Hydrodynamics of warps in the local model of astrophysical discs

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ABSTRACT

We show how the local approximation of astrophysical discs, which is the basis for the well known model of the shearing box, can be used to study many aspects of the dynamics of warped discs. In the local model, inclination of the orbit of a test particle with respect to the reference orbit corresponds to a vertical oscillation of the particle at the orbital frequency. Warping of a disc corresponds to a locally axisymmetric corrugation of the midplane of the disc that oscillates vertically at the orbital frequency, while evolution of the warp corresponds to a modulation of the complex amplitude of the vertical oscillation. We derive a conservation law for this amplitude that is the local equivalent of the conservation of angular momentum and therefore governs the evolution of the warp. For lengthscales that are long compared to the vertical scaleheight of the disc, the known non-resonant and resonant regimes of warp dynamics, including the diffusive and wavelike regimes of Keplerian discs, occur in the local model in the same way as in a global view of warped discs.

Key words: accretion, accretion discs – hydrodynamics – waves

1 INTRODUCTION

An astrophysical disc is warped when the plane of its orbital motion varies with distance from the centre. An initially flat disc will become warped if it interacts with a companion on a misaligned orbit (Papaloizou & Terquem 1995; Xiang-Gruess & Papaloizou 2013) or with a central object with a misaligned spin axis (Bardeen & Petterson 1975; Lai 1999). Instabilities can also generate warps spontaneously in initially aligned systems subject to tidal (Lubow 1992), radiation (Pringle 1996) or magnetic (Lai 1999) forces. The existence of warped or misaligned discs has been confirmed by observations of galactic nuclei (e.g. NGC 4258; Miyoshi et al. 1995), interacting binary stars (e.g. Her X-1; Gerend & Boynton 1976) and, increasingly, young stars (Sakai et al. 2019; Casassus et al. 2019; Bohn et al. 2022). The Milky Way itself has also been found to be warped (Skowron et al. 2019, and references therein).

The hydrodynamics of warped gaseous discs has been studied since the 1970s (Bardeen & Petterson 1975). A significant body of work has been directed at deriving an evolutionary equation for the tilt vector $\boldsymbol{l}(r,t)$ (a unit vector normal to the local orbital plane of the disc at radius r and time t) and, therefore, the shape of the disc (Petterson 1978; Papaloizou & Pringle 1983; Pringle 1992; Papaloizou & Lin 1995; Ogilvie 1999, 2006). This evolution is controlled by the conservation of angular momentum and therefore involves a calculation of the internal torque in the disc. In Keplerian discs, the transmission of the warp is relatively fast because the coincidence of the orbital and epicyclic frequencies leads to a resonant amplification of the internal flows and associated torques. In a sufficiently viscous disc, the warp evolves diffusively on a timescale that is typically much shorter than the viscous timescale, while in an inviscid disc the warp propagates as a bending wave at a significant fraction of the sound speed.

Numerical simulations of warped discs have been carried out since the 1990s (Larwood et al. 1996; Nelson & Papaloizou 1999), mainly using the SPH method but also, increasingly, with grid-based methods (e.g. Hawley & Krolik 2019; Liska et al. 2019). These simulations are very demanding because they are global and three-dimensional, involving a huge range of length- and timescales. It is extremely challenging at present to attempt to resolve both the global structure and long-term evolution of a thin, warped disc and the small-scale physics that may be occurring on scales less than the vertical scaleheight of the disc.

Partly in order to study this small-scale physics, Ogilvie & Latter (2013a) introduced a local model, the warped shearing box. This model generalizes the well known shearing box to incorporate the oscillatory geometry experienced by an observer orbiting within a warped disc, which is described by a single dimensionless parameter, the warp amplitude $|\psi|$. The warped

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shearing box has been used successfully to study the propagation of warps in magnetized discs (Paris & Ogilvie 2018) and the hydrodynamic instability of warped discs in both linear and nonlinear regimes (Ogilvie & Latter 2013b; Paardekooper & Ogilvie 2019).

The aim of this paper is to propose and develop an alternative way of studying the physics of warped discs using a local model. It is based on a standard shearing box and so can make use of existing numerical codes. Instead of a fixed warp being imposed on the system by an oscillatory deformation of the coordinate system, as in the warped shearing box, the warp is represented as part of the solution and evolves freely. Crucial to this approach is the idea that the warp can be identified, in the local model, with the modulation of a vertical oscillation of the disc. We develop the theory underlying this correspondence in Sections 2 and 3. Then, in Section 4, we derive evolutionary equations for the warp in the long-wavelength limit, showing the detailed correspondence with global asymptotic theories of warped discs. In Section 5 we discuss some relevant computational considerations, before concluding in Section 6.

2 THE LOCAL MODEL OF ASTROPHYSICAL DISCS

2.1 Construction of the local model

Let (r, ϕ, z) be cylindrical polar coordinates in an inertial frame of reference. We consider a gravitational potential $\Phi(r, z)$ that has both axial symmetry and reflectional symmetry in the plane $z = 0$ and admits a family of stable circular orbits in that plane. The angular velocity $\Omega(r)$ of the circular orbit of radius r is given by

$$r\Omega^2 = \Phi_r(r, 0), \tag{1}$$

where the subscript denotes a partial derivative, and the angular frequencies $\kappa(r)$ and $\nu(r)$ of horizontal and vertical oscillations about that orbit are given by

$$\kappa^2 = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}(r^4\Omega^2) = 4\Omega^2 + 2r\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r} = \Phi_{rr}(r, 0) + \frac{3}{r}\Phi_r(r, 0), \qquad \nu^2 = \Phi_{zz}(r, 0). \tag{2}$$

We assume that Ω^2, κ^2 and ν^2 are positive so that circular orbits exist and are stable. For any spherically symmetric potential (associated with a central force), $\nu = \Omega$. In the important special case of a point-mass potential $\Phi = -GM(r^2 + z^2)^{-1/2}$, we have $\kappa = \nu = \Omega = (GM/r^3)^{1/2}$.

The local model is based on a reference point that follows a selected circular orbit of radius r_0 and angular velocity $\Omega_0 = \Omega(r_0)$. We set up a Cartesian coordinate system (x, y, z) with origin at the reference point and axes pointing in the radial (x), azimuthal (y) and vertical (z) directions. The coordinate system therefore rotates with angular velocity Ω_0. The equation of motion of a test particle in the local approximation is

$$\ddot{x} - 2\Omega_0\dot{y} = 2q_0\Omega_0^2 x, \qquad \ddot{y} + 2\Omega_0\dot{x} = 0, \qquad \ddot{z} = -\nu_0^2 z, \tag{3}$$

with $\nu_0 = \nu(r_0)$ and $q_0 = q(r_0)$, where

$$q = -\frac{\mathrm{d}\ln\Omega}{\mathrm{d}\ln r} \tag{4}$$

is the dimensionless orbital shear rate. The terms in equation (3) proportional to $2\Omega_0$ come from the Coriolis force, while the terms proportional to x or z come from the expansion of the sum of the gravitational and centrifugal forces to first order about the reference point. The epicyclic frequency is

$$\kappa_0 = \sqrt{2(2 - q_0)}\Omega_0 = \kappa(r_0). \tag{5}$$

Henceforth we drop the subscript 0 on Ω_0, q_0, κ_0 and ν_0, so that Ω, q, κ and ν are regarded as constants and correspond to the values of those quantities on the reference orbit.

The general solution of equations (3) is

$$x = x_0 + \mathrm{Re}\left(X\,\mathrm{e}^{-\mathrm{i}\kappa t}\right), \qquad y = y_0 - q\Omega x_0 t + \mathrm{Re}\left(\frac{2\Omega}{\mathrm{i}\kappa}X\,\mathrm{e}^{-\mathrm{i}\kappa t}\right), \qquad z = \mathrm{Re}\left(Z\,\mathrm{e}^{-\mathrm{i}\nu t}\right), \tag{6}$$

where x_0 and y_0 are real constants and X and Z are complex constants. The horizontal part of the solution involves an elliptical oscillation at the epicyclic frequency around a guiding centre. The guiding centre has a fixed radial location and drifts uniformly in the azimuthal direction if $x_0 \neq 0$. Therefore the guiding centre follows the local representation of a circular orbit, the azimuthal drift being a consequence of the orbital shear. The vertical part of the solution is just a harmonic oscillation at the vertical frequency.

In the case of a spherically symmetric potential, for which $\nu = \Omega$, the vertical oscillation of a particle in the local model can be identified with the inclination of the particle's orbit with respect to the reference orbit. If we consider a circular orbit of radius r and angular velocity Ω in an inclined plane with unit normal vector $\boldsymbol{l}$, then the vertical coordinate oscillates harmonically in time, such that $z = \mathrm{Re}\left(Z\,\mathrm{e}^{-\mathrm{i}\Omega t}\right)$ with $Z = -r(l_x + \mathrm{i}l_y)$, where (in this paragraph only) (x, y, z) are Cartesian coordinates in an inertial frame with origin at the centre of the potential. Therefore $-Z/r$ can be identified with the complex tilt variable

$W = l_x + il_y$ used in numerous previous studies of warped discs (e.g. Hatchett, Begelman & Sarazin 1981; Kumar & Pringle 1985).

For a fluid disc, we replace the equation of motion (3) with the equivalent version for a continuous medium,

$$\mathrm{D}u_x - 2\Omega u_y = 2q\Omega^2 x + \frac{1}{\rho}(\partial_x T_{xx} + \partial_y T_{xy} + \partial_z T_{xz}), \tag{7}$$

$$\mathrm{D}u_y + 2\Omega u_x = \frac{1}{\rho}(\partial_x T_{yx} + \partial_y T_{yy} + \partial_z T_{yz}), \tag{8}$$

$$\mathrm{D}u_z = -\nu^2 z + \frac{1}{\rho}(\partial_x T_{zx} + \partial_y T_{zy} + \partial_z T_{zz}), \tag{9}$$

where $\boldsymbol{u}$ is the fluid velocity,

$$\mathrm{D} = \partial_t + u_x\partial_x + u_y\partial_y + u_z\partial_z \tag{10}$$

is the Lagrangian time-derivative following the fluid motion, ρ is the mass density and $\mathbf{T}$ is the stress tensor. We also require the equation of mass conservation,

$$\mathrm{D}\rho = -\rho(\partial_x u_x + \partial_y u_y + \partial_z u_z). \tag{11}$$

The divergence of the stress tensor gives the force per unit volume resulting from momentum transport within the fluid. The stress tensor could include a number of effects such as fluid pressure, viscosity, magnetic fields, self-gravity, radiation forces, turbulence, etc. To maintain generality we do not at this stage write down a constitutive or evolutionary equation for the stress, which would be required in order to close the system of equations.

It can be useful to separate the fluid velocity $\boldsymbol{u}$ into a part due to the orbital shear, $-q\Omega x\,\boldsymbol{e}_y$, and a residual velocity $\boldsymbol{v}$:

$$u_x = v_x, \qquad u_y = -q\Omega x + v_y, \qquad u_z = v_z. \tag{12}$$

In terms of $\boldsymbol{v}$, the governing equations read

$$\mathrm{D}v_x - 2\Omega v_y = \frac{1}{\rho}(\partial_x T_{xx} + \partial_y T_{xy} + \partial_z T_{xz}), \tag{13}$$

$$\mathrm{D}v_y + (2 - q)\Omega v_x = \frac{1}{\rho}(\partial_x T_{yx} + \partial_y T_{yy} + \partial_z T_{yz}), \tag{14}$$

$$\mathrm{D}v_z = -\nu^2 z + \frac{1}{\rho}(\partial_x T_{zx} + \partial_y T_{zy} + \partial_z T_{zz}), \tag{15}$$

$$\mathrm{D}\rho = -\rho(\partial_x v_x + \partial_y v_y + \partial_z v_z), \tag{16}$$

with

$$\mathrm{D} = (\partial_t - q\Omega x\partial_y) + v_x\partial_x + v_y\partial_y + v_z\partial_z. \tag{17}$$

These equations are horizontally homogeneous in the sense that their coefficients do not depend on x or y, except for the appearance of $x\partial_y$ in the operator D. This dependence can be removed either by considering 'locally axisymmetric' solutions that are independent of y, as we do in this paper, or by adopting a shearing coordinate system that follows the orbital shear, although in that case an explicit time-dependence appears in the equations.

The well known model of the shearing box (Hawley, Gammie & Balbus 1995) considers these equations in a cuboid together with boundary conditions on $\boldsymbol{v}$, ρ, etc., that are periodic in y and shearing-periodic in x (i.e. periodic in shearing coordinates). In this paper we consider the equations of the local model without necessarily applying the boundary conditions of the shearing box.

2.2 Axisymmetric linear waves in the local model

Locally axisymmetric solutions that are independent of y satisfy the equations

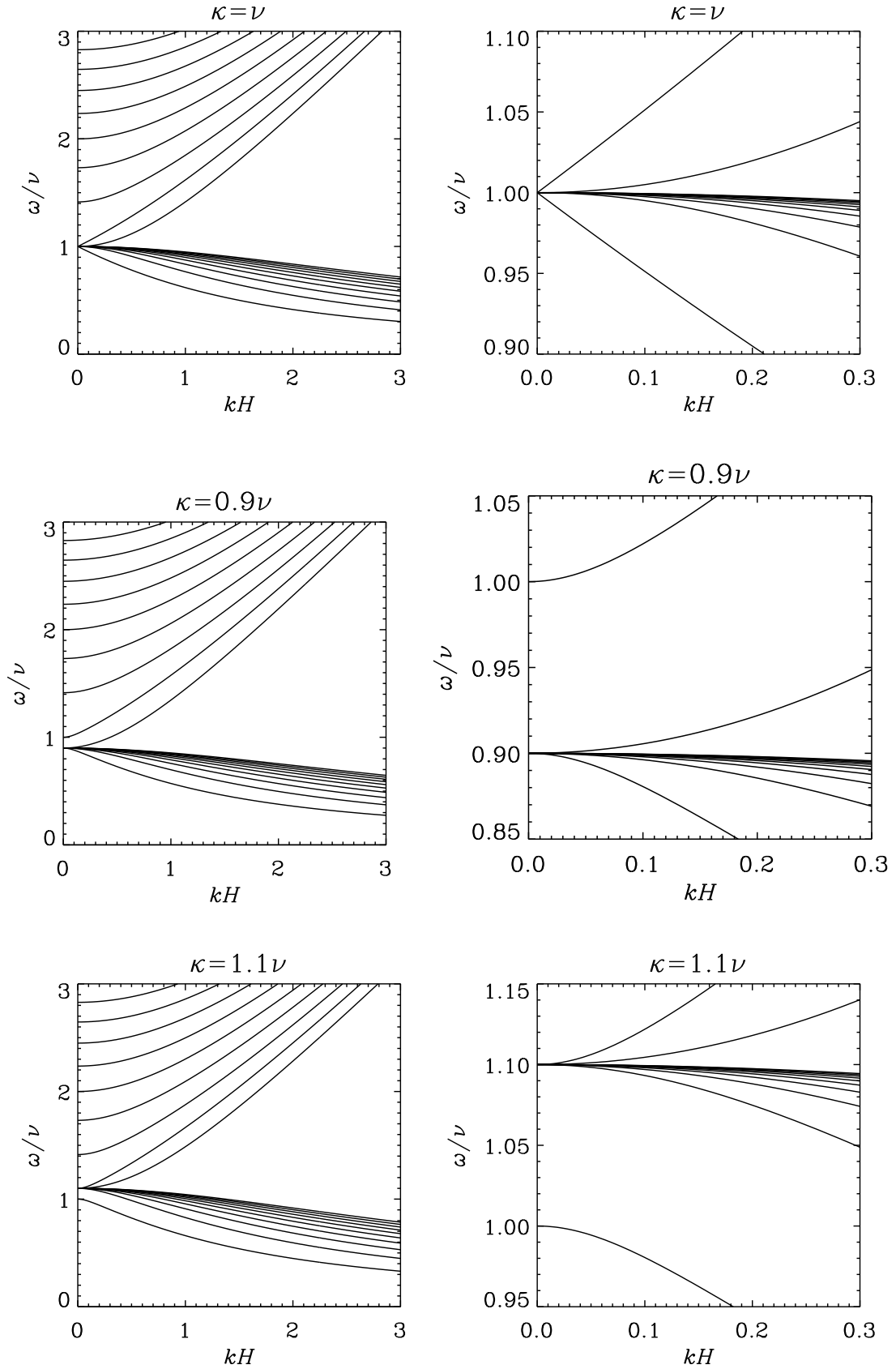$$\mathrm{D}v_x - 2\Omega v_y = \frac{1}{\rho}(\partial_x T_{xx} + \partial_z T_{xz}), \tag{18}$$

$$\mathrm{D}v_y + (2 - q)\Omega v_x = \frac{1}{\rho}(\partial_x T_{yx} + \partial_z T_{yz}), \tag{19}$$

$$\mathrm{D}v_z = -\nu^2 z + \frac{1}{\rho}(\partial_x T_{zx} + \partial_z T_{zz}), \tag{20}$$

$$\mathrm{D}\rho = -\rho(\partial_x v_x + \partial_z v_z), \tag{21}$$

where

$$\mathrm{D} = \partial_t + v_x\partial_x + v_z\partial_z. \tag{22}$$

Figure 1. Linear dispersion relations for axisymmetric waves in an isothermal disc in the local approximation. Modes up to $n = 10$ are plotted. The right panels zoom in to the region relevant for long-wavelength bending modes.

We consider here the simplest situation of an ideal fluid that is an isothermal gas, so that all components of **T** vanish except for an isotropic pressure that is proportional to the density. Thus

$$T_{xx} = T_{yy} = T_{zz} = -c_{\rm s}^2 \rho, \tag{23}$$

where $c_{\rm s} = $ constant is the isothermal sound speed.

The simplest solution of the equations is the vertically hydrostatic basic state

$$v_x = v_y = v_z = 0, \qquad \rho = \rho_0 \exp\left(-\frac{z^2}{2H^2}\right), \qquad H = \frac{c_{\rm s}}{\nu}, \tag{24}$$

in which there is no departure from circular orbital motion, and the density and pressure are Gaussian functions of z with scaleheight (or standard deviation) H.

Linear wave modes on this background that depend on x and t through the factor $\exp(\mathrm{i}kx - \mathrm{i}\omega t)$ have the following structure (Okazaki, Kato, & Fukue 1987; Ogilvie & Lubow 1999):

$$v_x \propto v_y \propto \frac{\rho'}{\rho} \propto \mathrm{He}_n\left(\frac{z}{H}\right), \qquad v_z \propto \mathrm{He}_{n-1}\left(\frac{z}{H}\right), \qquad n = 0, 1, 2, \dots, \tag{25}$$

involving the Hermite polynomials. The wave's angular frequency ω, radial wavenumber k and vertical mode number n satisfy the dispersion relation

$$(\omega^2 - n\nu^2)(\omega^2 - \kappa^2) = c_{\rm s}^2 k^2 \omega^2. \tag{26}$$

We focus on the modes with vertical mode number $n = 1$, which are related to warping or bending disturbances of the disc (e.g. Papaloizou & Lin 1995). The structure of these modes is such that v_z is independent of z, while $v_x, v_y \propto z$, with

$$\frac{v_x}{v_z} = \left(\frac{\omega^2}{\omega^2 - \kappa^2}\right)\mathrm{i}kz, \qquad \frac{v_y}{v_x} = \frac{(2-q)\Omega}{\mathrm{i}\omega}. \tag{27}$$

When $k = 0$ there is an $n = 1$ mode with frequency $\omega = \nu$, corresponding to a horizontally uniform vertical oscillation of the disc ($v_z = $ constant) at the vertical frequency. For $0 < kH \ll 1$ this mode becomes a long-wavelength bending mode with a frequency slightly different from ν.

Provided that $\kappa \neq \nu$, which we refer to as the *non-resonant case*, the $n = 1$ bending mode has the long-wavelength expansion

$$\omega = \nu\left\{1 + \frac{\nu^2}{2(\nu^2 - \kappa^2)}(kH)^2 + O\left[(kH)^4\right]\right\}. \tag{28}$$

The group velocity $\mathrm{d}\omega/\mathrm{d}k$ is proportional to k for sufficiently small values of kH, indicating that the waves are dispersive. We see from equation (27) that this mode involves some horizontal motion proportional to z in addition to the vertical motion independent of z. The horizontal motion is forced by radial pressure gradients associated with the radial variation of the vertical oscillation.

In the *resonant case* $\kappa = \nu$, which includes the important special case of a point-mass potential, a different expansion is required. We have instead

$$\omega = \nu\left\{1 \pm \frac{kH}{2} + O\left[(kH)^2\right]\right\}, \tag{29}$$

corresponding to a pair of waves with constant group velocities $\pm c_{\rm s}/2$ in the limit $kH \ll 1$.

Examples of the dispersion relations in resonant and non-resonant cases are plotted in Fig. 1. The right-hand panels zoom in to the region where $kH \ll 1$ and ω is close to ν. We can think of the solutions on the relevant branches as long-wavelength bending waves in which, to a first approximation, each column of the disc undergoes a harmonic oscillation at the vertical frequency; this oscillation is modulated on a longer timescale, resulting in a frequency slightly different from ν, because of the finite horizontal wavelength of the corrugation and the communication of vertical momentum between neighbouring columns of the disc.

2.3 Relation to a warped disc

It might seem obvious that a warped disc is a global, large-scale and non-axisymmetric phenomenon. The idea that the physics of warped discs can be captured in a local, axisymmetric model therefore requires some explanation. Fig. 2 illustrates the construction of a local model around a reference point in a circular orbit in a warped disc. The orbital motion at neighbouring radial locations in the disc is slightly misaligned with the reference orbit. The relative motion appears, in the local model, as a vertical oscillation of the disc with an amplitude and phase that depend on the radial location. To the extent that the warp is stationary in the non-rotating frame, each part of the disc oscillates vertically at the orbital frequency in the local frame; any slow evolution of the warp in the non-rotating frame would correspond to a temporal modulation of the local vertical oscillation. The vertical oscillation is also locally axisymmetric (independent of y), to the extent that the azimuthal length of the box is small compared to the circumference of the disc.
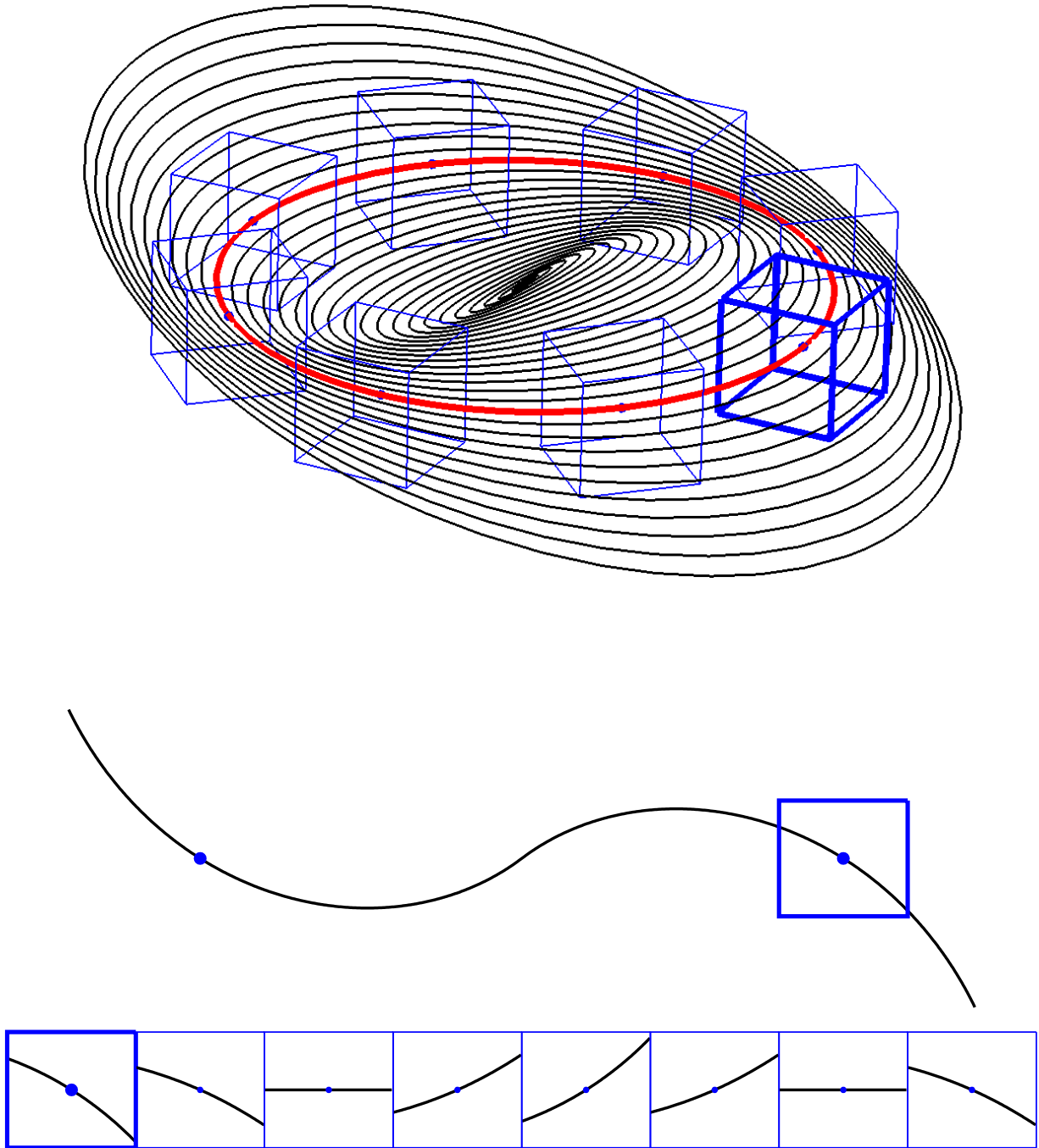
Figure 2. Appearance of a warped disc in the local model. The upper panel shows an example of a warped disc in which a reference orbit (red circle) is selected and used to construct a local model. The local frame (blue cube) is illustrated at eight equally spaced orbital phases. The lower panel presents a side-on view of the warped disc at the initial orbital phase; below this, a succession of boxes shows the appearance of the warped disc in the local frame at eight equally spaced orbital phases. To the extent that the warp is stationary in the non-rotating frame, each part of the disc oscillates vertically at the orbital frequency in the local frame. The illustrated warp is untwisted, so the phase of the vertical oscillation does not vary with radial location.

3 SYMMETRIES AND CONSERVATION LAWS OF THE LOCAL MODEL

3.1 Particle dynamics

The equation of motion (3) of a test particle in the local approximation is invariant under the addition of a uniform vertical oscillation of angular frequency ν and of arbitrary amplitude and phase. Thus if z is replaced by $z + \zeta$, where $\zeta(t)$ satisfies $\ddot{\zeta} + \nu^2\zeta = 0$, and all other variables remain unchanged, the equations are invariant. In the case of a spherically symmetric gravitational potential for which $\nu = \Omega$, the transformation $z \mapsto z + \zeta$ corresponds to a redefinition of the orientation of the reference orbit about which the local model is constructed.

Associated with this continuous symmetry is a conservation law for the complex quantity

$$Z = \left(z + \frac{\mathrm{i}\dot{z}}{\nu}\right)E_\nu, \qquad E_\nu = \mathrm{e}^{\mathrm{i}\nu t}, \tag{30}$$

which represents both the amplitude and phase of the vertical oscillation and agrees with the quantity Z in the general solution (6). This conservation law can be derived from Noether's Theorem, and is in fact just one of several conservation laws that hold in the local model when both horizontal and vertical motion are considered.

3.2 Continuum mechanics

In a similar way, the differential equations (7)–(11) and boundary conditions of a continuous medium in the local model are invariant under the addition of a uniform vertical oscillation of angular frequency ν and of arbitrary amplitude and phase. Thus if z and u_z are replaced by $z + \zeta$ and $u_z + \dot{\zeta}$, where $\zeta(t)$ satisfies $\ddot{\zeta} + \nu^2\zeta = 0$, and all other variables remain unchanged, the equations are invariant.

Associated with this continuous symmetry is a conservation law for the complex quantity

$$Z = \left(z + \frac{\mathrm{i}u_z}{\nu}\right)E_\nu, \qquad E_\nu = \mathrm{e}^{\mathrm{i}\nu t}, \tag{31}$$

which is clearly analogous to the complex amplitude of the vertical oscillation of a test particle, as considered above. Indeed, starting from equations (7)–(11) we can obtain the equation

$$\partial_t(\rho Z) + \partial_x\left(\rho Z u_x - \frac{\mathrm{i}T_{zx}E_\nu}{\nu}\right) + \partial_y\left(\rho Z u_y - \frac{\mathrm{i}T_{zy}E_\nu}{\nu}\right) + \partial_z\left(\rho Z u_z - \frac{\mathrm{i}T_{zz}E_\nu}{\nu}\right) = 0, \tag{32}$$

which is in conservative form.

We can relate the conservation of Z to the conservation of angular momentum. Let us consider a spherically symmetric gravitational potential and temporarily employ a Cartesian coordinate system in an inertial frame with origin at the centre of the potential. Then the conservative form of the angular momentum equation is

$$\partial_t(\rho\epsilon_{ijk}x_j u_k) + \partial_l(\rho\epsilon_{ijk}x_j u_k u_l - \epsilon_{ijk}x_j T_{kl}) = 0, \tag{33}$$

provided that the stress tensor is symmetric. The horizontal components of this equation are

$$\partial_t[\rho(yu_z - zu_y)] + \partial_l[\rho(yu_z - zu_y)u_l - yT_{zl} + zT_{yl}] = 0, \tag{34}$$
$$\partial_t[\rho(zu_x - xu_z)] + \partial_l[\rho(zu_x - xu_z)u_l - zT_{xl} + xT_{zl}] = 0. \tag{35}$$

Combining these components in the complex linear combination $x + \mathrm{i}y$, we obtain

$$\partial_t\left\{-\mathrm{i}\rho[(x+\mathrm{i}y)u_z - z(u_x+\mathrm{i}u_y)]\right\} + \partial_l\left\{-\mathrm{i}\rho[(x+\mathrm{i}y)u_z - z(u_x+\mathrm{i}u_y)]u_l + \mathrm{i}(x+\mathrm{i}y)T_{zl} - \mathrm{i}z(T_{xl}+\mathrm{i}T_{yl})\right\} = 0. \tag{36}$$

In cylindrical polar coordinates this reads

$$\partial_t\left\{-\mathrm{i}\rho[ru_z - z(u_r+\mathrm{i}u_\phi)]\,\mathrm{e}^{\mathrm{i}\phi}\right\} + \boldsymbol{\nabla}\cdot\left\{-\mathrm{i}\rho[ru_z - z(u_r+\mathrm{i}u_\phi)]\,\mathrm{e}^{\mathrm{i}\phi}\boldsymbol{u} + \mathrm{i}[r\boldsymbol{T}_z - z(\boldsymbol{T}_r+\mathrm{i}\boldsymbol{T}_\phi)]\,\mathrm{e}^{\mathrm{i}\phi}\right\} = 0, \tag{37}$$

where $\boldsymbol{T}_i = T_{ij}\,\boldsymbol{e}_j$ is (minus) the flux density of the i-component of momentum. If we now adopt the local approximation by selecting a reference orbit of radius r_0, going into a frame that rotates with that orbit and constructing a local Cartesian coordinate system, then equation (32) (multiplied by the constant $-r_0\Omega_0$) emerges as the leading approximation to this angular-momentum equation. To see this we must note that $|z| \ll r \approx r_0$ and $|u_r| \ll u_\phi \approx r_0\Omega_0$. The phase factor $\mathrm{e}^{\mathrm{i}\phi}$ translates into $\mathrm{e}^{\mathrm{i}\Omega_0 t}$ in the leading approximation, and we note that $\nu = \Omega$ for a spherically symmetric potential as considered here. Therefore the conservation of Z in the local model is directly related to the conservation of horizontal angular momentum in the global description. The reason for this association is that, by assumption, the angular momentum of the fluid is dominated by its orbital motion. The horizontal angular momentum derives from the inclination of the orbital motion, which appears in the local model as a vertical oscillation.

4 LONG-WAVELENGTH CORRUGATIONS AND CONNECTION WITH WARPED DISCS

The main aim of this section is to demonstrate a correspondence between the dynamics of long-wavelength corrugations in the local model and the known theories of warped discs that were derived from asymptotic analysis in spherical geometry. We first introduce a warped coordinate system that follows the corrugation, before making separate asymptotic analyses of the non-resonant case (allowing for nonlinearity and viscosity) and the resonant case.

4.1 Warped coordinates

We return to equations (18)–(21) governing locally axisymmetric solutions in the local model. To study the dynamics of a y-independent corrugation, which, as we have seen, is the local representation of a warp, we introduce the coordinate transformation

$$z' = z - \zeta(x,t), \tag{38}$$

where $\zeta(x,t)$ describes the corrugation whose dynamics is to be determined. The warped midplane corresponds to $z = \zeta(x,t)$ or $z' = 0$. The chain rule gives

$$\partial_x = \partial_x' - \zeta_x \partial_z', \qquad \partial_z = \partial_z', \qquad \partial_t = \partial_t' - \zeta_t \partial_z', \tag{39}$$

where ζ_x and ζ_t are the partial derivatives of $\zeta(x,t)$ with respect to x and t, and our shorthand notation for partial derivatives is

$$\partial_x = \left(\frac{\partial}{\partial x}\right)_{z,t}, \qquad \partial_z = \left(\frac{\partial}{\partial z}\right)_{x,t}, \qquad \partial_t = \left(\frac{\partial}{\partial t}\right)_{x,z}, \qquad \partial_x' = \left(\frac{\partial}{\partial x}\right)_{z',t}, \qquad \partial_z' = \left(\frac{\partial}{\partial z'}\right)_{x,t}, \qquad \partial_t' = \left(\frac{\partial}{\partial t}\right)_{x,z'}. \tag{40}$$

It can be helpful to introduce the relative vertical velocity

$$v_z' = v_z - (\zeta_t + v_x \zeta_x) = v_z - \mathrm{D}\zeta, \tag{41}$$

which differs from the absolute vertical velocity v_z by excluding the vertical velocity $\mathrm{D}\zeta$ associated with the time-dependent corrugation. The Lagrangian derivative is then

$$\mathrm{D} = \partial_t' + v_x \partial_x' + v_z' \partial_z', \tag{42}$$

and the velocity divergence is

$$\Delta = \partial_x' v_x + \partial_z' v_z'. \tag{43}$$

The basic equations become

$$\mathrm{D}v_x - 2\Omega v_y = \frac{1}{\rho}[\partial_x' T_{xx} + \partial_z'(T_{xz} - \zeta_x T_{xx})], \tag{44}$$

$$\mathrm{D}v_y + (2-q)\Omega v_x = \frac{1}{\rho}[\partial_x' T_{yx} + \partial_z'(T_{yz} - \zeta_x T_{yx})], \tag{45}$$

$$\mathrm{D}v_z = -\nu^2 z + \frac{1}{\rho}[\partial_x' T_{zx} + \partial_z'(T_{zz} - \zeta_x T_{zx})], \tag{46}$$

$$\mathrm{D}\rho = -\rho\Delta. \tag{47}$$

An alternative form of equation (46) that is more useful for some purposes is

$$\mathrm{D}v_z' = -\nu^2 z' - (\mathrm{D}^2 + \nu^2)\zeta + \frac{1}{\rho}[\partial_x' T_{zx} + \partial_z'(T_{zz} - \zeta_x T_{zx})]. \tag{48}$$

The conservative forms of these equations are

$$\partial_t' \rho + \partial_x'(\rho v_x) + \partial_z'(\rho v_z') = 0, \tag{49}$$

$$\partial_t'(\rho v_x) + \partial_x'(\rho v_x v_x - T_{xx}) + \partial_z'(\rho v_x v_z' - T_{xz} + \zeta_x T_{xx}) = 2\Omega\rho v_y, \tag{50}$$

$$\partial_t'(\rho v_y) + \partial_x'(\rho v_y v_x - T_{yx}) + \partial_z'(\rho v_y v_z - T_{yz} + \zeta_x T_{yx}) = -(2-q)\Omega\rho v_x, \tag{51}$$

$$\partial_t'(\rho Z') + \partial_x'\left(\rho Z' v_x - \frac{\mathrm{i}T_{zx}E_\nu}{\nu}\right) + \partial_z'\left[\rho Z' v_z' - \frac{\mathrm{i}(T_{zz} - \zeta_x T_{zx})E_\nu}{\nu}\right] = -\frac{\mathrm{i}\rho}{\nu}E_\nu(\mathrm{D}^2 + \nu^2)\zeta, \tag{52}$$

in which the right-hand sides represent source terms, and where

$$Z' = \left(z' + \frac{\mathrm{i}v_z'}{\nu}\right)E_\nu, \qquad E_\nu = \mathrm{e}^{\mathrm{i}\nu t}. \tag{53}$$

We integrate these conservative forms with respect to z' over the full vertical extent of the disc and assume that there are no net fluxes of mass or momentum through the vertical boundaries. (This important assumption should be reconsidered if the

disc has a significant mass outflow, is self-gravitating, or has an external magnetic field.) Thus

$$\partial'_t \int \rho \, dz' + \partial'_x \int \rho v_x \, dz' = 0, \tag{54}$$

$$\partial'_t \int \rho v_x \, dz' + \partial'_x \int (\rho v_x v_x - T_{xx}) \, dz' = 2\Omega \int \rho v_y \, dz', \tag{55}$$

$$\partial'_t \int \rho v_y \, dz' + \partial'_x \int (\rho v_y v_x - T_{yx}) \, dz' = -(2-q)\Omega \int \rho v_x \, dz', \tag{56}$$

$$\partial'_t \int \rho Z' \, dz' + \partial'_x \int \left(\rho Z' v_x - \frac{iT_{zx}E_\nu}{\nu} \right) \, dz' = -\frac{iE_\nu}{\nu} \int \rho(\mathrm{D}^2 + \nu^2)\zeta \, dz'. \tag{57}$$

4.2 Non-resonant case

4.2.1 Asymptotic expansions

So far the equations are valid for any y-independent solution, with $\zeta(x, t)$ being an arbitrarily specified function. In order to deduce the dynamical evolution of the corrugation, we make an asymptotic analysis of slowly modulated oscillating corrugations using the method of multiple scales (e.g. Bender & Orszag 1978). Taking the vertical scaleheight and orbital timescale as the characteristic scales of length and time, we write the corrugation as

$$\zeta = \epsilon^{-1}\zeta_0(X, t, T) = \epsilon^{-1}\mathrm{Re}\left[Z_0(X, T) \, e^{-i\nu t} \right], \tag{58}$$

where $\epsilon \ll 1$ is a small dimensionless parameter and X and T are slow space and time coordinates defined by

$$x = \epsilon^{-1}X, \qquad t = \epsilon^{-2}T. \tag{59}$$

The meaning of these expressions is as follows. We are making a formal separation between the fast orbital timescale (described by t) and the slow modulatory timescale (described by T). The corrugation is of large amplitude (ϵ^{-1}) compared to the vertical scaleheight. It consists of a harmonic oscillation on the orbital timescale, with angular frequency ν, and with an amplitude and phase (described by the complex amplitude Z_0) that vary on a radial lengthscale that is long (ϵ^{-1}) compared to the vertical scaleheight and on a timescale that is very slow (ϵ^{-2}) compared to the orbital timescale. The specific scaling of the very slow timescale adopted here, which is related to that used in Ogilvie (1999), is designed to capture the evolution of a large-scale warp due to both pressure and viscosity, allowing for the possibility that the viscosity parameter α is $O(1)$ in general.

The quantity Z_0 agrees with the complex amplitude Z introduced in Section 3 in the sense that $Z \sim \epsilon^{-1}Z_0$. The amplitude of the corrugation may be comparable to the radial lengthscale on which it varies. Thus the corrugation gradient

$$\zeta_x = \zeta_{0X} = \mathrm{Re}\left(Z_{0X} \, e^{-i\nu t} \right) \tag{60}$$

is of order unity, indicating that the corrugation is nonlinear. (Subscripts X and T will denote partial derivatives with respect to the slow variables.) We may write

$$\zeta_{0X} = \mathrm{Re}\left(-\psi \, e^{-i\nu t} \right) = -|\psi| \cos\tau, \tag{61}$$

where $\psi(X, T) = -Z_{0X}$ is the dimensionless complex warp amplitude used in previous work (Ogilvie 1999), $|\psi| = |Z_{0X}|$ is its magnitude and $\tau = \nu t - \arg\psi$ is a phase variable for the vertical oscillation.

The fluid variables associated with such a corrugation are generally required to have the following asymptotic expansions in order for the equations to balance:

$$v_x = v_{x0}(X, z', t, T) + \epsilon v_{x1}(X, z', t, T) + \epsilon^2 v_{x2}(X, z', t, T) + \cdots, \tag{62}$$

$$v_y = v_{y0}(X, z', t, T) + \epsilon v_{y1}(X, z', t, T) + \epsilon^2 v_{y2}(X, z', t, T) + \cdots, \tag{63}$$

$$v'_z = v'_{z0}(X, z', t, T) + \epsilon v'_{z1}(X, z', t, T) + \epsilon^2 v'_{z2}(X, z', t, T) + \cdots, \tag{64}$$

$$\rho = \rho_0(X, z', t, T) + \epsilon \rho_1(X, z', t, T) + \epsilon^2 \rho_2(X, z', t, T) + \cdots, \tag{65}$$

$$\mathbf{T} = \mathbf{T}_0(X, z', t, T) + \epsilon \mathbf{T}_1(X, z', t, T) + \epsilon^2 \mathbf{T}_2(X, z', t, T) + \cdots. \tag{66}$$

These expressions allow for internal velocities that are comparable to the sound speed and (generally anisotropic) stresses that are comparable to the pressure. Each term depends on t because the nonlinearity of the corrugation causes all quantities to oscillate on the orbital timescale at leading order. Note that the absolute vertical velocity $v_z = v'_z + \mathrm{D}\zeta$ has a different expansion,

$$v_z = \epsilon^{-1}\zeta_{0t} + v_{z0}(X, z', t, T) + \epsilon v_{z1}(X, z', t, T) + \epsilon^2 v_{z2}(X, z', t, T) + \cdots, \tag{67}$$

because it includes the large velocity associated with the time-dependent corrugation.

In the method of multiple timescales, t and T are regarded as independent variables. So when the operator ∂'_t (in which x

and z' are held constant) acts on any of the above quantities such as v_x, it has the action

$$\partial'_t + \epsilon^2 \partial'_T = \left(\frac{\partial}{\partial t}\right)_{X,z',T} + \epsilon^2 \left(\frac{\partial}{\partial T}\right)_{X,z',t}. \tag{68}$$

The Lagrangian time-derivative then has the expansion

$$D = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \tag{69}$$

$$D_0 = \partial'_t + v'_{z0}\partial'_z, \qquad D_1 = v_{x0}\partial'_X + v'_{z1}\partial'_z, \qquad D_2 = \partial'_T + v_{x1}\partial'_X + v'_{z2}\partial'_z. \tag{70}$$

When these expansions are substituted into the basic equations (44)–(47) and terms of the same order in ϵ are compared, we obtain a number of new equations. The one of lowest order comes from the vertical component (46) of the equation of motion at $O(\epsilon^{-1})$, which yields

$$\zeta_{0tt} = -\nu^2 \zeta_0. \tag{71}$$

This equation is satisfied by our assumption (58), which in fact is the general solution of this equation. Next, the four equations at $O(\epsilon^0)$ yield

$$D_0 v_{x0} - 2\Omega v_{y0} = \frac{1}{\rho_0}\partial'_z(T_{xz0} - \zeta_{0X}T_{xx0}), \tag{72}$$

$$D_0 v_{y0} + (2-q)\Omega v_{x0} = \frac{1}{\rho_0}\partial'_z(T_{yz0} - \zeta_{0X}T_{yx0}), \tag{73}$$

$$v_{x0}\zeta_{0Xt} + D_0 v_{z0} = -\nu^2 z' + \frac{1}{\rho_0}\partial'_z(T_{zz0} - \zeta_{0X}T_{zx0}), \tag{74}$$

$$D_0 \rho_0 = -\rho_0 \partial'_z v'_{z0}, \tag{75}$$

with

$$v'_{z0} = v_{z0} - \zeta_{0X}v_{x0}. \tag{76}$$

These form a closed system of equations (except for the specification of the stress tensor) that describe the nonlinear oscillations of the fluid variables on the orbital timescale in response to the corrugation. Note that the corrugation appears only through the terms (cf. equation 61)

$$\zeta_{0X} = -|\psi|\cos\tau, \qquad \zeta_{0Xt} = \nu|\psi|\sin\tau, \tag{77}$$

which involve the local warp amplitude $|\psi|$. Furthermore, the equations do not involve any derivatives with respect to X, so they are local in X. They are in fact equivalent to the equations for horizontally invariant solutions in the warped shearing box (Ogilvie & Latter 2013a) and also to Set A of the global asymptotic description of Ogilvie (1999). We develop this correspondence in Section 4.2.3 below.

We assume that the relevant solution of these equations is periodic in t, with the same period $2\pi/\nu$ as the vertical oscillation associated with the corrugation. (This is the equivalent in the local model of the assumption in a global model that the disc is stationary on the orbital timescale and evolves only on a slower timescale.) In a sufficiently dissipative disc we would expect the solution of these equations to converge towards such a periodic solution starting from general initial conditions. In a non-dissipative disc, additional free oscillation modes could persist unless the initial conditions are chosen correctly. The laminar oscillatory flows in a warped disc can be unstable (Gammie, Goodman & Ogilvie 2000; Ogilvie & Latter 2013b), leading to turbulent motion with a complicated dependence on x and t (Paardekooper & Ogilvie 2019). If such an instability is present, the present analysis could be taken to describe the oscillatory mean flows on the turbulent background, if the turbulent stresses are represented within **T**.

4.2.2 Evolutionary equations

In order to determine the evolution of the corrugation, we do require some information from higher orders in ϵ. It is more convenient for this purpose to use the conservative forms of the equations. When the asymptotic expansions are applied to equation (54) for mass conservation, we obtain, at $O(\epsilon^0)$, $O(\epsilon^1)$ and $O(\epsilon^2)$,

$$\partial'_t \int \rho_0 \, dz' = 0, \tag{78}$$

$$\partial'_t \int \rho_1 \, dz' + \partial'_X \int \rho_0 v_{x0} \, dz' = 0, \tag{79}$$

$$\partial'_T \int \rho_0 \, dz' + \partial'_t \int \rho_2 \, dz' + \partial'_X \int (\rho_1 v_{x0} + \rho_0 v_{x1}) \, dz' = 0. \tag{80}$$

Equations (55) and (56) for horizontal momentum conservation at $O(\epsilon^0)$ yield

$$\partial'_t \int \rho_0 v_{x0}\, \mathrm{d}z' = 2\Omega \int \rho_0 v_{y0}\, \mathrm{d}z', \tag{81}$$

$$\partial'_t \int \rho_0 v_{y0}\, \mathrm{d}z' = -(2-q)\Omega \int \rho_0 v_{x0}\, \mathrm{d}z', \tag{82}$$

and at $O(\epsilon^1)$ yield

$$\partial'_t \int (\rho_1 v_{x0} + \rho_0 v_{x1})\, \mathrm{d}z' + \partial'_X \int (\rho_0 v_{x0} v_{x0} - T_{xx0})\, \mathrm{d}z' = 2\Omega \int (\rho_1 v_{y0} + \rho_0 v_{y1})\, \mathrm{d}z', \tag{83}$$

$$\partial'_t \int (\rho_1 v_{y0} + \rho_0 v_{y1})\, \mathrm{d}z' + \partial'_X \int (\rho_0 v_{y0} v_{x0} - T_{yx0})\, \mathrm{d}z' = -(2-q)\Omega \int (\rho_1 v_{x0} + \rho_0 v_{x1})\, \mathrm{d}z'. \tag{84}$$

Finally, equation (57) for Z' at $O(\epsilon^0)$ and $O(\epsilon^1)$ yields

$$\partial'_t \int \rho_0 Z'_0\, \mathrm{d}z' = -\frac{\mathrm{i}E_\nu}{\nu} \int \rho_0 (\mathrm{D}_1\mathrm{D}_0 + \mathrm{D}_0\mathrm{D}_1)\zeta_0\, \mathrm{d}z', \tag{85}$$

$$\partial'_t \int (\rho_1 Z'_0 + \rho_0 Z'_1)\, \mathrm{d}z' + \partial'_X \int \left(\rho_0 Z'_0 v_{x0} - \frac{\mathrm{i}T_{zx0}E_\nu}{\nu} \right) \mathrm{d}z' = -\frac{\mathrm{i}E_\nu}{\nu} \int [\rho_1(\mathrm{D}_1\mathrm{D}_0 + \mathrm{D}_0\mathrm{D}_1) + \rho_0(\mathrm{D}_2\mathrm{D}_0 + \mathrm{D}_1\mathrm{D}_1 + \mathrm{D}_0\mathrm{D}_2)]\zeta_0\, \mathrm{d}z', \tag{86}$$

where

$$Z' = Z'_0 + \epsilon Z'_1 + \cdots, \qquad Z'_0 = \left(z' + \frac{\mathrm{i}v'_{z0}}{\nu} \right) E_\nu, \qquad Z'_1 = \frac{\mathrm{i}v'_{z1}}{\nu} E_\nu. \tag{87}$$

A sequence of deductions can be made from these equations. First, we see from equation (78) that the surface density at leading order,

$$\Sigma_0(X,T) = \int \rho_0\, \mathrm{d}z', \tag{88}$$

is independent of the fast time variable t. This makes sense because mass is conserved and the vertical oscillation does not cause any horizontal mass transport. Second, equations (81) and (82) imply that the mass-weighted mean horizontal velocities at leading order undergo an unforced and undamped epicyclic oscillation, because they can be combined into

$$(\partial'^2_t + \kappa^2) \int \rho_0 v_{x0}\, \mathrm{d}z' = 0, \qquad (\partial'^2_t + \kappa^2) \int \rho_0 v_{y0}\, \mathrm{d}z' = 0. \tag{89}$$

If the amplitude of this oscillation were non-zero, the disc could be considered to have a non-zero eccentricity. Therefore we assume the appropriate solution to be

$$\int \rho_0 v_{x0}\, \mathrm{d}z' = \int \rho_0 v_{y0}\, \mathrm{d}z' = 0. \tag{90}$$

Third, we assume that the solution is periodic in the variable t (i.e. purely oscillatory on the orbital timescale, as discussed above), and carry out the averaging operation

$$\langle \cdot \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cdot\, \mathrm{d}\tau \tag{91}$$

on equations (80), (84) and (86) to eliminate some of the higher-order variables and obtain

$$\partial_T \Sigma_0 + \partial_X (\Sigma_0 \bar{v}_{x1}) = 0, \tag{92}$$

$$\partial_X \int \langle \rho_0 v_{y0} v_{x0} - T_{yx0} \rangle\, \mathrm{d}z' = -(2-q)\Omega \Sigma_0 \bar{v}_{x1}, \tag{93}$$

$$\partial_X \int \left\langle \rho_0 Z'_0 v_{x0} - \frac{\mathrm{i}T_{zx0}E_\nu}{\nu} \right\rangle \mathrm{d}z' = -\frac{\mathrm{i}}{\nu} \int \langle E_\nu[\rho_1(\mathrm{D}_1\mathrm{D}_0 + \mathrm{D}_0\mathrm{D}_1) + \rho_0(\mathrm{D}_2\mathrm{D}_0 + \mathrm{D}_1\mathrm{D}_1 + \mathrm{D}_0\mathrm{D}_2)]\zeta_0 \rangle\, \mathrm{d}z', \tag{94}$$

where $\bar{v}_{x1}(X,T)$ is the mass-weighted mean radial velocity defined by

$$\Sigma_0 \bar{v}_{x1} = \int \langle \rho_1 v_{x0} + \rho_0 v_{x1} \rangle\, \mathrm{d}z'. \tag{95}$$

(We can safely write ∂'_X and ∂'_T as ∂_X and ∂_T here because they are acting on quantities that do not depend on z'.) Equations (83) and (85) can also be averaged in this way, but they do not yield any further information that we require.

Equations (92) and (93) are related to those of classical accretion-disc theory and can be combined into the 'diffusion equation'

$$\partial_T \Sigma_0 = \frac{1}{(2-q)\Omega} \partial^2_X \int \langle \rho_0 v_{y0} v_{x0} - T_{yx0} \rangle\, \mathrm{d}z' \tag{96}$$

for the surface density. (It has the character of a diffusion equation if the stress integral on the right-hand side is a positive and increasing function of the surface density.) The evolution of the surface density can be affected by the presence of a warp.

Equation (94) determines the evolution of the warp. After some integrations by parts on the right-hand side and use of the equation of mass conservation at $O(\epsilon^1)$, we obtain

$$\partial_X \int \left\langle E_\nu \left[\rho_0 z' v_{x0} - \frac{\mathrm{i}(T_{zx0} - \rho_0 v_{z0}' v_{x0})}{\nu} \right] \right\rangle \mathrm{d}z' = \left\langle \frac{E_\nu}{\mathrm{i}\nu} \int \left[2\rho_0 \zeta_{0tT} + (\rho_1 v_{x0} + \rho_0 v_{x1})(\partial_t - \mathrm{i}\nu)\zeta_{0X} + \partial_X'(\rho_0 v_{x0} v_{x0} \zeta_{0X}) \right] \mathrm{d}z' \right\rangle, \tag{97}$$

which can be rearranged into the form

$$\Sigma_0 (Z_{0T} + \bar{v}_{x1} Z_{0X}) + \partial_X \int \left\langle E_\nu \left[\rho_0 z' v_{x0} - \frac{\mathrm{i}(T_{zx0} - \rho_0 v_{z0} v_{x0})}{\nu} \right] \right\rangle \mathrm{d}z' = 0. \tag{98}$$

The integral in equation (98) is equivalent (apart from a factor of $-\nu$) to the horizontal torque integral in equation 54 of Ogilvie & Latter (2013a).

4.2.3 Nonlinear oscillation equations

Returning to the solution of the nonlinear oscillation equations (72)–(75) in the case of a viscous disc, we first rewrite them as

$$\mathrm{D}_0 v_{x0} - 2\Omega v_{y0} = \frac{1}{\rho_0} \partial_z'(T_{xz0} + |\psi| \cos\tau \, T_{xx0}), \tag{99}$$

$$\mathrm{D}_0 v_{y0} + (2-q)\Omega v_{x0} = \frac{1}{\rho_0} \partial_z'(T_{yz0} + |\psi| \cos\tau \, T_{yx0}), \tag{100}$$

$$\mathrm{D}_0 v_{z0} = -\nu^2 z' - \nu|\psi| \sin\tau \, v_{x0} + \frac{1}{\rho_0} \partial_z'(T_{zz0} + |\psi| \cos\tau \, T_{zx0}), \tag{101}$$

$$\mathrm{D}_0 \rho_0 = -\rho_0 \Delta_0, \tag{102}$$

with

$$\mathrm{D}_0 = \nu\partial_\tau + v_{z0}' \partial_z', \qquad \Delta_0 = \partial_z' v_{z0}', \qquad v_{z0}' = v_{z0} + |\psi| \cos\tau \, v_{x0}. \tag{103}$$

For a combination of isotropic pressure and viscous stress, we write

$$T_{ij} = -p\,\delta_{ij} + \mu(\partial_i u_j + \partial_j u_i) + (\mu_\mathrm{b} - \tfrac{2}{3}\mu)(\partial_k u_k)\delta_{ij}, \tag{104}$$

where μ and μ_b are the dynamic shear and bulk viscosities. The leading-order stress components that we require are then

$$T_{xx0} = -p_0 + 2\mu_0(|\psi| \cos\tau \partial_z' v_{x0}) + (\mu_\mathrm{b0} - \tfrac{2}{3}\mu_0)\Delta_0, \tag{105}$$

$$T_{xz0} = T_{zx0} = \mu_0(\nu|\psi| \sin\tau + |\psi| \cos\tau \, \partial_z' v_{z0} + \partial_z' v_{x0}), \tag{106}$$

$$T_{yx0} = \mu_0(-q\Omega + |\psi| \cos\tau \, \partial_z' v_{y0}), \tag{107}$$

$$T_{yz0} = \mu_0(\partial_z' v_{y0}), \tag{108}$$

$$T_{zz0} = -p_0 + 2\mu_0(\partial_z' v_{z0}) + (\mu_\mathrm{b0} - \tfrac{2}{3}\mu_0)\Delta_0. \tag{109}$$

For comparison with previous work, we simplify the thermal physics by considering an adiabatic flow, thereby neglecting viscous heating and radiative cooling. The differential identity

$$\frac{\mathrm{d}p}{\rho} = \mathrm{d}h - T\,\mathrm{d}s \tag{110}$$

is used to rewrite pressure gradients in terms of gradients of the specific enthalpy h and the specific entropy s (in this equation only, T denotes the temperature). The specific enthalpy and entropy of a perfect gas of adiabatic index γ evolve according to

$$\frac{\mathrm{D}h}{\mathrm{D}t} = -(\gamma-1)h\Delta, \qquad \frac{\mathrm{D}s}{\mathrm{D}t} = 0, \tag{111}$$

leading to

$$\mathrm{D}_0 h_0 = -(\gamma-1)h_0 \Delta_0, \qquad \mathrm{D}_0 s_0 = 0. \tag{112}$$

If we assume

$$\mu_0 = \frac{\alpha p_0}{\nu}, \qquad \mu_\mathrm{b0} = \frac{\alpha_\mathrm{b} p_0}{\nu}, \tag{113}$$

where α and α_b are dimensionless shear and bulk viscosity coefficients that are independent of z', then we can separate the variables to write

$$v_{x0} = u(\tau)\nu z', \qquad v_{y0} = v(\tau)\nu z', \qquad v_{z0} = w(\tau)\nu z', \qquad h_0 = f(\tau)\nu^2 - \tfrac{1}{2}g(\tau)\nu^2 z'^2, \qquad s_0 = s(\tau), \tag{114}$$

where u, v, w and g are dimensionless and satisfy the ordinary differential equations

$$d_\tau u + (w + |\psi| \cos \tau \, u)u - 2\beta v = |\psi| \cos \tau \, g - (\alpha_b + \tfrac{1}{3}\alpha)|\psi| \cos \tau \, g(w + |\psi| \cos \tau \, u) - \alpha g[|\psi| \sin \tau + (1 + |\psi|^2 \cos^2 \tau)u], \quad (115)$$

$$d_\tau v + (w + |\psi| \cos \tau \, u)v + (2 - q)\beta u = -\alpha g[-q\beta|\psi| \cos \tau + (1 + |\psi|^2 \cos^2 \tau)v], \quad (116)$$

$$d_\tau w + (w + |\psi| \cos \tau \, u)w + |\psi| \sin \tau \, u = g - 1 - (\alpha_b + \tfrac{1}{3}\alpha)g(w + |\psi| \cos \tau \, u) - \alpha g[|\psi|^2 \sin \tau \cos \tau + (1 + |\psi|^2 \cos^2 \tau)w], \quad (117)$$

$$d_\tau f = -(\gamma - 1)(w + |\psi| \cos \tau \, u)f, \quad (118)$$

$$d_\tau g = -(\gamma + 1)(w + |\psi| \cos \tau \, u)g, \quad (119)$$

$$d_\tau s = 0. \quad (120)$$

(Here we have suppressed the parametric dependence of the solution on X and T.) These are exactly equivalent to equations A37–A42 of Ogilvie & Latter (2013a), except for the inclusion of the factor $\beta = \Omega/\nu$, which here could in principle differ from unity. In Ogilvie & Latter (2013a) it is explained how these equations are in turn exactly equivalent to equations 105–109 of Ogilvie (1999). (Note that these equations remain invariant when f is multiplied by a constant, so they do not fix the normalization of f, which must instead be determined from the surface density and entropy.)

The orbital stress averages we require are

$$\int \langle \rho_0 v_{y0} v_{x0} - T_{yx0} \rangle \, dz' = -Q_1 \nu^2 \langle \mathcal{I}_0 \rangle, \quad (121)$$

$$\int \left\langle E_\nu \left[\rho_0 z' v_{x0} - \frac{i(T_{zx0} - \rho_0 v_{z0} v_{x0})}{\nu} \right] \right\rangle dz' = Q_4 \psi \nu^2 \langle \mathcal{I}_0 \rangle, \quad (122)$$

where

$$\mathcal{I}_0(X, T, t) = \int \rho_0 z'^2 \, dz' \quad (123)$$

is the second vertical moment of the density and Q_1 and $Q_4 = Q_2 + iQ_3$ are real and complex dimensionless coefficients given by

$$Q_1 = \langle f_6[-uv + \alpha g(-q\beta + |\psi| \cos \tau \, v)] \rangle, \qquad Q_4 |\psi| = \left\langle e^{i\tau} f_6[u(1 + iw) - i\alpha g(|\psi| \sin \tau + |\psi| \cos \tau \, w + u)] \right\rangle, \quad (124)$$

where $f_6 = fg^{-1}/\langle fg^{-1} \rangle$ describes the variation of $\mathcal{I}_0$ with orbital phase. As explained in Ogilvie & Latter (2013a), these expressions agree exactly with equations 112 and 120 in Ogilvie (1999).

An issue not discussed in Ogilvie (1999) is how to relate $\langle \mathcal{I}_0 \rangle$ to Σ_0 in a homentropic (or polytropic) disc. We may write

$$\rho_0 = Ch_0^n, \quad (125)$$

where C is a constant related to the specific entropy and n is the polytropic index given by $\gamma = 1 + 1/n$. (The dimensionless solution given above has a specific entropy s_0 that is independent of z', but which could in principle depend on X and T. In that case we should also solve an evolutionary equation for $s_0(X, T)$, which will just be advected by the mean radial velocity $\bar{v}_{x1}$ in the absence of non-adiabatic processes.) We therefore have

$$\Sigma_0 = C(f\nu^2)^n \left(\frac{2f}{g} \right)^{1/2} I_n, \qquad \mathcal{I}_0 = C(f\nu^2)^n \left(\frac{2f}{g} \right)^{3/2} J_n = \frac{2f}{g} \frac{\Sigma_0}{2n+3}, \quad (126)$$

where

$$I_n = \int_{-1}^{1} (1 - x^2)^n \, dx = \frac{\Gamma(n+1)\Gamma(\tfrac{1}{2})}{\Gamma(n+\tfrac{3}{2})}, \qquad J_n = \int_{-1}^{1} x^2 (1 - x^2)^n \, dx = \frac{I_n}{2n+3} \quad (127)$$

are two dimensionless numbers. The fact that Σ_0 is independent of τ is related to the fact that $f^{\gamma+1} \propto g^{\gamma-1}$, which can be seen from equations (118) and (119). Eliminating f, we obtain

$$\mathcal{I}_0 = \frac{\Sigma_0^{(3\gamma-1)/(\gamma+1)} g^{-2/(\gamma+1)}}{(n+\tfrac{3}{2})(CI_n\sqrt{2})^{2(\gamma-1)/(\gamma+1)} \nu^{4/(\gamma+1)}}. \quad (128)$$

Thus

$$\langle \mathcal{I}_0 \rangle = C_\mathcal{I} Q_5 \Sigma_0^{(3\gamma-1)/(\gamma+1)}, \quad (129)$$

where $C_\mathcal{I}$ is a dimensional constant that depends on the entropy, and

$$Q_5(|\psi|) = \left\langle g^{-2/(\gamma+1)} \right\rangle \quad (130)$$

is a dimensionless function of the warp amplitude such that $Q_5(0) = 1$. This power-law relation between $\langle \mathcal{I}_0 \rangle$ and Σ_0, involving (for reasonable values of γ) a power between 1 and 2 and a coefficient that depends on the warp amplitude, is similar to what is obtained for a radiative disc (Ogilvie 2000).

In the isothermal case $\gamma = 1$, we have instead $f_6 = g^{-1}/\langle g^{-1} \rangle$ (because f becomes a constant in the limit $\gamma \to 1$) and

$$\langle \mathcal{I}_0 \rangle = c_s^2 Q_5 \Sigma_0, \qquad Q_5(|\psi|) = \left\langle g^{-1} \right\rangle. \quad (131)$$

When Q_1 and Q_4 are combined with Q_5, the factor $\langle g^{-1} \rangle$ cancels out, leaving the expressions 91 and 92 in Ogilvie & Latter

(2013a). In the interests of consistency between the isothermal and polytropic cases, these expressions should really be regarded as definitions of Q_1Q_5 and Q_4Q_5 rather than Q_1 and Q_4, and we adopt this convention henceforth.

4.2.4 Summary of the non-resonant case

When we remove the asymptotic scalings and subscripts, the evolutionary equations we have derived take the form

$$\frac{\partial \Sigma}{\partial t} + \frac{\partial (\Sigma \bar{v})}{\partial x} = 0, \tag{132}$$

$$\Sigma \left(\frac{\partial Z}{\partial t} + \bar{v} \frac{\partial Z}{\partial x} \right) = \frac{\partial}{\partial x} \left(Q_4 \nu^2 \langle \mathcal{I} \rangle \frac{\partial Z}{\partial x} \right), \tag{133}$$

together with

$$(2-q)\Omega\Sigma\bar{v} = \frac{\partial}{\partial x} \left(Q_1 \nu^2 \langle \mathcal{I} \rangle \right), \tag{134}$$

$$\langle \mathcal{I} \rangle = C_{\mathcal{I}} Q_5 \Sigma^{(3\gamma-1)/(\gamma+1)}, \tag{135}$$

where Q_1 (real), Q_4 (complex) and Q_5 (real, positive) are nonlinear functions of

$$|\psi| = \left| \frac{\partial Z}{\partial x} \right|. \tag{136}$$

Overall we obtain a system of equations for the evolution of the surface density $\Sigma(x,t)$ and the vertical amplitude $Z(x,t)$ that are very similar to those of Ogilvie (1999) for warped discs, with Z playing the role of $-r(l_x + il_y)$, and with exactly the same coefficients Q_i. The only differences are that certain factors of r coming from the global, spherical geometry do not appear in the local model, and that terms involving $Q_2|\psi|^2$ are absent. Our equations are a consistent simplification of those of Ogilvie (1999) for a warp that varies on a lengthscale that is small compared to r; this makes sense because we derived them in a local approximation.

The local model admits a special solution in the form of a uniformly travelling (and generally decaying) bending wave,

$$Z = A \, e^{ikx}, \qquad \Sigma = \text{constant}, \qquad \bar{v} = 0, \tag{137}$$

where $A(t)$ is a complex amplitude and k is a constant real wavenumber. This solution has

$$\psi = -Z_x = -ikA \, e^{ikx}, \qquad |\psi| = |kA|, \tag{138}$$

so that $|\psi|$ and the coefficients Q_i are independent of x. It represents a twisted warp of uniform amplitude. The evolutionary equation for Z reduces to the first-order ordinary differential equation

$$\dot{A} = -\frac{Q_4 \nu^2 \langle \mathcal{I} \rangle k^2}{\Sigma} A. \tag{139}$$

The warp amplitude therefore decays according to

$$\frac{\mathrm{d}}{\mathrm{d}t} (\ln |\psi|) = -Q_2 Q_5 \nu^2 C_{\mathcal{I}} \Sigma^{2(\gamma-1)/(\gamma+1)} k^2, \tag{140}$$

while the phase evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t} (\arg \psi) = -Q_3 Q_5 \nu^2 C_{\mathcal{I}} \Sigma^{2(\gamma-1)/(\gamma+1)} k^2. \tag{141}$$

This pair of equations can be thought of as a nonlinear dispersion relation showing how the decay rate and angular frequency of a travelling wave depend on its wavenumber and amplitude. The decay is not exactly exponential because the decay rate depends on amplitude through the function $Q_2(|\psi|)Q_5(|\psi|)$.

It is known that the nonlinear diffusion of warps can be subject to an instability, which may cause a warp to steepen into a break (Ogilvie 2000; Doğan et al. 2018; Doğan & Nixon 2020; Raj, Nixon & Doğan 2021). The instability results from the dependence of the coefficients Q_i on $|\psi|$. The special solution described above has a uniform warp amplitude $|\psi|$ and therefore does not exhibit this behaviour, but it could be linearly unstable to perturbations that modulate the warp amplitude.

4.3 Resonant case

The analysis of Section 4.2 breaks down in the resonant case $\kappa = \nu$ if the disc has only an isotropic stress from gas pressure. In this case the forcing of horizontal oscillations by the slowly modulated corrugation is resonant and undamped. An alternative

asymptotic scaling that works in this case is

$$x = \epsilon^{-1}X, \qquad t = \epsilon^{-1}T, \tag{142}$$

$$\zeta = \zeta_0(X, t, T) = \text{Re}\left[Z_0(X, T)\,e^{-i\nu t}\right], \tag{143}$$

$$v_x = v_{x0}(X, z', t, T) + \epsilon v_{x1}(X, z', t, T) + \cdots, \tag{144}$$

$$v_y = v_{y0}(X, z', t, T) + \epsilon v_{y1}(X, z', t, T) + \cdots, \tag{145}$$

$$v_z' = \epsilon v_{z1}'(X, z', t, T) + \cdots, \tag{146}$$

$$\rho = \rho_0(z') + \epsilon \rho_1(X, z', t, T) + \cdots, \tag{147}$$

$$p = p_0(z') + \epsilon p_1(X, z', t, T) + \cdots, \tag{148}$$

so that

$$D = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots, \tag{149}$$

with

$$D_0 = \partial_t', \qquad D_1 = \partial_T' + v_{x0}\partial_X' + v_{z1}'\partial_z', \qquad D_2 = v_{x1}\partial_X' + v_{z2}'\partial_z'. \tag{150}$$

The meaning of these expressions is somewhat different from that of the non-resonant case. The corrugation is now of comparable amplitude to the vertical scaleheight. It consists of a harmonic oscillation on the orbital timescale, with angular frequency ν, and with an amplitude and phase (described by the complex amplitude Z_0) that vary on a radial lengthscale that is long (ϵ^{-1}) compared to the vertical scaleheight and on a timescale that is slow (ϵ^{-1}) compared to the orbital timescale. Despite the reduced amplitude of the corrugation, the horizontal internal velocities are still comparable to the sound speed because they are driven at resonance. Apart from being translated by the vertical oscillation, the density and pressure experience relatively small (ϵ) fractional perturbations.

We substitute these expansions again into the basic equations (44)–(47) and compare terms of the same order in ϵ. The vertical component (46) of the equation of motion at $O(\epsilon^0)$ yields

$$0 = -\nu^2 z' - (\partial_t'^2 + \nu^2)\zeta_0 - \frac{1}{\rho_0}\partial_z' p_0. \tag{151}$$

The assumed form of ζ_0 means that $(\partial_t'^2 + \nu^2)\zeta_0$ vanishes, leaving the standard equation of vertical hydrostatic equilibrium,

$$0 = -\nu^2 z' - \frac{1}{\rho_0}\partial_z' p_0, \tag{152}$$

involving quantities that depend only on z'.

The horizontal components (44) and (45) at $O(\epsilon^0)$ give

$$\partial_t' v_{x0} - 2\Omega v_{y0} = 0, \qquad \partial_t' v_{y0} + \frac{\kappa^2}{2\Omega}v_{x0} = 0, \tag{153}$$

which admit a free epicyclic motion of the form

$$v_{x0} = \text{Re}\left[U(X, z', T)\,e^{-i\kappa t}\right], \qquad v_{y0} = \text{Re}\left[-\frac{i\kappa}{2\Omega}U(X, z', T)\,e^{-i\kappa t}\right], \tag{154}$$

where U is a complex amplitude to be determined subsequently. The epicyclic motion appears to be free at this stage because it is in fact forced resonantly.

At $O(\epsilon^1)$ equations (44)–(47) yield (using the hydrostatic balance)

$$D_1 v_{x0} + \partial_t' v_{x1} - 2\Omega v_{y1} = -\nu^2 z' \zeta_{0X}, \tag{155}$$

$$D_1 v_{y0} + \partial_t' v_{y1} + \frac{\kappa^2}{2\Omega}v_{x1} = 0, \tag{156}$$

$$\partial_t' v_{z1}' = -(D_1 D_0 + D_0 D_1)\zeta_0 - \frac{\rho_1}{\rho_0}\nu^2 z' - \frac{1}{\rho_0}\partial_z' p_1, \tag{157}$$

$$v_{z1}'\partial_z'\rho_0 + \partial_t'\rho_1 = -\rho_0(\partial_X' v_{x0} + \partial_z' v_{z1}'). \tag{158}$$

For adiabatic flow we also have a corresponding equation for the pressure,

$$v_{z1}'\partial_z' p_0 + \partial_t' p_1 = -\gamma p_0(\partial_X' v_{x0} + \partial_z' v_{z1}'), \tag{159}$$

where γ is the adiabatic index. Note that

$$(D_1 D_0 + D_0 D_1)\zeta_0 = \zeta_{0X}\partial_t' v_{x0} + 2(\zeta_{0tT} + v_{x0}\zeta_{0Xt}). \tag{160}$$

Equations (155) and (156) can be combined into

$$(\partial_t'^2 + \kappa^2)v_{x1} = F_{h1}, \tag{161}$$

with horizontal forcing

$$F_{h1} = -\nu^2 z' \zeta_{0Xt} - \partial_t' D_1 v_{x0} - 2\Omega D_1 v_{y0}, \tag{162}$$

which evaluates to

$$F_{\mathrm{h}1} = \mathrm{Re}\left\{\mathrm{i}\nu^3 z' Z_{0X}\,\mathrm{e}^{-\mathrm{i}\nu t} + [2\mathrm{i}\kappa U_T - U_{z'}(\partial_t' - 2\mathrm{i}\kappa)v_{z1}']\,\mathrm{e}^{-\mathrm{i}\kappa t} + \frac{\mathrm{i}\kappa}{2}U^*U_X + \frac{3\mathrm{i}\kappa}{2}UU_X\,\mathrm{e}^{-2\mathrm{i}\kappa t}\right\}. \tag{163}$$

The linear operator on the left-hand side of equation (161) is self-adjoint and has null eigenfunctions $\mathrm{e}^{\mp\mathrm{i}\kappa t}$ representing free epicyclic oscillations with an arbitrary vertical structure. The corresponding solvability conditions are

$$\int F_{\mathrm{h}1}\,\mathrm{e}^{\pm\mathrm{i}\kappa t}\,\mathrm{d}t = 0, \tag{164}$$

where the integration is over one period of the epicyclic oscillation. Given that $\kappa = \nu$, the first term in $F_{\mathrm{h}1}$, which is the forcing of the epicyclic oscillations by the warp, is resonant and contributes to the solvability conditions, which become

$$U_T = -\frac{1}{2}\nu^2 z' Z_{0X} \tag{165}$$

and the complex conjugate of this equation. The term $-U_{z'}(\partial_t' - 2\mathrm{i}\kappa)v_{z1}'$ cannot contribute to the solvability conditions because to do so v_{z1}' would need to contain terms proportional to either $\mathrm{e}^{2\mathrm{i}\kappa t}$ or e^{0t}. In the first case the term vanishes on application of $(\partial_t' - 2\mathrm{i}\kappa)$. In the second case there would have to be a non-zero mean relative vertical velocity, which we exclude in the next paragraph.

Using the hydrostatic condition, equations (157)–(159) can be combined into

$$-\partial_z'(\gamma p_0\partial_z' v_{z1}') + \rho_0(\partial_t'^2 + \nu^2)v_{z1}' = F_{\mathrm{v}1}, \tag{166}$$

with vertical forcing

$$F_{\mathrm{v}1} = \partial_z'(\gamma p_0\partial_X' v_{x0}) + \rho_0\nu^2 z'\partial_X' v_{x0} - \rho_0\partial_t'(\mathrm{D}_1\mathrm{D}_0 + \mathrm{D}_0\mathrm{D}_1)\zeta_0. \tag{167}$$

Given that $\kappa = \nu$, this evaluates to

$$F_{\mathrm{v}1} = \mathrm{Re}\left\{[\partial_z'(\gamma p_0 U_X) + \rho_0\nu^2(z'U_X + 2Z_{0T})]\,\mathrm{e}^{-\mathrm{i}\nu t} + 3\rho_0\nu^2 U Z_{0X}\,\mathrm{e}^{-2\mathrm{i}\nu t}\right\}. \tag{168}$$

So there are no non-oscillatory contributions to $F_{\mathrm{v}1}$ or to v_{z1}'. This justifies the step taken above in deriving equation (165).

The linear operator on the left-hand side of equation (166) is also self-adjoint and has null eigenfunctions $\mathrm{e}^{\mp\mathrm{i}\nu t}$ representing free vertical oscillations independent of z'. The corresponding solvability conditions are

$$\iint F_{\mathrm{v}1}\,\mathrm{e}^{\pm\mathrm{i}\nu t}\,\mathrm{d}z'\,\mathrm{d}t = 0, \tag{169}$$

where the integration is over the full vertical extent of the disc and over one period of the vertical oscillation. Thus we obtain

$$\int \rho_0 Z_{0T}\,\mathrm{d}z' = -\frac{1}{2}\int \rho_0 z' U_X\,\mathrm{d}z' \tag{170}$$

and the complex conjugate of this equation. Combining this with equation (165), we obtain the wave equation

$$Z_{0TT} = \frac{1}{4}\nu^2 H^2 Z_{0XX}, \tag{171}$$

where the scaleheight H is defined by

$$\int \rho_0 z'^2\,\mathrm{d}z' = H^2\int \rho_0\,\mathrm{d}z'. \tag{172}$$

When the asymptotic scalings are removed, the equation takes the form

$$\frac{\partial^2 Z}{\partial t^2} = \frac{1}{4}\nu^2 H^2\frac{\partial^2 Z}{\partial x^2}. \tag{173}$$

This result shows that the warp propagates in the form of non-dispersive waves, with wave speeds $\pm\frac{1}{2}\nu H$. In the case of a spherically symmetric potential with $\nu = \Omega$, this agrees with the result obtained in cylindrical geometry by Papaloizou & Lin (1995). It also agrees with the dipsersion relation (29).

5 COMPUTATIONAL CONSIDERATIONS

We have made some preliminary numerical investigations of the local model for warped discs by using the PLUTO (Mignone et al. 2007) and Athena++ (Stone et al. 2020) codes to solve the equations of ideal gas dynamics for an isothermal gas in a 2D Cartesian domain (x and z coordinates, but with three velocity components) using a finite-volume method. A uniform kinematic viscosity can also be included. Equations (18)–(23) can be solved using the standard shearing-box modules of these codes. The use of periodic boundary conditions in the x direction allows propagating bending waves and warps to be studied without end-effects. Provided that the domain is sufficiently long in the x direction, it is possible to access the regime of astrophysical interest, in which the radial wavelength of the warp is long compared to the vertical scaleheight of the disc.

These preliminary investigations, which we do not report in detail here, confirm that the dynamics of warped discs and the propagation of bending waves can be studied using this model, and it is possible to observe the occurrence of the parametric

instability (Gammie, Goodman & Ogilvie 2000; Ogilvie & Latter 2013b; Paardekooper & Ogilvie 2019). We note here some of the considerations that will be important for a more detailed computational study.

The height at which the vertical boundaries are placed, and the nature of those boundaries, can have an important effect on the outcome. None of the standard boundary conditions (periodic, reflecting or outflow) is well suited to the desired solution in which the gas oscillates freely through the boundary. The vertical motion can be transonic if the amplitude is sufficiently large, and the presence of boundaries causes shocks. The damping of the oscillatory vertical motion resulting from these shocks depends on the location and nature of the boundaries and needs to be quantified. A Lagrangian method that can follow the free vertical oscillation of the disc would have a clear advantage here. However, for subsonic vertical motion and reflecting boundary conditions at several scaleheights from the midplane, we found that the damping was very small.

It can be useful to view the simulations stroboscopically, once per orbit. This method filters out the basic vertical oscillation and reveals the modulatory dynamics that corresponds to the slow evolution of the warp in the non-rotating frame.

Although the periodic radial boundaries are artificial, and mean that a propagating warp cycles through the domain, they have the advantage of being compatible with special solutions such as the twisted warp of uniform amplitude, discussed in Section 4.2.4.

It may be useful to compare the approach proposed in this paper, which represents a warp within a standard shearing box (SSB), with that of the warped shearing box (WSB) defined by Ogilvie & Latter (2013a) and used in nonlinear hydrodynamic simulations by Paardekooper & Ogilvie (2019). In the SSB the warp is represented explicitly and evolves freely as a result of the dynamics occurring within the box, whereas in the WSB a warp of fixed amplitude is imposed through the oscillatory coordinate system, and the evolution of the warp is to be deduced from the torques measured in the box. The SSB should be much larger in the radial (x) direction to incorporate the scale of the warp explicitly, whereas the WSB can zoom in to a region that is small compared to the scale of the warp. Simulations in the SSB can make use of existing publicly available codes, while the WSB requires the coding of a novel set of equations. Finally, the SSB has to deal with oscillatory flows through the vertical boundaries (as discussed above), while the WSB naturally follows this motion, although in nonlinear warp regimes it may still have to deal with strong vertical compressions of the disc.

6 CONCLUSION

In this paper we have shown that many aspects of the dynamics of warped discs can be studied in the local approximation, which is the basis for the well known model of the shearing box. We have demonstrated that the warping of a disc corresponds, in the local model, to a locally axisymmetric corrugation of the midplane of the disc that oscillates vertically at the orbital frequency, while evolution of the warp corresponds to a modulation of the complex amplitude of the vertical oscillation. We have derived a conservation law for this amplitude that is the local equivalent of the conservation of angular momentum. For lengthscales that are long compared to the vertical scaleheight, the non-resonant and resonant regimes of warp dynamics, including the diffusive and wavelike regimes of Keplerian discs, occur in the local model in the same way as in the global model. This opens the possibility of studying the local physics of warped discs at high resolution using standard computational methods.

DATA AVAILABILITY

No new data were generated or analysed in support of this research.

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