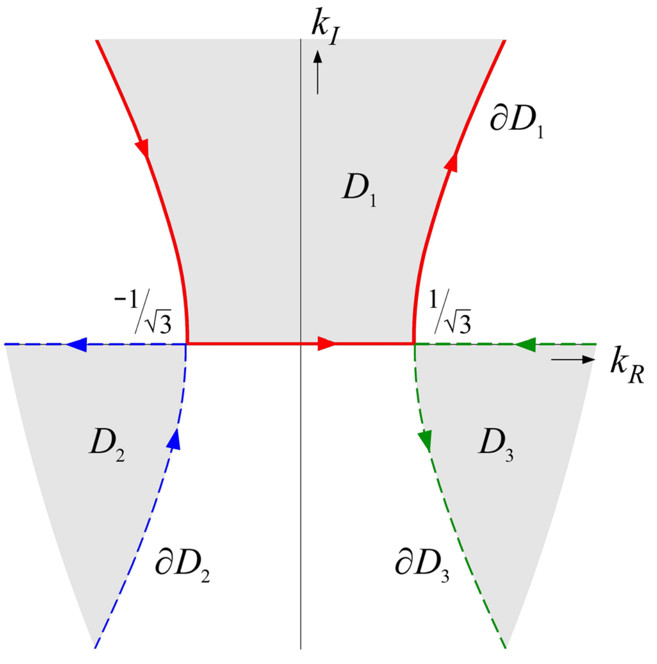

FIGURE 1 The contour $k_I(3k_R^2 - k_I^2 - 1) = 0$ appearing in the analysis of the Stokes equation (51). This contour, determined by Equation (54), consists of three branches ∂D_1, ∂D_2, and ∂D_3. On each of these branches lies precisely one root of $-k^3 + k + n\omega = 0$, as discussed in the text

For any root of $i(-k^3 + k + n\omega) = 0$, or equivalently of $-k^3 + k + n\omega = 0$, because $n\omega \in \mathbb{R}$, we have that $\mathrm{Re}(-ik^3 + ik) = 0$, i.e.,

$$k_I\left(3k_R^2 - k_I^2 - 1\right) = 0, \tag{54}$$

where $k = k_R + ik_I$ with $k_R, k_I \in \mathbb{R}$. The contour $k_I(3k_R^2 - k_I^2 - 1) = 0$ of the complex k-plane is depicted in Figure 1. Hence, for each n, the three roots of $-k^3 + k + n\omega = 0$ are positioned on the depicted contour. We will show that for each n, precisely *one* root of this equation lies on each of the branches ∂D_1, ∂D_2, and ∂D_3 of the contour. Indeed, there are two cases to distinguish:

- *First case*: When $(n\omega)^2 > 4/27$, then $-k^3 + k + n\omega = 0$ has one real root and a pair of nonreal complex conjugate roots. One of the complex roots will be in the upper half plane and, hence it will lie on one of the hyperbolic branches of ∂D_1. Its complex conjugate will lie on the corresponding hyperbolic branch in the lower half plane, so this root lies on either ∂D_2 or ∂D_3. Let us denote the roots of $-k^3 + k + n\omega = 0$ with $k_1(n)$, $k_2(n)$, and $k_3(n)$. Because the Vieta formulas tell us that $k_1(n) + k_2(n) + k_3(n) = 0$, we see that the real root must either be greater than $^2/_{\sqrt{3}}$ (when the second complex root lies on ∂D_2) and hence be positioned on ∂D_3, *or* be smaller than $-^2/_{\sqrt{3}}$ (when the second complex root lies on ∂D_3) and hence lie on ∂D_2.
- *Second case*: When $(n\omega)^2 < 4/27$, then $-k^3 + k + n\omega = 0$ has three real roots. The analysis of the graph of $f(k) = -k^3 + k + n\omega$, $k \in \mathbb{R}$, shows that exactly one root is smaller than $-^1/_{\sqrt{3}}$ (and hence lies on ∂D_2), one root lies between $-^1/_{\sqrt{3}}$ and $^1/_{\sqrt{3}}$ (on ∂D_1), and one root is greater than $^1/_{\sqrt{3}}$ (and thus lies on ∂D_3).

As before, we require the singularities of $q_n(k)$ on ∂D^- to be removable, where for the Stokes equation we have

$$D^- = \left\{k \in \mathbb{C} : k_I < 0,\ k_I\left(3k_R^2 - k_I^2 - 1\right) < 0\right\}. \tag{55}$$

We observe that $D^- = D_2 \cup D_3$, see Figure 1. Let us denote with $k_1(n)$, $k_2(n)$, and $k_3(n)$ the zeros of the denominator, which lie on the boundaries ∂D_1, ∂D_2, and ∂D_3, respectively. Thus, the singularities $k_2(n)$ and $k_3(n)$ are the ones that lie on ∂D^-. Because these two singularities should be removable, we need the numerator of (53) to vanish for $k = k_2(n)$ and $k = k_3(n)$, so we arrive at the following system for the coefficients $\alpha_n^{(0)}$, $\alpha_n^{(1)}$, and $\alpha_n^{(2)}$:

$$-\alpha_n^{(2)} - ik_2(n)\alpha_n^{(1)} + \left(k_2^2(n) - 1\right)\alpha_n^{(0)} = 0, \tag{56a}$$

$$-\alpha_n^{(2)} - ik_3(n)\alpha_n^{(1)} + \left(k_3^2(n) - 1\right)\alpha_n^{(0)} = 0. \tag{56b}$$

Let us assume that we consider the Dirichlet boundary problem of the Stokes equation and, hence, the coefficients $\alpha_n^{(0)}$, $n \in \mathbb{Z}$, are given. We can then proceed to solve the above system for the unknown Neumann coefficients $\alpha_n^{(1)}$ and $\alpha_n^{(2)}$ in terms of the Dirichlet coefficients $\alpha_n^{(0)}$. From Equation (56a) we get

$$-\alpha_n^{(2)} = ik_2(n)\alpha_n^{(1)} - \left(k_2^2(n) - 1\right)\alpha_n^{(0)}. \tag{57}$$

Substituting this expression into Equation (56b), we find the first set of Neumann coefficients $\alpha_n^{(1)}$ (in terms of the Dirichlet coefficients):

$$\alpha_n^{(1)} = \frac{\left(k_2^2(n) - 1\right)\alpha_n^{(0)} - \left(k_3^2(n) - 1\right)\alpha_n^{(0)}}{i(k_2(n) - k_3(n))} = -i(k_2(n) + k_3(n))\alpha_n^{(0)} = ik_1(n)\alpha_n^{(0)}, \quad n \in \mathbb{Z}, \tag{58}$$

where we have used the Vieta formula $k_1(n) + k_2(n) + k_3(n) = 0$. From Equation (56b) we have

$$-\alpha_n^{(2)} = ik_3(n)\alpha_n^{(1)} - \left(k_3^2(n) - 1\right)\alpha_n^{(0)}. \tag{59}$$

Substituting the value of $\alpha_n^{(1)}$ from (58) into Equations (57) and (59) and adding the resulting equations, we find

$$\begin{aligned} -2\alpha_n^{(2)} &= \left(2 - k_2^2(n) - k_3^2(n)\right)\alpha_n^{(0)} + i(k_2(n) + k_3(n))\left(ik_1(n)\alpha_n^{(0)}\right) \\ &= \left(2 - k_2^2(n) - k_3^2(n)\right)\alpha_n^{(0)} + (k_2(n) + k_3(n))^2\alpha_n^{(0)}, \end{aligned} \tag{60}$$

where we have again used that $k_1(n) + k_2(n) + k_3(n) = 0$. Hence, the second set of Neumann coefficients $\alpha_n^{(2)}$ (in terms of the Dirichlet coefficients) is given by

$$\alpha_n^{(2)} = -[1 + k_2(n)k_3(n)]\alpha_n^{(0)}, \quad n \in \mathbb{Z}. \tag{61}$$

Thus, we have determined the Dirichlet-to-Neumann map for the Stokes equation.

4 | THE UNIFIED TRANSFORM FOR THE LARGE *t* ASYMPTOTICS OF THE DIFFUSION–CONVECTION EQUATION

Proposition 1. *Let $u(x,t)$ satisfy the following Dirichlet problem for the diffusion–convection equation (6) on the half-line:*

$$u_t = u_{xx} + \beta u_x, \quad \beta \geq 0, \quad x > 0, \quad t > 0, \tag{62a}$$

$$u(x,0) = u_0(x), \quad x \geq 0, \tag{62b}$$

$$u(0,t) = g_0(t), \quad t \geq 0, \tag{62c}$$

where the initial and boundary data are compatible at the origin. The solution $u(x,t)$ of this problem is assumed to be sufficiently smooth (up to and including the boundary) and both $u(x,t)$ and its x-derivatives are assumed to decay sufficiently fast to zero as $x \to \infty$ for each $t \geq 0$. The initial condition $u_0(x)$ is therefore taken to have sufficient smoothness, and both $u_0(x)$ and its derivatives are taken to decay sufficiently fast to zero as $x \to \infty$. We further suppose that the Dirichlet boundary condition $g_0(t)$ is bounded, sufficiently smooth, and for large t asymptotes sufficiently fast toward a periodic function

$$g_0(t) - {\sum_{n=-\infty}^{\infty}}' \alpha_n^{(0)} \mathrm{e}^{in\omega t} \to 0, \quad t \to \infty, \tag{63}$$

where $\omega > 0$, the prime denotes that $\alpha_0^{(0)} = 0$, and the Fourier coefficients $\alpha_n^{(0)} \in \mathbb{C}$, $n \in \mathbb{Z}$, are such that

$${\sum_{n=-\infty}^{\infty}}' \sqrt{|n|}|\alpha_n^{(0)}| < \infty. \tag{64}$$

Then:

(i) For each $x \geq 0$, $u(x,t)$ and $u_x(x,t)$ become time-periodic for large t.
(ii) The Neumann boundary function $u_x(0,t) = g_1(t)$ for large t takes the form

$$g_1(t) \sim {\sum_{n=-\infty}^{\infty}}' \alpha_n^{(1)} \mathrm{e}^{in\omega t}, \quad t \to \infty, \tag{65}$$

with

$$\alpha_n^{(1)} = -(ik_2(n) + \beta)\alpha_n^{(0)}, \quad n \in \mathbb{Z}, \tag{66}$$

where the $k_2(n)$'s, $n \in \mathbb{Z}$, denote the roots with the minus sign given in Equation (40).

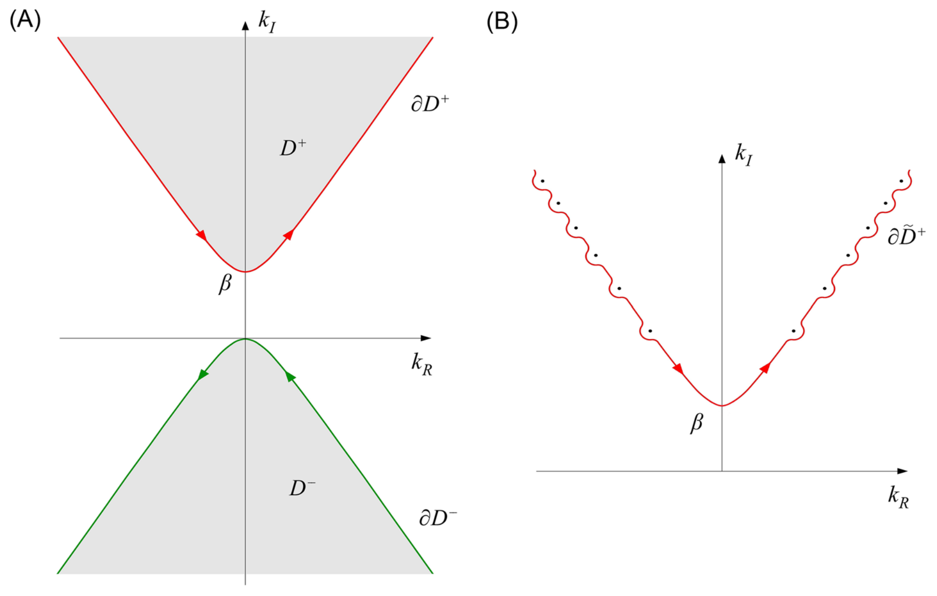

FIGURE 2 (A) The domains D^+ and D^- in the complex k-plane with their respective boundaries (∂D^+ and ∂D^-) for the diffusion–convection equation (62a). These boundaries are given by $\mathrm{Re}(k^2 - i\beta k) = 0$, i.e., the hyperbolas $k_R^2 - k_I^2 + \beta k_I = 0$. (B) The deformed contour $\partial\tilde{D}^+$ which avoids the singularities $k_1(n)$, $n \in \mathbb{Z}^*$, given by Equation (90)

Proof. It is shown in Ref. 62 that the Fokas method yields the following representation for the solution of Equation (62a):

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{ikx-(k^2-i\beta k)t}\hat{u}_0(k)\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D^+} \mathrm{e}^{ikx-(k^2-i\beta k)t}\left[\hat{u}_0(i\beta - k) + (2ik+\beta)\tilde{g}_0(k^2 - i\beta k, t)\right]\mathrm{d}k, \tag{67}$$

where the contour ∂D^+ is depicted in Figure 2(A) and $\hat{u}_0(k)$ and $\tilde{g}_0(k,t)$ are defined by

$$\hat{u}_0(k) := \int_0^{+\infty} \mathrm{e}^{-ikx}u_0(x)\mathrm{d}x, \quad \mathrm{Im}k \le 0, \quad \text{and} \quad \tilde{g}_0(k,t) := \int_0^t \mathrm{e}^{ks}g_0(s)\mathrm{d}s, \quad k \in \mathbb{C}. \tag{68}$$

We observe that the term $\mathrm{e}^{ikx-(k^2-i\beta k)t}$, appearing in Equation (67), is bounded and analytic when both $\mathrm{Re}(k^2 - i\beta k) \ge 0$ and $\mathrm{Re}(ik) \le 0$, i.e., when $k_R^2 - k_I^2 + \beta k_I \ge 0$ and $k_I \ge 0$, where $k = k_R + ik_I$ with $k_R, k_I \in \mathbb{R}$. Also, from the definition of $\hat{u}_0$, Equation (68), we have

$$\hat{u}_0(i\beta - k) = \int_0^{+\infty} \mathrm{e}^{(\beta+ik)x}u_0(x)\mathrm{d}x = O\left(\frac{1}{\beta + ik}\right), \quad \text{as } k \to \infty \quad \text{with } k_I \ge \beta. \tag{69}$$

This expression is bounded and analytic for $k_I \ge \beta$. Hence, in the part of the second integral of Equation (67) which involves $\hat{u}_0(i\beta - k)$, we can deform the contour ∂D^+ to $(-\infty + i\beta, +\infty + i\beta)$, i.e., to the horizontal line in the complex k-plane passing through $i\beta$. Performing this deformation in Equation (67) and using the definition of $\tilde{g}_0$ from Equation (68), we find

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{ikx-(k^2-i\beta k)t}\hat{u}_0(k)\mathrm{d}k-\frac{1}{2\pi}\int_{-\infty+i\beta}^{+\infty+i\beta}\mathrm{e}^{ikx-(k^2-i\beta k)t}\hat{u}_0(i\beta-k)\mathrm{d}k$$
$$-\frac{1}{2\pi}\int_{\partial D^+}(2ik+\beta)\mathrm{e}^{ikx-(k^2-i\beta k)t}\left(\int_0^t\mathrm{e}^{(k^2-i\beta k)s}g_0(s)\mathrm{d}s\right)\mathrm{d}k. \tag{70}$$

Because we are interested in finding the Neumann boundary value in terms of the Dirichlet boundary condition, we differentiate both sides of Equation (70) with respect to x and we find

$$u_x(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}ik\mathrm{e}^{ikx-(k^2-i\beta k)t}\hat{u}_0(k)\mathrm{d}k-\frac{1}{2\pi}\int_{-\infty+i\beta}^{+\infty+i\beta}ik\mathrm{e}^{ikx-(k^2-i\beta k)t}\hat{u}_0(i\beta-k)\mathrm{d}k$$
$$-\frac{1}{2\pi}\frac{\partial}{\partial x}\int_{\partial D^+}(2ik+\beta)\mathrm{e}^{ikx-(k^2-i\beta k)t}\left(\int_0^t\mathrm{e}^{(k^2-i\beta k)s}g_0(s)\mathrm{d}s\right)\mathrm{d}k. \tag{71}$$

In what follows, we will analyze separately the terms of Equation (71). Let us begin by analyzing the first two integrals appearing in Equation (71). Replacing k with $-k+i\beta$ in the second of these integrals, and using the definition of $\hat{u}_0$ from Equation (68), the sum of these integrals takes the form

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}ik\mathrm{e}^{ikx-(k^2-i\beta k)t}\hat{u}_0(k)\mathrm{d}k+\frac{1}{2\pi}\int_{-\infty}^{+\infty}(ik+\beta)\mathrm{e}^{-(ik+\beta)x-(k^2-i\beta k)t}\hat{u}_0(k)\,\mathrm{d}k$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}ik\left(\mathrm{e}^{ikx}+\mathrm{e}^{-(ik+\beta)x}\right)\mathrm{e}^{-(k^2-i\beta k)t}\int_0^{+\infty}\mathrm{e}^{-iky}u_0(y)\mathrm{d}y\,\mathrm{d}k$$
$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\beta\mathrm{e}^{-(ik+\beta)x-(k^2-i\beta k)t}\int_0^{+\infty}\mathrm{e}^{-iky}u_0(y)\mathrm{d}y\,\mathrm{d}k$$
$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}ik\left(\mathrm{e}^{ikx}+\mathrm{e}^{-(ik+\beta)x}\right)\mathrm{e}^{-(k^2-i\beta k)t}\left(\frac{1}{ik}u_0(0)+\int_0^{+\infty}\frac{\mathrm{e}^{-iky}}{ik}\dot{u}_0(y)\mathrm{d}y\right)\mathrm{d}k$$
$$+\frac{1}{2\pi}\int_{-\infty}^{+\infty}\beta\mathrm{e}^{-(ik+\beta)x-(k^2-i\beta k)t}\int_0^{+\infty}\mathrm{e}^{-iky}u_0(y)\mathrm{d}y\,\mathrm{d}k. \tag{72}$$

In the third step of the above computation we integrated by parts, so that in the next step we can now change the order of integration and obtain well-defined k-integrals. Changing the order of integration in Equation (72) and completing the squares in the exponents, we find that Equation (72) can be written as

$$\frac{u_0(0)}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{-\left(\sqrt{t}k+\frac{\beta t+x}{2i\sqrt{t}}\right)^2-\frac{(\beta t+x)^2}{4t}}+\mathrm{e}^{-\left(\sqrt{t}k+\frac{\beta t-x}{2i\sqrt{t}}\right)^2-\frac{(\beta t+x)^2}{4t}}\mathrm{d}k$$
$$+\frac{1}{2\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty}\mathrm{e}^{-\frac{(\beta t+x-y)^2}{4t}}\left[\mathrm{e}^{-\left(\sqrt{t}k+\frac{\beta t+x-y}{2i\sqrt{t}}\right)^2}\dot{u}_0(y)+\mathrm{e}^{-\left(\sqrt{t}k+\frac{\beta t-x-y}{2i\sqrt{t}}\right)^2-\frac{xy}{t}}(\dot{u}_0(y)+\beta u_0(y))\right]\mathrm{d}k\,\mathrm{d}y. \tag{73}$$

Using the following complex variant of the Gaussian integral:

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-p(k+c)^2} \mathrm{d}k = \sqrt{\frac{\pi}{p}}, \quad p, c \in \mathbb{C} \text{ with } \mathrm{Re}(p) > 0, \tag{74}$$

we find that Equation (73) is equal to

$$\frac{1}{2\sqrt{\pi t}} \left\{ 2\mathrm{e}^{-\frac{(\beta t+x)^2}{4t}} u_0(0) + \int_0^{+\infty} \mathrm{e}^{-\frac{(\beta t+x-y)^2}{4t}} \left[\dot{u}_0(y) + \mathrm{e}^{-\frac{xy}{t}} (\dot{u}_0(y) + \beta u_0(y)) \right] \mathrm{d}y \right\}. \tag{75}$$

The above expression vanishes as $t \to \infty$, by also taking into account that $u_0(y)$ and $\dot{u}_0(y)$ are assumed to have sufficient decay as $y \to \infty$, see the assumptions below (62). Thus, for $t \to \infty$ the x-derivative of u, given by Equation (71), reduces to

$$u_x(x,t) = U(x,t) + O(t^{-1/2}), \tag{76}$$

where

$$\begin{aligned} U(x,t) &= -\frac{1}{2\pi} \frac{\partial}{\partial x} \int_{\partial D^+} (2ik+\beta) \mathrm{e}^{ikx-(k^2-i\beta k)t} \left(\int_0^t \mathrm{e}^{(k^2-i\beta k)s} g_0(s) \mathrm{d}s \right) \mathrm{d}k \\ &= -\frac{1}{2\pi} \int_{\partial D^+} (-2k^2+i\beta k) \mathrm{e}^{ikx-(k^2-i\beta k)t} \left(\int_0^t \mathrm{e}^{(k^2-i\beta k)s} g_0(s) \mathrm{d}s \right) \mathrm{d}k, \quad \text{for } x > 0, \end{aligned} \tag{77}$$

where we can differentiate under the integral sign by Leibniz's rule, because we assume that g_0 is bounded and we consider strictly positive x. In what follows, to avoid convergence issues, we assume that $x > 0$ and only at the end of this section we will take the limit $x \to 0^+$ to arrive at the Neumann boundary value. Assuming that in the large time limit $g_0(t)$ tends to a periodic function sufficiently fast,

$$g_0(t) - \sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} \mathrm{e}^{in\omega t} \to 0, \quad t \to \infty, \tag{78}$$

where $\omega > 0$ and the prime denotes that $\alpha_0^{(0)} = 0$, we find

$$U(x,t) \sim -\frac{1}{2\pi} \int_{\partial D^+} (-2k^2+i\beta k) \mathrm{e}^{ikx} \left(\int_0^t \mathrm{e}^{-(k^2-i\beta k)(t-s)} \sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} \mathrm{e}^{in\omega s} \mathrm{d}s \right) \mathrm{d}k, \quad t \to \infty. \tag{79}$$

Let us rigorously justify the above formula. From Equation (78) we have that for any $\epsilon > 0$, there exists a $T(\epsilon) > 0$, such that

$$\left| g_0(t) - \sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} \mathrm{e}^{in\omega t} \right| \leq \epsilon, \quad \text{for all } t > T(\epsilon). \tag{80}$$

Let us now define the following:

(i) the uniform norm

$$\Delta g := \left\| g_0(t) - \sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} e^{in\omega t} \right\|_\infty ; \tag{81}$$

(ii) the integral

$$I_0 := \int_{\Gamma_D} |-2k^2 + i\beta k| e^{-k_I x} |dk|, \quad \text{where } k = k_R + ik_I \quad \text{with } k_R, k_I \in \mathbb{R}; \tag{82}$$

and

(iii) the contour in the complex k-plane, for $\beta > 0$,

$$\Gamma_D := \begin{cases} k_I = k_R + \frac{\beta}{2}, & k_R \geq 0, \\ k_I = -k_R + \frac{\beta}{2}, & k_R \leq 0. \end{cases} \tag{83}$$

For future use, we note that $\mathrm{Re}(k^2 - i\beta k) = \frac{\beta^2}{4}$ for $k \in \Gamma_D$.
For any $\epsilon' > 0$, we choose

$$\epsilon = \frac{\pi \beta^2 \epsilon'}{4 I_0} \tag{84}$$

and let $T \equiv T(\epsilon)$ be the time involved in Equation (80). Denoting the right-hand side of Equation (79) as $\tilde{U}(x,t)$, and deforming the original contour ∂D^+ to the more convenient contour Γ_D, we have

$$\begin{aligned}
|U(x,t) - \tilde{U}(x,t)| &= \frac{1}{2\pi} \left| \int_{\Gamma_D} (-2k^2 + i\beta k) e^{ikx} \left[\int_0^t e^{-(k^2 - i\beta k)(t-s)} \left(g_0(s) - \sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} e^{in\omega s} \right) ds \right] dk \right| \\
&\leq \frac{1}{2\pi} \left| \int_{\Gamma_D} (-2k^2 + i\beta k) e^{ikx} \left[\int_0^T e^{-(k^2 - i\beta k)(t-s)} \left(g_0(s) - \sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} e^{in\omega s} \right) ds \right] dk \right| \\
&\quad + \frac{1}{2\pi} \left| \int_{\Gamma_D} (-2k^2 + i\beta k) e^{ikx} \left[\int_T^t e^{-(k^2 - i\beta k)(t-s)} \left(g_0(s) - \sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} e^{in\omega s} \right) ds \right] dk \right| \\
&\leq \frac{\Delta g}{2\pi} \int_{\Gamma_D} |-2k^2 + i\beta k| e^{-k_I x} \left[\int_0^T e^{-\frac{\beta^2}{4}(t-s)} ds \right] |dk|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\epsilon}{2\pi}\int_{\Gamma_D}|-2k^2+i\beta k|e^{-k_I x}\left[\int_T^t e^{-\frac{\beta^2}{4}(t-s)}ds\right]|dk|\\
&=\frac{2\Delta g I_0}{\pi\beta^2}\left(e^{-\frac{\beta^2}{4}(t-T)}-e^{-\frac{\beta^2}{4}t}\right)+\frac{2\epsilon I_0}{\pi\beta^2}\left(1-e^{-\frac{\beta^2}{4}(t-T)}\right)\\
&\leq\frac{4\epsilon I_0}{\pi\beta^2}\\
&=\epsilon',\quad \text{for all } t>T'(\epsilon').
\end{aligned}\tag{85}
$$

Here we choose $T'(\epsilon')>T(\epsilon)$ large enough such that

$$
\Delta g\left(e^{-\frac{\beta^2}{4}(t-T)}-e^{-\frac{\beta^2}{4}t}\right)\leq\epsilon,\quad \text{for all } t>T'(\epsilon'). \tag{86}
$$

Thus, Equation (85) shows that

$$
U(x,t)\sim\tilde{U}(x,t),\quad t\to\infty, \tag{87}
$$

and therefore Equation (79) holds for $\beta>0$.

Note that Equation (79) can be justified in a similar way for $\beta=0$, i.e., the heat equation, but in this case the convenient contour to which we deform ∂D^+ is taken to be

$$
\Gamma_D' := \begin{cases} k_I=\sqrt{k_R^2-1}, & k_R>1,\\ k_I=0, & -1\leq k_R\leq 1,\\ k_I=\sqrt{k_R^2-1}, & k_R<-1,\end{cases} \tag{88}
$$

where $k=k_R+ik_I$ with $k_R,k_I\in\mathbb{R}$. For completeness, we note that for other equations (such as the linearized NLS equation) more caution may be needed for the justification of Equation (79).

From Equation (79) we find

$$
\begin{aligned}
U(x,t)&\sim-\frac{1}{2\pi}\int_{\partial D^+}(-2k^2+i\beta k)e^{ikx}\sum_{n=-\infty}^{\infty}{}'\,\alpha_n^{(0)}\left(\int_0^t e^{-(k^2-i\beta k)(t-s)+in\omega s}ds\right)dk\\
&=\frac{1}{2\pi}\int_{\partial D^+}(-2k^2+i\beta k)e^{ikx}\sum_{n=-\infty}^{\infty}{}'\,\alpha_n^{(0)}\frac{e^{-(k^2-i\beta k)t}-e^{in\omega t}}{k^2-i\beta k+in\omega}dk,\quad t\to\infty,
\end{aligned}\tag{89}
$$

where in the first line we were able to interchange the order of integration and summation because of the assumption in Equation (64).

The singularities of the expression in Equation (89) lie at $k = k_{1,2}(n)$, $n \in \mathbb{Z}^*$, as defined in Equation (40). We observe that the roots $k_1(n)$ with the plus sign of (40), i.e.,

$$k_1(n) = \begin{cases} \frac{1}{2}\left(-\sqrt{\frac{-\beta^2+\sqrt{\beta^4+16n^2\omega^2}}{2}} + i\beta + i\sqrt{\frac{\beta^2+\sqrt{\beta^4+16n^2\omega^2}}{2}}\right), & n \geq 0, \\ \frac{1}{2}\left(\sqrt{\frac{-\beta^2+\sqrt{\beta^4+16n^2\omega^2}}{2}} + i\beta + i\sqrt{\frac{\beta^2+\sqrt{\beta^4+16n^2\omega^2}}{2}}\right), & n < 0, \end{cases} \tag{90}$$

lie on ∂D^+ for all $n \neq 0$. However, they are removable singularities, because the following limit exists:

$$\lim_{k\to k_1(n)} \frac{\mathrm{e}^{-(k^2-i\beta k)t} - \mathrm{e}^{in\omega t}}{k^2 - i\beta k + in\omega} = \lim_{k\to k_1(n)} \frac{\frac{\mathrm{e}^{-(k^2-i\beta k)t} - \mathrm{e}^{in\omega t}}{k-k_1(n)}}{\frac{k^2-i\beta k+in\omega}{k-k_1(n)}} = -t\mathrm{e}^{in\omega t}. \tag{91}$$

Thus, we can deform the contour ∂D^+ to $\partial\tilde{D}^+$, which passes below the singularities $k = k_1(n)$, $n \in \mathbb{Z}^*$, see Figure 2(B).

We split the integral appearing in Equation (89) into two terms, which we call $I_1(x,t)$ and $I_2(x,t)$, as follows:

$$\begin{aligned} U(x,t) &\sim I_1(x,t) + I_2(x,t) \\ &= -\frac{1}{2\pi} \sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} \mathrm{e}^{in\omega t} \int_{\partial\tilde{D}^+} \frac{(-2k^2 + i\beta k)\mathrm{e}^{ikx}}{k^2 - i\beta k + in\omega} \mathrm{d}k \\ &\quad + \frac{1}{2\pi} \int_{\partial\tilde{D}^+} (-2k^2 + i\beta k)\mathrm{e}^{ikx} \sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} \frac{\mathrm{e}^{-(k^2-i\beta k)t}}{k^2 - i\beta k + in\omega} \mathrm{d}k, \quad t \to \infty. \end{aligned} \tag{92}$$

Note that we were able to interchange the order of integration and summation in the term $I_1(x,t)$ of Equation (92) because of the absolute convergence assumption on the Fourier coefficients $\alpha_n^{(0)}$, see Equation (64).

We first focus on $I_2(x,t)$ and show that it vanishes as $t \to \infty$. The term $I_2(x,t)$ involves an integral of the form

$$I_2(x,t) = \int_C f(k,x)\mathrm{e}^{t\Phi(k)}\mathrm{d}k, \tag{93}$$

where

$$f(k,x) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} \frac{(-2k^2 + i\beta k)\mathrm{e}^{ikx}}{k^2 - i\beta k + in\omega}, \quad \Phi(k) = -k^2 + i\beta k \quad \text{and} \quad C = \partial\tilde{D}^+. \tag{94}$$

Its large t-asymptotics (for each given x) can be computed via the method of steepest descent.[63] We deform the contour $\partial\tilde{D}^+$ to a new contour C' on which $\Phi(k)$ has a constant imaginary part and which passes through the saddle point $k_0 = i\frac{\beta}{2}$ of $\Phi(k)$. Thus, the contour C' is the horizontal line going through $k_0 = i\frac{\beta}{2}$. Observing that $k_0 = i\frac{\beta}{2}$ is a simple saddle point of Φ, meaning that $\Phi'(k_0) = 0$ and $\Phi''(k_0) \neq 0$, we find

$$\Phi(k) - \Phi\left(i\frac{\beta}{2}\right) \sim \frac{\left(k - i\frac{\beta}{2}\right)^2}{2!}\Phi''\left(i\frac{\beta}{2}\right), \quad \text{as } k \to i\frac{\beta}{2}. \tag{95}$$

It can easily be checked that $f(i\frac{\beta}{2}, x) = 0$ and $\partial_k f(i\frac{\beta}{2}, x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} \frac{-i\beta e^{-\beta x/2}}{\beta^2/4 + in\omega}$. Without loss of generality, the latter expression can be assumed (for $\beta > 0$) to converge to a nonzero value. Even if the $\alpha_n^{(0)}$'s would happen to be such that $\partial_k f(i\frac{\beta}{2}, x)$ converges to zero, we could proceed to compute higher order k-derivatives of $f(k, x)$ at $k_0 = i\frac{\beta}{2}$ until one of them would be nonzero; in this case we would get an even faster rate of decay in Equation (97) as $t \to \infty$. Hence, assuming that $\partial_k f(i\frac{\beta}{2}, x) \neq 0$, we have

$$f(k, x) \sim \left(k - i\frac{\beta}{2}\right)\partial_k f\left(i\frac{\beta}{2}, x\right), \quad \text{as } k \to i\frac{\beta}{2}. \tag{96}$$

Using the formula given by eq. (6.4.9) of Ref. 63, we find

$$I_2(x, t) \sim \frac{f_0(x)(m!)^{b/m} e^{ib\theta}}{m} \frac{e^{t\Phi(k_0)}\Gamma(b/m)}{(t|\Phi^{(m)}(k_0)|)^{b/m}}, \quad \text{as } t \to \infty, \tag{97}$$

where in our case $m = 2$, $b = 2$, $k_0 = i\frac{\beta}{2}$, and $f_0(x) = \partial_k f(i\frac{\beta}{2}, x)$. Substituting these values in the above equation, we find for $\beta > 0$:

$$I_2(x, t) \sim \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} \frac{-i\beta e^{-\frac{\beta x}{2}} e^{2i\theta}}{\frac{\beta^2}{4} + in\omega} \frac{e^{-\frac{\beta^2}{4}t}}{2t} \to 0, \quad t \to \infty. \tag{98}$$

Note that in the above analysis we have used the absolute convergence assumption on the Fourier coefficients $\alpha_n^{(0)}$, Equation (64), to assure convergence and to find the derivative of the infinite sum. Also, note that for the special case $\beta = 0$, which corresponds to the heat equation, we find $m = 2$, $b = 3$, $k_0 = 0$, and $f_0(x) = \frac{1}{2}\partial_k^2 f(0, x)$. Substituting these values in Equation (97), and noting that $\Phi(0) = 0$, we again find that $I_2(x, t)$ vanishes as $t \to \infty$.

So, from (92) and (98) we have

$$U(x, t) \sim -\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}{}' \alpha_n^{(0)} e^{in\omega t} \int_{\partial\tilde{D}^+} \frac{(-2k^2 + i\beta k)e^{ikx}}{k^2 - i\beta k + in\omega} dk, \quad \text{as } t \to \infty. \tag{99}$$

Hence, recalling Equation (76), we observe that $u_x(x, t)$ is asymptotically t-periodic. In an almost identical way, one can find an expression for the large t behavior of Equation (70) and show that $u(x, t)$ itself also becomes t-periodic for $t \to \infty$.

Let us consider the contour $C'' = \partial\tilde{D}_R^+ \cup C_R$, where $\partial\tilde{D}_R^+$ is the part of $\partial\tilde{D}^+$, which is bounded by a circle with centre at $k = 0$ and radius R (where $R \to \infty$ to include the singularities), and C_R

is the circular arc with radius R connecting the right and left branches of $\partial\tilde{D}_R^+$. We observe that

$$\int_{C''}\frac{(-2k^2+i\beta k)\mathrm{e}^{ikx}}{k^2-i\beta k+in\omega}\mathrm{d}k=\int_{\partial\tilde{D}_R^+}\frac{(-2k^2+i\beta k)\mathrm{e}^{ikx}}{k^2-i\beta k+in\omega}\mathrm{d}k+\int_{C_R}\frac{(-2k^2+i\beta k)\mathrm{e}^{ikx}}{k^2-i\beta k+in\omega}\mathrm{d}k. \tag{100}$$

By the Residue Theorem, we find that the above integral (over the closed contour C'') equals

$$\begin{aligned}2\pi i\operatorname*{Res}_{k=k_1(n)}\frac{(-2k^2+i\beta k)\mathrm{e}^{ikx}}{k^2-i\beta k+in\omega}&=2\pi i\lim_{k\to k_1(n)}\frac{(-2k^2+i\beta k)\mathrm{e}^{ikx}}{\frac{k^2-i\beta k+in\omega}{k-k_1(n)}}\\&=2\pi i\frac{\left(-2k_1^2(n)+i\beta k_1(n)\right)\mathrm{e}^{ik_1(n)x}}{2k_1(n)-i\beta}\\&=-2\pi ik_1(n)\mathrm{e}^{ik_1(n)x}.\end{aligned} \tag{101}$$

We observe that the second integral of the right-hand side of Equation (100) tends to zero as $R\to\infty$, because $x>0$. Indeed, writing $k=k_R+ik_I$ with $k_R,k_I\in\mathbb{R}$, we find that

$$\begin{aligned}\left|\int_{C_R}\frac{(-2k^2+i\beta k)\mathrm{e}^{ikx}}{k^2-i\beta k+in\omega}\mathrm{d}k\right|&\le\pi R\sup_{k\in C_R}\frac{|k|\,|-2k+i\beta|\mathrm{e}^{-k_Ix}}{|k^2-i\beta k+in\omega|}\\&\le\pi R^2\frac{(2R+\beta)\mathrm{e}^{-\frac{\beta+\sqrt{\beta^2+8R^2}}{4}x}}{(R-|k_1(n)|)(R-|k_2(n)|)},\end{aligned} \tag{102}$$

tends to zero as $R\to\infty$. The expression $(\beta+\sqrt{\beta^2+8R^2})/4$ is the imaginary part of the points at which $\partial\tilde{D}_R^+$ and C_R intersect and, hence, it is the value of k_I, which maximizes e^{-k_Ix} for $k\in C_R$. Hence, by taking $R\to\infty$ in Equation (100) and using Equations (99), (101), and (102), we find

$$U(x,t)\sim\sum_{n=-\infty}^{\infty}{}'\,\alpha_n^{(0)}\mathrm{e}^{in\omega t}ik_1(n)\mathrm{e}^{ik_1(n)x},\quad t\to\infty, \tag{103}$$

where the $k_1(n)$'s are given in Equation (90). Because our goal is to find the Neumann boundary value, we take the limit $x\to0^+$ in Equation (103) and, taking into account Equation (76), we arrive at the asymptotically periodic form of the Neumann boundary function (65) with Fourier coefficients

$$\alpha_n^{(1)}=ik_1(n)\alpha_n^{(0)},\quad n\in\mathbb{Z}, \tag{104}$$

where we were able to swap the limit $x\to0^+$ with the summation using the Weierstrass criterion for uniform convergence, under the assumption that

$$\sum_{n=-\infty}^{\infty}{}'\left|k_1(n)\alpha_n^{(0)}\right|<\infty, \tag{105}$$

which is equivalent to the assumption made in Equation (64):

$$\sum_{n=-\infty}^{\infty}{}' \sqrt{|n|}\left|\alpha_n^{(0)}\right| < \infty. \tag{106}$$

For the derivation of the desired form of the Dirichlet-to-Neumann correspondence (66), we use the Vieta formula $k_1(n) + k_2(n) = i\beta$, because $k_1(n)$ and $k_2(n)$ are the two roots of $k^2 - i\beta k + in\omega = 0$, and thus Equation (104) yields

$$\alpha_n^{(1)} = -(ik_2(n) + \beta)\alpha_n^{(0)}, \quad n \in \mathbb{Z}. \tag{107}$$

Note that the expression for $\alpha_n^{(1)}$ obtained here is the same as the one in Equation (42), as it should be.

5 | THE REMARKABLE RESULTS OF BOUTET DE MONVEL, KOTLYAROV, AND SHEPELSKY REVISITED

Let us consider the following initial-boundary value problem for the NLS equation on the half-line:

$$iu_t + u_{xx} - 2\lambda|u|^2u = 0, \quad \lambda = \pm 1, \quad x > 0, \quad t > 0, \tag{108a}$$

$$u(x,0) = u_0(x), \quad x \geq 0, \tag{108b}$$

$$u(0,t) = g_0(t), \quad t \geq 0, \tag{108c}$$

where the Dirichlet boundary condition $g_0(t)$ approaches the periodic function $\alpha e^{i\omega t}$ $(\alpha \in \mathbb{R})$ sufficiently fast as $t \to \infty$, and the initial condition $u_0(x)$ decays sufficiently fast to zero as $x \to \infty$. The solution $u(x,t)$ of the above problem is supposed to be sufficiently smooth (up to and including the boundary) and both $u(x,t)$ and its x-derivatives are taken to decay rapidly enough as $x \to \infty$ for each $t \geq 0$.

Here we will show how the expressions (3) for γ obtained in Ref. 55, can be derived in a direct manner via the Q-equation approach. Namely, assuming that the Neumann boundary function $u_x(0,t) = g_1(t)$ has the form $\gamma e^{i\omega t}$ for large t, we will reproduce Equations (3) with only a slight incompleteness in the range of ω values.

We begin by deriving the nonlinear version of Equation (18) for the case of the NLS equation. The Lax pair of the NLS is given by (see, e.g., Ref. 60)

$$\psi_x + ik\sigma_3\psi = U\psi, \tag{109a}$$

$$\psi_t + 2ik^2\sigma_3\psi = V\psi, \tag{109b}$$

where $k \in \mathbb{C}$ is the spectral parameter, $\psi(k,x,t)$ is a 2×2-matrix-valued function, and the matrices U, V, and σ_3 are defined by

$$U = \begin{pmatrix} 0 & u \\ \lambda\overline{u} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -i\lambda|u|^2 & 2ku + iu_x \\ 2\lambda k\overline{u} - i\lambda\overline{u}_x & i\lambda|u|^2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Evaluating the 12 and 22 components of Equation (109b) at $x = 0$, we find that $\psi_{12}(k,0,t)$ and $\psi_{22}(k,0,t)$ satisfy the following equations:

$$(\psi_{12})_t + 2ik^2\psi_{12} = -i\lambda|g_0|^2\psi_{12} + (2kg_0 + ig_1)\psi_{22}, \tag{110a}$$

$$(\psi_{22})_t - 2ik^2\psi_{22} = \lambda(2k\overline{g}_0 - i\overline{g}_1)\psi_{12} + i\lambda|g_0|^2\psi_{22}. \tag{110b}$$

Introducing the functions

$$\Phi_1(k,t) = \mathrm{e}^{-2ik^2t}\psi_{12}(k,0,t), \tag{111a}$$

$$\Phi_2(k,t) = \mathrm{e}^{-2ik^2t}\psi_{22}(k,0,t), \tag{111b}$$

the second column of the t-part of the Lax pair evaluated at $x = 0$ becomes

$$(\Phi_1)_t + 4ik^2\Phi_1 = -i\lambda|g_0|^2\Phi_1 + (2kg_0 + ig_1)\Phi_2, \tag{112a}$$

$$(\Phi_2)_t = \lambda(2k\overline{g}_0 - i\overline{g}_1)\Phi_1 + i\lambda|g_0|^2\Phi_2. \tag{112b}$$

We define $Q(k,t)$ by

$$Q(k,t) := \frac{\Phi_1(k,t)}{\Phi_2(k,t)}, \tag{113}$$

where it is assumed that we stay away from the possible poles. Equations (112) can now be rewritten as

$$(Q\Phi_2)_t + 4ik^2Q\Phi_2 = -i\lambda|g_0|^2Q\Phi_2 + (2kg_0 + ig_1)\Phi_2, \tag{114a}$$

$$(\Phi_2)_t = \lambda(2k\overline{g}_0 - i\overline{g}_1)Q\Phi_2 + i\lambda|g_0|^2\Phi_2. \tag{114b}$$

Multiplying Equation (114b) by Q and substituting $Q(\Phi_2)_t$ from Equation (114a), we find [for $\Phi_2(k,t) \neq 0$] the Q-equation

$$Q_t(k,t) + \lambda(2k\overline{g}_0(t) - i\overline{g}_1(t))Q^2(k,t) + (2i\lambda|g_0(t)|^2 + 4ik^2)Q(k,t) - (2kg_0(t) + ig_1(t)) = 0. \tag{115}$$

In the linear limit, i.e., when $Q(k,t) = \epsilon Q_1(-2k,t) + O(\epsilon^2)$, $g_0(t) = \epsilon g_{01}(t) + O(\epsilon^2)$, and $g_1(t) = \epsilon g_{11}(t) + O(\epsilon^2)$ with $\epsilon \to 0$, we observe that (after substituting k by $-k/2$) the $O(\epsilon)$ terms of Equation (115) give us Equation (18) for the particular case of the linearized NLS, with Q replaced by Q_1, g_0 by g_{01}, and g_1 by g_{11}.

We let $\lambda = -1$, i.e., we treat the *focusing* NLS, and we consider the physically significant case of asymptotically periodic single-exponential boundary functions. For large t, we assume that Q, g_0, and g_1 are of the following form:

$$Q(k,t) \sim q(k)\mathrm{e}^{i\omega t}, \quad g_0(t) \sim \alpha \mathrm{e}^{i\omega t}, \quad g_1(t) \sim \gamma \mathrm{e}^{i\omega t}, \quad \omega \in \mathbb{R}, \quad \alpha > 0, \quad \gamma \in \mathbb{C}, \quad t \to \infty. \tag{116}$$

After substituting these asymptotic expressions, Equations (114) in the large time limit become

$$q(\Phi_2)_t + i\omega q\Phi_2 + 4ik^2 q\Phi_2 = i\alpha^2 q\Phi_2 + (2\alpha k + i\gamma)\Phi_2, \tag{117a}$$

$$(\Phi_2)_t = -(2\alpha k - i\overline{\gamma})q\Phi_2 - i\alpha^2\Phi_2. \tag{117b}$$

We now distinguish the two cases below (assuming that we keep away from the possible zeros of $\Phi_2(k,t)$):

(a) If $(\Phi_2(k,t))_t \to 0$ for large t, then the above equations give rise to the following relation:

$$q(k) = \frac{-2i\alpha k + \gamma}{4k^2 - \alpha^2 + \omega} = \frac{\alpha^2}{2i\alpha k + \overline{\gamma}}. \tag{118}$$

The second equality implies

$$|\gamma|^2 + 2i\alpha k(\gamma - \overline{\gamma}) = \alpha^2(\omega - \alpha^2). \tag{119}$$

Thus, in this case γ is real (provided that $(\Phi_2(k,t))_t \to 0$, as $t \to \infty$, for at least one nonreal value of k) and hence

$$\gamma = \pm\alpha\sqrt{\omega - \alpha^2}, \quad \omega \geq \alpha^2, \tag{120}$$

in accordance with Equation (3a).

(b) More generally, also for nonvanishing $(\Phi_2(k,t))_t$, we arrive at Equation (115), which after the substitution of the asymptotic expressions from Equation (116) becomes, for large t,

$$(2\alpha k - i\overline{\gamma})q^2(k) - i(4k^2 - 2\alpha^2 + \omega)q(k) + (2\alpha k + i\gamma) = 0. \tag{121}$$

If γ is purely imaginary of the form $\gamma = i\Gamma$, where we will take $\Gamma \geq 0$, and using the identity $\gamma = -\overline{\gamma}$, we find that the Q-equation (121) simplifies to

$$q^2(k) - i\frac{4k^2 - 2\alpha^2 + \omega}{2\alpha k - \Gamma}q(k) + 1 = 0. \tag{122}$$

At this point, it is important to observe that $q(k)$ is bounded at $k = \frac{\Gamma}{2\alpha}$, where $\alpha > 0$ and $\Gamma \geq 0$. This can be seen by substituting the asymptotic forms of $g_0(t)$ and $g_1(t)$ from Equation (116) into the Q-equation (115), because one then finds $Q(\frac{\Gamma}{2\alpha}, t) = c\,\exp[i(2\alpha^2 - \Gamma^2/\alpha^2)t]$, $c \in \mathbb{R}$, when $t \to \infty$. So we have to require that $4k^2 - 2\alpha^2 + \omega = 0$ when the denominator is zero, i.e., when $k = \frac{\Gamma}{2\alpha}$. This can also be inferred from Equation (121), where for $\gamma = i\Gamma$ and $k = \frac{\Gamma}{2\alpha}$ (with Γ real and nonnegative) only the second term survives, which then necessarily must be equal to zero. Under the assumption that $q(\frac{\Gamma}{2\alpha})$ is nonzero, this implies that $4k^2 - 2\alpha^2 + \omega = 0$ when $k = \frac{\Gamma}{2\alpha}$. Hence $\Gamma = \alpha\sqrt{2\alpha^2 - \omega}$, and thus we find

$$\gamma = i\alpha\sqrt{2\alpha^2 - \omega}, \tag{123}$$

in agreement with Equation (3b). It may be noted that the inequality $\omega \leq -6\alpha^2$ of Boutet de Monvel, Kotlyarov, and Shepelsky is not reproduced, yet the present analysis provides an elementary and straightforward approach for determining the large t behavior of the Dirichlet-to-Neumann map in the case of periodic single-exponential boundary functions.

Boutet de Monvel et al.[55] showed that the coefficient γ of the asymptotically periodic Neumann boundary value is as presented in Equation (3), i.e., γ is either real or purely imaginary if ω is outside the range $(-6\alpha^2, \alpha^2)$. Inside the range $\omega \in (-6\alpha^2, \alpha^2)$ the asymptotic form of the Neumann boundary value is *not* of the simple form $\gamma e^{i\omega t}$,[64] so then Equation (116) would not be a valid substitution to make. Hence, the above analysis concerning real and purely imaginary γ (but not more general complex values of γ) covers all cases of interest for the focusing NLS with boundary data that asymptote to periodic single exponentials.

6 | CONCLUSIONS

The goal of this work is to elucidate the effectiveness of the use of the Q-formulation. For integrable nonlinear evolution equations, the Q-equation was introduced in Ref. 60 in connection with the NLS. In the present paper we have established the following: (i) For linear evolution equations, the analysis of the Q-equation yields the large t asymptotics form of the generalized Dirichlet-to-Neumann map for t-periodic boundary conditions, in a very simple, algebraic way. (ii) For the NLS, it reproduces the remarkable results of Ref. 55, again in a simple, algebraic manner. We expect our method to give useful results also for other integrable nonlinear PDEs, which are amenable to the Fokas method.

For linear evolution equations, the Q-equation is the t-part of the Lax pair evaluated at $x = 0$. It is interesting to note that when the unified transform was first introduced, it was implemented via the Lax pair formulation. Later, it was realized that (in the linear case) it could be derived in a straightforward manner, avoiding the Lax pair connection. Of course, the Lax pair approach remains indispensable for the application of the unified transform to nonlinear integrable PDEs. The results presented in this paper clearly show that, even for linear evolution equations, the connection with the Lax pair is very useful. Indeed, in terms of simplicity, the Q-approach (which is based entirely on the Lax pair formulation) provides the most unexpected results, so far, of the Fokas method. The effectiveness of this new approach becomes apparent by comparing it with the lengthy derivation in Section 4 and earlier implementations of the unified transform in Refs. 55, 60.

Taking into consideration that the classical investigations of linear PDEs did not take into account that linear PDEs admit a Lax pair formulation,[65] it is not surprising that the Q-approach was missed in these previous investigations.

The Q-approach is based on the assumption that the unknown boundary values become periodic for large t. Although it is rather complicated to determine their precise asymptotic form, it is often much simpler to show that they indeed become t-periodic as t tends to infinity. Using the general formulation of the Fokas method and the standard asymptotic technique of steepest descent, it can be shown in many instances that, if the given boundary conditions are t-periodic, the unknown boundary values also become t-periodic as t tends to infinity. Then, the Q-approach provides a most effective way for computing explicitly the Fourier series coefficients of the unknown periodic functions in terms of the Fourier series coefficients of the given boundary conditions.

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Data sharing not applicable to this article as no data sets were generated or analyzed during the current study. This article describes entirely mathematical/theoretical research and does not rely on any experimental data.

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