Combinatorics and Metric Geometry

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To my grandfather

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

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Abstract

This thesis consists of an introduction and seven chapters, each devoted to a different combinatorial problem.

In Chapter 2 and 3, we consider the main subject of this thesis; the sharp stability of the Brunn-Minkowski inequality (BM). This celebrated theorem from the 19th century asserts that for bodies $A, B \subset \mathbb{R}^k$, we have

$$|A+B|^{1/k} \geq |A|^{1/k} + |B|^{1/k},$$

where $|\cdot|$ is the Lebesgue measure and $A+B := \{a+b : a \in A, b \in B\}$ is the Minkowski sum. Moreover, we have equality if and only if A, B are homothetic convex sets. The stability question, studied in many papers, asks how the distance to equality in BM relates to the distance from A, B to homothetic convex sets. In particular, given Brunn-Minkowsi deficit

$$\delta := \frac{|A+B|^{1/k}}{|A|^{1/k} + |B|^{1/k}} - 1,$$

and normalized volume ratio

$$t := \frac{|A|^{1/k}}{|A|^{1/k} + |B|^{1/k}},$$

what is the best bound one can find on

$$\omega := \frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|},$$

where $K_A \supset A$, and $K_B \supset B$ are homothetic convex sets of minimal size? In Chapter 2, we prove a conjecture by Figalli and Jerison establishing the sharp stability for homothetic sets. In particular, we show that for homothetic sets, we have $\omega = O_k\left(\delta t^{-1}\right)$, for δ sufficiently small. In Chapter 3, we establish the sharp stability for planar sets, i.e. we show that for planar sets and δ sufficiently small, we have $\omega = O(\delta^{1/2} t^{-1/2})$. A crucial result in Chapter 3

shows that for any $\varepsilon > 0$, if δ is sufficiently small, then we have

$$|\operatorname{co}(A+B) \setminus (A+B)| \leq (1+\varepsilon)(|\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B|).$$

In Chapter 4, we consider a reconstruction problem for functions on graphs. Given a function $f : V(G) \to [k]$ on the vertices of a graph G and a random walk $(U_i)_{i=1}^{\infty}$ on that graph, can we reconstruct f (up to automorphisms) based on just $(f(U_i))_{i=1}^{\infty}$? Gross and Grupel showed this was not generally possible on the hypercube, by constructing non-isomorphic *locally p-biased* sets X, so that for each vertex v the fraction of neighbours which is in X is exactly p. Answering a question of Gross and Grupel, we construct uncountably many non-isomorphic partitions of $\mathbb{Z}^k$ into $2k$ parts such that every element of $\mathbb{Z}^k$ has exactly one neighbour in each part. As a result, we find *locally p-biased* sets for all $p = c/2n$ with $c \in \{0, \dots 2n\}$.

In Chapter 5, we prove the complete graph case of the bunkbed conjecture. Given a graph G, let the bunkbed graph $BB(G)$ be the graph $G \square K_2$, i.e. the graph obtained from considering two copies of G and connecting equivalent vertices with an edge. The bunkbed conjecture posed by Kasteleyn in 1985 asserts the very intuitive statement that when considering percolation with uniform parameter p, we have $\mathbb{P}(u_1 \leftrightarrow v_1) \geq \mathbb{P}(u_1 \leftrightarrow v_2)$, i.e. a vertex has a higher probability of being connected to a vertex in the same copy of G than being connected to the equivalent vertex in the other copy of G.

In Chapter 6, we consider the (t,r) broadcast domination number, a generalisation of the domination number in graphs. In this form of domination, we consider a set $\mathcal{T} \subset V(G)$ of towers which broadcast at strength t, where broadcast strength decays linearly with distance in the graph. A set of towers is (t,r) broadcast dominating if every vertex in the graph receives at least r signal from all towers combined. More formally, the (t,r) broadcast domination number of a graph G is the minimal cardinality of a set $\mathcal{T} \subset V(G)$ such that for every vertex $v \in V(G)$, we have

$$\sum_{u \in \mathcal{T}} \max\{t - d(u,v), 0\} \geq r.$$

Proving a conjecture by Drews, Harris, and Randolph, we establish that the minimal asymptotical density of $(t,3)$ broadcasting subset of $\mathbb{Z}^2$ is the same as the minimal asymptotical density of a $(t-1,1)$ broadcasting subset of $\mathbb{Z}^2$.

In Chapter 7, we consider the eternal game chromatic number, a version of the game chromatic number in which the game continues after all vertices have been coloured. We show that with high probability $\chi_g^{\infty}(G_{n,p}) = (p/2 + o(1))n$ for odd n, and also for even n when $p = 1/k$ for some $k \in \mathbb{N}$. The upper bound applies for even n and any other value of p

as well, but we conjecture in this case this upper bound is not sharp. Finally, we answer a question posed by Klostermeyer and Mendoza.

In Chapter 8, we consider the bridge-burning cops and robbers game, a version of the game where after a robber moves over an edge, the edge is removed from the graph. Proving a generalization of a conjecture by Kinnersley and Peterson, we establish the asymptotically maximal capture time in this game for graphs with bridge-burning cops number at least three. In particular, we show that this maximal capture time grows as

$$k^{-O(k)}n^{k+2},$$

where $k \geq 3$ is the bridge burning cop number and n is the number of vertices of the graph.

Table of contents

Nomenclature **xvii**

1 Introduction **1**

1.1 Sharp Stability of the Brunn-Minkowski inequality 1

1.1.1 History 2

1.1.2 Homothetic sets 3

1.1.3 Planar sets 4

1.2 Locally biased partitions of $\mathbb{Z}^k$ 4

1.3 The bunkbed conjecture on the complete graph 5

1.4 The (t,r) broadcast domination number of the infinite grid 5

1.5 The eternal game chromatic number of random graphs 6

1.6 Capture times in the Bridge-Burning Cops and Robbers game 7

2 Sharp stability of Brunn-Minkowski for homothetic regions **9**

2.1 Introduction 9

2.1.1 Main theorem 9

2.1.2 Outline 11

2.2 Initial Reduction 13

2.3 Setup and technical lemmas 15

2.4 Proof of Theorem 2.2.1 21

2.5 Sharpness of C_n 24

3 Sharp quantitative stability of the planar Brunn-Minkowski inequality **27**

3.1 Introduction 27

3.1.1 Outline of the chapter 29

3.2 Setup 31

3.2.1 Equal area reformulation 31

3.2.2 Preliminary affine transformation 32

3.2.3 Definitions . . . 33
3.2.4 General Observations . . . 34
3.2.5 Constants and their dependencies . . . 34
3.3 Initial structural results . . . 34
3.3.1 Showing $\mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t)$ contain a large scaled copy of K . . . 35
3.3.2 Showing points in ∂K, $\partial\,\mathrm{co}(A)$, $\partial\,\mathrm{co}(B)$, $\partial\,\mathrm{co}(D_t)$ are $(59^\circ, \frac{1}{3})$-bisecting 36
3.3.3 Showing D_t contains a large scaled copy of $\mathrm{co}(D_t)$ . . . 36
3.4 Decomposing $\partial\,\mathrm{co}(D_t)$ into good arcs, and bad arcs of small total angular size 38
3.5 Replacing $5t^{-1}\sqrt{\gamma}$ with $\xi\sqrt{\gamma}$ on arcs in $\mathcal{I}^{\mathrm{good}}_{2\ell}(\theta,\ell)$ . . . 40
3.6 Covering $\partial\,\mathrm{co}(D_t)$ with parallelograms . . . 48
3.6.1 Definitions . . . 49
3.6.2 Covering $\partial\,\mathrm{co}(D_t)$ with parallelograms . . . 50
3.7 Preimages of the $R_{\mathfrak{q}}$ associated to A and B. . . . 52
3.8 Far away weighted averages in $\partial\,\mathrm{co}(D_t)$ lie in $\mathcal{J}^{\mathrm{good}}_{3\ell}(\theta,\ell)$ . . . 53
3.9 Bound on parallelograms jutting out of $\mathrm{co}(A), \mathrm{co}(B)$ . . . 55
3.10 Bounding overlapping parallelograms . . . 58
3.11 Proof of Theorem 1.1.5 and Theorem 3.2.2 . . . 61
3.12 Proof that Theorem 1.1.5 implies Theorem 1.1.4 . . . 62
3.13 Equivalence of measures ω and α . . . 64

4 Locally biased partitions of $\mathbb{Z}^k$ 67
4.1 Introduction . . . 67
4.2 There is a locally biased partition of $\mathbb{Z}^n$ for every $n \in \mathbb{N}$ . . . 68
4.3 Counting locally biased partitions . . . 70
4.4 Open Problems . . . 76

5 The bunkbed conjecture on the complete graph 79
5.1 Introduction . . . 79
5.2 Proof of Theorem 5.1.1 . . . 80

6 The (t,r) broadcast domination number of some regular graphs 83
6.1 Introduction . . . 83
6.2 Proof of Theorem 1.4.1 . . . 83
6.3 Generalizations of the (t,r) broadcast number for grids . . . 92
6.4 Proof of Theorem 1.4.2 . . . 96
6.5 Proof of Theorem 1.4.3 . . . 97
6.6 Concluding Remarks . . . 98

7 The Eternal Game Chromatic Number of Random Graphs 101
7.1 Introduction 101
7.2 Upper bound 103
7.3 Lower bound for odd n 106
7.4 Generalization of the lower bound for odd n 110
7.5 Even n 113
7.6 Conjecture 7.1.3 115
7.7 Proofs of structural results in the random graph 116
7.8 Answer to a question of Klostermeyer and Mendoza 119

8 Capture times in the Bridge-burning Cops and Robbers game 121
8.1 Introduction 121
8.2 Catching Times 121
8.3 Proof of Theorem 1.6.1 123
8.4 $G_{n,p}$ has $c_b(G) = 1$ with high probability 131
8.5 Concluding remarks 132

References 133

Nomenclature

Roman Symbols

$\mathrm{co}(A)$	convex hull of A
$D(A;t)$	$tA+(1-t)A$
$t(A,B)$	normalized volume ratio $\frac{\lvert A\rvert^{1/k}}{\lvert A\rvert^{1/k}+\lvert B\rvert^{1/k}}$
e_i	i-th canonical basis vector
$\mathrm{capt}(G)$	capture time of G in the classic cops and robbers game
$\mathrm{capt}_b(G)$	capture time of G in the bridge-burning cops and robbers game
$c(G)$	cop number of G
$c_b(G)$	bridge-burning cop number of G
C_n	circular graph on n vertices
$G_{n,p}$	Erdős-Renyi random graph with paramaters $n \in \mathbb{N}$ and $p \in [0,1]$, i.e. the random subgraph of K_n retaining every edge independently with probability p.
K_n	complete graph on n vertices
$K_{m,n}$	complete bipartite graph on $m+n$ vertices
P_n	path graph on $n+1$ vertices with n edges

Greek Symbols

$\alpha(A,B)$	Fraenkel asymmetry index, $\inf_{x\in\mathbb{R}^n} \frac{\lvert A\Delta(s\cdot\mathrm{co}(B)+x)\rvert}{\lvert A\rvert}$, where $s=\frac{\lvert A\rvert^{1/k}}{\lvert\mathrm{co}(B)\rvert^{1/k}}$.
$\delta'(A;t)$	$\mid D(A;t)\setminus A\mid$
$\delta(A,B)$	Brunn-Minkowski deficit, $\frac{\lvert A+B\rvert^{1/k}}{\lvert A\rvert^{1/k}+\lvert B\rvert^{1/k}}-1$

$\gamma(A,B)$	$t^2 \mid \mathrm{co}(A) \setminus A \mid + (1-t)^2 \mid \mathrm{co}(B) \setminus B \mid$.
$\omega(A,B)$	$\frac{\lvert K_A \setminus A \rvert}{\lvert A \rvert} + \frac{\lvert K_B \setminus B \rvert}{\lvert B \rvert}$, where $K_A \supset A$, and $K_B \supset B$ are homothetic convex sets of minimal size.
$\chi(G)$	chromatic number of G
$\chi_g^{\infty}(G)$	eternal game chromatic number of G
$\chi_g(G)$	game chromatic number of G
$\delta_{t,r}(\mathbb{Z}^2)$	minimal asymptotic density of a (t,r) broadcasting subset of $\mathbb{Z}^2$
$\Gamma(v)$	neighbourhood of a vertex v
$\gamma_{t,r}(G)$	(t,r) broadcast domination number of G

Other Symbols

∂A	boundary of A
$A+B$	Minkowski sum, i.e. $\{a+b : a \in A, b \in B\}$
$\sqcup$	disjoint union
$[m,n]$	$\{m,\ldots,n\}$
$[n]$	$\{1,\ldots,n\}$

Acronyms / Abbreviations

whp	with high probability

Chapter 1

Introduction

This thesis is organised into eight chapters of which this introduction is the first. The chapters are devoted to different though sometimes related combinatorial problems. The first two chapters represent my main line of research. The other chapters contain results in more distinct directions all about questions in different branches of graph theory.

1.1 Sharp Stability of the Brunn-Minkowski inequality

The results in this section are all joint with Marius Tiba and Hunter Spink.

Given bodies $A, B \subset \mathbb{R}^n$ of positive measure, the Brunn-Minkowski inequality says

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}},$$

with equality for homothetic convex sets A and B (less a measure 0 set). Here $A+B = \{a+b \mid a \in A, b \in B\}$ is the *Minkowski sum*, and $|\cdot|$ refers to the Lebesgue measure. Stability results for the Brunn-Minkowski inequality quantify how close A, B are to homothetic[1] convex[2] sets in terms of

- $\delta = \delta(A,B) := \frac{|A+B|^{\frac{1}{n}}}{|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}} - 1$, the *Brunn-Minkowski deficit*,
- $t = t(A,B) := \frac{|A|^{\frac{1}{n}}}{|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}}$, the *normalized volume ratio*, and
- $\omega = \omega(A,B) := \frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|}$, where $K_A \supset A$, and $K_B \supset B$ are homothetic convex sets of minimal size.

[1] Sets $X, Y \subset \mathbb{R}^n$ are homothetic if $X = v + \lambda Y$, for some $v \in \mathbb{R}^k$ and $\lambda > 0$.

[2] A set X is convex if $x, y \in X$ implies $tx + (1-t)y \in X$ for all $t \in [0,1]$.

Much of the study of the stability of the Brunn-Minkowski inequality has focused on the following question.

Question 1.1.1. *For $n \geq 1$ do there exist exponents a_n, b_n such that the following is true, and if so what are the optimal exponents (prioritized in this order)? There is a constant C_n and constants $d_n(\tau) > 0$ for $\tau \in (0, \frac{1}{2}]$ such that whenever $A, B \subset \mathbb{R}^n$ are measurable sets with $t \in [\tau, 1-\tau]$ and $\delta \leq d_n(\tau)$, there exist homothetic convex sets $K_A \supset A$ and $K_B \supset B$ such that*

$$\frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|} \leq C_n \tau^{-b_n} \delta^{a_n}.$$

We prioritize the exponents a_n, b_n in this order since if the inequality holds for (a_n, b_n), then the inequality also holds for (a'_n, b'_n) whenever $a_n > a'_n$ by taking $d'_n(\tau)$ sufficiently small.

Question 1.1.1 is one of the central open problems in the study of geometric inequalities, and has been studied intensely in recent years by Barchiesi and Julin [4], Carlen and Maggi [18], Christ [20], Figalli and Jerison [28, 29, 31], Figalli, Maggi and Mooney [32], and Figalli, Maggi and Pratelli [33, 34].

1.1.1 History

In a landmark paper, Figalli and Jerison [29, Theorem 1.3] answered the first part of Question 1.1.1, with computable suboptimal exponents on τ and δ, and with the exponent of δ depending on τ (which we rephrase for the convenience of the reader).

Theorem 1.1.2. *There exist computable constants $a_n(\tau), b_n$ such that the following is true. There are computable constants C_n and $d_n(\tau) > 0$ such that whenever $A, B \subset \mathbb{R}^n$ with $t \in [\tau, 1-\tau]$ and $\delta \leq d_n(\tau)$, there exist homothetic convex sets $K_A \supset A$ and $K_B \supset B$ such that*

$$\frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|} \leq C_n \tau^{-b_n} \delta^{a_n(\tau)}.$$

This naturally gives rise to the second part of Question 1.1.1, asking for the optimal exponents of δ and τ, prioritized in this order. This question, with A, B restricted to various sub-classes of geometric objects, is the subject of a large body of research.

Before Figalli and Jerison in [29] introduced ω, another measure for quantifying how close A, B are to homothetic convex sets was used. The *Fraenkel asymmetry index* is defined to be

$$\alpha(A, B) = \inf_{x \in \mathbb{R}^n} \frac{|A \triangle (s \cdot \mathrm{co}(B) + x)|}{|A|}$$

where s satisfies $|A| = |s \cdot \mathrm{co}(B)|$ and $\mathrm{co}(B)$ is the *convex hull of* B, i.e. the smallest convex set containing B. Providing an upper bound for ω is stronger than providing an upper bound for α as we always have $\alpha \leq 2\omega$. Note that in $\mathbb{R}^2$ when A, B are both convex and δ is bounded, there is a reverse inequality (see Section 3.13).

Prior to [29], Christ [20] had proved via a compactness argument that for a given τ, $\omega \to 0$ as $\delta \to 0$. When A and B are convex, the optimal inequality $\alpha \leq C_n \tau^{-\frac{1}{2}} \delta^{\frac{1}{2}}$ was obtained by Figalli, Maggi, and Pratelli in [33, 34]. When B is a ball and A is arbitrary, the optimal inequality $\alpha \leq C_n \tau^{-\frac{1}{2}} \delta^{\frac{1}{2}}$ was obtained by Figalli, Maggi, and Mooney in [32]. We note that this particular case is intimately connected with stability for the isoperimetric inequality. When just B is convex the (non-optimal) inequality $\alpha \leq C_n \tau^{-(n+\frac{3}{4})} \delta^{\frac{1}{4}}$ was obtained by Carlen and Maggi in [18]. Finally, Barchiesi and Julin [4] showed the optimal inequality $\alpha \leq C_n \tau^{-\frac{1}{2}} \delta^{\frac{1}{2}}$ in the case that just B is convex, subsuming these previous results.

Before their general result for distinct sets A, B in [29], Figalli and Jerison [28] had considered the case $A = B$ and gave a polynomial upper bound $\omega \leq C_n \delta^{a_n}$. Later, in [31], they conjectured the sharp bound $\omega \leq C_n \delta$ when $A = B$, and proved it in dimensions 2 and 3 using an intricate analysis which unfortunately does not extend to higher dimensions. Afterwards, Figalli and Jerison suggested a stronger conjecture that $\omega \leq C_n \tau^{-1} \delta$ for A, B homothetic regions. [30]

1.1.2 Homothetic sets

Our first result in this direction [51] proves this last conjecture by Figalli and Jerison from [31], i.e. we establish the sharp quantitative stability of the Brunn-Minkowski inequality for homothetic sets.

Theorem 1.1.3. *For all $n \geq 2$, there is a constant $C_n > 0$ and constants $d_n(\tau) > 0$ for each $\tau \in (0, \frac{1}{2}]$ such that the following is true. If $A, B \subset \mathbb{R}^n$ are measurable homothetic sets such that $t \in [\tau, 1-\tau]$ and $\delta \leq d_n(\tau)$, then*

$$\omega \leq C_n \tau^{-1} \delta.$$

For these optimal exponents, we additionally show that $\exp(\Omega(n)) \leq C_n \leq \exp(O(n \log n))$ with explicit constants.

This result is the subject of Chapter 2, which is adapted from [51].

1.1.3 Planar sets

Our second result [50] solved the sharp stability question for planar regions $A, B \subset \mathbb{R}^2$, showing that the optimal exponents are $(a_2, b_2) = (\frac{1}{2}, \frac{1}{2})$.

Theorem 1.1.4. *There are constants $C, d(\tau) > 0$ such that if $A, B \subset \mathbb{R}^2$ are measurable sets with $t \in [\tau, 1-\tau]$ and $\delta \leq d(\tau)$, then*

$$\omega \leq C\tau^{-\frac{1}{2}}\delta^{\frac{1}{2}}.$$

Taking $A = [0,t] \times [0, t(1+\varepsilon)]$ and $B = [0, (1-t)(1+\varepsilon)] \times [0, 1-t]$ shows that the exponents a_2, and b_2 are optimal.

Our key result in proving Theorem 1.1.4 is a strong generalization to arbitrary sets A, B of the linear stability for homothetic sets from the previous subsection. The generalization we prove in Chapter 3 involves a completely different analysis to the one in Chapter 2, and we are unaware of a similar approach used previously.

Theorem 1.1.5. *For all $\varepsilon, \tau > 0$ there is a constant $d_\tau(\varepsilon) > 0$ such that the following is true. Suppose that $A, B \subset \mathbb{R}^2$ are measurable sets with $t \in [\tau, 1-\tau]$ and $\delta \leq d_\tau(\varepsilon)$. Then*

$$|\mathrm{co}(A+B) \setminus (A+B)| \leq (1+\varepsilon)\left(|\mathrm{co}(A) \setminus A| + |\mathrm{co}(B) \setminus B|\right).$$

Taking $A = B = \{(x,y) \in [0,1]^2 : x+y \leq 1\} \cup \{(0, 1+\lambda), (1+\lambda, 0)\}$ shows that the constant $1+\varepsilon$ is optimal.

This result is the subject of Chapter 3, which is adapted from [50].

1.2 Locally biased partitions of $\mathbb{Z}^k$

In Chapter 4, we consider the following reconstruction problem on graphs. This work has been adapted from [48].

Given a function f on the vertex set of some graph G, a scenery, let a simple random walk run over the graph and produce a sequence of values. Is it possible to, with high probability, reconstruct the scenery f from this random sequence?

To show this is impossible for some graphs, Gross and Grupel, in [42], call a function $f : V \to \{0,1\}$ on the vertex set of a graph $G = (V, E)$ *locally p-biased* if for each vertex v the fraction of neighbours on which f is 1 is exactly p. Clearly, two locally p-biased functions are indistinguishable based on their sceneries. Gross and Grupel construct locally p-biased functions on the hypercube $\{0,1\}^n$ and ask for what $p \in [0,1]$ there exist locally p-biased

functions on $\mathbb{Z}^k$ and additionally how many there are. In Chapter 4, we fully answer this question by giving a complete characterization of these values of p. In particular, we show that locally p-biased functions exist for all $p = c/2k$ with $c \in \{0, \ldots, 2k\}$ and that, in fact, there are uncountably many of them for every $c \in \{1, \ldots, 2k-1\}$. To this end, we construct uncountably many partitions of $\mathbb{Z}^k$ into $2k$ parts such that every element of $\mathbb{Z}^k$ has exactly one neighbour in each part. This additionally shows that not all sceneries on $\mathbb{Z}^k$ can be reconstructed from a sequence of values attained on a simple random walk.

1.3 The bunkbed conjecture on the complete graph

In Chapter 5, we prove the complete graph case of the bunkbed conjecture. The work in this chapter was done jointly with Piet Lammers and has been adopted from [49].

The bunkbed conjecture was first posed by Kasteleyn. If $G = (V, E)$ is a finite graph and H some subset of V, then the bunkbed of the pair (G, H) is the graph $G \times \{1, 2\}$ plus $|H|$ extra edges to connect for every $v \in H$ the vertices $(v, 1)$ and $(v, 2)$. The conjecture considers the independent bond percolation model, i.e. every edge of the graph is retained independently with probability p. The conjecture then asserts that $(v, 1)$ is more likely to connect with $(w, 1)$ than with $(w, 2)$ for any $v, w \in V$. This is intuitive because $(v, 1)$ is in some sense closer to $(w, 1)$ than it is to $(w, 2)$. The conjecture has however resisted several attempts of proof. This chapter settles the conjecture in the case of a constant percolation parameter and G the complete graph.

1.4 The (t, r) broadcast domination number of the infinite grid

In Chapter 6, we prove a conjecture by Drews, Harris, and Randolph on the (t, r) broadcast domination number of $\mathbb{Z}^2$. The work in this chapter was done jointly with Rebekah Herrman and has been adopted from [46].

The *domination number* of a graph G is the cardinality of the smallest dominating set of the graph, which is the smallest set S such that every vertex in $V(G) \setminus S$ is adjacent to a vertex of S. In 2014, Blessing, Insko, Johnson, and Mauretour generalized this notion to (t, r) broadcast domination [7]. In broadcast domination, there is a collection $\mathcal{T} \subset V(G)$ of vertices called towers that transmit a signal $t \in \mathbb{N}$ in the following manner. If $u \in \mathcal{T}$, and $v \in G$, then the signal at v from u is denoted $f_u(v)$ and is $f_u(v) = max\{0, t - d(u, v)\}$, where $d(u, v)$ is the distance between u and v. The set $\mathcal{T}$ is said to be (t, r) *broadcast dominating*

if each tower transmits a signal t and for all $v \in G$, $\Sigma_{u\in\mathcal{T}} f_u(v) \geq r$. The (t,r) broadcast domination number of G, $\gamma_{t,r}(G)$ is the minimum cardinality of a (t,r) broadcasting set $\mathcal{T}$. Note that when $t = r = 1$, the (t,r) broadcast domination number is the domination number.

The (t,r) broadcasting domination number has been studied for two-dimensional grids, paths, triangular grids, matchstick graphs, and n-dimensional grids [7, 22, 25, 45, 72]. Asymptotic bounds of the $(t,2)$ broadcast domination number on finite grids has been studied [69], as well.

To describe the (t,r) broadcast domination number of $\mathbb{Z}^2$, we consider the *density* of a set $\mathcal{T} \subset \mathbb{Z}^2$ defined as $\limsup_{n\to\infty} \frac{|\mathcal{T}\cap[-n,n]^2|}{(2n+1)^2}$. Accordingly, let $\delta_{t,r}(\mathbb{Z}^2)$ be the minimal density of a (t,r) broadcasting set in $\mathbb{Z}^2$. In 2019, Drews, Harris, and Randolph [25] showed that $\delta_{t,3}(\mathbb{Z}^2) \leq \delta_{t-1,1}(\mathbb{Z}^2) = \frac{1}{2t^2-6t+5}$ for grid graphs $\mathbb{Z}^2$ and conjectured $\delta_{t,3}(\mathbb{Z}^2) = \delta_{t-1,1}(\mathbb{Z}^2)$ for $t > 2$. We prove this conjecture for $t > 17$.

Theorem 1.4.1. *For $t \geq 17$, $\delta_{t,3}\left(\mathbb{Z}^2\right) = \delta_{t-1,1}\left(\mathbb{Z}^2\right)$*

Following the proof of Theorem 1.4.1, in Section 6.3, we explore other statements in this direction and suggest some conjectures.

Additionally, we extend a result by Crepeau, Harris, Hays, Loving, Rennie, Rojas Kirby, and Vasquez about the (t,r)-broadcast domination number of paths [22] to powers of paths. Recall that the kth power of a graph G, denoted $G^{(k)}$ is a graph with the same vertex set as G in which edges are drawn between all vertices that are distance at most k apart in G.

Theorem 1.4.2. *Let $n \geq 1$ and $t \geq r \geq 1$. Then $\gamma_{t,r}(P_n^{(k)}) = \left\lceil \frac{n+k(r-1)}{2kt-k(r+1)+1} \right\rceil$.*

Crepeau et. al. found $\gamma_{t,r}(C_n) \leq \lceil \frac{n+r-1}{2t-r} \rceil$ and asked if this bound could be improved [22]. We answer their question by giving the exact value for the (t,r) broadcast domination number for all powers of cycles:

Theorem 1.4.3. *Let $n \geq 1$ and $t \geq r \geq 1$. Then*

$$\gamma_{t,r}(C_n^{(k)}) = \begin{cases} 1 & \text{if } n \leq 2(t-r)k+1 \\ 2 & \text{if } 2(t-r)k+1 < n \leq (2t-r-1)k+1 \\ \left\lceil \frac{n}{(2t-r-1)k+1} \right\rceil & \text{if } n > (2t-r-1)k+1 \end{cases}$$

1.5 The eternal game chromatic number of random graphs

In Chapter 7, we compute the eternal game chromatic number of random graphs. The work in this chapter was done jointly with Vojtěch Dvořák, and Rebekah Herrman and has been adopted from [26].

The eternal vertex colouring game, recently introduced by Klostermeyer and Mendoza [60], is a version of the vertex colouring game, a well studied graph game. The game is played by two players, Alice and Bob on a graph G with a set of colours $\{1,\dots,k\}$. The game consists of rounds, such that in each round, every vertex is coloured exactly once. In the first round, players alternate turns with Alice playing first. During their turn, each player first picks yet uncoloured vertex and then colours it by any colour so that the resulting partial colouring of the graph is proper (if at least one such colour exists). During all further rounds, players keep choosing vertices alternately. After choosing a vertex, the player assigns a colour to the vertex which is distinct from its current colour such that the resulting colouring is still proper. Each vertex retains its colour between rounds until it is recoloured. Bob wins if at any point the chosen vertex does not have a legal recolouring, while Alice wins if the game is continued indefinitely. The *eternal game chromatic number* $\chi_g^\infty(G)$ is the smallest number k such that Alice has a winning strategy.

In this chapter, we study the eternal game chromatic number of random graphs. We show that with high probability $\chi_g^\infty(G_{n,p}) = \left(\frac{p}{2}+o(1)\right)n$ for odd n, and also for even n when $p=\frac{1}{k}$ for some $k\in\mathbb{N}$. The upper bound applies for even n and any other value of p as well, but we conjecture in this case this upper bound is not sharp. Finally, we answer a question posed by Klostermeyer and Mendoza.

1.6 Capture times in the Bridge-Burning Cops and Robbers game

In Chapter 8, we construct a graph with asymptotically maximal capture time in the Bridge-Burning Cops and Robbers game. The work in this chapter was done jointly with Rebekah Herrman, and Stephen G.Z. Smith and has been adopted from [47].

Cops and Robbers is a well-studied game on a graph G with two players, the *cops* and the *robber*. The game begins with the cops choosing their starting vertices, followed by the robber selecting his. The cops and robber then alternate turns moving from their current vertex to an adjacent one or choosing not to move. A *round* consists of a cop turn and the subsequent robber turn. The cops win, or capture the robber, if a cop and the robber occupy the same vertex, whereas the robber wins if he manages to avoid capture.

Several variants of the game have been introduced over the years, including those where the robbers can move more quickly than the cops [3, 38], where the cops have imperfect information [21], and where only one cop may move during the cops' turn [23]. Recently, Kinnersley and Peterson [59] introduced the variant *bridge-burning cops and robbers*. In

the bridge-burning version, each time the robber moves from a vertex u to a vertex v, the edge uv is erased from the graph. Using the notation introduced in [59], $c_b(G)$ is defined to be the *bridge burning cop number*, which is the minimum number of cops required to catch the robber on the graph G in the bridge-burning game. Kinnersley and Peterson studied the game on numerous graphs including paths P_n, cycles C_n, complete bipartite graphs $K_{m,n}$, hypercubes Q_n, and two dimensional finite grids $G_{m,n}$ [59].

A related notion to $c_b(G)$ is the *capture time* of G, denoted $capt_b(G)$. The bridge-burning capture time is the minimum number of rounds it takes for the cop to capture the robber in the bridge-burning variant. The capture time of the original cops and robbers game was introduced in 2009 by Bonato, Golovach, Hahn, and Kratochvíl [11] and has been studied on trees [75] and planar graphs [68] among various other classes of graphs [12, 40, 58, 66]. Counting the number of possible configurations shows quickly that for graphs with cop number $c(G) \leq k$, we have capture time $capt(G) = O(n^{k+1})$, where n is the number of vertices in G. In their original paper [11], Bonato, Golovach, Hahn, and Kratochvíl showed that for $c(G) = 1$, this can be improved to $capt(G) = O(n)$, which is tight as shown by the path graph P_n. Perhaps surprisingly, Brandt, Emek, and Uitto [14] showed that for $c(G) \geq 2$, the trivial upper bound is actually tight, i.e. there exist graphs G, with $capt(G) = \Omega(n^{c(G)+1})$, indicating a qualitative difference between graphs G with $c(G) = 1$ and $c(G) \geq 2$.

Returning our attention to the bridge-burning capture time, Kinnersley and Peterson [59] showed that if $c_b(G) = 1$, then $capt_b(G) = O(n^3)$ and conjectured that there exists a graph G such that $c_b(G) = 1$ and $capt_b(G) = \Omega(n^3)$. We generalise their result by showing that $capt_b(G) = O(n^{c_b(G)+2})$ and prove the matching lower bound analogous to the one in their conjecture for $c_b(G) \geq 3$.

Theorem 1.6.1. *There exists a universal constant $C > 0$ such that the following holds. For every $k \geq 3$ and n sufficiently large, there exists a graph G_n such that $v(G_n) = n$, $c_b(G_n) = k$, and*

$$C \frac{n^{k+2}}{k^{k+2}} \leq capt_b(G_n).$$

In fact, in Proposition 8.2.4 we show that for all G on n vertices $capt_b(G) \leq \frac{(2n)^{c_b(G)+2}}{c_b(G)!}$, which shows that the asymptotics in terms of $c_b(G)$ are almost tight.

Chapter 2

Sharp stability of Brunn-Minkowski for homothetic regions

The work in this section was done jointly with Marius Tiba and Hunter Spink. It has been adapted from [51].

2.1 Introduction

In this chapter we prove Theorem 1.1.3 the sharp stability result for the Brunn-Minkowski inequality in the particular case that A, B are homothetic sets. Taking $A = B$ resolves a conjecture of Figalli and Jerison [31]. For convenience, we restate Theorem 1.1.3.

Theorem 2.1.1. *For all $n \geq 2$, there is a (computable) constant $C_n' > 0$ and (computable) constants $d_n(\tau) > 0$ for each $\tau \in (0, \frac{1}{2}]$ such that the following is true. With the notation above, if $\tau \in (0, \frac{1}{2}]$ and $A, B \subset \mathbb{R}^n$ are measurable homothetic sets such that $t \in [\tau, 1-\tau]$ and $\delta \leq d_n(\tau)$, then*

$$\omega \leq C_n' \tau^{-1} \delta.$$

For these optimal exponents, we also show that $e^{\Omega(n)} \leq C_n' \leq e^{O(n \log n)}$ with explicit constants. We discuss this further in Section 2.5.

2.1.1 Main theorem

As we are considering homothetic regions A, B, we can replace A with tA and B with $(1-t)A$. Note that t retains its earlier meaning as $t = \frac{|tA|^{\frac{1}{n}}}{|tA|^{\frac{1}{n}} + |(1-t)A|^{\frac{1}{n}}}$. Define the *interpolated sumset* of A as

$$D(A;t) := tA + (1-t)A = \{ta_1 + (1-t)a_2 \mid a_1, a_2 \in A\}.$$

Note that we always have $A \subset D(A;t)$. To quantify how small $D(A;t)$ is, we introduce the expression

$$\delta'(A;t) := |D(A;t) \setminus A|.$$

As a further simplification, we note that

$$\delta = \frac{|D(A;t)|^{\frac{1}{n}}}{|A|^{\frac{1}{n}}} - 1 = \left(\frac{1}{n} + o(1)\right)\left(\frac{|D(A;t)|}{|A|} - 1\right) = \left(\frac{1}{n} + o(1)\right)\frac{\delta'(A;t)}{|A|}$$

where $o(1)$ depends on the upper bound on δ. Since the exponent of δ is always at most 1 (as shown by Example 2.1.4), we may work with $\frac{\delta'(A;t)}{|A|}$ in place of δ by absorbing the $\frac{1}{n} + o(1)$ term into C_n' to make a new constant C_n.

The following is the specialization of Theorem 1.1.2 (i.e. [29, Theorem 1.3]) to homothetic A, B, which we restate for the reader's convenience.

Theorem 2.1.2. *For $n \geq 2$ there are (computable) constants $b_n, C_n > 0$, and (computable) constants $a_n(\tau), d_n(\tau) > 0$ for each $\tau \in (0, \frac{1}{2}]$, such that the following is true. If $A \subset \mathbb{R}^n$ is a measurable set, $\tau \in (0, \frac{1}{2}]$ and $t \in [\tau, 1-\tau]$, then*

$$|\operatorname{co}(A) \setminus A| \leq C_n |A| \tau^{-b_n} \left(\frac{\delta'(A;t)}{|A|}\right)^{a_n(\tau)}$$

whenever $\delta'(A;t) \leq d_n(\tau)|A|$.

Our main result optimizes the exponents to be $a_n = b_n = 1$ in Theorem 2.1.2, verifying the conjecture of [31] and the further generalization to homothetic sets suggested by Figalli and Jerison. We will prove the following reformulation of Theorem 1.1.3.

Theorem 2.1.3. *For all $n \geq 2$, there is a (computable) constant $C_n > 0$ (we can take $C_n = (4n)^{5n}$), and (computable) constants $\Delta_n'(\tau) > 0$ for each $\tau \in (0, \frac{1}{2}]$ such that the following is true. If $A \subset \mathbb{R}^n$ is a measurable set, $\tau \in (0, \frac{1}{2}]$ and $t \in [\tau, 1-\tau]$, then*

$$|\operatorname{co}(A) \setminus A| \leq C_n \tau^{-1} \delta'(A;t)$$

whenever $\delta'(A;t) \leq \Delta_n'(\tau)|A|$.

Example 2.1.4. *To see that the exponents on δ' and τ are sharp, suppose we have some inequality of the form*

$$|\operatorname{co}(A) \setminus A| \leq C_n |A| \tau^{-\rho_1} (\delta'(A;t)|A|^{-1})^{\rho_2}$$

whenever $\delta'(A;t) \leq \Delta'_n(\tau)|A|$. *Take* $A = \{(0,0) \cup [\lambda, 1+\lambda] \times [0,1]\} \times [0,1]^{n-2}$, *with* $\lambda \ll \frac{\Delta'_n(\tau)}{2\tau}$, *and* $t = \tau$. *The inequality then becomes* $\frac{\lambda}{2} \leq C_n \tau^{-\rho_1}(\tau\lambda(2-3\tau))^{\rho_2}$. *Because we can take* λ *arbitrarily small, it follows that* $\rho_2 \leq 1$, *so* $\rho_2 = 1$ *would be the optimal exponent. Given* $\rho_2 = 1$, *we then have* $\rho_1 \geq 1$, *so* $\rho_1 = 1$ *would be the optimal exponent.*

Remark 2.1.5. *When* $n = 1$, *Theorem 1.1 from [28] (a corollary of Freiman's* $3k-4$ *theorem [36]) with A replaced with tA and B replaced with* $(1-t)A$ *shows that the optimal exponents are actually* $\tau^0\delta'(A;t)^1$ *in contrast to the case* $n \geq 2$.

Example 2.1.6. *Given exponents* $\rho_1 = \rho_2 = 1$, *the constant* C_n *grows at least exponentially as shown by the following example. Let* $R \geq 2$. *Consider the set* $A \subset \mathbb{R}^n$, $A = [0,2]^{n-1} \times [-R,0] \cup \{(0,\ldots,0,2)\}$. *Then* $\mathrm{co}(A) = A \cup \bigcup_{x\in[0,2]}[0,2-x]^{n-1} \times \{x\}$ *and* $\frac{A+A}{2} = A \cup [0,1]^n$. *Hence,* $\delta'(A,\frac{1}{2}) = 1$ *and* $|\mathrm{co}(A) \setminus A| = \int_{x\in[0,2]}(2-x)^{n-1}dx = \frac{2^n}{n}$. *This example shows that* $C_n \geq \frac{2^{n-1}}{n}$.

2.1.2 Outline

By replacing t with $1-t$ we may assume that $t \leq \frac{1}{2}$.

Initial Reduction

We first carry out a straightforward reduction along the lines of the reduction in [31] to [31, Lemma 2.2], reducing to the case that $\mathrm{co}(A)$ is a simplex T, so A contains all of the vertices of T. In this reduction we use Theorem 2.1.2, though we need only the following much weaker statement due to Christ [20]: $|\mathrm{co}(A) \setminus A||A|^{-1}$ is bounded above by a (computable) function of the parameters $\delta'(A;t)|A|^{-1}$ and τ which, for fixed τ, tends to 0 as $\delta'(A;t)|A|^{-1}$ tends to 0.

Fractal Structure

Next we show that if $\delta'(A;t)|A|^{-1}$ is small, then A contains an approximate fractal structure. For each i we recursively construct a nested sequence of families of simplices $\mathcal{T}_{i,0} \subset \mathcal{T}_{i,1} \subset \ldots$; each family $\mathcal{T}_{i,k}$ consists of translates of $(1-t)^iT$ contained inside T, and in the limit $\cup_k \mathcal{T}_{i,k}$ is dense among the translates of $(1-t)^iT$ contained inside T. We show that there exist universal constants $c_{i,k,n} = i+2k$ such that for translates $T' \in \mathcal{T}_{i,k}$,

$$|((1-t)^iA)_{T'} \cap A| \geq |((1-t)^iA)_{T'}| - c_{i,k,n}\delta'(A;t),$$

where $((1-t)^iA)_{T'}$ is the translate of $(1-t)^iA$ induced by the translation that identifies $(1-t)^iT$ with T'. Though we need this fractal structure in order to prove this inequality

recursively, we only use the corollary that $|T' \cap A| \geq \frac{|T'|}{|T|}|A| - c_{i,k,n}\delta'(A;t)$. This corollary quantitatively establishes that A becomes more homogeneous in T as $\delta'(A;t)|A|^{-1} \to 0$.

Covering a thickened ∂T with small total volume

Next, we consider a large homothetic scaled copy $R := (1-\zeta)T$ inside T for $\zeta \approx \frac{1}{n^4}$ and we produce a cover $\mathcal{A} \subset \mathcal{T}_{i,k}$ of $T \setminus R$ for $i \approx 5\log(n)/t$ and $k \approx n\log(n)/t$. The cover $\mathcal{A}$ consists of translates of $(1-t)^i T \approx \frac{1}{n^5}T$ and has the property that the size of $\mathcal{A}$ is at most $(2n)^{5n}$ and the total volume of the simplicies in $\mathcal{A}$ is less than $\frac{1}{2}|T|$. We note that $|\mathcal{A}|, i, k$ affect the complexity of C_n, whereas ζ affects only the complexity of $\delta'_n(\tau)$ and not C_n.

In order to produce the covering $\mathcal{A}$ above we proceed in two steps. First, we use a covering result of Rogers [71] to produce an efficient covering $\mathcal{B}$ of $T \setminus R$ with translates of $n^{-\frac{1}{n}}(1-t)^i T$ contained inside T. The covering $\mathcal{B}$ has the property that the size of $\mathcal{B}$ is at most $(2n)^{5n}$ and the total volume of the simplicies in $\mathcal{B}$ is less than $\frac{1}{2n}|T|$. Second, we show that for each translate T' of $n^{-\frac{1}{n}}(1-t)^i T$ contained inside T, there exists a simplex $T'' \in \mathcal{T}_{i,k}$ such that $T' \subset T''$. This naturally gives the desired cover $\mathcal{A}$.

Putting it all together

We may assume that $R \subset D(A;t)$ since a straightforward argument shows this holds whenever $|T \setminus A||A|^{-1}$ is sufficiently small, and $|T \setminus A||A|^{-1} \to 0$ as $\delta'(A;t)|A|^{-1} \to 0$ by Theorem 2.1.2. Rephrasing the homogeneity statement for A, for each $T' \in \mathcal{T}_{i,k}$ we have

$$|T' \setminus A| \leq \frac{|T \setminus A|}{|T|}|T'| + c_{i,k,n}\delta'(A;t).$$

Because $\mathcal{A}$ covers $T \setminus R$ and $A \subset D(A;t)$, we have $|T \setminus D(A;t)| \leq \sum_{T' \in \mathcal{A}} |T' \setminus A|$, and by construction $\sum_{T' \in \mathcal{A}} |T'| \leq \frac{1}{2}|T|$. Combining these facts, we immediately deduce

$$|T \setminus D(A;t)| \leq \frac{1}{2}|T \setminus A| + |\mathcal{A}|c_{i,k,n}\delta'(A;t),$$

i.e.

$$|T \setminus A| \leq 2(1 + |\mathcal{A}|c_{i,k,n})\delta'(A;t).$$

Because $|\mathcal{A}| \leq (2n)^{5n}$ and $c_{i,k,n} \approx \frac{n\log(n)}{t}$, we see that with $C_n = (4n)^{5n}$ we have

$$|T \setminus A| \leq C_n \tau^{-1} \delta'(A;t).$$

2.2 Initial Reduction

In this section, we will reduce Theorem 2.1.3 to Theorem 2.2.1, similar to the initial reduction in [31] to [31, Lemma 2.2].

Theorem 2.2.1. *For all $n \geq 2$ there are (computable) constants $C_n > 0$ (we can take $C_n = (4n)^{5n}$) and constants $0 < \delta'_n(\tau) < 1$ for each $\tau \in (0, \frac{1}{2}]$ such that the following is true. Let $\tau \in (0, \frac{1}{2}]$, $t \in [\tau, 1-\tau]$, and suppose $T \subset \mathbb{R}^n$ is a simplex with $|T| = 1$, $A \subset T$ a measurable subset containing all vertices of T, and $|A| = 1 - \delta'$ with $0 < \delta' \leq \delta'_n(\tau)$. Then*

$$|T \setminus A| \leq C_n \tau^{-1} \delta'(A;t).$$

We first need the following geometric lemma.

Lemma 2.2.2. *For every convex polytope P, there exists a point $o \in P$ (which we set to be the origin) such that the following is true. For any constant $b_n(\tau) \in (0,1)$, there exists a constant $\varepsilon_n(\tau)$ such that for any $A \subset P$, if $t \in [\tau, 1-\tau]$ and $|P \setminus A| \leq \varepsilon_n(\tau)|P|$, then $(1 - b_n(\tau))P \subset D(A;t)$.*

Proof. We may assume that $t \leq \frac{1}{2}$ as the statement is invariant under replacing t with $1-t$. Without loss of generality we may assume that $|P| = 1$. By a lemma of John [54], after a volume-preserving affine transformation, there exists a ball $B \subset P$ of radius n^{-1}. Denote o for the center of B, and set o to be the origin.

We will show that $(1 - b_n(\tau))P \subset D(A;t)$. Take $x \in (1 - b_n(\tau))P$, and let y be the intersection of the ray ox with ∂P. Note that the ratio $r = |xy|/|oy| \geq b_n(\tau)$.

Let H be the homothety with center y and ratio r. This homothety sends o to x and P to $H(P)$. Note that $H(P) \subset P$ because P is convex. Denoting

$$A' = A \cap H(P),$$

we have

$$|A'| \geq r^n - \varepsilon_n(\tau).$$

The statement $x \in D(A';t)$ is implied by the statement that $o \in D(C;t)$ for $C = H^{-1}(A') \subset P$, which we will now show (in fact we will show $o \in D(C \cap B;t)$).

Note that $|C| \geq 1 - r^{-n}\varepsilon_n(\tau)$, so $|B \setminus C| \leq r^{-n}\varepsilon_n(\tau)$. Consider the negative homothety H' scaling by a factor of $-\frac{t}{1-t} \in [-1,0)$ about o. If $o \notin D(C \cap B;t)$, then at least one of y and $H'(y)$ is not in $C \cap B$ for every $y \in B$. A simple volume argument shows that this would

imply $|B \setminus C| \geq \frac{1}{2}|H'(B)|$, and as B contains a cube of side length $2/\sqrt{n}$ we would have

$$r^{-n}\varepsilon_n(\tau) \geq |B \setminus C| \geq \frac{1}{2}|H'(B)| = \frac{1}{2}\left(\frac{t}{1-t}\right)^n |B| \geq \frac{1}{2}\left(\frac{\tau}{1-\tau}\right)^n \left(\frac{2}{\sqrt{n}}\right)^n.$$

Therefore as $b_n(\tau)^{-n} \geq r^{-n}$, taking

$$\varepsilon_n(\tau) < b_n(\tau)^n \frac{1}{2}\left(\frac{\tau}{1-\tau}\right)^n \left(\frac{2}{\sqrt{n}}\right)^n,$$

we deduce that $o \in D(C \cap B; t)$ and therefore in particular $x \in D(A'; t)$.

□

Observation 2.2.3. *If P is a (regular) simplex T, we can take o to be the barycenter of T.*

Proof that Theorem 2.2.1 implies Theorem 2.1.3. We may assume that $t \leq \frac{1}{2}$ since Theorem 2.1.3 is invariant under replacing t with $1-t$. By approximation, we can assume that A has polyhedral convex hull $\mathrm{co}(A)$ with the vertices of $\mathrm{co}(A)$ lying in A (see e.g. [31, p.3 footnote 2]).

Take $b_n(\tau)$ to be the minimum of τ and the constant such that

$$\delta_n'(\tau)^{-1}(1-(1-b_n(\tau))^n) = 1 - C_n^{-1}\tau,$$

and take $\varepsilon_n(\tau)$ as in Lemma 2.2.2.

From Theorem 2.1.2, we see that we can choose $\Delta_n'(\tau)$ sufficiently small so that

$$|\mathrm{co}(A) \setminus A| \leq \varepsilon_n(\tau)|A| \leq \varepsilon_n(\tau)|\mathrm{co}(A)|,$$

and therefore by Lemma 2.2.2 there is a translate of $(1-b_n(\tau))\,\mathrm{co}(A) \subset D(A;t)$. Let o be the center of homothety relating this translate of $(1-b_n(\tau))\,\mathrm{co}(A)$ and $\mathrm{co}(A)$. Because $b_n(\tau) \leq \tau$, the region $to + (1-t)\,\mathrm{co}(A)$ is contained in $D(A;t)$, so from this we deduce that $D(A \cup \{o\}; t) = D(A;t)$. Therefore we may assume without loss of generality that $o \in A$.

Note that the inequality in Theorem 2.1.3 that we want to deduce is equivalent to

$$|\mathrm{co}(A) \setminus D(A;t)| \leq (1 - C_n^{-1}\tau)|\mathrm{co}(A) \setminus A|.$$

Triangulate $\mathrm{co}(A)$ into simplices T_i by triangulating $\partial\,\mathrm{co}(A)$ and coning off each facet at o. Then in each simplex T_i, we claim that

$$|T_i \setminus D(A;t)| \leq (1 - C_n^{-1}\tau)|T_i \setminus A|.$$

Provided $|T_i \setminus A| \leq \delta_n'(\tau)|T_i|$, applying Theorem 2.2.1 to $T_i, A \cap T_i$ yields the stronger inequality

$$|T_i \setminus D(A \cap T_i; t)| \leq (1 - C_n^{-1}\tau)|T_i \setminus A|.$$

On the other hand, if $|T_i \setminus A| \geq \delta_n'(\tau)|T_i|$, then as $b_n(\tau)o + (1 - b_n(\tau))T_i \subset D(A;t) \cap T_i$, we have

$$|T_i \setminus D(A;t)| \leq |T_i|(1-(1-b_n(\tau))^n) \leq \delta_n'(\tau)^{-1}(1-(1-b_n(\tau))^n)|T_i \setminus A| \leq (1-C_n^{-1}\tau)|T_i \setminus A|.$$

We conclude by noting

$$|\operatorname{co}(A) \setminus D(A;t)| = \sum |T_i \setminus D(A;t)| \leq \sum (1 - C_n\tau^{-1})|T_i \setminus A| = (1 - C_n^{-1}\tau)|\operatorname{co}(A) \setminus A|.$$

□

2.3 Setup and technical lemmas

We take A to satisfy the hypotheses of Theorem 2.2.1. We may assume that $t \leq \frac{1}{2}$ since Theorem 2.2.1 is invariant under replacing t with $1-t$. It suffices to prove the statement for a particular choice of T since all simplices of volume 1 in $\mathbb{R}^n$ are equivalent under volume-preserving affine transformations. Hence we work in a fixed regular simplex $T \subset \mathbb{R}^n$ from now on. Let $x_0, \ldots, x_n$ denote the vertices of T, and define the *corner λ^i-scaled simplices* to be

$$S_i^j(\lambda) = (1 - \lambda^i)x_j + \lambda^i T \text{ for } 0 \leq j \leq n$$

and set

$$\mathcal{S}_i(\lambda) := \{S_i^0(\lambda), \ldots, S_i^n(\lambda)\}.$$

Define the *λ^i-scaled k-averaged simplices* $\mathcal{T}_{i,k}(\lambda)$ iteratively by

$$\begin{aligned} \mathcal{T}_{i,0}(\lambda) &= \mathcal{S}_i(\lambda) \\ \mathcal{T}_{i,k+1}(\lambda) &= \{\lambda B_1 + (1-\lambda)B_2 \mid B_1, B_2 \in \mathcal{T}_{i,k}(\lambda)\}. \end{aligned}$$

Note that all simplices in $\mathcal{T}_{i,k}(\lambda)$ are translates of $\lambda^i T$, and we have the inclusions

$$\mathcal{T}_{i,0}(\lambda) \subset \mathcal{T}_{i,1}(\lambda) \subset \mathcal{T}_{i,2}(\lambda) \subset \ldots$$

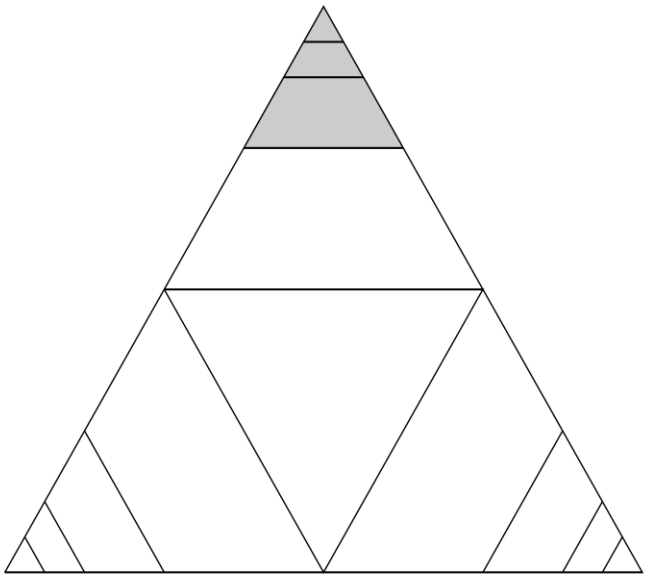

Fig. 2.1 T with different $S_i^j(\frac{1}{2})$'s indicated and a $S_2^j(\frac{1}{2})$ shaded.

For fixed i, λ, the simplices in the family $\mathcal{T}_{i,k}(\lambda)$ eventually cover all of T and heavily overlap each other as $k \to \infty$ (in fact the translates become dense among all possible translates of $\lambda^i T$ which lie inside T). Shaded below are the simplices in $\mathcal{T}_{2,1}(\frac{1}{2})$ when $n = 2$.

Lemma 2.3.1 is the crux of our argument. The proof of Lemma 2.3.1 shows that for all $T' \in \mathcal{T}_{i,k}(1-t)$, the set $|T' \cap A|$ contains a translated copy of $(1-t)^i A$ (up to a bounded error). This fractal structure allows us to conclude that $|T' \cap A|$ is bounded below by $|T'|(1-\delta')$ (up to a bounded error).

Lemma 2.3.1. *The constants $c_{i,k,n} = i + 2k$ are such that for every $T' \in \mathcal{T}_{i,k}(1-t)$ we have*

$$|T' \cap A| \geq |T'|(1-\delta') - c_{i,k,n}\delta'(A;t).$$

Proof. For the remainder of this proof, we will denote

$$\lambda = 1 - t,$$

and write for notational convenience S_i^j instead of $S_i^j(\lambda)$. The following notation will be useful for us: consider the translation that brings $\lambda^i T$ to T' and denote by $(\lambda^i A)_{T'}$ the shift of the set $\lambda^i A$ under this translation.

We shall actually show the stronger inequalities

$$|(\lambda^i A)_{T'} \setminus A| \leq c_{i,k,n}\delta'(A;t)$$

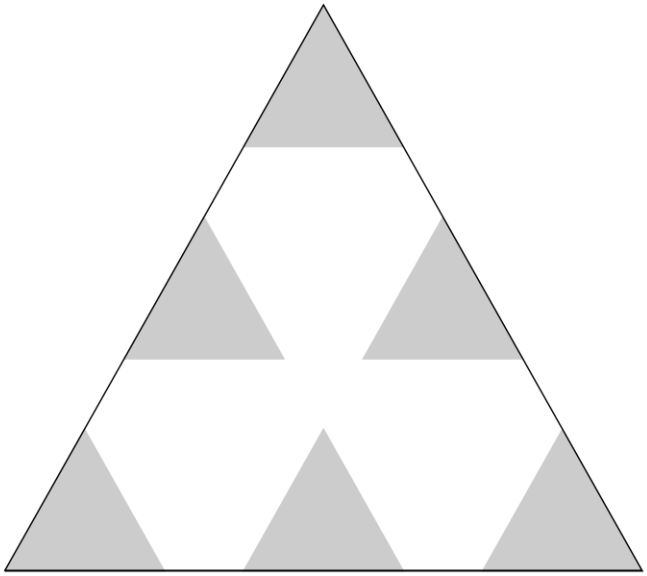

Fig. 2.2 T with the elements of $\mathcal{T}_{2,1}(\frac{1}{2})$ shaded

(which are stronger as $|(\lambda^i A)_{T'}| = |T'|(1-\delta')$).

First, we show the inequality when $k=0$. Recall that if $T' \in \mathcal{T}_{i,0}(\lambda)$ then $T' = S_i^j$ for some j. The inequality is trivial for $(i,k)=(0,0)$ by definition of δ'.

We now show the inequality for $(i,k)=(1,0)$. Note $(\lambda A)_{S_1^j} = (1-\lambda)x_j + \lambda A \subset D(A;t)$, so

$$|(\lambda A)_{S_1^j} \setminus A| \leq |D(A;t) \setminus A| = \delta'(A;t).$$

Suppose we know the result for $(i,0)$, we now prove the result for $(i+1,0)$. Then $(\lambda^{i+1}A)_{S_{i+1}^j} = (1-\lambda^{i+1})x_j + \lambda^{i+1}A$, and we have

$$\begin{aligned}
|(\lambda^{i+1}A)_{S_{i+1}^j} \setminus A| &\leq |(\lambda^{i+1}A)_{S_{i+1}^j} \setminus (\lambda A)_{S_1^j}| + |(\lambda A)_{S_1^j} \setminus A| \\
&= \lambda^n |(\lambda^i A)_{S_i^j} \setminus A| + |(\lambda A)_{S_1^j} \setminus A| \\
&\leq (\lambda^n c_{i,0,n} + c_{1,0,n})\delta'(A;t) \\
&\leq c_{i+1,0,n}\delta'(A;t).
\end{aligned}$$

Finally, we induct on k. We have proved the base case $k=0$, so assume the inequality for (i,k). We will now prove the inequality for $(i,k+1)$.

Thus we suppose that $T' \in \mathcal{T}_{i,k+1}$, which by definition means that there exists $T_1', T_2' \in \mathcal{T}_{i,k}$ such that

$$T' = \lambda T_1' + (1-\lambda)T_2'.$$

We now prove an easy claim before returning to the proof of the lemma.

Claim 2.3.2. *Let X, X' be translates of each other in $\mathbb{R}^n$ with common volume $V = |X| = |X'|$, and let $Y \subset X$, $Y' \subset X'$. Then if V' is a constant such that $|X \setminus Y|, |X' \setminus Y'| \leq V'$, we have*

$$|\lambda Y + (1-\lambda)Y'| \geq V - V'.$$

Proof. We have $|Y|, |Y'| \geq V - V'$, so the result follows from the Brunn-Minkowski inequality. □

Returning to the proof of the lemma, we have by the induction hypothesis that both

$$\begin{aligned} &|(\lambda^i A)_{T_1'} \setminus A| \leq c_{i,k,n}\delta'(A;t), \text{ and} \\ &|(\lambda^i A)_{T_2'} \setminus A| \leq c_{i,k,n}\delta'(A;t). \end{aligned}$$

Because $(\lambda^i A)_{T_1'}$ and $(\lambda^i A)_{T_2'}$ are translates of each other with common volume $(1-\delta')|T'|$, setting $X = (\lambda^i A)_{T_1'}$, $X' = (\lambda^i A)_{T_2'}$, $Y = A \cap (\lambda^i A)_{T_1'}$, $Y' = A \cap (\lambda^i A)_{T_2'}$ we deduce from the claim that

$$\left|\lambda(A \cap (\lambda^i A)_{T_1'}) + (1-\lambda)(A \cap (\lambda^i A)_{T_2'})\right| \geq |T'|(1-\delta') - c_{i,k,n}\delta'(A;t).$$

Because $D(A;t) = \lambda A + (1-\lambda)A$ and $(\lambda^i D(A;t))_{T'} = \lambda(\lambda^i A)_{T_1'} + (1-\lambda)(\lambda^i A)_{T_2'}$, we have

$$\begin{aligned} |D(A;t) \cap (\lambda^i A)_{T'}| &\geq |D(A;t) \cap (\lambda^i D(A;t))_{T'}| - |\lambda^i D(A;t) \setminus \lambda^i A| \\ &\geq |D(A;t) \cap (\lambda^i D(A;t))_{T'}| - \delta'(A;t) \\ &\geq |\lambda(A \cap (\lambda^i A)_{T_1'}) + (1-\lambda)(A \cap (\lambda^i A)_{T_2'})| - \delta'(A;t) \\ &\geq |T'|(1-\delta') - (c_{i,k,n}+1)\delta'(A;t), \end{aligned}$$

which as $|(\lambda^i A)_{T'}| = (1-\delta')|T'|$ is equivalent to

$$|(\lambda^i A)_{T'} \setminus D(A;t)| \leq (c_{i,k,n}+1)\delta'(A;t).$$

We conclude that

$$\begin{aligned} |(\lambda^i A)_{T'} \setminus A| &\leq |(\lambda^i A)_{T'} \setminus D(A;t)| + \delta'(A;t) \\ &\leq (c_{i,k,n}+2)\delta'(A;t) \\ &= c_{i,k+1,n}\delta'(A;t). \end{aligned}$$

□

The following lemma shows that given $\alpha < 1$ and $\frac{1}{2} \leq \lambda < 1$, any arbitrary covering of T by translates of $\alpha^n \lambda T$ contained inside T can be approximated by a covering consisting of elements of $\mathcal{T}_{i,k}(\lambda)$ for fixed small values i,k. The parameters i,k are positively correlated with λ, α.

Before we proceed, we need the following notation. Let $\mathcal{T}_k(\lambda;\lambda';T)$ be recursively defined by setting

$$\mathcal{T}_0(\lambda;\lambda';T) = \{\lambda' T + (1-\lambda')x_j \mid j \in \{0,\dots,n\}\}$$
$$\mathcal{T}_k(\lambda;\lambda';T) = \{\lambda B_1 + (1-\lambda)B_2 \mid B_1, B_2 \in \mathcal{T}_{k-1}(\lambda;\lambda';T)\}.$$

Note that by definition, $\mathcal{T}_{i,k}(\lambda) = \mathcal{T}_k(\lambda;\lambda^i;T)$.

Lemma 2.3.3. *For $\alpha, \mu \in (0,1), \lambda \in [\frac{1}{2},1)$, every translate $T' \subset T$ of $\alpha^n \mu T$ is completely contained in some element of $\mathcal{T}_{k'}(\lambda;\mu;T)$ with*

$$k' = \sum_{j=1}^{n} \lceil \log(\alpha^{j-1}(1-\alpha)\mu)/\log(\lambda) \rceil.$$

Proof. To prove this we need the following claim, which is essentially the result for $n = 1$.

Claim 2.3.4. *Every weighted average of two (corner) simplices in $\mathcal{T}_0(\lambda;\alpha\mu;T)$ lies in some simplex of $\mathcal{T}_\ell(\lambda;\mu;T)$ with $\ell = \lceil \log((1-\alpha)\mu)/\log(\lambda) \rceil$*

Proof. Suppose the two corner simplices are at the corners x_a and x_b. Then every homothetic copy $T' \subset T$ of T is determined by the corresponding edge $x'_a x'_b$. Thus the claim is implied by the one-dimensional version of the claim by intersecting all simplices with $x_a x_b$. Hence we may assume that $T = [0,1]$, so that $\mathcal{T}_0(\lambda;\mu;T) = \{[0,\mu],[1-\mu,1]\}$, and we want to show that every sub-interval of $[0,1]$ of length $\alpha\mu$ is contained in an element of $\mathcal{T}_\ell(\lambda;\mu;T)$.

We will now proceed by showing that the largest distance between consecutive midpoints of intervals in $\mathcal{T}_{j+1}(\lambda;\mu;T)$ is at most λ times the largest such distance in $\mathcal{T}_j(\lambda;\mu;T)$. Let I_1, I_2 be two consecutive intervals in $\mathcal{T}_j(\lambda;\mu;T)$ for some j. Then in $\mathcal{T}_{j+1}(\lambda;\mu;T)$ we also have the intervals $J = \lambda I_1 + (1-\lambda)I_2$ and $K = (1-\lambda)I_1 + \lambda I_2$, and the intervals I_1, J, K, I_2 appear in this order from left to right as $\lambda \geq \frac{1}{2}$. If d is the distance between the midpoints of I_1, I_2, then the distances between the consecutive midpoints of I_1, J, K, I_2 are $(1-\lambda)d, (2\lambda-1)d, (1-\lambda)d$ respectively. Therefore, the largest distance between two midpoints d_{j+1} in $\mathcal{T}_{j+1}(\lambda;\mu;T)$ is at most $\max(1-\lambda, 2\lambda-1, 1-\lambda)d_j \leq \lambda d_j$ where d_j is the largest distance between two consecutive midpoints in $\mathcal{T}_j(\lambda;\mu;T)$. Therefore, the distance between two consecutive midpoints in $\mathcal{T}_\ell(\lambda;\mu;T)$ is at most $\lambda^\ell \leq (1-\alpha)\mu$.

Given an interval I of length $\alpha\mu$, then either the midpoint lies in $[0,\mu/2]\cup[1-\mu/2,1]$, in which case I is already contained in one of $[0,\mu]$ or $[1-\mu,1]$ belonging to $\mathcal{T}_0(\lambda;\mu;T)$, or else we can find an interval $I'\in\mathcal{T}_\ell(\lambda;\mu;T)$ of length μ such that the distance between the midpoints of I and I' is at most $\frac{1}{2}(1-\alpha)\mu$, which implies $I\subset I'$. □

We prove our desired statement by induction on the dimension n. The claim above proves the base case $n=1$, so now assuming the statement is true for dimensions up to $n-1$, we will show it to be true for n.

Let $T'\subset T$ be a fixed translate of $\alpha^n\mu T$, with corresponding vertices $x_0',\dots,x_n'$. Denote by F the facet of T opposite x_n, and denote by F' the facet of T' opposite the corresponding vertex x_n'. Denote by H the hyperplane spanned by F'. Then $S=H\cap T$ is an $n-1$-simplex, with vertices $y_0,\dots,y_{n-1}$ such that y_i is on the edge of T connecting x_i to x_n.

If the common ratio $r:=|y_jx_n|/|x_jx_n|\le\alpha\mu$, then T' is already contained in an element of $\mathcal{T}_0(\lambda;\mu;T)$ and we are done. Otherwise, denote by $T_0,\dots,T_{n-1}\subset T$ the translates of $\alpha\mu T$ that sit on H and have corners at $y_0,\dots,y_{n-1}$ respectively. Denote the facet $T_i\cap H$ of T_i by F_i. We remark that each F_i is a translate of $\mu' S$ for some fixed $\mu'\ge\alpha\mu$.

By the claim, the simplices $T_0,\dots,T_{n-1}$ are completely contained in elements of $\mathcal{T}_\ell(\lambda;\mu;T)$ with

$$\ell=\lceil\log((1-\alpha)\mu)/\log(\lambda)\rceil.$$

By the induction hypothesis applied to the $n-1$-simplex S, F' is completely contained in a simplex from the family $\mathcal{T}_{\ell'}(\lambda;\mu';S)$ for

$$\ell':=\sum_{j=1}^{n-1}\left\lceil\log((1-\alpha)\alpha^{j-1}\mu')/\log(\lambda)\right\rceil\le\sum_{j=1}^{n-1}\left\lceil\log((1-\alpha)\alpha^{j}\mu)/\log(\lambda)\right\rceil,$$

as $\mu'\ge\alpha\mu$. Note that F' is contained in a certain iterated weighted average of the facets $F_0,\dots,F_{n-1}$ if and only if T' is contained in the analogously defined iterated weighted average of $T_0,\dots,T_{n-1}$. Therefore $T'\in\mathcal{T}_{\ell+\ell'}(\lambda;\mu;T)$.

Finally, we have that $\ell+\ell'\le k'$, so $T'\in\mathcal{T}_{k'}(\lambda;\mu;T)$ as desired. □

The following lemma helps to show that arbitrary coverings of T can be modified at no extra cost to coverings of T contained inside T.

Lemma 2.3.5. *Let $r\in(0,1)$ and $rT+x$ a translate of rT. Then there exists a y such that $(rT+x)\cap T\subset rT+y\subset T$.*

Proof. The intersection of any two copies of the simplex T is itself homothetic to T. Therefore $(rT+x)\cap T$ is homothetic to T, and so must be a translate of $r'T$ for some $r'\le r$.

Because T is convex and $(rT+x)\cap T$ is a homothetic copy of T lying inside T, the center of homothety between $(rT+x)\cap T$ and T lies inside $(rT+x)\cap T$, and all intermediate homotheties lie inside T. In particular there is a homothety which produces a translate of rT which lies inside T, and this translate by construction contains $(rT+x)\cap T$. □

2.4 Proof of Theorem 2.2.1

Recall that we may assume that $t \le \frac{1}{2}$.

Proof of Theorem 2.2.1. Let $i = \left\lceil \log\left(\frac{n^{\frac{1}{n}}}{(2n)^5}\right) / \log(1-t) \right\rceil$, so that $(1-t)^i \in \left[\frac{n^{\frac{1}{n}}}{2(2n)^5}, \frac{n^{\frac{1}{n}}}{(2n)^5}\right]$ (as $t \le \frac{1}{2}$). Note that

$$i \le 1 + \log((2n)^5)/t \le 6\log(2n)/t.$$

Let $\eta = n^{-\frac{1}{n}}(1-t)^i \in \left[\frac{1}{2(2n)^5}, \frac{1}{(2n)^5}\right]$, and let $\zeta = (n+1)\eta$.

Recall T is a regular simplex of volume 1, denote by o the barycenter. By Lemma 2.2.2, setting o to be the origin, if we choose $\delta_n'(\tau)$ sufficiently small, then $R := (1-\zeta)T$ is contained in $D(A;t)$. Let $L = (1+\eta(n+1))T \setminus (1-\zeta-\eta(n+1))T$. Note that $T \setminus R \subset L$ and for any T' a translate of ηT intersecting $T \setminus R$, we have $T' \subset L$.

Claim 2.4.1. *There exists a covering $\mathcal{B}$ of $T \setminus R$ by translates of ηT contained in T, such that $\sum_{T' \in \mathcal{B}} |T'| \le \frac{1}{2n}$ and $|\mathcal{B}| \le (2n)^{5n}$.*

Proof of claim. It follows from [71] that[1] for all $n \ge 2$, there exists $r \in \mathbb{R}$ and there exists a covering $\mathcal{F}$ of $\mathbb{R}^n/(r\mathbb{Z})^n$ by translates of ηT with average density at most $7n\log(n)$, i.e.

$$\frac{|\mathcal{F}|\eta^n}{r^n} \le 7n\log n.$$

Passing to a multiple of r, we may assume that $T, L \subset [-r/2, r/2]^n$. Consider a uniformly random translate $\mathcal{F}+\mathbf{x}$. For any $T' \in \mathcal{F}$ and any point $t' \in T'$, we have

$$\mathbb{P}(T'+\mathbf{x} \subset L) \le \mathbb{P}(t'+\mathbf{x} \in L) = \frac{|L|}{r^n}.$$

Therefore,

$$\mathbb{E}(|\{T'+\mathbf{x} \in \mathcal{F}+\mathbf{x} \mid T'+\mathbf{x} \subset L\}|) \le \frac{|L|}{r^n}|\mathcal{F}| \le |L|\eta^{-n}7n\log n,$$

[1] We note that $n=2$ is not mentioned explicitly in [71] but follows easily.

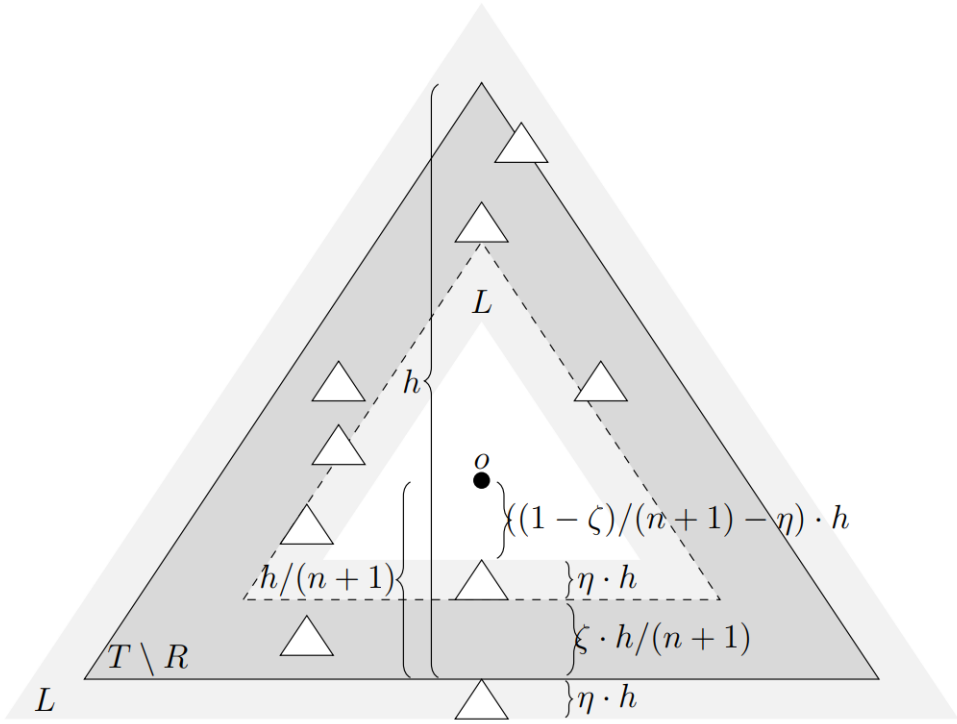

Fig. 2.3 T with L and R indicated and some elements of $\mathcal{B}$.

so there exists an $\mathbf{x_0}$ such that

$$|\{T'+\mathbf{x_0}\in\mathcal{F}+\mathbf{x_0}\mid T'+\mathbf{x_0}\subset L\}|\leq|L|\eta^{-n}7n\log n.$$

Define

$$\mathcal{B}'=\{T'+\mathbf{x_0}\in\mathcal{F}+\mathbf{x_0}\mid(T'+\mathbf{x_0})\cap(T\setminus R)\neq\emptyset\},$$

then by the above discussion we have $\mathcal{B}'$ is a covering of $T\setminus R$, and

$$|\mathcal{B}'|\leq|L|\eta^{-n}7n\log n.$$

By Lemma 2.3.5, for each element $T'+\mathbf{x_0}\in\mathcal{B}'$ we can find a translate $T'+\mathbf{y_{T'}}$ such that $(T'+\mathbf{x_0})\cap T\subset T'+\mathbf{y_{T'}}\subset T$. Define

$$\mathcal{B}=\{T'+\mathbf{y_{T'}}\mid T'+\mathbf{x_0}\in\mathcal{B}'\},$$

then $\mathcal{B}$ is a cover of $T\setminus R$ by translate of ηT contained in T with $|\mathcal{B}|\leq|L|\eta^{-n}7n\log n$.

We can calculate the upper bound

$$\begin{aligned}|L| &= (1+\eta(n+1))^n - (1-\zeta-\eta(n+1))^n \\ &= (1+\eta(n+1))^n - (1-2\eta(n+1))^n \\ &\le 1+2\eta n(n+1) - (1-2\eta n(n+1)) \\ &= 4\eta n(n+1).\end{aligned}$$

The inequality follows from the fact that $\eta^k(n+1)^k\binom{n}{k} \le (1/2)^k 2\eta(n+1)$ and the convexity of $(1-x)^n$ for $x \in (0,1)$.

Therefore,

$$|\mathcal{B}| \le 4\eta n(n+1)\eta^{-n}(7n\log n) \le \frac{1}{2n}\eta^{-n} \le (2n)^{5n}$$

and

$$\sum_{T'\in\mathcal{B}} |T'| = \eta^n|\mathcal{B}| \le \eta^n \frac{1}{2n}\eta^{-n} \le \frac{1}{2n}.$$

□

Claim 2.4.2. *There is a cover $\mathcal{A} \subset \mathcal{T}_{i,k}(1-t)$ of $T \setminus R$ with $k \le 8n\log(2n)/t$ such that $|\mathcal{A}| \le (2n)^{5n}$ and $\sum_{T''\in\mathcal{A}} |T''| \le \frac{1}{2}$.*

Proof. We apply Lemma 2.3.3 with $\alpha = n^{-\frac{1}{n}}$, $\lambda = 1-t$, $\mu = (1-t)^i$.

$$\begin{aligned}k = \sum_{j=1}^{n} \lceil \log(\alpha^{j-1}(1-\alpha)\mu)/\log(\lambda)\rceil &\le n\lceil \log(\alpha^n(1-\alpha)\mu)/\log(\lambda)\rceil \\ &\le n\left(1+\log\left(\frac{\log n}{2n^2}\mu\right)/\log(\lambda)\right) \\ &\le n\left(1+\log\left(\frac{\log n}{(2n)^7}\right)/(-t)\right) \\ &\le 8n\log(2n)/t.\end{aligned}$$

This shows that every translate of $\eta T = n^{-\frac{1}{n}}(1-t)^i T$ inside T is contained in some element of $\mathcal{T}_k((1-t);(1-t)^i;T) = \mathcal{T}_{i,k}(1-t)$. For each simplex $T' \in \mathcal{B}$, we can therefore choose a simplex $f(T') \in \mathcal{T}_{i,k}(1-t)$ such that $T' \subset f(T')$. Let

$$\mathcal{A} = \{f(T') \mid T' \in \mathcal{B}\}.$$

Note that $\mathcal{A}$ is a cover of $T \setminus R$,

$$|\mathcal{A}| \le |\mathcal{B}| \le (2n)^{5n},$$

and

$$\sum_{T''\in\mathcal{A}} |T''| = (n^{\frac{1}{n}})^n \sum_{T'\in\mathcal{B}} |T'| \leq \frac{1}{2}.$$

□

Returning to the proof of Theorem 2.2.1, note that since $\mathcal{A} \subset \mathcal{T}_{i,k}(1-t)$, Lemma 2.3.1 implies that for every $T'' \in \mathcal{A}$ we have

$$|T'' \setminus A| \leq \frac{|T \setminus A|}{|T|}|T''| + c_{i,k,n}\delta'(A;t).$$

Since $R \subset D(A;t)$, we have

$$\begin{aligned}|T \setminus D(A;t)| = |(T \setminus R) \setminus D(A;t)| &\leq \sum_{T''\in\mathcal{A}} |T'' \setminus D(A;t)| \leq \sum_{T''\in\mathcal{A}} |T'' \setminus A| \\ &\leq \frac{|T \setminus A|}{|T|} \sum_{T''\in\mathcal{A}} |T''| + |\mathcal{A}| \cdot c_{i,k,n}\delta'(A;t) \leq \frac{1}{2}|T \setminus A| + |\mathcal{A}| \cdot c_{i,k,n}\delta'(A;t),\end{aligned}$$

which after replacing $|T \setminus D(A;t)| = |T \setminus A| - \delta'(A;t)$ yields

$$|T \setminus A| \leq 2(1 + |\mathcal{A}| \cdot c_{i,k,n})\delta'(A;t).$$

We estimate

$$c_{i,k,n} = i + 2k \leq 6\log(2n)/t + 16n\log(2n)/t \leq 19n\log(2n)/t$$

Therefore

$$2(1 + |\mathcal{A}|c_{i,k,n}) \leq 2(1 + (2n)^{5n}(19n\log(2n)/t)) \leq (4n)^{5n}/\tau.$$

In conclusion, with $C_n = (4n)^{5n}$ we obtain

$$|T \setminus A| \leq C_n \tau^{-1} \delta'(A;t)$$

as desired. □

2.5 Sharpness of C_n

In studying the asymptotic behaviour of the optimal value of C_n in Theorem 2.1.3, we note that there is still a gap of order $\log(n)$ in the exponent between the upper and lower bounds.

Our proof shows the upper bound $C_n \leq (4n)^{5n} = e^{5n\log(4n)}$ and, the example mentioned in the introduction shows the lower bound $C_n \geq \frac{2^{n-1}}{n}$.

In our method the complexity of C_n is limited by the fact that $|\mathcal{A}| \leq C_n$, where $\mathcal{A}$ is a set of translates of ηT contained inside T with $\eta \leq \frac{1}{2}$ covering ∂T and satisfying $\sum_{T' \in \mathcal{A}} |T'| < |T|$. In fact, by a slight restructuring of our proof it is equivalent to covering just a single facet F of T. Taking $\mathcal{A}'$ to be the family of intersections of elements of $\mathcal{A}$ with the hyperplane containing F, we see that $|\mathcal{A}'| \leq C_n$ with $\mathcal{A}'$ a set of translates of ηF covering F and $\sum_{F' \in \mathcal{A}'} |F'| < \eta^{-1}|F|$.

Question 2.5.1. *Is it true that for every $0 < \eta_0 \leq \frac{1}{2}$, then for all sufficiently large n if $F \subset \mathbb{R}^n$ is a simplex and $\mathcal{A}'$ is a family of translates of $\eta_0 F$ covering F we have*

$$\sum_{F' \in \mathcal{A}'} |F'| > \eta_0^{-1}|F|?$$

Resolving this question would shed light on the correct growth rate of C_n. In particular, if the question has a negative answer with η_0^{-1} replaced with $\eta_0^{-1}(1-\varepsilon)$ for some fixed ε, then our methods would show that C_n has exponential growth.

Chapter 3

Sharp quantitative stability of the planar Brunn-Minkowski inequality

The work in this section was done jointly with Marius Tiba and Hunter Spink. It is adapted from [50].

3.1 Introduction

In this section, we prove Theorem 1.1.4, which solves the sharp stability question for planar regions $A, B \subset \mathbb{R}^2$, showing that the optimal exponents are $(a_2, b_2) = (\frac{1}{2}, \frac{1}{2})$. Recall the following concepts;

- $\delta = \delta(A,B) := \frac{|A+B|^{\frac{1}{n}}}{|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}} - 1$, the *Brunn-Minkowski deficit*,
- $t = t(A,B) := \frac{|A|^{\frac{1}{n}}}{|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}}$, the *normalized volume ratio*, and
- $\omega = \omega(A,B) := \frac{|K_A \setminus A|}{|A|} + \frac{|K_B \setminus B|}{|B|}$, where $K_A \supset A$, and $K_B \supset B$ are homothetic convex sets of minimal size.

We restate the theorem for convenience.

Theorem 3.1.1. *There are explicit constants $C, d(\tau) > 0$ such that if $A, B \subset \mathbb{R}^2$ are measurable sets with $t \in [\tau, 1-\tau]$ and $\delta \leq d(\tau)$, then there are homothetic convex sets $K_A \supset A$ and $K_B \supset B$ such that*

$$\omega \leq C\tau^{-\frac{1}{2}}\delta^{\frac{1}{2}}.$$

Our key result in proving Theorem 1.1.4 is the following generalization to arbitrary sets A,B of the result in from the previous chapter that for $A=B$, $|\mathrm{co}(A)\setminus A||A|^{-1}=O(\delta)$, as also stated in Theorem 1.1.5.

Theorem 3.1.2. *For all $\varepsilon,\tau>0$ there is an explicit constant $d_\tau(\varepsilon)>0$ such that the following is true. Suppose that $A,B\subset\mathbb{R}^2$ are measurable sets with $t\in[\tau,1-\tau]$ and $\delta\le d_\tau(\varepsilon)$. Then*

$$|\mathrm{co}(A+B)\setminus(A+B)|\le(1+\varepsilon)\left(|\mathrm{co}(A)\setminus A|+|\mathrm{co}(B)\setminus B|\right).$$

Taking $A=B=\{(x,y):0\le x,y\le x+y\le 1\}\cup\{(0,1+\lambda),(1+\lambda,0)\}$ shows that $1+o(1)$ is optimal. By taking $\varepsilon=\frac{\tau}{2}$, we will deduce in Section 3.12 the following corollary, used to prove Theorem 1.1.4.

Corollary 3.1.3. *There is a constant C' such that*

$$\frac{|\mathrm{co}(A)\setminus A|}{|A|}+\frac{|\mathrm{co}(B)\setminus B|}{|B|}\le C'\tau^{-1}\delta,\ \textit{and}\ \delta_{conv}:=\delta(\mathrm{co}(A),\mathrm{co}(B))\le\delta(A,B).$$

We make a note on how we apply Corollary 3.1.3 to conclude Theorem 1.1.4. We will estimate

$$\begin{aligned}\frac{|K_A\setminus A|}{|A|}+\frac{|K_B\setminus B|}{|B|}=&\frac{|K_A\setminus\mathrm{co}(A)|}{|\mathrm{co}(A)|}\cdot\frac{|\mathrm{co}(A)|}{|A|}+\frac{|K_B\setminus\mathrm{co}(B)|}{|\mathrm{co}(B)|}\cdot\frac{|\mathrm{co}(B)|}{|B|}+\frac{|\mathrm{co}(A)\setminus A|}{|A|}+\frac{|\mathrm{co}(B)\setminus B|}{|B|}\\ \le&C''\tau^{-\frac12}\delta_{conv}^{\frac12}+C'\tau^{-1}\delta\le C\tau^{-\frac12}\delta^{\frac12},\end{aligned}$$

where the first estimate uses [34], and separately [29] to show $|\mathrm{co}(A)||A|^{-1}\to 1$ as $\delta\to 0$. In particular, the error in approximating A and B with their convex hulls is quadratically smaller than the error in approximating $\mathrm{co}(A)$ and $\mathrm{co}(B)$ with homothetic convex sets.

In order to deduce Theorem 1.1.4 from Theorem 1.1.5, even for $\tau=\frac12$ it is insufficient to take say $1+\varepsilon=100$. In fact, with such a large ε the proof of Theorem 1.1.5 would be substantially easier. Showing the result for a suitably small ε is the primary challenge which we are able to overcome.

Example 3.1.4. *We note that Theorem 1.1.5 with $\mathbb{R}^2$ replaced with $\mathbb{R}^n$ is false for any fixed $\varepsilon>0$. To do this, we will give an example in $\mathbb{R}^3$ with equal volume sets A,B with δ arbitrarily small and with $|\mathrm{co}(A+B)\setminus(A+B)|>(1+\varepsilon)(|\mathrm{co}(A)\setminus A|+|\mathrm{co}(B)\setminus B|)$. Let T be the triangle with vertices $(0,0,0),(1,0,1),(2,0,0)$, and let I_A,I_B be the intervals connecting $(0,0,0)$ to $v_A=(-\eta,1,0)$ and $v_B=(\eta,1,0)$ respectively. Let $T'=(T\setminus\{z\ge 1-\lambda\})\cup(1,0,1)$, and define*

$$A=T'+I_A,\qquad B=T'+I_B.$$

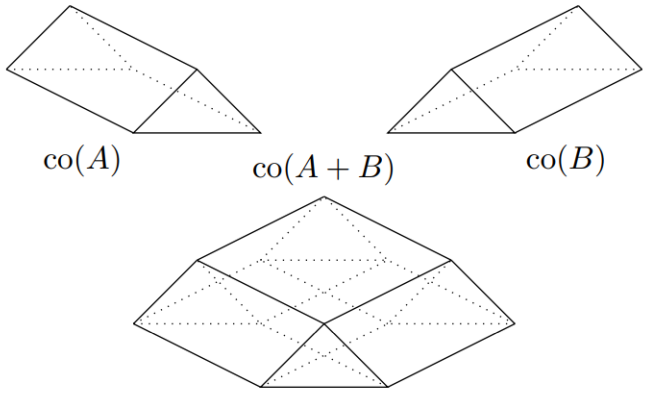

Fig. 3.1 The bodies in Example 3.1.4.

Note that $\delta \to 0$ *as* $\lambda, \eta \to 0$. *Also,* $A+B = (T'+T') + (I_A + I_B)$ *where* $T'+T' = 2T \setminus \{z \geq 2-\lambda\} \cup (2,0,2)$ *and* $I_A + I_B$ *is a parallelogram in the xy-plane determined by vectors* v_A, v_B. *Then*

$$|\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B| = 2\lambda^2$$

and

$$|\operatorname{co}(A+B) \setminus (A+B)| \geq |I_A + I_B| \cdot \lambda = 2\eta\lambda.$$

Therefore, choosing $\eta > (1+\varepsilon)\lambda$, *we obtain*

$$|\operatorname{co}(A+B) \setminus (A+B)| = 2\eta\lambda > (1+\varepsilon)2\lambda^2 = (1+\varepsilon)(|\operatorname{co}(A) \setminus A| + |\operatorname{co}(B) \setminus B|).$$

Finally, we note that the planar stability inequalities we consider are *not* Bonneson-style inequalities relating mixed volumes of planar convex K, L to the L-inradius and L-circumradius of K. See e.g. [13, Section 5] and separately [67] for an extensive survey of such inequalities.

3.1.1 Outline of the chapter

In Section 3.2, we give a reformulation of Theorem 1.1.5, make some simplifications and general observations, and give definitions which will be used throughout the remainder of the chapter. Simplifications include assuming A, B are finite unions of polygonal regions so the vertices of $\partial \operatorname{co}(A), \partial \operatorname{co}(B)$ are contained in A, B respectively, and that they are translated in a specific way so that $\operatorname{co}(A)$ and $\operatorname{co}(B)$ contain the origin o.

In Section 3.3, by an averaging argument we show that $(1-4\tau^{-1}\sqrt{\gamma})\,\mathrm{co}(A+B)\subset A+B$, where $\gamma=|\mathrm{co}(A)\setminus A|+|\mathrm{co}(B)\setminus B|$, i.e. for every $x\in\partial\,\mathrm{co}(A+B)$, we have $(1-4\tau^{-1}\sqrt{\gamma})ox\subset A+B$.

In Section 3.4, we introduce a partition of $\partial\,\mathrm{co}(A+B)$ into **good** arcs and **bad** arcs. We think of good arcs as being the parts of the boundary of $\mathrm{co}(A+B)$ which are straight (or close to straight). We show that a very small part of the boundary $\partial\,\mathrm{co}(A+B)$ is covered by bad arcs.

In Section 3.5, we show for x in a good arc of $\partial\,\mathrm{co}(A+B)$, we can in fact guarantee that $(1-\xi\sqrt{\gamma})ox$ lies in $A+B$ for any small ξ (provided small d_τ). Thus $\mathrm{co}(A+B)\setminus(A+B)$ lies in a thickened boundary Λ of $\partial\,\mathrm{co}(A+B)$, which is thinner near the good arcs.

In Sections 3.6 and 3.7, we set up the following method for proving $|\mathrm{co}(A+B)\setminus(A+B)|\le(1+\varepsilon)(|\mathrm{co}(A)\setminus A|+|\mathrm{co}(B)\setminus B|)$.

The edges of $\partial\,\mathrm{co}(A+B)$ are precisely the edges of $\partial\,\mathrm{co}(A)$ and $\partial\,\mathrm{co}(B)$ attached one after the other ordered by slope. Moreover, every edge of $\partial\,\mathrm{co}(A+B)$ is the Minkowski sum of an edge of $\partial\,\mathrm{co}(A)$ with a vertex of $\partial\,\mathrm{co}(B)$ or vice versa. We subdivide $\partial\,\mathrm{co}(A+B)$ into tiny straight arcs $\mathcal{J}$, and partition these arcs into collections $\mathcal{A}$ and $\mathcal{B}$ accordingly. We note that the arcs of $\mathcal{A}$ can be reassembled to $\partial\,\mathrm{co}(A)$ and the arcs of $\mathcal{B}$ can be reassembled to $\partial\,\mathrm{co}(B)$, in the same orders as they appear in $\partial\,\mathrm{co}(A+B)$.

We erect on each arc $\mathfrak{q}\in\mathcal{J}$ a parallelogram $R_\mathfrak{q}$ pointing roughly towards the origin such that these parallelograms cover the thickened boundary Λ. We ensure that we use a constant number of directions (1000 suffices), such that the $R_\mathfrak{q}$s with the same directions occur in contiguous arcs of $\partial\,\mathrm{co}(A+B)$. The heights of the parallelograms will be roughly on the order of $\sqrt{\gamma}$ if $\mathfrak{q}$ lies in a bad arc, and $\xi\sqrt{\gamma}$ if $\mathfrak{q}$ lies in a good arc. Each parallelogram $R_\mathfrak{q}$ with $\mathfrak{q}\in\mathcal{A}$ is the Minkowski sum of a parallelogram $R_{\mathfrak{q},A}$ erected on the corresponding segment of $\partial\,\mathrm{co}(A)$ with a vertex $p_{\mathfrak{q},B}\in\partial\,\mathrm{co}(B)\cap B$. Similarly for $\mathfrak{q}\in\mathcal{B}$.

This construction allows us to cover the thickened boundary Λ of $\partial\,\mathrm{co}(A+B)$ with translates of small regions erected on $\partial\,\mathrm{co}(A)$ and $\partial co(B)$ as follows:

$$\Lambda\subset\bigcup_{\mathfrak{q}\in\mathcal{A}}(R_{\mathfrak{q},A}+p_{\mathfrak{q},B})\cup\bigcup_{\mathfrak{q}\in\mathcal{B}}(p_{\mathfrak{q},A}+R_{\mathfrak{q},B}).$$

Therefore, we can cover $\mathrm{co}(A+B)\setminus(A+B)$ as follows:

$$\mathrm{co}(A+B)\setminus(A+B)\subset\bigcup_{\mathfrak{q}\in\mathcal{A}}\left((R_{\mathfrak{q},A}\setminus A)+p_{\mathfrak{q},B}\right)\cup\bigcup_{\mathfrak{q}\in\mathcal{B}}\left(p_{\mathfrak{q},A}+(R_{\mathfrak{q},B}\setminus B)\right).$$

If we have subsets $\mathcal{A}'\subset\mathcal{A}$ and $\mathcal{B}'\subset\mathcal{B}$ such that $\{R_{\mathfrak{q},A}\}_{\mathfrak{q}\in\mathcal{A}'}$ are disjoint and contained in $\mathrm{co}(A)$ and analogously $\{R_{\mathfrak{q},B}\}_{\mathfrak{q}\in\mathcal{B}'}$ are disjoint and contained in $\mathrm{co}(B)$, then we obtain an

inequality

$$|\operatorname{co}(A+B)\setminus(A+B)| \leq |\operatorname{co}(A)\setminus A| + |\operatorname{co}(B)\setminus B| + \sum_{\mathfrak{q}\in\mathcal{A}\setminus\mathcal{A}'} |R_{\mathfrak{q},A}| + \sum_{\mathfrak{q}\in\mathcal{B}\setminus\mathcal{B}'} |R_{\mathfrak{q},B}|.$$

Hence to prove Theorem 1.1.5, it suffices to show that we can find such $\mathcal{A}'$ and $\mathcal{B}'$ with

$$\sum_{\mathfrak{q}\in\mathcal{A}\setminus\mathcal{A}'} |R_{\mathfrak{q},A}| + \sum_{\mathfrak{q}\in\mathcal{B}\setminus\mathcal{B}'} |R_{\mathfrak{q},B}| \leq \varepsilon(|\operatorname{co}(A)\setminus A| + |\operatorname{co}(B)\setminus B|).$$

In Section 3.8 we show that bad arcs of $\partial\operatorname{co}(A+B)$ are close in angular distance to the corresponding arcs in $\partial\operatorname{co}(A)$ and $\partial\operatorname{co}(B)$. This result is crucial for Sections 3.9 and 3.10 where we bound the areas of the parallelograms we have to remove to create $\mathcal{A}'$ and $\mathcal{B}'$.

In Section 3.9, we use Section 3.8 to show that parallelograms $R_{\mathfrak{q},A} \not\subset \operatorname{co}(A)$ and $R_{\mathfrak{q},B} \not\subset B$ have $\mathfrak{q}$ on a good arc. This is then used to show that the area of parallelograms not contained in $\operatorname{co}(A)$ or $\operatorname{co}(B)$ is bounded roughly by $\xi^2\gamma$.

In Section 3.10 we use Section 3.8 to show that parallelograms $R_{\mathfrak{q},A}$ and $R_{\mathfrak{r},A}$ that intersect non-trivially have at least one of $\mathfrak{q}$ and $\mathfrak{r}$ on a good arc. This allows us to remove only good parallelograms to ensure disjointness. We conclude that the area of parallelograms we need to remove is bounded by roughly $\xi\gamma$.

In Section 3.11 we complete the proof of Theorem 1.1.5 by synthesizing our bounds to deduce the final inequality. In Section 3.12 we show how Theorem 1.1.5 implies Theorem 1.1.4. Finally, in Section 3.13 we add a proof that the measures α and ω are commensurate for small δ.

3.2 Setup

In this section, we collect together the preliminaries we need to start proving Theorem 1.1.5. In Section 3.2.1 we introduce an equal area reformulation of Theorem 1.1.5. In Section 3.2.2 we apply a preliminary affine transformation to $\mathbb{R}^2$ and collect facts about the resulting lengths and areas. In Section 3.2.3 we collect the main definitions which will be used throughout the body of the chapter. Finally, in Section 3.2.4 we collect general observations which we will use frequently throughout.

3.2.1 Equal area reformulation

We will primarily work with the equivalent equal area reformulation Theorem 3.2.2 of Theorem 1.1.5.

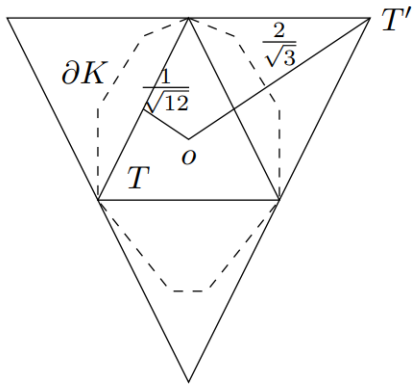

Fig. 3.2 Convex body K with the largest contained triangle T

Definition 3.2.1. *For $A,B \subset \mathbb{R}^2$ measurable sets and $t \in [0,1]$, define*

$$D_t = tA + (1-t)B.$$

Theorem 3.2.2. *For $\tau \in (0, \frac{1}{2}]$, there are constants $d_\tau = d_\tau(\varepsilon) > 0$ such that the following is true. Let $A,B \subset \mathbb{R}^2$ be measurable sets with $|A| = |B| = V$, let t a parameter satisfying $t \in [\tau, \frac{1}{2}]$, and suppose that $|D_t| \leq (1+d_\tau(\varepsilon))^2 V$. Then*

$$|\mathrm{co}(D_t) \setminus D_t| \leq (1+\varepsilon)\left(t^2|\mathrm{co}(A)\setminus A| + (1-t)^2|\mathrm{co}(B)\setminus B|\right).$$

In Theorem 3.2.2, t is a free parameter, which we note is the normalized volume ratio of tA and $(1-t)B$. Given the sets A,B in Theorem 1.1.5, A/t and $B/(1-t)$ have equal volumes, and Theorem 1.1.5 is equivalent to Theorem 3.2.2 applied with these equal volume sets.

In the equal area reformulation, we let K be the smallest convex set such that K contains a translate of A and B. We assume from now on that $A,B \subset K$. By approximation[1], we may assume that A,B,K are unions of polygons.

3.2.2 Preliminary affine transformation

Let $T \subset K$ be the maximal area triangle, and let o be the barycenter (which we will always take to be the origin). This maximal area triangle T has the property that $T \subset K \subset -2T := T'$, and by applying an affine transformation, we may assume that T is a unit equilateral triangle whose vertices are contained in K.

[1]It is easy to show that for any fixed $d_\tau(\varepsilon)$ we must have A,B bounded. Now, approximate A,B from the inside by a nested sequence of compact subsets $A_1 \subset A_2 \subset \ldots$ and $B_1 \subset B_2 \subset \ldots$. Then for each A_i, B_i approximate the pair from the outside by finite unions of polygons.

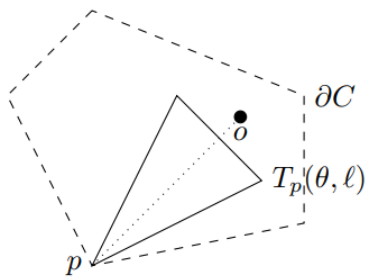

Fig. 3.3 Triangle $T_p(\theta,\ell)$ corresponding to point $p \in \partial C$.

Observation 3.2.3. *We make the following observations concerning lengths and areas.*

- *We have* $|T| = \frac{\sqrt{3}}{4}$, $|T'| = \sqrt{3}$, $|A|, |B| \in \left(0, \sqrt{3}\right]$ *and* $|K| \in \left[\frac{\sqrt{3}}{4}, \sqrt{3}\right]$.
- *For* $p \in T' \setminus T$ *we have* $|op| \in \left[\frac{1}{\sqrt{12}}, \frac{2}{\sqrt{3}}\right]$, *and this in particular holds for* $p \in \partial K$.

3.2.3 Definitions

We now collect definitions we will use for the remainder of the chapter.

Definition 3.2.4. *We define*

$$\gamma := \gamma(A,B) = t^2|\operatorname{co}(A) \setminus A| + (1-t)^2|\operatorname{co}(B) \setminus B|.$$

Definition 3.2.5. *In a convex set* C *containing* o, *given a point* $p \in \partial C$ *we say that* p *is* (θ,ℓ)-***bisecting*** *if the unique isosceles triangle* $T_p(\theta,\ell)$ *with angle* θ *at* p *and equal sides* ℓ *such that* po *internally bisects the corresponding angle is contained inside* C.

Definition 3.2.6. *Given a convex set* C, *and a point* $p \in \partial C$, *we say that* p *is* (θ,ℓ)-***good*** *if there are any points* $q, r \in C$ *such that* $|pq|, |pr| \geq \ell$ *and* $\angle qpr \geq 180^\circ - \theta$. *Any point in* ∂C *which is not* (θ,ℓ)-*good is* (θ,ℓ)-***bad***.

Definition 3.2.7. *Given a point* p *and a set* E *with* $o \in \operatorname{co}(E)$, *we denote* p_E *the intersection of the ray* op *with* $\partial \operatorname{co}(E)$.

3.2.4 General Observations

Observation 3.2.8. *Suppose we have subsets $R_A \subset \mathrm{co}(A), R_B \subset \mathrm{co}(B)$, and $z \in \mathbb{R}^2$. Let $H = H_{-\frac{1-t}{t},z}$ denote the negative homothety of ratio $-\frac{1-t}{t}$ through z. Then if $|R_A \cap H(R_B)| > t^{-2}\gamma$, or equivalently $|H^{-1}(R_A) \cap R_B| > (1-t)^{-2}\gamma$, then we have $z \in D_t$.*[2]

Observation 3.2.9. *For sets A, B with common volume V, Figalli and Jerison showed (see Theorem 1.1.2) that for fixed τ we have $|K \setminus A|V^{-1}$, $|K \setminus B|V^{-1} \to 0$ as $|D_t|V^{-1} \to 1$. In particular, as $V \in (0, \sqrt{3}]$ by Observation 3.2.3, we have*

$$|K \setminus A|,\ |K \setminus B|,\ |\mathrm{co}(A) \setminus A|, |\mathrm{co}(B) \setminus B|, \gamma \to 0 \text{ as } d_\tau \to 0.$$

3.2.5 Constants and their dependencies

Fix τ and ε. For the convenience of the reader, we describe roughly our choice of parameters throughout. First, we will take $M = 1000 \in 2\mathbb{N}$ to be a universal constant and $\alpha = \frac{720^\circ}{M}$. Next, we will take ξ such that $\varepsilon \geq (\tau^2 + (1-\tau)^2)(25\tau^{-1}M\xi^2 + 16000\tau^{-1}M\xi)$. Next, we take $\theta \leq \frac{1}{2}^\circ$ such that $\frac{1}{2}\xi^2 \sin(28^\circ)^6 / \sin(4\theta) \geq 1$, and we take ℓ such that $\left(\frac{1440^\circ}{\theta} + 3\right) 4(1 + 100t^{-1})\ell \frac{100}{99}\sqrt{12} < \frac{1}{3}\alpha$. Finally, take d_τ sufficiently small to make various statements true along the way.

3.3 Initial structural results

In this section, we will show three preliminary propositions which quantify how close we may assume A, B are to K, and how much of $\mathrm{co}(D_t)$ we can guarantee is covered by D_t without resorting to a finer analysis of the boundaries of the various regions.

- In Proposition 3.3.1 we show that for any constant $\eta \in (0,1)$, if d_τ is sufficiently small in terms of η then we have

$$(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t) \subset K.$$

- In Proposition 3.3.3 we show that if d_τ is sufficiently small, then for every $z \in \partial K$ we have that z, z_A, z_B, z_{D_t} are $(59^\circ, \frac{1}{3})$-bisecting.

- Finally, in Proposition 3.3.5 we show that if d_τ is sufficiently small, then

$$\left(1 - 4t^{-1}\sqrt{\gamma}\right) \mathrm{co}\,(D_t) \subset D_t.$$

[2]Note that $t^{-2}\gamma = |\mathrm{co}(A) \setminus A| + |\mathrm{co}(H(B)) \setminus H(B)|$, so there is at least one $x \in R_A \cap H(R_B) \subset \mathrm{co}(A) \cap H(\mathrm{co}(B))$ which is not in $(\mathrm{co}(A) \setminus A) \cup (\mathrm{co}(H(B)) \setminus H(B))$. Thus $x \in A \cap H(B)$, and $z = tx + (1-t)H^{-1}(x) \in D_t$.

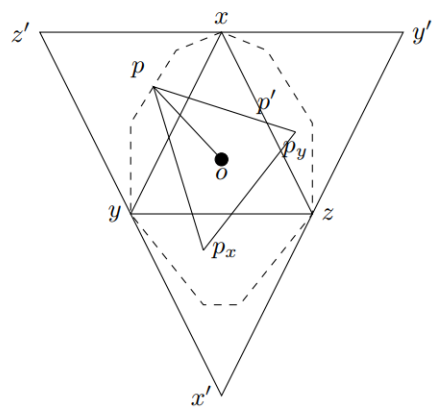

Fig. 3.4 The points p_x, p_y, and $p_z = p$ as in the proof of Lemma 3.3.2

3.3.1 Showing $\mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t)$ contain a large scaled copy of K

Proposition 3.3.1. *For any fixed $\eta \in (0,1)$, if d_τ is sufficiently small in terms of η, then $(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t) \subset K$.*

To prove Proposition 3.3.1, we need Lemma 3.3.2 which guarantees that ∂K behaves well under the notion of (θ, ℓ)-bisecting from Definition 3.2.5.

Lemma 3.3.2. *Every point $p \in \partial K$ is $(60^\circ, \frac{1}{2})$-bisecting.*

Proof. Note that the statement is trivially true if p is a vertex of ∂T (since then $T_p(60^\circ, 1) = T \subset K$), so assume otherwise. Let x, y, z be the vertices of T and $x' = -2x$, $y' = -2y$, $z' = -2z$ the corresponding vertices of T'. Let $p = p_z$ be in the triangle xyz'. Let $p_y \in xy'z$ and $p_x \in x'yz$ be the point p_z rotated by 120° and 240° clockwise around o respectively. Note that $p_x p_y p_z$ is an equilateral triangle with centre o, such that $\angle o p_z p_y = 30^\circ$. Let p' be the intersection between segments xz and $p_z p_y$.

Note that $pp' \subset K$. We will show that $|pp'| \geq \frac{1}{2}$. Note that the points o, p, p', x are concyclic as $\angle oxp' = 30^\circ = \angle opp'$. We have $\angle pxp' \in [60^\circ, 120^\circ]$, so by the law of sines, $2r = \frac{|pp'|}{sin \angle pxp'} \leq \frac{2}{\sqrt{3}}|pp'|$, where r is the circumradius of this circle. But $2r \geq |ox| = \frac{1}{\sqrt{3}}$, so $|pp'| \geq \frac{1}{2}$. By showing a similar result for $p_z p_x$, we conclude that $T_p(60^\circ, \frac{1}{2})$ lies in K. □

Proof of Proposition 3.3.1. We prove this for $\mathrm{co}(A)$, the identical proof works for $\mathrm{co}(B)$ and then because $\mathrm{co}(D_t) = t\,\mathrm{co}(A) + (1-t)\,\mathrm{co}(B)$ we deduce the final containments. By Observation 3.2.9, we can take d_τ sufficiently small in terms of η so that $|K \setminus A| < \frac{\sqrt{3}}{36}\eta^2$. Let $p \in \partial K$, let $p' \in op$ be such that $|pp'| = \eta |op|$, and suppose for the sake of contradiction that $p' \notin \mathrm{co}(A)$. Then as $|op| \in \left[\frac{1}{\sqrt{12}}, \frac{2}{\sqrt{3}}\right]$ by Observation 3.2.3, we have $|pp'| \in \left[\frac{\eta}{\sqrt{12}}, \frac{2\eta}{\sqrt{3}}\right] = \left[(\frac{2}{3}\eta)h, (\frac{8}{3}\eta)h\right]$ where $h = \frac{\sqrt{3}}{4}$ is the height of $T_p(60^\circ, \frac{1}{2})$. A line separating p from $\mathrm{co}(A)$

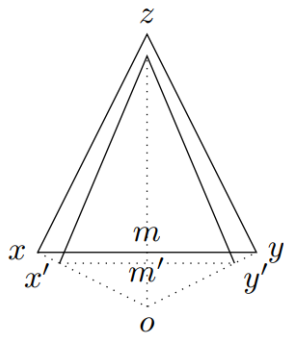

Fig. 3.5 Rescaling triangle xyz

through p' cuts off from $T_p(60^\circ, \frac{1}{2})$ an area of at least $\min(\frac{1}{2}, (\frac{2}{3}\eta)^2)\left|T_p(60^\circ, \frac{1}{2})\right| = \frac{\sqrt{3}}{36}\eta^2$ on the p-side, which lies in $K \setminus A$, contradicting $|K \setminus A| < \frac{\sqrt{3}}{36}\eta^2$. □

3.3.2 Showing points in ∂K, $\partial\,\mathrm{co}(A)$, $\partial\,\mathrm{co}(B)$, $\partial\,\mathrm{co}(D_t)$ are $(59^\circ, \frac{1}{3})$-bisecting

Proposition 3.3.3. *For d_τ sufficiently small, we have for every $z \in \partial K$ that z, z_A, z_B, z_{D_t} are $(59^\circ, \frac{1}{3})$-bisecting.*

Proof. By Proposition 3.3.1 we can take d_τ sufficiently small so that $(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B) \subset K$ with $\eta = 10^{-9}$. Let C be one of $K, \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t)$. We have $T_z(60^\circ, \frac{1}{2}) \subset K$. Let x, y denote the other two vertices of the triangle, and let $x' = (1-\eta)x$, $y' = (1-\eta)y$. Note that $x', y' \in (1-\eta)K \subset C$. Let m be the midpoint of xy and m' be the midpoint of $x'y'$. Then $|x'm'| = \frac{1}{4}(1-\eta)$, $|m'z_C| \le |mz_C| + |mm'| \le |mz| + \eta|om| \le \frac{\sqrt{3}}{4} + \eta\frac{2}{\sqrt{3}}$ by Observation 3.2.3, and similarly $|m'z_C| \ge |mz| - |zz_C| - |m'm| \ge |mz| - \eta(|oz| + |om|) \ge \frac{\sqrt{3}}{4} - 2\eta\frac{2}{\sqrt{3}}$ (these are true even if o is inside the triangle xyz). Thus, by inspecting the right-angled triangles $x'm'z_C$ and $y'm'z_C$, because $\tan(29.5^\circ)\left(\frac{\sqrt{3}}{4} + \eta\frac{2}{\sqrt{3}}\right) < \frac{1}{4}(1-\eta)$ and $\frac{1}{\cos(29.5^\circ)}\left(\frac{\sqrt{3}}{4} - 2\eta\frac{2}{\sqrt{3}}\right) > \frac{1}{3}$, the vertices of $T_{z_C}(59^\circ, \frac{1}{3})$ lie in the triangle $x'y'z_C \subset C$. □

Corollary 3.3.4. *Let C be $K, \mathrm{co}(A), \mathrm{co}(B)$ or $\mathrm{co}(D_t)$. For d_τ sufficiently small, given $z \in \partial C$ and a supporting line l to C at z, we have $\angle l, zo \in (29^\circ, 180^\circ - 29^\circ)$.*

3.3.3 Showing D_t contains a large scaled copy of $\mathrm{co}(D_t)$

Proposition 3.3.5. *For d_τ sufficiently small, we have*

$$(1 - 4t^{-1}\sqrt{\gamma})\,\mathrm{co}(D_t) \subset D_t.$$

In particular, if $z \in \partial\,\mathrm{co}(D_t)$ and $p \in oz$ has $|pz| \ge 5t^{-1}\sqrt{\gamma}$, then $p \in D_t$.

To show Proposition 3.3.5, we need the following lemma.

Lemma 3.3.6. *For every $\eta \in (0,1)$ and d_τ sufficiently small in terms of η, we have $(1-\eta)K \subset D_t$.*

Proof. We may assume that $\eta \leq 10^{-9}$. We take d_τ sufficiently small in terms of η such that $\frac{1-\eta}{1-\frac{\eta}{2}}K \subset \mathrm{co}(A), \mathrm{co}(B)$ by Proposition 3.3.1, and $t^{-2}\gamma < \pi(\frac{1}{100}\eta)^2$ by Observation 3.2.9. First, we show that for every $k \in K$ we have

$$B\left((1-\eta)k, \frac{1}{100}\eta\right) \subset \mathrm{co}(A), \mathrm{co}(B).$$

We show the $\mathrm{co}(A)$ containment, the other containment's proof is identical.

Write $k = \lambda k'$ with $k' \in \partial K$ and $\lambda \in [0,1]$. Because k' is $(60^\circ, \frac{1}{2})$-bisecting we see that

$$B\left(\left(1-\frac{\eta}{2}\right)k', \frac{\eta}{2\sqrt{12}}\sin(30^\circ)\right) \subset T_{k'}(60^\circ, \frac{1}{2}) \subset K,$$

as $|ok'| \geq \frac{1}{\sqrt{12}}$ by Observation 3.2.3. Thus

$$B\left((1-\eta)k', \frac{\eta}{20}\right) \subset B\left((1-\eta)k', \frac{1-\eta}{1-\frac{\eta}{2}}\frac{\eta}{2\sqrt{12}}\sin(30^\circ)\right) \subset \left(\frac{1-\eta}{1-\frac{\eta}{2}}\right)K \subset \mathrm{co}(A),$$

and so $B\left((1-\eta)k, \frac{\lambda}{20}\eta\right) \subset \mathrm{co}(A)$. If $\lambda \geq \frac{1}{5}$, then $B\left((1-\eta)k, \frac{1}{100}\eta\right) \subset \mathrm{co}(A)$, as desired.

Otherwise, assume $\lambda < \frac{1}{5}$. By Observation 3.2.3 we have $|k'| \leq \frac{2}{\sqrt{3}}$, so it follows that $\left|(1-\eta)\frac{100}{99}k\right| + \frac{1}{99} \leq \frac{1}{\sqrt{12}}$, the distance from o to ∂T, and hence $B((1-\eta)\frac{100}{99}k, \frac{1}{99}) \subset T$. Hence, we have $B\left((1-\eta)k, \frac{1}{100}\right) \subset \frac{99}{100}T \subset \mathrm{co}(A)$. Thus we always have $B((1-\eta)k, \frac{1}{100}\eta) \subset \mathrm{co}(A)$ as desired.

Let $k \in K$. To check that $z = (1-\eta)k = t(1-\eta)k + (1-t)(1-\eta)k \in D_t$, in the notation of Observation 3.2.8 we take $R_A = R_B = B((1-\eta)k, \frac{1}{100}\eta) \subset \mathrm{co}(A), \mathrm{co}(B)$. Then $|R_A \cap H_{-\frac{1-t}{t},z}(R_B)| = |R_A| = \pi(\frac{1}{100}\eta)^2 > t^{-2}\gamma$. Hence, we conclude by Observation 3.2.8 that $z \in D_t$. □

Proof of Proposition 3.3.5. Let $\eta = 10^{-9}$, and take d_τ sufficiently small so that Proposition 3.3.3 and Lemma 3.3.6 apply, and that $\gamma \leq \frac{t^2}{16}$ by Observation 3.2.9. Let $z = tx + (1-t)y \in \partial\,\mathrm{co}(D_t)$ where $x \in \partial\,\mathrm{co}(A)$ and $y \in \partial\,\mathrm{co}(B)$. We will show that $z' = (1-4\lambda t^{-1}\sqrt{\gamma})z$ lies in D_t for all $\lambda \in [1, \frac{t}{4\sqrt{\gamma}}]$.

By Proposition 3.3.3 we have x, y are $(59^\circ, \frac{1}{3})$-bisecting. Define x', y' analogously to z', and note that $tx' + (1-t)y' = z'$ and $|xx'|, |yy'|, |zz'| \in [\frac{4}{\sqrt{12}}\lambda t^{-1}\sqrt{\gamma}, \frac{8}{\sqrt{3}}\lambda t^{-1}\sqrt{\gamma}]$, $|oz| \leq \frac{2}{\sqrt{3}}$

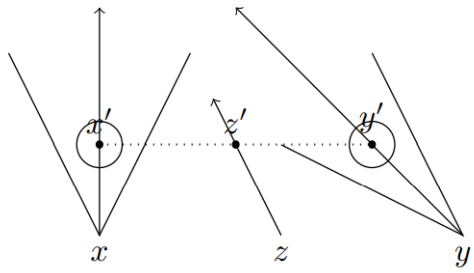

Fig. 3.6 Balls around x' and y', showing that z' is in D_t.

by Observation 3.2.3. Because $\frac{1}{4}|xx'|, \frac{1}{4}|yy'| \leq |zz'|$, if either $|xx'|$ or $|yy'|$ is at least $\frac{1}{100}$, then $|zz'| \geq \frac{1}{400}$, which by Lemma 3.3.6 implies

$$z' \in \left(1 - \frac{|zz'|}{|oz|}\right) K \subset \left(1 - \frac{\sqrt{3}}{50}\right) K \subset (1-\eta)K \subset D_t.$$

Assume now that $|xx'|, |yy'| < \frac{1}{100}$, so that the altitudes from x (resp. y) of $T_x(59^\circ, \frac{1}{3})$ (resp. $T_y(59^\circ, \frac{1}{3})$) exceed $2|xx'|$ (resp. $2|yy'|$). Because $\lambda \geq 1$ we have

$$|xx'|, |yy'| \geq \frac{4\sqrt{\gamma}}{\sqrt{12}} \lambda t^{-1} \geq 1.001 t^{-1} \sqrt{\frac{\gamma}{\pi}} / \sin(29.5^\circ).$$

Together the last two sentences show that $B\left(x', 1.001t^{-1}\sqrt{\frac{\gamma}{\pi}}\right) \subset T_x\left(59^\circ, \frac{1}{3}\right) \subset \mathrm{co}(A)$, and $B\left(y', 1.001t^{-1}\sqrt{\frac{\gamma}{\pi}}\right) \subset T_y\left(59^\circ, \frac{1}{3}\right) \subset \mathrm{co}(B)$. By applying Observation 3.2.8 with $R_A = B\left(x', 1.001t^{-1}\sqrt{\frac{\gamma}{\pi}}\right)$ and $R_B = B\left(y', 1.001t^{-1}\sqrt{\frac{\gamma}{\pi}}\right)$, we conclude that $z' \in D_t$. Finally, $|zz'| = 4t^{-1}\sqrt{\gamma}|oz| \leq \frac{8}{\sqrt{3}} t^{-1}\sqrt{\gamma} < |pz|$, so $p \in D_t$. □

3.4 Decomposing $\partial\,\mathrm{co}(D_t)$ into good arcs, and bad arcs of small total angular size

Recall that $M \in 2\mathbb{N}$ be some universal constant (1000 suffices), and set $\alpha = \frac{720^\circ}{M}$.

Definition 3.4.1. *For any s, we denote by $\mathcal{I}_s^{bad}(\theta, \ell)$ the collection of arcs formed by the set of all points in $\partial\,\mathrm{co}(D_t)$ within Euclidean distance s of a (θ, ℓ)-bad point (which is a union of arcs). We let $\mathcal{I}_s^{good}(\theta, \ell)$ denote the remaining arcs in $\partial\,\mathrm{co}(D_t)$, which we subdivide into arcs of angular length at most $\frac{1}{3}\alpha$.*

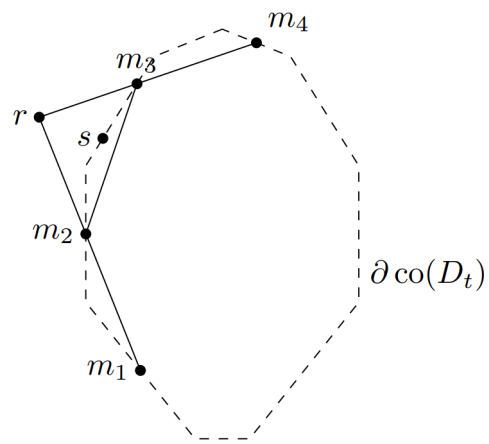

Fig. 3.7 s is a (θ,ℓ)-good point.

Proposition 3.4.2. *For d_τ sufficiently small, there exists an increasing function $\ell = \ell(\theta)$ for $\theta < 180^\circ$, such that the union of arcs $\bigcup \mathcal{I}^{bad}_{100t^{-1}\ell}(\theta,\ell)$ has total angular size at most $\frac{1}{3}\alpha$.*

Proof. Take d_τ sufficiently small so that $\frac{99}{100}K \subset \mathrm{co}(D_t)$ by Proposition 3.3.1.

Choose a point on $\partial\,\mathrm{co}(D_t)$, and form a polygon P inscribed in $\partial\,\mathrm{co}(D_t)$ by traveling around clockwise and picking the first vertex at distance ℓ from the previous vertex, all the way until the polygon would self-intersect, and then we simply join the first and last vertex with an edge. Then all sides are of length ℓ except one side of possibly smaller size. Moreover, each vertex of the polygon is within distance ℓ of every point of the next subtended arc of $\partial\,\mathrm{co}(D_t)$.

We let $\mathcal{S}^{\mathrm{good}}$ be the collection of arcs of $\mathrm{co}(D_t)$ which arise as the arc subtended by m_2m_3, where m_1,m_2,m_3,m_4 are four consecutive vertices of the polygon P, with $|m_1m_2| = |m_2m_3| = |m_3m_4| = \ell$ and $\angle m_1m_2m_3, \angle m_2m_3m_4 \geq 180^\circ - \frac{\theta}{2}$. We claim that every point $s \in \mathfrak{q} \in \mathcal{S}^{\mathrm{good}}$ is (θ,ℓ)-good. To see this, note that the angle condition in particular implies that $\angle m_1m_2m_3, \angle m_2m_3m_4 > 90^\circ$, so the rays m_1m_2 and m_4m_3 meet at a point r as shown in the figure below. We now show that m_1,m_4 realize s as a (θ,ℓ)-good point. First, note that $|m_1s| \geq \ell = |m_1m_2|$ because $\angle m_1m_2s \geq 90^\circ$. Similarly $|m_4s| \geq \ell = |m_3m_4|$. Finally, $\angle m_1sm_4 \geq \angle m_1rm_4 \geq 180^\circ - \theta$, where the first inequality follows as s lies inside the triangle m_1rm_4, and the second as $\angle rm_2m_3, \angle rm_3m_2 \leq \frac{\theta}{2}$.

Let $\mathcal{S}^{\mathrm{bad}}$ be the collection of remaining arcs of $\partial\,\mathrm{co}(D_t)$ subtended by sides of P which are not in $\mathcal{S}^{\mathrm{good}}$. As the sum of the exterior angles of P is 360°, the number of interior angles which are strictly less than $180^\circ - \frac{\theta}{2}$ is at most $\frac{720^\circ}{\theta}$. Thus, $|\mathcal{S}^{\mathrm{bad}}| \leq \frac{1440^\circ}{\theta} + 3$ (we add 3 for the arc subtended by the last side of the polygon and the two adjacent arcs). Note that every (θ,ℓ)-bad point is contained in an arc in $\mathcal{S}^{\mathrm{bad}}$.

For each arc $\mathfrak{q} \in \mathcal{S}^{\text{bad}}$ let $x_{\mathfrak{q}}$ denote its clockwise starting point and $I_{\mathfrak{q}} := \partial \operatorname{co}(D_t) \cap B(x_{\mathfrak{q}}, (1+100t^{-1})\ell)$ the set of all points of $\partial \operatorname{co}(D_t)$ within Euclidean distance at most $(1+100t^{-1})\ell$ of $x_{\mathfrak{q}}$. This includes the points within Euclidean distance at most $100t^{-1}\ell$ of $\mathfrak{q}$. Let $I := \bigcup I_{\mathfrak{q}}$, so that $\bigcup \mathcal{I}^{\text{bad}}_{100t^{-1}\ell}(\theta,\ell) \subset I$.

Recall that $\frac{99}{100}K \subset \operatorname{co}(D_t)$, so that $\partial \operatorname{co}(D_t) \subset T' \setminus \frac{99}{100}T$ and thus $|ox| \geq \frac{99}{100}\frac{1}{\sqrt{12}}$ by Observation 3.2.3. As $I_{\mathfrak{q}}$ is contained in $B(x_{\mathfrak{q}}, (1+100t^{-1})\ell)$ it has angular size at most $2\sin^{-1}((1+100t^{-1})\ell)\frac{100}{99}\sqrt{12}) \leq 4(1+100t^{-1})\ell\frac{100}{99}\sqrt{12}\frac{180^\circ}{\pi}$. We conclude that $\bigcup \mathcal{I}^{\text{bad}}_{100t^{-1}\ell}(\theta,\ell) \subset I$ has angular size at most

$$\left(\frac{1440^\circ}{\theta}+3\right)4(1+100t^{-1})\ell\frac{100}{99}\sqrt{12}\frac{180^\circ}{\pi},$$

which we can make smaller than $\frac{1}{3}\alpha$ by choosing ℓ sufficiently small.

□

Definition 3.4.3. *We will always denote by $\ell = \ell(\theta)$ the increasing function of θ produced by the lemma above.*

Observation 3.4.4. *Every point in an arc in $\mathcal{I}^{good}_s(\theta,\ell)$ has distance at least s to all (θ,ℓ)-bad points in $\partial \operatorname{co}(D_t)$, and we have the partition (up to a finite collection of endpoints)*

$$\bigsqcup \mathcal{I}^{good}_s(\theta,\ell) \sqcup \bigsqcup \mathcal{I}^{bad}_s(\theta,\ell) = \partial \operatorname{co}(D_t).$$

3.5 Replacing $5t^{-1}\sqrt{\gamma}$ with $\xi\sqrt{\gamma}$ on arcs in $\mathcal{I}^{\text{good}}_{2\ell}(\theta,\ell)$

This section is devoted to proving the following proposition.

Proposition 3.5.1. *For every $\xi \in (0,1)$ there exists $\theta > 0$, such that for d_τ sufficiently small in terms of ξ the following is true. For every $p \in \mathfrak{q} \in \mathcal{I}^{good}_{2\ell}(\theta,\ell)$ (recalling $\ell = \ell(\theta)$) and $p' \in op$ with $|pp'| \geq \xi\sqrt{\gamma}$, we have $p' \in D_t$.*

We outline the proof of Proposition 3.5.1. Suppose first that p is the t-weighted average of points x_A and y_B[3] which are distance at most ℓ apart. Then x_{D_t}, y_{D_t} are both close enough to p that by definition of $\mathcal{I}^{\text{good}}_{2\ell}(\theta,\ell)$, x_{D_t} is (θ,ℓ)-good in $\operatorname{co}(A)$ and y_{D_t} is (θ,ℓ)-good in $\operatorname{co}(B)$, which by Lemma 3.5.4 implies x_A, y_B are $(2\theta, \frac{\ell}{2})$-good, yielding certain angular regions at x_A and y_B lying in $\operatorname{co}(A)$ and $\operatorname{co}(B)$ respectively.

If instead the distance is at least ℓ, then the triangles ox_Ay_A and oy_Bx_B serve as the large angular regions at x_A and y_B respectively.

[3] Here and in the future we will be writing for example $x_{D_t} := (x_A)_{D_t}$ even if no point x has been defined.

In either case, the fact that $p \in \partial\,\text{co}(D_t)$ implies the angular regions are in suitable directions so that Lemma 3.5.5 applies, showing in either case these regions are suitable for an application of Observation 3.2.8, and we conclude.

Lemma 3.5.2. *If we perturb the endpoints of a line segment of length ℓ each by an amount $r < \frac{\ell}{2}$, then the newly created line segment is rotated by at most* $\sin^{-1}\frac{2r}{\ell}$.

Proof. Consider two circles of radius r around the two endpoints of the segment, then the maximally rotated segment is one of the interior bitangents to these circles. □

Lemma 3.5.3. *In a triangle with vertices a,b,c, suppose that $\angle acb \in (28^\circ, 180^\circ - 28^\circ)$. Then the distance from c to ab is at least* $\sin(14^\circ)\min(|ac|,|bc|)$.

Proof. Let z be the foot of the perpendicular from c to the line ab. We have either $\angle acz \leq 90^\circ - 14^\circ$ or $\angle bcz \leq 90^\circ - 14^\circ$. Suppose without loss of generality that $\angle azc \leq 90^\circ - 14^\circ$. Then $|cz| = (\cos\angle azc)|ac| \geq \sin(14^\circ)|ac|$. □

Lemma 3.5.4. *For d_τ sufficiently small in terms of θ, if x_{D_t} is (θ,ℓ)-good in* $\text{co}(D_t)$*, then x_A is $(2\theta,\ell/2)$-good in* $\text{co}(A)$ *and x_B is $(2\theta,\ell/2)$-good in* $\text{co}(B)$.

Proof. We prove the statement for x_A, the statement for x_B is proved identically. Let $\eta = \frac{\sqrt{3}\ell}{8}\sin(\theta/2)$ (recall ℓ is defined to be a function of θ), and take d_τ sufficiently small so that $(1-\eta)K \subset \text{co}(A),\text{co}(B),\text{co}(D_t) \subset K$ by Proposition 3.3.1. Let y,z be the other two points in $\text{co}(D_t)$ realizing x_{D_t} as (θ,ℓ)-good. Because $(1-\eta)K \subset \text{co}(A),\text{co}(D_t) \subset K$, we have $|x_{D_t}x_A| \leq \eta\frac{2}{\sqrt{3}}$. Defining $y' = (1-\eta)y \in \text{co}(A)$ and $z' = (1-\eta)z \in \text{co}(B)$ we have $|yy'|,|zz'| \leq \eta\frac{2}{\sqrt{3}}$. Thus by Lemma 3.5.2, as $\sin^{-1}\left(\frac{4\eta}{\sqrt{3}\ell}\right) < \theta/2$ we have $\angle y'x_Az' \geq 180^\circ - 2\theta$. As $|x_{D_t}x_A| + |yy'| \leq \frac{4\eta}{\sqrt{3}} < \frac{\ell}{2}$, by the triangle inequality $|x_Ay'| \geq \frac{\ell}{2}$. Similarly $|x_Az'| \geq \frac{\ell}{2}$, so we see that y',z' realize x_A as $(2\theta,\frac{\ell}{2})$-good. □

Lemma 3.5.5. *Let m,n be two points and let l_m^1,l_m^2 and l_n^1,l_n^2 be pairs of rays originating at m,n, respectively and label u,v,x,y as shown in Figure 3.8. Assume further that $\angle unv = \angle ymu \geq 28^\circ$. Denote $\angle num = \theta$ and $|mn| = r$. Then we have the area lower bound $|uvxy| \geq \frac{1}{2}r^2\sin(28^\circ)^6/\sin(\theta)$.*

Proof. First, we note that

$$|uvxy| \geq |uvy| = |umn| \cdot \frac{|uv|}{|um|} \cdot \frac{|uy|}{|un|}.$$

By the law of sines, we have $|um| = r\sin(\angle unm)/\sin(\theta)$ and $|un| = r\sin(\angle umn)/\sin(\theta)$. We have $\angle unm, \angle umn \geq 28^\circ$, so as the sum of the angles of the triangle umn is 180°, we

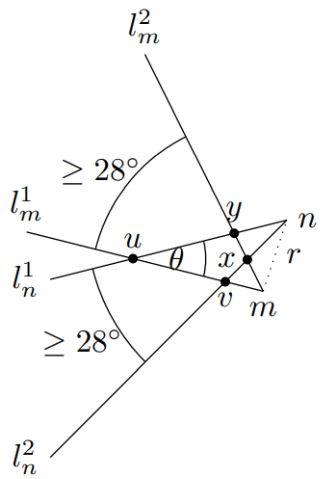

Fig. 3.8 The configuration in Lemma 3.5.5

have $\angle unm, \angle umn \in [28°, 180° - 28°]$. Therefore

$$\begin{aligned}|umn| &= \frac{1}{2}|um||un|\sin(\theta) = \frac{1}{2}r^2\sin(\angle unm)\sin(\angle umn)/\sin(\theta)\\ &\geq \frac{1}{2}r^2\sin(28)^2/\sin(\theta).\end{aligned}$$

Next, we have

$$\frac{|uv|}{|um|} = \frac{|unv|}{|unm|} = \frac{|nv|}{|nm|}\frac{\sin(\angle unv)}{\sin(\angle unm)} = \frac{\sin(\angle umn)\sin(\angle unv)}{\sin(\angle nvm)\sin(\angle unm)} \geq \sin(\angle umn)\sin(\angle unv) \geq \sin(28°)^2,$$

and by a symmetric argument we have $\frac{|uy|}{|un|} \geq \sin(28°)^2$. Multiplying the bounds, we obtain $|uvxy| \geq \frac{1}{2}r^2\sin(28°)^6/\sin(\theta)$ as desired. □

Proof of Proposition 3.5.1. We choose parameters as follows.

- $\theta \leq \frac{1}{2}^\circ$ such that $\frac{1}{2}\xi^2\sin(28°)^6/\sin(4\theta) \geq 1$ and $\ell = \ell(\theta) \leq \frac{1}{2}$.
- Next, take $\eta = \frac{\sqrt{3}}{8}\ell\sin(\theta)$ (note with this choice of η we have $(1-\eta)\frac{1}{\sqrt{12}} \geq \frac{1}{2}\ell$).
- Next, take γ_0 such that $5t^{-2}\sqrt{\gamma_0} \leq \frac{\ell}{20}\sin(4\theta)$.
- Finally, take $d = d_\tau$ sufficiently small so that
  - $\gamma \leq \gamma_0$ by Observation 3.2.9
  - $(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t) \subset K$ by Proposition 3.3.1,
  - $p' \in D_t$ if $|pp'| \geq 5t^{-1}\sqrt{\gamma_0}$ by Proposition 3.3.5

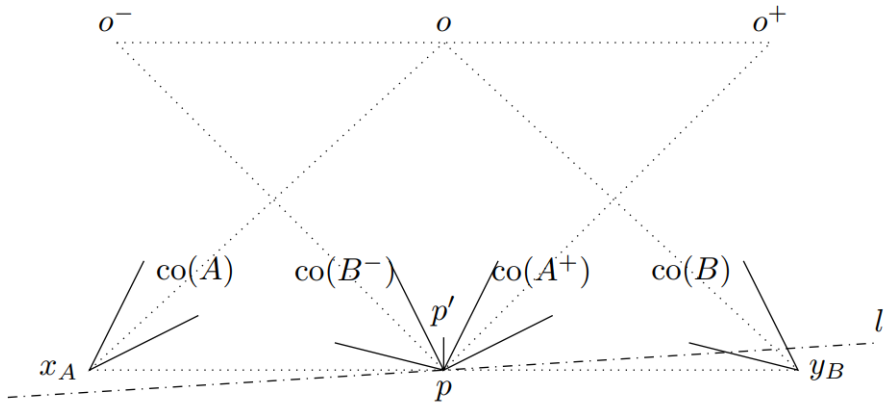

Fig. 3.9 o^+ and o^- as induced by the points x_A and y_B with $p = tx_A + (1-t)y_B$.

– Corollary 3.3.4 and Lemma 3.5.4 apply.

By our choice of d_τ we may assume that $|pp'| \in \left[\xi\sqrt{\gamma}, 5t^{-1}\sqrt{\gamma}\right]$. Write $p = tx_A + (1-t)y_B$, with $x_A \in \partial\,\mathrm{co}(A), y_B \in \partial\,\mathrm{co}(B)$. Construct

$$A^+ = A + \vec{x_A p} \qquad B^- = B + \vec{y_B p}$$
$$o^+ = o + \vec{x_A p} \qquad o^- = o + \vec{y_B p}.$$

Note that $o = to^+ + (1-t)o^-$ and hence p' is a point in triangle o^+po^- such that $|pp'| \in [\xi\sqrt{\gamma}, 5t^{-1}\sqrt{\gamma}]$. It is enough to show that for any such p' we have $p' \in tA^+ + (1-t)B^-$.

Because $p \in \partial\,\mathrm{co}(D_t)$, there is a supporting line l at p to $\mathrm{co}(D_t)$, and because $\mathrm{co}(D_t)$ is the Minkowski semisum $t\,\mathrm{co}(A) + (1-t)\,\mathrm{co}(B)$, this line also leaves $\mathrm{co}(A^+), \mathrm{co}(B^-)$ on this same side as well. By Corollary 3.3.4 we have that $\angle l, po^+, \angle l, po^- \in (29^\circ, 180^\circ - 29^\circ)$.

Our goal will be to produce points $g^+ \in \mathrm{co}(A^+), g^- \in \mathrm{co}(B^-)$ with $|g^+p|, |g^-p| \geq \frac{\ell}{10}$, fitting into the following diagram where the horizontal line is l, the points appear counterclockwise in the order g^+, o^+, p', o^-, g^-, and furthermore that pg^+ is rotated 2θ counterclockwise from ℓ about p, pg^- is rotated 2θ clockwise from ℓ about p, and $\angle g^-po^-, \angle g^+po^+ \geq 28^\circ$.

Claim 3.5.6. *If such points g^+, g^- exist then $p' \in D_t$.*

Proof. Note that $|o^+p| = |ox_A| \geq (1-\eta)\frac{1}{\sqrt{12}} \geq \frac{\ell}{2} > \frac{\ell}{10}$ by Observation 3.2.3, and similarly $|o^-p| \geq \frac{\ell}{10}$. Furthermore, $|pp'| \leq 5t^{-1}\sqrt{\gamma_0} \leq \frac{\ell}{20}\sin(4\theta)$.

Let S^- denote the triangle g^-po^- and S^+ denote the triangle g^+po^+. Let H denote the negative homothety $H = H_{p',-\frac{1-t}{t}}$ of ratio $-\frac{1-t}{t}$ at p'. Note that the inverse homothety H^{-1} is a negative homothety with ratio $-\frac{t}{1-t}$ about p'.

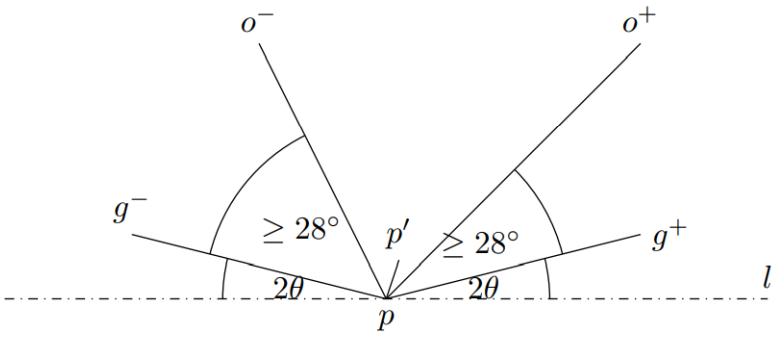

Fig. 3.10 Desired positions of g^+ and g^-

First, we show that

$$|H^{-1}(S^+)\cap S^-| \geq \frac{1}{2(1-t)^2}|pp'|^2\sin(28°)^6/\sin(4\theta).$$

This will be seen to follow from Lemma 3.5.5, applied with angle 4θ, $m=p$, $n=H^{-1}(p)$, $l_m^1 = pg^-$, $l_m^2 = po^-$, $l_n^1 = H^{-1}(pg^+)$ and $l_n^2 = H^{-1}(po^+)$. Let u,v,x and y be defined as in Lemma 3.5.5 such that $\angle num = 4\theta$.

In order to apply Lemma 3.5.5, we need to check that the intersection of the triangles $H^{-1}(S^+)$ and S^- contains the quadrilateral $uvxy$.

Indeed, we have that $|un| = \sin(\angle upn)\frac{|mn|}{\sin(4\theta)} \leq \frac{\ell}{20}\cdot\frac{t}{1-t}$, because $|mn| = \frac{1}{1-t}|pp'| \leq \frac{5}{t(1-t)}\sqrt{\gamma_0} \leq \frac{\sin(4\theta)\ell}{20}\cdot\frac{t}{1-t}$, and similarly $|up| \leq \frac{\ell}{20}\cdot\frac{t}{1-t}$. Then the triangle inequality shows that $|nv|,|py| \leq \frac{\ell}{10}\cdot\frac{t}{1-t}$ as well, and we conclude from the fact that $|H^{-1}(o^+p)|, |H^{-1}(o^-p)|, |g^+p|, |g^-p| \geq \frac{\ell}{10}\cdot\frac{t}{1-t}$.

Next, because $|pp'|^2 \geq \xi^2\gamma$, by our choice of θ_0 this implies that

$$|H^{-1}(S^+)\cap S^-| > (1-t)^{-2}\gamma.$$

Thus as

$$\begin{aligned}\frac{t^2}{(1-t)^2}|pg^+o^+\setminus A^+| + |pg^-o^-\setminus B^-| &\leq \frac{t^2}{(1-t)^2}|\mathrm{co}(A^+)\setminus A^+| + |\mathrm{co}(B^-)\setminus B^-| \\ &= \frac{t^2}{(1-t)^2}|\mathrm{co}(A)\setminus A| + |\mathrm{co}(B)\setminus B| \\ &= (1-t)^{-2}\gamma < |H^{-1}(S^+)\cap S^-|,\end{aligned}$$

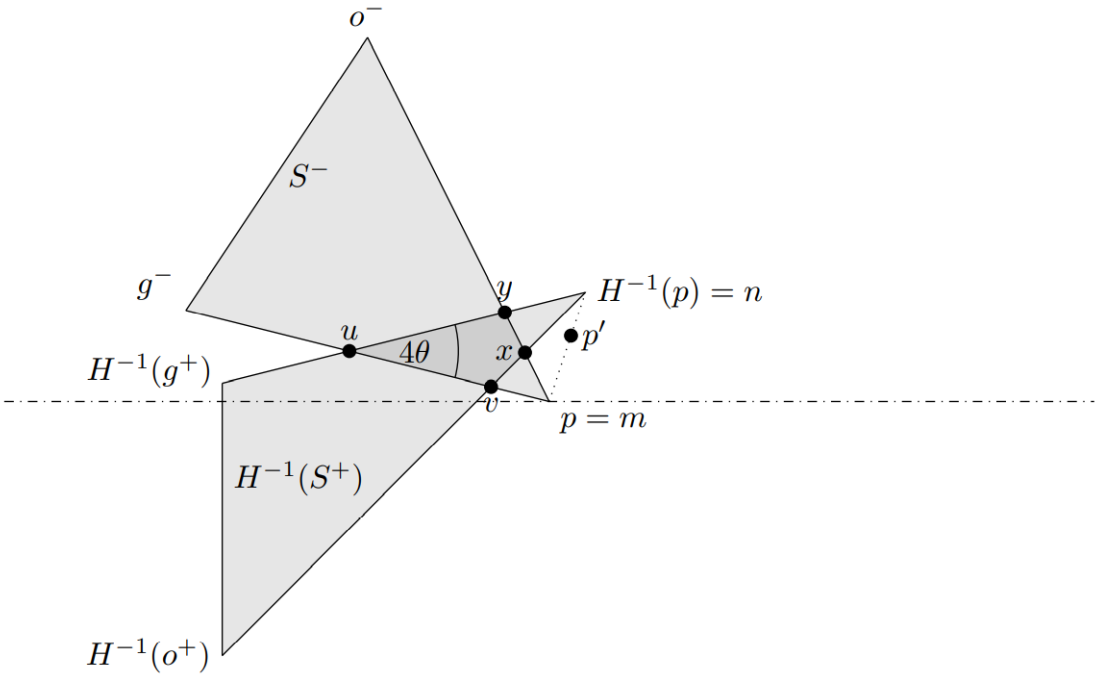

Fig. 3.11 The quadrilateral $uxvy$ as induced by S^- and $H^{-1}(S^+)$.

a suitable modification of Observation 3.2.8 shows $p' \in tA^+ + (1-t)B^-$ and hence $p' \in tA + (1-t)B$. □

Returning to the proof of the proposition, we note that exactly as in the start of Claim 3.5.6 we have $|po^+|, |po^-| \geq \frac{\ell}{2}$. We now distinguish two cases.

Case 1: $|x_A y_B| \geq \ell$. Recall the definitions of x_B and y_A from Definition 3.2.7. By Observation 3.2.3, we have that $|x_A x_B|, |y_A y_B| \leq \eta \frac{2}{\sqrt{3}} \leq \frac{\ell}{4}$ and hence by the triangle inequality $|x_A y_A|, |x_B y_B| \geq \frac{\ell}{2}$.

We also have $\angle x_A y_A, x_B y_B \leq \sin^{-1}\left(\frac{8\eta}{\sqrt{3}\ell}\right) \leq \theta$ by Lemma 3.5.2.

Define $y_A^+ := y_A + \vec{x_A p} \in A^+, x_B^- = x_B + \vec{y_B p} \in B^-$. We have that

$$|py_A^+| = |x_A y_A|, \quad |px_B^-| = |x_B y_B|,$$

and these are all $\geq \frac{\ell}{2}$ by the above discussion. Furthermore, $\angle y_A^+ p x_B^- = \angle x_A y_A, y_B x_B \geq \pi - \theta$, and the line l through p has $y_A^+, o^+, p', o^-, x_B^-$ on one side, appearing in this order counterclockwise above l. To see this, note that as p lies on the segment $x_A y_B$, $\vec{x_A p}$ lies on the same side of the line ox_A as y_A does, so $o \notin \angle y_A^+ p o_A^+$. In particular, this implies that $\angle l, py_A^+, \angle l, px_B^- \leq \theta$.

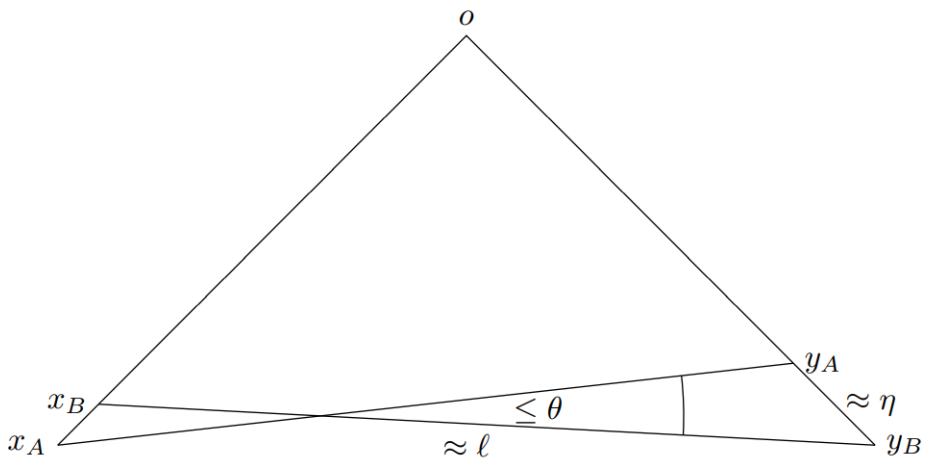

Fig. 3.12 If $|x_A y_B|$ is large, then the angle between $x_A y_A$ and $x_B y_B$ is small.

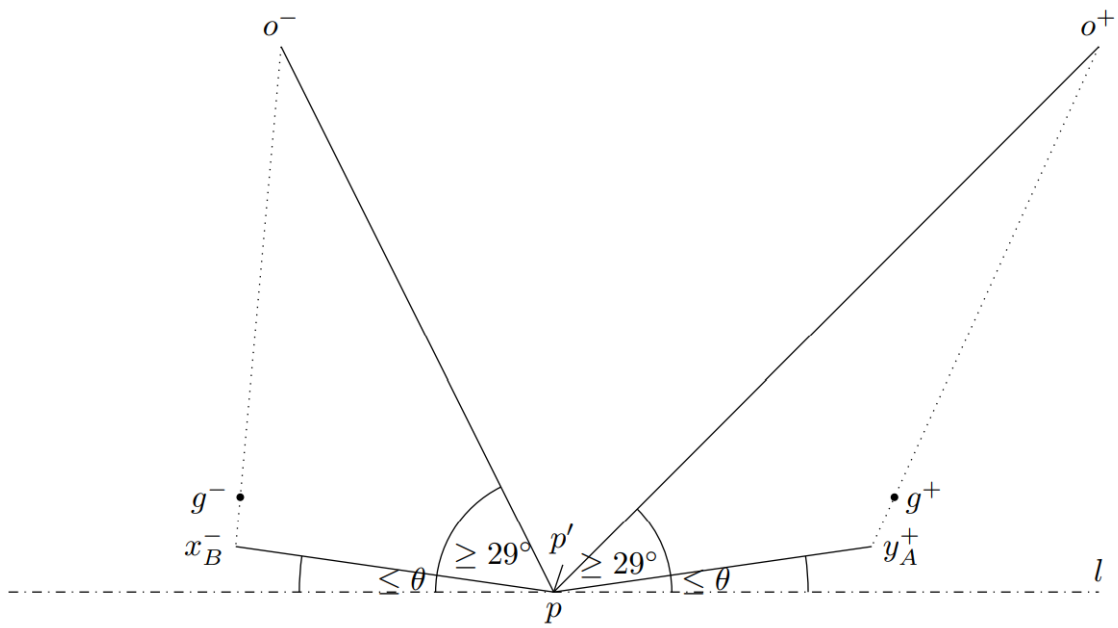

Fig. 3.13 The locations of g^+ and g^- in Case 1.

Because $\angle l, po^+, \angle l, po^- \geq 29^\circ$ and $2\theta < 29^\circ$, we have $\angle l, py_A^+ \leq 2\theta < \angle l, po^+$ and $\angle l, px_B^- \leq 2\theta < \angle l, po^-$. These imply the existence of points

$$g^+ \in y_A^+ o^+ \subset \text{co}(A^+), \text{ and } g^- \in x_B^- o^- \subset \text{co}(B^-),$$

such that $\angle l, pg^+, \angle l, pg^- = 2\theta$. Because $\angle l, py_A^+, \angle l, px_B^- \leq \theta$ and $2\theta \leq 1^\circ$, we have

$$\angle g^+ po^+, \angle g^- po^- \geq 29^\circ - 2\theta \geq 28^\circ.$$

It is clear from the construction that g^+, o^+, p', o^-, g^- also appear in this order counterclockwise above l. Finally, recall $|po^+| \geq \frac{\ell}{2}$, so by Lemma 3.5.3 as $\angle o^+ py_A^+ \in (28^\circ, 180^\circ - 28^\circ)$ we have

$$|pg^+| \geq \min\left(|py_A^+|, |po^+|\right)\sin(14^\circ) \geq \frac{\ell}{10},$$

and similarly $|pg^-| \geq \frac{\ell}{10}$.

Case 2: $|x_A y_B| \leq \ell$. Then $|x_A p|, |y_B p| \leq \ell$, and we have $|x_{D_t} x_A|, |y_{D_t} y_A| \leq \frac{2}{\sqrt{3}}\eta \leq \frac{\ell}{4}$ by Observation 3.2.3. Thus by the triangle inequality $|x_{D_t} p|, |y_{D_t} p| \leq \frac{5}{4}\ell < 2\ell$. By definition of $\mathcal{I}_{2\ell}^{\text{good}}(\theta,\ell)$, since $p \in \mathfrak{q} \in \mathcal{I}_{2\ell}^{\text{good}}(\theta,\ell)$ we have x_{D_t}, y_{D_t} are (θ,ℓ)-good. By Lemma 3.5.4 we have that $x_A \in \text{co}(A), y_B \in \text{co}(B)$ are $(2\theta, \frac{\ell}{2})$-good. Therefore, there exists

$$e_1, e_2 \in \text{co}(A), \text{ and } f_1, f_2 \in \text{co}(B)$$

such that

$$\angle e_1 x_A e_2, \angle f_1 y_B f_2 \geq 180 - 2\theta, \text{ and } |e_1 x_A|, |e_2 x_A|, |f_1 y_B|, |f_2 y_B| \geq \frac{\ell}{2}.$$

Let

$$\begin{aligned} e_1^+ &= e_1 + \vec{x_A p}, & e_2^+ &= e_2 + \vec{x_A p} \\ f_1^- &= f_1 + \vec{y_B p}, & f_2^- &= f_2 + \vec{y_B p} \end{aligned}$$

such that $e_1^+, e_2^+ \in \text{co}(A^+)$ and $f_1^-, f_2^- \in \text{co}(B^-)$. With this notation we have that $\angle e_1^+ pe_2^+, \angle f_1^- pf_2^- \geq 180 - 2\theta$ and $|e_1^+ p|, |e_2^+ p|, |f_1^- p|, |f_2^- p| \geq \frac{\ell}{2}$. Recall that $\angle l, po^+, \angle l, po^- \in (29^\circ, 180^\circ - 29^\circ)$.

Notice that the line l through p leaves $e_1^+, e_2^+, f_1^-, f_2^- o^+, o^-, p'$ on one side, and that up to relabelling the points, $e_2^+, o^+, p', o^-, f_1^-$ appear in this order counterclockwise above l. Note

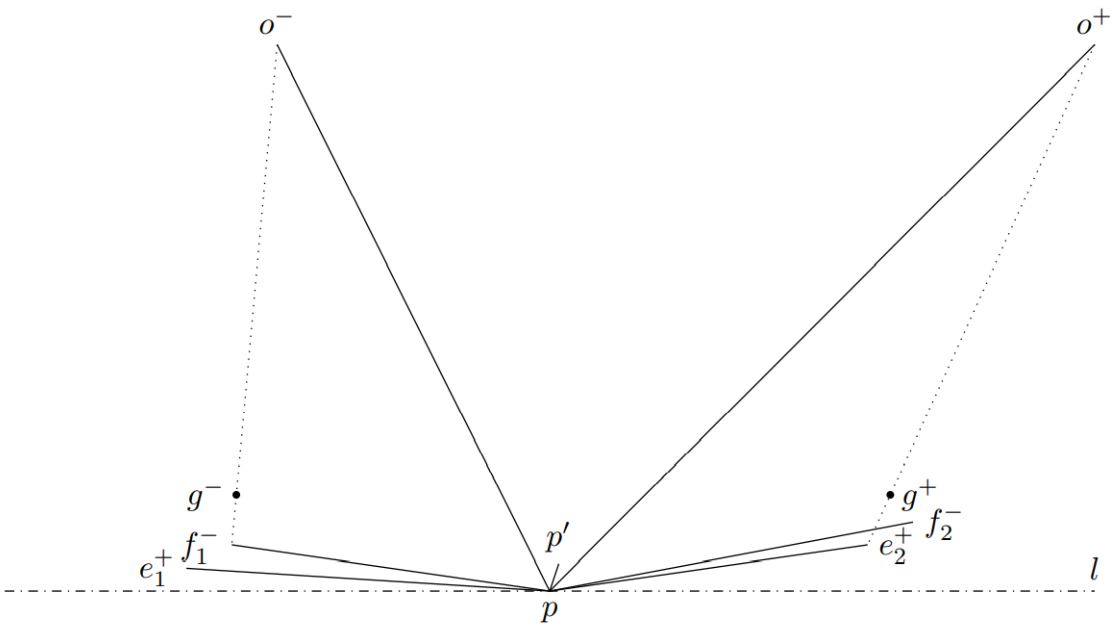

Fig. 3.14 The locations of g^+ and g^- in Case 2.

that $\angle l, e_2^+ p, \angle l, f_1^- p \leq 2\theta$. Construct points

$$g^+ \in e_2^+ o^+ \subset \mathrm{co}(A^+), \text{ and } g^- \in f_1^- o^- \subset \mathrm{co}(B^-)$$

such that $\angle l, pg^+, \angle l, pg^- = 2\theta$ and note that $\angle g^+ po^+, \angle g^- po^- \geq 28^\circ$ as $2\theta \leq 1^\circ$. We can see from the construction that the points g^+, o^+, p', o^-, g^- also appear in this order counterclockwise above l. Finally, recall $|po^+| \geq \frac{\ell}{2}$, so by Lemma 3.5.3 as $\angle o^+ pe_2^+ \in (28^\circ, 180 - 28^\circ)$, we have

$$|pg^+| \geq \min\left(|pe_2^+|, |po^+|\right) \sin(14^\circ) \geq \frac{\ell}{10},$$

and similarly that $|pg^-| \geq \frac{\ell}{10}$. □

3.6 Covering $\partial\,\mathrm{co}(D_t)$ with parallelograms

From now on, we let θ, ℓ depend on $\xi \in (0,1)$ as in Proposition 3.5.1, and always assume that d_τ is sufficiently small so that Proposition 3.5.1 holds. We will fix ξ in terms of ε, so

when we say to take d_τ sufficiently small, we implicitly will take it sufficiently small in terms of our choice of ξ.

In this section, we construct a partition $\mathcal{J}(\theta,\ell)$ of $\partial\,\mathrm{co}(D_t)$ into small straight arcs $\mathfrak{q}$, and parallelograms $R_\mathfrak{q}$ which have one side on $\mathfrak{q}$ such that

$$\mathrm{co}(D_t)\setminus D_t \subset \bigcup_{\mathfrak{q}\in\mathcal{J}(\theta,\ell)} R_\mathfrak{q}.$$

Recall that in Proposition 3.3.5 we showed that for d_τ sufficiently small D_t contains all points at radial distance $5t^{-1}\sqrt{\gamma}$ from $\partial\,\mathrm{co}(D_t)$. Furthermore, in Proposition 3.5.1 we improved the bound to $\xi\sqrt{\gamma}$ for points in $\partial\,\mathrm{co}(D_t)$ that belong to arcs in $\mathcal{I}^{\mathrm{good}}_{2\ell}(\theta,\ell)$.

We will for the remainder of the chapter be using $\mathcal{I}^{\mathrm{good}}_s(\theta,\ell),\mathcal{I}^{\mathrm{bad}}_s(\theta,\ell)$ exclusively for $s=2\ell,3\ell,100t^{-1}\ell$. Note that

$$\mathcal{I}^{\mathrm{bad}}_{2\ell}(\theta,\ell)\subset\mathcal{I}^{\mathrm{bad}}_{3\ell}(\theta,\ell)\subset\mathcal{I}^{\mathrm{bad}}_{100t^{-1}\ell}(\theta,\ell),$$

$$\mathcal{I}^{\mathrm{good}}_{2\ell}(\theta,\ell)\supset\mathcal{I}^{\mathrm{good}}_{3\ell}(\theta,\ell)\supset\mathcal{I}^{\mathrm{good}}_{100t^{-1}\ell}(\theta,\ell).$$

Thus Proposition 3.5.1 also applies to points that belong to arcs in $\mathcal{I}^{\mathrm{good}}_{3\ell}(\theta,\ell)$ and $\mathcal{I}^{\mathrm{good}}_{100t^{-1}\ell}(\theta,\ell)$, and Proposition 3.4.2 also shows that the total angular size of arcs in $\mathcal{I}^{\mathrm{bad}}_{2\ell}(\theta,\ell)$ and $\mathcal{I}^{\mathrm{bad}}_{3\ell}(\theta,\ell)$ is at most $\frac{1}{3}\alpha$. We remark in what follows that we use

- $\mathcal{I}^{\mathrm{good}}_{3\ell}(\theta,\ell)\cup\mathcal{I}^{\mathrm{bad}}_{3\ell}(\theta,\ell)$ to determine the heights of the $R_\mathfrak{q}$, and
- $\mathcal{I}^{\mathrm{good}}_{100t^{-1}\ell}(\theta,\ell)\cup\mathcal{I}^{\mathrm{bad}}_{100t^{-1}\ell}(\theta,\ell)$ to determine directions of the parallelograms $R_\mathfrak{q}$.

3.6.1 Definitions

We first refine the partitions $\mathcal{I}^{\mathrm{good}}_s(\theta,\ell)\cup\mathcal{I}^{\mathrm{bad}}_s(\theta,\ell)$ of $\partial co(D_t)$ for $s=2\ell,3\ell,100t^{-1}\ell$ into small straight segments.

Definition 3.6.1. *Let $\mathcal{J}(\theta,\ell)$ be a partition of $\partial\,\mathrm{co}(D_t)$ formed as a common refinement to all of the sets of arcs from the partitions*

$$\mathcal{I}^{good}_{2\ell}(\theta,\ell)\cup\mathcal{I}^{bad}_{2\ell}(\theta,\ell),\qquad \mathcal{I}^{good}_{3\ell}(\theta,\ell)\cup\mathcal{I}^{bad}_{3\ell}(\theta,\ell),\qquad \mathcal{I}^{good}_{100t^{-1}\ell}(\theta,\ell)\cup\mathcal{I}^{bad}_{100t^{-1}\ell}(\theta,\ell)$$

of $\partial\,\mathrm{co}(D_t)$, into straight line segments of length at most $\xi\sqrt{\gamma}$. For $s\in\{2\ell,3\ell,100t^{-1}\ell\}$, define the partition $\mathcal{J}^{good}_s(\theta,\ell)\cup\mathcal{J}^{bad}_s(\theta,\ell)$ of $\mathcal{J}(\theta,\ell)$ by setting $\mathfrak{q}\in\mathcal{J}^{good}_s(\theta,\ell)$ if and only if $\mathfrak{q}\subset\mathfrak{q}'\in\mathcal{I}^{good}_s(\theta,\ell)$.

We will now in Definition 3.6.2 choose the vectors $v_{\mathfrak{q}}$ for $\mathfrak{q} \in \mathcal{J}(\theta,\ell)$ with direction vectors $\widehat{v_{\mathfrak{q}}}$ determined by the partition $\mathcal{I}^{\text{bad}}_{100t^{-1}\ell}(\theta,\ell) \cup \mathcal{I}^{\text{good}}_{100t^{-1}\ell}(\theta,\ell)$, and with lengths determined by $\mathcal{I}^{\text{bad}}_{3\ell}(\theta,\ell) \cup \mathcal{I}^{\text{good}}_{3\ell}(\theta,\ell)$. We then in Definition 3.6.3 form parallelograms $R_{\mathfrak{q}}$ with sides $\mathfrak{q}$ and $v_{\mathfrak{q}}$.

Definition 3.6.2. *For an arc $\mathfrak{q} \in \mathcal{J}(\theta,\ell)$, we define a vector $v_{\mathfrak{q}}$ as follows.*

- *We choose the direction vector $\widehat{v_{\mathfrak{q}}}$ of $v_{\mathfrak{q}}$ as follows. Let $\mathfrak{q} \subset \mathfrak{q}' \in \mathcal{I}^{bad}_{100t^{-1}\ell}(\theta,\ell) \cup \mathcal{I}^{good}_{100t^{-1}\ell}(\theta,\ell)$. If $\mathfrak{q}'$ is contained inside an angular interval $[m\alpha,(m+1)\alpha]$, we take the direction vector $\widehat{v_{\mathfrak{q}}}$ to be the inward pointing direction at angle $(m+\frac{1}{2})\alpha$. Otherwise (recalling that $\mathfrak{q}' \in \mathcal{I}^{bad}_{100t^{-1}\ell}(\theta,\ell) \cup \mathcal{I}^{good}_{100t^{-1}\ell}(\theta,\ell)$ has angular length at most $\frac{1}{3}\alpha$) $\mathfrak{q}'$ overlaps a unique angle $m\alpha$, and we take $\widehat{v_{\mathfrak{q}}}$ to be the inward pointing vector at angle $m\alpha$.*

- *We choose the length of $v_{\mathfrak{q}}$ to be*

$$||v_{\mathfrak{q}}|| = \begin{cases} 15\sqrt{\gamma} & \mathfrak{q} \in \mathcal{J}^{bad}_{3\ell}(\theta,\ell), \text{ and} \\ 3\xi\sqrt{\gamma} & \mathfrak{q} \in \mathcal{J}^{good}_{3\ell}(\theta,\ell). \end{cases}$$

For $p \in \partial\,\mathrm{co}(D_t)$, we denote $v_p = v_{\mathfrak{q}}$, where $p \in \mathfrak{q} \in \mathcal{J}(\theta,\ell)$.

Definition 3.6.3. *For $\mathfrak{q} \in \mathcal{J}(\theta,\ell)$, let $R_{\mathfrak{q}}$ be a parallelogram with one side $\mathfrak{q}$ and one side $v_{\mathfrak{q}}$.*

By construction, the directions of each of the v_p for $p \in \partial\,\mathrm{co}(D_t)$ are one of $M = \frac{4\pi}{\alpha}$ directions, and the directions of the vectors are constant on arcs of $\partial\,\mathrm{co}(D_t)$ from $\mathcal{I}^{\text{bad}}_{100t^{-1}\ell}(\theta,\ell) \cup \mathcal{I}^{\text{good}}_{100t^{-1}\ell}(\theta,\ell)$.

Observation 3.6.4. *For every point $p \in \partial\,\mathrm{co}(D_t)$ we have $\angle po, v_p < \frac{1}{2}\alpha$.*

3.6.2 Covering $\partial\,\mathrm{co}(D_t)$ with parallelograms

Now we are able to state the main result of this section.

Proposition 3.6.5. *For d_τ sufficiently small, we have*

$$\mathrm{co}(D_t) \setminus D_t \subset \bigcup_{\mathfrak{q} \in \mathcal{J}(\theta,\ell)} R_{\mathfrak{q}}.$$

We need the following observation about the unit direction vectors $\widehat{v_{\mathfrak{q}}}$ of $v_{\mathfrak{q}}$.

Lemma 3.6.6. *Let $p \in \partial\,\mathrm{co}(D_t)$, and $p' \in op$. Then there exists $r \in \partial\,\mathrm{co}(D_t)$, with $\widehat{v_p} = \widehat{v_r}$ and this is parallel to rp'.*

Proof. Let z be the unique point on $\partial\,\mathrm{co}(D_t)$ with zo in the direction of $\widehat{v_p}$. By Observation 3.6.4, the angle between $\widehat{v_z}$ and zo (which is in the direction $\widehat{v_p}$) is strictly less than $\frac{1}{2}\alpha$. As the $\widehat{v}$ angles occur in multiples of $\frac{1}{2}\alpha$, this implies $\widehat{v_z} = \widehat{v_p}$.

Let r be the unique point on $\partial co(D_t)$ with rp' in the direction of v_p. Then r lies on the arc pz, so $\widehat{v_p} = \widehat{v_r}$ is parallel to rp'. □

Proof of Proposition 3.6.5. Assume that d_τ is sufficiently small so that Proposition 3.3.3 and Proposition 3.3.5 are true. Given a point $p \in \partial\,\mathrm{co}(D_t)$, define the interval

$$S_p(\theta,\ell;\xi) = pp'$$

where $p' \in op$ is such that

$$|pp'| = \begin{cases} 5\sqrt{\gamma} & p \in \mathfrak{q} \subset \mathcal{I}_{2\ell}^{\mathrm{bad}}(\theta,\ell), \text{ and} \\ \xi\sqrt{\gamma} & p \in \mathfrak{q} \subset \mathcal{I}_{2\ell}^{\mathrm{good}}(\theta,\ell). \end{cases}$$

By Proposition 3.3.5 and Proposition 3.5.1 we have $(\mathrm{co}(D_t) \setminus D_t) \cap op \subset S_p(\theta,\ell,\xi)$ for all $p \in \partial co(D_t)$. Thus denoting by

$$\Lambda(\theta,\ell;\xi) := \bigcup_{p \in \partial\,\mathrm{co}(D_t)} S_p(\theta,\ell;\xi),$$

we have

$$\mathrm{co}(D_t) \setminus D_t \subset \Lambda(\theta,\ell;\xi).$$

Fix a point $p \in \partial\,\mathrm{co}(D_t)$, and let $p' \in S_p(\theta,\ell;\xi) = op \cap \Lambda(\theta,\ell;\xi)$. It suffices to show that

$$p' \in \bigcup_{\mathfrak{q} \in \mathcal{J}(\theta,\ell)} R_{\mathfrak{q}}.$$

Note that by Lemma 3.6.6 there exists a point $r' \in \partial co(D_t)$ such that $r'p'$ is parallel to $\widehat{v_{r'}} = \widehat{v_p}$. Let r be the intersection of the line extended from the segment $\mathfrak{q}$ and the ray $p'r'$. Note that $\angle rpp' \in (29^\circ, 180^\circ - 29^\circ)$ by Corollary 3.3.4, and $\angle pp'r < \frac{1}{2}\alpha$ by Observation 3.6.4, so $\angle prp' \in (29^\circ - \frac{1}{2}\alpha, 180^\circ - 29^\circ)$. Thus by the law of sines $|r'p'| \le |rp'| = \frac{\sin(\angle rpp')}{\sin(\angle prp')}|pp'| \le 3|pp'|$.

If $\mathfrak{q} \in \mathcal{J}_{2\ell}^{\mathrm{good}}(\theta,\ell)$, then $|pp'| \le \xi\sqrt{\gamma}$, so $|r'p'| \le 3\xi\sqrt{\gamma}$, and letting $r' \in \mathfrak{r} \in \mathcal{J}(\theta,\ell)$ we have $p' \in R_{\mathfrak{r}} \subset \bigcup_{\mathfrak{q} \in \mathcal{J}(\theta,\ell)} R_{\mathfrak{q}}$.

Alternatively if $\mathfrak{q} \in \mathcal{J}_{2\ell}^{\mathrm{bad}}(\theta,\ell)$ then $|pp'| \le 5\sqrt{\gamma}$. Note that $|pr'| \le |pp'| + |rp'| \le 4|pp'| \le \ell$, so r' is in an arc $\mathfrak{r} \in \mathcal{J}_{3\ell}^{\mathrm{bad}}(\theta,\ell)$. Hence, $|r'p'| \le 15\sqrt{\gamma}$, implying $p' \in R_{\mathfrak{r}} \subset \bigcup_{\mathfrak{q} \in \mathcal{J}(\theta,\ell)} R_{\mathfrak{q}}$.

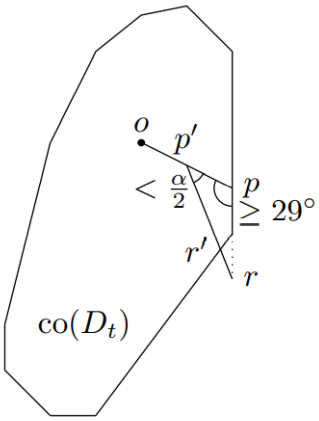

Fig. 3.15 The points r and r' for a given pair p, p'.

□

3.7 Preimages of the $R_{\mathfrak{q}}$ associated to A and B.

By Proposition 3.6.5, for d_τ sufficiently small we have

$$\mathrm{co}(D_t) \setminus D_t \subset \bigcup_{\mathfrak{q} \in \mathcal{J}(\theta,\ell)} (R_{\mathfrak{q}} \setminus D_t).$$

The boundary of $\mathrm{co}(D_t)$ is composed of translates of edges from $\partial\,\mathrm{co}(A)$ scaled by a factor of t and of edges from $\partial\,\mathrm{co}(B)$ scaled by a factor of $(1-t)$. If an edge of $\mathrm{co}(A)$ is parallel to an edge of $\mathrm{co}(B)$ then there is an ambiguity in how we do this; we fix one such decomposition from now on.

Definition 3.7.1. *Let $\mathcal{J}(\theta,\ell) = \mathcal{A} \sqcup \mathcal{B}$ be the partition defined as follows. For every arc $\mathfrak{q} \in \mathcal{J}(\theta,\ell)$ (which is straight by construction), we let $\mathfrak{q} \in \mathcal{A}$ if $\mathfrak{q}$ is on a translated t-scaled edge from $\partial\,\mathrm{co}(A)$, and we let $\mathfrak{q} \in \mathcal{B}$ if $\mathfrak{q}$ is on a translated $(1-t)$-scaled edge from $\partial\,\mathrm{co}(B)$.*

Definition 3.7.2. *For $\mathfrak{q} \in \mathcal{A}$, let $p_{\mathfrak{q},B} \in \partial\,\mathrm{co}(B)$ and $R_{\mathfrak{q},A} \subset \mathbb{R}^2$ be the parallelogram with edge $\mathfrak{q}_A \subset \partial\,\mathrm{co}(A)$ such that*

$$R_{\mathfrak{q}} = tR_{\mathfrak{q},A} + (1-t)p_{\mathfrak{q},B}.$$

Similarly, for $\mathfrak{q} \in \mathcal{B}$, let $p_{\mathfrak{q},A} \in \partial\,\mathrm{co}(A)$ and $R_{\mathfrak{q},B} \subset \mathbb{R}^2$ be the parallelogram with edge $\mathfrak{q}_B \subset \partial\,\mathrm{co}(B)$ such that

$$R_{\mathfrak{q}} = tp_{\mathfrak{q},A} + (1-t)R_{\mathfrak{q},B}.$$

Remark 3.7.3. *The parallelogram $R_{\mathfrak{q},A}$ (resp. $R_{\mathfrak{q},B}$) may not be entirely contained inside* $\mathrm{co}(A)$ *(resp.* $\mathrm{co}(B)$*), and the various $R_{\mathfrak{q},A}$ with $\mathfrak{q}\in\mathcal{A}$ (respectively $R_{\mathfrak{q},B}$ with $\mathfrak{q}\in\mathcal{B}$) may not be disjoint.*

Proposition 3.7.4. *For d_τ sufficiently small, we have*

$$|\mathrm{co}(D_t)\setminus D_t| \le t^2\sum_{\mathfrak{q}\in\mathcal{A}}|R_{\mathfrak{q},A}\setminus A| + (1-t)^2\sum_{\mathfrak{q}\in\mathcal{B}}|R_{\mathfrak{q},B}\setminus B|$$

Proof. Assume d_τ is sufficiently small that Proposition 3.6.5 holds. Then we have

$$\mathrm{co}(D_t)\setminus D_t \subset \bigcup_{\mathfrak{q}\in\mathcal{J}(\theta,\ell)}(R_{\mathfrak{q}}\setminus D_t).$$

The result then follows from the fact that if

- $\mathfrak{q}\in\mathcal{A}$ then $|R_{\mathfrak{q}}\setminus D_t| \le |R_{\mathfrak{q}}\setminus(tA+(1-t)p_{\mathfrak{q},B})| = t^2|R_{\mathfrak{q},A}\setminus A|$, and if
- $\mathfrak{q}\in\mathcal{B}$ then $|R_{\mathfrak{q}}\setminus D_t| \le |R_{\mathfrak{q}}\setminus(tp_{\mathfrak{q},A}+(1-t)B)| = (1-t)^2|R_{\mathfrak{q},B}\setminus B|$.

□

From Proposition 3.7.4, we see that if the preimages in A,B of these regions were disjoint and contained in $\mathrm{co}(A)$ and $\mathrm{co}(B)$, then we'd immediately obtain $|\mathrm{co}(D_t)\setminus D_t| \le t^2|\mathrm{co}(A)\setminus A| + (1-t)^2|\mathrm{co}(B)\setminus B|$.

Our goal will be to remove certain $R_{\mathfrak{q},A}$ and $R_{\mathfrak{q},B}$ to ensure that all remaining parallelograms are disjoint and are entirely contained in $\mathrm{co}(A)$ and $\mathrm{co}(B)$, such that the total area of the $R_{\mathfrak{q},A}$ with $\mathfrak{q}\in\mathcal{A}$ that were removed is at most $\varepsilon|\mathrm{co}(A)\setminus A|$, and the total area of the $R_{\mathfrak{q},B}$ with $\mathfrak{q}\in\mathcal{B}$ that were removed is at most $\varepsilon|\mathrm{co}(B)\setminus B|$. This will imply Theorem 3.2.2.

3.8 Far away weighted averages in $\partial\,\mathrm{co}(D_t)$ lie in $\mathcal{J}_{3\ell}^{\mathrm{good}}(\theta,\ell)$

We now show that points on the $\partial\,\mathrm{co}(D_t)$ which are the t-weighted average of points from $\partial\,\mathrm{co}(A)$, $\partial\,\mathrm{co}(B)$ that are at distance at least $20t^{-1}\ell$ lie in arcs from $\mathcal{J}_{3\ell}^{\mathrm{good}}(\theta,\ell)$.

The main application will be to show that for parallelograms $R_{\mathfrak{q}}$ with $\mathfrak{q}\in\mathcal{J}_{3\ell}^{\mathrm{bad}}(\theta,\ell)$, we know that the point and parallelogram or parallelogram and point in $co(A)$ and $co(B)$ whose t-weighted average gives $R_{\mathfrak{q}}$ are close to each other.

Proposition 3.8.1. *For d_τ sufficiently small, if $p\in\partial\,\mathrm{co}(D_t)$ with $p = tx_A+(1-t)y_B$, where $x_A\in\partial\,\mathrm{co}(A)$, $y_B\in\partial\,\mathrm{co}(B)$ and $|x_Ay_B|\ge 20t^{-1}\ell$, then $p\in\mathfrak{q}\in\mathcal{J}_{3\ell}^{good}(\theta,\ell)$.*

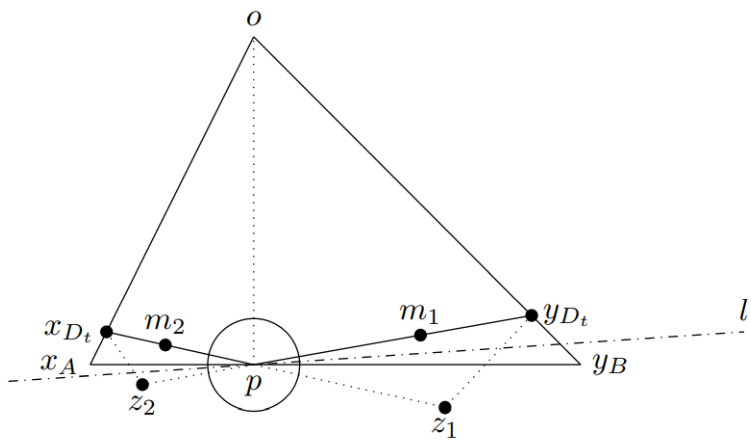

Fig. 3.16 The configuration in the proof of Proposition 3.8.1.

Proof. Let $\eta = \min(10\sqrt{3}\sin(\frac{\theta}{4}), \frac{\sqrt{3}}{2}\ell)$. Assume d_τ is sufficiently small so that Corollary 3.3.4 holds, and $(1-\eta)K \subset \mathrm{co}(A), \mathrm{co}(B), \mathrm{co}(D_t) \subset K$ by Proposition 3.3.1. We will first show that x_{D_t} and y_{D_t} realize p as a $(\frac{1}{2}\theta, 19\ell)$-good point. For the angle, note that by Observation 3.2.3 we have

$$\angle x_{D_t} p x_A \leq \sin^{-1}\left(\frac{|x_A x_{D_t}|}{|x_A p|}\right) \leq \sin^{-1}\left(\frac{\eta|ox_A|}{20t^{-1}\ell}\right) \leq \sin^{-1}\left(\frac{\eta}{10\sqrt{3}t^{-1}\ell}\right) \leq \frac{\theta}{4},$$

and similarly

$$\angle y_{D_t} p y_B \leq \sin^{-1}\left(\frac{|y_B y_{D_t}|}{|y_B p|}\right) \leq \frac{\theta}{4}.$$

For the lengths, notice that $|x_{D_t}x_A| \leq \eta|ox_A| \leq \frac{\sqrt{3}}{2}\ell|ox_A| \leq \ell$ and similarly $|y_{D_t}y_A| \leq \ell$, so by triangle inequality we have

$$|px_{D_t}| \geq |px_A| - |x_{D_t}x_A| = (1-t)|x_A y_B| - |x_{D_t}x_A| \geq 20\ell - \ell = 19\ell, \text{ and}$$
$$|py_{D_t}| \geq |py_B| - |y_{D_t}y_B| = t|x_A y_B| - |y_{D_t}y_A| \geq 20\ell - \ell = 19\ell.$$

Now, we show that $p \in \mathfrak{q} \in \mathcal{J}_{3\ell}^{\mathrm{good}}(\theta, \ell)$ by showing that if $p' \in \partial co(D_t)$ and $|pp'| \leq 3\ell$, then we have p' is (θ, ℓ)-good. Denote by l the supporting line to $\mathrm{co}(D_t)$ at p, and note by Corollary 3.3.4 that $\angle l, op \in (29^\circ, 180^\circ - 29^\circ)$. The line l intersects either the interior of the angle $\angle x_{D_t} p x_A$ or $\angle y_{D_t} p y_B$, so as we have already shown that $\angle x_{D_t} p x_A, \angle y_{D_t} p y_B \leq \frac{\theta}{4}$, we have that $x_A y_B$ makes an angle of at most $\frac{\theta}{4}$ with l. In particular, $\angle opx_A, \angle opy_B \in \left(29^\circ - \frac{\theta}{4}, 180^\circ - 29^\circ + \frac{\theta}{4}\right) \subset (28^\circ, 180^\circ - 28^\circ)$. Thus we may apply Lemma 3.5.3 to triangles $x_A po$ and $y_B po$ to conclude that the distance from p to the lines ox_A and oy_B is at

least $\sin(14^\circ)\min(|px_A|, |po|, |py_B|) \geq \sin(14^\circ)20\ell > 3\ell$. Because $ox_{D_t}py_{D_t} \subset \mathrm{co}(D_t)$, we conclude that p' lies outside of the angle $x_{D_t}py_{D_t}$ (and because $p' \in \mathrm{co}(D_t)$, it lies on the same side of l as x_{D_t}, y_{D_t}).

Let z_1 be in the ray $x_{D_t}p$ extended past p such that $|z_1p| = |z_1y_{D_t}|$. Note that as $pz_1y_{D_t}$ is isosceles, $\angle pz_1y_{D_t} \geq \pi - \theta$, and note that $\angle y_{D_t}pz_1 \leq \frac{\theta}{2}$. Analogously let z_2 be the point at py_{D_t} which has $|z_2x_{D_t}| = |z_2p|$, so that $\angle pz_2x_{D_t} \geq \pi - \theta$ and $\angle x_{D_t}pz_2 \leq \frac{\theta}{2}$. Finally, let m_1 be the midpoint of py_{D_t}, and let m_2 be the midpoint of px_{D_t}, so that $\angle pm_1z_1 = \angle pm_2z_2 = 90^\circ$.

We claim that $p' \in pm_1z_1 \cup pm_2z_2$. First, note that by the above we have shown that p' lies in either the angular region $\angle m_1pz_1$ or $\angle m_2pz_2$. Thus as pm_1z_1, pm_2z_2 are right triangles, it suffices to note that $|pm_1|, |pm_2| \geq \frac{19}{2}\ell > 3\ell$. Therefore, $p' \in pm_1z_1 \cup pm_2z_2 \subset py_{D_t}z_1 \cup px_{D_t}z_2$. Hence, $\angle y_{D_t}p'x_{D_t} \geq \pi - \theta$ and p' is (θ, ℓ)-good since $|p'x_{D_t}|, |p'y_{D_t}| \geq 19\ell - 3\ell > \ell$ by the triangle inequality. □

3.9 Bound on parallelograms jutting out of $\mathrm{co}(A), \mathrm{co}(B)$

We will now show that the $R_{\mathfrak{q},A}$ and $R_{\mathfrak{q},B}$ which are not entirely contained in $\mathrm{co}(A)$ and $\mathrm{co}(B)$ have negligible total area.

Proposition 3.9.1. *For d_τ sufficiently small, we have*

$$\sum_{\mathfrak{q}\in\mathcal{A} \text{ and } R_{\mathfrak{q},A}\not\subset \mathrm{co}(A)} \left|R_{\mathfrak{q},A}\right| \leq 25t^{-1}M\xi^2\gamma, \text{ and } \sum_{\mathfrak{q}\in\mathcal{B} \text{ and } R_{\mathfrak{q},B}\not\subset \mathrm{co}(B)} \left|R_{\mathfrak{q},B}\right| \leq 25t^{-1}M\xi^2\gamma.$$

To prove this proposition, we first use Proposition 3.8.1 to show that for such parallelograms we have $\mathfrak{q} \in \mathcal{J}_{3\ell}^{\mathrm{good}}(\theta, \ell)$.

Lemma 3.9.2. *For d_τ sufficiently small, if $\mathfrak{q} \in \mathcal{A}$ and $R_{\mathfrak{q},A} \not\subset \mathrm{co}(A)$ or $\mathfrak{q} \in \mathcal{B}$ and $R_{\mathfrak{q},B} \not\subset \mathrm{co}(B)$, then $\mathfrak{q} \in \mathcal{J}_{3\ell}^{good}(\theta, \ell)$.*

Proof. The cases $\mathfrak{q} \in \mathcal{A}$ and $\mathfrak{q} \in \mathcal{B}$ are proved identically, so we will now suppose that $\mathfrak{q} \in \mathcal{A}$. Assume d_τ is sufficiently small so that Proposition 3.3.3 and Proposition 3.8.1 are true. Recall that we defined $p_{\mathfrak{q},B} \in \partial\,\mathrm{co}(B)$ and $\mathfrak{q}_A \subset \mathrm{co}(A)$ such that $\mathfrak{q} = t\mathfrak{q}_A + (1-t)p_{\mathfrak{q},B}$.

We first show that there exists a point $p_A \in \mathfrak{q}_A$ such that $\angle p_Ao, v_\mathfrak{q} \geq 29^\circ$. Indeed, by Proposition 3.3.3 we know that every point in $x \in \mathfrak{q}_A$ is $(59^\circ, \frac{1}{3})$-bisecting in $\mathrm{co}(A)$. For $x \in \mathfrak{q}_A$, let $x' = x + t^{-1}v_\mathfrak{q}$, which lies on the opposite side of $\partial R_{\mathfrak{q},A}$ to x. Note that $|xx'| \leq \frac{1}{10}$, so if $\angle ox, v_q \leq 29^\circ$, then $xx' \subset T_x(58^\circ, \frac{1}{3})$. Hence, as $R_{\mathfrak{q},A} = \bigcup_{x\in\mathfrak{q}_A} xx' \not\subset co(A)$ but $\bigcup_{x\in\mathfrak{q}_A} T_x(58^\circ, \frac{1}{3}) \subset \mathrm{co}(A)$, we find a point $p_A \in \mathfrak{q}_A$ with $\angle p_Ao, v_\mathfrak{q} \geq 29^\circ$.

Let $z = tp_A + (1-t)p_{\mathfrak{q},B} \in \mathfrak{q}$. By Observation 3.6.4, $\angle zo, v_\mathfrak{q} \leq \frac{1}{2}\alpha$. Hence $\angle p_Aoz \geq 29^\circ - \frac{1}{2}\alpha \geq 28^\circ$, so $|p_Az| \geq \sin(28^\circ)|oz| > \frac{1}{100}$, so as z lies on the segment $p_Ap_{\mathfrak{q},B}$, we

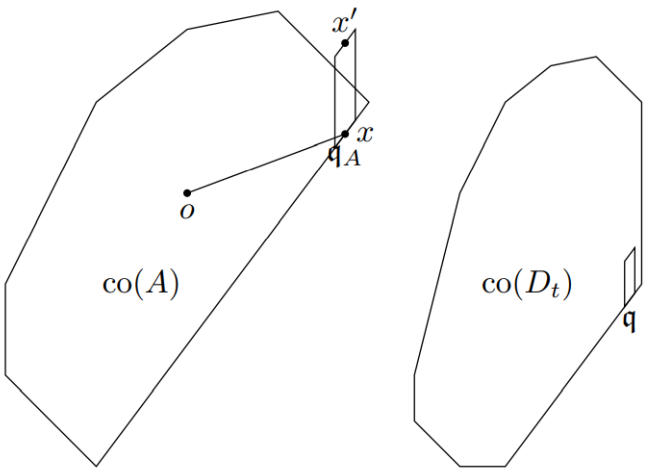

Fig. 3.17 An example where $\mathfrak{q} \in \mathcal{A}$ and $R_{\mathfrak{q},A} \not\subset \mathrm{co}(A)$.

have $|p_A p_{\mathfrak{q},B}| > \frac{1}{100}$. Note that by definition of $\ell = \ell(\theta)$ in Definition 3.4.3, we have $20t^{-1}\ell \leq \frac{1}{100}$. Therefore, by Proposition 3.8.1 applied with $x_A = p_A$ and $y_B = p_{\mathfrak{q},B}$, we have $z \in \mathfrak{q} \in \mathcal{J}_{3\ell}^{\mathrm{good}}(\theta,\ell)$.

□

We now know that parallelograms $R_{\mathfrak{q},A}$ and $R_{\mathfrak{q},B}$ which escape $\mathrm{co}(A)$ and $\mathrm{co}(B)$ have small height, since they are supported on arcs from $\mathcal{J}_{3\ell}^{\mathrm{good}}(\theta,\ell)$. By showing that such arcs with a constant direction v_p have small total length, we will obtain Proposition 3.9.1 (recalling M is the number of distinct v_p).

Proof of Proposition 3.9.1. The proof below works for the $\mathrm{co}(B)$ inequality verbatim, so we focus on proving the $\mathrm{co}(A)$ inequality. Take d_τ sufficiently small so that Proposition 3.3.3 holds, and so that $t^{-1}3\xi\sqrt{\gamma} \leq \frac{1}{4}\sin(1^\circ)$ by Observation 3.2.9.

By Lemma 3.9.2, all $\mathfrak{q} \in \mathcal{A}$ with $R_{\mathfrak{q},A} \not\subset co(A)$ are in $\mathcal{J}_{3\ell}^{\mathrm{good}}(\theta,\ell)$. Fix one of the $\leq M$ vectors v with $|v| = 3\xi\sqrt{\gamma}$. It suffices to show

$$\sum_{\mathfrak{q}\in\mathcal{A},\ v_{\mathfrak{q}}=v,\ \text{and } R_{\mathfrak{q},A}\not\subset \mathrm{co}(A)} |R_{\mathfrak{q},A}| \leq 25t^{-1}\xi^2\gamma.$$

Recall that by construction v was chosen so that it was not parallel to any edge of $\mathrm{co}(A)$. Let l, l' be the two lines in the direction v which are tangent to $\mathrm{co}(A)$, and let y and y' be the points of contact with $\mathrm{co}(A)$. Note that every line in the direction v between y and y' intersects each of the arcs $\partial\,\mathrm{co}(A) \setminus \{y, y'\}$ exactly once. As $\mathrm{co}(A)$ is convex, the cross-sectional slices in the v-direction satisfy unimodality. Hence there are exactly two pairs (x_1, x_2) and (x_1', x_2')

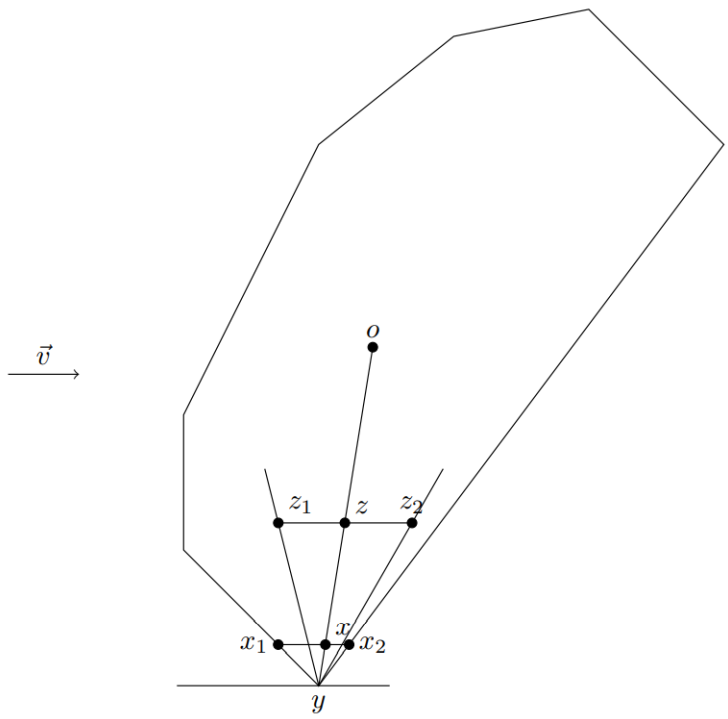

Fig. 3.18 The configuration in the proof of Proposition 3.9.1.

of points in the two different arcs of $\partial\,\mathrm{co}(A) \setminus \{y, y'\}$ such that we have the equality of vectors $x_1x_2 = x_1'x_2' = t^{-1}v$ — we let (x_1, x_2) be the pair closer to y.

We will show that the lengths of the two minor arcs in $\mathrm{co}(A)$ between x_1x_2 and between $x_1'x_2'$ are both of length at most $24t^{-1}\sqrt{\gamma}$. We show this for x_1x_2 as the other case will be identical.

Note that $T_y(56^\circ, \frac{1}{4}) \subset T_y(59^\circ, \frac{1}{3}) \subset \mathrm{co}(A)$. Let $z \in oy$ such that $|yz| = t^{-1}3\xi\sqrt{\gamma} \leq \frac{1}{4}\sin(1^\circ)$ and denote by z_1, z_2 the intersections of the extensions of the arms of $T_y(56^\circ, \frac{1}{4})$ with the line through z with direction vector v. We will show that the line x_1x_2 is closer to y than the line z_1z_2 by showing that $|z_1z_2| \geq |x_1x_2|$ and applying unimodality.

Note that $\angle z_1yz = 28^\circ$ and $\angle z_1zy \in (29^\circ, 180^\circ - 29^\circ)$. Hence $\angle yz_1z \in (1^\circ, 180^\circ - 57^\circ)$ so $\sin\angle yz_1z \geq \sin(1^\circ)$. Thus by the law of sines,

$$|yz_1| = \frac{\sin\angle z_1zy}{\sin\angle yz_1z}|yz| \leq \frac{|yz|}{\sin 1^\circ} \leq \frac{1}{4}.$$

Hence $z_1 \in T_y(56^\circ, \frac{1}{4})$ and by a similar argument we obtain $z_2 \in T_y(56^\circ, \frac{1}{4})$.

Now,

$$|z_1z_2| \geq |z_1z| = \frac{\sin 28^\circ}{\sin \angle yz_1z}|yz| \geq \sin(28^\circ)|yz| = t^{-1}3\xi\sqrt{\gamma} = |x_1x_2|.$$

Thus by the unimodality, the line x_1x_2 is closer than the line z_1z_2 to y, so denoting by $x = oy \cap x_1x_2$ we have x lies in the segment yz. Hence

$$|yx| \leq |yz| = t^{-1}3\xi\sqrt{\gamma}.$$

Note that there are up to 2 arcs $\mathfrak{q}_A$ which contain one of the points x_1, x_1', and as each arc in $\mathcal{J}(\theta,\ell)$ has length at most $\xi\sqrt{\gamma}$ by construction, the total length of these arcs is at most $2t^{-1}\xi\sqrt{\gamma}$.

If $v_{\mathfrak{q}} = v$ and $R_{\mathfrak{q},A} \not\subset \mathrm{co}(A)$, then $\mathfrak{q}_A$ is contained in the arc of $\partial\,\mathrm{co}(A) \setminus \{y,y'\}$ containing x_1, x_1', and $\mathfrak{q}_A$ intersects either the minor arc subtended by x_1y or by $x_1'y'$. Indeed, let $\widetilde{l}$ be the supporting line of $\mathfrak{q}$. Then for any point $p \in \mathfrak{q}$, by Proposition 3.3.3 the angle $\angle po, \widetilde{l} \in (29^\circ, 180^\circ - 29^\circ)$, and by Observation 3.6.4 $\angle po, v_{\mathfrak{q}} \leq \frac{\alpha}{2}$. Hence $v_{\mathfrak{q}}$ lies on the same side of $\widetilde{l}$ as $\mathrm{co}(D_t)$. Therefore $v_{\mathfrak{q}}$ lies on the same side of the supporting line $\widetilde{l_A}$ to $\mathfrak{q}_A$ as $\mathrm{co}(A)$, so $\mathfrak{q}_A$ lies in the arc of $\mathrm{co}(A) \setminus \{y,y'\}$ that contains x_1, x_1'. Now, if $\mathfrak{q}_A$ does not intersect the minor arcs x_1y or $x_1'y'$, then by unimodality, the v cross-sectional lengths of $\mathrm{co}(A)$ on the arc $\mathfrak{q}_A$ exceed $3\xi t^{-1}\sqrt{\gamma} = ||t^{-1}v||$, which implies $R_{\mathfrak{q}_A}$ is contained inside $\mathrm{co}(A)$.

Hence, the total width (measured in the direction $v^\perp$) of such parallelograms $R_{\mathfrak{q},A}$ in direction v which are not contained in $co(A)$ is at most $2 \cdot t^{-1}3\xi\sqrt{\gamma} + 2t^{-1}\xi\sqrt{\gamma} = 8t^{-1}\xi\sqrt{\gamma}$.

Because all of the arcs $\mathfrak{q}$ we are considering lie in $\mathcal{J}_{3\ell}^{\mathrm{good}}(\theta,\ell)$, the total area of such parallelograms is then at most

$$\left(8t^{-1}\xi\sqrt{\gamma}\right)\left(3\xi\sqrt{\gamma}\right) = 24t^{-1}\xi^2\gamma.$$

□

3.10 Bounding overlapping parallelograms

We will now show that the $R_{\mathfrak{q},A}$ and $R_{\mathfrak{q},B}$ which we remove to guarantee non-overlapping have negligible area.

Proposition 3.10.1. *For d_τ sufficiently small, if $\mathfrak{q}, \mathfrak{q}' \in \mathcal{J}_{3\ell}^{bad}(\theta,\ell) \cap \mathcal{A}$, then $|R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| = 0$, and if $\mathfrak{q}, \mathfrak{q}' \in \mathcal{J}_{3\ell}^{bad}(\theta,\ell) \cap \mathcal{B}$, then $|R_{\mathfrak{q},B} \cap R_{\mathfrak{q}',B}| = 0$.*

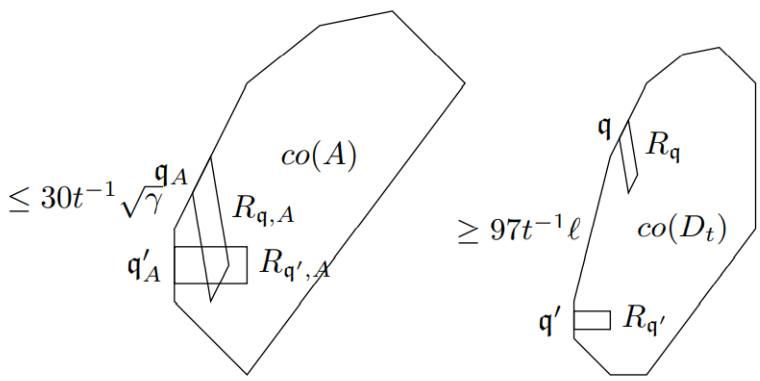

Fig. 3.19 If $|R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| > 0$, then not both $\mathfrak{q}$ and $\mathfrak{q}'$ are bad.

Because of Proposition 3.10.1, it will suffice to bound overlaps between parallelograms supported on arcs in $\mathcal{J}^{\text{good}}_{3\ell}(\theta,\ell)$ with all other parallelograms.

Proposition 3.10.2. *For d_τ sufficiently small, we have*

$$\sum_{\mathfrak{q}\in\mathcal{J}^{good}_{3\ell}(\theta,\ell)\cap\mathcal{A} \text{ and } \exists \mathfrak{q}'\in\mathcal{A}\setminus\{\mathfrak{q}\} \text{ with } |R_{\mathfrak{q},A}\cap R_{\mathfrak{q}',A}|>0} |R_{\mathfrak{q},A}| \leq 16000t^{-1}M\xi\gamma$$

and similarly with B and $\mathcal{B}$.

Proof of Proposition 3.10.1. The proof we give works verbatim for B and $\mathcal{B}$, so we focus on the case with A and $\mathcal{A}$. We take d_τ sufficiently small such that Proposition 3.8.1 holds, and such that $\sqrt{\gamma} \leq \ell$ by Observation 3.2.9. Because $\mathfrak{q},\mathfrak{q}' \in \mathcal{J}^{\text{bad}}_{3\ell}(\theta,\ell)$, we have $||v_{\mathfrak{q}}|| = ||v_{\mathfrak{q}'}|| = 15\sqrt{\gamma}$. Consider the arcs $\mathfrak{r},\mathfrak{r}' \in \mathcal{I}^{\text{bad}}_{100t^{-1}\ell}(\theta,\ell)$ such that $\mathfrak{q} \subset \mathfrak{r}$ and $\mathfrak{q}' \subset \mathfrak{r}'$. If $\mathfrak{r} = \mathfrak{r}'$ then $v_{\mathfrak{q}} = v_{\mathfrak{q}'}$ so $|R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| = 0$.

Assume now that $\mathfrak{r} \neq \mathfrak{r}'$. In this case, the distance between $\mathfrak{q}$ and $\mathfrak{q}'$ is at least $97t^{-1}\ell$. Indeed, otherwise there exists a point $p \in \mathfrak{q}$ and $p' \in \mathfrak{q}'$ such that $|pp'| \leq 97t^{-1}\ell$. Let x be a (θ,ℓ)-bad point such that $|xp| \leq 3\ell$. Then $B(x, 100t^{-1}\ell)$ contains p, and by the triangle inequality it also contains p'. This implies p, p' are contained in the same arc of $\mathcal{I}^{\text{bad}}_{100t^{-1}\ell}(\theta,\ell)$, so $\mathfrak{r} = \mathfrak{r}'$, a contradiction.

Assuming for the sake of contradiction that $|R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| > 0$, then there exists a point $z \in R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}$. Then because z is within distance $t^{-1}||v_{\mathfrak{q}}|| = 15t^{-1}\sqrt{\gamma}$ of $\mathfrak{q}_A$ and within distance $t^{-1}||v_{\mathfrak{q}'}|| = 15t^{-1}\sqrt{\gamma}$ of $\mathfrak{q}'_A$, we have by the triangle inequality that the distance between $\mathfrak{q}_A$ and $\mathfrak{q}'_A$ is at most $30t^{-1}\sqrt{\gamma} \leq 30t^{-1}\ell$.

By the above, there either exists $p \in \mathfrak{q}$ and $z_A \in \mathfrak{q}_A$ such that $|pz_A| \geq 33t^{-1}\ell$, or there exists $p' \in \mathfrak{q}'$ and $z'_A \in \mathfrak{q}'_A$ such that $|p'z'_A| \geq 33t^{-1}\ell$. Suppose without loss of generality the first case holds. Then $p = tx_A + (1-t)y_B$ for some point $x_A \in \mathfrak{q}$ and $y_B = p_{\mathfrak{q},B}$, and $|x_A z_A| \leq \xi t^{-1}\sqrt{\gamma}$ since this is an upper bound for the length of $\mathfrak{q}_A$. Therefore,

$$|x_A y_B| \geq |x_A p| \geq |pz| - |x_A z| \geq 20t^{-1}\ell,$$

so by Proposition 3.8.1, $p \in \mathfrak{q} \in \mathcal{J}_{3\ell}^{\text{good}}(\theta,\ell)$, a contradiction.

□

Proof of Proposition 3.10.2. The proof we give works verbatim for B and $\mathcal{B}$, so we focus on the case with A and $\mathcal{A}$. Assume d_τ is sufficiently small so that Corollary 3.3.4 is true, and such that $\frac{99}{100}K \subset \text{co}(A), \text{co}(B), \text{co}(D_t) \subset K$ by Proposition 3.3.1. Fix one of the M directions v. Consider all arcs $\mathfrak{q} \in \mathcal{J}(\theta,\ell) \cap \mathcal{A}$ with the direction vector $\widehat{v_{\mathfrak{q}}} = v$. Let $\mathfrak{r}_A$ be the union of all the corresponding arcs $\mathfrak{q}_A$. Note that $\mathfrak{r}_A$ forms a connected arc of $\partial co(A)$. Let x and x' be the endpoints of this arc.

For any point $z \in \mathfrak{r}_A$, we claim that $|xz| \leq \frac{9}{\sin(14^\circ)}\text{dist}(z, ox)$. Indeed, by Lemma 3.5.3, since $|xz| \leq 9|oz|$ (this follows as the diameter of $\text{co}(A) \subset T'$ is at most $\frac{2}{\sqrt{3}}$ by Observation 3.2.3, and $|oz| \geq \frac{99}{100}\frac{1}{\sqrt{12}}$) it suffices to show that $\angle ozx \in (28^\circ, 180^\circ - 28^\circ)$. By Corollary 3.3.4, we know that the supporting lines l_x, l_z to $\text{co}(A)$ at x, z make an angle of at most $180^\circ - 29^\circ$ with ox, oz respectively. Therefore, we have that $\angle ozx, oxz \leq 180^\circ - 29^\circ$. By Observation 3.6.4, ox, oz each make an angle of at most $\frac{1}{2}\alpha$ with v. Therefore, $\angle xoz \leq \alpha$. Because the sum of the angles in xoz is 180°, this implies that $\angle ozx \in (29^\circ - \alpha, 180^\circ - 29^\circ) \subset (28^\circ, 180^\circ - 28^\circ)$.

For every y outside of $\mathfrak{r}_A$, we have either y is on the opposite side of ox or y is on the opposite side of ox' to $\mathfrak{r}_A$. This implies that $\min(zx, zx') \leq \frac{9}{\sin(14^\circ)}|yz|$ as y lies either on the other side of ox or of ox' to z.

We claim that if $R_{\mathfrak{q},A}$ with $\mathfrak{q}_A \subset \mathfrak{r}_A$ intersects in positive area with some $R_{\mathfrak{q}',A}$, then $\mathfrak{q}_A, \mathfrak{q}'_A \subset (B(x, 1200t^{-1}\sqrt{\gamma}) \cup B(x', 1200t^{-1}\sqrt{\gamma}))$. Indeed, first note that if $\mathfrak{q}'_A \subset \mathfrak{r}_A$, then $\widehat{v_{\mathfrak{q}}} = \widehat{v_{\mathfrak{q}'}}$, forbidding a positive area intersection. Hence $\mathfrak{q}_A$ lies outside of $\mathfrak{r}_A$. Note that if $|R_{\mathfrak{q},A} \cap R_{\mathfrak{q}',A}| > 0$, then the distance between $\mathfrak{q}_A$ and $\mathfrak{q}'_A$ is at most $30t^{-1}\sqrt{\gamma}$ by the triangle inequality (as the heights of these parallelograms are each at most $15t^{-1}\sqrt{\gamma}$). From this, we conclude that

$$\min(\text{dist}(\mathfrak{q}_A, x), \text{dist}(\mathfrak{q}_A, x')) \leq \frac{9}{\sin(14^\circ)}30t^{-1}\sqrt{\gamma} \leq 1199t^{-1}\sqrt{\gamma}.$$

Because

$$|\mathfrak{q}_A| \leq \xi t^{-1}\sqrt{\gamma} \leq t^{-1}\sqrt{\gamma},$$

the conclusion follows.

We have the length of $\partial\,\mathrm{co}(A) \cap (B(x, 1200t^{-1}\sqrt{\gamma}) \cup B(x', 1200t^{-1}\sqrt{\gamma}))$ is at most $4800\pi t^{-1}\sqrt{\gamma}$, the sum of the perimeters of the two balls. Hence for each direction v we have that

$$\sum_{\mathfrak{q}\in\mathcal{J}_{3\ell}^{\mathrm{good}}(\theta,\ell)\cap\mathcal{A},\widehat{v_{\mathfrak{q}}}=v \text{ and } \exists\mathfrak{q}'\in\mathcal{A}\setminus\{\mathfrak{q}\} \text{ with } |R_{\mathfrak{q},A}\cap R_{\mathfrak{q}',A}|>0} \left|R_{\mathfrak{q},A}\right| \leq 4800\pi t^{-1}\sqrt{\gamma}\cdot\xi\sqrt{\gamma} = 16000t^{-1}\xi\gamma.$$

□

3.11 Proof of Theorem 1.1.5 and Theorem 3.2.2

With all the machinery in place, we are now ready to tackle Theorem 3.2.2. We note that Theorem 1.1.5 and Theorem 3.2.2 are formally equivalent by replacing A with $\frac{1}{t}A$ and B with $\frac{1}{1-t}B$.

Proof of Theorem 3.2.2. Fix $\varepsilon > 0$ and choose ξ such that $\varepsilon \geq (t^2 + (1-t)^2)(25t^{-1}M\xi^2 + 16000t^{-1}M\xi)$. Choose θ depending on ξ given by Proposition 3.5.1. Choose ℓ depending on θ given by Proposition 3.4.2. Recall that M, α are universal constants chosen above. Finally, take d_τ sufficiently small so that Proposition 3.4.2, Proposition 3.7.4, Proposition 3.9.1, Proposition 3.10.1 and Proposition 3.10.2 hold. Recall by Proposition 3.7.4 that

$$\left|\mathrm{co}\,(D_t)\setminus D_t\right| \leq t^2 \sum_{\mathfrak{q}\in\mathcal{A}} |R_{\mathfrak{q},A}\setminus A| + (1-t)^2 \sum_{\mathfrak{q}\in\mathcal{B}} |R_{\mathfrak{q},B}\setminus B|.$$

We split the first summand on the right into three parts; one for those $\mathfrak{q}$ such that $R_{\mathfrak{q},A} \not\subset co(A)$ (collect them in a set X_A), one for those $\mathfrak{q} \in \mathcal{J}_{3\ell}^{\mathrm{good}}(\theta,\ell)$ such that $R_{\mathfrak{q},A}$ intersects non trivially with $R_{\mathfrak{q},A}$ for some $\mathfrak{q}' \neq \mathfrak{q}$ (collect them in a set Y_A), and all the other $\mathfrak{q}$ (collect them in a set Z_A). Note that the $R_{\mathfrak{q},A}$ in the last sum are disjoint by Proposition 3.10.1 and contained in $co(A)$, so $\sum_{\mathfrak{q}\in Z_A} |R_{\mathfrak{q},A}\setminus A| \leq |\mathrm{co}(A)\setminus A|$. Combining Proposition 3.9.1 and Proposition 3.10.2 we find:

$$\begin{aligned}\sum_{\mathfrak{q}\in\mathcal{A}} |R_{\mathfrak{q},A}\setminus A| &\leq \sum_{\mathfrak{q}\in X_A} |R_{\mathfrak{q},A}| + \sum_{\mathfrak{q}\in Y_A} |R_{\mathfrak{q},A}| + \sum_{\mathfrak{q}\in Z_A} |R_{\mathfrak{q},A}\setminus A| \\ &\leq 25t^{-1}M\xi^2\gamma + 16000t^{-1}M\xi\gamma + |co(A)\setminus A|.\end{aligned}$$

We similarly obtain

$$\sum_{\mathfrak{q}\in\mathcal{B}} |R_{\mathfrak{q},B}\setminus B| \leq 25t^{-1}M\xi^2\gamma + 16000t^{-1}M\xi\gamma + |\operatorname{co}(B)\setminus B|.$$

Hence, (recalling $\gamma = t^2|\operatorname{co}(A)\setminus A| + (1-t)^2|\operatorname{co}(B)\setminus B|$), we have

$$\begin{aligned}\Big|\operatorname{co}(D_t)\setminus D_t\Big| &\leq (t^2+(1-t)^2)(25t^{-1}M\xi^2 + 16000t^{-1}M\xi)\gamma + t^2|\operatorname{co}(A)\setminus A| + (1-t)^2|\operatorname{co}(B)\setminus B| \\ &\leq (1+\varepsilon)\left(t^2|\operatorname{co}(A)\setminus A| + (1-t)^2|\operatorname{co}(B)\setminus B|\right).\end{aligned}$$

□

3.12 Proof that Theorem 1.1.5 implies Theorem 1.1.4

Finally, what remains is to deduce Theorem 1.1.4. Note that we now return to A and B with unequal areas. Along the way, we will show Corollary 3.1.3.

Proof that Theorem 1.1.5 implies Theorem 1.1.4. By [33, 34] and Section 3.13, there is a constant $\widetilde{C}$ such that

$$\frac{|K_A\setminus\operatorname{co}(A)|}{|\operatorname{co}(A)|} + \frac{|K_B\setminus\operatorname{co}(B)|}{|\operatorname{co}(B)|} \leq \widetilde{C}\tau_{conv}^{-\frac{1}{2}}\sqrt{\delta_{conv}}$$

where $\delta_{conv} = \frac{|\operatorname{co}(A+B)|^{\frac{1}{2}}}{|\operatorname{co}(A)|^{\frac{1}{2}}+|\operatorname{co}(B)|^{\frac{1}{2}}} - 1$, and $t_{conv} = \frac{|\operatorname{co}(A)|^{\frac{1}{2}}}{|\operatorname{co}(A)|^{\frac{1}{2}}+|\operatorname{co}(B)|^{\frac{1}{2}}} \in [\tau_{conv}, 1-\tau_{conv}]$. Also, by Theorem 1.1.2 by taking d_τ sufficiently small, we may assume that $\frac{|\operatorname{co}(A)|}{|A|}$, $\frac{|\operatorname{co}(B)|}{|B|}$, and $\frac{|\operatorname{co}(A+B)|}{|A+B|}$ are as close to 1 as we like, so in particular we may assume that $\tau_{conv}^{-1} \leq 2\tau^{-1}$. Thus

it suffices to prove that $\delta_{conv} \leq \delta$ and $\frac{|\mathrm{co}(A)\setminus A|}{|\mathrm{co}(A)|} + \frac{|\mathrm{co}(B)\setminus B|}{|\mathrm{co}(B)|} \leq 5\tau^{-1}\delta$. We have

$$
\begin{aligned}
&\delta - \delta_{conv} \\
&\geq \frac{|A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}}{|\mathrm{co}(A)|^{\frac{1}{2}} + |\mathrm{co}(B)|^{\frac{1}{2}}}\delta - \delta_{conv} \\
&= \frac{1}{|\mathrm{co}(A)|^{\frac{1}{2}} + |\mathrm{co}(B)|^{\frac{1}{2}}}\left(|\mathrm{co}(A)|^{\frac{1}{2}} - |A|^{\frac{1}{2}} + |\mathrm{co}(B)|^{\frac{1}{2}} - |B|^{\frac{1}{2}} - (|\mathrm{co}(A+B)|^{\frac{1}{2}} - |A+B|^{\frac{1}{2}})\right) \\
&= \frac{1}{|\mathrm{co}(A)|^{\frac{1}{2}} + |\mathrm{co}(B)|^{\frac{1}{2}}}\left(\frac{|\mathrm{co}(A)\setminus A|}{|\mathrm{co}(A)|^{\frac{1}{2}} + |A|^{\frac{1}{2}}} + \frac{|\mathrm{co}(B)\setminus B|}{|\mathrm{co}(B)|^{\frac{1}{2}} + |B|^{\frac{1}{2}}} - \frac{|\mathrm{co}(A+B)\setminus (A+B)|}{|\mathrm{co}(A+B)|^{\frac{1}{2}} + |A+B|^{\frac{1}{2}}}\right) \\
&\geq \frac{1}{|\mathrm{co}(A)|^{\frac{1}{2}} + |\mathrm{co}(B)|^{\frac{1}{2}}}\left(\frac{|\mathrm{co}(A)\setminus A|}{|\mathrm{co}(A)|^{\frac{1}{2}} + |A|^{\frac{1}{2}}} + \frac{|\mathrm{co}(B)\setminus B|}{|\mathrm{co}(B)|^{\frac{1}{2}} + |B|^{\frac{1}{2}}} - \frac{(1+\varepsilon)(|\mathrm{co}(A)\setminus A| + |\mathrm{co}(B)\setminus B|)}{|\mathrm{co}(A+B)|^{\frac{1}{2}} + |A+B|^{\frac{1}{2}}}\right).
\end{aligned}
$$

Suppose $t \leq \frac{1}{2}$ and take $\varepsilon = \frac{\tau}{2}$. We can write this last line as $m_A \frac{|\mathrm{co}(A)\setminus A|}{|\mathrm{co}(A)|} + m_B \frac{|\mathrm{co}(B)\setminus B|}{|\mathrm{co}(B)|}$ with

$$
\begin{aligned}
m_A =& t\frac{|\mathrm{co}(A)|}{|A|} \cdot \frac{|A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}}{|\mathrm{co}(A)|^{\frac{1}{2}} + |\mathrm{co}(B)|^{\frac{1}{2}}}\left(\frac{1}{\frac{|\mathrm{co}(A)|^{\frac{1}{2}}}{|A|^{\frac{1}{2}}} + 1} - \frac{1}{\frac{|\mathrm{co}(A+B)|^{\frac{1}{2}}}{|A+B|^{\frac{1}{2}}} + 1} \cdot \frac{(1+\varepsilon)t}{(1+\delta)}\right) \\
\geq& t\frac{|\mathrm{co}(A)|}{|A|} \cdot \frac{|A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}}{|\mathrm{co}(A)|^{\frac{1}{2}} + |\mathrm{co}(B)|^{\frac{1}{2}}}\left(\frac{1}{\frac{|\mathrm{co}(A)|^{\frac{1}{2}}}{|A|^{\frac{1}{2}}} + 1} - \frac{1}{\frac{|\mathrm{co}(A+B)|^{\frac{1}{2}}}{|A+B|^{\frac{1}{2}}} + 1} \cdot \frac{3}{4}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
m_B =& (1-t)\frac{|\mathrm{co}(B)|}{|B|} \cdot \frac{|A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}}{|\mathrm{co}(A)|^{\frac{1}{2}} + |\mathrm{co}(B)|^{\frac{1}{2}}}\left(\frac{1}{\frac{|\mathrm{co}(B)|^{\frac{1}{2}}}{|B|^{\frac{1}{2}}} + 1} - \frac{1}{\frac{|\mathrm{co}(A+B)|^{\frac{1}{2}}}{|A+B|^{\frac{1}{2}}} + 1} \cdot \frac{(1+\varepsilon)(1-t)}{(1+\delta)}\right) \\
\geq& (1-t)\frac{|\mathrm{co}(B)|}{|B|} \cdot \frac{|A|^{\frac{1}{2}} + |B|^{\frac{1}{2}}}{|\mathrm{co}(A)|^{\frac{1}{2}} + |\mathrm{co}(B)|^{\frac{1}{2}}}\left(\frac{1}{\frac{|\mathrm{co}(B)|^{\frac{1}{2}}}{|B|^{\frac{1}{2}}} + 1} - \frac{1}{\frac{|\mathrm{co}(A+B)|^{\frac{1}{2}}}{|A+B|^{\frac{1}{2}}} + 1} \cdot \left(1 - \frac{\tau}{2}\right)\right).
\end{aligned}
$$

Both of these are at least $\frac{1}{5}\tau$ assuming d_τ is sufficiently small. Thus we get $\delta - \delta_{conv} \geq \frac{1}{5}\tau\left(\frac{|\mathrm{co}(A)\setminus A|}{|\mathrm{co}(A)|} + \frac{|\mathrm{co}(B)\setminus B|}{|\mathrm{co}(B)|}\right)$, which shows $\delta_{conv} \leq \delta$ and $\frac{|\mathrm{co}(A)\setminus A|}{|\mathrm{co}(A)|} + \frac{|\mathrm{co}(B)\setminus B|}{|\mathrm{co}(B)|} \leq 5\tau^{-1}\delta$. $\square$

3.13 Equivalence of measures ω and α

In this section, we show that in two dimensions the measures ω and α are commensurate for convex sets when d_τ is sufficiently small. Recall from the introduction that we always have $\alpha \leq 2\omega$.

Proposition 3.13.1. *For all $\tau \in \left(0, \frac{1}{2}\right]$, there exists a $d_\tau > 0$ such that the following holds. If $E, F \subset \mathbb{R}^2$ are convex with $t(E,F) \in [\tau, 1-\tau]$ and $\delta(E,F) \leq d_\tau$, then*

$$\omega(E,F) \leq 21\alpha(E,F).$$

Proof. Let d_τ be sufficiently small so that by [33], $\alpha(E,F) \leq \frac{1}{10}$. We never use any other property of $\delta(E,F)$ or $t(E,F)$. The quantities ω, α are invariant under affine transformations of E and F separately, so by applying these transforms we can take E, F to have equal volumes, translated so that $\alpha(E,F) = \frac{|E \triangle F|}{|E|}$. After a further affine transformation, we may assume that the maximal triangle $T \subset E \cap F$ is a unit equilateral triangle. Note that because T is maximal, we have $T \subset E \cap F \subset -2T$. Take $K = \mathrm{co}(E \cup F)$. Note that $|E \triangle F| \leq \frac{1}{18}|E \cap F| \leq \frac{1}{18}|-2T| \leq \frac{1}{2}$.

First, we claim that $E, F \subset 10T$. Indeed, if any point $x \in E$ lies in in $\partial 10T$ then $|E \triangle F| \geq |\mathrm{co}(x \cup T) \setminus (-2T)| \geq 1$, a contradiction.

To show $\omega(E,F) \leq 11\alpha(E,F)$, it suffices to prove

$$|K \setminus (E \cup F)| \leq 10|E \triangle F|.$$

Indeed, if this is true, then

$$|E| \cdot \omega(E,F) \leq |K \setminus E| + |K \setminus F| = 2|K \setminus (E \cup F)| + |E \triangle F| \leq 21|E \triangle F| = |E| \cdot 21\alpha(E,F).$$

We consider the triangle opq with p, q consecutive vertices of K. These triangles partition the area of K, so it suffices to show for each such triangle that

$$|(K \setminus (E \cup F)) \cap opq| \leq 10|(E \triangle F) \cap opq|.$$

To obtain this, we note that if $p, q \in E$ or $p, q \in F$ then the left hand side is zero and the inequality holds. Suppose now that $p \in E$ and $q \in F$ (the other case is identical). Then there must be a point $i \in \partial\,\mathrm{co}(E) \cap \partial\,\mathrm{co}(F)$ which lies in the triangle opq. Let q' be the intersection of the ray pi with segment oq, and let p' be the intersection of the ray qi with op. Because $o, p \in E$ we also have $p' \in E$, and similarly $q' \in F$. Note that $p' \notin F$ and $q' \notin E$. We note that $E, F \subset 10T$ implies $|op'| \geq \frac{1}{10}|oq|$ and $|oq'| \geq \frac{1}{10}|oq|$. If any point x in the strict interior

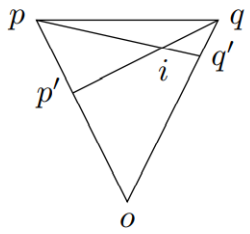

Fig. 3.20 The points p' and q' as defined by p, q, i and o.

$(qiq')^\circ$ lies in E, then i lies in the strict interior of $xpo \subset E$, contradicting that i lies on ∂E. Also, $qiq' \subset oqi \subset F$. Thus $(qiq')^\circ \subset E \triangle F$. Similarly $(pip')^\circ \subset E \triangle F$. Finally, we note that $(K \setminus (E \cup F)) \cap opq \subset piq$, so it suffices to show that

$$|piq| \leq 10(|pip'| + |qiq'|).$$

To show this, suppose without loss of generality that $|oiq| \leq |oip|$. Then $\frac{|piq|}{|oiq|} = \frac{|pip'|}{|oip'|}$ so

$$|piq| = |pip'|\frac{|oiq|}{|oip'|} \leq |pip'|\frac{|oip|}{|oip'|} = |pip'|\frac{|op|}{|op'|} \leq 10|pip'|. \quad \square$$

Chapter 4

Locally biased partitions of $\mathbb{Z}^k$

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4.1 Introduction

For a graph $G = (V, E)$, we call a function f on V a *scenery*. Let $\tilde{X} = (X_n)_{n=0}^{\infty}$ be a simple random walk on G. We associate with $\tilde{X}$ the sequence $(f(X_n))_{n=0}^{\infty}$ of values attained by the scenery. Is it possible to, with high probability, reconstruct the scenery f from this random sequence? This question has an extensive history, in particular, the case where $G = \mathbb{Z}$ has been studied in many papers, see e.g. [5, 35, 52, 56, 62, 63] . In [52], Howard showed that periodic sceneries on $\mathbb{Z}$, or equivalently sceneries on finite cycles, can always be reconstructed. Matzinger and Lember [63] extended this result on periodic sceneries to random walks on $\mathbb{Z}$ which include steps of different sizes. In turn, Finuncane, Tamuz and Yaar [35] refined this idea and extended it to more general Cayley graphs of finite abelian groups. The question how many steps of observation are needed to distinguish sceneries was addressed in [64].

Focusing on the question for $G = \mathbb{Z}$ (and $G = \mathbb{Z}^2$), Benjamini and Kesten[5] showed that almost all sceneries can in fact be distinguished, in the sense that any given scenery on G can be distinguished with probability 1 from a scenery chosen randomly in the product measure on all sceneries. Benjamini and, independently, Keane and Den Hollander conjectured that in fact all pairs of sceneries on $\mathbb{Z}$ are distinguishable (unpublished) [53]. This, however, was soon disproved by Lindenstrauss [62] who constructed a collection of uncountably many distinct yet indistinguishable sceneries.

In a recent paper, Gross and Grupel [42] showed that for 0-1 functions on the hypercube, i.e. functions of the form $f\colon \{0,1\}^n \to \{0,1\}$, this reconstruction is not possible in general. To this end, they defined a locally p-biased function to be a function which takes the value 1

on exactly a fraction p of the neighbours of each vertex in the graph. Note that if there exist two non-isomorphic locally p-biased functions on a graph, they will be indistinguishable as the sequence of values for both will look like a sequence of independent Bernoulli random variables with success probability p.

Gross and Grupel [42] extended their construction from the hypercube to $\mathbb{Z}^n$ to find locally p-biased functions for $p = c/2^k$ where $2^k|n$. They asked for what $p \in [0,1]$ such locally p-biased functions exist and how many there are for each p. Our main aim in this chapter is to give a complete characterization of all the values $p \in [0,1]$ for which a locally p-biased function on $\mathbb{Z}^n$ exists and the exact number for each of those p.

As every element of $\mathbb{Z}^n$ has $2n$ neighbours, if there is a locally p-biased function, then p is of the form $p = c/2n$ for some integer c. In fact, we will find that for all $c \in \{0,\dots,2n\}$ there exists a locally p-biased function with $p = c/2n$. Note that the sum of a locally p-biased function and a locally q-biased function on disjoint supports is a locally $p+q$-biased function. Hence, to show that these locally $c/2n$-biased functions exist, it suffices to construct locally $1/2n$-biased functions the supports of which partition $\mathbb{Z}^n$.
We proceed by showing that for all $n > 1$ there are uncountably many non-isomorphic such partitions and uncountably many indistinguishable locally p-biased functions, answering the second question from [42]. Our result extends the result of Lindenstrauss[62] that there exist uncountably many indistinguishable sceneries on $\mathbb{Z}$.

We consider $\mathbb{Z}^n$ to be the graph on that set with edge set $E = \{\{x, x+e_i\} : x \in \mathbb{Z}^n, i \in [n]\}$. We identify a 0-1 function with its support, so that we can talk of sets rather than functions. A set $X \subset \mathbb{Z}^n$ is *locally p-biased* if for all $x \in \mathbb{Z}^n$, we have $|\Gamma(x) \cap X| = 2pn$. A partition $\{X_i\}_{i\in[2n]}$ of $\mathbb{Z}^n$ is *locally biased* if each of its elements is locally $1/2n$-biased. Hence, as noted in the introduction, if there is a locally biased partition of $\mathbb{Z}^n$, then there are locally p-biased functions on $\mathbb{Z}^n$ for all $p = c/2n$.

4.2 There is a locally biased partition of $\mathbb{Z}^n$ for every $n \in \mathbb{N}$

Theorem 4.2.1. *For every $n \in \mathbb{N}$, there is a locally biased partition of $\mathbb{Z}^n$.*

This result follows directly by considering the function $f\colon \mathbb{Z}^n \to [2n], x \mapsto \sum_{i=1}^{n-1} ix_i + n\lfloor x_n/n \rfloor$. However, for the benefit of the counting result, we present a different construction of locally biased partitions. In particular, we will consider a recursive construction. Accordingly, we start by observing that $\mathbb{Z} = \{m : m \equiv 0,1 \mod 4\} \sqcup \{m : m \equiv 2,3 \mod 4\}$ is a locally biased partition. For the recursive construction of the locally biased partitions, we define a closely related notion. Let $m,n \in \mathbb{N}$ such that m is a multiple of n and let $\{X_j^i\}_{i\in[\frac{m+n}{n}],j\in[2n]}$ be a family of disjoint subsets of $\mathbb{Z}^m$. We write $X^i = \bigsqcup_j X_j^i$. We say the family $\{X_j^i\}_{i\in[\frac{m+n}{n}],j\in[2n]}$

is (m,n)-*filling* if the family partitions $\mathbb{Z}^m$ and if, for each $i \in [\frac{m+n}{n}]$ and $j \in [2n]$, we have that if $x \in \mathbb{Z}^m \setminus X^i$, then $|\Gamma(x) \cap X^i_j| = 1$ and if $x \in X^i$, then $\Gamma(x) \cap X^i = \emptyset$.
The following lemma is the basis for our recursive construction.

Lemma 4.2.2. *If there is a locally biased partition of $\mathbb{Z}^n$ and there exists an (m,n)-filling family of subsets of $\mathbb{Z}^m$, then there is a locally biased partition of $\mathbb{Z}^{m+n}$.*

Proof. Let $\{X^i_j\}_{i\in[\frac{m+n}{n}],j\in[2n]}$ be an (m,n)-filling family of subsets of $\mathbb{Z}^m$ and let $\{Y_i\}_{i\in[2n]}$ be a locally biased partition of $\mathbb{Z}^n$. Let, for $i \in [\frac{m+n}{n}]$ and $l \in \{0,\dots,2n-1\}$;

$$Z^i_l = \bigsqcup_{j\in[2n]} X^i_{j+l} \times Y_j \tag{4.1}$$

where the indices are considered modulo $2n$. We claim this set is locally $\frac{1}{2(m+n)}$-biased in $\mathbb{Z}^{m+n}$. For notational convenience, we take $i = 1$ and $l = 0$.

We claim that every $z = (x,y) \in \mathbb{Z}^m \times \mathbb{Z}^n$ has a unique neighbour in $Z = Z^1_0$.
If $x \in X^1$, then z cannot have a neighbour in Z in the first m coordinates, as $\Gamma(x) \cap X^1 = \emptyset$. Let $x \in X^1_j$. By definition of Y_j, y has a unique neighbour in Y_j, say y', which gives z a unique neighbour in the last n coordinates, i.e. $(x,y') \in Z$.
If, on the other hand, $x \notin X^1$, then z cannot have a neighbour in Z in the last n coordinates. Then x has a neighbour in X^1_j for all j. In particular for the j with $y \in Y_j$. Let x' be this unique element of $\Gamma(x) \cap X^1_j$. Now z has a unique neighbour in Z, i.e. (x',y).

Analogously, Z^i_l is a locally $\frac{1}{2(m+n)}$-biased set for each $i \in [\frac{m+n}{n}], l \in \{0,\dots,2n-1\}$. Moreover, these sets partition $\mathbb{Z}^{m+n}$, showing that there is a locally biased partition of $\mathbb{Z}^{m+n}$. □

For the proof of Theorem 3.1, it remains to find suitable (m,n)-filling families.

Lemma 4.2.3. *Given $n \in \mathbb{N}$, define for $l \in [2], j \in [2n]$ the following subsets of $\mathbb{Z}^n$:*

$$X^l_j = \begin{cases} \{x \in \mathbb{Z}^n : \sum_{i\in[n]} x_i \equiv l \mod 4, \quad \sum_{i\in[n]} ix_i \equiv j \mod n\} & \text{if } j \le n \\ \{x \in \mathbb{Z}^n : \sum_{i\in[n]} x_i \equiv l+2 \mod 4, \ \sum_{i\in[n]} ix_i \equiv j \mod n\} & \text{if } j > n \end{cases}$$

Then the family $\{X^l_j\}_{l\in[2],j\in[2n]}$ is an (n,n)-filling family.

Proof. Note that $X^l = \bigsqcup_j X^l_j = \{x : \sum_i x_i \equiv l \mod 2\}$, so the sets X^l_j partition $\mathbb{Z}^n$.
Let $l \in [2]$ and $x \in X^l$. Then $x \pm e_i$, the neighbours of x, are in X^{3-l} for all $i \in [n]$. It remains to show that these neighbours are all in distinct X^{3-l}_k. We find that the neighbour $y = x + e_j$ is such that $\sum_i y_i \equiv 1 + \sum_i x_i \mod 4$ and $\sum_i iy_i \equiv j + \sum_i ix_i \mod n$. For distinct $j \in [n]$ the second sums are clearly distinct modulo n. If we compare y to $z = x - e_k$ for some $k \in [n]$, we find that $\sum_i z_i \equiv -1 + \sum_i x_i \mod 4$, so y and z belong to different parts of the partition. □

These lemmas imply that if there is a locally biased partition of $\mathbb{Z}^n$, then there also is a locally biased partition of $\mathbb{Z}^{2n}$. Hence, there is a locally biased partition of $\mathbb{Z}^{2^k}$ for all $k \in \mathbb{N}$. To extend this to the natural numbers with odd prime divisors, we use the following construction.

Lemma 4.2.4. *There exists a $(2tn,n)$-filling family of subsets of $\mathbb{Z}^{2tn}$ for all $t,n \in \mathbb{N}$.*

Proof. Define for $l \in [2t+1]$ and $k \in [2n]$;

$$X^l = \left\{ x \in \mathbb{Z}^{2tn} : \sum_{j=1}^{t} \sum_{i=2(j-1)n+1}^{2nj} jx_i \equiv l \mod 2t+1 \right\},$$
$$X^l_k = \left\{ x \in \mathbb{Z}^{2tn} : x \in X^l, \ \sum_{i=1}^{2tn} ix_i \equiv k \mod 2n \right\}$$

We will show that these sets X^l_k form a $(2tn,n)$-filling family. To this end, note that the sets X^l partition $\mathbb{Z}^{2tn}$ and that $\{X^l_k\}_{k\in[2n]}$ is a partition of X^l into $2n$ parts. It remains to check that each element of $\mathbb{Z}^{2tn}$ is either in X^l with no neighbours in X^l or is not in X^l with exactly one neighbour in each of the parts X^l_k for $k \in [2n]$. Fix some $l \in [2t+1]$.

If $y \in X^l$, then $\sum_{j=1}^{t} \sum_{i=2jn+1}^{2n(j+1)} jy_i \equiv l \mod 2t+1$, so any neighbour of y is not in X^l as changing any coordinate would change this sum.

If $y \notin X^l$, then $\sum_{j=1}^{t} \sum_{i=2jn+1}^{2n(j+1)} jx_i \equiv l-p \mod 2t+1$, for some $p \in [2t]$. We distinguish between two cases; either $p \in [t]$ or $p \in \{t+1,\dots,2t\}$.
If $p \in [t]$, then we know that $y+e_i \in X^l$ for every $i \in \{2pn+1,\dots,2(p+1)n\}$. Note that each of these $2n$ vectors is in a different set X^l_k.
If, on the other hand, $p \in \{t+1,\dots,2t\}$, then let $h = 2t+1-p$. Then $y-e_i \in X^l$ for every $i \in \{2hn+1,\dots,2(h+1)n\}$. Again, each of these $2n$ vectors is in a different set X^l_p. Note that in both cases, these are the only neighbours of y in X^l. □

This is the last ingredient needed for the proof of Theorem 4.2.1.

Proof. As noted in the beginning of the section, there is a locally biased partition of $\mathbb{Z}$. In combination with Lemma 4.2.3 and Lemma 4.2.2, this implies that there is a locally biased partition of $\mathbb{Z}^{2^k}$ for every $k \in \mathbb{Z}_{\geq 0}$. To extend this to all of $\mathbb{N}$, let $n = 2^k(2l+1)$ for some $k,l \in \mathbb{Z}_{\geq 0}$. By Lemma 4.2.4, we can find a $(2l2^k, 2^k)$-filling family, which by Lemma 4.2.2 implies that there is a locally biased partition of $\mathbb{Z}^{2^k+2l2^k} = \mathbb{Z}^n$. □

4.3 Counting locally biased partitions

In [42], Gross and Grupel ask, besides a characterisation of p values for which locally p-biased functions exist, for a count of the number of non-isomorphic such locally p-biased

functions. In fact, they provide some finite lower bounds on this number for $p = 1/n$ and $p = 1/2$, based on an extension of their construction on the hypercube. We find the following complete characterization.

Theorem 4.3.1. *For all $p = c/2n$ with $n > 1$ and $c \in \{1, \ldots, 2n-1\}$, there are $2^{\aleph_0}$ non-isomorphic locally p-biased functions in $\mathbb{Z}^n$.*

We say two locally biased partitions $\{X_i\}_{i\in[2n]}$ and $\{Y_i\}_{i\in[2n]}$ of $\mathbb{Z}^n$ are isomorphic if there exist a graph isomorphism $\phi : \mathbb{Z}^n \to \mathbb{Z}^n$ and permutation $\sigma : [2n] \to [2n]$ such that $\phi(X_i) = Y_{\sigma(i)}$ for all $i \in [2n]$.

Theorem 4.3.2. *For all $n > 1$, there are $2^{\aleph_0}$ non-isomorphic locally biased partitions of $\mathbb{Z}^n$.*

As there are only $2^{\aleph_0}$ subsets of $\mathbb{Z}^n$, the upper bound for both Theorems 4.3.1 and 4.3.2 are immediate.

In fact, our construction proving Theorems 4.3.1 and 4.3.2 will be almost identical to the one in the previous section. By introducing a degree of freedom, we produce an uncountable collection of distinct locally biased partitions. An automorphism $\phi : \mathbb{Z}^n \to \mathbb{Z}^n$ consists of three components; a permutation of the coordinates ($n!$), a reflection of some of the coordinates (2^n) and a translation ($|\mathbb{Z}^n| = \aleph_0$). There are only countably many combinations of these operations and hence there are only countably many automorphisms $\phi : \mathbb{Z}^n \to \mathbb{Z}^n$.

To use the previous construction, we need to have a notion of isomorphism for (m,n)-filling families. We say two (m,n)-filling families $\{A^i_j\}_{i\in[\frac{m+n}{n}],j\in[2n]}$ and $\{B^i_j\}_{i\in[\frac{m+n}{n}],j\in[2n]}$ are isomorphic if there exist a graph isomorphism $\phi : \mathbb{Z}^m \to \mathbb{Z}^m$, a permutation $\sigma : [\frac{m+n}{n}] \to [\frac{m+n}{n}]$ and a family of permutations $\tau_i : [2n] \to [2n]$ for all $i \in [\frac{m+n}{n}]$, such that $\phi(A^i_j) = B^{\sigma(i)}_{\tau_i(j)}$. Note that by the above observation, each (m,n)-filling family is isomorphic to at most countably many other (m,n)-filling families.

It is interesting to note the following. Given an (m,n)-filling family, we can define a function, taking any element of $\mathbb{Z}^m$ to the the part of the partition that element is in. Two functions arising in such a way from two non-isomorphic (m,n)-filling families will be indistinguishable by a simple random walk.

Lemma 4.3.3. *If there is a collection of pairwise non-isomorphic (m,n)-filling families of size $2^{\aleph_0}$ and a locally biased partition of $\mathbb{Z}^n$, then there are $2^{\aleph_0}$ pairwise non-isomorphic locally biased partitions of $\mathbb{Z}^{m+n}$.*

Proof. In fact, the construction in Lemma 4.2.2 produces such a collection. Let $\{X^i_{j,x}\}_{i\in[\frac{m+n}{n}],j\in[2n]}$ be an uncountable family of pairwise non-isomorphic (m,n)-filling families indexed by some $x \in \mathbb{R}$ and let $\{Y_i\}_{i\in[2n]}$ be a locally biased partition of $\mathbb{Z}^n$. Let $\{Z^i_{l,x}\}_{i\in[\frac{m+n}{n}],l\in[2n]}$ be the locally biased partitions as defined in equation (4.1).

As only countably many of the families $\{Z^i_{l,x}\}_{i\in[\frac{m+n}{n}],l\in[2n]}$ are pairwise isomorphic, it suffices to prove there are $2^{\aleph_0}$ distinct locally biased partitions.

We claim that all (m,n)-filling families $\{X^i_{j,x}\}_{i\in[\frac{m+n}{n}],j\in[2n]}$ produce distinct locally biased partitions. For a contradiction, suppose $\{Z^i_{l,x}\}_{i\in[\frac{m+n}{n}],l\in[2n]} = \{Z^i_{l,y}\}_{i\in[\frac{m+n}{n}],l\in[2n]}$ for some $x \neq y$. Then we find that for any $i \in [\frac{m+n}{n}]$ and $l \in [2n]$, we can find some $i' \in [\frac{m+n}{n}]$ and $l' \in [2n]$ such that; $\bigsqcup_j X^i_{j+l,x} \times Y_j = \bigsqcup_j X^{i'}_{j+l',y} \times Y_j$. Since $\{Y_j\}$ is a partition, this implies that for all $j \in [2n]$; $X^i_{j+l,x} = X^{i'}_{j+l',y}$ and $\{X^i_{j,x}\}_{i\in[\frac{m+n}{n}],j\in[2n]} = \{X^i_{j,y}\}_{i\in[\frac{m+n}{n}],j\in[2n]}$. However, we assumed that the (m,n)-filling families were distinct; this contradiction proves the lemma. □

The construction of Lemma 4.2.4 works in such a way that for any element not in X^l all $2n$ neighbours in X^l lie in the same hyperplane in $\mathbb{Z}^{2tn}$ defined by $\sum_{j=1}^{t}\sum_{i=2jn+1}^{2n(j+1)} jx_i = l + (2t+1)h$ for some h. This has the advantage that the construction used to make sure that each set X^l_k contains exactly one of those neighbours need not be the same on distinct planes, giving the following construction.

Lemma 4.3.4. *For any $t,n \in \mathbb{N}$, there are $2^{\aleph_0}$ $(2tn,n)$-filling families.*

Proof. Consider the following slight alteration of the construction in Lemma 4.2.4.

$$X^l_{k,f} = \left\{ x \in \mathbb{Z}^{2tn} : \exists h \in \mathbb{Z}; \sum_{j=1}^{t}\sum_{i=2jn+1}^{2n(j+1)} jx_i = l + h(2t+1) \text{ and } \sum_{i=1}^{2tn} ix_i \equiv k + f(h) \mod 2n \right\} \tag{4.2}$$

where $k \in [2n]$, $l \in [2t+1]$ and $f : \mathbb{Z} \to [2n]$ is any function. This family is $(2tn,n)$-filling by the proof of Lemma 4.2.4. There are $\left|[2n]^{\mathbb{Z}}\right| = 2^{\aleph_0}$ such functions, and thus such $(2tn,n)$-filling families. As each of these locally biased partitions is isomorphic to at most countably many others, we must have $2^{\aleph_0}$ non-isomorphic locally biased partitions among these. □

Similarly we can extend the construction from Lemma 4.2.3.

Lemma 4.3.5. *For $n > 1$, there are $2^{\aleph_0}$ (n,n)-filling families*

Proof. Consider the following construction for $l \in [2]$, $p \in \{0,1\}$, $q \in [n]$ and $f : \mathbb{Z} \to [n]$:

$$X^l_{p,q,f} = \left\{ x \in \mathbb{Z}^n : \exists h \in \mathbb{Z}; \sum_{i=1}^{n} x_i = l + 2p + 4h \text{ and } \sum_{i=1}^{n} ix_i \equiv q + f(h) \mod n \right\} \tag{4.3}$$

Writing it in proper form, let $X^l_{k,f} = X^l_{p,q,f}$ with $p = \begin{cases} 0 \text{ if } k \leq n \\ 1 \text{ if } k > n \end{cases}$, and $q \in [n]$ with $q \equiv k \mod n$. To check that this is in fact an (n,n)-filling family, note that $X^l_f = \bigsqcup_k X^l_{k,f} = \{x :$

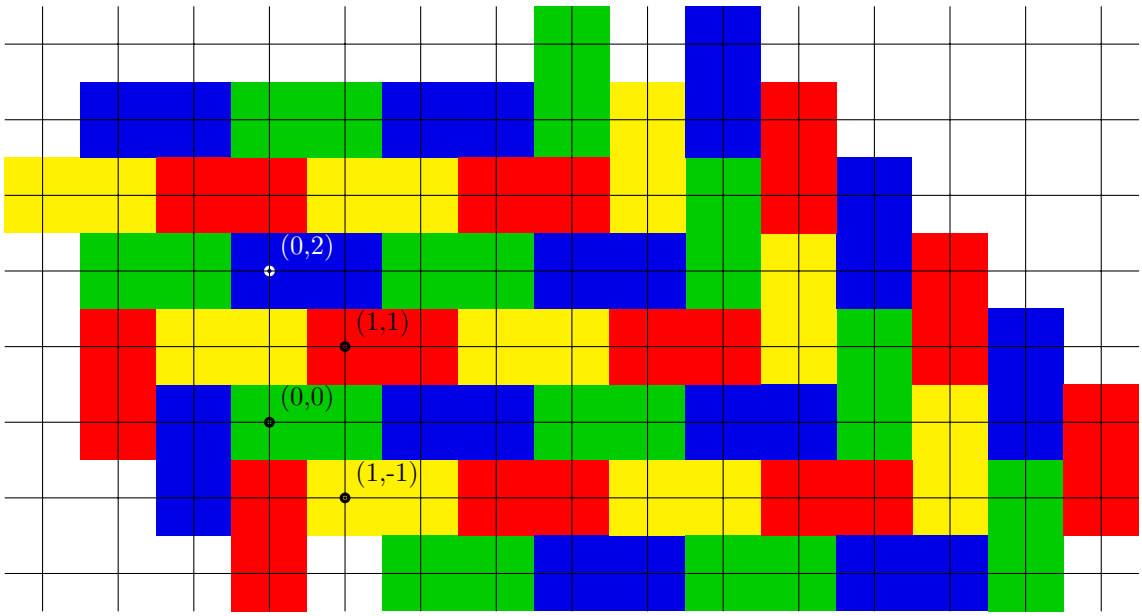

Fig. 4.1 Part of an example of a locally biased partition of $\mathbb{Z}^2$, where the elements of X_f^1, X_f^2, X_f^3 and X_f^4 are contained in the green, yellow, red and blue regions respectively. Note that $f(-1) = f(2) = 2$ and $f(0) = f(1) = 1$

$\sum_i x_i \equiv l \mod 2\}$, so the $X_{k,f}^l$ form a partition of $\mathbb{Z}^n$ and $\Gamma(X_f^l) \cap X_f^l = \emptyset$. We proceed to check that each element of $\mathbb{Z}^n \setminus X_f^l$ has a unique neighbour in $X_{k,f}^l$.

Let $x \in \mathbb{Z}^n \setminus X_f^l$, i.e. $\sum_i x_i \equiv l+1 \mod 2$. Write $X_{p,f}^l = \bigsqcup_q X_{p,q,f}^l$, then $x - e_j \in X_{p,f}^l$ and $x + e_j \in X_{1-p,f}^l$ for all $j \in [n]$ for some $p \in \{0,1\}$. Let $y = x + e_j$, then we have $\sum_i y_i = 1 + \sum_i x_i = l + 2(1-p) + 4h$ for some h not dependent on j. Thus, for different $j \in [n]$, we find $\sum_i i y_i = j + \sum_i i x_i \equiv q + f(h) \mod n$ with distinct q. Hence, $y = x + e_j \in X_{1-p,q,f}^l$ with distinct q for distinct j.

Analogously $x - e_j \in X_{p,q,f}^l$ with distinct q for distinct j.

As in the proof of Lemma 4.3.4, we find that for $n > 1$, there are $2^{\aleph_0}$ functions $f : \mathbb{Z} \to [n]$ and thus sets $X_{p,q,f}^l$. At most $\aleph_0$ of those can be pairwise isomorphic, so there must be $2^{\aleph_0}$ pairwise non-isomorphic (n,n)-filling families. □

What remains is to count the number of non-isomorphic locally biased partitions of $\mathbb{Z}^2$.

Lemma 4.3.6. *There are $2^{\aleph_0}$ non-isomorphic locally biased partitions of $\mathbb{Z}^2$.*

Proof. Consider the following set $S = \{(2k, 2k) : k \in \mathbb{Z}\}$. Note that every element of the set $\{x \in \mathbb{Z}^2 : x_1 - x_2 \in \{-1,0,1,2\}\}$, i.e. four upward diagonals around the origin, has exactly one neighbour in the set $S + \{0, e_i\}$ for both $i = 1$ and $i = 2$.

Given some function $f : \mathbb{Z} \to [2]$, let $X_f = S + \{(2n, -2n), (2n, -2n) + e_{f(n)} : n \in \mathbb{Z}\}$. Now consider the following locally biased partition:

$$X_f^k = \begin{cases} X_f & \text{if } k = 1 \\ X_f + (1, -1) & \text{if } k = 2 \\ X_f + (1, 1) & \text{if } k = 3 \\ X_f + (0, 2) & \text{if } k = 4 \end{cases}$$

For an example, cf. Figure 4.1. These 4 sets partition $\mathbb{Z}^2$. Each of the sets is locally $1/4$-biased, by the note above. Finally this produces $2^{\aleph_0}$ non-isomorphic locally biased partitions by the same argument as for the previous two lemmas. □

Proof of Theorem 4.3.2. .
By Theorem 4.3.6, we find uncountably many non-isomorphic locally biased partitions of $\mathbb{Z}^2$, and by Lemmas 4.3.4 and 4.3.5 combined with Lemma 4.3.3, we find uncountably many non-isomorphic locally biased partitions of $\mathbb{Z}^n$ for $n > 2$. □

Extending the theorem on locally biased partitions to locally p-biased functions is not immediate as different locally biased partitions might give rise to the same locally p-biased functions. Consider for instance the locally $1/2$-biased functions on $\mathbb{Z}^2$ which has support on $X_f^1 \cup X_f^2$ from the proof of Lemma 4.3.6; this function is the same for all choices of f, as can easily be seen in Figure 4.1. We will see that for dimensions bigger than two this is not a problem. We consider the case for $\mathbb{Z}^2$ separately.

Lemma 4.3.7. *For all $p \in \{1/4, 1/2, 3/4\}$, there exist uncountable non-isomorphic locally p-biased functions in $\mathbb{Z}^2$.*

Proof. For $p = 1/4$ and $p = 3/4$, this follows immediately from Lemma 4.3.6, so consider $p = 1/2$. Let $f : \mathbb{Z} \to [2]$ be any function, and let

$$X_f = \{x \in \mathbb{Z}^2 : x_1 \equiv f(x_1 + x_2) \mod 2\}$$

Note that for any $x \in \mathbb{Z}^2$ either $x + e_1$ or $x + e_2$ is in X_f, and similarly either $x - e_1$ or $x - e_2$ is in X_f. Hence, X_f is locally $\frac{1}{2}$-biased. As there are $2^{\aleph_0}$ different functions $f : \mathbb{Z} \to [2]$, there are as many different locally $\frac{1}{2}$-biased functions, at most countably many of which are isomorphic. □

All that remains is to prove Theorem 4.3.1

Proof of Theorem 4.3.1. Note that for $p = 1/2n$ and $p = (2n-1)/2n$ the statement follows immediately from Theorem 4.3.2. The case $n = 2$ follows from Lemma 4.3.7. Fix some $n > 2$ and $p = c/2n$.

We use the constructions of equations 4.2 and 4.3 to produce locally biased partitions indexed by some function f, which we then feed into the construction in equation 4.1. This produces the sets $Z^l_{k,f}$ with $l \in [r]$ and $k \in \{0,\dots,q-1\}$ with r,q integers depending on the last step of the recursive construction of locally biased partitions. That is; $r = 2$ and $q = n$ if n is some power of 2 and $r = m+1$ and $q = 2^{k+1}$ if $n = 2^k(2m+1)$ for some $k \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{N}$.

Recall that for $l \in [r]$ and $k \in [q]$;

$$Z^l_{k,f} = \bigsqcup_{j\in[q]} X^l_{j+k,f} \times Y_j$$

where the indices are considered modulo q. Now $\{Z^l_{k,f} : l \in [r], k \in [q]\}$ is a locally biased partition, so any union of c elements of $\{Z^l_{k,f} : l \in [r], k \in [q]\}$ is a locally $c/2n = p$-biased set. We know that distinct functions f give rise to distinct locally biased partitions. However, as noted before, it might be the case that when taking a union of c parts of the locally biased partition, the differences introduced by different functions f disappear. We choose our c parts in such a way that this does not happen, in the following way;

Let $I \subset [r] \times [q]$ be such that $|I| = c$ and there is an $l_0 \in [r]$ such that $|\{k \in [q] : (l_0,k) \in I\}|$ is either 1 or $n-1$. Let

$$S_f = \bigsqcup_{(l,k)\in I} Z^l_{k,f} = \bigsqcup_{(l,k)\in I} \bigsqcup_{j\in[q]} X^l_{j+k,f} \times Y_j$$

Note that S_f is a locally p-biased set. We will find shortly that the choice of I ensures that the original structure of the locally biased partition remains visible in the created set.

We claim that $\{S_f : f \in [r]^{\mathbb{Z}}\}$ is a family of uncountably many non-isomorphic locally p-biased sets. As before, only countably many distinct sets can be isomorphic, so it suffices to show that $f \mapsto S_f$ is injective. Let $S_f = S_g$, then

$$\bigsqcup_{(l,k)\in I} \bigsqcup_{j\in[q]} X^l_{j+k,f} \times Y_j = \bigsqcup_{(l,k)\in I} \bigsqcup_{j\in[q]} X^l_{j+k,g} \times Y_j$$

Consider $S_f \cap X^{l_0} \times Y_q = S_g \cap X^{l_0} \times Y_q$. Taking complements if $|\{k \in [q] : (l_0,k) \in I\}| = q-1$, this is equal to $X^{l_0}_{k,f} \times Y_q = X^{l_0}_{k,g} \times Y_q$ for some k, by the choice of l_0. Thus, $X^{l_0}_{k,f} = X^{l_0}_{k,g}$ and $f = g$. Hence, $f \mapsto S_f$ is injective.

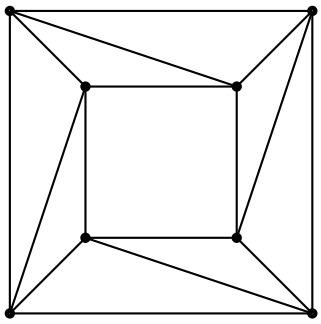

Fig. 4.2 An example of a 4-regular (vertex transitive) graph with 8 vertices which does not allow a locally biased partition.

Thence, there are uncountably many sets S_f and only countably many of those can be pairwise isomorphic, so there are uncountably many non-isomorphic locally p-biased sets and functions. □

4.4 Open Problems

The concept of a locally biased partition introduced in this paper raises the question what graphs contain them. Evidently a graph needs to be d-regular for some d, and we can easily identify the following additional condition.

Proposition 4.4.1. *If d-regular graph G, with $|V(G)| = n$ allows a locally biased partition, then $2d|n$*

Proof. Let $\{A_i : i \in [d]\}$ be a locally biased partition of G. Choose $v \sim \mathcal{U}(V)$ and choose $u \sim \mathcal{U}(\Gamma(v))$. As v has exactly one neighbour in each part A_i, the probability that u is in A_i is $\mathbb{P}(u \in A_i) = 1/d$.
On the other hand, as G is regular, choosing a random vertex in the way we choose u is equivalent to choosing a vertex uniformly at random from $V(G)$. Therefore, we find $1/d = \mathbb{P}(u \in A_i) = |A_i|/n$ and $|A_i| = n/d$. Additionally, we know that the elements of A_i appear in pairs, so $2|A_i = n/d$. Hence, $2d|n$. □

This condition implies that the only hypercubes Q_n that allow a locally biased partition are those with $n = 2^k$ for some natural k. The construction of locally p-biased functions on the hypercube described in [42] uses exactly that locally biased partition.

Note however that this is not a sufficient condition as illustrated by the graph in Figure 4.2, which does not allow a locally biased partition as every edge is part of a triangle.

Question 4.4.2. *What graphs allow a locally biased partitions? If a graph allows a locally biased partition, how many non-isomorphic locally biased partitions does it allow?*

Additional open questions on the topic of scenery reconstruction can be found in [35] and specifically on the topic of locally biased functions and partitions in [42].

Chapter 5

The bunkbed conjecture on the complete graph

The work in this section was done jointly with Piet Lammers, and is published in the European Journal of Combinatorics. [49]

5.1 Introduction

The bunkbed conjecture is an intuitive statement in percolation theory. In rough terms the conjecture asserts that – in a specific setting and in a specific sense – two vertices of a graph are more likely to remain connected after randomly removing some edges if the graph distance between the vertices is smaller. The conjecture is appealing because it is intuitive yet difficult to prove. In this chapter we prove the conjecture for the case that the underlying graph is symmetrical. The conjecture was first posed by Kasteleyn (in 1985), as was remarked by Van den Berg and Kahn [6]. Before stating the conjecture, we introduce the notion of the bunkbed of a graph and we introduce the percolation model. Given a graph G and a subset $H \subset V(G)$, the *bunkbed* of the pair (G,H), or $BB(G,H)$, is the graph $G \times \{1,2\}$ plus $|H|$ extra edges to connect for every $v \in H$ the vertices $(v,1)$ and $(v,2)$. For any vertex $v \in V$, write $v^- := (v,1)$ and $v^+ := (v,2)$. Any vertex of $BB(G,H)$ is of the form v^- or v^+. Equivalently, if $e \in E$, then write $e^\pm$ for the two corresponding edges in the bunkbed graph. Now introduce the bond percolation model for the bunkbed graph. Pick a percolation parameter $p \in [0,1]$. In the percolation model, every edge of the form $e^\pm$ is declared open with probability p and closed with probability $1-p$, independently of the other edges. The edges of the form $\{v^-, v^+\}$ are always declared open. Write $\mathbb{P}_p$ for the measure corresponding to the states of the edges. For $v,w \in V$, write $v \sim w$ if $\{v,w\}$ is an

open edge, and write $v \leftrightarrow w$ if v and w are joined by an open path. Furthermore, if $v \in V$ and $W \subset V$, then write $v \sim W$ if there is a vertex $w \in W$ with $v \sim w$. See [41] for a more elaborate introduction into the percolation model. The general bunkbed conjecture asserts that $\mathbb{P}_p(v^- \leftrightarrow w^-)$ is greater than or equal to $\mathbb{P}_p(v^- \leftrightarrow w^+)$, for any $v, w \in V$. We prove the conjecture in the case that G is the complete graph.

Theorem 5.1.1. *Pick $n \in \mathbb{N}$ and $H \subset [n]$. Consider independent bond percolation on $BB(K_n, H)$ with parameter $p \in [0,1]$ for the edges of the form $e^{\pm}$, and with the edges of the form $\{v^-, v^+\}$ always open. Then for any pair of vertices $v, w \in [n]$ we have*

$$\mathbb{P}_p(v^- \leftrightarrow w^-) \geq \mathbb{P}_p(v^- \leftrightarrow w^+). \tag{5.1}$$

De Buyer independently proved the theorem for the special case $p = \frac{1}{2}$ in [15], which he later extended to $p \geq \frac{1}{2}$ in [16]. The proof presented here draws on a different method. It has been proved that the connection probability of two vertices of a graph is the same in the percolation model with parameter $p = \frac{1}{2}$ as it is in the model in which every edge is assigned a direction uniformly at random [55, 65].The conjecture has been proved for any p for wheel graphs and some small other graphs by Leander [61]. A statement similar to the bunkbed conjecture has been studied on bunkbed graphs. Bollobás and Brightwell [10] considered a continuous time random walk on a bunkbed graph, such that the jump rate to any neighbour of the current state is one. They conjectured that for every $t > 0$, this random walk started at v^- is more likely to have hit w^- than w^+ before time t. Häggström proved this conjecture in [44].

The idea of the proof will be to fix v, condition on the subgraph on all vertices outside of v, choose $w \neq v$ uniformly random, sample the edges between v and the rest of the graph, and finally use the symmetry between the upper and the lower bunk.

5.2 Proof of Theorem 5.1.1

Proof of Theorem 5.1.1. We prove the theorem for $n+1$ instead of n for notational convenience (the conjecture is trivial for $n = 1$). It will be assumed that $w = n+1$, without loss of generality. If $w \in H$ then $w^- \sim w^+$ and the two events in Equation (5.1) are the same. If $v = w$ then the left side of Equation (5.1) equals one. Therefore we only need to consider the case that $w \notin H$ and $v \neq w$. If $v \in H$ then both sides of Equation (5.1) are equal (by symmetry of the bunkbed), and if $v \notin H$, then both sides of Equation (5.1) do not depend on the actual choice of $v \in [n] \setminus H$ (by symmetry of the complete graph). Therefore it is sufficient to prove the inequality for v chosen uniformly at random in the

set $[n]$, independently of the percolation. By choosing v uniformly at random in $[n]$, we make optimal use of the symmetry of the graph K_{n+1}. Now write $(V,E) := BB(K_{n+1},H)$ and note that $V = [n+1] \times \{1,2\} = ([n] \times \{1,2\}) \cup \{w^-, w^+\}$. Write O for the open subgraph of $BB(K_{n+1},H)$ induced by the set $[n] \times \{1,2\}$. This means that the vertex set of O is $[n] \times \{1,2\} = V \setminus \{w^-, w^+\}$, and that every edge $e \in E$ is an edge of O if and only if its endpoints are in $[n] \times \{1,2\}$ and if e is open in the percolation measure $\mathbb{P}_p$. The edge set of O is thus random in the measure $\mathbb{P}_p$. Moreover, O determines the configuration of all edges incident to neither w^- nor w^+, and the configuration of the edges incident to either w^- or w^+ and the value of v are independent of O. Write c for the partition of O into connected components, and label these $c = \{c_1, \ldots, c_k\}$ (where k is the number of connected components, also random). In order to calculate the difference between the two probabilities in Equation (5.1), we define the events

$$A := \{w^- \leftrightarrow v^- \not\leftrightarrow w^+\} = \bigsqcup_i \left(\{v^- \in c_i\} \cap \{w^- \sim c_i\} \cap \{w^+ \not\sim c_i\} \cap \{\nexists j \neq i, w^- \sim c_j \sim w^+\}\right),$$
$$B := \{w^+ \leftrightarrow v^- \not\leftrightarrow w^-\} = \bigsqcup_i \left(\{v^- \in c_i\} \cap \{w^+ \sim c_i\} \cap \{w^- \not\sim c_i\} \cap \{\nexists j \neq i, w^- \sim c_j \sim w^+\}\right).$$

In each of these two equations the four events within the disjoint union are, conditional on O and for fixed i, mutually independent. This is because the last three events (in each of the two lines) depend on the states of different edges, and because (for the first event) the value of v is chosen independently of the percolation. Write $\tilde{\mathbb{P}}$ for the measure $\mathbb{P}_p$ conditioned on O. Now

$$\mathbb{P}_p(A|O) = \sum_i \tilde{\mathbb{P}}(v^- \in c_i)\tilde{\mathbb{P}}(w^- \sim c_i)\tilde{\mathbb{P}}(w^+ \not\sim c_i)\tilde{\mathbb{P}}(\nexists j \neq i, w^- \sim c_j \sim w^+),$$
$$\mathbb{P}_p(B|O) = \sum_i \tilde{\mathbb{P}}(v^- \in c_i)\tilde{\mathbb{P}}(w^+ \sim c_i)\tilde{\mathbb{P}}(w^- \not\sim c_i)\tilde{\mathbb{P}}(\nexists j \neq i, w^- \sim c_j \sim w^+),$$
$$\mathbb{P}_p(A|O) - \mathbb{P}_p(B|O) = \sum_i \tilde{\mathbb{P}}(v^- \in c_i)\left(\tilde{\mathbb{P}}(w^+ \not\sim c_i) - \tilde{\mathbb{P}}(w^- \not\sim c_i)\right)\tilde{\mathbb{P}}(\nexists j \neq i, w^- \sim c_j \sim w^+).$$

The difference between the two sides of Equation (5.1) is $\mathbb{P}_p(A) - \mathbb{P}_p(B)$, which equals the expectation of the final line of the display over O. The probabilities in Equation (5.1) are invariant under simultaneously replacing v^- by v^+ and interchanging w^- and w^+. Taking

the average over the original expression and the permuted one gives

$$\begin{aligned}
&\mathbb{P}_p(v^- \leftrightarrow w^-) - \mathbb{P}_p(v^- \leftrightarrow w^+) = \mathbb{P}_p(A) - \mathbb{P}_p(B) = \mathbb{E}_p(\mathbb{P}_p(A|O) - \mathbb{P}_p(B|O)) \\
&\quad = \frac{1}{2}\mathbb{E}_p \sum_i \Big(\tilde{\mathbb{P}}(v^- \in c_i) \left(\tilde{\mathbb{P}}(w^+ \not\sim c_i) - \tilde{\mathbb{P}}(w^- \not\sim c_i) \right) \tilde{\mathbb{P}}(\nexists j \neq i, w^- \sim c_j \sim w^+) \\
&\qquad\qquad + \tilde{\mathbb{P}}(v^+ \in c_i) \left(\tilde{\mathbb{P}}(w^- \not\sim c_i) - \tilde{\mathbb{P}}(w^+ \not\sim c_i) \right) \tilde{\mathbb{P}}(\nexists j \neq i, w^+ \sim c_j \sim w^-) \Big) \\
&\quad = \frac{1}{2}\mathbb{E}_p \sum_i \left(\tilde{\mathbb{P}}(v^- \in c_i) - \tilde{\mathbb{P}}(v^+ \in c_i) \right) \left(\tilde{\mathbb{P}}(w^+ \not\sim c_i) - \tilde{\mathbb{P}}(w^- \not\sim c_i) \right) \tilde{\mathbb{P}}(\nexists j \neq i, w^- \sim c_j \sim w^+).
\end{aligned}$$

We claim that the two differences in the final sum always have the same sign, so that the product is always non-negative. Write c_i^- and c_i^+ for the number of vertices in c_i of the form u^- and u^+ respectively, so that, for example, $\sum_i c_i^- = \sum_i c_i^+ = n$. We explicitly calculate that (writing $q = 1 - p$)

$$\left(\tilde{\mathbb{P}}(v^- \in c_i) - \tilde{\mathbb{P}}(v^+ \in c_i) \right) \left(\tilde{\mathbb{P}}(w^+ \not\sim c_i) - \tilde{\mathbb{P}}(w^- \not\sim c_i) \right) = \frac{1}{n} \left(c_i^- - c_i^+ \right) \left(q^{c_i^+} - q^{c_i^-} \right) \geq 0,$$

where the final inequality is due to $(a-b)(q^b - q^a) \geq 0$ for any $a, b \in \mathbb{Z}_{\geq 0}$ and $q \in [0,1]$. $\square$

Chapter 6

The (t,r) broadcast domination number of some regular graphs

The work in this chapter was done jointly with Rebekah Herrman. It has been published in Discrete Applied Mathematics. [46]

6.1 Introduction

The main aim of this chapter will be to prove Theorem 1.4.1, a conjecture by Drews, Harris, and Randolph [25]. The proof is included in Section 6.2. In Section 6.3, we explore other statements in this direction and suggest some conjectures. In Section 6.4, and Section 6.5, we prove Theorem 1.4.2, and Theorem 1.4.3 respectively. Finally, in Section 6.6, we suggest some research questions for future research.

6.2 Proof of Theorem 1.4.1

First consider the following $(t,1)$ broadcasting set of vertices with minimal density $\mathcal{T}_0 = \{ma+nb : m,n \in \mathbb{Z}\}$ where $a = (t,t-1)$ and $b = (t-1,-t)$. Part of this configuration is shown in Figure 6.1.

Note that the previously described $\mathcal{T}_0$ is also a configuration that provides a $(t+1,3)$ broadcast. We find that four vertices within distance $t-1$ of any tower receive signal 4 rather than the required 3. In Figure 6.1, the bold vertices are the one with extra signal. We would like to study this extra signal for more general configurations of towers.

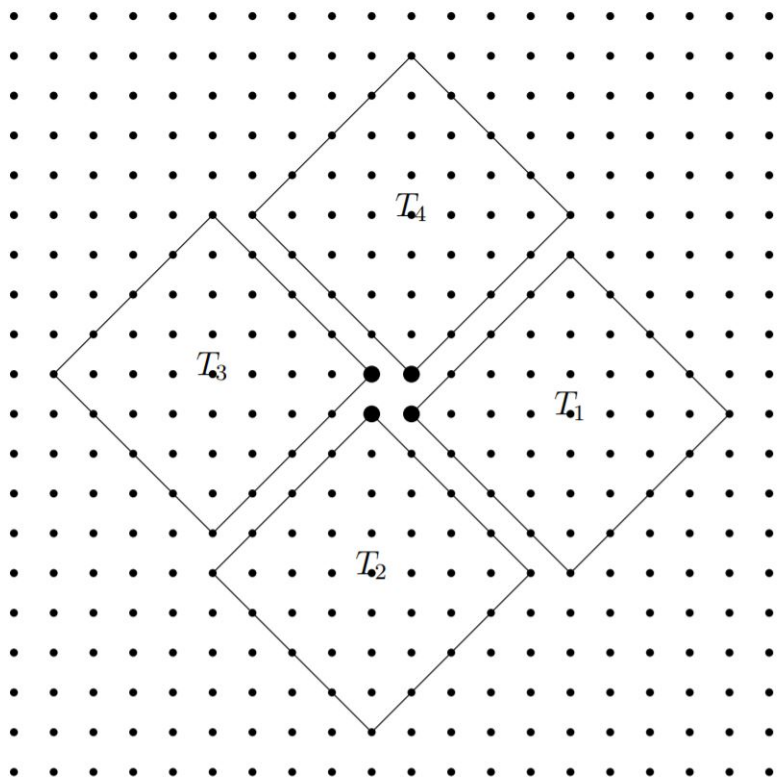

Fig. 6.1 An example of a $(5,1)$ broadcasting set. When considered as a $(6,3)$ broadcasting set, the four large vertices in the middle receive excess signal.

We consider for every tower the usable transmission, which is the sum over the amount transmitted to all the vertices not exceeding r. For a tower at vertex v this is

$$\sum_{u\in\mathbb{Z}^2:d(u,v)\leq t-1} \min\{r,t-d(u,v)\}.$$

Note that in a vertex transitive graph, this amount is the same for all vertices v. In line with this notion of usable transmission, consider the following notion of usable signal arriving at a vertex v; given a (t,r) broadcasting set $\mathcal{T}$, let $signal(v) := \sum_{u\in\mathcal{T}:d(u,v)\leq t-1} \min\{r,t-d(u,v)\}$ To formalise the notion of extra signal, let $excess(v) := signal(v) - r$ be the *excess signal* received by a vertex v in a given (t,r)-broadcasting set of towers.

We would like to attribute the amount of excess to a given tower T to use the following bound; In a vertex transitive graph G with n vertices, if for every tower T we have $\sum_{u\in\mathbb{Z}^2:d(u,T)\leq t-1} \min\{r,t-d(u,T)\} = X$ and in any (t,r) broadcasting set $\mathcal{T}$ we can attribute at least Y excess to every tower, then we find the following inequality:

$$nr + |\mathcal{T}|Y \leq \sum_{v\in V} signal(v) = \sum_{T\in\mathcal{T}} \sum_{u\in\mathbb{Z}^2:d(u,T)\leq t-1} \min\{r,t-d(u,T)\} = |\mathcal{T}|X.$$

Note that if we find a (t,r) broadcasting set $\mathcal{T}$ so that the total amount of excess satisfies $\sum_{v\in V(G)} excess(v) = |\mathcal{T}|Y$, this set satisfies $|\mathcal{T}| = \gamma_{t,r}(G)$.

Our goal is to show $\delta_{t,3}(\mathbb{Z}^2) \geq \delta_{t-1,1}(\mathbb{Z}^2)$ for $t > 17$. In the configuration $\mathcal{T}_0$ considered in the beginning of the section, we have exactly 4 excess attributed to each tower. We want to show that the excess attributed to each tower must be at least 4 in any $(t,3)$ broadcasting configuration, so that the configuration $\mathcal{T}_0$ minimises the excess, following the above heuristic.

Henceforth fix some $(t,3)$ broadcasting set of towers $\mathcal{T}$ of the finite grid $G_{2n+1,2n+1}$. We will prove the following lemma.

Lemma 6.2.1. *For any tower at $(x,y)\in\mathcal{T}$, if $(x,y)+[t-4,t+2]\times[-4,4]\subset V(G_{2n+1,2n+1})$, then there is at least four excess within the vertices $(x,y)+[t-4,t+2]\times[-4,4]$.*

Proof. Without loss of generality consider a tower T, that will be fixed throughout the argument, at $(-t+2,0)$. As the signal at $(0,0)$ from T is 2, there must be another tower, T', within distance $t-1$ from $(0,0)$. We shall consider the following three cases;

1. T' is at most distance $t-2$ from the origin,
2. T is of distance $t-1$ from the origin and $T' \notin \{(0,t-1),(1,t-2),(0,1-t),(1,2-t)\}$,
3. T is of distance $t-1$ from the origin and $T' \in \{(0,t-1),(1,t-2),(0,1-t),(1,2-t)\}$

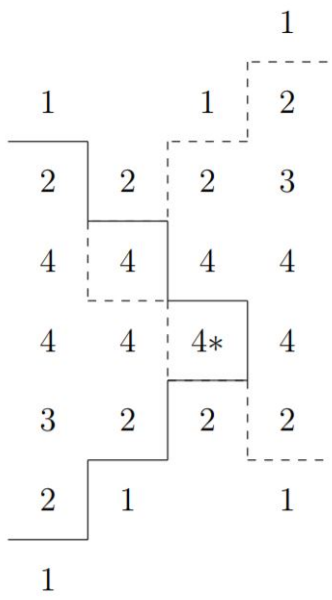

Fig. 6.2 The signal received from T and T' in Subcase 1.1, where second tower T' is located at $(t-3,1)$. The line (dashed line resp.) denote the boundary of those vertices receiving at least 2 signal from T (T'' resp.). For a minimal $(t-1,1)$ broadcasting set, these regions partition the plane. The $*$ marks the origin.

Figures 6.2, 6.3, and 6.4 correspond to Case 1, while Figure 6.5 corresponds to Case 2 and Figure 6.6 corresponds to Case 3.

Case 1. *There is another tower T' with $d(T',(0,0)) \leq t-2$.*

Subcase 1.1. *T' is not on the x-axis.*

Without loss of generality assume T' is above the x-axis, then T' is closer to $(0,1)$ than to $(0,0)$, so $t-d(T',(0,1)) \geq 3$ and similarly $t-d(T',(-1,1)) \geq 2$ and $t-d(T',(-1,0)) \geq 1$. Hence, we find that the excess on $(0,0),(0,1),(-1,0)$ and $(-1,1)$ alone is already more than four, as seen in Figure 6.2.

Subcase 1.2. *T' is at $(x,0)$ for $x \leq t-3$.*

If T' is at $(x,0)$ for $x \leq t-3$, the vertices $(-1,0)$ and $(0,0)$ both have excess at least 2, as seen in Figure 6.3.

Subcase 1.3. *T' is at $(t-2,0)$.*

Note that $(-1,0),(0,0)$ and $(1,0)$ all receive at least one excess from T and T' combined. The points $(-1,1),(0,1)$ and $(1,1)$ receive 2 signal from T and T' combined, so they need another tower to supply at least one signal. If this is the same tower for two of these, one must

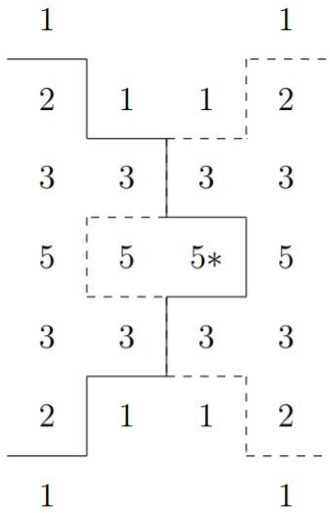

Fig. 6.3 The signal received from T and T' in Subcase 1.2, where second tower $T' = (t-3,0)$.

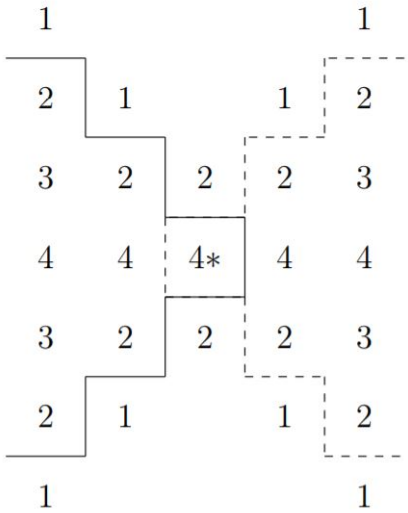

Fig. 6.4 The signal received from T and T' in Subcase 1.3, where the second tower is at $T'(t-2,0)$.

1				
2	1			1
3	2	1	1	2
4	3	3*	3	3
3	3	3	3	4
3	3	3	4	5
1	1	2	3	4

Fig. 6.5 The signal received from T and T' in Case 2 for the specific example $t = 5$ where second tower $T' = (2,4)$.

get excess signal. On the other hand consider they receive one signal from three different towers. Either $(-2,1),(2,1)$ or $(0,0)$ must receive excess signal from these towers, or $(0,2)$ receives at least signal 4 from the three towers combined, as seen in Figure 6.4.

This concludes Case 4.1.

We now distinguish two possible configurations for the tower T' giving additional signal to vertex $(0,0)$. Note that this tower has distance exactly $t-1$ to the origin. Consider whether $T' \in \{(0,t-1),(1,t-2),(0,1-t),(1,2-t)\}$ or not. Note that up to reflection, if $T' \in \{(0,t-1),(1,t-2),(0,1-t),(1,2-t)\}$, we are in the realm of Figure 6.5.

Case 2. $T' \notin \{(0,t-1),(1,t-2),(0,1-t),(1,2-t)\}$

Reflecting if necessary, assume T' is somewhere on $y = x-(t-1)$.

Note that in this case both $(0,1)$ and $(1,1)$ receive 1 signal from T and T' combined. Hence, they both need signal from an additional tower.

Subcase 2.1. *One additional tower covers both* $(0,1)$ *and* $(1,1)$.

This tower will transmit at least a combined signal of three to $(0,0)$ and $(1,0)$, causing a total excess of at least 4 on these four vertices combined.

Subcase 2.2. *The points* $(0,1)$ *and* $(1,1)$ *receive additional signal from two distinct towers.*

Consider the tower T'' giving additional signal to $(-1,1)$. If that tower gives signal at least 2 to $(-2,1)$ or $(-1,0)$, we immediately find the excess. As we additionally know there is no tower at $(0,t-1)$, we find that it must be at $(-1,t)$.

3	3	4	3	2
3	3	3	2	1
3	3	3	1	
4	3	3*	1	
3	2	1		
2	1			
1				

Fig. 6.6 The signal received from T and T' in Case 3, where second tower $T' = (1, t-2)$.

Note that more specifically we know that $(0,1)$ must receive signal from two additional towers. A tower that gives signal 1 to $(0,1)$ must give at least 1 signal to one of $(-1,2)$ and $(1,0)$ and to one of $(-2,3)$ and $(2,-1)$. All of those points already receive 3 signal, so the two additional towers for $(0,1)$ give rise to at least 4 excess on these vertices.

Case 3. $T' \in \{(0,t-1),(1,t-2),(0,1-t),(1,2-t)\}$

Without loss of generality $T' = (1,2-t)$. Note that $(-1,1)$ receives only signal 2 from T and T', so receives additional signal from another tower T''. By Case 1, we only need to consider towers at distance $t-1$ from $(-1,1)$. There are only two significant cases. If T'' has x-coordinate at least 1, then the excess signal on $(0,0),(1,0)$ and $(1,-1)$ is at least 4 already. Hence, T'' is either $(0,t-1)$ or $(-1,t)$.

Subcase 3.1. $T'' = (0,t-1)$

Note that $(1,1)$ and $(1,2)$ only receive 2 signal from towers T, T' and T''. If these two were reached by the same tower say T''', then one of the two must receive signal 2 from T'''. If that is $(1,1)$, note that $(0,0),(0,1),(1,0)$ and $(1,1)$ all receive excess at least 1. If it is $(1,2)$, note that $(0,0),(0,2),(1,2)$ and $(1,3)$ all receive excess at least 1, as seen in Figure 6.6.

Subcase 3.2. $T'' = (-1,t)$

This case is completely analogous to Subcase 2.2.

On the other hand, suppose the points $(1,1)$ and $(1,2)$ receive signal 1 from two distinct towers. If either of these towers transmits 2 signal to $(0,1),(0,2),(1,3)$ or $(1,0)$, the excess

is immediately more than 4. The towers transmit 2 to $(1,1)$ and $(1,2)$ respectively, then $(2,2)$ receives 1 excess signal and $(3,1)$ receives 2 excess signal. □

The next goal is to show that for large t, we have excess at least four times the number of towers.

Lemma 6.2.2. *Let $t \geq 17$. For any $(t,3)$ broadcasting set $\mathcal{T}$ of $G = G_{2n+1,2n+1}$, there is at least $4|\mathcal{T}| - O(n)$ excess.*

Proof. We devise a way to attribute at least 4 excess to towers $T \in \mathcal{T}$ such that $T + [t-4, t+2] \times [-4,4] \subset V(G)$. Note that there are $O(n)$ towers such that $T + [t-4,t+2] \times [-4,4] \not\subset V(G)$. Hence, if we can attribute at least 4 excess to all other towers, the lemma follows.

First to all towers T with no other towers within $T + [-6,6] \times [-8,8]$, assign 4 excess from the rectangle $T + [t-4,t+2] \times [-4,4]$. Note that this excess exists by Lemma 6.2.1 and that these rectangles are disjoint. Let $\mathcal{T}'$ be the set of all these towers and all those towers with $T + [t-4,t+2] \times [-4,4] \not\subset V(G)$. What remains is to attribute at least 4 excess to the towers in $\mathcal{T} \setminus \mathcal{T}'$.

For every tower $T \in \mathcal{T} \setminus \mathcal{T}'$, we know that there exists a tower T' such that

$$(T + [t-4,t+2] \times [-4,4]) \cap \big(T' + [t-4,t+2] \times [-4,4]\big) \neq \emptyset,$$

so $T' \in T + [-6,6] \times [-8,8]$. Consider the set $S = \lfloor \frac{T+T'}{2} \rfloor + [-3,2] \times [-4,4]$. Note that $d(T, \lfloor \frac{T+T'}{2} \rfloor), d(T', \lfloor \frac{T+T'}{2} \rfloor) \leq 7$, so that for a vertex $x \in S$, we have

$$d(T,x), d(T',x) \leq 7 + d\left(x, \left\lfloor \frac{T+T'}{2} \right\rfloor\right) \leq 14.$$

Hence, as $t \geq 17$, all $x \in S$ receive 3 signal from both T and T'. Let $f : \mathcal{T} \setminus \mathcal{T}' \to \mathcal{P}(V(G))$ assign to every $T \in \mathcal{T} \setminus \mathcal{T}'$ the corresponding set S.

Claim 6.2.3. *In the set $\bigcup_{T \in \mathcal{T} \setminus \mathcal{T}'} f(T)$, there is at least $54|\mathcal{T} \setminus \mathcal{T}'|$ excess.*

Proof of Claim. We'll show that for all $x \in V(G)$, we have $excess(x) \geq |f^{-1}(x)|$, where we use $|f^{-1}(x)|$ to denote the number of $T \in \mathcal{T} \setminus \mathcal{T}'$ such that $x \in f(T)$. If $|f^{-1}(x)| = 1$, then by the above discussion, we have that x receives 3 signal from at least two towers and hence has $excess(x) \geq 3 > |f^{-1}(x)|$. If $|f^{-1}(x)| > 1$, then x receives 3 signal from each of the towers such that $x \in f(T)$. Thus we find $excess(x) \geq 3|f^{-1}(x)| - 3 > |f^{-1}(x)|$. Now we find that the amount of excess in the set $\bigcup_{T \in \mathcal{T} \setminus \mathcal{T}'} f(T)$ is at least $\sum_{T \in \mathcal{T} \setminus \mathcal{T}'} |f(T)| = 54|\mathcal{T} \setminus \mathcal{T}'|$. □

Of course, some of this excess may have already been attributed to some tower in $\mathcal{T}'$. However, note that for every tower $T \in \mathcal{T} \setminus \mathcal{T}'$, we have that $f(T)$ can intersect at most 4

disjoint translates of $[t-4,t+2]\times[-4,4]$. Hence at most $4\cdot 4\cdot|\mathcal{T}\setminus\mathcal{T}'|$ of the excess in $\bigcup_{T\in\mathcal{T}\setminus\mathcal{T}'} f(T)$ can already have been attributed.

We conclude that there is at least

$$4|\mathcal{T}'|-O(n)+(54-16)|\mathcal{T}\setminus\mathcal{T}'|\geq 4|\mathcal{T}|-O(n)$$

excess in the graph.

□

We are now ready to prove Theorem 1.4.1.

Proof. Let $G_{2n+1,2n+1}$ be the $2n+1$ by $2n+1$ grid. Let $\mathcal{T}_0$ be a $(t,3)$ broadcasting subset of $\mathbb{Z}^2$ and let $\mathcal{T}:=\mathcal{T}_0\cap V(G_{2n+1,2n+1})$. We then need at least $3(2n+1)^2$ signal to be transmitted. A tower $T\in\mathcal{T}$ can transmit at most

$$\sum_{u\in\mathbb{Z}^2:d(u,T)\leq t-1}\min\{3,t-d(u,T)\}=6t^2-18t+19=3(2t^2-6t+5)+4.$$

At most $O(n)$ towers in $\mathcal{T}_0\setminus\mathcal{T}$ can transmit signal to $G_{2n+1,2n+1}$, let $\mathcal{T}'$ be this collection of towers.

By Lemma 6.2.2, the $(t,3)$-broadcasting set $\mathcal{T}\cup\mathcal{T}'$ of towers can transmit at most

$$(|\mathcal{T}|+|\mathcal{T}'|)3(2t^2-6t+5)+O(n)=|\mathcal{T}|3(2t^2-6t+5)+O(n)$$

signal effectively. Therefore $|\mathcal{T}|\geq\frac{(2n+1)^2-O(n)}{2t^2-6t+5}$, so we find

$$\begin{aligned}\delta_{t,3}(\mathbb{Z}^2)&\geq\lim_{n\to\infty}\frac{\left(\frac{(2n+1)^2-O(n)}{2t^2-6t+5}\right)}{(2n+1)^2}\\&=\frac{1}{2t^2-6t+5}-\lim_{n\to\infty}\frac{O(n)}{(2n+1)^2(2t^2-6t+5)}\\&=\frac{1}{2t^2-6t+5}\\&=\delta_{t-1,1}(\mathbb{Z}^2)\end{aligned}$$

The theorem follows. □

6.3 Generalizations of the (t,r) broadcast number for grids

The proof of Theorem 1.4.1 suggests that the result may be extended to any odd value of r. Note first the following simple, though seemingly unobserved fact;

Proposition 6.3.1. *For all $t,k \geq 1$;*

$$\delta_{t,1}(\mathbb{Z}^2) \geq \delta_{t+k,1+2k}(\mathbb{Z}^2)$$

Proof. It suffices to show a $(t,1)$ broadcasting set of towers $\mathcal{T}$ is also $(t+k,1+2k)$ broadcasting. Consider a vertex $v \in \mathbb{Z}^2$. As $\mathcal{T}$ is $(t,1)$-broadcasting, there exists $T \in \mathcal{T}$ with $d(T,v) < t$. Find a vertex $u \in \mathbb{Z}^2$ with $d(T,u) = d(T,v) + d(u,v) = t$, which is possible in the plane. Again, as $\mathcal{T}$ is $(t,1)$ broadcasting, there is a $T' \in \mathcal{T}$ with $d(T',u) < t$. Now note that if all towers transmitted $t+k$ of signal, then v receives $t+k-d(T,v) = k+d(u,v)$ signal from tower T and

$$t+k-d(T',v) \geq t+k-d(u,v)-d(T',u) \geq k+1-d(u,v)$$

from tower T'. In total v thus receives signal at least

$$k+d(u,v)+k+1-d(u,v) = 2k+1.$$

Hence, $\mathcal{T}$ is also $(t+k,1+2k)$ broadcasting. □

Similarly, we have

Proposition 6.3.2. *For all $t,k \geq 1$;*

$$\delta_{t,2}(\mathbb{Z}^2) \geq \delta_{t+k,2+2k}(\mathbb{Z}^2)$$

Proof. As before, consider $\mathcal{T} \subset \mathbb{Z}^2$ to be $(t,2)$-broadcasting and $v \in \mathbb{Z}^2$. We will show that if the towers in $\mathcal{T}$ transmitted $t+k$ signal, then all vertices would receive at least $2+2k$ signal. If there is a $T \in \mathcal{T}$ with $d(T,v) \leq t-2$ the proof of the previous lemma suffices completely analogously. If there is no such T, there must be $T,T' \in \mathcal{T}$ with $d(T,v) = d(T',v) = t-1$. That implies that v receives signal $k+1$ from both towers and thus $2k+2$ in total. □

In [7], Blessing et al. conjectured that in general equality holds, i.e. that $\delta_{t+1,r+2}(\mathbb{Z}^2) = \delta_{t,r}(\mathbb{Z}^2)$. However, Drews, Harris, and Randolph in [25], showed by computing these quantities that, in fact, $\delta_{t+1,r+2}(\mathbb{Z}^2) < \delta_{t,r}(\mathbb{Z}^2)$ for several values of t and r. Consequently, they formulated a stronger conjecture on the value of $\delta_{t,r}(\mathbb{Z}^2)$ for $r \leq 10$. We believe the

improved bounds suggested in [25] are an artifact of the small values of t used in the simulation run by Drews, Harris, and Randolph, as results for $t \leq 15$ were reported in the paper. We propose the following weakening of the conjecture proposed by Blessing, et al.

Conjecture 6.3.3. *For all $r \geq 2$, there exists $t_0 = t_0(r)$ such that for all $t \geq t_0$;*

$$\delta_{t+1,r+2}(\mathbb{Z}^2) = \delta_{t,r}(\mathbb{Z}^2).$$

In the hopes of proving this result along the line of the proof of Theorem 1.4.1, we compute the average amount of excess per tower in an optimally $(t,1)$ broadcasting configuration when viewed as a $(t+k,2k+1)$ broadcasting configuration. The task of showing that one cannot achieve a configuration with a smaller average amount of excess per tower remains open, but a proof along the same lines as Lemma 6.2.1 seems reasonable. Our attempts have resulted in impenetrable casework, and more ideas to improve elegance would be needed.

Lemma 6.3.4. *Let $t > k$. The average excess per tower in the optimal $(t,1)$ broadcasting configuration $\mathcal{T}_0$ as defined at the beginning of Section 2 when viewed as a $(t+k,2k+1)$ broadcasting configuration is $\frac{4}{6}k(k+1)(2k+1)$.*

Proof. Consider four towers around the origin at $T_1 = (t,0), T_2 = (0,1-t), T_3 = (1-t,1)$ and $T_4 = (1,t)$ and call the square formed by these towers S. This configuration provides a $(t,1)$- broadcast. Note that regions like S partition the plane for this configuration, with four regions adjacent to any tower and four towers adjacent to any region. Hence, it suffices to establish that the amount of excess in S is $\frac{4}{6}k(k+1)(2k+1)$.

Let R to be the collection of points $(x,y) \in \mathbb{Z}^2$ with $1-k \leq x+y \leq k+1$, and $-k \leq x-y \leq k$ (cf. Figure 6.7).

Claim 6.3.5. *The vertices in $S \setminus R$ receive signal exactly $2k+1$, i.e. they receive no excess.*

Proof. Note the configuration is rotationally symmetric around the point $(\frac{1}{2},\frac{1}{2})$, so that we need only check that there is no excess in S above the line $x+y = k+1$. Above the line $x+y = k+1$, no vertex receives any signal from T_2 and T_3. Consider a vertex (x,y) in this region. If $x-y \geq k+1$ or below $x-y \leq -k$, it will receive signal from only one tower. This will be signal at least $2k+1$ but will have no excess as it lies in the broadcast zone of exactly one tower. Otherwise, this vertex will receive signal $t+k-(t-x+y)$ from T_1 and $t+k-(x+t-y)$ from T_4, which amounts to a total signal of $2k+1$. □

To compute the amount of excess, partition R into four sets

- $Q_1 := R \cap \{x,y \geq 1\} = \{(x,y) \in \mathbb{Z}^2 : x,y \geq 1,\ x+y \leq k+1\}$

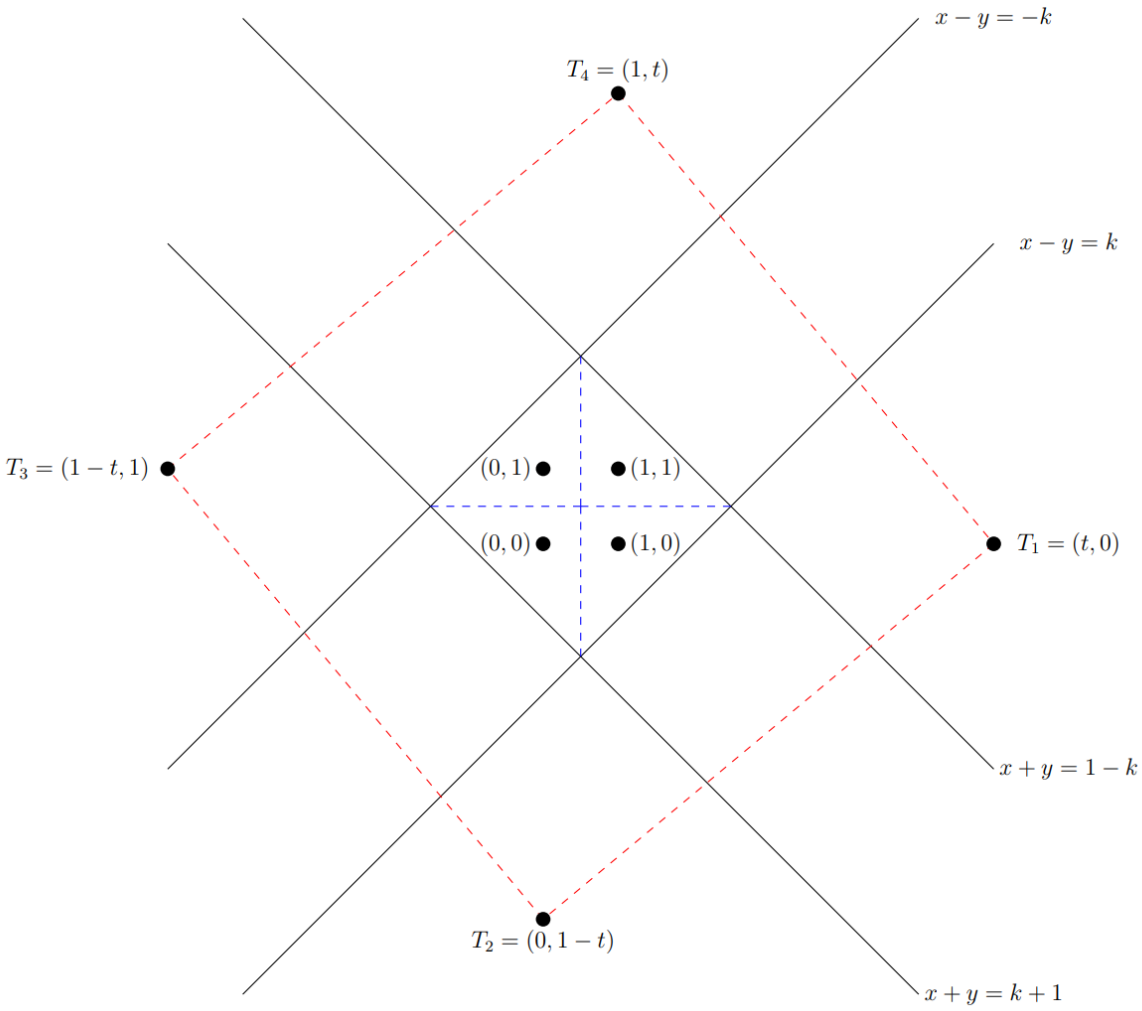

Fig. 6.7 An example of the regions described in the proof of Lemma 9. The red dotted square is S, the diamond outlined in black is R, and Q_i for $i \in [4]$ is any of the blue dashed triangles that shares a border with R.

- $Q_2 := R \cap \{x \geq 1, y \leq 0\} = \{(x,y) \in \mathbb{Z}^2 : x \geq 1, y \leq 0,\ x-y \leq k\}$
- $Q_3 := R \cap \{x \leq 0, y \geq 1\} = \{(x,y) \in \mathbb{Z}^2 : x \leq 0, y \geq 1,\ x-y \geq k\}$
- $Q_4 := R \cap \{x, y \leq 0\} = \{(x,y) \in \mathbb{Z}^2 : x, y \leq 0,\ x+y \geq 1-k\}$

Note that by symmetry the excess in each of these four parts is the same. Hence, consider the excess in $Q = Q_1$.

Claim 6.3.6. *The excess in Q is $\frac{1}{6}k(k+1)(2k+1)$.*

Proof. In fact we note that for every $2 \leq i \leq k$, a vertex $(x,y) \in Q$ with $x+y=i$, receives an excess of $2k-2i-5$. We proceed by induction on i. For $i=2$, note that $(1,1)$ receives

$$(t-(t'-1))+(t-t')+(t-(t'+1))+(t-t') = 4k$$

signal, which corresponds to $2k-1$ excess. For a vertex $v=(x,y) \in Q$ with $x+y \geq 3$, note that at least one of $v-e_1$ and $v-e_2$ was in Q. Fix one of these to be v'. Now the distances to three towers increases, while to one tower it decreases.
In particular, if $v = v'+e_1$, then

$$\begin{aligned} d(v,T_2) &= d(v',T_2)+1, \\ d(v,T_3) &= d(v',T_3)+1, \\ d(v,T_4) &= d(v',T_4)+1, \\ d(v,T_1) &= d(v',T_1)-1. \end{aligned}$$

On the other hand, if $v = v'+e_2$, then

$$\begin{aligned} d(v,T_1) &= d(v',T_1)+1, \\ d(v,T_2) &= d(v',T_2)+1, \\ d(v,T_3) &= d(v',T_3)+1, \\ d(v,T_4) &= d(v',T_4)-1. \end{aligned}$$

Either way the signal received by v is 2 less than the signal received by v', finishing the induction.

The number of vertices $(x,y) \in Q$ with $x+y=i$ is $i-1$, so we find total excess:

$$\sum_{i=2}^{k+1}(i-1)(2k-2i-5) = \frac{1}{6}k(k+1)(2k+1).$$

□

Adding up the amount of excess for each of the four sets Q_1, Q_2, Q_3, and Q_4 proves the lemma. □

6.4 Proof of Theorem 1.4.2

Proof. We will consider the power of a path, $G = P_n^{(k)}$ on vertex set $\{0,\dots,n-1\}$ with v_iv_j an edge if and only if $|i-j| \le k$. For the lower bound we consider the potentially useful amount of signal transmitted by a tower. Note that from the signal submitted to a vertex at distance at most $t-r$ from a tower, only r can be used to exceed the signal threshold. Hence, the total amount of potentially useful signal transmitted by a tower is at most

$$(2k(t-r)+1)r + 2k((r-1)+(r-2)+\cdots+1) = ((2t-r-1)k+1)r.$$

Moreover, as the vertex v_0 receives signal at least r, there must be a tower at v_i for some $i \le (t-r)k$. This tower wastes

$$k((r-1)+(r-2)+\cdots+1) = kr(r-1)/2$$

of its potentially useful amount of transmitted signal. Similarly, v_n receives signal at least r. We may conclude that the total amount of transmitted signal needed is at least $nr + kr(r-1)$. This gives the lower bound $\left\lceil \frac{n+k(r-1)}{(2t-r-1)k+1} \right\rceil$.

For the upper bound consider

$$\mathcal{T} = \{v_i : 0 \le i \le n-1, i \equiv (t-r)k \mod (2t-r-1)k+1\}$$

if $(n-1) \mod (2t-r-1)k+1$ is between $(t-r)k$ and $2(t-r)k+1$. Otherwise, let

$$\mathcal{T} = \{v_i : 0 \le i \le n-1, i \equiv (t-r)k \mod (2t-r-1)k+1\} \cup \{v_{n-1}\}$$

(cf. Figure 6.8).

Note that vertices v_i with $i \le (t-r)k$ all receive enough signal from the tower at $v_{(t-r)k}$. By construction, the last tower is at distance at most $(t-r)$ away from the vertex v_{n-1}, so all the vertices not between two towers receive enough signal.

Now consider a vertex v_i between two towers, say $i = l((2t-r-1)k+1)+(t-r)k+p$ where $0 \le p < (2t-r-1)k+1$ and both $v_{l((2t-r-1)k+1)+(t-r)k}$ and $v_{\min\{n,(l+1)((2t-r-1)k+1)+(t-r)k\}}$

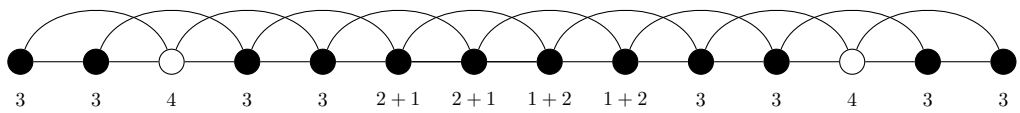

Fig. 6.8 A $(4,3)$ broadcast on $P_{14}^{(2)}$. The unfilled vertices denote the positions of towers.

are in $\mathcal{T}$. Then

$$\begin{aligned} d\left(v_i, v_{l((2t-r-1)k+1)+(t-r)k}\right) + d\left(v_i, v_{\min\{(l+1)((2t-r-1)k+1)+(t-r)k,n\}}\right) \\ &\leq \left\lceil \frac{p}{k} \right\rceil + \left\lceil \frac{(2t-r-1)k+1-p}{k} \right\rceil \\ &= (2t-r-1) + \left\lceil \frac{p}{k} \right\rceil + \left\lceil \frac{1-p}{k} \right\rceil \\ &\leq (2t-r-1)+1 \\ &= 2t-r \end{aligned}$$

Thus, the broadcast received by vertex v_i is

$$\begin{aligned} \max\{t-d(v_i, v_{l((2t-r-1)k+1)+(t-r)k}), 0\} + \max\{t-d(v_i, v_{\min\{(l+1)((2t-r-1)k+1)+(t-r)k,n\}}), 0\} \\ &\geq 2t - \left(d\left(v_i, v_{l((2t-r-1)k+1)+(t-r)k}\right) + d\left(v_i, v_{\min\{(l+1)((2t-r-1)k+1)+(t-r)k,n\}}\right)\right) \\ &\geq 2t-(2t-r) = r \end{aligned}$$

Thence, all vertices receive sufficient signal.

□

When $k=1$, we are left with a path, and obtain $\gamma_{t,r}(P_n) = \left\lceil \frac{n+r-1}{2t-r} \right\rceil$, agreeing with the result by Crepeau, et al.

6.5 Proof of Theorem 1.4.3

Proof. If $n \leq 2(t-r)k+1$, then any vertex is at most distance $(t-r)$ from any other vertex, so a tower at any vertex is (t,r)-broadcasting. If, on the other hand, $n > 2(t-r)k+1$ we find that for all $0 \leq i < n$,

$$d(v_i, v_{i+(t-r)k+1}) = (t-r)+1.$$

Hence, no one tower can be (t,r)-broadcasting. For $n \leq (2t-r-1)k+1$, we have that $\mathcal{T} = \left\{0, \left\lfloor \frac{n}{2} \right\rfloor\right\}$ is (t,r)-broadcasting.

First we will show the upper bound. When $2(t-r)k+1<n$, consider the set

$$\mathcal{T}=\{v_i : i\equiv 0 \mod (2t-r-1)k+1\}\cap\{v_0,\dots,v_n\}$$

(cf. Figure 6.9). Evidently, $|\mathcal{T}|=\left\lceil\frac{n}{(2t-r-1)k+1}\right\rceil$. Moreover, we will show that these towers are (t,r)-broadcasting. Consider vertex v_i. Choose l and p such that $p\in\{0,\dots,(2t-r-1)k\}$ and $i=l((2t-r-1)k+1)+p$. Note that the two towers closest to v_i are $v_{l((2t-r-1)k+1)}$ and $v_{\min\{(l+1)((2t-r-1)k+1),n\}}$. We find that the sum of the distance between each tower and v_i is

$$\begin{aligned}
d(v_i,v_{l((2t-r-1)k+1)})+d(v_i,v_{\min\{(l+1)((2t-r-1)k+1),n\}}) &\le \left\lceil\frac{p}{k}\right\rceil+\left\lceil\frac{(2t-r-1)k+1-p}{k}\right\rceil\\
&=(2t-r-1)+\left\lceil\frac{p}{k}\right\rceil+\left\lceil\frac{1-p}{k}\right\rceil\\
&\le(2t-r-1)+1\\
&=2t-r
\end{aligned}$$

Thus, the broadcast received by vertex v_i is

$$\begin{aligned}
&\max\{t-d(v_i,v_{l((2t-r-1)k+1)}),0\}+\max\{t-d(v_i,v_{\min\{(l+1)((2t-r-1)k+1),n\}}),0\}\\
&\quad\ge 2t-(d(v_i,v_{l((2t-r-1)k+1)}+d(v_i,v_{\min\{(l+1)((2t-r-1)k+1),n\}}))\\
&\quad\ge 2t-(2t-r)=r
\end{aligned}$$

Note that from the signal submitted to a vertex at distance at most $t-r$ from a tower, only r is used to exceed the signal threshold. Hence, the total amount of potentially useful signal submitted by a tower is at most $(2k(t-r)+1)r+2k((r-1)+(r-2)+\cdots+1)=((2t-r-1)k+1)r$. The total signal needed to saturate all the vertices is at least nr. Hence, $\gamma_{t,r}(C_n^{(k)})\ge\left\lceil\frac{nr}{r((2t-r-1)k+1)}\right\rceil=\left\lceil\frac{n}{(2t-r-1)k+1}\right\rceil$. □

6.6 Concluding Remarks

A natural next direction would be to consider n-dimensional generalizations. Analogously to the 2 dimensional definitions, let the density of a set $\mathcal{T}\subset\mathbb{Z}^n$ be defined to be $\limsup_{m\to\infty}\frac{|\mathcal{T}\cap[-m,m]^n|}{(2m+1)^n}$ and let $\delta_{t,r}(\mathbb{Z}^n)$ be the minimal density of a (t,r) broadcasting set $\mathcal{T}\subset\mathbb{Z}^n$.

Question 6.6.1. *Is there a relationship between $\delta_{t,r}(\mathbb{Z}^n)$ and $\delta_{t-1,r-2}(\mathbb{Z}^n)$ for some t, and r?*

In complete parallel to Propositions 6.3.1 and 6.3.2, we have that $\delta_{t+k,1+2k}(\mathbb{Z}^n)\le\delta_{t,1}(\mathbb{Z}^n)$ and $\delta_{t+k,2+2k}(\mathbb{Z}^n)\le\delta_{t,2}(\mathbb{Z}^n)$ by an analogous proof. Note that in dimensions $n>2$, unlike

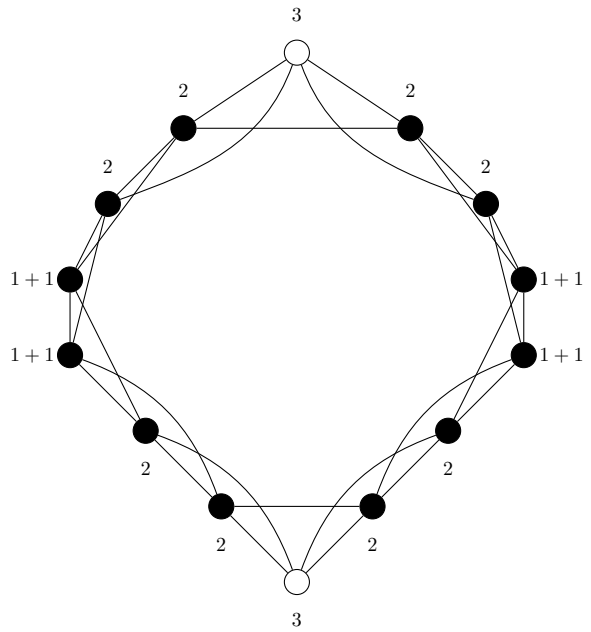

Fig. 6.9 A $(3,2)$ broadcast on $C_{14}^{(2)}$. The unfilled vertices denote the positions of towers.

in dimensions one and two, l_1-balls of constant radius do not partition $\mathbb{Z}^n$, so even the exact value of $\delta_{t,1}(\mathbb{Z}^n)$ can be hard to obtain. In 3 dimensions this amounts to efficiently covering space with octahedrons.

In another direction, the continuous generalization of Conjecture 6.3.3 might provide a lot of insight. We say a discrete set of towers $\mathcal{T} \subset \mathbb{R}^2$ is (t,r) broadcasting if all points in points $v \in \mathbb{R}^2$ satisfy that

$$\sum_{T \in \mathcal{T}} \max\{t - d(T,v), 0\} \geq r$$

where d is some metric on $\mathbb{R}^2$. It is natural to look for the minimal density $\limsup_{x\to\infty} \frac{\mathrm{card}(\mathcal{T}\cap[-x,x]^2)}{4x^2}$ of a (t,r)-broadcasting set $\mathcal{T}$, where $\mathrm{card}(X)$ is the cardinality of the set X. For d the Euclidean ℓ_2 distance, this problem is intimately related to efficient sphere packing. To stay as close to the discrete context as possible, let d be the ℓ_1 distance. Let $\delta'_{t,r}\left(\mathbb{R}^2\right)$ be the smallest density of a (t,r) broadcasting set in $\mathbb{R}^2$. Note that in this definition being (t,r) broadcasting and being $\left(1,\frac{r}{t}\right)$ broadcasting are equivalent. In fact for $\alpha > 0$, $\delta'_{t,r}\left(\mathbb{R}^2\right) = \delta'_{\alpha t,\alpha r}\left(\mathbb{R}^2\right)$. Analogously to Conjecture 6.3.3, we believe

Conjecture 6.6.2. *There exists $\gamma_0 > 0$ such that for all $\gamma \leq \gamma_0$,*

$$\delta'_{1,\gamma}\left(\mathbb{R}^2\right) = \lim_{\varepsilon\to 0} \delta'_{1-\gamma/2,\varepsilon}\left(\mathbb{R}^2\right) = \frac{1}{4\left(1-\frac{\gamma}{2}\right)^2}$$

The right equality follows from the fact that the set $\mathcal{T}_\varepsilon = \{ma + nb : m,n \in \mathbb{Z}\}$ with $a = \left(1 - \frac{\gamma}{2} - \varepsilon, 1 - \frac{\gamma}{2} - \varepsilon\right)$ and $b = \left(1 - \frac{\gamma}{2} - \varepsilon, \frac{\gamma}{2} + \varepsilon - 1\right)$ is $(1 - \gamma/2, \varepsilon)$ broadcasting and has asymptotic density $\frac{1}{4(1-\frac{\gamma}{2}-\varepsilon)^2}$, which tends to $\frac{1}{4\left(1-\frac{\gamma}{2}\right)^2}$ as $\varepsilon \to 0$. Moreover, the set $\mathcal{T}_0$ immediately shows $\delta'_{1,\gamma} \leq \frac{1}{4\left(1-\frac{\gamma}{2}\right)^2}$.

When viewing the discrete setting as an approximation of the continuous setting, Conjecture 6.6.2 would indicate that the minimal t_0 as a function of r in Conjecture 6.3.3 should be at most linear, i.e. $t_0 = O(r)$.

Chapter 7

The Eternal Game Chromatic Number of Random Graphs

The work in this chapter was done jointly with Vojtěch Dvořák, and Rebekah Herrman. It has been published in the European Journal of Combinatorics.[26]

7.1 Introduction

The *vertex colouring game* was introduced by Brams [39] in 1981; it was later rediscovered by Bodlaender [8]. In this game, two players, Alice and Bob, take turns choosing uncoloured vertices from a graph, G, and assigning a colour from a predefined set $\{1,\dots,k\}$, such that the resulting partial colouring of G is proper. Bob wins, if at some stage, he or Alice chooses a vertex that cannot be properly coloured. Alice wins if each chosen vertex can be properly coloured. The game chromatic number $\chi_g(G)$ is the smallest integer k such that if there are k colours, Alice has a winning strategy in the vertex colouring game. This number is well defined, as Alice can win if the number of colours is at least the number of vertices. The vertex colouring game has been well studied [1, 2, 17, 19, 24, 27, 43, 70, 73, 74, 76]. In particular, Bohman, Frieze and Sudakov [9] studied the game chromatic number of random graphs $G_{n,p}$ and found that with high probability, $(1-\varepsilon)\frac{n}{\log(pn)} \le \chi_g(G_{n,p}) \le (2+\varepsilon)\frac{n}{\log(pn)}$, where all logarithms have base $\frac{1}{1-p}$. Keusch and Steger [57] improved the result to $\chi_g(G_{n,p}) = (1+o(1))\frac{n}{\log(pn)}$ with high probability. A classic result of Bollobás showed that $\chi(G_{n,p}) = (\frac{1}{2}+o(1))\frac{n}{\log(n)}$ with high probability for p constant, so $\chi_g(G_{n,p}) = (2+o(1))\chi(G_{n,p})$ with high probability for p constant. Both of the results require lower bounds on p decaying with n slowly. Frieze, Haber and Lavrov [37] studied the game on sparse random graphs, finding that for $p = d/n$, $\chi(G_{n,p}) = \Theta\left(\frac{d}{\ln(d)}\right)$, where $d \le n^{-1/4}$ is at least a large constant.

This vertex colouring game requires Alice and Bob to colour the vertices once, attaching no value to the colouring that is produced at the end of the round. In a variant of the game called the *eternal vertex colouring game* recently introduced by Klostermeyer and Mendoza [60], the focus is shifted by continuing the game after a colouring is produced.

Fix a graph G. In the eternal vertex colouring game, there is a fixed set of colours $\{1,\dots,k\}$. The game consists of rounds, such that in each round, every vertex is coloured exactly once. The first round proceeds precisely the same way as the vertex colouring game, with Alice taking the first turn. During all further rounds, players keep choosing vertices alternately. After choosing a vertex, the player assigns a colour to the vertex which is distinct from its current colour such that the resulting colouring is proper. Each vertex retains its colour between rounds until it is recoloured. Bob wins if at any point the chosen vertex does not have a legal recolouring, while Alice wins if the game is continued indefinitely. The eternal game chromatic number $\chi_g^\infty(G)$ is the smallest number k such that Alice has a winning strategy. Note that if $k \geq \Delta(G)+2$, there will always be a colour available for every vertex, so $\chi_g^\infty(G)$ is well-defined.

As Alice and Bob alternate their turns, the parity of the order of the graph determines whose turn it is at the beginning of the second round. For even order, Alice always has the first move, while for odd order Bob gets to play first in all even rounds.

This game has not been well studied, but Klostermeyer and Mendoza [60] obtained some basic results pertaining to paths, cycles, and balanced bipartite graphs.

In this chapter, we determine $\chi_g^\infty(G_{n,p})$ for n odd with general p and for n even with $p \in \left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\right\}$. Moreover, we provide an upper bound for n even and $p \notin \left\{\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\right\}$.

Theorem 7.1.1. *For all $p \in (0,1)$ constant, for odd n, with high probability,*

$$\chi_g^\infty(G_{n,p}) = (1+o(1))\frac{pn}{2}.$$

Theorem 7.1.2. *For all $p \in (0,1)$ constant, for even n, with high probability,*

$$\chi_g^\infty(G_{n,p}) \leq (1+o(1))\frac{pn}{2}.$$

Moreover, when $p = \frac{1}{l}$ for some $l \in \mathbb{N} \setminus \{0,1\}$,

$$\chi_g^\infty(G_{n,p}) = (1+o(1))\frac{pn}{2}.$$

The difference in the even and odd cases is because when n is odd, Bob moves first in the second round. Also, note that we made no efforts to optimize $o(n)$ terms. For the unresolved case when n is even and $p \notin \{\frac{1}{2},\frac{1}{3},\dots\}$, we conjecture the following.

Conjecture 7.1.3. $\forall p \in (0,1) \setminus \{\frac{1}{2}, \frac{1}{3}, \dots\}, \exists \varepsilon > 0$ *such that for even n with high probability,*

$$\chi_g^\infty(G_{n,p}) \leq (1-\varepsilon)\frac{pn}{2}.$$

The structure of the chapter is as follows. In Section 2, we prove the upper bound for $\chi_g^\infty(G_{n,p})$. In this proof, we make no distinction between odd and even values of n. In Section 3, we prove the corresponding lower bound for odd n. In Section 4, we prove a generalization of the result in Section 3, which we then use in Section 5 to get the lower bound for the case $p = 1/l$ for some $l \in \mathbb{N}$. Along the way, we use various structural results about the random graph $G_{n,p}$. As the proofs of these are usually quite easy but technical, we collect all of them in Section 7.7. Finally in Section 7.8, we provide answer to one of the questions posed in the paper of Klostermeyer and Mendoza.

Throughout this chapter, we say that a result holds in $G_{n,p}$ with high probability (*whp*) if the probability that it holds tends to 1 as $n \to \infty$.

7.2 Upper bound

In this section, we show the following proposition.

Proposition 7.2.1. *For any fixed* $p, \varepsilon \in (0,1)$*, whp* $\chi_g^\infty(G_{n,p}) \leq (\frac{p}{2} + \varepsilon)n$.

To prove this, we formulate a deterministic strategy for Alice and prove that whp this strategy enables her to prevent Bob from winning when the game is played with $(\frac{p}{2} + \varepsilon)n$ colours.

The biggest danger facing Alice is that at the end of some round, Bob would manage to introduce all the colours in the neighbourhood of at least one vertex. He could then win by choosing one of those vertices at the beginning of the next round. Thus, her strategy should be to ensure that at any point of each round, she has coloured roughly as many vertices in the neighbourhood of any single vertex as Bob has, and she should use few colours on them. If, at some point during a round, a vertex has many colours in its neighbourhood compared to other vertices, Alice might be forced to colour it so Bob cannot win by choosing it later that same round. Fortunately for Alice, the number of times she is forced to colour a vertex with many different coloured neighbours is so few that she can still follow her strategy.

Sample a random $G \in G_{n,p}$. Consider the following four properties of G.

(i) Every vertex of G has degree at most $(p + \frac{\varepsilon}{100})n$

(ii) There exists a constant $K = K(\varepsilon, p)$ such that G does not contain sets $A, B, S \subset V(G)$, with $|A \cap B| = 0$, $|A| = |B| \geq \frac{\varepsilon}{200}n$, $|S| = K$, such that every $v \in S$ is adjacent to at least $\frac{\varepsilon}{200}n$ more vertices in B than in A.

(iii) There exist constants $\beta = \beta(\varepsilon, p)$, $\frac{\varepsilon}{100} > \beta > 0$ and $C = C(\varepsilon, p)$ such that the following holds: for any colouring of G by $(\frac{p}{2} + \varepsilon)n$ colours, the number of vertices that have all but at most $2\beta n$ colours in their neighbourhood is at most $C \log n$.

(iv) There exist constants $\gamma = \gamma(\varepsilon, p) > 0$ and $D = D(\varepsilon, p)$ such that, in any colouring of G by $(\frac{p}{2} + \varepsilon)n$ colours, the number of vertices that have all but at most γn of the colours $1, 2, ..., \frac{\varepsilon}{200}n$ in their neighbourhood is at most $D \log n$.

We prove in Section 7.7 that each of these holds whp for $G \in G_{n,p}$. For the remainder of this paragraph, we will assume i through iv hold for the graph G. Note that as we are assuming finitely many properties, each of which holds whp, then whp all of them hold simultaneously.

For a particular round of the game, let A_i and B_i denote the sets of vertices played by Alice and Bob respectively in the first i moves of that round. We shall define the vertices that threaten Alice's chance of winning as dangerous.

Definition 7.2.2. *For a fixed round of play, let D_i denote the set of* dangerous vertices at i moves*, denoted D_i. A vertex v belongs to D_i if for some number of moves $j \leq i$, Bob has played at least $\frac{\varepsilon}{100}n$ times more in the neighbourhood of v than Alice has, i.e. $|\Gamma(v) \cap B_j| \geq |\Gamma(v) \cap A_j| + \frac{\varepsilon}{100}n$.*

We additionally define vertices that Alice can colour to maintain some symmetry in the game as follows.

Definition 7.2.3. *Let S be a finite subset of vertices of a graph, G. For a vertex $v \notin S$, we say that a vertex w* mirrors *v* with respect to *S if $w \notin S$ and for any $t \in S$, G contains an edge vt if and only if it contains an edge wt.*

Let $C = \{1, 2, ..., (\frac{p}{2} + \varepsilon)n\}$ be the set of colours used in the game. We call a colour *large* if it is at least $\frac{\varepsilon}{200}n$, and *small* otherwise.

Alice will use the following strategy at the ith move of a round: from the list below, she chooses the first point that applies, and colours the corresponding vertex with smallest colour available to that vertex. If there are multiple vertices for the same point on the list, she chooses one of these arbitrarily.

1. If there is a vertex v that misses less than βn colours in its neighborhood and such that v has not yet been coloured in the current round, she chooses v.

2. If Bob played a vertex w for his previous move, w is not dangerous, and there is a vertex v which mirrors w with respect to D_i, she chooses v.

3. She chooses an arbitrary vertex.

We shall prove that because i, ii, iii, and iv hold, selecting vertices in order of priority will ensure that Bob can never win the eternal vertex colouring game for sufficiently large n. To show Bob cannot win, we prove the following lemma.

Lemma 7.2.4. *For n sufficiently large, at the beginning of the kth round of play for $k \geq 2$, the following two conditions hold:*

• *During the $(k-1)^{st}$ round, there was no vertex v such that the number of times Bob played in neighbourhood of v was more than $\frac{\varepsilon}{50}n$ greater than the number of times Alice played in the neighbourhood of v.*

• *Alice used no more than $\frac{\varepsilon}{100}n$ colours in the $(k-1)^{st}$ round.*

Then, by playing according to the above described strategy, Alice ensures the following:

• *Bob does not win during the kth round*

• *During the kth round, there is no vertex v such that the number of times Bob played in the neighbourhood of v is more than $\frac{\varepsilon}{50}n$ greater than the number of times Alice played in the neighbourhood of v.*

• *Alice uses no more than $\frac{\varepsilon}{100}n$ colours in the kth round.*

Note that Lemma 7.2.4 implies that, if Alice plays according to the strategy described above, Bob can never win the eternal graph colouring game.

Proof of Lemma 7.2.4. The first step is to establish that at the beginning of the round, each vertex misses more than $2\beta n$ colours in its neighbourhood, so that there is no immediate threat to Alice. In the first round, this is immediate, as no colour is used yet. When $k \geq 2$, Alice uses at most $\frac{\varepsilon}{100}n$ colours in the neighbourhood of any vertex v. Bob played at most $\frac{\varepsilon}{50}n$ more moves in the neighbourhood of v than Alice did, so by property i, Bob played at most $(\frac{p}{2} + \frac{\varepsilon}{200} + \frac{\varepsilon}{100})n$ colours in the neighborhood of v. Hence, at least $\frac{39\varepsilon}{40}n > 2\beta n$ colours are missing from the neighbourhood of v.

Now, if some vertex misses at most βn colours at any point during the round, then in particular at least one of the times $\beta n, 2\beta n, ..., \lfloor \beta^{-1} \rfloor \beta n$, this vertex missed at most $2\beta n$ colours. By property iii, we conclude there are at most $C\beta^{-1}\log n = o(n)$ vertices that, at some point in this round, have missed at most βn colours. Recall that colouring vertices that miss at most βn colours is of the highest priority in Alice' strategy. If Bob were to create a vertex seeing all colours that was not yet played in this round, Alice must have spend the previous βn moves playing in other vertices missing at most βn colours in their

neighbourhoods. However, this contradicts the fact that there were at most $C\beta^{-1}\log n < \beta n$ such vertices. Hence, Alice can colour all such vertices in time.

Next, note that Alice uses $o(n)$ (and in fact only constantly many) moves that are arbitrary. If Alice colours an arbitrary vertex, then either Bob played a dangerous vertex for his previous move or she cannot mirror Bob on the current set of dangerous vertices. By property ii, there are at most K vertices declared dangerous during the round, so Bob can play in a dangerous vertex no more than K times. On the other hand, consider if Bob did not play a dangerous vertex and Alice cannot mirror his move on D, the set of dangerous vertices. If we partition the rest of the graph into $2^{|D|} \leq 2^K$ classes according to which vertices of D they are adjacent to, Bob just played the last vertex from one of these classes. Hence, Alice can play at most $K + 2^K = O(1)$ arbitrary moves in any particular round.

Following this strategy, Alice also ensures that Bob will play in the neighbourhood of any vertex at most $\frac{\varepsilon}{50}n$ more than Alice does. Indeed, once Bob has played $\frac{\varepsilon}{100}n$ more colours in the neighbourhood of any vertex, v, it is declared dangerous. She then plays in neighbourhood of v whenever Bob does, except $o(n)$ times when she plays a move of type 1 or an arbitrary vertex.

Finally, note that Alice uses large colours only if the vertex she wants to colour is adjacent to all the small colours. If at any point during the round a vertex is adjacent to all the small colours, then at least one of the times $\gamma n, 2\gamma n, .., \lfloor \gamma^{-1} \rfloor \gamma n$, this vertex must have been missing at most γn small colours. By property iv, there could have only been at most $D\gamma^{-1}\log n = o(n)$ vertices that were adjacent to all small colours at some point in this round. Hence, as she can colour all other vertices with small colours, Alice uses at most $\frac{\varepsilon}{200}n + D\gamma^{-1}\log n < \frac{\varepsilon}{100}n$ colours during the round. □

7.3 Lower bound for odd n

In this section, we prove the lower bound for the eternal game chromatic number on a graph with an odd number of vertices.

Proposition 7.3.1. *For any $p, \varepsilon \in (0,1)$ fixed, whp $\chi_g^\infty(G_{2m+1,p}) \geq (\frac{p}{2} - \varepsilon)(2m+1)$.*

For convenience, we shall let $n = 2m+1$. Sample $G \in G_{n,p}$ and fix any vertex v of the graph G.

We will show that whp, Bob can ensure that in the first round, all $(\frac{p}{2} - \varepsilon)n$ colours are in the closed neighbourhood of v in G. Bob then wins in the first move of the second round, by choosing v.

Bob can introduce all colours in $\Gamma(v)$ by playing in $\Gamma(v)$ whenever Alice does, thus ensuring he plays in at least roughly half of the vertices in $\Gamma(v)$ while introducing a new

colour every time. Some set of vertices X outside $\Gamma(v)$ might at some point be adjacent to all unplayed vertices of $\Gamma(v)$. If Alice were to play some colour not appearing in $\Gamma(v)$ in all these vertices, this colour could no longer be introduced to $\Gamma(v)$. Fortunately for Bob, the number of such sets will be very limited, and thus Bob can take care of them in time.

Consider following two properties of a random graph.

(i) Whp, every vertex of $G_{n,p}$ has degree at least $(p - \frac{\varepsilon}{100})n$.

(ii) For all $\varepsilon > 0$, and $p \in (0,1)$, there exist positive constants $\delta = \delta(\varepsilon, p), K = K(\varepsilon, p)$, such that in $G_{n,p}$

- Whp, for all distinct vertices u, v, w, we have $|\Gamma(u) \setminus (\Gamma(v) \cup \Gamma(w))| > \delta n$.
- Whp, for any set S of size $\frac{\varepsilon}{100}n$ in the graph, there exist at most K mutually disjoint pairs of vertices $\{a_i, b_i\}$ such that at most δn vertices of S are not in $\Gamma(a_i) \cup \Gamma(b_i)$

Henceforth, we assume our graph $G \in G_{n,p}$ has both properties, and fix $\delta = \delta(\varepsilon, p)$ and $K = K(\varepsilon, p)$. In Section 7.7, we show that indeed whp G has these properties.

Note that if at some stage there exists a colour c that does not appear in $\Gamma(v)$ and all vertices not yet played in $\Gamma(v)$ are adjacent to a vertex of colour c, then c will not appear in $\Gamma(v)$, which is contrary to Bob's goal.

We introduce the ideas of a *double block* and being α *away from becoming a double block* in order to describe a strategy Bob should take to achieve his goal of filling the neighbourhood of a vertex with all colours.

Definition 7.3.2. *A pair of vertices a and b is called a* double block *if at some stage in the round, all uncoloured vertices in* $\Gamma(v)$ *are in the neighbourhood of either a or b and neither a nor b (if coloured) is coloured with a colour appearing in* $\Gamma(v)$.

Definition 7.3.3. *A pair of vertices a and b is said to be* α *away from becoming a double block, if all but at most* α *of the uncoloured vertices in* $\Gamma(v)$ *are in the neighbourhood of either a or b and neither a or b (if coloured) is coloured with a colour appearing in* $\Gamma(v)$.

Bob will play according to the following strategy. From the list below, he picks the highest point that applies.

1. If there exists a colour that appears at least twice outside of $\Gamma(v)$ but not in $\Gamma(v)$, then Bob plays it in $\Gamma(v)$ if it is a valid colouring.

2. If at least $10K$ colours appear nowhere in the graph and there are at least $\frac{\varepsilon}{100}n$ uncoloured vertices in the neighbourhood of v, then Bob does the blocking moves in the

chronological order they were called for, if legally possible. Blocking moves are called for if a pair $\{a,b\}$ of vertices disjoint from $\Gamma(v)$ is δn away from becoming a double block, and at most one out of a,b was played. Blocking moves consist of the following steps. If vertex a is uncoloured, we first colour it with a colour not appearing in G, say c_a. If it is already coloured, we skip this step. Next:

- If Alice plays in vertex b with a colour appearing in $\Gamma(v)$, introduce c_a in $\Gamma(v)$, at which point the entire process ends.
- If Alice plays in vertex b with a colour c_b not appearing in $\Gamma(v)$, play c_b in $\Gamma(v)$, and finish by playing c_a in $\Gamma(v)$ on the next move unless Alice already introduced c_a there. If she introduced c_a, do nothing more.
- If Alice introduces c_a in $\Gamma(v)$, colour vertex b with a colour c_b not yet appearing in the graph, and introduce c_b to $\Gamma(v)$ in the next step, unless Alice introduces it right before.
- In any other case, play c_a in $\Gamma(v)$ and ensure b is coloured with some c_b that is also in $\Gamma(v)$ in the following two turns, unless Alice makes some of these moves for us, in that case alter the moves in the same way as we did for a.

3. If legally possible, Bob introduces colours appearing once outside of $\Gamma(v)$ into $\Gamma(v)$.
4. If legally possible, Bob introduces new colours to $\Gamma(v)$.
5. Otherwise Bob does anything.

Note that (2) might involve up to four moves for any pair close to becoming a double block. If in between these four moves a situation as in (1) arises, situation (1) takes priority.

Claim 7.3.4. *There are no more than* $4K$ *moves of type (2) used in the first round.*

Proof. Let U denote the set of vertices that are uncoloured in $\Gamma(v)$ when the last blocking moves were played. By the definition of type (2) moves, $|U| \geq \frac{\varepsilon}{100}n$, so property ii gives the result. □

Let T be the first move after which precisely $10K-1$ colours are missing in G during the first round. We collect the following observations about T:

• T exists and at T, at least εn vertices in $\Gamma(v)$ are uncoloured

We shall show that $T \leq (p-2\varepsilon)n$. After Bob's first $(\frac{p}{2}-\varepsilon)n$ moves, Alice has also played $(\frac{p}{2}-\varepsilon)n$ moves. As $|\Gamma(v)| \geq (p-\frac{\varepsilon}{100})n$, at this stage at least $\frac{199}{100}\varepsilon n$ vertices in $\Gamma(v)$ are uncoloured. Bob spent at most $4K$ moves playing according to (2), and when he did not,

he always introduced a new colour in $\Gamma(v)$, if he legally could. Note that if there were still colours missing from G and there were uncoloured vertices in $\Gamma(v)$, moves of type (4) were always legal. Hence, unless all colours appear in the graph, Bob played at least $(\frac{p}{2}-\varepsilon)n-4K$ colours in $\Gamma(v)$ and hence in G.

• At T, at most $18K$ colours are missing in $\Gamma(v)$

Between two consecutive moves of Bob before T, the number of colours appearing outside of $\Gamma(v)$ but not in $\Gamma(v)$ can increase by at most 2. In fact it only increases if Bob makes a move of type (2). Hence, there are at most $8K$ such colours at time T, and the result follows.

• At T, Bob has played at most $8K$ (1) moves.

Let c be the number of colours appearing outside $\Gamma(v)$ and not in $\Gamma(v)$. Note that between two consecutive moves of Bob up to time T, c increases only if Bob plays a (2) move, in which case it increases by at most 2. On the other hand, note that Bob only plays (1) moves directly after Alice plays a colour already appearing outside $\Gamma(v)$. Hence, c decreases whenever Bob plays a (1) move. By 7.3.4, there were at most $4K$ (2) moves, so there were at most $8K$ (1) moves.

• No pair of vertices is closer than $\frac{\delta}{2}n$ to becoming a double-block at any point up to T

We know from ii that at the beginning of the game, no pair is closer than δn to becoming a double block. Up to time T, whenever a pair gets closer than δn to becoming a double block, no (3)-(5) moves are played until this pair is eliminated. However, there are at most $12K$ (1) and (2) moves played until T. Hence, no pair can be closer than $\delta n-24K \geq \frac{\delta}{2}n$ to becoming a double-block up to T.

• Every colour that does not appear in $\Gamma(v)$ at T appears at most once in G

Note that the only time before T that there is a colour appearing twice outside $\Gamma(v)$, but not inside, is directly after Alice has played this colour. In response, Bob immediately plays that same colour in $\Gamma(v)$, which is possible as no pair of vertices is a double block. Hence, if Bob made the last move before T, the statement follows. If the last move before T was by Alice, she must have introduced a new colour into the graph by the definition of T, which again implies the statement.

Next we claim:

Claim 7.3.5. *In the* $18K$ *moves of Bob following* T*, he will introduce all colours in* $\Gamma(v)$.

Proof. Moves of type (2) are no longer played after T by their definition. In the next $36K$ moves, $18K$ of which are made by Bob and the $18K$ by Alice, no complete double-block can be created, as all are at least $\frac{\delta}{2}n > 36K$ moves away. Since at T at least εn vertices of $\Gamma(v)$ are still uncoloured, during the next $36K < \varepsilon n$ moves, there are ample uncoloured vertices in

$\Gamma(v)$. Hence, Bob can and will introduce a new colour to $\Gamma(v)$ every move until all colours appear there. □

Thus we see that Bob will ensure that all colours appear in $\Gamma(v)$ during the first round and he will win in the first move of the second round by picking v.

7.4 Generalization of the lower bound for odd n

Proposition 7.3.1 doesn't trivially extend to even n, as it is not enough for Bob to let all colours appear in the neighbourhood of a fixed vertex because Alice could use her first move in the second round to remove one of the colours from this neighbourhood.

If Bob can manage to play all colours in the neighbourhood of two vertices, with no colour appearing uniquely in the intersection of the neighbourhoods, then Alice can not. This limits how Bob can colour the intersection of two neighbourhoods of vertices. By increasing the number of vertices that simultaneously see all colours, the size of this intersection can be reduced. Our aim is to show that if $p = 1/k$ for some $k \in \mathbb{N}$, then for any fixed l, Bob can choose l vertices and play all of $(p/2 - p^l/2 - \varepsilon)n$ colours in the neighbourhoods of these vertices. The $(p^l/2)n$ correction term comes from the intersection of the neighbourhoods of these l vertices. As l can be taken arbitrarily large, this shows $\chi_g^\infty(G_n, p) = (1 + o(1))\frac{pn}{2}$ for even n and $p = 1/k$.

In this section, we prove a generalization of Proposition 7.3.1, showing that if $V(G)$ is partitioned into constantly many parts and each of the parts is assigned a set of colours of size roughly half the size of the part, Bob can guarantee all these colours to appear in the parts by the end of the first round. This generalizes the notion that Bob could achieve this in the single set $\Gamma(v)$. In the next section, we will fix some set of vertices X of constant size and induce partition $\{A_I : I \subset [l]\}$, where $A_I = \{v \in V : \Gamma(v) \cap X = I\}$. We show in Lemma 7.5.2, that for the special case $p = \frac{1}{k}$, there exists an appropriate way of assigning colours to the A_I's such that each vertex in X will see all colours after the first round.

Proposition 7.4.1. *$\forall \varepsilon, \eta, \gamma > 0, l \in \mathbb{N}$, and $p \in (0,1)$, if $X_i \subset V$ $(i \in [l])$ are disjoint sets of vertices of the graph $G_{n,p}$ chosen before sampling the edges with $|X_i| \geq \gamma n$ and $Y_i \subset [(p/2 - \varepsilon)n]$ with $|Y_i| \leq \frac{(1-\eta)|X_i|}{2}$, then whp Bob can guarantee that at the the end of round all of the colours in Y_i appear in X_i.*

To prove Proposition 7.4.1 we will use the following generalization of the structural result in Item ii.

Lemma 7.4.2. *For all $m \in \mathbb{N}$, $\alpha > 0$ and $p \in (0,1)$, there exist positive constants $\delta = \delta(\alpha, p, m), K = K(\alpha, p, m)$, such that*

- *For any set S of size at least αn in the graph, whp there exist no m-sets of vertices $\{a_1, \dots a_m\}$ such that at most δn vertices of S are not in $\bigcup_j N(a_j)$*
- *Whp, for any set S of size at least αn in the graph, there exist at most K mutually disjoint m-sets of vertices $\{a_{i,1}, a_{i,2}, \dots a_{i,m}\}$ such that at most δn vertices of S are not in $\bigcup_j N(a_{i,j})$*

Refer to the Lemma 7.7.4 in Section 7.7 for the proof of Lemma 7.4.2.

In order to prove Propositon 7.4.1, we will define the concepts of an *end stage*, *m-block* and α *away from becoming an m-block*.

Definition 7.4.3. *Let T_i be the first move after which at most $10K$ of the colours in Y_i are missing from X_i, if this exists. After T_i, say X_i is in its* end stage.

Definition 7.4.4. *Given disjoint sets $X_i \subset V$, at some stage of the round we say a set $\{a_1, \dots, a_m\}$ is an m-*block *if for some i, X_i is not in its end stage, every uncoloured vertex in X_i is in the neighbourhood of some a_j, and no a_j is coloured in some colour also appearing in X_i.*

Definition 7.4.5. *Given disjoint sets $X_i \subset V$, at some stage of the round we say a set $\{a_1, \dots, a_m\}$ is α* away from becoming a *m-*block *if for some i, X_i is not in its end stage, all but α of the uncoloured vertices in X_i is in the neighbourhood of some a_j, and no a_j is coloured in some colour also appearing in X_i.*

Proof of Proposition 7.4.1. Let $C_l = \frac{12l}{\eta\gamma} + 4$ and let $\delta = \delta(\eta/4, p, 100C_l l)$ and $K = K(\eta/4, p, 100C_l l)$ as in Lemma 7.4.2. Bob will play the move of the highest priority that he legally can according to the following list:

1. If for some $q \in [l]$, some colour c appears $C_l q$ times in the graph, but it is missing from more than $l - q$ of the X_i's for which $c \in Y_i$, Bob plays it in any of X_i's where it does not yet appear.
2. If for some i, X_i is in its end stage, Bob plays the missing colours into it, copying the colour Alice played if it was missing.
3. If there is $100C_l l$ block closer than δn moves away from becoming an m-block and at least $\eta/4n$ vertices in the corresponding X_i are uncoloured, Bob kills it. By killing it, we mean the following sequence of moves. Colour the first vertex of our $100C_l l$-set by some colour that appears less than $C_l q$ times in the graph and is missing from at most $l - q$ of the relevant X_i's. Then make sure in the next moves that this colour also appears in all of its designated X_i's. Repeat this procedure for all vertices of our $100C_l l$-set.

4. If for some $q \in [l]$, some colour appears $C_l q$ times in the graph, but it is still missing from more than $l-q+1$ of X_i's, Bob plays it in any of X_i's where it does not yet appear.

5. Bob plays any colour in Y_i not yet used in X_i to that X_i, if possible in the same X_i as Alice played in the previous move.

6. Bob plays anything anywhere.

Note that (3) might involve up to $100C_l l(l+1)$ moves. If in between these moves a situation as in (1) or (2) arises, those are resolved first.

Claim 7.4.6. *Let $C = 100C_l l^2(l+1)$. There were no more than CK (3) moves called for.*

Proof. Let $U \subset X_i$ denote the set of vertices that are still uncoloured in X_i when the last blocking moves were called for, for this X_i. Lemma 7.4.2 says X_i called for at most $100C_l l(l+1)K$ (3) moves. Hence, in total at most $100C_l l^2(l+1)K = CK$ (3) moves are called for. □

Claim 7.4.7. *There were no more than $\frac{2l}{C_l}n$ moves of types (1),(2),(3) and (4) during the first round of the game.*

Proof. Note that at most $\frac{n}{C_l}$ colours appear at least C_l times in the graph. Moreover, these colours prompt a (1) or (4) move at most l times. Finally, there are at most $10Kl$ (2) moves. Hence, there are at most $\frac{l}{C_l}n + 10Kl + CK \leq \frac{2l}{C_l}n$, given $n \geq \frac{C_l K}{l}(10l + C)$. □

We collect the following observations about T_i:

• T_i exists and, at T_i, at least $\frac{\eta}{4}n$ vertices in X_i are still uncoloured

At the end of round one there were $|X_i|$ moves in X_i. Moreover, by Claim 7.4.7 there were at most $\frac{2l}{C_l}n$ (1)-(4) moves. After Bob's first $|Y_i| + \frac{2l}{C_l}n$ in X_i, he has played at most $\frac{2l}{C_l}n$ (1)-(4) moves. He also played in X_i after every move of Alice in that set, except the times when he played (1)-(4) moves. Thus, at least $\eta|X_i| - \frac{6l}{C_l}n \geq \frac{\eta\gamma}{2}n$ of the vertices in X_i are uncoloured. As $C_l \geq \frac{12l}{\eta\gamma}$, this gives the result.

• Let $C' = C + 10l$. At T_i, Bob has played at most $C'K$ (1) moves.

For a colour j, let q_j be the number such that colour j is missing from $l - q_j$ of its designated sets. Let r_j be the number of times j appears in the graph. If $r_j - q_j C_l > 0$, then Bob is forced to play a (1) move. If $r_j - (q_j - 1)C_l > 0$, then this induces a (4) move. Let $D = \sum_j \max\{r_j - (q_j - 1)C_l, 0\}$. Note that if $D > 0$, then Bob must play a (1),(2),(3) or (4) move. If D increases between consecutive moves of Bob, he must have played a (2) or (3) move. Moreover, D increases by at most 2 in that case. On the other hand, if Bob is prompted to play a (1) move, D decreases by at least $C_l - 1 > 2$. Hence, there are at most as many (1) moves as there are (2) and (3) moves, i.e. at most $CK + 10Kl$ (1) moves.

• No pair of vertices is closer than $\frac{\delta}{2}n$ to becoming a $100C_l l$ block at any point up to T_i

By Lemma 7.4.2, at the beginning of the game no $100C_l l$-set of vertices is closer than δn to becoming a $100C_l l$ block. Whenever, up to time T_i, a $100C_l l$-set gets closer than δn to becoming a $100C_l l$ block, no (4)-(6) moves are played until this pair is eliminated. However, there are at most $(2C+10l)K$ (1) and (3) moves played until T_i. Hence, no $100C_l l$-set gets closer than $\delta n-(2C+10l)K \geq \frac{\delta}{2}n$ to becoming a $100C_l l$ block up to T_i.

• Every designated colour that does not appear in X_i at T_i appears at most $C_l l+2$ times in our graph

By the definition of (1) moves, some colour c can never appear $C_l l+2$ times in our graph, yet not appear in some of X_i's with $c \in Y_i$.

Next we claim:

Claim 7.4.8. *In the* $10Kl+C'K$ *moves of Bob following* T_i*, he will introduce all colours in* X_i*.*

Proof. Note that while there are still colours missing from X_i in its end stage, Bob only plays (1) and (2) moves, both of which copy the colour Alice played. Hence, the colours missing from X_i can be played at most $2l$ times before being played into X_i. At that stage, the colour is played at most $C_l l+2+2l < 100C_l l$ times and no $100C_l l$-set is closer than $\frac{\delta}{2}n$ to becoming a $100C_l l$ block, so no $100C_l l$ block will be formed in the endstage of X_i. Hence, we can still play this colour in X_i. As we can introduce all the missing colours and we play at most $C'K$ (1) moves, we need at most $10Kl+C'K$ moves to introduce them all. □

Thus, since Bob can introduce all colours into X_i during the end game, the proof of Proposition 7.4.1 is complete. □

Having proven Proposition 7.4.1, we are ready to look at even n.

7.5 Even n

In this section, we shall prove that for particular values of p, we can achieve the same lower bound for even n as for odd n.

Proposition 7.5.1. *Let* $p=1/k$ *for some* $k \in \mathbb{N}$*, and* $\varepsilon > 0$*. Then whp* $\chi_g^{\infty}(G_{2m,p}) \geq (p/2-\varepsilon)2m$*.*

For convenience write $n=2m$. For given $p,\varepsilon > 0$, fix $l \in \mathbb{N}$, such that $p^l < \varepsilon/100$.

Lemma 7.5.2. *Let $X \subset V(G)$ be a set of l vertices and $p = 1/k$ for some $k \in \mathbb{N}$. There exists $\eta > 0$, and a function $f : \mathcal{P}(X) \mapsto \mathcal{P}([(p/2 - p^l/2 - \varepsilon)n])$, assigning to every subset $X' \subsetneq X$, $p^{|X'|}(1-p)^{l-|X'|}(1-\eta)\frac{n}{2}$ colours, such that $\bigcup_{X':x\in X'\subsetneq X} f(X') = [(p/2 - p^l/2 - \varepsilon)n]$ for every $x \in X$.*

To prove this lemma we will use the following auxiliary lemma.

Let $\mathcal{B}(X)$ be the set of all partitions of the set X.

Lemma 7.5.3. *Consider any $k \in \mathbb{N}$. Let $p = 1/k$ and $|X| = l$, then there exists $g : \mathcal{B}(X) \to [0,1]$, such that for all $\emptyset \neq A \subsetneq X$;*

$$\sum_{T:A\in T\in\mathcal{B}(X)} g(T) = p^{|A|}(1-p)^{l-|A|}$$

Proof. Define g as

$$g(T) = \begin{cases} k^{-l}\frac{(k-1)!}{(k-|T|)!} & \text{if } |T| \leq k \\ 0 & \text{else} \end{cases}$$

Fix $A \subset X$ and evaluate $\sum_{T:A\in T\in\mathcal{B}(X)} g(T)$. Consider ordered partitions of $X \setminus A$ into $k-1$ potentially empty sets. Each of these contributes exactly k^{-l} to this sum.

To see this, consider a particular ordered partition of $X \setminus A$ into $k-1$ sets, $m-1$ of which are non-empty. This corresponds to a partition T of X into m parts, which has weight $g(T) = k^{-l}\frac{(k-1)!}{(k-m)!}$. Note that a given partition T of $X \setminus A$ into $m-1$ parts gives rise to $\frac{(k-1)!}{(k-m)!}$ ordered partitions into $k-1$ (potentially empty) sets. Hence, every ordered partition of $X \setminus A$ into $k-1$ potentially empty sets contributes weight exactly k^{-l} to the sum.

Noting that there are exactly $(k-1)^{l-|A|}$ ordered partitions of $X \setminus A$ into $k-1$ potentially empty sets, we can evaluate the sum as

$$\sum_{T:A\in T\in\mathcal{B}(X)} g(T) = (k-1)^{l-|A|}k^{-l} = \left(\frac{1}{k}\right)^{|A|}\left(\frac{k-1}{k}\right)^{l-|A|} = p^{|A|}(1-p)^{l-|A|}$$

□

Proof of Lemma 7.5.2. Let $g : \mathcal{B}(X) \to [0,1]$ as in Lemma 7.5.3 and set $\mathcal{B}'(X) = \mathcal{B}(X) \setminus \{\{X\}\}$. Note that $\sum_{T\in\mathcal{B}'(X)} g(T) = p(1-p^{l-1})$. Consider any linear order on $\mathcal{B}'(X)$ and let

$$f' : \mathcal{B}'(X) \to \mathcal{P}([(p/2 - p^l/2 - \varepsilon)n]), T \mapsto \left\{ \left\lfloor \sum_{T'<T} g(T') \right\rfloor \frac{(1-\eta)n}{2} + 1, \ldots, \left\lfloor \sum_{T'\leq T} g(T') \right\rfloor \frac{(1-\eta)n}{2} \right\}$$

where η is such that $\left\lfloor \sum_{T\in\mathcal{B}'(X)} g(T) \right\rfloor \frac{(1-\eta)n}{2} = (p/2 - p^l/2 - \varepsilon)n$. Let

$$f : \mathcal{P}(X) \to \mathcal{P}([(p/2 - p^l/2 - \varepsilon)n]); X' \mapsto \bigcup_{T:X'\in T} f'(T)$$

Hence;

$$\begin{aligned}\bigcup_{X':x\in X'\subsetneq X} f(X') &= \bigcup_{X':x\in X'\subsetneq X} \bigcup_{T:X'\in T} f'(T)\\ &= \bigcup_{T\in\mathcal{B}'(X)} f'(T)\end{aligned}$$

□

Proof of Proposition 7.5.1. Fix $X \subset V$ with $|X| = l$. Sample all edges incident to X. For $I \subsetneq X$, let $X_I = \{v \in V \setminus X : \Gamma(v) \cap X = I\}$. Note that whp $|X_I| \geq p^{|I|}(1-p)^{l-|I|}(1-\eta/10)n$ for any $\eta > 0$. Use Lemma 7.5.2 to find $Y_I = f(I)$, such that $|Y_I| \leq \frac{(1-\eta/10)|X_I|}{2}$. Now sample all the other edges in the graph. By Proposition 7.4.1, whp Bob can guarantee that at the end of round one the colours in Y_I appears in X_I. By construction of Y_I, all vertices in X will see all colours in $\bigcup_{I:\emptyset\neq I\subsetneq X} X_I$. Hence, regardless of Alice' first move in the second round, Bob can choose a vertex that sees all colours in his first move in the second round. Indeed, if Alice recolours a vertex outside of $\bigcup_{I:\emptyset\neq I\subsetneq X} X_I$, then any vertex in X sees all colours. On the other hand, if Alice recolours a vertex in $\bigcup_{I:\emptyset\neq I\subsetneq X} X_I$, then for at least one vertex $x \in X$, the colouring of neighbourhood $N(x)$ is not affected. Thus, the proposition follows. □

7.6 Conjecture 7.1.3

Note that the condition $p = 1/k$ is essential in Proposition 7.5.1. In Conjecture 7.1.3 we conjecture that this points to a fundamental structural difference, i.e. that for all $p \in (0,1) \setminus \{\frac{1}{2}, \frac{1}{3}, \dots\}, \exists \varepsilon > 0$ such that whp $\chi_g^\infty(G_{n,p}) \leq (1-\varepsilon)\frac{pn}{2}$.

Note that, if $p \notin \{\frac{1}{2}, \frac{1}{3}, \dots\}$, then there exists a k, so that for any k vertices $v_1, \dots, v_k$, it is impossible to assign colours Y_I to $X_I = \{v : \Gamma(v) \cap \{v_1, \dots, v_k\} = I\}$, such that $|Y_I| \leq (1/2 + \eta)|X_I|$ and $\bigcup_{I:i\in I\subset[k]} Y_I = [(p/2 - \varepsilon)n]$ for every $i \in [k]$. Crucially, Lemma 7.5.3 fails to hold. Hence, given that Alice can play in roughly half the vertices in $\Gamma(v)$ for all $v \in V$, as suggested by Lemma 7.2.4, at most two vertices at the end of every round can see all colours, say u, and v. In particular, some colours will appear only once in $\Gamma(u) \cup \Gamma(v)$ viz in $\Gamma(u) \cap \Gamma(v)$, so Alice can recolour one of these in the first move of the second round. Hence,

we cannot expect Bob to win at the beginning of round two. The crucial question remains whether Bob can reintroduce this colour into the neighbourhood of u or v.

We believe the answer to be *No* because of the following heuristic. As all but two vertices will see at most $(p/2-c_p)n$ colours at the end of a round, for some definite constant $c_p > 0$ depending on p, Alice will be able to determine which two vertices these are $\Omega(n)$ moves before the end of the round. At that stage she can first make sure all vertices in $\Gamma(u)\cup\Gamma(v)$ have been played, after which $\Omega(n)$ moves still remain in the round. Next she chooses a colour c appearing uniquely in $\Gamma(u)\cup\Gamma(v)$ viz in $\Gamma(u)\cap\Gamma(v)$. Finally, she colours $\Theta(\log(n))$ of the remaining vertices with c in order to guarantee that all points in $\Gamma(u)\cup\Gamma(v)$, except the unique vertex of colour c, are adjacent to some vertex of colour c. Because there are $\Omega(n)$ vertices remaining she'll have ample time and choice to achieve this.

7.7 Proofs of structural results in the random graph

In this section, we provide proofs of various structural results about $G_{n,p}$ that were used in earlier proofs. Some of them will be shown in a more general form. One of our main tools will be the following well-known form of Hoeffding's Inequality.

Lemma 7.7.1. *For any $\varepsilon > 0, n \in \mathbb{N}$, and $p \in (0,1)$, $\mathbb{P}(Bin(n,p) \geq (p+\varepsilon)n) \leq \exp(-2\varepsilon^2 n)$ and $\mathbb{P}(Bin(n,p) \leq (p-\varepsilon)n) \leq \exp(-2\varepsilon^2 n)$.*

Note that Hoeffding's inequality implies i from Section 2 and i from Section 3.

To prove iii and iv from Section 2, we first prove the following result.

Lemma 7.7.2. *For all $\alpha > 0$, $p \in (0,1)$, there exist constants $K = K(\alpha,p), \beta = \beta(\alpha,p) > 0$ such that whp the following holds. For any colouring of $G_{n,p}$ with αn colours, the number of vertices that have all but at most βn colours in their neighbourhood is at most $K \log n$.*

Proof. Let $q = 1-(1-p)^{2/\alpha}$, $\beta = \frac{\alpha(1-q)}{8}$, and $K = \frac{4}{(1-q)^2}$.

Note that in any colouring of $G_{n,p}$ by αn colours, we have $\frac{\alpha}{2}n$ colours appearing at most $\frac{2}{\alpha}$ times each.

Assume there exists a set S of $K\log n$ vertices missing at most βn colours each. For n satisfying $K\log n < \frac{\alpha}{4}n$, there exists a set C of $\frac{\alpha}{4}n$ colours appearing at most $\frac{2}{\alpha}$ times each such that no vertex in S has any colour from C. In particular, there must be mutually disjoint sets of vertices $S, T_1, \ldots, T_{\frac{\alpha}{4}n}$, such that $|T_i| \leq \frac{2}{\alpha}$ for each i, $|S| = K\log n$ and each vertex in S is joined to at least $(\frac{\alpha}{4}-\beta)n$ sets T_i in our graph.

Now, we consider the probability that such structure exists in $G_{n,p}$. For n sufficiently large, we find there are $\sum_{i=1}^{2/\alpha}\binom{n}{i} \leq 2\binom{n}{2/\alpha}$ ways of choosing each of the sets T_i. So for such large n, we have at most

$$\binom{n}{K\log n}\left(2\binom{n}{2/\alpha}\right)^{\frac{\alpha n}{4}} \leq n^{K\log n}2^{\frac{\alpha n}{4}}\left(\frac{e\alpha n}{2}\right)^{\frac{\alpha n}{4}}$$
$$= \exp\left(K(\log n)^2 + \frac{\alpha n}{4}\log n + \frac{\alpha}{4}n\log(e\alpha)\right)$$

ways to choose sets $S, T_1, ..., T_{\alpha n/4}$. Now for any such fixed choice, the probability that these sets satisfy the conditions is at most

$$\mathbb{P}\left(\mathrm{Bin}\left(\frac{\alpha n}{4}, q\right) \geq \left(q + \frac{1-q}{2}\right)\frac{\alpha n}{4}\right)^{K\log n} \leq \exp\left(-2\left(\frac{(1-q)}{2}\right)^2\frac{\alpha n}{4}K\log n\right)$$

The union bound then gives that the probability of finding appropriate $S, T_1, \ldots, T_{\alpha n/4}$

$$\exp\left(K(\log n)^2 + \frac{\alpha n}{4}\log n + \frac{\alpha}{4}n\log(e\alpha) - 2\left(\frac{(1-q)}{2}\right)^2\frac{\alpha n}{4}K\log n\right)$$
$$= \exp\left(K(\log n)^2 + \frac{\alpha}{4}n\log(e\alpha) + \left(1 - K\frac{(1-q)^2}{2}\right)\frac{\alpha n}{4}\log n\right)$$
$$= \exp\left(K(\log n)^2 + \frac{\alpha}{4}n\log(e\alpha) - \frac{\alpha n}{4}\log n\right) = o(1)$$

The result follows. □

To conclude iii, simply plug in $\alpha = (\frac{p}{2} + \varepsilon)$, and note that if some value of $\beta > 0$ works, then any smaller one does too, so we can insist on β being not too large.

To conclude iv, plug in $\alpha = \frac{\varepsilon}{200}$ and note that presence of other colours only helps us, as the result would still hold even if all the other vertices were also coloured in $\frac{\varepsilon}{200}n$ small colours.

The following implies ii from section 2.

Lemma 7.7.3. *Fix any ε, and δ greater than 0. Assume $K \in \mathbb{N}$ is fixed, such that $K > \frac{6\varepsilon}{\delta^2}$. Whp, $\forall A, B \subset V(G)$ are disjoint subsets with $|A| = |B| \geq \varepsilon n$, then there are less than K vertices adjacent to at least δn more vertices in B than in A.*

Proof. Let $n \geq \frac{2K}{\varepsilon}$, and assume for a contradiction that there exist A, B, as stated in the lemma such that there are at least K vertices adjacent to at least δn more vertices in B than in A. Let S be a collection of K such vertices. Let $A' = A \setminus S$ and $B' = B \setminus S$. Note that $e(A', S) \leq e(A, S)$ and $e(B', S) \geq e(B, S) - K^2$, so that

$$e(B', S) - e(A', S) \geq e(B, S) - e(A, S) - K^2 \geq K\delta n - K^2 \geq K\delta n/2.$$

Hence, either

$$e(B',S) \geq (|B'| \cdot |S|p) + \delta Kn/8,$$

or

$$e(A',S) \leq (|B'| \cdot |S|p) - 3\delta Kn/8 \leq (|A'| \cdot |S|p) - \delta Kn/8.$$

The probability of the former (the latter follows analogously) is given by;

$$\begin{aligned}\mathbb{P}\left(\mathrm{Bin}(|B'| \cdot |S|, p) \geq (|B'|Kp) + \delta Kn/8\right) &\leq \exp\left(-2\left(\frac{\delta n}{8|B'|}\right)^2 |B'|K\right) \\ &\leq \exp\left(-\frac{\delta^2 nK}{2\varepsilon}\right)\end{aligned}$$

We can choose sets A, B, and S in at most

$$\binom{n}{|A|}^2 \binom{n}{K} \leq 2^{3n}$$

ways. Thus, the probability that any such sets A, B and S exist is at most

$$2\exp\left(-\frac{\delta^2}{2\varepsilon}nK + 3n\log 2\right) \to 0$$

provided $K > \frac{6\varepsilon}{\delta^2} > \frac{6\varepsilon}{\delta^2}\log 2$. □

The proof of ii from section 3 follows from the fact that for $\delta(p)$ sufficiently small and positive, whp there exists no three vertices u, v, w such that the number of vertices in the neighbourhood of u, but not in the neighbourhood of v or w is at most δn by Hoeffding's Inequality. ii follows directly from the following setting $m = 2$.

Lemma 7.4.2 follows in the same manner, this time using the particular m we need.

Lemma 7.7.4. *Fix $m \in \mathbb{N}$, $\gamma > 0$, $p \in (0,1)$. Then for any $K > \frac{4}{\gamma(1-p)^{2m}}$, whp $G_{n,p}$ does not contain any collection of sets $S, T_1, ..., T_K$ such that $T_1, ..., T_K$ are all mutually disjoint, $|S| \geq \gamma n$, $|T_1| = ... = |T_K| = m$ and for every T_i, all but at most $\frac{(1-p)^m}{4}\gamma n$ vertices of S are adjacent to at least one vertex in T_i.*

Proof. Provided $n > \frac{2Km}{\gamma}$, we can find $S' \subset S$ such that $|S'| = \frac{\gamma n}{2}$ and S' is disjoint from all of $T_1, ..., T_K$. For any such fixed $S', T_1, ..., T_K$, by Hoeffding's Inequality, the probability that for each T_i, all but at most $\frac{(1-p)^m}{4}\gamma n$ vertices of S' are adjacent to at least one vertex in T_i is at most

$$\P\left(\mathrm{Bin}\left(\frac{\gamma}{2}n,(1-p)^m\right)\leq\frac{(1-p)^m}{2}\gamma n-\frac{(1-p)^m}{4}\gamma n\right)^K\leq\exp\left(-2\frac{(1-p)^{2m}}{4}\frac{\gamma}{2}nK\right)$$
$$=\exp\left(-\frac{(1-p)^{2m}\gamma Kn}{4}.\right)$$

There are at most

$$\binom{n}{m}^K\binom{n}{\gamma n/2}\leq n^{mK}2^n$$
$$=\exp\left(mK\log n+n\log(2)\right)$$

ways to choose such sets $S',T_1,\ldots,T_K$. So, as long as $K>\frac{4}{\gamma(1-p)^{2m}}>\frac{4}{\gamma(1-p)^{2m}}\log(2)$, the result follows. □

7.8 Answer to a question of Klostermeyer and Mendoza

We conclude the chapter by answering a question posed by Klostermeyer and Mendoza in their original paper.

They define other variants of the eternal chromatic game on graph. One of them is greedy colouring game, where Bob must colour whatever vertex he chooses with the smallest colour possible. Let $\chi_g^{\infty 2}(G)$ be the smallest number k such that when this game is played with k colours on G, Alice is guaranteed to win. Further, they consider the variant of game when not only Bob, but also Alice, must use the smallest colour available for each vertex she chooses, and define $\chi_g^{\infty 3}(G)$ to be eternal number of the game played with these rules. Note that clearly $\chi_g^{\infty 2}(G)\leq\chi_g^{\infty 3}(G)$ since Alice can, if she wishes so, choose the smallest colour for each vertex she chooses in any variant of the game and analogously $\chi_g^{\infty 2}(G)\leq\chi_g^{\infty}(G)$.

Klostermeyer and Mendoza pose the following question about these new variants of the game.

Question 7.8.1. *Let G be a graph with subgraph or induced subgraph H. Is it necessarily true that $\chi_g^{\infty 2}(G)\geq\chi_g^{\infty 2}(H)$? Is it necessarily true that $\chi_g^{\infty 3}(G)\geq\chi_g^{\infty 3}(H)$?*

Indeed, it is not true. Consider the following example.

Proposition 7.8.2. *For $n\geq 2$, $\chi_g^{\infty 3}(K_{1,2n+1})=3$ and $\chi_g^{\infty 2}(K_{1,2n})\geq 4$.*

Proof. For $\chi_g^{\infty 3}(K_{1,2n+1})=3$ note that Alice starts every round as the number of vertices is even. Every round she will first play in the central vertex which will become the unique

element from [3] not yet appearing in the graph. All the other vertices will now become the former colour of the central vertex.

For $\chi_g^{\infty 3}(K_{1,2n}) \geq 4$. assume for a contradiction 3 colours suffice and note that Bob begins the second round. Let x for the central vertex. Then x is either adjacent to two different colours or $N(x)$ is monochromatic. In the former case, Bob plays in x and finds that there is no colour available, a contradiction.
In the latter case, Bob plays in $N(x)$, bringing the number of colours in $N(x)$ to two. Hence, Alice cannot play in x. She can also not bring down the number of colours in $N(x)$ as it contains at least three vertices. Thence, when Bob gets to play his second move in the second round, and plays x, he finds no colours available, again a contradiction. □

Note that $H = K_{1,2n}$ is an (induced) subgraph of $G = K_{1,2n+1}$, and

$$\chi_g^{\infty 2}(G) \leq \chi_g^{\infty 3}(G) \leq \chi_g^{\infty 2}(H) \leq \chi_g^{\infty 3}(H)$$

This answers all the subquestions in the negative.

Finally, note that while there is no clear relationship between $\chi_g^{\infty 3}(G)$ and $\chi_g^{\infty}(G)$ for general graphs G, in our definition of strategy of Alice in section 2, we let her always play the smallest colour available, and so in particular we have $\chi_g^{\infty 3}(G_{n,p}) \leq (\frac{p}{2} + o(1))n$ whp.

Chapter 8

Capture times in the Bridge-burning Cops and Robbers game

The work in this chapter was done jointly with Rebekah Herrman and Stephen G.Z. Smith. [47]

8.1 Introduction

The main aim of this chapter will be to prove Theorem 1.6.1, establishing the asymptotic maximal capture time in the bridge-burning cops and robbers game.

In Section 8.2, we present some preliminary results on catching times in the bridge-burning game and in Section 8.3 we prove Theorem 1.6.1.

The asymptotics in this chapter are with respect to the number of vertices n, assuming fixed burning-bridge cop number, unless explicitly stated to be otherwise.

8.2 Catching Times

In this chapter, we will show that the graph G on n vertices with cop number $k \geq 3$ which maximizes the capture time satisfies

$$C\,\frac{n^{k+2}}{k^{k+2}} \leq \max\{capt_b(G) : c_b(G) = k, v(G) = n\} \leq C'\,\frac{(2n)^{k+2}}{k!}$$

for some universal constants $C, C' > 0$.

First, we show that the capture time of $K_{n,n}$ is $\Theta(n^2)$. Our proof significantly simplifies the proof given in [59, Theorem 5.2] to demonstrate that there are graphs with $c(G) = 1$ and

$capt_b(G) = \Omega(n^2)$. In order to prove the result, we need the following slight strengthening of a theorem from Kinnersley and Peterson [59, Theorem 2.2].

Lemma 8.2.1. *If $\exists\, X \subset V(G)$, such that $G[X]$ is a clique and $X \cup \Gamma(X) = V(G)$, then $c_b(G) = 1$ and $capt_b(G) = O(n^2)$, where $\Gamma(X)$ is the neighbourhood of X.*

Proof. Place the cop on any vertex in X. Subsequently, always move the cop to a vertex in X adjacent to the position of the robber. Note that the robber can never move onto a vertex in X and, thus, can never remove an edge incident to X. Hence, $X \cup \Gamma(X) = V(G)$ remains constant throughout the game. After each round, the cop is adjacent to the robber, so the robber must move in every round. Given that the robber removes one edge in every round, eventually he must move into X, as all the other possible edges have been removed. As there are $O(n^2)$ edges, this must happen within $O(n^2)$ moves. □

This lemma provides the cop number and an upper bound in the following proposition.

Proposition 8.2.2. *$K_{n,n}$ has capture time $\Theta(n^2)$*

Proof. As any two adjacent vertices in $K_{n,n}$ satisfy the conditions in Lemma 8.2.1, we find that $c_b(K_{n,n}) = 1$ and $capt_b(G) = O(n^2)$.

On the other hand, we consider the following strategy for the robber to delay capture. First, we find an Euler cycle of $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$ (or $K_{\lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor - 1}$ if $\lfloor \frac{n}{2} \rfloor$ is odd). Next, we traverse the following route through $K_{n,n}$; to each vertex in $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$, assign a distinct pair of vertices in $K_{n,n}$ such that the pairs of vertices from the same part of $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$ are in the same part of $K_{n,n}$. Now, the robber follows the Euler cycle through $K_{n,n}$ in the sense that every time he is forced to move, he goes to an element in the corresponding pair which is available. As the cop is only able to occupy one vertex of a given pair, there is no way for the cop to obstruct the robber's path. This route has length $\Omega(n^2)$. □

In fact Lemma 8.2.1 implies the following result for random graphs $G_{n,p}$.

Proposition 8.2.3. *Consider $G = G_{n,p}$ with p constant. Then whp $c_b(G) = 1$ and $capt_b(G) = O(n^2)$.*

Proof. By Lemma 8.2.1, it suffices to show that G contains a dominating clique with high probability. This follows from a second moment argument included in Lemma 8.4.1 in the Appendix. □

For general graphs, we find the following generalization of a result from [59, Theorem 5.1] which showed this proposition in the case $c_b(G) = 1$.

Proposition 8.2.4. *Let G be a graph on n vertices, then $capt_b(G) \leq \frac{(2n)^{c_b(G)+2}}{c_b(G)!}$.*

Proof. Note that as the robber removes an edge with every move, the robber can make at most $e(G) \leq \binom{n}{2}$ moves before getting caught. Between two moves of the robber, the cops move around. Without the robber moving, there is no point in the cops returning twice to the exact same configuration. As there are at most $\binom{n+c_b(G)-1}{c_b(G)} \leq \frac{(2n)^{c_b(G)}}{c_b(G)!}$ configurations of the cops on the vertices, it can take at most $\frac{(2n)^{c_b(G)+2}}{c_b(G)!}$ moves before the robber is caught. □

The remainder of the chapter is dedicated to proving Theorem 1.6.1.

8.3 Proof of Theorem 1.6.1

We first prove the result for $k = 3$ and then extend the construction to larger k. We claim the following graph G has $c_b(G) = 3$ and $capt_b(G) = \Theta(n^5)$.

Let the vertex set of G be the following union of sets;

$$\begin{aligned} V(G) = \{p_i, q_i : i \in [3n]\} &\cup \{x_1, x_2\} \cup \{d_x, h_x\} \cup X \cup Y \\ &\cup \{d_{X,1}, d_{X,2}, d_{Y,1}, d_{Y,2}, h_{X,1}, h_{X,2}, h_{Y,1}, h_{Y,2}\} \\ &\cup \{a_i : i \in [3n]\} \cup \{d_a, h_a\} \cup \{d_{a,v,1}, d_{a,2}, h_{a,1}, h_{a,2}\} \\ &\cup \{b_i : i \in [3n]\} \cup \{d_{b,1}, d_{b,2}, h_{b,1}, h_{b,2}\} \\ &\cup \{d_{a,i,1}, d_{a,i,2}, h_{a,i,1}, h_{a,i,2} : i \in [3]\} \cup \{d_{b,i,1}, d_{b,i,2}, h_{b,i,1}, h_{b,i,2} : i \in [3]\} \end{aligned}$$

with $|X| = |Y| = 3n$. Let the edge set of G be the following union of sets;

$$
\begin{aligned}
E(G) = \{&p_ip_{i+1}, q_iq_{i+1} : i \in [3n-1]\} \cup \{p_1x_1, q_1x_1, p_nx_2, q_nx_2\} \\
&\cup \{p_id_x, q_id_x, x_1d_x, x_2d_x : i \in [3n]\} \\
&\cup \{x_1v : v \in X\} \cup \{x_2v : v \in Y\} \cup \{uv : u \in X, v \in Y\} \\
&\cup \{p_id, q_id, x_1d, x_2d : i \in [3n], d \in \{d_{X,1}, d_{X,2}, d_{Y,1}, d_{Y,2}\}\} \\
&\cup \{vd_{X,1}, vd_{X,2} : v \in X\} \cup \{vd_{Y,1}, vd_{Y,3} : v \in Y\} \\
&\cup \{a_ia_{i+1} : i \in [3n]\} \cup \{a_id_a : i \in [3n]\} \cup \{a_id_{a,1}, a_id_{a,2} : i \in [3n]\} \\
&\cup \{vd_{a,1}, vd_{a,2} : v \in X \cup Y\} \cup \{a_ix_1 : i \in [3n]\} \\
&\cup \{b_ib_{i+1} : i \in [3n]\} \cup \{b_id_b : i \in [3n]\} \cup \{b_id_{b,1}, b_id_{b,2} : i \in [3n]\} \\
&\cup \{vd_{b,1}, vd_{b,2} : v \in X \cup Y\} \cup \{b_ix_2 : i \in [3n]\} \\
&\cup \{vd_{a,i,1}, vd_{a,i,2} : v \in X \cup Y, i \in [3]\} \\
&\cup \{x_1d_{a,i,1}, x_1d_{a,i,2}, x_2d_{a,i,1}, x_2d_{a,i,2} : i \in [3]\} \\
&\cup \{p_jd_{a,i,1}, p_jd_{a,i,2}, q_jd_{a,i,1}, q_jd_{a,i,2} : j \not\equiv i \mod 3\} \\
&\cup \{a_{(i-1)n+j}d_{a,i,1}, a_{(i-1)n+j}d_{a,i,2} : j \in [n], i \in [3]\} \\
&\cup \{vd_{b,i,1}, vd_{b,i,2} : v \in X \cup Y, i \in [3]\} \cup \{a_jd_{b,i,1}, a_jd_{b,i,2} : j \not\equiv i \mod 3\} \\
&\cup \{b_{(i-1)n+j}d_{b,i,1}, b_{(i-1)n+j}d_{b,i,2} : j \in [n], i \in [3]\} \\
&\cup \{h_id_i : \text{all } i, \text{ such that } d_i \in V(G)\}
\end{aligned}
$$

For an illustration of G, see Figure 8.1.

In the remainder of the chapter we shall use the notation $\{a_i\}_i$ to denote the set of a_i's with i ranging over all possible values in the given context, so e.g. $\{p_i, q_i, x_i\}_i = \{p_i, q_i, x_j : i \in [3n], j = 1, 2\}$.

This graph G consists of three cycles, $\{a_i\}_i$, $\{b_i\}_i$ and $\{x_i, p_i, q_i\}_i$, a complete bipartite graph on the sets X and Y and a great number of *doors*, d_i's, and *holes*, h_i's. Holes are vertices with degree one and doors are their unique neighbours.

One of the cycles, viz $\{x_i, p_i, q_i\}_i$, contains two special vertices, x_1 and x_2, each of which is complete to one of the parts of the bipartite graph and to one of the cycles. In particular, we have $\{a_i\}_i \cup X \subset \Gamma(x_1)$ and $\{b_i\}_i \cup Y \subset \Gamma(x_2)$. In fact, these are the only edges between vertices that are not doors or holes.

The doors and holes restrict the freedom of the cops; if the robber manages to move to a door that is not adjacent to a cop, he will move to the corresponding hole in the next move, disconnecting himself from the rest of the graph and thus winning the game.

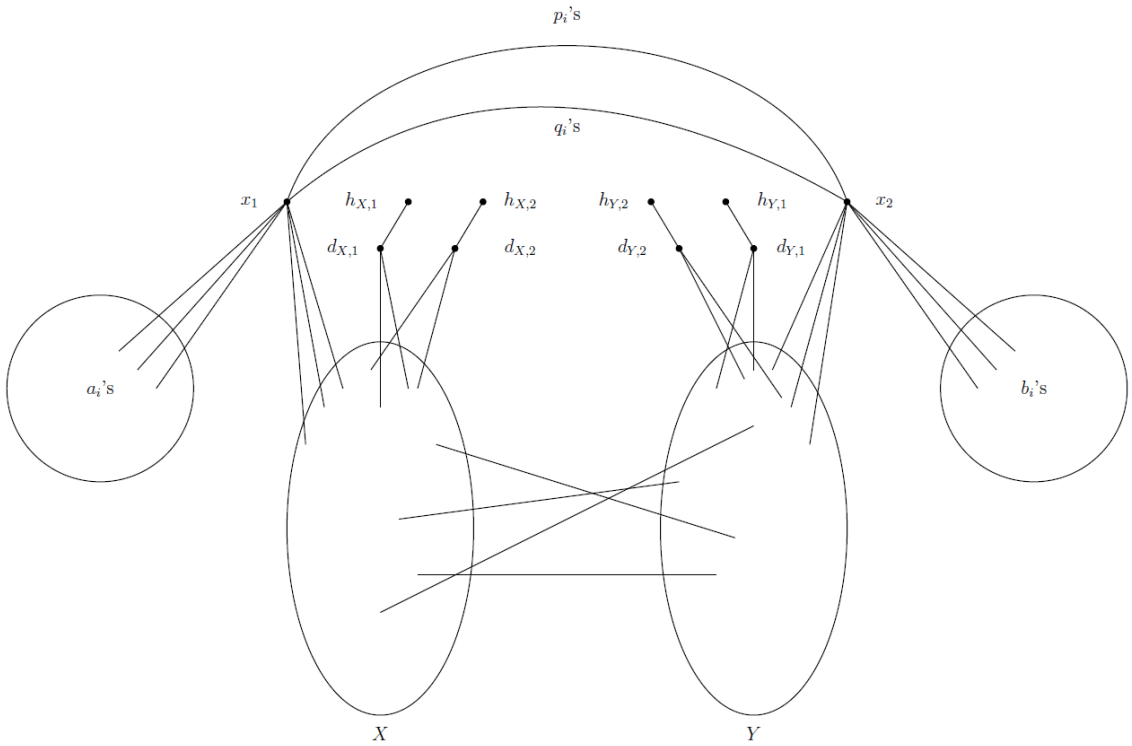

Fig. 8.1 The graph G described in the proof of Theorem 1.6.1. Most of the doors and holes are omitted though all the other vertices and edges are displayed.

Clearly, G has $O(n)$ vertices. We first aim to establish that $c_b(G) = 3$, starting with the lower bound.

Lemma 8.3.1. *$c_b(G) \geq 3$. Moreover, if $c_b(G) = 3$, then one cop starts in $\{a_i\}_i$, one cop starts in $\{b_i\}_i$ and one cop starts in $\{x_i, p_i, q_i\}_i$.*

Proof. To see that $c_b(G) \geq 3$, note that we initially need a cop next to, or on, every door. In particular, doors d_a, d_b and d_x. Since $\Gamma(d_a) = \{a_i\}_i \cup \{h_a\}$, $\Gamma(d_b) = \{b_i\}_i \cup \{h_b\}$ and $\Gamma(d_x) = \{x_i, p_i, q_i\}_i \cup \{h_x\}$, there is no vertex next to or on more than one of these, so we need at least three cops. If $c_b(G) = 3$, then evidently one cop must start in each of $\{d_a\} \cup \Gamma(d_a)$, $\{d_b\} \cup \Gamma(d_b)$, and $\{d_x\} \cup \Gamma(d_x)$. If one of these cops doesn't start in the corresponding cycle ($\{a_i\}_i$,$\{b_i\}_i$ and $\{x_i, p_i, q_i\}_i$ respectively), then no cop is adjacent to some other door ($d_{a,1}$, $d_{b,1}$ or d_{X_1} respectively). □

To see that three cops suffice to catch the robber, consider the following strategy for the cops. Start one cop on a_1, say Alex, one on b_1, say Blake, and one on x_1, say Charlie. We will refer to this starting position as the *standard position*. Note that these three vertices cover all the doors. Each of the cops will stay on their respective cycles unless the robber moves onto a vertex adjacent to them, in which case they catch him.

Lemma 8.3.2. *If the cops start in the standard position, then every cop can reach any vertex in their cycle, while guarding all doors at each of the intermediate steps.*
Moreover, if the robber starts and remains in $X \cup Y$ and the cops start in standard position and remain in their cycles always guarding all the doors, then it takes Charlie $\Omega(n^3)$ moves to get from x_1 to x_2 and from x_2 to x_1.

Proof. We will show that every cop can move to a neighbouring vertex in at most $O(n^2)$ steps. Recall that each cycle has diameter $O(n)$.

We first consider Blake's moves. Blake can move freely between the vertices in $\{b_{kn+j} : j \in [n]\}$ for any fixed $k \in \{0, 1, 2\}$, as each of these vertices has the same neighbourhood outside $\{b_i\}_i$. When changing k, Blake's neighbourhood in $\{d_{b,i,j}\}_{i,j}$ changes, so in order to keep guarding all the doors, Alex must move in parallel to cover Blake's old neighbourhood (cf. Figure 8.2). This, in turn, affects Alex's neighbourhood in $\{d_{a,i,j}\}_{i,j}$, which would then have to be compensated by Charlie. Thus, Blake can move anywhere in $\{b_i\}_i$ in $O(n)$ steps.

For Alex, the concerns are very similar. Two adjacent vertices in $\{a_i\}_i$ have different neighbourhoods in $\{d_{b,i,j}\}_{i,j}$, so for every consecutive step Alex takes, Blake has to take n steps. Hence, Alex can move anywhere in $O(n^2)$ steps. Moreover, to move to a vertex at distance $\Omega(n)$ in the cycle $\{a_i\}$ takes $\Omega(n^2)$ moves.

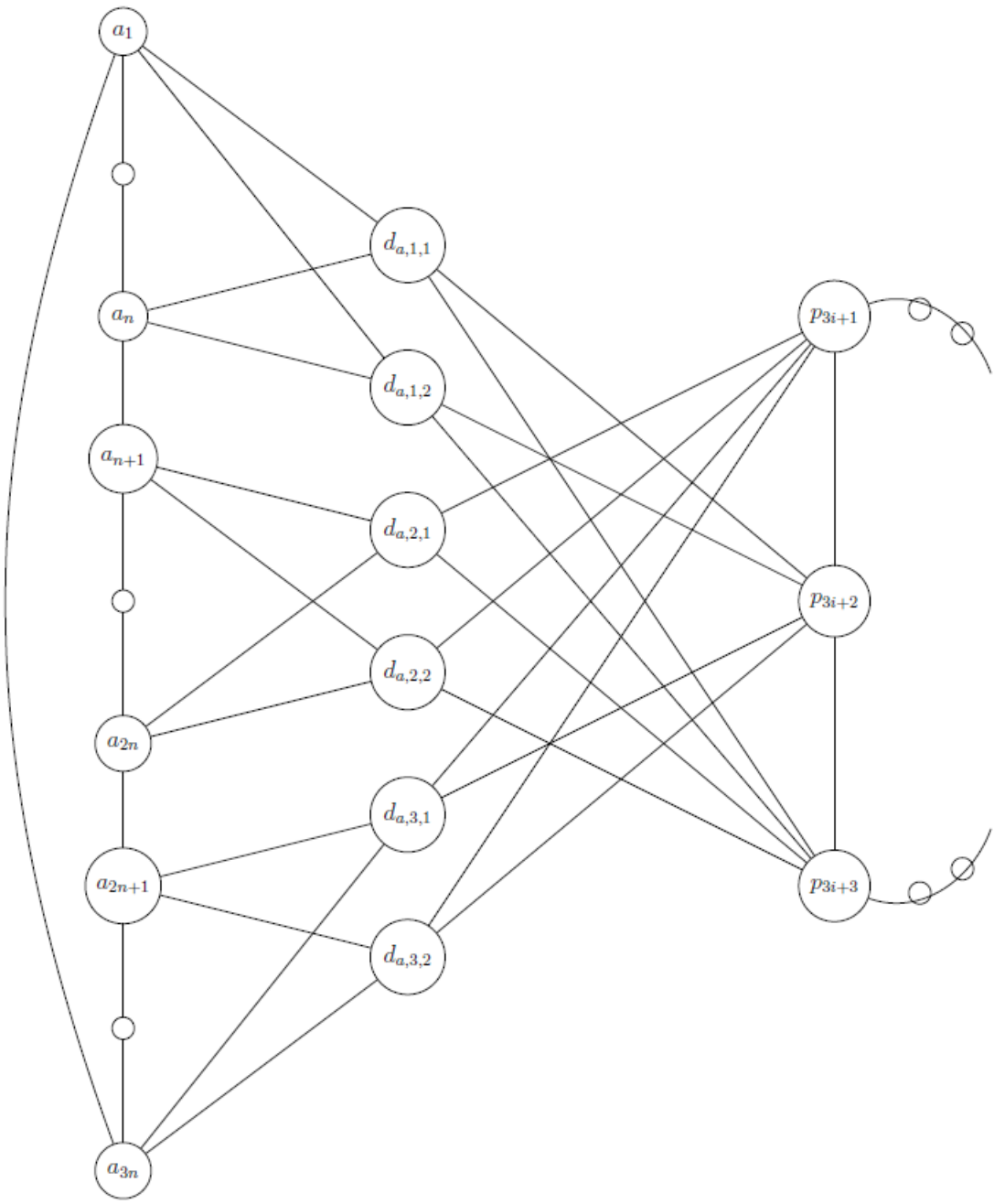

Fig. 8.2 The graph described in the proof of Lemma 8.3.2. The central vertices are doors, and the other vertices form cycles $\{a_i\}_i$ and $\{p_i, q_i, x_i\}$ patrolled by Alex and Charlie respectively which watch the doors.

Finally, every move by Charlie requires $O(n)$ steps of Alex, which in turn requires $O(n^2)$ steps by Blake. Hence, Charlie can move anywhere in $O(n^3)$. Moreover, to move from x_1 to x_2 and back takes $\Omega(n^3)$ moves. □

We need to exclude the case that the robber doesn't start in $X \cup Y$.

Lemma 8.3.3. *If the cops start in the standard position and the robber starts on any vertex that is not in $X \cup Y$, then the robber is caught in $O(n^3)$ moves.*

Proof. If the robber starts on a door or hole or x_1 or x_2, then the cops can immediately catch or corner him.

Alternatively, suppose the robber starts in one of the cycles. If the cops stay in their cycles, they can move along the cycles while guarding all the doors as shown in the previous lemmas. This implies that the robber cannot leave the cycle he starts in. It is easy to catch a robber on a cycle in $O(n)$ moves. Every step by the cop can require up to $O(n^2)$ moves by the other cops, so the cops need $O(n^3)$ moves to catch the robber. □

Now that we may assume the robber starts in $X \cup Y$, we are ready to show that Alex, Blake and Charlie will succeed in catching the robber.

Lemma 8.3.4. $c_b(G) = 3$

Proof. Lemma 8.3.1 shows we need at least three cops, so we only need provide a bound from above.

Consider the cops starting in the standard position. If the robber starts outside $X \cup Y$, then the cops can catch the robber according to Lemma 8.3.3. Hence, we may assume the robber starts in $X \cup Y$.

The cops will move in such a way that all doors are guarded at all times. Moreover, Alex and Blake will stay on $\{a_i\}_i$ and $\{b_i\}_i$ respectively at all times. Hence, if at any point the robber leaves the set $X \cup Y$, either to a door or to one of x_1, x_2, then the cops can immediately seize him. Hence, the robber has to stay inside $X \cup Y$.

Finally, to show that the cops can actually capture the robber, it suffices to show that they can force the robber to keep moving, as he can make at most $|X| \cdot |Y|$ moves staying on the vertices of $X \cup Y$. To this end, Charlie will move between x_1 and x_2, which by Lemma 8.3.2 is possible while ensuring the cops guard all the doors at every intermediate step. As x_1 is complete to X and x_2 is complete to Y, this forces the robber to keep moving. Hence, the cops will eventually capture the robber. □

To find the lower bound on the capture time, we need to be sure that the cops do not have a better strategy.

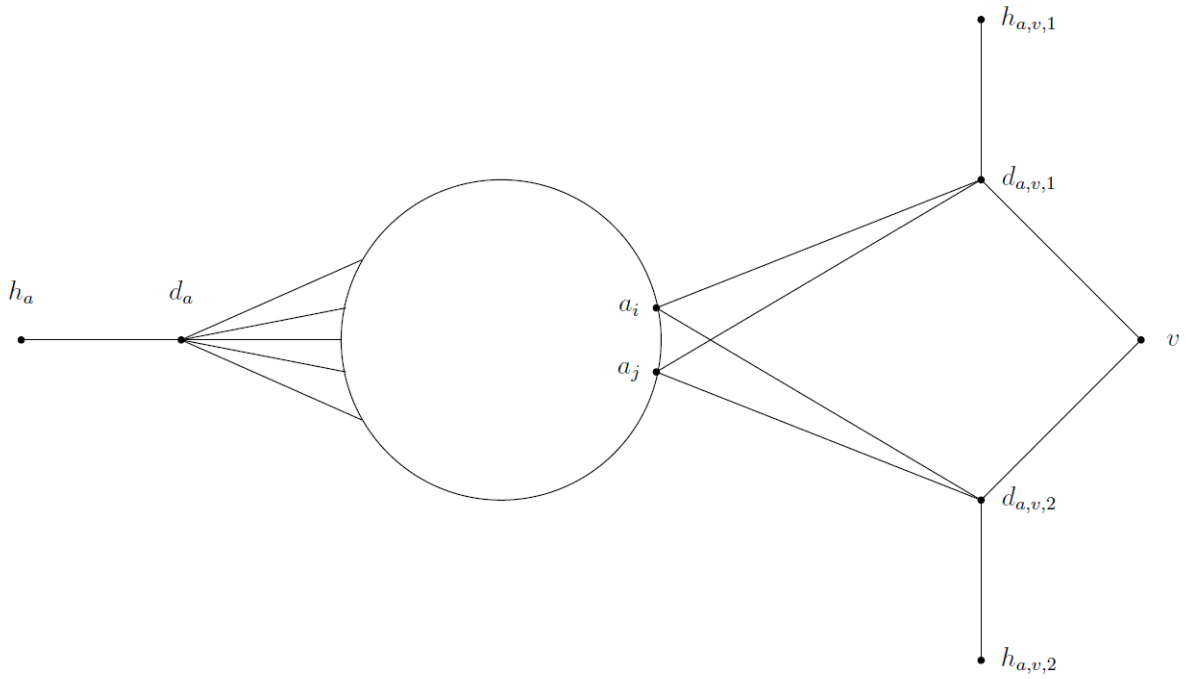

Fig. 8.3 The graph described in Lemma 8.3.5, where v is a vertex in $X \cup Y$.

Lemma 8.3.5. *If the cops start in the standard position and the robber starts in $X \cup Y$, then the cops need to stay on their respective cycles for as long as the robber stays in $X \cup Y$, unless they can capture the robber directly.*

Proof. For Charlie, let the robber be on $v \in X$, without loss of generality. If Charlie leaves the cycle still guarding $d_{X,1}$ and $d_{X,2}$, then Charlie must have moved into X. However, that would imply the cop was previously on x_1, so Charlie could have caught the robber immediately. Hence, Charlie cannot leave the cycle without allowing the robber to escape.

For Alex (resp. Blake), note that if the robber is on $v \in X \cup Y$, then leaving their cycles would mean leaving either $d_{a,v,1}$ or $d_{a,v,2}$ (resp. $d_{b,v,1}$ or $d_{b,v,2}$) unguarded, providing an escape route for the robber, as seen in Figure 8.3. □

Lemma 8.3.6. $capt_b(G) = \Omega(n^5)$

Proof. By Lemma 8.3.1, the cops must start in standard position or equivalent.

The robber will follow the following strategy. He will fix a walk of length $\Omega(n^2)$ through the induced complete bipartite graph on vertex set $X \cup Y$, which he can trivially do. He will proceed to follow this walk as slowly as possible, i.e. only proceeding to the next vertex when a cop is adjacent to him.

The robber will only move through $X \cup Y$, so by Lemma 8.3.5 the cops are confined to their cycles. Only Charlie can be adjacent to $X \cup Y$ without leaving his cycle, so it is up to Charlie to walk up and down between x_1 and x_2 to force the robber to move. By Lemma 8.3.2, it thus takes the cops $\Omega(n^3)$ moves to make the robber move once. Hence, the robber manages to stay out of the cops hands for $\Omega(n^5)$ moves. □

The construction slowing down Charlie can be extended in a natural way to higher cop numbers. Consider the following construction for cop number k. For $k \geq 3$, let G_k be the graph constructed as follows.

$$\begin{aligned}
V(G_k) = V(G) &\cup \{u_i^j : i \in [3n], j \in [k-3]\} \cup \{d_{u^j}, h_{u^j} : j \in [k-3]\} \\
&\cup \{d_{u^j,i,l}, h_{u^j,i,l} : i \in [3], l \in [2], j \in [k-3]\} \\
E(G_k) = E(G) &\cup \{u_i^j u_{i+1}^j : i \in [3n], j \in [k-3]\} \\
&\cup \{u_i^j d_{u^j} : i \in [3n], j \in [k-3]\} \\
&\cup \{v d_{u^j,i,l} : v \in X \cup Y, i \in [3], l \in [2], j \in [k-3]\} \\
&\cup \{u_l^{j-1} d_{u^j,i,k} : l \not\equiv i \mod 3, k \in [2], j \in [k-3]\} \\
&\cup \{u_{(i-1)n+l}^j d_{u^j,i,k} : l \in [n], i \in [3], k \in [2], j \in [k-3]\}
\end{aligned}$$

where $u_i^0 = b_i$. The $\{u_i^j\}_i$ form cycles, which are similar to cycles $\{a_i\}_i$ and $\{b_i\}_i$. The doors $\{d_{u^j,i,l}\}_{i,l}$ are connected to respective cycles in the same fashion $\{a_i\}_i$ and $\{b_i\}_i$ are connected to the doors $\{d_{b,i,l}\}_{i,l}$.

Proposition 8.3.7. *$c_b(G_k) = k$ and $capt_b(G_k) \geq Cv(G_k)^{k+2}k^{-(k+2)}$ for some universal constant C.*

Sketch of proof. As in Lemma 8.3.1, each of the doors d_x, d_a, d_b and d_{u^j} (with $j \in [k-3]$) must be guarded initially, so $c_b(G_k) \geq k$. Moreover, if $c_b(G_k) = k$, then one cop must start in each of the cycles; $\{p_i, q_i, x_i\}_i$, $\{a_i\}_i$, $\{b_i\}_i = \{u_i^0\}_i$ and $\{u_i^j\}_i$ for $j \in [k-3]$. As in lemma 8.3.3, if the robber starts in one of the cycles, he will be captured quickly. If the robber starts in the bipartite graph $X \cup Y$, the cops can prevent him from leaving it. Moreover, by Charlie moving between x_1 and x_2 the robber can be forced to use up all the edges between X and

Y and thus be forced out of the bipartite graph, leading to his immediate capture. Hence, $c_b(G_k) \leq k$.

The robber will follow the same strategy as before; planning out a Eulerian walk (assume for convenience that n is even) through complete bipartite graph $X \cup Y$ and only proceeding through the walk when Charlie is directly adjacent to him. Note that this walk has length $(3n)^2$. As in Lemma 8.3.5, the cops are restricted to their cycles as long as the robber stays in $X \cup Y$. As in Lemma 8.3.2, for Charlie to move once from x_1 to x_2 and back, the cops must make $2(3n)^k + o(n)$ moves. Thus, the robber can avoid the cops for at least $(3n)^{k+2}$ rounds.

Note that the graph has $(k+2)3n + 18k$ vertices. Hence,

$$\begin{aligned} capt_b(G_k) &\geq (3n)^{k+2} \\ &= \left(\frac{v(G_k) - 18k}{k+2}\right)^{k+2} \\ &\geq \left(\frac{1 - \frac{6}{n}}{1 + 2/k}\right)^{k+2} \left(\frac{v(G_k)}{k}\right)^{k+2} \\ &\geq C \left(\frac{v(G_k)}{k}\right)^{k+2} \end{aligned}$$

for some constant C. □

This completes the sketch of the proof of Theorem 1.6.1.

8.4 $G_{n,p}$ has $c_b(G) = 1$ with high probability

By Lemma 8.2.1, it suffices to show that whp $G_{n,p}$ contains a dominating clique. We shall abbreviate $\log_{\frac{1}{1-p}}(x)$ to $\log(x)$.

Lemma 8.4.1. *Consider $G_{n,p}$ with $p \in (0,1]$ constant, then with high probability, $\exists X \subset V(G_{n,p})$ such that $G_{n,p}[X]$ is a complete graph and $X \cup \Gamma(X) = V(G_{n,p})$.*

Proof. We use a second moment argument to show the result. Fix some small $\varepsilon \in (0, \frac{1}{2})$ and let $k = (1+\varepsilon)\log(n)$.

Let S be the number of sets $X \subset V(G_{n,p})$ such that $|X| = k$, X induces a clique and $X \cup \Gamma(X) = V(G_{n,p})$. Note that the events that X is a clique and that $X \cup \Gamma(X) = V(G_{n,p})$ are dependent on disjoint edges.

$$\mathbb{E}[S] = \binom{n}{k} p^{\binom{k}{2}} \left(1 - (1-p)^k\right)^{n-k}$$

To compute the second moment of S, let A and B be two k-sets of vertices. We will use the law of total expectation to condition on the size of $A \cap B$. Note that the probability that A and B both satisfy the conditions is at most the probability that they are both independent sets.

$$\begin{aligned}\mathbb{E}[S^2] &\leq \sum_{i=0}^{k} \binom{n}{k-i, i, k-i} p^{2\binom{k}{2}-\binom{i}{2}} \\ &\leq \left(\binom{n}{k} p^{\binom{k}{2}}\right)^2 \left[1 + \sum_{i=1}^{k} \frac{\binom{n}{k-i,i,k-i}}{\binom{n}{k}^2} p^{-\binom{i}{2}}\right]\end{aligned}$$

Each of these last terms is bounded as:

$$\frac{\binom{n}{k-i,i,k-i}}{\binom{n}{k}^2} p^{-\binom{i}{2}} \leq \frac{k^{2i} p^{-\binom{i}{2}}}{n^i},$$

so for the entire sum we find

$$\sum_{i=1}^{k} \frac{\binom{n}{k-i,i,k-i}}{\binom{n}{k}^2} p^{-\binom{i}{2}} \leq \max_{i \in [k]} \left\{ \frac{k^{2i+1} p^{-\binom{i}{2}}}{n^i} \right\} = o(1),$$

and thus

$$\mathbb{E}[S^2] \leq \left(\binom{n}{k} p^{\binom{k}{2}}\right)^2 (1 + o(1)).$$

Now we find by Chebyshev's inequality:

$$\mathbb{P}(S > 0) \geq \frac{\left[\binom{n}{k} p^{\binom{k}{2}} \left(1 - (1-p)^k\right)^{n-k}\right]^2}{\left(\binom{n}{k} p^{\binom{k}{2}}\right)^2 (1 + o(1))} \to 1$$

Hence, the probability that there is a dominating clique tends to one. □

8.5 Concluding remarks

The natural question remaining is the asymptotic maximal capture time for $c_b(G) = 1, 2$. Kinnersley and Peterson [59] conjectured that there exists an n-vertex graph, G, with $c_b(G) = 1$ and $capt_b(G) = \Omega(n^3)$, which we leave open.

Additionally, we are interested in the exact asymptotics in terms of $c_b(G)$. The results in this chapter show the function to lie between $\frac{1}{c_b(G)^{c_b(G)+2}}$ and $\frac{2^{c_b(G)}}{c_b(G)!}$. We expect the correct answer to be $\frac{1}{c_b(G)!}$.

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