The Kijowski–Liu–Yau quasi-local mass of the Kerr black hole horizon[*]

Maciej Dunajski[1],[**] and Paul Tod[2]

[1] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom
[2] The Mathematical Institute, Oxford University, Woodstock Road, Oxford OX2 6GG, United Kingdom

E-mail: m.dunajski@damtp.cam.ac.uk and tod@maths.ox.ac.uk

CrossMark

Abstract

We use an isometric embedding of the cross-over surface of the outer horizon of a rapidly rotating Kerr black hole in a hyperbolic space to compute the quasi-local mass of the horizon for any allowed value of the spin parameter $j = J/m^2$. The mass is monotonically decreasing from twice the ADM mass at $j = 0$ to $1.765\,69m$ at $j = \sqrt{3}/2$. It then monotonically increases to a maximum around $j = 0.999\,07$, and finally decreases to $2.019\,66m$ for $j = 1$ which corresponds to the extreme Kerr black hole.

Keywords: quasi local mass, hyperbolic embeddings, Kerr horizon

(Some figures may appear in colour only in the online journal)

1. Introduction

There is no local concept of energy density in general relativity, which makes defining the mass of a local system problematic. On the other hand the ADM and the Trautman–Bondi masses of asymptotically flat space times can be defined at, respectively, spacelike and null infinities, and their positivity can be established given appropriate energy and regularity conditions. A

---

[*]Dedicated to Gary Gibbons on the occasion of his 75th birthday.
[**]Author to whom any correspondence should be addressed.

quasi-local energy aims be a compromise, as it associates a number to any closed space-like two-surface Σ in space time. There are several approaches to the subject which are reviewed in [10]. In particular the prescriptions of Kijowski [3] and Liu–Yau [4], which are related to a class of definitions proposed by Brown and York [1], are applicable to surfaces with non-negative Gaussian curvature. This condition guarantees, by Nirenberg's solution [7] of the Weyl embedding problem, the existence of a unique (up to an isometry of $\mathbb{R}^3$) global isometric embedding of Σ in $\mathbb{R}^3$. The resulting mass is then defined, up to a constant factor, to be an integral of the difference between the mean curvature of this flat embedding, and the norm of the mean curvature vector of Σ regarded as a surface in the space-time. This prescription is therefore not applicable to a space-like section of the horizon of rapidly-rotating Kerr black hole. This is because if the dimensionless spin parameter $j \equiv J/m^{-2}$ (where J is the angular momentum, and m is the ADM mass) is between $\sqrt{3}/2$ and 1, then the Gaussian curvature is negative near north and south poles. It is this problem which we resolve in this note. It will be done by embedding the Kerr horizons with $j > \sqrt{3}/2$ in a hyperbolic space and choosing the hyperbolic radius so that the embedding continuously matches with that in $\mathbb{R}^3$ for $j = \sqrt{3}/2$.

In section 2 we shall compute the mean curvatures of both the hyperbolic and flat embeddings, and in section 3 we shall construct the corresponding quasi-local mass as a function of $j \in [0, 1]$. This function is monotonically decreasing from twice the ADM mass m at $j = 0$ to $1.765\,69m$ at $j = \sqrt{3}/2$. The energy then monotonically increases to the value of $2.022\,23m$ reached around $j = 0.999\,07$, and decreases down to $2.019\,66m$ which is the quasi local mass of the extreme Kerr horizon where $j = 1$.

2. Isometric embeddings of Kerr black hole horizons

2.1. Hyperbolic embedding of rapidly rotating horizons

Let (Σ, g) be a two-dimensional Riemannian manifold with the Gaussian curvature bounded below by a negative constant $-L^{-2}$. The Pogorelov theorem [9] states that there is a global isometric embedding of Σ into hyperbolic three-space $\mathbb{H}^3$ with Ricci scalar less than or equal to $-6L^{-2}$. We shall consider the upper half-space model of $\mathbb{H}^3$, with the hyperbolic metric

$$G_L = \frac{L^2}{z^2} \left(\mathrm{d}z^2 + \mathrm{d}r^2 + r^2 \, \mathrm{d}\phi^2 \right), \quad z > 0, r > 0, \phi \in [0, 2\pi). \tag{2.1}$$

In [2] a global isometric embedding of (Σ, g) was explicitly constructed in the case where g admits a $U(1)$ isometric action, and such that this isometry preserves the embedding. This result was then used in [2] to construct an isometric embedding of the spatial sections of Kerr black hole horizons. We shall first reproduce this embedding, and then compute its extrinsic properties: the second fundamental form, and the mean curvature.

Consider an embedding $\iota : \Sigma \to \mathbb{H}^3$, where (Σ, g) is a surface of revolution with coordinates $x \in [-1, 1], \phi \in [0, 2\pi)$ and

$$g = \rho^2(B^{-1} \, \mathrm{d}x^2 + B \, \mathrm{d}\phi^2), \quad B = B(x), \quad \rho = \mathrm{const}. \tag{2.2}$$

If $z = Z(x), r = R(x)$, then $g = \iota^*(G_L)$ iff

$$Z(x) = \exp \left(\int \left(\frac{-\rho^2 BB' \pm \rho \sqrt{B(4\rho^2 B + 4L^2 - L^2(B')^2)}}{2B(\rho^2 B + L^2)} \right) \mathrm{d}x \right), \tag{2.3}$$

$$R(x) = \frac{\rho}{L} \sqrt{B(x)} Z(x).$$

The mean curvature H of this embedding with respect to the outward pointing unit normal vector field N to Σ in $\mathbb{H}^3$, where $N = \frac{Z}{L\sqrt{(R')^2+(Z')^2}} \left(R'\frac{\partial}{\partial Z} - Z'\frac{\partial}{\partial R}\right)$, is a half of the g-trace of the second fundamental form h. Using the definition $h(X, Y) = G_L(N, \nabla_X Y)$, where (X, Y) are the elements of $T\Sigma$, and ∇ is the Levi–Civita connection of G_L we find

$$H = \frac{-RZZ'R'' + RZR'Z'' + 2((R')^2 + (Z')^2)(RR' + ZZ'/2)}{2LR((R')^2 + (Z')^2)^{3/2}}. \tag{2.4}$$

We aim to match (2.2) with the general form of the Kerr horizon metric with the ADM mass m, and the angular momentum $0 \leqslant J \leqslant m^2$. To do it, consider the Kerr metric written in the Boyer–Lindquist coordinates (see, e.g. [12]), and restrict it to a surface of constant time on the outer event horizon, which gives

$$g = S\,\mathrm{d}\theta^2 + \left(r_+{}^2 + \frac{J^2}{m^2} + \frac{2J^2 r_+}{mS}\sin^2\theta\right)\sin^2\theta\,\mathrm{d}\phi^2, \tag{2.5}$$

where

$$r_+ = m + \sqrt{m^2 - J^2/m^2}, \quad S = r_+{}^2 + (J^2/m^2)\cos^2\theta.$$

The metric (2.2) then arises from (2.5) by setting $x = \cos\theta$, adopting (x, ϕ) as coordinates, and taking

$$B = \frac{(1 + c^2)(1 - x^2)}{1 + c^2 x^2}, \quad \rho^2 = 2m(m + \sqrt{m^2 - J^2/m^2}), \quad c = \frac{2J}{\rho^2} \in [0, 1], \quad x \in [-1, 1].$$

The constant ρ is twice the irreducible mass of the Kerr black hole (so that it is also proportional to the square root of the area of the outer horizon), and we choose a plus sign in (2.3).

The Gaussian curvature K of g is bounded from below

$$K = \frac{(c^2 + 1)^2(1 - 3c^2 x^2)}{\rho^2(1 + c^2 x^2)^2} \geqslant \frac{(1 - 3c^2)}{\rho^2(c^2 + 1)} = K_{\min}. \tag{2.6}$$

For $c \in (\sqrt{3}^{-1}, 1]$ we take $L = \rho\sqrt{\frac{1+c^2}{3c^2-1}}$, which is the largest hyperbolic radius for which the embedding is global. This, after some calculations, gives the mean curvature (2.4) as

$$H = \frac{c\sqrt{1 - x^2}(x^4(2c^4 - 5c^6) + x^2(-4c^6 - 12c^4 + 6c^2) + c^4 + 3c^2 + 9)}{\rho\sqrt{(1 + c^2)(1 + c^2 x^2)^3}\sqrt{x^4(c^4 - 2c^6) + x^2(-c^6 - 4c^4 + 3c^2) + 3}}. \tag{2.7}$$

2.2. Flat embedding of slowly rotating horizons

The hyperbolic embedding (2.3) with the critical choice of the hyperbolic radius is well defined as long as $\sqrt{3}^{-1} < c \leqslant 1$, which corresponds to the spin parameter $j \in [\sqrt{3}/2, 1]$. If $c \leqslant \sqrt{3}^{-1}$ then the Kerr horizon can be globally isometrically embedded in $\mathbb{R}^3$. If the flat metric on $\mathbb{R}^3$ is

$$G = \mathrm{d}\zeta^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2,$$
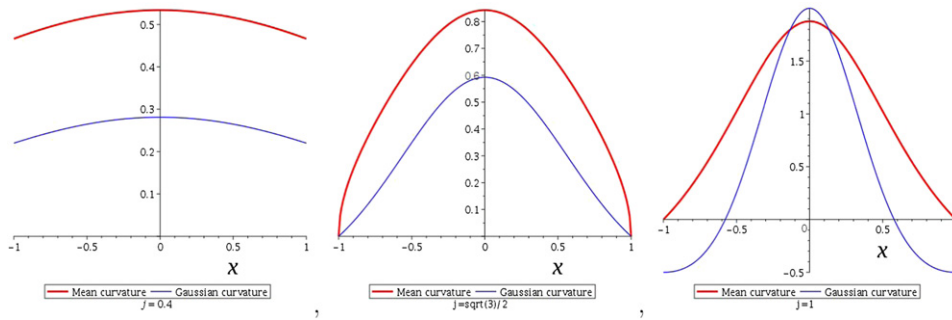
Figure 1. Mean (in red) and Gaussian (in blue) curvatures of the Kerr horizons for various values of the spin parameter j.

then the embedding is given by

$$\zeta = \pm \frac{\rho}{2} \int \frac{\sqrt{B(4 - (B')^2)}}{B} \, \mathrm{d}x, \quad r = \rho \sqrt{B}. \tag{2.8}$$

The formulae (2.8) can also be obtained as a limiting case of the hyperbolic embedding when the hyperbolic radius tends to infinity. To see this, set $z = L\,\mathrm{e}^{\zeta/L}, L > 0$ and find

$$G = \lim_{L \to \infty} G_L, \quad \text{where } G_L = \mathrm{d}\zeta^2 + \mathrm{e}^{-2\zeta/L}(\mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2).$$

For (2.8) to be well defined we need $B(4 - (B')^2) \geqslant 0$ which gives

$$c^8(x^6 + x^4 + x^2) + 4c^6(x^4 + x^2) + 6c^4x^2 - 1 \leqslant 0$$

which should hold for all $x \in [0, 1]$. Evaluating the expression above at $x^2 = 1$ gives a polynomial with two real roots $c = \pm\sqrt{3}/3$, and the inequality holds when $0 \leqslant c \leqslant 1/\sqrt{3}$. This is the same condition which guarantees $B'' \leqslant 0$, which holds iff the Gaussian curvature is non-negative. Thus, although $K \geqslant 0$ is necessary and sufficient for (2.8) to be a global isometric embedding, there can be regions on the Kerr horizon where the Gaussian curvature is negative, and yet the embedding still exists (although it does not extend to the whole horizon).

Repeating the steps leading to (2.7) we find that the mean curvature of the embedding (2.8) is given by

$$H_0 = -\frac{c^8x^6 + (c^8 + 4c^6)x^4 + (4c^8 + 13c^6 + 15c^4 + 3c^2)x^2 - (c^6 + 3c^4 + 2c^2 + 2)}{2\rho(c^2x^2 + 1)\sqrt{(1 + c^2)(1 + c^2x^2)(1 - (x^3 + x^2 + x)c^4 - 2c^2 x)(1 + (x^3 + x^2 + x)c^4 + 2c^2x)}}.$$

To sum up, the Kerr horizon with any $c \in [0, 1]$ can be globally and isometrically embedded as a hypersurface in the space of constant curvature (figure 1). If $c \leqslant 1/\sqrt{3}$ then the embedding is in $\mathbb{R}^3$ and when $c > 1/\sqrt{3}$ then it is in $\mathbb{H}^3$. The mean curvatures of the flat and hyperbolic embeddings are equal at $c = 1/\sqrt{3}$, but their derivatives w.r.t. c are different. This will play a role in analysing the behaviour of the quasi-local energy as a function of c. We shall do this in the next section.

3. Modified Kijowski–Liu–Yau mass

The Kijowski–Liu–Yau [3, 4] definition of quasi-local mass of a space-like closed two-surface Σ in a space-time M is

$$E_{\text{KLY}} = \frac{1}{4\pi} \int_{\Sigma} (H - |\hat{H}|) \text{vol}_{\Sigma}, \tag{3.1}$$

where H is the mean curvature of the embedding of Σ in $\mathbb{R}^3$, and $|\hat{H}|$ is the space-time norm of the mean curvature vector $\hat{H}$ of the surface Σ embedded in M. This is well defined only if $\hat{H}$ is space-like or zero, and if the Gaussian curvature of Σ is non-negative, as then a global embedding of Σ in $\mathbb{R}^3$ exists.

In what follows, we modify the KLY definition replacing the embedding in $\mathbb{R}^3$ by the embedding in $\mathbb{H}^3$, where the hyperbolic radius L of $\mathbb{H}^3$ is maximal for which the embedding exists, i.e. such that $K_{\min} = -L^2$, where $K_{\min}$ given by (2.6) is the lower bound for the Gaussian curvature of Σ. If Σ is taken to be the surface of the horizon, then $\hat{H} = 0$, and the mass is proportional to the integral of the mean curvature. This modification of the KLY mass also leads to a non-negative expression (theorem 3.1 in [13]).

Set $j = J/m^2$. We first restrict the range of j to $(\sqrt{3}/2, 1]$, which corresponds to the hyperbolic radius between ∞ and $\sqrt{2}m$, the latter case corresponding to the extremal Kerr metric, and the former case corresponding to $K_{\min} = 0$. For the mean curvature (2.7) we compute the modified mass to be

$$E(m, j) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} H(x)\rho^2 \, d\phi \, dx$$

$$= m \int_{-1}^{1} \frac{\sqrt{1-x^2}(x^4(2c^5 - 5c^7) + x^2(-4c^7 - 12c^5 + 6c^3) + c^5 + 3c^3 + 9c)}{2(1+c^2)(1+c^2x^2)^{3/2}\sqrt{x^4(c^4 - 2c^6) + x^2(-c^6 - 4c^4 + 3c^2) + 3}} \, dx$$

$$\text{where } c = \frac{1 - \sqrt{1 - J^2}}{j}, \quad \rho = \frac{2m}{\sqrt{1+c^2}}. \tag{3.2}$$

The limiting values of the mass are

$$E(m, \sqrt{3}/2) = m \int_{-1}^{1} \frac{3\sqrt{1-x^2}(x^4 + 14x^2 + 273)}{8(x^2+3)^{3/2}\sqrt{3x^4 + 42x^2 + 243}} \, dx \approx 1.765\,69m, \tag{3.3}$$

$$E(m, 1) = m \int_{-1}^{1} \frac{13 - 10x^2 - 3x^4}{\sqrt{(x^2+3)(x^2+1)^3}} \, dx \approx 2.019\,66m. \tag{3.4}$$

Using the flat embedding in the range $c \in [0, \sqrt{3}^{-1}]$ we find

$$E(m, j) = m \int_{-1}^{1} \frac{-c^8x^6 - (c^8 + 4c^6)x^4 + (4c^8 + 13c^6 + 15c^4 + 3c^2)x^2 + c^6 + 3c^4 + 3c^2 + 2}{2(c^2x^2 + 1)^{3/2}(c^2 + 1)^{3/2}\sqrt{(1 - c^4(x^3 + x^2 + x) - 2c^2x)(1 + c^4(x^3 - x^2 + x) + 2c^2x)}} \, dx. \tag{3.5}$$

The limiting value at $j = \sqrt{3}/2$ agrees with (3.3). The other limit is $E(m, 0) = 2m$.

The mean curvature as a function of the spin parameter $j = J/m^2$ is continuous but not smooth at $j = \sqrt{3}/2$ which separates the flat and the hyperbolic embeddings. For
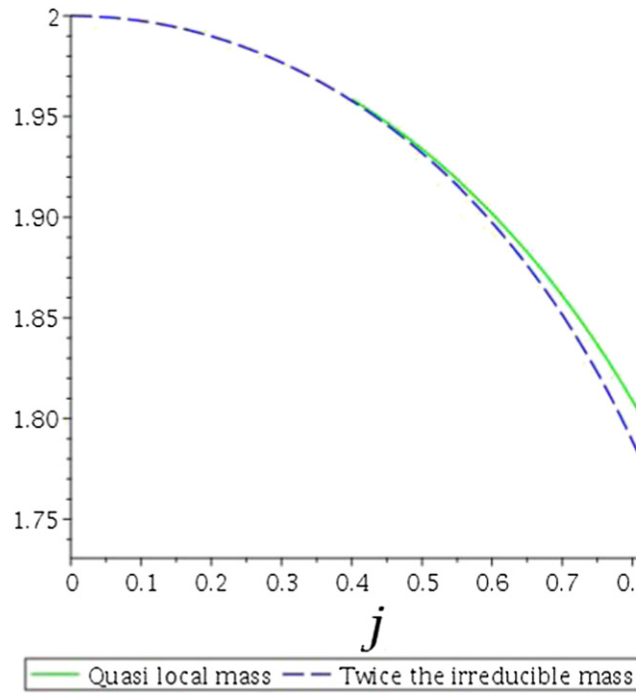
Figure 2. The quasi local mass is greater than twice the irreducible mass.

$0 \leqslant j \leqslant \sqrt{3}/2$ the quasi local mass is a decreasing function of j, and is very well approximated by the first four terms of the series

$$E(m, j) = m \left(2 - \frac{1}{4}j^2 - \frac{17}{320}j^4 - \frac{407}{17\,920}j^6 - \dots \right). \tag{3.6}$$

In the Schwarzschild case $j = 0$ the quasi-local energy is equal to twice the ADM mass in agreement with the results of Martinez [5], who (unlike us) additionally assumed that $j \ll 1$, and only derived the first two terms in the series (3.6). Our findings also disprove the conjecture of Martinez, that the quasi-local energy is equal to twice the irreducible mass. Expanding the latter quantity (which is equal to our ρ) we find

$$\rho = m \left(2 - \frac{1}{4}J^2 - \frac{25}{320}j^4 - \frac{735}{17\,920}j^6 - \dots \right) < E(m, j) \quad \text{if } j \in \left(0, \frac{\sqrt{3}}{2} \right].$$

Thus, in this range of j, the quasi-local mass is always greater than twice the irreducible mass (figure 2) which is in agreement with the Minkowski inequality

$$\frac{1}{4\pi} \int_\Sigma H \mathrm{vol}_\Sigma \geqslant \frac{1}{4\pi} \sqrt{4\pi \mathrm{area}(\Sigma)} = \rho.$$

As the spin parameter increases to $\sqrt{3}/2$, the energy decreases to $1.765\,69m$. For j above this value, the original Brown–York–Kijowski–Liu–Yau prescription breaks down as the global isometric embedding in $\mathbb{R}^3$ does not exist. This was noted by Martinez, who states in [5]
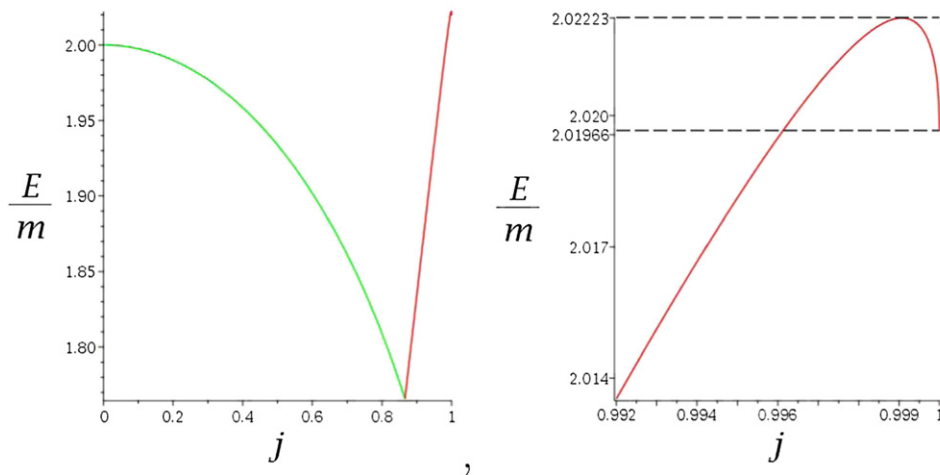
Figure 3. The quasi-local energy as a function of the spin parameter $j = J^2/m^1$.
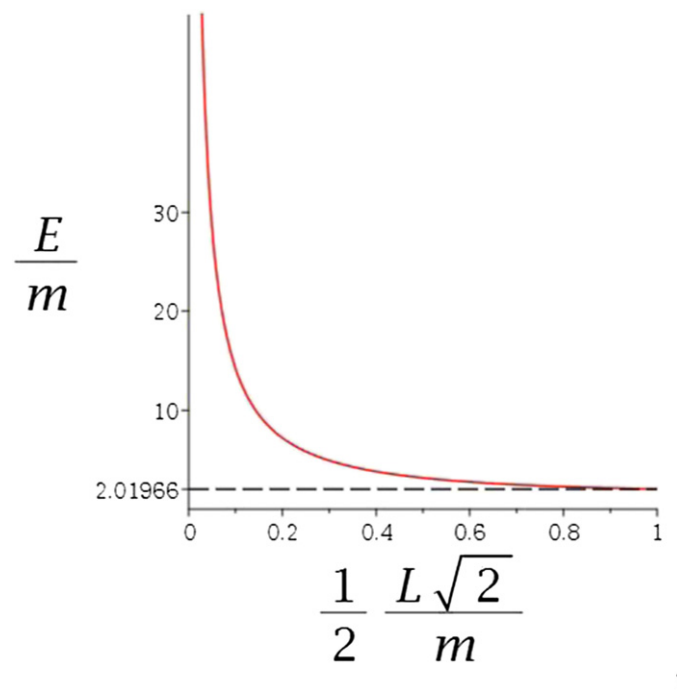
Figure 4. The quasi-local energy of the extreme Kerr horizon computed using the hyperbolic embedding as a function of the hyperbolic radius.

that his calculations are applicable only to slowly spinning Kerr black holes. In the range $j \in (\sqrt{3}/2, 1]$ we use the hyperbolic embedding. This has a free parameter—the hyperbolic

radius L—constrained by the inequality $0 < L \leqslant \frac{2m}{\sqrt{3c^2-1}}$, and, for each c we choose the maximal value of L which makes the embedding global. We then numerically integrate (3.2) to compute the energy as a function of j. We have used the Clenshaw–Curtis quadrature method implemented on MAPLE 2020, and have verified that increasing the precision, and at the same adding more digits to j does not change the first six digits in $E(j)$. We have independently verified the result using a Python code and, applying Simpson's rule, which we tested for convergence. We found the convergence between 1st and 2nd order, and so significantly lower than the 4th order expected from the method. This slow convergence is caused by the presence of square roots in the integrand which makes the function not differentiable at the boundaries. Indeed, repeating the test, but instead integrating between ±0.9 yields a convergence factor of 15.4 which is close enough to 16 expected from 4th order Simpson's method. The numerical value of energy appears to be very precise for all values of j. Changing the resolution between 500 and 10 000 points does not change the leading six digits.

The energy increases in an almost linear way (a closer analysis shows that the graph deviates from a line). Zooming near $j = 1$ shows that the energy reaches a maximum of $2.022\,23m$ around $j = 0.999\,07$, and then decreases to $2.019\,66m$ for the extremal Kerr horizon corresponding to $j = 1$ (figure 3). To show that this maximum is not a numerical artefact we can expand the integrand in (3.2) near $j = 1$, and find the integral to the lowest order in $(j-1)$:

$$E(j) \approx E(1) + \sqrt{1-j}\,0.174\,59m$$

so that $E(j)$ is indeed a decreasing function of j near $j = 1$.

In our computation of E_{KLY} from the hyperbolic embedding, we have chosen the hyperbolic radius of the ambient $\mathbb{H}^3$ to be maximal such the embedding is global. This choice has ensured the continuity of the mean curvature as well as the resulting energy at $j = \sqrt{3}/2$. We could instead leave L as a positive parameter, and regard (for each value of j) the energy as a function of L. It turns out that this function is monotonically decreasing from $L = 0$ to the critical value (figure 4), which again suggests that our initial choice was a right one.

4. Conclusions

We have used a combination of flat and hyperbolic embeddings to compute the quasi-local mass of the Kerr black hole horizon for any allowed value of the angular momentum. The hyperbolic embedding of the rapidly rotating horizons allowed us to overcome the difficulty arising from the Gaussian curvature not being positive everywhere on Σ. There is another, by now well established, way around this problem due to Wang and Yau [14, 15], who used an embedding of Σ in $\mathbb{R}^{3,1}$ to construct a reference frame. As well as allowing for non-positive Gaussian curvature, the Wang–Yau approach addresses a problem (pointed out in [6]) that E_{KLY} is positive on some closed two-surfaces in Minkowski space which do not lie in a space-like hyper-plane. The computation of the Wang–Yau quasi local mass involves taking an infimum of all mean curvature integrals over all 'time functions' τ such that the Gaussian curvature of $\hat{g} = g + \mathrm{d}\tau^2$ is positive. It is therefore difficult to implement for concrete examples. This has nevertheless been attempted in [16], where the authors noted the difficulty of finding a non-zero admissible time function as in general they lead to complex energies. By examining the boundary separating the complex and real energies they have confirmed the result of Martinez [5] in the range [0, 0.4], and improved it up to $j \leqslant \sqrt{3}/2$. For $j \in (\sqrt{3}/2, 1]$ the numerical computations of [16] suggest that the mass is increasing which agrees with our findings. The

analysis of [16] has not however revealed the global maximum[3] of mass just before $j = 1$. Additionally, the Wang–Yau mass is not defined if the mean curvature vector of Σ in the space-time vanishes, which is the case for the cross-over surface of the outer horizon. In [16] the mass was therefore calculated at a constant radius, and the outer-horizon limit was taken. It is however not clear whether this limit is unique.

Acknowledgments

Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Maciej Dunajski ⦿ https://orcid.org/0000-0002-6477-8319
Paul Tod ⦿ https://orcid.org/0000-0002-6086-4850

References

[1] Brown J D and York J W 1993 Quasilocal energy and conserved charges derived from the gravitational action *Phys. Rev.* D **47** 1407–19
[2] Gibbons G W, Herdeiro C A R and Rebelo C 2009 Global embedding of the Kerr black hole event horizon into hyperbolic three-space *Phys. Rev.* D **80** 044014
[3] Kijowski J 1997 A simple derivation of canonical structure and quasi-local Hamiltonians in general relativity *Gen. Relativ. Gravit.* **29** 307–43
[4] Liu C-C M and Yau S-T 2003 Positivity of quasilocal mass *Phys. Rev. Lett.* **90** 231102
[5] Martinez E A 1994 Quasilocal energy for a Kerr black hole *Phys. Rev.* D **50** 4920
[6] Murchadha N Ó, Szabados L B and Tod K P 2004 Comment on 'positivity of quasilocal mass' *Phys. Rev. Lett.* **92** 259001
[7] Nirenberg L 1953 The Weyl and Minkowski problems in differential geometry in the large *Commun. Pure Appl. Math.* **6** 337–94
[8] Page D 2021 Photon boomerang in a nearly extreme Kerr metric (arXiv:2106.13262)
[9] Pogorelov A V 1964 Some results on surface theory in the large *Adv. Math.* **1** 191–264
[10] Szabados L B 2004 Quasi-local energy–momentum and angular momentum in GR *Living Rev. Relativ.* **7** 4
[11] Thorne K S 1974 Disk-accretion onto a black hole: II. Evolution of the hole *Astrophys. J.* **191** 507–20
[12] Wald R M 1984 *General Relativity* (Chicago, IL: University of Chicago Press)

---

[3] While we cannot offer any physical explanation of this maximum, there exist at least two other occurrences of near extremal, but not extremal values of j in astrophysics, and general relativity: the value $j \approx 0.998$ is needed for the equilibrium of a black hole absorbing matter and radiation from an accretion disk [11]. In the context of rotating photon orbits in the Kerr solution it has been shown that if $j \approx 0.994\,34$, then a photon sent out in a constant radial direction from the north polar axis returns to the north polar axis in the opposite direction. An effect which page calls 'a photon boomerang' [8].

[13] Wang M-T and Yau S-T 2007 A generalization of Liu–Yau's quasi-local mass *Commun. Anal. Geom.* **15** 249–82

[14] Wang M T and Yau S T 2009 Quasilocal mass in general relativity *Phys. Rev. Lett.* **102** 021101

[15] Wang M-T and Yau S-T 2009 Isometric embeddings into the Minkowski space and new quasi-local mass *Commun. Math. Phys.* **288** 919–42

[16] Warner A M, Ray S, Wang M T and Yau S T 2018 Wang and Yau's quasi-local energy for an extreme Kerr spacetime *Class. Quantum Grav.* **35** 055007