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Clubs and Networks

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Abstract

A recurring theme in the study of society is the concentration of influence and power that is driven through unequal membership of groups and associations. In some instances these bodies constitute a small world while in others they are fragmented into distinct cliques. This paper presents a new model of clubs and networks to understand the sources of individual marginalization and the origins of different club networks.

In our model, individuals seek to become members of clubs while clubs wish to have members. Club value is increasing in its size and in the strength of ties with other clubs. We show that a stable membership profile exhibits marginalization of individuals and that this is generally not welfare maximizing. Our second result shows that if returns from strength of ties are convex (concave) then stable memberships support fragmented networks with strong ties (small worlds held together by weak ties).

We illustrate the value of these theoretical results through case studies of inter-locking directorates, boards of editors of journals, and defence and R&D alliances.

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1 Introduction

Economists study group formation using the theory of coalitions/clubs and the theory of network formation. In the coalitions approach individual payoffs are defined on the partition of players into mutually exclusive groups and in the networks literature individuals can join any number of groups but each of the groups is of size 2. However, in some important instances – examples include inter-locking directorates, R&D alliances, boards of editors of journals, and defence alliances – groups have size larger than 2 and individuals typically join multiple groups. Importantly, the productivity of a group depends on both its size and how it is connected to other groups through overlapping memberships. In these contexts, a major concern is that a few individuals take up most memberships while everyone else is left out thereby giving rise to a very unequal distribution of payoffs.[1] A second and related concern is that groups may be fragmented into cliques when a few individuals join them and that this may undermine openness and the performance of the system as a whole. Our paper proposes a new model of clubs and networks to examine these concerns.[2]

In our model, individuals seek to become members of clubs while clubs wish to have members. Clubs have capacity constraints (due to congestion effects) and individuals can only join up to a certain number of clubs (due to time limitations). Links between two clubs arise when an individual joins both clubs. The value of joining a club is increasing in the number of members (until the capacity is reached) and it may be increasing or decreasing in the strength of ties with other clubs. Individual utility is increasing in the sum of the productivity of the clubs they join. We define a notion of stable memberships that takes into account the incentives of individuals and clubs. Our interest is in understanding patterns of individual memberships and on the network of connections across clubs.

The main body of the analysis focuses on a setting where club value is increasing in link strength: in this case, a club prefers individuals who are members of more clubs and an individual prefers a club that links with more clubs. We show that stable outcomes exhibit a strong marginalization property: when club capacity is the binding constraint, a few individuals exhaust their membership capacity, while all others join no clubs; when individual availability is the binding constraint, a few clubs are fully occupied while all others go empty.[3]

[1] Durlauf and Young (2004) present an influential account of the groups based perspective on inequality and poverty. In section 6 we present case studies on a number of empirical contexts.

[2] There is a small set of papers that allow for membership of multiple groups, e.g. Page and Wooders (2010) and Fershtman and Persitz (2021); we discuss these papers in detail later in the introduction after presenting our model and results.

[3] For concreteness suppose that the number of individuals is 8, the number of clubs is 4, every individual

We next show that this marginalization is not always in line with efficiency: when individual utility is strongly concave, this marginalization is inefficient. Similarly, when club productivity is a concave function of membership size, the marginalization of clubs is inefficient. Thus, incentives of individuals and clubs and the collective interest is generally not aligned.

We then study the network of connections among the clubs. When the returns to link strength are linear, the distribution of link strength across clubs is not important for the productivity of clubs: as a result, a variety of club networks are stable. In applications, however, the marginal returns from link strength are likely to be non-linear. For instance, in case club links are used for information sharing, we would expect marginal returns to decline with link strength. On the other hand, if links help members coordinate activities of the clubs then the marginal returns may be increasing in link strength. We show that if the marginal returns from link strength are increasing, i.e., they are convex, then incentives of clubs and individuals push towards disconnected cliques of clubs with full strength links. If, on the other hand, the marginal returns from link strength are decreasing, i.e., they are concave, then the club network entails larger components that are held together by weak links.[4]

We also consider a setting where club value is decreasing in link strength with other clubs: a club prefers individuals who are not members of other clubs. In this setting, when club capacity is the binding constraint, stable outcomes entail isolated clubs. On the other hand, if individual availability is the binding constraint then clubs may be obliged to accept individuals who are also members of other clubs.

The paper closes with brief case studies on inter-locking directorates, defence alliances, R&D alliances and editorial boards of journals. There is a large and distinguished body of work on inter-locking directorates, see e.g., Brandeis (1915), Brandeis (2009), Mizruchi (1996), Levine (1977), Useem (1984), and Davis, Yoo and Baker (2003); for a recent networks perspective on this literature see Kogut (2012). This literature argues that a major function of boards is to encourage best practices and that this is facilitated when a board member also

can join up to 4 clubs and every club has capacity 4. The total club capacity is 16, so in principle every individual could belong to 2 clubs each. We will say that a membership profile exhibits marginalization when 4 individuals become members of 4 clubs each while the other four individuals are completely left out.

4 For concreteness suppose that number of individuals is 16, number of clubs is 6, every individual can join up to 2 clubs and every club has capacity 5. If returns are convex in link strength then the unique club-efficient and stable outcome is three cliques of two clubs each, and the links have maximal strength with 5 common members. If returns are concave in link strength then the unique club-efficient and stable membership profile is a connected network where every club has a link with one common member with every other club. These networks of clubs are illustrated in Figure 4 in section 4 below).

has ties with other firms' boards. If information sharing is important then it is reasonable to suppose that the marginal returns from the strength of links is declining. In this setting the theory predicts that the stable (and efficient) club network will contain weak ties and exhibit high connectivity. This is in line with the empirical evidence: Baker, Davis and Yoo (2001) and Kogut (2012) show that inter-locking directorates exhibit a small-world property – weak ties form the basis for a large connected network.[5]

In the context of defence alliances, a general presumption is that memberships bring additional resources but that overlaps in memberships could be detrimental for the security of an alliance, as a member may share valuable information with potentially adversarial alliances; for overviews of the literature on defence alliances see Bloch and Dutta (2012), Bloch, Sánchez-Pagés and Soubeyran (2006), and Jackson and Nei (2015). The potential negative impact of common memberships leads us to a model in which the value of a club is falling in link strength. Our model then predicts that defence alliances will have exclusive membership: this prediction is in line with the empirical evidence.

R&D alliances among firms have become increasingly common since the 1980's (Hagedoorn (2002) and Gulati (2007)). The empirical research suggests that there is great inequality in the number of alliances firms participate in, the degree distribution has a power law (Powell et al. (2005) and König et al. (2019)). This unequal degree distribution is consistent with our marginalization results and the connectivity of the network of alliances is consistent with the prediction of our model in case of concave returns from links.

Our final application pertains to editorial boards of journals. We draw on the work of Ductor and Visser (2021) to study the membership of authors in these boards and the connections between boards defined by common editors. There exists very significant inequality in editorial memberships: a very small fraction of authors become editors. Moreover, most editors serve only on one or two boards, but there exists a core group of editors who serve on 4 or more journals. The network of the editorial boards is connected that is held together

[5] The work on inter-locking directorates is also related to a more general study of elites and power structures in sociology. In the nineteenth century, the Italian school of sociology proposed a theory of elites defined in terms of the membership of the top echelons of different – government and non-government – organizations (Pakulski (2018)). Building on this tradition, in his well-known study of mid-twentieth-century American society, Wright Mills (1956) argued that the power to make major decisions was highly concentrated: a very small group of individuals moved between the top levels of the Federal government, a few hundred largest corporations, and the military. He referred to these individuals as the *power Elite*. Similar claims have been made about the concentration of power and influence in other societies. For an overview of the theory of elites, see Bottomore (1993), and for a critique of theories of elite power and control, see Dahl (1958). Our model and case studies draw attention to economic forces that push toward concentration of power in modern society.

with (mostly) weak links. These patterns are consistent with our theoretical predictions on marginalization and on club networks (in the presence of concave returns from link strength).

There is a voluminous literature on coalitions and networks; for surveys of this work see e.g., Demange and Wooders (2005), Bloch and Dutta (2012), Bramoullé, Galeotti and Rogers (2016) and Goyal (2022). Our model draws on the theory of clubs and the theory of networks to explain phenomena such as marginalization, the small world of interlocking directorates, and power elites. Specifically, we combine the ideas of congestion and capacity constraints from club theory (Buchanan, 1965; Cornes, 1996; Demange and Wooders, 2005) with the ideas of multiple memberships and returns from links from the theory of networks (Bala and Goyal, 2000; Jackson and Wolinsky, 1996; Bloch and Dutta, 2012). We now discuss two earlier papers that seek in different ways to combine networks and clubs.

In an early paper Page and Wooders (2010) study a setting of bipartite networks in which individuals decide on which clubs to join. Individual utility depends on own choices as well as the choices of others. Page and Wooders (2010) focus on the conditions under which the game of club memberships has a potential function (and this allows them to study the existence of Nash equilibrium). In our approach the clubs are owned by players who can choose to admit and expel members; these owners seek to maximize club productivity. The interaction between players and club owners gives rise to different incentives and strategic effects and hence to a different solution concept. Moreover, the focus of the paper is on the characterization of stable membership profiles. In particular we derive a marginalization result and a mapping between marginal returns to link strength and club networks. While we consider more specific functional forms and pay-off structures these results go beyond the Page and Wooders (2010) paper.

A recent paper by Fershtman and Persitz (2021) also studies a model of clubs and networks. At a general level, there are similarities – both papers study a memberships model. But the motivation of the two papers is different and so the models and the main insights are also different. For Fershtman and Persitz (2021) the principal object of interest is the social network among individuals; by contrast, our interest is in understanding the membership profile of individuals in clubs. We explore questions such as who joins which club and what is the network of clubs that arises. This gives rise to very different types of results. Fershtman and Persitz (2021) highlight a trade-off between the size of clubs, the depreciation of indirect connections, and the membership fee. By contrast, we develop a marginalization property of stable outcomes and show why this is socially inefficient. We also draw attention to how the marginal returns from link strength – whether they are increasing or decreasing – determine the architecture of club networks.

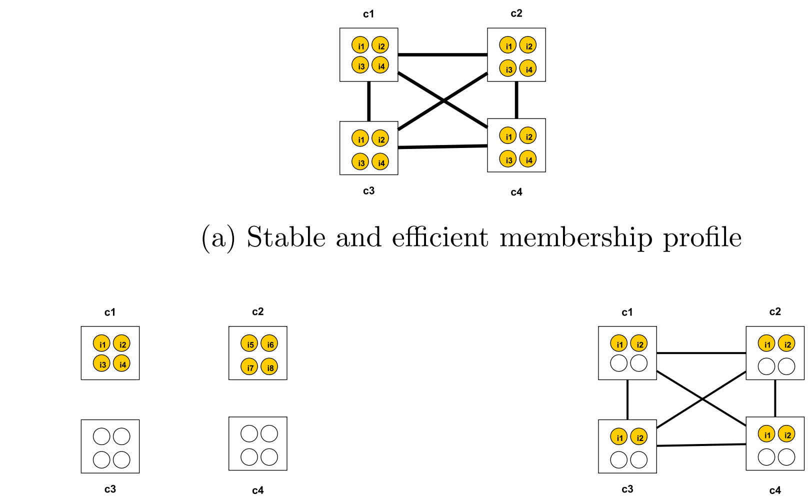
To clarify the relation between our approach and the coalitions and networks approaches it is instructive to lay out the basic notation and then work through an example. In our model, there are n individuals and m clubs, each individual can join up to D clubs and every club can admit up to S members. It is assumed that club productivity is increasing in club size and in strength of links with other clubs. In our approach we allow D and S to take arbitrary values. In a coalitions model, the outcome is a partition, so individuals can join only one club, so $D = 1$. Similarly, networks constitute a special case where every club can have exactly 2 members, roughly this means $S = 2$ and the payoff to the club from a single member is 0.

Example 1. Suppose there are 8 individuals and 4 clubs, with individuals able to join 4 clubs and a club having a capacity of 4. In our model (so long as utilities are not too concave) the stable and welfare maximizing membership profile involves 4 clubs that are occupied by the same 4 members. This means that 4 individuals are marginalized. In the coalition framework, the efficient and stable partition involves every individual joining one club each (thus two clubs are occupied by 8 members), while the remaining 2 clubs remain unoccupied.[6] In contrast to our result there is no marginalization and the network of clubs is empty. In the networks framework, a relation is bilateral; so clubs consist of exactly 2 members. An efficient and stable network involves all 4 clubs being occupied by the same 2 members. Thus six individuals are left out of clubs and marginalization is even greater than in our model and clubs are smaller (less connected and hence less productive). □

We close the introduction with a few words on the relation with the matching literature. A key underlying motivation for the matching literature is that individuals (or firms) have preferences over the individuals that are matched (see Roth and Sotomayor (1992)). This is the driving force for the original one-to-one matching models and remains a central feature of many-to-many matching models (see e.g., Hatfield and Kominers (2015), Rostek and Yoder (2019), Bando and Hirai (2021), Echenique and Oviedo (2006), Klaus and Walzl (2009)). By contrast, the focus in our paper is on the size of memberships (both for individuals and clubs) and on the structure of connections between the clubs. The methods of analysis and the results (on marginalization and on the structure of club networks) are therefore quite different.[7]

[6] For our definitions of stability and efficiency see section 2 below.

[7] We have also extended our model to a setting where individuals have preference for same-type club mates. Our methods of analysis can be extended in a straightforward manner to cover this case: indeed a small inclination for homophily leads to a strong division of individuals into distinct groups, and this further exacerbate payoffs inequality and undermine overall efficiency.



(a) Stable and efficient membership profile

(b) Stable and efficient coalition

(c) Stable and efficient network

Figure 1: Comparison of our approach with coalitions and networks

Section 2 presents the model, section 3 presents an analysis of the marginalization and section 4 presents our results on network structure of clubs. We show how research alliances among competing firms can be studied using similar methods in section 5. Section 6 presents case studies on defence alliances, inter-locking directorates, R&D alliances and boards of editors of journals. All the proofs are presented in the Online Appendix.

2 The Model

There is a set of individuals $I = \{i_1, \ldots, i_n\}$ and a set of clubs $C = \{c_1, \ldots, c_m\}$. We use i to denote a typical individual and c to denote a typical club. Individuals join clubs to become members. A *membership profile* is represented by a matrix $\boldsymbol{a} = (a_{ic})_{i \in I, c \in C}$ where $a_{ic} \in \{0, 1\}$ indicates whether individual i is a member of club c.

We define a few notions based on a membership profile $\boldsymbol{a}$. The *degree* of individual i, given a membership profile $\boldsymbol{a}$, is the number of clubs joined by i:

$$d_i(\boldsymbol{a}) = \sum_{c \in C} a_{ic}.$$

The *membership size* of club c, given a membership profile $\boldsymbol{a}$, is the number of individuals who join c:

$$s_c(\boldsymbol{a}) = \sum_{i \in I} a_{ic}.$$

There is a *link* between two clubs if they share common members. The *link strength* between clubs c and c', given a membership profile $\boldsymbol{a}$, is the number of common members they share:

$$g_{cc'}(\boldsymbol{a}) = \sum_{i \in I} a_{ic} a_{ic'}.$$

Following the large literature in club theory, we shall assume that there are strong congestion effects that set limits to club capacity (see Buchanan (1965) and Page and Wooders (2010)). Similarly, we assume that individuals can only join a certain number of clubs; this is because they have a fixed amount of time and participating in a club has a minimum time commitment. Formally, we assume that $d_i(\boldsymbol{a}) \leq D$, for all $i \in I$, and $s_c(\boldsymbol{a}) \leq S$, for all $c \in C$, where D and S are two positive integers. The set of feasible membership profiles is $A = \{\boldsymbol{a} \in \{0,1\}^{n \times m} : d_i(\boldsymbol{a}) \leq D, s_c(\boldsymbol{a}) \leq S\}$. We also assume that $2 \leq S \leq n$ and $2 \leq D \leq m$: this ensures that at least one club can be fully occupied and at least one person can join the maximum number of clubs.

A club provides goods and services to its members. The productivity of a club depends on its size and on the links it has with other clubs. We assume that until the capacity is reached, club productivity increases in the number of its members. And we assume that the productivity of a club is increasing in the strength of the ties it maintains with other clubs.[8] In different contexts, we can interpret clubs as different institutions. For example, a club can be a board of a firm: links between boards, created by overlapping directors, may help the transmission of best practices and the coordination of corporate strategies. A club can also be a research alliance between firms: links between alliances, generated by shared participating firms, can facilitate knowledge spillovers. Depending on the roles links serve, the marginal returns from link strength vary. If the link helps to convey factual information then the marginal returns from link strength may be declining. On the other hand, if the information concerns complex issues such as new technologies or standards then marginal returns to link

[8] In some contexts, club productivity may be falling in links with other clubs. This happens for instance if the clubs are in a competitive setting and when individuals belong to many clubs, they allocate limited time to each of their clubs and that lowers their productivity. The analysis of clubs and networks with negative spillovers can be carried out using the same methods as we develop for the case of positive spillovers across clubs. We comment on the implications of negative spillovers after presenting the results for positive spillovers.

strength may be increasing. Similarly, if we are in a context of developing common standards (technological or social) then there may be value in significant overlap of membership.

With these ideas in mind, let us define the productivity of club $c \in C$ in profile $\boldsymbol{a}$ as

$$\pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a})) + \sum_{c' \neq c} h\left(g_{cc'}(\boldsymbol{a})\right), \tag{1}$$

where returns from membership size, f, are strictly increasing with $f(0) = 0$, and the externality from links, h, is increasing with $h(0) = 0$. The next section studies the benchmark case of linear increasing returns case: $h(x) = \alpha x$, with $\alpha \geq 0$. We take up the case of convex and concave returns in Section 4.

Turning to individual utility, we assume that an individual enjoys benefits from the productivity of clubs she joins. Given a profile $\boldsymbol{a}$, the utility of individual $i \in I$ is

$$u_i(\boldsymbol{a}) = v\left(\sum_{c \in C} a_{ic}\pi_c(\boldsymbol{a})\right), \tag{2}$$

where v is strictly increasing with $v(0) = 0$. In situations where individuals are directors of boards, it is natural to assume that their utility increases at a decreasing rate with the aggregate productivity of clubs they are in, so $v''(\cdot) \leq 0$. When individuals are firms that participate in research alliances, their utility (profit) increases at an increasing rate with the aggregate clubs they join. To see how, assume that firms reduce production costs by joining research alliances; let the cost of firm i under membership profile $\boldsymbol{a}$ be $\gamma_0 - \gamma \sum_{c \in C} a_{ic}\pi_c(\boldsymbol{a})$, where γ_0 and γ are two positive real numbers. Suppose firms are monopolies that operate in markets with an inverse demand function $p = \beta - q_i$, where $\beta > \gamma_0$, then the equilibrium output of firm i is $(\beta - \gamma_0 + \gamma \sum_{c \in C} a_{ic}\pi_c(\boldsymbol{a}))/2$ and the equilibrium profit of firm i is $(\beta - \gamma_0 + \gamma \sum_{c \in C} a_{ic}\pi_c(\boldsymbol{a}))^2/4$, which is a convex function of $\sum_{c \in C} a_{ic}\pi_c(\boldsymbol{a})$.[9]

We study efficient and stable memberships. We consider two standards for a membership profile to be efficient: maximizing the utilitarian welfare of individuals and maximizing the aggregate productivity of clubs.

[9] We also study the case where firms engage in Cournot competition. In that case, the profit of a firm depends not only on the aggregate productivity of research alliances it joins, but also on the productivity of alliances joined by other firms. We demonstrate, in Section 5, that our key results can be extended to cover this case.

Definition 1. A membership profile $\boldsymbol{a} \in A$ is the *utilitarian optimum* if for all $\boldsymbol{a'} \in A$,

$$\sum_{i \in I} u_i(\boldsymbol{a}) \geq \sum_{i \in I} u_i(\boldsymbol{a'}).$$

A membership profile $\boldsymbol{a} \in A$ is *clubs-efficient* if for all $\boldsymbol{a'} \in A$,

$$\sum_{c \in C} \pi_c(\boldsymbol{a}) \geq \sum_{c \in C} \pi_c(\boldsymbol{a'}).$$

Turning to strategic stability, it seems reasonable to require that individuals should be able to quit clubs if that increases their utility and clubs should be able to expel members if that raises their productivity. In addition, it seems reasonable to require that an individual and a club cannot coordinate on a deviation that makes them both strictly better off. i.e., no pair of individual i and club c can both benefit from a joint deviation where i is allowed to quit any clubs she is in, c is allowed to exile any members it has, and i joins c. We propose a notion of stability that reflects these ideas.

Formally, let $a_i = (a_{ic})_{c \in C}$ and $a_c = (a_{ic})_{i \in I}$ be the vectors recording the clubs i joins and the members c has, and let $a_{-i} = (a_{i'c})_{i' \neq i, c \in C}$ and $a_{-c} = (a_{ic'})_{i \in I, c' \neq c}$ denote the club joining of individuals other than i and member admission of clubs other than c. Moreover, we use $a_{-i,c} = (a_{i'c'})_{i' \neq i, c' \neq c}$ to represent the membership profile excluding individual i and club c, and we use $a_{-ic} = (a_{i'c'})_{i'c' \neq ic}$ to represent the membership profile excluding the relationship between individual i and club c. We write $a \geq a'$ if a is element-wise greater than or equal to a'.

Definition 2. A membership profile $\boldsymbol{a} \in A$ is *stable* if

1. $\forall i \in I, c \in C$: there is no $\boldsymbol{a'} \in A$ with $a'_i \leq a_i$ and $a'_{-i} = a_{-i}$ such that $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$, or $a'_c \leq a_c$ and $a'_{-c} = a_{-c}$ such that $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$, and

2. $\forall i \in I, c \in C$: there is no $\boldsymbol{a'} \in A$ with $a'_{ic} = 1$, $a'_{-ic} \leq a_{-ic}$, and $a'_{-i,c} = a_{-i,c}$ such that $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$ and $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$.

3 Marginalization

This section presents an analysis of a benchmark model in which returns from links take a linear form, $h(x) = \alpha x$, where $\alpha \geq 0$. So, there is a positive externality from links with other clubs when $\alpha > 0$.

We first investigate stable membership profiles. Substituting the linear functional form for $h(\cdot)$ in the club productivity function in (1), we see that the productivity of a club $c \in C$ under a membership profile $\boldsymbol{a}$ is

$$\Pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a})) + \alpha \sum_{i \in C} a_{ic}(d_i(\boldsymbol{a}) - 1). \tag{3}$$

Observe that a club prefers an individual who is also a member of other clubs. Similarly, given their utility in (2), individuals prefer clubs with higher productivity. These two incentives press in the same direction: clubs like well-connected individuals and individuals prefer well-connected clubs. Thus, in this model, the incentives of clubs and individuals press toward marginalizing poorly connected clubs and poorly connected individuals.

To make this precise, let us define a partition of individuals and clubs. Let π^* be the highest productivity a club can achieve and u^* be the highest utility an individual can enjoy. Observe that in our benchmark model,

$$\pi^* = f(S) + \alpha S(D-1) \text{ and } u^* = v(D\pi^*). \tag{4}$$

Next note that for a membership profile $\boldsymbol{a}$, the set of individuals I can be partitioned into four parts: a first group $I_1(\boldsymbol{a})$ that consists of individuals who join D clubs and obtain utility u^*; a second group $I_2(\boldsymbol{a})$ that consists of individuals who join D clubs but do not obtain utility u^*; a third group, $I_3(\boldsymbol{a})$, who join some but not D clubs; and a fourth group, $I_4(\boldsymbol{a})$, that consists of individuals who join no clubs.

$$\begin{aligned}
I_1(\boldsymbol{a}) &= \{i \in I : d_i(\boldsymbol{a}) = D, u_i(\boldsymbol{a}) = u^*\} \\
I_2(\boldsymbol{a}) &= \{i \in I : d_i(\boldsymbol{a}) = D, u_i(\boldsymbol{a}) < u^*\} \\
I_3(\boldsymbol{a}) &= \{i \in I : 0 < d_i(\boldsymbol{a}) < D\} \\
I_4(\boldsymbol{a}) &= \{i \in I : d_i(\boldsymbol{a}) = 0\}
\end{aligned}$$

Similarly, the set of clubs can be partitioned into three parts. The first group, $C_1(\boldsymbol{a})$, consists of clubs with productivity π^*; the second group, $C_2(\boldsymbol{a})$, consists of clubs with positive productivity less than π^*; and a third group, $C_3(\boldsymbol{a})$, that consists of clubs with zero productivity.

$$C_1(\boldsymbol{a}) = \{c \in C : \pi_c(\boldsymbol{a}) = \pi^*\}$$
$$C_2(\boldsymbol{a}) = \{c \in C : 0 < \pi_c(\boldsymbol{a}) < \pi^*\}$$
$$C_3(\boldsymbol{a}) = \{c \in C : \pi_c(\boldsymbol{a}) = 0\}$$

Let us say that a membership profile exhibits **marginalization of individuals** if some individuals become members of D clubs while all other individuals are excluded from clubs altogether. In other words, $I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a}) \cup I_4(\boldsymbol{a}) = I$. Our initial remarks suggest that if there are enough individuals around as compared to club capacity, then a stable membership profile must exhibit this marginalization of individuals. Let us work through some examples to develop a feel for the different issues at work in this model.

The first point to note is that integer constraints may come in the way of this marginalization. This is easily seen in the following example: suppose that $n = 8$, $m = 4$, $D = 3$ and $S = 4$. A membership profile in which five individuals exhaust their membership availability while a sixth individual joins one club is stable. The following example shows that even in the absence of integer constraints – i.e., even if mS/D is an integer – the pure marginalization property may fail to hold.

Example 2. Suppose that $n \geq 12$, $m \geq 10$, and $D = S = 6$. In this case, mS/D is an integer. We show that there exists a stable club membership profile with individuals joining some but less than D clubs. Consider the following membership profile. Let i_x, i_y and i_z be three individuals who join 4 clubs. For other individuals, let $m-2$ of them, which we denote with i_1,...,i_{m-2}, join $D = 6$ clubs and the rest of them join no clubs. Allocate i_x, i_y, i_z, i_1, i_2, i_3 and i_4 to four clubs c_1, c_2, c_3 and c_4 in the way depicted in Figure 2. Also, let individuals i_1 to i_4 join any three other clubs and let individuals i_5 to i_{m-2} join any six other clubs. This membership profile, with 3 individuals joining some but less than D clubs, is stable. To see how, under this memberhip profile, all clubs are full and clubs other than c_1 to c_4 reach the highest productivity possible and would not want any deviations. For clubs c_1 to c_4, they wish to make deviations. For example, c_1 wants to admit i_4 instead of i_x, i_y or i_z. If i_4 joins c_1, the productivity of c_1 would raise by 2α and be higher than that of c_2, c_3 and c_4 she is currently in. With this logic, it seems that i_4 would want to quit c_2, c_3 or c_4 and join c_1. However, note that with the deviation, the degree of i_x, i_y or i_z drops by 1, making the productivity of c_2, c_3 and c_4 drop by α. Although i_4 leaves one of c_2, c_3 and c_4, she is still in two of them. The aggregate productivity i_4 enjoys from clubs drops by 2α, which cancels out the productivity gain from c_1. Hence, i_4 has no incentive to make the deviation. Using the same logic, we can show that c_2, c_3 and c_4 cannot attract a higher-degree individual to replace i_x, i_y or i_z as well and the membership profile is stable. □

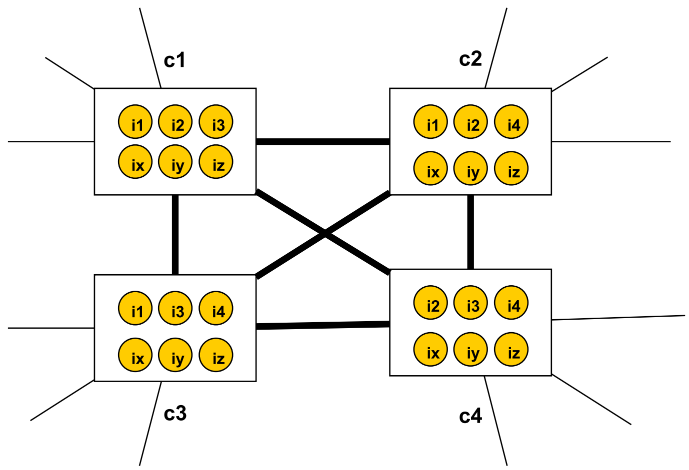


Figure 2: Coordination problem.

This example draws attention to a coordination problem among individuals and clubs: note that the m clubs and individuals i_1 to i_{m-2}, i_x and i_y would be better off in the membership profile where the clubs are exactly filled by those individuals so that all those individuals have degree D.

The combination of integer constraints and coordination problems gives rise to a number of complications that inform the characterization of stable membership profiles that is presented below.

Proposition 1. *Assume that $h(x) = \alpha x$, where $\alpha \geq 0$. There exists a stable membership profile. A membership profile $\boldsymbol{a} \in A$ is stable if and only if*

(i) for every individual $i \in I$ and club $c \in C$, if i is not a member of c, then either $d_i(\boldsymbol{a}) = D$ or $s_c(\boldsymbol{a}) = S$,

(ii) for every club c with fewer than S members, every individual i, and every club c' that i

joins, if i is not a member of c, then

$$\pi_c(\boldsymbol{a}) + f(s_c(\boldsymbol{a})+1) - f(s_c(\boldsymbol{a})) + \alpha(D-1) \leq \pi_{c'}(\boldsymbol{a}),$$

In addition, if $\alpha > 0$, then

(iii) for every individual i who joins fewer than D clubs, every club c that i does not join, every individual i' in club c must have with $d_{i'}(\boldsymbol{a}) > d_i(\boldsymbol{a})$, and

(iv) for every individual i who joins D clubs, every club c that i does not join and every individual i' that is a member of c, if $d_i' < D$, then

$$\pi_c(\boldsymbol{a}) + \alpha(D - d_{i'}(\boldsymbol{a})) - \alpha \sum_{c'' \neq c'} a_{ic''} a_{i'c''} \leq \pi_{c'}(\boldsymbol{a}), \text{ for all } c' \text{ that } i \text{ joins.}$$

The proof is presented in the Online Appendix. Let us briefly elaborate on the content of the conditions so that we can appreciate some of the arguments that are involved. The four conditions ensure that there is no profitable deviation for a pair of individual i and a club c in four different cases that together exhaust all possible situations.

Point (i) considers deviation where $d_i < D$ and $s_c < S$. We require that there does not exist such a pair as otherwise i can join c and both are better off. Point (ii) considers deviation where $d_i = D$ and $s_c < S$. We require that i does not want to quit an existing club to join c. The condition states that the productivity of c, taking into account the change resulting from i's joining, must not be greater than that of any club c' that i is currently a member of.

In points (i) and (ii) we assume $s_c < S$. So, they are not about c replacing a low-degree individual with a higher-degree one, but concern i joining a higher-productivity club. Hence, the two conditions are needed both when $\alpha = 0$ and when $\alpha > 0$. For the next two situations we look at, $s_c = S$. They are only needed when $\alpha > 0$.

Point (iii) considers deviation where $d_i < D$ and $s_c = S$. We require that c does not want to replace an existing member with i. The requirements leads to the characterization that for i, i' with degree less than D, if $d_i \geq d_{i'}$, then the set of clubs i joins is a superset of clubs i' joins.

Point (iv) considers deviation where $d_i = D$ and $s_c = S$. Now, for i to join c, i needs to quit a club c' and c needs to expel a member i'. A profitable deviation does not exist if either (1) $d_i \leq d_{i'}$, so that the club has no replacement incentive, or (2) the individual has no wish to switch clubs. Note, however, the condition for i to not want to change does not only require that the productivity of c, taking into account the change resulting from i's joining, is not greater than that of c', as in the case of (ii). There is an additional consideration that i hopes c's exiling of i' does not hurt her utility (this is key to the stability of non-marginal membership profile in Example 2). This is captured by the term $\alpha \sum_{c'' \neq c'} a_{ic''} a_{i'c''}$.

Equipped with this characterization, we can provide a fairly complete description of the partition of individuals and clubs in a stable membership profile. This will allow us to answer the question of whether or not stability implies marginalization of individuals and clubs. It is helpful to define egalitarian and unequal membership profiles. A membership profile is egalitarian if there is minimal difference in the degrees between individuals. As opposed to an egalitarian membership profile, a membership profile marginalizes individuals if almost all individuals join D clubs or no clubs at all. Similarly, a membership profile marginalizes clubs if all clubs have either S members or 0 members. Define a measure of marginalization for individuals as

$$\mathcal{M}_{\mathcal{I}} = \frac{|I_1(\boldsymbol{a})| + |I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| - mS/D}{mS - mS/D}. \tag{5}$$

Observe that $\mathcal{M}_{\mathcal{I}} \in [0,1]$; it takes on value 0 when the number of individuals who join clubs is equal to mS/D and it is equal to 1 when the number of individuals who join clubs is equal to the aggregate club capacity, mS.

Likewise we may define marginalization of clubs with the measure

$$\mathcal{M}_{\mathcal{C}} = \frac{|C_1(\boldsymbol{a})| + |C_2(\boldsymbol{a})| - nD/S}{nD - nD/S}. \tag{6}$$

Definition 3. A membership profile $\boldsymbol{a} \in A$ is egalitarian if there is minimal difference between the membership level of individuals: $|d_i(a) - d_j(a)| \leq 1$ for any pair $i, j \in I$.
A membership profile $\boldsymbol{a} \in A$ marginalizes individuals if $\mathcal{M}_{\mathcal{I}}$ is close to 0.
A membership profile $\boldsymbol{a} \in A$ is said to marginalize clubs if $\mathcal{M}_{\mathcal{C}}$ is close to 0.

Proposition 2. *Assume that $h(x) = \alpha x$, where $\alpha \geq 0$. When $\alpha = 0$, an egalitarian membership profile is stable. When $\alpha > 0$, for a stable $\boldsymbol{a}$,*

- *if $nD \geq mS$, then*

$$\frac{mS}{D} - \frac{S(D+3)}{2} \leq |I_1(\boldsymbol{a})| \leq |I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \leq \frac{mS}{D} \text{ and}$$
$$n - \frac{mS}{D} - S \leq |I_4(\boldsymbol{a})| \leq n - \frac{mS}{D}.$$

Therefore, $\mathcal{M}_{\mathcal{I}} \leq \frac{D}{(D-1)m}$: every stable membership marginalizes individuals for large m.

- *if $nD < mS$, then*

$$\frac{nD}{S} - D \leq |C_1(\boldsymbol{a})| \leq \frac{nD}{S} \text{ and}$$
$$m - \frac{nD}{S} - D \leq |C_3(\boldsymbol{a})| \leq m - \frac{nD}{S}.$$

Therefore, $\mathcal{M}_{\mathcal{C}} \leq \frac{S}{(S-1)n}$: every stable membership profile marginalizes clubs for large n.

The proof is presented in the Online Appendix.

In the absence of network externalities, it is fairly straightforward to see that an egalitarian club profile is stable. Given $nD > mS$, assign the mS club slots to distinct individuals, this is clearly stable as there is no advantage of having common membership in clubs (the difference in degree between the maximally connected and minimally connected individual is 1). Given $nD < mS$, assign the nD membership capacity across the nD/S clubs. Everyone has an equal number of memberships equal to D.[10]

Turning to the setting with positive externalities, let us comment on the expressions for the bounds. Clearly, mS/D is the maximal number of individuals who can be a member of D clubs each. So the upper bound on $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|$ is fairly immediate. Let us comment on the lower bound for $|I_1(\boldsymbol{a})|$. To do this we derive an upper bound on $|I_2(\boldsymbol{a})|$ and $|I_3(\boldsymbol{a})|$. To derive a bound on the number of individuals in $|I_2(\boldsymbol{a})|$, note that all of them must join a club in $C_2(\boldsymbol{a})$. The number of clubs in $C_2(\boldsymbol{a})$ is limited by D because the member who has the highest degree in the least productive club of $C_2(\boldsymbol{a})$ must join all clubs in $C_2(\boldsymbol{a})$, otherwise, she would deviate to join another $C_2(\boldsymbol{a})$ club and the club is willing to take her. Therefore, the number of available slots for $I_2(\boldsymbol{a})$ individuals from $C_2(\boldsymbol{a})$ clubs is (weakly) smaller than $(S-1)D$. In the proof we show that the number of $I_2(\boldsymbol{a})$ individuals who only join one $C_2(\boldsymbol{a})$ club is limited by $S-1$: putting together these numbers we arrive at the bound of $S - 1 + [(S-1)D - (S-1)]/2 = (S-1)(D+1)/2$ for the number of individuals in $I_2(\boldsymbol{a})$. Turning to $|I_3(\boldsymbol{a})|$, observe that for individuals in $I_3(\boldsymbol{a})$, if an individual i's degree is greater than or equal to the degree of another individual i', then the set of clubs i joins must be a superset of the set of clubs i' joins. Otherwise, i can crowd out i' and join one more club.

[10] Matters are slightly more complicated when nD/S is not an integer: in that case, let $\lfloor nD/S \rfloor$ clubs have S members and one club have $(nD) \bmod S$ members. The structure is stable and every individual has degree D.

Thus, for a club that hosts the individual with the lowest degree in $I_3(\boldsymbol{a})$, it must be the case that it hosts all individuals in $I_3(\boldsymbol{a})$. Since a club can host at most S members, $|I_3(\boldsymbol{a})| \leq S$. The expression in the Proposition follow by noting that $S(D+3)/2 > S + (S-1)(D+1)/2$.

We now turn to the marginalization results. Note that, when $nD > mS$, we can derive an upper bound on $|I_3(\boldsymbol{a})|$: this set is at most of size S. Fixing D and S, for large n our measure $\mathcal{M}_{\mathcal{I}}$ approximates 0: in other words, if $nD > mS$, and n is large, then every stable membership profile marginalizes individuals. Similarly, if $nD < mS$, and m is large, then every stable membership profile marginalizes clubs.

We next turn to welfare properties of membership profiles. We have shown that in the presence of a connection externality, a stable membership profile always marginalizes individuals or clubs. Are such membership profiles desirable? We show that the answer depends on whether we look at clubs-efficiency or at the utilitarian optimum. In our study of utilitarian optimum, we will make use of the following condition on the concavity of the utility function.

$$v\left(f(S)\right) - v(0) > (n-1)\left(v\left(f(S) + \frac{2\alpha S(D-1)}{n-1}\right) - v\left(f(S)\right)\right). \tag{7}$$

Proposition 3. *Suppose $\alpha > 0$. Assume $nD \geq mS$ and that mS/D is an integer.*[11]

- *A membership profile is clubs-efficient if and only if mS/D individuals join D clubs and the remaining individuals join no clubs ($\mathcal{M}_{\mathcal{I}} = 0$).*

- *If $v''(\cdot) \geq 0$, then a membership profile is an utilitarian optimum if and only if it is clubs-efficient. If $v''(\cdot) < 0$ and satisfies condition (7), then in any utilitarian optimum membership profile, either $d_i(\boldsymbol{a}) \leq 1$ for all $i \in I$ or $d_i(\boldsymbol{a}) \geq 1$ for all $i \in I$ ($\mathcal{M}_{\mathcal{I}} = 1$).*

Assume $nD < mS$ and that nD/S is an integer.

- *If $f''(\cdot) > 0$, then a membership profile is clubs-efficient if and only if nD/S clubs admit S members and the remaining clubs admit no members ($\mathcal{M}_{\mathcal{C}} = 0$). If $f''(\cdot) < 0$, then a membership profile is clubs-efficient if and only if $(nD) \bmod m$ clubs admit $\lceil \frac{nD}{m} \rceil$ members and the remaining clubs admit $\lfloor \frac{nD}{m} \rfloor$ members ($\mathcal{M}_{\mathcal{C}} = 1$).*

- *A membership profile is an utilitarian optimum if and only if nD/S clubs admit S members and the remaining clubs admit no members ($\mathcal{M}_{\mathcal{C}} = 0$).*[12]

[11] In the Online Appendix, we provide characterizations of clubs-efficient and utilitarian optimal membership profiles without the integer condition.

[12] If the integer condition (S divides nD) does not hold, then the utilitarian optimum characterization for

The proof is presented in the Online Appendix.

Consider first the case where $nD > mS$. Proposition 3 tells us that a membership profile that maximizes the aggregate output of the clubs features a strong form of marginalization: (modulo integer restrictions) club-efficient profile allocates exactly mS/D individuals into memberships, all other individuals join no clubs. This is because this marginalization ensures maximal overlap of members between clubs.

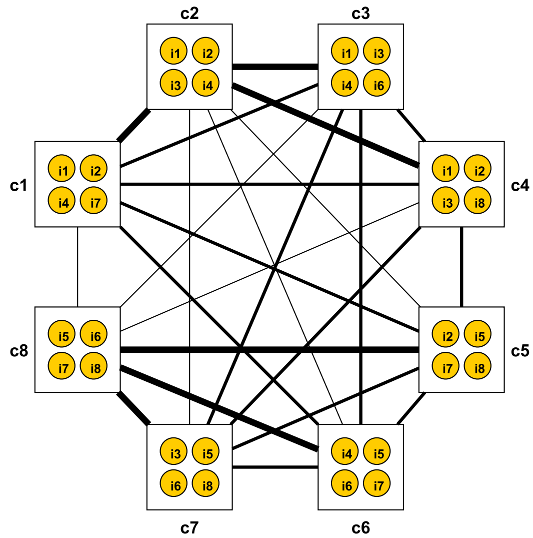
Turning to the utilitarian optimum, if the utility of individuals rises at an increasing or constant rate with the productivity of clubs they join, i.e., if $v''(\cdot) \geq 0$, then the profile that is utility-maximizing is the same as the profile that is productivity-maximizing. This is because when $v''(\cdot) = 0$, the aggregate utility of individuals is simply the number of individuals club can admit, S, times the aggregate productivity of clubs, and when $v''(\cdot) > 0$, utilitarian optimality pushes toward marginalization of individuals, which coincides with the outcome generated by clubs-efficiency. If, on the other hand, the marginal utility is decreasing, i.e., $v''(\cdot) < 0$, then that opens up a potential trade-off: although a concentration of memberships maximizes the total output of clubs, it comes at the expense of entirely excluding $n - mS/D$ individuals from memberships. If the utility function is sufficiently concave – the marginal utility is declining sufficiently rapidly (a condition that is formalized in inequality condition (7), then the welfare benefit from picking more members outweighs the loss to aggregate productivity. We present an example that brings out the difference between clubs-efficiency and utilitarian optimum when we move from a convex/linear to a concave utility function.

Example 3. Suppose $n = 16$, $D = 4$, $m = 8$ and $S = 4$. Figure 3a depicts a membership profile that is clubs-efficient and utilitarian optimum when $v(\cdot)$ is linear. Notice that in this membership profile, 8 individuals (i_1 to i_8) exhaust their membership availability while the other 8 individuals (i_9 to i_{16}) join no clubs. To appreciate the role of concave $v(\cdot)$ is concave, set

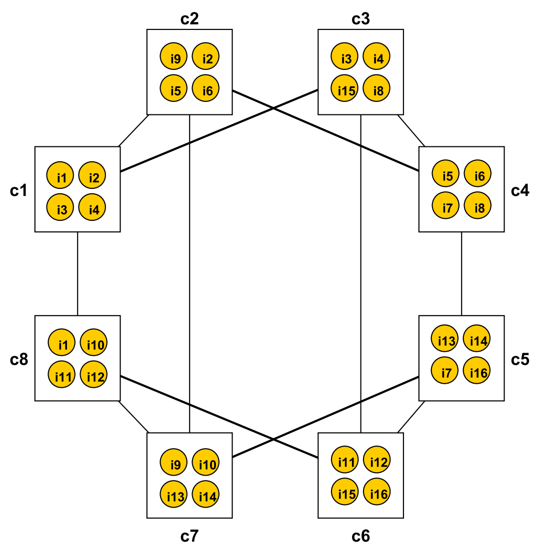
$$v(x) = \begin{cases} 10x & \text{when } x \leq 2f(4) + 8\alpha, \\ 10\,(2f(4) + 8\alpha) + 0.1\,(x - 2f(4) - 8\alpha) & \text{when } x > 2f(4) + 8\alpha. \end{cases}$$

In this case, the clubs-efficient outcomes remains unchanged and is as in Figure 3a, while the utilitarian optimal profile, which features all 16 individuals joining 2 clubs, is given in Figure 3b. □

when $v''(\cdot) \geq 0$ and when $v''(\cdot) < 0$ could be different. When $v''(\cdot) \geq 0$, there is one club that hosts some but less than S members. When $v''(\cdot) < 0$, the number of clubs that admit some but less than S members ranges from 1 to $S - 1$.



(a) clubs-efficient and utilitarian optimal profile: convex or linear $v(\cdot)$



(b) utilitarian optimal profile: highly concave $v(\cdot)$

Figure 3: Efficient membership profiles.

Let us next take up the case where $nD < mS$. On club efficiency, note that there are enough club capacity to cover the individuals, so every person will join D clubs: keeping anyone out of clubs is clearly dominated for clubs. Moreover, as spillovers are linear, there is a constant spillover irrespective of how the individuals are allocated across clubs. So the issue of how to allocate individuals turns on the f function. If f is convex, then it is better to allocate individuals to few clubs, i.e., nD/S clubs; if on the other hand, f is concave then you allocate as evenly as possible across clubs, subject to integer constraints.

Regarding utilitarian optimum profiles, no matter what the f function, the optimal profile entails marginalization of clubs. This is because to maximize the aggregate utility of individuals, it is clearly better to allocate more individuals to high-productivity clubs and fewer individuals to low-productivity clubs. This taken in tandem with the assumption that the productivity of a club rises with its size implies the marginalization of clubs.

When we compare Propositions 2 with 3, we see that there exists a tension between the incentives toward marginalization (created by the increasing club productivity from membership and from the strength of links with other clubs) and the demands of inclusiveness (created by the concave utility function and concave club production function).

We conclude our study of the benchmark model with a brief remark on stable and efficient membership profiles when spillovers across clubs are negative. This happens when $\alpha < 0$ in the benchmark model. Observe that when spillovers are negative, a club would like to only admit members who have no other memberships. So in a world with many individuals relative to club capacity, i.e., $n > mS$, any stable membership profile must involve exactly mS individuals filling the aggregate club capacity, i.e., every person joins at most one club and the resulting club network is an empty network. However, when the number of individuals is small the clubs face a trade-off: on the one hand, their productivity grows with membership (up to their capacity size). On the other hand, expanding membership may necessitate bringing in individuals who are already members of other clubs, and this lowers their productivity. We can apply methods developed above to show that whatever the outcome of the tradeoff is, a stable profile and an aggregate productivity maximizing profile both feature an egalitarian membership profile, i.e., there does not exist two individuals i and i' with $|d_i(\boldsymbol{a}) - d_{i'}(\boldsymbol{a})| > 1$.

The argument goes as follows: suppose there exist two individuals i and i' where $d_{i'}(a) \geq d_i(a) + 2$. If so, then there exists a club c which i' joins but not i. Clearly, this club c would want to expel i' and recruit i. We show that i is also willing to join c. By joining an additional club, the productivity of the clubs i currently joins drops by α. Nevertheless, the productivity of c, after i's joining must be greater than $d_i(a)\alpha$, as otherwise, c would not want to admit

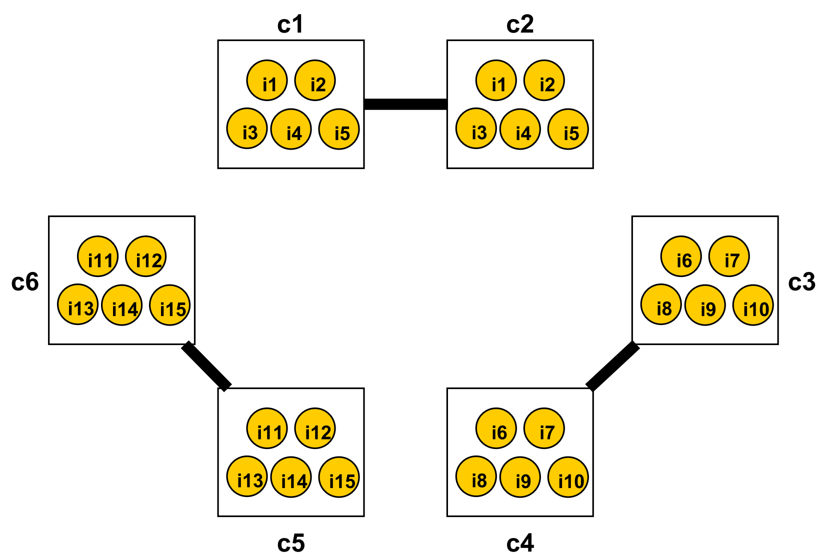
i. There is therefore a profitable deviation for the club-individual pair i and c. Turning to maximizing the aggregate productivity of clubs, note that the same deviation also improves the situation: it reduces the productivity of clubs i joins by α and raises the productivity of clubs i' joins by at least α. The result then follows given that i' is in more clubs than i does.

4 Small worlds, fragmented cliques, and strength of ties

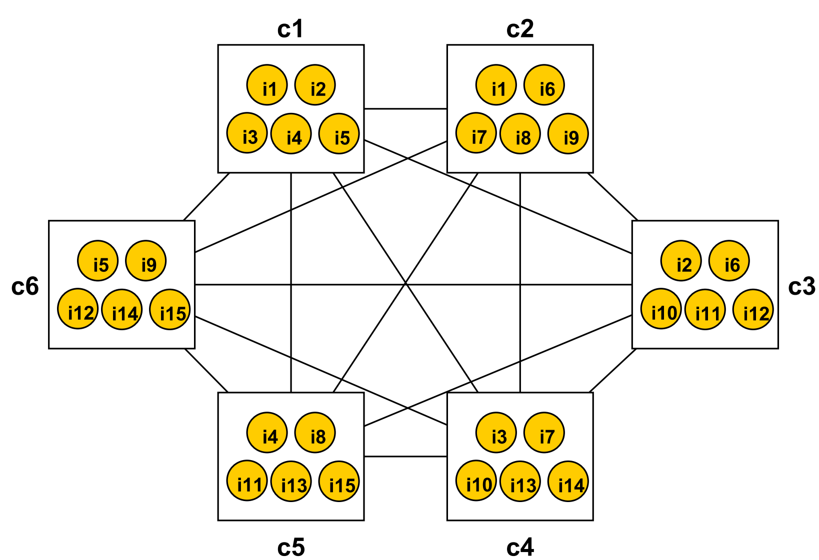
In this section, we examine the network of clubs and the strength of ties that support this network.[13] In our study of membership profiles so far, we have assumed that returns were linear in link strength. We start by showing that in this case, a variety of club networks are stable. In some prominent instances, however, the returns from link strength are likely to be non-linear. For example, in case club links are used for information sharing then we would expect marginal returns to decline with link strength. On the other hand, if links help members coordinate activities of the clubs, then the marginal returns may be increasing in link strength. With these observations in mind, we examine the implications of non-linear returns from link strength. We show that if the marginal return from link strength is increasing, then incentives of clubs and individuals push toward disconnected cliques of clubs with full strength links. If, on the other hand, the marginal return from link strength is decreasing then the club network entails larger components that are connected through weak links.

Example 4. Suppose that $n > 15$, $m = 6$, $D = 2$ and $S = 5$. Figure 4 depicts two clubs-efficiency and stable membership profiles when returns from links rise linearly. Note that the two profiles lead to the same degree distribution of individuals (the first 15 individuals all join two clubs while the others join no clubs) and the same aggregate link strength clubs have (each club shares five membership overlaps with other clubs). However, the resulting club networks take very different forms: one consists of three separate cliques where all links are of strength 5 while the other is a complete network where all links are of strength 1. This indicates that linear spillovers from links always lead to marginalization of individuals/clubs but the resulting club networks can be very different.

[13] A membership profile can be projected both into a network of clubs and a network of individuals. In this section, we focus on the club network. Nevertheless, the individual network, since originated from the same membership profiles as the club network, shares some important properties with the latter. For example, there exists a strong link (link with strength greater than 1) in the club network if and only if there exists a strong link in the individual network, and the club network is connected iff the individual network is connected. Therefore, we can infer properties of the individual network with analysis on the club network.



(a) Convex return from link strength: $h''(\cdot) > 0$



(b) Concave returns from link strength: $h''(\cdot) < 0$

Figure 4: Clubs-efficient and stable membership profiles.

When $h(\cdot)$ is convex, there is a unique clubs-efficient membership profile which is depicted in Figure 4a. We know that when $h(\cdot)$ is convex, the productivity of a club is maximized if the number of membership overlaps it has with other clubs is maximized and concentrated in as few clubs as possible. This can only be achieved when the club network takes the form depicted in Figure 4a.

On the other hand, when $h(\cdot)$ is concave, Figure 4b depicts the unique club-efficient network. When $h(\cdot)$ is concave, clubs want to maximize their membership overlaps with other clubs and spread them as evenly as possible. In this example, for this to be the case, the club network has to be complete with all links being weak.

Turning to stability, the structure depicted in Figure 4a is stable when $h(\cdot)$ is convex, since all clubs have reached the highest productivity possible and have no incentives to deviate. It is not stable when $h(\cdot)$ is concave: there is a profitable deviation for individual i_6 and club c_1 where c_1 exiles i_1 to admit i_6 and i_6 leaves c_3 to join c_1.

Similarly, the structure depicted in Figure 4b is stable when $h(\cdot)$ is concave but not so when $h(\cdot)$ is convex. Stability under a concave $h(\cdot)$ is obvious since all clubs have reached the highest productivity possible; instability under a convex $h(\cdot)$ can be verified by again considering the deviation by individual i_6 and club c_1 where c_1 exiles i_2 to admit i_6 and i_6 leaves c_3 to join c_1. □

The above example shows that the curvature of the returns from links has a significant influence on the structure of club networks. A convex $h(\cdot)$ function results in fragmented club networks with strong links while a concave $h(\cdot)$ function leads to connected club networks with weak links. Proposition 4 generalizes this example to cover a broader range of group size and capacity configurations. Formally, we say a club network $g = g(\boldsymbol{a})$ is clubs-efficient/utilitarian optimum/stable if it is created with a clubs-efficient/utilitarian optimum/stable membership profile $\boldsymbol{a}$. Additionally, we define a k-clique as a subnetwork that has k mutually linked clubs and a k-regular network as a network where all clubs have k links. The complete network is a special kind of regular network where all clubs are linked to each other ($k = m - 1$).

Proposition 4. *Assume $nD \geq mS$, $D = 2$, m is even, and $2 \leq S \leq m - 1$.*

- *When $h(\cdot)$ is convex, the clubs-efficient club network consists of m/D separate 2-cliques where all links are of strength S. This network is stable when $h(\cdot)$ is convex and unstable when $h(\cdot)$ is concave.*

- *When $h(\cdot)$ is concave, the clubs-efficient club network is an S-regular network (a complete* *network when $S = m - 1$) where all links are of strength 1. This network is stable when $h(\cdot)$ is concave and unstable when $h(\cdot)$ is convex.*

Assume $nD < mS$, $D = 2$, S divides n, and $2 \leq S \leq 2n/S - 1$.

- *When $h(\cdot)$ is convex, the utilitarian optimum club network consists of n/S separate 2-cliques where all links are of strength S. This network is stable when $h(\cdot)$ is convex and unstable when $h(\cdot)$ is concave.*
- *When $h(\cdot)$ is concave, the utilitarian optimum club network is an S-regular network (a complete network when $S = 2n/S - 1$) where all links are of strength 1. This network is stable when $h(\cdot)$ is concave and unstable when $h(\cdot)$ is convex.*

The proof is presented in the Online Appendix.

As mentioned earlier, the club network and the individual network generated by a membership profile share some important properties. The club network mentioned in Proposition 4 can be mapped into individual networks. When $h(\cdot)$ is convex, our characterization involves 2-cliques with strength S links for the club network; the corresponding individual network consists of S-cliques with strength 2 links. When $h(\cdot)$ is concave, our characterization features a S-regular club network with strength 1 links; the corresponding individual network is a $D(S-1)$-regular network with strength 1 links.

5 Variation on the model: alliances among competing firms

In Section 2 we showed that our model can be used to study the formation of research alliances of monopolies. We now turn to the study of research alliances between competing firms. The difference is that when firms compete in the same market they care not only about their own costs but also the costs of other firms, which is influenced by their club (research alliance) joining choices. In this section, we modify our model to address this complication and show that our arguments can be extended to this setting and that our main results are robust.

We assume that firms engage in Cournot competition and the demand of the market follows a linear inverse demand function $p = \beta - \sum_{i \in I} q_i$. The cost of firm i under membership profile

$\boldsymbol{a}$, as in the monopoly case, is $\gamma_0 - \gamma \sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a})$. Given a project profile $\boldsymbol{a}$, the Cournot equilibrium output can be written as

$$q_i(\boldsymbol{a}) = \frac{(\beta - \gamma_0) + \gamma n \sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}) - \gamma \sum_{i'\neq i}\sum_{c\in C} a_{i'c}\pi_c(\boldsymbol{a})}{n+1}, \tag{8}$$

and the Cournot profit of firm i is given by $q_i(\boldsymbol{a})^2$. In the language of our model, by treating a firm as an individual, the utility of individual i, given membership profile $\boldsymbol{a}$, is

$$u_i(\boldsymbol{a}) = v\left(\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}), \sum_{i'\neq i}\sum_{c\in C} a_{i'c}\pi_c(\boldsymbol{a})\right) = q_i(\boldsymbol{a})^2.$$

From (8), we can see that the objective of a firm is rising in the productivity of research alliances it joins and decreasing in the productivity of research alliances other firms join. Nevertheless, note that the term $a_{ic}\pi_c(\boldsymbol{a})$, which captures the aggregate productivity of clubs i joins, is multiplied by γn, while the term $\sum_{c\in C} a_{i'c}\pi_c(\boldsymbol{a})$, which is the aggregate productivity of clubs other firms join, is multiplied only by γ. So, the incentive of a firm to maximize the aggregate productivity of clubs it joins dominates other considerations. The structure of stable membership profiles in this extended setup is thus analogous to those characterized for the standard model.

To be more specific, let us first consider the case where the productivity of a research alliance rises linearly with the strength of links it has with other alliances ($h(x) = \alpha x$, where $\alpha \geq 0$). We partition firms (individuals) and research alliances (clubs) in a similar way as done in Section 3, so that $I_1(\boldsymbol{a})$ is the set of firms that join D alliances and have the highest profit among all firms, $I_2(\boldsymbol{a})$ is the set of firms that join D alliances but have lower profit than those in $I_1(\boldsymbol{a})$, $I_3(\boldsymbol{a})$ is the set of firms that join some but not D alliances, $I_4(\boldsymbol{a})$ is the set of firms that join no alliances, $C_1(\boldsymbol{a})$ is the set of alliances that reach the highest productivity possible, $C_2(\boldsymbol{a})$ is the set of alliances that have positive productivity that is lower than those achieved by alliances in $C_1(\boldsymbol{a})$, and $C_3(\boldsymbol{a})$ is the set of alliances with zero productivity. We show that we arrive at the same marginalization results as characterization in Proposition 2.

Proposition 5. *Assume that firms engage in Cournot competition and that $h(x) = \alpha x$, where $\alpha \geq 0$. When $\alpha = 0$, an egalitarian membership profile is stable. When $\alpha > 0$, for a stable $\boldsymbol{a}$,*

- *if $nD \geq mS$, then*

$$\frac{mS}{D} - \frac{S(D+3)}{2} \leq |I_1(\boldsymbol{a})| \leq |I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \leq \frac{mS}{D} \text{ and}$$
$$n - \frac{mS}{D} - S \leq |I_4(\boldsymbol{a})| \leq n - \frac{mS}{D}.$$

  Therefore, $\mathcal{M}_{\mathcal{I}} \leq \frac{D}{(D-1)m}$: every stable membership marginalizes firms for large m.

- *if $nD < mS$, then*

$$\frac{nD}{S} - D \leq |C_1(\boldsymbol{a})| \leq \frac{nD}{S} \text{ and}$$
$$m - \frac{nD}{S} - D \leq |C_3(\boldsymbol{a})| \leq m - \frac{nD}{S}.$$

  Therefore, $\mathcal{M}_{\mathcal{C}} \leq \frac{S}{(S-1)n}$: every stable membership profile marginalizes research alliances for large n.

The proof is presented in the Online Appendix.

Turning to the effect of having increasing/decreasing marginal returns from links, we show that, as in the standard model, increasing marginal returns leads to an R&D network that consists of strongly linked fragmented cliques while decreasing marginal returns leads to a weakly linked connected network.

Proposition 6. *Let $x = \min\{m, nD/S\}$ so that x equals m if $nD \geq mS$ and nD/S if $nD < mS$. Assume $D = 2$, x is even, and $2 \leq S \leq x - 1$.*

- *There exists a membership profile $\boldsymbol{a}$ where the resulting alliance network consists of separate 2-cliques with strength S links and the resulting firm network consists of separate S-cliques with strength 2 links. This membership profile is stable when $h(\cdot)$ is convex and unstable when $h(\cdot)$ is concave.*

- *There exists a membership profile $\boldsymbol{a}$ where the resulting alliance network is an S-regular network (a complete network when $S = x-1$) with strength 1 links and the resulting firm network is a $D(S-1)$-regular network with strength 1 links. This membership profile is stable when $h(\cdot)$ is concave and unstable when $h(\cdot)$ is convex.*

The proof is presented in the Online Appendix.

6 Case Studies

In this section we present four case studies that map our theory onto empirical context of defence alliances, inter-locking directorates, R&D alliances and editorial boards of directors.

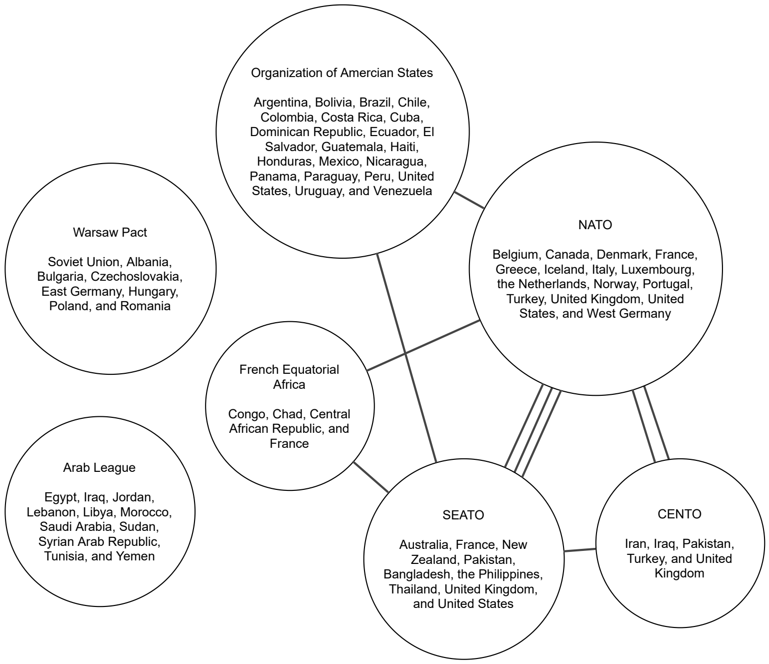
Defence Alliances: In terms of the terminology of our model, we may think of a defence alliance as a club and countries as individuals. This is a reasonable application for our model as members of the defence alliances decide on whether to let in new countries (and they may sometimes expel a member). A country can also choose to leave an alliance or to ask to join a new alliance. Links between alliances generate negative spillovers: a defence alliance will be very reluctant to admit a member from a competing defence alliance, as admitting such a member may compromise the security of the entire alliance. The empirical evidence presented here is taken from Jackson and Nei (2015).

Figures 5a and 5b present the club network of strategic and defence alliances in the years 1960 and 2000. Observe that in 1960, the network of alliances exhibits clear fragmentation along lines of geography and ideology. Indeed, there are few common members across the clubs (other than the USA and Canada). Turning to 2000, the major change that happened is the dissolution of the Warsaw Pact, but the club network remains fragmented. This fragmentation of the network matches with the prediction of our model with negative spillovers across clubs.

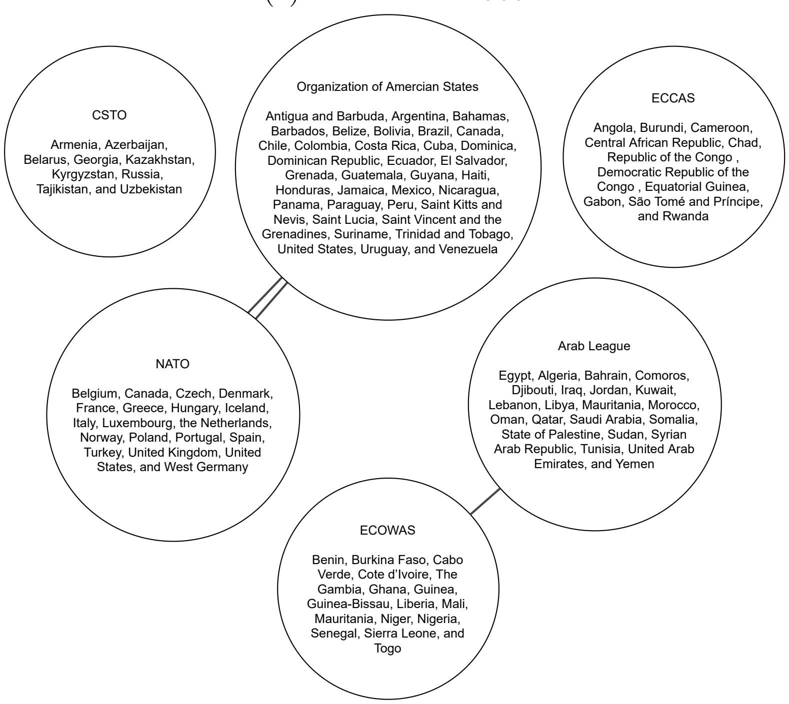
Interlocking Directorates: It is widely recognized that the board-to-board ties serve as a mechanism for the diffusion of corporate practices, strategies, and structures (Mizruchi (1996)). We may consider boards as clubs and directors as individuals; links between clubs raise productivity. In what follows, we discuss empirical studies on interlocking directorates and explain how our model sheds light on the understandings of the empirical findings.

Consider first the degree distribution of board directors. Conyon and Muldoon (2006) study the affiliations of board directors who hold positions in 1,733 firms in the United States in 2003. They find that 80.37% of the directors sit only on one board, 13.02% of them sit on two boards, and the remaining 6.61% of the directors sit on 8.6 boards on average. Thus most directors hold only one or two positions, but there are a small fraction of directors who occupy many positions. The authors show that similar patterns hold in Germany and the UK. This inequality in degrees of directors is in line with the marginalization result (Proposition 2)

Consider next the structure of board networks. Mizruchi (1982) provides a historical analysis of the US board network among 167 firms at seven points from 1904 to 1974, finding that



(a) Alliances in 1960



(b) Alliances in 2000

Figure 5: Club network of defence alliances.

almost all nodes were within distance 4. More recently, with the increased availability in data and advancement in analyzing techniques, Davis et al. (2003) study the largest manufacturing and service firms in the US over the period 1982 to 1999. They show that despite the major changes in the nature of economic activities, the structure of the board network remained relatively unchanged: the average geodesic distance between boards was 3.38, 3.46, and 3.46 in 1982, 1991 and 2001.

Turning finally to the strength of ties among boards: Battiston and Catanzaro (2004) investigate the board networks of the Fortune 1000 firms in 1999 and show that they consist mostly of weak links (the number of strength 1 links is about 10 times that the number of stronger links) and that they have a small world feature (the largest connected component includes 87% of all firms). Given that links between boards serve as information diffusion channels, the marginal returns from board-to-board ties are likely to be decreasing. Proposition 4 shows that in this case the club network is likely to be held together by weak links. The empirical patterns are consistent with our theoretical analysis.

Heath Care Organizations: Willems and Jegers (2011) study the interlocking boards of 92 Belgian health-care organizations. One of their main findings is that the board network is fragmented with strong links: the 92 organizations are divided into 23 components; 24 pairs of organizations share exactly the same set of board members and the heaviest link in the network is of strength 10.

Woo (2017) and Hansson et al. (2018) suggest that health care organizations often need to collaborate with each other to treat multi-diseased and vulnerable patients. To achieve smooth coordination, it is more efficient for organizations to have multiple shared directors with their partners. In the language of our model, this suggests that marginal returns from links are increasing in overlaps. In this case, the theory predicts that the resulting board network is fragmented with strong links. This is consistent with the empirical finding of Willems and Jegers (2011).

R&D Alliances: Research agreements among firms that involve technology development and sharing have become increasingly common since the 1980's (Hagedoorn (2002)). These arrangements have profound effects on the firms and on the functioning of the markets they operate in (Gulati (2007)). Here we very briefly summarize some of the empirical patterns on collaboration.

First, there appears to exist a great disparity in the participation of alliances among firms. Powell et al. (2005) and König et al. (2019) show that the distribution of the number of alliances a firm joins has a power-law degree distribution. In a similar vein, Rosenkopf and Schilling (2007) and Kitsak et al. (2010) show that R&D networks have a core-periphery structure. This unequal degree distribution is consistent with our marginalization characterization.

Next, let us consider the network of clubs (where a club is an alliance). A number of studies show that across most industries the network is connected (Owen-Smith and Powell, 2004; Roijakkers and Hagedoorn, 2006; Hanaki et al., 2010; Schilling and Phelps, 2007). Our view is that there are positive spillovers from shared memberships but as these are generally information sharing arrangements the marginal returns from strength of ties is falling. Proposition 4 suggests that in this case the stable network will exhibit high connectivity. This is consistent with the empirical evidence.

The empirical research also shows that the network of alliances is fragmented in the case of chemical and petroleum refining. Our view is that this may be due to the relative importance of information and competitive elements involved. If strategic and competitive elements are dominant then our analysis of negative spillovers across clubs may be more relevant. Our discussion of that setting suggests that the network will be fragmented. This may help explain the pattern in these industries.

Boards of Editors of Journals: The editorial board of a journal along with its set of referees shapes the research papers that are published in it. The collection of prestigious journals in a discipline taken together therefore can have a profound influence on the directions of research in that discipline. In economics, there has existed a concern for some time now that the leading journals are dominated by members from a few economics departments based in the United States. This concentration of editors has some to suggest that the discipline may be a risk of becoming too conformist and losing its innovativeness. This question has become more pressing over the last few decades as the profession has grown greatly and there has been a massive increase in the number of journals: this has resulted in a massive increase in the relative prestige of publishing in a few core journals. A leading economist has termed this phenomenon 'Top5ites' (see Serrano (2018) and **?**) and in a recent paper the emphasis on top few journals in the career prospects of economists has been referred to as the 'Curse of the top-5' (Heckman and Moktan (2020)). We may view authors as individuals and boards of journals as clubs. In this case study we draw on a recent paper by Ductor and Visser (2021) to document some facts about editorships and the relationship between the boards of leading journals and then relate them to our theoretical predictions.

Ductor and Visser (2021) study a set of 106 leading economics and finance journals over the period 1990-2011. They find that there were 79533 authors publishing in these journals but that only 6069 became editors, i.e. only 7.63%. Moreover, within the set of editors, over 75% were editors of just one journal but over 1.6% of these editors were editors at 4 or more journals. We recognize that the model assumes individuals are ex-ante homogenous while economics authors clearly differ in their abilities and productivity and their suitability for editorial roles. But, at a high level, these two facts are broadly consistent with the model's prediction on the marginalization of individuals (that can arise even if all individuals are similar).

Turning next to the links between the boards of different journals, for concreteness let us discuss the empirical situation in 2010. The network contains the 106 journals as nodes; a link between two journals reflects common editors. An inspection of this network reveals a number of interesting facts. The largest component contains 101 nodes, suggesting that it is more or less connected. The network is sparse with roughly 11% of all possible links being present. These links have uneven strength but the vast majority of the links are weak – over 82% have only one or two common editors. These facts suggest that the network is a small world that is held together with mostly weakly ties.

To illustrate these patterns, we present the network of editorial boards of leading economics journals from the year 2010 in Figure 6. The network covers 28 leading economics journals.[14] We see that the network is connected and that most of the links are relatively weak. Interestingly the network is held together through a hierarchical structure – the general interest journals share common editors with field journals; there are relatively few ties among the general interest journals and the field journals, respectively.

[14] These journals are Journal of Health Economics (JHE), Review of Economics and Statistics (REStat), Review of Economic Studies (REStud), Econometric Theory (ET), Journal of Monetary Economics (JME), Quarterly Journal of Economics (QJE), Journal of Economic Literature (JEL), Journal of Business and Economic Statistics (JBES), Econometrica (ECMA), Review of Financial Studies (RFS), RAND Journal of Economics (RAND), Economic Journal (EJ), Journal of Environmental Economics and Management (JEEM), Journal of Finance (JoF), Journal of Econometrics (JoE), Journal of International Economics (JIE), European Economic Review (EER), World Bank Economic Review (WBER), International Economic Review (IER), American Economic Review (AER), Journal of Human Resources (JHR), Journal of Labor Economics (JLE), Journal of Political Economy (JPolE) Journal of Public Economics (JPubE), Games and Economic Behavior (GEB), Journal of Economic Theory (JET), Journal of Economic Perspectives (JEP), Journal of Financial Economics (JFE).

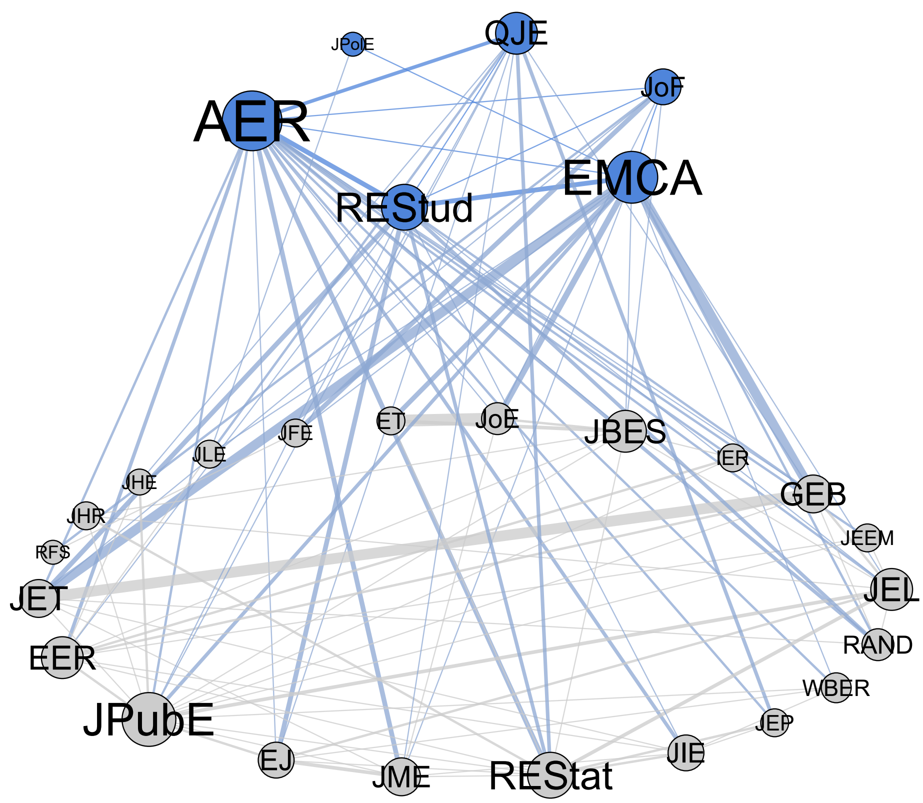


Figure 6: The editorial boards of economic journals 2010. Node size reflects number of editors; link thickness indicates number of common editors. Courtesy of Lorenzo Ductor and Bauke Visser

References

Baker, W., Gerald F. Davis, and Mina Yoo, "The Small World of the American Corporate Elite, 1991-2001," *Strategic Organization*, 2001, *1.*

Bala, Venkatesh and Sanjeev Goyal, "A noncooperative model of network formation," *Econometrica*, 2000, *68* (5), 1181–1229.

Bando, Keisuke and Toshiyuki Hirai, "Stability and Venture Structures in Multilateral Matching," *Journal of Economic Theory*, 2021, p. 105292.

Battiston, Stefano and Michele Catanzaro, "Statistical Properties of Corporate Board and Director Networks," *The European Physical Journal B*, 2004, *38* (2), 345–352.

Bloch, F. and B. Dutta, "Formation of Networks and Coalitions," in J. Benhabib A. Bisin and M. Jackson, eds., *Handbook of Social Economics*, North Holland. Amsterdam, 2012.

Bloch, Francis, Santiago Sánchez-Pagés, and Raphael Soubeyran, "When does universal peace prevail? Secession and group formation in conflict," *Economics of Governance*, 2006, *7* (1), 3–29.

Bottomore, Tom, *Elites and Society*, London: Routledge, 1993.

Bramoullé, Yann, Andrea Galeotti, and Brian Rogers, *The Oxford handbook of the economics of networks*, Oxford University Press, 2016.

Brandeis, Louis D, "Interlocking directorates," *The ANNALS of the American Academy of Political and Social Science*, 1915, *57* (1), 45–49.

_ , *Other people's money and how the bankers use it*, Cosimo, Inc., 2009.

Buchanan, James M., "An Economic Theory of Clubs," *Economica*, 1965, *32*, 1–14.

Conyon, Martin J and Mark R Muldoon, "The Small World of Corporate Boards," *Journal of Business Finance & Accounting*, 2006, *33* (9-10), 1321–1343.

Cornes, R., *The Theory of Externalities, Public Goods, and Club Goods*, Cambridge. UK: Cambridge University Press, 1996.

Dahl, Robert A., "A Critique of the Ruling Elite Model," *The American Political Science Review*, 1958, *52* (2), 463–469.

Davis, Gerald F, Mina Yoo, and Wayne E Baker, "The Small World of the American Corporate Elite, 1982-2001," *Strategic organization*, 2003, *1* (3), 301–326.

Demange, G. and M. Wooders, *Group Formation in Economics: Networks, Clubs, and Coalitions*, Cambridge. UK.: Cambridge University Press, 2005.

Ductor, Lorenzo and Bauke Visser, "Concentration of Power at the Editorial Boards of Economics Journals," *Working Paper*, 2021.

Durlauf, Steven N and H Peyton Young, *Social dynamics*, Vol. 4, Mit Press, 2004.

Echenique, Federico and Jorge Oviedo, "A Theory of Stability in Many-to-many Matching Markets," *Theoretical Economics*, 2006, *1* (2), 233–273.

Fershtman, Chaim and Dotan Persitz, "Social Clubs and Social Networks," *American Economic Journal: Microeconomics*, 2021, *13* (1), 224–51.

Goyal, Sanjeev, *Principles of Networks*, MIT Press: forthcoming, 2022.

Gulati, Ranjay., *Managing Network Resources: Alliances, Affiliations, and Other Relational Assets*, Oxford University Press, 2007.

Hagedoorn, John, "Inter-firm R&D Partnerships: an Overview of Major Trends and Patterns Since 1960," *Research Policy*, 2002, *31* (4), 477–492.

Hanaki, Nobuyuki, Ryo Nakajima, and Yoshiaki Ogura, "The Dynamics of R&D Network in the IT Industry," *Research Policy*, 2010, *39* (3), 386–399.

Hansson, Anders, Ann Svensson, Britt Hedman Ahlström, Lena G Larsson, Berit Forsman, and Pia Alsén, "Flawed Communications: Health Professionals' Experience of Collaboration in the Care of Frail Elderly Patients," *Scandinavian Journal of Public Health*, 2018, *46* (7), 680–689.

Hatfield, John William and Scott Duke Kominers, "Multilateral Matching," *Journal of Economic Theory*, 2015, *156*, 175–206.

Heckman, James J and Sidharth Moktan, "Publishing and promotion in economics: the tyranny of the top five," *Journal of Economic Literature*, 2020, *58* (2), 419–70.

Jackson, Matthew O and Stephen Nei, "Networks of Military Alliances, Wars, and International Trade," *Proceedings of the National Academy of Sciences*, 2015, *112* (50), 15277–15284.

Jackson, M.O. and A. Wolinsky, "A Stratgegic Model of Social and Economic Networks," *Journal of Economic Theory*, 1996, *71* (1), 44–74.

Kitsak, Maksim, Massimo Riccaboni, Shlomo Havlin, Fabio Pammolli, and H Eugene Stanley, "Scale-free Models for the Structure of Business Firm Networks," *Physical Review E*, 2010, *81* (3), 036117.

Klaus, Bettina and Markus Walzl, "Stable Many-to-many Matchings with Contracts," *Journal of Mathematical Economics*, 2009, *45* (7-8), 422–434.

Kogut, Bruce, *The Small Worlds of Corporate Governance*, Cambridge, Massachusetts: MIT Press, 2012.

König, Michael D, Xiaodong Liu, and Yves Zenou, "R&D networks: Theory, Empirics, and Policy Implications," *Review of Economics and Statistics*, 2019, *101* (3), 476–491.

Levine, Joel H, "The Sphere of Influence," in "Social Networks," Elsevier, 1977, pp. 433–446.

Mills, C. Wright, *The Power Elite*, Oxford: Oxford University Press, 1956.

Mizruchi, Mark S, "What do Interlocks Do? An Analysis, Critique, and Assessment of Research on Interlocking Directorates," *Annual Review of Sociology*, 1996, *22* (1), 271–298.

Mizruchi, M.S., *The American Corporate Network: 1904-1974*, Beverly Hills, CA: Sage Publications, 1982.

Owen-Smith, Jason and Walter W Powell, "Knowledge Networks as Channels and Conduits: The Effects of Spillovers in the Boston Biotechnology Community," *Organization Science*, 2004, *15* (1), 5–21.

Page, Frank H and Myrna Wooders, "Club networks with multiple memberships and noncooperative stability," *Games and Economic Behavior*, 2010, *70* (1), 12–20.

Pakulski, Jan, "Classical Elite Theory: Pareto and Weber," in "The Palgrave Handbook of Political Elites," Springer, 2018, pp. 17–24.

Powell, Walter W, Douglas R White, Kenneth W Koput, and Jason Owen-Smith, "Network Dynamics and Field Evolution: The Growth of Interorganizational Collaboration in the Life sSiences," *American Journal of Sociology*, 2005, *110* (4), 1132–1205.

Roijakkers, Nadine and John Hagedoorn, "Inter-firm R&D Partnering in Pharmaceutical Biotechnology since 1975: Trends, Patterns, and Networks," *Research Policy*, 2006, *35* (3), 431–446.

Rosenkopf, Lori and Melissa A Schilling, "Comparing Alliance Network Structure across industries: Observations and Explanations," *Strategic Entrepreneurship Journal*, 2007, *1* (3-4), 191–209.

Rostek, Marzena J and Nathan Yoder, "Matching with Multilateral Contracts," *Available at SSRN 2997223*, 2019.

Roth, Alvin E and Marilda Sotomayor, "Two-sided matching," *Handbook of game theory with economic applications*, 1992, *1*, 485–541.

Schilling, Melissa A and Corey C Phelps, "Interfirm Collaboration Networks: The Impact of Large-scale Network Structure on Firm Innovation," *Management Science*, 2007, *53* (7), 1113–1126.

Serrano, Roberto, "Top5itis," Technical Report, Working Paper 2018.

Useem, M., *The Inner Circle*, New York: Oxford University Press, 1984.

Willems, Jurgen and Marc Jegers, "Analyzing Board-Director Networks as Small Worlds: Theory and Applications," in "Macroeconomics and beyond, Liber Amicorum Wim Meeusen," Garant, 2011, pp. 260–279.

Woo, Jean, "Designing Fit for Purpose Health and Social Services for Ageing Populations," *International Journal of Environmental Research and Public Health*, 2017, *14* (5), 457.

7 Appendix: For Online Publication

Proofs

Proof of Proposition 1

We first take up the characterization – the sufficient and necessary conditions – for stability. We then prove existence.

From the production function of clubs and the utility function of individuals, we know that there cannot be any $i \in I, c \in C$ and $\boldsymbol{a'} \in A$ with $a'_i \leq a_i$ and $a'_{-i} = a_{-i}$ such that $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$, or $a'_c \leq a_c$ and $a'_{-c} = a_{-c}$ such that $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$. Hence, the deviations we need to consider are joint deviation by i and c such that both of them are better off. Such deviation can be divided into four types: individual i joins club c and nothing else is changed; individual i quits some clubs and joins club c; club c dismisses some members and admits individual i; and individual i quits some clubs, club c dismisses some members, and i joins c. Notice that for the last three kinds of deviations, if quitting two or more clubs and dropping two or more members is profitable, then quitting only one club and dropping only one member is also profitable given our utility and productivity specification. So, we only consider deviations with one quitting and (or) one dropping. We show that conditions (i)–(iii) are necessary and sufficient for the four kinds of deviations not to be jointly profitable.

For the necessity of condition (i), suppose it does not hold and there exists an individual $i \in I$ with $d_i(\boldsymbol{a}) < D$ and a club $c \in C$ with $s_c(\boldsymbol{a}) < S$, such that i is not a member of c. But then i joining c is strictly improving for both parties, which contradicts stability of $\boldsymbol{a}$.

We also show that if condition (i) holds, then there is no jointly profitable deviation for i and c where i joins c and nothing else changes since such deviation is not feasible.

For the necessity of condition (ii), suppose, to the contrary, that there exists a club c with $s_c(\boldsymbol{a}) < S$, an individual $i \in I$ who is not a member of c, and a club $c' \in C$ that i joins, such that

$$\pi_c(\boldsymbol{a}) > \pi_{c'}(\boldsymbol{a}) - f(s_c(\boldsymbol{a}) + 1) + f(s_c(\boldsymbol{a})) - \alpha(D - 1). \quad (9)$$

Notice that, by condition (i), $d_i(\boldsymbol{a}) = D$. Let $\boldsymbol{a'}$ be a membership profile obtained from $\boldsymbol{a}$ by i leaving c' and joining c and c accepting i. First, it is obvious that $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$. The difference in productivity of c between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\pi_c(\boldsymbol{a'}) - \pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a}) + 1) - f(s_c(\boldsymbol{a})) + \alpha(D - 1), \quad (10)$$

so the difference in utility of i between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\begin{aligned} u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a}) =& v(\pi_c(\boldsymbol{a}) + f(s_c(\boldsymbol{a})+1) - f(s_c(\boldsymbol{a})) + \alpha(D-1) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) \\ & - v(\pi_{c'}(\boldsymbol{a}) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})), \end{aligned} \tag{11}$$

which has the same sign as

$$\pi_c(\boldsymbol{a}) - \pi_{c'}(\boldsymbol{a}) + f(s_c(\boldsymbol{a})+1) - f(s_c(\boldsymbol{a})) + \alpha(D-1), \tag{12}$$

which is positive since v is increasing. The deviation by individual i and club c from $\boldsymbol{a}$ to $\boldsymbol{a'}$ makes them both better off. A contradiction with stability of $\boldsymbol{a}$.

We also show that if conditions (i) and (ii) hold, then there is no jointly profitable deviation for i and c where i quits a club to join c and nothing else changes. If there is such a deviation, it must be that $s_c(\boldsymbol{a}) < S$. Since i is not a member of c so, by condition (i), $d_i(\boldsymbol{a}) = D$. Let c' be the club that i leaves when joining c. Then, by (11) and (12) and condition (ii), utility of i does not increase and so the deviation is not profitable to i.

For the necessity of condition (iii), suppose, to the contrary, that there exist individuals $i \in I$ and $i' \in I$ such $D > d_i(\boldsymbol{a}) \geq d_{i'}(\boldsymbol{a})$ and a club $c \in C$ such that i' is a member of and i is not. Let $\boldsymbol{a'}$ be a membership profile obtained from $\boldsymbol{a}$ by i joining c and c accepting i and dropping i'. The difference in productivity of c between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\pi_c(\boldsymbol{a'}) - \pi_c(\boldsymbol{a}) = \alpha(d_i(\boldsymbol{a}) - d_{i'}(\boldsymbol{a}) + 1), \tag{13}$$

which is positive if and only if $\alpha > 0$. Also, since v is increasing, both individual i and club c and strictly benefit deviating from $\boldsymbol{a}$ to $\boldsymbol{a'}$ when $\alpha > 0$. A contradiction with stability of $\boldsymbol{a}$.

We also show that if conditions (i) and (iii) hold, then there is no jointly profitable deviation for i and c where c drops a member to admit i and nothing else changes. If there is such a deviation, it must be that $d_i(\boldsymbol{a}) < D$. Then from condition (i), it mush be $s_c(\boldsymbol{a}) = S$. Let i' be the individual that club c drops. Then, by (13) and condition (iii), productivity of club c does not increase and so the deviation is not profitable to c.

For the necessity of condition (iv), suppose that $\alpha > 0$ and suppose, to the contrary, that there exists two individuals $i, i' \in I$ with $d_i(\boldsymbol{a}) = D$ and $d_{i'}(\boldsymbol{a}) < D$, a club $c \in C$ that has member i' but not i, and a club c' that i joins, such that

$$\pi_c(\boldsymbol{a}) > \pi_{c'}(\boldsymbol{a}) - \alpha\left(D - d_{i'}(\boldsymbol{a}) - \sum_{c''\neq c'} a_{ic''}a_{i'c''}\right). \tag{14}$$

Let $\boldsymbol{a'}$ be a membership profile obtained from $\boldsymbol{a}$ by i joining c and leaving c', and c accepting i and dropping i'. The difference in productivity of c between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\pi_c(\boldsymbol{a'}) - \pi_c(\boldsymbol{a}) = \alpha(D - d_{i'}(\boldsymbol{a})), \tag{15}$$

which is positive if and only if $\alpha > 0$. The difference in utility of i between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\begin{aligned} u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a}) =& v(\pi_c(\boldsymbol{a'}) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) - v(\pi_{c'}(\boldsymbol{a}) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) \\ =& v(\pi_c(\boldsymbol{a}) + \alpha\,(D - d_{i'}(\boldsymbol{a})) - \alpha \sum_{c''\neq c'} a_{ic''}a_{i'c''} + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) \\ & - v(\pi_{c'}(\boldsymbol{a}) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})), \end{aligned} \tag{16}$$

which has the same sign as

$$\pi_c(\boldsymbol{a}) - \pi_{c'}(\boldsymbol{a}) + \alpha\left(D - d_{i'}(\boldsymbol{a}) - \sum_{c''\neq c'} a_{ic''}a_{i'c''}\right), \tag{17}$$

which is positive since v is increasing. The deviation by individual i and club c from $\boldsymbol{a}$ to $\boldsymbol{a'}$ makes them both better off. A contradiction with stability of $\boldsymbol{a}$.

We also show that if conditions (i)–(iv) hold, then there is no jointly profitable deviation for i and c where i leaves a club, c drops a member, and i joins c. Suppose there is such a deviation, if $d_i(\boldsymbol{a}) < D$ or $s_c(\boldsymbol{a}) < S$, since the deviation is profitable with i quitting a club and c dismissing a member, it is also profitable if i does not quit the club and c does not dismiss the member. We know conditions (i)–(iii) guarantee that there is no such mutually beneficial deviation. So, here we consider the deviations of i and c when $d_i(\boldsymbol{a}) = D$ and $s_c(\boldsymbol{a}) = S$. In this case, by (16) and (17) and condition (iv), utility of i does not increase and so the deviation is not profitable to i.

We finally turn to the existence of stable membership profile. We provide a proof by construction.

Suppose $nD \geq mS$. Let $m' \leq m$ and $n' \leq n$ be the largest integers such that $m'S = n'D$. Notice that since $m \geq D$ and $n \geq S$ so $m' \geq D$ and $n' \geq S$. Construct a membership profile $\boldsymbol{a}$ as follows. First, select n' individuals and m' clubs, let all selected individuals join D clubs we select so that all m' clubs have S members. This profile can be obtained by letting clubs admit individuals in sequence: make each club admit S individuals that have the smallest degree in its turn before moving to the next club. If $n - n' \geq S$, take S out of $n - n'$ remaining individuals and put each of them in each of $m - m'$ remaining clubs. Otherwise, put each of $n - n'$ remaining individuals in each of $m - m'$ clubs. It is easy to verify that this profile is stable.

Suppose $mS > nD$: consider a membership profile $\boldsymbol{a}$ where all individuals join D clubs, and $\lfloor \frac{nD}{S} \rfloor$ clubs have S members, one club has $(nD) \bmod S$ members, and the remaining clubs have 0 members. This profile can be obtained by letting clubs admit individuals in sequence. Make each club admit S individuals that has the smallest degree in its turn before moving to the next club. Stop when all individuals have degree D. This profile is always stable as it satisfies all four conditions in Proposition 1. Condition (i) is satisfied obviously. Conditions (iv) and (iii) are automatically satisfied as no individual joins less than D clubs. For condition (ii), if a club c has less than S members, then either it is the one club with $(nD) \bmod S$ members or it has 0 members. In both cases, for an individual i that is not in c and for any club that i is in, c' must have more members than c does and all members of c' join D clubs, making condition (ii) satisfied. ■

Proof of Proposition 2

We first take up the egalitarian outcome result in the absence of network effects. When $\alpha = 0$, the membership profile generated with the following algorithm is stable. Let clubs admit individuals sequentially. Fill a club with S individuals that currently have the lowest degrees and then move to the next club. Stop until all clubs are full or all individuals have joined D clubs. Since $n \geq S$, this algorithm is feasible. If the algorithm terminates when all clubs have S members, then all clubs have productivity $f(S)$ which is the highest productivity a club can get. Hence the membership profile is stable. If the algorithm terminates when all individuals are in D clubs, then there are $\lfloor \frac{mS}{D} \rfloor$ clubs that have productivity $f(S)$, one club that has productivity $f((mS) \mod D)$, and the rest clubs have productivity 0. The only possible profitable deviation from one individual is to quit the club with productivity $f((mS)$ mod $D)$ and join a club with productivity $f(S)$, but no club with productivity $f(S)$ want to deviate. Hence the membership profile is stable. Given the way we construct the membership profile, we have $|d_i(\boldsymbol{a}) - d_{i'}(\boldsymbol{a})| \leq 1$ for all $i, i' \in I$.

When $\alpha > 0$, we develop the conditions for the sizes of the different groups. For the cardinality of $I_3(\boldsymbol{a})$, take any individual $i \in I_3(\boldsymbol{a})$ with minimal $d_i(\boldsymbol{a})$ and let $c \in C$ be any club that i members. By condition (iii) of Proposition 1, all individuals in $I_3(\boldsymbol{a})$ are members of c and, by condition (i) of Proposition 1, $s_c(\boldsymbol{a}) \leq S$. Hence $|I_3(\boldsymbol{a})| \leq S$.

For cardinality of $C_2(\boldsymbol{a})$, suppose that $C_2(\boldsymbol{a}) \neq \varnothing$, we will show that there exists an individual i that is a member of all clubs in $C_2(\boldsymbol{a})$. We consider the cases of $I_3(\boldsymbol{a}) = \varnothing$ and $I_3(\boldsymbol{a}) \neq \varnothing$ separately. If $I_3(\boldsymbol{a}) = \varnothing$, then members of the clubs in $C_2(\boldsymbol{a})$ are of degree D and, for any $c \in C_2(\boldsymbol{a})$, $s_c(\boldsymbol{a}) < S$ (as c does not achieve maximal productivity). Take any $c' \in C_2(\boldsymbol{a})$ with minimal productivity, $\pi_{c'}(\boldsymbol{a})$, and any member i of c'. Take any $c \in C_2(\boldsymbol{a}) \setminus \{c'\}$. Since $s_c(\boldsymbol{a}) < S$ and since $\pi_c(\boldsymbol{a}) \geq \pi_{c'}(\boldsymbol{a})$ so, by condition (ii) of Proposition 1, i is a member of c. Hence i is a member of all clubs in $C_2(\boldsymbol{a})$. If $I_3(\boldsymbol{a}) \neq \varnothing$ then take any $i \in I_3(\boldsymbol{a})$ with maximal degree. Take any club $c \in C_2(\boldsymbol{a})$. Since c does not achieve the highest productivity so either $s_c(\boldsymbol{a}) < S$ or c has a member in $I_3(\boldsymbol{a})$. In the first case, i is a member of c by condition (i) of Proposition 1. In the second case, i is a member of c by condition (iii) of Proposition 1. Hence i is a member of all clubs in $C_2(\boldsymbol{a})$. By condition (i) of Proposition 1, $d_i(\boldsymbol{a}) \leq D$. Hence $|C_2(\boldsymbol{a})| \leq D$.

For cardinality of $I_2(\boldsymbol{a})$, notice first that, by definition, every individual in $I_2(\boldsymbol{a})$ members at least one club in $C_2(\boldsymbol{a})$. Thus the aggregate membership of individuals in $I_2(\boldsymbol{a})$ in the clubs in $C_2(\boldsymbol{a})$ is at least $x + 2(|I_2(\boldsymbol{a})| - x)$, where x is the number of individuals from $I_2(\boldsymbol{a})$ who member exactly one club from $C_2(\boldsymbol{a})$. On the other hand, since $|C_2(\boldsymbol{a})| \leq D$ and, for all $c \in C_2(\boldsymbol{a})$, either $s_c(\boldsymbol{a}) \leq S - 1$ or c has a member in $I_3(\boldsymbol{a})$, so aggregate club capacity of the clubs in $C_2(\boldsymbol{a})$ for individuals in $I_2(\boldsymbol{a})$ is at most $(S-1)D$. Hence

$$x + 2(|I_2(\boldsymbol{a})| - x) = 2|I_2(\boldsymbol{a})| - x \leq (S-1)D. \tag{18}$$

The number of individuals in $I_2(\boldsymbol{a})$ who member exactly one club in $C_2(\boldsymbol{a})$ is at most $S-1$. To see that, suppose that an individual $i \in I_2(\boldsymbol{a})$ members exactly one club $c' \in C_2(\boldsymbol{a})$. Let $c \in C_2(\boldsymbol{a}) \setminus \{c'\}$ be another club in $C_2(\boldsymbol{a})$. Since i is not a member of c so, by condition (ii) of Proposition 1, $\pi_c(\boldsymbol{a}) < \pi_{c'}(\boldsymbol{a})$. Hence c' must achieve the highest productivity of all clubs in $C_2(\boldsymbol{a})$ and must be unique such. Since all individuals in $C_2(\boldsymbol{a})$ who member exactly one club in $C_2(\boldsymbol{a})$ must be members of the same club from $C_2(\boldsymbol{a})$ and since, as we observed above, this club can host at most $S-1$ members from $I_2(\boldsymbol{a})$, so there can be at most $S-1$ such individuals. This shows that $x \leq S-1$ and from (18) it follows that $|I_2(\boldsymbol{a})| \leq (S-1)(D+1)/2$.

We now use these derivations on the size of the different groups to derive bounds on the size of $I_1(\boldsymbol{a})$ and $I_2(\boldsymbol{a})$ and $I_4(\boldsymbol{a})$.

We begin with the case $nD \geq mS$: Suppose that in a stable membership profile $\boldsymbol{a}$, all clubs are full, then we have $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|D + \sum_{i \in I_3(\boldsymbol{a})} d_i(\boldsymbol{a}) = mS$, and hence $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|D + |I_3(\boldsymbol{a})|D > mS$. Since $|I_3(\boldsymbol{a})| \leq S$,

$$|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| > \frac{mS}{D} - S.$$

Suppose that in a stable membership profile $\boldsymbol{a}$, not all clubs are full, then we know $|I_4(\boldsymbol{a})| = 0$ as otherwise there is a jointly profitable deviation for an individual in $I_4(\boldsymbol{a})$ and a club that is not full where the individual joins the club. Therefore,

$$|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| = n \geq \frac{mS}{D},$$

and so $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \geq \frac{mS}{D} - S$ given $|I_3(\boldsymbol{a})| \leq S$.

Now, since $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \geq \frac{mS}{D} - S$ and $|I_2(\boldsymbol{a})| \leq \frac{(S-1)(D+1)}{2}$, we have $|I_1(\boldsymbol{a})| \geq \frac{mS}{D} - \frac{S(D+3)}{2}$. For the upper bound of $|I_1(\boldsymbol{a})|$, since aggregate club capacity is mS, we must have $|I_1(\boldsymbol{a})|D \leq mS$, and so $|I_1(\boldsymbol{a})| \leq \frac{mS}{D}$.

Regarding the bounds for $|I_4(\boldsymbol{a})|$. Since $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| + |I_4(\boldsymbol{a})| = n$, $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \leq \frac{mS}{D}$, and $|I_3(\boldsymbol{a})| \leq S$, so $|I_4(\boldsymbol{a})| \geq n - \frac{mS}{D} - S$. Moreover, if $|I_4(\boldsymbol{a})| > n - \frac{mS}{D}$, then $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| < \frac{mS}{D}$. The club capacity is not exhausted and there must exist a club c that is not full. There is a jointly profitable deviation for an individual i in $I_4(\boldsymbol{a})$ and club c where i joins c. A contradiction.

Next consider the case when $nD < mS$: We first show the lower bound for $|C_1(\boldsymbol{a})|$ is $nD/S - D$. Suppose that in a stable membership profile $\boldsymbol{a}$, all individuals exhaust their membership availability, then we have $|C_1(\boldsymbol{a})|S + \sum_{c \in C_2(\boldsymbol{a})} s_c(\boldsymbol{a}) = nD$, and hence $|C_1(\boldsymbol{a})|S + |C_W(\boldsymbol{a})|S \geq nD$. Since $|C_2(\boldsymbol{a})| \leq D$, $|C_1(\boldsymbol{a})| \geq \frac{nD}{S} - D$. Suppose that in a a stable membership profile $\boldsymbol{a}$, not all individuals exhaust their membership availability, then we know $|C_3(\boldsymbol{a})| = 0$ as otherwise there is a jointly profitable deviation for the individual who joins less than D clubs and a club in $C_3(\boldsymbol{a})$ where the individual joins the club. Therefore,

$$|C_1(\boldsymbol{a})| + |C_2(\boldsymbol{a})| = m \geq \frac{nD}{S},$$

and so $|C_1(\boldsymbol{a})| \geq \frac{nD}{S} - D$ given $|C_2(\boldsymbol{a})| \leq D$.

For the upper bound of $|C_1(\boldsymbol{a})|$, since aggregate membership availability is nD, we must have $|C_1(\boldsymbol{a})|S \leq nD$, and so $|C_1(\boldsymbol{a})| \leq \frac{nD}{S}$.

Regarding the bounds for $|C_3(\boldsymbol{a})|$. Since $|C_1(\boldsymbol{a})| + |C_2(\boldsymbol{a})| + |C_3(\boldsymbol{a})| = m$, $|C_1(\boldsymbol{a})| \leq \frac{nD}{S}$, and $|c_2(\boldsymbol{a})| \leq D$, so $|C_3(\boldsymbol{a})| \geq m - \frac{nD}{S} - D$. Moreover, if $|C_3(\boldsymbol{a})| > m - \frac{nD}{S}$, then $|C_1(\boldsymbol{a})| + |C_2(\boldsymbol{a})| < \frac{nD}{S}$. The aggregate membership availability is not exhausted and there must exist an individual i that joins less than D clubs. There is a jointly profitable deviation for i and a club c in $C_3(\boldsymbol{a})$ and club c where i joins c. A contradiction. ■

Proof of Proposition 3

We prove a more general characterization of efficient membership profiles, without the parity conditions in Proposition 3.

Lemma 1. *Suppose $\alpha > 0$. Assume $nD \geq mS$.*

- *A membership profile is clubs-efficient if and only if there are $\lfloor \frac{mS}{D} \rfloor$ individuals that join D clubs, one individual that joins $(mS) \bmod D$ clubs, and the remaining individuals join no clubs.*

- *If $v''(\cdot) \geq 0$, then a membership profile is utilitarian optimum if and only if it is clubs-efficient. If $v''(\cdot) < 0$ and satisfies condition (7), then in any utilitarian optimum membership profile, either $d_i(\boldsymbol{a}) \leq 1$ for all $i \in I$ or $d_i(\boldsymbol{a}) \geq 1$ for all $i \in I$.*

Assume $nD < mS$.

- *If $f''(\cdot) > 0$, then a membership profile is clubs-efficient if and only if $\lfloor \frac{nD}{S} \rfloor$ clubs admit S members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members. If $f''(\cdot) = 0$, then a membership profile is clubs-efficient if and only if each individual join D clubs. If $f''(\cdot) < 0$, then a membership profile is clubs-efficient if and only if $(nD) \bmod m$ admit $\lceil \frac{nD}{m} \rceil$ members and the remaining clubs admit $\lfloor \frac{nD}{m} \rfloor$ members.*

- *If $v''(\cdot) \geq 0$, then a membership profile is utilitarian optimum if and only if $\lfloor \frac{nD}{S} \rfloor$ clubs admit S members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members. If $v''(\cdot) < 0$ and $(nD) \bmod S = 0$, then membership profile is utilitarian optimum if and only if nD/S clubs admit S members and the remaining clubs admit no members. If $v''(\cdot) < 0$ and $(nD) \bmod S > 0$, then in any utilitarian optimum membership profile, the number of clubs that admit some but less than S members is not more than $S - 1$.*

For the case when $nD \geq mD$. First, given a membership profile $\boldsymbol{a}$, the aggregate productivity of clubs is

$$\sum_{c\in C}\pi_c(\boldsymbol{a}) = \sum_{c\in C} f(s_c(\boldsymbol{a})) + \alpha\sum_{c\in C}\sum_{i\in I} a_{ic}(d_i(\boldsymbol{a})-1)$$
$$\leq mf(S) + \alpha\sum_{i\in I} d_i(\boldsymbol{a})(d_i(\boldsymbol{a})-1),$$

where the equality is obtained only when $s_c(\boldsymbol{a}) = S$ for all $c \in C$. Now we solve the following maximization problem:

$$\max\sum_{i\in I} d_i(\boldsymbol{a})(d_i(\boldsymbol{a})-1) \text{ s.t. } d_i(\boldsymbol{a}) \in \{0,1,...,D\} \text{ for all } i \in I \text{ and } \sum_{i\in I} d_i(\boldsymbol{a}) \leq mS.$$

Since $g(x) = x(x-1)$ is superadditive on the set of non-negative integers and this is strict on positive integers, the solution to the maximization problem is a vector $(d_i^*(\boldsymbol{a}))_{i\in I}$ such that $d_i^*(\boldsymbol{a}) = D$ for all $i \in I'$ where $I' \subset I$ and $|I'| = \lfloor\frac{mS}{D}\rfloor$, $d_i^*(\boldsymbol{a}) = (mS) \bmod D$ for some $i = k \in I\backslash I'$ (in the case of $(mS) \bmod D \geq 1$) and $d_i^*(\boldsymbol{a}) = 0$ for all $i \in I\backslash(I' \cup \{k\})$.

We now show that when $nD \geq mS$, there always exists a club membership structure where there are $\lfloor\frac{mS}{D}\rfloor$ individuals that join D clubs, one individual that joins $(mS) \bmod D$ clubs, and the remaining individuals join no clubs (which makes $s_c(\boldsymbol{a}) = S$ for all $c \in C$), so that a structure $\boldsymbol{a} \in A$ is clubs-efficient if and only if it satisfies such a club joining pattern. Construct a membership structure as follows. Consider $\lfloor mS/D\rfloor$ individuals first, in a sequence. Make each such i join D clubs that have the smallest membership size at her turn before moving to the next individual. If $(mS) \bmod D \geq 1$ so that there are clubs that do not have S members at the end of the process, take one more individual and make him join those $(mS) \bmod D$ clubs. Since $\lfloor mS/D\rfloor D + (mS) \bmod D = mS$, the construction is valid and results in the desired membership structure.

For utilitarian optimal structures, given a membership profile $\boldsymbol{a}$, the aggregate utility of individuals is

$$\sum_{i\in I} u_i(\boldsymbol{a}) = \sum_{i\in I} v(\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a})).$$

We know that $\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}) \leq D(f(S) + S(D-1))$ for all $i \in I$ and $\sum_{i\in I}\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}) = \sum_{c\in C} s_c(\boldsymbol{a})\pi_c(\boldsymbol{a}) \leq S\sum_{c\in C}\pi_c(\boldsymbol{a})$, where the equality is obtained only when $s_c(\boldsymbol{a}) = S$ for all $c \in C$. Given that a clubs-efficient structure that maximizes $\sum_{c\in C}\pi_c(\boldsymbol{a})$ features $s_c(\boldsymbol{a}) = S$ for all $c \in C$, $\sum_{i\in I}\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a})$ is maximized if and only if $\boldsymbol{a}$ is clubs-efficient. We also know that a clubs-efficient structure makes $\lfloor \frac{mS}{D} \rfloor$ individuals have utility $v(D(f(S)+S(D-1)))$, at most one individual have positive but less than $v(D(f(S)+S(D-1)))$ utility, and the rest individuals have zero utility. Hence, when $v''(\cdot) \geq 0$, the clubs-efficient membership profile is the solution to the maximization profile of $\max_{\boldsymbol{a}\in A} u_i(\boldsymbol{a})$. We have shown that when $v''(\cdot) \geq 0$, a membership profile is utilitarian optimum if and only if it is clubs-efficient.

Turning to when $v''(\cdot) < 0$ and satisfies

$$v(f(S)) - v(0) > (n-1)\left(v\left(f(S) + \frac{2\alpha S(D-1)}{n-1}\right) - v\left(f(S)\right)\right),$$

we show that suppose in a club membership structure $\boldsymbol{a} \in A$, there exists two individuals $i, i' \in I$ such that $d_i(\boldsymbol{a}) > 1$ and $d_{i'}(\boldsymbol{a}) = 0$, then $\boldsymbol{a}$ cannot be utilitarian optimum. Suppose such a structure $\boldsymbol{a}$ is utilitarian optimum. Note first that it must be $s_c(\boldsymbol{a}) = S$ for all $c \in C$, as otherwise making individual i' join a club that is not full strictly raises aggregate welfare. Let $c \in C$ be a club where $a_{ic} = 1$. Consider another club membership structure $\boldsymbol{a}'$ where c drops i and admits i'. The difference of aggregate utility between the two structures is

$$\sum_{i\in I}(u_i(\boldsymbol{a}') - u_i(\boldsymbol{a})) \geq v(f(S)) - v(0) + \sum_{i\neq i'}(u_i(\boldsymbol{a}') - u_i(\boldsymbol{a})),$$

since $u_{i'}(\boldsymbol{a}') \geq v(f(S))$ and $u_{i'}(\boldsymbol{a}) = v(0)$. Given that i' replaces i in club c, the productivity of club c and clubs that i members decreases:

$$\pi_c(\boldsymbol{a}) - \pi_c(\boldsymbol{a}') = \alpha(d_i(\boldsymbol{a}) - 1), \text{ and}$$
$$\pi_{c'}(\boldsymbol{a}) - \pi_{c'}(\boldsymbol{a}') = \alpha \text{ for all } c' \neq c \text{ with } a_{ic'} = 1.$$

So, the aggregate productivity drop is at most $2\alpha(D-1)$, which is obtained when $d_i(\boldsymbol{a}) = D$. Since $v''(\cdot) < 0$ and the minimal utility an individual obtains when he is in a club is $v(f(S))$,

$$\begin{aligned}\sum_{i\neq i'}(u_i(\boldsymbol{a}) - u_i(\boldsymbol{a}')) &\leq \sum_{i\neq i'}\left[v\left(f(S) + \sum_{c\in C} a_{ic}(\pi_c(\boldsymbol{a}) - \pi_c(\boldsymbol{a}'))\right) - v(f(S))\right] \\ &\leq (n-1)\left[v\left(f(S) + \frac{2\alpha(D-1)S}{n-1}\right) - v(f(S))\right].\end{aligned}$$

Hence,

$$\sum_{i\in I}(u_i(\boldsymbol{a}') - u_i(\boldsymbol{a})) \geq v(f(S)) - v(0) - (n-1)\left[v\left(f(S) + \frac{2\alpha(D-1)S}{n-1}\right) - v(f(S))\right] > 0,$$

contradicting structure $\boldsymbol{a}$ being utilitarian optimum. This completes the proof.

For the case when $nD < mS$, given a membership profile $\boldsymbol{a}$, the aggregate productivity of clubs is

$$\begin{aligned}\sum_{c\in C}\pi_c(\boldsymbol{a}) &= \sum_{c\in C} f(s_c(\boldsymbol{a})) + \alpha\sum_{c\in C}\sum_{i\in I} a_{ic}(d_i(\boldsymbol{a}) - 1)\\ &\leq \sum_{c\in C} f(s_c(\boldsymbol{a})) + \alpha n D(D-1),\end{aligned}$$

where the equality is obtained only when $d_i(\boldsymbol{a}) = D$ for all $i \in I$. Now we look at the problem of $\max \sum_{c\in C} f(s_c(\boldsymbol{a}))$, s.t. $s_c(\boldsymbol{a}) \in \{0, 1, ..., S\}$ for all $c \in C$ and $\sum_{c\in C} s_c(\boldsymbol{a}) \leq nD$. When $f(\cdot)$ is convex, the solution to the maximization problem is a vector $(s_c^*(\boldsymbol{a}))_{c\in C}$ such that $s_c^*(\boldsymbol{a}) = S$ for all $c \in C'$ where $C' \subset C$ and $|C'| = \lfloor \frac{nD}{S} \rfloor$, $s_c^*(\boldsymbol{a}) = (nD) \bmod S$ for some $c = k \in C \backslash C'$ (in the case of $(nD) \bmod S \geq 1$) and $s_c^*(\boldsymbol{a}) = 0$ for all $c \in C\backslash(C' \cup \{k\})$. When $f(\cdot)$ is linear, the solution to the maximization problem is any $(s_c^*(\boldsymbol{a}))_{c\in C}$ where $s_c(\boldsymbol{a}) \in \{0, 1, ..., S\}$ for all $c \in C$ and $\sum_{c\in C} s_c(\boldsymbol{a}) = nD$. When $f(\cdot)$ is concave, the solution to the maximization problem is a vector $(s_c^*(\boldsymbol{a}))_{c\in C}$ such that $s_c^*(\boldsymbol{a}) = \lceil \frac{nD}{m} \rceil$ for all $c \in C'$ where $C' \subset C$ and $|C'| = (nD) \bmod m$, and $s_c^*(\boldsymbol{a}) = \lfloor \frac{nD}{m} \rfloor$ for all $c \in C\backslash C'$. This proves the characterization for clubs-efficient membership profiles.

For utilitarian optimum membership profiles, given a membership profile $\boldsymbol{a}$, we know the aggregate utility of individuals is $\sum_{i\in I} u_i(\boldsymbol{a}) = \sum_{i\in I} v(\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}))$ where $\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}) \leq D(f(S) + S(D-1))$ for all $i \in I$ and

$$\begin{aligned}\sum_{i\in I}\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}) &= \sum_{c\in C} s_c(\boldsymbol{a}) f(s_c(\boldsymbol{a})) + \alpha\sum_{c\in C} s_c(\boldsymbol{a})\sum_{i\in I} a_{ic}(d_i(\boldsymbol{a}) - 1)\\ &\leq \sum_{c\in C} s_c(\boldsymbol{a}) f(s_c(\boldsymbol{a})) + \alpha(D-1)s_c(\boldsymbol{a})^2,\end{aligned}$$

where the equality is obtained only when $d_i(\boldsymbol{a}) = D$ for all $i \in I$. Since $g(x) = xf(x) + \alpha(D-1)x^2$ is superadditive on non-negative integers and strictly superadditive on positive integers, for any $\alpha \geq 0$, $D \geq 1$, and strictly increasing f with $f(0) = 0$, $\sum_{i\in I}\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a})$ is maximized if and only if $\lfloor \frac{nD}{S} \rfloor$ clubs admit S members, one club that admits $(nD) \bmod S$

members, and the remaining clubs admit no members. When $v''(\cdot) \geq 0$, it is easy to see that this membership profile is also the solution to the maximization problem of $\max_{\boldsymbol{a} \in A} u_i(\boldsymbol{a})$. We have shown that when $v''(\cdot) \geq 0$, a membership profile is utilitarian optimum if and only if $\lfloor \frac{nD}{S} \rfloor$ clubs admit S members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members.

Turning to when $v''(\cdot) < 0$, consider the utilitarian optimum structure where $v''(\cdot) \geq 0$. Under this structure, the utility of $(nD) \bmod S$ individuals is

$$v[(D-1)(f(S) + \alpha S(D-1)) + f\left((nD) \bmod S\right) + \alpha \left((nD) \bmod S\right)(D-1)], \tag{19}$$

while the utility of all other individuals is $v[(D)(f(S) + \alpha S(D-1))]$. If this structure is utilitarian optimum, we have finished the proof. If the structure is not utilitarian optimum, then $(nD) \bmod S \neq 0$ and in a utilitarian optimum club membership profile, the lowest utility of an individual is greater than (19), implying that the smallest size of a club is greater than $(nD) \bmod S$. Suppose the smallest club size is $s_c(\boldsymbol{a}) = (nD) \bmod S + k$ where $k \in \{1, ..., S - (nD) \bmod S - 1\}$. For the structure to be utilitarian optimal, the number of unfull clubs is at most k, where the bound k is reached when we reduce the club size of k clubs by 1 to increase the size of the smallest club. So, the number of clubs with size greater than 0 and lower than S is at most $1 + k \leq S - (nD) \bmod S \leq S - 1$.

■

Proof of Proposition 4

First, we consider when $nD \geq mS$.

When $h(\cdot)$ is convex, for any membership profile $\boldsymbol{a} \in A$, the productivity of a club $\pi_c(\boldsymbol{a})$ satisfies

$$\pi_c(\boldsymbol{a}) \leq f(S) + h(S(D-1)) = f(S) + h(S),$$

where the equality is obtained only when the club has one strength-S link with another club. For every club to reach this highest level of productivity, the club network consists of m/D separate 2-cliques where all links are of strength S. We now show such a structure exists by construction: Allocate the first S individuals to clubs c_1 and c_2, the next S individuals to clubs c_3 and c_4,..., and the $\frac{m}{2}^{th}$ group of S individuals (individuals $i_{mS/2-S+1}$ to $i_{mS/2}$) to clubs c_{m-1} and c_m.

Since all clubs have reached the highest productivity with the membership profile when $h(\cdot)$ is convex, it is also stable when $h(\cdot)$ is convex. To show the profile is not stable when

$h(\cdot)$ is concave, consider a deviation by club c_1 and individual i_{S+1} where c_1 exiles i_1 to admit i_{S+1} and i_{S+1} leaves c_3 to join c_1. It is straight-forward to verify that the deviation benefits both c_1 and i_{S+1}.

When $h(\cdot)$ is concave, for any membership profile $\boldsymbol{a} \in A$, the productivity of a club $\pi_c(\boldsymbol{a})$ satisfies

$$\pi_c(\boldsymbol{a}) \leq f(S) + S(D-1)h(1) = f(S) + S \cdot h(1),$$

where the equality is obtained only when the club has S strength-1 links with other clubs. For every club to reach this highest level of productivity, the club network is an S-regular network where all links are of strength 1. We now show such a structure exists by construction with the following algorithm: At each step, pick the club with the maximum number of empty slots, fill the slots with different individuals, and then allocate each of those individuals to a different club with the maximum number of empty slots. Stop when all clubs are full.

Since all clubs have reached the highest productivity with the membership profile when $h(\cdot)$ is concave, it is also stable when $h(\cdot)$ is concave. Now we show the profile is not stable when $h(\cdot)$ is convex. In this profile, for each club c, it must has at least 2 strength-1 links with c' and c''. Let i_1 be the common member of c and c' and i'' be the common member of c and c''. There must also exist an indiviual, call him i_3, who is in c' and $c''' \neq c$. Consider the deviation by club c and individual i_3 where c exiles i_2 to admit i_3 and i_3 leaves c''' to join c. This deviation benefits both c and i_3.

Turning to when $nD < mS$, let π^* be the highest productivity a club can obtain, note that for any membership profile $\boldsymbol{a} \in A$, the utility of an individual $u_i(\boldsymbol{a})$ satisfies

$$u_i(\boldsymbol{a}) \leq v(D \cdot \pi^*),$$

where the equality is obtained only when all clubs i joins has productivity π^*. For any individual to reach this highest level of utility, the subnetwork of clubs that contains all non-empty clubs must consist of n/S separate 2-cliques where all links are of strength S when $h(\cdot)$ is convex and be an S-regular network (a complete network when $S = 2n/S - 1$) where all links are of strength 1 when $h(\cdot)$ is concave. Such subnetworks can be constructed in the same way we construct we construct the clubs-efficieny networks when $nD \geq mS$.

For the statements on stability, since all individuals have reached the highest level of utility, they have no incentives to deviate. We consider the same deviations examined for the case when $nD \geq mS$ to show that the utilitarian optimum club network under a convex (concave)

$h(\cdot)$ is unstable when $h(\cdot)$ is concave (convex).

■

Proof for Proposition 5

Let us use the terms 'individuals' and 'clubs' interchangeably.

When $\alpha = 0$, we can always construct an egalitarian membership profile with the same algorithm mentioned in the proof of Proposition 2. It is straight-forward to verify that this membership profile is stable.

When $\alpha > 0$, we first show that for a membership profile $\boldsymbol{a}$ to be stable, it must satisfy:

(i) for any pair of i and c, if i is not in c, then either $d_i(\boldsymbol{a}) = D$ or $s_c(\boldsymbol{a}) = S$,

(ii) for any two individuals i and i' with $D > d_i(\boldsymbol{a}) \geq d_{i'}(\boldsymbol{a}) > 0$, i must join all clubs i' does,

(iii) for an individual i with degree D, there cannot exist two clubs $c, c' \in C_2(\boldsymbol{a})$ where $\pi_c(\boldsymbol{a}) \geq \pi_{c'}(\boldsymbol{a})$, $s_c(\boldsymbol{a}) < S$, and i is in c' but not c,

(iv) for an individual i with degree D, if i joins one and only one club in $C_2(\boldsymbol{a})$, c', then $\pi_{c'}(\boldsymbol{a}) > \pi_c(\boldsymbol{a})$ for all $c \in C_2(\boldsymbol{a}) \backslash \{c'\}$.

Suppose condition (i) is not satisfied, consider the deviation by i and c where i joins c. With that deviation, the productivity of c and all the clubs i is in increase; the production costs of i and all the firms in club c and clubs that have i decrease. The aggregate cost reduction of firms other than i is not greater than

$$d_i(\boldsymbol{a})(S-1)\alpha + (S-1)\left(f(s_c(\boldsymbol{a})+1) - f(s_c(\boldsymbol{a})) + \alpha d_i(\boldsymbol{a})\right),$$

while the cost reduction of i is not less than

$$d_i(\boldsymbol{a})\alpha + f(s_c(\boldsymbol{a})+1) - f(s_c(\boldsymbol{a})) + \alpha d_i(\boldsymbol{a}).$$

The utility of firm i is

$$u_i(\boldsymbol{a}) = \left(\frac{(\beta - \gamma_0) + \gamma n \sum_{c \in C} a_{ic}\pi_c(\boldsymbol{a}) - \gamma \sum_{i' \neq i} \sum_{c \in C} a_{i'c}\pi_c(\boldsymbol{a})}{n+1}\right)^2.$$

Since $n \geq S$, the utility of i rises as a result of the deviation. Both i and c are strictly better off.

Suppose condition (ii) is not satisfied, let c be a club that has i' but not i, consider a deviation by i and c where c exiles i' to admit i. With the deviation, the productivity of c

and all the clubs i joins increase; the costs of i and all the firms in club c and clubs that have i decrease. The aggregate cost reduction of firms other than i is not greater than

$$d_i(\boldsymbol{a})(S-1)\alpha + (S-1)\alpha\left(d_i(\boldsymbol{a}+1-d_{i'}(\boldsymbol{a}))\right),$$

while the cost reduction of i is not less than

$$d_i(\boldsymbol{a})\alpha + \alpha\left(d_i(\boldsymbol{a}+1-d_{i'}(\boldsymbol{a}))\right).$$

Since $n \geq S$, the utility of i rises as a result of the deviation. Both i and c are strictly better off.

Suppose condition (iii) does not hold, consider a deviation by i and c where i quits c' to join c. With the deviation, the output of c increases; the costs of i and all the firms in club c decrease. The aggregate cost reduction of firms other than i is not greater than

$$(S-1)\left(f(s_c(\boldsymbol{a})+1) - f(s_c(\boldsymbol{a})) + \alpha(D-1)\right),$$

while the cost reduction of i is not less than

$$f(s_c(\boldsymbol{a})+1) - f(s_c(\boldsymbol{a})) + \alpha(D-1).$$

Since $n \geq S$, the utility of i rises as a result of the deviation. Both i and c are strictly better off.

Suppose condition (iv) does not hold. Since $c \in C_2(\boldsymbol{a})$, either $s_c(\boldsymbol{a}) < S$ or there exists an $i' \in I_3(\boldsymbol{a})$ who is in c. Condition (iii) shows that it cannot be $s_c(\boldsymbol{a}) < S$, so there exists an $i' \in I_3(\boldsymbol{a})$ who is in c. Consider a deviation by i and c where i quits c', c exiles i' and i joins c. With the deviation, the productivity of c increases; the costs of i and all the firms tin club c decrease. The aggregate cost reduction of firms other than i is not greater than $(S-1)\alpha\left(D-d_{i'}(\boldsymbol{a})\right)$, while the cost reduction of i is not less than $\alpha\left(D-d_{i'}(\boldsymbol{a})\right)$. Since $n \geq S$, the utility of i rises as a result of the deviation. Both i and c are strictly better off.

We can then prove Proposition 5 in the same way we prove Proposition 2.

First, we show $|I_3(\boldsymbol{a})| \leq S$. Take any $i \in I_3(\boldsymbol{a})$ with the minimal degree and let $c \in C$ be any club that i joins. By condition (ii), all firms in $I_3(\boldsymbol{a})$ are in c. Since $s_c(\boldsymbol{a}) \leq S$, $|I_3(\boldsymbol{a})| \leq S$.

Second, we show $|C_2(\boldsymbol{a})| \leq D$. We consider the cases of $I_3(\boldsymbol{a}) = \varnothing$ and $I_3(\boldsymbol{a}) \neq \varnothing$

separately. If $I_3(\boldsymbol{a}) = \varnothing$, then all individuals who member clubs in $C_2(\boldsymbol{a})$ are of degree D and, for any $c \in C_2(\boldsymbol{a})$, $s_c(\boldsymbol{a}) < S$ (as c does not achieve maximal productivity). Take any $c' \in C_2(\boldsymbol{a})$ with minimal productivity and any member i of c'. Take any $c \in C_2(\boldsymbol{a}) \setminus \{c'\}$, by condition (iii), i is a member of c. Hence i is in all clubs in $C_2(\boldsymbol{a})$. Since $d_i(\boldsymbol{a}) \leq D$, $|C_2(\boldsymbol{a})| \leq D$. If $I_3(\boldsymbol{a}) \neq \varnothing$, then take any $i \in I_3(\boldsymbol{a})$ with maximal degree and any $c \in C_2(\boldsymbol{a})$. Since c does not achieve maximal productivity, either $s_c(\boldsymbol{a}) < S$ or c has a member in $I_3(\boldsymbol{a})$. In the first case, i is in c by condition (i). In the second case, i is a member of c by condition (ii). Hence, i is a member of all clubs in $C_2(\boldsymbol{a})$. Since $d_i(\boldsymbol{a}) \leq D$, $|C_2(\boldsymbol{a})| \leq D$.

Third, we show $|I_2(\boldsymbol{a})| \leq (S-1)(D+1)/2$. Notice first that, by definition, every firm in $I_2(\boldsymbol{a})$ joins at least one club in $C_2(\boldsymbol{a})$. Thus, the aggregate membership of firms in $I_2(\boldsymbol{a})$ in the clubs in $C_2(\boldsymbol{a})$ is at least $x + 2(|I_2(\boldsymbol{a})| - x)$, where x is the number of firms from $I_2(\boldsymbol{a})$ that join exactly one club in $C_2(\boldsymbol{a})$. On the other hand, since $|C_2(\boldsymbol{a})| \leq D$ and, for all $c \in C_2(\boldsymbol{a})$, either $s_c(\boldsymbol{a}) \leq S - 1$ or c has a member in $I_3(\boldsymbol{a})$, so aggregate club capacity of the clubs in $C_2(\boldsymbol{a})$ for firms in $I_2(\boldsymbol{a})$ is at most $(S-1)D$. Hence

$$x + 2(|I_2(\boldsymbol{a})| - x) = 2|I_2(\boldsymbol{a})| - x \leq (S-1)D.$$

The number of firms in $I_2(\boldsymbol{a})$ that are in exactly one club in $C_2(\boldsymbol{a})$ is at most $S-1$. To see that, suppose that an $i \in I_2(\boldsymbol{a})$ joins exactly one $c' \in C_2(\boldsymbol{a})$. Let $c \in C_2(\boldsymbol{a}) \setminus \{c'\}$ be another club in $C_2(\boldsymbol{a})$. Since i is not in c so, by condition (iv), $\pi_c(\boldsymbol{a}) < \pi_{c'}(\boldsymbol{a})$. Hence c' must achieve the highest productivity of all clubs in $C_2(\boldsymbol{a})$ and must be unique such. Since all individuals in $C_2(\boldsymbol{a})$ who member exactly one club in $C_2(\boldsymbol{a})$ must be members of the same club from $C_2(\boldsymbol{a})$ and since, this club can host at most $S-1$ members from $I_2(\boldsymbol{a})$, $x \leq S-1$. From the equation above, it follows that $|I_2(\boldsymbol{a})| \leq (S-1)(D+1)/2$.

We then use these derivations on the size of the different groups to derive bounds on the size of $I_1(\boldsymbol{a})$ and $I_2(\boldsymbol{a})$ and $I_4(\boldsymbol{a})$.

■

Proof for Proposition 6

We first show by construction that there exists a membership profile $\boldsymbol{a}$ where the resulting alliance network consists of separate 2-cliques with strength S links and the resulting firm network consists of separate S-cliques with strength 2 links. Allocate the first S firms to clubs c_1 and c_2, the next S firms to c_3 and c_4, etc. Stop when all clubs are full (this happens when $x = m$) or all firms join $D = 2$ clubs (this happens when $x = nD/S$).

When $h(\cdot)$ is convex, with this constructed membership profile, all clubs that have members reach the highest productivity possible, which is $f(S)+h(S(D-1))$. Thus, clubs with members have no incentive to deviate. For the clubs with no members, no firm is willing to quit its club to join such a club, since then the production cost of the firm rises by

$$[f(S) + h(S) - f(1) - h(1)] + [h(S) - h(S-1) - h(1)],$$

while the aggregate cost of other firms rise by

$$(S-1)[f(S) - f(S-1) + h(S) - h(S-1)] + (S-1)[h(S) - h(S-1) - h(1)].$$

With this deviation, given that $n \geq S$, $f(1) \leq f(S-1)$ and $h(1) \leq h(S-1)$, the profit of the firm drops. Therefore, the constructed membership profile is stable when $h(\cdot)$ is convex.

When $h(\cdot)$ is concave, consider a deviation by club c_1 and individual i_{S+1} where c_1 exiles i_1 to admit i_{S+1} and i_{S+1} quits c_3 to join c_1. It is straightforward to verify that the deviation benefits both c_1 and i_{S+1}. Therefore, the constructed membership profile is not stable when $h(\cdot)$ is concave.

Now we show, by construction, that there exists a membership profile $\boldsymbol{a}$ where the resulting alliance network is an S-regular network with strength 1 links and the resulting firm network is a $D(S-1)$-regular network with strength 1 links. Consider the following algorithm: at each step, pick a club with the maximum number of empty slots, fill the slots with different firms, and then allocate each of those firms to a different club with the maximum number of empty slots. Stop when all the clubs are full or all the firms join $D = 2$ clubs.

When $h(\cdot)$ is concave, with this constructed membership profile, all clubs that have members reach the highest productivity possible, which is $f(S) + S(D-1)h(1)$. Thus, clubs with members have no incentive to deviate. For the clubs with no members, no firm is willing to quit its club to join any of them, because then the production cost of the firm rises by

$$f(S) + Sh(1) - f(1) - h(1),$$

while the aggregate costs of other firms rise by

$$(S-1)[f(S) - f(S-1) + Sh(1) - (S-1)h(1)].$$

With this deviation, given that $n \geq S$, $f(1) \leq f(S-1)$ and $h(1) \leq h(S-1)$, the profit of the firm drops. Therefore, the constructed membership profile is stable when $h(\cdot)$ is concave.

When $h(\cdot)$ is convex, in the constructed profile, for each c, it must has at least 2 strength 1 links with c' and c''. Let i_1 be the common member of c and c' and i'' be the common member of c and c''. There must also exist a firm, call it i_3, that is in c' and $c''' \neq c$. Consider the deviation by c and i_3 where c exiles i_2 to admit i_3 and i_3 quits c''' to join c. This deviation benefits both c and i_3. Therefore, the constructed membership profile is not stable when $h(\cdot)$ is convex. ■