A Mathematical Study of Hawking Radiation on Collapsing, Spherically Symmetric Spacetimes

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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. I further state that no substantial part of my thesis has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

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Summary

In this thesis, we give a mathematical treatment of the late time Hawking radiation of massless bosons emitted by a family of collapsing, spherically symmetric, charged models of black hole formation, including both extremal and sub-extremal black holes. We further bound the rate at which the late time behaviour is approached. This treatment relies heavily on analysing the behaviour of the linear scattering map for massless bosons (solutions to the wave equation), which will be discussed further in this thesis. The thesis will be split into three chapters.

The first chapter will be an introduction and derivation of the underlying spacetime models, known as Reissner–Nordström Oppenheimer–Snyder (RNOS) models. We will discuss the derivation of the Oppenheimer–Snyder model [39], before moving on to the more general charged case. We will then summarise the interesting and useful properties of these models.

The second chapter will cover the analysis of the scattering map for the wave equation on RNOS backgrounds. The main results will be the forward boundedness and backwards non-boundedness of the scattering map on the original Oppenheimer–Snyder space-time [39], and then the subsequent generalisation of this to RNOS models. These results will be achieved primarily by using vector field methods: by considering different energy currents and how they interact with the collapsing dust cloud, we will show that solutions of the linear wave equation have bounded energy when going from past null infinity up until a spacelike hypersurface which intersects the point of collapse of the dust cloud. Previous works allow us to extend this result to one on the whole spacetime.

The final chapter of this thesis will apply the above results to the calculation first considered by Stephen Hawking [27, 28], in order to obtain the rate of radiation emitted by collapsing black holes. This result will further make use of some high frequency approximations and also an r^{*p} weighted energy estimate. In particular, we will prove that for late times, the radiation given off by any RNOS model approaches its predicted Hawking radiation limit, that of a black body of fixed temperature. We will also prove a bound on the rate at which this limit is approached.

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Contents

Introduction

In this thesis, we will be studying the behaviour of solutions to the linear wave equation

$$\Box_g \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = 0, \tag{1}$$

on spherically symmetric solutions to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{2}$$

In general in this thesis, $T_{\mu\nu}$ in (2) will be given by either the Maxwell energy momentum tensor (see already (1.31)), or the energy momentum tensor of dust (1.2).

The overall goal of this thesis will be to give a mathematically rigorous treatment of Hawking radiation, first considered in [27, 28]. This quantity is the rate at which radiation is emitted by black holes predicted by quantum field theories on curved background spaces.

Classically, black holes, once formed, are permanent and (conjecturally) stable. The discovery of Hawking Radiation was therefore a major breakthrough in understanding how black holes can vary radically over time, as it describes a mechanism to decrease the mass of black holes and potentially cause them to evaporate entirely. Since Hawking proposed this phenomenon in 1974 [27], there have been hundreds of papers on the topic within the physics literature. For an overview of the physical aspects of Hawking radiation, we refer the reader to [46] . Concerning a mathematically rigorous treatment of Hawking radiation, however, there are substantially fewer works (see already [6] and Section 3.2 for a discussion of further references), and the mathematical status of Hawking radiation still leaves much to be desired. As a result, it has not been possible yet to ask more quantitative questions about Hawking radiation, which are necessary if one wants to eventually understand this phenomenon in the non-perturbative setting. This thesis hopes to contribute towards solving this problem by giving a new physical space approach to Hawking's calculation allowing one to obtain also a rigorous bound on the rate at which emission approaches black body radiation.

0.1 Summary of the Thesis

We begin in Chapter 1, by considering the possible spherically symmetric models of collapsing spacetimes. We first consider the original model studied by Hawking, the Oppenheimer–Snyder model [39]. This model considers chargeless, pressureless dust collapsing to form a black hole. We then proceed to add charge to the dust and consider the effects on the behaviour of the surface of the dust cloud. This allows us to generalise to the RNOS (Reissner–Nordström Oppenheimer–Snyder) models, which will be the topic of this thesis. The main result of this chapter is the derivation itself of the metric of these models, and of the behaviour of the surface of the dust cloud.

In Chapter 2, we construct the Scattering map. That is, we define the map taking initial data of solutions to the wave equation from past null infinity, $\mathscr{I}^-$, to future null infinity and the event horizon, $\mathscr{I}^+ \cup \mathscr{H}^+$. The main result of this Chapter, Theorem 2.7.2 is that this map is a linear, bounded map (with respect to an energy norm defined in Section 2.3), but that its inverse (where it is defined) is not bounded. In the process of proving this, we will also determine exactly where this inverse map 'goes wrong', which will allow us to proceed with our treatment of Hawking radiation, despite the difficulty that non-invertibility imposes.

In Chapter 3, we will perform the Hawking radiation calculation originally done in [27, 28] in a mathematically rigorous manner. This calculation determines the change in frequency of a solution from $\mathscr{I}^+$ to $\mathscr{I}^-$. In particular, the calculation supposes the solution on $\mathscr{I}^+$ contains only positive frequencies, and determines the size of the negative frequency components of the solution on $\mathscr{I}^-$. Here 'size' is with respect to the particle current, which we will explain in more detail in Section 0.2. This chapter will culminate in Theorem 4, showing that the size of these negative frequency components approaches a fixed limit, and will also prove a bound on the rate at which this limit is approached.

There is also one appendix, concerned with pure Reissner–Nordström spacetime. Appendix A derives a result bounding reflection coefficients, which is used in the Hawking calculation.

Throughout this thesis, we will be using the signature convention $\{-,+,+,+\}$, with summation convention (repeated indices are summed over unless otherwise stated). Summations over Greek characters and early Latin characters (a,b,c) will be over all 4 dimensions, where as summations over i, j, k are summations over the 3 spatial dimensions. The Fourier transform of a function f will be denoted by $\hat{f}$, and will use the convention

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} f(x) dx. \tag{3}$$

Fourier transform of a function on a cylinder will always be with respect to the non-angular variable.

Other conventions and notations are covered in Section 2.3.

0.2 Physical Derivation of Hawking Radiation

Before turning to the mathematical study of Hawking radiation, we briefly review the physical derivation of the Hawking calculation. This section is not intended to be rigorous, it is only intended to give an overview of the motivation for this thesis. For more on making the framework of quantum field theories on curved backgrounds rigorous, we refer the reader to [30], [46]. In this section, we will be imposing $\hbar = 1$, as well as the usual $G = c = 1$. This section will be closely following Chapter 10 of [40].

0.2.1 Quantum Field Theory on a Curved Background

Let us consider a metric of the form

$$g = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{4}$$

known as a $3+1$ decomposition.

On this background, let us consider a massless scalar field, with action

$$S = \int_{\mathscr{M}} \frac{\sqrt{-g}}{2} \nabla_a \psi \nabla^a \psi dt d^3x. \tag{5}$$

This has equation of motion given by (1). We can consider the momentum conjugate of this scalar field (using our t coordinate) to obtain

$$\Pi = \frac{\delta S}{\delta(\partial_t \psi)} = \sqrt{-g} g^{t\mu} \partial_\mu \psi. \tag{6}$$

As in all quantum mechanics, we now promote ψ and Π to operators, and impose the following commutation relations:

$$[\psi(t,x), \Pi(t,x')] = i\delta(x-x') \qquad [\psi(t,x), \psi(t,x')] = 0 = [\Pi(t,x), \Pi(t,x')]. \tag{7}$$

We now consider what these operators act on. Assuming our manifold is globally hyperbolic, we know that any solution is uniquely determined by data on $\Sigma_0 = \{t = 0\}$. Let $\alpha, \beta \in \mathbf{S}$, where $\mathbf{S}$ is a space of suitable complex solutions to (1). We define the particle current "inner product" as follows:

$$(\alpha,\beta) = -i\int_{\Sigma_0} \sqrt{-g}g^{t\mu}(\bar{\alpha}\nabla_\mu\beta - \beta\nabla_\mu\bar{\alpha})d^3x. \tag{8}$$

Note we have not defined the function space on which this is an inner product yet.

We note from (1) that this current is conserved, as

$$\nabla^\mu\left(\bar{\alpha}\nabla_\mu\beta - \beta\nabla_\mu\bar{\alpha}\right) = 0. \tag{9}$$

Thus, the integral (8) over any surface of constant t, Σ_t, is independent of t. Also note the following properties:

$$(\alpha,\beta) = -(\bar{\beta},\bar{\alpha}) \tag{10}$$

$$(\alpha,\beta) = 0 \quad \forall\beta\in\mathbf{S} \implies \alpha = 0. \tag{11}$$

However, as $(\alpha,\alpha) = -(\bar{\alpha},\bar{\alpha})$, we have that this inner product (8) is not positive definite on $\mathbf{S}$. We would like to consider a subset of $\mathbf{S}_p$ on which $(,)$ is positive definite, denoted $\mathbf{S}_p$. By (10), we know that $(,)$ is negative definite on $\bar{\mathbf{S}}_p = \{\bar{\alpha} : \alpha\in\mathbf{S}_p\}$. We would like to pick an $\mathbf{S}_p$ such that

$$\mathbf{S} = \mathbf{S}_p \oplus \bar{\mathbf{S}}_p. \tag{12}$$

In general, there will be many ways to do this. For now, we will just pick one, though we will return to this choice later.

In quantum theory, we define creation and annihilation operators (associated to $f\in\mathbf{S}_p$) of a real scalar field ϕ by

$$a(f) = (f,\phi) \qquad a(f)^\dagger = -(\bar{f},\phi). \tag{13}$$

We then have the following commutation relations

$$[a(f),a(g)^\dagger] = (f,g) \qquad [a(f),a(g)] = 0 = [a(f)^\dagger,a(g)^\dagger] = 0. \tag{14}$$

As usual in quantum theories, we define the vacuum state, $|0\rangle$, by

$$a(f)|0\rangle = 0 \quad \forall f\in\mathbf{S}_p \qquad \langle 0|0\rangle = 1. \tag{15}$$

An N-particle state is defined by

$$a^\dagger(f_1)a^\dagger(f_2)...a^\dagger(f_n)|0\rangle, \quad f_i\in\mathbf{S}_p. \tag{16}$$

We now finally define our Hilbert space, $\mathbf{H}_p$, to be the Fock space spanned by the vacuum state, the 1-particle states, the 2-particle states, etc.

If we consider particles created by the creation operator $a(f)^\dagger$, the expected number of these particles measured in state $|\varphi\rangle$ is

$$\langle\varphi|N_f|\varphi\rangle = \langle\varphi|a(f)^\dagger a(f)|\varphi\rangle. \tag{17}$$

0.2.2 Bogoliubov Coefficients

We have defined $\mathbf{H}_p$ using $\mathbf{S}_p$, for which there are many options. Therefore, there are in fact many different options for this Fock space (these spaces would be isomorphic, but we already have an obvious map between them, and they are not the same space using this map). We will now consider how to transform between two different sets of creation and annihilation operators.

Let $\mathbf{S}_p$ and $\mathbf{S}'_p$ be two different choices for a space of positive definite solutions to (12). Let $\{\phi_i\} \subset \mathbf{S}_p$ be an orthonormal basis for $\mathbf{S}_p$, i.e.

$$(\phi_i, \phi_j) = \delta_{ij}, \tag{18}$$

and let $\{\phi'_i\}$ be an orthonormal basis for $\mathbf{S}'_p$ (these exist, as $\mathbf{S}_p$ and $\mathbf{S}'_p$ restricted to any Cauchy surface are subsets of $L^2(\mathbb{R}^3)$).

As $\{\phi_i\} \cup \{\bar{\phi}_i\}$ is a basis of $\mathbf{S}$, we can write

$$\phi'_i = \sum_j \left(A_{ij}\phi_j + B_{ij}\bar{\phi}_j \right), \tag{19}$$

for some A, B, known as *Bogoliubov coefficients*. We can rearrange for the B_{ij} coefficient to get

$$B_{ij} = -(\bar{\phi}_j, \phi'_i). \tag{20}$$

Annihilation and creation operators are then related by

$$a(\phi'_i) = \sum_j \left(\bar{A}_{ij} a(\phi_j) - \bar{B}_{ij} a(\phi_j)^\dagger \right). \tag{21}$$

Now we wish to consider the number of particles given by $a(\phi'_i)$ expected to be in the vacuum state given by $\mathbf{S}_p$.

$$\langle 0| a(\phi'_i)^\dagger a(\phi'_i) |0\rangle = \sum_{j,k} \langle 0| (-B_{ij}) a(\phi_j) (-\bar{B}_{ik}) a(\phi_k)^\dagger |0\rangle = \sum_{j,k} B_{ij}\bar{B}_{ik} = (BB^\dagger)_{ii}. \tag{22}$$

0.2.3 The Collapsing Setting

As stated previously, we have many choices of $\mathbf{S}_p$. However, if we have a preferential choice of timelike coordinate, say t, we have a preferential choice of $\mathbf{S}_p$: Let ϕ_p be the eigenfunction of the operator ∂_t with positive imaginary eigenvalue, that is

$$\partial_t \phi_p = i p_0 \phi_p \quad p_0 > 0. \tag{23}$$

Thus, if we have a preferred timelike derivative, we define $\mathbf{S}_p$ to be the span of all such positive imaginary eigenfunctions of ∂_t.

We now consider an asymptotically flat gravitational collapse. Then near to $\mathscr{I}^-$ and $\mathscr{I}^+$, we have a preferential choice of timelike coordinate, given by the definition of asymptotically flat. This allows us to define a canonical choice of $\mathbf{S}_p^\pm$ on $\mathscr{I}^\pm$. These will now take the part of $\mathbf{S}_p$ and $\mathbf{S}'_p$ in equations (18) to (22).

Now suppose we wish to calculate the expected number of particles of frequency ω emitted by this gravitational collapse. Let ψ'_ω be a function on future null infinity with frequency approximately ω. Note that here, approximately means that $\hat{\psi}'$ is supported on $[\omega - \varepsilon, \omega + \varepsilon] \times S^2$ for some small ε. Here S^2 is the 2-sphere. This is required, as if $\hat{\psi}'$ was only supported at ω, then $\psi'_\omega \propto e^{i\omega u}$, and we cannot have $(\psi'_\omega, \psi'_\omega) = 1$.

Let ϕ'_ω be the solution to (1) with future radiation field ψ'_ω, and which vanishes at $\mathscr{H}^+$ (as we are not interested in the expected number of particles crossing the event horizon). Denote the past radiation field of ψ'_ω by $\psi_\omega^{\mathscr{I}^-}$. We can split $\psi_\omega^{\mathscr{I}^-}$ up into positive and negative frequency components:

$$\psi_\omega^{\mathscr{I}^-} = \psi_\omega^{\mathscr{I}^-+} + \psi_\omega^{\mathscr{I}^--}, \tag{24}$$

Where $\hat{\psi}_\omega^{\mathscr{I}^-+}$ is supported in $[0, \infty] \times S^2$, and $\hat{\psi}_\omega^{\mathscr{I}^--}$ is supported in $[-\infty, 0] \times S^2$.

Let ϕ_i be a basis of $\mathbf{S}_p^-$. Then the number of expected particles emitted by the formation of the black hole is

$$\begin{aligned}\sum_{j,k} B_{ij}\bar{B}_{ik} &= (\sum_j \bar{B}_{ij}\phi_j, \sum_k \bar{B}_{ik}\phi_k) = (\bar{\psi}_\omega^{\mathscr{I}^{--}}, \bar{\psi}_\omega^{\mathscr{I}^{--}}) \\ &= -i\int \psi_\omega^{\mathscr{I}^{--}}\partial_v\bar{\psi}_\omega^{\mathscr{I}^{--}} - \bar{\psi}_\omega^{\mathscr{I}^{--}}\partial_v\psi_\omega^{\mathscr{I}^{--}} \sin\theta dv d\theta d\varphi \\ &= 2\int_{\sigma=-\infty}^{\infty} \sigma|\hat{\psi}_\omega^{\mathscr{I}^{--}}|^2 \sin\theta d\sigma d\theta d\varphi \\ &= 2\int_{\sigma=-\infty}^{0} \sigma|\hat{\psi}_\omega^{\mathscr{I}^{-}}|^2 \sin\theta d\sigma d\theta d\varphi.\end{aligned} \tag{25}$$

Here σ is the Fourier space variable.

0.2.4 The Hawking Calculation

Let ψ_+ be a Schwartz function on the cylinder, with $\hat{\psi}_+$ supported in $[\omega-\varepsilon, \omega+\varepsilon]$, with $(\psi_+, \psi_+) = 1$. Let ϕ be the solution to (1) which vanishes on the future event horizon, and has future radiation field equal to ψ_+. Let $\psi_{\mathscr{I}^-}$ be the past radiation field of ϕ. Then we must calculate

$$2\int_{\sigma=-\infty}^{0} \sigma|\hat{\psi}_{\mathscr{I}^-}|^2 \sin\theta d\sigma d\theta d\varphi. \tag{26}$$

In his original paper [27], Hawking argued that at late times (to be defined more rigorously later) in the formation of the Schwarzschild black hole the rate of radiation of frequency ω emitted would tend towards that of a black body of temperature $\kappa/2\pi$, where κ is the surface gravity of the black hole. Since then, there have been many heuristic arguments for this result to extend to all Reissner–Nordström black holes, along with several more rigorous papers exploring this phenomenon (see Section 3.2 for a further discussion of these).

There are two ways to obtain similar results by considering quantum states on a non-collapsing Reissner–Nordström background. Firstly, one can construct the Unruh state [44]. If one considers quantum states on the permanent Reissner–Nordström black hole, one can show that there is a unique state that coincides with the vacuum state on $\mathscr{I}^-$ and is well behaved at $\mathscr{H}^+$ (i.e. is a Hadamard state). This state evaluated on $\mathscr{I}^+$ is a thermal state of temperature $\kappa/2\pi$ (see [17], for example).

The second similar result can be obtained by constructing the Hartle–Hawking–Israel state. If one again considers quantum states on the permanent Reissner–Nordström black hole, one can show that there is a unique state that is well behaved (Hadamard) at $\mathscr{I}^+$, $\mathscr{I}^-$, $\mathscr{H}^+$, $\mathscr{H}^-$ [31]. This state is that of a thermal black body, again of temperature $\kappa/2\pi$. The interpretation of this is that the black hole is in equilibrium with this level of thermal radiation, and is therefore emitting the radiation of a black body of this temperature. This result has been considered in a mathematically rigorous manner on Schwarzschild [29, 17], and more recently in a more general setting [43, 23]. This thesis, however, will be focused on the collapsing setting, as it is believed that this method will generalise more readily, as the Hartle–Hawking–Israel state has been shown not to exist in Kerr spacetimes [31].

One of the reasons this result has gained so much traction in both the mathematics and physics communities is that it provides the only known mechanism for black holes to lose mass. Without the ability to lose mass, any black hole that has formed would be a permanent fixture of the universe, and could only grow in size. However, if the black holes are able to emit radiation at a fixed rate, even if only in very small doses, then given enough time, isolated black holes will disappear entirely, known as black hole evaporation.

0.2.5 Goal of this Thesis

In this thesis, we take (26) to be our starting point. We will impose the radiation field $\psi_+(u-u_0, \theta, \varphi)$ at $\mathscr{I}^+$, and 0 at $\mathscr{H}^+$. We will rigorously define $\hat{\psi}_{\mathscr{I}^-}$ in terms of classical scattering theory, using results from Chapters 1 and 2. We will then rigorously calculate the limit that (26) approaches as $u_0 \to \infty$, on a family of collapsing models forming Reissner–Nordström blackholes. We will further include a rigorous bound on the rate at which this result is approached, obtaining the result of Theorem 4:

Theorem 4 (Late Time Emission of Hawking Radiation)**.** *Let $\psi_+(u,\theta,\varphi)$ be a Schwartz function on the* 3*-cylinder, with $\hat{\psi}_+$ only supported on positive frequencies. Let ϕ be the solution of* (2.1)*, as given by Theorem 2.4.1, such that*

$$\lim_{v\to\infty} r(u,v)\phi(u,v,\theta,\varphi) = \psi_+(u-u_0,\theta,\varphi) \tag{27}$$

$$\lim_{u\to\infty} r(u,v)\phi(u,v,\theta,\varphi) = 0 \quad \forall v \geq v_c, \tag{28}$$

Define the function ψ_{-,u_0} by

$$\lim_{u\to-\infty} r(u,v)\phi(u,v,\theta,\varphi) = \psi_{-,u_0}(v,\theta,\varphi). \tag{29}$$

Then for all $|q|<1$, $n\in\mathbb{N}$, there exist constants $A_n(M,q,T^*,\psi_+)$ such that

$$\left| \int_{\omega=-\infty}^{0}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} |\omega||\hat{\psi}_{-,u_0}(\omega,\theta,\varphi)|^2 \sin\theta d\omega d\theta d\varphi - \int_{\mathscr{H}^-} \frac{|\omega||\hat{\psi}_{\mathscr{H}^-}(\omega,\theta,\varphi)|^2}{e^{\frac{2\pi|\omega|}{\kappa}}-1}\sin\theta d\omega d\theta d\varphi \right| \leq A_n u_0^{-n}, \tag{30}$$

for sufficiently large u_0.

Here, $\psi_{\mathscr{H}^-}$ is the reflection of ψ_+ in pure Reissner–Nordström spacetime (as will be discussed in Section 3.3), and κ is the surface gravity of the Reissner–Nordström black hole.

In the case $|q|=1$, there exists a constant $A(M,q,T^*)$ such that

$$\left| \int_{\omega=-\infty}^{0}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} |\omega||\hat{\psi}_{-,u_0}(\omega,\theta,\varphi)|^2 \sin\theta d\omega d\theta d\varphi \right| \leq \frac{A}{u_0^{3/2}}, \tag{31}$$

for sufficiently large u_0.

This result is restated in Theorem 3.4.1 and Corollary 3.4.6, including the precise relationship of A_n,A on ψ_+.

In terms of the Unruh state mentioned previously, one can view this result as the following statement: Let us impose the vacuum state on past null infinity and evolve forward to future null infinity. Then at late times, any number operator acting on this future null infinity state approaches the same number operator acting on the Unruh state at future null infinity (and again a rate is included).

This result is especially physically relevant for the extremal case. As the surface gravity of the extremal Reissner–Nordström black hole is 0, this is the only current model of black hole which may be stable to Hawking radiation (though it may be unstable to classical perturbations [4, 5]). That is, any black hole with non-zero surface gravity will emit radiation at a constant rate, and will therefore, if left alone, eventually evaporate. However, this thesis will show that the extremal Reissner–Nordström black hole will emit only a finite amount of radiation over time, and may therefore not evaporate, becoming a permanent fixture to the universe.

Chapter 1

RNOS Spacetimes

1.1 The Oppenheimer–Snyder Spacetime

Any discussion of collapsing spacetimes must start with the Oppenheimer–Snyder spacetime [39]. The Oppenheimer–Snyder spacetime is a homogeneous, spherically symmetric collapsing dust star. That is to say, a spherically symmetric solution of the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\mathbf{T}_{\mu\nu} \tag{1.1}$$

where for dust, we have

$$\begin{aligned} \mathbf{T}_{\mu\nu} &= \rho u_\mu u_\nu \\ \nabla_\mu \mathbf{T}^{\mu\nu} &= 0. \end{aligned} \tag{1.2}$$

Here the vector u^μ is the 4-velocity of the dust, and ρ is the density of the dust. On our initial timelike hypersurface, this density is a positive constant inside the star, but 0 outside the star. The case of the non-homogeneous dust cloud was studied by Christodoulou in [11].

As this density is not continuous across the boundary of the star, the Oppenheimer–Snyder model is only a global solution of the Einstein equations in a weak sense. However, it is a classical solution on both the interior and the exterior of the star.

We therefore have two specific regions of the space-time to consider: inside the star (section 1.1.2), and outside the star (section 1.1.1). We will finally give the definition of our manifold and global coordinates in section 1.1.3. If the reader is uninterested in the derivation of the metric, they may want to skip to that section. Finally in section 1.1.4, we discuss the Penrose diagram for this space-time.

1.1.1 Exterior

We will first consider the exterior of the star. This region is a spherically symmetric vacuum space-time, thus by Birkhoff's theorem, [40], this is a region of Schwarzschild space-time. It is bounded by the timelike hypersurface $r = r_b(t^*)$. This hypersurface will be referred to in this thesis as the boundary of the star. We will be using the following coordinate system in the exterior of the star:

$$g = -\left(1 - \frac{2M}{r}\right)dt^{*2} + \frac{4M}{r}dt^*dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2 g_{S^2} \qquad (t^*, r, \theta, \varphi) \in \mathbb{R} \times [R^*(t^*), \infty) \times S^2 \tag{1.3}$$

where g_{S^2} is the usual metric on the unit sphere, and t^* is defined by

$$t^* = t + 2M\log\left(\frac{r}{2M} - 1\right), \tag{1.4}$$

in terms of the usual t and r coordinates on Schwarzschild.

As the surface of the star is itself free-falling and massive, we may assume that the surface of the star follows timelike geodesics. This assumption is true in the Oppenheimer–Snyder model, but also generalises to other models, provided the matter remains well behaved. Thus if a particle on the surface has space-time coordinates $x^\alpha(\tau)$, then these coordinates satisfy

$$\begin{aligned} -1 = g_{ab}\left(\frac{dx^a}{d\tau}\right)\left(\frac{dx^b}{d\tau}\right) &= -\left(1-\frac{2M}{r}\right)\left(\frac{dt^*}{d\tau}\right)^2 + \frac{4M}{r}\frac{dt^*}{d\tau}\frac{dr}{d\tau} + \left(1-\frac{2M}{r}\right)\left(\frac{dr}{d\tau}\right)^2 \\ &= \left(-\left(1-\frac{2M}{r}\right) + \frac{4M}{r}\dot{r_b} + \left(1-\frac{2M}{r}\right)\dot{r_b}^2\right)\left(\frac{dt^*}{d\tau}\right)^2. \end{aligned} \tag{1.5}$$

Here we are using the t and r coordinates in equation (1.3), and using the fact that this space-time is spherically symmetric to ignore $\frac{d\theta}{d\tau}$ and $\frac{d\varphi}{d\tau}$ terms. Note that $\dot{r_b} = \frac{dr_b}{dt^*}$.

Now, as $r_b(t^*)$ is to be timelike and is the surface of a collapsing star, we assume $\dot{r_b} < 0$, and that the surface emanates from past timelike infinity. Again, this is true in Oppenheimer–Snyder space-time, but also in many other models of gravitational collapse. At some time, t_c^*, we have $r_b(t_c^*) = 2M$ (note that $r_b(t^*)$ does not cross $r = 2M$ in finite t coordinate, as $t \to \infty$ as $r \to 2M$ on any timelike curve). For $t^* > t_c^*$ and $r \geq 2M$, we have that the space-time is standard exterior Schwarzschild space-time, with event horizon at $r = 2M$.

In the exterior region, we define our outgoing and ingoing null coordinates as follows:

$$v = t^* + r \tag{1.6}$$

$$u = t^* - r - 4M\log\left(\frac{r}{2M} - 1\right) \tag{1.7}$$

$$g = -\left(1-\frac{2M}{r}\right)dudv + r(u,v)^2 g_{S^2}. \tag{1.8}$$

1.1.2 Interior

We now move on to considering the interior of the star. One thing that is important to note here is that as we go from considering the exterior of the star to considering the interior, i.e. as our coordinates cross the boundary of our star, our metric changes from solving the vacuum Einstein equations to solving the Einstein equations with matter. Thus across the boundary, our metric will not be smooth, so we must be careful when wishing to take derivatives of the metric. This will have implications on the regularity of solutions of (2.1) across the surface of the star.

This derivation will closely follow the original Oppenheimer–Snyder paper, [39].

We first consider taking a spatial hypersurface in our space time, which is preserved under the spherical symmetry SO_3 action. We can therefore parametrise this by some R, θ, φ, where θ and φ are our spherical angles. Then we locally extend this coordinate system to the space-time off this surface by constructing the radial geodesics through each point with initial direction normal to the surface. In these coordinates, our metric must be of the form

$$g = -d\tau^2 + e^{\bar{\omega}}dR^2 + e^{\omega}g_{S^2}. \tag{1.9}$$

for $\omega = \omega(\tau, R)$ and $\bar{\omega} = \bar{\omega}(\tau, R)$.

Now our matter is moving along lines of constant R, θ and ϕ, as we assume our matter follows radial geodesics which are normal to our initial surface. In these coordinates the dust's velocity u^μ is therefore proportional to ∂_τ. Thus, we have from equation (1.2) that $T^\tau_\tau = -\rho$, for density ρ. We also have that all other components of the energy momentum tensor T vanish. Then the Einstein equations (1.1) imply that the following is a solution:

$$e^{\bar{\omega}} = \frac{1}{4}\omega'^2 e^{\omega} \tag{1.10}$$

$$e^{\omega} = (F\tau + G)^{\frac{4}{3}}, \tag{1.11}$$

where $'$ denotes derivative with respect to R, and F, G are arbitrary functions of R. Then we can rescale R to choose $G = R^{\frac{3}{2}}$. We now assume that at $\tau = 0$, ρ is a constant density ρ_0 inside the star, and vacuum outside the star, i.e.

$$\rho(0,R) = \begin{cases} \rho_0 & R \leq R_b \\ 0 & R > R_b \end{cases}, \tag{1.12}$$

for $R_b > 0$ constant. Then the equation for T^{τ}_{τ} gives:

$$FF' = \begin{cases} 9\pi\rho_0 R^2 & R \leq R_b \\ 0 & R > R_b \end{cases} \tag{1.13}$$

where, in these coordinates, $\{R = R_b\}$ is the boundary of the star. This has the particular solution

$$F = \begin{cases} -\frac{3}{2}\sqrt{2M}\left(\frac{R}{R_b}\right)^{\frac{3}{2}} & R \leq R_b \\ -\frac{3}{2}\sqrt{2M} & R > R_b \end{cases}, \tag{1.14}$$

for $M = 4\pi\rho_0 R_b^3/3$. This gives us a range for which our coordinate system is valid, as the angular part of the metric, e^{ω} has to be greater than or equal to 0. Thus we obtain $\tau \leq \frac{2R}{3\sqrt{2M}}$. Now, if we transform to a new radial coordinate, $r = e^{\frac{\omega}{2}}$, then we obtain a metric of the form:

$$g = \begin{cases} -\left(1 - \frac{2Mr^2}{r_b^3}\right) d\tau^2 + 2\sqrt{\frac{2Mr^2}{r_b^3}} dr d\tau + dr^2 + r^2 g_{S^2} & r < r_b \\ -\left(1 - \frac{2M}{r}\right) d\tau^2 + 2\sqrt{\frac{2M}{r}} dr d\tau + dr^2 + r^2 g_{S^2} & r \geq r_b \end{cases} \tag{1.15}$$

where

$$r_b(\tau)^{\frac{3}{2}} = R_b^{\frac{3}{2}} - \frac{3\tau}{2}\sqrt{2M}. \tag{1.16}$$

In the region $r \geq r_b$, (1.15) are known as Gullstrand–Painlevé coordinates.

Once $r_b(\tau) \leq 2M$, i.e. $\tau \geq \tau_c = \frac{4M}{3}\left(\left(\frac{R_b}{2M}\right)^{\frac{3}{2}} - 1\right)$, we have $r = 2M$ is the surface of an event horizon, and the $r \geq 2M$ section of our space-time is exterior Schwarzschild space-time.

Thus any point which can be connected by a future directed null geodesic to a point outside $r = 2M$ at $\tau \geq \tau_c$ is outside our black hole, and any point which cannot reach $r > 2M$ at $\tau \geq \tau_c$ is inside our black hole. The future directed, outgoing radial, null geodesic which passes through $r = 2M$, $\tau = \tau_c$ is given by:

$$r = r_b(\tau)\left(3 - 2\sqrt{\frac{r_b(\tau)}{2M}}\right). \tag{1.17}$$

Thus the set of coordinates obeying both (1.17) and $\tau \in [\tau_{c^-}, \tau_c]$ is part of the boundary of our black hole, where

$$\tau_{c^-} = 2M\left(\frac{2}{3}\left(\frac{R_b}{2M}\right)^{\frac{3}{2}} - \frac{9}{4}\right). \tag{1.18}$$

Before τ_{c^-}, no part of the star is within a black hole, and for $\tau > \tau_c$, all of the collapsing star is inside the black hole region.

Thus, we define our ingoing and outgoing null geodesics by defining their derivative:

$$du = \begin{cases} d\tau - (1-\sqrt{2M/r})^{-1}dr & r \geq r_b \\ \alpha(d\tau - (1-\sqrt{2Mr^2/r_b^3})^{-1}dr) & r < r_b \end{cases} \tag{1.19}$$

$$dv = \begin{cases} d\tau + (1+\sqrt{2M/r})^{-1}dr & r \geq r_b \\ \beta(d\tau + (1+\sqrt{2Mr^2/r_b^3})^{-1}dr) & r < r_b \end{cases}. \tag{1.20}$$

These coordinates exist, thanks to Frobenius' theorem (see for example [40]) with α and β real functions on the manifold, bounded both above and away from 0. However, we may not be able to write α and β explicitly.

Remark 1.1.1. *Note that when using different coordinates across the boundary of the star, $r = r_b(\tau)$, such as in* (1.19) *and* (1.20) *compared to* (1.15)*, one should be concerned that these coordinates may define different smooth structures on $\mathscr{M}$. For example, the function $f(\tau, r, \theta, \varphi) = r - r_b(\tau)$ is smooth on $r = r_b(\tau)$ with respect to $(\tau, r, \theta, \varphi)$, but is not smooth with respect to coordinates $(\tau, x := (r - r_b(\tau))^3, \theta, \varphi)$.*

However, when considering (in the exterior) the coordinates in (1.3) *compared to* (1.15)*, the change of coordinates is smooth with bounded (above and away from* 0*) Jacobian. Thus a function is smooth with respect to* (1.3) *if and only if it is smooth with respect to* (1.15)*, so this is not a concern in this case.*

1.1.3 Global Coordinates and the Definition of the Oppenheimer–Snyder Manifold

We summarise the work of the previous sections by defining our manifold and metric with respect to global coordinates. Fix $M > 0, R_b \geq 0$, let $\tau_{c^-} = \sqrt{\frac{2R_b^3}{3M}}$, and consider $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$. Here $\mathbb{R}$ is parametrised by τ and $\mathbb{R}^3$ is parametrised by the usual spherical polar coordinates. We then define $\mathscr{M}$ by:

$$\mathscr{M} := \mathbb{R}^4 \backslash \{\tau \in [\tau_{c^-}, \infty), r = 0\}. \tag{1.21}$$

In these coordinates, we then have the metric:

$$g_{M,R_b} = \begin{cases} -\left(1 - \frac{2Mr^2}{r_b^3}\right) d\tau^2 + 2\sqrt{\frac{2Mr^2}{r_b^3}} dr d\tau + dr^2 + r^2 g_{S^2} & r < r_b(\tau) \\ -\left(1 - \frac{2M}{r}\right) d\tau^2 + 2\sqrt{\frac{2M}{r}} dr d\tau + dr^2 + r^2 g_{S^2} & r \geq r_b(\tau) \end{cases} \tag{1.22}$$

where $r_b(\tau)$ is defined by

$$r_b(\tau) = \left(R_b^{3/2} - \frac{3\tau}{2}\sqrt{2M}\right)^{2/3}. \tag{1.23}$$

Note that choice of R_b is equivalent to choosing when $\tau = 0$. Also note the $r = 0$ line (as a subset of $\mathbb{R}^4$) ceases to be part of the manifold $\mathscr{M}$ when the singularity "forms" at τ_{c^-}, where $r_b = 0$. For $\tau < \tau_{c^-}$, $r = 0$ is included in the manifold, as the metric extends regularity to this line.

We define our future event horizon by:

$$\mathscr{H}^+ = \left\{ r = r_b(\tau)\left(3 - 2\sqrt{\frac{r_b(\tau)}{2M}}\right), \tau \in [\tau_{c^-}, \tau_c] \right\} \cup \{r = 2M, \tau \geq \tau_c\}. \tag{1.24}$$

Note that geometrically, this family of space-times (H^1_{loc} Lorentzian manifolds), $(\mathscr{M}, g_{M,R_b})$, is a one parameter family of space-times. The geometry depends only on M, as R_b just corresponds to the coordinate choice of where $\tau = 0$. Thus constants which only depend on the overall geometry of the space-time only depend on M.

We can also explicitly calculate ρ in these coordinates for $r < r_b(\tau)$:

$$\rho(\tau) = \frac{3M}{4\pi r_b^3(\tau)} = \frac{3M}{4\pi\left(R_b^{3/2} - \frac{3\tau}{2}\sqrt{2M}\right)^2} = \frac{R_b^3}{r_b^3(\tau)}\rho_0. \tag{1.25}$$

In the exterior of the space-time, we have one timelike Killing field, $\partial_{t^*} = \partial_\tau$, which is not Killing in the interior. Throughout the whole space-time, we have 3 angular Killing fields, $\{\Omega_i\}_{i=1}^3$, which between them span all angular derivatives. When given in the usual θ, φ coordinates, these take the form:

$$\begin{aligned}\Omega_1 &= \partial_\varphi \\ \Omega_2 &= \cos\varphi\partial_\theta - \sin\varphi\cot\theta\partial_\varphi \\ \Omega_3 &= -\sin\varphi\partial_\theta - \cos\varphi\cot\theta\partial_\varphi\end{aligned} \tag{1.26}$$

1.1.4 Penrose Diagram of $(\mathscr{M}, g)$

We now look to derive the Penrose diagram for the space-time $(\mathscr{M}, g)$. Recall that the Penrose diagram corresponds to the range of globally defined radial double null coordinates. Using the original R and τ coordinates in (1.9), we obtain that the interior of the dust cloud has metric

$$g = -d\tau^2 + \left(1 - \frac{3\sqrt{2M}\tau}{2R_b^{\frac{3}{2}}}\right)^{\frac{4}{3}}\left(dR^2 + R^2 g_{S^2}\right) \tag{1.27}$$

for $R \leq R_b$ and $\tau \leq \tau_c = \frac{2R_b^{\frac{3}{2}}}{3\sqrt{2M}}$. We then choose a new time coordinate, η such that

$$\eta(\tau) = \int_{\tau'=0}^{\tau}\left(1 - \frac{3\sqrt{2M}\tau'}{2R_b^{\frac{3}{2}}}\right)^{-\frac{2}{3}} d\tau'. \tag{1.28}$$

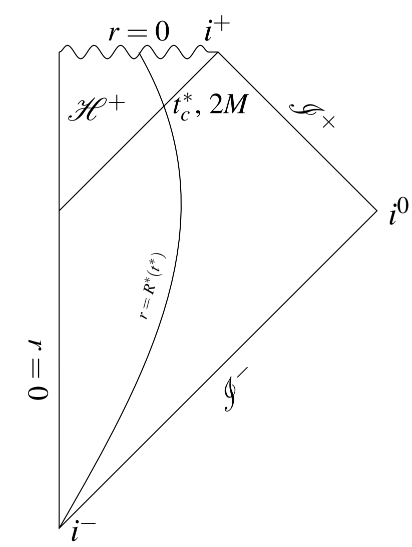


Figure 1.1 Penrose diagram of Oppenheimer–Snyder space-time

Then we change to coordinates $u = \eta - R$, and $v = \eta + R$. Thus we obtain the metric to be of the form

$$g = \left(1 - \frac{3\sqrt{2M}\tau(u,v)}{2R_b^{\frac{3}{2}}}\right)^{\frac{4}{3}}\left(-dudv + R(u,v)^2 g_{S^2}\right). \tag{1.29}$$

In this coordinate system, the range of u and v is given by $u + v \leq 2\tau_c$ and $0 \leq v - u \leq 2R$. Thus the interior of the star is conformally flat. Hence the Penrose diagram for the interior is that of Minkowski space-time, subject to the above ranges of $u + v$ and $v - u$. We also note that we have that $R_{abcd}R^{abcd}$ blows up as η approaches $\eta_c = \eta(\tau_c)$, so this corresponds to a singular boundary of space-time.

On the exterior of the dust cloud, our solution is a subregion of Schwarzschild space-time. The boundary of this region is given by a timelike curve going from past timelike infinity to $r = 0$. Matching these two diagrams across the relevant boundary, we obtain the Penrose diagram shown in Figure 1.1. Again, remember the metric is only a piecewise smooth and H^1_{loc} function of u and v.

1.2 Generalising to the RNOS model

In this section, we begin the novel work of this thsis. We will generalise background models of spherically symmetric dust cloud collapse to include charged matter. In section 1.2.1, we derive the metric of Reissner–

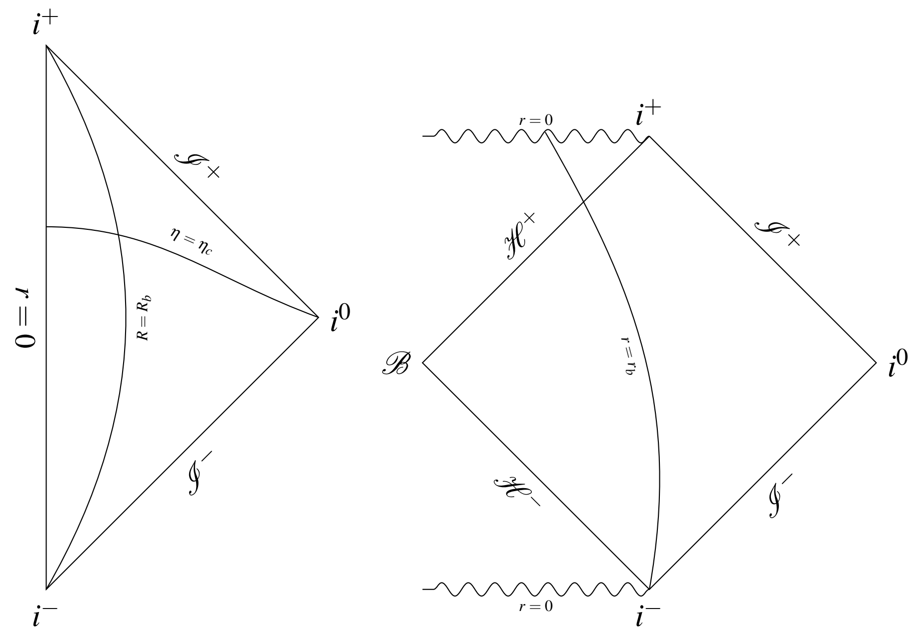


Figure 1.2 Penrose diagram of Minkowski (left) and Schwarzschild (right) space-times, with appropriate boundaries.

Nordstrom Oppenheimer–Snyder (RNOS) spacetimes in the exterior of our collapsing dust cloud. If the reader is not interested in this derivation, they may skip straight to section 1.2.4, where the background manifold is defined, with some interesting and/or useful properties stated.

1.2.1 Derivation

In this section, we derive our metric under the following assumptions: We assume our manifold is a spherically symmetric solution of the Einstein–Maxwell equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.30}$$

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu}\right) \tag{1.31}$$

$$\nabla_{\nu}F^{\nu\mu} = 0 \tag{1.32}$$

$$\nabla_{\mu}F_{\nu\alpha} + \nabla_{\nu}F_{\alpha\mu} + \nabla_{\alpha}F_{\mu\nu} = 0, \tag{1.33}$$

with coordinates $t^* \in (-\infty, \infty)$, $(\theta, \varphi) \in S^2$, $r \in [\tilde{r}_b(t^*), \infty)$. We define

$$\tilde{r}_b := \max\{r_b(t^*), r_+\} \tag{1.34}$$

$$r_+ := M(1 + \sqrt{1 - q^2}). \tag{1.35}$$

Here $r = r_b(t^*)$ is a hypersurface generated by a family of timelike, ingoing radial curves such that, for any fixed θ, φ, the curve $\{t^*, r_b(t^*), \theta, \varphi\}$ describes the motion of a particle moving only under the electromagnetic force, with charge to mass ratio matching that of the black hole. That is, we assume that the surface of the cloud is itself massive and charged, with the same charge density as the cloud itself. For our results, we will actually only require certain bounds on r_b and $\dot{r}_b$ (see already Remarks 2.7.2, 3.4.8), but here we will derive the behaviour of r_b in full. We also note that we are looking solely at the exterior of the black hole. Thus, we will not be considering the region $r_b(t^*) < r_+$. We will instead use $\tilde{r}_b(t^*) = \max\{r_b(t^*), r_+\}$ as the boundary of our manifold. The topology of our manifold $\{t^*, r \geq \tilde{r}_b(t^*), \theta, \varphi\}$ is that of a cylinder in $3+1D$

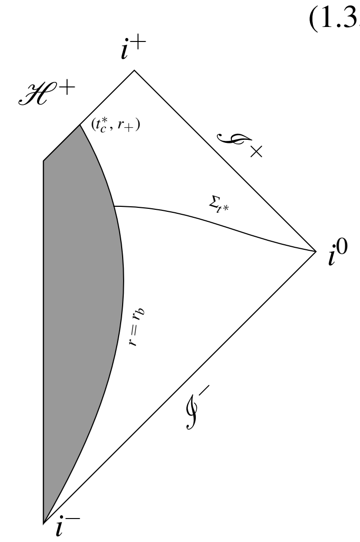


Figure 1.3 Penrose Diagram of RNOS Model, with space-like hyper surface Σ_{t^*}.

Lorentzian space. As this is simply connected, equation (1.33) means we can choose an A such that

$$F = dA. \tag{1.36}$$

Given an asymptotically flat, spherically symmetric solution of the Einstein–Maxwell equations, we know that our solution is a subset of a Reissner–Nordström spacetime (see for example [42]). This gives the first two parameters of our spacetime; M, the mass of the Reissner–Nordström black hole spacetime our manifold is a subset of, and $q = Q/M$, the charge density of our underlying Reissner–Nordström spacetime. We will assume q has modulus less than or equal to 1, as otherwise our dust cloud will either not collapse, or will form a naked singularity rather than a black hole.

Exterior Reissner–Nordström spacetime has global coordinates:

$$g = -\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)dt^{*2} + 2\left(\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)dt^*dr + \left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)dr^2 + r^2 g_{S^2} \tag{1.37}$$

$$A = \frac{qM}{r}dt^*, \tag{1.38}$$

where g_{S^2} is the metric on the unit 2-sphere.

We now proceed to calculate the path moved by a radially moving charged test particle, with charge density q. The motion of this particle extremises the following action:

$$\begin{aligned} S &= \int_{r=r_b} L d\tau = m\int_{r=r_b} \frac{1}{2} g_{ab}v^a v^b - q v^a A_a d\tau \\ &= m\int_{r=r_b}\left(\frac{1}{2}\left(-\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\left(\frac{dt^*}{d\tau}\right)^2 + 2\left(\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\frac{dt^*}{d\tau}\frac{dr}{d\tau} + \left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\left(\frac{dr}{d\tau}\right)^2\right)\right. \\ &\quad \left. - \frac{q^2M}{r}\frac{dt^*}{d\tau}\right) \end{aligned} \tag{1.39}$$

for v^a the velocity of the particle with respect to τ, A as defined in (1.36), and τ the proper time for the particle, i.e. normalised such that $g_{ab}v^a v^b = -1$.

We can then use first integrals of the Euler–Lagrange equations to find constants of the motion. Firstly, L is independent of explicit τ dependence, and so $g_{ab}v^a v^b$ is constant. By rescaling τ, we choose $g_{ab}v^a v^b$ to be 1. The second constant we obtain is from L being independent of t^*. Thus

$$T^* = \left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\frac{dt^*}{d\tau} - \left(\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\frac{dr}{d\tau} + \frac{q^2M}{r} \tag{1.40}$$

is constant.

Using (1.40) to remove dependence of $g_{ab}v^a v^b$ on $\frac{dt^*}{d\tau}$, we obtain

$$\begin{aligned} \left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)^{-1}\left(T^*-\frac{q^2M}{r}\right)^2 &= \left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\left(\frac{dt^*}{d\tau}\right)^2 - 2\left(\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\frac{dt^*}{d\tau}\frac{dr}{d\tau} \\ &\quad + \frac{\left(\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}\left(\frac{dr}{d\tau}\right)^2 \\ &= -g_{ab}v^a v^b + \left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}+\frac{\left(\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}\right)\frac{dr}{d\tau}^2 \\ &= 1+\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2, \end{aligned} \tag{1.41}$$

which rearranges to

$$\begin{aligned}\left(\frac{dr}{d\tau}\right)^2 &= \left(T^* - \frac{q^2M}{r}\right)^2 - \left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right) \\ &= \left(T^* - \frac{q^2M}{r}\right)^2 - \left(1 - \frac{M}{r}\right)^2 + \frac{(1-q^2)M^2}{r^2}.\end{aligned} \tag{1.42}$$

From (1.40), we can see that if this particle's velocity is to be future directed, we need $T^* > 0$.

For $|q| \le 1$, $T^* \ge 1$, and $(|q|, T^*) \neq (1,1)$, equation (1.42) tells us that $\frac{dr}{d\tau}$ is positive. As $\frac{dr}{d\tau}$ is a continuous function of r, this implies that r_b must tend to ∞ as $t^* \to -\infty$. If $q = \pm 1$, i.e. the extremal case, and $T^* = 1$, then we have $\frac{dr}{d\tau} \equiv 0$. Thus the dust cloud will not collapse, so we will not be considering $|q| = 1 = T^*$.

We can also see, from writing out the statement $g_{ab}v^av^b = -1$, that

$$\left(-\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right) + 2\left(\frac{2M}{r} - \frac{q^2M^2}{r^2}\right)\frac{dr}{dt^*} + \left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right)\left(\frac{dr}{dt^*}\right)^2\right)\left(\frac{dt^*}{d\tau}\right)^2 = -1 \tag{1.43}$$

which tells us that $\frac{dr}{dt^*} > -1$.

1.2.2 $T^* < 1$

We now look at the behaviour of r_b in the case where $T^* < 1$.

If $T^* < 1$, then looking at $r \to \infty$ we can see $\frac{dr}{d\tau}$ vanishes at a finite radius, so the dust cloud will tend to that radius, either reaching it at a finite time, or as $t \to -\infty$.

We therefore look at integrating equation (1.42) to obtain $\tau(r)$, which gives us

$$\tau = \frac{\left(\left(T^*r - q^2M\right)^2 - \left(r^2 - 2Mr + q^2M^2\right)\right)^{\frac{1}{2}}}{1 - T^{*2}} - \frac{2M(1-q^2T^*)}{2(1-T^{*2})^{3/2}}\sin^{-1}\left(\frac{1-T^{*2}}{D}r - \frac{2M(1-q^2T^*)}{D}\right) \tag{1.44}$$

where $D = M\sqrt{(1-q^2T^*)^2 - q^2(1-q^2)(1-T^{*2})}$ is a constant.

Equation (1.44) tells us that in the case $T^* < 1$, the dust cloud's radius obtains its limit within a finite (and therefore compact) proper time interval. As t^* is a continuous increasing function of τ, r_b obtains its limit in finite coordinate (t^*) time. We will call this finite time t^*_-. At this point, the curve would collapse back into the black hole, hitting the past event horizon. Therefore, in order to have a collapsing model, we will, in the $T^* < 1$ case, assume that the radius of the dust cloud, $r_b(t^*)$ remains at $r_b(t^*_-)$ for all $t^* \le t^*_-$.

1.2.3 $T^* \ge 1$

Here we have that our dust cloud radius tends to ∞, as $\tau \to -\infty$. Thus the main part we will need to concern ourselves with is what happens to the surface of the dust cloud as $\tau \to -\infty$, $r_b \to \infty$. Equations (1.42) and (1.40) give us that

$$\dot{r}_b := \frac{dr}{dt^*} = \frac{\frac{dr}{d\tau}}{\frac{dt^*}{d\tau}} \to -\frac{\sqrt{T^{*2}-1}}{T^*} =: -a \qquad \text{as} \quad t^* \to -\infty \tag{1.45}$$

where we will refer to $a \in [0,1)$ as the asymptotic speed of the surface of the dust cloud.

1.2.4 Definition of the RNOS Manifold and Global Coordinates

We have a 3 parameter family of collapsing spacetimes, with parameters $M \geq 0$, $q \in [-1,1]$, and $T^* \in (0,\infty)$. However, we exclude the points with $T^* = 1 = q$. The topologies of the underlying manifolds are all given by:

$$\begin{aligned}\mathscr{M} &= \mathbb{R} \times [1,\infty) \times S^2 \\ &= \{(t^*, x, \theta, \varphi)\}.\end{aligned} \tag{1.46}$$

We scale the second coordinate in (1.46) to define $r = x\tilde{r}_b(t^*)$, so that the boundary is not at 1, but is at $\tilde{r}_b(t^*) = \max\{r_+, r_b(t^*)\}$, where t^* is the first coordinate. Then we have metric

$$g = -\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right) dt^{*2} + 2\left(\frac{2M}{r} - \frac{q^2M^2}{r^2}\right) dt^* dr + \left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right) dr^2 + r^2 g_{S^2} \tag{1.47}$$

$$t^* \in \mathbb{R} \qquad r \in [\tilde{r}_b(t^*), \infty)$$

where $M \geq 0$, $q \in [-1,1]$, r_+ is given by (1.35), and g_{S^2} is the Euclidean metric on the unit sphere.

Note that $\tilde{r}_b(t^*)$ is not a smooth function of t^*. Thus our manifold's smooth structure, as given by (t^*, r) coordinates, is not the same as the smooth structure given by (t^*, x) coordinates.

We have derived the following statements about r_b:

$$\dot{r}_b(t^*) := \frac{dr}{dt^*} \in (-1, 0] \tag{1.48}$$

$$\exists t_c^* \text{ s.t. } r_b(t_c^*) = r_+, r_b(t^*) > r_+ \quad \forall t^* < t_c^*, \tag{1.49}$$

where r_+ is the black hole horizon for the Reissner–Nordström spacetime given by (1.35).

We also have 2 possible past asymptotic behaviours for r_b. If $T^* < 1$, we have

$$r_b(t^*) = r_0 \quad \forall t^* \leq t_-^*, \tag{1.50}$$

and if $T^* \geq 1$, we have

$$\frac{dr_b}{dt^*} \to -a := \frac{\sqrt{T^{*2} - 1}}{T^*} \in (-1, 0] \quad \text{as} \quad t^* \to -\infty. \tag{1.51}$$

If $T^* = 1$, then $r_b \sim (-t^*)^{2/3}$ as $t^* \to -\infty$.

The RNOS models have the same exterior Penrose diagram as the original Oppenheimer–Snyder model, see Figure 2.1, derived in [2], for example.

We will also be using the double null coordinates given by:

$$u = t^* - \int_{s=3M}^{r} \frac{1 + \frac{2M}{s} - \frac{q^2M^2}{s^2}}{1 - \frac{2M}{s} + \frac{q^2M^2}{s^2}} ds \tag{1.52}$$

$$v = t^* + r \tag{1.53}$$

$$\partial_u = \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right)(\partial_{t^*} - \partial_r) \tag{1.54}$$

$$\partial_v = \frac{1}{2}\left(\left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right)\partial_{t^*} + \left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right)\partial_r\right) \tag{1.55}$$

$$g = -\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right) du dv + r(u,v)^2 g_{S^2}. \tag{1.56}$$

Much of the later discussion will be concerning u and v coordinates. Therefore, we will find it useful to parameterise the surface of the cloud by u and v. That is, given any u, define $v_b(u)$ to be the unique solution to

$$r(u, v_b(u)) = r_b(t^*(u, v_b(u))). \tag{1.57}$$

We will also define u_b in the domain $v \leq v_c$ as the inverse of v_b, *i.e.*

$$u_b(v) := v_b^{-1}(v). \tag{1.58}$$

We will be making use of the following properties of v_b:

$$v_b(u) \to v_c := v(t_c^*, r_+) \qquad \text{as } u \to \infty \tag{1.59}$$

$$v_c - v_b(u) = \begin{cases} Ae^{-\kappa u} + O(e^{-2\kappa u}) & |q| < 1 \\ \frac{A}{u} + O(u^{-3}) & |q| = 1. \end{cases} \tag{1.60}$$

$$v_b'(u) = \begin{cases} Ae^{-\kappa u} + O(e^{-2\kappa u}) & |q| < 1 \\ \frac{A}{u^2} + O(u^{-4}) & |q| = 1 \end{cases}. \tag{1.61}$$

These are straightforward calculations, once we note in the extremal case we can choose where $u = 0$ to remove the u^{-2} term to be zero in $v_c - v_b$. Here, κ is the surface gravity of the Reissner–Nordström black hole that our cloud is collapsing to form, given by

$$k^a \nabla_a k^b = \kappa k^b, \tag{1.62}$$

where k^a is the null Killing vector field tangent to the horizon. In Reissner–Nordström, $k^a = \partial_{t^*}$, and we have

$$\kappa = \frac{\sqrt{1-q^2}}{\left(2 + 2\sqrt{1-q^2} - q^2\right) M}. \tag{1.63}$$

Finally, we have four linearly independent Killing vector fields in our space time. The timelike Killing field, ∂_{t^*} does not preserve the boundary $\{r = r_b(t^*)\}$. However, we have 3 angular Killing fields, $\{\Omega_i\}_{i=1}^3$ (as given in (1.26)) which span all angular derivatives and are tangent to the boundary of the dust cloud.

Chapter 2

The Scattering Map

2.1 Introduction and Overview of Main Theorems

In this chapter we will be studying energy boundedness of solutions to the linear wave equation

$$\Box_g \phi = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\phi) = 0 \tag{2.1}$$

on both Oppenheimer–Snyder space-time $(\mathscr{M}, g)$ [39] and RNOS [1] backgrounds, as discussed in the previous chapter.

In the case of Oppenheimer–Snyder, we will further consider two different sets of boundary conditions: *reflective*, where we will impose the condition

$$\phi = 0 \ \texttt{on} \ r = r_b(t^*), \tag{2.2}$$

where this is understood in a trace sense, and *permeating*, where we will be solving the linear wave equation throughout the whole space-time, including the interior of the star. In the case of the RNOS backgrounds, we will only be considering reflective boundary conditions, as the interior will depend entirely on one's choice of matter model. We will then be using these results to define a scattering theory for these space-times.

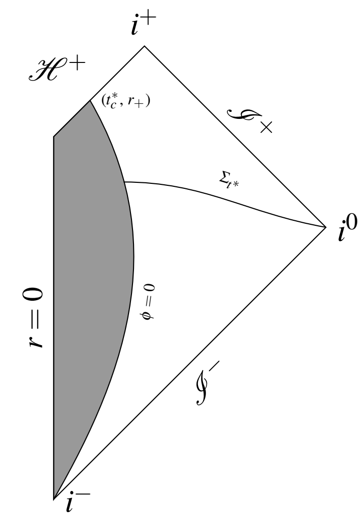


Figure 2.1 Penrose diagram of RNOS space-time with reflective boundary conditions, with spacelike hypersurface Σ_{t^*}.

The first main theorem dealing with solutions of (2.1) in the bulk of the space-time is informally stated below:

Theorem 1 (Non-degenerate Energy (N-energy) boundedness)**.** *In both RNOS space-times (with reflective boundary conditions) and Oppenheimer–Snyder space-time (with permeating boundary conditions), let the map $\mathscr{F}_{(t_0^*, t_1^*)}$ take the solution of* (2.1) *on a time slice $\Sigma_{t_0^*}$ (or Σ_{τ_0}), forward to the same solution on a later time slice, $\Sigma_{t_1^*} \cup (\mathscr{H}^+ \cap \{t^* \in [t_0^*, t_1^*]\})$ (or $\Sigma_{\tau_1} \cup (\mathscr{H}^+ \cap \{\tau \in [\tau_0, \tau_1]\})$). Then $\mathscr{F}_{(t_0^*, t_1^*)}$ is uniformly bounded in time with respect to the non-degenerate energy. Furthermore, for $t_1^* \leq t_c^*$ (or $\tau_1 \leq \tau_c$), its inverse is also bounded with respect to this non-degenerate energy.*

The contents of this theorem are stated more precisely across Theorems 2.5.3, 2.5.4, 2.5.5.

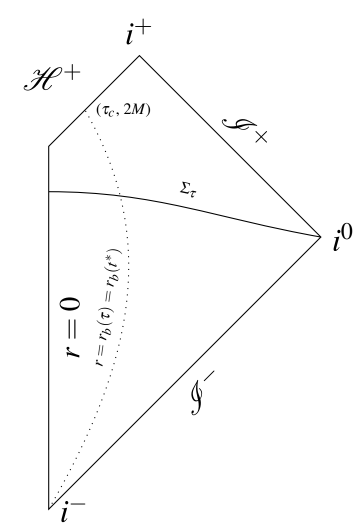


Figure 2.2 Penrose Diagram of Oppenheimer–Snyder space-time with permeating boundary conditions, with spacelike hypersurface Σ_τ.

The sphere $(t_c^*, 2M)$ and the time slice Σ_{t^*} (for $t^* < t_c^*$) are shown in Figure 2.1. The sphere $(\tau_c, 2M)$ and the time slice Σ_τ ($\tau < \tau_c$) are shown in Figure 2.2.

Non-degenerate energy will be defined more accurately later in this chapter, but it can be defined as the energy with respect to an everywhere timelike vector field (including on the horizon $\mathscr{H}^+$) which coincides with the timelike Killing vector in a neighbourhood of null infinity $\mathscr{I}^{\pm}$. This energy controls the L^2 norm of each 1^{st} derivative of the field, ϕ.

In the reflective case of Oppenheimer–Snyder, we also go on to show forwards and backwards boundedness of higher order derivatives, see Theorem 2.6.1 and 2.6.2. In the permeating case we go on to show forwards and backwards boundedness of 2^{nd} order derivatives, see Theorem 2.6.3.

We then consider the limiting process to look at the radiation field on past null infinity $\mathscr{I}^-$, and obtain the following result.

Theorem 2 (Existence and Non-degenerate Energy Boundedness of the Past Radiation Field)**.** *In both RNOS space-times (with reflective boundary conditions) and Oppenheimer–Snyder space-time (with permeating boundary conditions), we define the map $\mathscr{F}^-$ as taking the solution of* (2.1) *on $\Sigma_{t_0^*}$, $t^* \leq t_c^*$ (or $\Sigma_{\tau_0} \cup (\mathscr{H}^+ \cap \{\tau \leq \tau_0\})$) to the radiation field on $\mathscr{I}^-$. $\mathscr{F}^-$ is well-defined, and is bounded with respect to the non-degenerate energy.*

This theorem is stated more precisely as Theorem 2.7.1.

On Reissner–Nordström space-times, we know that the future radiation field exists, so the map $\mathscr{G}^+$ from data on Σ_{t^*} to $\mathscr{I}^+ \cup \mathscr{H}^+$ exists (see [35] for example). It is also bounded in terms of the N-energy, [13, 3]. It is, however, unbounded, going backwards, in terms of the N-energy (see for example [16]). This is stated more precisely as Proposition 2.7.2. This result immediately applies to Oppenheimer–Snyder space-time. Together with Theorem 2 and a new result about decay towards the past on asymptotically null foliations (see Lemma 2.7.1), this allows us to define the inverse of $\mathscr{F}^-$, $\mathscr{F}^+$ (see Proposition 2.7.1). This combination also gives us the final theorem:

Theorem 3 (Boundedness but non-surjectivity of the scattering map)**.** *We define the scattering map,*

$$\begin{aligned} &\mathscr{S}^+ : \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-} \to \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^+} \times \mathscr{E}^{N}_{\mathscr{H}^+} \\ &\mathscr{S}^+ := \mathscr{G}^+ \circ \mathscr{F}^+ \end{aligned} \tag{2.3}$$

on RNOS space-times (with reflective boundary conditions) and Oppenheimer–Snyder space-time (with permeating boundary conditions) from data on $\mathscr{I}^-$ to data on $\mathscr{I}^+ \cup \mathscr{H}^+$. $\mathscr{S}^+$ is injective and bounded, with respect to the non-degenerate energy (L^2 norms of $\partial_v(r\phi)$ on $\mathscr{I}^-$ and $\mathscr{H}^+$ and $\partial_u(r\phi)$ on $\mathscr{I}^+$). One can then define the inverse, $\mathscr{S}^-$, of (2.3)*, going backwards from $\mathscr{S}^+(\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-})$ (dense in $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^+} \times \mathscr{E}^{N}_{\mathscr{H}^+}$), in either the reflective or permeating case. However, $\mathscr{S}^-$ is not bounded with respect to the non-degenerate energy. It follows that $\mathscr{S}^+$ is not surjective. Moreover, $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^+} \times \{0\}_{\mathscr{H}^+}$ is not a subset of* $\mathrm{Im}(\mathscr{S}^+)$.

This Theorem is stated more precisely as Theorem 2.7.2.

In proving Lemma 2.7.1, we obtain a result on the rate at which our solution decays (towards i^-, with respect to this asymptotically null foliation) for data decaying sufficiently quickly towards spatial infinity. However we do not look at optimising this rate, as only very weak decay is required for Theorem 3.

The non-invertibility of $\mathscr{S}^+$ is inherited from that of $\mathscr{G}^+$. This ultimately arises from the red-shift effect along $\mathscr{H}^+$, which for backwards time evolution corresponds to a blue-shift instability. It is the existence of the map $\mathscr{F}^+$ mapping into the space of non-degenerate energy however, that extends this non-invertibility to data on $\mathscr{I}^-$. Note that for $\mathscr{I}^-$, the notion of energy is completely canonical. This is in contrast to the pure Reissner–Nordström case, where no such $\mathscr{F}^+$ exists.

It remains an open problem to precisely characterise the image of the scattering map $\mathscr{S}^+$.

2.2 Previous Work

There has been a substantial amount of work done concerning the scattering map on Schwarzschild. However there has been considerably less concerning the scattering map for collapsing space-times such as Oppenheimer–Snyder. The exterior of the star is a vacuum spherically symmetric space-time and therefore has the Schwarzschild metric by Birkhoff's Theorem, [40]. We will thus be using a couple of results in this region from previous papers. However, we will not be discussing the scattering map on Schwarzschild very much beyond this. For a more complete discussion of the wave equation on Schwarzschild, see [14].

Most previous works on scattering in gravitational collapse, such as [8, 26, 6, 34], assume that the star/dust cloud is at a finite radius from infinite past up to a certain time and then proceed to let this cloud collapse, as in the RNOS model with $T^* < 1$. Thus these models are stationary in all but a compact region of space-time. This model allows these previous works to avoid the difficulty of allowing the star to tend to infinite radius towards the past, as happens in the original Oppenheimer–Snyder model that we will be studying here. Also, dynamics on the interior of the star have not been examined, and so only the case of reflective boundary conditions has been studied previously. The energy current techniques we will be using here can, with relatively little difficulty, also be applied to these finite-radius models. These energy current methods are also more easily generalisable to other space-time models: for example, to obtain boundedness of the forward scattering map, all that is required to apply these techniques is that the star is collapsing. Nonetheless, in this thesis, the only interior we will consider is that of the Oppenheimer–Snyder model.

In this thesis, we look at defining the scattering map $\mathscr{S}^+$ geometrically as a map from data on $\mathscr{I}^-$ to data on $\mathscr{H}^+ \cup \mathscr{I}^+$ (equation (2.3)). This is treating scattering in terms of the Friedlander radiation formalism (as in [21]). In the above papers ([8, 26, 6, 34]), their solution is evolved forward a finite time, then evolved back to $t = 0$ with respect to either Schwarzschild metric (for the horizon radiation field) or Minkowski metric (for the null infinity radiation field). Then the authors show that the limit as that time tends to infinity exists. All this is done using the language of wave operators. For a comparison of these two approaches to scattering theory, the reader may wish to refer to Section 4 of [38].

Let us discuss two related works in more detail. The work [8] studies the Klein–Gordon equation ((2.1) is the massless Klein–Gordon equation, thus is studied as a special case) on the finite-radius model discussed above. In this context, the author obtains what can be viewed as a partial result towards the analogue of Theorem 1 for each individual spherical harmonic. However they do not find a bound independent of angular frequency.

Again in the finite-radius model, [26] studies the Dirac equation for spinors. However, as this has a 0^{th} order conserved current, this allows a Hilbert space to be defined such that the propagator through time is a unitary operator. Thus there is no need for the (first order) energy currents we will be using. This also allows questions of surjectivity to be answered with relative ease.

There have been no mathematical works considering the scattering map in the charged case. There have, however, been other works considering the underlying models of charged collapse, and there have been other works considering scattering on Reissner–Nordström backgrounds.

Most papers modelling collapsing models focus on the interior of the collapsing star. This thesis, however, will not focus on the specifics of interior models such as these in the charged case. There are many such models, which entirely depend on what equation of state is chosen for the interior of the dust cloud.

For simplicity in the charged case, this thesis will assume only that the surface of the collapsing cloud follows the motion of charged particles in the exterior spacetime, as discussed in Section 1.2. In fact the techniques in this chapter do not require an assumption this substantial, and will allow the results to be generalised beyond this. However, to avoid becoming mired in the details of the interior of the dust cloud, we will restrict to these RNOS models.

Generally, study of the scattering map in the exterior sub-extremal Reissner–Nordström spacetime is paired together with that of Schwarzschild, as it has similar behaviour (see [14]). The extremal case has been studied in detail separately, see [3]. Behaviour in this case differs substantially from the sub-extremal case. Scattering in the interior has also been studied independently, [32]. The exterior of the RNOS Models (see Section 1.2) is a spherically symmetric, vacuum solution to the Einstein–Maxwell equations, and thus has the Reissner–Nordström metric, by uniqueness (see [42], for example). However, this thesis will not be discussing the scattering map on Reissner–Nordström much beyond this, and instead will quote results from [14] (in the sub-extremal case) and [3] (in the extremal case). We refer the reader to these for a more complete discussion of scattering in Reissner–Nordström spacetimes.

2.3 Notation

In this section, we will be using similar notation to [2, 1].

We will be considering the following hypersurfaces in our manifold, equipped with the stated normals and volume forms. Note these normals will not neccessarily be unit normals, but have been chosen such that divergence theorem can be applied without involving additional factors.

$$\Sigma_{t_0^*} := \{(t^*,r,\theta,\varphi) :, t^* = t_0^*\} \qquad dV = r^2 dr d\omega \qquad dn = -dt^* \tag{2.4}$$

$$\Sigma_{u_0} := \{(t^*,r,\theta,\varphi) : u(t^*,r) = u_0\} \qquad dV = \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right) r^2 dv d\omega \qquad dn = -du \tag{2.5}$$

$$\Sigma_{v_0} := \{(t^*,r,\theta,\varphi) : v(t^*,r) = v_0\} \qquad dV = \frac{1}{2}\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right) r^2 du d\omega \qquad dn = -dv \tag{2.6}$$

$$S_{[t_0^*,t_1^*]} = \{(t^*,\tilde{r}_b(t^*),\theta,\varphi) \text{ s.t. } t^* \in [t_0^*,t_1^*]\} \quad dV = r^2 dt^* d\omega \qquad dn = d\rho := dr - \dot{\tilde{r}}_b dt^*, \tag{2.7}$$

where $d\omega$ is the Euclidean volume form on the unit sphere *i.e.*

$$d\omega = \sin\theta d\theta d\varphi. \tag{2.8}$$

Note that $d\omega$ will not be used as the volume form on the unit sphere in Chapter 3, to avoid confusion with ω being a frequency.

We will also later be using, for the permeating case, the eventually null foliation, $\tilde{\Sigma}_{\tau_0}$. These are the set of points with $\tau = \tau_0$ for $r < r_b$, and $v = v_0$ for $r \geq r_b$. Here v_0 is the value of v at $(\tau_0, r_b(\tau_0))$.

$$\tilde{\Sigma}_{\tau_0} = (\Sigma_{\tau_0} \cap \{r < r_b(\tau_0)\}) \cup (\Sigma_{v_0} \cap \{r \geq r_b(\tau_0)\}). \tag{2.9}$$

This will have the same volume form as Σ_{τ_0} for $r < r_b(\tau_0)$ and the same volume form as Σ_{v_0} for $r \geq r_b(\tau_0)$.

We will finally make use of

$$\bar{\Sigma}_{t_0,R} := (\Sigma_{u=t_0+R} \cap \{r^* \leq -R\}) \cup (\Sigma_{t_0} \cap \{r^* \in [-R,R]\}) \cup (\Sigma_{v=t_0+R} \cap \{r^* \geq R\}). \tag{2.10}$$

The volume form of $\bar{\Sigma}_{t_0,R}$ matches that of Σ_{u_0}, Σ_{v_0} and Σ_{t_0} in each section.

Whenever considering the reflective case, we will restrict these surfaces to $r \geq \tilde{r}_b(t^*)$. However, in the permeating case, or when considering pure Reissner–Nordström spacetime, this will not be required.

We define future/past null infinity by:

$$\mathscr{I}^+ := \mathbb{R} \times S^2 \qquad dV = du d\omega \qquad \mathscr{I}^- := \mathbb{R} \times S^2 \qquad dV = dv d\omega. \tag{2.11}$$

Past null infinity is viewed as the limit of Σ_{u_0} as $u_0 \to \infty$. For an appropriate function $f(u,v,\theta,\varphi)$, we will define the function "evaluated on $\mathscr{I}^+$" to be

$$f(v,\theta,\varphi)|_{\mathscr{I}^-} := \lim_{u\to-\infty} f(u,v,\theta,\varphi). \tag{2.12}$$

Similarly, $\mathscr{I}^+$ is considered to be the limit of Σ_{v_0} as $v_0 \to \infty$. For an appropriate function $f(u,v,\theta,\varphi)$, we will define the function "evaluated on $\mathscr{I}^+$" to be

$$f(u,\theta,\varphi) := \lim_{v\to\infty} f(u,v,\theta,\varphi). \tag{2.13}$$

From here onwards, any surface integral that is left without a volume form will be assumed to have the relevant volume form listed above, and all space-time integrals will be assumed to have the usual volume form $\sqrt{-det(g)}$.

We will be considering solutions of (2.1) which vanish on the surface $r = r_b(t^*)$ (in a trace sense). We will generally be considering these solutions to arise from initial data on a spacelike surface. Initial data will consist of imposing the value of the solution and its normal derivative, with both smooth and compactly supported.

We will then consider the following seminorms of a spacetime function f, given by:

$$\|f\|^2_{L^2(\Sigma)} = \int_\Sigma |f|^2 dV. \tag{2.14}$$

We will also define the $\dot{H}^1$ norm as:

$$\|f\|^2_{\dot{H}^1(\Sigma_{t_0^*})} := \int_{\Sigma_{t_0^*}} |\partial_{t^*} f|^2 + |\partial_r f|^2 + \frac{1}{r^2}\|\mathring{\nabla} f\|^2 dV \tag{2.15}$$

$$\|f\|^2_{\dot{H}^1(\Sigma_{u_0})} := \int_{\Sigma_{u_0}} \frac{|\partial_v f|^2}{\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)^2} + \frac{1}{r^2}\|\mathring{\nabla} f\|^2 dV \tag{2.16}$$

$$\|f\|^2_{\dot{H}^1(\Sigma_{v_0})} := \int_{\Sigma_{v_0}} \frac{|\partial_u f|^2}{\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)^2} + \frac{1}{r^2}\|\mathring{\nabla} f\|^2 dV, \tag{2.17}$$

where $\mathring{\nabla}$ is the induced gradient on the unit sphere. This is a tensor on the unit sphere, and we define the norm of such a tensor by

$$\|T\|^2 = \sum_{a_1,a_2,\dots a_m=1}^{n} |T_{a_1,a_2,\dots a_n}|^2 \tag{2.18}$$

for T an m tensor on S^n, in any orthonormal basis tangent to the sphere at that point.

Note that we have not yet defined the spaces for which the $\dot{H}^1$ norms will actually be norms.

The generalisation of the $\dot{H}^1$ norm are the $\dot{H}^n$ norms, which we will define on Σ_{t^*} by

$$\|f\|^2_{\dot{H}^n(\Sigma_{t_0^*})} := \sum_{\substack{n_1,n_2,n_3 \\ 1\le n_1+n_2+n_3\le n \\ n_1,n_2,n_3\ge 0}} \int_{\Sigma_{t_0^*}} \frac{1}{r^{2n_3}} \|\mathring{\nabla}^{n_3}\partial_r^{n_2}\partial_{t^*}^{n_1} f\|^2 dV. \tag{2.19}$$

Let $C_0^\infty(S)$ be the space of compactly supported functions on surface S, which vanish on $\{r = r_b(t^*)\} \cap S$. We will define the $\dot{H}^1(\Sigma_{t_0^*})$ norm on a pair of functions $\phi_0, \phi_1 \in C_0^\infty(\Sigma_{t_0^*})$ as follows:

$$\|(\phi_0,\phi_1)\|_{\dot{H}^1(\Sigma_{t_0^*})} := \|\phi\|_{\dot{H}^1(\Sigma_{t_0^*})} \qquad \text{for any } \phi \text{ s.t. } (\phi|_{\Sigma_{t_0^*}}, \partial_{t^*}\phi|_{\Sigma_{t_0^*}}) = (\phi_0,\phi_1). \tag{2.20}$$

We similarly define the $\dot{H}^1(\Sigma_{u_0})$ and $\dot{H}^1(\Sigma_{v_0})$ on $\phi_0 \in C_0^\infty(\Sigma_{u_0,v_0})$ as follows:

$$\|\phi_0\|_{\dot{H}^1(\Sigma_{u_0,v_0})} := \|\phi\|_{\dot{H}^1(\Sigma_{u_0,v_0})} \qquad \text{for any } \phi \text{ s.t. } \phi|_{\Sigma_{u_0,v_0}} = \phi_0. \tag{2.21}$$

We will also need to consider what functions we will be working with. For this, we will be using the same notation as [15, 2]. We first need to look at the notions of energy momentum tensors and energy currents (note this energy momentum tensor will be expressed as T, and is different from $\mathbf{T}$ in (1.1)).

$$T_{\mu\nu}(\phi) = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi \tag{2.22}$$
$$J^X_\mu = X^\nu T_{\mu\nu} \tag{2.23}$$
$$K^X = \nabla^\mu J^X_\mu \tag{2.24}$$
$$J^{X,w}_\mu = X^\nu T_{\mu\nu} + w\nabla_\mu(\phi^2) - \phi^2\nabla_\mu w \tag{2.25}$$
$$K^{X,w} = \nabla^\nu J^{X,w}_\nu = K^X + 2w\nabla_\mu\phi\nabla^\mu\phi - \phi^2\Box_g w \tag{2.26}$$
$$X\text{-energy}(\phi,S) = \int_S dn(J^X). \tag{2.27}$$

Here, dn is the normal to S. It should be noted that applications of divergence theorem do not introduce any additional factors with our choice of volume form and normal, i.e.

$$\int_{t^*\in[t^*_0,t^*_1]} K^{X,\omega} = -\int_{\Sigma_{t^*_1}} dn(J^{X,\omega}) + \int_{\Sigma_{t^*_0}} dn(J^{X,\omega}) - \int_{S_{[t^*_0,t^*_1]}} dn(J^{X,\omega}), \tag{2.28}$$

with similar equations holding for $\Sigma_{u,v}$ and in the permeating case.

For any $T_{\mu\nu}$ obeying the dominant energy condition, X future pointing and causal, and S spacelike, then the X-energy is non-negative.

For any pair of functions, $\phi_0, \phi_1 \in C^\infty_0(\Sigma_{t^*_0})$, and X a causal, future pointing vector, we define the X norm by

$$\|(\phi_0,\phi_1)\|^2_{X,\Sigma_{t^*_0}} := X\text{-energy}(\phi,\Sigma_{t^*_0}) \qquad \text{for any } \phi \text{ s.t. } (\phi|_{\Sigma_{t^*_0}}, \partial_{t^*}\phi|_{\Sigma_{t^*_0}}) = (\phi_0,\phi_1). \tag{2.29}$$

We similarly define for $\phi_0 \in C^\infty_0(\Sigma_{u_0,v_0})$

$$\|\phi_0\|^2_{X,\Sigma_{u_0,v_0}} := X\text{-energy}(\phi,\Sigma_{u_0,v_0}) \qquad \text{for any } \phi \text{ s.t. } \phi|_{\Sigma_{u_0,v_0}} = \phi_0. \tag{2.30}$$

Note that for any causal, future pointing X which coincides with the timelike Killing vector field ∂_{t^*} in a neighbourhood of $\mathscr{I}^\pm$, we have that the X norm is Lipschitz equivalent to the $\dot{H}^1$ norm.

For causal timelike vector X, we define the following function spaces

$$\mathscr{E}^X_{\Sigma_{t^*_0}} := Cl_{X,\Sigma_{t^*_0}}(C^\infty_0(\Sigma_{t^*_0}) \times C^\infty_0(\Sigma_{t^*_0})) \tag{2.31}$$
$$\mathscr{E}^X_{\Sigma_{u_0,v_0}} := Cl_{X,\Sigma_{u_0,v_0}}(C^\infty_0(\Sigma_{u_0,v_0})) \tag{2.32}$$

where these closures are in H^1_{loc} with respect to the subscripted norms.

For $\psi_0 \in C^\infty_0(\mathscr{I}^\pm)$, we define

$$\|\psi_0\|^2_{\partial_{t^*},\mathscr{I}^+} := \int_{\mathscr{I}^+} |\partial_v\psi_0|^2 dv d\omega \tag{2.33}$$
$$\|\psi_0\|^2_{\partial_{t^*},\mathscr{I}^-} := \int_{\mathscr{I}^-} |\partial_u\psi_0|^2 du d\omega. \tag{2.34}$$

We define the energy spaces $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^\pm}$ by

$$\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^\pm} := Cl_{\partial_{t^*},\mathscr{I}^\pm}(C^\infty_0(\mathscr{I}^\pm)). \tag{2.35}$$

We will finally be considering "well behaved" functions to prove results, and extending our results by density arguments. If a function on $\mathscr{M}$ is smooth and compactly supported on Σ_{t^*} for every t^* (or compactly supported on Σ_τ and in $H^2_{loc}(\Sigma_\tau)$ for every τ), then we will refer to this function as being in $C^\infty_{0\forall t^*}$ (or $H^2_{0\forall\tau}$).

2.4 Existence and Uniqueness of Solutions to the Wave Equation

Consider initial data given by $\phi = \phi_0$ and $\partial_{t^*}\phi = \phi_1$ on the spacelike hypersurface $\Sigma_{t_0^*}$ (or $\partial_\tau \phi = \phi_1$ on Σ_{τ_0}). For the reflective case, we also impose Dirichlet conditions on the surface of the star, $\phi = 0$ on $r = R(t)$. We first show existence of a solution to the forced wave equation,

$$\Box_g \phi = \frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\phi) = F \tag{2.36}$$

for g as given in (1.47) and (1.22).

These are standard results which can be taken from literature, but there is no elementary reference. For completeness we will include a proof here.

2.4.1 The Reflective Case

In the reflective case, we initially prove existence and uniqueness for smooth, compactly supported initial data. We will be proving existence up until the surface of the star passes through the horizon. For later t^* times, we are then in exterior Schwarzschild space-time with the usual boundary ar $r = 2M$, so can refer to standard existing proofs of existence and uniqueness (see for example proposition 3.1.1 in [14]). The proof below closely follows that of Theorems 4.6 and 5.3 of Jonathan Luk's notes on Nonlinear Wave Equations [33], and comes in two parts:

We proceed by first proving uniqueness via the following lemma:

Lemma 2.4.1 (Uniqueness of Solution to the Forced Wave Equation)**.** *Let $\phi \in C^\infty_{0\forall t^*}$ be a solution to equation* (2.36) *in some region $r \geq r_b(t^*) \geq 2M$, $t^*_{-1} \leq t^*_0 \leq t^*_1$, with*

$$\begin{gathered} \phi = 0 \text{ on } r = r_b(t^*) \\ \left.\begin{aligned} \phi &= \phi_0 \\ \partial_{t^*}\phi &= \phi_1 \end{aligned}\right\} \text{ on } \Sigma_{t_0^*} \end{gathered} \tag{2.37}$$

with g_{ab} given by (1.47).

It follows that $\exists A, C > 0$ s.t.

$$\sup_{t^*\in[t^*_{-1},t^*_1]} \|\phi\|_{\dot{H}^1(\Sigma_{t^*})} \leq C\left(\|(\phi_0,\phi_1)\|_{\dot{H}^1(\Sigma_{t_0^*})} + \int_{t^*_{-1}}^{t^*_1} \|F\|_{L^2(\Sigma_{t^*})}(t^*)dt^*\right)\exp\left(A|t_1^* - t^*_{-1}|\right). \tag{2.38}$$

In particular, if $\phi, \phi' \in C^\infty_{0\forall t^*}$ are both solutions to the above problem, then consider $\zeta = \phi - \phi'$. We have that ζ solves equation (2.36) *with $F = 0$ and has* 0 *initial data. Thus, $\zeta = \phi - \phi' = 0$ everywhere, and we have uniqueness.*

Proof. We first consider coordinates $(t^*, \rho, \theta, \varphi)$, where $\rho = r - r_b(t^*) + r_+$. This causes ∂_{t^*} to be tangent to the boundary $r = r_b(t^*)$. The metric then takes the form

$$\begin{aligned} g = &-\left(\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right) - \frac{2M}{r}\left(2 - \frac{q^2M}{r}\right)\dot{r_b} - \left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right)\dot{r_b}^2\right)dt^{*2} \\ &+ \left(\frac{2M}{r} - \frac{q^2M^2}{r^2} + \left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right)\dot{r_b}\right)2d\rho dt^* + \left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right)d\rho^2 \\ &+ r(t^*,\rho)^2 g_{S^2}. \end{aligned} \tag{2.39}$$

We integrate the following identity:

$$\partial_0\phi(\partial_a(\bar{g}^{ab}\partial_b\phi) - \bar{F})) = 0 \tag{2.40}$$

for

$$\bar{g}^{ab} = \sqrt{-g}g^{ab} \tag{2.41}$$

$$\bar{F} = \sqrt{-g}F. \tag{2.42}$$

We look at the cases $a = b = 0$, $a = i, b = j$, and $\{a,b\} = \{0,i\}$ separately, where $i, j \in \{1,2,3\}$.

$$\int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} \partial_0\phi\partial_0(\bar{g}^{00}\partial_0\phi)d\rho d\omega^2 dt^* = \frac{1}{2}\left(\int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} (\partial_0\bar{g}^{00})(\partial_0\phi)^2 d\rho d\omega^2 dt^* + \left(\int_{\Sigma_{t_1^*}} - \int_{\Sigma_{t_0^*}}\right)\bar{g}^{00}(\partial_0\phi)^2 d\rho d\omega^2\right). \tag{2.43}$$

$$\int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} \partial_0\phi\partial_i(\bar{g}^{ij}\partial_j\phi)d\rho d\omega^2 dt^* = \frac{1}{2}\left(\int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} (\partial_0\bar{g}^{ij})(\partial_i\phi\partial_j\phi)d\rho d\omega^2 dt^* - \left(\int_{\Sigma_{t_1^*}} - \int_{\Sigma_{t_0^*}}\right)\bar{g}^{ij}(\partial_i\phi\partial_j\phi)d\rho d\omega^2\right). \tag{2.44}$$

$$\int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} \partial_0\phi(\partial_i(\bar{g}^{i0}\partial_0\phi) + \partial_0(\bar{g}^{i0}\partial_i\phi))d\rho d\omega^2 dt^* = \int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} (\partial_0\bar{g}^{i0})(\partial_0\phi\partial_i\phi)d\rho d\omega^2 dt^*. \tag{2.45}$$

Here we have integrated by parts. Using the fact that as $\phi = 0$ on our boundary and ∂_0 is tangent to our boundary, we can see that $\partial_0\phi = 0$ on our boundary. We have used this to simplify the above boundary terms. Using

$$\bar{g}^{ab} = \begin{pmatrix} -\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right) & \frac{2M}{r}-\frac{q^2M^2}{r^2}+\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\dot{r_b} \\ \frac{2M}{r}-\frac{q^2M^2}{r^2}+\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\dot{r_b} & \left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)-\frac{2M}{r}\left(2-\frac{q^2M}{r}\right)\dot{r_b}-\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\dot{r_b}^2 \end{pmatrix} r^2\sin\theta \tag{2.46}$$

for $a,b = 0,1$, we see that

$$|\partial_0\bar{g}^{ab}|\begin{cases} \leq A'r^2 & a,b = 0,1 \\ = 0 & a,b = 2,3. \end{cases} \tag{2.47}$$

(Note that coordinate singularities have been removed from g^{ab} by multiplying by $\sqrt{-g}$.)

We then have, by summing (2.43), (2.44) and (2.45) together, that

$$\begin{aligned} \int_{\Sigma_{t_1^*}} \bar{g}^{ij}\partial_i\phi\partial_j\phi - \bar{g}^{00}(\partial_0\phi)^2 d\rho d\omega^2 &= \int_{\Sigma_{t_0^*}} \bar{g}^{ij}\partial_i\phi\partial_j\phi - \bar{g}^{00}(\partial_0\phi)^2 d\rho d\omega^2 + \frac{1}{2}\int_{t_0^*}^{t_1^*}(\partial_0\bar{g}^{ab})\partial_a\phi\partial_b\phi - \bar{F}\partial_0\phi d\rho d\omega^2 dt^* \\ &\leq \int_{\Sigma_{t_0^*}} \bar{g}^{ij}\partial_i\phi\partial_j\phi - \bar{g}^{00}(\partial_0\phi)^2 d\rho d\omega^2 \\ &\quad + \frac{1}{2}\int_{t_0^*}^{t_1^*} \|\phi\|_{\dot{H}^1(\Sigma_{t^*})}\|F\|_{L^2(\Sigma_{t^*})} + A'\|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}dt^*. \end{aligned} \tag{2.48}$$

Note that the bar is removed from F as the factor of $\sqrt{-g}$ is absorbed into the volume form in the norms of F and ϕ.

Then we define

$$E(t_1^*) = \int_{\Sigma_{t_1^*}} \bar{g}^{ij}\partial_i\phi\partial_j\phi - \bar{g}^{00}(\partial_0\phi)^2 d\rho d\omega^2. \tag{2.49}$$

As the surface of the star is timelike, we have that $-g_{00} = g^{11}$ is bounded above and below by positive constants independent of time (from equation (1.5)). We note that the r^2 term from using $\bar{g}$ instead of g is identical to the volume form in $\|\phi\|^2_{\dot{H}^1(\Sigma_{t_1^*})}$. This implies $E(t_1^*) \sim \|\phi\|^2_{\dot{H}^1(\Sigma_{t_1^*})}$. Thus, using the fact that the RHS of (2.48) is increasing in t_1^*, we have

$$\begin{aligned} f(t_1^*) := \sup_{t^*\in[t_0^*,t_1^*]} \|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} &\leq C \sup_{t^*\in[t_0^*,t_1^*]} E(t^*) \leq C'\|\phi\|^2_{\dot{H}^1(\Sigma_0)} + C\int_{t_0^*}^{t_1^*} \|\phi\|_{\dot{H}^1(\Sigma_{t^*})}\|F\|_{L^2(\Sigma_{t^*})} + A'\|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} dt^* \\ &\leq C' f(t_0^*) + C\sqrt{f(t_1^*)}\int_{t_0^*}^{t_1^*} \|F\|_{L^2(\Sigma_{t^*})} dt^* + C\int_{t_0^*}^{t_1^*} A' f(t^*) dt^* \\ &\leq C' f(t_0^*) + \frac{C^2}{2}\left(\int_{t_0^*}^{t_1^*} \|F\|_{L^2(\Sigma_{t^*})} dt^*\right)^2 + \frac{f(t_1^*)}{2} + C\int_{t_0^*}^{t_1^*} A' f(t^*) dt^*. \end{aligned} \tag{2.50}$$

We can then subtract the $f(t_1^*)/2$ term from both sides to end up with an inequality of the form

$$f(t_1^*) \leq A(t_0^*, t_1^*) + \int_{t_0^*}^{t_1^*} f(t^*) h(t^*) dt^*. \tag{2.51}$$

An application of Gronwall's inequality gives our result, but with t_{-1}^* replaced with t_0^*. We then repeat the same argument with time reversed to obtain the final result. □

Note we have written out the above argument explicitly in coordinates. It could be written out using the energy momentum tensor and a suitable vector field multiplier, as we have done in Section 2.5.

Next we need to deal with existence. To do this, we prove the following theorem:

Theorem 2.4.1 (Existence of Reflective Solutions)**.** *Let $F \in C^k([t_{-1}^*, t_1^*]; C_0^\infty(\Sigma_{t^*}))$ $\forall k \in \mathbb{N}$ and g_{ab} as above. Let also ϕ_0 and ϕ_1 smooth, compactly supported functions on $\Sigma_{t_0^*}$ such that $\phi_0(r_b(t_0^*), \theta, \varphi) = 0$, and such that these initial conditions are not incompatible with the wave equation at $r = r_b(t^*)$*. There exists a $C^\infty_{0\forall t^*}$ solution to equation* (2.36) *subject to* (2.37)*.*

*By this, we mean that setting $\phi|_{\Sigma_{t_0^*}} = \phi_0$ and $\partial_{t^*}\phi_1|_{\Sigma_{t_0^*}} = \phi_1$ imposes the value of all first and second order derivatives of ϕ on $\Sigma_{t_0^*}$, other than $\partial^2_{t^*}\phi$. As we also require ϕ to vanish on the surface $r = r_b(t^*)$, this boundary condition imposes that $\partial_{t^*}\phi + \dot{r}_b\partial_r\phi = 0$ and $(\partial_{t^*} + \dot{r}_b\partial_r)^2\phi = 0$ on $r = r_b(t^*)$. We require that setting $\phi|_{\Sigma_{t_0^*}} = \phi_0$ and $\partial_{t^*}\phi_1|_{\Sigma_{t_0^*}} = \phi_1$ does not cause a contradiction between this boundary condition and the wave equation at $(t_0^*, r_b(t_0^*), \theta, \varphi)$.

Proof. We begin the proof with the case $(\phi_0, \phi_1) = (0, 0)$. Let the set $C_0 \subset C^\infty_{0\forall t^*}$ be the image under the map $\Box_g$ of $C_0^\infty(\mathscr{M})$. We define the map W by:

$$\begin{aligned} W : C_0 &\to \mathbb{R} \\ \Box_g\psi &\mapsto \int_{t_{-1}^*}^{t_1^*}\int_{\Sigma_{t^*}} \psi F\sqrt{-g}\,d\rho\, d\theta\, d\varphi\, dt^* =: \langle F, \psi\rangle \end{aligned}$$

This is well defined by our previous uniqueness lemma: suppose two functions $\psi_1, \psi_2 \in C_0$ have $\Box_g\psi_1 = \Box_g\psi_2$. Then we can choose t_0^* to be far back enough that $\Sigma_{t_0^*}$ does not intersect the support of either ψ_1 or ψ_2. Thus $\psi_1 - \psi_2$ solves (2.1) with vanishing initial data. Lemma 2.4.1 then gives $\psi_1 - \psi_2 = 0$ everywhere, i.e. they are equal.

We then proceed by quoting Lemma 5.2 in [33], which relies on definitions of H^{-k} spaces. The space $H^{-k}(\Sigma_{t^*})$ is defined to be the dual of $H^k(\Sigma_{t^*})$ (the space of bounded linear maps from $H^k(\Sigma_{t^*})$ to $\mathbb{R}$). Note also that, as a Hilbert space, $H^k(\Sigma_{t^*})$ is reflexive, i.e. the dual of $H^{-k}(\Sigma_{t^*})$ is $H^k(\Sigma_{t^*})$. In the permeating case, we define $H^{-k}(\Sigma_\tau)$ in an identical manner.

Lemma 2.4.2. *Suppose $\psi \in C_0^\infty((-\infty, t_1^*) \times \Sigma_{t^*})$, supported away from $r = r_b(t^*)$, and g as above. Fix $t_0^* \in (-\infty, t_1^*)$. Then for any $m \in \mathbb{Z}$, $\exists C = C(m, t_0^*, t_1^*, g) > 0$ s.t.*

$$\|\psi\|_{H^m(\Sigma_{t_2^*})} \leq C\int_{t_0^*}^{t_1^*} \|\Box_g\psi\|_{H^{m-1}(\Sigma_{t^*})}(s)dt^* \quad \forall t_2^* \in [t_0^*, t_1^*]. \tag{2.52}$$

Remark 2.4.1. *To see this from [33], one must first "Euclideanise", i.e. replace angular and r coordinates with some x,y,z in order for these coordinates to be everywhere regular. We can then extend our metric smoothly to inside the star. Using the result of Lemma 2.4.1 allows the proof to proceed exactly as in [33]. Note that linear maps on the space extended inside the star are also linear maps when restricted to functions on the outside of the star.*

Lemma 2.4.2 then gives the bound

$$\begin{aligned}|W(\Box_g\psi)| &= \left|\int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}}\psi F\sqrt{-g}d\rho d\theta d\varphi dt^*\right| \\ &\leq C\left(\int_{t_{-1}^*}^{t_1^*}\|F\|_{H^{k-1}(\Sigma_{t^*})}(t^*)dt^*\right)\left(\sup_{t^*\in[t_{-1}^*,t_1^*]}\|\psi\|_{H^{-k+1}(\Sigma_{t^*})}\right)\leq C\int_{t_{-1}^*}^{t_1^*}\|\Box_g\psi\|_{H^{-k}(\Sigma_s)}(s)ds, \end{aligned} \tag{2.53}$$

for smooth and compactly supported functions away from the horizon. We then take the closure of such functions with respect to the H^k norm, for which W is linear and bounded. Thus by Hahn–Banach (Theorem 5.1, [33]), there exists a function $\phi\in(L^1((-\infty,T);H^{-k}(\Sigma_{t^*})))^*=L^\infty((-\infty,T);H^k(\Sigma_{t^*}))\ \forall k$, which extends W as a linear map. This means

$$\langle F,\psi\rangle=\langle\phi,\Box_g\psi\rangle\qquad\forall\psi\in C_0^\infty(\mathscr{M}). \tag{2.54}$$

Now $\Box_g$ obeys

$$\langle\Box_g\psi_1,\psi_2\rangle=\langle\psi_1,\Box_g\psi_2\rangle\quad\forall\psi_{1,2}\in C_0^\infty((-\infty,t_1^*)\times\Sigma_{t^*}). \tag{2.55}$$

Thus equation (2.54) means that ϕ is a solution of (2.36) in the sense of distributions.

We then consider the following equation which $\partial_{t^*}\phi$ solves, in a distributional sense:

$$v^\mu\nabla_\mu(\dot\phi)=h\dot\phi+F_1, \tag{2.56}$$

for

$$v^\mu=(-g^{00},-2g^{01},-2g^{02},-2g^{03}) \tag{2.57}$$

$$h=\frac{1}{\sqrt{-g}}\partial_\nu\left(g^{\nu0}\sqrt{-g}\right) \tag{2.58}$$

$$F_1=\frac{1}{\sqrt{-g}}\partial_\nu\left(g^{\nu i}\sqrt{-g}\right)\partial_i\phi+g^{ij}\partial_i\partial_j\phi-F. \tag{2.59}$$

We explicitly have h and F_1, as we have ϕ and its spacelike derivatives. We can then easily solve this along integral curves of v^μ to obtain that $\dot\phi$ exists as a function and is continuous.

We then look at the difference between equation (2.56) and the wave equation (2.36). Here we are considering everything as distributions rather than functions. This gives us that

$$v^\mu\nabla_\mu\left(\dot\phi-\partial_{t^*}\phi\right)-h\left(\dot\phi-\partial_{t^*}\phi\right)=0. \tag{2.60}$$

Applying the zero distribution is the same as integrating against the zero function. We also know $\dot\phi-\partial_{t^*}\phi$ is zero on the initial surface. It is then zero along all integral curves of v^μ, and is therefore the zero function everywhere. Thus $\partial_{t^*}\phi$ exists everywhere and is continuous.

Then, by considering equation (2.36) and its derivatives, we can determine further weak derivatives with respect to time. If $F\in C^k([t_{-1}^*,t_1^*];C_0^\infty(\Sigma_{t^*}))\ \forall k$, then our final solution has finite $H^k(\Sigma_{t^*})$ norm for all k and all $t^*\in[t_{-1}^*,t_1^*]$, This means it is smooth. Due to finite speed of propagation of the wave equation it is also compactly supported on each Σ_{t^*}.

Finally, we show ϕ is a classical solution. Let ψ be an arbitrary function in $C_0^\infty(\mathscr{M})$, supported away from the boundary. We can then integrate (2.54) by parts. Using the fact ϕ is smooth, we can see that $\Box_g\phi=F$.

By choosing $k=1$, we note that ϕ extends W to the closure of $C_0^\infty(\mathscr{M})$ under the H^1 norm. In particular, this includes functions which are smooth with non-vanishing derivative at the horizon. From this set, we can choose any arbitrary smooth compactly supported function ψ. Let us choose one which is zero at the boundary, but with non-zero normal derivative at the boundary. The boundary term we obtain when integrating (2.54) by parts gives that $\phi = 0$ on $r = r_b(t^*)$, as required, provided $F = 0$ on the boundary.

Now let (ϕ_0, ϕ_1) be smooth, as in the statement of the theorem. Let $u \in C_0^\infty([0,t_1^*] \times \Sigma_{t^*})$ be any function with $(u, \partial_t u) = (\phi_0, \phi_1)$ on $t = t_0^*$, and $u = 0$ on the boundary $r = r_b(t^*)$. Then if we solve

$$\Box_g v = F_2 = F - \Box_g u \tag{2.61}$$
$$(v, \partial_t v) = (0,0) \text{ on } t = t_0^*, \tag{2.62}$$

then $\phi := v + u$ is our required solution. The existence of such a u for which F_2 vanishes at $r = r_b$ makes use of * in the statement of the Theorem. □

Remark 2.4.2. *Theorem 2.4.1 will allow us to extend other results. Suppose we obtain any result on boundedness between times slices Σ_{t^*} in the $H^1(\Sigma_{t^*})$ norm (not necessarily uniform in time). We can use a density argument to obtain that given initial data in $H^1(\Sigma_{t_0^*})$, there exists an $H^1(\Sigma_{t^*})$ $\forall t^*$ solution. Again, this would be a solution in the sense of distributions (see already Theorem 2.5.2).*

2.4.2 The Permeating Case

The proof for the permeating case follows almost identical lines to that of the reflective case. There are fewer concerns about the boundary, but the solution itself cannot be shown to be smooth for smooth initial data.

We still have Lemma 2.4.1 applying in this case, with almost no change to the proof. Lemma 2.4.2 also remains the same for all $m \leq 2$. This can be seen by considering $(\tau, R, \theta, \varphi)$ coordinates ((1.27) in chapter 1). We can then commute with both angular derivatives and ∂_τ, and also rearranging (2.36) for $\partial_R^2 \phi$. This just leaves the analogue of Theorem 2.4.1:

Proposition 2.4.1 (Existence of Permeating Solutions with Initial Data Constraints)**.** *Suppose $F \in C^1([\tau_{-1}, \tau_1]; H^1(\Sigma_\tau))$, and g_{ab} as above. Suppose also we are given $(\phi_0, \phi_1) \in H^1(\Sigma_\tau)$ such that there exists a function $u \in \Box_g^{-1}(H^1(\mathscr{M})) \cap H^2(\mathscr{M})$ with $(\phi_0, \phi_1) = (u, \partial_\tau u)$ on Σ_{τ_0}. Then there exists an $H^2([\tau_{-1}, \tau_1] \times \Sigma_\tau)$ weak solution to equation* (2.36)*, subject to:*

$$\left.\begin{aligned} \phi &= \phi_0 \\ \partial_\tau \phi &= \phi_1 \end{aligned}\right\} \text{ on } \Sigma_{\tau_0}. \tag{2.63}$$

Proof. Again, we begin with the $(\phi_0, \phi_1) = (0,0)$ case. We define the map W exactly as in the reflecting case. We define it on C_0', the image under the map $\Box_g$ on $C_0^\infty(\mathscr{M})$. Note the components of g are H^1_{loc} functions, so have weak derivatives in L^2_{loc}. Thus this operator still exists. As before, this operator is well defined, is linear, and is bounded.

Thus, again by Hahn–Banach, there exists a function $\phi \in (L^1((-\infty, \tau_1); H^{-k}(\Sigma_{t^*})))^* = L^\infty((-\infty, \tau_1); H^k(\Sigma_{t^*}))$ $\forall k \leq 2$ such that

$$\langle F, \psi \rangle = \langle \phi, \Box_g \psi \rangle \qquad \forall \psi \in C_0^\infty(\mathscr{M}). \tag{2.64}$$

As before, we can show ϕ has a τ derivative by considering the equation obeyed by $\partial_\tau \phi$ as a distribution. If F is in H^1, then ϕ has two weak spatial derivatives. It also has a τ derivative with spacelike weak derivatives. By integrating (2.64), we obtain that it also has a second weak time derivative. Thus it is H^2, and thus our solution is a weak solution of (2.36).

We then proceed with the final section in exactly the same way. Note that given our function, u, we can take $F_2 = F - \Box_g u$ well behaved. Thus our solution is H^2, and therefore is a solution in a weak sense. However, this sense is sufficient for the applications listed in later sections. □

The final thing we need in order to complete existence of solutions is the following: we need to show that initial data matching our condition $(\phi_0, \phi_1) = (u, \partial_\tau u)$ on Σ_{τ_0} is dense in $H^1(\Sigma_{\tau_0})$:

Proposition 2.4.2. *Let (ϕ_0,ϕ_1) be any pair of functions in $C_0^\infty(\Sigma_{\tau_0})\times C_0^\infty(\Sigma_{\tau_0})$. Then there exists a sequence of globally defined functions $u_n\in C_0^\infty(\mathscr{M})\cap\Box_g^{-1}(H^1(\mathscr{M}))$ such that*

$$(u_n|_{\Sigma_{\tau_0}},\partial_\tau u_n|_{\Sigma_{\tau_0}})\xrightarrow{H^1(\Sigma_{\tau_0})}(\phi_0,\phi_1). \tag{2.65}$$

Proof. We first remove the region over which g is not smooth by defining a smooth sequence $(\phi_{0,n},\phi_{1,n})$ such that

$$(\phi_{0,n},\phi_{1,n})\xrightarrow{H^1(\Sigma_{\tau_0})}(\phi_0,\phi_1) \tag{2.66}$$

and $\partial_r\phi_{0,n}=\phi_{1,n}=\partial_r\phi_{1,n}=0$ for the region $[r_b-1/n,r_b+1/n]$.

Let χ be a smooth cut-off function which is 1 outside $[-2,2]$ and 0 inside $[-1,1]$. We first construct $\phi_{1,n}$: Let $\phi_{1,n}=\phi_1\chi(n(r-r_b))$. It is clear that this tends to ϕ_1 in the H^0 norm. It is also clear that it and its r derivative are 0 in the required region. Then we choose $\phi_{0,n}=\chi(n(r-r_b))\phi_0+(1-\chi(n(r-r_b)))\phi_0(r_b)$.

It is clear to see that the r derivative vanishes while $\chi=0$. All that remains is to show that

$$\|((1-\chi(n(r-r_b)))(\phi_0-\phi_0(r_b)),0)\|_{H^1(\Sigma_{\tau_0})}\to 0. \tag{2.67}$$

It is easy to see that the L^2 norm of this tends to 0. Similarly the angular derivatives tend to 0 in L^2. Then all that is left to prove is that the r derivative tends to 0.

$$\begin{aligned}\|\partial_r(\phi_0-\phi_{0,n})\|_{L^2(\Sigma_{\tau_0})}&=\|(1-\chi(n(r-r_b)))\partial_r\phi_0-n\chi'(n(r-r_b))(\phi_0-\phi_0(r_b))\|_{L^2(\Sigma_{\tau_0})}\\&\le\|(1-\chi(n(r-r_b)))\partial_r\phi_0\|_{L^2(\Sigma_{\tau_0})}+\|n\chi'(n(r-r_b))(\phi_0-\phi_0(r_b))\|_{L^2(\Sigma_{\tau_0})}\\&\le\|1-\chi(n(r-r_b))\|_{L^2(\Sigma_{\tau_0})}\sup|\partial_r\phi_0|\\&\quad+\sup_{r\in[r_b-1/n,r_b+1/n]}|n(\phi_0-\phi_0(r_b))|\|\chi'(n(r-r_b))\|_{L^2(\Sigma_{\tau_0})}.\end{aligned} \tag{2.68}$$

The first term in the RHS tends to 0, as $(1-\chi(n(r-r_b)))\in[0,1]$ is only supported in $r\in[r_b-2/n,r_b+2/n]$. The supremum in the second terms tends to $|\partial_r\phi_0(r_b)|$, so is bounded. The χ' in the second term is bounded and only non-zero in a region whose volume tends to 0. Therefore the whole second term also tends to 0.

Now, given the pair $(\phi_{0,n},\phi_{1,n})$, we define $u_n:=(\phi_{0,n}+\tau\phi_{1,n})(1-\chi((2n\tau)))$.

As $\partial_\tau r_b\le 1$, we have that $\partial_r u_n=\partial_\tau u_n=\partial_r\partial_\tau u_n=0$ for all $r\in[r_b-1/2n,r_b+1/2n]$. We also have $(u_n,\partial_\tau u_n)=(\phi_{0,n},\phi_{1,n})$ at $\tau=0$. Thus we can see

$$\Box_g u_n=\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b u_n)=H^1\text{-terms}+\sum_{a,b\in\{r,\tau\}}(\partial_a g^{ab})\partial_b u_n. \tag{2.69}$$

The only terms in the sum where $\partial_a g^{ab}\notin H^1_{loc}$ is at $r=r_b$. However, in that region we have $\partial_r u_n=\partial_0\tau u_n=0$. Thus $\Box_g u_n\in H^1_{loc}$. As ϕ_0 and ϕ_1 are compactly supported, we then obtain that $\Box_g u_n\in H^1$. □

Combining Propositions 2.4.1 and 2.4.2 gives us the following Theorem:

Theorem 2.4.2 (Existence of Permeating Solutions)**.** *There exists a dense subset D of $H^1(\Sigma_{\tau_0})$ with the following property. Given initial data $(\phi_0,\phi_1)\in D$, there exists an $H^2_{0\forall\tau}$ solution to* (2.1) *and* (2.63) *in the permeating case.*

Proof. We use the subset given by

$$D=\{(u,\partial_\tau u),u\in C_0^\infty(\mathscr{M})\cap\Box_g^{-1}(H^1(\mathscr{M}))\}. \tag{2.70}$$

This is dense, by Proposition 2.4.2, and has an $H^2_{0\forall\tau}$ solution by Proposition 2.4.1. □

Remark 2.4.3. *As previously, Theorem 2.4.2 allows us to extend other results. Suppose we obtain a result on boundedness between times slices Σ_τ in the $H^1(\Sigma_\tau)$ norm (not necessarily uniform in time). Then we can use a* *density argument to obtain the following: given initial data in $H^1(\Sigma_{\tau_0})$), there exists an $H^1(\Sigma_\tau)\ \forall\tau$ solution, in the sense of distributions (see already Theorem 2.5.6).*

2.5 Boundedness of the Wave Equation

In this section we work towards proving Theorem 1, as stated in the overview. We will prove the result stated in that Theorem firstly for the RNOS model with reflective boundary conditions, then we will prove the result in the Oppenheimer–Snyder case with permeating boundary conditions.

2.5.1 The Reflective Case

We will prove this boundedness result in two sections. We will first prove that going forwards we have a uniform bound on the $\dot{H}^1$ norm, i.e. there exists a constant $C(M,q,T^*)$ such that

$$\|\phi\|_{\dot{H}^1(\Sigma_{t_1^*})} \leq C\|\phi\|_{\dot{H}^1(\Sigma_{t_0^*})} \quad \forall t_1^* \geq t_0^*. \tag{2.71}$$

In the second section we will prove the analogous statement going backwards in time:

$$\|\phi\|_{\dot{H}^1(\Sigma_{t_0^*})} \leq C\|\phi\|_{\dot{H}^1(\Sigma_{t_1^*})} \quad \forall t_0^* \leq t_1^* \leq t_c^*. \tag{2.72}$$

Note the backwards in time version includes a condition on $t_1^* \leq t_c^*$, as, were $t_1^* > t_c^*$, we can lose arbitrarily large amounts of energy across the event horizon.

From here on in this thesis, when we say *solution*, unless stated otherwise, we mean ϕ which has finite $\dot{H}^1(\Sigma_{t^*})$ norm for all t^*, and is a solution of (2.1) in a distributional sense, i.e.

$$\int_{\mathscr{M}} g^{ab}\partial_b f \partial_a \phi = 0 \quad \forall f \in C_0^\infty(\mathscr{M}). \tag{2.73}$$

Again, note that smooth compactly supported solutions of (2.1) are dense within these functions with respect to the $\dot{H}^1(\Sigma_{t^*})$ norm. The methods in this section will closely follow [2].

We begin by proving a local in time bound on solutions of (2.1) which are compactly supported on each Σ_{t^*}, though this can be extended to all $\dot{H}^1$ functions by a density argument.

Theorem 2.5.1 (Finite in Time Energy Bound)**.** *Given an RNOS model given by M,q,T^*, ϕ a solution of the wave equation* (2.1) *with reflective boundary conditions* (2.2)*, and a time interval $t_0^* \leq t_1^* \leq t_c^*$, we have that there exists a constant $C = C(M,q,T^*,t_0^*) > 0$ such that*

$$C^{-1}\|\phi\|_{\dot{H}^1(\Sigma_{t_0^*})} \leq \|\phi\|_{\dot{H}^1(\Sigma_{t_1^*})} \leq C\|\phi\|_{\dot{H}^1(\Sigma_{t_0^*})} \tag{2.74}$$

Proof. We start by choosing a vector field in the region $t^* \in [t_0^*, t_1^*]$ which is everywhere timelike, including on the surface of the dust cloud. We will also choose this vector field to be tangent to the surface of the dust cloud. For example

$$X = \partial_{t^*} + \dot{r_b}(t^*)\partial_r. \tag{2.75}$$

Then we have that

$$-dt^*(J^X) = \frac{1}{2}\Bigg(\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)(\partial_{t^*}\phi)^2 + 2\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\dot{r_b}(t^*)\partial_{t^*}\phi\partial_r\phi + \left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}+\frac{2M}{r}\left(2-\frac{q^2M}{r}\right)|\dot{r_b}(t^*)|\right)(\partial_r\phi)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\Bigg). \tag{2.76}$$

Note, in every RNOS model, when the dust cloud crossed $r = r_+$, $\dot{r_b}(t^*) \neq 0$, and as $\dot{r_b}(t^*) \in (-1,0]$, we have that there exists a time independent constant $A = A(M,q,T^*)$ such that

$$A^{-1}\|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} \leq -\int_{\Sigma_{t^*}} dt^*(J^X) \leq A\|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} \tag{2.77}$$

Then we look at the energy current through the surface of the dust cloud

$$d\rho(J^X) = 0 \tag{2.78}$$

once we notice that on the surface of the dust cloud $X^\nu\nabla_\nu\phi = 0$ and $d\rho(X) = 0$ for $d\rho$ the normal to the surface of the dust cloud.

If we then calculate K^X, we get:

$$\begin{aligned}|K^X| = \Bigg| &\left(1+\frac{M}{r}\right)\frac{\dot{r_b}(t^*)}{r}(\partial_{t^*}\phi)^2 - \left(\frac{2M\dot{r_b}(t^*)}{r^2} + \left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\ddot{r_b}(t^*)\right)\partial_{t^*}\phi\partial_r\phi \\ &-\left(\left(1-\frac{M}{r}\right)\frac{\dot{r_b}(t^*)}{r} - \left(\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\ddot{r_b}(t^*)\right)(\partial_r\phi)^2\Bigg| \leq B\|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}\end{aligned} \tag{2.79}$$

for $B = B(t^*)$ a continuous function of t^*.

Define

$$f(t^*) := -\int_{\Sigma_{t^*}} dt^*(J^X) = \|\phi\|^2_{X,\Sigma_{t^*}}. \tag{2.80}$$

Now, if we integrate K^X in the space $t^* \in [t_0^*, t_1^*]$ and apply (2.28), we get

$$f(t_1^*) - \int_{t_0^*}^{t_1^*} B(t^*)f(t^*)dt^* \leq f(t_0^*) \leq f(t_1^*) + \int_{t_0^*}^{t_1^*} B(t^*)f(t^*)dt^*. \tag{2.81}$$

Then an application of Gronwall's Inequality to each of the inequalities in (2.81), and applying (2.77) to the result gives us that

$$\left(Ae^{\int_{t_0^*}^{t_1^*}B(t^*)dt^*}\right)^{-1}\|\phi\|^2_{\dot{H}^1(\Sigma_{t_0^*})} \leq \|\phi\|^2_{\dot{H}^1(\Sigma_{t_1^*})} \leq \left(Ae^{\int_{t_0^*}^{t_1^*}B(t^*)dt^*}\right)\|\phi\|^2_{\dot{H}^1(\Sigma_{t_0^*})}. \tag{2.82}$$

Letting $C^2 = Ae^{\int_{t_0^*}^{t_c^*}B(t^*)dt^*}$ gives us the required result. □

Theorem 2.5.1 gives us the conditions mentioned in Remark 2.4.2, so we have the following Theorem:

Theorem 2.5.2 (H^1 Existence of Reflective Solutions). *Let $(\phi_0,\phi_1) \in H^1(\Sigma_{t_0^*})$, where $t_0^* \leq t_c^*$. There exists a solution ϕ to the wave equation* (2.1) *with reflective boundary conditions such that*

$$(\phi|_{\Sigma_{t_0^*}}, \partial_{t^*}\phi|_{\Sigma_{t_0^*}}) = (\phi_0,\phi_1). \tag{2.83}$$

Here this restriction holds in a trace sense, and ϕ is a solution in the sense of distributions. Finally, $\phi \in \dot{H}^1(\Sigma_{t^*})$ for all $t^* \leq t_c^*$.

Proof. This is a result of Theorem 2.4.1 and a density argument. This density argument relies on the bounds given by Theorem 2.5.1. □

Remark 2.5.1. *Note that our existence result, Theorem 2.5.2 allows us to define the forwards map:*

$$\mathscr{F}_{(t_0^*,t_1^*)} : \mathscr{E}^X_{\Sigma_{t_0^*}} \to \mathscr{E}^X_{\Sigma_{t_1^*}} \tag{2.84}$$

$$\mathscr{F}_{(t_0^*,t_1^*)}(\phi_0,\phi_1) := \left(\phi|_{\Sigma_{t_1^*}}, \partial_{t^*}\phi|_{\Sigma_{t_1^*}}\right) \quad \textit{where } \phi \textit{ is the solution to } (2.1) \textit{ and } (2.37). \tag{2.85}$$

Then Theorem 2.5.1 gives boundedness of $\mathscr{F}_{(t_0^*,t_1^*)}$*:*

$$\|\mathscr{F}_{(t_0^*,t_1^*)}(\phi_0,\phi_1)\|_X \leq C\|(\phi_0,\phi_1)\|_X \quad \forall t_0^* \leq t_1^* \leq t_c^* \tag{2.86}$$

for some $C = C(M,t_0^*,t_1^*) > 0$*.*

Now we wish to obtain a bound for our solution which does not depend on the time interval we are looking in. We first prove such a uniform boundedness result for the case $T^* < 1$:

Theorem 2.5.3 (Uniform in Time Energy Bound for $T^* < 1$)**.** *Given an RNOS model given by M, q and* $T^* < 1$*, and* ϕ *a solution of the wave equation* (2.1) *with reflective boundary conditions* (2.2)*, we have that there exists a constant* $C = C(M,q,T^*) > 0$ *such that*

$$C^{-1}\|\phi\|_{\dot{H}^1(\Sigma_{t_0^*})} \leq \|\phi\|_{\dot{H}^1(\Sigma_{t_1^*})} \leq C\|\phi\|_{\dot{H}^1(\Sigma_{t_0^*})} \quad \forall t_0^* \leq t_1^* \leq t_c^*. \tag{2.87}$$

Proof. This proof is identical to the proof for Theorem 2.5.1. Once we note that for $T^* < 1$ and $t^* < t_-^*$, $\dot{r}_b(t^*) = \ddot{r}_b(t^*) = 0$, we can take $B(t^*) = 0$ for $t^* \leq t_-^*$. Therefore the constant given by Theorem 2.5.1 is actually a uniform in time bound for all $t_0^* < t_-^*$, i.e. take

$$C^2 = Ae^{\int_{t_-^*}^{t_c^*} B(t^*)dt^*}. \tag{2.88}$$

□

The $T^* \geq 1$ case is much more difficult than the $T^* < 1$ case, so we will break this boundedness result into two different Theorems. We will start with the forward bound:

Theorem 2.5.4 (Uniform Forward in Time Energy Bound for $T^* \geq 1$)**.** *Given an RNOS model with parameters q and* $T^* \geq 1$*, and a solution* ϕ *of the wave equation* (2.1) *with reflective boundary conditions* (2.2)*, we have that there exists a constant* $C = C(M,q,T^*) > 0$ *such that*

$$\|\phi\|_{\dot{H}^1(\Sigma_{t_1^*})} \leq C\|\phi\|_{\dot{H}^1(\Sigma_{t_0^*})} \quad \forall t_0^* \leq t_1^* \leq t_c^*. \tag{2.89}$$

Proof. We will proceed similarly to Theorem 2.5.1, but we will take the vector field

$$X = \partial_{t^*}. \tag{2.90}$$

Then we obtain the following results:

$$K^X = 0 \tag{2.91}$$

$$-dt^*(J^X) = \frac{1}{2}\left(\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)(\partial_{t^*}\phi)^2 + \left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)(\partial_r\phi)^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\right) \tag{2.92}$$

$$d\rho(J^X) = -\frac{\dot{r}_b(t^*)}{2}(1+\dot{r}_b(t^*))\left(\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right) - \left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\dot{r}_b(t^*)\right)(\partial_r\phi)^2 \geq 0, \tag{2.93}$$

recalling that $\dot{r}_b \in (-1,0]$.

Now if we take an arbitrary $t_0^* < t_c^*$, then in the region $t^* \leq t_0^*$, $r \geq r_b(t^*) \geq r_b(t_0^*) > r_+$. Thus there exists an $\varepsilon > 0$ such that

$$1-\frac{2M}{r}+\frac{q^2M^2}{r^2} \geq \varepsilon. \tag{2.94}$$

Therefore in the region $t^* \leq t_0^*$

$$\varepsilon\|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} \leq -\int_{\Sigma_{t^*}} dt^*(J^X) \leq \|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}. \tag{2.95}$$

Then, as before, we integrate K^X in a region $t^* \in [t_1^*, t_2^*]$ for $t_1^* \le t_2^* \le t_0^*$. Once we note that the boundary term has the correct sign, we have

$$\|\phi\|^2_{\dot{H}^1(\Sigma_{t_2^*})} \le \varepsilon^{-1} \|\phi\|^2_{\dot{H}^1(\Sigma_{t_1^*})}. \tag{2.96}$$

An application of Theorem 2.5.1 will then allow us to extend our bound over the remaining finite interval $[t_0^*, t_c^*]$ to obtain the required result. □

Now we look at obtaining the backward in time bound:

Theorem 2.5.5 (Uniform Backward in Time Energy Bound for $T^* \ge 1$)**.** *Given an RNOS model given by M, q and $T^* \ge 1$, and a solution ϕ of the wave equation* (2.1) *with reflective boundary conditions* (2.2)*, we have that there exists a constant $C = C(M, q, T^*) > 0$ such that*

$$\|\phi\|_{\dot{H}^1(\Sigma_{t_0^*})} \le C \|\phi\|_{\dot{H}^1(\Sigma_{t_1^*})} \quad \forall t_0^* \le t_1^* \le t_c^*. \tag{2.97}$$

Proof. For this proof, we will need to use the modified currents, as defined in (2.25). Given asymptotic speed $a < 1$ (see (1.45)), let $b \in (a, 1)$. Looking in the region $t^* < 0$, we will use the vector field and modifier

$$X = f(t^*)\partial_{t^*} - b\partial_r \tag{2.98}$$

$$f(t^*) := \left(1 + \frac{1}{\log\left(\frac{|t^*|}{2M}\right)}\right) \tag{2.99}$$

$$w = -\frac{b}{2r}. \tag{2.100}$$

We then calculate

$$\begin{aligned} K^{X,w} = &-\left(\frac{1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}}{2|t^*|\left(\log\left(\frac{|t^*|}{2M}\right)\right)^2} - \frac{bM}{r^2}\left(1 - \frac{q^2M}{r}\right)\right)(\partial_{t^*}\phi)^2 \\ &-\left(\frac{1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}}{2|t^*|\left(\log\left(\frac{|t^*|}{2M}\right)\right)^2} - \frac{bM}{r^2}\left(1 - \frac{q^2M}{r}\right)\right)(\partial_r\phi)^2 - \frac{2bM}{r^2}\left(1 - \frac{q^2M}{r}\right)\partial_{t^*}\phi\partial_r\phi \\ &-\frac{1}{r^2}\left(\frac{b}{r} + \frac{1}{2|t^*|\left(\log\left(\frac{|t^*|}{2M}\right)\right)^2}\right)|\mathring{\nabla}\phi|^2 - \frac{b}{r^4}\left(1 - \frac{q^2M}{r}\right)\phi^2 \end{aligned} \tag{2.101}$$

$$d\rho(J^{X,w}) = -\frac{1}{2}\left(b + f(t^*)\dot{r_b}(t^*)\right)\left(1 + \dot{r_b}(t^*)\right)\left(\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right) - \left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right)\dot{r_b}(t^*)\right)(\partial_r\phi)^2 \tag{2.102}$$

$$\begin{aligned} -dt^*(J^{X,w}) = \frac{1}{2}\Bigg(&\left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right)f(t^*)(\partial_{t^*}\phi)^2 - 2b\left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right)\partial_{t^*}\phi\partial_r\phi \\ &+\left(\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right)f(t^*) + \frac{2bM}{r}\left(2 - \frac{q^2M}{r}\right)\right)(\partial_r\phi)^2 + \frac{1}{r^2}f(t^*)|\mathring{\nabla}\phi|^2 \\ &+\frac{2bM}{r}\left(2 - \frac{q^2M}{r}\right)\frac{\phi}{r}\partial_r\phi - 2b\left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right)\frac{\phi}{r}\partial_{t^*}\phi + \frac{bM}{r}\left(2 - \frac{q^2M}{r}\right)\frac{\phi^2}{r^2}\Bigg). \end{aligned} \tag{2.103}$$

We now note that either $a \ne 0$, or $\dot{r_b}(t^*) \sim \sqrt{r_b}$ as $t^* \to -\infty$. In either of these cases, terms of order $\left(|t^*|\left(\log\left(\frac{|t^*|}{2M}\right)\right)^2\right)^{-1}$ dominate terms of order r^{-2} as $t^* \to -\infty$. Thus for sufficiently negative t^*, $K^{X,w} \le 0$. Similarly we have $d\rho(J^X, w) \ge 0$, for sufficiently negative t^*, since $b + f(t^*)\dot{r_b}(t^*) \to b - a > 0$ as $t^* \to \infty$.

Finally, we consider $-dt^*(J^{X,w})$. Integrating over Σ_{t^*}, we obtain:

$$
\begin{aligned}
-\int_{\Sigma_{t^*}} dt^*(J^{X,w}) = \frac{1}{2}\int_{\Sigma_{t^*}} \Bigg(&\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right) f(t^*)(\partial_{t^*}\phi)^2 - 2b\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\partial_{t^*}\phi\partial_r\phi \\
&+\left(\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)f(t^*)+\frac{2bM}{r}\left(2-\frac{q^2M}{r}\right)\right)(\partial_r\phi)^2+\frac{1}{r^2}f(t^*)|\mathring{\nabla}\phi|^2 \\
&+\frac{2bM}{r}\left(2-\frac{q^2M}{r}\right)\frac{\phi}{r}\partial_r\phi - 2b\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\frac{\phi}{r}\partial_{t^*}\phi+\frac{bM}{r}\left(2-\frac{q^2M}{r}\right)\frac{\phi^2}{r^2}\Bigg) \\
= \frac{1}{2}\int_{\Sigma_{t^*}} \Bigg(&\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)(f(t^*)-b)(\partial_{t^*}\phi)^2 \\
&+b\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\left(\left(\partial_{t^*}\phi-\partial_r\phi-\frac{\phi}{r}\right)^2-2\frac{\phi}{r}\partial_r\phi\right) \\
&+\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)(f(t^*)-b)(\partial_r\phi)^2+\frac{1}{r^2}f(t^*)|\mathring{\nabla}\phi|^2+\frac{bM}{r}\left(2-\frac{q^2M}{r}\right)\left(\frac{\phi}{r}+\partial_r\phi\right)^2\Bigg) \\
= \frac{1}{2}\int_{\Sigma_{t^*}} \Bigg(&\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)(f(t^*)-b)(\partial_{t^*}\phi)^2+b\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\left(\partial_{t^*}\phi-\partial_r\phi-\frac{\phi}{r}\right)^2 \\
&+b\left(1+\frac{q^2M^2}{r^2}\right)\frac{\phi^2}{r^2}+\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)(f(t^*)-b)(\partial_r\phi)^2+\frac{1}{r^2}f(t^*)|\mathring{\nabla}\phi|^2 \\
&+\frac{bM}{r}\left(2-\frac{q^2M}{r}\right)\left(\frac{\phi}{r}+\partial_r\phi\right)^2\Bigg).
\end{aligned}
\tag{2.104}
$$

We then note a version of Hardy's inequality. If h is a differentiable function of one variable, with $h(0)=0$, then

$$
\exists C>0 \; s.t. \int_{\Sigma_{t^*}} \left(\frac{h(r)}{r}\right)^2 \leq C\int_{\Sigma_{t^*}} (\partial_r h(r))^2, \tag{2.105}
$$

providing the right hand side is finite.

Using (2.105), we have that there exists a t^* independent constant A such that

$$
\begin{aligned}
0 \leq \int_{\Sigma_{t^*}} \Bigg(b\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\left(\partial_{t^*}\phi-\partial_r\phi-\frac{\phi}{r}\right)^2+b\left(1+\frac{q^2M^2}{r^2}\right)\frac{\phi^2}{r^2}+\frac{1}{r^2}f(t^*)|\mathring{\nabla}\phi|^2 \\
+\frac{bM}{r}\left(2-\frac{q^2M}{r}\right)\left(\frac{\phi}{r}+\partial_r\phi\right)^2\Bigg) \leq A\|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}
\end{aligned}
\tag{2.106}
$$

Note $f(t^*)\to 1$ as $t^*\to-\infty$, and $b\leq 1$. Thus for sufficiently large negative t^*, there exists a t^* independent ε such that

$$
\varepsilon\|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} \leq -\int_{\Sigma_{t^*}} dt^*(J^{X,w}) \leq \varepsilon^{-1}\|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}. \tag{2.107}
$$

Finally, for negative and large enough t^*, $t^*\leq t_2^*$, say, we have $-\int_{\Sigma_{t^*}} dt^*(J^{X,w})$ non-decreasing in t^*. Therefore

$$
\|\phi\|^2_{\dot{H}^1(\Sigma_{t_0^*})} \leq A^2\|\phi\|^2_{\dot{H}^1(\Sigma_{t_1^*})} \quad \forall t_0^*\leq t_1^*\leq t_2^*. \tag{2.108}
$$

Combining (2.108) with Theorem 2.5.1 for the interval $[t_2^*, t_c^*]$ gives us the final result. □

2.5.2 The Permeating Case

We now look at the permeating case. In the Oppenheimer–Snyder model for the interior of the star, we have that our metric is C^0, but piecewise smooth. Thus, as given by Theorem 2.4.2, we are dealing with a weak solution rather than a classical solution, i.e. $\phi \in H^1(\Sigma_\tau)$ a solution to

$$\int_{\tau=-\infty}^{\infty}\int_{\Sigma_\tau}\sqrt{-g}g^{ab}\partial_a\phi\partial_b\left(\frac{\psi}{\sqrt{-g}}\right)dVd\tau=0 \qquad \forall\psi\in C_0^\infty(\mathscr{M}). \tag{2.109}$$

(Note that in the coordinates chosen below, the determinant of $\sqrt{-g}$ is $r^2\sin\theta$)

The metric in the interior of our star has the form (see Section 1.1.2):

$$ds^2=\begin{cases}-\left(1-\frac{2Mr^2}{r_b^3}\right)d\tau^2+2\sqrt{\frac{2Mr^2}{r_b^3}}d\tau dr+dr^2+r^2g_{S^2} & r<r_b:=(R_b^{\frac{3}{2}}-\frac{3\tau}{2}\sqrt{2M})^{\frac{2}{3}}\\ -\left(1-\frac{2M}{r}\right)d\tau^2+2\sqrt{\frac{2M}{r}}d\tau dr+dr^2+r^2g_{S^2} & r\geq r_b\end{cases} \tag{2.110}$$

for constants R_b and M. Here, r_b is the boundary of the star.

We note that the null hypersurface given by $r=r_b\left(3-2\sqrt{\frac{r_b}{2M}}\right)$ is part of our event horizon. This means when we construct the backwards scattering map, we will require data on this as well as the $r=2M$, $\tau>\tau_c$ surface.

We begin our study of boundedness by noticing that our usual ∂_τ-energy does not give the same bound as before. This is due to the fact that ∂_τ is no longer a Killing vector. We therefore obtain a term arising from K^{∂_τ} inside the star. We can still obtain a bound from integrating K^{∂_τ}, however it is now exponentially growing in τ:

Lemma 2.5.1 (Finite in Time Forwards Bound in the Permeating Case)**.** *Let $\phi\in H^2_{0\forall\tau}$ be a weak solution to the wave equation* (2.1) *with permeating boundary conditions. There exists a constant $B=B(M)>0$ such that*

$$\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_1})}\leq -2\int_{\Sigma_{\tau_1}}d\tau(J^X[\phi])\leq -2\int_{\Sigma_{\tau_0}}d\tau(J^X[\phi])e^{B(\tau_1-\tau_0)}\leq 4\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_0})}e^{B(\tau_1-\tau_0)} \qquad \forall\tau_1\geq\tau_0 \tag{2.111}$$

for suitably chosen future directed timelike X.

Proof. Choose $f(r)$ to be a smooth cut off function

$$f(r)\begin{cases}=-\frac{1}{2} & r\in[\frac{3M}{2},\frac{5M}{2}]\\ =0 & r\notin[M,3M]\\ \in[-\frac{1}{2},0] & r\in(M,\frac{3M}{2})\cup(\frac{5M}{2},3M)\end{cases}. \tag{2.112}$$

Note f has bounded derivatives. Then if we let $X=\partial_\tau+f(r)\partial_r$ we have that:

$$-d\tau(J^X)=\begin{cases}\frac{1}{2}\left(\frac{1}{r^2}|\mathring{\nabla}\phi|^2+(\partial_\tau\phi)^2+2f\partial_\tau\phi\partial_r\phi+\left(1-\frac{2Mr^2}{r_b^3}-2f\sqrt{\frac{2Mr^2}{r_b^3}}\right)(\partial_r\phi)^2\right) & r<r_b\\ \frac{1}{2}\left(\frac{1}{r^2}|\mathring{\nabla}\phi|^2+(\partial_\tau\phi)^2+2f\partial_\tau\phi\partial_r\phi+\left(1-\frac{2M}{r}-2f\sqrt{\frac{2M}{r}}\right)(\partial_r\phi)^2\right) & r\geq r_b\end{cases}. \tag{2.113}$$

$$K^X=\begin{cases}\frac{1}{2}\Bigg(-\frac{f'}{r^2}|\mathring{\nabla}\phi|^2+(\frac{2f}{r}+f')(\partial_\tau\phi)^2-\left(\frac{6Mr}{r_b^3}+\frac{6\sqrt{2M}f}{\sqrt{r_b^3}}\right)\partial_\tau\phi\partial_r\phi\\ \qquad+\left((1-\frac{2Mr^2}{r_b^3})f'-(1-\frac{4Mr^2}{r_b^3})\frac{2f}{r}+\frac{3\sqrt{2M}^3r^2}{\sqrt{r_b^9}}\right)(\partial_r\phi)^2\Bigg) & r<r_b\\ \frac{1}{2}\Bigg(-\frac{f'}{r^2}|\mathring{\nabla}\phi|^2+(\frac{2f}{r}+f')(\partial_\tau\phi)^2-\frac{3f}{r}\sqrt{\frac{2M}{r}}\partial_\tau\phi\partial_r\phi\\ \qquad+\left((1-\frac{2M}{r})f'-(1-\frac{M}{r})\frac{2f}{r}\right)(\partial_r\phi)^2\Bigg) & r\geq r_b\end{cases}. \tag{2.114}$$

Thus K^X can always be bounded by multiples of $-d\tau(J^X)$.

$$-\int_{\Sigma_{\tau_0}}d\tau(J^X)=-\int_{\Sigma_{\tau_1}}d\tau(J^X)+\int_{\tau=\tau_0}^{\tau_1}\int_{\Sigma_\tau}K^X-\int_{\mathscr{H}^+\cap\{r<2M\}}dn(J^X). \tag{2.115}$$

We can also note that the contribution from the part of the horizon in (2.115) is of the form $-T_{ab}X^a n^b$ for future directed normal n. By the dominant energy condition, we have that this term has the correct sign. Thus letting $g(\tau) = -\int_{\Sigma_\tau} d\tau(J^X)$, we have that

$$g(\tau) \leq g(\tau_0) + A\int_{s^*=\tau_0}^{\tau} g(s)ds, \tag{2.116}$$

which gives us our result by Gronwall's Inequality. □

Remark 2.5.2. *For the purposes of the scattering map however, we will not want to disregard the surface term from the event horizon. Instead we will want to consider a norm on the horizon such that the map from a surface Σ_τ to $\Sigma_{\tau_c} \cup (\mathscr{H} \cap \{r < 2M\})$ is bounded in both directions. Letting $X = \partial_\tau + f(r)\partial_r$, then we have*

$$-dn(J^X) = \frac{1}{2}\left(3\sqrt{\frac{2M}{r_b}} - 1 + f(r)\right)\left(\partial_\tau\phi + 3\left(1-\sqrt{\frac{2M}{r_b}}\right)\partial_r\phi\right)^2 + \left(3\left(1-\sqrt{\frac{2M}{r_b}}\right) - f\right)\frac{1}{2r^2}|\mathring{\nabla}\phi|^2. \tag{2.117}$$

If we then use the f from Lemma 2.5.1, we have all these terms being positive definite. Therefore the norm we will consider on the surface contains only the L^2 norms of the angular derivatives and the derivative with respect to the vector $\partial_\tau + 3\left(1-\sqrt{\frac{2M}{r_b}}\right)\partial_r$.

Lemma 2.5.2 (Finite-in-Time Backward Non-degenerate Energy Boundedness for the Permeating Case)**.** *Let $\phi \in H^2_{0\forall\tau}$ be a weak solution to the wave equation* (2.1) *with permeating boundary conditions. There exists a constant $B = B(M) > 0$ such that for all $\tau_0 \leq \tau_1 \leq \tau_{c^-}$, we have*

$$\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_0})} \leq 4\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_1})} e^{B(\tau_1-\tau_0)}. \tag{2.118}$$

Here $(\tau_{c^-}, r = 0)$ is defined by equation (1.18).

Proof. This is proved identically to the previous lemma. We bound K^X below instead of above, and we ignore the boundary term, as $\mathscr{H}^+ \cap \{\tau < \tau_{c^-}\} = 0$. □

Lemmas 2.5.1 and 2.5.2 give us the conditions mentioned in Remark 2.4.3. Thus we have the following Theorem:

Theorem 2.5.6 (H^1 Existence of Permeating Solutions)**.** *Let $(\phi_0, \phi_1) \in H^1(\Sigma_{\tau_0})$, where $\tau_0 \leq \tau_{c^-}$. There exists a solution ϕ to the wave equation* (2.1) *with permeating boundary conditions, such that*

$$(\phi|_{\Sigma_{\tau_0}}, \partial_\tau\phi|_{\Sigma_{\tau_0}}) = (\phi_0, \phi_1) \tag{2.119}$$

Here this restriction holds in a trace sense, and ϕ is a solution in the sense of distributions. Finally, $\phi \in \dot{H}^1(\Sigma_\tau)$ for all $\tau \leq \tau_c$.

Proof. This is a result of Theorem 2.4.2 and a density argument. This density argument relies on the bounds given by Lemmas 2.5.1 and 2.5.2. □

Remark 2.5.3. *As in the permeating case, our existence result Theorem 2.5.6 allows us to define the forwards map:*

$$\mathscr{F}_{(\tau_0,\tau_1)} : \mathscr{E}^X_{\Sigma_{\tau_0}} \to \mathscr{E}^X_{\Sigma_{\tau_1}} \tag{2.120}$$

$$\mathscr{F}_{(\tau_0,\tau_1)}(\phi_0,\phi_1) := \left(\phi|_{\Sigma_{\tau_1}}, \partial_{t^*}\phi|_{\Sigma_{\tau_1}}\right) \quad \text{where } \phi \text{ is the solution to (2.1) and (2.63).} \tag{2.121}$$

We can use Lemmas 2.5.1 and 2.5.2 to bound the solution over any finite time interval. Thus we can now consider only the case where $r_b \gg 2M$, i.e. $\frac{2M}{r_b} < \varepsilon$ for some small, fixed epsilon. Once we have a uniform bound for $\frac{2M}{r_b} < \varepsilon$, we can bound solutions of the wave equation for $\tau \leq \tau_c$ using Lemmas 2.5.1 and 2.5.2. Previous work on the external Schwarzschild space-time gives us the required bounds for $\tau > \tau_c$.

This brings us to our next result:

Proposition 2.5.1 (Forward Non-degenerate Energy Boundedness for the Permeating Case, Sufficiently Far Back)**.** *Let ϕ be a solution to the wave equation* (2.1) *with permeating boundary conditions (as given by Theorem 2.5.6). There exists a constant, $A = A(M) > 0$, and a time, τ^* such that*

$$\|\phi\|_{\dot{H}^1(\Sigma_{\tau_0})} \leq A\|\phi\|_{\dot{H}^1(\Sigma_{\tau_1})} \qquad \forall \tau_0 < \tau_1 \leq \tau^*. \tag{2.122}$$

Proof. For this proof, we choose a time dependent vector field. Let $Y = h(\tau)\partial_\tau$. Then we have that

$$-d\tau(J^Y) = \begin{cases} \frac{h}{2}\left(\frac{1}{r^2}|\mathring{\nabla}\phi|^2 + (\partial_\tau\phi)^2 + \left(1 - \frac{2Mr^2}{r_b^3}\right)(\partial_r\phi)^2\right) & r < r_b \\ \frac{h}{2}\left(\frac{1}{r^2}|\mathring{\nabla}\phi|^2 + (\partial_\tau\phi)^2 + \left(1 - \frac{2M}{r}\right)(\partial_r\phi)^2\right) & r \geq r_b \end{cases}. \tag{2.123}$$

$$K^Y = \begin{cases} -\frac{1}{2}\Bigg(\frac{h'}{r^2}|\mathring{\nabla}\phi|^2 + h'(\partial_\tau\phi)^2 - \frac{2Mr}{r_b^3}3h\partial_\tau\phi\partial_r\phi \\ \qquad + \left(h'\left(1 - \frac{2Mr^2}{r_b^3}\right) - \frac{2Mr^2}{r_b^3}\frac{3h}{r}\right)(\partial_r\phi)^2\Bigg) & r < r_b \\ \frac{h'}{2}\left(\frac{1}{r^2}|\mathring{\nabla}\phi|^2 + (\partial_\tau\phi)^2 + \left(1 - \frac{2M}{r}\right)(\partial_r\phi)^2\right) & r \geq r_b \end{cases}. \tag{2.124}$$

Now, we would like both of these to be everywhere positive definite. For this, we need to pick $h > 0$ and bounded. We also need $h' < 0$, with $-h' > \frac{3M}{r_b^2}h$. Thus we can choose, for example,

$$h(\tau) = 1 - \left(\frac{2M}{r_b}\right)^{1/4} \in [1 - \left(\frac{2M}{r_b(\tau^*)}\right)^{1/4}, 1] \tag{2.125}$$

$$h'(\tau) = -\left(\frac{r_b}{2M}\right)^{1/4}\frac{2M}{4r_b^2} \leq -\left(\frac{r_b(\tau^*)}{2M}\right)^{1/4}\frac{2M}{4r_b^2} < -\frac{3M}{r_b^2}h \tag{2.126}$$

where we have chosen τ^* s.t. $\left(\frac{r_b(\tau^*)}{2M}\right)^{1/4} > 6$. This choice also gives us

$$\left(1 - \frac{2M}{r_b(\tau^*)}\right)\left(1 - \left(\frac{2M}{r_b(\tau^*)}\right)^{\frac{1}{4}}\right)\|\phi\|^2_{\dot{H}^1(\Sigma_\tau)} \leq -2\int_{\Sigma_\tau} d\tau(J^X) \leq \|\phi\|^2_{\dot{H}^1(\Sigma_\tau)}. \tag{2.127}$$

Finally, we apply these inequalities to divergence theorem (the permeating equivalent of (2.28)) in the region $\tau \in [\tau_0, \tau_1]$ to obtain

$$\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_0})} \geq -2\int_{\Sigma_{\tau_0}} d\tau(J^Y) = -2\int_{\Sigma_{\tau_1}} d\tau(J^Y) + 2\int_{\tau=\tau_0}^{\tau_1}\int_{\Sigma_\tau} K^Y \geq -2\int_{\Sigma_{\tau_1}} d\tau(J^Y) \geq A\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_1})} \tag{2.128}$$

as required. □

Theorem 2.5.7 (Forward Non-degenerate Energy Boundedness for the Permeating Case)**.** *Let ϕ be a solution to the wave equation* (2.1) *with permeating boundary conditions (as given by Theorem 2.5.6). There exists a constant $A = A(M)$ such that*

$$\|\phi\|_{\dot{H}^1(\Sigma_{\tau_1})} \leq A\|\phi\|_{\dot{H}^1(\Sigma_{\tau_0})} \qquad \forall \tau_0 \leq \tau_1. \tag{2.129}$$

Proof. Previous works on Schwarzschild exterior space time (e.g. [14]), gives us that (2.129) holds for $\tau_c \leq \tau_0 \leq \tau_1$. Thus if we prove the result for the case $\tau_1 \leq \tau_c$, we can combine these results to obtain (2.129) for all $\tau_0 \leq \tau_1$.

Let A and τ^* be defined as in Proposition 2.5.1. Let B be as defined in Lemma 2.5.1. Remember τ_c is defined to be the time at which the surface of the star crosses $r = 2M$. We then have that

$$\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_1})} \leq 4Ae^{B(\tau_c-\tau^*)}\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_0})} \qquad \forall \tau_0 \leq \tau_1 \leq \tau_c. \tag{2.130}$$

□

In a similar way, we can obtain a boundedness statement for the reverse direction:

Theorem 2.5.8 (Backwards Non-degenerate Energy Boundedness for the Permeating Case)**.** *Let ϕ be a solution to the wave equation* (2.1) *with permeating boundary conditions (as given by Theorem 2.5.6). There exists a constant $A = A(M)$, and a time τ^{*-} such that*

$$\|\phi\|_{\dot{H}^1(\Sigma_{\tau_0})} \leq A\|\phi\|_{\dot{H}^1(\Sigma_{\tau_1})} \qquad \forall \tau_0 \leq \tau_1 \leq \tau^{*-}. \tag{2.131}$$

Proof. As before, we consider $Y = h(\tau)\partial_\tau$. However, this time we require the sign of K^Y to be non-positive. Thus, we choose

$$h(\tau) = 1 + \left(\frac{2M}{r_b}\right)^{\frac{1}{4}} \in [1,2]. \tag{2.132}$$

$$h'(\tau) = \left(\frac{r_b}{2M}\right)^{1/4}\frac{2M}{4r_b^2} \geq \left(\frac{r_b(\tau^{*-})}{2M}\right)^{1/4}\frac{2M}{4r_b^2} > \frac{3M}{r_b^2}h. \tag{2.133}$$

Therefore, if we choose τ^{*-} s.t. $\left(\frac{r_b(\tau^{*-})}{2M}\right)^{1/4} > 12$, we have $K^Y \leq 0$ (see equation (2.124)). Then, as before we obtain:

$$\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_1})} \geq -\int_{\Sigma_{\tau_1}} d\tau(J^Y) = -\int_{\Sigma_{\tau_0}} d\tau(J^Y) - \int_{\tau=\tau_0}^{\tau_1}\int_{\Sigma_\tau} K^Y \geq -\int_{\Sigma_{\tau_0}} d\tau(J^Y) \geq \frac{1}{2}\left(1 - \frac{2M}{r_b(\tau^{*-})}\right)\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_0})}. \tag{2.134}$$

□

Corollary 2.5.1 (Uniform Boundedness for the Permeating Case)**.** *Let ϕ be a solution to the wave equation* (2.1) *with permeating boundary conditions (as given by Theorem 2.5.6). There exist constants $B = B(M) > 0, b = b(M) > 0$ such that*

$$b\|\phi\|_{\dot{H}^1(\Sigma_{\tau_1})} \leq \|\phi\|_{\dot{H}^1(\Sigma_{\tau_0})} \leq B\|\phi\|_{\dot{H}^1(\Sigma_{\tau_1})} \qquad \forall \tau_0, \tau_1 \leq \tau_{c^-}. \tag{2.135}$$

Proof. We have the forward bound due to Theorem 2.5.7. The backwards bound is done by combining Theorem 2.5.8 and Corollary 2.5.2 over the finite time interval $[\tau^{*-}, \tau_{c^-}]$. □

Corollary 2.5.2. *Let ϕ be a solution to the wave equation* (2.1) *with permeating boundary conditions (as given by Theorem 2.5.6). There exists a constant, $C = C(M) > 0$, such that*

$$-\int_{\tilde{\Sigma}_{v_0}} dn(J^Y) \leq -\int_{\tilde{\Sigma}_{v_1}} dn(J^Y) \leq C\|\phi\|^2_{\dot{H}^1(\Sigma_{\tau_0})} \quad \forall v_0 \leq v_1 \leq \tau_0 + r_b(\tau_0) \tag{2.136}$$

where n is the normal to $\tilde{\Sigma}_v$.

Proof. We integrate K^Y between the relevant surfaces and use Stokes' theorem to obtain these bounds. □

Remark 2.5.4. *The first order energy results from this section can be given using the forwards map and the energy space notation from* (2.27) *and* (2.31)*. Let X be strictly timelike everywhere (for example, as in Lemma*

2.5.1). We then have that there exist $A_1 = A_1(M) > 0$ and $A_2 = A_2(M) > 0$ such that

$$A_1^{-1}\|(\phi_0,\phi_1)\|_X \leq \|\mathscr{F}_{t_0^*,t_1^*}(\phi_0,\phi_1)\|_X \leq A_1\|(\phi_0,\phi_1)\|_X \qquad \forall t_0^* \leq t_1^* \leq t_c^* \tag{2.137}$$

$$A_2^{-1}\|(\phi_0,\phi_1)\|_X \leq \|\mathscr{F}_{\tau_0,\tau_1}(\phi_0,\phi_1)\|_X \leq A_2\|(\phi_0,\phi_1)\|_X \qquad \forall \tau_0 \leq \tau_1 \leq \tau_{c^-} \tag{2.138}$$

$$\|\mathscr{F}_{\tau_0,\tau_2}(\phi_0,\phi_1)\|_X \leq A_2\|(\phi_0,\phi_1)\|_X \qquad \forall \tau_0 \leq \tau_{c^-} \leq \tau_2 \leq \tau_c. \tag{2.139}$$

2.6 Higher Order Boundedness

We now try to extend Theorems 2.5.3, 2.5.4 and 2.5.5 to $\dot{H}^n$ norms. However, it turns out that this cannot be done unless the surface of the dust cloud has asymptotic velocity 0, i.e. $T^* \leq 1$. We begin with the following 3-part Lemma:

Lemma 2.6.1. *Given an RNOS model given by $M,q,T^* \leq 1$, and a solution $\phi \in C^\infty_{0\forall t^*}$ of the wave equation* (2.1) *with reflective boundary conditions* (2.2)*, we have the following results:*

1. *Let Ω_i be the angular Killing vector fields earlier (see* (1.26)*). Then $\Box_g\left(\frac{1}{r^{|p|}}\Omega^p\partial_r^m\partial_{t^*}^{n-1-m-|p|}\phi\right)$ only contains at most n^{th} order derivatives. Furthermore, all coefficients of these derivatives are smooth and have all their derivatives bounded. Thus there exists a constant $D = D(M,n) > 0$ such that*

$$\left\|\Box_g\left(\frac{1}{r^{|p|}}\Omega^p\partial_r^m\partial_{t^*}^{n-1-m-|p|}\phi\right)\right\|_{L^2(\Sigma_{t^*})} \leq D\|\phi\|_{\dot{H}^n(\Sigma_{t^*})}. \tag{2.140}$$

2. *There exists a $t_0^* \leq t_c^*$ and a constant $C = C(M,t_0^*) > 0$ such that*

$$C\left(\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} + \|\phi\|^2_{\dot{H}^{n-1}(\Sigma_{t^*})}\right) \geq \|\phi\|^2_{\dot{H}^n(\Sigma_{t^*})} \quad \forall t^* \leq t_0^*. \tag{2.141}$$

   Here $\bar{\partial}_{t^*}$ is the t^* derivative with respect to $(t^*, \rho = r - r_b(t^*) + 2M, \theta, \varphi)$ coordinates, as given in (2.39).

3. *Given any finite time $t_0^* \leq t_1^* \leq t_c^*$, there exists a constant $A = A(n,t_0^*,t_1^*,M)$ such that*

$$\frac{1}{A}\|\phi\|^2_{\dot{H}^n(\Sigma_{t_0^*})} \leq \|\phi\|^2_{\dot{H}^n(\Sigma_{t_1^*})} \leq A\|\phi\|^2_{\dot{H}^n(\Sigma_{t_0^*})}. \tag{2.142}$$

Remark 2.6.1. *Note that when calculating $\|\psi\|^2_{\dot{H}^1(\Sigma_{t^*})}$, we can use the $\bar{\partial}_{t^*}$ derivative in place of the ∂_{t^*} derivative. This is due to the fact that these norms can differ by at most a factor of* 2*, since*

$$\|\partial_{t^*}\psi\|_{L^2(\Sigma_{t^*})} - \|\partial_r\psi\|_{L^2(\Sigma_{t^*})} \leq \|(\partial_{t^*} + \dot{r_b}\partial_r)\psi\|_{L^2(\Sigma_{t^*})} = \|\bar{\partial}_{t^*}\psi\|_{L^2(\Sigma_{t^*})} \leq \|\partial_{t^*}\psi\|_{L^2(\Sigma_{t^*})} + \|\partial_r\psi\|_{L^2(\Sigma_{t^*})}. \tag{2.143}$$

This in turn implies

$$\frac{1}{2}\left(\|\partial_{t^*}\psi\|^2_{L^2(\Sigma_{t^*})} + \|\partial_r\psi\|^2_{L^2(\Sigma_{t^*})}\right) \leq \|\bar{\partial}_{t^*}\psi\|^2_{L^2(\Sigma_{t^*})} + \|\partial_r\psi\|^2_{L^2(\Sigma_{t^*})} \leq 2\left(\|\partial_{t^*}\psi\|^2_{L^2(\Sigma_{t^*})} + \|\partial_r\psi\|^2_{L^2(\Sigma_{t^*})}\right). \tag{2.144}$$

Proof. 1. Note Ω_i and ∂_{t^*} commute with $\Box_g$. Thus for this part we only need to check $\Box_g\left(\frac{1}{r^{|p|}}\partial_r^{n-1}\phi\right)$ explicitly. Using the fact that $\Box_g\phi = 0$, we obtain:

$$\Box_g\left(\frac{1}{r^{|p|}}\partial_r^{n-1}\phi\right) = \frac{n-1}{r^{2+|p|}}\left((n-2)\partial_{t^*}^2\partial_r^{n-3}\phi + 2(r-M)\partial_{t^*}^2\partial_r^{n-2}\phi - 4M\partial_{t^*}\partial_r^{n-1}\phi - n\partial_r^{n-1}\phi - 2(r-M)\partial_r^n\phi\right)$$
$$-\frac{|p|}{r^{1+|p|}}\nabla^\mu r\nabla_\mu\partial_r^{n-1}\phi. \tag{2.145}$$

Given in the above case, $|p| \leq n-1$, then we have our result.

2. We first look at how the wave operator commutes with $\bar{\partial}_{t^*}$:

$$\Box_g\left(\frac{1}{r^{|p|}}\Omega^p\bar{\partial}_{t^*}^{n-|p|}(\phi)\right)=\Box_g\left(\frac{1}{r^{|p|}}\Omega^p\left(\partial_{t^*}+\dot{r_b}\partial_r\right)^{n-|p|}\phi\right) \tag{2.146}$$
$$=\sum_{m=0}^{n-|p|}\binom{n-|p|}{m}\frac{\dot{r_b}^m}{r^{|p|}}\Box_g\left(\Omega^p\partial_{t^*}^{n-|p|-m}\partial_r^m\phi\right)+(\text{Bounded lower order terms}).$$

As ∂_{t^*} and Ω commute with $\Box_g$, we can ignore the $m=0$ term in the sum. Then, by the first part of the Lemma, we can bound the right hand side of (2.146). It is bounded by $|\dot{r_b}|$ times a constant multiple of the $\dot{H}^{n+1}$ norm, plus lower order terms:

$$\left\|\Box_g\left(\frac{1}{r^{|p|}}\Omega^p\bar{\partial}_{t^*}^{n-|p|}(\phi)\right)\right\|^2_{L^2(\Sigma_{t^*})}\leq D\left(\|\phi\|^2_{\dot{H}^n(\Sigma_{t^*})}+\dot{r_b}^2\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t^*})}\right). \tag{2.147}$$

We also have that $\Omega^p\bar{\partial}_{t^*}^n(\phi)=0$ on the boundary of the star. Thus we can then proceed by using an elliptic estimate (such as in [37]) on $\bar{\partial}_{t^*}^n\phi$.

We consider the elliptic operator, L, given by

$$L\psi:=\left(1-\frac{2M}{r}-\frac{4M\dot{r_b}}{r}-\left(1+\frac{2M}{r}\right)\dot{r_b}^2\right)\partial_r^2\psi+\frac{1}{r^2}\mathring{\Delta}\psi=f(t^*,r)\partial_r^2\psi+\frac{1}{r^2}\mathring{\Delta}\psi. \tag{2.148}$$

Thus we have

$$\begin{aligned}\int_{\Sigma_{t^*}}(L\psi)^2&=\int_{\Sigma_{t^*}}f^2(\partial_r\psi)^2+\frac{2f}{r^2}\partial_r^2\psi\mathring{\Delta}\psi+\frac{1}{r^4}(\mathring{\Delta}\psi)^2\\&=\int_{\Sigma_{t^*}}f^2(\partial_r^2\psi)^2+\frac{2f}{r^2}|\partial_r\mathring{\nabla}\psi|^2+\frac{2\partial_rf}{r^2}\mathring{\nabla}\psi.\partial_r\mathring{\nabla}\psi+\frac{1}{r^4}|\mathring{\nabla}\mathring{\nabla}\psi|^2-\int_{S_{t^*}}\frac{2f}{r^2}\mathring{\nabla}\psi.\partial_r\mathring{\nabla}\psi\\&\geq\int_{\Sigma_{t^*}}\frac{1}{2}(\partial_r^2\psi)^2+\frac{1}{2r^2}|\partial_r\mathring{\nabla}\psi|^2-\frac{C}{r^2}|\mathring{\nabla}\psi|^2+\frac{1}{r^4}|\mathring{\nabla}\mathring{\nabla}\psi|^2-\int_{S_{t^*}}\frac{2f}{r^2}\mathring{\nabla}\psi.\partial_r\mathring{\nabla}\psi\\&\geq\frac{1}{2}\|\psi\|^2_{\dot{H}^2(\Sigma_{t^*})}-\frac{1}{2}\|\partial_{t^*}\psi\|^2_{\dot{H}^1(\Sigma_{t^*})}-C\|\psi\|^2_{\dot{H}^1(\Sigma_{t^*})}-\int_{S_{t^*}}\frac{2f}{r^2}\mathring{\nabla}\psi.\partial_r\mathring{\nabla}\psi.\end{aligned} \tag{2.149}$$

By rearranging equation (2.1) in coordinates given by (2.39), we have that

$$\|L\psi\|^2_{L^2(\Sigma_{t^*})}\leq C\left(\|\bar{\partial}_{t^*}\psi\|^2_{\dot{H}^1(\Sigma_{t^*})}+\|\Box_g\psi\|^2_{L^2(\Sigma_{t^*})}+\|\psi\|^2_{\dot{H}^1(\Sigma_{t^*})}\right). \tag{2.150}$$

Combining (2.149) and (2.150) with $\psi=\bar{\partial}_{t^*}^n\phi$ ($=0$ on S_{t^*}), and noting that

$$\|\partial_{t^*}\psi\|^2_{\dot{H}^1(\Sigma_{t^*})}\leq C\left(\|\bar{\partial}_{t^*}\psi\|^2_{\dot{H}^1(\Sigma_{t^*})}+\|\psi\|^2_{\dot{H}^1(\Sigma_{t^*})}\right), \tag{2.151}$$

we obtain

$$\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^2(\Sigma_{t^*})}\leq C\left(\|\bar{\partial}_{t^*}^{n+1}\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}+\|\phi\|^2_{\dot{H}^n(\Sigma_{t^*})}+\dot{r_b}^2\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t^*})}\right). \tag{2.152}$$

We then look at $\psi=\frac{1}{r^{|p|}}\Omega^p\bar{\partial}_{t^*}^{n-1}(\phi)$, where p is a multi-index of size 1 (as this also vanishes on S_{t^*}). As the L^2 norms of $\bar{\partial}_{t^*}^2\psi$ and $\bar{\partial}_{t^*}\partial_r\psi$ are bounded by the left hand side of (2.152), we repeat the above argument to get that

$$\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^2(\Sigma_{t^*})}+\|\frac{1}{r^{|p|}}\Omega^p\bar{\partial}_{t^*}^{n-|p|}(\phi)\|^2_{\dot{H}^2(\Sigma_{t^*})}\leq C\left(\|\bar{\partial}_{t^*}^{n+1}\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}+\|\phi\|^2_{\dot{H}^n(\Sigma_{t^*})}+\dot{r_b}^2\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t^*})}\right), \tag{2.153}$$

for $|p|=1$.

We repeat this argument n times to obtain that (2.153) is true for all $|p|\leq n$. The coefficient of ∂_r^2 in (2.1) (with respect to the coordinates in (2.39)) is bounded above and away from 0. This means, we can rearrange (2.1), to bound all r derivatives to obtain:

$$\|\phi\|^2_{\dot{H}^{n+2}(\Sigma_{t^*})} \leq C\left(\|\bar{\partial}^{n+1}_{t^*}\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} + \|\phi\|^2_{\dot{H}^n(\Sigma_{t^*})} + \dot{r_b}^2\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t^*})}\right). \tag{2.154}$$

If we then choose t_0^* such that $\dot{r_b}^2 C < 1$, then we can rearrange the above to get the required result.

3. We proceed in a very similar way to our previous results for finite-in-time boundedness; we use energy currents, Stokes' theorem, and then Gronwall's inequality. For this case, our energy currents will be

$$\sum_{n=1}^{n=N}\sum_{|p|=0}^{n-1} J^{\bar{\partial}_{t^*}}\left(\frac{1}{r^{|p|}}\Omega^p\bar{\partial}^{n-1-|p|}_{t^*}\phi\right), \tag{2.155}$$

where $\bar{\partial}_{t^*}$ is timelike, so $-\int_{\Sigma_{t^*}} dt^*(J^{\bar{\partial}_{t^*}}) \sim \|.\|^2_{\dot{H}^1(\Sigma_{t^*})}$. Note here that Ω are our angular Killing vector fields, and p is a multi-index. Now, as $\Box_g \frac{1}{r^{|p|}}\Omega^p\bar{\partial}^{n-1-|p|}_{t^*}\phi \neq 0$, we obtain an extra term in our bulk integral:

$$\begin{aligned}\int_{t^*=t_0^*}^{t_1^*}&\left(K^{\bar{\partial}_{t^*}} + \bar{\partial}_{t^*}\left(\frac{1}{r^{|p|}}\Omega^p\bar{\partial}^{n-1-|p|}_{t^*}\phi\right)\Box_g\left(\frac{1}{r^{|p|}}\Omega^p\bar{\partial}^{n-1-|p|}_{t^*}\phi\right)\right)\\ &= \int_{\Sigma_{t_0^*}}(-dt^*(J^{\bar{\partial}_{t^*}})) - \int_{\Sigma_{t_1^*}}(-dt^*(J^{\bar{\partial}_{t^*}})) - \int_{S_{[t_0^*,t_1^*]}} d\rho(J^{\bar{\partial}_{t^*}}).\end{aligned} \tag{2.156}$$

As in part 2, we have that the coefficients of ∂_r^2 in (2.1) are bounded above and away from 0. Suppose we have bounded the L^2 norms of all derivatives up to N^{th} order that have fewer that 2 r derivatives. We can then use (2.1) to bound the remaining derivatives up to N^{th} order.

Now we consider the new second term in (2.156). The first part of this Lemma gives us that the sum of these additional term can be bounded by

$$\begin{aligned}\left|\sum_{n=1}^{n=N}\sum_{|p|=0}^{n-1}\bar{\partial}_{t^*}\left(\frac{1}{r^{|p|}}\Omega^p\bar{\partial}^{n-1-|p|}_{t^*}\phi\right)\Box_g\left(\frac{1}{r^{|p|}}\Omega^p\bar{\partial}^{n-1-|p|}_{t^*}\phi\right)\right| &\leq C\int_{t^*=t_0^*}^{t_1^*}\|\phi\|^2_{\dot{H}^N(\Sigma_{t^*})}dt^*\\ &\leq -C'\sum_{n=1}^{n=N}\sum_{|p|=0}^{n-1}\int_{t^*=t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} dt^*\left(J^{\bar{\partial}_{t^*}}\left(\frac{1}{r^{|p|}}\Omega^p\bar{\partial}^{n-1-|p|}_{t^*}\phi\right)\right),\end{aligned} \tag{2.157}$$

where we have used that $\Box_g$ commutes with ∂_{t^*} and each Ω_i. As usual, we can then bound the $K^{\bar{\partial}_{t^*}}$ terms by a multiple of this.

Finally, we note that $\Omega^p\bar{\partial}^{n-1-|p|}_{t^*}\phi = 0$ on $S_{[t_0^*,t_1^*]}$, and $\bar{\partial}_{t^*}$ is tangent to this surface. Therefore $d\rho(J^{\bar{\partial}_{t^*}})$ also vanishes. Thus from equation (2.156), we obtain

$$g(t_1^*) := \sum_{n=1}^{n=N}\sum_{|p|=0}^{n-1}\int_{\Sigma_{t_0^*}}(-dt^*(J^{\bar{\partial}_{t^*}}(\Omega^p\bar{\partial}^{n-1-|p|}_{t^*}\phi)) \leq g(t_0^*) + C\int_{t^*=t_0^*}^{t_1^*} g(t^*)dt^*, \tag{2.158}$$

$$g(t_0^*) \leq g(t_1^*) + C\int_{t^*=t_0^*}^{t_1^*} g(t^*)dt^*. \tag{2.159}$$

Then, in a similar manner to Gronwall's inequality, we will show $g(t^*) \leq e^{c(t^*-t_0^*)}g(t_0^*)$ for $t^* \geq t_0^*$.

The $g(t_0^*) = 0$ is trivial. We proceed to prove that if $g(t_0^*)$ is non-zero, then $g(t^*) < (1+\delta)e^{c(t^*-t_0^*)}g(t_0^*)$ for all $\delta > 0$. Suppose that there exists a t_2^* such that $g(t_2^*) = (1+\delta)e^{C(t_2^*-t_0^*)}g(t_0^*)$, but up to this point,

$g(t_2^*) < (1+\delta)e^{C(t_2^*-t_0^*)}g(t_0^*)$. Then we obtain

$$\begin{aligned} g(t_2^*) < (1+\delta)g(t_0^*) + C\int_{t^*=t_0^*}^{t_2^*} g(t^*)dt^* &\leq (1+\delta)g(t_0^*) + C\int_{t^*=t_0^*}^{t_2^*}(1+\delta)e^{C(t^*-t_0^*)}g(t_0^*)dt^* \\ &= (1+\delta)g(t_0^*) + [(1+\delta)e^{C(t^*-t_0^*)}g(t_0^*)]_{t_0^*}^{t_2^*} \\ &= (1+\delta)e^{C(t_2^*-t_0^*)}g(t_0^*) = g(t_2^*), \end{aligned} \tag{2.160}$$

which gives us a contradiction.

We similarly have $g(t_0^*) \leq e^{C(t_1^*-t_0^*)}g(t_1^*)$.

Thus by letting $A = e^{C(t_1^*-t_0^*)}$ in the statement of the Lemma, we are done.

□

The above lemma then allows us to come to our n^{th} energy uniform boundedness results:

Theorem 2.6.1 (Forward n^{th} order Non-degenerate Energy Boundedness for the Reflective Case)**.** *Given an RNOS model given by* $M,q,T^* \leq 1$, *and a solution* $\phi \in C^\infty_{0\forall t^*}$ *to the wave equation* (2.1) *with reflective boundary conditions* (2.2)*, there exists a constant* $E = E(n,M)$ *such that*

$$\|\phi\|_{\dot{H}^n(\Sigma_{t_1^*})} \leq E\|\phi\|_{\dot{H}^n(\Sigma_{t_0^*})} \qquad \forall \phi \in C^\infty_{0\forall t^*} \quad \forall t_0^* \leq t_1^* \leq t_c^*. \tag{2.161}$$

Proof. As with previous uniform boundedness results, we look at bounding the energy uniformly for sufficiently far back in time. Then we use our local result (part 3 of Lemma 2.6.1) to obtain a uniform bound for all t^*.

If $T^* < 1$, then for sufficiently negative times, $\dot{r_b} = 0$, and therefore $\partial_{t^*}^n\phi = \bar{\partial}_{t^*}\phi$ is a solution to the wave equation (2.1) with reflective boundary conditions (2.2). Therefore we have uniform boundedness of $\|\partial_{t^*}^n\phi\|_{\dot{H}^1(\Sigma_{t^*})}$. Then by Lemma 2.6.1 part 2, we have $\dot{H}^n$ boundedness, as required. Therefore we now focus on the $T^* = 1$ case.

We proceed inductively, by considering $\bar{\partial}_{t^*}^n(\phi)$. Here $\bar{\partial}_{t^*} = \partial_{t^*} + \dot{r_b}\partial_r$ is the partial t^* derivative with respect to the coordinates given in (2.39).

$$\begin{aligned} \Box_g\left(\bar{\partial}_{t^*}^n(\phi)\right) &= \Box_g\left((\partial_{t^*}+\dot{r_b}\partial_r)^n\phi\right) \\ &= \sum_{m=0}^n \binom{n}{m}\dot{r_b}^m\Box_g\left(\partial_{t^*}^{n-m}\partial_r^m\phi\right) + \ddot{r_b}\,(\text{Lower order terms with bounded coefficients}) \\ &= \sum_{m=0}^n \binom{n}{m}\dot{r_b}^m m\Bigg(\frac{2}{r}\left(1-\frac{M}{r}\right)\left(\partial_{t^*}^{n-m+2}\partial_r^{m-1}\phi - \partial_{t^*}^{n-m}\partial_r^{m+1}\phi\right) - \frac{4M}{r^2}\partial_{t^*}^{m-n+1}\partial_r^m\phi \\ &\quad + \frac{m-1}{r^2}\partial_{t^*}^{n-m+2}\partial_r^{m-2}\phi - \frac{m+1}{r^2}\partial_{t^*}^{n-m}\partial_r^m\phi\Bigg) \\ &\quad + \ddot{r_b}(\text{Lower order terms with bounded coefficients}). \end{aligned} \tag{2.162}$$

In RNOS models with $T^* = 1$, $\ddot{r_b} < 0$ for sufficiently negative t^*. We have, by the induction hypothesis, that for some $A = A(M,n) > 0$

$$\int_{t^*=t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} |\ddot{r_b}(\text{Lower order terms with bounded coefficients})|^2\, dt^* \leq \int_{t_0^*}^{t_1^*} A\|\phi\|^2_{\dot{H}^n(\Sigma_{t_0^*})}|\ddot{r_b}|^2 dt^* \tag{2.163}$$

We also have that if we fix t^*_- to be large and negative enough, then $r_b(t^*) \geq A|t^*|^{2/3} \geq 0$, $0 \leq -\dot{r_b} \leq B|t^*|^{-1/3}$ for all $t^* \leq t^*_-$. This means that if $t^*_0, t^*_1 \leq t^*_-$,

$$\int_{\Sigma_{t^*}} \left| \sum_{m=0}^{n} \binom{n}{m} \dot{r_b}^m m \left(\frac{m-1}{r^2} \partial_{t^*}^{n-m+2} \partial_r^{m-2} \phi - \frac{m+1}{r^2} \partial_{t^*}^{n-m} \partial_r^m \phi \right) \right|^2 \leq \frac{1}{{r_b}^4} C' \|\phi\|^2_{\dot{H}^n(\Sigma_{t_0^*})} \tag{2.164}$$
$$\leq \frac{C'^2}{3A^4} |t^*|^{-8/3} \|\phi\|^2_{\dot{H}^n(\Sigma_{t_0^*})} \leq C|t^*|^{-8/3} \|\phi\|^2_{\dot{H}^n(\Sigma_{t_0^*})},$$

for some $C = C(M, n, t^*_-) > 0$.

So now we consider the modified current $J^{X,\varepsilon/2r}(\bar{\partial}^n_{t^*}(\phi))$, as given by (2.25). Here $X = \partial_{t^*} + \varepsilon \partial_r$ and $0 < \varepsilon \ll 1$ is a fixed, small constant. Given we are already restricting ourselves to $t^*_0, t^*_1 \leq t^*_-$, we can calculate

$$-\int_{\Sigma_{t^*}} dt^*(J^{X,\varepsilon/2r}(\bar{\partial}^n_{t^*}(\phi))) \geq c \|\bar{\partial}^n_{t^*}\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} \tag{2.165}$$

for some positive constant, $c = c(M, n, t^*_-) > 0$. We also have

$$d\rho(J^T(X^n(\phi))) \geq 0. \tag{2.166}$$

Thus applying generalised Stokes' theorem, we obtain

$$-\int_{\Sigma_{t_1^*}} dt^*(J^{X,\varepsilon/2r}(\bar{\partial}^n_{t^*}\phi)) \leq -\int_{\Sigma_{t_0^*}} dt^*(J^{X,\varepsilon/2r}(\bar{\partial}^n_{t^*}\phi)) - \int_{t_0^*}^{t_1^*} \int_{\Sigma_{t^*}} \partial_{t^*}\left(\bar{\partial}^n_{t^*}\phi\right) \Box_g(\bar{\partial}^n_{t^*}(\phi)) + K^{X,\varepsilon/2r} dt^*$$
$$\leq -\int_{\Sigma_{t_0^*}} dt^*(J^{X,\varepsilon/2r}(\bar{\partial}^n_{t^*}\phi)) \tag{2.167}$$
$$+ \int_{t_0^*}^{t_1^*} \|\partial_{t^*}\bar{\partial}^n_{t^*}\phi\|_{L^2(\Sigma_{t^*})} \left(A|\ddot{r_b}| + C|t^*|^{-4/3}\right) \|\phi\|_{\dot{H}^n(\Sigma_{t_0^*})} - K^{X,\varepsilon/2r} dt^*$$
$$- \int_{t_0^*}^{t_1^*} \int_{\Sigma_{t^*}} \partial_{t^*}\bar{\partial}^n_{t^*}(\phi) \left(\sum_{m=0}^{n} \binom{n}{m} \dot{r_b}^m m \right.$$
$$\left. \left(\frac{2}{r}\left(1 - \frac{M}{r}\right) \left(\partial_{t^*}^{n-m+2}\partial_r^{m-1}\phi - \partial_{t^*}^{n-m}\partial_r^{m+1}\phi\right) - \frac{4M}{r^2} \partial_{t^*}^{m-n+1} \partial_r^m \phi \right) \right) dt^*.$$

We now note that in the case $m \geq 2$, every term has a coefficient which can be bounded by $A\dot{r_b}^2/r_b \leq B|t^*|^{-4/3}$. We can similarly bound any terms with a $1/r^2$ coefficient. Thus we have that

$$-\int_{t_0^*}^{t_1^*} \int_{\Sigma_{t^*}} \partial_{t^*}\bar{\partial}^n_{t^*}\phi \left(\sum_{m=0}^{n} \binom{n}{m} \dot{r_b}^m m \left(\frac{2}{r}\left(1 - \frac{M}{r}\right) \left(\partial_{t^*}^{n-m+2}\partial_r^{m-1}\phi - \partial_{t^*}^{n-m}\partial_r^{m+1}\phi\right) - \frac{4M}{r^2} \partial_{t^*}^{m-n+1} \partial_r^m \phi \right) \right) dt^*$$
$$\leq \int_{t_0^*}^{t_1^*} B|t^*|^{-4/3} \|\bar{\partial}^n_{t^*}\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} dt^* - \int_{t_0^*}^{t_1^*} \int_{\Sigma_{t^*}} \frac{n\dot{r_b}}{r} \left(\partial_{t^*}^{n+1}\phi - \partial_{t^*}^{n-1}\partial_r^2\right) \partial_{t^*}\bar{\partial}^n_{t^*}(\phi) dt^*, \tag{2.168}$$

Here, we have used part 2 of Lemma 2.6.1 to bound $\partial_{t^*}\bar{\partial}^n_{t^*}(\phi)$ by $\|\bar{\partial}^n_{t^*}\phi\|_{\dot{H}^1(\Sigma_{t^*})}$.

In order to bound the final term, it is useful to note that swapping between ∂_{t^*} and $\bar{\partial}_{t^*}$ introduces terms with a factor of $\dot{r_b}$. Any derivative that now has an extra factor of $\dot{r_b}$ can be absorbed into the B term in (2.168). Thus we can freely swap between the two derivatives when bounding this final term. We can similarly ignore any $\bar{\partial}_{t^*} r$

terms.

$$\begin{aligned}\int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} \frac{n\dot{r}_b}{r}\left(\partial_{t^*}^{n+1}\phi-\partial_{t^*}^{n-1}\partial_r^2\right)\partial_{t^*}\bar{\partial}_{t^*}^n(\phi)dt^* &\geq \int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} \frac{n\dot{r}_b}{r}\bar{\partial}_{t^*}^{n-1}\left(\partial_{t^*}^2\phi-\partial_r^2\phi\right)\bar{\partial}_{t^*}^{n+1}\phi\, dt^* \\ &\quad -\int_{t_0^*}^{t_1^*} B|t^*|^{-4/3}\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}dt^* \qquad (2.169)\\ &\geq \int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} \frac{n\dot{r}_b}{r^3}\bar{\partial}_{t^*}^{n-1}\mathring{\slashed{\Delta}}\phi\bar{\partial}_{t^*}^{n+1}\phi\, dt^* - \int_{t_0^*}^{t_1^*} B|t^*|^{-4/3}\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}dt^* \\ &\geq -\int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} \frac{n\dot{r}_b}{r^3}\bar{\partial}_{t^*}^{n-1}\mathring{\slashed{\nabla}}\phi.\bar{\partial}_{t^*}^{n+1}\mathring{\slashed{\nabla}}\phi\, dt^* - \int_{t_0^*}^{t_1^*} B|t^*|^{-4/3}\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}dt^* \\ &\geq \int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} \frac{n\dot{r}_b}{r^3}|\bar{\partial}_{t^*}^n\mathring{\slashed{\nabla}}\phi|^2 dt^* - \int_{t_0^*}^{t_1^*} B|t^*|^{-4/3}\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}dt^* \\ &\quad -\left|\frac{n\dot{r}_b}{r_b}\right|_{t_0^*}\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t_0^*})}-\left|\frac{n\dot{r}_b}{r_b}\right|_{t_1^*}\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t_1^*})}.\end{aligned}$$

We then have, that

$$\int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}} K^{X,\varepsilon/2r}dt^* \geq \int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}}\frac{\varepsilon}{r^3}|\bar{\partial}_{t^*}^n\mathring{\slashed{\nabla}}\phi|^2dt^* - \int_{t_0^*}^{t_1^*} D|t^*|^{-4/3}\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}dt^*, \qquad (2.170)$$

for some fixed constant $D = D(M,n,t_-^*) > 0$.

Finally, we note that

$$\begin{aligned}&\int_{t_0^*}^{t_1^*}\|\partial_{t^*}\bar{\partial}_{t^*}^n\phi\|_{L^2(\Sigma_{t^*})}\left(A|\ddot{r}_b|+C|t^*|^{-4/3}\right)\|\phi\|_{\dot{H}^n(\Sigma_{t_0^*})}dt^* \\ &\quad\leq A\int_{t_0^*}^{t_1^*}\left(|\ddot{r}_b|+|t^*|^{-4/3}\right)\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}+\left(|\ddot{r}_b|+|t^*|^{-4/3}\right)\|\phi\|^2_{\dot{H}^n(\Sigma_{t_0^*})}dt^* \\ &\quad\quad -\int_{t_0^*}^{t_1^*}\int_{\Sigma_{t^*}}\frac{\varepsilon+n\dot{r}_b}{r^3}|\bar{\partial}_{t^*}\mathring{\slashed{\nabla}}\phi|^2dt^* \qquad (2.171)\\ &\quad\leq A\int_{t_0^*}^{t_1^*}\left(|\ddot{r}_b|+|t^*|^{-4/3}\right)\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}dt^*+E\|\phi\|^2_{\dot{H}^n(\Sigma_{t_0^*})},\end{aligned}$$

for $E = E(M,n,t_-^*) > 0$. This is given t_-^* negative enough that $|\dot{r}_b| \leq \varepsilon/n$ and that we can apply part 2 of Lemma 2.6.1.

Adding these all together, we get

$$\begin{aligned}\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t_1^*})} &\leq C\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t_0^*})}+D\|\phi\|_{\dot{H}^n(\Sigma_{t_0^*})}+A\int_{t_0^*}^{t_1^*}\left(|\ddot{r}_b|+|t^*|^{-4/3}\right)\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}dt^* \qquad (2.172)\\ &\leq C\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t_0^*})}+A\int_{t_0^*}^{t_1^*}\left(|\ddot{r}_b|+|t^*|^{-4/3}\right)\|\bar{\partial}_{t^*}^n\phi\|^2_{\dot{H}^1(\Sigma_{t^*})}dt^*\end{aligned}$$

where constants C, D, A all only depend on M, n and t_-^*.

Thus by Gronwall's inequality, we have

$$\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t_1^*})} \leq C\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t_0^*})}\exp\left(A\int_{t_0^*}^{t_1^*}\left(|\ddot{r}_b|+|t^*|^{-4/3}\right)\right)dt^* \leq C\|\phi\|^2_{\dot{H}^{n+1}(\Sigma_{t_0^*})}\exp\left(A\left(|\dot{R}(t_-^*)|+|t_-^*|^{-1/3}\right)\right), \qquad (2.173)$$

for all $t_0^* \leq t_1^* \leq t_-^*$. We can then proceed to cover the interval $[t_-^*, t_c^*]$ by using part 3 of Lemma 2.6.1. Thus we obtain our result. □

The last theorem we then prove in this section is backwards n^{th} order energy boundedness.

Theorem 2.6.2 (Backwards n^{th} order Non-degenerate Energy Boundedness for the Reflective Case)**.** *Let $\phi \in C^\infty_{0\forall t^*}$ be a solution to the wave equation* (2.1) *with reflective boundary conditions. There exists a constant*

$E = E(n,M)$ *such that*

$$\|\phi\|_{\dot{H}^n(\Sigma_{t_0^*})} \leq E\|\phi\|_{\dot{H}^n(\Sigma_{t_1^*})} \qquad \forall \phi \in C^\infty_{0\forall t^*} \quad \forall t_1^* \in [t_0^*, t_c^*]. \tag{2.174}$$

Proof. This is proved identically to Theorem 2.6.1, but let $X = \partial_{t^*} - \varepsilon\partial_r$, and we are done (for positive definiteness of the surface terms, see Theorem 2.5.5). □

We finally look to extend the boundedness result in the permeating case to the $\dot{H}^2$ norm. Note we cannot extend the result beyond this, as we do not know that solutions with higher order derivatives even exist.

Theorem 2.6.3. *Let $\phi \in H^2_{0\forall\tau}$ be a solution to the wave equation* (2.1) *(as given by Theorem 2.4.2). There exists a constant $C = C(M) > 0$ such that*

$$\|\phi\|^2_{\dot{H}^2(\Sigma_{\tau_1})} \leq C\|\phi\|^2_{\dot{H}^2(\Sigma_{\tau_0})} \qquad \forall \tau_0 < \tau_1 < \tau_c \tag{2.175}$$

$$\|\phi\|^2_{\dot{H}^2(\Sigma_{\tau_0})} \leq C\|\phi\|^2_{\dot{H}^2(\Sigma_{\tau_1})} \qquad \forall \tau_0 < \tau_1 < \tau_{c^-}. \tag{2.176}$$

Proof. As in the reflective case, we first prove the local in time case. Let

$$X = \partial_\tau + \chi\left(\frac{r}{r_b}\right)\dot{r}_b\partial_r, \tag{2.177}$$

where χ is a smooth cut-off function which vanishes outside $[1/2, 3/2]$ and is identically 1 inside $[2/3, 4/3]$.

Note that X is tangent to the boundary over which irregularities of g occur. This means that derivatives of components of g in the X direction are still H^1_{loc}.

We then proceed to write (2.1) in terms of this X:

$$\Box_g\phi = \begin{cases} -\partial_\tau(X\phi) + \left(2\sqrt{\frac{2Mr^2}{r_b^3}} + \chi\dot{r}_b\right)\partial_r(X\phi) + \left(1 - \frac{2Mr^2}{r_b^3} - \dot{r}_b\left(2\sqrt{\frac{2Mr^2}{r_b^3}} + \chi\dot{r}_b\right)\right)\partial_r^2\phi + \frac{1}{r^2}\mathring{\Delta}\phi \\ +\text{Lower Order Terms} \qquad r \leq r_b \\ -\partial_\tau(X\phi) + \left(2\sqrt{\frac{2M}{r}} + \chi\dot{r}_b\right)\partial_r(X\phi) + \left(1 - \frac{2M}{r} - \dot{r}_b\left(2\sqrt{\frac{2M}{r}} + \chi\dot{r}_b\right)\right)\partial_r^2\phi + \frac{1}{r^2}\mathring{\Delta}\phi \\ +\text{Lower Order Terms} \qquad r \geq r_b. \end{cases} \tag{2.178}$$

Here note:

$$1 - \frac{2Mr^2}{r_b^3} - \dot{r}_b\left(2\sqrt{\frac{2Mr^2}{r_b^3}} + \chi\dot{r}_b\right) = 1 - \frac{2Mr^2}{r_b^3} + \sqrt{\frac{2M}{r_b}}\left(2\sqrt{\frac{2Mr^2}{r_b^3}} - \chi\sqrt{\frac{2M}{r_b}}\right) \tag{2.179}$$
$$= 1 - \frac{2Mr^2}{r_b^3} + \frac{2Mr}{r_b^2}\left(2 - \chi\frac{r_b}{r}\right) > 0$$

$$1 - \frac{2M}{r} - \dot{r}_b\left(2\sqrt{\frac{2M}{r}} + \chi\dot{r}_b\right) = 1 - \frac{2M}{r} + \sqrt{\frac{2M}{r_b}}\left(2\sqrt{\frac{2M}{r}} - \chi\sqrt{\frac{2M}{r_b}}\right) \tag{2.180}$$
$$= 1 - \frac{2M}{r} + \frac{2M}{\sqrt{rr_b}}\left(2 - \chi\sqrt{\frac{r}{r_b}}\right) > 0.$$

We can approximate ϕ on Σ_τ by smooth functions, and then manipulate (2.178) in an identical way to (2.149) to obtain:

$$\int_{\Sigma_\tau}(\partial_\tau X\phi)^2 \geq \varepsilon\|\phi\|^2_{\dot{H}^2(\Sigma_{t^*})} - C\|X\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} - C\|\phi\|^2_{\dot{H}^1(\Sigma_{t^*})} - \int_{S_{\tau,r=0}} 2f\mathring{\nabla}\phi.\partial_r\mathring{\nabla}\phi d\omega^2. \tag{2.181}$$

Here $\varepsilon > 0$ and $C > 0$ may depend on the time interval we are considering. One can then show that for smooth approximations to ϕ, the final term in (2.181) vanishes.

Thus to prove the local in time result, we can consider the following:

$$f(\tau) := \|\phi\|^2_{\dot{H}^1(\Sigma_\tau)} - \int_{\Sigma_\tau} d\tau\left(J^X(X\phi)\right) \sim \|\phi\|^2_{\dot{H}^2(\Sigma_\tau)}. \tag{2.182}$$

Looking at $\Box_g(X\phi)$ as a distribution, we obtain:

$$\int_{\Sigma_\tau} |(\Box_g(X\phi))XX\phi| = \int_{\Sigma_\tau} |(\Box_g(X\phi) - X(\Box_g\phi))XX\phi| \leq C\|\phi\|^2_{\dot{H}^2(\Sigma_\tau)}. \tag{2.183}$$

As in previous cases, we have that $|K^X(X\phi)| \leq -Cd\tau\left(J^X(X\phi)\right)$ for some $C > 0$. Applying Stokes theorem and boundedness of $\dot{H}^1(\Sigma)$ norms, we obtain:

$$\frac{1}{C}f(\tau_1) - \int_{\tau_0}^{\tau_1} f(\tau)d\tau \leq f(\tau_0) \leq Cf(\tau_1) + \int_{\tau_0}^{\tau_1} Cf(\tau)d\tau \tag{2.184}$$

for some $C = C(M, \tau_0, \tau_1) \geq 0$. Then an application of Gronwall's lemma gives the local result, in either direction.

Once we are sufficiently far back in time, we can consider

$$Y^{\pm} = \left(1 \pm \left(\frac{2M}{r_b}\right)^{1/4}\right)\partial_\tau, \tag{2.185}$$

as given in the proof of Proposition 2.5.1 and Theorem 2.5.8. Define

$$g(\tau)^{\pm} := \|\phi\|^2_{\dot{H}^1(\Sigma_\tau)} - \int_{\Sigma_\tau} d\tau\left(J^{Y^\pm}(T\phi)\right) \sim \|\phi\|^2_{\dot{H}^2(\Sigma_\tau)}. \tag{2.186}$$

$$\int_{\Sigma_\tau} |(\Box_g(T\phi))Y^{\pm}T\phi| = \int_{\Sigma_\tau} |(\Box_g(T\phi) - T(\Box_g\phi))Y^{\pm}T\phi| \leq \frac{C}{r_b^2}\|\phi\|^2_{\dot{H}^2(\Sigma_\tau)} \leq \frac{C'}{r_b^2}g(\tau)^{\pm}. \tag{2.187}$$

Now we can choose $\tau \leq \tau^*$ negative enough that $\mp K^{Y^\pm}(T\phi) \geq 0$. Thus

$$\mp K^{Y^\pm}(T\phi) + |(\Box_g(T\phi))Y^{\pm}T\phi| \leq \frac{C'}{r_b^2}g(\tau)^{\pm}, \tag{2.188}$$

In this region we can use the boundedness of $\dot{H}^1$ norms given by Theorems 2.5.7 and 2.5.8 to obtain

$$g(\tau_1)^+ \leq Cg(\tau_0)^+ + \int_{\tau_0}^{\tau_1} \frac{C}{r_b^2}g(\tau)^+ d\tau \tag{2.189}$$

$$g(\tau_0)^- \leq Cg(\tau_1)^- + \int_{\tau_0}^{\tau_1} \frac{C}{r_b^2}g(\tau)^- d\tau. \tag{2.190}$$

An application of Gronwall's lemma completes the proof, on noting that $\int_{-\infty}^{\tau^*} r_b(\tau)^{-2}d\tau < \infty$. □

2.7 The Scattering Map

We now consider bounds on the radiation fields. We will be considering the maps $\mathscr{G}^+$ and $\mathscr{F}^-$, which take data from $\Sigma_{t_c^*}$ to data on $\mathscr{I}^-$ and $\mathscr{I}^+$ respectively. We will also consider their inverses (where defined), $\mathscr{G}^-$ and $\mathscr{F}^+$, which take data from $\mathscr{I}^+$ and $\mathscr{I}^-$ respectively to $\Sigma_{t_c^*}$. We will look at obtaining boundedness or non-boundedness for these.

Finally, we will define the scattering map, $\mathscr{S}^+ := \mathscr{G}^+ \circ \mathscr{F}^+$, and consider boundedness results for this.

2.7.1 Existence of Radiation Fields

To look at these maps, we will first need a definition of radiation field. We will then need to show it exists for all finite energy solutions of the wave equation.

Proposition 2.7.1 (Existence of the Backwards Radiation Field)**.** *Given ϕ a solution to the wave equation* (2.1) *with boundary conditions* (2.2)*, there exist $\psi_{+,-}$ such that*

$$r(u,v)\phi(u,v,\theta,\varphi) \xrightarrow[u\to-\infty]{H^1_{loc}} \psi_-(v,\theta,\varphi) \tag{2.191}$$

$$r(u,v)\phi(u,v,\theta,\varphi) \xrightarrow[v\to\infty]{H^1_{loc}} \psi_+(u,\theta,\varphi). \tag{2.192}$$

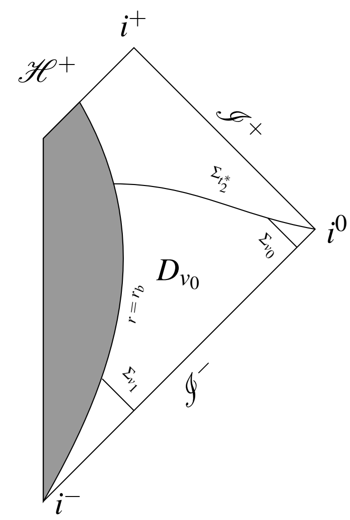
Proof. This existence has been done many times before, see for example [35, 2]. □

2.7.2 Backwards Scattering from $\Sigma_{t_c^*}$

Now we have existence of the radiation field, we define the following map:

$$\begin{aligned} \mathscr{F}^- : \quad \mathscr{E}^X_{\Sigma_{t_c^*}} &\longrightarrow \mathscr{F}^-\left(\mathscr{E}^X_{\Sigma_{t_c^*}}\right) \subset H^1_{loc}(\mathscr{I}^-) \\ (\phi|_{\Sigma_{t_c^*}}, \partial_{t^*}\phi|_{\Sigma_{t_c^*}}) &\mapsto \psi_- \end{aligned} \tag{2.193}$$

where the ψ_- is as defined in Proposition 2.7.1, and the X is any everywhere timelike vector field (including on the event horizon) which coincides with the timelike Killing vector field ∂_{t^*} for sufficiently large r. An X with these properties is chosen, so that the X norm is equivalent to the $\dot{H}^1$ norm.



We define the inverse of $\mathscr{F}^-$ (once injectivity is established on the image of $\mathscr{F}^-$) as $\mathscr{F}^+$.

- Firstly, we will show $\mathscr{F}^-$ is bounded. (Proposition 2.7.2)
- Then we will show that $\mathscr{F}^+$, if it can be defined, would be bounded, which gives us that $\mathscr{F}^-$ is injective. (Proposition 2.7.3)
- Finally, we show that $Im(\mathscr{F}^-)$ is dense in $\mathscr{E}^T_{\mathscr{I}^-}$. (Proposition 2.7.4)

We will then combine these results in Theorem 2.7.1 to obtain that $\mathscr{F}^-$ is a linear, bounded bijection with bounded inverse between the spaces $\mathscr{E}^X_{\Sigma_{t_c^*}}$ and $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$.

We will begin with the following:

Proposition 2.7.2 (Boundedness of $\mathscr{F}^-$)**.** *There exists a constant $A(M,q,T^*)$ such that*

$$\|\mathscr{F}^-(\phi)\|^2_{\partial_{t^*},\mathscr{I}^-} = \int_{\mathscr{I}^-} (\partial_v(\mathscr{F}^-(\phi)))^2 dv d\omega \le A\|\phi\|^2_{\dot{H}^1(\Sigma_{t_c^*})}. \tag{2.194}$$

Proof. We will first prove this for compactly supported smooth functions, and then extend to H^1 functions using a density argument.

Let X, w and t_2^* be as in the proof of Theorem 2.5.5. Let ϕ be smooth and compactly supported on $\Sigma_{t_2^*}$. Take v_0 large enough such that on $\Sigma_{t_2^*}$, ϕ is only supported on $v \le v_1$. We integrate $K^{X,w}$, in the region $D_{v_0} = \{v \in [v_0, v_1], t^* \le t_2^*\}$, for any $v_0 \le v_1$.

We then apply generalised Stokes' Theorem in D_{v_0} to obtain the following boundary terms:

$$\begin{aligned}-\int_{\Sigma_{t_2^*}} dt^*(J^{X,w}) &= \int_{D_{v_0}} K^{X,w} - \int_{\{v=v_0\}} dv(J^{X,w}) + \int_{S_{[t_0^*,t_2^*]}} d\rho(J^{X,w}) - \lim_{u_0\to-\infty}\int_{\{u=u_0\}\cap[v_0,v_1]} du(J^X,w)\\ &\geq -\int_{\{v=v_0\}} dv(J^{X,w}) - \lim_{u_0\to-\infty}\int_{\{u=u_0\}\cap[v_0,v_1]} du(J^{X,w})\end{aligned} \tag{2.195}$$

where t_0^* is the value of t^* at the sphere where $\{v=v_0\}$ intersects $\{r=r_b(t^*)\}$.

$$\begin{aligned}-\int_{\{v=v_0\}} dv(J^{X,w}) &= \int_{\{v=v_0\}} \frac{b}{2}\left(\partial_{t^*}\phi - \partial_r\phi - \frac{\phi}{r}\right)^2\\ &+\frac{1}{2}\left(\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)f(t^*)+\left(\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)b\right)(\partial_{t^*}\phi-\partial_r\phi)^2 \geq 0\end{aligned} \tag{2.196}$$

$$\begin{aligned}&-\int_{\{u=u_0\}\cap[v_0,v_1]} du(J^{X,w})\\ &= \int_{\{u=u_0\}\cap[v_0,v_1]} \frac{f(t^*)-b}{2\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)}\left(\left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)\partial_{t^*}\phi - \left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\partial_r\phi\right)^2\\ &+\frac{b}{r}\phi\left(\frac{1+\frac{2M}{r}-\frac{q^2M^2}{r^2}}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}\partial_{t^*}\phi - \partial_r\phi\right) - \frac{b\phi^2}{2r^2}\\ &+\frac{\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)f(t^*) - \left(1+\frac{2M}{r}-\frac{q^2M^2}{r^2}\right)b}{2r^2\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)}|\mathring{\nabla}\phi|^2\end{aligned} \tag{2.197}$$

However, as we know that $r\phi$ tends to an H^1_{loc} function, and the volume form on $\{u=u_0\}$ is r^2, we can see that the terms in (2.197) with a factor of ϕ tend to 0 as $u_0\to\infty$. Similarly, by applying the rotational Killing fields Ω_i (defined in (1.26)) to ϕ, we can see $r\Omega_i\phi$ has an H^1_{loc} limit. Thus terms in (2.197) involving $\mathring{\nabla}\phi$ will also tend to 0 in the limit $u_0\to\infty$.

Thus in the limit $u_0\to\infty$ (and therefore $r\to\infty$, $t^*\to-\infty$) we obtain:

$$\begin{aligned}-\lim_{u_0\to-\infty}\int_{\{u=u_0\}\cap[v_0,v_1]} du(J^{X,w}) &= \lim_{u_0\to-\infty}\int_{\{u=u_0\}\cap[v_0,v_1]} \frac{1-b}{4}\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)(\partial_{t^*}\phi-\partial_r\phi)^2 r^2 dv d\omega\\ &= \lim_{u_0\to-\infty}\int_{\{u=u_0\}\cap[v_0,v_1]} \frac{1-b}{4}\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)(\partial_{t^*}(r\phi)-\partial_r(r\phi))^2 dv d\omega\\ &= \int_{\mathscr{I}^-\cap[v_0,v_1]} \frac{1-b}{4}(\partial_{t^*}\psi_- - \partial_r\psi_-)^2 dv d\omega \geq \varepsilon\int_{\mathscr{I}^-\cap[v_0,v_1]}(\partial_v\psi_-)^2 dv d\omega\end{aligned} \tag{2.198}$$

where to get from the first line to the second, we have ignored terms of order ϕ, as these tend to 0.

Substituting (2.198) and (2.196) into (2.195), and noting that $-\int_{\Sigma_{t_2^*}} dt^*(J^{X,w})$ can be bounded by the $\dot{H}^1$ norm, we have that:

$$\int_{\mathscr{I}^-\cap[v_0,v_1]} (\partial_v(\mathscr{F}^-(\phi)))^2 dv d\omega \leq A\|\phi\|^2_{\dot{H}^1(\Sigma_{t_2^*})}, \tag{2.199}$$

where A is independent of v_0. Thus taking a limit as $v_0\to-\infty$ and imposing Theorem 2.5.1 in the region $[t_0^*,t_2^*]$ gives us the result of the theorem. □

We then move on to showing $\mathscr{F}^+$, if it exists, would be bounded:

Proposition 2.7.3 (Boundedness of $\mathscr{F}^+$). *There exists a constant A such that*

$$\|\phi\|^2_{\dot{H}^1(\Sigma_{t_c^*})} \leq A\int_{\mathscr{I}^-}(\partial_v(\mathscr{F}^-(\phi)))^2 dv d\omega. \tag{2.200}$$

To prove this, we will first need to show a decay result:

Lemma 2.7.1 (Decay of Solutions Along a Null Foliation)**.** *Let ϕ be a solution to* (2.1) *with reflective boundary conditions. Then*

$$\lim_{v_0\to-\infty}\int_{\Sigma_{v_0}} dv(J^{\partial_{t^*}})=0. \tag{2.201}$$

Similarly, let $\phi\in H^2_{0\forall\tau}$ be a solution to (2.1) *with permeating boundary conditions and $\Omega_i\phi\in H^2_{0\forall\tau}$. Then*

$$\lim_{v_0\to-\infty}\int_{\tilde{\Sigma}_{v_0}} dv(J^{Y})=0, \tag{2.202}$$

for J^Y as in Theorem 2.5.1.

Proof. We first deal with the reflective case by showing the result for ϕ compactly supported on some Σ_{v_1}, and then extend the result by a density argument.

Firstly, we calculate $-dv(J^{\partial_{t^*}})$ and $-du(J^{\partial_{t^*}})$.

$$\begin{aligned}-\int_{\Sigma_{v_0}} dv(J^{\partial_{t^*}})&=\frac{1}{2}\int_{\Sigma_{v_0}}\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)(\partial_{t^*}\phi-d_r\phi)^2+\frac{1}{r^2}|\mathring{\nabla}\phi|^2\\&=\int_{\Sigma_{v_0}}2\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)^{-1}(\partial_u\phi)^2+\frac{1}{2r^2}|\mathring{\nabla}\phi|^2=\int_{\Sigma_{v_0}}(\partial_u\phi)^2+\frac{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}{4r^2}|\mathring{\nabla}\phi|^2r^2dud\omega\end{aligned} \tag{2.203}$$

$$-\int_{\Sigma_{u_0}\cap[v_0,v_1]} du(J^{\partial_{t^*}})=\frac{1}{2}\int_{\Sigma_{u_0}\cap[v_0,v_1]}\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\left(\left(\frac{1+\frac{2M}{r}-\frac{q^2M^2}{r^2}}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}\right)\partial_{t^*}\phi-\partial_r\phi\right)^2+\frac{1}{r^2}|\mathring{\nabla}\phi|^2 \tag{2.204}$$

$$=\int_{\Sigma_{u_0}\cap[v_0,v_1]}(\partial_v\phi)^2+\frac{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}{4r^2}|\mathring{\nabla}\phi|^2r^2dvd\omega\geq 0$$

Integrating $K^{\partial_{t^*}}$ in the area $D_{u_0}:=\{v\in[v_0,v_1]\}\cap\{u\geq u_0\}$, using (2.91), (2.93), and then letting $u_0\to-\infty$ gives us that

$$-\int_{\Sigma_{v_1}} dv(J^{\partial_{t^*}})\leq-\int_{\Sigma_{v_0}} dv(J^{\partial_{t^*}}). \tag{2.205}$$

We then proceed to use the r^p method [12]. We consider the wave operator applied to $r\phi$:

$$\frac{4\partial_u\partial_v(r\phi)}{1-\frac{2M}{r^2}+\frac{q^2M^2}{r^2}}=\frac{1}{r^2}\mathring{\triangle}(r\phi)-\frac{2M}{r^3}\left(1-\frac{q^2M}{r}\right)r\phi. \tag{2.206}$$

We apply (2.206) to the following integral over D_{u_0}:

$$
\begin{aligned}
\int_{\Sigma_{v_1}} \left(\frac{2r}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}(\partial_u(r\phi))^2\right) du d\omega &\geq \left(\int_{\Sigma_{v_1}} - \int_{\Sigma_{v_0}} - \int_{S_{[v_0,v_1]}}\right)\left(\frac{2r}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}(\partial_u(r\phi))^2\right) du d\omega \\
&= \int_{D_{u_0}} \frac{4r\partial_u(r\phi)\partial_u\partial_v(r\phi)}{1-\frac{2M}{r^2}+\frac{q^2M^2}{r^2}} + 2(\partial_u(r\phi))^2\partial_v\left(\frac{1}{r}-\frac{2M}{r^2}+\frac{q^2M^2}{r^2}\right)^{-1} dudvd\omega \\
&= \int_{D_{u_0}} \left(\frac{1}{r}\mathring{\Delta}(r\phi) - \frac{2M}{r^2}\left(1-\frac{q^2M}{r}\right)r\phi\right)\partial_u(r\phi) + \frac{\left(1-\frac{4M}{r}+\frac{3q^2M^2}{r^2}\right)(\partial_u(r\phi))^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} dudvd\omega \\
&= \int_{D_{u_0}} \frac{\left(1-\frac{4M}{r}+\frac{3q^2M^2}{r^2}\right)(\partial_u(r\phi))^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} - \frac{1}{2r}\partial_u\left(|\mathring{\nabla}r\phi|^2\right) - \frac{M}{r^2}\left(1-\frac{q^2M}{r}\right)\partial_u((r\phi)^2) dudvd\omega \\
&\geq \int_{D_{u_0}} \frac{\left(1-\frac{4M}{r}+\frac{3q^2M^2}{r^2}\right)(\partial_u(r\phi))^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} \\
&\quad + \left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\left(\frac{1}{4r^2}|\mathring{\nabla}r\phi|^2 + \left(1-\frac{3q^2M}{2r}\right)\frac{M}{r^3}(r\phi)^2\right) dudvd\omega \\
&\geq \frac{1}{2}\int_{D_{u_0}} (\partial_u\phi)^2 + \frac{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}{2r^2}|\mathring{\nabla}\phi|^2 r^2 dudvd\omega \geq \frac{1}{2}\int_{v_0}^{v_1}\left(-\int_{\Sigma_{v_0}} dv(J^{\partial_{t^*}})\right) dv
\end{aligned}
\tag{2.207}
$$

In order to obtain the last line, we have used that for r large enough,

$$
\begin{aligned}
\int_{\Sigma_v} \frac{1-\frac{4M}{r}+\frac{3q^2M^2}{r^2}}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}(\partial_u(r\phi))^2 dud\omega &\geq \frac{1}{2}\int_{\Sigma_v}(\partial_u(r\phi))^2 dud\omega = \frac{1}{2}\int_{\Sigma_v} r^2(\partial_u\phi)^2 + \partial_u\left(r\partial_u r\phi^2\right) - \partial_u^2 r\frac{(r\phi)^2}{r} dud\omega \\
&\geq \frac{1}{2}\int_{\Sigma_v} r^2(\partial_u\phi)^2 - \frac{1}{2}\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\left(1-\frac{q^2M}{r}\right)\frac{M}{r^2}\phi^2 dud\omega \\
&\geq \frac{1}{2}\int_{\Sigma_v} r^2(\partial_u\phi)^2 - \frac{M}{2r^2}\phi^2 dud\omega.
\end{aligned}
\tag{2.208}
$$

The left hand side of (2.207) is independent of v_0, so if we choose ϕ to be compactly supported on Σ_{v_1} (these functions are dense in the set of H^1 functions), then we can let v_0 tend to $-\infty$ to obtain

$$
\int_{-\infty}^{v_1}\left(-\int_{\Sigma_{v_0}} dv(J^{\partial_{t^*}})\right) dv \leq \int_{\Sigma_{v_1}}\left(\frac{4r}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}(\partial_u(r\phi))^2\right) dud\omega. \tag{2.209}
$$

Thus there exists a sequence $v_i \to -\infty$ such that

$$
-\int_{\Sigma_{v_i}} dv(J^{\partial_{t^*}}) \to 0 \quad \text{as } i \to \infty. \tag{2.210}
$$

We then note that given any $\varepsilon > 0$, and a solution ϕ to (2.1) with finite ∂_{t^*} energy on Σ_{v_0}, there exists a smooth compactly supported function ϕ_ε such that

$$
\|\phi - \phi_\varepsilon\|_{\partial_{t^*},\Sigma_{v_0}} \leq \varepsilon/2. \tag{2.211}
$$

Furthermore, by (2.205), we know that for all $v_1 \leq v_0$, we have

$$
\|\phi - \phi_\varepsilon\|_{\partial_{t^*},\Sigma_{v_1}} \leq \varepsilon/2. \tag{2.212}
$$

By (2.210), there exists a $v_1 \leq v_0$ such that

$$\|\phi_\varepsilon\|_{\partial_{t^*},\Sigma_{v_1}} \leq \varepsilon/2. \tag{2.213}$$

By combining (2.212) (2.213), and (2.205) again, we obtain that

$$\|\phi\|_{\partial_{t^*},\Sigma_v} \leq \|\phi\|_{\partial_{t^*},\Sigma_{v_1}} \leq \|\phi_\varepsilon\|_{\partial_{t^*},\Sigma_{v_1}} + \|\phi - \phi_\varepsilon\|_{\partial_{t^*},\Sigma_{v_1}} \leq \varepsilon, \tag{2.214}$$

for all $v \leq v_1$.

Thus given any solution ϕ to (2.1) with finite ∂_{t^*} energy, and given any $\varepsilon > 0$, there exists a v_1 such that

$$\|\phi\|_{\partial_{t^*},\Sigma_v} \leq \varepsilon, \tag{2.215}$$

for all $v \leq v_1$.

Next we look at the permeating case. For this case, let $Y = h(\tau)\partial_\tau$. We then have

$$-dn(J^Y) = \begin{cases} \frac{h}{2}\left(\frac{1}{r^2}|\mathring{\nabla}\phi|^2 + \left(1-\sqrt{\frac{2M}{r}}\right)\left(1+\sqrt{\frac{2M}{r}}\right)^{-1}\left(\partial_\tau\phi - \left(1+\sqrt{\frac{2M}{r}}\right)\partial_r\phi\right)^2\right) & r \geq r_b \\ \frac{h}{2}\left(\frac{1}{r^2}|\mathring{\nabla}\phi|^2 + (\partial_\tau\phi)^2 + \left(1-\frac{2Mr^2}{r_b^3}\right)(\partial_r\phi)^2\right) & r < r_b \end{cases} \tag{2.216}$$

where n is the normal to $\tilde{\Sigma}_\tau$.

We then perform something similar to the reflective case above. However, as we do not have an explicit coordinate system u,v, we will use ∂_τ and ∂_r. Let f be given by

$$f(\tau,r) = \begin{cases} \sqrt{\frac{2Mr^2}{r_b^3}} & r < r_b \\ \sqrt{\frac{2M}{r}} & r \geq r_b \end{cases} \tag{2.217}$$

so our metric is of the form

$$g = -(1-f^2)d\tau^2 + 2f\,d\tau dr + dr^2 + r^2 g_{S^2}. \tag{2.218}$$

Again, let $\psi := r\phi$. We obtain

$$(\partial_\tau + (1-f)\partial_r)(\partial_\tau - (1+f)\partial_r)\psi = f'(\partial_\tau - (1+f)\partial_r)\psi + (2ff' - \dot{f})\frac{\psi}{r} + \frac{1}{r^2}\mathring{\Delta}\psi. \tag{2.219}$$

Note that $(2ff' - \dot{f})$ is not continuous over $r = r_b$.

Then, in a similar way to the reflective case, we obtain that:

$$
\begin{aligned}
C\int_{\Sigma_{v_1}} r((\partial_\tau-(1+f)\partial_r)\psi)^2 du d\omega^2 &\geq \iint_{v_0}^{v_1}(\partial_\tau+(1-f)\partial_r)(r(\partial_\tau-(1+f)\partial_r)\psi)^2 dudvd\omega^2\\
&=\iint_{v_0}^{v_1}\Bigg(-(\partial_\tau-(1+f)\partial_r)\left(\frac{1}{r}|\mathring{\nabla}\psi|^2-(2ff'-\dot f)\psi^2\right)+(1+f)\frac{1}{r^2}|\mathring{\nabla}\psi|^2\\
&\quad -[(\partial_\tau-(1+f)\partial_r)(2ff'-\dot f)]\psi^2\\
&\quad +(1+2rf'-f)((\partial_\tau-(1+f)\partial_r)\psi)^2\Bigg)dudvd\omega^2\\
&\geq \iint_{v_0}^{v_1}\frac{1}{r^2}|\mathring{\nabla}\psi|^2+\frac{1}{2}((\partial_\tau-(1+f)\partial_r)\psi)^2 dudvd\omega^2\\
&\quad +\iint_{r<r_b}\frac{M}{2r_b^3}\psi^2 dudvd\omega^2+\iint_{r\geq r_b}\frac{2M}{r^3}\psi^2 dudvd\omega^2\\
&\quad +\int_{r=r_b}(2ff'-\dot f)|_{r_b^-}^{r_b^+}\psi^2 dvd\omega^2+\int_{r=0}(2ff'-\dot f)\psi^2 dvd\omega^2\\
&\geq \iint_{v_0}^{v_1}\frac{1}{r^2}|\mathring{\nabla}\psi|^2+\frac{1}{2}((\partial_\tau-(1+f)\partial_r)\psi)^2 dudvd\omega^2\\
&\quad +\iint_{r<r_b}\frac{M}{2r_b^3}\psi^2 dudvd\omega^2+\iint_{r\geq r_b}\frac{2M}{r^3}\psi^2 dudvd\omega^2-\int_{r=r_b}\frac{3M}{r_b^2}\psi^2 dvd\omega^2.
\end{aligned}
\tag{2.220}
$$

We will only be using this r^p method to bound the Y-energy for the exterior, so we note

$$
\begin{aligned}
\int_{\Sigma_v\cap\{r\geq r_b\}}(\partial_u\psi)^2 dud\omega^2 &= \int_{\Sigma_v\cap\{r\geq r_b\}} r^2(\partial_u\phi)^2+\partial_u\left(r\partial_u r\phi^2\right)-\partial_u^2 r\frac{\psi^2}{r}dud\omega^2\\
&\geq \int_{\Sigma_v\cap\{r\geq r_b\}} r^2(\partial_u\phi)^2 dud\omega^2-\int_{r=r_b}\frac{2}{r_b}\psi^2 dud\omega^2\\
&= \int_{\Sigma_v\cap\{r\geq r_b\}} r^2(\partial_u\phi)^2-\int_{r=r_b}\frac{2}{r_b}\psi^2 dud\omega^2.
\end{aligned}
\tag{2.221}
$$

Now, we need to bound the ψ/r_b surface term, and therefore also bound the $3M\psi/r_b^2$ surface term in (2.220). We proceed by noting

$$
\begin{aligned}
\int_{r=r_b}\frac{1}{r}\psi^2 dud\omega^2 &= -\int_{r<r_b}(\partial_\tau-(1+f)\partial_r)\left(\frac{1}{r}\chi\left(\frac{2Mr^2}{r_b^3}\right)\psi^2\right)dudvd\omega^2\\
&=\int_{r<r_b}-\frac{2}{r}\psi(\partial_\tau-(1+f)\partial_r)\psi-(1+f)\frac{1}{r^2}\psi^2+\left((1+f)\frac{4M}{r_b^3}+\frac{2Mr}{r_b^4}\dot r_b\right)\chi'\psi^2 dudvd\omega^2\\
&\leq \int_{r<r_b}((\partial_\tau-(1+f)\partial_r)\psi)^2-\left(\frac{1}{r}\psi+(\partial_\tau-(1+f)\partial_r)\psi\right)^2+\frac{CM}{r_b^3}\psi^2 dudvd\omega^2\\
&\leq \int_{r<r_b}((\partial_\tau-(1+f)\partial_r)\psi)^2+\frac{CM}{r_b^3}\psi^2 dudvd\omega^2.
\end{aligned}
\tag{2.222}
$$

Dividing equations (2.220) and (2.222) by $4C$, say, we can absorb the ψ^2 term in (2.222) by our ψ^2 bulk term in (2.220). This gives us

$$
\int_{\Sigma_{v_1}} r((\partial_\tau-(1+f)\partial_r)\psi)^2 dud\omega^2 \geq c\int_{v_0}^{v_1}\int_{\Sigma_v\cap\{r\geq r_b\}}\left(\frac{1}{r^2}|\mathring{\nabla}\phi|^2+(\partial_u\phi)^2\right)\geq c\int_{v_0}^{v_1}\int_{\Sigma_v\cap\{r\geq r_b\}}(-dn(J^T)).
\tag{2.223}
$$

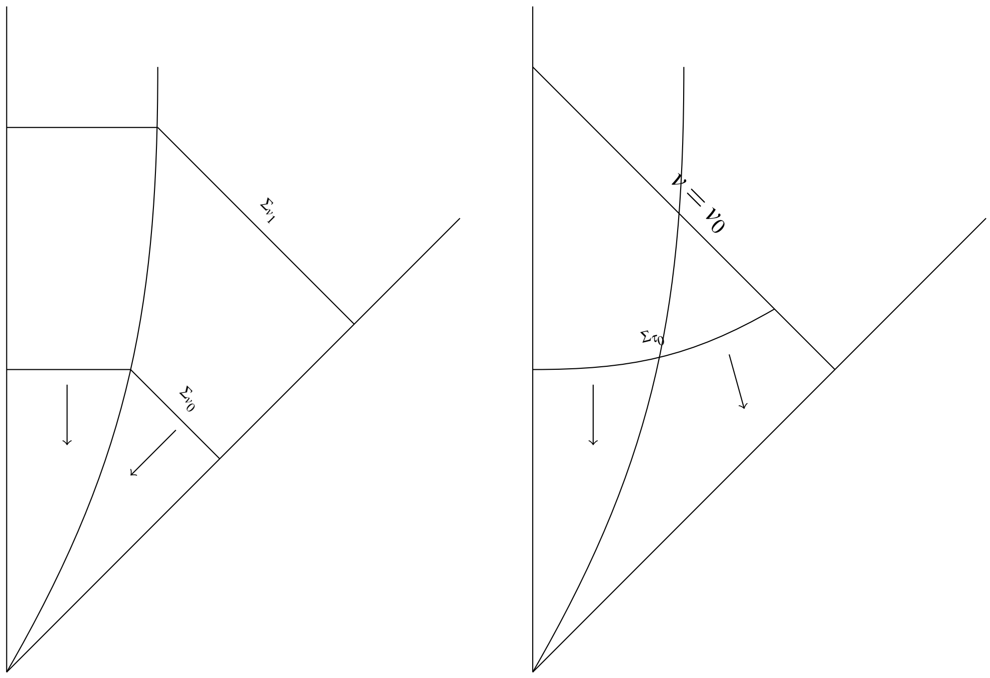


Figure 2.3 Left: Region for integrating in the r^p method as above. Right: Region for integrating for interior decay see below.

Finally, we need to consider the interior of our star. To do this, we will restrict ourselves to $v \leq v_1 < 0$. In this region, we will consider

$$X = -\sqrt{\frac{2Mr^2}{r_b^3}}\partial_r, \qquad w = -\frac{1}{2}\sqrt{\frac{2M}{r_b^3}}. \tag{2.224}$$

We will integrate the modified current $J^{X,w}$, as in (2.25).

This has the properties that:

$$-\nabla.J^{X,w} \geq 0 \tag{2.225}$$

$$-\nabla.J^{X,w}|_{r<r_b} \geq c\sqrt{\frac{2M}{r_b^3}}(-d\tau(J^Y)) = \partial_\tau(\log r_b)c(-d\tau(J^Y)) \tag{2.226}$$

$$|d\tau(J^{X,w})| \leq A(-d\tau(J^Y)) + \sqrt{\frac{2M}{r_b^3}}\psi d\tau(\phi) + \frac{6M}{4{r_b}^3}\phi^2. \tag{2.227}$$

We now integrate $\nabla.J^{X,w}$ in the region $R_{\tau_0,v_1} := \{v \leq v_1, \tau \geq \tau_0\}$ to obtain:

$$\int_{\Sigma_{v_1}\cap\{\tau\leq\tau_0\}} -dn(J^{X,w}) - \int_{\Sigma_{\tau_0}\cap\{v\leq v_1\}} -d\tau(J^{X,w}) = c\int_{R_{\tau_0,v_1}} -\nabla.J^{X,w} \geq \int_{[\tau_0,\tau_1]\cap\{r<r_b\}} \sqrt{\frac{2M}{r_b^3}}(-d\tau(J^Y)). \tag{2.228}$$

Here τ_1 is chosen such that $(\tau_1, r_b(\tau_1))$ has $v \leq v_1$.

The terms on the left hand side of equation (2.228) can both be bounded by data on Σ_{v_1}.

As $\frac{2M}{r_b^3} \to 0$, by choosing τ_1 sufficiently far back, we can combine equations (2.220) and (2.228) to see

$$C \geq \int_{\tau_0}^{\tau_1} \sqrt{\frac{2M}{r_b^3}} \int_{\tilde{\Sigma}_\tau} (-dn(J^Y)). \tag{2.229}$$

This C only depends on our initial data on Σ_{v_1}, and is finite for compactly supported data. As $\sqrt{\frac{2M}{r_b^3}}$ is not integrable, there must be a sequence of $v_i \to \infty$ such that the required result holds. The same logic as before then gives the required result.

□

Proof of Proposition 2.7.3. Fix $t_1^* < t_c^*$, and let ϕ be a solution of (2.1) with boundary conditions (2.2) such that ϕ has finite ∂_{t^*} energy on $\Sigma_{t_1^*}$. Note as $t_1^* < t_c^*$ fixed, finite ∂_{t^*} energy is equivalent to having finite X energy. An explicit calculation gives

$$\left(1-\frac{2M}{r_b(t_1^*)}+\frac{q^2M^2}{r_b(t_1^*)^2}\right)\|\phi\|^2_{\dot{H}^1(\Sigma_{t_1^*})} \leq -\int_{\Sigma_{t_1^*}} dt^*(J^{\partial_{t^*}}) \leq \|\phi\|^2_{\dot{H}^1(\Sigma_{t_1^*})} \tag{2.230}$$

We prove Proposition 2.7.3 by simply integrating $K^{\partial_{t^*}}$ in the region $D_{u_0,v_0} = \{u > u_0, v \geq v_0, t^* < t_1^*\}$. We will then let $u_0 \to -\infty$ to get:

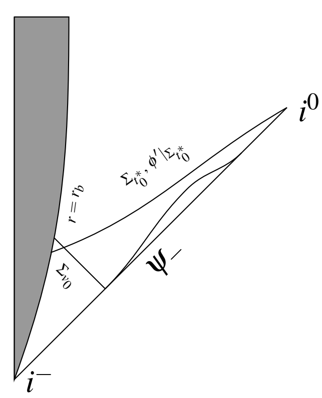
$$\begin{aligned}
-\int_{\Sigma_{t_1^*}} dt^*(J^{\partial_{t^*}}) &= \lim_{u_0\to-\infty}\left(-\int_{\Sigma_{t_1^*}\cap\{u\geq u_0\}} dt^*(J^{\partial_{t^*}})\right) \qquad (2.231)\\
&= -\lim_{u_0\to-\infty}\int_{\Sigma_{u_0}\cap\{v\geq v_0\}} du(J^{\partial_{t^*}}) - \lim_{u_0\to-\infty}\int_{\Sigma_{v_0}\cap\{u\geq u_0\}} dv(J^{\partial_{t^*}}) - \int_{S[v_0,t_1^*]} d\rho(J^{\partial_{t^*}})\\
&\leq \int_{\mathscr{I}^-\cap\{v\geq v_0\}} (\partial_v\psi_-)^2 dv d\omega - \int_{\Sigma_{v_0}} dv(J^{\partial_{t^*}}).
\end{aligned}$$

Here we have used (2.93) to ignore the $S[v_0,t_1^*]$ term. Letting $v_0 \to -\infty$, and using Lemma 2.7.1, we obtain

$$\left(1-\frac{2M}{r_b(t_1^*)}+\frac{q^2M^2}{r_b(t_1^*)^2}\right)\|\phi\|^2_{\dot{H}^1(\Sigma_{t_1^*})} \leq -\int_{\Sigma_{t_1^*}} dt^*(J^{\partial_{t^*}}) \leq \int_{\mathscr{I}^-} (\partial_v\psi_-)^2 dv d\omega. \tag{2.232}$$

Theorem 2.5.1 on the interval $t^* \in [t_1^*, t_c^*]$ then gives us our result. □

We now have that $\mathscr{F}^-$ is bounded and injective, so the inverse is well defined. We also have that $\mathscr{F}^+$ is bounded where it is defined. The final result needed to define the scattering map on the whole space $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$ is that the image of $\mathscr{F}^-$ is dense in $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$:



Proposition 2.7.4 (Density of $Im(\mathscr{F}^-)$ in $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$). *$Im(\mathscr{F}^-)$ is dense in $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$.*

Proof. We prove this using existing results on the scattering map on the full exterior of Reissner–Nordström spacetime. We show that compactly supported smooth functions on $\mathscr{I}^-$ are in the image of $\mathscr{F}^-$. These are dense in $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$.

Given any smooth compactly supported function $\psi_- \in \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$, supported in $v \geq v_0$, we can find a t_0^* such that the sphere $(t_0^*, r_b(t_0^*))$ is in the region $v \leq v_0$. Using previous results from [25], there exists a solution, ϕ' in Reissner–Nordström with radiation field ψ and vanishing on the past horizon. By finite speed of propagation, ϕ' will be supported in $v \geq v_0$. Thus both ϕ' and its derivatives on $\Sigma_{t_0^*}$ vanishes around $r = r_b$.

We then evolve $(\phi'|_{\Sigma_{t_0^*}}, \partial_{t^*}\phi'|_{\Sigma_{t_0^*}})$ from $\Sigma_{t_0^*}$ in RNOS, call this solution ϕ. By finite speed of propagation and uniqueness of solutions, we must have $\phi = \phi'$ for $t^* \leq t_0^*$. By boundedness of $\mathscr{F}^+$ (Proposition 2.7.3) we have that ϕ is in $\mathscr{E}^X_{\Sigma_{t_c^*}}$, and so the radiation field, ψ_-, is in the image of $\mathscr{F}^-$. □

Theorem 2.7.1 (Bijectivity and Boundedness of $\mathscr{F}^-$). *$\mathscr{F}^-$ is a linear, bounded bijection with bounded inverse between the spaces $\mathscr{E}^X_{\Sigma_{t_c^*}}$ and $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$.*

Proof. $\mathscr{F}^+$ is continuous (linear and bounded, by Proposition 2.7.3), and its image, $\mathscr{E}^X_{\Sigma_{t_c^*}}$, is a closed set. Therefore $(\mathscr{F}^+)^{-1}\left(\mathscr{E}^X_{\Sigma_{t_c^*}}\right)$ is closed. Thus

$$\mathscr{F}^-(\mathscr{E}^X_{\Sigma_{t_c^*}}) = (\mathscr{F}^+)^{-1}\left(\mathscr{E}^X_{\Sigma_{t_c^*}}\right) = Cl\left((\mathscr{F}^+)^{-1}\left(\mathscr{E}^X_{\Sigma_{t_c^*}}\right)\right) \supset \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}, \tag{2.233}$$

as $\mathscr{F}^-(\mathscr{E}^X_{\Sigma_{t_c^*}})$ is dense in $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$ (Proposition 2.7.4).

However, as $\mathscr{F}^-$ is also bounded, we have

$$\mathscr{F}^-(\mathscr{E}^X_{\Sigma_{t_c^*}}) \subset \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-} \tag{2.234}$$

Therefore

$$\mathscr{F}^-(\mathscr{E}^X_{\Sigma_{t_c^*}}) = \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}. \tag{2.235}$$

Thanks to Propositions 2.7.2 and 2.7.3, $\mathscr{F}^-$ and $\mathscr{F}^+$ are bounded, and thus we have the required result. □

2.7.3 Forward Scattering from $\Sigma_{t_c^*}$

In a similar manner to Section 2.7.2, we define the map taking initial data on $\Sigma_{t_c^*}$ to radiation fields on $\mathscr{H}^+ \cup \mathscr{I}^+$:

$$\begin{aligned} \mathscr{G}^+ : \quad \mathscr{E}^X_{\Sigma_{t_c^*}} &\longrightarrow \mathscr{G}^+\left(\mathscr{E}^X_{\Sigma_{t_c^*}}\right) \subset H^1_{loc}(\mathscr{H}^+ \cup \mathscr{I}^+) \\ (\phi|_{\Sigma_{t_c^*}}, \partial_{t^*}\phi|_{\Sigma_{t_c^*}}) &\mapsto (\phi|_{\mathscr{H}^+}, \psi_+) \end{aligned} \tag{2.236}$$

where ψ_+ is as in Proposition 2.7.1, and X is again any everywhere timelike vector field (including on the event horizon) which coincides with the timelike Killing vector field ∂_{t^*} for sufficiently large r. We will define the inverse of $\mathscr{G}^+$ (only defined on the image of $\mathscr{G}^+$) as

$$\begin{aligned} \mathscr{G}^- : \mathscr{G}^+\left(\mathscr{E}^X_{\Sigma_{t_c^*}}\right) &\longrightarrow \mathscr{E}^X_{\Sigma_{t_c^*}} \\ \mathscr{G}^+(\phi|_{\Sigma_{t_c^*}}, \partial_{t^*}\phi|_{\Sigma_{t_c^*}}) &\mapsto (\phi|_{\Sigma_{t_c^*}}, \partial_{t^*}\phi|_{\Sigma_{t_c^*}}). \end{aligned} \tag{2.237}$$

Remark 2.7.1. *Note that $\mathscr{G}^\pm$ are defined using scattering in pure Reissner–Nordström. Thus they have been studied extensively already, see for example [14] for the sub-extremal case ($|q| < 1$) and [3] for the extremal case ($|q| = 1$).*

We will be using the following facts about $\mathscr{G}^+$:

Lemma 2.7.2.
- *$\mathscr{G}^+$ is injective.*
- *For the sub-extremal case ($|q| < 1$), $\mathscr{G}^+$ is bounded with respect to the X norm on $\Sigma_{t_c^*}$ and $\mathscr{H}^+$ and the ∂_{t^*} norm on $\mathscr{I}^+$. In the extremal case ($|q| = 1$), we use the weaker result that $\mathscr{G}^+$ is bounded with respect to the X norm on $\Sigma_{t_c^*}$ and the ∂_{t^*} norm on $\mathscr{I}^+$ and $\mathscr{H}^+$.*
- *$\mathscr{G}^+$ is not surjective into $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^+}$, for both sub-extremal and extremal Reissner–Nordström.*
- *$\mathscr{G}^-$ is not bounded, again with respect to the X norm on $\Sigma_{t_c^*}$ and $\mathscr{H}^+$, and the ∂_{t^*} norm on $\mathscr{I}^+$.*

Proof. Thanks to T energy conservation, we have that $\mathscr{G}^+$ is injective.

For $\mathscr{G}^+$ bounded in the sub-extremal case, we apply the celebrated red-shift vector [13] in order to obtain boundedness of the X energy on $\mathscr{H}^+$.

In the extremal case, we do not have the red-shift effect. In this case, the best we can do is apply conservation of T energy which immediately gives the weaker extremal result.

For $\mathscr{G}^+$ not surjective, we can look at any solution with finite ∂_{t^*} energy on $\Sigma_{t_0^*}$, but infinite X energy, such as $\phi = \sqrt{r - r_+}$, $\partial_{t^*}\phi = 0$. Let $\mathscr{G}^+(\phi) = (\phi_+, \psi_+)$, which has finite X and ∂_{t^*} energy respectively (the angular component vanishes by spherical symmetry). $\mathscr{G}^+$ is injective from the ∂_{t^*} energy space, thus no other finite ∂_{t^*} energy data on $\Sigma_{t_c^*}$ can map to (ϕ_+, ψ_+). Therefore no finite X energy solution can map to (ϕ_+, ψ_+), and thus $\mathscr{G}^+\left(\mathscr{E}^X_{\Sigma_{t_c^*}}\right) \neq \mathscr{E}^X_{\mathscr{H}^+} \times \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^+}$. For a more detailed discussion of non-surjectivity in the sub-extremal case see [16] (note this proves non-surjectivity for Kerr, but the proof can be immediately applied to Reissner–Nordström). For the extremal case, again see [3].

By taking a series of smooth compactly supported functions approximating (ϕ_+, ψ_+) in the above paragraph, we can see that $\mathscr{G}^-$ is not bounded. □

2.7.4 The Scattering Map

We are finally able to define the forwards Scattering Map:

$$\begin{gathered}\mathscr{S}^{+} : \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-} \longrightarrow \mathscr{S}^{+}\left(\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}\right) \\ \mathscr{S}^{+} := \mathscr{G}^{+} \circ \mathscr{F}^{+}\end{gathered} \tag{2.238}$$

and similarly with the backwards scattering map:

$$\begin{gathered}\mathscr{S}^{-} : \mathscr{S}^{+}\left(\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}\right) \longrightarrow \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-} \\ \mathscr{S}^{-} := \mathscr{F}^{-} \circ \mathscr{G}^{-}.\end{gathered} \tag{2.239}$$

Note $\mathscr{S}^-$ is defined only on the image of $\mathscr{S}^+$.

Theorem 2.7.2 (The Scattering Map)**.** *The sub-extremal (*$|q| < 1$*) forward scattering map* $\mathscr{S}^+$ *defined by* (2.238) *is an injective linear bounded map from* $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$ *to* $\mathscr{E}^{X}_{\mathscr{H}^+} \cup \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^+}$*. The extremal (*$|q| = 1$*) forward scattering map* $\mathscr{S}^+$*, again defined by* (2.238)*, is an injective linear bounded map from* $\mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^-}$ *to* $\mathscr{E}^{\partial_{t^*}}_{\mathscr{H}^+} \cup \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^+}$*. In both cases,* $\mathscr{S}^+$ *is not surjective, and its image does not even contain* $0 + \mathscr{E}^{\partial_{t^*}}_{\mathscr{I}^+}$*. When defined on the image of* $\mathscr{S}^+$*, its inverse* $\mathscr{S}^-$ *is injective but not bounded.*

Proof. This is an easy consequence of Theorem 2.7.1 and Lemma 2.7.2. □

Remark 2.7.2. *[Generality of Theorem 2.7.2] It should be noted that in the reflective case, proving Theorem 2.7.2 only uses the following two behaviours of* r_b*:*

- *The tangent vector* $(1, \dot{r}_b(t^*), 0, 0)$ *at the point* $(t^*, r_b(t^*), \theta, \varphi)$ *is timelike (for all* t^*, θ, φ*, including at* $t^* = t^*_c$*).*
- *There exists a* t^*_- *and an* $\varepsilon > 0$ *such that for all* $t^* \leq t^*_-$*,* $\dot{r}_b(t^*) \in (-1+\varepsilon, 0)$*.*

Provided r_b *obeys these points, then Theorem 2.7.2 remains true.*

This is in immediate contrast with Reissner–Nordström spacetime. The scattering map in Reissner–Nordström spacetime is an isometry with respect to the T energy, and this immediately follows from the fact that T is a global Killing vector field. Moreover, this imposes the canonical choice of energy on $\mathscr{I}^{\pm}$.

However, in the RNOS model, if one considers the T energy on $\mathscr{I}^-$, then $\mathscr{F}^+$ gives an isometry between $\mathscr{E}^{T}_{\mathscr{I}^-}$ and $\mathscr{E}^{X}_{\Sigma_{t^*_c}}$. Thus, we are forced to consider the non-degenerate X energy, when considering the solution on $\Sigma_{t^*_c}$. This is the main contrast with Reissner–Nordström spacetime, where we consider T energy throughout the whole spacetime.

In both Reissner–Nordström and RNOS spacetimes, we can consider the backwards reflection map, which takes finite energy solutions from $\mathscr{I}^+$ to $\mathscr{I}^-$. On both these surfaces, choice of energy is canonically given by the existence of Killing vector fields in the region around $\mathscr{I}^{\pm}$. In Reissner–Nordström this map is bounded, however in RNOS, this map does not even exist as a map between finite energy spaces.

Chapter 3

Hawking Radiation

3.1 Overview

The problem of Hawking radiation for massless zero-spin bosons can be formulated as follows. We study solutions of the linear wave equation (2.1) on the exteriors of collapsing spherically symmetric spacetimes. In this exterior, the spacetime is a subset of Reissner–Nordström spacetime [41]. Therefore we have coordinates t^*, r, θ, φ for which the metric takes the form

$$g = -\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right) dt^{*2} + 2\left(\frac{2M}{r} - \frac{q^2M^2}{r^2}\right) dt^* dr \qquad (3.1)$$
$$+ \left(1 + \frac{2M}{r} - \frac{q^2M^2}{r^2}\right) dr^2 + r^2 g_{S^2} \qquad t^* \in \mathbb{R} \quad r \in [\max(r_b(t^*), r_+), \infty),$$

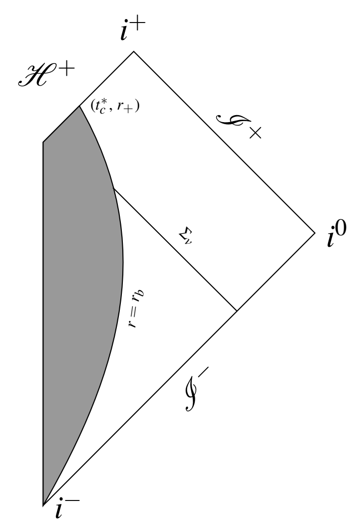


Figure 3.1 Penrose Diagram of RNOS Model, with space-like hyper surface Σ_v.

where g_{S^2} is the metric on the unit 2-sphere, and $r = r_+$ is the horizon of the underlying Reissner–Nordström metric, as given by (1.35). Here $r_b(t^*)$ is the area radius of the boundary of the collapsing dust cloud as a function of t^*. See Section 1.2 for further details of this. We refer to M as the mass of the underlying Reissner–Nordström spacetime, and $q \in [-1, 1]$ as the charge to mass ratio. In particular, we are allowing the extremal case, $|q| = 1$.

We will be imposing Dirichlet conditions on the boundary of the dust cloud, i.e. $\phi = 0$ on $\{r = r_b(t^*)\}$.

These collapsing charged models will include the Oppenheimer–Snyder Model [39] (for which $q = 0$), and we will refer to these more general models as Reissner–Nordström Oppenheimer–Snyder (RNOS) models, as defined in Chapter 1.

The main Theorem of this chapter is informally stated below:

Theorem 4 (Late Time Emission of Hawking Radiation)**.** *Let $\psi_+(u, \theta, \varphi)$ be a Schwartz function on the* 3*-cylinder, with $\hat{\psi}_+$ only supported on positive frequencies. Let ϕ be the solution of* (2.1)*, as given by Theorem 2.4.1, such that*

$$\lim_{v \to \infty} r(u, v)\phi(u, v, \theta, \varphi) = \psi_+(u - u_0, \theta, \varphi) \qquad (3.2)$$

$$\lim_{u \to \infty} r(u, v)\phi(u, v, \theta, \varphi) = 0 \quad \forall v \geq v_c, \qquad (3.3)$$

Define the function ψ_{-,u_0} by

$$\lim_{u \to -\infty} r(u, v)\phi(u, v, \theta, \varphi) = \psi_{-,u_0}(v, \theta, \varphi). \qquad (3.4)$$

Then for all $|q|<1$*,* $n\in\mathbb{N}$*, there exist constants* $A_n(M,q,T^*,\psi_+)$ *such that*

$$\left|\int_{\omega=-\infty}^{0}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}|\omega||\hat{\psi}_{-,u_0}(\omega,\theta,\varphi)|^2\sin\theta d\omega d\theta d\varphi-\int_{\mathscr{H}^-}\frac{|\omega||\hat{\psi}_{\mathscr{H}^-}(\omega,\theta,\varphi)|^2}{e^{\frac{2\pi|\omega|}{\kappa}}-1}\sin\theta d\omega d\theta d\varphi\right|\leq A_n u_0^{-n}, \tag{3.5}$$

for sufficiently large u_0*.*

Here, $\psi_{\mathscr{H}^-}$ *is the reflection of* ψ_+ *in pure Reissner–Nordström spacetime (as will be discussed in Section 3.3), and* κ *is the surface gravity of the Reissner–Nordström black hole.*

In the case $|q|=1$*, there exists a constant* $A(M,q,T^*)$ *such that*

$$\left|\int_{\omega=-\infty}^{0}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}|\omega||\hat{\psi}_{-,u_0}(\omega,\theta,\varphi)|^2\sin\theta d\omega d\theta d\varphi\right|\leq\frac{A}{u_0^{3/2}}, \tag{3.6}$$

for sufficiently large u_0*.*

This result is restated in Theorem 3.4.1 and Corollary 3.4.6, including the precise relationship of A_n, A on ψ_+.

This proof will rely on certain scattering results for solutions of (2.1), which are proven in our Chapter 2. It will also make use of scattering results in pure Reissner–Nordström spacetime, some high frequency approximations, and finally will use an r^{*p} weighted energy estimate, based very closely on the estimates given in [3].

Physically, in order to be normalised, any massless boson cannot have frequency equal to ω. Instead, we will be considering a ψ_+ which has $\hat{\psi}_+$ peaked sharply about ω. The integral $\int_{-\infty}^{0}|\sigma||\psi_{\mathscr{I}^-}|^2 d\sigma$ represents the number of such particles emitted by the black hole (see Section 0.2, or [28, 40] for a discussion of why this is the case).

The limits (3.5) and (3.6) demonstrate that a sub-extremal Reissner–Nordström black hole forming in the collapse of a dust cloud gives off radiation approaching that of a black body with temperature $\frac{\kappa}{2\pi}$ (in units where $\hbar=G=c=1$). In the extremal case, this limit demonstrates that the amount of radiation emitted by a forming, extremal Reissner–Nordström black hole tends towards 0. This is thus a rigorous result confirming Hawking's original calculation in both extremal and subextremal settings.

Equation (3.5) and (3.6) give estimates for the rate at which the limit is approached. In the case of (3.5), this rate is very fast. In the case of (3.6), however, we note that the bound for the rate obtained is integrable. This means that, as the 'final' temperature is zero, the total radiation emitted by an extremal Reissner–Nordström black hole that forms from collapse is finite. Thus, extremal black holes are indeed stable to Hawking radiation.

3.2 Previous Works

Hawking radiation on collapsing spacetimes has been mathematically studied in several other settings, for example [8, 25, 18, 20]. Each of these papers primarily work in frequency space, and work in different contexts to this thesis. Let us discuss some of these differences in more detail.

The original mathematical study of Hawking radiation, [6], considered Hawking radiation of massive or massless non-interacting bosons for a spherically symmetric uncharged collapsing model, and performs this calculation almost entirely in frequency space. Thus [6] obtains what can be viewed as a partial result towards Theorem 4, where the surface of the collapsing star is assumed to remaining at a fixed radius for all sufficiently far back times, and no rate at which the limit is approached is calculated.

In [25], Hawking radiation of fermions is studied for sub-extremal charged, rotating black holes. The Dirac equation itself has a 0^{th} order conserved current, which avoids many of the difficulties of considering the linear wave equation, for which no such current exists. The extremal case is not considered in [25].

The paper [18] considers the full Klein–Gordon equation, but on the Schwarzschild de-Sitter metric. This paper is also the first paper in this setting to obtain a rate at which the limit is obtained, independent of the angular mode. The asymptotically flat case we will be studying introduces several of new issues, due to the lack of a cosmological horizon at a finite radius.

The paper [19] looks at calculating the Hawking radiation of extremal and subextremal Reissner–Nordström black holes in one fewer dimension, with no rate obtained. This paper also considers Hawking radiation in the context of the Unruh-type vacuum rather than Hawking radiation generated from a collapsing spacetime.

Hawking radiation on a charged background has been considered in several other papers in the physics literature, the most relevant being [22, 9]. The second of these, [9], considers Hawking radiation emitted by extremal black holes in the style of Hawking's original paper. Many papers also make use of the surprising fact that the extremal Reissner–Nordström Hawking radiation calculation is very similar to an accelerated mirror in Minkowski space [24].

A more thorough discussion of the physical derivation of Hawking radiation in general, along with a full explanation of Hawking's original method for the calculation, can be found in chapter 14.4 of General Relativity by Wald, [45].

As mentioned in the introduction, one can also rigorously consider the Hartle–Hawking state in order to determine the thermal temperature of a black hole [29, 17], but we will be considering the collapsing model derivation of Hawking radiation in this thesis, as this derivation is more generalisable.

In contrast to many of the above works, the considerations of this thesis are almost entirely in physical space. We will be using the Friedlander radiation formalism, [21], for the radiation field, and we hope this will make the proof more transparent to the reader.

3.3 Classical Scattering and Transmission and Reflection Coefficients of Reissner–Nordström Spacetimes

We will begin this section by first stating a well known result which will be used frequently in this chapter:

Theorem 3.3.1 (Existence of Scattering Solution in pure Reissner–Nordström)**.** *Let $\psi_+(u,\theta,\varphi)$ be a smooth function, compactly supported in $[u_-,u_+]\times S^2$ on the 3-cylinder. Then there exists a unique finite energy smooth solution, $\phi(u,v,\theta,\varphi)$ to* (2.1) *in the region $r\geq r_+$ such that*

$$\lim_{v\to\infty} r(u,v)\phi(u,v,\theta,\varphi)=\psi_+(u,\theta,\varphi) \tag{3.7}$$

$$\lim_{u\to\infty} r(u,v)\phi(u,v,\theta,\varphi)=0. \tag{3.8}$$

Further, there exist functions ψ_{RN} and $\psi_{\mathscr{H}^-}$ such that

$$\lim_{v\to-\infty} r(u,v)\phi(u,v,\theta,\varphi)=\psi_{\mathscr{H}^-}(u,\theta,\varphi) \tag{3.9}$$

$$\lim_{u\to-\infty} r(u,v)\phi(u,v,\theta,\varphi)=\psi_{RN}. \tag{3.10}$$

Finally, $\phi(u,v,\theta,\varphi)=0$ for all $u\geq u_+$.

We refer the reader to [15] (sub-extremal) and [3] (extremal) for this result.

Another result we will frequently use is the existence of a domain of dependence:

Theorem 3.3.2 (Domain of Dependence of the wave equation)**.** *Let $\phi(t_0^*,r,\theta,\varphi)$ be a smooth solution of* (2.1)*, such that on surface $\Sigma_{t_0^*}$, $\phi(t_0^*,r,\theta,\varphi)$ is supported on $r\in[r_0,r_1]$.*

Then ϕ vanishes in the 4 *regions $\{t^*>t_0^*\}\cap\{v\leq v(t_0^*,r_0)\}$, $\{t^*>t_0^*\}\cap\{u\leq u(t_0^*,r_1)\}$, $\{t^*<t_0^*\}\cap\{v\geq v(t_0^*,r_1)\}$ and $\{t^*<t_0^*\}\cap\{u\geq u(t_0^*,r_0)\}$.*

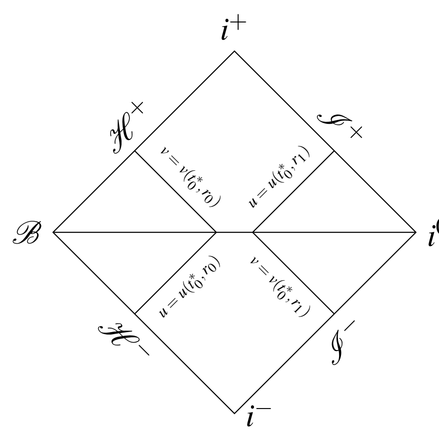


Figure 3.2 The Domain of Dependence

This result is a trivial consequence of T-energy conservation.

An important part of the Hawking radiation calculation is the use of transmission and reflection coefficients, so we will discuss their definitions and useful properties here. For a more full discussion of these, we refer the reader to [15, 10].

We will define the transmission and reflection coefficients in the same way as [15]. We first change coordinates to the tortoise radial function, r^*, and then consider fixed frequency solutions of the wave equation, $\psi = e^{i\omega t}u_{\omega,m,l}(r^*)Y_{l,m}(\theta,\varphi)$. The equation obeyed by this $u_{\omega,l,m}(r^*)$ is

$$u'' + (\omega^2 - V_l)u = 0, \tag{3.11}$$

where

$$V_l(r) = \frac{1}{r^2}\left(l(l+1) + \frac{2M}{r}\left(1 - \frac{q^2M}{r}\right)\right)\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right). \tag{3.12}$$

Considering asymptotic behaviour of possible solutions, there exist unique solutions U_{hor} and U_{inf} [10], characterised by

$$U_{hor} \sim e^{-i\omega r^*} \text{ as } r^* \to -\infty \tag{3.13}$$

$$U_{inf} \sim e^{i\omega r^*} \text{ as } r^* \to \infty. \tag{3.14}$$

We can also see that $\bar{U}_{hor}$ and $\bar{U}_{inf}$ are solutions to (3.11). Moreover U_{inf} and $\bar{U}_{inf}$ are linearly independent, so we can write U_{hor} in terms of U_{inf} and $\bar{U}_{inf}$:

$$\tilde{T}_{\omega,l,m}U_{hor} = \tilde{R}_{\omega,l,m}U_{inf} + \bar{U}_{inf}. \tag{3.15}$$

Here $\tilde{R}$ and $\tilde{T}$ are what we refer to as the reflection and transmission coefficients, respectively.

Now we consider Reissner–Nordström spacetime again. For a Schwartz function $\psi_+(u)$, we can impose the future radiation field $\psi_+(u)Y_{l,m}(\theta,\varphi)$ on $\mathscr{I}^+$, and 0 on $\mathscr{H}^+$. Therefore we can consider only one spherical harmonic, and our solution of the wave equation is of the form $\phi = \frac{Y_{l,m}(\theta,\varphi)}{r}\psi$. We then rewrite the wave equation in terms of ψ, and take a Fourier transform with respect to the timelike coordinate t, where ∂_t is our timelike Killing vector field. We obtain that $\hat{\phi}(\omega,r^*)$ obeys (3.11) for each fixed value of ω. By considering $r^* \to -\infty$, we know that

$$\psi(r^*,t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \hat{\psi}(\omega,r^*)e^{i\omega t}d\omega \sim \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \hat{\psi}_{\mathscr{H}^+}(\omega)e^{i\omega(t+r^*)} + \hat{\psi}\mathscr{H}^-(\omega)e^{i\omega(t-r^*)}d\omega \text{ as } r^* \to -\infty. \tag{3.16}$$

Similarly we can consider $r^* \to \infty$:

$$\psi(r^*,t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \hat{\psi}(\omega,r^*)e^{i\omega t}d\omega \sim \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \hat{\psi}_{\mathscr{I}^-}(\omega)e^{i\omega(t+r^*)} + \hat{\psi}_{\mathscr{I}^+}(\omega)e^{i\omega(t-r^*)}d\omega \text{ as } r^* \to \infty. \tag{3.17}$$

Using that $\psi_{\mathscr{H}^+} = 0$ and $\psi_{\mathscr{I}^+} = \psi_+(u)$, we obtain

$$\hat{\psi}(r^*,\omega) = \hat{\psi}_+(\omega)\tilde{T}_{\omega,l,m}U_{hor}(r^*) = \hat{\psi}_+(\omega)\left(\tilde{R}_{\omega,l,m}U_{inf}(r^*) + \bar{U}_{inf}(r^*)\right). \tag{3.18}$$

Therefore we can obtain our solution on $\mathscr{H}^-$ and $\mathscr{I}^-$:

$$\psi_{\mathscr{H}^-}(v) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \hat{\psi}_+(\omega)\tilde{T}_{\omega,l,m}e^{i\omega v}d\omega \tag{3.19}$$

$$\psi_{\mathscr{I}^-}(u) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \hat{\psi}_+(\omega)\tilde{R}_{\omega,l,m}e^{i\omega u}d\omega. \tag{3.20}$$

We will only be using two properties of $\tilde{R}$ and $\tilde{T}$. Firstly, as a result of conservation of T energy, we have

$$|\tilde{R}_{\omega,l,m}|^2 + |\tilde{T}_{\omega,l,m}|^2 = 1. \tag{3.21}$$

Secondly we will consider the decay of the reflection coefficient for large ω. Corollary A.0.1 states there exists a constant C (independent of M,q,l,m,ω) such that

$$|\tilde{R}_{\omega,l,m}|^2 \leq \frac{C(l+1)^2}{1+M^2\omega^2}. \tag{3.22}$$

The final Theorem we will be using is (part of) Proposition 7.4 in [3], which we will restate here:

Proposition 3.3.1. *Let ϕ be a solution to* (2.1) *in an extremal Reissner–Nordström spacetime, with radiation field $\psi = r\phi$. Let $M < r_0 < 2M$. Then there exists a constant, $C = C(M,r_0) > 0$ such that:*

$$\begin{aligned}\int_{\Sigma_{u_1}\cap\{r\leq r_0\}} \left(1-\frac{r}{M}\right)^{-2} |\partial_v\psi|^2 \sin\theta d\theta d\varphi dv + \int_{\Sigma_{v=u_1+r_0^*}\cap\{r^*>0\}} r^2|\partial_u\psi|^2 \sin\theta d\theta d\varphi du \\ \leq C\int_{\mathscr{H}^-\cap\{u\geq u_1\}} \left(M^2+(u-u_1)^2\right)|\partial_u\psi|^2 + \left|\mathring{\nabla}\psi\right|^2 \sin\theta d\theta d\varphi du \\ + C\int_{\mathscr{I}^-\cap\{v\leq u_1\}} (M^2+(v-u_1-r_0^*)^2)|\partial_v\psi|^2 + \left|\mathring{\nabla}\psi\right|^2 \sin\theta d\theta d\varphi dv.\end{aligned} \tag{3.23}$$

Proof. To prove this, we have taken the $r_{\mathscr{I}^+}$ in the original statement of the theorem to be where $r^* = 0$. □

3.4 The Hawking Radiation Calculation

In this section we will be proving Theorem 4 from the overview, which is a combination of Theorem 3.4.1 and Corollary 3.4.6.

Theorem 3.4.1 (Hawking Radiation)**.** *Let $\psi_+(u,\theta,\varphi)$ be a Schwartz function on the $3-$cylinder. Let ϕ be the solution of* (2.1)*, as given by Theorem 2.4.1, such that*

$$\lim_{v\to\infty} r(u,v)\phi(u,v,\theta,\varphi) = \psi_+(u-u_0,\theta,\varphi) \tag{3.24}$$

$$\lim_{u\to\infty} r(u,v)\phi(u,v,\theta,\varphi) = 0 \quad \forall v\geq v_c, \tag{3.25}$$

Define the function $\psi_{\mathscr{I}^-,u_0}$ by

$$\lim_{u\to-\infty} r(u,v)\phi(u,v,\theta,\varphi) = \psi_{\mathscr{I}^-,u_0}(v,\theta,\varphi). \tag{3.26}$$

Then for $|q|<1$, there exists a constant $A(M,q,T^*)$ such that

$$\begin{aligned}\Bigg|\int_{\sigma=-\infty}^{\infty}\int_{S^2} |\sigma||\hat{\psi}_{\mathscr{I}^-,u_0}(\sigma,\theta,\varphi)|^2 \sin\theta d\sigma d\theta d\varphi \\ -\int_{\sigma=-\infty}^{\infty}\int_{S^2} |\sigma|\left(\coth\left(\frac{\pi}{\kappa}|\sigma|\right)|\psi_{\mathscr{H}^-}(\sigma,\theta,\varphi)|^2 + |\psi_{RN}(\sigma,\theta,\varphi)|^2\right) 2\sin\theta d\sigma d\theta d\varphi\Bigg| \\ \leq A\left(e^{-\kappa u_1} I.T.[\psi_+] + e^{2\kappa u_1} I.E[\psi_+,v_c,u_1,u_0]\right),\end{aligned} \tag{3.27}$$

for sufficiently large u_0.

Here, $\psi_{\mathscr{H}^-}$ and ψ_{RN} are the transmission and reflection of ψ_+ in pure Reissner–Nordström spacetime, as defined by Theorem 3.3.1, κ is the surface gravity of the Reissner–Nordström black hole. Finally, $I.T.$ and $I.E.$

are given by

$$
\begin{aligned}
I.E.[\psi_+, v_c, u_1, u_0] = & \int_{-\infty}^{u_1}\int_{S^2}\left[(M+u_0-v_c)(M+u_0-u_1)|\partial_u\psi_{\mathscr{H}^-}|^2+|\psi_{\mathscr{H}^-}|^2\right]\sin\theta d\theta d\varphi du \\
& +(M+u_0-u_1)\int_{-\infty}^{v_c}\int_{S^2}\left[(M^2+(v_c-u)^2)|\partial_u\psi_{\mathscr{H}^-}|^2\right]\sin\theta d\theta d\varphi du \qquad (3.28)\\
& +(M+u_0-u_1)\int_{-\infty}^{v_c}\int_{S^2}\left[(M^2+M(u_0-v_c)+(v_c-v)^2)|\partial_v\psi_{RN}|^2+|\mathring{\nabla}|^2|\psi_{RN}|^2\right]\sin\theta d\theta d\varphi dv. \\
& +\left[\int_{u=u_0-u_1}^{\infty}\int_{S^2}(M^2+(u-u_0+u_1)^2)|\partial_u\psi_+(u)|^2\sin\theta d\theta d\varphi du\right]^*
\end{aligned}
$$

$$
I.T.[\psi_+] = \int_{-\infty}^{\infty}\int_{S^2}(M^2+u^2)(1+|\mathring{\nabla}|^4)\left(|\partial_u\psi_+(u)|^2+|\psi_+(u)|^2\right)\sin\theta d\theta d\varphi du. \qquad (3.29)
$$

Here, $\mathring{\nabla}$ is the derivative on the unit sphere, and we write $|\mathring{\nabla}|^4|f|^2$ to mean $\left|\mathring{\nabla}^2 f\right|^2$. Note that $I.T.[\psi_+]$ controls similarly weighted norms of $\psi_{\mathscr{H}^-}$ and ψ_{RN}, thanks to reflection and transmission coefficients being bounded above by 1 *(see Section 3.3). Finally, it should be noted that the final term in I.E. (marked by $[\,]^*$), is only required in the extremal ($|q|=1$) case.*

In the case $|q|=1$, there exists a constant $A(M,q,T^*)$ such that

$$
\left|\int_{\mathscr{I}^-}|\sigma||\hat{\psi}_{\mathscr{I}^-,u_0}|^2 d\sigma d\theta d\varphi-\int_{-\infty}^{\infty}|\sigma||\hat{\psi}_+|^2 d\sigma\right|\leq A\left(\frac{I.T.[\psi_+]}{u_1^{3/2}}+u_0^{5/2}I.E[\psi_+,v_c,u_0-\sqrt{Mu_0},u_0]\right), \qquad (3.30)
$$

for sufficiently large u_0.

Furthermore, let us fix $\delta>0$. If we suppose that $|q|<1$, and that ψ_+ be such that all $I.E.[\psi_+,v_c,(1-\delta)u_0,u_0]$ terms decay faster than $e^{-3\kappa(1-\delta)u_0}$. Then there exists a constant $B(M,q,T^*,\psi_+,\delta)$ such that

$$
\left|\int_{\mathscr{I}^-}|\sigma||\hat{\psi}|^2 d\sigma-\int_{-\infty}^{\infty}|\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)|\hat{\psi}_{\mathscr{H}^-}|^2 d\sigma-\int_{-\infty}^{\infty}|\sigma||\hat{\psi}_{RN}|^2 d\sigma\right|\leq Be^{-\kappa(1-\delta)u_0}, \qquad (3.31)
$$

for sufficiently large u_0.

3.4.1 The Set-up and Reduction to Fixed l

We will prove Theorem 3.4.1 by first restricting to a fixed spherical harmonic, as these are orthogonal. We further restrict $\psi_+(x)$ to be a smooth compactly supported function in one variable. Let ϕ be the solution to (2.1), subject to $\psi=0$ on $r=r_b(t^*)$, with future radiation field $Y_{l,m}(\theta,\varphi)\psi_+(u-u_0)$, and $\phi=0$ on $\mathscr{H}^+$, as given by Theorem 2.4.1. Here $Y_{l,m}$ is spherical harmonic, see for example [36]. Note that this result will then immediately generalise to Schwartz functions by an easy density argument.

We will generally be considering $\psi(u,v)$, given by

$$
\psi(u,v)Y_{l,m}(\theta,\varphi) := r(u,v)\phi(u,v,\theta,\varphi) \qquad (3.32)
$$

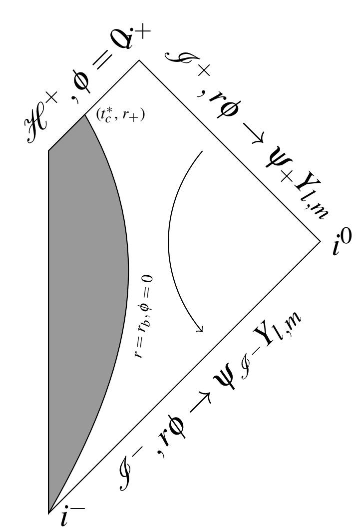


Figure 3.3 The set-up for the Hawking radiation calculation

rather than ϕ itself. Note $\psi(u,v)$ is independent of θ,φ, as spherical symmetry of our system implies that if we restrict scattering data in Theorems 3.3.1 and 2.4.1 to one spherical harmonic, the solution will also be restricted to that harmonic.

Re-writing the wave equation for fixed l,m in terms of ψ, we obtain:

$$
4\partial_u\partial_v\psi = -\frac{1}{r^2}\left(1-\frac{2M}{r^2}+\frac{q^2M^2}{r^2}\right)\left(l(l+1)+\frac{2M}{r}\left(1-\frac{q^2M}{r}\right)\right)\psi =: -4V(r)\psi. \qquad (3.33)
$$

$$
\psi(u,v_b(u)) = 0, \qquad (3.34)
$$

where v_b is as given in (1.57).

3.4.2 Summary of the Proof

This proof will be broken up into 5 sections.

1. Firstly we will consider the evolution of the solution determined by scattering data $(0, \psi_+ Y_{l,m})$ on $\mathscr{H}^+ + \mathscr{I}^+$ through the region $R_1 := \{v \in [v_c, \infty)\}$, where v_c is the value of the v coordinate at $(t_c^*, 2M)$ given by (1.53). Note that evolution in R_1 is entirely within a region of Reissner–Nordström spacetime, so is relatively easy to compute.

   We will obtain that
   $$\hat{\psi}(\sigma, v) \approx \tilde{T}_{\sigma,l,m} \hat{\psi}_+(\sigma), \tag{3.35}$$
   where $\tilde{T}_{\omega,l,m}$ is the transmission coefficient (section 3.3) from $\mathscr{I}^+$ back to $\mathscr{H}^-$ in Reissner–Nordström for the spherical harmonic $Y_{l,m}$ (again see [36]). Here, when we say "$\approx$", we mean to leading order for large u_0, and the exact nature of these error terms will be covered in more detail in the full statement of Corollary 3.4.2.

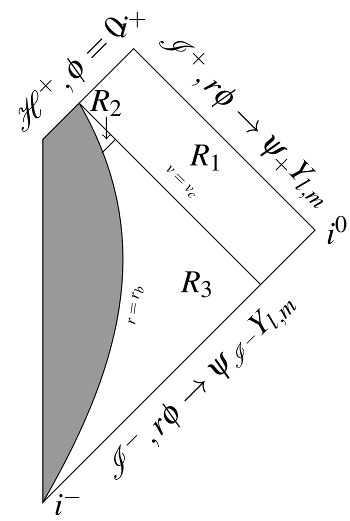


Figure 3.4 The regions we will consider in the Hawking radiation calculation

2. Secondly we will consider the reflection of the solution off the surface of the dust cloud. This will occur in the region $R_2 := \{v \leq v_c, u \leq u_1\} \cap \{r \geq r_b\}$ for the same u_0 in the definition of ψ_+.

   We will obtain that, for $v \leq v_c$,
   $$\psi(u_1, v) \approx \psi(u_b(v), v_c), \tag{3.36}$$
   where here $u = u_b(v)$ is parametrising the surface $r = r_b(t^*)$ in terms of the null coordinates. See Corollary 3.4.3 for a precise statement of this.

3. Thirdly we will consider the high frequency transmission of the solution from near the surface of the dust cloud to $\mathscr{I}^-$. This will occur in a region we will call $R_3 := \{v \leq v_c, u \geq u_1\}$.

   In a very similar manner to (3.35), we will obtain
   $$\hat{\psi}_{\mathscr{I}^-}(\sigma) \approx T_{\sigma,l,m} \hat{\psi}(u_1, \sigma), \tag{3.37}$$
   where $T_{\sigma,l,m}$ is the transmission coefficient from $\mathscr{H}^+$ to $\mathscr{I}^-$ (or equivalently from $\mathscr{H}^-$ to $\mathscr{I}^+$).

   However, as $\psi|_{\Sigma_{u_1} \cap \{v \leq v_c\}}$ is supported in a small region, we will also obtain that
   $$\hat{\psi}_{\mathscr{I}^-}(\sigma) \approx T_{\sigma,l,m} \hat{\psi}(u_1, \sigma) \approx \hat{\psi}(u_1, \sigma). \tag{3.38}$$
   See Corollary 3.4.4 for the precise statement of this.

4. Before the final calculation, we will consider the integrated error terms, $I.E.$, and how they behave, depending on our future radiation field, ψ_+. We will look to show that, provided $\hat{\phi}_+$ is only supported on $\omega \geq 0$,
   $$I.E.[\psi_+, v_c, u_1, u_0] \leq A_n (u_0 - u_1)^{-n}, \tag{3.39}$$
   for all n, provided $M \leq u_1 \leq u_0$. Here A_n depends on n, M, q and ψ_+ itself.

5. Finally, we will consider the actual calculation of $I[\psi_+, l, u_0]$ on $\mathscr{I}^-$. We will use a conserved current to show that
   $$\int_{\sigma \in \mathbb{R}} \sigma |\hat{\psi}_{\mathscr{I}^-}|^2 d\sigma = \int_{\sigma \in \mathbb{R}} \sigma |\hat{\psi}_+|^2 d\sigma. \tag{3.40}$$

Using a useful Lemma by Bachelot (Lemma II.6 in [7], Lemma 3.4.2 here), we obtain for the sub-extremal case

$$\int_{-\infty}^{\infty} |\sigma||\hat{\psi}_{\mathscr{I}^-}|^2 d\sigma \approx \int_{-\infty}^{\infty} |\sigma||\tilde{R}_{\sigma l,m}|^2|\hat{\psi}_+|^2 + |\tilde{T}_{\sigma,l,m}|^2|\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)|\hat{\psi}_+|^2 d\sigma. \tag{3.41}$$

Here, κ is the surface gravity of the Reissner–Nordström black hole, as given in (1.63). We obtain the equivalent result on the extremal case, using Lemma 3.4.3.

Thus combining (3.40) and (3.41), we will obtain the final result:

$$\lim_{u_0\to\infty} I[\psi_+, l, u_0] \approx \int_{-\infty}^{\infty} \frac{|\tilde{T}_{\sigma,l,m}|^2}{2}\left(\coth\left(\frac{\pi\omega}{\kappa}\right) - 1\right)|\sigma||\hat{\psi}_+|^2 d\sigma, \tag{3.42}$$

subject to an extra condition on the support of $\hat{\psi}_+$.

We also obtain the extremal equivalent:

$$\lim_{u_0\to\infty} I[\psi_+, l, u_0] \approx \int_{-\infty}^{\infty} \frac{|\tilde{T}_{\sigma,l,m}|^2}{2}(1-1)|\sigma||\hat{\psi}_+|^2 d\sigma = 0. \tag{3.43}$$

See Theorem 3.4.1 and Corollary 3.4.6 for the precise statement of this.

We will prove this result almost entirely in physical space rather than frequency space, which will hopefully be a more transparent proof.

3.4.3 Evolution in Pure Reissner–Nordström

In this section we will be considering the following problem: In Reissner–Nordström spacetime, if we impose radiation field $\psi_+(u)Y_{l,m}(\theta,\varphi)$ on $\mathscr{I}^+$ and that our solution vanishes on $\mathscr{H}^+$, what happens to the solution on a surface of constant v as we let $v \to -\infty$? By transporting our solution along the Killing vector, T, this is equivalent to considering a solution with radiation field $\psi_+(u-u_0)Y_{l,m}(\theta,\varphi)$ on $\mathscr{I}^+$ on a surface of fixed v, and allowing $u_0 \to \infty$. We obtain the following result:

Proposition 3.4.1 (Reissner-Nordström Transmission)**.** *Let $\psi_+ : \mathbb{R} \to \mathbb{C}$ be a smooth, compactly supported function. Let ψ be the solution of* (3.33)*, as given by Theorem 3.3.1, with radiation field on $\mathscr{I}^+$ equal to ψ_+, and which vanishes on $\mathscr{H}^+$. Let $v_c, u_1 \in \mathbb{R}$. Define*

$$\hat{\psi}_{\mathscr{H}^-}(\sigma) := \tilde{T}_{\sigma,l,m}\hat{\psi}_+(\sigma), \tag{3.44}$$

Then there exists a constant $A(M,q)$ such that

$$\int_{u=u_1}^{\infty} |\partial_u\psi(u,v_c) - \partial_u\psi_{\mathscr{H}^-}(u)|^2 du \le A I.T.[\psi_+](r(u_1,v_c) - r_+)^2. \tag{3.45}$$

$$(1+l)^4 \sup_{v\le v_c}\left(\int_{u_1}^{\infty}|\psi(u,v)|^2 du\right) \le A I.T.[\psi_+], \tag{3.46}$$

provided $r(u_1,v_c) \le 3M$.

Moreover, if $|q| < 1$, then we also have a constant $B(M,q)$ such that

$$\int_{u=u_1}^{\infty} |\psi(u,v_c) - \psi_{\mathscr{H}^-}(u)|^2 du \le A I.T.[\psi_+](r(u_1,v_c) - r_+)^2 + A|\psi(u_1,v_c)|^2, \tag{3.47}$$

again provided $r(u_1,v_c) \le 3M$.

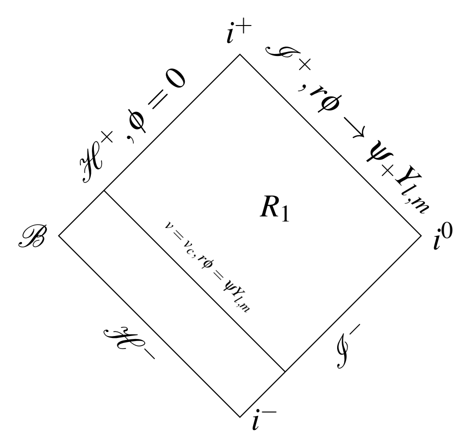


Figure 3.5 The first region we will consider in the Hawking radiation calculation

In the case $|q|=1$, we have

$$\begin{aligned}\int_{u=u_1}^{\infty}|\psi(u,v_c)-\psi_{\mathscr{H}^-}(u)|^2du\leq AI.T.[\psi_+](u_0-u_1)^2(r(u_1,v_c)-r_+)^2\\+A(M^2+(u_0-u_1)^2)|\psi(u_1,v_c)-\psi_{\mathscr{H}^-}(u_1)|^2\\+4I.E.[\psi_+,v_c,u_1,u_0].\end{aligned}\tag{3.48}$$

Here $I.T.$ and $I.E.$ are as defined in Theorem 3.4.1.

Remark 3.4.1. *We can also define the past radiation field in pure RN by ψ_{RN}, which is given by*

$$\hat{\psi}_{RN}:=\tilde{R}_{\sigma,l,m}\hat{\psi}_+,\tag{3.49}$$

where $\tilde{R}_{\sigma,l,m}$ are the reflection coefficients from $\mathscr{I}^+$ to $\mathscr{I}^-$ in Reissner–Nordström for the spherical harmonic $Y_{l,m}$ (again see [36]).

Proof. We know from many previous works on Reissner-Nordström (see [35] for example) that

$$\lim_{v\to-\infty}\psi(u,v)=\psi_{\mathscr{H}^-}(u),\tag{3.50}$$

for $\psi_{\mathscr{H}^-}$ as in the statement of the Proposition.

The proof of (3.45) is fairly straightforward:

$$\begin{aligned}\int_{u=u_1}^{\infty}|\partial_u\psi(u,v_c)-\partial_u\psi_{\mathscr{H}^-}(u)|^2du&=\int_{u=u_1}^{\infty}\left|\int_{-\infty}^{v_c}\partial_v\partial_u\psi dv\right|^2du=\int_{u=u_1}^{\infty}\left|\int_{-\infty}^{v_c}V\psi dv\right|^2du\\&\leq\left(\int_{-\infty}^{v_c}\left(\int_{u_1}^{\infty}V^2|\psi|^2du\right)^{1/2}dv\right)^2\\&\leq\sup_{v\leq v_c}\left(\int_{u_1}^{\infty}|\psi(u,v)|^2du\right)\left(\int_{-\infty}^{v_c}V(u_1,v)dv\right)^2\\&\leq A(l+1)^4(r(u_1,v_c)-r_+)^2\sup_{v\leq v_c}\left(\int_{u_1}^{\infty}|\psi(u,v)|^2du\right).\end{aligned}\tag{3.51}$$

Here, we have used Minkowski's integral inequality to reach the second line.

In the case of $|q|<1$, we first show that to prove (3.47), it is sufficient to bound $\int_{u=u_1}^{\infty}(u-u_1)^2|\partial_u\psi(u,v_c)-\partial_u\psi_{\mathscr{H}^-}(u)|^2du$.

Let χ be a smooth cut off function such that

$$\chi(x)\begin{cases}=0 & x\geq 1\\ \in[0,1] & x\in[0,1]\\ =1 & x\leq 0\end{cases}.\tag{3.52}$$

Then for any function ϕ,

$$\begin{aligned}
\int_{u=u_1}^{\infty} |\phi(u,v)|^2 du &\leq 2\int_{u=u_1}^{\infty} \left|\phi(u,v) - \phi(u_1,v_c)\chi\left(\frac{u-u_1}{M}\right)\right|^2 + \left|\phi(u_1,v_c)\chi\left(\frac{u-u_1}{M}\right)\right|^2 du \qquad (3.53)\\
&\leq 8\int_{u=u_1}^{\infty} (u-u_1)^2 \left|\partial_u\phi(u,v) - \phi(u_1,v_c)M^{-1}\chi'\left(\frac{u-u_1}{M}\right)\right|^2 du + 2M|\phi(u_1,v_c)|^2 \int_{x=0}^{\infty} |\chi(x)|^2 dx\\
&\leq 16\int_{u=u_1}^{\infty} (u-u_1)^2 |\partial_u\phi(u,v)|^2 + (u-u_1)^2 \left|\phi(u_1,v_c)M^{-1}\chi'\left(\frac{u-u_1}{M}\right)\right|^2 du\\
&\quad + 2M|\phi(u_1,v_c)|^2 \int_{x=0}^{\infty} |\chi(x)|^2 dx\\
&\leq 16\int_{u=u_1}^{\infty} (u-u_1)^2 |\partial_u\phi(u,v)|^2 du + 2M|\phi(u_1,v_c)|^2 \int_{x=0}^{\infty} |\chi(x)|^2 + 8x^2 |\chi'(x)|^2 dx\\
&\leq 16\int_{u=u_1}^{\infty} (u-u_1)^2 |\partial_u\phi(u,v)|^2 du + A(M)|\phi(u_1,v_c)|^2,
\end{aligned}$$

Here, we have used Hardy's inequality.

Now we prove (3.47), in a similar way to (3.45):

$$\begin{aligned}
\int_{u=u_1}^{\infty} (u-u_1)^2 |\partial_u\psi(u,v_c) - \partial_u\psi_{\mathscr{H}^-}(u)|^2 du &= \int_{u=u_1}^{\infty} (u-u_1)^2 \left|\int_{-\infty}^{v_c} \partial_v\partial_u\psi dv\right|^2 du = \int_{u=u_1}^{\infty} (u-u_1)^2 \left|\int_{-\infty}^{v_c} V\psi dv\right|^2 du\\
&\leq \left(\int_{-\infty}^{v_c}\left(\int_{u_1}^{\infty}(u-u_1)^2 V^2|\psi|^2 du\right)^{1/2} dv\right)^2 \qquad (3.54)\\
&\leq \sup_{v\leq v_c}\left(\int_{u_1}^{\infty}|\psi(u,v)|^2 du\right)\left(\int_{-\infty}^{v_c}\sup_{u\geq u_1}\{(u-u_1)V(u,v)\} dv\right)^2\\
&\leq B(l+1)^4 (r(u_1,v_c)-r_+)^2 \sup_{v\leq v_c}\left(\int_{u_1}^{\infty}|\psi(u,v)|^2 du\right).
\end{aligned}$$

Here we have used that there exists a constant $C(M,q)$ such that

$$C^{-1}(l+1)^2 e^{\kappa(v-u)} \leq V(u,v) \leq C(l+1)^2 e^{\kappa(v-u)}, \qquad (3.55)$$

for $r \leq 3M$.

In order to prove (3.46), we use similar logic to (3.53) to show

$$\int_{u=-\infty}^{\infty} |\phi(u,v)|^2 du \leq 16\int_{u=-\infty}^{\infty} (u-u_0)^2 |\partial_u\phi(u,v)|^2 du + A(M)|\phi(u_0,v_c)|^2, \qquad (3.56)$$

And we then consider

$$\phi = \begin{cases} \psi & u \geq u_1 \\ \frac{u_0-u_1}{u_0-u}\psi(u_1,v_c) & u < u_1 \end{cases}, \qquad (3.57)$$

to obtain

$$\begin{aligned}
\int_{u_1}^{\infty} |\psi(u,v)|^2 du \leq \int_{u=-\infty}^{\infty} |\phi(u,v)|^2 du &\leq 16\int_{u=-\infty}^{\infty} (u-u_0)^2 |\partial_u\phi(u,v)|^2 du + A(M)|\phi(u_0,v_c)|^2 \qquad (3.58)\\
&\leq 16\int_{u=u_1}^{\infty} (u-u_0)^2 |\partial_u\psi(u,v)|^2 du + A(M)|\psi(u_0,v_c)|^2 + (u_0-u_1)|\psi(u_1,v_c)|^2.
\end{aligned}$$

In order to bound $\psi(u_0, v_c)$, we look at

$$|\psi(u_0,v_c)|^2 \leq \left|\int_{u_0}^{\infty} \partial_u \psi(u,v_c) du\right|^2 \\ \leq \left(\int_{u_0}^{\infty} \frac{1}{M^2+(u_0-u)^2} du\right)\left(\int_{u_0}^{\infty} \left(M^2+(u_0-u)^2\right)|\partial_u\psi(u,v_c)|^2 du\right) \\ \leq \frac{\pi}{2M}\int_{u_0}^{\infty}\left(M^2+(u_0-u)^2\right)|\partial_u\psi(u,v_c)|^2 du. \tag{3.59}$$

We then consider the conserved T-energy. In u,v coordinates, this is given by:

$$\text{T-energy}(\phi,\Sigma_v) = \int_{-\infty}^{\infty} |\partial_u\psi(u,v)|^2 + V(r)|\psi(u,v)|^2 du. \tag{3.60}$$

We apply (3.33) to a weighted version of the T-energy in the region $u \geq u_0$:

$$\int_{u_0}^{\infty}\left(M^2+(u-u_0)^2\right)\left(|\partial_u\psi(u,v)|^2+V|\psi(u,v)|^2\right)du = \int_{u_0}^{\infty}\left(M^2+(u-u_0)^2\right)|\partial_u\psi_+|^2 du \\ -\int_{u\geq u_0, v'\geq v} 2(u-u_0)\left(|\partial_u\psi(u,v')|^2+V|\psi(u,v')|^2\right)dv'du \\ \leq \int_{u_0}^{\infty}\left(M^2+(u-u_0)^2\right)|\partial_u\psi_+|^2 du. \tag{3.61}$$

We bound the $u \leq u_0$ in a similar way:

$$\int_{u_1}^{u_0}\left(M^2+(u-u_0)^2\right)\left(|\partial_u\psi(u,v)|^2+V|\psi(u,v)|^2\right)du \\ = \int_{u_1}^{u_0}\left(M^2+(u-u_0)^2\right)|\partial_u\psi_{\mathcal{H}^-}|^2 du \\ + \int_{-\infty}^{v_c}\left(M^2+(u_0-u_1)^2\right)\left(|\partial_u\psi(u_1,v)|^2+V|\psi(u_1,v)|^2\right)dv \\ - \int_{u\in[u_1,u_0], v'\geq v} 2(u_0-u)\left(|\partial_u\psi(u,v')|^2+V|\psi(u,v')|^2\right)dv'du \\ \leq \int_{-\infty}^{u_0}\left(M^2+(u-u_0)^2\right)|\partial_u\psi_{\mathcal{H}^-}|^2 du + \int_{-\infty}^{v_c}\left(M^2+(u_0-u_1)^2\right)|\partial_v\psi_{RN}|^2. \tag{3.62}$$

For the extremal case, we simply use Poincaré's inequality to bound

$$\int_{u=u_1}^{2u_0-u_1}|\psi-\psi_{\mathcal{H}^-}|^2 du \leq A(u_0-u_1)^2\int_{u=u_1}^{2u_0-u_1}|\partial_u\psi-\partial_u\psi_{\mathcal{H}^-}|^2 du + (M^2+(u_0-u_1)^2)|\psi(u_1,v_c)-\psi_{\mathcal{H}^-}(u_1)|^2 \\ \leq AI.T.[\psi_+](u_0-u_1)^2(r(u_1)-r_+)^2 + (M^2+(u_0-u_1)^2)|\psi(u_1,v_c)-\psi_{\mathcal{H}^-}(u_1)|^2. \tag{3.63}$$

We are then left to bound

$$\int_{u=2u_0-u_1}^{\infty}|\psi-\psi_{\mathcal{H}^-}|^2 du = \int_{x=0}^{\infty}\frac{1}{x^2}|\psi-\psi_{\mathcal{H}^-}|^2 dx \leq 4\int_{x=0}^{\infty}|\partial_x\psi-\partial_x\psi_{\mathcal{H}^-}|^2 dx \\ \leq 4\int_{u=2u_0-u_1}^{\infty}(u-2u_0+u_1)^2|\partial_u\psi-\partial_u\psi_{\mathcal{H}^-}|^2 du. \tag{3.64}$$

This can then be bounded in exactly the same way as (3.61) to obtain

$$\int_{u=2u_0-u_1}^{\infty}(u-2u_0+u_1)^2|\partial_u\psi-\partial_u\psi_{\mathcal{H}^-}|^2 du \leq 2\int_{u=2u_0-u_1}^{\infty}(u-2u_0+u_1)^2\left(|\partial_u\psi_{\mathcal{H}^-}|^2+|\partial_u\psi_{\mathcal{H}^-}|^2\right)du \\ \leq 4\int_{u=2u_0-u_1}^{\infty}(u-2u_0+u_1)^2|\partial_u\psi_+|^2 du, \tag{3.65}$$

as required. □

We will also need to calculate the r-weighted energy of our solution on Σ_{v_c}.

Proposition 3.4.2 (Reissner–Nordström Weighted Bounds)**.** *Let $\psi_+ : \mathbb{R} \to \mathbb{C}$ be a smooth, compactly supported function. Let ψ be the solution of* (3.33)*, as given by Theorem 3.3.1, with radiation field on $\mathscr{I}^+$ equal to ψ_+, and which vanishes on $\mathscr{H}^+$. Let χ be a smooth function such that*

$$\chi(x) \begin{cases} = 1 & x \geq 1 \\ \in [0,1] & x \in [0,1] \\ = 0 & x \leq 0 \end{cases}, \tag{3.66}$$

Let $r_0 > r_+$ and v_c be fixed. Define $\psi_{\mathscr{H}^-}$ and ψ_{RN} as in Theorem 3.3.1.

Then there exists constants $A(M,q,r_0,\chi)$ and $B(M,q,r_0,\chi)$ such that

$$\begin{aligned} \int_{\Sigma_{v_c}} \chi\left(\frac{r-r_0}{M}\right) r^2 |\partial_u \psi|^2 du \leq A\Bigg(&\int_{u=-\infty}^{v_c - 2r_0^*} \left(1 + (v_c - 2r_0^* - u)^2\right) |\partial_u \psi_{\mathscr{H}^-}|^2 du \\ &+ \int_{v=-\infty}^{v_c} \left(1 + (v_c - v)^2\right) |\partial_v \psi_{RN}|^2 + l(l+1)|\psi_{RN}|^2 dv \Bigg), \end{aligned} \tag{3.67}$$

for $l \neq 0$, and

$$\begin{aligned} \int_{\Sigma_{v_c}} \chi\left(\frac{r-r_0}{M}\right) r^3 |\partial_u \psi|^2 du \leq B\Bigg(&\int_{u=-\infty}^{v_c - 2r_0^*} \left(1 + (v_c - 2r_0^* - u)^3\right) |\partial_u \psi_{\mathscr{H}^-}|^2 du \\ &+ \int_{v=-\infty}^{v_c} \left(1 + (v_c - v)^3\right) |\partial_v \psi_{RN}|^3 + 2M|\psi_{RN}|^2 dv \Bigg), \end{aligned} \tag{3.68}$$

for $l = 0$. Here, $r_0^* = \frac{1}{2}(v-u)$ when $r = r_0$.

Remark 3.4.2. *In the extremal ($q = 1$) case,* (3.67) *follows easily from Proposition 3.3.1, but the proof we will offer below will not distinguish between the extremal and sub-extremal cases.*

Proof. We first write the conserved $T-$energy flux through in terms of ψ, though the surface

$$\bar{\Sigma}_{v_c,0} := (\Sigma_{u=v_c-2r_0^*} \cap \{r \leq r_0\}) \cup (\Sigma_{v_c} \cap \{r > r_0\}) \tag{3.69}$$

An explicit calculation gives:

$$\begin{aligned} T-\text{energy}(\phi, \bar{\Sigma}_{v_c,0}) &= \int_{\Sigma_{u=v_c-2r_0^*} \cap \{r \leq r_0\}} \left[2|\partial_v \phi|^2 + \frac{1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}}{2r^2} l(l+1)|\phi|^2 \right] \sin\theta d\theta d\varphi dv \\ &\quad + \int_{\Sigma_{v_c} \cap \{r > r_0\}} \left[2|\partial_u \phi|^2 + \frac{1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}}{2r^2} l(l+1)|\phi|^2 \right] \sin\theta d\theta d\varphi dv \\ &= \int_{\Sigma_{u=v_c-2r_0^*} \cap \{r \leq r_0\}} 2\left[|\partial_v \psi|^2 + V(r)|\psi|^2\right] dv + \int_{\Sigma_{v_c} \cap \{r > r_0\}} 2\left[|\partial_u \psi|^2 + V(r)\psi^2\right] du \\ &= \int_{\bar{\Sigma}_{v_c,0}} 2\left[|\tilde{\partial}_{r^*} \psi|^2 + V(r)|\psi^2|\right] dr^*, \end{aligned} \tag{3.70}$$

where here we define $\tilde{\partial}_{r^*}$ to be the r^* derivative along $\bar{\Sigma}_{v,0}$, i.e.

$$\tilde{\partial}_{r^*} = \begin{cases} \partial_v & r \leq r_0 \\ \partial_u & r > r_0 \end{cases}. \tag{3.71}$$

We define $\tilde{\partial}_r := \left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right) \tilde{\partial}_{r^*}$.

We note here that even if $l=0$, the $T-$energy bounds $(\chi\left(\frac{r-r_0}{M}\right)\psi)^2/r^2$ for any $r_0>2M$:

$$\begin{aligned}\int_{\bar{\Sigma}_{v,0}}\frac{\chi\left(\frac{r-r_0}{M}\right)^2|\psi|^2}{r^2}dr^* &\leq \left(1-\frac{2M}{r_0}+\frac{q^2M^2}{r_0^2}\right)^{-1}\int_{\bar{\Sigma}_{v,0}}\frac{\chi\left(\frac{r-r_0}{M}\right)^2|\psi|^2}{r^2}dr\\ &\leq A(r_0)\int_{\bar{\Sigma}_{v,0}}|\tilde{\partial}_r(\chi\psi)|^2dr = A(r_0)\int_{\bar{\Sigma}_{v,0}}|\chi\tilde{\partial}_r\psi|^2-\chi\chi''\frac{|\psi|^2}{M^2}dr \qquad (3.72)\\ &\leq B(M,r_0,\chi)\int_{\bar{\Sigma}_{v,0}}|\tilde{\partial}_{r^*}\psi|^2+V(r)|\psi|^2dr^*.\end{aligned}$$

Here we have used Hardy's inequality.

We look at the integral of $\chi\left(\frac{r-r_0}{M}\right)r^p(\partial_u\psi)^2$ on Σ_{v_c}:

$$\begin{aligned}\int_{\Sigma_{v_c}}\chi\left(\frac{r-r_0}{2M}\right)r^p|\partial_u\psi|^2du &= \int_{v\leq v_c}\left[\frac{1}{2}\left(1-\frac{2M}{r^2}+\frac{q^2M^2}{r^2}\right)\left(pr^{p-1}\chi+\frac{r^p}{M}\chi'\right)|\partial_u\psi|^2+2\chi r^pV(r)\partial_u(|\psi|^2)\right]dvdu\\ &= \int_{v\leq v_c}\left[\left(1-\frac{2M}{r^2}+\frac{q^2M^2}{r^2}\right)\left(\frac{1}{2}\left(pr^{p-1}\chi+\frac{r^p}{M}\chi'\right)|\partial_u\psi|^2+\partial_r(\chi r^pV(r))|\psi|^2\right)\right]dvdu\\ &\quad+\int_{\mathscr{I}^-\cap\{v\leq v_c\}}2r^{p-2}l(l+1)|\psi|^2dv \qquad (3.73)\\ &\leq \int_{v\leq v_c}\frac{pr^{p-1}}{2}\chi|\partial_u\psi|^2dvdu+A\int_{v\leq v_c,r\geq r_0}|\partial_{r^*}\psi|^2+V(r)|\psi|^2dvdu\\ &\quad+\int_{\mathscr{I}^-\cap\{v\leq v_c\}}2r^{p-2}l(l+1)|\psi|^2dv.\end{aligned}$$

For $p=1$, we obtain

$$\int_{\Sigma_{v_c}}\chi\left(\frac{r-r_0}{M}\right)r|\partial_u\psi|^2du\leq A\int_{v\leq v_c,r\geq r_0}|\partial_{r^*}\psi|^2+V(r)|\psi|^2dvdu. \qquad (3.74)$$

Then for $p=2$, we obtain

$$\begin{aligned}\int_{\Sigma_{v_c}}\chi\left(\frac{r-r_0}{M}\right)r^2|\partial_u\psi|^2du &\leq A\int_{v=-\infty}^{v_c}\left(A\int_{v'\leq v,r\geq r_0}|\partial_{r^*}\psi|^2+V(r)|\psi|^2dv'du\right)dv \qquad (3.75)\\ &\quad+\int_{v\leq v_c,r\geq r_0}|\partial_{r^*}\psi|^2+V(r)|\psi|^2dvdu+\int_{\mathscr{I}^-\cap\{v\leq v_c\}}2l(l+1)|\psi|^2dv.\end{aligned}$$

By T - energy conservation, we have that

$$\begin{aligned}\int_{v=-\infty}^{v_c}\left(\int_{v'\leq v_c,r\geq r_0}|\partial_{r^*}\psi|^2+V(r)|\psi|^2dv'du\right)dv &= \int_{v=-\infty}^{v_c}\left(\int_{v'=-\infty}^{v}T-\text{energy}(\phi,\bar{\Sigma}_{v',0})dv'\right)dv \qquad (3.76)\\ &= \int_{\mathscr{H}^-,u\leq v_c-2r_0^*}\int_{u'\leq u}\int_{u''\leq u'}|\partial_u\psi|^2du''du'du+\int_{\mathscr{I}^-,v\leq v_c}\int_{v'\leq v}\int_{v''\leq v'}|\partial_v\psi|^2dv''dv'dv\\ &= \int_{\mathscr{H}^-,u\leq v_c-2r_0^*}(v_c-2r_0^*-u)^2|\partial_u\psi|^2du+\int_{\mathscr{I}^-,v\leq v_c}(v_c-v)^2|\partial_v\psi|^2dv,\end{aligned}$$

where we have integrated by parts to obtain the last line. (Note ψ on $\mathscr{I}^-$ or $\mathscr{H}^-$ is Schwartz, so we have arbitrarily large polynomial decay.)

Combining (3.75) and (3.76), we obtain:

$$\begin{aligned}\int_{\Sigma_{v_c}}\chi\left(\frac{r-r_0}{M}\right)r^2|\partial_u\psi|^2du\leq A\Bigg(&\int_{\mathscr{H}^-,u\leq v_c-2r_0^*}\left(1+(v_c-2r_0^*-u)^2\right)|\partial_u\psi|^2du \qquad (3.77)\\ &+\int_{\mathscr{I}^-,v\leq v_c}\left(1+(v_c-v)^2\right)|\partial_v\psi|^2+l(l+1)|\psi|^2dv\Bigg).\end{aligned}$$

for $l \neq 0$, and

$$\int_{\Sigma_{v_c}} \chi\left(\frac{r-r_0}{M}\right) r^3 |\partial_u \psi|^2 du \leq A\Bigg(\int_{\mathscr{H}^-, u\leq v_c - 2r_0^*} \left(1+(v_c-2r_0^*-u)^3\right)|\partial_u \psi|^2 du \\ + \int_{\mathscr{I}^-, v\leq v_c} \left(1+(v_c-v)^3\right)|\partial_v \psi|^3 + 2M|\psi|^2 dv\Bigg), \tag{3.78}$$

for $l = 0$, as required. □

Corollary 3.4.1 (Pointwise Bounds)**.** *Let $\psi_+ : \mathbb{R} \to \mathbb{C}$ be a smooth, compactly supported function. Let ψ be the solution of* (3.33)*, as given by Theorem 3.3.1, with radiation field on $\mathscr{I}^+$ equal to ψ_+, and which vanishes on $\mathscr{H}^+$. Let $r_0 > r_+$ and v_c be fixed. Then there exists a constant $A(M,q,r_0)$ such that*

$$|\psi(u_1, v_c)|^2 \leq AI.E[\psi_+, v_c, u_1, u_0], \tag{3.79}$$

for any $u_1 > v_c - r_0^*$. Here, $r_0^* = \frac{1}{2}(v-u)$ on $r = r_0$. Here we define $\psi_{\mathscr{H}^-}$ and ψ_{RN} as in Theorem 3.3.1.

Proof. This is a fairly straight forward consequence of Proposition 3.4.2.

$$\begin{aligned} |\psi(u_1,v_c)|^2 &\leq 2\left|\int_{-\infty}^{v_c} \partial_v \psi_{RN}(v) dv\right|^2 + 2\left|\int_{-\infty}^{v_c - r_0^*} \partial_u \psi(u,v_c) du\right|^2 + 2\left|\int_{v_c - r_0^*}^{u_1} \partial_u \psi(u,v_c) du\right|^2 \\ &\leq 2\left(\int_{-\infty}^{v_c} \frac{1}{M^2+(v_c-v)^2} dv\right)\left(\int_{-\infty}^{v_c} (M^2+(v_c-v)^2)|\partial_v \psi_{RN}(v)|^2 dv\right) \\ &\quad + 2\left(\int_{-\infty}^{v_c - r_0^* - M} r^{-2} dv\right)\left(\int_{-\infty}^{v_c} r^2 |\partial_u \psi(u,v_c)|^2 dv\right) + 2\left(\int_{v_c - r_0^* - M}^{u_1} dv\right)\left(\int_{v_c - r_0^* - M}^{u_1} |\partial_u \psi(u,v_c)|^2 dv\right) \\ &\leq A\Bigg(\int_{u=-\infty}^{v_c - 2r_0^*} \left(M^2+(v_c-2r_0^*-u)^2\right)|\partial_u \psi_{\mathscr{H}^-}|^2 du + \int_{u=-\infty}^{u_1} (u_1 - v_c + r_0^*)|\partial_u \psi_{\mathscr{H}^-}|^2 du \\ &\quad + \int_{v=-\infty}^{v_c} \left(M^2 + M(u_1 - v_c + r_0^*) + (v_c - v)^2\right)|\partial_v \psi_{RN}|^2 + l(l+1)|\psi_{RN}|^2 dv\Bigg), \end{aligned} \tag{3.80}$$

as required. □

Proposition 3.4.3 (Extremal Weighted Energy Bounds)**.** *Let $\psi_+ : \mathbb{R} \to \mathbb{C}$ be a smooth, compactly supported function. Let ψ be the solution of* (3.33)*, as given by Theorem 3.3.1, on an Extremal Reisnner–Nordström background, with radiation field on $\mathscr{I}^+$ equal to ψ_+, and which vanishes on $\mathscr{H}^+$. Let u_1, v_c be such that $r(u_1, v_c) < \frac{3}{2}M$. Then there exists a constant $C = C(M) > 0$ such that*

$$\int_{\Sigma_{u_1} \cap \{v \leq v_c\}} \left(1 - \frac{M}{r}\right)^{-2} |\partial_v \psi|^2 \sin\theta d\theta d\varphi dv \leq C(u_1 - v_c)^2 I.E.[\psi_+, v_c, u_1, u_0]. \tag{3.81}$$

Proof. We will be considering the solution to the wave equation (2.1), $\tilde{\phi}$, with radiation field $\tilde{\psi}$, given by

$$\tilde{\psi}|_{\mathscr{H}^-} = \psi_{\mathscr{H}^-} \tag{3.82}$$

$$\tilde{\psi}|_{\mathscr{I}^-} = \begin{cases} \chi\left(\frac{v-v_c}{M}\right)\psi_{RN}(v_c) & v > v_c \\ \psi_{RN} & v \leq v_c \end{cases}, \tag{3.83}$$

where χ is as defined in Proposition 3.4.2.

By a standard domain of dependence argument, we can see that

$$\int_{\Sigma_{u_1} \cap \{v \leq v_c\}} \left(1 - \frac{M}{r}\right)^{-2} |\partial_v \psi|^2 dv = \int_{\Sigma_{u_1} \cap \{v \leq v_c\}} \left(1 - \frac{M}{r}\right)^{-2} |\partial_v \tilde{\psi}|^2 dv. \tag{3.84}$$

Here we will be making use of Proposition 3.3.1. By choosing $r_0 = \frac{3}{2}M$, we can then bound

$$\begin{aligned}\int_{\Sigma_{u_1}\cap\{v\le v_c\}} \left(1-\frac{M}{r}\right)^{-2} |\partial_v\tilde{\psi}|^2 \sin\theta d\theta d\varphi dv \le C\int_{\mathscr{H}^-\cap\{u\ge u_1\}} \left(M^2+(u-u_1)^2\right)|\partial_u\tilde{\psi}|^2+\left|\mathring{\nabla}\tilde{\psi}\right|^2 \sin\theta d\theta d\varphi du\\ +C\int_{\mathscr{I}^-\cap\{v\le u_1\}} (M^2+(v-u_1)^2)|\partial_v\tilde{\psi}|^2+\left|\mathring{\nabla}\tilde{\psi}\right|^2 \sin\theta d\theta d\varphi dv\\ \le C(u_1-v_c)^2 I.E.[\psi_+,v_c,u_1,u_0].\end{aligned} \tag{3.85}$$

□

Corollary 3.4.2 (Hawking Radiation Error from Reissner–Nordström Transmission)**.** *Let f be a smooth compactly, supported function with $f(0)=1$. Let $\psi_+:\mathbb{R}\to\mathbb{C}$ be a Schwartz function. Let ψ be the solution of* (3.33)*, as given by Theorem 3.3.1, with radiation field on $\mathscr{I}^+$ equal to ψ_+, and which vanishes on $\mathscr{H}^+$. Let $v_c, u_1\in\mathbb{R}$, both tending to $-\infty$, with $v_c\le u_1$. Let u_0 be fixed. Define*

$$\psi_0(u) := \begin{cases}\psi(u,v_c)-f(u-u_1)\psi(u_1,v_c) & u\ge u_1\\ 0 & u<u_1\end{cases}. \tag{3.86}$$

In the extremal case, we also restrict $\hat{\psi}_+$ to be supported on positive frequencies, and $u_0-u_1\le u_1-v_c$. Then there exists a constant $A(M,q,f)>0$ such that

$$\left|\int_{\sigma\in\mathbb{R}}(\kappa+|\sigma|)\left(|\hat{\psi}_{\mathscr{H}^-}|^2-|\hat{\psi}_0|^2\right)d\sigma\right| \le \begin{cases} A\left(e^{\kappa(v_c-u_1)}I.T.[\psi_+]+I.E.[\psi_+,v_c,u_1,u_0]^{1/2}I.T.[\psi_+]^{1/2}\right) & |q|<1\\ A\left(\frac{\left(\ln\left(\frac{u_1-v_c}{M}\right)+u_0-u_1\right)I.T.[\psi_+]}{(u_1-v_c)^2}\right.\\ \qquad\left.+(u_1-v_c)^{1/2}I.E.[\psi_+,v_c,u_1,u_0]^{1/2}I.T.[\psi_+]^{1/2}\right) & |q|=1\end{cases}, \tag{3.87}$$

where κ is given by (1.63)*. Here $I.E.[\psi_+,v_c,u_1,u_0]$ are "Integrated Errors" due to the tail of ψ_+, $\psi_{\mathscr{H}^-}$ and ψ_{RN}, and $I.T.[\psi_+,v_c,u_1,u_0]$ are "Integrated Terms", both given in the statement of Theorem 3.4.1. where $\psi_{RN},\psi_{\mathscr{H}^-}$ are as defined in Proposition 3.4.1 and Remark 3.4.1.*

Remark 3.4.3. *We have chosen the above form of ψ_0 to ensure that it is a weakly differentiable function. If ψ_0 were less well behaved, then we do not know for certain that the integral in* (3.87) *would converge.*

Remark 3.4.4. *Wherever v_c,u_1,u_0 occur in Corollary 3.4.2, they occur as a difference. Thus when we propagate our solution along T, these differences remain the same.*

Proof. We will consider $|q|<1$ first:

$$\begin{aligned}\int_{\sigma\in\mathbb{R}}(\kappa+|\sigma|)\left||\hat{\psi}_{\mathscr{H}^-}|^2-|\hat{\psi}_0|^2\right|d\sigma &\le A\left(\int_{\sigma\in\mathbb{R}}|\hat{\psi}_{\mathscr{H}^-}-\hat{\psi}_0|^2d\sigma\right)^{1/2}\left(\int_{\sigma\in\mathbb{R}}(\kappa^2+\sigma^2)\left(|\hat{\psi}_{\mathscr{H}^-}|^2+|\hat{\psi}_0|^2\right)d\sigma\right)^{1/2}\\ &\le A\left(\int_{u_1}^{\infty}|\psi_{\mathscr{H}^-}-\psi(u,v_c)|^2du+\int_{-\infty}^{u_1}|\psi_{\mathscr{H}^-}|^2du+|\psi(u_1,v_c)|^2\right)^{1/2}\\ &\quad\left(\int_{u_1}^{\infty}\kappa^2\left(|\psi_{\mathscr{H}^-}|^2+|\psi_0|^2\right)+|\partial_u\psi_{\mathscr{H}^-}|^2+|\partial_u\psi_0|^2du+\int_{-\infty}^{u_1}\kappa^2|\psi_{\mathscr{H}^-}|^2+|\partial_u\psi_{\mathscr{H}^-}|^2du\right)^{1/2}\\ &\le A\left(I.T.[\psi_+](r(u_1,v_c)-r_+)^2+I.E.[\psi_+,u_0]\right)^{1/2}\\ &\quad\left(\int_{u_1}^{\infty}\kappa^2|\psi_+|^2+|\partial_u\psi_+|^2du+\int_{-\infty}^{u_1}\kappa^2|\psi_{\mathscr{H}^-}|^2+|\partial_u\psi_{\mathscr{H}^-}|^2du+|\psi(u_1,v_c)|^2\right)^{1/2}\\ &\le A\left((r(u_1,v_c)-r_+)^2I.T.[\psi_+]+I.E.[\psi_+,v_c,u_1,u_0]\right)^{1/2}(I.T.[\psi_+]))^{1/2},\end{aligned} \tag{3.88}$$

as required. We have used Proposition 3.4.1 to bound $\int_{u_1}^{\infty}|\psi_{\mathscr{H}^-}-\psi_0|^2du$, and Corollary 3.4.1 to bound $|\psi_1(u_1,v)|^2$.

For the $|q|=1$ case, we have $\kappa=0$. We then proceed slightly differently to obtain our result, by first noting:

$$-i\sigma\partial_v\hat{\psi}=\widehat{V\psi}. \tag{3.89}$$

Here, $\hat{\psi}$ is the Fourier transform of ψ with respect to u. While this transform may not exist in an L^2 sense, as $V\psi$ is an L^2 function on Σ_u, this implies that $\partial_v\hat{\psi}$ exists in a distributional sense.

We will write

$$\psi_0(u,v):=\begin{cases}\psi(u,v)-f(u-u_1)\psi(u_1,v) & u\geq u_1\\ 0 & u<u_1\end{cases}. \tag{3.90}$$

Substituting (3.90) into (3.89), we obtain

$$-i\sigma\widehat{\partial_v\psi_0}=-\widehat{V\psi_0}-\widehat{f'\partial_v\psi}-\psi(u_1,v)\widehat{Vf\mathbb{I}_{u\geq u_1}}, \tag{3.91}$$

where

$$\mathbb{I}_{u\geq u_1}=\begin{cases}1 & u\geq u_1\\ 0 & u\leq u_1\end{cases}. \tag{3.92}$$

We therefore obtain:

$$\begin{aligned}\left|\int_{\sigma\in\mathbb{R}}|\sigma|\left(|\hat{\psi}_{\mathscr{H}^-}|^2-|\hat{\psi}_0|^2\right)d\sigma\right| &\leq 2\left|\int_{\sigma\in\mathbb{R}}\frac{\sigma}{|\sigma|}\int_{v=-\infty}^{v_c}\mathbb{R}\left(i\bar{\hat{\psi}}_0\left(\widehat{V\psi_0}+\widehat{f'\partial_v\psi}+\psi(u_1,v)\widehat{Vf\mathbb{I}_{u\geq u_1}}\right)\right)dvd\sigma\right|\\ &\quad+\left|\int_{\sigma\in\mathbb{R}}|\sigma|\left(|\hat{\psi}_{\mathscr{H}^-}|^2-|\hat{\psi}_0(u,-\infty)|^2\right)d\sigma\right|\\ &\leq 2\left|\int_{\sigma\in\mathbb{R}}\frac{\sigma}{|\sigma|}\int_{v=-\infty}^{v_c}\mathbb{R}\left(i\bar{\hat{\psi}}_0\widehat{V\psi_0}\right)dvd\sigma\right|\\ &\quad+2\left(\int_u\int_{v\leq v_c}\left(1-\frac{M}{r}\right)^2|\psi_0|^2dudv\right)^{1/2}\left(\int_u\int_{v\leq v_c}f'^2\frac{|\partial_v\psi|^2}{\left(1-\frac{M}{r}\right)^2}dudv\right)^{1/2}\\ &\quad+2\sup_{v\leq v_c}|\psi(u_1,v)|\left(\int_u\int_{v\leq v_c}\left(1-\frac{M}{r}\right)^2|\psi_0|^2dudv\right)^{1/2}\\ &\quad\left(\int_{u\geq u_1}\int_{v\leq v_c}\frac{V^2}{\left(1-\frac{M}{r}\right)^2}f^2dudv\right)^{1/2}+AI.E.[\psi_+,v_c,u_1,u_0].\end{aligned} \tag{3.93}$$

We note that

$$\int_u\int_{v\leq v_c}\left(1-\frac{M}{r}\right)^2|\psi_0|^2dudv\leq\frac{A(r(u_1,v_c)-M)}{(l+1)^4}I.T.[\psi_+], \tag{3.94}$$

using Proposition 3.4.1, and given f is a compactly supported function, we can bound

$$\int_{u\geq u_1}\int_{v\leq v_c}\frac{V^2}{\left(1-\frac{M}{r}\right)^2}f^2dudv\leq A(l+1)^4(r(u_1,v_c)-M). \tag{3.95}$$

We can also bound

$$\int_u\int_{v\leq v_c}f'^2\frac{|\partial_v\psi|^2}{\left(1-\frac{M}{r}\right)^2}dudv\leq A\sup_{f'\neq 0}\int_{v\leq v_c}\frac{|\partial_v\psi|^2}{\left(1-\frac{M}{r}\right)^2}dv\leq C(u_1-v_c)^2I.E.[\psi_+,v_c,u_1,u_0], \tag{3.96}$$

using Proposition 3.4.3.

Thus we have

$$\begin{aligned}\left|\int_{\sigma\in\mathbb{R}}|\sigma|\left(|\hat{\psi}_{\mathscr{H}^-}|^2-|\hat{\psi}_0|^2\right)d\sigma\right| &\leq 2\left|\int_{\sigma\in\mathbb{R}}\frac{\sigma}{|\sigma|}\int_{v=-\infty}^{v_c}\mathbb{R}\left(i\bar{\hat{\psi}}_0\widehat{V\psi_0}\right)dvd\sigma\right|\\ &\quad+\left((u_1-v_c)I.T.[\psi_+]I.E.[\psi_+,v_c,u_1,u_0]\right)^{1/2}.\end{aligned} \tag{3.97}$$

Given that we know in some sense that $\phi \to \phi_{\mathscr{H}^-}$, and $V \sim \frac{l(l+1)}{r^{*2}}$ as $v \to -\infty$, we will replace $\phi_0 = \phi_{0\mathscr{H}^-} + \delta\phi(u,v)$ and $V = \frac{l(l+1)}{r^{*2}} + \delta V$.

Then we obtain

$$\begin{aligned}
\left|\int_{\sigma\in\mathbb{R}} \frac{\sigma}{|\sigma|}\int_{v=-\infty}^{v_c} \mathbb{R}\left(i\bar{\hat{\psi}}_0 \widehat{V\psi_0}\right) dv d\sigma\right| &\leq l(l+1)\left|\int_{\sigma\in\mathbb{R}} \frac{\sigma}{|\sigma|}\int_{v=-\infty}^{v_c} \mathbb{R}\left(i\bar{\hat{\psi}}_{0\mathscr{H}^-}\widehat{\left(\frac{\psi_{0\mathscr{H}^-}}{(u-v)^2}\right)}\right) dv d\sigma\right| \qquad (3.98)\\
&\quad + \int_{v=-\infty}^{v_c} |\delta V(u_1,v)| \|\psi_0\|^2_{\Sigma_v} dv\\
&\quad + 2\int_{v=-\infty}^{v_c} V(u_1,v)\|\psi_0\|_{\Sigma_v}\|\delta\psi_0\|_{\Sigma_v} dv\\
&\leq l(l+1)\left|\int_{\sigma\in\mathbb{R}} \frac{\sigma}{|\sigma|}\mathbb{R}\left(i\bar{\hat{\psi}}_{0\mathscr{H}^-}\widehat{\left(\frac{\psi_{0\mathscr{H}^-}}{u-v_c}\right)}\right) d\sigma\right|\\
&\quad + AI.T.[\psi_+]\left((r(u_1,v_c)-M)^2\ln\left(\frac{r(u_1,v_c)}{M}-1\right)\right.\\
&\quad \left. + \frac{u_0-u_1}{u_1-v_c}(r(u_1,v_c)-M)\right)\\
&\quad + \frac{M+u_0-u_1}{u_1-v_c} I.T.[\psi_+]^{1/2} I.E.[\psi_+,v_c,u_1,u_0]^{1/2}.
\end{aligned}$$

We have bounded $\|\delta\psi_0\|_{\Sigma_v}$ using Proposition 3.4.1, and $|\delta V(u_1,v) \leq A(l+1)^2\left(1-\frac{M}{r}\right)^3\ln\left(\frac{r}{M}-1\right)$, by an explicit calculation. Also, as $\psi_{0\mathscr{H}^-}$ is compactly supported, we can bring the integral over v inside the Fourier transform. Denoting $\delta\psi_{\mathscr{H}^-} = \psi_{\mathscr{H}^-} - \psi_{0\mathscr{H}^-}$, we then obtain

$$\begin{aligned}
\left|\int_{\sigma\in\mathbb{R}}\frac{\sigma}{|\sigma|}\mathbb{R}\left(i\bar{\hat{\psi}}_{0\mathcal{H}^-}\left(\widehat{\frac{\psi_{0\mathcal{H}^-}}{u-v_c}}\right)\right)d\sigma\right| \leq{}& \left|\int_{\sigma\in\mathbb{R}}\frac{\sigma}{|\sigma|}\mathbb{R}\left(i\bar{\hat{\psi}}_{\mathcal{H}^-}\left(\widehat{\frac{\psi_{\mathcal{H}^-}}{u-v_c}}\right)\right)d\sigma\right| \\
&+\left|\int_{\sigma\in\mathbb{R}}\frac{\sigma}{|\sigma|}\mathbb{R}\left(i\delta\bar{\hat{\psi}}_{\mathcal{H}^-}\left(\frac{\psi_{\mathcal{H}^-}-\widehat{\psi_{\mathcal{H}^-}}(v_c)}{u-v_c}\right)\right)d\sigma\right| \\
&+\left|\int_{\sigma\in\mathbb{R}}\frac{\sigma}{|\sigma|}\mathbb{R}\left(i\delta\bar{\hat{\psi}}_{\mathcal{H}^-}\left(\frac{\widehat{\psi_{\mathcal{H}^-}(v_c)}}{u-v_c}\right)\right)d\sigma\right| \\
&+\left|\int_{\sigma\in\mathbb{R}}\frac{\sigma}{|\sigma|}\mathbb{R}\left(i\left(\bar{\hat{\psi}}_{\mathcal{H}^-}+\delta\bar{\hat{\psi}}_{\mathcal{H}^-}\right)\left(\frac{\delta\psi_{\mathcal{H}^-}-\widehat{\delta\psi}_{\mathcal{H}^-}(v_c)}{u-v_c}\right)\right)d\sigma\right| \\
&+\left|\int_{\sigma\in\mathbb{R}}\frac{\sigma}{|\sigma|}\mathbb{R}\left(i\left(\bar{\hat{\psi}}_{\mathcal{H}^-}+\delta\bar{\hat{\psi}}_{\mathcal{H}^-}\right)\left(\frac{\widehat{\delta\psi_{\mathcal{H}^-}(v_c)}}{u-v_c}\right)\right)d\sigma\right| \\
\leq{}& \left|\int_{\sigma\in\mathbb{R}}\frac{\sigma}{|\sigma|}\mathbb{R}\left(i\bar{\hat{\psi}}_{\mathcal{H}^-}\left(\widehat{\frac{\psi_{\mathcal{H}^-}}{u-v_c}}\right)\right)d\sigma\right|+\|\delta\psi_{\mathcal{H}^-}\|_{L^2}\left\|\frac{\psi_{\mathcal{H}^-}-\psi_{\mathcal{H}^-}(v_c)}{u-v_c}\right\|_{L^2} \\
&+\sqrt{\frac{\pi}{2}}\left|\int_{\sigma\in\mathbb{R}}\mathbb{R}\left(\widehat{\delta\bar{\psi}}_{\mathcal{H}^-}(-\sigma)\psi_{\mathcal{H}^-}(v_c)e^{-iv_c\sigma}\right)d\sigma\right| \qquad (3.99)\\
&+\|\bar{\psi}_{\mathcal{H}^-}+\delta\bar{\psi}_{\mathcal{H}^-}\|_{L^2}\left\|\frac{\delta\psi_{\mathcal{H}^-}-\delta\psi_{\mathcal{H}^-}(v_c)}{u-v_c}\right\|_{L^2} \\
&+\sqrt{\frac{\pi}{2}}\left|\int_{\sigma\in\mathbb{R}}\mathbb{R}\left(\left(\bar{\hat{\psi}}_{\mathcal{H}^-}+\delta\bar{\hat{\psi}}_{\mathcal{H}^-}\right)(-\sigma)\delta\psi_{\mathcal{H}^-}(v_c)e^{-iv_c\sigma}\right)d\sigma\right| \\
\leq{}& \sqrt{\frac{\pi}{2}}\left|\int_{\sigma\in\mathbb{R}}\int_{\sigma'\in\mathbb{R}}\frac{\sigma}{|\sigma|}\frac{\sigma'}{|\sigma'|}\mathbb{R}\left(\bar{\hat{\psi}}_{\mathcal{H}^-}(\sigma)\hat{\psi}_{\mathcal{H}^-}(\sigma-\sigma')e^{-iv_c\sigma'}\right)d\sigma'd\sigma\right| \\
&+2\|\delta\psi_{\mathcal{H}^-}\|_{L^2}\|\partial_u\psi_{\mathcal{H}^-}\|_{L^2} \\
&+|\delta\bar{\psi}_{\mathcal{H}^-}(v_c)\psi_{\mathcal{H}^-}(v_c)|+2\|\bar{\psi}_{\mathcal{H}^-}+\delta\bar{\psi}_{\mathcal{H}^-}\|_{L^2}\|\partial_u\delta\psi_{\mathcal{H}^-}\|_{L^2} \\
&+|(\bar{\psi}_{\mathcal{H}^-}(v_c)+\delta\bar{\psi}_{\mathcal{H}^-}(v_c))\delta\psi_{\mathcal{H}^-}(v_c)| \\
\leq{}& \sqrt{\frac{\pi}{2}}\left|\int_{\sigma\in\mathbb{R}}\int_{\sigma'\in\mathbb{R}}\frac{\sigma}{|\sigma|}\frac{\sigma'}{|\sigma'|}\mathbb{R}\left(\bar{\hat{\psi}}_{\mathcal{H}^-}(-\sigma)\hat{\psi}_{\mathcal{H}^-}(\sigma-\sigma')e^{-iv_c\sigma'}\right)d\sigma'd\sigma\right| \\
&+\frac{AI.T.[\psi_+]^{1/2}I.E.[\psi_+,v_c,u_1,u_0]^{1/2}}{(l+1)^2}.
\end{aligned}$$

Here we have used that the Fourier transform of u^{-1} is $\sqrt{\pi/2}\,\sigma/|\sigma|$, and we have used Hardy's Inequality to bound

$$\left\|\frac{f(u)-f(v_c)}{u-v_c}\right\|_{L^2}\leq 2\|\partial_u f\|_{L^2}. \tag{3.100}$$

Thus we have

$$\begin{aligned}
&\left|\int_{\sigma\in\mathbb{R}}|\sigma|\left(|\hat{\psi}_{\mathcal{H}^-}|^2-|\hat{\psi}_0|^2\right)d\sigma\right| \\
&\quad\leq\sqrt{\frac{\pi}{2}}\left|\int_{\sigma\in\mathbb{R}}\int_{\sigma'\in\mathbb{R}}\frac{\sigma}{|\sigma|}\frac{\sigma'}{|\sigma'|}\mathbb{R}\left(\bar{\hat{\psi}}_{\mathcal{H}^-}(-\sigma)\hat{\psi}_{\mathcal{H}^-}(\sigma-\sigma')e^{-iv_c\sigma'}\right)d\sigma'd\sigma\right| \qquad (3.101)\\
&\quad+A\left(\ln\left(\frac{u_1-v_c}{M}\right)+u_0-u_1\right)\frac{I.T.[\psi_+]}{(u_1-v_c)^2} \\
&\quad+\left((u_1-v_c)I.T.[\psi_+]I.E.[\psi_+,v_c,u_1,u_0]\right)^{1/2}.
\end{aligned}$$

If $\hat{\psi}$ is only supported on positive frequencies, then we can simplify the following

$$\begin{aligned}\sqrt{\frac{\pi}{2}}\left|\int_{\sigma\in\mathbb{R}}\int_{\sigma'\in\mathbb{R}}\frac{\sigma}{|\sigma|}\frac{\sigma'}{|\sigma'|}\mathbb{R}\left(\hat{\bar{\psi}}_{\mathscr{H}^-}(-\sigma)\hat{\psi}_{\mathscr{H}^-}(\sigma-\sigma')e^{-iv_c\sigma'}\right)d\sigma' d\sigma\right|&\\ =\sqrt{\frac{\pi}{2}}\left|\int_{\sigma\in\mathbb{R}}\int_{\sigma'\in\mathbb{R}}\frac{\sigma'}{|\sigma'|}\mathbb{R}\left(\hat{\bar{\psi}}_{\mathscr{H}^-}(-\sigma)\hat{\psi}_{\mathscr{H}^-}(\sigma-\sigma')e^{-iv_c\sigma'}\right)d\sigma' d\sigma\right|&\\ =\sqrt{\frac{\pi}{2}}\left|\int_{\sigma'\in\mathbb{R}}\frac{\sigma'}{|\sigma'|}\mathbb{R}\left(\widehat{|\psi_{\mathscr{H}^-}|^2}(-\sigma')e^{-iv_c\sigma'}\right)d\sigma'\right|&\\ =\sqrt{\frac{\pi}{2}}\left|\int_{\sigma'\in\mathbb{R}}\mathbb{R}\left(\widehat{|\psi_{\mathscr{H}^-}|^2}(\sigma')e^{iv_c\sigma'}\right)d\sigma'\right| = |\psi_{\mathscr{H}^-}(v_c)|^2&\\ \leq I.E.[\psi_+,v_c,u_1,u_0].&\end{aligned} \tag{3.102}$$

□

3.4.4 The Reflection

In this section we will consider evolving our solution in the small compact (in r, t^* coordinates) region, given by $R_2 := \{v\leq v_c, u\leq u_1\}\cap\{r\geq r_b\}$.

We will consider the surface $r = r_b(t^*)$ to be instead parametrised by $v = v_b(u)$, or equivalently by $u = u_b(v) = v_b^{-1}(v)$, as in (1.57).

Proposition 3.4.4 (Reflection Energy Bounds)**.** *Let ψ be a smooth solution to* (3.33) *subject to* (3.34)*. Define the function ψ_0 by*

$$\psi_0 := \psi(u,v_c) - \psi(u_b(v),v_c). \tag{3.103}$$

Then there exists a constant $A(M,q,T^*)$ such that

$$\int_{v_b(u_1)}^{v_c}\left|\frac{du_b}{dv}\right|^{-1}|\partial_v\psi(u_1,v)-\partial_v\psi_0(u_1,v)|^2dv \leq \begin{cases} Ae^{-3\kappa u_1}I.T.[\psi_+] & |q|<1\\ \frac{Au_0^2 I.T.[\psi_+]}{u_1^8} & |q|=1\end{cases}, \tag{3.104}$$

for any sufficiently large u_1.

Furthermore, there exists a constant $B(M,q,T^*)$ such that

$$\int_{v_b(u_1)}^{v_c}\left|\frac{du_b}{dv}\right||\psi(u_1,v)-\psi_0(u_1,v)|^2dv \leq \begin{cases} Bu_1e^{-3\kappa u_1}I.T.[\psi_+] & |q|<1\\ \frac{Bu_0^2 I.T.[\psi_+]}{u_1^6} & |q|=1\end{cases}. \tag{3.105}$$

Finally, there exists a constant $C(M,q,T^*)$ such that

$$\int_{v_b(u_1)}^{v_c}|\psi(u_1,v)|^2dv \leq \begin{cases} C\frac{I.T.[\psi_+]}{(l+1)^4}e^{-\kappa u_1} & |q|<1\\ C\frac{I.T.[\psi_+]u_0^2}{(l+1)^4u_1^4} & |q|=1.\end{cases} \tag{3.106}$$

If ψ_+ decays quickly enough as $u\to\infty$, then

$$\int_{v_b(u_1)}^{v_c}|\psi(u_1,v)|^2dv \leq e^{\kappa(u_0-3u_1)}\int_{u=-\infty}^{\infty}e^{\kappa u}|\partial_u\psi_+|^2du. \tag{3.107}$$

Remark 3.4.5. *Note the form of ψ_0 here is the solution to the equation*

$$\partial_u\partial_v\psi_0 = 0, \tag{3.108}$$

with initial conditions

$$\psi_0(u,v_c) = \psi(u,v_c). \tag{3.109}$$

Therefore, ψ_0 is reflected as if it were in Minkowski spacetime. Thus, Proposition 3.4.4 gives a bound on how much the solution differs from a reflection in $1+1$ dimensional Minkowski.

This solution ψ_0 takes the form:

$$\psi_0(u,v) = \psi(u,v_c) - \psi(u_b(v),v_c). \tag{3.110}$$

Proof. We begin by considering how the derivatives of ψ and ψ_0 vary on the surface of the dust cloud, and applying (3.33):

$$\begin{aligned}
\int_{S_{[u_1,\infty)}} (u-u_1)^p |\partial_u\psi - \partial_u\psi_0|^2 du &\leq \int_{S_{[u_1,\infty)}} (u-u_1)^p \left|\int_{v_b(u_1)}^{v_c} \partial_u\partial_v\psi dv\right|^2 du = \int_{S_{[u_1,\infty)}} (u-u_1)^p \left|\int_{v_b(u_1)}^{v_c} V\psi dv\right|^2 du \\
&\leq \int_{S_{[u_1,\infty)}} (u-u_1)^p \|1\|^2_{L^2(\Sigma_u)} \|V\psi\|^2_{L^2(\Sigma_u)} du \\
&\leq \int_{u=u_1}^{\infty}\int_{v=v_b(u)}^{v_c} (u-u_1)^p (v-v_b(u)) V^2|\psi|^2 dv du.
\end{aligned} \tag{3.111}$$

We then proceed to do the same to compare the derivatives of ψ on the surface of the dust cloud to the derivatives of ψ on Σ_{u_1}:

$$\begin{aligned}
&\int_{\Sigma_{u_1}} |u_b'|^{-1}(u_b(v)-u_1)^p |\partial_u\psi(u_1,v) - \partial_u\psi(u_b(v),v)|^2 du \\
&\leq \int_{\Sigma_{u_1}} |u_b'|^{-1}(u_b(v)-u_1)^p \left|\int_{v_b(u_1)}^{v_c} \partial_u\partial_v\psi dv\right|^2 dv \\
&\leq \int_{\Sigma_{u_1}} |u_b'|^{-1}(u_b(v)-u_1)^p \left|\int_{v_b(u_1)}^{v_c} V\psi dv\right|^2 dv \\
&\leq \int_{\Sigma_{u_1}} |u_b'|^{-1}(u_b(v)-u_1)^p \|\sqrt{V}\|^2_{L^2(\Sigma_v)} \left\|\sqrt{V}\psi\right\|^2_{L^2(\Sigma_v)} dv \\
&\leq A\int_{u=u_1}^{\infty}\int_{v=v_b(u)}^{v_c} |u_b'|^{-1}(u_b(v)-u_1)^p \left(\int_{u_1}^{\infty} V(u',v_c)du'\right) V|\psi|^2 dv du.
\end{aligned} \tag{3.112}$$

Combining equations (3.111) and (3.112), we obtain

$$\begin{aligned}
&\int_{\Sigma_u} |u_b'|^{-1}(u_b(v)-u_1)^p |\partial_v\psi - \partial_v\psi_0|^2 dv \\
&\leq 2\int_{u=u_1}^{\infty}\int_{v=v_b(u)}^{v_c} \left((v-v_b(u))(u-u_1)^p V + |u_b'|^{-1}(u_b(v)-u_1)^p \left(\int_{u_1}^{\infty} V(u',v_c)du'\right)\right) V|\psi|^2 dv du \\
&\leq \begin{cases} A(l+1)^2 \int_{u=u_1}^{\infty}\int_{v=v_b(u)}^{v_c} \left(u_b(v)^{p-2} + u_b(v)^{p-1}e^{-\kappa u_1}\right) V|\psi|^2 dv du & |q|<1 \\ A(l+1)^2 \int_{u=u_1}^{\infty}\int_{v=v_b(u)}^{v_c} \left((v-v_b(u))u^{p-2} + \frac{u_b(v)^{p-2}}{u_1-v_c}\right) V|\psi|^2 dv du & |q|=1 \end{cases},
\end{aligned} \tag{3.113}$$

where we have used the behaviour of u_b and v_b for late times to obtain the final line.

We first consider the extremal case, and the non-extremal case for sufficiently good decay. We will be using energy boundedness results from [1] (Theorem 1).

This tells us that the non-degenerate energy of ϕ on Σ_{v_c} bounds the non-degenerate energy of ϕ on Σ_u, *i.e.* there exists a constant $A(M,q,T^*)$ such that for all $u' > u_1$

$$\begin{aligned}
&\int_{u_1}^{\infty} \frac{|\partial_u\phi(u,v_c)|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} + l(l+1)\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)|\phi(u,v_c)|^2 du \\
&\geq A\int_{v_b(u')}^{v_c} \frac{|\partial_v\phi(u',v)|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} + l(l+1)\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)|\phi(u',v)|^2 dv \\
&\geq \frac{A}{\left(1-\frac{2M}{r(u',v_c)}+\frac{q^2M^2}{r(u',v_c)^2}\right)(v_c-v_b(u'))^2}\int_{v_b(u')}^{v_c} |\phi(u',v)|^2 dv.
\end{aligned} \tag{3.114}$$

Here we have used that ϕ vanishes on the surface of the dust cloud in order to apply Poincaré's inequality. In the extremal case, we can bound this non-degenerate energy as follows:

$$\begin{aligned}\int_{u_1}^{\infty} \frac{|\partial_u\phi(u,v_c)|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} &+ l(l+1)\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)|\phi(u,v_c)|^2du \\ &\leq A\int_{u_1}^{\infty}(M^2+u^2)|\partial_u\phi(u,v_c)|^2+l(l+1)\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)|\phi(u,v_c)|^2du \\ &\leq A\int_{u_1}^{\infty}(M^2+(u-u_0)^2+u_0^2)|\partial_u\phi(u,v_c)|^2+l(l+1)\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)|\phi(u,v_c)|^2du.\end{aligned} \tag{3.115}$$

We can use (3.61) and (3.62) to bound this by $I.T.[\psi_+]$ plus u_0^2 times T energy. Cambining (3.114) and (3.115), we obtain (3.106) for the extremal case. We then proceed by combining this with (3.113):

$$\begin{aligned}A(l+1)^2&\int_{u=u_1}^{\infty}\int_{v=v_b(u)}^{v_c}\left((v-v_b(u))u^{p-2}+\frac{u_b(v)^{p-2}}{u_1-v_c}\right)V|\psi|^2dvdu \\ &\leq A(l+1)^2\sup_{u\geq u_1,v<v_c}\left(\left((v-v_b(u))u^{p-2}+\frac{u_b(v)^{p-2}}{u_1-v_c}\right)V\right)\int_{u_1}^{\infty}\int_{v_b(u)}^{v_c}|\phi(u,v)|^2 \\ &\leq A(l+1)^{-2}\sup_{u\geq u_1,v<v_c}\left(\left((v-v_b(u))u^{p-2}+\frac{u_b(v)^{p-2}}{u_1-v_c}\right)V\right) \\ &\qquad\int_{u_1}^{\infty}\left(\left(1-\frac{2M}{r(u',v_c)}+\frac{q^2M^2}{r(u',v_c)^2}\right)(v_c-v_b(u'))^2\right)u_0^2I.T.[\psi_+]du \\ &\leq \frac{A(l+1)^{-2}u_0^2I.T.[\psi_+]}{u_1^3}\sup_{u\geq u_1,v<v_c}\left(\left((v-v_b(u))u^{p-2}+\frac{u_b(v)^{p-2}}{u_1-v_c}\right)V\right).\end{aligned} \tag{3.116}$$

The two cases we will be considering are the cases $p=0$ and $p=2$. These give the following results:

$$\int_{\Sigma_u}|u_b'|^{-1}|\partial_v\psi-\partial_v\psi_0|^2dv \leq \frac{Au_0^2I.T.[\psi_+]}{u_1^8} \tag{3.117}$$

$$\int_{\Sigma_u}|u_b'|^{-1}(u_b(v)-u_1)^2|\partial_v\psi-\partial_v\psi_0|^2dv \leq \frac{Au_0^2I.T.[\psi_+]}{u_1^6}. \tag{3.118}$$

In the less well behaved sub-extremal case, we instead consider energy currents in order to bound

$$\int_{u=u_1}^{\infty}\int_{v=v_b(u)}^{v_c}V|\psi|^2dvdu \leq A\int_{u=u_1}^{\infty}\int_{v=v_b(u)}^{v_c}V|\phi|^2dvdu. \tag{3.119}$$

We apply divergence theorem to the following vector field

$$J:=M\partial_u+\alpha(v-v_c)\partial_v-\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)(M-\alpha(v-v_c))\frac{\phi\nabla\phi}{2r}+\phi^2\nabla\left(\frac{M-\alpha(v-v_c)}{4r}\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\right), \tag{3.120}$$

in the region $u\geq u_1, v\in[v_b(u),v_c]$. Here, $\alpha=\alpha(M,q,T^*)>0$ is chosen such that$\alpha^{-1}\geq 8(1-q^2)^{-1}$ and $\alpha^{-1}\geq(v-v_c)u_b'(v)$ for $v-v_c$ sufficiently small.

A simple calculation reveals the following results:

$$\begin{aligned}
\nabla . J = &\left(\frac{\alpha}{2r^2} + \frac{M-\alpha(v_c-v)}{2r^3}\left(\frac{M}{r}\left(1-\frac{q^2M}{r}\right) - \left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\right)\right) l(l+1)|\phi|^2 \quad (3.121)\\
&+\left(\frac{M^4(1-q^2)}{r^6}\left(2-\frac{Mq^2}{r}\right) - \left(\frac{q^2M^4(8-q^2)}{r^6}+\frac{q^2M^5(4-3q^2)}{r^7}\right)\alpha\right)|\phi|^2\\
&+\left(O\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)+O(v_c-v)\right)|\phi|^2 \geq 0\\
-du(J) = &\frac{2\alpha(v_c-v)|\partial_v\phi|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} + \frac{Ml(l+1)|\phi|^2}{2r^2} - \partial_v\left(\frac{(M-\alpha(v_c-v))|\phi|^2}{2r}\right)\\
&+\left(\frac{M^2}{2r^3}\left(1-\frac{q^2M}{r}\right)+\frac{\alpha}{2r}+\frac{\alpha M^2}{2r^3}\left(4-q^2-\frac{q^2M}{r}\right)\right)|\phi|^2\\
&+\left(O\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)+O(v_c-v)\right)|\phi|^2\\
-dv(J)|_{v=v_c} = &\frac{2M|\partial_u\phi|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} - \frac{M\phi\partial_u\phi}{r} + \frac{M}{4r^2}\left(\frac{2M}{r}\left(1-\frac{q^2M}{r}\right)+\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\right)|\phi|^2 \quad (3.122)\\
\leq &\frac{4M|\partial_u\phi|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}\\
(du-u_b'dv)(J)|_S = &\frac{2u_b'\left(1-\alpha(v-v_c)u_b'\right)|\partial_u\phi|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} \geq 0, \quad (3.123)
\end{aligned}$$

for $v_c - v$ sufficiently small. Here 'sufficiently small' only depends on m, q, T^*.

We then apply divergence theorem:

$$\int \nabla . J + \int_{\Sigma_{u_0}\cap\{v\leq v_c\}} (-du(J)) + \int ((du - v_b' dv)(J)) = \int_{\Sigma_{v_c}} (-dv(J)), \tag{3.124}$$

to obtain

$$\int_{\Sigma_{u_0}\cap\{v\leq v_c\}} (v_c - v)|\partial_v\phi|^2 dv \leq \frac{2M}{\alpha}\int_{\Sigma_{v_c}} |\partial_u\phi|^2 du. \tag{3.125}$$

An application of Hardy's inequality to the function $f\left(\frac{v_c-v}{M}\right)\phi$, for f a smooth function which vanishes at 0 yields

$$\int_{v_b}^{v_c} \frac{f^2|\phi|^2}{(v_c-v)^2} dv \leq 4\int_{v_b}^{v_c} f^2|\partial_v\phi|^2 - \frac{ff'}{M}\partial_v\left(|\phi|^2\right) + \frac{f'^2}{M^2}|\phi|^2 dv \tag{3.126}$$

$$\leq 4\int_{v_b}^{v_c} f^2|\partial_v\phi|^2 - \frac{ff''}{M^2}|\phi|^2 dv. \tag{3.127}$$

Choosing $f(x) = x\sqrt{-\log(x)}$ gives

$$\int_{v_b}^{v_c} -\log\left(\frac{v_c-v}{M}\right)|\phi|^2 dv \leq 4\int_{v_b}^{v_c} -\log\left(\frac{v_c-v}{M}\right)(v_c-v)^2|\partial_v\phi|^2 + \left(\frac{1}{2}+\frac{1}{4\left(-\log\left(\frac{v_c-v}{M}\right)\right)}\right)\frac{|\phi|^2}{M^2}dv. \tag{3.128}$$

Provided $v_c - v \leq \frac{M}{e}$, then this can be rearranged for

$$\int_{v_b}^{v_c} -\log\left(\frac{v_c-v}{M}\right)|\phi|^2 dv \leq 16\int_{v_b}^{v_c} -\log\left(\frac{v_c-v}{M}\right)(v_c-v)^2|\partial_v\phi|^2, \tag{3.129}$$

and as we know the form u_b' takes for v close to v_b, we have that

$$\begin{aligned}\int_{\Sigma_u\cap\{v\leq v_c\}} u_b'(v)|\phi|^2 dv &\leq A\int_{\Sigma_u\cap\{v\leq v_c\}} -\log\left(\frac{v_c-v}{M}\right)(v_c-v)^2|\partial_v\phi|^2 \\ &\leq -A\log\left(\frac{v_c-v_b(u_1)}{M}\right)(v_c-v_b(u_1))\int_{\Sigma_{v_c}}|\partial_u\phi|^2 du.\end{aligned} \tag{3.130}$$

This can be used to immediately obtain (3.106), and we can also apply it to (3.113) to obtain

$$\begin{aligned}&\int_{\Sigma_u}|u_b'|^{-1}(u_b(v)-u_1)^p|\partial_v\psi-\partial_v\psi_0|^2 dv \\ &\qquad\leq A(l+1)^2e^{-\kappa u_1}\left|\log\left(\frac{v_c-v_b(u)}{M}\right)\right|^{p-1}(v_c-v_b(u_1))\int_{\Sigma_{v_c}}|\partial_u\phi|^2du\int_{u_1}^{\infty}V(u,v_c)du \\ &\qquad\leq\begin{cases}Ae^{-3\kappa u_1}I.T.[\psi_+] & p=0\\ Au_1e^{-3\kappa u_1}I.T.[\psi_+] & p=2.\end{cases}\end{aligned} \tag{3.131}$$

Finally, we can use Hardy's inequality on the function $\psi(u_1,v_b(u))-\psi_0(u_1,v_b(u))$, as this vanises on $u=u_1$, to get

$$\begin{aligned}\int_{v_b(u_1)}^{v_c}u_b'|\psi(u_1,v)-\psi_0(u_1,v)|^2dv &= \int_{u_1}^{\infty}|\psi(u_1,v_b(u))-\psi_0(u_1,v_b(u))|^2du \\ &\leq 4\int_{u_1}^{\infty}(u-u_1)^2|\partial_u\psi(u_1,v_b(u))-\partial_v\psi_0(u_1,v_b(u))|^2du \\ &= 4\int_{v_b(u_1)}^{v_c}(u_b'(v))^{-1}(u_b(u)-u_1)^2|\partial_v\psi(u_1,v)-\partial_v\psi_0(u_1,v)|^2du.\end{aligned} \tag{3.132}$$

The proof of (3.107) requires using a weighted $T-$energy estimate to obtain

$$\begin{aligned}\int_{u=-\infty}e^{\kappa(u-u_0)}|\partial_u\psi_+(u-u_0)|^2du &\geq \int_{u=-\infty}e^{\kappa(u-u_0)}|\partial_u\psi(u,v_c)|^2du \\ &\geq ae^{-\kappa u_0}\int_{u=u_1}^{\infty}\frac{|\partial_u\psi(u,v_c)|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}du \\ &\geq ae^{-\kappa u_0}\int_{v=v_b(u_1)}^{v_c}\frac{|\partial_v\psi(u_1,v)|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}dv \\ &\geq ae^{-\kappa(u_0-u_1)}|\partial_v\psi(u_1,v)|^2dv\geq ae^{-\kappa(u_0-3u_1)}|\psi(u_1,v)|^2dv.\end{aligned} \tag{3.133}$$

As required. Here we have again used Theorem 1 from [1], followed by Poincaré's inequality. □

Note we still have other error occurring across the rest of Σ_{v_c}:

$$\int_{u=-\infty}^{u_1}|\partial_u\psi(u,v_c)-\partial_u\psi_{\mathscr{H}^-}(u)|^2du\leq\begin{cases}Ae^{\kappa u_1}I.E.[\psi_+,v_c,u_1,u_0] & |q|<1\\ AI.E.[\psi_+,v_c,u_1,u_0] & |q|=1\end{cases}. \tag{3.134}$$

Corollary 3.4.3 (Hawking Radiation Error from the Reflection)**.** *Let $\psi_+:\mathbb{R}\to\mathbb{C}$ be a smooth, compactly supported function. Let ψ be the solution of* (3.33)*, as given by Theorem 2.4.1, with radiation field on $\mathscr{I}^+$ equal to ψ_+, and which vanishes on $\mathscr{H}^+$. Let f be a smooth compactly supported function such that $f(0)=1$, and define*

$$\psi_0(u):=\begin{cases}\psi(u,v_c)-f(u-u_1)\psi(u_1,v_c) & u\geq u_1\\ 0 & u<u_1\end{cases}. \tag{3.135}$$

$$\psi_1(v):=\begin{cases}\psi(u_1,v)-(1-f(u_b(v)-u_1))\,\psi(u_1,v_c) & v\in[v_b(u_1),v_c]\\ 0 & v\notin[v_b(u_1),v_c]\end{cases}. \tag{3.136}$$

Then there exists a constant $A(M,q,T^*)$ such that

$$\int_{\sigma\in\mathbb{R}}(\kappa+|\sigma|)\left||\hat{\psi}_0|^2-|\widehat{\psi_1\circ v_b}|^2\right|d\sigma\le\begin{cases}A\sqrt{u_1}e^{-\frac{3\kappa}{2}u_1}I.T.[\psi_+] & |q|<1\\ \frac{A}{u_1^4}I.T.[\psi_+] & |q|=1\end{cases},\tag{3.137}$$

as $u_0,u_1\to\infty$ with $u_1<u_0$. Here κ is the surface gravity, as in (1.63) *and $I.T.[\psi_+]$ is as defined in the statement of Theorem 3.4.1*

Proof. This proof is similar to that of Corollary 3.4.2. We consider $|q|<1$ first

$$\begin{aligned}\int_{\sigma\in\mathbb{R}}(\kappa+|\sigma|)\left||\hat{\psi}_0|^2-|\widehat{\psi_1\circ v_b}|^2\right|d\sigma &\le A\left(\int_{\sigma\in\mathbb{R}}\left|\hat{\psi}_0-\widehat{\psi_1\circ v_b}\right|^2d\sigma\right)^{1/2}\left(\int_{\sigma\in\mathbb{R}}(\kappa^2+\sigma^2)\left(|\widehat{\psi_1\circ v_b}|^2+|\hat{\psi}_0|^2\right)d\sigma\right)^{1/2}\\ &\le A\left(\int_{u_1}^{\infty}|\psi_0-\psi_1\circ v_b|^2du\right)^{1/2}\\ &\quad\left(\int_{u_1}^{\infty}|\psi_1\circ v_b|^2+|\psi_0|^2+|\partial_u(\psi_1\circ v_b)|^2+|\partial_u\psi_0|^2du\right)^{1/2}\\ &\le A\left(\int_{u_1}^{\infty}|\psi(u_1,v_c)-\psi(u,v_c)-\psi(u_1,v_b(u))|^2du\right)^{1/2}\\ &\quad\left(\int_{u_1}^{\infty}|\psi_0-\psi_1\circ v_b|^2+|\psi_0|^2+|\partial_u(\psi_0-\psi_1\circ v_b)|^2+|\partial_u\psi_0|^2du\right)^{1/2}\\ &\le A\left(I.T.[\psi_+]u_1e^{-3\kappa u_1}\right)^{1/2}\left(I.T.[\psi_+]\right)^{1/2}\\ &\le A\sqrt{u_1}e^{-\frac{3\kappa}{2}u_1}I.T.[\psi_+].\end{aligned}\tag{3.138}$$

As required. Here, we have used Proposition 3.4.4 to reach the penultimate line.

We next consider $|q|=1$, where $\kappa=0$

$$\begin{aligned}\int_{\sigma\in\mathbb{R}}|\sigma|\left||\hat{\psi}_0|^2-|\widehat{\psi_1\circ v_b}|^2\right|d\sigma &\le A\left(\int_{\sigma\in\mathbb{R}}\sigma^2\left|\hat{\psi}_0-\widehat{\psi_1\circ v_b}\right|^2d\sigma\right)^{1/2}\left(\int_{\sigma\in\mathbb{R}}\left(|\widehat{\psi_1\circ v_b}|^2+|\hat{\psi}_0|^2\right)d\sigma\right)^{1/2}\\ &\le A\left(\int_{u_1}^{\infty}|\partial_u\psi_0-\partial_u(\psi_1\circ v_b)|^2du\right)^{1/2}\left(\int_{u_1}^{\infty}|\psi_1\circ v_b|^2+|\psi_0|^2du\right)^{1/2}\\ &\le A\left(\frac{I.T.[\psi_+]}{u_1^8}\right)^{1/2}\left(I.T.[\psi_+]\right)^{1/2}\\ &\le\frac{A}{u_1^4}I.T.[\psi_+].\end{aligned}\tag{3.139}$$

□

3.4.5 High Frequency Transmission

We now consider how our solution on Σ_{u_1} is transmitted to $\mathscr{I}^-$. We first look to bound the energy through the surface $\Sigma_{v_b(u_1)}$, as all other energy is transmitted to $\mathscr{I}^-$. However, the map taking solutions on space-like surfaces back to their past radiation fields is bounded with respect to the *non-degenerate* energy (Theorem 7.1 in [1]). Thus we need to look at non-degenerate energy through $\Sigma_{v_b(u_1)}$. The non-degenerate energy on a surface Σ_v takes the form

$$\int_{\Sigma_v}\frac{|\partial_u\psi|^2}{1-\frac{2M}{r}+\frac{M^2q^2}{r^2}}du,\tag{3.140}$$

where we can absorb the ψ term using Hardy's inequality:

$$\int_{\Sigma_v}\left(1-\frac{2M}{r}+\frac{M^2q^2}{r^2}\right)\frac{|\psi^2|}{(r-r_b)^2}du = 2\int_{\Sigma_v}\frac{|\psi|^2}{(r-r_b)^2}dr \leq 8\int_{\Sigma_v}|\partial_r\psi|^2 dr = 4\int_{\Sigma_v}\frac{|\partial_u\psi|^2}{1-\frac{2M}{r}+\frac{M^2q^2}{r^2}}du. \quad (3.141)$$

We have the following proposition:

Proposition 3.4.5 (High Frequency Reflection in Pure Reissner–Nordström). *Let ψ be a smooth solution to* (3.33)*, and let $v_2 \in \mathbb{R}$. Then there exists a constant $A(M,q,T^*)$ such that*

$$\int_{\Sigma_{v_b(u_1)}}\frac{1}{V}|\partial_u\psi|^2 du \leq A\left(\int_{\Sigma_{v_c}\cap\{u\leq u_1\}}\frac{1}{V}|\partial_u\psi|^2 du + \int_{v=v_b(u_1)}^{v_c}\left||\psi(u_1,v)|^2-|\psi_{\mathscr{I}^-}(v)|^2\right|dv\right). \quad (3.142)$$

There also exists a constant $B(M,q,T^*)$ such that

$$\int_{\Sigma_{v_b(u_1)}}\frac{|\partial_u\psi|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}du \leq B\left(\int_{\Sigma_{v_c}\cap\{u\leq u_1\}}\frac{|\partial_u\psi|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}du + (l+1)^2\int_{\Sigma_{u_1}\cap\{v\leq v_c\}}|\psi|^2 dv\right). \quad (3.143)$$

Furthermore, there exists a constant $C(M,q,T^*)$ such that

$$\int_{\mathscr{I}^-\cap\{v\leq v_b(u_1)\}}|\psi|^2 dv \leq C\left(\int_{\Sigma_{v_c}\cap\{u\leq u_1\}}\frac{1}{V}|\partial_u\psi|^2 du + \int_{v=v_b(u_1)}^{v_c}\left||\psi(u_1,v)|^2-|\psi_{\mathscr{I}^-}(v)|^2\right|dv\right). \quad (3.144)$$

Proof. Given that $r_b \to \infty$ as $v \to -\infty$, there exists some v^* such that $V' \leq 0$ for all $v \leq v^*$. We then proceed using (3.33):

$$\begin{aligned}
E_V(v) := \int_{\Sigma_v\cap\{u\leq u_1\}}\frac{1}{V}|\partial_u\psi|^2 du &= E_V(v_c) + \int_v^{v_c}\int_{\Sigma_{v'}\cap\{u\leq u_1\}}\partial_u(|\psi|^2) - \partial_v\left(\frac{1}{V}\right)|\partial_u\psi|^2 du dv' \\
&\leq E_V(v_c) + \int_v^{v_c}\frac{\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)V'}{V^2}|\partial_u\psi|^2 dv' + \int_{\Sigma_{u_1}\cap\{v'\in[v,v_c]\}}|\psi|^2 dv' - \int_{\mathscr{I}^-\cap\{v'\in[v,v_c]\}}|\psi|^2 dv' \\
&\qquad (3.145) \\
&\leq E_V(v_c) + A\int_{\max\{v,v^*\}}^{v_c}E_V(v')dv' + \int_{\Sigma_{u_1}\cap\{v'\in[v,v_c]\}}|\psi|^2 dv' - \int_{\mathscr{I}^-\cap\{v'\in[v,v_c]\}}|\psi|^2 dv' \\
&\leq E_V(v_c) + A\int_{\max\{v,v^*\}}^{v_c}E_V(v')dv' + \int_{v'=v_b(u_1)}^{v_c}\left||\psi(u_1,v')|^2-|\psi_{\mathscr{I}^-}(v')|^2\right|dv' - \int_{\mathscr{I}^-\cap\{v\leq v_b(u_1)\}}|\psi_{\mathscr{I}^-}(v')|^2 dv
\end{aligned}$$

To reach the penultimate line, we have used (3.141).

We then apply Gronwall's Inequality to obtain

$$\int_{\Sigma_{v_2}}\frac{1}{V}|\partial_u\psi|^2 du \leq \left(\int_{\Sigma_{v_c}}\frac{1}{V}|\partial_u\psi|^2 du + \int_{v=v_b(u_1)}^{v_c}\left||\psi(u_1,v)|^2-|\psi_{\mathscr{I}^-}(v)|^2\right|dv\right)e^{\int_{v^*}^{v_c}A dv}, \quad (3.146)$$

for all $v_2 \leq v_b(u_1)$.

By keeping the $\mathscr{I}^-$ term in (3.145), we can then bound

$$\int_{\mathscr{I}^-\cap\{v\leq v_b(u_1)\}}|\psi|^2 dv \leq B\left(\int_{\Sigma_{v_c}\cap\{u\leq u_1\}}\frac{1}{V}|\partial_u\psi|^2 du + \int_{v=v_b(u_1)}^{v_c}\left||\psi(u_1,v)|^2-|\psi_{\mathscr{I}^-}(v)|^2\right|dv\right), \quad (3.147)$$

as required.

We perform a similar calculation for the remaining result, (3.143)

$$E(v) := \int_{\Sigma_v \cap \{u \le u_1\}} \frac{|\partial_u \psi|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} du = E(v_c) + \int_v^{v_c} \int_{\Sigma_{v'} \cap \{u \le u_1\}} \frac{V\partial_u(|\psi|^2)}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} - \partial_v\left(\frac{1}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}\right)|\partial_u\psi|^2 du dv'$$
$$\le E(v_c) + A\int_{v'=v}^{v_c} E(v')dv' + \int_{\Sigma_{u_1}\cap\{v\le v_c\}} \frac{V|\psi|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}dv - \int_v^{v_c}\int_{\Sigma_{v'}\cap\{u\le u_1\}} \partial_u\left(\frac{V}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}\right)|\psi|^2 du dv' \tag{3.148}$$
$$\le E(v_c) + A\int_{v'=v}^{v_c} E(v')dv' + A(l+1)^2\int_{\Sigma_{u_1}\cap\{v\le v_c\}} |\psi|^2 dv + \int_v^{v_c}\int_{\Sigma_{v'}\cap\{u\le u_1, r\le 2M\}}\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)|\psi|^2 du dv'$$
$$\le E(v_c) + A\int_{v'=v}^{v_c} E(v')dv' + A(l+1)^2\int_{\Sigma_{u_1}\cap\{v\le v_c\}} |\psi|^2 dv,$$

to which another application of Gronwall's Inequality obtains the result. □

The final Proposition in this section is:

Proposition 3.4.6 (High Frequency Transmission)**.** *Let ψ be a smooth solution to* (3.33), (3.34) *such that*

$$\int_{\Sigma_{v_c}\cap\{u\le u_1\}} \frac{|\partial_u\psi|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} du =: E_{v_c} < \infty \tag{3.149}$$

$$\int_{\Sigma_{u_1}} |\psi|^2 du =: L_{u_1} < \infty. \tag{3.150}$$

Then we have that there exists a constant $A(M,q,T^*)$ such that

$$\int_{v\le v_c} \left|\partial_v\psi|_{\mathscr{I}^-} - \partial_v\psi|_{\Sigma_{u_1}}\right|^2 dv \le A\left((l+1)^6 L_{u_1} + (l+1)^4 E_{v_c}\right). \tag{3.151}$$

Proof. We will start this result by considering the interval $[v_b(u_1), v_c]$. This section is done in a similar manner to Proposition 3.4.4.

$$\int_{v_b(u_1)}^{v_c} |\partial_v\psi_{\mathscr{I}^+}(v) - \partial_v\psi(u_1,v)|^2 dv = \int_{v=v_b(u_1)}^{v_c}\left|\int_{u=-\infty}^{u_1}\partial_u\partial_v\psi(u,v)du\right|^2 dv = \int_{v=v_b(u_1)}^{v_c}\left|\int_{u=-\infty}^{u_1} V\psi(u,v)du\right|^2 dv \tag{3.152}$$
$$\le \left\|\sqrt{V}\right\|^2_{L^2(\Sigma_{v_c})}\int_{v=v_b(u_1)}^{v_c}\int_{u=-\infty}^{u_1} V|\psi(u,v)|^2 du dv$$
$$\le A(l+1)^4\int_{v=v_b(u_1)}^{v_c}\int_{u=-\infty}^{u_1}\left(1-\frac{2M}{r}+\frac{q^2M^2}{r^2}\right)\frac{|\psi(u,v)|^2}{r^2} du dv$$
$$\le A(l+1)^4\int_{v=v_b(u_1)}^{v_c}|\psi(u_1,v)|^2 + \left(\int_{u=\infty}^{u_1}\frac{|\partial_u\psi|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}}du\right)dv$$
$$\le A(l+1)^4\left(L_{u_1} + (v_c - v_b(u_1))\left(E_{v_c} + (l+1)^2 L_{u_1}\right)\right)$$
$$\le A\left((l+1)^6 L_{u_1} + (l+1)^4 E_{v_c}\right).$$

For the region $v \le v_c$, we will use Theorem 2 from [1], which gives us energy boundedness of the scattering map, that is

$$\int_{v=-\infty}^{v_b(u_1)} |\partial_v\psi_{\mathscr{I}^-}|^2 dv \le A\int_{u=-\infty}^{u=u_1}\frac{|\partial_u\psi(u,v_b(u_1))|^2}{1-\frac{2M}{r}+\frac{q^2M^2}{r^2}} + V|\psi|^2 du \le A(l+1)^2\left(E_{v_c} + (l+1)^2 L_{u_1}\right), \tag{3.153}$$

which gives us our result. □

Propositions 3.4.5 and 3.4.6 give rise to the following corollary.

Corollary 3.4.4 (Hawking Radiation Error from High Frequency Transmission)**.** *Let $\psi_+ : \mathbb{R} \to \mathbb{C}$ be a smooth, compactly supported function. Let ψ be the solution of* (3.33)*, as given by Theorem 2.4.1, with radiation field on $\mathscr{I}^+$ equal to $\psi_+(u-u_0)$, and which vanishes on $\mathscr{H}^+$. Let f be a smooth compactly supported function such that $f(0)=1$, and define*

$$\psi_1(v) := \begin{cases} \psi(u_1,v) - \left(1 - f\left((l+1)^2(u_b(v)-u_1)\right)\right)\psi(u_1,v_c) & u \geq u_1 \\ 0 & u < u_1 \end{cases}. \tag{3.154}$$

Let $\psi_{\mathscr{I}^-}$ be the past radiation field. Then there exists a constant $A(M,q,T^*)$ such that

$$\int_{\sigma\in\mathbb{R}} |\sigma| \left||\hat{\psi}_{\mathscr{I}^-}|^2 - |\hat{\psi}_1|^2 - |\hat{\psi}_{RN}|^2\right| d\sigma \leq \begin{cases} A\left(I.T.[\psi_+]e^{-\kappa u_1} + e^{2\kappa u_1} I.E.[\psi_+,v_c,u_1,u_0]\right) & |q|<1 \\ A\left(\frac{I.T.[\psi_+]u_0}{u_1^{5/2}} + u_1^{7/2} I.E.[\psi_+,v_c,u_1,u_0]\right) & |q|=1 \end{cases}, \tag{3.155}$$

as $u_0, u_1 \to \infty$ with $u_1 < u_0$. Here κ is the surface gravity, as in (1.63) *and $I.T.[\psi_+]$, $I.E.[\psi_+,v_c,u_1,u_0]$ are as defined in the statement of Theorem 3.4.1. In the extremal case, we will also required $u_1 > u_0/2$.*

Suppose further that $|q|<1$, and that ψ_+, $\psi_{\mathscr{H}^-}$ and ψ_{RN} decay sufficiently fast that all $I.E.[\psi_+,v_c,(1-\delta)u_0,u_0]$ terms decay faster than $e^{-3\kappa(1-\delta)u_0}$. Then for all $\delta>0$, there exists a constant $B(M,q,T^*,\delta,\psi_+)$ such that

$$\int_{\sigma\in\mathbb{R}} |\sigma| \left||\hat{\psi}_{\mathscr{I}^-}|^2 - |\psi_1|^2 - |\hat{\psi}_{RN}|^2\right| d\sigma \leq Be^{-\kappa(1-\delta)u_0}. \tag{3.156}$$

Proof. Define the following

$$\psi_2 := \psi_{\mathscr{I}^-} - \psi_1 - \psi_{RN}. \tag{3.157}$$

Note that ψ_2 is only supported in $v \leq v_c$, as $v > v_c$ is out of the past light cone of the collapsing cloud. Thus, the solution in $v > v_c$ coincides with that of Reissner–Nordström.

We can expand (3.155) to get:

$$\int_{-\infty}^{\infty} |\sigma| \left||\hat{\psi}_{\mathscr{I}^-}|^2 - |\hat{\psi}_1|^2 - |\hat{\psi}_{RN}|^2\right| d\sigma = \int_{-\infty}^{\infty} |\sigma| \left||\hat{\psi}_2|^2 + 2\Re\left((\hat{\psi}_1+\hat{\psi}_2)\bar{\hat{\psi}}_{RN} + \hat{\psi}_1\bar{\hat{\psi}}_2\right)\right| d\sigma. \tag{3.158}$$

We can then bound

$$\int_{\infty}^{\infty} |\sigma| Re\left(\hat{\psi}_2\bar{\hat{\psi}}_{RN}\right) d\sigma \leq \|\psi_2\|_{L^2(\mathscr{I}^-)} \|\psi_{RN}\|_{\dot{H}^1(\mathscr{I}^-)} \tag{3.159}$$

$$\int_{\infty}^{\infty} |\sigma| Re\left(\hat{\psi}_1\bar{\hat{\psi}}_{RN}\right) d\sigma \leq \left\|\frac{\sigma\hat{\psi}_1}{1+M^2\sigma^2}\right\|_{L^2(\mathscr{I}^-)} \left\|\left(1+M^2\sigma^2\right)\hat{\psi}_{RN}\right\|_{L^2(\mathscr{I}^-)} \tag{3.160}$$

$$\int_{\infty}^{\infty} |\sigma| Re\left(\hat{\psi}_1\bar{\hat{\psi}}_2\right) d\sigma \leq \|\psi_1\|_{L^2(\mathscr{I}^-)} \|\psi_2\|_{\dot{H}^1(\mathscr{I}^-)} \tag{3.161}$$

$$\int_{-\infty}^{\infty} |\sigma||\hat{\psi}_2|^2 d\sigma \leq \|\psi_2\|_{L^2(\mathscr{I}^-)} \|\psi_2\|_{\dot{H}^1(\mathscr{I}^-)}. \tag{3.162}$$

We already have a bounds on $\|\psi_1\|_{L^2(\mathscr{I}^-)}$, given by Proposition 3.4.4:

$$\begin{aligned} \|\psi_1\|^2_{L^2(\mathscr{I}^-)} &\leq \|\psi(u_1,v)\|^2_{L^2(\{v\in[v_b(u_1),v_c]\})} + |\psi(u_1,v_c)|^2 \|f\|^2_{L^2(\mathscr{I}^-)} \\ &\leq \begin{cases} A\left(\frac{I.T.[\psi_+]}{(l+1)^4}e^{-\kappa u_1} + \frac{I.E.[\psi_+,v_c,u_1,u_0]}{(l+1)^2}e^{-\kappa u_1}\right) & |q|<1 \\ A\left(\frac{I.T.[\psi_+]u_0^2}{(l+1)^4 u_1^4} + \frac{I.E.[\psi_+,v_c,u_1,u_0]}{(l+1)^2 u_1^2}\right) & |q|=1 \end{cases}. \end{aligned} \tag{3.163}$$

We also have a bound on $\|\psi_2\|_{\dot{H}^1(\mathscr{I}^-)}$, thanks to Proposition 3.4.6:

$$\begin{aligned}
\|\psi_2\|^2_{\dot{H}^1(\mathscr{I}^-)} &\leq 2\|\psi-\psi_1\|^2_{\dot{H}^1(\mathscr{I}^-)\cap\{v\leq v_c\}} + 2\|\psi_{RN}\|^2_{\dot{H}^1(\mathscr{I}^-)\cap\{v\leq v_c\}} \\
&\leq 4\|f\psi(u_1,v_c)\|^2_{\dot{H}^1(\mathscr{I}^-)\cap\{v\leq v_c\}} + \int_{v\leq v_c}\left|\partial_v\psi|_{\mathscr{I}^-} - \partial_v\psi|_{\Sigma_{u_1}}\right|^2 dv + 2\|\psi_{RN}\|^2_{\dot{H}^1(\mathscr{I}^-)\cap\{v\leq v_c\}} \\
&\leq A\Bigg(\frac{(l+1)^2|\psi(u_1,v_c)|^2}{v_b'(u_1)} + \frac{(l+1)^4}{V(u_1,v_c)}\int_{\Sigma_{v_c}\cap\{u\leq u_1\}}|\partial_u\psi|^2 du \\
&\qquad + (l+1)^6\int_{\Sigma_{u_1}\cap\{v\leq v_c\}}|\psi|^2 dv + I.E.[\psi_+,v_c,u_1,u_0]\Bigg) \\
&\leq \begin{cases} A\left((l+1)^2 I.T.[\psi_+]e^{-\kappa u_1} + (l+1)^2 e^{\kappa u_1} I.E.[\psi_+,v_c,u_1,u_0]\right) & |q|<1 \\ A\left(\frac{(l+1)^2u_0^2}{u_1^4} I.T.[\psi_+] + (l+1)^2 u_1^2 I.E.[\psi_+,v_c,u_1,u_0]\right) & |q|=1 \end{cases}.
\end{aligned} \tag{3.164}$$

We bound $\|\psi_2\|_{L^2(\mathscr{I}^-)}$ as follows:

$$\begin{aligned}
\|\psi_2\|^2_{L^2(\mathscr{I}^-)} &\leq 2\|\psi-\psi_1\|^2_{L^2(\mathscr{I}^-)\cap\{v\leq v_c\}} + 2\|\psi_{RN}\|^2_{L^2(\mathscr{I}^-)\cap\{v\leq v_c\}} \\
&\leq 2(v_c - v_b(u_1))^2\|\partial_v\psi(u_1,v) - \partial_v\psi_{\mathscr{I}^-}(v)\|^2_{L^2(\{v\in[v_b(u_1),v_c]\})} \\
&\qquad + 2\|\psi\|^2_{L^2(\mathscr{I}^-)\cap\{v\leq v_b(u_1)\}} + 2I.E.[\psi_+,v_c,u_1,u_0] \\
&\leq 2(v_c-v_b(u_1))^2\left(\int_{\Sigma_{v_c}\cap\{u\leq u_1\}}\frac{(l+1)^4}{V}|\partial_u\psi|^2 du + (l+1)^6\int_{\Sigma_{u_1}\cap\{v\leq v_c\}}|\psi|^2 dv\right) \\
&\qquad + B\left(\int_{\Sigma_{v_c}\cap\{u\leq u_1\}}\frac{1}{V}|\partial_u\psi|^2 du + \int_{v=v_b(u_1)}^{v_c}\left||\psi(u_1,v)|^2 - |\psi_{\mathscr{I}^-}(v)|^2\right| dv\right) + 2I.E.[\psi_+,v_c,u_1,u_0] \\
&\leq A\Bigg((l+1)^6(v_c-v_b(u_1))^2\|\psi(u_1,v)\|^2_{L^2(\{v\in[v_b(u_1),v_c]\})} + \frac{(l+1)^4}{V(u_1,v_c)} I.E[\psi_+,v_c,u_1,u_0] \\
&\qquad + (v_c - v_b(u_1))\|\psi(u_1,v)-\psi_{\mathscr{I}^-}\|_{\dot{H}^1(\{v\in[v_b(u_1),v_c]\})} \\
&\qquad \left(\|\psi(u_1,v)\|_{L^2(\{v\in[v_b(u_1),v_c]\})} + \|\psi_{\mathscr{I}^-}(v)\|_{L^2(\{v\in[v_b(u_1),v_c]\})}\right)\Bigg) \\
&\leq A\left((l+1)^6(v_c-v_b(u_1))\|\psi(u_1,v)\|^2_{L^2(\{v\in[v_b(u_1),v_c]\})} + \frac{(l+1)^4}{V(u_1,v_c)} I.E[\psi_+,v_c,u_1,u_0]\right) \\
&\leq \begin{cases} A(l+1)^2\left(I.T.[\psi_+]e^{-2\kappa u_1} + I.E.[\psi_+,v_c,u_1,u_0]e^{\kappa u_1}\right) & |q|<1 \\ A(l+1)^2\left(\frac{I.T.[\psi_+]u_0^2}{u_1^5} + I.E.[\psi_+,v_c,u_1,u_0]u_1^2\right) & |q|=1 \end{cases}.
\end{aligned} \tag{3.165}$$

Finally, we consider $(1+M^2\sigma^2)\hat{\psi}_1$, for which we will use the following Lemma:

Lemma 3.4.1. *Let f be a smooth function supported in the interval $[0,\varepsilon]$, for $\varepsilon<1$. Then there exists a constant A such that:*

$$\left\|\frac{\sigma\hat{f}(\sigma)}{1+\sigma^2}\right\|^2_{L^2} \leq A\varepsilon\|f\|^2_{L^2}. \tag{3.166}$$

Proof. Let f_{-1} be defined by $\hat{f}_{-1}(\sigma) = \sigma(1+\sigma^2)^{-1}\hat{f}(\sigma)$. Then f_{-1} is an L^2 solution to the equation:

$$-f''_{-1}(x) + f_{-1}(x) = f'(x). \tag{3.167}$$

For $x<0$ and $x>\varepsilon$, we have that $f_{-1} = Ae^x + Be^{-x}$. In order for f_{-1} to be L^2, this means that

$$f_{-1} = \begin{cases} Ae^{x-\varepsilon} & x<0 \\ Be^{\varepsilon - x} & x>\varepsilon \end{cases}. \tag{3.168}$$

The solution to (3.167) in the interval $[0,\varepsilon]$ is therefore

$$f_{-1}(x) = Ae^{x-\varepsilon} - g(x), \tag{3.169}$$

where

$$g(x) = \int_{x_1=0}^{x} f(x_1)dx_1 + \int_{x_1=0}^{x}\int_{x_2=0}^{x_1}\int_{x_3=0}^{x_2} f(x_3)dx_3dx_2dx_1 + ..., \tag{3.170}$$

assuming such a sequence converges. To show that such a sequence converges, we write:

$$\begin{aligned} |g(x)| &\le \int_{x_1=0}^{x} |f(x_1)|dx_1 + \int_{x_1=0}^{x}\int_{x_2=0}^{x_1}\int_{x_3=0}^{x_2} |f(x_3)|dx_3dx_2dx_1 + ... \\ &\le \|1\|_{L^2([0,\varepsilon])}\|f\|_{L^2([0,\varepsilon])} + \int_{x_1=0}^{x}\int_{x_2=0}^{x_1} \|1\|_{L^2([0,\varepsilon])}\|f\|_{L^2([0,\varepsilon])}dx_2dx_1 + ... \\ &\le \sqrt{\varepsilon}\|f\|_{L^2([0,\varepsilon])}\left(1 + \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} + ...\right) \le \cosh(\varepsilon)\sqrt{\varepsilon}\|f\|_{L^2([0,\varepsilon])}. \end{aligned} \tag{3.171}$$

We can similarly bound the derivative of $g(x)$:

$$\begin{aligned} |g'(x)| &\le |f(x)|dx_2 + \int_{x_2=0}^{x}\int_{x_3=0}^{x_2} |f(x_3)|dx_3dx_2 + ... \\ &\le |f(x)| + \int_{x_2=0}^{x_1} \|1\|_{L^2([0,\varepsilon])}\|f\|_{L^2([0,\varepsilon])}dx_2 + ... \\ &\le |f(x)| + \sqrt{\varepsilon}\|f\|_{L^2([0,\varepsilon])}\left(\varepsilon + \frac{\varepsilon^3}{3!} + \frac{\varepsilon^5}{5!} + ...\right) \le |f(x)| + \sinh(\varepsilon)\sqrt{\varepsilon}\|f\|_{L^2([0,\varepsilon])}. \end{aligned} \tag{3.172}$$

We then need to consider the values of A and B which allow this function to be twice weakly differentiable:

$$f_{-1}(\varepsilon) = A + g(\varepsilon) = B \tag{3.173}$$

$$f'_{-1}(\varepsilon) = A + g'(\varepsilon) = -B. \tag{3.174}$$

Solving these gives

$$A = -\frac{g(\varepsilon) + g'(\varepsilon)}{2} \tag{3.175}$$

$$B = \frac{g(\varepsilon) - g'(\varepsilon)}{2}. \tag{3.176}$$

Then the L^2 norm of f_{-1} can be bounded by:

$$\begin{aligned} \|f_{-1}\|^2_{L^2(\mathbb{R})} &\le \int_{\varepsilon}^{\infty} |B|^2 e^{2(\varepsilon-x)}dx + 2\int_{-\infty}^{\varepsilon} |A|^2 e^{2(x-\varepsilon)}dx + 2\int_0^{\varepsilon} |g(x)|^2 dx \\ &\le A\left(|g(\varepsilon)|^2 + |g'(\varepsilon)|^2 + \int_0^{\varepsilon} |g(x)|^2 dx\right) \\ &\le A\left(\cosh(\varepsilon)^2 + \sinh(\varepsilon)^2 + \varepsilon\cosh(\varepsilon)^2\right)\varepsilon\|f\|^2_{L^2}, \end{aligned} \tag{3.177}$$

as required. □

We can apply this Lemma to $\sigma(1+M^2\sigma^2)^{-1}\hat{\psi}_1$ to get that

$$\left\|\frac{\sigma\hat{\psi}_1}{1+M^2\sigma^2}\right\|^2_{L^2(\mathscr{I}^-)} \le \frac{v_c - v_b(u_1)}{M}\|\psi_1\|^2_{L^2(\mathscr{I}^-)}. \tag{3.178}$$

We then use (3.22) to obtain

$$\|(1+M^2\sigma^2)\hat{\psi}_{RN}\|^2_{L^2(\mathscr{I}^-)} = \int_{-\infty}^{\infty}(1+M^2\sigma^2)^2|\tilde{R}_{\sigma,l,m}|^2|\hat{\psi}_+|^2 d\sigma \leq A\int_{-\infty}^{\infty}(1+M^2\sigma^2)^2\frac{(l+1)^2}{1+M^2\sigma^2}|\hat{\psi}_+|^2 d\sigma$$
$$\leq A\int_{-\infty}^{\infty}(l+1)^2(1+M^2\sigma^2)|\hat{\psi}_+|^2 d\sigma \leq A(l+1)^{-2}I.T.[\psi_+] \tag{3.179}$$

Substituting into (3.158) gives the required results.

For the result with sufficiently fast decay of ψ_+, we can use (3.107) to bound $\int_{v_b(u_1)}^{v_c}|\psi(u_1,v)|^2 dv$ more accurately. Setting $u_1 = (1-\frac{3}{4}\delta)u_1$, all $I.E.$ terms will decay sufficiently fast to obtain our result. □

3.4.6 Treatment of the I.E. Terms

In this section we show the arbitrary polynomial decay of the $I.E.$ terms, provided that $\hat{\psi}_+$ vanishes and has all derivatives vanishing at $\omega = 0$. This result has been largely done in the extremal ($|q| = 1$) case, in [3]. This gives our first Theorem:

Theorem 3.4.2. *[Decay of the I.E. terms in the $|q| = 1$ case] Let ψ_+ be a Schwartz function on the cylinder, with $\hat{\psi}_+$ compactly supported on $\sigma \geq 0$. Then for each n, there exists an $A_n(M,\psi_+)$ such that*

$$I.E.[\psi_+, v_c, u_1, u_0] \leq A_n(u_0 - u_1)^{-n}, \tag{3.180}$$

as $u_0 - v_c, u_1 - v_c \to \infty$, with $u_0 \geq u_1$. Here, I.E. is as defined in Theorem 3.4.1, in the case of an extremal ($|q| = 1$) RNOS model.

Proof. As $\hat{\phi}_+$ and all it's ω derivatives vanish at $\omega = 0$, then $\hat{\psi}_{-n} := \omega^{-n}\psi_+$ is also a Schwartz function. Instead of imposing ψ_+ as our radiation field on $\mathscr{I}^+$, we can use ψ_{-n}. The resulting solution has the property

$$\partial_{t^*}^n \psi_{-n} = \psi. \tag{3.181}$$

We then apply Theorem 4.2 (with u_0 as the origin) from [3] to ψ_{-n}, to see

$$\int_{\mathscr{H}^-}(1+(u-u_0)^2)^n|\partial_u\psi_{\mathscr{H}^-}|^2 + (1+(u-u_0)^2)^n|\mathring{\nabla}\psi_{\mathscr{H}^-}|^2\sin\theta d\theta d\varphi du \tag{3.182}$$
$$+\int_{\mathscr{I}^-}(1+(v-u_0-R)^2)^n|\partial_u\psi_{\mathscr{R}\mathscr{N}}|^2 + (1+(v-u_0-R)^2)^n|\mathring{\nabla}\psi_{RN}|^2\sin\theta d\theta d\varphi du$$
$$\leq A_n[\psi_+].$$

Restricting the integral to $u \leq u_1$, we can see that

$$I.E.[\psi_+, v_c, u_1, u_0] \leq A\int_{u=-\infty}^{u_1}(1+(u-u_0)^2)^{3/2}|\partial_u\psi_{\mathscr{H}^-}|^2 + (1+(u-u_0)^2)^{3/2}|\mathring{\nabla}\psi_{\mathscr{H}^-}|^2\sin\theta d\theta d\varphi du$$
$$+\int_{\mathscr{I}^-}(1+(v-u_0-R)^2)^{3/2}|\partial_u\psi_{\mathscr{R}\mathscr{N}}|^2 + (1+(v-u_0-R)^2)^{3/2}|\mathring{\nabla}\psi_{RN}|^2\sin\theta d\theta d\varphi du$$
$$\leq A_n[\psi_+](1+(u_1-u_0)^2)^{-n+3/2}, \tag{3.183}$$

giving our result. □

We now look to extend this result to the sub-extremal case. The following section will closely follow that of the extremal case [3].

The next ingredient needed for the r^{*p} method is integrated local energy decay, or *ILED*. This will be done in a manner similar to [13].

Proposition 3.4.7 (ILED for sub-extremal Reissner–Nordström)**.** *Let ϕ be a solution of* (2.1) *on a sub-extremal* ($|q| < 1$) *Reissner–Nordström background. Let t_0 be a fixed value of t, and let R be a large fixed constant. Then* *there exists a constant $A = A(M,q,R,n)$ such that*

$$\int_{-\infty}^{t_0}\left(\int_{\Sigma_t\cap\{|r^*|\le R\}}|\partial_r\phi|^2\right)dt \le A\int_{\bar{\Sigma}_{t_0,R}} dn(J^{\partial_t}) \le A\int_{\Sigma_{t_0}} -dt(J^{\partial_t}) \tag{3.184}$$

$$\int_{-\infty}^{t_0}\left(\int_{\Sigma_t\cap\{|r^*|\le R\}}-dt(J^{\partial_t})\right)dt \le A\sum_{|\alpha|+j\le 1}\int_{\bar{\Sigma}_{t_0,R}} dn(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi]) \le A\sum_{|\alpha|+j\le 1}\int_{\Sigma_{t_0}} -dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi]). \tag{3.185}$$

Proof. Consider Reisnner–Nordström spacetime in r^*, t, θ, φ coordinates. For ease of writing, we will denote

$$D(r^*) = 1 - \frac{2M}{r} + \frac{M^2q^2}{r^2}. \tag{3.186}$$

As done so far in this chapter, we will restrict to a spherical harmonic. We will first consider the case $l \ge 1$. We choose $\omega = h'/4 + hD/2r$, $h'(r^*) = (A^2 + (r^* - R)^2)^{-1}$, and consider divergence theorem applied to

$$\begin{aligned} J^{\mathbf{X}} &:= J^{X,\omega} + \frac{h'}{D}\beta\phi^2\partial_{r^*} \\ \beta &= \frac{D}{r} - \frac{r^* - R}{A^2 + (r^* - R)^2}, \end{aligned} \tag{3.187}$$

for A and R yet to be chosen.

Note that the flux of this current through any $t = const$ surface is bounded by the $\partial_{r^*}\phi$ and $\partial_t\phi$ terms of the T energy. The bulk term of this is given by

$$K^{\mathbf{X}} = \nabla^\nu J^{\mathbf{X}}_\nu = \frac{h'}{D}(\partial_r\phi + \beta\phi)^2 + \left(\frac{(r^*-R)^2 - A^2}{2D((r^*-R)^2 + A^2)^3} + \left(\frac{l(l+1)}{r^2}\left(\frac{D}{r} - \frac{D'}{2D}\right) + \frac{D'}{2r^2} - \frac{D''}{2Dr}\right)h\right)\phi^2. \tag{3.188}$$

Calculating the coefficient of $h\phi^2$ gives us

$$\begin{aligned} &\frac{l(l+1)}{r^2}\left(\frac{D}{r} - \frac{D'}{2D}\right) + \frac{D'}{2r^2} - \frac{D''}{2Dr} \\ &\quad = \frac{M^4}{r^7}\left(l(l+1)\left(\frac{r}{M}\right)^4 - 3(l(l+1)-1)\left(\frac{r}{M}\right)^3\right. \\ &\qquad \left. + (2q^2l(l+1) - 4q^2 - 8)\left(\frac{r}{M}\right)^2 + 15q^2\left(\frac{r}{M}\right) - 6q^4\right) \\ &\quad = \frac{M^4}{r^7}\left((x-1)(x-2)(l(l+1)x^2 + 3(x-1)) - (1-q^2)((2l(l+1)-4)x^2 + 15x - 6(1+q^2))\right) \\ &\quad = l(l+1)(x-3)x^3 + (3x-8)x^2 + q^2((2l(l+1)-4)x^2 + 15x - 6q^2), \end{aligned} \tag{3.189}$$

where $x = r/M$. Searching for roots of this, we can see there is a root at $r = M$, but this is strictly less than 0 for $r < 2M$, and strictly greater than 0 for $r > 3M$. In this interval, we consider the function

$$f(x) = l(l+1)x - 3(l(l+1)-1) + (2q^2l(l+1) - 4q^2 - 8)x^{-1} + 15q^2x^{-2} - 6q^4x^{-3} \tag{3.190}$$

$$f'(x) = l(l+1)(1 - 2q^2x^{-2}) + (8 + 4q^2)x^{-2} - 30q^2x^{-3} + 18q^4x^{-4} > 0, \tag{3.191}$$

for $x > 2$. Therefore the coefficient of $h\phi^2$ in (3.188) has exactly one root, in a bounded region of r^*. We label this point r^*_0, and we let

$$h(r^*_0) = 0. \tag{3.192}$$

As h has a positive gradient, this means that $f(x)h \ge 0$, with a single quadratic root at r^*_0. Provided $R > r^*_0$, we also know $h > \pi/2A$ for sufficiently large values of r^*. Thus to ensure $K^{\mathbf{X}}$ is positive definite, it is sufficient to show that R and A can be chosen such that

$$\frac{(r^*-R)^2-A^2}{2D((r^*-R)^2+A^2)^3}+\frac{M}{r^4}f\left(\frac{r}{M}\right)h>0. \tag{3.193}$$

We only need to consider the region $|r^*-R|<A$. By choosing $R-r_0^*-A>>M$, we can ensure that in this region, $D>1-\varepsilon$, $\frac{M}{r}f\left(\frac{r}{M}\right)\geq l(l+1)(1-\varepsilon)$, and $r\leq r^*(1-\varepsilon)$. Thus it is sufficient to choose R and A, with $R-r_0^*-A>>M$, such that

$$l(l+1)\pi(1-\varepsilon)-\frac{A\left(A^2-(r^*-R)^2\right)r^{*3}}{((r^*-R)^2+A^2)^3}>0. \tag{3.194}$$

Let $y=\frac{r^*-R}{A}$, then we are looking for the maximum of

$$\frac{(1-y^2)(y+\frac{R}{A})^3}{(1+y^2)^3}. \tag{3.195}$$

If we choose $R-r_0^*=1.001A$, and choose $0.001A>>M$, then

$$\sup_{-1\leq y\leq 1}\frac{(1-y^2)(y+1.001)^3}{(1+y^2)^3}<\frac{\pi}{2}, \tag{3.196}$$

and we have $K^{\mathbf{X}}$ is positive definite.

$$\begin{aligned}K^{\mathbf{X}}&\geq\frac{\varepsilon|\partial_r\phi+\beta\phi|^2}{D(M^2+r^{*2})}+\varepsilon\left(\frac{l(l+1)D\tanh\left(\frac{r^*-r_0^*}{M}\right)^2}{r^3}+\frac{1}{D(M^2+r^{*2})^2}\right)|\phi|^2\\&\geq\frac{\varepsilon|\partial_r\phi|^2}{D(M^2+r^{*2})}+\varepsilon\left(\frac{l(l+1)D\tanh\left(\frac{r^*-r_0^*}{M}\right)^2}{r^3}+\frac{1}{D(M^2+r^{*2})^2}\right)|\phi|^2.\end{aligned} \tag{3.197}$$

To bound the T-energy locally, we can thus consider

$$A\sum_{|\alpha|+j\leq 1}K^{\mathbf{X}}[\partial_t^j\Omega^\alpha\phi]\geq\frac{-dt(J^{\partial_t})}{M^2+r^{*2}} \tag{3.198}$$

$$A\sum_{|\alpha|+j\leq 1}K^{\mathbf{X}}[\partial_t^j\Omega^\alpha\phi]\geq A(-dt(J^{\partial_t}))\qquad\forall|r^*|\leq R, \tag{3.199}$$

where Ω are the angular Killing Fields, as given by (1.26).

For the $l=0$ case, we again follow the example of [13] and take $X=\partial_{r^*}$. Given that all angular derivatives vanish, applying divergence theorem to J^X in the interval $r^*\in(-\infty,r_0^*)$, we obtain

$$\begin{aligned}\int_{-\infty}^{t_0}(\partial_t\phi(r_0^*))^2+(\partial_{r^*}\phi(r_0^*))^2r^2\sin\theta d\theta d\varphi dt+\int_{r^*=-\infty}^{r_0^*}\frac{2D}{r}\int_{-\infty}^{t_0}\left(-(\partial_t\phi)^2+(\partial_{r^*}\phi)^2\right)r^2\sin\theta d\theta d\varphi dr^*dt\\\leq 4T\text{-energy}(\Sigma_{t_0^*}).\end{aligned} \tag{3.200}$$

Let

$$F(r^*):=\int_{r^*=-\infty}^{r_0^*}\frac{2D}{r}\int_{-\infty}^{t_0}(\partial_t\phi)^2r^2\sin\theta d\theta d\varphi dr^*dt. \tag{3.201}$$

Then (3.200) implies

$$F'(r^*)\leq\frac{2D}{r}F(r^*)+\frac{8D}{r}T\text{-energy}(\Sigma_{t_0^*}). \tag{3.202}$$

Noting that $\int_{r^*=-\infty}^{r_0^*} \frac{2D}{r} dr^* = 2\log\left(\frac{r}{r^+}\right)$, an application of Gronwall's inequality yields

$$F(r^*) \leq A\left(\frac{r^2}{r_+^2}\right) T\text{-energy}(\Sigma_{t_0^*}). \tag{3.203}$$

By applying this to (3.200), we can obtain

$$\left(\frac{r_+^2}{r_0^2}\right)\int_{t_0}^{\infty}\int_{r^*=-\infty}^{r_0^*} \frac{2D}{r}\left((\partial_t\phi)^2 + (\partial_{r^*}\phi)^2\right)\sin\theta d\theta d\varphi dr^* dt \leq AT\text{-energy} \tag{3.204}$$

$$\int_{-\infty}^{t_0}\left(\int_{\Sigma_t\cap\{|r^*|\leq R\}} -dt(J^{\partial_t})\right) dt \leq AT\text{-energy} \tag{3.205}$$

We now have the result for all l using Σ_{t_0}.

Once we note that the region $\{t \leq t_0, |r^*| \leq R\}$ is entirely in the domain of dependence of $\bar{\Sigma}_{t_0,R}$, we can consider the alternative solution, $\tilde{\phi}$, given by the data of ϕ on $\bar{\Sigma}_{t_0,R}$, but vanishing on $\mathscr{H}^-$ and $\mathscr{I}^-$ to the future of $\bar{\Sigma}_{t_0,R}$. We evolve this forward to Σ_{t_0}, we can apply the above result. As $\tilde{\phi} = \phi$ to the past of $\bar{\Sigma}_{t_0,R}$, we have the result. □

Remark 3.4.6 (Degeneracy at the Photon Sphere)**.** *For the $l \geq 1$ case, as $l \to \infty$, the root of the h function chosen tends towards the root of*

$$1 - \frac{3M}{r} + \frac{2M^2q^2}{r^2} = 0, \tag{3.206}$$

known as the photon sphere, $r = r_p$. If we do not require control of the T-energy at this particular value, then we do not need to include angular derivatives

$$\int_{t_0}^{\infty}\left(\int_{\Sigma_t\cap\{\varepsilon\leq|r^*-r_p^*|\leq R\}} -dt(J^{\partial_t})\right) dt \leq A\sum_{j=0}^{1}\int_{\Sigma_{t_0}} -dt(J^{\partial_t}[\partial_t^j\phi]). \tag{3.207}$$

Remark 3.4.7 (Forward and higher order ILED)**.** *By sending $t \to -t$, Proposition 3.4.7 immediately gives us the result in the forward direction:*

$$\int_{t_0}^{\infty}\left(\int_{\Sigma_t\cap\{|r^*|\leq R\}} -dt(J^{\partial_t})\right) dt \leq A\sum_{j+|\alpha|\leq 1}\int_{\Sigma_{t_0}} -dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi]). \tag{3.208}$$

We can also apply the Proposition 3.4.7 to $\partial_t^j\Omega^\alpha\phi$ to obtain

$$\int_{t_0}^{\infty}\left(\int_{\Sigma_t\cap\{|r^*|\leq R\}} |\nabla^n\phi|^2\right) dt \leq A\sum_{j+|\alpha|\leq n}\int_{\Sigma_{t_0}} -dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi]), \tag{3.209}$$

where we have rewritten terms in $\nabla^n\phi$ involving more than one r^* derivative using (2.1).

Proposition 3.4.8 (Boundedness of r^* Weighted Energy)**.** *Let ψ_+ be a Schwartz function. Let ψ be the solution to* (3.33) *on a sub-extremal Reissner–Nordström background, with radiation field on $\mathscr{I}^+$ equal to ψ_+, and which vanishes on $\mathscr{H}^+$. Let R be a constant, and let t_0 be a fixed value of t. Then for each $n \in \mathbb{N}_0$, we have the following bounds:*

$$\sum_{j+|\alpha|\leq n}\int_{\bar{\Sigma}_{t_0,R}} (M^{2+2j} + |r^*|^{2+2j}) dn(J^{\partial_t}[\Omega^\alpha\partial_t^j\phi]) \leq A_n \sum_{1\leq j+|m|\leq n+1}\int_{-\infty}^{\infty}\left(M^{2j} + u^{2j}\right)(l+1)^{2m}\left|\partial_u^j\psi_+\right|^2 du, \tag{3.210}$$

where $A_n = a_n(M, q, t_0, R, n)$.

Proof. We start by bounding an r^p weighted norm on $\Sigma_{u_0} \cap r^* \leq -R$ for some $u_0 \in \mathbb{R}$ and R large.

$$\begin{aligned}\int_{\Sigma_{u_0}\cap r^*\leq -R}(-R-r^*)^p|\partial_v\psi|^2dv &= \int_{u\geq u_0, r^*\leq -R} -p(-r^*-R)^{p-1}|\partial_v\psi|^2+(-R-r^*)^pV\partial_v(|\psi|^2)dudv \\ &\leq -\int_{u\geq u_0, r^*\leq -R}\partial_v((-R-r^*)^pV)|\psi|^2dudv \\ &\leq A\int_{u=u_0}^{\infty}\int_{\Sigma_v}V|\psi|^2dvdu \\ &\leq A\int_{u=u_0}^{\infty}\int_{u'=u}^{\infty}|\partial_u\psi_+|^2du'du = A\int_{u=u_0}^{\infty}(u-u_0)|\partial_u\psi_+|^2du.\end{aligned} \tag{3.211}$$

Here we have used T energy boundedness to reach the last line, along with an explicit calculation to show that $-\partial_v((-R-r^*)^pV) \leq AV$. A is a constant which depends on M, q, and the choice of R. Note this calculation applies for all $p \in \mathbb{N}$ for sub-extremal Reissner–Nordström, but in the extremal case this only applies up to $p = 2$. By applying this result to $\partial_t^j\Omega^\alpha\phi$, we obtain the required bound for $\bar{\Sigma}_{t_0,R} \cap \{r^* \leq -R\}$.

For $r^* \in [-R, R]$, we note that T-energy boundedness of $\partial_t^j\Omega^\alpha\phi$ is sufficient for our result, as the constant A_n may depend on our choice of R.

For the equivalent result on $\Sigma_{v_0} \cap r^* \geq R$, a similar approach does not work, as the T energy on Σ_v does not approach 0 as $v \to \infty$. Instead, we will make use of the vector field multiplier $u^2\partial_u + v^2\partial_v$. Let $u_0 \leq v_0 - R$. This will closely follow the proof of Proposition 8.1 in [3].

$$\begin{aligned}\int_{\Sigma_{v_0}\cap u\leq v_0-R} u^2|\partial_u\psi|^2+v^2V|\psi|^2du + \int_{\Sigma_{u_0}\cap\{v\geq v_0\}} v^2|\partial_v\psi|^2+u^2V|\psi|^2du \\ = \int_{\mathscr{I}^+\cap\{u\leq v_0-R\}} u^2|\partial_u\psi|^2+v^2V|\psi|^2du \\ + \int_{\Sigma_{u=v_0-R}\cap\{v\geq v_0\}} v^2|\partial_v\psi|^2+u^2V|\psi|^2du \\ + \int_{u\in[u_0,v_0-R], v\geq v_0}\left(\partial_v(v^2V)+\partial_u(u^2V)\right)|\psi|^2dudv.\end{aligned} \tag{3.212}$$

We then note

$$\partial_v(v^2V)+\partial_u(u^2V) = 2tV+tr^*V' = t(2V+r^*V') \leq \begin{cases}\frac{A|t|}{r^3} \leq Ar^{-2} & l=0 \\ \frac{AV|t|\log\left(\frac{r}{M}\right)}{r} \leq AV\log\left(\frac{r}{M}\right) & l\neq 0\end{cases}, \tag{3.213}$$

using that $|t| \leq r^* + \max(v_0 - R, -v_0)$ in the region we are considering. Here A depends on the choice of v_0 and R. We can then take a supremum of (3.212) over $u_0 \leq v_0 - R$ and $v \geq v_0$ to obtain

$$\begin{aligned}\sup_{v\geq v_0}\int_{\Sigma_v\cap u\leq v_0-R} u^2|\partial_u\psi|^2+v^2V|\psi|^2du + \sup_{u\leq v_0-R}\int_{\Sigma_u\cap\{v\geq v_0\}} v^2|\partial_v\psi|^2+u^2V|\psi|^2dv \\ \leq \int_{\mathscr{I}^+\cap\{u\leq v_0-R\}} u^2|\partial_u\psi|^2+v^2V|\psi|^2du \\ + \int_{\Sigma_{u=v_0-R}\cap\{v\geq v_0\}} v^2|\partial_v\psi|^2+u^2V|\psi|^2dv \\ + A\int_{u\leq v_0-R, v\geq v_0}\left(V\log\left(\frac{r}{M}\right)+r^{-2}\right)|\psi|^2dudv.\end{aligned} \tag{3.214}$$

We can bound the final integral using the following:

$$\begin{aligned}
\int_{u=-\infty}^{v_0-R}\int_{v=v_0}^{\infty}\left(V\log\left(\frac{r}{M}\right)+r^{-2}\right)|\psi|^2dvdu &\leq \int_{u=-\infty}^{v_0-R}\int_{v=v_0}^{\infty}\left(V\log\left(\frac{-u}{M}\right)+V\log\left(\frac{v}{M}\right)+u^{-2}\right)|\psi|^2dvdu \\
&\leq A\int_{u=-\infty}^{v_0-R}u^{-2}\log\left(\frac{-u}{M}\right)\int_{v=v_0}^{\infty}u^2V|\psi|^2dvdu \\
&\quad +A\int_{v=v_0}^{\infty}v^{-2}\log\left(\frac{v}{M}\right)\int_{u=-\infty}^{v_0-R}v^2V|\psi|^2dudv \\
&\quad +A\int_{u=-\infty}^{v_0-R}u^{-2}\int_{v=v_0}^{\infty}|\psi|^2dudv \\
&\leq \varepsilon\sup_{u\leq v_0-R}\int_{\Sigma_u\cap\{v\geq v_0\}}v^2|\partial_v\psi|^2+u^2V|\psi|^2du \\
&\quad +\varepsilon\sup_{v\geq v_0}\int_{\Sigma_v\cap u\leq v_0-R}u^2|\partial_u\psi|^2+v^2V|\psi|^2du \\
&\quad +\varepsilon\sup_{u\leq v_0-R}\int_{\Sigma_u\cap\{v\geq v_0\}}|\psi|^2dv,
\end{aligned} \tag{3.215}$$

where v_0 and R are sufficiently large.

We can then apply Hardy's inequality to $\chi(1+R/M)\psi$ (χ as in Proposition 3.4.2) to get

$$\sup_{u\leq v_0-R}\int_{\Sigma_u\cap\{v\geq v_0\}}|\psi|^2dv\leq A\sup_{u\leq v_0-R}\int_{\Sigma_u\cap\{v\geq v_0\}}V|\psi|^2dv+\sup_{u\leq v_0-R}\int_{\Sigma_u\cap\{v\geq v_0\}}v^2|\partial_v\psi|^2dv. \tag{3.216}$$

We can then rearrange (3.214) to see

$$\begin{aligned}
\sup_{v\geq v_0}\int_{\Sigma_v\cap u\leq v_0-R}u^2|\partial_u\psi|^2+v^2V|\psi|^2du+\sup_{u\leq v_0-R}\int_{\Sigma_u\cap\{v\geq v_0\}}v^2|\partial_v\psi|^2+u^2V|\psi|^2dv \\
\leq A\int_{\mathscr{I}^+\cap\{u\leq v_0-R\}}u^2|\partial_u\psi|^2+v^2V|\psi|^2du \\
+A\int_{\Sigma_{u=v_0-R}\cap\{v\geq v_0\}}v^2|\partial_v\psi|^2+u^2V|\psi|^2dv.
\end{aligned} \tag{3.217}$$

By taking an appropriate limit of this, we can see that

$$\begin{aligned}
\int_{\Sigma_{v_0}\cap u\leq v_0-R}u^2|\partial_u\psi|^2+v^2V|\psi|^2du+\int_{\mathscr{I}^-\cap\{v\geq v_0\}}v^2|\partial_v\psi|^2+u^2V|\psi|^2dv \\
\leq A\int_{\mathscr{I}^+\cap\{u\leq v_0-R\}}u^2|\partial_u\psi|^2+v^2V|\psi|^2du \\
+A\int_{\Sigma_{u=v_0-R}\cap\{v\geq v_0\}}v^2|\partial_v\psi|^2+u^2V|\psi|^2dv.
\end{aligned} \tag{3.218}$$

We can also consider a time reversal of this statement to get

$$\begin{aligned}
\int_{\mathscr{I}^+\cap\{u\leq v_0-R\}}u^2|\partial_u\psi|^2+v^2V|\psi|^2du+\int_{\Sigma_{u=v_0-R}\cap\{v\geq v_0\}}v^2|\partial_v\psi|^2+u^2V|\psi|^2dv \\
\leq A\int_{\Sigma_{v_0}\cap u\leq v_0-R}u^2|\partial_u\psi|^2+v^2V|\psi|^2du \\
+A\int_{\mathscr{I}^-\cap\{v\geq v_0\}}v^2|\partial_v\psi|^2+u^2V|\psi|^2dv.
\end{aligned} \tag{3.219}$$

In order to add more u and v weighting to this, we commute with the vector field $S=u\partial-U+v\partial_v$.

$$(\partial_u\partial_v+V)S(f)=S[(\partial_u\partial_v+V)f]+(2V-r^*\partial_{r^*}V)f+2(\partial_u\partial_v+V)f. \tag{3.220}$$

Thus an easy induction argument gives

$$|(\partial_u\partial_v+V)S^n\psi| \le A|V-r^*\partial_{r^*}V|\sum_{k=0}^{n-1}S^k\psi, \tag{3.221}$$

noting that

$$|\partial_{r^*}^n(V-r^*\partial_{r^*}V)| \le A|V-r^*\partial_{r^*}V| \le \frac{A(l+1)^2\log\left(\frac{r}{M}\right)}{r^3}. \tag{3.222}$$

Repeating (3.214), but applied to $S^n\psi$, we obtain

$$\begin{aligned}
F_n := \sup_{v\ge v_0}\int_{\Sigma_v\cap u\le v_0-R} u^2|\partial_u S^n\psi|^2+v^2V|S^n\psi|^2du + \sup_{u\le v_0-R}\int_{\Sigma_u\cap\{v\ge v_0\}} v^2|\partial_v S^n\psi|^2+u^2V|S^n\psi|^2du \\
\le \int_{\mathscr{I}^+\cap\{u\le v_0-R\}} u^2|\partial_u S^n\psi|^2+v^2V|S^n\psi|^2du + \int_{\Sigma_{u=v_0-R}\cap\{v\ge v_0\}} v^2|\partial_v S^n\psi|^2+u^2V|S^n\psi|^2du \\
+A\int_{u\le v_0-R,v\ge v_0}\left(V\log\left(\frac{r}{M}\right)+r^{-2}\right)|S^n\psi|^2dudv \\
+A\sum_{k=0}^{n-1}\int_{u\le v_0-R,v\ge v_0}\frac{(l+1)^2\log\left(\frac{r}{M}\right)}{r^3}|S^k\psi|\left|u^2\partial_u S^n\psi+v^2\partial_v S^n\psi\right|dudv \\
\le \int_{\mathscr{I}^+\cap\{u\le v_0-R\}} u^2|\partial_u S^n\psi|^2+v^2V|S^n\psi|^2du + \int_{\Sigma_{u=v_0-R}\cap\{v\ge v_0\}} v^2|\partial_v S^n\psi|^2+u^2V|S^n\psi|^2du \\
+A\varepsilon F_n + AF_n^{1/2}\sum_{k=0}^{n-1}\left(\int_{u=-\infty}^{v_0-R}\frac{(l+1)^4\log\left(\frac{r(u,v_0)}{M}\right)^2}{u^2V(u,v_0)r(u,v_0)^6}du\right)^{1/2}F_k^{1/2} \\
\le A\int_{\mathscr{I}^+\cap\{u\le v_0-R\}} u^2|\partial_u S^n\psi|^2+v^2V|S^n\psi|^2du + A\int_{\Sigma_{u=v_0-R}\cap\{v\ge v_0\}} v^2|\partial_v S^n\psi|^2+u^2V|S^n\psi|^2du \\
+AF_n^{1/2}\sum_{k=0}^{n-1}\left(\int_{u=-\infty}^{v_0-R}\frac{(l+1)^4\log\left(\frac{r(u,v_0)}{M}\right)^2}{u^2V(u,v_0)r(u,v_0)^6}du\right)^{1/2}F_k^{1/2} \\
\le A\int_{\mathscr{I}^+\cap\{u\le v_0-R\}} u^2|\partial_u S^n\psi|^2+v^2V|S^n\psi|^2du + A\int_{\Sigma_{u=v_0-R}\cap\{v\ge v_0\}} v^2|\partial_v S^n\psi|^2+u^2V|S^n\psi|^2du \\
+A\varepsilon(l+1)F_n^{1/2}\sum_{k=0}^{n-1}F_k^{1/2} \\
\le A\int_{\mathscr{I}^+\cap\{u\le v_0-R\}} u^2|\partial_u S^n\psi|^2+v^2V|S^n\psi|^2du + A\int_{\Sigma_{u=v_0-R}\cap\{v\ge v_0\}} v^2|\partial_v S^n\psi|^2+u^2V|S^n\psi|^2du \\
+A(l+1)^2\sum_{k=0}^{n-1}F_k.
\end{aligned} \tag{3.223}$$

As F_0 is bounded by (3.218), we can inductively obtain

$$\begin{aligned}
\sum_{k+m\le n}\Bigg(\int_{\Sigma_{v_0}\cap u\le v_0-R} u^2(l+1)^{2m}|\partial_u(S^k\psi)|^2+v^2V(l+1)^m|S^k\psi|^2du \\
+\int_{\mathscr{I}^-\cap\{v\ge v_0\}} v^2(l+1)^{2m}|\partial_v((v\partial_v)^k\psi)|^2+u^2V(l+1)^{2m}|((v\partial_v)^k\psi)|^2dv\Bigg) \\
\le A\sum_{k+m\le n}\Bigg(\int_{\mathscr{I}^+\cap\{u\le v_0-R\}} u^2(l+1)^{2m}|\partial_u((u\partial_u)^k\psi)|^2+v^2V(l+1)^{2m}|((u\partial_u)^k\psi)|^2du \\
+\int_{\Sigma_{u=v_0-R}\cap\{v\ge v_0\}} v^2(l+1)^{2m}|\partial_v(S^k\psi)|^2+u^2V(l+1)^{2m}|(S^k\psi)|^2dv\Bigg),
\end{aligned} \tag{3.224}$$

along with the time reversed result

$$\sum_{k+m\leq n}\left(\int_{\mathscr{I}^+\cap\{u\leq v_0-R\}} u^2(l+1)^{2m}|\partial_u((u\partial_u)^k\psi)|^2+v^2V(l+1)^{2m}|((u\partial_u)^k\psi)|^2du\right.$$
$$\left.+\int_{\Sigma_{u=v_0-R}\cap\{v\geq v_0\}} v^2(l+1)^{2m}|\partial_v(S^k\psi)|^2+u^2V(l+1)^{2m}|(S^k\psi)|^2dv\right)$$
$$\leq A\sum_{k+m\leq n}\left(\int_{\Sigma_{v_0}\cap u\leq v_0-R} u^2(l+1)^{2m}|\partial_u(S^k\psi)|^2+v^2V(l+1)^{2m}|(S^k\psi)|^2du\right. \tag{3.225}$$
$$\left.+\int_{\mathscr{I}^-\cap\{v\geq v_0\}} v^2(l+1)^{2m}|\partial_v((v\partial_v)^k\psi)|^2+u^2V(l+1)^{2m}|((v\partial_v)^k\psi)|^2dv\right).$$

All that is now left for the result is to bound

$$\sum_{k+m\leq n}\int_{\Sigma_{u=v_0-R}\cap\{v\geq v_0\}} v^2(l+1)^{2m}|\partial_v(S^k\psi)|^2+u^2V(l+1)^{2m}|(S^k\psi)|^2dv$$
$$\leq\sum_{k+m+j\leq n}\int_{\Sigma_{u=v_0-R}\cap\{v\geq v_0\}} v^{2+2k}(l+1)^{2m}|\partial_v^{k+1}\partial_t^j\psi|^2 \tag{3.226}$$
$$+v^{2k}V(l+1)^{2m}|(\partial_v^k\partial_t^j\psi)|^2dv,$$

for fixed and arbitrarily large R, v_0. We have used (3.33) to remove any $\partial_u\partial_v$ derivatives, and have replaced any ∂_u derivatives with $\partial_t+\partial_v$ derivatives. As ∂_t and Ω are Killing fields, it is sufficient to bound

$$\int_{\Sigma_{u=v_0-R}\cap\{v\geq v_0\}} v^{2k+2}|\partial_v^{k+1}\psi|^2dv\leq A\int_{\Sigma_{u=v_0-R}}\chi\left(\frac{r-R}{M}-1\right)r^{2(k+1)}|\partial_v^{k+1}\psi|^2dv, \tag{3.227}$$

for $k\geq 0$.

We can immediately apply Proposition 3.4.2 (with time reversed) to obtain the $k=0$ case

$$\int_{\Sigma_{u=v_0-R}}\chi r^2|\partial_v\psi|^2du\leq A\int_{\mathscr{I}^+}(M^2+u^2)|\partial_u\psi_+|^2+l(l+1)|\psi_+|^2du. \tag{3.228}$$

Here the constant A depends on choice of v_0 and R. We would now like to generalise this to the following result (closely based on Proposition 7.7 in [3]).

$$\int_{\Sigma_{u=v_0-R}}\chi\left(\frac{r-R}{M}-1\right)r^{2k}|\partial_v^k\psi|^2dv\leq A\sum_{1\leq m+j\leq k}\int_{u=v_0-R}^{\infty}\left(M^{2m}+(u-u_R)^{2m}\right)(l+1)^{2j}|\partial_u^m\psi_+|^2du, \tag{3.229}$$

where A depends on M,q,n,R. From here, we will denote $v_0-R=u_R$.

We will prove this inductively. First, we consider commuting (3.33) with ∂_v to obtain

$$\partial_u\partial_v(\partial_v^n\psi)+V\partial_v^n\psi=-\partial_v^n(V\psi)+V\partial_v^n\psi=-\sum_{j=0}^{n-1}\binom{n}{j}\partial_{r^*}^{n-j}V\partial_v^j\psi\leq A\sum_{j=0}^{n-1}\frac{(l+1)^2}{r^{2+n-j}}|\partial_v^j\psi|. \tag{3.230}$$

We then look at applying this to the following generalisation of the right hand side of (3.229)

$$\begin{aligned}\int_{\Sigma_{u=u_R}}\chi r^p|\partial_v^k\psi|^2dv=&\int_{u\geq u_R}D(\chi' r^p+p\chi r^{p-1})|\partial_v^k\psi|^2+2\mathbb{R}\left(\chi r^p\partial_v^k\bar{\psi}\sum_{j=0}^{k-1}\binom{k}{j}\partial_v^{k-j}V\partial_v^j\psi\right)\\&-D\partial_{r^*}\left(\chi r^pV\right)|\partial_v^{k-1}\psi|dudv+\int_{\mathscr{I}^+}r^{p-2}l(l+1)|\partial_v^{k-1}\psi_+|^2du \qquad (3.231)\\&\leq A\int_{u\geq u_R,r^*\in[R,R+M]}\sum_{m+j\leq k-1}-dt((l+1)^{2j}J^{\partial_t}[\partial_t^m\psi])dudv+A\int_{u\geq u_R}\chi\sum_{j=0}^{k}\frac{(l+1)^{2j}}{r^{1+2j-p}}|\partial_v^{k-j}\psi|^2dudv\\&\leq A\int_{u=u_R}\sum_{m+j\leq k-1}(l+1)^{2j}|\partial_u^{m+1}\psi_+|^2du+A\int_{u\geq u_R}\chi\sum_{j=0}^{k}\frac{(l+1)^{2j}}{r^{1+2j-p}}|\partial_v^{k-j}\psi|^2dudv,\end{aligned}$$

where we have used Proposition 3.4.7.

For our induction argument, we will assume we have proved (3.229) for $k\leq n$, where $n\geq 1$. We first consider 3.231, with $k=n+1$ and $p=1+2n$.

$$\begin{aligned}\int_{\Sigma_{u=u_R}}\chi r^{1+2n}|\partial_v^{n+1}\psi|^2dv\leq&\int_{u=u_R}\sum_{m+j\leq n}(l+1)^{2j}|\partial_u^{m+1}\psi_+|^2du+\int_{u\geq u_R}\chi\sum_{j=0}^{n+1}\frac{(l+1)^{2j}}{r^{2(j-n)}}|\partial_v^{1+n-j}\psi|^2dudv \qquad (3.232)\\&\leq A\int_{u=u_R}\sum_{m+j\leq n}(l+1)^{2j}|\partial_u^{m+1}\psi_+|^2du\\&\quad+A\int_{u\geq u_R}\chi\sum_{j=0}^{n}\frac{(l+1)^{2j}}{r^{2(j-n)}}|\partial_v^{1+n-j}\psi|^2+(l+1)^{2n}\chi|\partial_v\psi|^2dudv\\&\leq A\int_{u=u_R}\sum_{m+j\leq n}(l+1)^{2j}|\partial_u^{m+1}\psi_+|^2du+A\int_{u\geq u_R}\chi\sum_{j=0}^{n}(l+1)^{2(n-j)}r^{2j}|\partial_v^j(\partial_t+\partial_u)\psi|^2dudv\\&\leq A\int_{u=u_R}\sum_{m+j\leq n}(l+1)^{2j}|\partial_u^{m+1}\psi_+|^2du+A\int_{u\geq u_R}\chi\sum_{j=0}^{n}(l+1)^{2(n-j)}r^{2j}|\partial_v^{j-1}(V\psi)|^2dudv\\&\quad+A\int_{u=u_R}\sum_{1\leq m+j\leq n}\int_{u'=u}^{\infty}\left(M^{2m}+(u-u_R)^{2m}\right)(l+1)^{2j}|\partial_u^m\partial_t\psi_+|^2du'du\\&\leq A\sum_{0\leq m+j\leq n}\int_{u=u_R}^{\infty}\left(M^{2m}+(u-u_R)^{2m+1}\right)(l+1)^{2j}|\partial_u^{m+1}\psi_+|^2du\\&\quad+A\int_{u\geq u_R}\chi\sum_{j=0}^{n}(l+1)^{2(n-j)+2}r^{2j}|\partial_v^{j-1}(V\psi)|^2dudv\\&\leq A\int_{u=u_R}\sum_{1\leq m+j\leq n+1}(l+1)^{2j}|\partial_u^m\psi_+|^2du,\end{aligned}$$

where we have used that ∂_t is a Killing field along with our induction hypothesis in the final three lines.

We then proceed to prove (3.229):

$$\begin{aligned}\int_{\Sigma_{u=u_R}}\chi r^{2+2n}|\partial_v^{n+1}\psi|^2dv\leq& A\int_{u=u_R}\sum_{m+j\leq n}(l+1)^{2j}|\partial_u^{m+1}\psi_+|^2du \qquad (3.233)\\&+A\int_{u\geq u_R}\chi\sum_{j=0}^{n}\frac{(l+1)^{2j}}{r^{2(j-n)-1}}|\partial_v^{1+n-j}\psi|^2+(l+1)^{2n}r\chi|\partial_v\psi|^2dudv\\&\leq A\int_{u=u_R}\sum_{m+j\leq n}(M+(u-u_R))(l+1)^{2j}|\partial_u^{m+1}\psi_+|^2du\\&+A\int_{u\geq u_R}\chi\sum_{j=0}^{n}(l+1)^{2(n-j)}r^{2j+1}|\partial_v^j(\partial_t+\partial_u)\psi|^2dudv\\&\leq A\sum_{1\leq m+j\leq n+1}\int_{u=v_0-R}^{\infty}\left(M^{2m}+(u-u_R)^{2m}\right)(l+1)^{2j}|\partial_u^m\psi_+|^2du,\end{aligned}$$

applying (3.232), along with identical reasoning as used in (3.232). □

Proposition 3.4.9 (Integrated Decay of Higher Order Energy)**.** *Let ψ_+ be a Schwartz function. Let ψ be the solution of* (3.33) *on a sub-extremal Reissner–Nordström background, as given by Theorem 3.3.1, with radiation field on $\mathscr{I}^+$ equal to ψ_+, and which vanishes on $\mathscr{H}^+$. Let R be a constant, and let t_0 be a fixed value of t. Then for each $n \in \mathbb{N}_0$, we have the following bounds:*

$$\begin{aligned}
&\int_{t_{2n+1}=-\infty}^{t_0}\int_{t_{2n}=-\infty}^{t_n}\dots\int_{t_1=-\infty}^{t_2}\int_{t=-\infty}^{t_1}\left(\int_{\bar{\Sigma}_{t,R}} -dt(J^{\partial_t}[\partial_t^n\phi])\right)dt dt_1 dt_2..dt_{2n+1} \qquad (3.234)\\
&\quad+\sum_{j+|\alpha|+m\le n}\int_{v=t_0+R,r^*\ge R}\int_{v\le t_0+R,r^*\ge R} r^{1+2j}\left(|\partial_u^{1+j}\partial_t^m\Omega^\alpha\psi|^2+jV|\partial_u^j\partial_t^m\Omega^\alpha\psi|^2\right)dudv\\
&\quad+\sum_{j+|\alpha|+m\le n}\int_{u=t_0+R,r^*\le -R}(-r^*)^{1+2j}\left(|\partial_v^{1+j}\partial_t^m\Omega^\alpha\psi|^2+(-r^*)V|\partial_v^j\partial_t^m\Omega^\alpha\psi|^2\right)dudv\\
&\le A_n\sum_{j+|\alpha|+m\le n}\int_{v=t_0+R,r^*\ge R} r^{2+2j}|\partial_u^{1+j}\partial_t^m\Omega^\alpha\psi|^2du\\
&\quad+A_n\sum_{j+|\alpha|+m\le n}\int_{u=t_0+R,r^*\le -R}(-r^*)^{2+2j}|\partial_v^{1+j}\partial_t^m\Omega^\alpha\psi|^2dv\\
&\quad+A_n\sum_{j+|\alpha|\le 2n+2}\int_{\Sigma_{t_0}} -dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi]),
\end{aligned}$$

where $A_n = A_n(M,q,n,R)$.

Proof. This proof again closely follows that of [3]. We will consider T energy through a null foliation, $\bar{\Sigma}_{t,R}$ (see (2.10)).

We first look at how the wave operator commutes with both ∂_u and ∂_v:

$$\partial_u\partial_v(\partial_u^n\psi)+V\partial_u^n\psi = -\partial_u^n(V\psi)+V\partial_u^n\psi = \sum_{j=0}^{n-1}\binom{n}{j}(-1)^{n-j+1}\partial_{r^*}^{n-j}V\partial_u^j\psi \le A\sum_{j=0}^{n-1}\frac{V}{r^{n-j}}|\partial_u^j\psi| \qquad (3.235)$$

$$\partial_u\partial_v(\partial_v^n\psi)+V\partial_v^n\psi = -\partial_v^n(V\psi)+V\partial_v^n\psi = -\sum_{j=0}^{n-1}\binom{n}{j}\partial_{r^*}^{n-j}V\partial_v^j\psi \le A\sum_{j=0}^{n-1}V\kappa^{n-j}|\partial_v^j\psi| \qquad (3.236)$$

We apply the r^p and r^{*p} methods to the null segments of $\bar{\Sigma}_{t_0,R}$ to obtain:

$$\begin{aligned}
&\int_{v=t_0+R,r^*\ge R} r^p\chi\left(\frac{r^*-R}{M}\right)|\partial_u^k\psi|^2du \qquad (3.237)\\
&=\int_{v\le t_0+R,r^*\ge R}\left(pr^{p-1}D\chi+\frac{r^p}{M}\chi'\right)|\partial_u^k\psi|^2 - r^p\chi V\partial_u\left(|\partial_u^{k-1}\psi|^2\right)dvdu\\
&\quad+\int_{v\le t_0+R,r^*\ge R}2r^p\chi\mathbb{R}\left(\partial_u^k\bar{\psi}\sum_{j=0}^{k-2}\binom{k-1}{j}(-1)^{k-j}\partial_{r^*}^{k-1-j}V\partial_u^j\psi\right)dudv\\
&\ge\int_{v\le t_0+R,r^*\ge R}\left(pr^{p-1}D\chi+\frac{r^p}{M}\chi'\right)|\partial_u^k\psi|^2-\partial_{r^*}(r^p\chi V)\left(|\partial_u^{k-1}\psi|^2\right)dvdu\\
&\quad-A\int_{v\le t_0+R,r^*\ge R} r^p\chi|\partial_u^k\psi|\sum_{j=0}^{k-2}Vr^{1-k+j}|\partial_u^j\psi|dudv+\int_{\mathscr{I}^-}r^pV|\partial_u^{k-1}\psi|^2dv\\
&\ge a\int_{v\le t_0+R,r^*\ge R}\chi r^{p-1}\left(p|\partial_u^k\psi|^2+(p-2)V|\partial_u^{k-1}\psi|^2\right)dudv\\
&\quad-A\int_{R\le r^*\le M+R,t\le t_0}|\partial_u^k\psi|^2d+V|\partial_u^{k-1}\psi|^2dr^*dt\\
&\quad-A\sum_{j=0}^{k-2}\int_{v\le t_0+R,r^*\ge R}\chi V^2r^{3-2k+2j+p}|\partial_u^j\psi|^2dudv.
\end{aligned}$$

$$
\begin{aligned}
&\int_{u=t_0+R}(-r^*)^p\chi\left(\frac{-r^*-R}{M}\right)|\partial_v^k\psi|^2dv\\
&\quad=\int_{u\le t_0+R}\left(p(-r^*)^{p-1}\chi+\frac{(-r^*)^p}{M}\chi'\right)|\partial_v^k\psi|^2\\
&\qquad-(-r^*)^p\chi V\partial_v\left(|\partial_v^{k-1}\psi|^2\right)-2\chi(-r^*)^p\mathbb{R}\left(\partial_v^k\bar{\psi}\sum_{j=0}^{k-2}\binom{k-1}{j}\partial_{r^*}^{k-1-j}V\partial_v^j\psi\right)dvdu\\
&\quad\ge\int_{u\le t_0+R}\left(p(-r^*)^{p-1}\chi+\frac{(-r^*)^p}{M}\chi'\right)|\partial_v^k\psi|^2+\partial_{r^*}\left((-r^*)^p\chi V\right)\left(|\partial_v^{k-1}\psi|^2\right)\\
&\qquad-A\chi(-r^*)^p|\partial_v^k\psi|\sum_{j=0}^{k-2}V\kappa^{k-1-j}|\partial_v^j\psi|dvdu \qquad (3.238)\\
&\quad\ge a\int_{u\le t_0+R}\chi(-r^*)^{p-1}\left(p|\partial_v^k\psi|^2+(-r^*\kappa-p)V|\partial_v^{k-1}\psi|^2\right)dudv\\
&\qquad-A\int_{-r^*\ge M+R,t\le t_0}|\partial_v^k\psi|^2+V|\partial_v^{k-1}\psi|^2dr^*dt\\
&\qquad-A\sum_{j=0}^{k-2}\int_{u\le t_0+R}\chi(-r^*)^{p+1}\kappa^{2k-2j-2}V^2|\partial_v^j\psi|^2dudv.
\end{aligned}
$$

By summing (3.237) and (3.238) when $p=1$, $k=1$ (as then the two summations vanish), we obtain:

$$
\begin{aligned}
&\int_{v=t_0+R,r^*\ge R}r\chi\left(\frac{r^*-R}{M}\right)|\partial_u\psi|^2du+\int_{u=t_0+R}(-r^*)\chi\left(\frac{-r^*-R}{M}\right)|\partial_v\psi|^2dv+\int_{\Sigma_{t_0}}\sum_{j=0}^{1}-(l+1)^{2-2j}dt(J^{\partial_t}[\partial_t^j\phi])\\
&\qquad\ge a\int_{t=-\infty}^{t_0}T\text{-energy}(\bar{\Sigma}_{t,R})dt \qquad (3.239)
\end{aligned}
$$

Here we have used Proposition 3.4.7.

We then consider the $p=2$, $k=1$ case to obtain

$$
\begin{aligned}
&\int_{v=t_0+R,r^*\ge R}r^2\chi\left(\frac{r^*-R}{M}\right)|\partial_u\psi|^2du+\int_{u=t_0+R}(-r^*)^2\chi\left(\frac{-r^*-R}{M}\right)|\partial_v\psi|^2dv+\int_{\Sigma_{t_0}}\sum_{j=0}^{2}-(l+1)^{4-2j}dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi])\\
&\qquad\ge a\int_{t=-\infty}^{t_0}\left(\int_{v=t+R}r\chi\left(\frac{r^*-R}{M}\right)|\partial_u\psi|^2du\right. \qquad (3.240)\\
&\qquad\quad+\int_{u=t+R}(-r^*)\chi\left(\frac{-r^*-R}{M}\right)\left(|\partial_v\psi|^2+((-r^*)\kappa-2)V|\psi|^2\right)dv\\
&\qquad\quad\left.+\int_{\Sigma_t}\sum_{j=0}^{1}-(l+1)^{2-2j}dt(J^{\partial_t}[\partial_t^j\phi])\right)\\
&\qquad\ge a\int_{t=-\infty}^{t_0}\int_{t'=-\infty}^{t}T\text{-energy}(\bar{\Sigma}_{t',R})dt'dt.
\end{aligned}
$$

By using mean value theorem and T-energy boundedness (see [12] for an example of this), one can thus obtain

$$
\int_{\bar{\Sigma}_{t,R}}T\text{-energy}\le A(-t)^{-2}\int_{\mathscr{I}^+}(M^2+u^2)|\partial_u\psi_+|^2+l(l+1)|\psi_+|^2du. \qquad (3.241)
$$

By considering T-energy boundedness between $\bar{\Sigma}_{t,R}$ and $\mathscr{H}^-\cup\mathscr{I}_-$, we can also obtain:

$$
\begin{aligned}
&\int_{t=-\infty}^{t_0}\int_{t'=-\infty}^{t}\left(\int_{u=-\infty}^{t'+R}|\partial_u\psi_{\mathscr{H}^-}|^2du+\int_{v=-\infty}^{t'+R}|\partial_v\psi_{\mathscr{I}^-}|^2dv\right) \qquad (3.242)\\
&\quad=\int_{u=-\infty}^{t_0+R}(u-t_0-R)^2|\partial_u\psi_{\mathscr{H}^-}|^2du+\int_{v=-\infty}^{t_0+R}(v-t_0-R)^2|\partial_v\psi_{RN}|^2dv\\
&\quad\le A\int_{\mathscr{I}^+}(M^2+u^2)|\partial_u\psi_+|^2+l(l+1)|\psi_+|^2du.
\end{aligned}
$$

We now proceed to prove the result inductively, given the case $n=0$ is (3.240) (Provided $R>3\kappa$). We first look to bound the r and r^* weighted summations. We take $p=2+2n, k=n+1$ in (3.237)

$$\begin{aligned}
&\int_{v\leq t_0+R}\chi r^{1+2n}\left(|\partial_u^{1+n}\psi|^2+V|\partial_u^n\psi|^2\right)dudv \qquad (3.243)\\
&\leq A\int_{v=t_0+R}\chi r^{2+2n}|\partial_u^{1+n}\psi|^2du\\
&\quad+A\int_{R\leq r^*\leq M+R,t\geq t_0}|\partial_u^{n+1}\psi|^2+V|\partial_u^n\psi|^2dr^*dt\\
&\quad+A\sum_{m=1}^{n-1}\int_{v\leq t_0+R}\chi(l+1)^2r^{1+2m}V|\partial_u^m\psi|^2dudv\\
&\quad+\int_{v\leq t_0+R}2\chi r^{2+2n}\mathbb{R}\left(\partial_u^{n+1}\bar{\psi}(-1)^n\partial_{r^*}^nV\psi\right)dudv\\
&\leq A\int_{v=t_0+R}\chi r^{2+2n}|\partial_u^{1+n}\psi|^2du+A\sum_{m+|\alpha|\leq n+1}(l+1)^2\int_{\Sigma_{t_0}}-dt(J^{\partial_t}[\partial_t^m\psi])\\
&\quad+A\sum_{j+k+m\leq n}\int_{v=t_0+R,r^*\geq R}r^{2+2j}(l+1)^{2k}|\partial_u^{1+j}\partial_t^m\psi|^2du\\
&\quad+A\sum_{j+k+m\leq n}\int_{u=t_0+R,r^*\leq -R}(-r^*)^{2+2j}(l+1)^{2k}|\partial_v^{1+j}\partial_t^m\psi|^2dv\\
&\quad+A\sum_{j+m\leq 2n+2}\int_{\Sigma_{t_0}}-dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi])+\int_{v\geq t_0+R,r^*\geq R}\chi(l+1)^2rV|\psi|^2dudv\\
&\quad+\int_{v\leq t_0+R}2\chi r^{2+2n}\mathbb{R}\left(\partial_u^{n+1}\bar{\psi}(-1)^n\partial_{r^*}^nV\psi\right)dudv\\
&\leq A\sum_{j+k+m\leq n}\int_{v=t_0+R,r^*\geq R}r^{2+2j}(l+1)^{2k}|\partial_u^{1+j}\partial_t^m\psi|^2du\\
&\quad+A\sum_{j+k+m\leq n}\int_{u=t_0+R,r^*\leq -R}(-r^*)^{2+2j}(l+1)^{2k}|\partial_v^{1+j}\partial_t^m\psi|^2dv\\
&\quad+A\sum_{j+m\leq 2n+2}\int_{\Sigma_{t_0}}-dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi])\\
&\quad+\int_{v\leq t_0+R}2\chi r^{2+2n}\mathbb{R}\left(\partial_u^{n+1}\bar{\psi}(-1)^n\partial_{r^*}^nV\psi\right)dudv
\end{aligned}$$

In order to bound the final term in (3.243), we first note that the usual method of separating does not work:

$$\int_{v=t+R}\chi r^{2+2n}\mathbb{R}\left(\partial_u^{n+1}\bar{\psi}(-1)^n\partial_{r^*}^nV\psi\right)du\leq A\int_{v=t+R}\chi r^{2n+1}|\partial_u^{n+1}\psi|^2+r(l+1)^2V|\psi|^2du. \qquad (3.244)$$

Unfortunately, we have no way to bound $rV|\psi|^2$. If we consider lower order terms in r, we can use Hardy's inequality.

$$\int_{v=t+R}V|\psi|^2\leq A\int_{v=t+R}(l+1)^2|\partial_u\psi|^2+A\int_{v=t+R,|r^*|\leq R}V|\psi|^2du. \qquad (3.245)$$

Thus, the only term we need to be concerned about in (3.243) is the leading order in r behaviour of the final term. This behaves as follows:

$$\begin{aligned}\int_{v=t+R} 2l(l+1)\chi r^n \mathbb{R}\left(\partial_u^{n+1}\bar{\psi}\psi\right) du &= \int_{v=t+R} 2l(l+1)\mathbb{R}\left(\partial_u\bar{\psi}\sum_{j=0}^{n}\binom{n}{j}\partial_{r^*}^{n-j}(-1)^j(\chi r^n)\partial_u^j\psi\right) du \\ &\leq A\int_{v=t+R} l(l+1)r|\partial_u\psi|^2 + l(l+1)\sum_{j=1}^{n-1} r^{2j-1}|\partial_u^j\psi|^2 du \\ &\quad + \int_{v=t+R} l(l+1)\partial_{r^*}^n(\chi r^n)\partial_u\left(|\psi|^2\right) du \\ &\leq A\int_{v=t+R} l(l+1)r|\partial_u\psi|^2 + l(l+1)\sum_{j=1}^{n-1} r^{2j-1}|\partial_u^j\psi|^2 du \\ &\quad + \int_{v=t+R} l(l+1)\partial_{r^*}^{n+1}(\chi r^n)|\psi|^2 du - l(l+1)n!(-1)^n|\psi|^2_{\mathscr{I}^-}.\end{aligned} \tag{3.246}$$

As $\partial_{r^*}^{n+1}(r^n)\leq Ar^{-2}$, we can use (3.245) to bound this. Combining (3.243), (3.245) and (3.246), we obtain

$$\begin{aligned}\int_{v\leq t_0+R}\chi r^{1+2n}\left(|\partial_u^{1+n}\psi|^2 + V|\partial_u^n\psi|^2\right) dudv \leq A &\sum_{j+k+m\leq n}\int_{v=t_0+R,r^*\geq R} r^{2+2j}(l+1)^{2k}|\partial_u^{1+j}\partial_t^m\psi|^2 du \\ &+ A\sum_{j+k+m\leq n}\int_{u=t_0+R,r^*\leq -R}(-r^*)^{2+2j}(l+1)^{2k}|\partial_v^{1+j}\partial_t^m\psi|^2 dv \\ &+ A\sum_{j+m\leq 2n+2}\int_{\Sigma_{t_0}} -dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi]),\end{aligned} \tag{3.247}$$

as required.

The $(-r^*)^p$ section is made much more easy by the exponential behaviour of the potential. We take $p=2+2n, k=n+1$ in (3.238):

$$\begin{aligned}&\int_{u\leq t_0+R}\chi(-r^*)^{1+2n}\left(|\partial_v^{n+1}\psi|^2 + (-r^*)V|\partial_v^n\psi|^2\right) dudv \\ &\leq A\int_{u=t_0+R}\chi(-r^*)^{2+2n}|\partial_v^{n+1}\psi|^2 dv + A\int_{u\leq t_0+R,-r^*\leq 2(2+2n)/\kappa}\chi(-r^*)^{2+2n}V|\partial_v^n\psi|^2 dudv \\ &\quad + A\int_{R\leq -r^*\leq R+M,t\leq t_0}|\partial_v^{n+1}\psi|^2 + V|\partial_v^k\psi|^2 dr^* dt \\ &\quad + A\sum_{j=0}^{n-1}\int_{u\leq t_0+R}\chi(-r^*)^{3+2n}\kappa^{2n-2j}V^2|\partial_v^j\psi|^2 dudv \\ &\leq A\int_{u=t_0+R}\chi(-r^*)^{2+2n}|\partial_v^{n+1}\psi|^2 dv \\ &\quad + A\int_{R\leq -r^*\leq \max\{R+M,2(2+2n)/\kappa\},t\leq t_0}|\partial_v^{n+1}\psi|^2 + V|\partial_v^n\psi|^2 dr^* dt \\ &\quad + A\frac{(l+1)^2}{M^2}\sum_{j=0}^{n-1}\int_{u\leq t_0+R}\chi(-r^*)^{1+2j}\kappa^{2n-2j}V|\partial_v^j\psi|^2 dudv \\ &\leq A\sum_{j+|\alpha|+m\leq n}\int_{v=t_0+R,r^*\geq R} r^{2+2j}|\partial_u^{1+j}\partial_t^m\Omega^\alpha\psi|^2 du \\ &\quad + A\sum_{j+|\alpha|+m\leq n}\int_{u=t_0+R,r^*\leq -R}(-r^*)^{2+2j}|\partial_v^{1+j}\partial_t^m\Omega^\alpha\psi|^2 dv \\ &\quad + A\sum_{j+|\alpha|\leq 2n+2}\int_{\Sigma_{t_0}} -dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi]),\end{aligned} \tag{3.248}$$

as required. In the final inequality, we have used the induction hypothesis.

By repeating the above argument, we can also show that

$$\int_{u\le t_0+R}\chi(-r^*)^{2n}\left(|\partial_v^{n+1}\psi|^2+(-r^*)V|\partial_v^n\psi|^2\right)dudv+\int_{v\le t_0+R}\chi r^{2n}\left(|\partial_u^{1+n}\psi|^2+V|\partial_u^n\psi|^2\right)dudv \tag{3.249}$$
$$\le A\sum_{j+|\alpha|+m\le n}\int_{v=t_0+R,r^*\ge R}r^{1+2j}|\partial_u^{1+j}\partial_t^m\Omega^\alpha\psi|^2du$$
$$+A\sum_{j+|\alpha|+m\le n}\int_{u=t_0+R,r^*\le -R}(-r^*)^{1+2j}|\partial_u^{1+j}\partial_t^m\Omega^\alpha\psi|^2dv$$
$$+A\sum_{j+|\alpha|\le 2n+1}\int_{\Sigma_{t_0}}-dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi])$$

We now look to prove the final part. Assuming the result is true in the n case, apply (3.234) to $\partial_t\psi$. We then integrate twice with respect to t_0 to obtain

$$\int_{t_{2n+3}=-\infty}^{t_0}\int_{t_{2n+2}=-\infty}^{t_{2n+3}}\cdots\int_{t_1=-\infty}^{t_2}\int_{t=-\infty}^{t_1}\left(\int_{\bar{\Sigma}_{t,R}}-dt(J^{\partial_t}[\partial_t^{n+1}\phi])\right)dtdt_1dt_2..dt_{2n+3} \tag{3.250}$$
$$\le A\sum_{j+k+m\le n}\int_{-\infty}^{t_0}\int_{-\infty}^{t_{2n+3}}\int_{v=t+R,r^*\ge R}r^{2+2k}(l+1)^{2k}|\partial_u^{1+k}\partial_t^{m+1}\psi|^2dudt_{2n+2}dt_{2n+3}$$
$$+A\sum_{j+k+m\le n}\int_{-\infty}^{t_0}\int_{-\infty}^{t_{2n+3}}\int_{u=t+R,r^*\le -R}(-r^*)^{2+2k}(l+1)^{2k}|\partial_v^{1+k}\partial_t^{m+1}|^2dvdt_{2n+2}dt_{2n+3}$$
$$+A\sum_{j+|\alpha|\le 2n+2}\int_{-\infty}^{t_0}\int_{-\infty}^{t_{2n+3}}\left(\int_{\Sigma_t}-dt(J^{\partial_t}[\partial_t^{j+1}\Omega^\alpha\phi])\right)dt_{2n+2}dt_{2n+3}.$$

The final term here can be immediately bounded using Proposition 3.4.7. To bound the earlier terms, we note that $\partial_u+\partial_v=\partial_t$, and we can use (3.33) to remove any mixed u,v derivatives.

$$\sum_{j+k+m\le n}\int_{-\infty}^{t_0}\int_{-\infty}^{t_{2n+3}}\int_{v=t+R,r^*\ge R}r^{2+2j}(l+1)^{2k}|\partial_u^{1+j}\partial_t^{m+1}\psi|^2dudt_{2n+2}dt_{2n+3} \tag{3.251}$$
$$\le A\sum_{j+k+m\le n}\int_{-\infty}^{t_0}\int_{-\infty}^{t_{2n+3}}\int_{v=t+R,r^*\ge R}r^{2+2j}(l+1)^{2k}|\partial_u^{2+j}\partial_t^{m}\psi|^2dudt_{2n+2}dt_{2n+3}$$
$$+A\sum_{j+k+m\le n}\int_{-\infty}^{t_0}\int_{-\infty}^{t_{2n+3}}\int_{v=t+R,r^*\ge R}r^{2+2j}(l+1)^{2k}|\partial_u^{j}\partial_t^{m}(V\psi)|^2dudt_{2n+2}dt_{2n+3}$$
$$\le A\sum_{j+k+m\le n}\int_{-\infty}^{t_0}\int_{-\infty}^{t_{2n+3}}\int_{v=t+R,r^*\ge R}r^{2+2j}(l+1)^{2k}|\partial_u^{2+j}\partial_t^{m}\psi|^2dudt_{2n+2}dt_{2n+3}$$
$$+A\sum_{j+k+m\le n}\int_{-\infty}^{t_0}\int_{-\infty}^{t_{2n+3}}\int_{v=t+R,r^*\ge R}r^{2j}(l+1)^{2k+2}V|\partial_u^{j}\partial_t^{m}\psi|^2dudt_{2n+2}dt_{2n+3}$$
$$\le A\sum_{j+k+m\le n+1}\int_{-\infty}^{t_0}\int_{v=t+R,r^*\ge R}r^{1+2j}(l+1)^{2k}|\partial_u^{1+j}\partial_t^{m}\psi|^2dudt_{2n+3}$$
$$+A\sum_{j+k+m\le n+1}\int_{-\infty}^{t_0}\int_{u=t+R,r^*\le R}(-r^*)^{1+2j}(l+1)^{2k}|\partial_v^{1+j}\partial_t^{m}\psi|^2dudt_{2n+3}$$
$$+A\sum_{j+|\alpha|\le 2n+3}\int_{-\infty}^{t_0}\left(\int_{\Sigma_{t_0}}-dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi])\right)dt_{2n+3}$$
$$\le A\sum_{j+k+m\le n+1}\int_{v=t+R,r^*\ge R}r^{2+2j}(l+1)^{2k}|\partial_u^{1+j}\partial_t^{m}\psi|^2du$$
$$+A\sum_{j+k+m\le n+1}\int_{u=t+R,r^*\le R}(-r^*)^{2+2j}(l+1)^{2k}|\partial_v^{1+j}\partial_t^{m}\psi|^2dv$$
$$+A\sum_{j+|\alpha|\le 2n+4}\int_{\Sigma_{t_0}}-dt(J^{\partial_t}[\partial_t^j\Omega^\alpha\phi]),$$

as required. An identical argument follows for the $-r^* \geq R$ region. □

Theorem 3.4.3 (Boundedness of the u and v Weighted Energy)**.** *Let ψ_+ be a Schwartz function on the cylinder. Let ϕ be a solution to* (2.1) *on a sub-extremal Reissner–Nordström background. Further, let ϕ vanish on $\mathscr{H}^+$ and have future radiation field equal to ψ_+. Then there exists a constant $A_n = A_n(M,q,n)$ (which also depends on the choice of origin of u,v) such that*

$$\begin{aligned}\sum_{k=0}^{2}\sum_{2j+m+2|\alpha|\leq 2n}\int_{\mathscr{H}^-\cap\{u\leq 0\}}(M^{2(j+1)-k}+u^{2(j+1)-k})|\partial_u^{j+m+k+1}\Omega^\alpha\psi_{\mathscr{H}^-}|^2du \\ +\sum_{k=0}^{2}\sum_{1\leq j+|\alpha|,2j+2|\alpha|+m\leq 2n+2}\int_{\mathscr{I}^-\cap\{v\leq 0\}}(M^{2j-k}+v^{2j-k})|\partial_v^{j+k+m}\Omega^\alpha\psi_{\mathscr{I}^-}|^2dv \\ \leq A\sum_{k=0}^{2}\sum_{1\leq j+|\alpha|,2j+2|\alpha|+m\leq 2n+2}\int_{\mathscr{I}^-}(M^{2j-k}+u^{2j-k})|\partial_u^{j+k+m}\Omega^\alpha\psi_+|^2du\end{aligned} \tag{3.252}$$

Proof. This result again follows closely that of [3]. It is an easy combination of Propositions 3.4.8 and 3.4.9, applied to $T^m\Omega^\alpha$, for $\alpha \leq n-j$ and $m \leq 2n-2k-2\alpha$. All that remains is to note

$$\begin{aligned}&\int_{t_{2n+1}=-\infty}^{t_0}\int_{t_{2n}=-\infty}^{t_n}\cdots\int_{t_1=-\infty}^{t_2}\int_{t=-\infty}^{t_1}\left(\int_{\bar{\Sigma}_{t,R}}-dt(J^{\partial_t}[\partial_t^n\phi])\right)dt\,dt_1dt_2..dt_{2n+1} \\ &=\int_{t_{2n+1}=-\infty}^{t_0}\int_{t_{2n}=-\infty}^{t_n}\cdots\int_{t_1=-\infty}^{t_2}\int_{t=-\infty}^{t_1}\left(\int_{-\infty}^{t+R}|\partial_u\psi_{\mathscr{H}^-}|^2\sin\theta d\theta d\varphi du+\int_{-\infty}^{t+R}|\partial_v\psi_{\mathscr{I}^-}|^2\sin\theta d\theta d\varphi dv\right)dt\,dt_1dt_2..dt_{2n+1} \\ &=\frac{1}{(2n+2)!}\left(\int_{-\infty}^{t+R}(u-t_0-R)^{2n+2}|\partial_u\psi_{\mathscr{H}^-}|^2\sin\theta d\theta d\varphi du+\int_{-\infty}^{t+R}(v-t_0-R)^{2n+2}|\partial_v\psi_{\mathscr{I}^-}|^2\sin\theta d\theta d\varphi dv\right),\end{aligned} \tag{3.253}$$

by repeated integration by parts. □

Corollary 3.4.5 (Arbitrary polynomial decay of I.E. Terms)**.** *Let ψ_+ be a Schwartz function on the cylinder, with $\hat{\psi}_+$ supported on $\omega \geq 0$. Then for each n, there exists an $A_n(M,q,\psi_+)$ such that*

$$I.E.[\psi_+,v_c,u_1,u_0] \leq A_n(u_0-u_1)^{-n}, \tag{3.254}$$

as $u_0 - v_c, u_1 - v_c \to \infty$, with $u_0 \geq u_1$. Here $I.E.$ is as defined in Theorem 3.4.1.

Proof. This proof is identical to that of Theorem 3.4.2. □

3.4.7 Final Calculation

We now have all the tools we need to calculate the final result. We wish to calculate:

$$I[\psi_+,l,u_0] := \int_{-\infty}^{\infty}|\sigma||\hat{\psi}_{\mathscr{I}^-}|^2d\sigma, \tag{3.255}$$

where $\psi_{\mathscr{I}^-}$ is the radiation field on $\mathscr{I}^-$.

Proof of Theorem 3.4.1. We will define ψ_0 and ψ_1 as in Corollary 3.4.3, that is

$$\psi_0(u) := \begin{cases}\psi(u,v_c)-f(u-u_1)\psi(u_1,v_c) & u\geq u_1 \\ 0 & u<u_1\end{cases}, \tag{3.256}$$

$$\psi_1(v) := \begin{cases}\psi(u_1,v)-(1-f(u_b(v)-u_1))\,\psi(u_1,v_c) & v\in[v_b(u_1),v_c] \\ 0 & v\notin[v_b(u_1),v_c]\end{cases}, \tag{3.257}$$

where f is a smooth compactly supported function with $f(0)=1$.

Note this coincides with the definition of ψ_0 in Corollary 3.4.2 and the definition of ψ_1 in Corollary 3.4.4.

For this final calculation, we will be using Lemma $II.6$ from [7]:

Lemma 3.4.2. *For $\beta > 0$, $u \in C_0^\infty(\mathbb{R})$, we define*

$$F(\xi) = \int_{\mathbb{R}} e^{i\xi e^{\beta x}} u'(x)dx. \tag{3.258}$$

Then we have

$$\int_{\mathbb{R}} |\xi|^{-1}|F(\xi)|^2 d\xi = \int_{\mathbb{R}} |\xi| \coth\left(\frac{\pi}{\beta}|\xi|\right) |\hat{u}(\xi)|^2 d\xi. \tag{3.259}$$

We also have a similar Lemma for the extremal case:

Lemma 3.4.3. *Let $A \in \mathbb{R}_{>0}$, $v_c \in \mathbb{R}$ be constants. Define*

$$p(v) = \frac{A}{v_c - v}. \tag{3.260}$$

Then for all $u \in C_0^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |\sigma||\hat{u}|^2 d\sigma = \int_{\mathbb{R}} |\sigma||\widehat{u\circ p}|^2 d\sigma. \tag{3.261}$$

Proof. This proof proceeds in an almost identical way to the proof of Lemma 3.4.2 (see [7]).

$$\begin{aligned}
\int_{\sigma\in\mathbb{R}} |\sigma||\widehat{u\circ p}|^2 d\sigma &= \lim_{\varepsilon\to 0}\left(\int_{\sigma\in\mathbb{R}} |\sigma| e^{-\varepsilon|\sigma|}|\widehat{u\circ p}|^2 d\sigma\right)\\
&= \lim_{\varepsilon\to 0}\left(\iiint_{x,x',\sigma\in\mathbb{R}} |\sigma| e^{-\varepsilon|\sigma|} e^{i\sigma(x'-x)} u\circ p(x)\overline{u\circ p}(x')dxdx'd\sigma\right)\\
&= \lim_{\varepsilon\to 0}\iint_{u,u'\in\mathbb{R}}\left(\int_{\sigma\in\mathbb{R}} |\sigma| e^{-\varepsilon|\sigma|} e^{i\sigma(\frac{A}{y}-\frac{A}{y'})}\right) u(y)\bar{u}(y')\frac{A^2}{y^2y'^2}dydy'\\
&= \lim_{\varepsilon\to 0}\iint_{y,y'\in\mathbb{R}}\left(\frac{2\left(\varepsilon^2-\left(\frac{A}{y}-\frac{A}{y'}\right)^2\right)}{\left(\varepsilon^2+\left(\frac{A}{y}-\frac{A}{y'}\right)^2\right)^2}\right) u(y)\bar{u}(y')\frac{A^2}{y^2y'^2}dydy'\\
&= \lim_{\varepsilon\to 0}\iint_{y,w\in\mathbb{R}}\left(\frac{2\left(\frac{\varepsilon^2y^2(y-w)^2}{A^2}-w^2\right)}{\left(\frac{\varepsilon^2y^2(y-w)^2}{A^2}+w^2\right)^2}\right) u(y)\bar{u}(y-w)dydw\\
&= \iint_{y,w\in\mathbb{R}}\lim_{\alpha\to 0}\left(\frac{2\left(\alpha^2-w^2\right)}{\left(\alpha^2+w^2\right)^2}\right) u(y)\bar{u}(y-w)dydw = \int_{w\in\mathbb{R}}\lim_{\alpha\to 0}(\widehat{|\sigma|e^{-\alpha|\sigma|}})(\widehat{|\hat{u}|^2})dw\\
&= \int_{\sigma\in\mathbb{R}} |\sigma||\hat{u}|^2 d\sigma,
\end{aligned} \tag{3.262}$$

as required. □

In order to use Lemmas 3.4.2 and 3.4.3, we take a sequence of functions in $C_0^\infty(\mathbb{R})$ which approximate ψ_1 with respect to the L^2 and $\dot{H}^1$ norms.

By considering $\hat{\psi}_1$, we can see

$$-ie^{i\sigma v_c}\sigma\hat{\psi}_1 := -i\int_{\Sigma^-_{u_1}\cap\{v\le v_c\}} \sigma e^{i\sigma(v_c-v)}\psi_1 dv = -i\int_{\mathbb{R}} \sigma e^{i\sigma(v_c-v_b(u))}\psi_1(v_b(u))\frac{dv_b}{du}du = \int_{\mathbb{R}} e^{i\sigma(v_c-v_b(u))}(\psi_1\circ v_b)'du. \tag{3.263}$$

In the extremal case, this is similar to the form of F in (3.258), once we note that $v_c - v_b(u) = Ae^{-\kappa u} + O(e^{-2\kappa u})$ for large u.

Thus, we define

$$\gamma_{SE}(u) := -\frac{1}{\kappa}\log\left(\frac{v_c - v_b(u)}{A}\right) = u - \frac{1}{\kappa}\log\left(1+O(e^{-\kappa u})\right) = u + O(e^{-\kappa u}), \tag{3.264}$$

as $u \to \infty$. Then combining (3.263) and (3.264) gives

$$-ie^{i\sigma v_c}\sigma\hat{\psi}_1 = \int_{\mathbb{R}} e^{i\sigma(v_c - v_b(u))}(\psi_1 \circ v_b)'\,du = \int_{\mathbb{R}} e^{i\sigma A e^{-\kappa\gamma_{SE}}}(\psi_1 \circ v_b)'\,du = \int_{\mathbb{R}} e^{i\sigma A e^{-\kappa w}}\left(\psi_1 \circ v_b \circ \gamma_{SE}^{-1}\right)'dw. \tag{3.265}$$

Then we can apply Lemma 3.4.2 to obtain:

$$\int_{\mathbb{R}} |\sigma||\hat{\psi}_1|^2 d\sigma = \int_{\mathbb{R}} |\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)|\widehat{\psi_1 \circ v_b \circ \gamma_{SE}^{-1}}|^2 d\sigma. \tag{3.266}$$

Note that $|\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right) \leq \frac{\kappa}{\pi} + |\sigma|$.

In the extremal case, $v_c - v_b(u) = A_0 u^{-1} + O(u^{-3})$, so we define

$$\gamma_E(u) := \frac{A_0}{v_c - v_b(u)} = \frac{A_0}{A_0 u^{-1} + O(u^{-3})} = u + O(u^{-2}), \tag{3.267}$$

as $u \to \infty$. For p as in Lemma 3.4.3, we have

$$p(v) = \frac{A}{v_c - v} = \frac{A}{v_c - v_b(u_b(v))} = \gamma_E \circ u_b. \tag{3.268}$$

Thus we can apply Lemma 3.4.3 to $\psi_1 \circ v_b \circ \gamma_E^{-1}$ to obtain

$$\int_{\sigma\in\mathbb{R}} |\sigma||\widehat{\psi_1 \circ v_b \circ \gamma_E^{-1}}|^2 d\sigma = \int_{\sigma\in\mathbb{R}} |\sigma||\psi_1 \circ v_b \circ \widehat{\gamma_E^{-1} \circ \gamma_E \circ u_b}|^2 d\sigma = \int_{\sigma\in\mathbb{R}} |\sigma||\hat{\psi}_1|^2 d\sigma, \tag{3.269}$$

as in the sub-extremal case.

We now note that in both the extremal and sub-extremal cases, we have:

$$\begin{aligned}
&\int_{\sigma\in\mathbb{R}} (\kappa + |\sigma|)\left||\widehat{\psi_1 \circ v_b \circ \gamma^{-1}}|^2 - |\widehat{\psi_1 \circ v_b}|^2\right| d\sigma \\
&\quad \leq A\left(\int_{u_1}^{\infty} |\partial_u(\psi_1 \circ v_b)|^2 + \kappa^2|\psi_1 \circ v_b|^2 du\right)^{1/2}\left(\int_{u_1}^{\infty} |\psi_1 \circ v_b \circ \gamma^{-1} - \psi_1 \circ v_b|^2 du\right)^{1/2} \\
&\quad \leq A\,(I.T.[\psi_+])^{1/2}\left(\int_{u_1}^{\infty}\left|\int_u^{\gamma(u)} \partial_u(\psi_1 \circ v_b)du\right|^2 du\right)^{1/2} \\
&\quad \leq A\,(I.T.[\psi_+])^{1/2}\left(\int_{u_1}^{\infty}\left|\int_{u_1}^{\gamma(u_1)} |\partial_u(\psi_1 \circ v_b)|du'\right|^2 du\right)^{1/2} \\
&\quad \leq A\,(I.T.[\psi_+])^{1/2}\int_{u_1}^{\gamma(u_1)}\left(\int_{u_1}^{\infty} |\partial_u(\psi_1 \circ v_b)|^2 du\right)^{1/2} du' \\
&\quad \leq A|\gamma(u_1) - u_1| I.T.[\psi_+],
\end{aligned} \tag{3.270}$$

where we have dropped the subscript from γ_{SE}, γ_E. We have used Minkowski's integral inequality to reach the penultimate line.

By using (3.266) and (3.269), we obtain

$$\begin{aligned}
&\left|\int_{\sigma\in\mathbb{R}} |\sigma|\left(|\hat{\psi}_{\mathscr{I}^-}|^2 - \coth\left(\frac{\pi}{\kappa}|\sigma|\right)|\hat{\psi}_{\mathscr{H}^-}|^2 - |\hat{\psi}_{RN}|^2\right) d\sigma\right| \\
&\qquad \leq A\int_{\sigma\in\mathbb{R}} |\sigma|\left||\hat{\psi}_{\mathscr{I}^-}|^2 - |\hat{\psi}_1|^2 - |\hat{\psi}_{RN}|^2\right| d\sigma \\
&\qquad\quad + A\int_{\sigma\in\mathbb{R}} |\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)\left||\widehat{\psi_1 \circ v_b \circ \gamma^{-1}}|^2 - |\widehat{\psi_1 \circ v_b}|^2\right| d\sigma \\
&\qquad\quad + A\int_{\sigma\in\mathbb{R}} |\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)\left||\widehat{\psi_1 \circ v_b}|^2 - |\hat{\psi}_0|^2\right| d\sigma \\
&\qquad\quad + A\int_{\sigma\in\mathbb{R}} |\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)\left||\hat{\psi}_0|^2 - |\hat{\psi}_{\mathscr{H}^-}|^2\right| d\sigma.
\end{aligned} \tag{3.271}$$

In the extremal case, we set $u_1 = u_0 - \sqrt{Mu_0}$. Then we can apply (3.270) and Corollaries 3.4.2, 3.4.3 and 3.4.4 to obtain the required result, again noting $|\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right) \leq \frac{\kappa}{\pi} + |\sigma|$.

If we have sufficient decay in the non-extremal case, then we set $u_1 = \left(1 - \frac{2}{3}\delta\right)$, and again use Corollaries 3.4.2, 3.4.3 and 3.4.4. □

Corollary 3.4.6 (Particle Emission by the RNOS Models)**.** *Let f be a Schwartz function on the cylinder, with $\hat{f}$ supported in $[-1,1]\times S_2$, and such that*

$$\int_{-\infty}^{\infty} |f(x,\theta,\varphi)|^2 dx = 1. \tag{3.272}$$

Then let ϕ be the solution to (2.1), (3.34) *with data on $\mathscr{H}^+$ vanishing, and radiation field on $\mathscr{I}^+$ given by*

$$\lim_{v\to\infty} r(u,v)\phi(u,v,\theta,\varphi) = f(\omega(u-u_0))e^{i\omega(u-u_0)}, \tag{3.273}$$

as given by Theorem 2.4.1.

Let $\psi_{\mathscr{I}^-}$ be the past radiation field, and let $n\in\mathbb{N}$. Then there exist constants $A_n(M,q,T^*,f,\omega)$ and $A(M,T^*,f,\omega)$ such that

$$\left|\int_{\sigma=-\infty}^{0} |\sigma||\hat{\psi}_{\mathscr{I}^-}|^2 \sin\theta d\theta d\varphi d\sigma - \int_{-\infty}^{\infty}\left(e^{\frac{2\pi\omega|x|}{\kappa}}-1\right)^{-1}\sum_{l,m}|\tilde{T}_{\omega x,l,m}|^2\left|\hat{f}_{l,m}(x-1)\right|^2 dx\right| \leq A_n u_0^{-n} \qquad |q|\neq 1$$

$$\int_{\sigma=-\infty}^{0}|\sigma||\hat{\psi}_{\mathscr{I}^-}|^2 d\sigma \leq \frac{A}{u_0^{3/2}} \qquad |q|=1, \tag{3.274}$$

where

$$\hat{f}_{l,m} = \int_{S^2}\hat{f}Y_{l,m}\sin\theta d\theta d\varphi, \tag{3.275}$$

are the projection of $\hat{f}$ onto spherical harmonics, $Y_{l,m}$.

Proof. This result follows easily from Theorem 3.4.1 and Corollary 3.4.5. In the sub-extremal case, one can choose u_1 such that $e^{-\kappa u_1} = u_0^{-n}$ to obtain

$$\begin{aligned}\left|\int_{\mathscr{I}^-}|\sigma||\hat{\psi}|^2\sin\theta d\theta d\varphi d\sigma - \int_{-\infty}^{\infty}|\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)\sum_{l,m}|\tilde{T}_{\sigma,l,m}|^2|\hat{\psi}_{+l,m}|^2 d\sigma - \int_{-\infty}^{\infty}|\sigma|\sum_{l,m}|\tilde{R}_{\sigma,l,m}|^2|\hat{\psi}_{+l,m}|^2 d\sigma\right|\\ \leq A_n u_0^{-n} + u_0^{2n} I.E.[\psi_+, v_c, \frac{n}{\kappa}\ln(u_0), u_0]\\ \leq A_n u_0^{-n}.\end{aligned} \tag{3.276}$$

Note also that

$$\frac{i}{2}\int_S \bar{\phi}\nabla\phi - \phi\nabla\bar{\phi}dn \tag{3.277}$$

is a conserved quantity, therefore by taking appropriate limits, we obtain

$$\int_{\sigma=-\infty}^{\infty}\sigma|\hat{\psi}_+|^2 d\sigma = \frac{i}{2}\int_{\mathscr{I}^+}\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}du = \frac{i}{2}\int_{\mathscr{I}^-}\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}dv = \int_{\sigma=-\infty}^{\infty}\sigma|\hat{\psi}_{\mathscr{I}^-}|^2 d\sigma. \tag{3.278}$$

Thus, we have

$$\left|\int_{\mathscr{I}^-}|\sigma||\hat{\psi}|^2\sin\theta d\theta d\varphi d\sigma-\int_{-\infty}^{\infty}|\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)\sum_{l,m}|\tilde{T}_{\sigma,l,m}|^2|\hat{\psi}_{+l,m}|^2d\sigma-\int_{-\infty}^{\infty}|\sigma|\sum_{l,m}|\tilde{R}_{\sigma,l,m}|^2|\hat{\psi}_{+l,m}|^2d\sigma\right|$$

$$=\left|\sum_{l,m}\left(\int_{\mathscr{I}^-}(|\sigma|-\sigma)|\hat{\psi}_{+l,m}|^2d\sigma-\int_{-\infty}^{\infty}|\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)|\tilde{T}_{\sigma,l,m}|^2|\hat{\psi}_{+l,m}|^2d\sigma\right.\right. \tag{3.279}$$

$$\left.\left.-\int_{-\infty}^{\infty}\left(|\sigma||\tilde{R}_{\sigma,l,m}|^2-\sigma\right)|\hat{\psi}_{+l,m}|^2d\sigma\right)\right|$$

$$=\left|\sum_{l,m}\left(\int_{-\infty}^{\infty}\left(\sigma-|\sigma||\tilde{R}_{\sigma,l,m}|^2-|\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)|\tilde{T}_{\sigma,l,m}|^2\right)|\hat{\psi}_{+l,m}|^2d\sigma-2\int_{\sigma=-\infty}^{0}|\sigma||\hat{\psi}_{+l,m}|^2d\sigma\right)\right|.$$

Finally, we note that $|\tilde{T}_{\sigma,l,m}|^2=1-|\tilde{R}_{\sigma,l,m}|^2$, which allows us to simplify:

$$\begin{aligned}\int_{-\infty}^{\infty}&\left(\sigma-|\sigma|+|\sigma|-|\sigma||\tilde{R}_{\sigma,l,m}|^2-|\sigma|\coth\left(\frac{\pi}{\kappa}|\sigma|\right)|\tilde{T}_{\sigma,l,m}|^2\right)|\hat{\psi}_{+l,m}|^2d\sigma\\
&=\int_{-\infty}^{\infty}\left(\sigma-|\sigma|+|\sigma|\left(1-\coth\left(\frac{\pi}{\kappa}|\sigma|\right)\right)|\tilde{T}_{\sigma,l,m}|^2\right)|\hat{\psi}_{+l,m}|^2d\sigma\\
&=2\int_{-\infty}^{\infty}\left(\frac{x-|x|}{2}+\left(e^{\frac{2\pi\omega|x|}{\kappa}}-1\right)^{-1}|\tilde{T}_{\omega x,l,m}|^2\right)\left|\hat{f}_{l,m}(x-1)\right|^2dx\\
&=2\int_{-\infty}^{\infty}\left(e^{\frac{2\pi\omega|x|}{\kappa}}-1\right)^{-1}|\tilde{T}_{\omega x,l,m}|^2\left|\hat{f}_{l,m}(x-1)\right|^2dx,\end{aligned} \tag{3.280}$$

as required. We have used that $\hat{f}(x-1)$ is only supported on $\sigma\geq 0$. The calculation follows identically for the extremal case. □

Remark 3.4.8 (Generality of Theorem 3.4.1 and Corollary 3.4.6)**.** *Similarly to Remark 2.7.2, the only behaviour of r_b required to prove Theorem 3.4.1 and Corollary 3.4.6 are*

- *The tangent vector $(1,\dot{r}_b(t^*),0,0)$ at the point $(t^*,r_b(t^*),\theta,\varphi)$ is timelike (for all t^*,θ,φ, including at $t^*=t_c^*$).*
- *There exists a t_-^* and an $\varepsilon>0$ such that for all $t^*\leq t_-^*$, $\dot{r}_b(t^*)\in(-1+\varepsilon,0)$.*

This allows Theorem 3.4.1 and Corollary 3.4.6 to be generalised more easily using these methods.

As discussed in the introduction, Corollary 3.4.6 is the calculation of radiation of frequency ω given off by the RNOS model of a collapsing black hole, see [28] for a full discussion of this. We will however, comment that the quantity of particles emitted by Extremal RNOS models is integrable. This means that the total number of particles given off by the forming extremal black hole is finite, and thus the black hole itself may never evaporate.

3.5 Future Work

In this thesis, we have considered Hawking radiation in collapsing Reissner–Nordström, with reflective boundary conditions on the surface of the collapse. There are several ways in which we would like to generalise this result going forward.

Firstly, we could change the background for the collapse to a Kerr black hole. This result would be of special significance for two reasons. On the one hand, this has physical application - all black holes that have been observed have some angular momentum. On the other hand, other methods for calculating black hole emission, like the Hartle–Hawking–Israel state mentioned in the introduction, do not generalise to this rotating case. We would then proceed to considering the collapsing Kerr–Newman case, where we allow both rotation and charge.

Secondly, we could consider other equations of motion beyond the wave equation. The most obvious example of this would be the Klein–Gordon equation (of which the wave equation is a special case). Further generalising this to the charged Klein–Gordon case would allow us to consider what charges are emitted by the black hole, and thus consider whether extremal black holes could become sub-extremal, or whether sub-extremal black holes become closer to extremal through this emission of charge.

Finally, extending this result beyond Dirichlet boundary conditions to include the interior of the collapse would allow us to understand whether different matter models could possibly have any influence in the emission of particles from the black hole, or whether this emission is, as theorised, independent of the matter collapsing.

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Appendix A

High Frequency Behaviour of the Reflection Coefficient

In this appendix, we will be considering solutions to the equation

$$u'' + (\omega^2 - V_l)u = 0, \tag{A.1}$$

where

$$V_l(r) = \frac{1}{r^2}\left(l(l+1) + \frac{2M}{r}\left(1 - \frac{q^2M}{r}\right)\right)\left(1 - \frac{2M}{r} + \frac{q^2M^2}{r^2}\right). \tag{A.2}$$

Throughout this appendix, we will be denoting $' = \frac{d}{dr^*}$.

We define solutions U_{hor} and U_{inf} as follows:

$$U_{hor} \sim e^{-i\omega r^*} \qquad r^* \to -\infty \tag{A.3}$$

$$U_{inf} \sim r^{i\omega r^*} \qquad r^* \to \infty. \tag{A.4}$$

We define the coefficients $\mathfrak{U}_{\mathscr{I}^-}$ and $\mathfrak{U}_{\mathscr{I}^+}$ as follows

$$U_{hor} = \mathfrak{U}_{\mathscr{I}^+} U_{inf} + \mathfrak{U}_{\mathscr{I}^-} \bar{U}_{inf}. \tag{A.5}$$

Note that these coefficients are related to the reflection and transmission coefficients mentioned in 3.3 by the following

$$\tilde{R}_{\omega,l,m} = \frac{\mathfrak{U}_{\mathscr{I}^+}}{\mathfrak{U}_{\mathscr{I}^-}} \tag{A.6}$$

$$\tilde{T}_{\omega,l,m} = \frac{1}{\mathfrak{U}_{\mathscr{I}^-}} \tag{A.7}$$

We then prove the following result:

Theorem A.0.1. *There exist constants A, C, independent of M, q, l, m, ω such that for all $\omega \geq C(l+1)^2 M^{-2}$,*

$$|\mathfrak{U}_{\mathscr{I}^+}|^2 \leq \frac{A(l+1)^2}{M^2\omega^2}. \tag{A.8}$$

Proof. We first consider the red-shift energy current

$$Q_{red} := |u' + i\omega u|^2 - V|u|^2 \tag{A.9}$$

$$Q_{red}{}' = -V'|u|^2.$$

by integrating $Q_{red}[U_{hor}]$, we obtain that

$$4\omega^2|\mathfrak{U}_{\mathscr{I}^+}|^2 = -\int_{-\infty}^{\infty} V'|U_{hor}|^2 dr^*. \tag{A.10}$$

Therefore, all that is left to show is

$$-\int_{-\infty}^{\infty} \frac{V'}{(l+1)^2}|U_{hor}|^2 dr^* \leq AM^{-2}, \tag{A.11}$$

for some constant A. To show this, we consider the Morawetz energy current given by

$$\begin{aligned} Q^y_{mor} &:= y(|u'|^2 + (\omega^2 - V)|u|^2) \\ {Q^y_{mor}}' &= y'|u'|^2 + \left((\omega^2 - V)y\right)'|u|^2, \end{aligned} \tag{A.12}$$

with

$$y = -\exp\left(-\int_{s=-\infty}^{r^*} \frac{M^2|V'|}{(l+1)^2} ds\right). \tag{A.13}$$

Integrating ${Q^y_{mor}}'$, we obtain

$$\begin{aligned} 2\omega^2 &\geq 2\omega^2 - e^{-\int_{-\infty}^{\infty} \frac{M^2|V'|}{(l+1)^2} dr^*}\left(4\omega^2|\mathfrak{U}_{\mathscr{I}^+}|^2 + 2\omega^2\right) \\ &= \int_{-\infty}^{\infty} e^{-\int_{s=-\infty}^{\infty} \frac{M^2|V'|}{(l+1)^2} ds}\left(\frac{M^2|V'|}{(l+1)^2}|U'_{hor}|^2 + \left(\frac{M^2|V'|}{(l+1)^2}(\omega^2 - V) - V'\right)|U_{hor}|^2\right) dr^* \\ &\geq \int_{-\infty}^{\infty} e^{-\int_{s=-\infty}^{\infty} \frac{M^2|V'|}{(l+1)^2} ds}\left(\frac{M^2\omega^2}{(l+1)^2} - \frac{M^2 V}{(l+1)^2} - 1\right)|V'||U_{hor}|^2 dr^* \\ &\geq e^{-\int_{-\infty}^{\infty} \frac{M^2|V'|}{(l+1)^2} dr^*}\left(\frac{M^2\omega^2}{(l+1)^2} - \frac{M^2 V_{max}}{(l+1)^2} - 1\right)\int_{-\infty}^{\infty} |V'||U_{hor}|^2 dr^*, \end{aligned} \tag{A.14}$$

provided that

$$\frac{M^2\omega^2}{(l+1)^2} \geq \frac{M^2 V_{max}}{(l+1)^2} + 1. \tag{A.15}$$

Now, if we assume that $\frac{M^2\omega^2}{(l+1)^2} \geq \frac{M^2 V_{max}}{(l+1)^2} + 1 + \delta$, for any fixed $\delta > 0$, we have

$$-\int_{-\infty}^{\infty} V'|U_{hor}|^2 dr^* \leq \frac{2}{M^2\delta} e^{\frac{M^2 V_{max}}{(l+1)^2}}, \tag{A.16}$$

as required. Note that

$$\sup_{|q|\leq 1} \frac{M^2 V_{max}}{(l+1)^2} < \infty. \tag{A.17}$$

□

Corollary A.0.1. *Let $\tilde{R}_{\omega,l,m}$ be the reflection coefficient of a Reissner–Nordström spacetime, as defined above. Then there exists a constant B, independent of M, q, l, m such that*

$$|\tilde{R}_{\omega,l,m}|^2 \leq \frac{B(l+1)^2}{1 + M^2\omega^2}. \tag{A.18}$$

Proof. Let A, C be as given by Theorem A.0.1, and without loss of generality, assume $A \geq 1$. From T-energy conservation (in this formalism, this is given by $\mathbb{I}(u'\bar{u}) = const$), we know

$$|\mathfrak{U}_{\mathscr{I}^-}|^2 = |\mathfrak{U}_{\mathscr{I}^+}|^2 + 1 \geq 1. \tag{A.19}$$

Therefore, in the region $\omega^2 \geq C(l+1)^2 M^{-2}$,

$$|\tilde{R}_{\omega,l,m}|^2 = \frac{|\mathfrak{U}_{\mathscr{I}^+}|^2}{|\mathfrak{U}_{\mathscr{I}^-}|^2} = \frac{|\mathfrak{U}_{\mathscr{I}^+}|^2}{|\mathfrak{U}_{\mathscr{I}^+}|^2 + 1} \leq \frac{A(l+1)^2}{A(l+1)^2 + M^2\omega^2} \leq \frac{A(l+1)^2}{1 + M^2\omega^2} \tag{A.20}$$

Then in the region $\omega^2 \geq C(l+1)^2M^{-2} \geq M^{-2}$, we have

$$|\tilde{R}_{\omega,l,m}|^2 \leq \frac{A(l+1)^2}{M^2\omega^2} \leq \frac{2A(l+1)^2}{1+M^2\omega^2}, \tag{A.21}$$

and in the region $\omega^2 < C(l+1)^2M^{-2}$, we have

$$|\tilde{R}_{\omega,l,m}|^2 \leq 1 \leq \frac{(1+C)(l+1)^2}{1+C(l+1)^2} \leq \frac{(1+C)(l+1)^2}{1+M^2\omega^2}. \tag{A.22}$$

Therefore letting $B = \max\{1+C, 2A\}$ gives the result.

□