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Radius, girth and minimum degree

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Abstract

The objective of the present paper is to study the maximum radius r of a connected graph of order n, minimum degree $\delta \geq 2$ and girth at least $g \geq 4$. Erdős, Pach, Pollack and Tuza proved that if $g = 4$, that is, the graph is triangle-free, then $r \leq \frac{n-2}{\delta} + 12$, and noted that up to the value of the additive constant, this upper bound is tight. In this paper we shall determine the exact maximum. For larger values of g little is known. We settle the order of the maximum r for $g = 6, 8$ and 12, and prove an upper bound for every even g, which we conjecture to be tight up to a constant factor. Finally, we show that our conjecture implies the so-called Erdős girth conjecture.

KEYWORDS

extremal graph theory, girth, minimum degree, radius, triangle-free graphs

1 | INTRODUCTION

The girth of a graph G is the length of the shortest cycle in G; we set the girth to be ∞ if no cycle exists. The radius r of a connected graph G is the smallest integer such that there exists some $v \in V(G)$ with $d(v, w) \leq r$ for every $w \in V(G)$.

Consider the following question: given a connected graph G on n vertices, with minimum degree $\delta \geq 2$ and girth at least $g \geq 4$, what is the maximum radius r this graph can have (note that the connectedness condition is superfluous if we let r be the biggest radius of a connected component)? Denote this maximum value of the radius as $r(n, \delta, g)$.

Erdős, Pach, Pollak and Tuza [2] studied $r(n, \delta, 4)$, and proved that it is at most $\frac{n-2}{\delta} + 12$. They also noted that, up to the additive constant 12, this bound is tight. We improve this to a best possible bound.

Theorem 1.1. *Fix integer $\delta \geq 2$.*

- *If $2\delta \leq n \leq 2\delta + 1$, then $r(n, \delta, 4) = 2$.*
- *If $2\delta + 2 \leq n < 4\delta$, then $r(n, \delta, 4) = 3$.*
- *If $n \geq 4\delta$, then*

$$r(n, \delta, 4) = \begin{cases} \frac{n}{\delta} - 1 & \text{if } \delta \text{ is odd and } n = k\delta \text{ for } k \text{ odd}, \\ \left\lfloor \frac{n}{\delta} \right\rfloor & \text{otherwise}. \end{cases}$$

Observe that every graph of order n and minimum degree greater than $n/2$ has a triangle, so in the study of $r(n, \delta, 4)$, we may assume that $n \geq 2\delta$.

Next we consider the case when the girth g is bigger than 4. We shall prove the following upper bound.

Theorem 1.2. *Let $n, \delta \geq 2$ and $g = 2k \geq 4$. Then*

$$r(n, \delta, 2k) \leq \frac{nk(\delta - 2)}{2((\delta - 1)^k - 1)} + 3k.$$

In the cases $g = 6, 8, 12$, we shall prove the following lower bound.

Theorem 1.3. *Let $\delta \geq 2$ be such that $\delta - 1$ is a prime power. Then there exists sequences $(n_i), (n_i'), (n_i'')$ with $n_i, n_i', n_i'' \to \infty$ such that*

- $r(n_i, \delta, 6) \geq \dfrac{3n_i}{2(\delta^2 - \delta + 1)} - 3 = \dfrac{3n_i(\delta - 2)}{2((\delta - 1)^3 - 1)} - 3,$
- $r(n_i', \delta, 8) \geq \dfrac{2n_i'}{\delta^3 - 2\delta^2 + 2\delta} - 4 = \dfrac{2n_i'(\delta - 2)}{(\delta - 1)^4 - 1} - 4,$
- $r(n_i'', \delta, 12) \geq \dfrac{3n_i''}{((\delta - 1)^3 + 1)(\delta^2 - \delta + 1)} - 6 = \dfrac{3n_i''(\delta - 2)}{(\delta - 1)^6 - 1} - 6.$

We note that the results for the girth 6, 8 and 12 are optimal up to the value of the additive constant, as established by Theorem 1.2. We are very grateful to the anonymous referee for pointing out that our previous approach can be optimised to obtain the correct bounds even for all lower-order terms.

It would be interesting to see whether the upper bound from Theorem 1.2 is tight, at least up to some constant factor. We believe it is and hence make the following conjecture.

Conjecture 1.4. *Let $g = 2k \geq 4$. Then there exist infinitely many values δ for which the following holds.*

There exists a sequence $(n(\delta)_i)$ with $n(\delta)_i \to \infty$ and a positive constant $c(\delta)$ such that $r(n(\delta)_i, \delta, 2k) \geq c(\delta)\frac{n(\delta)_i}{\delta^{k-1}}$.

As our final result, we obtain the following theorem.

Theorem 1.5. *Let $r, c > 0$, $g = 2k$ and $n \leq c(r+1)\delta^{k-1}$, so that $r(n, \delta, g) \geq r$. Then there exists a connected graph of girth at least $2k$ on at most $(2k+1)c\delta^{k-1}$ vertices with at least $\frac{1}{2}\delta^2(\delta-1)^{k-2}$ edges.*

This theorem is related to the following girth conjecture of Erdős.

Conjecture 1.6 (Erdős [1]). *For any positive integers l, n, there exists a graph with girth $2l+1$, n vertices and $\Omega(n^{1+\frac{1}{l}})$ edges.*

If the upper bound from Theorem 1.2 was tight up to a constant factor for some fixed $g = 2k$, then we could find graphs G_i with $\delta_i \to \infty$ and $n_i \leq c(r(n_i, \delta_i, 2k)+1)\delta_i^{k-1}$ for some fixed c. By Theorem 1.5, that would verify the girth conjecture of Erdős for $l = k-1$.

We note that some similar problems relating to various parameters in a graph have been studied in the literature—for instance, the analogous problem for the diameter instead of the radius [2,4], and problems involving more detailed information about the degree sequence of the graph [5].

The structure of the paper is as follows: In Lemma 2.1, we establish a general tool that gives a lower bound on n in terms of r and δ that is tight in many cases. This lemma unfortunately is not strong enough to handle all cases, so we prove the additional Lemma 3.6. We use these two lemmas (the key ingredients of the proof) to prove Lemma 3.4 which establishes the upper bound on r in Theorem 1.1. Together with Lemmas 3.1 and 3.3, which establish the lower bound on r, this completes our proof. Finally, in Section 4, we consider the case of general girth.

Throughout the paper, for a vertex v in a graph, we will denote by $N(v)$ its open neighbourhood, and by $N[v]$ its closed neighbourhood. The difference between closed and open neighbourhoods is that the closed one contains also the vertex v itself.

2 | STRATEGY

The following lemma will be used throughout our paper. It tells us that if we can find a large collection of vertices in our graph such that any two vertices are either neighbours or sufficiently far away from each other, then our graph must in fact have many vertices.

We thank the anonymous referee for pointing out to us how to improve the lemma.

Lemma 2.1. *Assume G is a graph on n vertices of girth $g \geq 2k$ (where $k \geq 2$) with minimum degree δ. Let $T \subset V(G)$ be such that all pairs of nonadjacent vertices in* T *have distance at least $2k-1$ from each other. Then we have $n \geq |T|\left(\frac{(\delta-1)^k-1}{\delta-2}\right)$. Moreover if $|T|$ is odd, this inequality is strict.*

Note that for $k = 2$, that is, the triangle-free case, this means that $n \geq |T|\delta$ and if $|T|$ is odd, then $n > |T|\delta$.

Proof. Since G contains no triangles, we know we can label

$$T = \{x_1, y_1, \dots, x_i, y_i, z_1, \dots, z_j\},$$

with $d(x_l, y_l) = 1$ for $1 \le l \le i$, while all other pairs of vertices in T have distance at least $2k - 1$.

For $v \in T$, let

$$S(v) = \{w \in V(G) \mid d(v, w) = k - 1\}$$

and

$$B(v) = \{w \in V(G) \mid d(v, w) \le k - 1\}.$$

Consider the sets $B(x_1), S(y_1) \backslash B(x_1), \dots, B(x_i), S(y_i) \backslash B(x_i), B(z_1), \dots, B(z_j)$.

Note that by the distance condition, all these sets are disjoint. Moreover, by the girth condition, for any $v \in T$, we have

$$|B(v)| \ge 1 + \delta(1 + (\delta - 1) + \dots + (\delta - 1)^{k-2}) = 1 + \delta\frac{(\delta - 1)^{k-1} - 1}{\delta - 2}$$

and that for any $1 \le l \le i$,

$$|S(y_l) \backslash B(x_l)| \ge (\delta - 1)^{k-1}.$$

We conclude

$$\begin{aligned} n = |V(G)| &\ge \left| \bigcup_{l=1}^{i} B(x_l) \cup \bigcup_{l=1}^{i} S(y_l) \backslash B(x_l) \cup \bigcup_{l=1}^{j} B(z_l) \right| \\ &= \sum_{l=1}^{i} |B(x_l)| + \sum_{l=1}^{i} |S(y_l) \backslash B(x_l)| + \sum_{l=1}^{j} |B(z_l)| \\ &\ge (|T| - i)\left(1 + \delta\frac{(\delta - 1)^{k-1} - 1}{\delta - 2}\right) + i(\delta - 1)^{k-1} \\ &\ge \frac{|T|}{2}\left(1 + \delta\frac{(\delta - 1)^{k-1} - 1}{\delta - 2} + (\delta - 1)^{k-1}\right) \\ &= |T|\left(\frac{(\delta - 1)^k - 1}{\delta - 2}\right). \end{aligned}$$

Moreover, when $|T|$ is odd, the third inequality above is strict (as $|B(v)| > |S(v)|$ for any v and we cannot have $j = 0$ in the odd case). Hence, the proof is finished. □

To find such large collections of points T with restricted distances, we shall use several observations. We formulate these observations used throughout the proof in the following general setting.

Let G be a graph with n vertices and radius r. We take v_0 to be some fixed centre of G. We let v_r be a vertex such that $d(v_0, v_r) = r$, and let $v_0, v_1, \dots, v_r$ be a path of length r from v_0 to v_r.

Fix an integer $m \in \{1, ..., r-1\}$, and let v' be a vertex such that $d(v_m, v') \geq r$. Then let $t \geq 0$ be such that $d(v_0, v') = r - t$, and let $v_0 = v_0', v_1', ..., v_{r-t}' = v'$ be a path of length $r - t$ from v_0 to $v' = v_{r-t}'$.

Observation 2.2. We have $t \leq m$.

Proof of Observation 2.2. Assume for contradiction that we had $t > m$. Then by a triangle inequality

$$d\left(v_m, v_{r-t}'\right) \leq d(v_m, v_0) + d\left(v_0, v_{r-t}'\right) = m + (r-t) < r,$$

which is a contradiction. □

Observation 2.3. For any $m \leq i \leq r$ and any $0 \leq j \leq r - t$, we have

$$d\left(v_i, v_j'\right) \geq d\left(v_m, v_{r-t}'\right) + m + t + j - r - i$$

and for any $i < m$ and any $0 \leq j \leq r - t$, we have

$$d\left(v_i, v_j'\right) \geq d\left(v_m, v_{r-t}'\right) + i + j + t - m - r.$$

Moreover, in either of these cases, we also have

$$d\left(v_i, v_j'\right) \geq |i - j|.$$

Proof of Observation 2.3. For the case $m \leq i \leq r$, note that

$$d\left(v_m, v_{r-t}'\right) \leq d(v_m, v_i) + d\left(v_i, v_j'\right) + d\left(v_j', v_{r-t}'\right) = (i-m) + d\left(v_i, v_j'\right) + (r - t - j).$$

Rearranging gives the result.

For the case $i < m$, note that

$$d\left(v_m, v_{r-t}'\right) \leq d(v_m, v_i) + d\left(v_i, v_j'\right) + d\left(v_j', v_{r-t}'\right) = (m-i) + d\left(v_i, v_j'\right) + (r - t - j).$$

Rearranging gives the result.

For the last claim, note that by triangle inequality

$$d\left(v_i, v_j'\right) \geq |d(v_i, v_0) - d\left(v_0, v_j'\right)| = |i - j|.$$

□

Observation 2.4. We cannot have $v_i = v_i'$ for any $i > \frac{m+r-t-d(v_m, v_{r-t}')}{2}$, and we cannot have $v_i = v_j'$ for any $i \neq j$.

Proof of Observation 2.4. Assume that $v_i = v_i'$ for some $r - t \geq i > \frac{m+r-t-d(v_m, v_{r-t}')}{2}$. Then we obtain contradiction, as $d(v_i, v_i') > 0$ by Observation 2.3.

We cannot have $v_i = v_j'$ for any $i \neq j$, since

$$d(v_0, v_i) = i \neq j = d\left(v_0, v_j'\right). \qquad \square$$

Now we are ready to move on to the case of the triangle-free graphs.

3 | TRIANGLE-FREE GRAPHS

To prove Theorem 1.1, we will establish the following four lemmas.

Lemma 3.1. *Fix integers $n \geq 4, \delta \geq 2$. If $n \geq 2\delta$, there exists a connected triangle-free graph on n vertices with minimum degree δ and radius 2. If $n \geq 2\delta + 2$, there exists a connected triangle-free graph on n vertices with minimum degree δ and radius 3.*

Lemma 3.2. *Every connected triangle-free graph on n vertices with minimum degree $\delta \geq 2$ and radius r satisfies $r \geq 2$ and $n \geq 2\delta$. Moreover, if $r = 3$, we have $n \geq 2\delta + 2$.*

Lemma 3.3. *Fix integers $r \geq 4, \delta \geq 2, c \geq 0$. There exists a connected triangle-free graph with $2\left\lceil \frac{r\delta}{2} \right\rceil + c$ vertices, minimum degree δ and radius r.*

Lemma 3.4. *Every connected triangle-free graph on n vertices with minimum degree $\delta \geq 2$ and radius $r \geq 4$ satisfies $n \geq 2\left\lceil \frac{r\delta}{2} \right\rceil$.*

Let us first see how Theorem 1.1 follows from these.

Proof of Theorem 1.1 *assuming Lemma* 3.1, 3.2, 3.3, 3.4.

Case: $2\delta \leq n \leq 2\delta + 1$.

As $2\delta \leq n$, Lemma 3.1 shows $r(n, \delta, 4) \geq 2$. As $n < 2\delta + 2 \leq 4\delta$, Lemma 3.4 shows $r(n, \delta, 4) < 4$ and Lemma 3.2 shows $r(n, \delta, 4) \neq 3$. We conclude $r(n, \delta, 4) = 2$.

Case: $2\delta + 2 \leq n \leq 4\delta - 1$.

As $2\delta + 2 \leq n$, Lemma 3.1 shows that $r(n, \delta, 4) \geq 3$. As $n < 4\delta$, Lemma 3.4 shows $r < 4$. We conclude $r(n, \delta, 4) = 3$.

Case: $4\delta \leq n$.

In this case, we consider two subcases depending on the precise form of n.

Subcase: $n = k\delta$ with δ and k both odd.

We set $r = \frac{n}{\delta} - 1$, $c = \delta$ and we show that $r(n, \delta, 4) = r$.

By Lemma 3.3, there exists a connected triangle-free graph with $2\left\lceil \frac{r\delta}{2} \right\rceil + c = n$ vertices, minimum degree δ and radius r, and hence $r(n, \delta, 4) \geq r$.

First consider the case $r(n, \delta, 4) < 4$. As $n \geq 4\delta$, we also have $r(n, \delta, 4) < \frac{n}{\delta}$ and hence $r(n, \delta, 4) \leq r$, finishing this case. So further assume $r(n, \delta, 4) \geq 4$. By Lemma 3.4, every connected triangle-free graph on n vertices and of minimum degree $\delta \geq 2$ and radius $r(n, \delta, 4) \geq 4$ satisfies $n \geq 2\left\lceil \frac{r(n, \delta, 4)\delta}{2} \right\rceil$. As n is odd integer and $2\left\lceil \frac{r(n, \delta, 4)\delta}{2} \right\rceil$ is an even integer, we must even have $n - 1 \geq 2\left\lceil \frac{r(n, \delta, 4)\delta}{2} \right\rceil$. So we get $n - 1 \geq r(n, \delta, 4)\delta$. Therefore, $r(n, \delta, 4) < \frac{n}{\delta}$ and hence $r(n, \delta, 4) \leq r$.

Subcase: n is not of the form $k\delta$ with δ and k both odd.

We set $r = \left\lfloor \frac{n}{\delta} \right\rfloor$ and $c = n - 2\left\lceil \frac{r\delta}{2} \right\rceil$ and show that $r(n, \delta, 4) = r$.

By Lemma 3.3, there exists a connected triangle-free graph with $2\lceil \frac{r\delta}{2} \rceil + c = n$ vertices, minimum degree δ and radius r, and hence $r(n, \delta, 4) \geq r$.

First once again consider the case $r(n, \delta, 4) < 4$. As $n \geq 4\delta$, we also have $r(n, \delta, 4) \leq \frac{n}{\delta}$ and hence $r(n, \delta, 4) \leq r$, completing the proof in this case. Hence further assume $r(n, \delta, 4) \geq 4$. By Lemma 3.4, every connected triangle-free graph on n vertices and of minimum degree $\delta \geq 2$ and radius $r(n, \delta, 4) \geq 4$ satisfies $n \geq 2\left\lceil \frac{r(n, \delta, 4)\delta}{2} \right\rceil$. Therefore, $r(n, \delta, 4) \leq \frac{n}{\delta}$ and hence $r(n, \delta, 4) \leq r$. □

In the rest of the section, we will prove Lemmas 3.1–3.4 and thus prove Theorem 1.1. The section will be divided into five subsections—in Section 3.1 we prove Lemma 3.3; in Section 3.2 we prove a technical lemma and we will need to prove Lemma 3.4; in Section 3.3 we prove Lemmas 3.1 and 3.2; in Section 3.4 we prove Lemma 3.4 when $r \in \{4k, 4k + 1, 4k + 2\}$; and in Section 3.5 we prove Lemma 3.4 when $r = 4k + 3$.

3.1 | Proof of Lemma 3.3

It suffices to prove the lemma for $c = 0$. Indeed, given a triangle-free graph G, we can add a vertex while preserving both the radius and the minimum degree: if $v \in V(G)$ is such that $d(v) = \delta$, then add a vertex v' to $V(G)$, which is connected precisely to the same vertices as v is.

For $c = 0$, consider the following example.

Partition $V(G)$ into $2r$ sets labelled $B_0, \ldots, B_{2r-1}$ such that

$$|B_i| = \begin{cases} \left\lceil \dfrac{\delta}{2} \right\rceil & \text{if } i \equiv 0, 1 \bmod 4, \\ \left\lfloor \dfrac{\delta}{2} \right\rfloor & \text{if } i \equiv 2, 3 \bmod 4. \end{cases}$$

Connect all vertices in B_i to all vertices in B_j whenever $i - j \equiv \pm 1 \mod 2r$. Example of such a graph with $r = 5$ and $\delta = 6$ is depicted in Figure 1.

It is easy to see that this is a connected triangle-free graph with $2\left\lceil \frac{r\delta}{2} \right\rceil$ vertices, minimum degree δ and radius r.

3.2 | Technical lemma

First, recall Lemma 2.1 which implies the following result for triangle-free graphs.

Lemma 3.5. *Let G be a triangle-free graph on n vertices and with minimum degree δ. Then for any subset $T \subset V(G)$ such that no two vertices of T are at distance 2, we have $n \geq 2\left\lceil \frac{\delta\,|T|}{2} \right\rceil$.*

We will also need another lemma of similar flavour here.

Lemma 3.6. *Let G be a triangle-free graph on n vertices and with minimum degree δ. Assume for some $r \geq 4$, we have a subset $U \subset V(G)$ such that $|U| = 2r$ and U is as follows:*

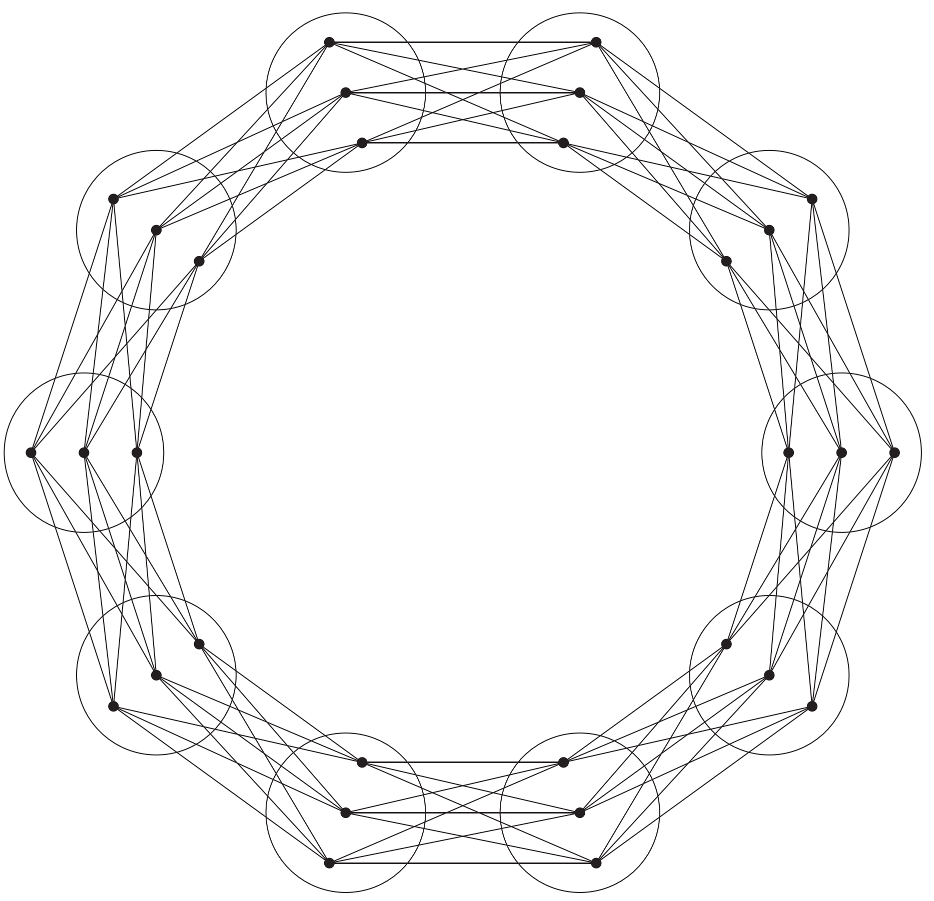

FIGURE 1 Construction from Lemma 3.3 for $r = 5, \delta = 6$

if we consider auxiliary graph H such that $V(H) = U$ and in which we connect two vertices if their distance in G is precisely 2, then H is a disjoint union of two cycles of length r. Then we have $n \geq 2\left\lceil \frac{r\delta}{2} \right\rceil$.

Proof. Let $c_1, \ldots, c_r$ and $d_1, \ldots, d_r$ be our two cycles of length r in H. Consider the open neighbourhoods $N(c_1), \ldots, N(c_r)$. On the one hand, we have $|N(c_i)| \geq \delta$ for $1 \leq i \leq r$. On the other hand, each $v \in V(G)$ can be contained in the neighbourhood of at most two vertices from $\{c_1, \ldots, c_r\}$ by our triangle-free condition.

Further, as no c_i and d_j are at distance 2, and G is triangle-free, no vertex can be contained both in an $N(c_i)$ and in an $N(d_j)$.

Let $B = \bigcup_i N(c_i)$, so that by the above discussion

$$2|B| \geq |N(c_1)| + \cdots + |N(c_r)| \geq r\delta.$$

Since $|B|$ is an integer, we have $|B| \geq \left\lceil \frac{r\delta}{2} \right\rceil$.

Let $B' = \bigcup_j N(d_j)$, so that similarly we get $|B'| \geq \left\lceil \frac{r\delta}{2} \right\rceil$. As B, B' are disjoint, we conclude $n \geq 2\left\lceil \frac{r\delta}{2} \right\rceil$. □

3.3 | Proof of Lemmas 3.1 and 3.2

Here we handle the small radius cases.

Proof of Lemma 3.1. If $n \geq 2\delta$, note that $K_{\delta,n-\delta}$ is a connected triangle-free graph on n vertices of radius 2 and minimum degree δ.

If $n \geq 2\delta + 2$, start with a complete bipartite graph $K_{\delta+1,n-\delta-1}$ with vertex classes $\{v_1, \ldots, v_{\delta+1}\}$ and $\{w_1, \ldots, w_{n-\delta-1}\}$. Erase the edges

$$v_1w_1, v_2w_2, \ldots, v_{\delta+1}w_{\delta+1}, v_{\delta+1}w_{\delta+2}, \ldots, v_{\delta+1}w_{n-\delta-1}.$$

The resulting graph is a connected triangle-free graph on n vertices of radius 3 and minimum degree δ. □

Proof of Lemma 3.2. Consider a connected triangle-free graph G on n vertices of radius r and minimum degree $\delta \geq 2$. We must have $r \geq 2$, since the only connected triangle-free graphs of radius 1 are star graphs, which have minimum degree 1.

Consider adjacent vertices $a, b \in V(G)$. It follows from Lemma 3.5 applied to $T = \{a, b\}$ that $n \geq 2\delta$.

If $r = 3$, then we can take a, b which instead satisfy $d(a, b) \geq 3$. But then even their closed neighbourhoods are disjoint, which implies

$$|V(G)| \geq |N[a] \cup N[b]| = |N[a]| + |N[b]| \geq 2\delta + 2.$$ □

3.4 | Proof of Lemma 3.4 for $r = 4k, 4k + 1, 4k + 2$

In this subsection, we prove the following.

Lemma 3.7. *If G is a connected triangle-free graph on n vertices with minimum degree $\delta \geq 2$ and radius $r \geq 4$ such that $r = 4k + i$ for some k and some $i \in \{0, 1, 2\}$, then we have $n \geq 2\left\lceil \frac{r\delta}{2} \right\rceil$.*

Proof. Let v_0 be a centre of our graph G with a minimal number of vertices at distance r. Let v_r be any vertex such that $d(v_0, v_r) = r$. Let $v_0, v_1, \ldots, v_r$ be a path of length r from v_0 to v_r.

Let v'_{r-t} be the following vertex: if v_3 is not a centre of G, then let v'_{r-t} be any vertex such that $d(v_3, v'_{r-t}) \geq r + 1$. If v_3 is a centre of G, then let v'_{r-t} be a vertex such that $d(v_3, v'_{r-t}) = r$ and $d(v_0, v'_{r-t}) < r$ (such a vertex exists by a choice of v_0).

Let t be so that $d(v_0, v'_{r-t}) = r - t$. Let $v_0 = v'_0, v'_1, \ldots, v'_{r-t}$ be a path of length $r - t$ from v_0 to v'_{r-t}. It follows from Observation 2.2 that $t \leq 3$.

Claim 3:8. If $r = 4k + 1$ and $0 \leq t \leq 2$, or $r \in \{4k, 4k + 2\}$, then $n \geq 2\left\lceil \frac{r\delta}{2} \right\rceil$.

Proof of Claim 3.8. We show that in each of these cases, we can find a collection C of r vertices in G such that no two are at distance 2. The result then follows from Lemma 3.5.

Depending on the values of r and t, choose C to be the following collection.

$r = 4k$	$t = 0$	$v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}, v'_3, v'_4, v'_7, v'_8, \ldots, v'_{4k-1}, v'_{4k}$
$r = 4k$	$t = 1$	$v_0, v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}, v'_3, v'_4, v'_7, v'_8, \ldots, v'_{4k-5}, v'_{4k-4}, v'_{4k-1}$
$r = 4k$	$t = 2$	$v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}, v'_1, v'_2, v'_5, v'_6, \ldots, v'_{4k-3}, v'_{4k-2}$
$r = 4k$	$t = 3$	$v_0, v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}, v'_1, v'_4, v'_5, v'_8, v'_9, \ldots, v'_{4k-4}, v'_{4k-3}$
$r = 4k + 1$	$t = 0$	$v_0, v_4, v_5, v_8, v_9, \ldots, v_{4k}, v_{4k+1}, v'_4, v'_5, v'_8, v'_9, \ldots, v'_{4k}, v'_{4k+1}$
$r = 4k + 1$	$t = 1$	$v_0, v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}, v'_3, v'_4, v'_7, v'_8, \ldots, v'_{4k-1}, v'_{4k}$
$r = 4k + 1$	$t = 2$	$v_0, v_1, v_4, v_5, v_8, v_9, \ldots, v_{4k}, v_{4k+1}, v'_3, v'_4, v'_7, v'_8, \ldots, v'_{4k-5}, v'_{4k-4}, v'_{4k-1}$
$r = 4k + 2$	$t = 0$	$v_0, v_1, v_5, v_6, v_9, v_{10}, \ldots, v_{4k+1}, v_{4k+2}, v'_5, v'_6, v'_9, v'_{10}, \ldots, v'_{4k+1}, v'_{4k+2}$
$r = 4k + 2$	$t = 1$	$v_0, v_1, v_4, v_5, v_8, v_9, \ldots, v_{4k}, v_{4k+1}, v'_4, v'_5, v'_8, v'_9, \ldots, v'_{4k}, v'_{4k+1}$
$r = 4k + 2$	$t = 2$	$v_0, v_1, v_5, v_6, v_9, v_{10}, \ldots, v_{4k+1}, v_{4k+2}, v'_3, v'_4, v'_7, v'_8, \ldots, v'_{4k-1}, v'_{4k}$
$r = 4k + 2$	$t = 3$	$v_1, v_2, v_5, v_6, v_9, v_{10}, \ldots, v_{4k+1}, v_{4k+2}, v'_2, v'_3, v'_6, v'_7, \ldots, v'_{4k-2}, v'_{4k-1}$

Subclaim: *$|C| = r$, and if $v_i, v_j, v'_i, v'_j \in C$, then $d(v_i, v_j), d(v'_i, v'_j) \neq 2$.*

Proof of subclaim. None of the collections above contains both v_1 and v'_1. For all other pairs v_i, v'_j, it follows from Observation 2.4 that $v_i \neq v'_j$. Hence, C consists of r distinct vertices.

Note that $v_0, \ldots, v_r$ is a path of length r and $v'_0, \ldots, v'_{r-t}$ is an induced path of length $r-t$. Hence, C contains no two vertices of the form v_i, v_j such that $d(v_i, v_j) = 2$ and no two vertices of the form v'_i, v'_j such that $d(v'_i, v'_j) = 2$. □

Subclaim: *If $v_i, v'_j \in C$, then $d(v_i, v'_j) \neq 2$.*

Proof of subclaim. If $|i-j| > 2$, the claim follows from Observation 2.4. Henceforth assume $|i-j| \leq 2$.

Case: $i = 1$.

It follows from Observation 2.3 that it suffices to ensure that if our collection contains v_1, then it does not contain:

- v'_1 in the case $v_1 \neq v'_1$,
- v'_2 in the case $d(v_3, v'_{r-t}) + t \leq r + 2$,
- v'_3 in the case $d(v_3, v'_{r-t}) + t \leq r + 1$.

Recall that if $t = 0$, then $d(v_3, v'_{r-t}) \geq r + 1$. Hence, we easily verify that C satisfies these conditions.

Case: $i = 2$.

It follows from Observation 2.3 that it suffices to ensure that if our collection contains v_2, then it does not contain:

- v'_1 in the case $d(v_3, v'_{r-t}) + t \leq r + 2$,
- v'_2 in the case $d(v_3, v'_{r-t}) + t \leq r + 1$.

We again can verify easily that all of the collections above satisfy these conditions.

Case: $i \geq 3$.

Note that by our choice of v_0, v'_{r-t}, we always have either $t \geq 1$ or $d(v_3, v'_{r-t}) \geq r + 1$. If $j \geq i - 1$, it follows from Observation 2.3 that $d(v_i, v'_j) \geq 3$. If $j = i - 2$, $d(v_i, v'_j) \geq 3$ follows from Observation 2.3 under additional assumption that $d(v_3, v'_{r-t}) + t \geq r + 2$. Hence, for $i \geq 3$, it is enough if C does not contain both v_i and v'_{i-2} in the case when we have $d(v_3, v'_{r-t}) + t \leq r + 1$. But it is easy to check this condition is satisfied for all of the collections above. □

The claim follows. □

Claim 3.9. If $r = 4k + 1$ and $t = 3$, we have $n \geq 2\left\lceil \frac{r\delta}{2} \right\rceil$.

Proof of Claim 3.9. We let v''_{r-s} be such a vertex that $d(v_1, v''_{r-s}) \geq r$, then $d(v_0, v''_{r-s}) = r - s$ for some $0 \leq s \leq 1$. We consider two cases based on the value of $d(v''_{r-s}, v'_{4k-2})$.

Case: $d(v''_{r-s}, v'_{4k-2}) \geq 3$.

Let

$$T = \left\{v_2, v_3, v_6, v_7, \ldots, v_{4k-2}, v_{4k-1}, v_1', v_2', v_5', v_6', \ldots, v_{4k-3}', v_{4k-2}', v_{r-s}''\right\}.$$

Assume for a contradiction two vertices of T have distance 2. It follows from Observation 2.3 that one of them has to be v_{r-s}''. Since for any $v, w \in V(G)$, we have

$$d(v, w) \geq |d(v, v_0) - d(w, v_0)|$$

and as also $d(v_{r-s}'', v_{4k-2}') \geq 3$ by assumption, it further follows that the other vertex would have to be v_{4k-2} or v_{4k-1}. Note that if $d(v_i, v_{r-s}'') \leq 2$ for some $1 \leq i \leq 4k-1$, then

$$d\left(v_1, v_{r-s}''\right) \leq d(v_1, v_i) + d\left(v_i, v_{r-s}''\right) \leq (4k-2) + 2 < r$$

yielding a desired contradiction. Hence, no two vertices of T have distance 2 while $|T| = r$. The result then follows from Lemma 3.5.

Case: $d(v_{r-s}'', v_{4k-2}') < 3$.

Since

$$d\left(v_{r-s}'', v_{4k-2}'\right) \geq |d\left(v_{r-s}'', v_0\right) - d\left(v_{4k-2}', v_0\right)| \geq 3 - s \geq 2,$$

this means $s = 1$ and $d(v_{r-1}'', v_{4k-2}') = 2$. Hence there exists a vertex a, such that a is neighbour of both v_{r-1}'' and v_{4k-2}'. Moreover, clearly $d(a, v_0) = r - 2$.

Consider two cases based on the value of $d(a, v_{4k+1})$.

Subcase: $d(a, v_{4k+1}) \geq 3$.

Take

$$T = \left\{v_1, v_2, v_5, v_6, \ldots, v_{4k-3}, v_{4k-2}, v_{4k+1}, v_2', v_3', v_6', v_7', \ldots v_{4k-6}', v_{4k-5}', v_{4k-2}', a\right\}.$$

Assume for a contradiction two vertices of T are at distance 2. It follows from Observation 2.3 one of them has to be a. Since for any v, w in G, we have

$$d(v, w) \geq |d(v, v_0) - d(w, v_0)|$$

as well as $d(a, v_{4k+1}) \geq 3$ and $d(a, v_{4k-2}') = 1$, the other vertex has to be v_{4k-3} or v_{4k-2}. Note that if $d(a, v_i) \leq 2$ for some $3 \leq i \leq 4k-2$, then

$$d\left(v_3, v_{4k-2}'\right) \leq d(v_3, v_i) + d(v_i, a) + d(a, v_{4k-2}) \leq (4k-5) + 2 + 1 < r,$$

a contradiction. Hence, no two vertices of T are at distance 2 and $|T| = r$. The result follows from Lemma 3.5.

Subcase: $d(a, v_{4k+1}) < 3$.

By the triangle inequality, we have

$$d(a, v_{4k+1}) \geq |d(a, v_0) - d(v_0, v_{4k+1})| = 2,$$

so that $d(a, v_{4k+1}) = 2$. Hence, there exists a vertex b such that b is neighbour of both a and v_{4k+1}. Consider

$$U = \left\{v_0, v_1, v_2, v_3, \ldots, v_{4k+1}, v_1', v_2', \ldots, v_{4k-2}', a, b\right\}.$$

We have $|U| = 8k + 2 = 2r$. Consider auxiliary graph H on $V(H) = U$ in which we connect two vertices if their distance in G is precisely 2. H is union of two disjoint cycles of length r, first being $v_0, v_2, \ldots, v_{4k}, b, v_{4k-2}', \ldots, v_2'$, and second being $v_1, v_3, \ldots, v_{4k+1}, a, v_{4k-3}', \ldots, v_1'$. The result then follows from Lemma 3.6. The only nontrivial relationships needed to prove that H is union of two disjoint cycles of length r are

$$d(b, v_{4k-1}), d(b, v_{4k-2}), d(a, v_{4k}), d(a, v_{4k-1}), d(a, v_{4k-2}), d(a, v_{4k-3}) \geq 3.$$

If any of these distances was at most 2, we could find a path of length at most $r - 1$ from v_3 to v_{4k-2}'. That would be a contradiction. □

Putting Claims 3.8 and 3.9 together now finishes the proof of Lemma 3.7. □

3.5 | Proof of Lemma 3.4 for $r = 4k + 3$

In this subsection, we prove the following.

Lemma 3.10. *If G is a connected triangle-free graph on n vertices with minimum degree $\delta \geq 2$ and radius $r \geq 4$ such that $r = 4k + 3$ for some k, then we have $n \geq 2\left\lceil \frac{r\delta}{2} \right\rceil$.*

We use a slightly weaker and more general set-up than we did in the proof of Lemma 3.7. This will have the advantage that we have more freedom in our choice of a centre v_0 as well as in the choice of v_{r-t}'.

Proof. Take v_0 to be any centre of our graph G. Let v_r be any vertex such that $d(v_0, v_r) = r$. Let $v_0, v_1, \ldots, v_r$ be any path of length r from v_0 to v_r. Let v_{r-t}' be any vertex such that $d(v_3, v_{r-t}') \geq r$. Then we have $d(v_0, v_{r-t}') = r - t$ for some $t \geq 0$. Let $v_0 = v_0', v_1', \ldots, v_{r-t}'$ be a path of length $r - t$ from v_0 to v_{r-t}'. By Observation 2.2, we have $t \leq 3$. Moreover, consider a vertex v_{r-s}'' such that $d(v_4, v_{r-s}'') \geq r$.

As before, we have $d(v_0, v_{r-s}'') = r - s$ for some $s \geq 0$. Let $v_0 = v_0'', v_1'', \ldots, v_{r-s}''$ be a path of length $r - s$ from v_0 to v_{r-s}''. By Observation 2.2, we have $s \leq 4$.

We will consider four cases depending on the value of t.

Case: $t = s = 0$.

Let

$$T = \left\{ v_0, v_3, v_3', v_6, v_6'', v_7, v_7'', v_{10}, v_{10}'', v_{11}, v_{11}'', \ldots, v_{r-5}, v_{r-5}'', v_{r-4}, v_{r-4}'', v_{r-1}, v_{r-1}'', v_r, v_r'' \right\}.$$

By Observation 2.3, no two vertices in T have distance 2. The result follows from Lemma 3.5.

Case: $t = 0, 1 \le s \le 4$.

We claim that we can find four vertices z_1, z_2, z_3, z_4 such that no two out of z_1, z_2, z_3, z_4 have distance 2, and for $i = 1, 2, 3, 4$;

$$r - 4 \le d(v_0, z_i) \le r - 3.$$

Set $z_1 = v_{r-4}$, $z_2 = v_{r-3}$ and $z_3 = v_{r-4}''$. By Observation 2.3, we immediately see $d(v_{r-4}, v_{r-4}'') \ge 5$ and $d(v_{r-3}, v_{r-4}'') \ge 4$. If we have a vertex x such that x is neighbour of v_{r-4}'' and $d(v_0, x) \ge r - 4$, we can set $z_4 = x$ and are done. If on the other hand there exists no such x, that implies $d(v_{r-3}', v_{r-4}'') \ge 3$. By Observation 2.3 we have $d(v_{r-4}, v_{r-3}') \ge 4$, $d(v_{r-3}, v_{r-3}') \ge 3$, so we can set $z_4 = v_{r-3}'$. Hence, we can always find suitable z_1, z_2, z_3, z_4.

Let

$$T = \left\{ v_0, v_3, v_3'', v_4, v_4'', v_7, v_7'', v_8, v_8'', \ldots, v_{r-8}, v_{r-8}'', v_{r-7}, v_{r-7}'', z_1, z_2, z_3, z_4, v_r, v_r' \right\}.$$

It follows from Observation 2.3 that no two vertices in T have distance 2. The result follows from Lemma 3.5.

Case: $t = 2$.

Let

$$T = \left\{ v_0, v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}, v_{4k+3}, v_1', v_4', v_5', v_8', v_9', \ldots, v_{4k}'. \; v_{4k+1}' \right\}.$$

It follows from Observation 2.3 that no two vertices in T have distance 2. The result follows from Lemma 3.5.

Case: $t = 3$.

Let w_{r-u} be so that $d(v_1, w_{r-u}) \ge r$ and $d(v_0, w_{r-u}) = r - u$ for some $0 \le u \le 1$. We consider subcases based on the value of $d(w_{r-u}, v_{4k}')$.

Subcase: $d(w_{r-u}, v_{4k}') \ge 3$.

Let

$$T = \left\{ v_0, v_1, v_4, v_5, \ldots, v_{4k}, v_{4k+1}, v_3', v_4', v_7', v_8', \ldots, v_{4k-1}', v_{4k}', w_{r-u} \right\}.$$

Assume for a contradiction that two vertices of T are at distance 2. It follows from Observation 2.3 that one of them has to be w_{r-u}. Since for any v, w in G, we have

$$d(v, w) \geq |d(v, v_0) - d(w, v_0)| \quad \text{and} \quad d\left(w_{r-u}, v'_{4k}\right) \geq 3,$$

it further follows that the other vertex would have to be v_{4k} or v_{4k+1}. If we had $d(v_i, w_{r-u}) \leq 2$ for some $1 \leq i \leq 4k + 1$, then

$$d(v_1, w_{r-u}) \leq d(v_1, v_i) + d(v_i, w_{r-u}) \leq 4k + 2 < r,$$

a contradiction. Hence, no two vertices of T are at distance 2, and $|T| = r$. The result follows from Lemma 3.5.

Subcase: $d(w_{r-u}, v'_{4k}) < 3$.

Since

$$d\left(w_{r-u}, v'_{4k}\right) \geq |d(w_{r-u}, v_0) - d\left(v'_{4k}, v_0\right)| \geq 3 - u \geq 2,$$

we have $u = 1$ and $d(w_{r-1}, v'_{4k}) = 2$. Hence there exists a vertex a such that a is neighbour of both w_{r-1} and v'_{4k}. Moreover, clearly $d(a, v_0) = r - 2$.

Consider two cases based on the value of $d(a, v_{4k+3})$.

Subsubcase: $d(a, v_{4k+3}) \geq 3$

Let

$$T = \left\{v_0, v_3, v_4, v_7, v_8, \ldots, v_{4k-1}, v_{4k}, v_{4k+3}, v'_1, v'_4, v'_5, v'_8, v'_9, \ldots, v'_{4k-4}, v'_{4k-3}, v'_{4k}, a\right\}.$$

Assume for a contradiction that two vertices of T are at distance 2. It follows from Observation 2.3 that one of them has to be a. Since for any v, w in G, we have

$$d(v, w) \geq |d(v, v_0) - d(w, v_0)|, \quad d(a, v_{4k+3}) \geq 3 \quad \text{and} \quad d\left(a, v'_{4k}\right) = 1,$$

the other has to be v_{4k-1} or v_{4k}. Since $d(a, v_i) \leq 2$ for some $3 \leq i \leq 4k$, we find

$$d\left(v_3, v'_{4k}\right) \leq d(v_3, v_i) + d(v_i, a) + d(a, v_{4k}) \leq (4k - 3) + 2 + 1 < r,$$

a contradiction. Hence, no two vertices of T are at distance 2 while $|T| = r$. The result follows from Lemma 3.5.

Subsubcase: $d(a, v_{4k+3}) < 3$.

By the triangle inequality, we have $d(a, v_{4k+3}) \geq 2$, so that $d(a, v_{4k+3}) = 2$. Hence, there exists a vertex b, such that b is neighbour of both a and v_r. Consider

$$U = \left\{v_0, v_1, v_2, v_3, \ldots, v_{4k+3}, v_1', v_2', \ldots, v_{4k}', a, b\right\}.$$

We have $|U| = 8k + 6 = 2r$. Consider the auxiliary graph H on $V(H) = U$ in which two vertices are connected if their distance in G is precisely 2. H is the union of two disjoint cycles of length r, the first being $v_0, v_2, \ldots, v_{4k+2}, b, v_{4k}', \ldots, v_2'$, and the second being $v_1, v_3, \ldots, v_{4k+3}, a, v_{4k-1}', \ldots, v_1'$. Indeed, the only nontrivial relationships needed to prove that H is the union of two disjoint cycles of length r are

$$d(b, v_{4k+1}), d(b, v_{4k}), d(a, v_{4k+2}), d(a, v_{4k+1}), d(a, v_{4k}), d(a, v_{4k-1}) \geq 3.$$

If any of these distances was at most 2, we could find a path of length at most $r - 1$ from v_3 to v_{4k}'. The result follows from Lemma 3.6. This concludes the case $t = 3$.

Case: $t = 1$.

We start with the following useful claim.

Claim 3.11. Assume $r \geq 4$, $r = 4k + 3$ and $t = 1$. Further assume there are four distinct vertices y_1, y_2, y_3, y_4 such that no two out of them have distance 2, and $d(v_0, y_i) \leq 3$ for $i = 1, 2, 3, 4$. Then we have $n \geq 2\left\lceil \frac{r\delta}{2} \right\rceil$.

Proof of Claim 3.11. Let

$$T = \left\{y_1, y_2, y_3, y_4, v_6, v_7, v_{10}, v_{11}, \ldots, v_{4k+2}, v_{4k+3}, v_6', v_7', v_{10}', v_{11}', \ldots, v_{4k-2}', v_{4k-1}', v_{4k+2}'\right\}.$$

It follows by Observation 2.3 that no two vertices of T have distance 2. Moreover, we have $|T| = r$. The result follows by Lemma 3.5. □

We return to the proof of Lemma 3.10 in the case $t = 1$.

Subcase: v_2 is not a centre of G.

There exists a vertex c such that $d(v_2, c) \geq r + 1$, and by the triangle inequality $d(v_0, c) \geq r - 1$ and $d(v_3, c) \geq r$. We consider two cases: if $d(v_0, c) = r$, we could have chosen c in place of v_{r-t}' (as $d(v_3, c) \geq r$) and pass to a case $t = 0$ which we already solved. If, on the other hand, $d(v_0, c) = r - 1$, then let $v_0 = v_0''', v_1''', \ldots, v_{r-1}''' = c$ be a path of length $r - 1$ from v_0 to c. No two out of v_3, v_2, v_3''', v_2''' can have distance 2 by Observation 2.3, using that $d(v_2, c) \geq r + 1$. Hence, we conclude by using Claim 3.11 for $y_1 = v_3$, $y_2 = v_2$, $y_3 = v_3'''$, $y_4 = v_2'''$.

Subcase: v_2 is a centre of G.

The vertices v_0, v_1, v_4, v_5 all have distance at most three to v_2 and no two have distance 2. Now start the proof again with $v_0^\dagger := v_2$ instead of v_0 (choosing some vertices $v_r^\dagger$ and $(v'_{r-t^\dagger})^\dagger$ in place of v_r, and v'_{r-t}). If $t^\dagger \neq 1$, then the conclusion follows as before. If $t^\dagger = 1$, then we can find four distinct vertices

$$y_1 = v_0, \quad y_2 = v_1, \quad y_3 = v_4, \quad y_4 = v_5$$

such that no two out of them have distance 2, and $d(v_2, y_i) \leq 3$ for $i = 1, 2, 3, 4$. We conclude with Claim 3.11.

This finishes the proof of Lemma 3.10. □

4 | GENERAL PROBLEM FOR GIRTH $g \geq 5$

We first establish Theorem 1.2 using Lemma 2.1.

Proof of Theorem 1.2. We will find a large enough collection of vertices T such that no two nonadjacent vertices of T are at distance less than $2k - 1$. The result then follows by Lemma 2.1.

Let v_0 be a centre of G, v_r a vertex with $d(v_0, v_r) = r$, and $v_0, v_1, \ldots, v_{r-1}, v_r$ a path of length r in G from v_0 to v_r. If $r \leq 2k$, we know the inequality holds, so assume $r \geq 2k$. We let v'_{r-t} be a vertex such that $d(v_{2k}, v'_{r-t}) \geq r$ and denote $d(v_0, v'_{r-t}) = r - t$ for some $0 \leq t \leq 2k$. Further, let $v_0 = v'_0, v'_1, \ldots, v'_{r-t}$ be a path of length $r - t$ from v_0 to v'_{r-t}.

Let

$$T = \left\{v_{2ki} \mid 0 \leq i \leq \left\lfloor \frac{r}{2k} \right\rfloor\right\} \cup \left\{v_{2ki+1} \mid 0 \leq i \leq \left\lfloor \frac{r}{2k} \right\rfloor - 1\right\}$$
$$\cup \left\{v'_{2ki} \mid 1 \leq i \leq \left\lfloor \frac{r}{2k} \right\rfloor - 1\right\} \cup \left\{v'_{2ki+1} \mid 1 \leq i \leq \left\lfloor \frac{r}{2k} \right\rfloor - 2\right\}.$$

It follows from Observation 2.4 that the above is a disjoint union. It follows from Observation 2.3 that no two nonadjacent vertices of T are at distance less than $2k - 1$. Hence, we conclude by Lemma 2.1 that

$$n \geq |T| \left(\frac{(\delta - 1)^k - 1}{\delta - 2}\right) \geq \left(\frac{2r}{k} - 6\right) \left(\frac{(\delta - 1)^k - 1}{\delta - 2}\right). \qquad \square$$

We prove the next lemma using an idea similar to one of Erdős, Pollack, Pach and Tuza [2]. Its most important corollary is Theorem 1.3.

Lemma 4.1. *Denote by $f(g, \delta)$ for $\delta \geq 2$, $g \geq 3$ the minimum number of vertices in the graph of girth at least g and minimum degree δ. Then for any $r > \frac{g}{2}$, there exists a* *connected graph G on $n = \left\lceil \frac{2r}{g} \right\rceil f(g, \delta)$ vertices of girth at least g, minimum degree δ and radius at least r.*

Proof. Let H be a connected graph with $f(g, \delta)$ vertices, minimum degree δ and girth at least g. As $\delta > 1$, we know H contains a cycle. Let v, w be two neighbouring vertices of H such that the edge vw is part of a cycle. Let H' be the (still connected) graph obtained by deleting the edge vw from H. By the girth condition, we have $d_{H'}(v, w) \geq g - 1$.

Take $\left\lceil \frac{2r}{g} \right\rceil$ identical disjoint copies of H', called $H'_1, \ldots, H'_{\lceil \frac{2r}{g} \rceil}$, with vertices $v_1, \ldots, v_{\lceil \frac{2r}{g} \rceil}$ and $w_1, \ldots, w_{\lceil \frac{2r}{g} \rceil}$, and connect v_i to w_{i+1}, where $w_{\lceil \frac{2r}{g} \rceil + 1} = w_1$. The resulting graph has $\left\lceil \frac{2r}{g} \right\rceil f(g, \delta)$ vertices, radius at least r, girth at least g, and minimum degree δ. □

Theorem 1.3 follows easily.

Proof of Theorem 1.3. We know (see [3]) that when $\delta - 1$ is a prime power, then

$$\begin{aligned} f(6, \delta) &\leq 2(\delta^2 - \delta + 1), \\ f(8, \delta) &\leq 2(\delta^3 - 2\delta^2 + 2\delta), \\ f(12, \delta) &\leq 2((\delta - 1)^3 + 1)(\delta^2 - \delta + 1). \end{aligned}$$

Hence the result follows directly from Lemma 4.1 by taking

$$n_i = \left\lceil \frac{i}{3} \right\rceil f(6, \delta), \quad n_i' = \left\lceil \frac{i}{4} \right\rceil f(8, \delta), \quad n_i'' = \left\lceil \frac{i}{6} \right\rceil f(12, \delta).$$

Finally, we prove Theorem 1.5. □

Proof of Theorem 1.5. Let v_0 be a centre of our graph, v_r a vertex with $d(v_0, v_r) = r$ and $v_0, \ldots, v_r$ a path of length r.

For $0 \leq i \leq r$, let

$$Q(v_i) := \{v \in V(G) : d(v, v_i) \leq k\}.$$

Every vertex in our graph is in at most $2k + 1$ of these sets, so in particular there is an i_0 so that

$$|Q(v_{i_0})| \leq (2k + 1)c\delta^{k-1}.$$

We easily find using the girth condition that v_{i_0} has at least $\delta(\delta - 1)^{k-2}$ vertices at distance at most $k - 1$ from it. Hence, as all edges adjacent to these vertices are inside $Q(v_{i_0})$, we get that for every i the subgraph induced by $Q(v_{i_0})$ has at least $\frac{1}{2}\delta^2(\delta - 1)^{k-2}$ edges.

We conclude that the subgraph induced by $Q(v_{i_0})$ is a connected graph of girth at least $2k$ on at most $(2k + 1)c\delta^{k-1}$ vertices with at least $\frac{1}{2}\delta^2(\delta - 1)^{k-2}$ edges. □

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REFERENCES

1. P. Erdős, *Extremal problems in graph theory*, Theory of Graphs and its Applications, Proc. Sympos. Smolenice, 1964, pp. 29–36.
2. P. Erdős, J. Pach, R. Pollack, and Z. Tuza, *Radius, diameter and minimum degree*, J. Combin. Theory Ser. B. **47** (1989), 73–79.
3. G. Exoo and R. Jajcay, *Dynamic cage survey*, Electron. J. Combin. **DS** (2013), 16.
4. B. Kim, Y. Rho, B. Song, and W. Hwang, *The maximum radius of graphs with given order and minimum degree*, Discrete Math. **312** (2012), 207–212.
5. J. P. Mazorodze and S. Mukwembi, *Radius, diameter and the degree sequence of graph*, Math. Slovaca. **65** (2015), 1223–1236.

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