mathematics

Article

Stability for Representations of Hecke Algebras of Type A

Kun Wang [1], Shanshan Qi [2,*], Haitao Ma [3] and Zhujun Zheng [4]

1 College of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China; wangkun@zknu.edu.cn
2 Department of Mathematics, Harbin Cambridge University, Harbin 150001, China
3 College of Mathematics Science, Harbin Engineering University, Harbin 150001, China; hmamath@hrbeu.edu.cn
4 School of Mathematics, South China University of Technology, Guangzhou 510641, China; zhengzj@scut.edu.cn
* Correspondence: sqimath@163.com

Abstract: In this paper, we introduce the concept of the stability of a sequence of modules over Hecke algebras. We prove that a finitely generated consistent sequence of the representations of Hecke algebras is representation stable.

Keywords: Hecke algebra; representations; representation stable

MSC: Primary 20C08; 20C99

Citation: Wang, K.; Qi, S.; Ma, H.; Zheng, Z. Stability for Representations of Hecke Algebras of Type *A*. *Mathematics* **2022**, *10*, 32. https://doi.org/10.3390/math10010032

Academic Editor: Xiao-Wu Chen

Received: 21 November 2021
Accepted: 20 December 2021
Published: 22 December 2021

1. Introduction

Representation theory uses a class of matrices to represent abstract algebraic objects and makes the operations in this original structure correspond to matrix operations. This method can be applied to many kinds of algebraic structures, such as group, associative algebra, and Lie algebra. Recently, inspired by the work of Church, Ellenberg, Farb, and Nagpal in References [1,2], people have been studying many problems in basic mathematics, such as algebra, algebraic geometry, and number theory, through the theory of FI-modules, where FI is the category of finite sets and injective maps. Essentially, FI-module over a commutative ring R is a functor from the category FI to the category of R-modules. Thus, a sequence of representations of symmetric groups can be described by a functor between categories, that is, an FI-module. Many important results have been obtained, such as homological properties of FI-modules, Noetherian properties, and representation stability properties, in a series of articles [3–8].

Let G_n be one of the symmetric group $\mathfrak{S}_n$, the general linear group $GL_n(\mathbb{Q})$, or the special linear group $SL_n(\mathbb{Q})$. In 2013, Church and Farb introduced the theory of representation stability to study the representations V_n of these natural groups G_n [9]. Let us review some basic concepts before introducing relevant background. The idea is mainly to study the consistent sequence of representations V_n of groups G_n as follows. A sequence of G_n-representations,

$$V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} \cdots,$$

is called consistent sequence if, for all $n \geq 0$ and $g \in G_n$, the diagram

$$\begin{array}{ccc} V_n & \xrightarrow{\phi_n} & V_{n+1} \\ g\downarrow & & g\downarrow \\ V_n & \xrightarrow{\phi_n} & V_{n+1} \end{array}$$

is commutative, where each ϕ_n is a linear map.

A consistent sequence is called representation stable if, for sufficiently large n, it satisfies

- Injectivity: The map ϕ_n is injective;
- Surjectivity: The span of the image of ϕ_n is equal to V_{n+1};
- Multiplicities: Decompose V_n into irreducible representations as

$$V_n = \bigoplus_{\lambda} c_{\lambda,n} V(\lambda)_n,$$

with multiplicities $0 \leq c_{\lambda,n} \leq \infty$. For each partition λ, $c_{\lambda,n}$ is independent of n.

Church, Ellenberg, and Farb used the theory of FI-modules to study the consistent sequence of representations of the symmetric groups $\mathfrak{S}_n$ over a commutative ring R [1]. It has been proved that, if a consistent sequence of representations of symmetric group over a field of characteristic zero as FI-module is finitely generated, then this consistent sequence is representation stable [1].

In Reference [2], Church, Ellenberg, Farb, and Nagpal proved the Noetherian property for FI-modules over Noetherian rings. The consistent sequence of representations of symmetric groups given by a finitely generated FI-module turned out to be representation stable.

In Reference [10], Gan and Watterlond researched VI-modules, where VI is the category of finite dimensional vector spaces and injective linear maps. They proved a result about the representation stability for the family of finite general linear groups $GL_n(\mathbb{F}_q)$, where q is a power of prime. That is, a VI-module over an algebraically closed field of characteristic zero is finitely generated if and only if the consistent sequence obtained from V is representation stable and $\dim(V_n) < \infty$ for each n. In Reference [11], d'Andecy and Walker introduced the cyclotomic quotients of affine Hecke algebras of type D. They studied the finite dimensional representations of affine Hecke algebras of type D through the representation theory of some cyclotomic quotients. In Reference [12], Laudone investigated a similar category with FI for the 0-Hecke algebra $\mathscr{H}_n(0)$ called the 0-Hecke category. They obtained a new type of representation stability in this setting and proved that it is implied by a finitely generated $\mathscr{H}$-module.

Hecke algebra $\mathscr{H}_n$ of type A is the deformation of group algebra $\mathbb{C}[\mathfrak{S}_n]$, and the representations of $\mathscr{H}_n$ is also the deformation of representations of $\mathfrak{S}_n$. In the light of works of Church, Ellenberg, and Farb, the relation between the representation categories of $\mathbb{C}[\mathfrak{S}_n]$ and $\mathscr{H}_n$ strongly suggests the existence of representation stability of Hecke algebras $\mathscr{H}_n$. This is the original motivation of this paper. The rest of the paper is organized as follows. Firstly, we review some basic concepts of Hecke algebras, partitions, and give the definition of stability for representations of Hecke algebras. Secondly, we discuss the injective degree, surjective degree, stability degree, weight, and Noetherian property of $FI_{\mathscr{H}}$-modules. Finally, we give the main theorem of this paper. We prove that a finitely generated consistent sequence of the representations of Hecke algebras is representation stable.

2. Preliminaries

Let us first list some basic results about Hecke algebras and some notations.

Let k be a field of characteristic 0. $q \in k$ is not a root of unity. Let $n \geq 2$ and $\mathfrak{S}_n$ be the symmetric group. It is easy to show that $\mathfrak{S}_n$ is generated by the $n-1$ transpositions $\{(1,2),(2,3),\cdots,(n-1,n)\}$. For the convenience of writing, we denote the transposition $(i,i+1)$ by s_i for $1 \leq i \leq n-1$. The Iwahori-Hecke algebra (or simply the Hecke algebra) $\mathscr{H}_n = \mathscr{H}_{k,q}(\mathfrak{S}_n)$ of the symmetric group $\mathfrak{S}_n$ is the unital associative k-algebra with generators $\{T_{s_1}, T_{s_2}, \cdots, T_{s_{n-1}}\}$ and relations

$(T_{s_i} - q)(T_{s_i} + 1) = 0$, for $i = 1, 2, \cdots, n-1$;

$T_{s_i} T_{s_j} = T_{s_j} T_{s_i}$, for $1 \leq i < j - 1 \leq n-2$;

$T_{s_i} T_{s_{i+1}} T_{s_i} = T_{s_{i+1}} T_{s_i} T_{s_{i+1}}$, for $i = 1, 2, \cdots, n-2$.

We shall denote $\mathscr{H}_0 = \mathscr{H}_1 = k$.

Suppose $\sigma \in \mathfrak{S}_n$. Then, σ can be expressed as the product of some transpositions. The expression that uses the minimum number of transpositions is called the reduced expression. This number about σ is denoted by $l(\sigma)$; thus, we have

$$l(\sigma) = \min\{m | \sigma = s_{i_1} \cdots s_{i_m}\}.$$

Denote $T_\sigma := T_{s_{i_1}} \cdots T_{s_{i_l}}$ which is an element of the Hecke algebra $\mathscr{H}_n$. Then, $\mathscr{H}_n$ is free with k-basis $\{T_\sigma | \sigma \in \mathfrak{S}_n\}$. If $\sigma_1 \in \mathfrak{S}_n, \sigma_2 \in \mathfrak{S}_n$ and $l(\sigma_1\sigma_2) = l(\sigma_1) + l(\sigma_2)$, then it is not difficult to obtain that $T_{\sigma_1\sigma_2} = T_{\sigma_1}T_{\sigma_2}$.

We now review the concept of composition (partition) of natural numbers, which will be used in later sections. Let $n \in \mathbb{N}$. A sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ which satisfies $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$ is called a composition of n. A composition λ is called a partition of n when $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. Composition (partition) λ of n is generally denoted by $\lambda \models n$ ($\lambda \vdash n$). Let $1 \leq k, l \leq n$ and $k + l \leq n$. Obviously, the set

$$S_{\{k+1,k+2,\cdots,k+l\}} \triangleq \{\sigma \in \mathfrak{S}_n | \sigma(i) = i, i \in \{1, 2, \cdots, n\} - \{k+1, k+2, \cdots, k+l\}\}$$

is a subgroup of $\mathfrak{S}_n$. For each composition λ of n, the standard Young subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}_n$ is defined by the direct product $S_{\{1,2,\cdots,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,\cdots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{\lambda_1+\lambda_2+\cdots+\lambda_{n-1}+1,\cdots,n\}}$. For example, if $\lambda = (3, 2, 1)$ is a composition of 6, then the corresponding Young subgroup is $\mathfrak{S}_3 \times \{(1), (4, 5)\}$. For more details of Young subgroups, please refer to Reference [13]. The subalgebra $\mathscr{H}_\lambda$ of $\mathscr{H}$ generated by $\{T_\sigma | \sigma \in \mathfrak{S}_\lambda\}$ is k-free with basis $\{T_\sigma | \sigma \in \mathfrak{S}_\lambda\}$. The subalgebras of these types are the so-called Young subalgebras.

Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ be a partition of some natural number. For any $n \geq |\lambda| + \lambda_1$, the padded partition is defined as

$$\lambda[n] := (n - |\lambda|, \lambda_1, \lambda_2, \cdots),$$

where $|\lambda| = \sum_{i=1}^{\infty} \lambda_i$.

In the case of the Hecke algebra $\mathscr{H}_n$ that we have assumed, the simple module of Hecke algebra can be labeled (up to isomorphism) by the set of all partitions of n. For a partition λ and $n \geq |\lambda| + \lambda_1$, we denote S^λ the corresponding simple $\mathscr{H}_{|\lambda|}$-module and set $S(\lambda)_n := S^{\lambda[n]}$.

Since $\mathfrak{S}_n$ is naturally regarded as a subgroup of $\mathfrak{S}_{n+1}$, then the generators of Hecke algebra $\mathscr{H}_n$ is a subset of generators of Hecke algebra $\mathscr{H}_{n+1}$. That is to say, $\mathscr{H}_n$ is a subalgebra of $\mathscr{H}_{n+1}$. The natural injective k-algebra map from $\mathscr{H}_n$ to $\mathscr{H}_{n+1}$ is denoted by τ.

Definition 1. *A sequence $V = (V_n, \phi_n)_{n\in\mathbb{N}}$ is called consistent sequence or $FI_{\mathscr{H}}$-module if it satisfies: for all $h \in \mathscr{H}_n$, the diagram*

$$\begin{array}{ccc} V_n & \xrightarrow{\phi_n} & V_{n+1} \\ h\downarrow & & \downarrow\tau(h) \\ V_n & \xrightarrow{\phi_n} & V_{n+1} \end{array}$$

is commutative, where V_n is $\mathscr{H}_n$-module, and $\phi_n : V_n \to V_{n+1}$ is k-linear map.

Let $V = (V_n, \phi_n)$ and $W = (W_n, \psi_n)$ be two $FI_{\mathscr{H}}$-modules. A morphism $f = (f_n)$ from V to W is a sequence of homomorphisms $f_n : V_n \to W_n$ for all $n \geq 0$ such that the following diagram

$$\begin{array}{ccc} V_n & \xrightarrow{\phi_n} & V_{n+1} \\ f_n \big\downarrow & & f_{n+1} \big\downarrow \\ W_n & \xrightarrow{\psi_n} & W_{n+1} \end{array}$$

is commutative.

Remark 1. *$FI_{\mathscr{H}}$-modules, together with the above morphisms, are an abelian category. Notions, such as kernel, cokernel, injection, and surjection, are defined by pointwise. The direct sum and tensor product also make sense. For example, let $f = (f_n) : V \to W$ be a morphism. We need to find an $FI_{\mathscr{H}}$-module, and the kernel of this morphism satisfies the kernel's universal property of this category. As usual, this $FI_{\mathscr{H}}$-module is denoted by $\ker f$. In addition, the construction of the $FI_{\mathscr{H}}$-module $\ker f$ is through with defining at each point n. That is, $(\ker f)_n$ is defined by $\ker(f_n)$, and the map from $(\ker f)_n$ to $(\ker f)_{n+1}$ is the restriction of ϕ_n. Now, it is easy to check that, for all $h \in \mathscr{H}_n$, we have commutative composite mappings, i.e., $\tau(h)\phi_n|_{\ker(f_n)} = \phi_n|_{\ker(f_n)}h$, which makes $\ker f$ an $FI_{\mathscr{H}}$-module. Similarly, we can also derive the tensor product of two $FI_{\mathscr{H}}$-modules straightforward. The tensor product $V \bigotimes W$ of two $FI_{\mathscr{H}}$-modules $V = (V_n, \phi_n)$ and $W = (W_n, \psi_n)$ is defined by $(V \bigotimes W)_n = V_n \bigotimes_k W_n$, and the map from $(V \bigotimes W)_n$ to $(V \bigotimes W)_{n+1}$ is defined by $\phi_n \otimes \psi_n$.*

Definition 2. *Let $W = (W_n)_{n\geq 0}$, where W_n is an $\mathscr{H}_n$-module. Denote $\lambda_{m,n}$ the composition $(m, n-m)$ for $m \leq n$. Then, W_m will be an $\mathscr{H}_{\lambda_{m,n}}$-module under the action $T_\sigma w = T_{\sigma_1\sigma_2} w = T_{\sigma_1} w$, where $\sigma = \sigma_1\sigma_2$ for $\sigma_1 \in S_{\{1,2,\ldots,m\}}$ and $\sigma_2 \in S_{\{m+1,\ldots,n-m\}}$ and $w \in W_m$. We define a sequence $M(W)$ for $n \geq 0$,*

$$M(W)_n := \bigoplus_{m\leq n} (\mathscr{H}_n \bigotimes_{\mathscr{H}_{\lambda_{m,n}}} W_m),$$

with the linear map $\phi_n : M(W)_n \to W(W)_{n+1}$ determined by $h \otimes w_m \mapsto \tau(h) \otimes w_m$, where $h \in \mathscr{H}_n$ and $w_m \in W_m$.

It is easy to check that $M(W)$ is an $FI_{\mathscr{H}}$-module. If $W = (W_n)_{n\geq 0}$ satisfies $W_m = \mathscr{H}_m$ for some m and $W_n = 0$ for $n \neq m$, then we denote $M(W)$ by $M(m)$.

Let $V = (V_n, \phi_n)$ be an $FI_{\mathscr{H}}$-module. Denote $\phi_{n-1,m} = \phi_{n-1}\phi_{n-2}\cdots\phi_m : V_m \to V_n$ for $0 \leq m \leq n-2$ and $\phi_{n-1,n-1} = \phi_{n-1}$. For a disjoint union, $S = \bigsqcup_{n=0}^{\infty} S_n$, where $S_n \subseteq V_n$. Next, we define the concept of generation through the span of linear space. We define $span(S)_n$ to be the submodule of V_n generated by $\big(\bigcup_{i=0}^{n-1} \phi_{n-1,i}(S_i)\big) \cup S_n$. In addition, the map $\widetilde{\phi}_n : span(S)_n \to span(S)_{n+1}$ is the restriction of ϕ_n. It is trivial to check from the above discussion that $span(S)$ is an $FI_{\mathscr{H}}$-module.

Definition 3. *We say that an $FI_{\mathscr{H}}$-module V is generated in degree $\leq n$ if V is generated by elements of V_m for $m \leq n$.*

According to the above definition, if V is generated in degree $\leq n$, then we have

$$span(\bigsqcup_{m\leq n} V_m) = V.$$

Definition 4. *We say that an $FI_{\mathscr{H}}$-module V is finitely generated if there is a finite set S of the disjoint union $\bigsqcup V_n$ such that $span(S) = V$.*

Definition 5 (Representation stability of consistent sequence)**.** *Let $(V_n, \phi_n)_{n\in\mathbb{N}}$ be an $FI_{\mathscr{H}}$-module. This sequence is called uniformly representation stable if there exists an integer N such that for each with $n \geq N$, the following conditions hold*

- *Injectivity: $\phi_n : V_n \to V_{n+1}$ is injective;*
- *Surjectivity: V_{n+1} is generated by the image of ϕ_n;*
- *Multiplicities: $V_n = \bigoplus_\lambda c_{\lambda,n} S(\lambda)_n$, where the multiplicities $c_{\lambda,n}$ is independent of n.*

This definition of representation stability is in the sense of Church and Farb [9]. They introduced the idea of representation stability for a sequence of representations of groups. An important result in Reference [1] is that a consistent sequence of $\mathfrak{S}_n$-representations is representation stable can be converted to a finite generation property for an FI-module.

3. Stability Degree

In this section, we define the concepts of stability degree, injective degree and surjective degree for an $FI_{\mathscr{H}}$-module. For $W = (W_n)_{n\geq 0}$, we will compute these numbers for the $FI_{\mathscr{H}}$-module $M(W)$. The following lemma shows that $M(W)$ is similar to the "free object".

Lemma 1. *If an $FI_{\mathscr{H}}$-module $V = (V_n, \phi_n)$ is generated in degree $\leq d$, then there exists an epimorphism from $\bigoplus_{i=0}^{d} M(V_i)$ to V.*

Proof. For every $0 \leq n \leq d$, V_n is a summand of $(\bigoplus_{i=0}^{d} M(V_i))_n$ since $M(V_n)_n = V_n$ by definition. So, the functions from $\left(\bigoplus_{i=0}^{d} M(V_i)\right)_n$ to V_n are defined by projections. Now, let $n > d$, and then $\left(\bigoplus_{i=0}^{d} M(V_i)\right)_n = \bigoplus_{i=0}^{d} \left(\mathscr{H}_n \otimes_{\mathscr{H}_{\lambda_i}} V_i\right)$. If $v_i \in V_i$ for $0 \leq i \leq d$, then $f_n(1 \otimes v_i) = \phi_{n-1} \cdots \phi_{i+1}\phi_i(v_i)$. The fact $\phi_n f_n = f_{n+1}$ implies that $f = (f_n)_{n\geq 0}$ is a morphism. From the construction of f and V generated in degree $\leq d$, we know f is an epimorphism. □

Given $a \geq 0$ and an $FI_{\mathscr{H}}$-module $V = (V_n, \phi_n)$, we define a sequence $\Phi_a(V)$ as follows. Denote by Q_n the subspace of V_{a+n} spanned by $\{T_\sigma v - v | \sigma \in S_{\{a+1,a+2,\ldots,a+n\}}, v \in V_{a+n}\}$ for $n \geq 0$ and then define $\Phi_a(V)_n := V_{a+n}/Q_n$. In addition, the map $T : \Phi_a(V)_n \to \Phi_a(V)_{n+1}$ is defined by $[v] \mapsto [\phi_{a+n}(v)]$. The map T is well-defined since, for $T_\sigma v - v \in Q_n$, $\phi_{a+n}(T_\sigma v - v) = T_\sigma \phi_{a+n}(v) - \phi_{a+n}(v) \in Q_{n+1}$.

Definition 6. *If there exists a natural number s such that for all $a \geq 0$, the map defined above is an isomorphism for $n \geq s$, then V is called to have stability degrees. If the map T is injective or surjective, then V is called to have injective degree s or surjective degree s. These facts are denoted by stab-deg(V)(or inj-deg(V), sur-deg(V))$= s$.*

The following lemma is about double cosets of symmetric group of two certain Young subgroups. It is an interesting consequence of group theory. This result will be used in the proof of the next lemmas.

Lemma 2. *Given $a \geq 0$. For $n \geq 0$ and $m \leq a+n$, let $\lambda_n = (m, a+n-m)$ and $\mu_n = (1, 1, \ldots, 1, n)$ be two compositions of $a+n$. Denote $\mathscr{D}_{\mu_n,\lambda_n}$ the representative elements of minimal length of the double cosets $\mathfrak{S}_{\mu_n} \backslash \mathfrak{S}_{a+n} / \mathfrak{S}_{\lambda_n}$. Then, $\mathscr{D}_{\mu_n,\lambda_n} \subseteq \mathscr{D}_{\mu_{n+1},\lambda_{n+1}}$ and if $n \geq m$, the set $\mathscr{D}_{\mu_n,\lambda_n}$ is independent of n.*

Proof. By Lemma 1.7 in Reference [14], there is a bijection from the set of row-standard tableaux in $\mathscr{T}(\lambda_n, \mu_n)$ to $\mathscr{D}_{\mu_n,\lambda_n}$. These row-stantard λ_n-tableaux of type μ_n are determined by the subset of $\{1, 2, \ldots, a\}$ which is in the first row of λ_n. Denote the number of the set of row-standard tableaux in $\mathscr{T}(\lambda_n, \mu_n)$ by $\mathscr{T}_n$. So, we have $\mathscr{T}_n \leq \mathscr{T}_{n+1}$. If $\sigma \in \mathscr{D}_{\mu_n,\lambda_n}$ is the element of minimal length in a double coset $\mathfrak{S}_{\mu_n}\sigma\mathfrak{S}_{\lambda_n}$, then σ's length is also minimal in the double coset $\mathfrak{S}_{\mu_{n+1}}\sigma\mathfrak{S}_{\lambda_{n+1}}$. We obtain that $\sigma \in \mathscr{D}_{\mu_n,\lambda_n}$ implies $\sigma \in \mathscr{D}_{\mu_{n+1},\lambda_{n+1}}$. If $n \geq m$, the number $\mathscr{T}_n$ is independent of n. So, we complete the proof of the lemma. □

For $\alpha \in \mathfrak{S}_{a+n}$, there exist $\sigma \in \mathscr{D}_{\mu_n,\lambda_n}$, $\beta \in \mathfrak{S}_{\mu_n}$, and $\gamma \in \mathfrak{S}_{\lambda_n}$ such that $\alpha = \beta\sigma\gamma$. According to the Lemma 1.6 in Reference [14], these elements can be chosen to satisfy $l(\alpha) = l(\beta) + l(\sigma) + l(\gamma)$. This property will be used in the following lemma.

Lemma 3. *Assume $W = (W_n)_{n\geq 0}$, where W_n is an $\mathscr{H}_n$-module for $n \geq 0$. Then, $M(W)$ has injective degree 0; if $W_i = 0$ for all $i > m$, then we have surj-deg$\big(M(W)\big) \leq m$. Therefore, $M(m)$ has injective degree 0 and surjective degree m.*

Proof. At first, we compute $M(W)_{a+n}$ by definition.

$$\begin{aligned} M(W)_{a+n} &= \bigoplus_{m\leq a+n} (\mathscr{H}_{a+n} \bigotimes_{\mathscr{H}_{\lambda_n}} W_m) \\ &= \bigoplus_{m\leq a+n} \Big(\bigoplus_{\sigma\in\mathscr{D}_{\mu_n,\lambda_n}} (\mathscr{H}_{\mu_n} T_\sigma \bigotimes_{\mathscr{H}_{\lambda_n}} W_m)\Big). \end{aligned}$$

The map $\phi_{a+n} : M(W)_{a+n} \to M(W)_{a+n+1}$ is the direct sum of the maps: For $m \leq a + n, \sigma \in \mathscr{D}_{\mu_n,\lambda_n}$, $\mathscr{H}_{\mu_n} T_\sigma \otimes_{\mathscr{H}_{\lambda_n}} W_m \to \mathscr{H}_{\mu_{n+1}} T_\sigma \otimes_{\mathscr{H}_{\lambda_{n+1}}} W_m$ defined by $hT_\sigma \otimes w \mapsto \tau(h)T_\sigma \otimes w$, where $h \in \mathscr{H}_{\mu_n}$ and $w \in W_m$. In addition, so

$$\begin{aligned} \Phi_a\big(M(W)\big)_n &= M(W)_{a+n}/Q_n \\ &\cong \bigoplus_{m\leq a+n} \Big(\bigoplus_{\sigma\in\mathscr{D}_{\mu_n,\lambda_n}} \big((\mathscr{H}_{\mu_n} T_\sigma \bigotimes_{\mathscr{H}_{\lambda_n}} W_m)/Q_n\big)\Big) \\ &\cong \bigoplus_{m\leq a+n} \Big(\bigoplus_{\sigma\in\mathscr{D}_{\mu_n,\lambda_n}} (T_\sigma \bigotimes_{\mathscr{H}_{\lambda_n}} W_m)\Big). \end{aligned}$$

The map $T : \Phi_a(M(W))_n \to \Phi_a(M(W))_{n+1}$ restricted in one factor in the above decomposition is the identity. So, we have obtained that the inj-deg$(M(W)) = 0$. If $W_i = 0$ for $i > m$ and $n \geq m$, then $\mathscr{H}_{\mu_n} T_\sigma \otimes_{\mathscr{H}_{\lambda_n}} W_i = 0$. So, T is surjective for all $n \geq m$. We have proved this lemma. □

From the above lemma we have the following corollary.

Corollary 1. *The surjective degree of a consistent sequence is less than its generated degree.*

4. Weight

Let us recall some basic results about the modules of Hecke algebra [15]. Assume that q is not a root of unity; then, the Hecke algebra $\mathscr{H}$ is semisimple, and all the non-isomorphic simple $\mathscr{H}$-modules are indexed by all partitions λ. For each partition λ, the corresponding irreducible module is denoted by S^λ. By the Corollary 6.2 in Reference [15], the branching rule of Hecke algebra is the same as the classical branching rule for $\mathfrak{S}_n$-representations. The results about the branching rule for $\mathfrak{S}_n$-representations, stated in Reference [1], are also true for the modules of Hecke algebra. For $\lambda \vdash n$ and $m \in \mathbb{N}$, write $\lambda \to \mu$ if μ is a partition of $n + m$ such that $[\mu]$ is obtained from $[\lambda]$ by adding m boxes from different columns. We write that lemma of the branching rule in the following:

Lemma 4 (The branching rule [15])**.** *Let λ be a partition of n and $m \in \mathbb{N}$. Q_n is defined in the Stability degree Section 3 under Lemma 1. Then,*

- $\mathrm{Ind}_{\mathscr{H}_n \bigotimes \mathscr{H}_m}^{\mathscr{H}_{n+m}} S^\lambda \cong \bigoplus_{\lambda\to\mu} S^\mu$;

- $\left(\mathrm{Res}^{\mathscr{H}_n}_{\mathscr{H}_{n-m}\otimes\mathscr{H}_m} S^\lambda\right)/Q_m \cong \bigoplus_{\mu\to\lambda} S^\mu.$

Lemma 5 ([1]). *Let λ be a partition of n and $a \le n$. Then,*

- $S(\lambda)_n/Q_{n-a} = 0 \Longleftrightarrow n-a > n-|\lambda|.$
- $S(\lambda)_n/Q_{n-|\lambda|} \cong S^\lambda.$
- *If $n \ge a+|\lambda|$, then $S(\lambda)_n/Q_{n-a}$ is independent of n.*
- *Let V be a consistent sequence with weight less or equal to d. If $W_n/Q_{n-d} = 0$ for any subquotient W_n of V_n, then $W_n = 0$.*

Definition 7. *Let V be an $FI_{\mathscr{H}}$-module. We say a partition λ occurs in V, if there exists $n \ge 0$, $S(\lambda)_n$ occurs in the $\mathscr{H}_n$-module V_n. The weight of V is defined the maximum of $|\lambda|$ which λ occurs in V.*

Lemma 6. *For any partition $\lambda \vdash m$, the $FI_{\mathscr{H}}$-module $M(S^\lambda)$ has weight m.*

Proof. By the definition of $M(S^\lambda)$, $M(S^\lambda)_n = \mathscr{H}_n \bigotimes_{\mathscr{H}_{m,n-m}} S^\lambda = \mathrm{Ind}\, S^\lambda$. So, for any $n \ge m+\lambda_1$, adding $n-m$ boxes in different columns we obtain the partition $\lambda[n]$ occurs in $M(S^\lambda)$. So, the weight of $M(S^\lambda)$ is at least m. On the other hand, for any partition μ that occurs in $M(S^\lambda)_n$ satisfies $\mu_1 \ge n-m$. If we write $\mu = \nu[n]$ for some partition ν, then we have $|\nu| = n-\mu_1 \le m$. Then, the weight of $M(S^\lambda)$ is at most m. We, thus, complete the proof of this lemma. □

If V is generated in degree $\le d$, then the weight of V is less or equal to d by the above lemma.

Lemma 7. *For any partition $\lambda = (\lambda_1, \lambda_2, \cdots)$, the consistent sequence $M(S^\lambda)$ has stability degree λ_1.*

Proof. From our previous results, we know the stability degree of $M(S_\lambda)$ is less than or equal to $|\lambda|$, and we have

$$\Phi_a M(S_\lambda)_n \cong \bigoplus_{\sigma\in\mathscr{D}_{\mu_n,\lambda_n}} \left((\mathscr{H}_{\mu_n} T_\sigma \bigotimes_{\mathscr{H}_{\lambda_n}} S^\lambda)/Q_n\right), \tag{1}$$

where $\mu_n = (1,1,\cdots,1,n)$ and $\lambda_n = (|\lambda|, a+n-|\lambda|)$.

By Lemma 2 and an easily computing, for $\sigma \in \mathscr{D}_{\mu_n,\lambda_n}$ there is some $k \le |\lambda|$ such that

$$\sigma^{-1} S_{\mu_n} \sigma \cap S_{\{1,2,\ldots,|\lambda|\}} = S_{\{|\lambda|-k+1,\ldots,|\lambda|\}}.$$

For any $w \in S_{\{|\lambda|-k+1,\cdots,|\lambda|\}}$, there exists $u \in S_{\mu_n}$ such that $w = \sigma^{-1}u\sigma \in \sigma^{-1}S_{\mu_n}\sigma \cap S_{\lambda_n}$. Since $\sigma \in \mathscr{D}_{\mu_n,\lambda_n}$, so $l(\sigma w) = l(\sigma) + l(w)$ and $l(u\sigma) = l(u) + l(\sigma)$. Assume x is an element in S^λ, and then we have

$$[T_\sigma \otimes T_w x] = [T_\sigma(T_\sigma^{-1} T_u T_\sigma) \otimes x] = [T_u T_\sigma \otimes x] = [T_\sigma \otimes x].$$

By Lemma 5, for $k > \lambda_1$, we know $S^\lambda/Q_k = 0$. So, the summands in the decomposition of (1) are stable when $n \ge \lambda_1$. Since $S^\lambda/Q_{\lambda_1} \ne 0$, then T is not surjective for $n = \lambda_1 - 1$. We have completed the proof of this lemma. □

5. Noetherian Property

In this section, we will study the Noetherian property of $FI_{\mathscr{H}}$-modules. Assume that $FI_{\mathscr{H}}$-module V is finitely generated, and then the degree of V is also finite. Conversely, let V have degree $\le d$. If, in addition, V_n for $n \le d$ is finitely generated $\mathscr{H}_n$-modules, then V is finitely generated.

Definition 8. *Given $a \geq 0$. Let $V = (V_n, \phi_n)$ be an $FI_{\mathscr{H}}$-module. We define an $FI_{\mathscr{H}}$-module $S_{+a}V$ by $(S_{+a}V)_n = V_{n+a}$.*

After a simple calculation, we derive that $(S_{+a}V)_n$ is isomorphic to $\mathrm{Res}^{\mathscr{H}_{n+a}}_{\mathscr{H}_n} V_{n+a}$ as $\mathscr{H}_n$-modules.

Lemma 8. *Let $m \geq 1$ and $a \geq 0$. Then, there exists a decomposition of $S_{+a}M(m)$*

$$S_{+a}M(m) = M(m) \oplus C_a,$$

where C_a is finitely generated in degree $\leq m-1$.

Proof. As we have computed in the previous sections

$$\begin{aligned}\big(S_{+a}M(m)\big)_n &= M(m)_{n+a} \\ &= \mathscr{H}_{a+n} \bigotimes_{\mathscr{H}_{\lambda_n}} \mathscr{H}_m \\ &= \bigoplus_{\sigma \in \mathscr{D}_{\mu_n,\lambda_n}} (\mathscr{H}_{\mu_n} T_\sigma \bigotimes_{\mathscr{H}_{\lambda_n}} \mathscr{H}_m),\end{aligned}$$

where $\mu_n = (1,1,\cdots,1,n)$ and $\lambda_n = (m, a+n-m)$.

Let $\sigma \in \mathscr{D}_{\mu_n,\lambda_n} - \{(1)\}$. There are many choices for the representative elements of the double coset $\mathscr{D}_{\mu_n,\lambda_n}$. Since the number of elements of symmetric group $\mathfrak{S}_n$ is finite, we can always choose the representative elements with minimum length. There exists $m' \in \{1,2,\cdots,m-1\}$ such that $\sigma^{-1}\mathfrak{S}_{\mu_n}\sigma \cap \mathfrak{S}_m = S_{\{m'+1,\cdots,m\}}$. Then, we can decompose $\mathfrak{S}_m = \bigsqcup_{\sigma' \in \mathscr{D}'} S_{\{m'+1,\cdots,m\}}\sigma'\mathfrak{S}_{m'}$, where $\mathscr{D}'$ is the corresponding representative elements of minimal length. So, we have

$$\begin{aligned}\bigoplus_{\sigma \in \mathscr{D}_{\mu_n,\lambda_n}} (\mathscr{H}_{\mu_n} T_\sigma \bigotimes_{\mathscr{H}_{\lambda_n}} \mathscr{H}_m) &= (\mathscr{H}_{\mu_n} \bigotimes_{\mathscr{H}_{\lambda_n}} \mathscr{H}_m) \bigoplus \Big(\bigoplus_{\sigma \in \mathscr{D}_{\mu_n,\lambda_n}-\{(1)\}} \big(\bigoplus_{\sigma' \in \mathscr{D}'} (\mathscr{H}_{\mu_n} T_\sigma T_{\sigma'} \bigotimes_{\mathscr{H}_{\lambda_n}} \mathscr{H}_{m'})\big)\Big) \\ &\cong M(m)_n \oplus (C_a)_n,\end{aligned}$$

where we denote $(C_a)_n = \Big(\bigoplus_{\sigma \in \mathscr{D}_{\mu_n,\lambda_n}-\{(1)\}} \big(\bigoplus_{\sigma' \in \mathscr{D}'} (\mathscr{H}_{\mu_n} T_\sigma T_{\sigma'} \otimes_{\mathscr{H}_{\lambda_n}} \mathscr{H}_{m'})\big)\Big)$.

Now, it is not difficult to obtain that C_a is finitely generated in degree $\leq m-1$ and we complete the proof of this lemma. □

From the above lemma, we know the degree of $S_{+a}M(m)$ is finite and by Lemma 1 $M(m)$ can be replaced by a consistent sequence with finite degree. Conversely, according to our definition of $S_{+a}V$, we have:

Lemma 9. *Let V be a consistent sequence. If $S_{+a}V$ is of finite degree, then so is V.*

Theorem 1 (Noetherian Property)**.** *Let $V = (V_n, \phi_n)$ be finitely generated and $W \subseteq V$. Then, W is also finitely generated.*

Proof. Suppose V is generated in degree m. By Lemma 1, V is a quotient of a finite direct sum of $M(m_i)$ for $m_i \leq m$. So, we need to prove $M(m_i)$ satisfies the Noetherian property. Let $W \subseteq M(m)$. Since the injective degree of $M(m)$ is zero, we can assume that $M(m)_n \subseteq M(m)_{n+1}$ for all n. Since every W_n is finite dimensional, then W has finite degree implies W is finitely generated. To show $\deg(W) < \infty$, it suffices to prove it for $S_{+a}W$ for some a.

By Lemma 8, $S_{+a}M(m)$ can be decomposed as two summand, one of degree m. In addition, so is $S_{+a}W$. Since the map $\phi|_{W_n} : W_n \to W_{n+1}$ is injective, we can assume $W_n \subseteq W_{n+1}$. For any a, $\mathscr{H}_m = M(m)_m$ is contained in $S_{+a}M(m)_m$, which is exactly the summand of degree m. Let W_m^a be the summand of $S_{+a}W_m$. Then, $W_m^a \subseteq W_m^{a+1} \subseteq \mathscr{H}_m$. Since $\mathscr{H}$ is finite dimensional, there exists N such that $W_m^N = W_m^{N+1}$. For any $a \geq 0$, W_m^{N+a} generates W_{m+a}^N which implies that W^N is finitely generated. For the degree $\leq m-1$ of $S_{+N}W$, we use induction, which implies $S_{+N}W$ is finitely generated of degree. $\square$

6. Representation Stability

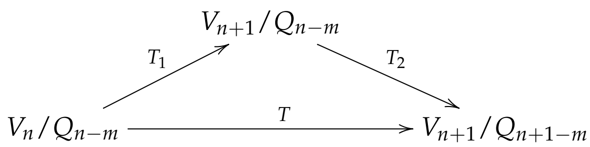
The following lemma shows that the representation stable range $N < \infty$ if the stability degree and weight are finite.

Lemma 10. *Let $V = (V_n, \phi_n)$ be an $FI_{\mathscr{H}}$-module with stability degree s and weight m. Then, V is uniformly representation stable with stable range $n \geq s+m$.*

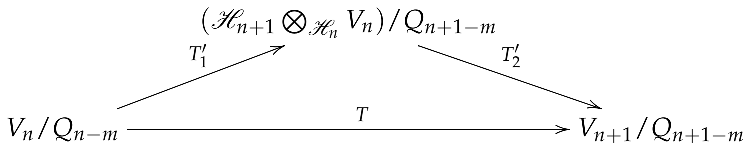
Proof. Let $\phi'_n : \mathscr{H}_{n+1} \bigotimes_{\mathscr{H}_n} V_n \to V_{n+1}$.

In order to indicate the injective and surjectivity, that is to say, $\ker \phi_n = 0$ and $\operatorname{coker} \phi'_n = 0$ for $n \geq s+m$, it suffices to prove $\ker \phi_n / Q_{n-m} = 0$ and $\operatorname{coker} \phi'_n / Q_{n+1-m} = 0$ by Lemma 5.

Since $n - m \geq s$, the map $T : \Phi_m(V)_{n-m} \to \Phi_m(V)_{n+1-m}$ is bijective. Moreover, notice that the diagrams



and



commute. So, T_1 is injective and T'_2 is surjective. We have obtained that $\ker \phi_n / Q_{n-m} = 0$ and $\operatorname{coker} \phi'_n / Q_{n+1-m} = 0$.

Assume that $V_n = \bigoplus_\lambda c_{\lambda,n} S(\lambda)_n$, and we need to show $c_{\lambda,n}$ is independent of n when $n \geq s+m$. Since the weight of V is m, there is no λ with $|\lambda| > m$ occurred in V. By Lemma 5, $S(\lambda)_n / Q_{n-m} = 0$ when $|\lambda| > m$. So,

$$\begin{aligned} V_n/Q_{n-m} &= \bigoplus_{|\lambda|\leq m} c_{\lambda,n}(S(\lambda)_n/Q_{n-m}) \\ &= \bigoplus_{|\lambda|<m} c_{\lambda,n}(S(\lambda)_n/Q_{n-m}) \bigoplus \Big(\bigoplus_{|\lambda|=m} c_{\lambda,n}(S(\lambda)_n/Q_{n-m}) \Big) \\ &= \bigoplus_{|\lambda|<m} c_{\lambda,n}(S(\lambda)_n/Q_{n-m}) \bigoplus (\bigoplus_{|\lambda|=m} c_{\lambda,n} S^\lambda). \end{aligned}$$

Because $n - m$ is greater or equal to the stability degree and for the first term of the above decomposition by the induction, we obtain $c_{\lambda,n}$ is independent of n. $\square$

Now, we state and prove our main theorem about representation stability of the representations of Hecke algebras.

Theorem 2. *An $FI_{\mathscr{H}}$-module $V = (V_n, \phi_n)$ is finitely generated if and only if the sequence (V_n, ϕ_n) is uniformly representation stable and each V_n is finite-dimensional.*

Proof. Assume V is uniformly stable with range N. Because of the surjectivity of V, we have $V_n = span(\operatorname{im} \phi_{n-1})_n$ for all $n - 1 \geq N$. So, $span(\bigsqcup_{n=0}^N V_n) = V$ and, together with the finite-dimensional, implies that V is finitely generated.

For the converse, by Lemma 10, we only need to show V has finite stability degree and weight. According to Corollary 1, the surjective degree is finite. Assume that V is generated in degree $\leq d$. There exists an epimorphism $g : M \to V$, where $M = M(W)$ for some $W = (W_n)_{n \geq 0}$. Let K be the kernel of g. By Theorem 1 and Corollary 1, K is also finitely generated, and the surjective degree of K is finite, say s.

For given $a \geq 0$ and for any $n \geq s$, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_a(K)_n & \longrightarrow & \Phi_a(M)_n & \longrightarrow & \Phi_a(V)_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T_2} & & \downarrow{\scriptstyle T_1} & & \downarrow{\scriptstyle T} & & \\ 0 & \longrightarrow & \Phi_a(K)_{n+1} & \longrightarrow & \Phi_a(M)_{n+1} & \longrightarrow & \Phi_a(V)_{n+1} & \longrightarrow & 0. \end{array}$$

By the Snake Lemma, we have an exact sequence:

$$\ker T_1 \longrightarrow \ker T \longrightarrow \operatorname{coker} T_2 \longrightarrow \operatorname{coker} T_1 \longrightarrow \operatorname{coker} T \longrightarrow 0\,.$$

So, $\ker T_1 = 0$ and $\operatorname{coker} T_2 = 0$ imply $\ker T = 0$. This shows the injective degree of V is finite. We, thus, have completed the proof. □

Remark 2. *As an application of Theorem 2, we point out that the result for representation stability of symmetric group representations of Theorem 1.13 in Reference [1] can be derived by our theorem as q tends to 1.*

7. Conclusions

We study the free object, projective object, surjective degree, injective degree, and stability degree of the abelian category $FI_{\mathscr{H}}$-modules. We prove that $FI_{\mathscr{H}}$-modules satisfies the Noetherian property. In addition, we obtain that an $FI_{\mathscr{H}}$-module $V = (V_n, \phi_n)$ is finitely generated if and only if the sequence (V_n, ϕ_n) is uniformly representation stable and each V_n is finite-dimensional. As q tends to 1, our results can derive the stability for representations of symmetric groups. As for the subsequent work of $FI_{\mathscr{H}}$-modules, one can further study homological stability of the representation category of Hecke algebras. Moreover, the stabilities for representations of quantum groups and Hecke algebras of other types are also worth studying. In addition, then, one can investigate the relationship with Schur-Weyl duality.

Author Contributions: Conceptualization and formal analysis, K.W., S.Q. and H.M.; writing—review and methodology, K.W., S.Q., H.M. and Z.Z.; writing—original draft, K.W. All authors have read and agreed to the published version of the manuscript.

Funding: The authors was supported in part by National Natural Science Foundation of China of Grant No.11571119 and 11475178; Zhoukou Normal University, China, high level talents research start funding project under No.ZKNUC2020005.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Acknowledgments: The authors are very grateful to the associate editor and referees for their useful comments and suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Church, T.; Ellenberg, J.S.; Farb, B. FI-modules and stability for representations of symmetric groups. *Duke Math. J.* **2015**, *164*, 1833–1910. [CrossRef]
2. Church, T.; Ellenberg, J.S.; Farb, B.; Nagpal, R. FI-modules over Noetherian rings. *Geom. Topol.* **2014**, *18*, 2951–2984. [CrossRef]
3. Gan, W.L.; Li, L. Coinduction functor in representation stability theory. *J. Lond. Math. Soc.* **2015**, *92*, 689–711. [CrossRef]
4. Gan, W.L. A long exact sequence for homology of FI-modules. *N. Y. J. Math.* **2016**, *22*, 1487–1502.
5. Gan, W.L. On the negative-one shift functor for FI-modules. *J. Pure Appl. Algebra* **2017**, *221*, 1242–1248. [CrossRef]

6. Li, W.; Guan, B.; Ouyang, B. FI_G-modules over coherent rings. *J. Algebra* **2017**, *474*, 116–125. [CrossRef]
7. Li, L. Homological degrees of representations of categories with shift functors. *Trans. Am. Math. Soc.* **2018**, *370*, 2563–2587. [CrossRef]
8. Ramos, E. Homological invariants of FI-modules and FI_G-modules. *J. Algebra* **2018**, *502*, 163–195. [CrossRef]
9. Church, T.; Farb, B. Representation theory and homological stability. *Adv. Math.* **2013**, *245*, 250–314. [CrossRef]
10. Gan, W.L.; Watterlond, J. A representation stability theorem for VI-modules. *Algebr. Represent. Theory* **2018**, *21*, 47–60. [CrossRef]
11. d'Andecy, L.P.; Walker, R. Affine Hecke algebras of type D and generalisations of quiver Hecke algebras. *J. Algebra* **2020**, *552*, 1–37.
12. Laudone, R.P. Representation stability for sequences of 0-Hecke modules. *Algebr. Comb.* **2021**, *4*, 619–661. [CrossRef]
13. Jones, A. A combinatorial approach to the double cosets of the symmetric group with respect to Young subgroups. *Eur. J. Comb.* **1996**, *17*, 647–655. [CrossRef]
14. Dipper, R.; James, G. Representations of Hecke algebras of general linear groups. *Proc. Lond. Math. Soc.* **1986**, *s3-52*, 20–52. [CrossRef]
15. Mathas, A. *Iwahori-Hecke Algebras and Schur Algebras of the Symmetric Group*; American Mathematical Society: Providence, RI, USA, 1999.