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Reference Details

2175	Cambridge Working Papers in Economics
2021/09	Janeway Institute Working Paper Series
Published	25 October 2021
Revised	13 April 2022
Websites	www.econ.cam.ac.uk/cwpe
	www.inet.econ.cam.ac.uk/working-papers

Clubs and Networks

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April 13, 2022

Abstract

A recurring theme in the study of society is the concentration of influence and power that is driven through unequal membership of groups and associations. In some instances these bodies constitute a small world while in others they are fragmented into distinct cliques. This paper presents a new model of clubs and networks to understand the sources of individual marginalization and the origins of different club networks.

In our model, individuals seek to become members of clubs while clubs wish to have members. Club value is increasing in its size and in the strength of ties with other clubs. We show that a stable membership profile exhibits marginalization of individuals and that this is generally not welfare maximizing. Our second result shows that if returns from strength of ties are convex (concave) then stable memberships support fragmented networks with strong ties (small worlds held together by weak ties).

We illustrate the value of these theoretical results through case studies of inter-locking directorates and boards of editors of journals.

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1 Introduction

Economists study group formation using the theory of coalitions/clubs and the theory of network formation. In the coalitions approach individual payoffs are defined on the partition of players into mutually exclusive groups and in the networks literature individuals can join any number of groups but each of the groups is of size 2. However, in some important instances – examples include inter-locking directorates and boards of editors of journals – groups have sizes larger than 2 and individuals typically join multiple groups. Importantly, the productivity of a group depends on both its size and how it is connected to other groups through overlapping memberships. In these contexts, a major concern is that a few individuals take up most memberships while everyone else is left out thereby giving rise to a very unequal distribution of payoffs.[1] A second and related concern is that groups may be fragmented into cliques when a few individuals join them and that this may undermine openness and the performance of the system as a whole. Our paper proposes a new model of clubs and networks to examine these concerns.[2]

In our model, individuals seek to become members of clubs while clubs wish to have members. Clubs have capacity constraints (due to congestion effects) and individuals can only join up to a certain number of clubs (due to time limitations). Links between two clubs arise when an individual joins both clubs. The value of joining a club is increasing in the number of members (until the capacity is reached) and it may be increasing or decreasing in the strength of ties with other clubs. Individual utility is increasing in the sum of the productivity of the clubs they join. We define a notion of stable memberships that takes into account the incentives of individuals and clubs. Our interest is in understanding patterns of individual memberships and on the network of connections across clubs.

The main body of the analysis focuses on a setting where club value is increasing in link strength: in this case, a club prefers individuals who are members of more clubs and an individual prefers a club that links with more clubs. We show that stable outcomes exhibit a strong marginalization property: when club capacity is the binding constraint, a few individuals exhaust their membership capacity, while all others join no clubs; when individual availability is the binding constraint, a few clubs are fully occupied while all others go empty.[3]

1 Durlauf and Young (2004) present an influential account of the groups based perspective on inequality and poverty. In Section 6 we present case studies on a number of empirical contexts.

2 There is a small set of papers that allow for membership of multiple groups, e.g. Page and Wooders (2010) and Fershtman and Persitz (2021); we discuss these papers in detail later in the introduction after presenting our model and results.

3 For concreteness suppose that the number of individuals is 8, the number of clubs is 4, every individual

We next show that this marginalization is not always in line with efficiency: when individual utility is strongly concave, this marginalization is inefficient. Similarly, when club productivity is a concave function of membership size, the marginalization of clubs is inefficient. Thus, incentives of individuals and clubs and the collective interest are generally not aligned.

We then study the network of connections among the clubs. When the returns to link strength are linear, the distribution of link strength across clubs is not important for the productivity of clubs: as a result, a variety of club networks are stable. In applications, however, the marginal returns from link strength are likely to be non-linear. For instance, in case club links are used for information sharing, we would expect marginal returns to decline with link strength. On the other hand, if links help members coordinate activities of the clubs then the marginal returns may be increasing in link strength. We show that if the marginal returns from link strength are increasing, i.e., they are convex, then incentives of clubs and individuals push towards disconnected cliques of clubs with full strength links. If, on the other hand, the marginal returns from link strength are decreasing, i.e., they are concave, then the club network entails larger components that are held together by weak links.[4]

We also consider a setting where club value is decreasing in link strength with other clubs: a club prefers individuals who are not members of other clubs. In this setting, when club capacity is the binding constraint, stable outcomes entail isolated clubs. On the other hand, if individual availability is the binding constraint then clubs may be obliged to accept individuals who are also members of other clubs.

We check the robustness of our results with three extensions. We show that a strengthening of the solution concept from stability to strong stability does not affect our main results on marginalization. We then allow for heterogeneity across individuals in their productivity and we show that this may help select individuals for clubs but that it does not alter the marginalization property of stable outcomes. Finally, we extend the model by letting the benefits clubs receive from links depend on the sizes of their neighbouring clubs. We show

can join up to 4 clubs and every club has capacity 4. The total club capacity is 16, so in principle every individual could belong to 2 clubs each. We will say that a membership profile exhibits marginalization when 4 individuals become members of 4 clubs each while the other four individuals are completely left out.

[4] For concreteness suppose that the number of individuals is 16, the number of clubs is 6, every individual can join up to 2 clubs and every club has capacity 5. If returns are convex in link strength then the unique club-efficient and stable outcome is three cliques of two clubs each, and the links have maximal strength with 5 common members. If returns are concave in link strength then the unique club-efficient and stable membership profile is a connected network where every club has a link with one common member with every other club. These networks of clubs are illustrated in Figure 5 in Section 4 below).

that our main insights concerning marginalization are robust to this generalization.

The theoretical analysis is complemented with case studies on inter-locking directorates and editorial boards of journals. There is a large and distinguished body of work on inter-locking directorates, see e.g., Brandeis (1915), Brandeis (2009), Mizruchi (1996), Levine (1977), Useem (1984), and Davis, Yoo and Baker (2003); for a recent networks perspective on this literature see Kogut (2012). This literature argues that a major function of boards is to encourage best practices and that this is facilitated when a board member also has ties with other firms' boards. If information sharing is important then it is reasonable to suppose that the marginal returns from the strength of links are declining. In this setting, the theory predicts that the stable (and efficient) club network will contain weak ties and exhibit high connectivity. This is in line with the empirical evidence: Baker, Davis and Yoo (2001) and Kogut (2012) show that inter-locking directorates exhibit a small-world property – weak ties form the basis for a large connected network.[5]

Our second case study pertains to editorial boards of journals. We draw on the work of Ductor and Visser (2021) to study the membership of authors in these boards and the connections between boards defined by common editors. There exists very significant inequality in editorial memberships: a very small fraction of authors become editors. Moreover, most editors serve only on one or two boards, but there exists a core group of editors who serve on 4 or more journals. The network of the editorial boards is held together with (mostly) weak links. These patterns are consistent with our theoretical predictions on marginalization and on club networks (in the presence of concave returns from link strength).

There is a voluminous literature on coalitions and networks; for surveys of this work see e.g., Demange and Wooders (2005), Bloch and Dutta (2012), Bramoullé, Galeotti and Rogers (2016) and Goyal (2022). Our model draws on the theory of clubs and the theory of networks to explain phenomena such as marginalization, the small world of interlocking directorates, and power elites. Specifically, we combine the ideas of congestion and capacity constraints

[5] The work on inter-locking directorates is also related to a more general study of elites and power structures in sociology. In the nineteenth century, the Italian school of sociology proposed a theory of elites defined in terms of the membership of the top echelons of different – government and non-government – organizations (Pakulski (2018)). Building on this tradition, in his well-known study of mid-twentieth-century American society, Wright Mills (1956) argued that the power to make major decisions was highly concentrated: a very small group of individuals moved between the top levels of the Federal government, a few hundred largest corporations, and the military. He referred to these individuals as the *power Elite*. Similar claims have been made about the concentration of power and influence in other societies. For an overview of the theory of elites, see Bottomore (1993), and for a critique of theories of elite power and control, see Dahl (1958). Our model and case studies draw attention to economic forces that push toward concentration of power in modern society.

from club theory (Buchanan, 1965; Cornes, 1996; Demange and Wooders, 2005) with the ideas of multiple memberships and returns from links from the theory of networks (Bala and Goyal, 2000; Jackson and Wolinsky, 1996; Bloch and Dutta, 2012). We now discuss two earlier papers that seek in different ways to combine networks and clubs.

In an early paper, Page and Wooders (2010) study a setting of bipartite networks in which individuals decide on which clubs to join. Individual utility depends on own choices as well as the choices of others. Page and Wooders (2010) focus on the conditions under which the game of club memberships has a potential function (and this allows them to study the existence of Nash equilibrium). In our approach the clubs have governing bodies or owners who can choose to admit and expel members; these owners seek to maximize club productivity. The interaction between players and club owners gives rise to different incentives and strategic effects and hence to a different solution concept. Moreover, the focus of the paper is on the characterization of stable membership profiles. In particular, we derive a marginalization result and a mapping between marginal returns to link strength and club networks. While we consider more specific functional forms and pay-off structures these results go beyond the Page and Wooders (2010) paper.

A recent paper by Fershtman and Persitz (2021) also studies a model of clubs and networks. At a general level, there are similarities – both papers study a memberships model. But the motivation of the two papers is different and so the models and the main insights are also different. For Fershtman and Persitz (2021) the principal object of interest is the social network among individuals; by contrast, our interest is in understanding the membership profile of individuals in clubs. We explore questions such as who joins which club and what is the network of clubs that arises. This gives rise to very different types of results. Fershtman and Persitz (2021) highlight a trade-off between the size of clubs, the depreciation of indirect connections, and the membership fee. By contrast, we develop a marginalization property of stable outcomes and show why this is socially inefficient. We also draw attention to how the marginal returns from link strength – whether they are increasing or decreasing – determine the architecture of club networks.

To clarify the relation between our approach and the coalitions and networks approaches it is instructive to lay out the basic notation and then work through an example. In our model, there are n individuals and m clubs, each individual can join up to D clubs and every club can admit up to S members. It is assumed that club productivity is increasing in club size and in the strength of links with other clubs. In our approach we allow D and S to take arbitrary values. In a coalitions model, the outcome is a partition, so individuals can join only one club, so $D = 1$. Similarly, networks constitute a special case where every club can have exactly 2 members, roughly this means $S = 2$ and the payoff to the club from a single member is 0.

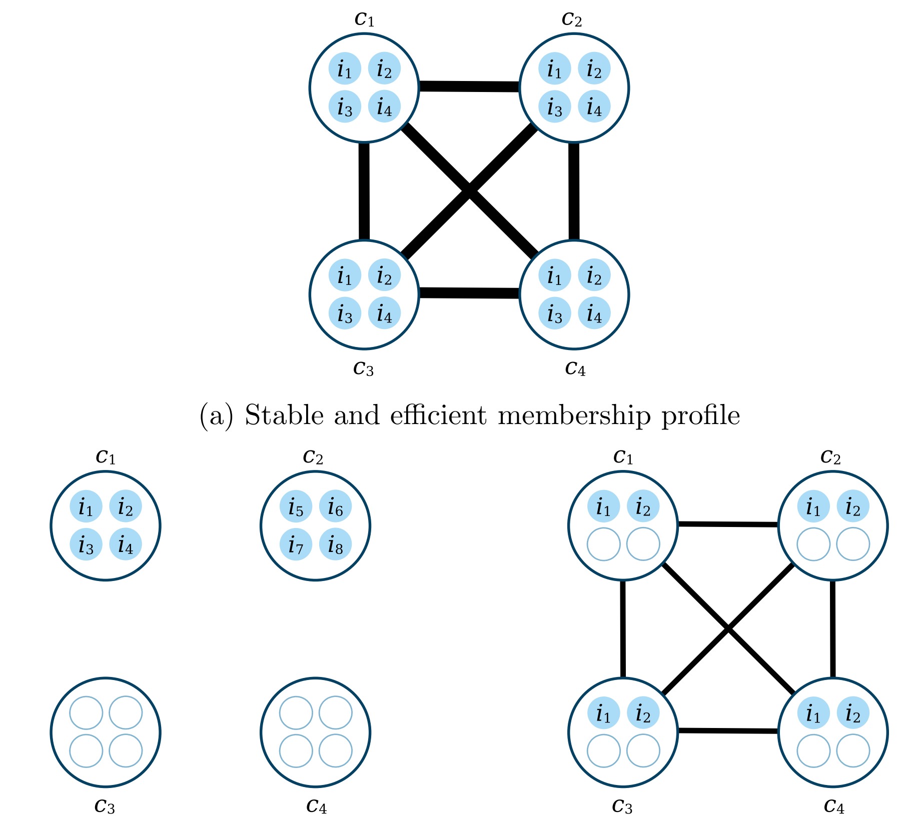
Example 1. Suppose there are 8 individuals and 4 clubs, with individuals able to join 4 clubs and a club having a capacity of 4. In our model (so long as utilities are not too concave) the stable and welfare maximizing membership profile involves 4 clubs that are occupied by the same 4 members. This means that 4 individuals are marginalized. In the coalition framework, the efficient and stable partition involves every individual joining one club each (thus two clubs are occupied by 4 members each), while the remaining 2 clubs remain unoccupied.[6] In contrast to our result there is no marginalization and the network of clubs is empty. In the networks framework, a relation is bilateral; so clubs consist of exactly 2 members. An efficient and stable network involves all 4 clubs being occupied by the same 2 members. Thus six individuals are left out of clubs and marginalization is even greater than in our model and clubs are smaller (less connected and hence less productive). □

We close the introduction with a few words on the relation with the matching literature. A key underlying motivation for the matching literature is that individuals (or firms) have preferences over the individuals that are matched (see Roth and Sotomayor (1992)). This is the driving force for the original one-to-one matching models and remains a central feature of many-to-many matching models (see e.g., Hatfield and Kominers (2015), Rostek and Yoder (2019), Bando and Hirai (2021), Echenique and Oviedo (2006), Klaus and Walzl (2009)). By contrast, the focus of our paper is on the size of memberships (both for individuals and clubs) and on the structure of connections between the clubs. The methods of analysis and the results (on marginalization and on the structure of club networks) are therefore quite different.[7]

Section 2 presents the model, Section 3 presents an analysis of the marginalization and Section 4 presents our results on network structure of clubs. Section 6 presents case studies on inter-locking directorates and boards of editors of journals. Section 7 concludes. All the proofs are presented in the Online Appendix.

[6] For our definitions of stability and efficiency see Section 2 below.

[7] We have also extended our model to a setting where individuals have a preference for same-type club mates. Our methods of analysis can be extended in a straightforward manner to cover this case: indeed a small inclination for homophily leads to a strong division of individuals into distinct groups, and this further exacerbates payoffs inequality and undermine overall efficiency.



(a) Stable and efficient membership profile

(b) Stable and efficient coalition

(c) Stable and efficient network

Figure 1: Comparison of our approach with coalitions and networks

2 The Model

There is a set of individuals $I = \{i_1, \ldots, i_n\}$ and a set of clubs $C = \{c_1, \ldots, c_m\}$. We use i to denote a typical individual and c to denote a typical club. Individuals join clubs to become members. A *membership profile* is represented by a matrix $\boldsymbol{a} = (a_{ic})_{i \in I, c \in C}$ where $a_{ic} \in \{0, 1\}$ indicates whether individual i is a member of club c.

We define a few notions based on a membership profile $\boldsymbol{a}$. The *degree* of individual i, given a membership profile $\boldsymbol{a}$, is the number of clubs joined by i:

$$d_i(\boldsymbol{a}) = \sum_{c \in C} a_{ic}.$$

The *membership size* of club c, given a membership profile $\boldsymbol{a}$, is the number of individuals who join c:

$$s_c(\boldsymbol{a}) = \sum_{i \in I} a_{ic}.$$

There is a *link* between two clubs if they share common members. The *link strength* between clubs c and c', given a membership profile $\boldsymbol{a}$, is the number of common members they share:

$$g_{cc'}(\boldsymbol{a}) = \sum_{i \in I} a_{ic} a_{ic'}.$$

Following the large literature in club theory, we shall assume that there are strong congestion effects that set limits to club capacity (see Buchanan (1965) and Page and Wooders (2010)). Similarly, we assume that individuals can only join a certain number of clubs; this is because they have a fixed amount of time and participating in a club has a minimum time commitment. Formally, we assume that $d_i(\boldsymbol{a}) \leq D$, for all $i \in I$, and $s_c(\boldsymbol{a}) \leq S$, for all $c \in C$, where D and S are two positive integers. The set of feasible membership profiles is $A = \{\boldsymbol{a} \in \{0, 1\}^{n \times m} : d_i(\boldsymbol{a}) \leq D, s_c(\boldsymbol{a}) \leq S\}$. We also assume that $2 \leq S \leq n$ and $2 \leq D \leq m$: this ensures that at least one club can be fully occupied and at least one person can join the maximum number of clubs.

A club provides goods and services to its members. The productivity of a club depends on its size and on the links it has with other clubs. We assume that until the capacity is reached, club productivity increases in the number of its members. And we assume that the productivity of a club is increasing in the strength of the ties it maintains with other clubs.[8]

[8] In some contexts, club productivity may be falling in links with other clubs. This happens for instance if

In different contexts, we can interpret clubs as different institutions. For example, a club can be a board of a firm: links between boards, created by overlapping directors, may help the transmission of best practices and the coordination of corporate strategies. A club can also be an editorial board: links between boards, generated by shared editors, can facilitate knowledge spillovers. Depending on the roles links serve, the marginal returns from link strength vary. If the link helps to convey factual information then the marginal returns from link strength may be declining. On the other hand, if the information concerns complex issues such as new technologies or standards then marginal returns to link strength may be increasing. Similarly, if we are in a context of developing common standards (technological or social) then there may be value in significant overlap of membership.

With these ideas in mind, let us define the productivity of club $c \in C$ in profile $\boldsymbol{a}$ as

$$\pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a})) + \sum_{c' \neq c} h\left(g_{cc'}(\boldsymbol{a})\right), \tag{1}$$

where returns from membership size, f, are strictly increasing with $f(0) = 0$, and the externality from links, h, is increasing with $h(0) = 0$. The next section studies the benchmark case of linear increasing returns case: $h(x) = \alpha x$, with $\alpha \geq 0$. We take up the case of convex and concave returns in Section 4.

Turning to individual utility, we assume that an individual enjoys benefits from the productivity of clubs she joins. Given a profile $\boldsymbol{a}$, the utility of individual $i \in I$ is

$$u_i(\boldsymbol{a}) = v\left(\sum_{c \in C} a_{ic}\pi_c(\boldsymbol{a})\right), \tag{2}$$

where v is strictly increasing with $v(0) = 0$. In situations where individuals are directors of boards, it is natural to assume that their utility increases at a decreasing rate with the aggregate productivity of clubs they are in, so $v''(\cdot) \leq 0$.

We study efficient and stable memberships. We consider two standards for a membership profile to be efficient: maximizing the utilitarian welfare of individuals and maximizing the aggregate productivity of clubs.

the clubs are in a competitive setting and when individuals belong to many clubs, they allocate limited time to each of their clubs and that lowers their productivity. The analysis of clubs and networks with negative spillovers can be carried out using the same methods as we develop for the case of positive spillovers across clubs. We comment on the implications of negative spillovers after presenting the results for positive spillovers.

Definition 1. A membership profile $\boldsymbol{a} \in A$ is the *utilitarian optimum* if for all $\boldsymbol{a'} \in A$,

$$\sum_{i \in I} u_i(\boldsymbol{a}) \geq \sum_{i \in I} u_i(\boldsymbol{a'}).$$

A membership profile $\boldsymbol{a} \in A$ is *clubs-efficient* if for all $\boldsymbol{a'} \in A$,

$$\sum_{c \in C} \pi_c(\boldsymbol{a}) \geq \sum_{c \in C} \pi_c(\boldsymbol{a'}).$$

Turning to strategic stability, it seems reasonable to require that individuals should be able to quit clubs if that increases their utility and clubs should be able to expel members if that raises their productivity. In addition, it seems reasonable to require that an individual and a club cannot coordinate on a deviation that makes them both strictly better off. i.e., no pair of individual i and club c can both benefit from a joint deviation where i is allowed to quit any clubs she is in, c is allowed to exile any members it has, and i joins c. We propose a notion of stability that reflects these ideas.

Formally, let $a_i = (a_{ic})_{c \in C}$ and $a_c = (a_{ic})_{i \in I}$ be the vectors recording the clubs i joins and the members c has, and let $a_{-i} = (a_{i'c})_{i' \neq i, c \in C}$ and $a_{-c} = (a_{ic'})_{i \in I, c' \neq c}$ denote the club joining of individuals other than i and member admission of clubs other than c. Moreover, we use $a_{-i,c} = (a_{i'c'})_{i' \neq i, c' \neq c}$ to represent the membership profile excluding individual i and club c, and we use $a_{-ic} = (a_{i'c'})_{i'c' \neq ic}$ to represent the membership profile excluding the relationship between individual i and club c. We write $a \geq a'$ if a is element-wise greater than or equal to a'.

Definition 2. A membership profile $\boldsymbol{a} \in A$ is *stable* if

1. $\forall i \in I, c \in C$: there is no $\boldsymbol{a'} \in A$ with $a'_i \leq a_i$ and $a'_{-i} = a_{-i}$ such that $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$, or $a'_c \leq a_c$ and $a'_{-c} = a_{-c}$ such that $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$, and

2. $\forall i \in I, c \in C$: there is no $\boldsymbol{a'} \in A$ with $a'_{ic} = 1$, $a'_{-ic} \leq a_{-ic}$, and $a'_{-i,c} = a_{-i,c}$ such that $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$ and $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$.

The definition of stability assumes that clubs have objectives that may be independent of the utility of individuals members. Such an assumption seems appropriate in applications where the clubs have some "governing body" or an owner who makes decision on behalf of the club. In Section 6 we present three case studies where such an assumption is justified.

3 Marginalization

This section presents an analysis of a benchmark model in which returns from links take a linear form, $h(x) = \alpha x$, where $\alpha \geq 0$. So, there is a positive externality from links with other clubs when $\alpha > 0$. We take up non-linear functions $h(.)$ in section 3.1 below.

We first investigate stable membership profiles. Substituting the linear functional form for $h(\cdot)$ in the club productivity function in (1), we see that the productivity of a club $c \in C$ under a membership profile $\boldsymbol{a}$ is

$$\Pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a})) + \alpha \sum_{i \in C} a_{ic}(d_i(\boldsymbol{a}) - 1).$$

Observe that a club prefers an individual who is also a member of other clubs. Similarly, given their utility in (2), individuals prefer clubs with higher productivity. These two incentives press in the same direction: clubs like well-connected individuals and individuals prefer well-connected clubs. Thus, in this model, the incentives of clubs and individuals press toward marginalizing poorly connected clubs and poorly connected individuals.

To make this precise, let us define a partition of individuals and clubs. Let π^* be the highest productivity a club can achieve and u^* be the highest utility an individual can enjoy. Observe that in our benchmark model,

$$\pi^* = f(S) + \alpha S(D-1) \text{ and } u^* = v(D\pi^*).$$

Next note that for a membership profile $\boldsymbol{a}$, the set of individuals I can be partitioned into four parts: a first group $I_1(\boldsymbol{a})$ that consists of individuals who join D clubs and obtain utility u^*; a second group $I_2(\boldsymbol{a})$ that consists of individuals who join D clubs but do not obtain utility u^*; a third group, $I_3(\boldsymbol{a})$, who join some but not D clubs; and a fourth group, $I_4(\boldsymbol{a})$, that consists of individuals who join no clubs.

$$\begin{aligned}
I_1(\boldsymbol{a}) &= \{i \in I : d_i(\boldsymbol{a}) = D, u_i(\boldsymbol{a}) = u^*\} \\
I_2(\boldsymbol{a}) &= \{i \in I : d_i(\boldsymbol{a}) = D, u_i(\boldsymbol{a}) < u^*\} \\
I_3(\boldsymbol{a}) &= \{i \in I : 0 < d_i(\boldsymbol{a}) < D\} \\
I_4(\boldsymbol{a}) &= \{i \in I : d_i(\boldsymbol{a}) = 0\}
\end{aligned}$$

Similarly, the set of clubs can be partitioned into three parts. The first group, $C_1(\boldsymbol{a})$, consists of clubs with productivity π^*; the second group, $C_2(\boldsymbol{a})$, consists of clubs with positive productivity less than π^*; and a third group, $C_3(\boldsymbol{a})$, that consists of clubs with zero productivity.

$$C_1(\boldsymbol{a}) = \{c \in C : \pi_c(\boldsymbol{a}) = \pi^*\}$$
$$C_2(\boldsymbol{a}) = \{c \in C : 0 < \pi_c(\boldsymbol{a}) < \pi^*\}$$
$$C_3(\boldsymbol{a}) = \{c \in C : \pi_c(\boldsymbol{a}) = 0\}$$

Notice that if total club capacity is less than the number of individuals, $mS < n$, then $n - mS$ individuals are necessarily left out of clubs. We are interested in marginalization that results from an unfair assignment of club memberships to agents, that is in the number of individuals who are unnecessarily left out of clubs. This is the number of individuals that could be assigned to some clubs if clubs membership was distributed more fairly. These are individuals who are *unfairly left out of clubs*. Let us say that a membership profile $\boldsymbol{a}$ exhibits *marginalization of individuals* if some individuals become members of clubs, some individuals are not members of any club, and it is possible to reassign club memberships increasing the number of agents in clubs and keeping the total number of club memberships unchanged. The total number of club memberships under $\boldsymbol{a}$ is equal to $\sum_{i\in I} d_i(\boldsymbol{a})$ and if there exist agents joining clubs then $\sum_{i\in I} d_i(\boldsymbol{a}) > 0$. If the total number of individuals $n \leq \sum_{i\in I} d_i(\boldsymbol{a})$ then each individual could be assigned a club membership keeping the total number of club memberships unchanged. Hence there are $|I_4(\boldsymbol{a})|$ agents who are unfairly left out of clubs and so in this case $\boldsymbol{a}$ exhibits marginalization of individuals if $|I_4(\boldsymbol{a})| > 0$. If the total number of individuals $n > \sum_{i\in I} d_i(\boldsymbol{a}) > 0$ then, keeping the total club membership unchanged, $n - \sum_{i\in I} d_i(\boldsymbol{a})$ individuals must be left out of clubs and only $|I_4(\boldsymbol{a})| - n + \sum_{i\in I} d_i(\boldsymbol{a})$ individuals who are out of clubs could be assigned a club membership. Thus in this case the latter is the number of individuals who are left out of clubs unfairly and $\boldsymbol{a}$ exhibits marginalization of individuals if $|I_4(\boldsymbol{a})| - n + \sum_{i\in I} d_i(\boldsymbol{a}) > 0$ (notice that the left hand side of this inequality is always non-negative).

Notice that the minimal number of individuals needed to take $\sum_{i\in I} d_i(\boldsymbol{a})$ club memberships is equal to $\lceil \sum_{i\in I} d_i(\boldsymbol{a})/D \rceil$. Therefore, in the case of $\sum_{i\in I} d_i(\boldsymbol{a}) \geq n > 0$, the maximum number of individuals who are left out of clubs unfairly is equal to $n - \lceil \sum_{i\in I} d_i(\boldsymbol{a})/D \rceil$ and, in the case of $n > \sum_{i\in I} d_i(\boldsymbol{a}) > 0$, the maximum number of individuals who are left out of clubs unfairly is equal to $\sum_{i\in I} d_i(\boldsymbol{a}) - \lceil \sum_{i\in I} d_i(\boldsymbol{a})/D \rceil$. Based on these observations we define a *measure of marginalization for individuals* as

$$\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = \begin{cases} \frac{|I_4(\boldsymbol{a})|-n+\min\left(n,\sum_{i\in I} d_i(\boldsymbol{a})\right)}{\min\left(n,\sum_{i\in I} d_i(\boldsymbol{a})\right)-\left\lceil\frac{\sum_{i\in I} d_i(\boldsymbol{a})}{D}\right\rceil}, & \text{if } \sum_{i\in I} d_i(\boldsymbol{a}) > 0 \text{ and } I_4(\boldsymbol{a}) \neq \varnothing \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Notice that for any membership profile $\boldsymbol{a}$, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \in [0, 1]$. The maximal value of $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$ is attained when club memberships are assigned to the minimal number of agents needed to exhaust the total number of club memberships. In this case we would say that $\boldsymbol{a}$ exhibits *extreme marginalization*. In addition, if membership profiles $\boldsymbol{a}$ and $\boldsymbol{a'}$ have the same number of club memberships, $\sum_{i\in I} d_i(\boldsymbol{a}) = \sum_{i\in I} d_i(\boldsymbol{a'})$, and there are more individuals unfairly left out of clubs under $\boldsymbol{a}$ than under $\boldsymbol{a'}$ then $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) > \mathcal{M}_{\mathcal{I}}(\boldsymbol{a'})$.

Let us work through some examples to illustrate how the measure of marginalization works and to develop a feel for the different issues at work in our model. Consider the following example. Suppose that $n \geq 7$, $m = 4$, $D = 3$ and $S = 4$. A membership profile $\boldsymbol{a}$ in which five individuals exhaust their membership availability while a sixth individual joins one club is stable. The total number of club memberships under $\boldsymbol{a}$ is equal to 16. Notice that $\lceil 16/3 \rceil = 6$ so the 16 club memberships are taken by the minimal number of individuals needed to take all of them. There are $|I_4(\boldsymbol{a})| = n - 6 \geq 1$ individuals left out of clubs and $\min(|I_4|, |I_4| - n + 16) = \min(n - 6, 10) \geq 1$ of them are left out of clubs unfairly. Since the maximal number of individuals that can be left out of clubs unfairly is equal to $\min(n, \sum_{i\in I} d_i(\boldsymbol{a})) - \lceil \sum_{i\in I} d_i(\boldsymbol{a})/D \rceil = \min(n, 16) - 6 = \min(n - 6, 10)$ so we have $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = \min(n - 6, 10)/\min(n - 6, 10) = 1$ and so $\boldsymbol{a}$ exhibits an extreme marginalization.

The example above might suggest that any stable club membership exhibits extreme marginalization. This, however, is not the case as the following example illustrates.

Example 2. Suppose that $m = 10$, $n \geq m + 2$, and $D = S = 6$. Consider the following membership profile, $\boldsymbol{a}$. Let i_x, i_y and i_z be three individuals who join 4 clubs. For other individuals, let 8 of them, whom we denote by $i_1, \ldots, i_8$, join $D = 6$ clubs and the rest of them join no club. Allocate i_x, i_y, i_z, i_1, i_2, i_3 and i_4 to four clubs c_1, c_2, c_3 and c_4 in the way depicted in Figure 2. Also, let individuals i_1 to i_4 join any three other clubs and let individuals i_5 to i_8 join clubs $c_5, \ldots, c_{10}$. This membership profile is stable. To see how, under this membership profile, all clubs are full and clubs other than c_1 to c_4 reach the highest productivity possible and would not want any deviations. For clubs c_1 to c_4, they wish to make deviations. For example, c_1 wants to admit i_4 instead of i_x, i_y or i_z. If i_4 joins c_1, the productivity of c_1 would raise by 2α and be higher than that of c_2, c_3 and c_4 she is currently in. With this logic, it seems that i_4 would want to quit c_2, c_3 or c_4 and join c_1. However, note that with the deviation, the degree of i_x, i_y or i_z drops by 1, making the productivity of c_2, c_3 and c_4 drop by α. Although i_4 leaves one of c_2, c_3 and c_4, she is still in two of them. The aggregate productivity i_4 enjoys from clubs drops by 2α, which cancels out the productivity gain from c_1. Hence, i_4 has no incentive to make the deviation. Using the same logic, we can show that c_2, c_3 and c_4 cannot attract a higher-degree individual to replace i_x, i_y or i_z as well and the membership profile is stable. $\square$

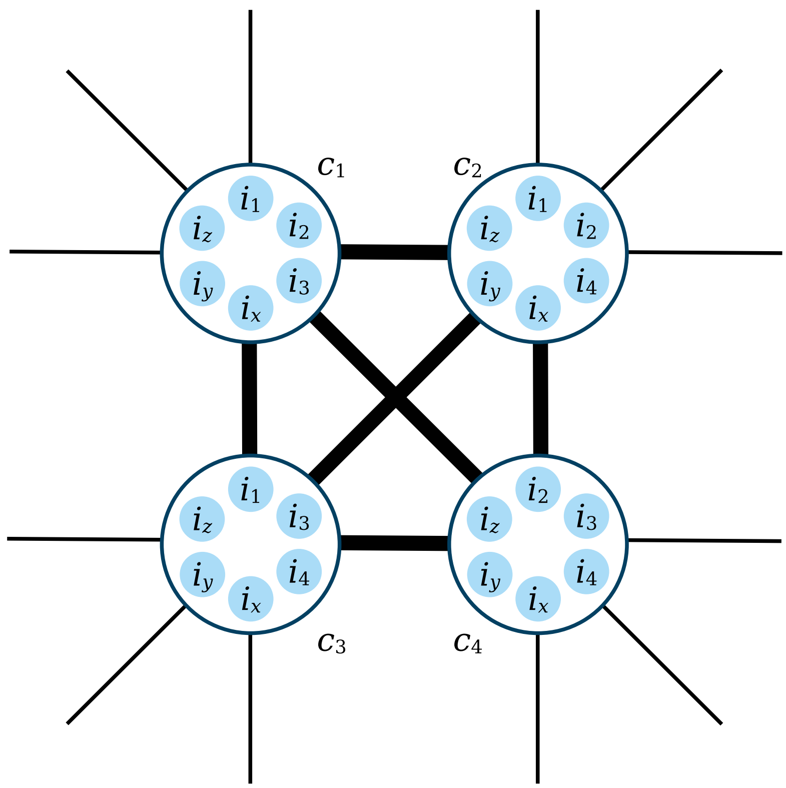


Figure 2: The coordination problem.

There are 60 total club memberships and $|I_4(\boldsymbol{a})| = n - 11 > 0$ individuals left out of clubs under $\boldsymbol{a}$. The number of individuals left out of clubs unfairly under $\boldsymbol{a}$ is $|I_4| - n + \min(n, 60) = \min(n, 60) - 11 > 0$ and the maximal number of individuals who could be left out of clubs when there are 60 club memberships is $\min(n, 60) - \lceil 60/6 \rceil = \min(n, 60) - 10$. Thus the value of the measure of marginalization for membership profile $\boldsymbol{a}$ is $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = 1 - \frac{1}{\min(n,60)-10} < 1$. Hence $\boldsymbol{a}$ does not exhibit an extreme marginalization of individuals.

This example draws attention to a coordination problem among individuals and clubs: note that the 10 clubs and individuals i_1 to i_8, i_x and i_y would be better off in the membership profile where the clubs are exactly filled by those individuals so that all those individuals have degree D, that is a membership profile featuring extreme marginalization. The combination of integer constraints and coordination problems gives rise to a number of complications that inform the characterization of stable membership profiles that is presented below.

Proposition 1. *Assume that $h(x) = \alpha x$, where $\alpha \geq 0$. There exists a stable membership profile. A membership profile $\boldsymbol{a} \in A$ is stable if and only if*

(i) for every individual $i \in I$ and club $c \in C$, if i is not a member of c, then either $d_i(\boldsymbol{a}) = D$ or $s_c(\boldsymbol{a}) = S$,

(ii) for every club c with fewer than S members, every individual i, and every club c' that i joins, if i is not a member of c, then

$$\pi_c(\boldsymbol{a}) + f(s_c(\boldsymbol{a}) + 1) - f(s_c(\boldsymbol{a})) + \alpha(D - 1) \leq \pi_{c'}(\boldsymbol{a}),$$

In addition, if $\alpha > 0$, then

(iii) for every individual i who joins fewer than D clubs, every club c that i does not join, every individual i' in club c must have with $d_{i'}(\boldsymbol{a}) > d_i(\boldsymbol{a})$, and

(iv) for every individual i who joins D clubs, every club c that i does not join and every individual i' that is a member of c, if $d_i' < D$, then

$$\pi_c(\boldsymbol{a}) + \alpha(D - d_{i'}(\boldsymbol{a})) - \alpha \sum_{c'' \neq c'} a_{ic''} a_{i'c''} \leq \pi_{c'}(\boldsymbol{a}), \textit{ for all } c' \textit{ that } i \textit{ joins.}$$

The proof is presented in the Online Appendix. Let us briefly elaborate on the content of the conditions so that we can appreciate some of the arguments that are involved. The four conditions ensure that there is no profitable deviation for a pair of individual i and a club c in four different cases that together exhaust all possible situations.

Point (i) considers deviation where $d_i < D$ and $s_c < S$. We require that there does not exist such a pair as otherwise i can join c and both are better off. Point (ii) considers deviation where $d_i = D$ and $s_c < S$. We require that i does not want to quit an existing club to join c. The condition states that the productivity of c, taking into account the change resulting from i's joining, must not be greater than that of any club c' that i is currently a member of.

In points (i) and (ii) we assume $s_c < S$. So, they are not about c replacing a low-degree individual with a higher-degree one, but concern i joining a higher-productivity club. Hence, the two conditions are needed both when $\alpha = 0$ and when $\alpha > 0$. For the next two situations we look at, $s_c = S$. They are only needed when $\alpha > 0$.

Point (iii) considers deviation where $d_i < D$ and $s_c = S$. We require that c does not want to replace an existing member with i. The requirements leads to the characterization that for i, i' with degree less than D, if $d_i \geq d_{i'}$, then the set of clubs i joins is a superset of clubs i' joins.

Point (iv) considers deviation where $d_i = D$ and $s_c = S$. Now, for i to join c, i needs to quit a club c' and c needs to expel a member i'. A profitable deviation does not exist if either (1) $d_i \leq d_{i'}$, so that the club has no replacement incentive, or (2) the individual has no wish to switch clubs. Note, however, the condition for i to not want to change does not only require that the productivity of c, taking into account the change resulting from i's joining, is not greater than that of c', as in the case of (ii). There is an additional consideration that i hopes c's exiling of i' does not hurt her utility (this is key to the stability of non-marginal membership profile in Example 2). This is captured by the term $\alpha \sum_{c'' \neq c'} a_{ic''} a_{i'c''}$.

Equipped with this characterization, we can provide a fairly complete description of the partition of individuals and clubs in a stable membership profile. This will allow us to answer the question of whether or not stability implies high marginalization of individuals and clubs. Analogous to the measure of marginalization of individuals we define a *measure of marginalization of clubs* for a given membership profile $\boldsymbol{a}$:

$$\mathcal{M}_C(\boldsymbol{a}) = \begin{cases} \frac{|C_3(\boldsymbol{a})| - m + \min\left(m, \sum_{c \in C} s_c(\boldsymbol{a})\right)}{\min\left(m, \sum_{c \in C} s_c(\boldsymbol{a})\right) - \left\lceil \frac{\sum_{c \in C} s_c(\boldsymbol{a})}{S} \right\rceil}, & \text{if } \sum_{c \in C} s_c(\boldsymbol{a}) > 0 \text{ and } C_3(\boldsymbol{a}) \neq \varnothing \\ 0, & \text{otherwise.} \end{cases}$$

We call a membership profile $\boldsymbol{a}$ *egalitarian* if there is minimal difference in the degrees between individuals, $\max_{i,j \in I} |d_i(\boldsymbol{a}) - d_j(\boldsymbol{a})| \leq 1$.

Proposition 2. *Assume that $h(x) = \alpha x$, where $\alpha \geq 0$. When $\alpha = 0$, an egalitarian membership profile is stable. When $\alpha > 0$, for a stable $\boldsymbol{a}$,*

- *if* $nD \geq mS$, *then*

$$\frac{mS}{D} - \frac{S(D+3)}{2} \leq |I_1(\boldsymbol{a})| \leq |I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \leq \frac{mS}{D} \text{ and}$$
$$n - \frac{mS}{D} - S \leq |I_4(\boldsymbol{a})| \leq n - \frac{mS}{D}.$$

Therefore, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$.

- *if* $nD < mS$, *then*

$$\frac{nD}{S} - D \leq |C_1(\boldsymbol{a})| \leq \frac{nD}{S} \text{ and}$$
$$m - \frac{nD}{S} - D \leq |C_3(\boldsymbol{a})| \leq m - \frac{nD}{S}.$$

Therefore, $\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) \geq 1 - \frac{S}{\min\left(\frac{mS-nD}{D},(S-1)n\right)}$.

The proof is presented in the Online Appendix.

In the absence of network externalities, it is fairly straightforward to see that an egalitarian club profile is stable. Given $nD > mS$, assign the mS club slots to distinct individuals, this is clearly stable as there is no advantage of having common membership in clubs (the difference in degree between the maximally connected and minimally connected individuals is 1). Given $nD < mS$, assign the nD membership capacity across the nD/S clubs. Everyone has an equal number of memberships equal to D.[9]

Turning to the setting with positive externalities, let us comment on the expressions for the bounds. Clearly, mS/D is the maximal number of individuals who can be a member of D clubs each. So the upper bound on $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|$ is fairly immediate. Let us comment on the lower bound for $|I_1(\boldsymbol{a})|$. To do this we derive an upper bound on $|I_2(\boldsymbol{a})|$ and $|I_3(\boldsymbol{a})|$. To derive a bound on the number of individuals in $|I_2(\boldsymbol{a})|$, note that all of them must join a club in $C_2(\boldsymbol{a})$. The number of clubs in $C_2(\boldsymbol{a})$ is limited by D because the member who has the highest degree in the least productive club of $C_2(\boldsymbol{a})$ must join all clubs in $C_2(\boldsymbol{a})$, otherwise, she would deviate to join another $C_2(\boldsymbol{a})$ club and the club is willing to take her. Therefore, the number of available slots for $I_2(\boldsymbol{a})$ individuals from $C_2(\boldsymbol{a})$ clubs is (weakly) smaller than $(S-1)D$. In the proof we show that the number of $I_2(\boldsymbol{a})$ individuals who only join one

[9] Matters are slightly more complicated when nD/S is not an integer: in that case, let $\lfloor nD/S \rfloor$ clubs have S members and one club have (nD) mod S members. The structure is stable and every individual has degree D.

$C_2(\boldsymbol{a})$ club is limited by $S-1$: putting together these numbers we arrive at the bound of $S-1+[(S-1)D-(S-1)]/2=(S-1)(D+1)/2$ for the number of individuals in $I_2(\boldsymbol{a})$. Turning to $|I_3(\boldsymbol{a})|$, observe that for individuals in $I_3(\boldsymbol{a})$, if an individual i's degree is greater than or equal to the degree of another individual i', then the set of clubs i joins must be a superset of the set of clubs i' joins. Otherwise, i can crowd out i' and join one more club. Thus, for a club that hosts the individual with the lowest degree in $I_3(\boldsymbol{a})$, it must be the case that it hosts all individuals in $I_3(\boldsymbol{a})$. Since a club can host at most S members, $|I_3(\boldsymbol{a})|\leq S$. The expression in the Proposition follow by noting that $S(D+3)/2>S+(S-1)(D+1)/2$.

We now turn to the marginalization results. When $nD>mS$ then we can derive a lower bound on the number of individuals that are left out of clubs in any stable membership profile, $|I_4(\boldsymbol{a})|\geq n-mS/D-S$. This means that there are at most $mS/D+S$ individuals who join at least one club in any such profile. This is $mS/D+S-\lceil mS/D\rceil\leq(D-1)/D+S$ greater than the minimal number of individuals needed to take all mS club memberships. Thus the number of individuals who are left out of clubs unfairly under any stable profile differs from the maximal such number by a constant and because of that any stable membership profile features high marginalization. To see this more precisely, if $n>mS$, so that the number of individuals exceeds total clubs capacity, then $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})=1-D/((D-1)m)$. Thus if m is large and n is sufficiently large so that $n>mS$, then every stable membership profile exhibits marginalization of individuals which is close to 1. Similarly, if $nD<mS$, and $m>nD$, $\mathcal{M}_{\mathcal{C}}(\boldsymbol{a})=1-S/((S-1)n)$. Thus if n is large and $m>nD$, every stable membership profile marginalizes clubs. The argument above relies on the fact that the capacity constraints D and S are constant and independent of the number of clubs and the number of individuals. This assumption makes sense when such constraints can be interpreted as time limitations (in the case of individuals) or some physical space restrictions (in the case of clubs).

We next turn to the welfare properties of membership profiles. We have shown that in the presence of a connection externality, a stable membership profile marginalizes individuals or clubs. Are such membership profiles desirable? We show that the answer depends on whether we look at clubs-efficiency or at the utilitarian optimum. In our study of utilitarian optimum, we will make use of the following condition on the concavity of the utility function.

$$v\left(f(S)\right)-v(0)>(n-1)\left(v\left(f(S)+\frac{2\alpha S(D-1)}{n-1}\right)-v\left(f(S)\right)\right). \tag{4}$$

Proposition 3. *Suppose $\alpha>0$. Assume $nD\geq mS$ and that mS/D is an integer.*[10]

[10] In the Online Appendix, we provide characterizations of clubs-efficient and utilitarian optimal membership

- *A membership profile is clubs-efficient if and only if mS/D individuals join D clubs and the remaining individuals join no clubs ($\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = 1$).*

- *If $v''(\cdot) \geq 0$, then a membership profile is an utilitarian optimum if and only if it is clubs-efficient ($\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = 1$). If $v''(\cdot) < 0$ and satisfies condition (4), then in any utilitarian optimum membership profile, either $d_i(\boldsymbol{a}) \leq 1$ for all $i \in I$ or $d_i(\boldsymbol{a}) \geq 1$ for all $i \in I$ ($\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = 0$).*

Assume $nD < mS$ and that nD/S is an integer.

- *If $f''(\cdot) > 0$, then a membership profile is clubs-efficient if and only if nD/S clubs admit S members and the remaining clubs admit no members ($\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) = 1$). If $f''(\cdot) < 0$, then a membership profile is clubs-efficient if and only if $(nD) \bmod m$ clubs admit $\lceil \frac{nD}{m} \rceil$ members and the remaining clubs admit $\lfloor \frac{nD}{m} \rfloor$ members ($\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) = 0$).*

- *A membership profile is an utilitarian optimum if and only if nD/S clubs admit S members and the remaining clubs admit no members ($\mathcal{M}_{\mathcal{C}}(\boldsymbol{a}) = 1$).*[11]

The proof is presented in the Online Appendix.

Consider first the case where $nD > mS$. Proposition 3 tells us that a membership profile that maximizes the aggregate output of the clubs exhibits extreme marginalization: a club-efficient profile allocates exactly mS/D individuals into memberships, all other individuals join no clubs. This is because this marginalization ensures maximal overlap of members between clubs.

Turning to the utilitarian optimum, if the utility of individuals rises at an increasing or constant rate with the productivity of clubs they join, i.e., if $v''(\cdot) \geq 0$, then the profile that is utility-maximizing is the same as the profile that is productivity-maximizing. This is because when $v''(\cdot) = 0$, the aggregate utility of individuals is simply the number of individuals a club can admit, S, times the aggregate productivity of clubs, and when $v''(\cdot) > 0$, utilitarian optimality pushes toward marginalization of individuals, which coincides with the outcome generated by clubs-efficiency. If, on the other hand, the marginal utility is decreasing, i.e., $v''(\cdot) < 0$, then that opens up a potential trade-off: although a concentration of memberships

profiles without the integer condition.

[11] If the integer condition (S divides nD) does not hold, then the utilitarian optimum characterization for when $v''(\cdot) \geq 0$ and when $v''(\cdot) < 0$ could be different. When $v''(\cdot) \geq 0$, there is one club that hosts some but less than S members. When $v''(\cdot) < 0$, the number of clubs that admit some but less than S members ranges from 1 to $S - 1$.

maximizes the total output of clubs, it comes at the expense of entirely excluding $n - mS/D$ individuals from memberships. If the utility function is sufficiently concave – the marginal utility is declining sufficiently rapidly (a condition that is formalized in inequality condition (4), then the welfare benefit from picking more members outweighs the loss to aggregate productivity. We present an example that brings out the difference between clubs-efficiency and utilitarian optimum when we move from a convex/linear to a concave utility function.

Example 3. Suppose $n = 16$, $D = 4$, $m = 8$ and $S = 4$. Figure 3a depicts a membership profile that is clubs-efficient and utilitarian optimum when $v(\cdot)$ is linear. Notice that in this membership profile, 8 individuals (i_1 to i_8) exhaust their membership availability while the other 8 individuals (i_9 to i_{16}) join no clubs. To appreciate the role of concave $v(\cdot)$ is concave, set

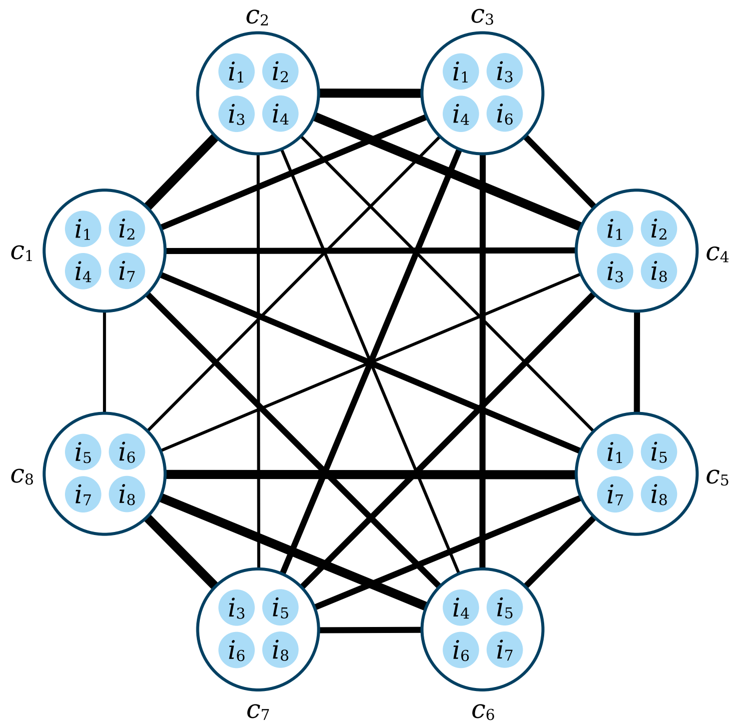
$$v(x) = \begin{cases} 10x & \text{when } x \leq 2f(4) + 8\alpha, \\ 10\,(2f(4) + 8\alpha) + 0.1\,(x - 2f(4) - 8\alpha) & \text{when } x > 2f(4) + 8\alpha. \end{cases}$$

In this case, the clubs-efficient outcomes remains unchanged and is as in Figure 3a, while the utilitarian optimal profile, which features all 16 individuals joining 2 clubs, is given in Figure 3b. □

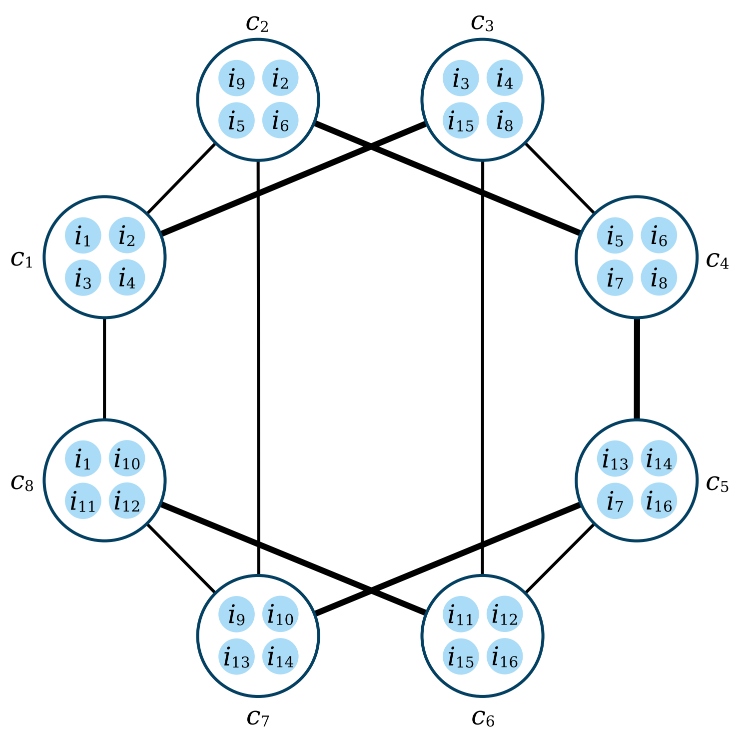
Let us next take up the case where $nD < mS$. On club efficiency, note that there is enough club capacity to cover the individuals, so every person will join D clubs: keeping anyone out of clubs is clearly dominated for clubs. Moreover, as spillovers are linear, there is a constant spillover irrespective of how the individuals are allocated across clubs. So the issue of how to allocate individuals turns on the f function. If f is convex, then it is better to allocate individuals to fewer clubs, i.e., nD/S clubs; if on the other hand, f is concave then you allocate as evenly as possible across clubs, subject to integer constraints.

Regarding utilitarian optimum profiles, no matter what the f function, the optimal profile entails marginalization of clubs. This is because to maximize the aggregate utility of individuals, it is clearly better to allocate more individuals to high-productivity clubs and fewer individuals to low-productivity clubs. This taken in tandem with the assumption that the productivity of a club rises with its size implies the marginalization of clubs.

When we compare Propositions 2 with 3, we see that there exists a tension between the incentives toward marginalization (created by the increasing club productivity from membership and from the strength of links with other clubs) and the demands of inclusiveness (created by the concave utility function and concave club production function).



(a) clubs-efficient and utilitarian optimal profile: convex or linear $v(\cdot)$



(b) utilitarian optimal profile: highly concave $v(\cdot)$

Figure 3: Efficient membership profiles.

We conclude our study of the benchmark model with a brief remark on stable and efficient membership profiles when spillovers across clubs are negative. This happens when $\alpha < 0$ in the benchmark model. Observe that when spillovers are negative, a club would like to only admit members who have no other memberships. So in a world with many individuals relative to club capacity, i.e., $n > mS$, any stable membership profile must involve exactly mS individuals filling the aggregate club capacity, i.e., every person joins at most one club and the resulting club network is an empty network. However, when the number of individuals is small the clubs face a trade-off: on the one hand, their productivity grows with membership (up to their capacity size). On the other hand, expanding membership may necessitate bringing in individuals who are already members of other clubs, and this lowers their productivity. We can apply the methods developed above to show that whatever the outcome of the tradeoff is, a stable profile and an aggregate productivity maximizing profile both feature an egalitarian membership profile, i.e., there does not exist two individuals i and i' with $|d_i(\boldsymbol{a}) - d_{i'}(\boldsymbol{a})| > 1$.

The argument goes as follows: suppose there exist two individuals i and i' where $d_{i'}(\boldsymbol{a}) \geq d_i(\boldsymbol{a}) + 2$. If so, then there exists a club c which i' joins but not i. Clearly, this club c would want to expel i' and recruit i. We show that i is also willing to join c. Since i' is willing to join c, it must be that $(d_{i'}(\boldsymbol{a}) - 1)|\alpha| \leq \pi_c(\boldsymbol{a})$, as otherwise i' would be better off leaving c, which makes the productivity of other clubs i' joins raise by $|\alpha$. It follows that $(d_i(\boldsymbol{a'}) - 1)|\alpha| \leq \pi_c(\boldsymbol{a'})$, where $\boldsymbol{a'}$ is the profile where c admits i instead of i' but is otherwise the same as $\boldsymbol{a}$. This implies i is willing to join c. There is therefore a profitable deviation for the club-individual pair i and c. Turning to maximizing the aggregate productivity of clubs, note that the same deviation also improves the situation: it reduces the productivity of clubs i joins by α and raises the productivity of clubs i' joins by at least α. The result then follows given that i' is in more clubs than i does.

3.1 Non-linear returns from links and marginalization

We have so far considered the case where returns are linearly increasing in link strength. In some prominent instances the returns from link strength are likely to be non-linear. For example, in case club links are used for information sharing then we would expect marginal returns to decline with link strength. On the other hand, if links help members coordinate activities of the clubs, then the marginal returns may be increasing in link strength. With these observations in mind, we examine the implications of non-linear returns from link strength. In this section, we study how robust is the marginalization result when we depart from the linear returns setting. We first show that when the marginal returns from links do not vary too much as links strengthen, then a stable membership profile always features marginalization that is asymptotically close to 1 (like in the case of linear h). We then turn to the case when h is very convex or concave. When h is very convex, we show that there exist stable membership profiles that feature the same level of marginalization as in the case when h is linear. However, there also exist stable profiles in which marginalization is bounded from above by a value smaller than 1 as long as each individual can join at least 3 clubs. When h is very concave, we show that a stable profile always features marginalization. Nonetheless, M_I converges to 1 at a slower rate than when h is linear.

Proposition 4. *Assume that h is strictly increasing.*

(i) If $nD > mS$ and

$$\min_{x\in[0,S-1]}(h(x+1)-h(x)) > \frac{D-1}{D}\max_{x\in[0,S-1]}(h(x+1)-h(x)),$$

then in any stable membership profile $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$.

(ii) Suppose h is convex, if $nD > mS$ then there exists a stable membership profile $\boldsymbol{a}$ with $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$. Suppose h is very convex in the sense that $h(S)-h(S-1) > 2(h(1)-h(0))$, if $n > mS$ and m is even then there exists a stable membership profile $\boldsymbol{a}$ with $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \leq \frac{D}{2(D-1)}$.

(iii) Suppose h is concave, if $n > mS$ then for any stable membership profile $\boldsymbol{a}$, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) > 1 - \frac{D^2S}{(D-1)m}$.

We briefly explain why structures that do not feature marginalization can be stable when h is very convex but not when h is very concave. When h is sufficiently convex, a club wants to form links with certain other clubs that it currently has strong links with, and it is thus willing to keep an individual with a low degree as a member if the individual happens to be in the right clubs. A non-marginalized profile emerges when each individual who joins some clubs has a low degree and happens to be in clubs that are strongly linked to each other. Figure 4 provides an example of such a profile.

The same reasoning does not apply when h is concave and the number of clubs m is large. In this case, suppose there are many individuals of low-degree individuals, for a club c that admits a low-degree individual i, even if i is in clubs that c has weak links with, there exists another individual i' who is in clubs that c does not have links with, which makes c want to exile i and admit i'. Therefore marginalization is robust when h is concave.

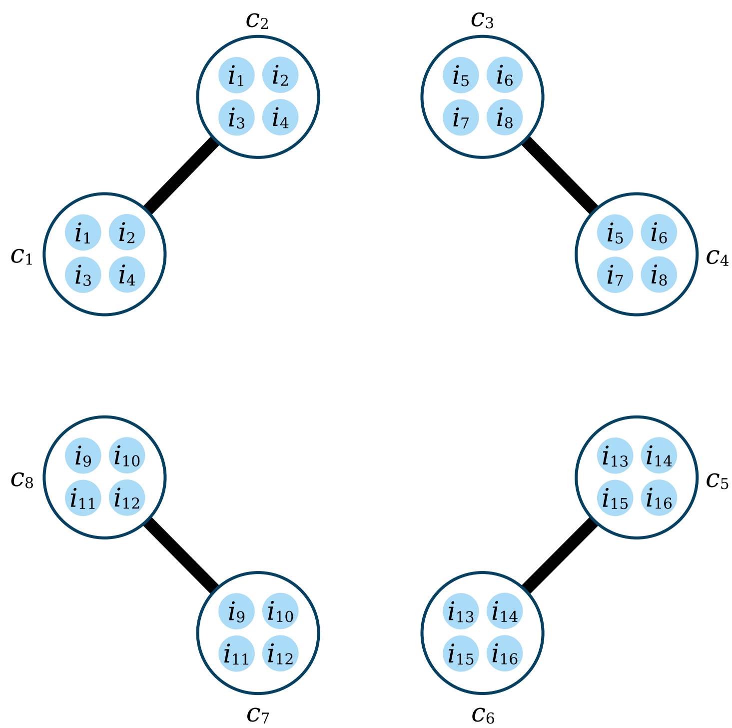


Figure 4: No marginalization when $h(\cdot)$ is very convex.

4 Small worlds, fragmented cliques, and strength of ties

In this section, we examine the network of clubs and the strength of ties that support this network.[12] We start by showing that if returns from link strength are linear then a variety of club networks are stable. We then turn to nonlinear returns and show that if the marginal return from link strength is increasing, then incentives of clubs and individuals push toward disconnected cliques of clubs with full strength links. If, on the other hand, the marginal

[12] A membership profile can be projected both into a network of clubs and a network of individuals. In this section, we focus on the club network. Nevertheless, the individual network, since originated from the same membership profiles as the club network, shares some important properties with the latter. For example, there exists a strong link (link with strength greater than 1) in the club network if and only if there exists a strong link in the individual network, and the club network is connected iff the individual network is connected. Therefore, we can infer the properties of the individual network with an analysis of the club network.

return from link strength is decreasing then the club network entails larger components that are connected through weak links.

Example 4. Suppose that $n > 15$, $m = 6$, $D = 2$ and $S = 5$. Figure 5 depicts two clubs-efficiency and stable membership profiles when returns from links rise linearly. Note that the two profiles lead to the same degree distribution of individuals (the first 15 individuals all join two clubs while the others join no clubs) and the same aggregate link strength clubs have (each club shares five membership overlaps with other clubs). However, the resulting club networks take very different forms: one consists of three separate cliques where all links are of strength 5 while the other is a complete network where all links are of strength 1. This indicates that linear spillovers from links always lead to marginalization of individuals/clubs but the resulting club networks can be very different.

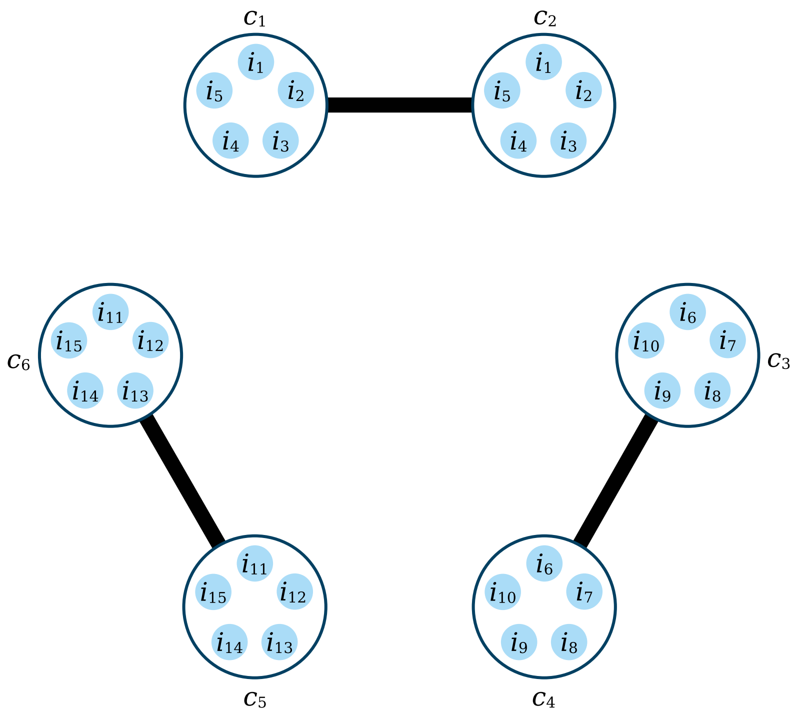
If $h(\cdot)$ is convex, there is a unique clubs-efficient membership profile which is depicted in Figure 5a. When $h(\cdot)$ is convex, the productivity of a club is maximized if the number of membership overlaps it has with other clubs is maximized and concentrated in as few clubs as possible. This can only be achieved when the club network takes the form depicted in Figure 5a.

On the other hand, when $h(\cdot)$ is concave, Figure 5b depicts the unique club-efficient network. When $h(\cdot)$ is concave, clubs want to maximize their membership overlaps with other clubs and spread them as evenly as possible. In this example, for this to be the case, the club network has to be complete with all links being weak.

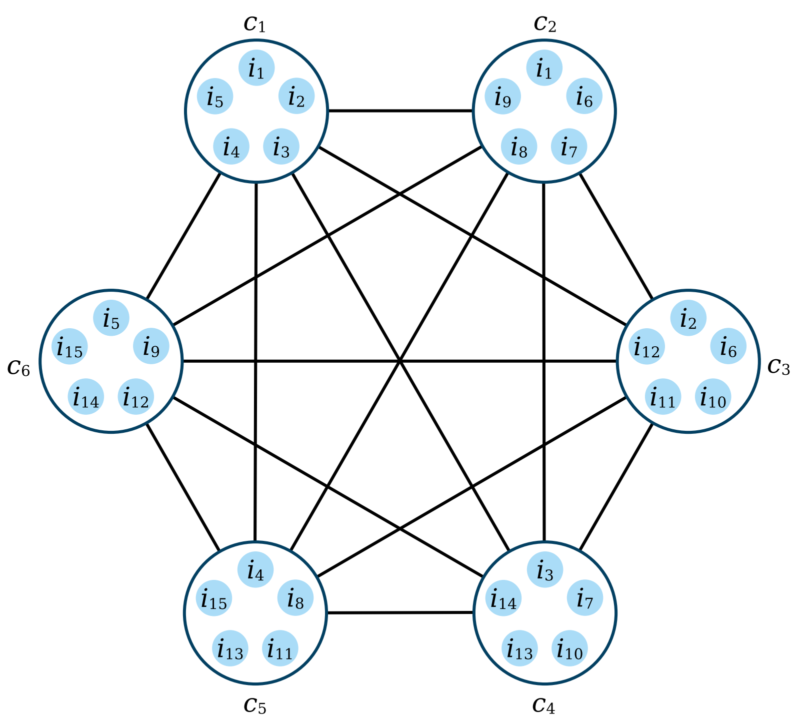
Turning to stability, the structure depicted in Figure 5a is stable when $h(\cdot)$ is convex, since all clubs have reached the highest productivity possible and have no incentives to deviate. It is not stable when $h(\cdot)$ is concave: there is a profitable deviation for individual i_6 and club c_1 where c_1 exiles i_1 to admit i_6 and i_6 leaves c_3 to join c_1.

Similarly, the structure depicted in Figure 5b is stable when $h(\cdot)$ is concave but not so when $h(\cdot)$ is convex. Stability under a concave $h(\cdot)$ is obvious since all clubs have reached the highest productivity possible; instability under a convex $h(\cdot)$ can be verified by again considering the deviation by individual i_6 and club c_1 where c_1 exiles i_2 to admit i_6 and i_6 leaves c_3 to join c_1. $\square$

The above example shows that the curvature of the returns from links has a significant influence on the structure of club networks. To get an intuition about the origin of that influence, notice that when $h(\cdot)$ is 0 at 0, strictly increasing and convex/concave then it is



(a) Convex returns from link strength: $h''(\cdot) > 0$



(b) Concave returns from link strength: $h''(\cdot) < 0$

Figure 5: Clubs-efficient and stable membership profiles.

superadditive/subadditive.[13] If h is superadditive then a solution to the optimisation problem

$$\max_{\boldsymbol{g} \in \mathbb{Z}_{\geq 0}^{C \times C}} \quad \sum_{c' \neq c} h\left(g_{cc'}(\boldsymbol{a})\right) \tag{5}$$

$$\text{s.t.} \quad \sum_{c' \neq c} g_{cc'}(\boldsymbol{a}) \leq T, \tag{6}$$

$$g_{cc'} \leq S, \tag{7}$$

where $T \in \mathbb{Z}_{\geq 0}$ is a constant, is any $\boldsymbol{g}$ such that $g_{cc'} = S$ for $\lfloor T/S \rfloor$ pairs $(c, c') \in C \times C$ with $c \neq c'$, $g_{cc'} = T \bmod S$ for one $(c, c') \in C \times C$ with $c \neq c'$, and $g_{cc'} = 0$ for all the remaining pairs $(c, c') \in C \times C$ with $c \neq c'$. On the other hand, if h is subadditive then a solution to the optimisation problem (5) is any $\boldsymbol{g}$ such that $g_{cc'} = 1$ for T pairs $(c, c') \in C \times C$ with $c \neq c'$ and $g_{cc'} = 0$ for all the remaining pairs $(c, c') \in C \times C$ with $c \neq c'$. Suppose that $nD > mS$ and consider clubs-efficient structures. Clearly any clubs-efficient membership structure must have S members in this case and the strength of a link between an two clubs is at most S. Each agent can contribute at most $D(D-1)/2$ to the total weight of links between clubs and so the total weight of links between clubs is at most $mSD(D-1)/2$. Hence finding a clubs-efficient structure amounts to solving the optimisation problem (5) with $T = mSD(D-1)/2$ and subject to an additional constraint:

$$\boldsymbol{g} \text{ is a membership structure.} \tag{8}$$

Constraint (8) makes the problem of characterising club-efficient membership structures difficult, especially in the case of subadditive (or even concave) h. For some values of parameters S, D, m and n the constraint is not binding and there exist membership structures attaining the maximum of problem (5) without constraint (5). In the case of superadditive h the constraint is not binding when D divides m and $n > mS/D$. In this case an optimal club-membership can be constructed by partitioning clubs in m/D groups $(c_1^i, \ldots, c_D^i)_{i=1}^{m/D}$, choosing m/D groups $(a_1^i, \ldots, a_S^i)_{i=1}^{m/D}$ of size S of individuals, and then assigning all the individuals in group i to all clubs in group i, for each $i \in \{1, \ldots, m/D\}$. In the case of subadditive h establishing the conditions for which the constraint is not binding is an open combinatorial problem (Chee et al., 2013). In such cases an optimal membership structure is related to a so called $(2, 1)$-packing

[13] Recall that a function $h : \mathbb{R} \to \mathbb{R}$ is supperadditive if for all $x, y \in \mathbb{R}$, $h(x + y) \geq h(x) + h(y)$ and it is subadditive if for all $x, y \in \mathbb{R}$, $h(x + y) \leq h(x) + h(y)$.

of the clique over the set of all clubs. In the cases when constraint (8) is binding the problem of finding optimal membership structures is not easier. Given the difficulties with obtaining characterization of optimal membership structures, we restrict attention to the cases of $D = 2$ and m even and provide a characterization of optimal structures then. Even these cases allow for contrasting the optimal structures under super- and subadditive functions h.

Formally, we say a club network $g = g(\boldsymbol{a})$ is clubs-efficient/utilitarian optimum/stable if it is created with a clubs-efficient/utilitarian optimum/stable membership profile $\boldsymbol{a}$. Additionally, we define a k-clique as a subnetwork that has k mutually linked clubs and a k-regular network as a network where all clubs have k links. The complete network is a special kind of regular network where all clubs are linked to each other ($k = m - 1$).

Proposition 5. *Assume $nD \geq mS$, $D = 2$, m is even, and $2 \leq S \leq m - 1$.*

- *When $h(\cdot)$ is superadditive, the clubs-efficient club network consists of m/D separate 2-cliques where all links are of strength S. This network is stable when $h(\cdot)$ is superadditive and unstable when $h(\cdot)$ is subadditive.*

- *When $h(\cdot)$ is subadditive, the clubs-efficient club network is an S-regular network (a complete network when $S = m - 1$) where all links are of strength 1. This network is stable when $h(\cdot)$ is subadditive and unstable when $h(\cdot)$ is superadditive.*

Assume $nD < mS$, $D = 2$, S divides n, and $2 \leq S \leq 2n/S - 1$.

- *When $h(\cdot)$ is superadditive, the utilitarian optimum club network consists of n/S separate 2-cliques where all links are of strength S. This network is stable when $h(\cdot)$ is superadditive and unstable when $h(\cdot)$ is subadditive.*

- *When $h(\cdot)$ is subadditive, the utilitarian optimum club network is an S-regular network (a complete network when $S = 2n/S - 1$) where all links are of strength 1. This network is stable when $h(\cdot)$ is subadditive and unstable when $h(\cdot)$ is superadditive.*

The proof is presented in the Online Appendix.

As mentioned earlier, the club network and the individual network generated by a membership profile share some important properties. The club network mentioned in Proposition 5 can be mapped into individual networks. When $h(\cdot)$ is convex, our characterization involves 2-cliques with strength S links for the club network; the corresponding individual network consists of S-cliques with strength 2 links. When $h(\cdot)$ is concave, our characterization features a S-regular club network with strength 1 links; the corresponding individual network is a $D(S-1)$-regular network with strength 1 links.

5 Extensions

5.1 Strong Stability

In the previous sections, our stability notion checks whether a membership profile is robust to deviations by a single agent (an individual or a club) and a pair of agents (an individual and a club). We can strengthen the stability notion by allowing deviations by subsets of agents of any size.

Definition 3. A membership profile $\boldsymbol{a} \in A$ is strongly stable if $\forall I' \subseteq I$ and $C' \subseteq C$: there is no $\boldsymbol{a'} \in A$ with $a'_{ic} \leq a_{ic}$ for all $(i,c) \in I \times C$ where $i \in I', c \notin C'$ or $i \notin I', c \in C$ and $a'_{ic} = a_{ic}$ for all $i \notin I', c \notin C'$ such that $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$ for all $i \in I'$ and $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$ for all $c \in C'$.

In words, strong stability tests deviation by any group of individuals and clubs where agents within the group can freely add or terminate memberships with each other subject to capacity constraints and terminate memberships with agents outside the group.

We limit our analysis to the case where the aggregate individual availability is non-trivially greater than the aggregate club capacity ($nD > mD + DS$) and show that strong stability does not lead to stronger results than the stability notion we use in previous sections which only considers unilateral and pairwise deviations.[14]

Proposition 6. *Assume that $h(x) = \alpha x$ where $\alpha \geq 0$ and $nD > mS + DS$. A membership profile $\boldsymbol{a} \in A$ is strongly stable if and only if it is stable.*

Proposition 6 shows that our stability characterization cannot be refined by allowing more deviations from agents. Note that both our stability notion and strong stability notion concern deviations that make all deviating agents strictly better off. We define stability in this way to ensure the existence of stable membership profiles. The following example shows why an alternative stability notion that requires no deviations that make all deviating agents weakly better off and some deviating agents strictly better off can lead to the non-existence of stable

[14] The restriction of $nD > mS + DS$ ensures that all clubs are full in a stable membership profile which simplifies our analysis.

profiles. Suppose there are one club that can admit one member and two individuals i_1 and i_2. There are three possible membership profiles here: the club admits no one; the club admits i_1; and the club admits i_2. The first profile is clearly not stable. The second and third profiles are not stable if we adopt the alternative stability concept: suppose the club admits i_x, the deviation by the club and i_y where the club exiles i_x to admit i_y makes i_y strictly better off and the productivity of the club unchanged.

5.2 Heterogeneous Individuals

Our model assumes homogeneous individuals so that the preference of clubs over individuals is completely based on the memberships of the individuals. In reality, individuals differ in other aspects that clubs could care about. To account for ths consideration, let $\theta_i \in \mathbb{R}_{>0}$ be the type of individual i. We assume the productivity of club c in profile $\boldsymbol{a}$ to be

$$\pi_c(\boldsymbol{a}) = f\left(\sum_{i \in I} a_{ic}\theta_i\right) + \sum_{c' \neq c} h(g_{cc'}(\boldsymbol{a})). \tag{9}$$

Note that the production function of clubs under the specification of homogeneous individuals is a special case of function (9) when $\theta_i = 1$ for all $i \in I$. We do not change our assumption about individual utility, that is, individuals have the same utility function.

With the introduction of individual heterogeneity, our partition of clubs into $C_1(\boldsymbol{a})$, $C_2(\boldsymbol{a})$ and $C_3(\boldsymbol{a})$ is not sensible anymore since clubs most likely have different productivities. As a result, the partition of individuals who join D clubs into $I_1(\boldsymbol{a})$ and $I_2(\boldsymbol{a})$ based on what kind of clubs they join is also not sensible. To study whether a stable profile is egalitarian or features marginalization, we use a simpler partition that categorizes individuals solely on their degrees:

$$\begin{aligned}
\hat{I}_1(\boldsymbol{a}) &= \{i \in I : d_i(\boldsymbol{a}) = D\} \\
\hat{I}_2(\boldsymbol{a}) &= \{i \in I : 0 < d_i(\boldsymbol{a}) < D\} \\
\hat{I}_3(\boldsymbol{a}) &= \{i \in I : d_i(\boldsymbol{a}) = 0\}
\end{aligned}$$

To measure marginalization of individuals, we simple replace $|I_4(\boldsymbol{a})|$ in equation (3) with $|\hat{I}_3(\boldsymbol{a})|$. We focus on the more natural situation where $nD \geq mS$ and obtain the following results for stable membership profiles.

Proposition 7. *Suppose all individuals are different from each other: $\theta_i \neq \theta_j$ $\forall i \neq j$, and the productivity of clubs follows* (9)*. Assume that $h(x) = \alpha x$ where $\alpha \geq 0$ and $nD \geq mS$. Then for any stable profile $\boldsymbol{a}$,*

$$\frac{mS}{D} - (D-1)S \leq |\hat{I}_1(\boldsymbol{a})| \leq \frac{mS}{D},$$
$$|\hat{I}_2(\boldsymbol{a})| \leq (D-1)S, \text{ and}$$
$$n - \frac{mS}{D} - (D-1)S \leq |\hat{I}_3(\boldsymbol{a})| \leq n - \frac{mS}{D}.$$

Therefore, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{(D-1)D}{\min\left(\frac{nD-mS}{S}, (D-1)m\right)}$. Additionally,

- *when $\alpha = 0$, $d_i(\boldsymbol{a}) \geq d_j(\boldsymbol{a})$ for all i, j with $\theta_i > \theta_j$ in a stable membership profile $\boldsymbol{a}$.*
- *when $\alpha > 0$, the difference between individuals is relatively less compared to the benefit from connections: $\theta_i - \theta_j < \alpha$ $\forall i, j \in I$, and D does not divide $m + 1$, there exists a stable membership profile $\boldsymbol{a}$ where $d_i(\boldsymbol{a}) \leq d_j(\boldsymbol{a})$ for all i, j with $\theta_i > \theta_j$.*

The proposition shows that in the presence of individual heterogeneity, the connection benefit is not needed to arrive at marginalization. Nonetheless, connection externality influences who gets marginalized. Without it, a stable profile always sorts better types into (weakly) more clubs, so those marginalized must be the low types. When there is connection externality and the difference between individuals is small, a stable profile could reverse the sorting completely and marginalize high types. This is because, with connection externality, club memberships are self-fulfilling: low types are admitted by clubs because of their memberships in other clubs. The condition of D not dividing $m + 1$ is imposed to ensure there does not exist a high type individual who joins only one club less than a lower type individual and thus will be wanted by a club the lower type is in for substitution.

5.3 Richer interdependence between clubs

In some applications, it may be reasonable to suppose that there is a greater value to a club to connecting to large clubs as compared to small clubs. In the basic model, we abstract away from this consideration. Here we discuss how introducing size of neighbouring club shapes stable networks. Let the productivity of a club be

$$\pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a})) + \sum_{c' \neq c} h(g_{cc'}(\boldsymbol{a}))q(s_c(\boldsymbol{a})), \tag{10}$$

where $q'(\cdot) \geq 0$. Note that when $q(x) = 1$ for $x = 1, ..., S$, so that $q'(\cdot) = 0$, we are back to the case where clubs only care about the aggregate strength of their connections.

We again investigate the case of $nD \geq mS$ and study whether a stable profile features marginalization.

Proposition 8. *Suppose the productivity of clubs follows* (10)*. Assume that* $h(x) = \alpha x$*, where* $\alpha \geq 0$ *and* $nD \geq mS$*. When* $\alpha = 0$*, an egalitarian membership profile is stable. When* $\alpha > 0$*, for a stable* $\boldsymbol{a}$*,*

$$\begin{aligned} \frac{mS}{D} - S \leq & |\hat{I}_1(\boldsymbol{a})| \leq \frac{mS}{D}, \\ & |\hat{I}_2(\boldsymbol{a})| \leq S, \text{ and} \\ n - \frac{mS}{D} - S \leq & |\hat{I}_3(\boldsymbol{a})| \leq n - \frac{mS}{D}. \end{aligned}$$

Therefore, $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S}, (D-1)m\right)}$.

Proposition 8 shows that even in this richer model of cross club benefits stable networks will exhibit marginalization. Intuitively, when $mS \geq nD$, in a stable profile most clubs, if not all, will be of the same size (full). Hence, it is obvious that adding the size of neighbouring clubs into the production function does not affect our characterization of marginalization. For strictness, we note that some clubs may not be full in a stable profile when nD is close to mS and prove Proposition 8 using an argument different from the very straightforward one above.

6 Case Studies

In this section, we present two case studies that map our theory onto empirical context of inter-locking directorates and editorial boards of directors.

Interlocking Directorates: It is widely recognized that the board-to-board ties serve as a mechanism for the diffusion of corporate practices, strategies, and structures (Mizruchi (1996)). We may consider boards as clubs and directors as individuals; links between clubs raise productivity. In what follows, we discuss empirical studies on interlocking directorates and explain how our model sheds light on the understanding of the empirical findings.

Consider first the degree distribution of board directors. Conyon and Muldoon (2006) study the affiliations of board directors who hold positions in 1,733 firms in the United States in 2003. They find that 80.37% of the directors sit only on one board, 13.02% of them sit on two boards, and the remaining 6.61% of the directors sit on 8.6 boards on average. Thus most directors hold only one or two positions, but there is a small fraction of directors who occupy many positions. The authors show that similar patterns hold in Germany and the UK. This inequality in degrees of directors is in line with the marginalization result (Proposition 2)

Consider next the structure of board networks. Mizruchi (1982) provides a historical analysis of the US board network among 167 firms at seven points from 1904 to 1974, finding that almost all nodes were within distance 4. More recently, with the increased availability in data and advancement in analyzing techniques, Davis et al. (2003) study the largest manufacturing and service firms in the US over the period 1982 to 1999. They show that despite the major changes in the nature of economic activities, the structure of the board network remained relatively unchanged: the average geodesic distance between boards was 3.38, 3.46, and 3.46 in 1982, 1991 and 2001.

Turning finally to the strength of ties among boards: Battiston and Catanzaro (2004) investigate the board networks of the Fortune 1000 firms in 1999 and show that they consist mostly of weak links (the number of strength 1 links is about 10 times that the number of stronger links) and that they have a small world feature (the largest connected component includes 87% of all firms). Given that links between boards serve as information diffusion channels, the marginal returns from board-to-board ties are likely to be decreasing. Proposition 5 shows that in this case, the club network is likely to be held together by weak links. The empirical patterns are consistent with our theoretical analysis.

Interlocking directorates among Health Care Organizations: Willems and Jegers (2011) study the interlocking boards of 92 Belgian healthcare organizations. One of their main findings is that the board network is fragmented with strong links: the 92 organizations are divided into 23 components; 24 pairs of organizations share exactly the same set of board members and the heaviest link in the network is of strength 10.

Woo (2017) and Hansson et al. (2018) suggest that health care organizations often need to collaborate with each other to treat multi-diseased and vulnerable patients. To achieve smooth coordination, it is more efficient for organizations to have multiple shared directors with their partners. In the language of our model, this suggests that marginal returns from links are increasing in overlaps. In this case, the theory predicts that the resulting board network is fragmented with strong links. This is consistent with the empirical finding of Willems and Jegers (2011).

Boards of Journal Editors: The editorial board of a journal along with its set of referees shapes the research papers that are published in it. The collection of prestigious journals in a discipline taken together therefore can have a profound influence on the directions of research in that discipline. In economics, there has existed a concern for some time now that the leading journals are dominated by members from a few economics departments based in the United States. This concentration of editors has some to suggest that the discipline may be a risk of becoming too conformist and losing its innovativeness. This question has become more pressing over the last few decades as the profession has grown greatly and there has been a massive increase in the number of journals: this has resulted in a massive increase in the relative prestige of publishing in a few core journals. A leading economist has termed this phenomenon 'Top5ites' (see Serrano (2018) and Ductor and Visser (2021)) and in a recent paper, the emphasis on the top few journals in the career prospects of economists has been referred to as the 'Curse of the top-5' (Heckman and Moktan (2020)). We may view authors as individuals and boards of journals as clubs. In this case study, we draw on a recent paper by Ductor and Visser (2021) to document some facts about editorships and the relationship between the boards of leading journals and then relate them to our theoretical predictions.

Ductor and Visser (2021) study a set of 106 leading economics and finance journals over the period 1990-2011. They find that there were 79533 authors publishing in these journals but that only 6069 became editors, i.e. only 7.63%. Moreover, within the set of editors, over 75% were editors of just one journal but over 1.6% of these editors were editors at 4 or more journals. We recognize that the model assumes individuals are ex-ante homogenous while economics authors clearly differ in their abilities and productivity and their suitability for editorial roles. But, at a high level, these two facts are broadly consistent with the model's prediction on the marginalization of individuals (that can arise even if all individuals are similar).

Turning next to the links between the boards of different journals, for concreteness let us discuss the empirical situation in 2010. The network contains the 106 journals as nodes; a link between two journals reflects common editors. An inspection of this network reveals a number of interesting facts. The largest component contains 101 nodes, suggesting that it is more or less connected. The network is sparse with roughly 11% of all possible links being present. These links have uneven strength but the vast majority of the links are weak – over 82% have only one or two common editors. These facts suggest that the network is a small world that is held together with mostly weakly ties.

To illustrate these patterns, we present the network of editorial boards of leading economics journals from the year 2010 in Figure 6. The network covers 28 leading economics journals.[15] We see that the network is connected and that most of the links are relatively weak. Interestingly the network is held together through a hierarchical structure – the general interest journals share common editors with field journals; there are relatively few ties among the general interest journals and the field journals, respectively.

7 Conclusions

Empirical research has documented a tendency for decision making power to be concentrated in a few persons at the head of large organizations. This phenomenon is termed the 'power elite' or the 'interlocking directorates', depending on the type of positions concerned. This paper proposed a simple model of club membership to study the circumstances that would lead to power elites and interlocking directorates and their welfare properties. The model has two types of active agents: individuals seeking to join clubs and club owners. The analysis shows that if club productivity is increasing in links with other clubs, then the stable network tends to be exclusive: a small subset of individuals are members of several clubs, while the vast majority are excluded from all membership. Whether such a structure is efficient depends on the utility specification of individuals. When the returns to common membership are convex, stability dictates strong ties and a fragmented club network. By contrast, when returns are concave, stability presses toward weak ties and more extensive connectivity of the club network.

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[15] These journals are Journal of Health Economics (JHE), Review of Economics and Statistics (REStat), Review of Economic Studies (REStud), Econometric Theory (ET), Journal of Monetary Economics (JME), Quarterly Journal of Economics (QJE), Journal of Economic Literature (JEL), Journal of Business and Economic Statistics (JBES), Econometrica (ECMA), Review of Financial Studies (RFS), RAND Journal of Economics (RAND), Economic Journal (EJ), Journal of Environmental Economics and Management (JEEM), Journal of Finance (JoF), Journal of Econometrics (JoE), Journal of International Economics (JIE), European Economic Review (EER), World Bank Economic Review (WBER), International Economic Review (IER), American Economic Review (AER), Journal of Human Resources (JHR), Journal of Labor Economics (JLE), Journal of Political Economy (JPolE) Journal of Public Economics (JPubE), Games and Economic Behavior (GEB), Journal of Economic Theory (JET), Journal of Economic Perspectives (JEP), Journal of Financial Economics (JFE).

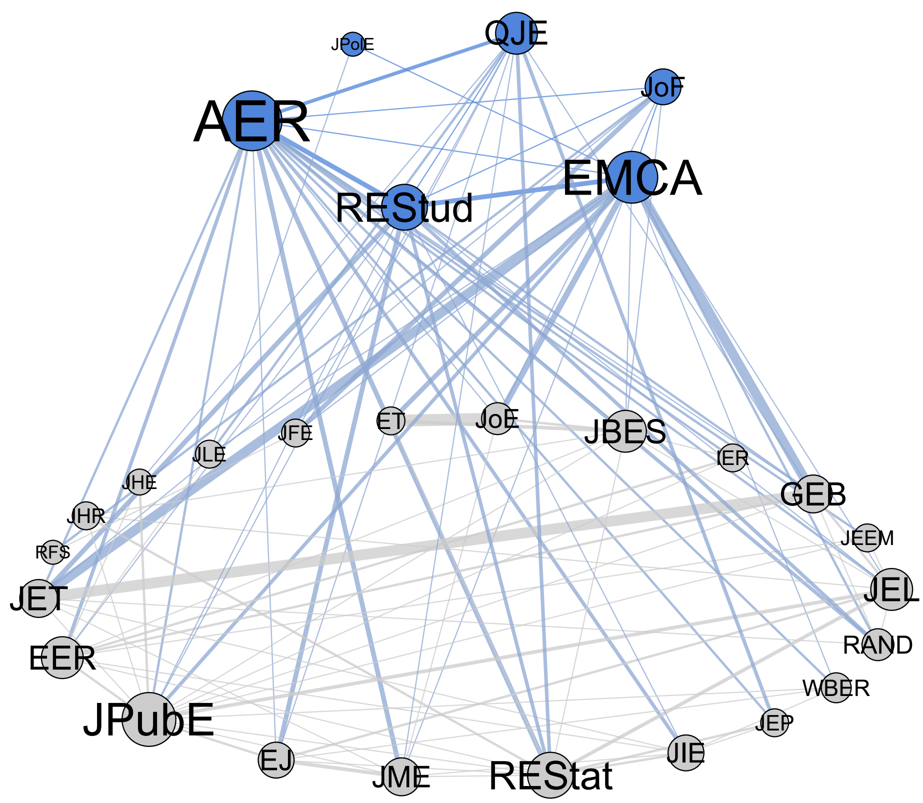

Figure 6: The editorial boards of economic journals 2010. Node size reflects number of editors; link thickness indicates number of common editors. Courtesy of Lorenzo Ductor and Bauke Visser

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8 Appendix: For Online Publication

Proofs

Proof of Proposition 1

We first take up the characterization – the sufficient and necessary conditions – for stability. We then prove existence.

From the production function of clubs and the utility function of individuals, we know that there cannot be any $i \in I, c \in C$ and $\boldsymbol{a'} \in A$ with $a'_i \leq a_i$ and $a'_{-i} = a_{-i}$ such that $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$, or $a'_c \leq a_c$ and $a'_{-c} = a_{-c}$ such that $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$. Hence, the deviations we need to consider are joint deviation by i and c such that both of them are better off. Such deviation can be divided into four types: individual i joins club c and nothing else is changed; individual i quits some clubs and joins club c; club c dismisses some members and admits individual i; and individual i quits some clubs, club c dismisses some members, and i joins c. Notice that for the last three kinds of deviations, if quitting two or more clubs and dropping two or more members is profitable, then quitting only one club and dropping only one member is also profitable given our utility and productivity specification. So, we only consider deviations with one quitting and (or) one dropping. We show that conditions (i)–(iii) are necessary and sufficient for the four kinds of deviations not to be jointly profitable.

For the necessity of condition (i), suppose it does not hold and there exists an individual $i \in I$ with $d_i(\boldsymbol{a}) < D$ and a club $c \in C$ with $s_c(\boldsymbol{a}) < S$, such that i is not a member of c. But then i joining c is strictly improving for both parties, which contradicts stability of $\boldsymbol{a}$.

We also show that if condition (i) holds, then there is no jointly profitable deviation for i and c where i joins c and nothing else changes since such deviation is not feasible.

For the necessity of condition (ii), suppose, to the contrary, that there exists a club c with $s_c(\boldsymbol{a}) < S$, an individual $i \in I$ who is not a member of c, and a club $c' \in C$ that i joins, such that

$$\pi_c(\boldsymbol{a}) > \pi_{c'}(\boldsymbol{a}) - f(s_c(\boldsymbol{a}) + 1) + f(s_c(\boldsymbol{a})) - \alpha(D - 1). \tag{11}$$

Notice that, by condition (i), $d_i(\boldsymbol{a}) = D$. Let $\boldsymbol{a'}$ be a membership profile obtained from $\boldsymbol{a}$ by i leaving c' and joining c and c accepting i. First, it is obvious that $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$. The difference in productivity of c between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\pi_c(\boldsymbol{a'}) - \pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a}) + 1) - f(s_c(\boldsymbol{a})) + \alpha(D - 1), \tag{12}$$

so the difference in utility of i between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a}) = v(\pi_c(\boldsymbol{a}) + f(s_c(\boldsymbol{a}) + 1) - f(s_c(\boldsymbol{a})) + \alpha(D-1) + \sum_{c'' \neq c, c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) \\ - v(\pi_{c'}(\boldsymbol{a}) + \sum_{c'' \neq c, c'} a_{ic''}\pi_{c''}(\boldsymbol{a})), \tag{13}$$

which has the same sign as

$$\pi_c(\boldsymbol{a}) - \pi_{c'}(\boldsymbol{a}) + f(s_c(\boldsymbol{a}) + 1) - f(s_c(\boldsymbol{a})) + \alpha(D-1), \tag{14}$$

which is positive since v is increasing. The deviation by individual i and club c from $\boldsymbol{a}$ to $\boldsymbol{a'}$ makes them both better off. A contradiction with stability of $\boldsymbol{a}$.

We also show that if conditions (i) and (ii) hold, then there is no jointly profitable deviation for i and c where i quits a club to join c and nothing else changes. If there is such a deviation, it must be that $s_c(\boldsymbol{a}) < S$. Since i is not a member of c so, by condition (i), $d_i(\boldsymbol{a}) = D$. Let c' be the club that i leaves when joining c. Then, by (13) and (14) and condition (ii), utility of i does not increase and so the deviation is not profitable to i.

For the necessity of condition (iii), suppose, to the contrary, that there exist individuals $i \in I$ and $i' \in I$ such $D > d_i(\boldsymbol{a}) \geq d_{i'}(\boldsymbol{a})$ and a club $c \in C$ such that i' is a member of and i is not. Let $\boldsymbol{a'}$ be a membership profile obtained from $\boldsymbol{a}$ by i joining c and c accepting i and dropping i'. The difference in productivity of c between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\pi_c(\boldsymbol{a'}) - \pi_c(\boldsymbol{a}) = \alpha(d_i(\boldsymbol{a}) - d_{i'}(\boldsymbol{a}) + 1), \tag{15}$$

which is positive if and only if $\alpha > 0$. Also, since v is increasing, both individual i and club c and strictly benefit deviating from $\boldsymbol{a}$ to $\boldsymbol{a'}$ when $\alpha > 0$. A contradiction with stability of $\boldsymbol{a}$.

We also show that if conditions (i) and (iii) hold, then there is no jointly profitable deviation for i and c where c drops a member to admit i and nothing else changes. If there is such a deviation, it must be that $d_i(\boldsymbol{a}) < D$. Then from condition (i), it mush be $s_c(\boldsymbol{a}) = S$. Let i' be the individual that club c drops. Then, by (15) and condition (iii), productivity of club c does not increase and so the deviation is not profitable to c.

For the necessity of condition (iv), suppose that $\alpha > 0$ and suppose, to the contrary, that there exists two individuals $i, i' \in I$ with $d_i(\boldsymbol{a}) = D$ and $d_{i'}(\boldsymbol{a}) < D$, a club $c \in C$ that has member i' but not i, and a club c' that i joins, such that

$$\pi_c(\boldsymbol{a}) > \pi_{c'}(\boldsymbol{a}) - \alpha\left(D - d_{i'}(\boldsymbol{a}) - \sum_{c''\neq c'} a_{ic''}a_{i'c''}\right). \tag{16}$$

Let $\boldsymbol{a'}$ be a membership profile obtained from $\boldsymbol{a}$ by i joining c and leaving c', and c accepting i and dropping i'. The difference in productivity of c between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\pi_c(\boldsymbol{a'}) - \pi_c(\boldsymbol{a}) = \alpha(D - d_{i'}(\boldsymbol{a})), \tag{17}$$

which is positive if and only if $\alpha > 0$. The difference in utility of i between $\boldsymbol{a'}$ and $\boldsymbol{a}$ is equal to

$$\begin{aligned} u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a}) =& v(\pi_c(\boldsymbol{a'}) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) - v(\pi_{c'}(\boldsymbol{a}) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) \\ =& v(\pi_c(\boldsymbol{a}) + \alpha\,(D - d_{i'}(\boldsymbol{a})) - \alpha \sum_{c''\neq c'} a_{ic''}a_{i'c''} + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})) \\ & - v(\pi_{c'}(\boldsymbol{a}) + \sum_{c''\neq c,c'} a_{ic''}\pi_{c''}(\boldsymbol{a})), \end{aligned} \tag{18}$$

which has the same sign as

$$\pi_c(\boldsymbol{a}) - \pi_{c'}(\boldsymbol{a}) + \alpha\left(D - d_{i'}(\boldsymbol{a}) - \sum_{c''\neq c'} a_{ic''}a_{i'c''}\right), \tag{19}$$

which is positive since v is increasing. The deviation by individual i and club c from $\boldsymbol{a}$ to $\boldsymbol{a'}$ makes them both better off. A contradiction with stability of $\boldsymbol{a}$.

We also show that if conditions (i)–(iv) hold, then there is no jointly profitable deviation for i and c where i leaves a club, c drops a member, and i joins c. Suppose there is such a deviation, if $d_i(\boldsymbol{a}) < D$ or $s_c(\boldsymbol{a}) < S$, since the deviation is profitable with i quitting a club and c dismissing a member, it is also profitable if i does not quit the club and c does not dismiss the member. We know conditions (i)–(iii) guarantee that there is no such mutually beneficial deviation. So, here we consider the deviations of i and c when $d_i(\boldsymbol{a}) = D$ and $s_c(\boldsymbol{a}) = S$. In this case, by (18) and (19) and condition (iv), utility of i does not increase and so the deviation is not profitable to i.

We finally turn to the existence of stable membership profile. We provide a proof by construction.

Suppose $nD \geq mS$. Let $m' \leq m$ and $n' \leq n$ be the largest integers such that $m'S = n'D$. Notice that since $m \geq D$ and $n \geq S$ so $m' \geq D$ and $n' \geq S$. Construct a membership profile $\boldsymbol{a}$ as follows. First, select n' individuals and m' clubs, let all selected individuals join D clubs we select so that all m' clubs have S members. This profile can be obtained by letting clubs admit individuals in sequence: make each club admit S individuals that have the smallest degree in its turn before moving to the next club. If $n-n' \geq S$, take S out of $n-n'$ remaining individuals and put each of them in each of $m-m'$ remaining clubs. Otherwise, put each of $n-n'$ remaining individuals in each of $m-m'$ clubs. It is easy to verify that this profile is stable.

Suppose $mS > nD$: consider a membership profile $\boldsymbol{a}$ where all individuals join D clubs, and $\lfloor \frac{nD}{S} \rfloor$ clubs have S members, one club has $(nD) \bmod S$ members, and the remaining clubs have 0 members. This profile can be obtained by letting clubs admit individuals in sequence. Make each club admit S individuals that has the smallest degree in its turn before moving to the next club. Stop when all individuals have degree D. This profile is always stable as it satisfies all four conditions in Proposition 1. Condition (i) is satisfied obviously. Conditions (iv) and (iii) are automatically satisfied as no individual joins less than D clubs. For condition (ii), if a club c has less than S members, then either it is the one club with $(nD) \bmod S$ members or it has 0 members. In both cases, for an individual i that is not in c and for any club that i is in, c' must have more members than c does and all members of c' join D clubs, making condition (ii) satisfied. ■

Proof of Proposition 2

We first take up the egalitarian outcome result in the absence of network effects. When $\alpha = 0$, the membership profile generated with the following algorithm is stable. Let clubs admit individuals sequentially. Fill a club with S individuals that currently have the lowest degrees and then move to the next club. Stop until all clubs are full or all individuals have joined D clubs. Since $n \geq S$, this algorithm is feasible. If the algorithm terminates when all clubs have S members, then all clubs have productivity $f(S)$ which is the highest productivity a club can get. Hence the membership profile is stable. If the algorithm terminates when all individuals are in D clubs, then there are $\lfloor \frac{mS}{D} \rfloor$ clubs that have productivity $f(S)$, one club that has productivity $f((mS) \mod D)$, and the rest clubs have productivity 0. The only possible profitable deviation from one individual is to quit the club with productivity $f((mS)$ mod $D)$ and join a club with productivity $f(S)$, but no club with productivity $f(S)$ want to deviate. Hence the membership profile is stable. Given the way we construct the membership profile, we have $|d_i(\boldsymbol{a}) - d_{i'}(\boldsymbol{a})| \leq 1$ for all $i, i' \in I$.

When $\alpha > 0$, we develop the conditions for the sizes of the different groups. For the cardinality of $I_3(\boldsymbol{a})$, take any individual $i \in I_3(\boldsymbol{a})$ with minimal $d_i(\boldsymbol{a})$ and let $c \in C$ be any club that i members. By condition (iii) of Proposition 1, all individuals in $I_3(\boldsymbol{a})$ are members of c and, by condition (i) of Proposition 1, $s_c(\boldsymbol{a}) \leq S$. Hence $|I_3(\boldsymbol{a})| \leq S$.

For cardinality of $C_2(\boldsymbol{a})$, suppose that $C_2(\boldsymbol{a}) \neq \varnothing$, we will show that there exists an individual i that is a member of all clubs in $C_2(\boldsymbol{a})$. We consider the cases of $I_3(\boldsymbol{a}) = \varnothing$ and $I_3(\boldsymbol{a}) \neq \varnothing$ separately. If $I_3(\boldsymbol{a}) = \varnothing$, then members of the clubs in $C_2(\boldsymbol{a})$ are of degree D and, for any $c \in C_2(\boldsymbol{a})$, $s_c(\boldsymbol{a}) < S$ (as c does not achieve maximal productivity). Take any $c' \in C_2(\boldsymbol{a})$ with minimal productivity, $\pi_{c'}(\boldsymbol{a})$, and any member i of c'. Take any $c \in C_2(\boldsymbol{a}) \setminus \{c'\}$. Since $s_c(\boldsymbol{a}) < S$ and since $\pi_c(\boldsymbol{a}) \geq \pi_{c'}(\boldsymbol{a})$ so, by condition (ii) of Proposition 1, i is a member of c. Hence i is a member of all clubs in $C_2(\boldsymbol{a})$. If $I_3(\boldsymbol{a}) \neq \varnothing$ then take any $i \in I_3(\boldsymbol{a})$ with maximal degree. Take any club $c \in C_2(\boldsymbol{a})$. Since c does not achieve the highest productivity so either $s_c(\boldsymbol{a}) < S$ or c has a member in $I_3(\boldsymbol{a})$. In the first case, i is a member of c by condition (i) of Proposition 1. In the second case, i is a member of c by condition (iii) of Proposition 1. Hence i is a member of all clubs in $C_2(\boldsymbol{a})$. By condition (i) of Proposition 1, $d_i(\boldsymbol{a}) \leq D$. Hence $|C_2(\boldsymbol{a})| \leq D$.

For cardinality of $I_2(\boldsymbol{a})$, notice first that, by definition, every individual in $I_2(\boldsymbol{a})$ members at least one club in $C_2(\boldsymbol{a})$. Thus the aggregate membership of individuals in $I_2(\boldsymbol{a})$ in the clubs in $C_2(\boldsymbol{a})$ is at least $x + 2(|I_2(\boldsymbol{a})| - x)$, where x is the number of individuals from $I_2(\boldsymbol{a})$ who member exactly one club from $C_2(\boldsymbol{a})$. On the other hand, since $|C_2(\boldsymbol{a})| \leq D$ and, for all $c \in C_2(\boldsymbol{a})$, either $s_c(\boldsymbol{a}) \leq S - 1$ or c has a member in $I_3(\boldsymbol{a})$, so aggregate club capacity of the clubs in $C_2(\boldsymbol{a})$ for individuals in $I_2(\boldsymbol{a})$ is at most $(S-1)D$. Hence

$$x + 2(|I_2(\boldsymbol{a})| - x) = 2|I_2(\boldsymbol{a})| - x \leq (S-1)D. \tag{20}$$

The number of individuals in $I_2(\boldsymbol{a})$ who member exactly one club in $C_2(\boldsymbol{a})$ is at most $S-1$. To see that, suppose that an individual $i \in I_2(\boldsymbol{a})$ members exactly one club $c' \in C_2(\boldsymbol{a})$. Let $c \in C_2(\boldsymbol{a}) \setminus \{c'\}$ be another club in $C_2(\boldsymbol{a})$. Since i is not a member of c so, by condition (ii) of Proposition 1, $\pi_c(\boldsymbol{a}) < \pi_{c'}(\boldsymbol{a})$. Hence c' must achieve the highest productivity of all clubs in $C_2(\boldsymbol{a})$ and must be unique such. Since all individuals in $C_2(\boldsymbol{a})$ who member exactly one club in $C_2(\boldsymbol{a})$ must be members of the same club from $C_2(\boldsymbol{a})$ and since, as we observed above, this club can host at most $S-1$ members from $I_2(\boldsymbol{a})$, so there can be at most $S-1$ such individuals. This shows that $x \leq S-1$ and from (20) it follows that $|I_2(\boldsymbol{a})| \leq (S-1)(D+1)/2$.

We now use these derivations on the size of the different groups to derive bounds on the size of $I_1(\boldsymbol{a})$ and $I_2(\boldsymbol{a})$ and $I_4(\boldsymbol{a})$.

We begin with the case $nD \geq mS$: Suppose that in a stable membership profile $\boldsymbol{a}$, all clubs are full, then we have $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|D + \sum_{i \in I_3(\boldsymbol{a})} d_i(\boldsymbol{a}) = mS$, and hence $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|D + |I_3(\boldsymbol{a})|D > mS$. Since $|I_3(\boldsymbol{a})| \leq S$,

$$|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| > \frac{mS}{D} - S.$$

Suppose that in a stable membership profile $\boldsymbol{a}$, not all clubs are full, then we know $|I_4(\boldsymbol{a})| = 0$ as otherwise there is a jointly profitable deviation for an individual in $I_4(\boldsymbol{a})$ and a club that is not full where the individual joins the club. Therefore,

$$|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| = n \geq \frac{mS}{D},$$

and so $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \geq \frac{mS}{D} - S$ given $|I_3(\boldsymbol{a})| \leq S$.

Now, since $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \geq \frac{mS}{D} - S$ and $|I_2(\boldsymbol{a})| \leq \frac{(S-1)(D+1)}{2}$, we have $|I_1(\boldsymbol{a})| \geq \frac{mS}{D} - \frac{S(D+3)}{2}$. For the upper bound of $|I_1(\boldsymbol{a})|$, since aggregate club capacity is mS, we must have $|I_1(\boldsymbol{a})|D \leq mS$, and so $|I_1(\boldsymbol{a})| \leq \frac{mS}{D}$.

Regarding the bounds for $|I_4(\boldsymbol{a})|$. Since $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| + |I_4(\boldsymbol{a})| = n$, $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \leq \frac{mS}{D}$, and $|I_3(\boldsymbol{a})| \leq S$, so $|I_4(\boldsymbol{a})| \geq n - \frac{mS}{D} - S$. Moreover, if $|I_4(\boldsymbol{a})| > n - \frac{mS}{D}$, then $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| < \frac{mS}{D}$. The club capacity is not exhausted and there must exist a club c that is not full. There is a jointly profitable deviation for an individual i in $I_4(\boldsymbol{a})$ and club c where i joins c. A contradiction.

Next consider the case when $nD < mS$: We first show the lower bound for $|C_1(\boldsymbol{a})|$ is $nD/S - D$. Suppose that in a stable membership profile $\boldsymbol{a}$, all individuals exhaust their membership availability, then we have $|C_1(\boldsymbol{a})|S + \sum_{c \in C_2(\boldsymbol{a})} s_c(\boldsymbol{a}) = nD$, and hence $|C_1(\boldsymbol{a})|S + |C_W(\boldsymbol{a})|S \geq nD$. Since $|C_2(\boldsymbol{a})| \leq D$, $|C_1(\boldsymbol{a})| \geq \frac{nD}{S} - D$. Suppose that in a a stable membership profile $\boldsymbol{a}$, not all individuals exhaust their membership availability, then we know $|C_3(\boldsymbol{a})| = 0$ as otherwise there is a jointly profitable deviation for the individual who joins less than D clubs and a club in $C_3(\boldsymbol{a})$ where the individual joins the club. Therefore,

$$|C_1(\boldsymbol{a})| + |C_2(\boldsymbol{a})| = m \geq \frac{nD}{S},$$

and so $|C_1(\boldsymbol{a})| \geq \frac{nD}{S} - D$ given $|C_2(\boldsymbol{a})| \leq D$.

For the upper bound of $|C_1(\boldsymbol{a})|$, since aggregate membership availability is nD, we must have $|C_1(\boldsymbol{a})|S \leq nD$, and so $|C_1(\boldsymbol{a})| \leq \frac{nD}{S}$.

Regarding the bounds for $|C_3(\boldsymbol{a})|$. Since $|C_1(\boldsymbol{a})| + |C_2(\boldsymbol{a})| + |C_3(\boldsymbol{a})| = m$, $|C_1(\boldsymbol{a})| \leq \frac{nD}{S}$, and $|c_2(\boldsymbol{a})| \leq D$, so $|C_3(\boldsymbol{a})| \geq m - \frac{nD}{S} - D$. Moreover, if $|C_3(\boldsymbol{a})| > m - \frac{nD}{S}$, then $|C_1(\boldsymbol{a})| + |C_2(\boldsymbol{a})| < \frac{nD}{S}$. The aggregate membership availability is not exhausted and there must exist an individual i that joins less than D clubs. There is a jointly profitable deviation for i and a club c in $C_3(\boldsymbol{a})$ and club c where i joins c. A contradiction.

Finally, we turn to the statements on $M_I(\boldsymbol{a})$ and $M_C(\boldsymbol{a})$. Consider the case of $nD > mS$ and let $\boldsymbol{a}$ be a stable membership profile. Suppose first that $nD > mS + DS$. By the lower bound on $I_4(\boldsymbol{a})$, $|I_4(\boldsymbol{a})| \geq (nD - mS - SD)/D > 0$. Hence there exists at least one individual without a club under $\boldsymbol{a}$. It follows that $\sum_{i \in I} d_i(\boldsymbol{a}) = mS$. For if $\sum_{i \in I} d_i(\boldsymbol{a}) < mS$ then there exists a club c with $s_c(\boldsymbol{a}) < S$ and, by $|I_4(\boldsymbol{a})| > 0$, there exists an individual i with $d_i(\boldsymbol{a}) = 0$. They would both benefit from i joining c, contradicting the stability of $\boldsymbol{a}$. Using $\sum_{i \in I} d_i(\boldsymbol{a}) = mS$ and $|I_4(\boldsymbol{a})| \geq n - \frac{mS}{D} - S$ we get

$$\begin{aligned}
\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) &= \frac{|I_4(\boldsymbol{a})| - n + \min(n, mS)}{\min(n, mS) - \left\lceil \frac{mS}{D} \right\rceil} \geq \frac{|I_4(\boldsymbol{a})| - n + \min(n, mS)}{\min(n, mS) - \frac{mS}{D}} \\
&\geq \frac{\min(n, mS) - \frac{mS}{D} - S}{\min(n, mS) - \frac{mS}{D}} = 1 - \frac{S}{\min(n, mS) - \frac{mS}{D}} \\
&= 1 - \frac{D}{\min\left(\frac{nD - mS}{S}, (D-1)m\right)}
\end{aligned}$$

Second, suppose that $nD \leq mS + DS$. Then

$$1 - \frac{D}{\min\left(\frac{nD - mS}{S}, (D-1)m\right)} \leq 1 - \frac{D}{\min\left(\frac{DS}{S}, (D-1)m\right)} \leq 0 \leq \mathcal{M}_{\mathcal{I}}(\boldsymbol{a}).$$

The upper bound on $M_C(\boldsymbol{a})$ can be obtained by analogous arguments.

■

Proof of Proposition 3

We prove a more general characterization of efficient membership profiles, without the parity conditions in Proposition 3.

Lemma 1. *Suppose $\alpha > 0$. Assume $nD \geq mS$.*

- *A membership profile is clubs-efficient if and only if there are* $\lfloor \frac{mS}{D} \rfloor$ *individuals that join* D *clubs, one individual that joins* $(mS) \bmod D$ *clubs, and the remaining individuals join no clubs.*

- *If* $v''(\cdot) \geq 0$, *then a membership profile is utilitarian optimum if and only if it is clubs-efficient. If* $v''(\cdot) < 0$ *and satisfies condition (4), then in any utilitarian optimum membership profile, either* $d_i(\boldsymbol{a}) \leq 1$ *for all* $i \in I$ *or* $d_i(\boldsymbol{a}) \geq 1$ *for all* $i \in I$.

Assume $nD < mS$.

- *If* $f''(\cdot) > 0$, *then a membership profile is clubs-efficient if and only if* $\lfloor \frac{nD}{S} \rfloor$ *clubs admit* S *members, one club that admits* $(nD) \bmod S$ *members, and the remaining clubs admit no members. If* $f''(\cdot) = 0$, *then a membership profile is clubs-efficient if and only if each individual join* D *clubs. If* $f''(\cdot) < 0$, *then a membership profile is clubs-efficient if and only if* $(nD) \bmod m$ *admit* $\lceil \frac{nD}{m} \rceil$ *members and the remaining clubs admit* $\lfloor \frac{nD}{m} \rfloor$ *members.*

- *If* $v''(\cdot) \geq 0$, *then a membership profile is utilitarian optimum if and only if* $\lfloor \frac{nD}{S} \rfloor$ *clubs admit* S *members, one club that admits* $(nD) \bmod S$ *members, and the remaining clubs admit no members. If* $v''(\cdot) < 0$ *and* $(nD) \bmod S = 0$, *then membership profile is utilitarian optimum if and only if* nD/S *clubs admit* S *members and the remaining clubs admit no members. If* $v''(\cdot) < 0$ *and* $(nD) \bmod S > 0$, *then in any utilitarian optimum membership profile, the number of clubs that admit some but less than* S *members is not more than* $S - 1$.

For the case when $nD \geq mD$. First, given a membership profile $\boldsymbol{a}$, the aggregate productivity of clubs is

$$\begin{aligned}\sum_{c \in C} \pi_c(\boldsymbol{a}) &= \sum_{c \in C} f(s_c(\boldsymbol{a})) + \alpha \sum_{c \in C} \sum_{i \in I} a_{ic}(d_i(\boldsymbol{a}) - 1) \\ &\leq mf(S) + \alpha \sum_{i \in I} d_i(\boldsymbol{a})(d_i(\boldsymbol{a}) - 1),\end{aligned}$$

where the equality is obtained only when $s_c(\boldsymbol{a}) = S$ for all $c \in C$. Now we solve the following maximization problem:

$$\max \sum_{i \in I} d_i(\boldsymbol{a})(d_i(\boldsymbol{a}) - 1) \text{ s.t. } d_i(\boldsymbol{a}) \in \{0, 1, ..., D\} \text{ for all } i \in I \text{ and } \sum_{i \in I} d_i(\boldsymbol{a}) \leq mS.$$

Since $g(x) = x(x-1)$ is superadditive on the set of non-negative integers and this is strict on positive integers, the solution to the maximization problem is a vector $(d_i^*(\boldsymbol{a}))_{i \in I}$ such that $d_i^*(\boldsymbol{a}) = D$ for all $i \in I'$ where $I' \subset I$ and $|I'| = \lfloor \frac{mS}{D} \rfloor$, $d_i^*(\boldsymbol{a}) = (mS) \bmod D$ for some $i = k \in I \backslash I'$ (in the case of $(mS) \bmod D \geq 1$) and $d_i^*(\boldsymbol{a}) = 0$ for all $i \in I \backslash (I' \cup \{k\})$.

We now show that when $nD \geq mS$, there always exists a membership structure where there are $\lfloor \frac{mS}{D} \rfloor$ individuals that join D clubs, one individual that joins $(mS) \bmod D$ clubs, and the remaining individuals join no clubs (which makes $s_c(\boldsymbol{a}) = S$ for all $c \in C$), so that a structure $\boldsymbol{a} \in A$ is clubs-efficient if and only if it satisfies such a club joining pattern. Construct a membership structure as follows. Consider $\lfloor mS/D \rfloor$ individuals first, in a sequence. Make each such i join D clubs that have the smallest membership size at her turn before moving to the next individual. If $(mS) \bmod D \geq 1$ so that there are clubs that do not have S members at the end of the process, take one more individual and make him join those $(mS) \bmod D$ clubs. Since $\lfloor mS/D \rfloor D + (mS) \bmod D = mS$, the construction is valid and results in the desired membership structure.

For utilitarian optimal structures, given a membership profile $\boldsymbol{a}$, the aggregate utility of individuals is

$$\sum_{i \in I} u_i(\boldsymbol{a}) = \sum_{i \in I} v(\sum_{c \in C} a_{ic} \pi_c(\boldsymbol{a})).$$

We know that $\sum_{c \in C} a_{ic} \pi_c(\boldsymbol{a}) \leq D(f(S) + S(D-1))$ for all $i \in I$ and $\sum_{i \in I} \sum_{c \in C} a_{ic} \pi_c(\boldsymbol{a}) = \sum_{c \in C} s_c(\boldsymbol{a}) \pi_c(\boldsymbol{a}) \leq S \sum_{c \in C} \pi_c(\boldsymbol{a})$, where the equality is obtained only when $s_c(\boldsymbol{a}) = S$ for all $c \in C$. Given that a clubs-efficient structure that maximizes $\sum_{c \in C} \pi_c(\boldsymbol{a})$ features $s_c(\boldsymbol{a}) = S$ for all $c \in C$, $\sum_{i \in I} \sum_{c \in C} a_{ic} \pi_c(\boldsymbol{a})$ is maximized if and only if $\boldsymbol{a}$ is clubs-efficient. We also know that a clubs-efficient structure makes $\lfloor \frac{mS}{D} \rfloor$ individuals have utility $v(D(f(S) + S(D-1)))$, at most one individual have positive but less than $v(D(f(S) + S(D-1)))$ utility, and the rest individuals have zero utility. Hence, when $v''(\cdot) \geq 0$, the clubs-efficient membership profile is the solution to the maximization profile of $\max_{\boldsymbol{a} \in A} u_i(\boldsymbol{a})$. We have shown that when $v''(\cdot) \geq 0$, a membership profile is utilitarian optimum if and only if it is clubs-efficient.

Turning to when $v''(\cdot) < 0$ and satisfies

$$v(f(S)) - v(0) > (n-1) \left(v \left(f(S) + \frac{2\alpha S(D-1)}{n-1} \right) - v \left(f(S) \right) \right),$$

we show that suppose in a membership structure $\boldsymbol{a} \in A$, there exists two individuals $i, i' \in I$ such that $d_i(\boldsymbol{a}) > 1$ and $d_{i'}(\boldsymbol{a}) = 0$, then $\boldsymbol{a}$ cannot be utilitarian optimum. Suppose such a structure $\boldsymbol{a}$ is utilitarian optimum. Note first that it must be $s_c(\boldsymbol{a}) = S$ for all $c \in C$, as otherwise making individual i' join a club that is not full strictly raises aggregate welfare. Let $c \in C$ be a club where $a_{ic} = 1$. Consider another membership structure $\boldsymbol{a'}$ where c drops i and admits i'. The difference of aggregate utility between the two structures is

$$\sum_{i \in I}(u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a})) \geq v(f(S)) - v(0) + \sum_{i \neq i'}(u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a})),$$

since $u_{i'}(\boldsymbol{a'}) \geq v(f(S))$ and $u_{i'}(\boldsymbol{a}) = v(0)$. Given that i' replaces i in club c, the productivity of club c and clubs that i members decreases:

$$\begin{aligned} \pi_c(\boldsymbol{a}) - \pi_c(\boldsymbol{a'}) &= \alpha(d_i(\boldsymbol{a}) - 1), \text{ and} \\ \pi_{c'}(\boldsymbol{a}) - \pi_{c'}(\boldsymbol{a'}) &= \alpha \text{ for all } c' \neq c \text{ with } a_{ic'} = 1. \end{aligned}$$

So, the aggregate productivity drop is at most $2\alpha(D-1)$, which is obtained when $d_i(\boldsymbol{a}) = D$. Since $v''(\cdot) < 0$ and the minimal utility an individual obtains when he is in a club is $v(f(S))$,

$$\begin{aligned} \sum_{i \neq i'}(u_i(\boldsymbol{a}) - u_i(\boldsymbol{a'})) &\leq \sum_{i \neq i'} \left[v\left(f(S) + \sum_{c \in C} a_{ic}(\pi_c(\boldsymbol{a}) - \pi_c(\boldsymbol{a'})) \right) - v(f(S)) \right] \\ &\leq (n-1)\left[v\left(f(S) + \frac{2\alpha(D-1)S}{n-1} \right) - v(f(S)) \right]. \end{aligned}$$

Hence,

$$\sum_{i \in I}(u_i(\boldsymbol{a'}) - u_i(\boldsymbol{a})) \geq v(f(S)) - v(0) - (n-1)\left[v\left(f(S) + \frac{2\alpha(D-1)S}{n-1} \right) - v(f(S)) \right] > 0,$$

contradicting structure $\boldsymbol{a}$ being utilitarian optimum. This completes the proof.

For the case when $nD < mS$, given a membership profile $\boldsymbol{a}$, the aggregate productivity of clubs is

$$\begin{aligned} \sum_{c \in C} \pi_c(\boldsymbol{a}) &= \sum_{c \in C} f(s_c(\boldsymbol{a})) + \alpha \sum_{c \in C} \sum_{i \in I} a_{ic}(d_i(\boldsymbol{a}) - 1) \\ &\leq \sum_{c \in C} f(s_c(\boldsymbol{a})) + \alpha n D(D-1), \end{aligned}$$

where the equality is obtained only when $d_i(\boldsymbol{a}) = D$ for all $i \in I$. Now we look at the problem of $\max \sum_{c \in C} f(s_c(\boldsymbol{a}))$, s.t. $s_c(\boldsymbol{a}) \in \{0, 1, ..., S\}$ for all $c \in C$ and $\sum_{c \in C} s_c(\boldsymbol{a}) \leq nD$. When $f(\cdot)$

is convex, the solution to the maximization problem is a vector $(s_c^*(\boldsymbol{a}))_{c\in C}$ such that $s_c^*(\boldsymbol{a}) = S$ for all $c \in C'$ where $C' \subset C$ and $|C'| = \lfloor \frac{nD}{S} \rfloor$, $s_c^*(\boldsymbol{a}) = (nD) \bmod S$ for some $c = k \in C \backslash C'$ (in the case of $(nD) \bmod S \geq 1$) and $s_c^*(\boldsymbol{a}) = 0$ for all $c \in C \backslash (C' \cup \{k\})$. When $f(\cdot)$ is linear, the solution to the maximization problem is any $(s_c^*(\boldsymbol{a}))_{c\in C}$ where $s_c(\boldsymbol{a}) \in \{0, 1, ..., S\}$ for all $c \in C$ and $\sum_{c\in C} s_c(\boldsymbol{a}) = nD$. When $f(\cdot)$ is concave, the solution to the maximization problem is a vector $(s_c^*(\boldsymbol{a}))_{c\in C}$ such that $s_c^*(\boldsymbol{a}) = \lceil \frac{nD}{m} \rceil$ for all $c \in C'$ where $C' \subset C$ and $|C'| = (nD) \bmod m$, and $s_c^*(\boldsymbol{a}) = \lfloor \frac{nD}{m} \rfloor$ for all $c \in C \backslash C'$. This proves the characterization for clubs-efficient membership profiles.

For utilitarian optimum membership profiles, given a membership profile $\boldsymbol{a}$, we know the aggregate utility of individuals is $\sum_{i\in I} u_i(\boldsymbol{a}) = \sum_{i\in I} v(\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}))$ where $\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}) \leq D(f(S) + S(D-1))$ for all $i \in I$ and

$$\begin{aligned}\sum_{i\in I}\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a}) =& \sum_{c\in C} s_c(\boldsymbol{a}) f(s_c(\boldsymbol{a})) + \alpha \sum_{c\in C} s_c(\boldsymbol{a}) \sum_{i\in I} a_{ic}(d_i(\boldsymbol{a}) - 1) \\ \leq& \sum_{c\in C} s_c(\boldsymbol{a}) f(s_c(\boldsymbol{a})) + \alpha(D-1)s_c(\boldsymbol{a})^2,\end{aligned}$$

where the equality is obtained only when $d_i(\boldsymbol{a}) = D$ for all $i \in I$. Since $g(x) = xf(x) + \alpha(D-1)x^2$ is superadditive on non-negative integers and strictly superadditive on positive integers, for any $\alpha \geq 0$, $D \geq 1$, and strictly increasing f with $f(0) = 0$, $\sum_{i\in I}\sum_{c\in C} a_{ic}\pi_c(\boldsymbol{a})$ is maximized if and only if $\lfloor \frac{nD}{S} \rfloor$ clubs admit S members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members. When $v''(\cdot) \geq 0$, it is easy to see that this membership profile is also the solution to the maximization problem of $\max_{\boldsymbol{a}\in A} u_i(\boldsymbol{a})$. We have shown that when $v''(\cdot) \geq 0$, a membership profile is utilitarian optimum if and only if $\lfloor \frac{nD}{S} \rfloor$ clubs admit S members, one club that admits $(nD) \bmod S$ members, and the remaining clubs admit no members.

Turning to when $v''(\cdot) < 0$, consider the utilitarian optimum structure where $v''(\cdot) \geq 0$. Under this structure, the utility of $(nD) \bmod S$ individuals is

$$v[(D-1)(f(S) + \alpha S(D-1)) + f\left((nD) \bmod S\right) + \alpha\left((nD) \bmod S\right)(D-1)], \tag{21}$$

while the utility of all other individuals is $v[(D)(f(S) + \alpha S(D-1))]$. If this structure is utilitarian optimum, we have finished the proof. If the structure is not utilitarian optimum, then $(nD) \bmod S \neq 0$ and in a utilitarian optimum membership profile, the lowest utility of an individual is greater than (21), implying that the smallest size of a club is greater than $(nD) \bmod S$. Suppose the smallest club size is $s_c(\boldsymbol{a}) = (nD) \bmod S + k$ where $k \in \{1, ..., S - (nD) \bmod S - 1\}$. For the structure to be utilitarian optimal, the number of unfull clubs is at most k, where the bound k is reached when we reduce the club size of k clubs by 1 to increase the size of the smallest club. So, the number of clubs with size greater than 0 and lower than S is at most $1 + k \leq S - (nD) \bmod S \leq S - 1$.

■

Proof of Proposition 5

First, we consider when $nD \geq mS$.

When $h(\cdot)$ is convex, for any membership profile $\boldsymbol{a} \in A$, the productivity of a club $\pi_c(\boldsymbol{a})$ satisfies

$$\pi_c(\boldsymbol{a}) \leq f(S) + h(S(D-1)) = f(S) + h(S),$$

where the equality is obtained only when the club has one strength-S link with another club. For every club to reach this highest level of productivity, the club network consists of m/D separate 2-cliques where all links are of strength S. We now show such a structure exists by construction: Allocate the first S individuals to clubs c_1 and c_2, the next S individuals to clubs c_3 and c_4,..., and the $\frac{m}{2}^{th}$ group of S individuals (individuals $i_{mS/2-S+1}$ to $i_{mS/2}$) to clubs c_{m-1} and c_m.

Since all clubs have reached the highest productivity with the membership profile when $h(\cdot)$ is convex, it is also stable when $h(\cdot)$ is convex. To show the profile is not stable when $h(\cdot)$ is concave, consider a deviation by club c_1 and individual i_{S+1} where c_1 exiles i_1 to admit i_{S+1} and i_{S+1} leaves c_3 to join c_1. It is straight-forward to verify that the deviation benefits both c_1 and i_{S+1}.

When $h(\cdot)$ is concave, for any membership profile $\boldsymbol{a} \in A$, the productivity of a club $\pi_c(\boldsymbol{a})$ satisfies

$$\pi_c(\boldsymbol{a}) \leq f(S) + S(D-1)h(1) = f(S) + S \cdot h(1),$$

where the equality is obtained only when the club has S strength-1 links with other clubs. For every club to reach this highest level of productivity, the club network is an S-regular network where all links are of strength 1. We now show such a structure exists by construction with the following algorithm: At each step, pick the club with the maximum number of empty slots, fill the slots with different individuals, and then allocate each of those individuals to a different club with the maximum number of empty slots. Stop when all clubs are full.

Since all clubs have reached the highest productivity with the membership profile when

$h(\cdot)$ is concave, it is also stable when $h(\cdot)$ is concave. Now we show the profile is not stable when $h(\cdot)$ is convex. In this profile, for each club c, it must has at least 2 strength-1 links with c' and c''. Let i_1 be the common member of c and c' and i'' be the common member of c and c''. There must also exist an indiviual, call him i_3, who is in c' and $c''' \neq c$. Consider the deviation by club c and individual i_3 where c exiles i_2 to admit i_3 and i_3 leaves c''' to join c. This deviation benefits both c and i_3.

Turning to when $nD < mS$, let π^* be the highest productivity a club can obtain, note that for any membership profile $\boldsymbol{a} \in A$, the utility of an individual $u_i(\boldsymbol{a})$ satisfies

$$u_i(\boldsymbol{a}) \leq v(D \cdot \pi^*),$$

where the equality is obtained only when all clubs i joins has productivity π^*. For any individual to reach this highest level of utility, the subnetwork of clubs that contains all non-empty clubs must consist of n/S separate 2-cliques where all links are of strength S when $h(\cdot)$ is convex and be an S-regular network (a complete network when $S = 2n/S - 1$) where all links are of strength 1 when $h(\cdot)$ is concave. Such subnetworks can be constructed in the same way we construct we construct the clubs-efficieny networks when $nD \geq mS$.

For the statements on stability, since all individuals have reached the highest level of utility, they have no incentives to deviate. We consider the same deviations examined for the case when $nD \geq mS$ to show that the utilitarian optimum club network under a convex (concave) $h(\cdot)$ is unstable when $h(\cdot)$ is concave (convex).

■

Proof of Proposition 4

For point (i) we will show that if

$$\min_{x \in [0,S-1]} (h(x+1) - h(x)) > \frac{D-1}{D} \max_{x \in [0,S-1]} (h(x+1) - h(x)),$$

then the number of agents who join no clubs under a stable membership structure is bounded from above by $|I_4(\boldsymbol{a})| \geq n - \frac{mS}{D} - S$, like in the case of linear returns from links. Then the derivation of the value of marginalization measure follows by the same steps as in proof of Proposition 2. We first show that the cardinality of $|I_3(\boldsymbol{a})|$ is not greater than S in any stable membership structure. Take any indvidual $i \in I_3(\boldsymbol{a})$ with minimal degree and let $c \in C$ be any clubs that i members. We show that all individuals in $I_3(\boldsymbol{a})$ are members of c, and since $s_c(\boldsymbol{a}) \leq S$, $|I_3(\boldsymbol{a})| \leq S$. Suppose that there exists an $i' \in I_3(\boldsymbol{a})$ where $d_{i'}(\boldsymbol{a}) \geq d_i(\boldsymbol{a})$ and

$a_{i'c} = 0$. There is a jointly profitable deviation for i' and c where c drops i and admits i'. The deviation obviously improves the utility of i' since none of the clubs she is already in suffers from a loss of productivity and she is in a new club. The productivity change of club c is positive since

$$\begin{aligned}
\pi_c(\boldsymbol{a'}) = \pi_c(\boldsymbol{a}) &= f(s_c(\boldsymbol{a'})) - f(s_c(\boldsymbol{a})) + \sum_{c' \neq c} h(g_{cc'}(\boldsymbol{a'})) - \sum_{c' \neq c} h(g_{cc'}(\boldsymbol{a})) \\
&= \sum_{c \neq c, a_{ic}=1, a_{i'c}=0} [h(g_{cc'}(\boldsymbol{a}) - 1) - h(g_{cc'}(\boldsymbol{a}))] + \sum_{c \neq c, a_{ic}=0, a_{i'c}=1} [h(g_{cc'}(\boldsymbol{a}) + 1) - h(g_{cc'}(\boldsymbol{a}))] \\
&\geq -(d_i(\boldsymbol{a}) - 1) \max_{x \in [0, S-1]} (h(x+1) - h(x)) + d_{i'}(\boldsymbol{a}) \min_{x \in [0, S-1]} (h(x+1) - h(x)) \\
&> 0
\end{aligned}$$

given $\min_{x \in [0,S-1]}(h(x+1) - h(x)) > \frac{D-1}{D} \max_{x \in [0,S-1]}(h(x+1) - h(x))$ and $d_i(\boldsymbol{a}) \leq d_{i'}(\boldsymbol{a}) \leq D$.

Given that $|I_3(\boldsymbol{a})| \leq S$, we can prove the lower bound for $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|$. Suppose that in a stable membership structure $\boldsymbol{a}$, all clubs are full, then we have $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|D + \sum_{i \in I_3(\boldsymbol{a})} d_i(\boldsymbol{a}) = mS$, and hence $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|D + |I_3(\boldsymbol{a})|D > mS$. Since $|I_3(\boldsymbol{a})| \leq S$,

$$|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| > \frac{mS}{D} - S.$$

Suppose that in a stable membership structure $\boldsymbol{a}$, not all clubs are full, then we know $|I_4(\boldsymbol{a})| = 0$ as otherwise there is a jointly profitable deviation for an individual in $I_4(\boldsymbol{a})$ and a club that is not full where the individual joins the club. Therefore,

$$|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| = n \geq \frac{mS}{D},$$

and so $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \geq \frac{mS}{D} - S$ given $|I_3(\boldsymbol{a})| \leq S$.

For the upper bound of $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|$, since aggregate club capacity is mS, we must have $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})|D \leq mS$ and so $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \leq \frac{mS}{D}$.

Turning to the bounds for $|I_4(\boldsymbol{a})|$. Since $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| + |I_4(\boldsymbol{a})| = n$, $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| \leq \frac{mS}{D}$, and $|I_3(\boldsymbol{a})| \leq S$, so $|I_4(\boldsymbol{a})| \geq n - \frac{mS}{D} - S$. Moreover, if $|I_4(\boldsymbol{a})| > n - \frac{mS}{D}$, then $|I_1(\boldsymbol{a}) \cup I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})| < \frac{mS}{D}$. The club capacity is not exhasted and there must exist a club c that is not full. There is a jointly profitable deviation for an individual i in $I_4(\boldsymbol{a})$ and club c where i joins c. A contradiction.

For the first part of point (ii), we show that there exists a stable membership profile $\boldsymbol{a}$ with

$\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) \geq 1 - \frac{D}{\min\left(\frac{nD-mS}{S},(D-1)m\right)}$. Let $\boldsymbol{a}$ be a membership structure constructed as follows. If D divides m then equally divide clubs into m/D groups. Let all D clubs in the same group admit the same S individuals. If D does not divide m then divide the clubs into $\lceil m/D \rceil$ groups where the first $\lfloor m/D \rfloor$ groups have D clubs and the last group has m mod D clubs. Let all D clubs in the first $\lfloor m/D \rfloor$ groups admit the same S individuals; individuals who join these clubs have degree D. Let the m mod D clubs also admit the same S individuals; individuals who join these clubs have degree m mod D. It is easy to see that the structure is stable. The sum of degrees of individuals under $\boldsymbol{a}$ is $\sum_{i\in I} d_i(\boldsymbol{a}) = mS$. There are $\lceil m/D \rceil S$ individuals in clubs and $|I_4(\boldsymbol{a})| = n - \lceil m/D \rceil S$ individuals who join no club. Therefore

$$\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = \frac{\min(n, mS) - \lceil \frac{m}{D} \rceil S}{\min(n, mS) - \lceil \frac{mS}{D} \rceil} \geq \frac{\min(n, mS) - \frac{mS}{D} - S}{\min(n, mS) - \frac{mS}{D}} = 1 - \frac{D}{\min(\frac{nD-mS}{S}, (D-1)m)}.$$

For the first part of point (ii), assume that $h(S) - h(S-1) > 2(h(1) - h(0))$, $n > mS$ and m is even. Consider a membership structure $\boldsymbol{a}$ constructed as follows. Group clubs into $m/2$ pairs and select $m/2$ groups of individuals containing S individuals each. Match each pair of clubs with a unique group of individuals and make all the individuals assigned to each pair members of both clubs in the pair (an example of this constructed membership structure with $m = 8$ clubs having capacity $S = 4$ each is illustrated in Figure 4). The structure is stable because any deviation by a club weakens its existing link with another club from S to $S-1$ and creates at most two new links. Notice that the sum of degrees of individuals under $\boldsymbol{a}$ is $\sum_{i\in I} d_i(\boldsymbol{a}) = 2 \cdot mS/2 = mS$ and there are $|I_4(\boldsymbol{a})| = n - mS/2$ individuals who are not members of any club. Hence the value of marginalisation measure for $\boldsymbol{a}$ is

$$\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) = \frac{\frac{mS}{2}}{mS - \lceil \frac{mS}{D} \rceil} \leq \frac{\frac{mS}{2}}{mS - \frac{mS+D-1}{D}} = \frac{D}{2(D-1)}.$$

For point (iii), first note that if $n > mS$ then all clubs are full in a stable profile $\boldsymbol{a}$, as otherwise there exists an individual $i \in I_4(\boldsymbol{a})$ and a club c that is not full, which implies that there is a profitable deviation by i and c where i joins c. We then provide the point by showing that in a stable profile $\boldsymbol{a}$, $|I_3(\boldsymbol{a})| < S^2D$, which leads to

$$\begin{aligned}\mathcal{M}_{\mathcal{I}}(\boldsymbol{a}) =& \frac{mS - (|I_1(\boldsymbol{a})| + |I_2(\boldsymbol{a})| + |I_3(\boldsymbol{a})|)}{mS - \lceil \frac{mS}{D} \rceil} \geq \frac{mS - \frac{mS}{D} - S^2D}{mS - \frac{mS}{D}} \\ &> 1 - \frac{S^2D^2}{mS(D-1)} = 1 - \frac{D^2S}{(D-1)m}.\end{aligned}$$

To derive the upper bound on $|I_3(\boldsymbol{a})|$, suppose $I_3(\boldsymbol{a}) \neq \varnothing$, let $i \in I_3(\boldsymbol{a})$ be the individual with the lowest degree in $I_3(\boldsymbol{a})$ and let c be a club that hosts i. The number of clubs c has links with is at most $(D-1)S$, since each member creates at most $D-1$ links for c and there are at most S members in c. The number of club memberships provided by c and its neighbouring clubs is thus at most $((D-1)S+1)S < S^2D$. Suppose $|I_3(\boldsymbol{a})| \geq S^2D$. Then there exists an individual $i' \in I_3(\boldsymbol{a})$ who is neither in club c nor in its neighbours clubs. Consider a deviation by c and i' where c exiles i and admits i'. This deviation is feasible for i' because being in $I_3(\boldsymbol{a})$, i' is in less than D clubs under $\boldsymbol{a}$. It is also strictly profitable to i'. The deviation is also weakly profitable to c. This is because exiling i reduces $d_i(\boldsymbol{a})$ links which have strength at least 1 under $\boldsymbol{a}$. Admitting i' creates $d_{i'}(\boldsymbol{a}) \geq d_i(\boldsymbol{a})$ links between c and new clubs. Because h is 0 at 0, strictly increasing, and concave, the benefits from creating a new link are weakly higher than the losses from reducing a weight of a link. Hence replacing i with i' is weakly profitable to c, a contradiction with $\boldsymbol{a}$ being stable. ■

Proof of Proposition 6

By definition, a strongly stable profile is stable. We prove that when $nD > mS + SD$, a stable profile is strongly stable.

First, note that when $nD > mS + SD$, all clubs are full in a stable profile. This is because when $nD > mS+SD$, $|I_4(\boldsymbol{a})| > 0$, which means there exists an individual $i \in I$ with $d_i(\boldsymbol{a}) = 0$ by Proposition 2. If there exists a club c that is not full, there is a blocking pair (i, c) where i joins c. A controdiction. This observation indicates that for all stable profiles, $C_3(\boldsymbol{a}) = \emptyset$ and a club in $C_2(\boldsymbol{a})$ must admit an individual with degree less than D.

Before we proceed to complete proving Proposition 6, we give a lemma about the membership structure for clubs in $C_2(\boldsymbol{a})$.

Lemma 2. *When $nD > mS + DS$, consider a stable profile $\boldsymbol{a}$. Suppose there are k different levels of degrees for individuals in $I_3(\boldsymbol{a})$. Let $d_1, ..., d_k$, where $d_1 > ... > d_k$ be the levels of the degrees and let $I_3^l(\boldsymbol{a}) = \{i \in I_3(\boldsymbol{a}) : d_i(\boldsymbol{a}) = d_l\}$ for $l = 1, ..., k$; let $C_2^l(\boldsymbol{a}) = \{c \in C_2(\boldsymbol{a}) : \pi_c(\boldsymbol{a})$ has the l^{th} highest level of productivity in $C_2(\boldsymbol{a})\}$. For any club $c \in C_2^l(\boldsymbol{a})$, the set of* *members of c includes all individuals in $I_3^1(\boldsymbol{a}) \cup ... \cup I_3^l(\boldsymbol{a})$. If c is not full with individuals in $I_3(\boldsymbol{a})$, then the rest of its members are elements of $I_2(\boldsymbol{a})$. Moreover, for an individual $i \in I_2(\boldsymbol{a})$ who is in $C_2^l(\boldsymbol{a})$, i must also be in any club in $C_2^1(\boldsymbol{a}) \cup ... \cup C_2^l(\boldsymbol{a})$.*

Proof. The statement that 'if c is not full with individuals in $I_3(\boldsymbol{a})$, then the rest of its members come from $I_2(\boldsymbol{a})$' follows directly from the fact that all clubs are full in a stable profile when $nD > mS + DS$ and the definition of sets $I_1(\boldsymbol{a})$, $I_2(\boldsymbol{a})$, and $I_4(\boldsymbol{a})$.

To prove the statement that for an individual $i \in I_2(\boldsymbol{a})$ who is in $C_2^l(\boldsymbol{a})$, i must also be in any club in $C_2^1(\boldsymbol{a}) \cup ... \cup C_2^l(\boldsymbol{a})$. Suppose the statement does not hold and there exists an individual $i \in I_2(\boldsymbol{a})$ who is in club $c \in C_2(\boldsymbol{a})$ but not $c' \in C_2(\boldsymbol{a})$ where $\pi_{c'}(\boldsymbol{a})(a) \geq \pi_c(\boldsymbol{a})$. Since $c' \in C_2(\boldsymbol{a})$ and all clubs are full, c' must have a member $i' \in I_3(\boldsymbol{a})$. There is a blocking pair of (i, c') where c' replaces i with i' and i' quits club c to join c'.

To prove the claim about the club members from set $I_3(\boldsymbol{a})$, recall that by point (iii) of Proposition 1, if individual $i \in I_3(\boldsymbol{a})$ is in a club c, then all individuals with degree d where $d_i(\boldsymbol{a}) \leq d < D$ must be in c. This combined with the fact that all clubs are full indicates that the productivity of a club is defined by its member with the lowest degree. The argument then follows from the definition of $C_2^l(\boldsymbol{a})$, and we also know additionally that there are k different productivity levels for clubs in $C_2(\boldsymbol{a})$. $\square$

By Lemma 2, we can see that $|\cup_{x=l}^k C_2^x(\boldsymbol{a})| = d_l$, where k is the number of degree levels of individuals in $I_3(\boldsymbol{a})$.

Now, suppose profile $\boldsymbol{a}$ is stable but there exists $I' \subseteq I$ and $C' \subseteq C$ such that there is $\boldsymbol{a'} \in A$ with $a'_{ic} \leq a_{ic}$ for all $(i, c) \in I \times C$, where $i \in I', c \notin C'$ or $i \notin I', c \in C$ and $a'_{ic} = a_{ic}$ for all $i \notin I', c \notin C'$, and $u_i(\boldsymbol{a'}) > u_i(\boldsymbol{a})$ for all $i \in I'$ and $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$ for all $c \in C'$. Since $\pi_c(\boldsymbol{a'}) > \pi_c(\boldsymbol{a})$ for all $c \in C'$, $C' \subseteq C_2(\boldsymbol{a})$. Similarly, $I' \subseteq I_2(\boldsymbol{a}) \cup I_3(\boldsymbol{a}) \cup I_4(\boldsymbol{a})$. We show, by induction, that $C_2^l \nsubseteq C'$ for $l = 1, 2, \ldots, k$, hence completing the proof.

First, we show that if $c \in C_2^1(\boldsymbol{a})$ then $c \notin C'$. This is because if $c \in C_2^1(\boldsymbol{a})$, then c has all individuals in $I_2(\boldsymbol{a}) \cup I_3^1(\boldsymbol{a})$ by Lemma 2. Those in $I_2(\boldsymbol{a})$ already have the highest degree possible and those in $I_3^1(\boldsymbol{a})$ cannot attain higher degree with a deviation that only involves clubs in $C_2(\boldsymbol{a})$ as those in $I_3^1(\boldsymbol{a})$ are members of all clubs in $C_2(\boldsymbol{a})$. This means that there is no way for c to improve its productivity by deviating from $\boldsymbol{a}$ to $\boldsymbol{a'}$.

Now, we show that if $C_2^1(\boldsymbol{a}) \cup \ldots \cup C_2^l(\boldsymbol{a}) \nsubseteq C'$, then $C_2^{l+1} \nsubseteq C'$. With a deviation that only invokes clubs in $C_2^{l+1} \cup \ldots \cup C_2^k(\boldsymbol{a})$, individuals in $I_3^1(\boldsymbol{a}) \cup \ldots \cup I_3^{l+1}$ cannot raise their degree, sine they are already in all clubs in $C_2^{l+1} \cup \ldots \cup C_2^k(\boldsymbol{a})$, and individuals in $I_3^{l+2}(\boldsymbol{a}) \cup \ldots \cup I_3^k(\boldsymbol{a})$ cannot raise their degree to a level greater than d_{l+1}. This means that a club in $C_2^{l+1} \cup \ldots \cup C_2^k(\boldsymbol{a})$

must have raised its productivity by deviating from $\boldsymbol{a}$ to $\boldsymbol{a'}$ by replacing a member in $I_3(\boldsymbol{a})$ with a member in $I_2(\boldsymbol{a})$. Let x be the number of individuals in $I_2(\boldsymbol{a})$ that deviates and join c, there must exists an $y \in \{2, 3, \ldots, l+2\}$ such that

$$|I_3^y(\boldsymbol{a}) \cup I_3^{y+1}(\boldsymbol{a}) \cup \ldots \cup I_3^{l+1}(\boldsymbol{a})| < x \leq |I_3^{y-1}(\boldsymbol{a}) \cup I_3^{y}(\boldsymbol{a}) \cup \ldots \cup I_3^{l+1}(\boldsymbol{a}),$$

where we set $|I_3^y(\boldsymbol{a}) \cup I_3^{y+1}(\boldsymbol{a}) \cup \ldots \cup I_3^{l+1}(\boldsymbol{a})| = 0$ when $y = l+2$.

Since $x > |I_3^y(\boldsymbol{a}) \cup I_3^{y+1}(\boldsymbol{a}) \cup \ldots \cup I_3^{l+1}(\boldsymbol{a})|$, one of the individual from $I_2(\boldsymbol{a})$, say i, who joins c can only have memberships in clubs that have productivity not less than those in $C_2^{y-1}(\boldsymbol{a})$. Since $< x \leq |I_3^{y-1}(\boldsymbol{a}) \cup I_3^{y}(\boldsymbol{a}) \cup \ldots \cup I_3^{l+1}(\boldsymbol{a})|$, the productivity of c does not exceed the productivity of a club in $C_2^{y-1}(\boldsymbol{a})$ under $\boldsymbol{a'}$, contradicting the requirement that i improves her utility by deviating from $\boldsymbol{a}$ to $\boldsymbol{a'}$. ■

Proof of Proposition 7

We start by deriving the upperbound on $|\hat{I}_2(\boldsymbol{a})|$. Suppose there exist individuals with degree $0 < d < D$, let i be an individual with the lowest type whose degree is d. It can be shown that for any club c that i joins, c hosts all individuals with degree d. Suppose not so that there exists i' with $d_{i'}(\boldsymbol{a}) = d_i(\boldsymbol{a})$, $\theta_{i'} > \theta_i$ and $a_{i'c} = 0$. Then c would benefit by replacing i with i' and i' would want to join c. A blocking pair is formed. Since $s_c(\boldsymbol{a}) \leq S$, so $|\{i \in I : 0 < d_i(\boldsymbol{a}) < D\}| \leq S$. Given that there are at most $D-1$ degrees greater than 0 and less than D, $|\hat{I}_2(\boldsymbol{a})| \leq (D-1)S$.

For the cardinality of $\hat{I}_1(\boldsymbol{a})$. $|\hat{I}_1(\boldsymbol{a})| \leq \frac{mS}{D}$ follows directly from $|\hat{I}_1(\boldsymbol{a})|D \leq mS$. For the lower bound, first suppose that in a stable profile $\boldsymbol{a}$, all clubs are full. Then we have $|\hat{I}_1(\boldsymbol{a})|D + \sum_{i \in \hat{I}_2(\boldsymbol{a})} d_i(\boldsymbol{a}) = mS$, which implies $(|\hat{I}_1(\boldsymbol{a})| + |\hat{I}_2(\boldsymbol{a})|)D > mS$. Since $|\hat{I}_2(\boldsymbol{a})| \leq (D-1)S$, so $|\hat{I}_1(\boldsymbol{a})| > \frac{mS}{D} - (D-1)S$. Now consider the cases where not all clubs are full. Then $|\hat{I}_3(\boldsymbol{a})| = 0$, as otherwise there is a jointly profitable deviation for an individual in $\hat{I}_3(\boldsymbol{a})$ and a club that is not full where the individual joins the club. Therefore, $|\hat{I}_1(\boldsymbol{a}) \cup \hat{I}_2(\boldsymbol{a})| = n \geq \frac{mS}{D}$. Since $|\hat{I}_2(\boldsymbol{a})| \leq (D-1)S$, $|\hat{I}_1(\boldsymbol{a})| \geq \frac{mS}{D} - (D-1)S$.

Finally, for $|\hat{I}_3(\boldsymbol{a})|$, since $|\hat{I}_1(\boldsymbol{a})| + |\hat{I}_2(\boldsymbol{a})| + |\hat{I}_3(\boldsymbol{a})| = n$, $|\hat{I}_1(\boldsymbol{a})| \leq \frac{mS}{D}$, and $|\hat{I}_2(\boldsymbol{a})| \leq (D-1)S$, so $|\hat{I}_3(\boldsymbol{a})| \geq n - \frac{mS}{D} - (D-1)S$. Moreover, if $|\hat{I}_3(\boldsymbol{a})| > n - \frac{mS}{D}$, then $|\hat{I}_1(\boldsymbol{a}) \cup \hat{I}_2(\boldsymbol{a})| < \frac{mS}{D}$. The club capacity is not exhausted and there must exist club c that is not full. There is a jointly profitable deviation for an individual i in $I_4(\boldsymbol{a})$ and club c where i joins c. A contradiction.

The bound on $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$ is derived following the same steps as in the proof for Proposition 2.

Turning to the relationship between an individual's type and memberships. When $\alpha = 0$, suppose there exists individual i and j where $d_i(\boldsymbol{a}) < d_j(\boldsymbol{a})$ and $\theta_i > \theta_j$. Then there exists club c that admits j but not i. There is a jointly profitable deviation for i and c where c replaces j with i and i quits no club to join c-a contradiction.

When $\alpha > 0$ and $\theta_i - \theta_j < \alpha$, $\forall i \neq j$, consider a membership profile constructed in the following way. W.L.O.G., let $\theta_1 < \theta_2 < ... < \theta_n$. Assign individuals i_1 to i_S to clubs c_1 to c_D, i_{S+1} to i_{2S} to clubs c_{D+1} to c_{2D},..., and $i_{\lfloor m/D \rfloor (S-1)+1}$ to $i_{\lfloor m/D \rfloor S}$ to clubs $c_{\lfloor m/D \rfloor (D-1)+1}$ to $c_{\lfloor m/D \rfloor D}$. If $\lfloor m/D \rfloor = m/D$, the construction is completed. If $\lfloor m/D \rfloor \neq m/D$ and $n - \lfloor m/D \rfloor S \geq S$, let the rest $m \bmod D$ clubs admit individuals $i_{\lfloor m/D \rfloor S+1}$ to $i_{(\lfloor m/D \rfloor +1)S}$. If $\lfloor m/D \rfloor \neq m/D$ and $n - \lfloor m/D \rfloor S < S$, let the rest $m \bmod D$ clubs admit individuals admit individuals $i_{\lfloor m/D \rfloor S+1}$ to i_n. In this constructed profile, $d_i(\boldsymbol{a}) \geq d_j(\boldsymbol{a})$, $\forall i, j \in I$, where $\theta_i < \theta_j$. It is easy to verify that this profile is stable when $\theta_i - \theta_j < \alpha$, $\forall i \neq j$, and $m \bmod D \neq D - 1$. ■

Proof of Proposition 8 We start by deriving the upperbound on $|\hat{I}_2(\boldsymbol{a})|$. If there does not exist two 2 individuals in $\hat{I}_2(\boldsymbol{a})$, the proof is done. Suppose there exist at least 2 such individuals, let i and j be two individuals such that $0 < d_j(\boldsymbol{a}) \leq d_i(\boldsymbol{a}) < D$.

We show that the set of clubs i joins is a superset of that j joins. Suppose not and there exists clubs c that admits j but not i. Since $d_i(\boldsymbol{a}) \geq d_j(\boldsymbol{a})$, there also exists club c' that admits i but not j. This structure is not stable because there is either a blocking pair (i, c) where c replaces j with i or a blocking pair (j, c') where c' replaces i with j. To prove this statement, note that the production function of firms can be represented by

$$\pi_c(\boldsymbol{a}) = f(s_c(\boldsymbol{a})) + \sum_{i \in I: a_{ic}=1, c' \neq c} a_{ic'} q(s_{c'}(\boldsymbol{a})).$$

Hence, the productivity change of c if it replaces j with i is $\sum_{c'' \neq c, c'} a_{ic''} q(s_{c''}(\boldsymbol{a})) + q(s_{c'}(\boldsymbol{a})) - \sum_{c'' \neq c, c'} a_{jc''} q(s_{c''}(\boldsymbol{a}))$ and the productivity change of c' if it replaces i with j is $\sum_{c'' \neq c, c'} a_{jc''} q(s_{c''}(\boldsymbol{a})) + q(s_c(\boldsymbol{a})) - \sum_{c'' \neq c, c'} a_{ic''} q(s_{c''}(\boldsymbol{a}))$. It is easy to see that one of the two expressions must be greater than 0, so that one of the clubs has an incentive to change. Individuals i and i' both want to accept invitations from clubs.

With this result, take any individual $i \in \hat{I}_2(\boldsymbol{a})$ with minimal degree and let c be any club that i members, all individuals in $\hat{I}_2(\boldsymbol{a})$ must be in c. Since $s_c(\boldsymbol{a}) \leq S$, so $|\hat{I}_2(\boldsymbol{a})| \leq S$.

The bounds for $\hat{I}_1(\boldsymbol{a})$ and $\hat{I}_3(\boldsymbol{a})$ can then be derived with similar arguments as in the proof for Proposition 7. The bound on $\mathcal{M}_{\mathcal{I}}(\boldsymbol{a})$ is derived following the same steps as in the proof for Proposition 2. ■