Combinatorial methods in the representation theory of the symmetric group

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This dissertation is submitted for the degree of
Doctor of Philosophy

Trinity College

February 2022

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as specified in the text. It is not substantially the same as any work that has been submitted, or is currently being submitted, for any degree or other qualification at the University of Cambridge or any other institution.

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Abstract

The study of the representation theory of the symmetric group can be carried out from a combinatorial point of view, avoiding the machinery of the representation theory of algebraic groups. This approach has the benefit of providing more insight into the subject as the study remains in the setting of the symmetric group. A number of combinatorial tools are explored throughout this dissertation, which as well as making advances in the representation theory of the symmetric group, also contributes to topics of a purely combinatorial interest.

The most important objects to be understood in the representation theory of the symmetric group are the *Specht modules*, which over fields of characteristic 0 are a complete set of simple modules. These can be defined combinatorially and thus allow an explicit combinatorial approach to be taken to their study. This dissertation, as with the modern study of the representation theory of the symmetric group, is concerned with modular representation theory - the theory over fields of positive characteristic. In this setting the Specht modules are not necessarily simple, however a complete set of simple modules can be found amongst their cosocles. A complete understanding of the Specht modules would, therefore, reveal the details of the simple modules.

To understand the structure of the Specht modules it is first necessary to understand the *decomposition numbers*, which count the multiplicity of each simple module as a factor of a Specht module. We shall examine the filtration introduced by Schaper, which remains the most powerful tool for determining decomposition numbers, and begin to classify Specht modules by the number of empty layers in this filtration. This knowledge improved the utility of Schaper's sum formula and we shall demonstrate how our classification leads to new decomposition numbers.

We also study extensions of Specht modules by the trivial module, giving a number of classes of modules for which no such extensions exist and giving an upper bound on the dimension of the Ext-group for all Specht modules. In the case of two part partitions we see that this bound is obtained and we explicitly construct these extensions. Our methods here are entirely combinatorial and remain in the setting of the symmetric group. This work motivates the study of modular combinatorial designs.

Combinatorial designs over the integers are well studied and have numerous applications while designs over fields of positive characteristic, modular, or p-ary designs, have received less attention. In this dissertation universal p-ary designs are defined and classified. Like their integral counterparts we anticipate a broad range of applications for the

theory of modular designs, amongst which are the applications to extensions of Specht modules seen in this dissertation.

for Steph,

with love

Acknowledgements

There are many people and organisations that have supported me while I worked towards my PhD and without whom this dissertation would not have come into existence. Firstly I would like to thank my supervisor, Dr Stuart Martin, who has been a constant source of ideas and encouragement throughout my time as a PhD student.

My understanding of the topics in this dissertation, and many others, were greatly enhanced by many discussions with Robert Spencer. He has fielded numerous silly questions, and perhaps even some insightful ones, and provided in house tech support in C.0.02. I am also grateful for his company and friendship.

I would like to thank Freddie Illingworth for a number of fruitful conversations, both cricketing and mathematical. In particular, I am grateful to him for insight which led to the proof of the rank formula for inclusion matrix in Chapter 6. I would like to acknowledge the many other PhD students in the department for their company and encouragement over the past several years.

I would like to thank the School of Mathematics and Statistics at Victoria University of Wellington. This is where my mathematical journey began as an undergraduate in 2013, and where I spent much of 2020 and 2021 as a visiting scholar as I waited out the Covid-19 pandemic.

My study in Cambridge, both for my Masters and PhD have been generously supported by the Woolf Fisher Trust. I am grateful to the trust for this, and in particular to Dr Nigel Evans who has been great personal and professional support to me and the other Woolf Fisher scholars. I would also like to thank Trinity College for providing funding in my final term.

I would like to acknowledge my family who have been an amazing support network for me throughout my life. If you only understand one thing in this thesis, then I hope it is that I love you all and am grateful for everything you have done for me. To my parents, Amanda and Greg; thank you for providing all of the opportunities which have led me to this point. To my brother, Corbin; for always making me laugh and smile growing up, and not letting me take myself too seriously. I am grateful for the time we had together and I miss you every day. To my younger siblings, Victoria and Patrick; thank you for being there for me, and for each other, and for never failing to make me smile.

Finally, to my wife, Stephanie, who moved to the other side of the world with me, twice, so I could complete this thesis. I am grateful to you for the proof-reading and editing, both of this thesis and numerous other papers and research statements. Most of all I am grateful for your patience and for your love and support throughout this journey, without which I would not have made it to this point. I look forward to beginning the next chapter of our life together.

Contents

Chapter 1

Introduction

The study of the representation theory of finite groups dates back to the end of the 19th century to the work of Frobenius, Burnside, Schur, and others, although some earlier work of Lagrange, Legendre, and Gauss, for example, may also be identified as paving the way for this subject. Two foundational papers were published in 1897; a paper by Frobenius which was the first paper on representations of finite groups, and a paper of Burnside which was the first treatise in English on the theory of finite groups. Frobenius and Burnside independently developed the theory of representations of finite groups and its applications to group theory. The early work on representation theory was done over fields of characteristic zero, until Brauer introduced 'modular representation theory', the theory over fields of positive characteristic, in the 1930's. This modular representation theory is much richer than the ordinary representation theory, as the group algebra is not necessarily semi-simple. This thesis is dedicated to the modular representation theory of the symmetric group.

The representation theory of the symmetric group has been studied since the dawn of representation theory. Alfred Young introduced Young tableaux and explicitly constructed matrices for the irreducible representations of the symmetric group. The characters had already been calculated by Frobenius. Schur discovered the link between the representation theory of the symmetric group and the general linear group, via what are known known as the Schur algebra and the Schur functor. The symmetric group is an important case study in the representation theory of finite groups, as it is a rare example of a group whose representation theory is well understood. Much of modern finite group theory is focused on the study of simple groups, or their close relatives, such as the almost simple groups, of which the symmetric group is an example. This means that the symmetric group is a useful test case for a number of general conjectures. It is also the cornerstone of much of combinatorial representation theory and has links with the representation theory of other objects, such as diagrammatic algebras.

The simple modules for the symmetric group algebra over fields of characteristic 0 were understood in the early 20th century; they are indexed by partitions of n and are known as the Specht modules. These modules have a combinatorial description via Young tableaux. When reduced mod p, these modules may not remain simple, however the simple modules all appear as the head of a certain Specht module. Despite knowing where to find these simple modules, much is unknown about them. In particular, their dimensions are not known in general. To understand the simple modules we shall focus our study on the Specht modules, as a full description of their structure would result in a full description of the irreducible modules. Even this would not complete our understanding of the representation theory of the symmetric group, as it would remain to understand when non-split extensions of simple modules can exist. In the third and fifth chapters of this thesis we shall attempt to understand the Specht modules and their extensions, which in turn will improve the understanding of the simple modules.

The simple modules which appear as composition factors of a given Specht module are known, due to a sum formula of Schaper, however their composition multiplicity remains unknown in general. Calculating these composition multiplicities has been the focus of much research over decades, but they are only known in some special cases, and we have a few ad hoc methods for calculating others. Schaper's sum formula gives upper bounds on these decomposition numbers, and the utility of this formula can be improved if the Schaper number of a partition is known. In the third chapter of this thesis we begin the characterisation of partitions by their Schaper number and demonstrate how new decomposition numbers will be found using this new data.

The fourth chapter of this thesis is entirely combinatorial in scope, although motivated by questions in the representation theory of the symmetric group, and so is self contained. It has been placed here as the combinatorial tools introduced here are used extensively in the fifth chapter. In the fourth chapter we study p-ary designs, as defined by Richard Wilson. A p-ary t-design is a generalisation of the usual notion of an integral t-design, as the number of blocks containing each set of size t is only required to be constant modulo p. The theory of integral and p-ary designs differ significantly; most notably while an integral t-design is necessarily a j-design for $j < t$, Wilson showed this does not need to be true for p-ary designs. The main result of this chapter is a complete classification of 'universal' p-ary designs: those t-designs which are j-designs for all $j < t$.

In the fifth chapter we shall study extensions of the trivial $k\mathcal{S}_n$ module by the Specht modules, which motivated the study of universal p-ary designs. We shall see how universal p-ary designs are related to extensions of the trivial $k\mathcal{S}_n$ module by a Specht module. The theory of p-ary designs allows a classification of these extensions when the Specht module is indexed by a two part partition, and gives upper bounds on the number of extensions in the more general case. Although these upper bounds were recently determined by Donkin and Geranios using the algebraic group approach, our new proof in this chapter is significant as it remains in the setting of the symmetric group and gives insight into how the extensions are constructed.

The final chapter of this thesis gives an overview of the theory of **FI**-modules, which encode the structure of a sequence of $k\mathcal{S}_n$ modules. These were first introduced by Church Ellenberg and Farb and studying these objects may lead to new structural information for certain families of $k\mathcal{S}_n$-modules. We construct a number of new examples of sub-modules of representable **FI**-modules, the modules which play the role of free modules in this context. The study of these inspired a representation theoretic interpretation of the rank formula for the inclusion matrices and a new, very short, proof of this formula which is also in the final chapter.

Chapter 2

Background

Throughout this thesis we shall assume that the reader is familiar with general concepts from modular representation theory. Some of these will be recalled briefly in the first section of this chapter in order to introduce notation. Further information can be found in the books of Alperin [Alp93] and of Webb [Web16], for example. Our overview of the representation theory of the symmetric group will be more comprehensive, providing all of the background required to read the chapters which follow.

2.1 Group representation theory

Throughout this section we shall fix a finite group G and a field k of characteristic $p \geq 0$. Let M be a kG-module, a *composition series* for M is a (finite) series of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$$

where M_i is a maximal submodule of M_{i+1}, and thus the quotients $S_i := \frac{M_i}{M_{i-1}}$ are simple. These S_i are the *composition factors* of M. The Jordan-Hölder theorem ensures that, as long as a composition series for M exists, all composition series for M have the same composition factors, including multiplicities. If S is a simple kG-module then we denote the composition multiplicity of S in M, that is the number of times a simple module isomorphic to S occurs as a composition factor of M, by $[M : S]$. If a series of submodules is such that the factors are not necessarily simple, then we call it a *filtration* of M.

If

$$0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$$

is a filtration of M with factors $N_i = \frac{M_i}{M_{i-1}}$, then we write:

$$M \sim \begin{array}{l} N_l \\ N_{l-1} \\ \vdots \\ N_2 \\ N_1 \end{array}.$$

We shall write $\mathrm{soc}(M)$ for *socle* of a module M; that is the maximal semi-simple submodule of M, $\mathrm{cosoc}(M)$ for *cosocle* of M; its maximal semisimple quotient, and $\mathrm{rad}(M)$ for *radical* of M; the intersection of all of its maximal submodules.

2.1.1 Block theory

Let R be a ring with identity. A *block* of R is an indecomposable two sided ideal which is a direct summand of R. The Krull-Schmidt theorem ensures that the group algebra kG has a unique decomposition (up to ordering) into blocks,

$$kG = b_1 \oplus \cdots \oplus b_r.$$

Each of these blocks is an algebra in its own right. Such a decomposition corresponds to an expression

$$1 = e_1 + \cdots + e_r,$$

where the e_i are orthogonal centrally primitive idempotents in kG. In this situation $b_i = e_i kG$. There are a number of approaches that one can take to block theory, for example some authors, [Web16], refer to the e_i as blocks. We may also think of blocks as equivalence classes of kG-modules as follows:

Definition 2.1.1. *If M is a kG-module, then $M = e_1M + \cdots + e_rM$ is a decomposition of submodules. If there is a unique i such that e_i does not annihilate M, then we say that M* lies in *or* belongs to *the block b_i. In particular, every simple module and every indecomposable module lie in a block.*

If two kG modules lie in distinct blocks, then they have no composition factors in common. Any kG module can be expressed as a direct sum of modules which each lie in a distinct block. The *principal block* of kG, denoted $B_0(kG)$, is the block which contains the trivial module, k.

2.1.2 Induction and restriction

If H is a subgroup of G, $H \leq G$, and M is a kG-module, then we may consider M as a kH-module, simply by only considering the action of the elements in kH. This module will be denoted by $M \downarrow_H^G$, or often just $M \downarrow_H$, and is called the *restriction* of M to H. It is clear that if $K \leq H \leq G$, then

$$M \downarrow_H^G \downarrow_K^H = M \downarrow_K^G,$$

which is to say that restriction is transitive.

On the other hand, if $H \leq G$ and M is a kH-module, then

$$M \uparrow_H^G = kG \otimes_{kH} M$$

is the kG-module *induced* from H to G. We call this process *induction* and say that $M \uparrow_H^G$ is an *induced module*. Induction is also transitive, which is to say if $K \leq H \leq G$, then

$$M \uparrow_K^H \uparrow_H^G = M \uparrow_K^G .$$

If $b = ekG$ is a block of kG and M a kH-module, then we denote $e(M \uparrow_H^G)$, the summands of $M \uparrow_H^G$ lying in b by $M \uparrow^b$. Similarly if $\tilde{b} = \tilde{e}kH$ is a block of kH and M a kG-module then we denote $\tilde{e}(M \downarrow_H^G)$ by $M \downarrow_{\tilde{b}}$.

Theorem 2.1.2 (Eckmann-Shapiro)**.** *Let $H \leq G$ be a subgroup, let M be a kH-module lying in a block A, and let N be a kG-module lying in a block B. Then for any $i \geq 0$ we have*

- $$\mathrm{Ext}^i_A(M, N \downarrow_A) \cong \mathrm{Ext}^i_B(M \uparrow^B, N)$$

- $$\mathrm{Ext}^i_A(N \downarrow_A, M) \cong \mathrm{Ext}^i_B(N, M \uparrow^B)$$

When $i = 0$, Theorem 2.1.2 is usually referred to as *Frobenius reciprocity.*

2.2 Symmetric group

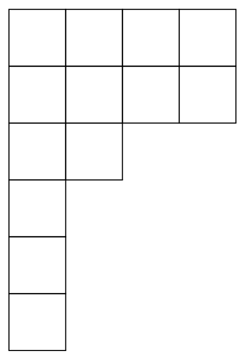
In this section we shall review the representation theory of the symmetric group and provide all of the background required for the remainder of this thesis. Much of this comes from James' book [Jam78], or from the book of James and Kerber [JK81]. We shall denote the symmetric group on n elements by $\mathcal{S}_n$ and write the permutations,

$\sigma \in \mathcal{S}_n$ in cycle notation. For example $(356) \in \mathcal{S}_6$ is the permutation which sends 3 to 5, 5 to 6, and 6 to 3, while fixing 1,2, and 4. The *signature* of σ, denoted $(-1)^\sigma$, is

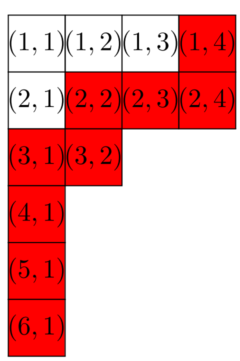
$$(-1)^\sigma = \prod_{1 \leq i < j \leq n} \frac{\sigma(i) - \sigma(j)}{i - j}.$$

2.2.1 Partitions and Young diagrams

A *partition* of the positive integer n is a tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ of non-increasing, non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ with $\sum_{i=1}^{r} \lambda_i = n$. We call the λ_i the *parts* of the partition and we draw attention to our convention that all parts are non-zero. If parts are repeated then we may abbreviate by writing as an exponent the multiplicity of each part; for example the partition of 13, $\lambda = (4, 4, 2, 1, 1, 1)$ may be written as $(4^2, 2, 1^3)$. To each partition we may associate its *Young diagram*, $Y(\lambda)$, which in the English convention, is a left justified array of boxes with λ_i boxes appearing in the ith row.

$Y((4^2, 2, 1^3)) =$ 

We will often abuse notation and identify a partition with its Young diagram. It is useful to have a coordinate system to label the boxes of $Y(\lambda)$, so we shall consider the diagram to be drawn on the Cartesian plane with x axis pointing down and y axis pointing right. For example; with this convention the coordinates of the boxes are:



We shall also often use compass direction when talking about Young diagrams. For example, in the diagram above we observe that there is no box southeast of the box at $(2, 2)$, however there is are boxes south and east of $(2, 2)$, namely the boxes $(3, 2)$ and $(2, 3)$, respectively. The set of all boxes which have no box to their southeast is called the *rim* of a Young diagram or, by abuse of notation, the rim of a partition and is denoted $\mathcal{R}(\lambda)$. The rim of the partition above is highlighted in red. More formally:

$$\mathcal{R}(\lambda) = \{(i, j) \in Y(\lambda) : (i + 1, j + 1) \notin Y(\lambda)\}.$$

A box which has no box to its south or east is called *removable* as if it was removed from the Young diagram the boxes which remain would be the Young diagram of a partition. In the partition λ above the boxes $(2, 4)$, $(3, 2)$, and $(6, 1)$ are the only removable boxes.

The set of partitions of n has a partial ordering induced by:

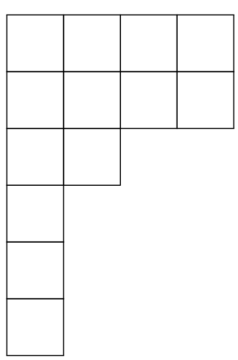
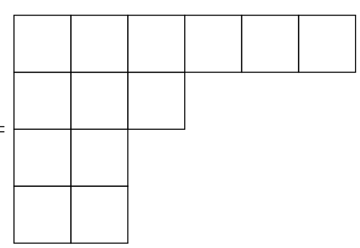
Definition 2.2.1. *Let λ and μ be partitions of n. We say that λ dominates μ, and write $\lambda \trianglerighteq \mu$, if*

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i,$$

for all j.

The partition (n) is the unique maximal element under this ordering, while the partition (1^n) is the unique minimal element. Observe that this is not a total ordering, as, for example, the partition above, $(4^2, 2, 1^3)$, is incomparable with the partition $(5, 3, 1^5)$.

The *conjugate* of a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, which we denote by λ', is the partition $(\mu_1, \ldots, \mu_{\lambda_1})$, where $\mu_j = \mid \{i \in \{1, \ldots, r\} : \lambda_i \geq j\} \mid$. We may think of this as reflection of the Young diagram in the diagonal $y = x$, for example if $\lambda = (4, 4, 2, 1, 1, 1)$ as before then $\lambda' = (6, 3, 2^2)$.

$Y(\lambda) =$  $Y(\lambda') =$ 

We shall call a partition *l-singular* if any part is repeated l (or more) times, and l-

regular otherwise. The l consecutive rows shall be referred to as an *$l-$singularity*. The partition λ above is l-regular for any $l \geq 4$, but is both 2-singular and 3-singular.

Given partitions $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ and $\mu = (\mu_1, \ldots, \mu_s) \vdash m$ we will denote by $\lambda \star \mu$ the partition of $n + m$ who has parts $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s$, sorted so that they are in non-increasing order. If $\lambda_r \geq \mu_1$ then this is the partition $\lambda \star \mu$ is simply obtained by concatenation and will be denoted $\lambda \# \mu$.

A *hook*, $H_B(\lambda)$, of a Young diagram $Y(\lambda)$ at a box $B \in Y(\lambda)$ is a subset of the boxes consisting of the box, B, together with all the boxes in the same column which are south of B, and all the boxes in the same row which are east of B. We will omit the explicit reference to λ when it is clear which partition is being considered. The *hand* and *foot* of a hook are the easternmost and southernmost box in the hook, respectively, while the *arm* is the collection of boxes in the hook in the same row as the hand and the *leg* is the collection of boxes in the same column as the foot. The *leg length* of a hook is the number of boxes in the leg and will be denoted $l(H_B)$, while the total number of boxes in the hook, denoted h_B, shall be called the *hook length*. If H is a hook with hook length l we call H an *l-hook*. The *rim hook*, $\mathcal{R}_B(\lambda)$ corresponding to a hook $H_B(\lambda)$ is the collection of boxes in the rim of the partition between the hand and the foot of the hook. The rim hook is the same size as its corresponding hook:

Proposition 2.2.2. *Let $\lambda \vdash n$ and let $B \in Y(\lambda)$. Then*

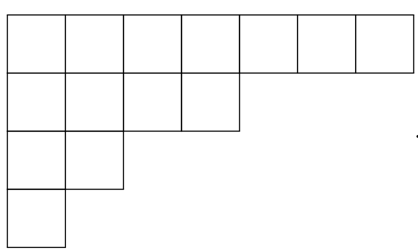
$$| \mathcal{R}_B(\lambda) | = h_B(\lambda).$$

We shall call a partition with no p-hooks a *p-core partition*.

Proposition 2.2.3. *[JK81][Theorem 2.7.16] Let $\lambda \vdash n$. Then there is a unique p-core partition which can be obtained from λ by repeatedly removing rim-hooks from $Y(\lambda)$ corresponding to p-hooks until a p-core partition remains. This partition is called the* p-core *of λ and is denoted $C_p(\lambda)$.*

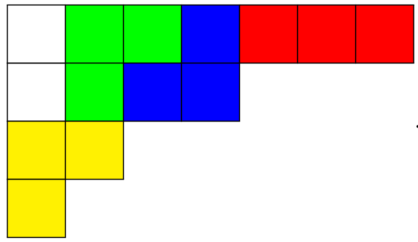
Example 2.2.4. *Let*

$Y(\lambda) =$



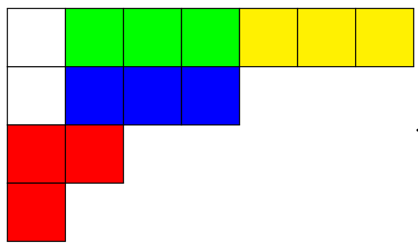
.

We shall determine $C_3(\lambda)$. Then we may remove the rim-hooks highlighted below:

$Y(\lambda) =$  .

These can be removed in any order as long as the red rim hook is removed before the blue one and the green rim hook is removed last. The resulting Young diagram corresponds to the partition (1^2), which is the 3-core of $(7, 4, 2, 1)$.

Note that there is another possible set of rim-hooks which can be removed:

$Y(\lambda) =$  .

Again, these rim-hooks can be removed in any order as long as the red rim hook is removed before the blue one and the green rim hook is removed last. Observe that the partition which remains is still (1^2), as guaranteed by Proposition 2.2.3.

We conclude this subsection by defining an operation on partitions called *p-regularisation*, which was introduced by James in [Jam76b]. To a partition λ, given a prime p, we associate a p-regular partition, denoted λ^R in a natural way. For $l \geq 1$ we define the lth *ladder* to be the subset of boxes $(i, j) \in \mathbb{N}^2$ such that $i + (p-1)(j-1) = l$. The p-regularisation of λ is the unique p-regular partition whose Young diagram has the same number of boxes on each ladder as $Y(\lambda)$. We may visualise $Y(\lambda^R)$ as being obtained from $Y(\lambda)$ by sliding boxes up the ladders as far as possible.

Example 2.2.5. *Let $p = 2$ and $\lambda = (4^2, 2, 1^3)$. The p-regularisation of λ is $(6, 4, 2, 1)$. The Young diagrams of these two partitions are below, and each box is labelled by the* *ladder it lies on.*

$$Y(\lambda) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} \qquad Y(\lambda^R) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 2 & 3 & 4 & 5 \\ \cline{1-4} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

The 3-regularisation of λ is $(4,4,2,2,1)$, while for $p > 3$, λ is p-regular and thus is its own p-regularisation.

2.2.2 Tableaux, tabloids and permutation modules

In this subsection we shall relax the condition that λ is a partition and consider compositions of n instead. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a tuple of non-negative integers which sum to n, but fails to be a partition, then we call λ a *composition* of n, and write $\lambda \vDash n$. In this case we can still define the Young diagram $Y(\lambda)$ as before, but observe that some rows may have more boxes than the row above. A λ-*tableau* is a bijection between $Y(\lambda)$ the set $[n] := \{1, 2, \ldots, n\}$, which we shall think of as a way of filling the boxes of $Y(\lambda)$ with the entries $1, 2, \ldots, n$. The following is an example of a $(3, 2, 1^2)$-tableau;

$$t = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array} .$$

There is an obvious action of the symmetric group $\mathcal{S}_n$ on the set of all λ-tableaux by permuting the entries. We shall denote by $R_i(t)$ the set of entries which appear in the boxes in the ith row of t. Two tableaux s and t are *row equivalent*, $s \sim_{row} t$, if $R_i(s) = R_i(t)$ for all $i \leq r$, the number of parts in the partition. *Column equivalence*, $s \sim_{col} t$, is defined similarly and the column stabiliser of t is defined to be the set $C(t) = \{\sigma \in \mathcal{S}_n \mid \sigma t \sim_{col} t\}$. The $(3, 3, 1^2)$-tableau, s, below, is row equivalent, but not column equivalent, to the tableau t, above.

$$s = \begin{array}{|c|c|c|}\hline 3 & 6 & 1 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

We call a λ-tableau *standard* if the entries increase along each row and down each column. The tableau t above is standard, while s is not. The set of standard λ-tableaux is denoted by $\mathrm{std}(\lambda)$.

Theorem 2.2.6 (Hook length formula)**.** *[FRT54] Let $\lambda \vdash n$, then the number of standard λ-tableaux is given by:*

$$|\,\mathrm{std}(\lambda)\,| = \frac{n!}{\prod_{(i,j)\in Y(\lambda)} h_{(i,j)}}.$$

A λ-*tabloid* is a row equivalence class of λ-tableaux, and will be denoted by writing the tableau in braces or by drawing the Young diagram without vertical lines separating boxes.

$$\{t\} = \{s\} = \begin{array}{ccc}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}.$$

For R, a commutative ring with 1, the R span of all λ tabloids is the *permutation module*, M_R^λ, although we often will omit the explicit reference to the ring R. An element of M^λ is a then a formal sum of λ-tabloids. If $u \in M^\lambda$ then we shall say that a tabloid whose coefficient in u is non-zero is *involved* in u. There is an inner product on M^λ by linearly extending the following:

$$\langle \{s\}, \{t\} \rangle = \begin{cases} 1 & \text{if } s \sim_{row} t \\ 0 & \text{otherwise} \end{cases}.$$

Proposition 2.2.7. *Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$. Then M^λ is a cyclic module, generated by any λ-tabloid, and*

$$\dim(M^\lambda) = \frac{n!}{\Pi_{i=1}^r \lambda_i!}.$$

We now define a family of $R\mathcal{S}_n$-morphisms between permutation modules. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$. For $0 \leq i < r$ and $0 \leq v \leq \lambda_{i+1}$ we define a map

$$\psi_{i,v} : M^\lambda \to M^{\hat{\lambda}},$$

where

$$\hat{\lambda}_j = \begin{cases} \lambda_i + \lambda_{i+1} - v & \text{if } j = i \\ v & \text{if } j = i+1 \\ \lambda_j & \text{otherwise} \end{cases}$$

by sending a λ-tabloid $\{t\}$ to the sum of all $\hat{\lambda}$-tabloids $\{t'\}$ which have $R_j(\{t\}) = R_j(\{t'\})$ for $j \notin \{i, i+1\}$ and $R_i(\{t\}) \subseteq R_i(\{t'\})$. That is; $\{t'\}$ is obtained from $\{t\}$ by moving all but v entries in the $(i+1)$th row of $\{t\}$ to the ith row. For example, if we take $\{t\}$ to be the $(4,3,3,1)$-tabloid,

$$\{t\} = \begin{array}{llll} \hline 1 & 4 & 8 & 10 \\ \hline 2 & 3 & 7 & \\ \hline 5 & 6 & 9 & \\ \hline 11 & & & \\ \hline \end{array} ,$$

then $\psi_{2,2}(\{t\})$ is the sum of all $(4,4,2,1)$-tabloids which are obtained by moving all but two of the (three) entries which appear in the third row of $\{t\}$ to the second row. There are three possible ways this can be done, and thus,

$$\psi_{2,2}(\{t\}) = \begin{array}{llll} \hline 1 & 4 & 8 & 10 \\ \hline 2 & 3 & 5 & 7 \\ \hline 6 & 9 & & \\ \hline 11 & & & \\ \hline \end{array} + \begin{array}{llll} \hline 1 & 4 & 8 & 10 \\ \hline 2 & 3 & 6 & 7 \\ \hline 5 & 9 & & \\ \hline 11 & & & \\ \hline \end{array} + \begin{array}{llll} \hline 1 & 4 & 8 & 10 \\ \hline 2 & 3 & 7 & 9 \\ \hline 5 & 6 & & \\ \hline 11 & & & \\ \hline \end{array} .$$

Observe that even if λ is a partition, $\hat{\lambda}$ may not be.

2.2.3 Specht modules

The Specht module S^λ is an important submodule of the permutation module M^λ and is the main object of study in this thesis. We shall give two alternate characterisations of S^λ and state some important properties. Given a λ-tableau t we define the *column symmetriser* of t to be the element of the group algebra, $R\mathcal{S}_n$, given by

$$\kappa_t = \sum_{\sigma \in C(t)} (-1)^\sigma \sigma,$$

and define the *polytabloid* $e_t = \kappa_t\{t\}$. The *Specht module* $S^\lambda_R \subseteq M^\lambda_R$ is the R span of polytabloids. In fact $S^\lambda_R = \langle e_t \mid t \in \mathrm{std}(\lambda)\rangle_R$, and so the dimension of S^λ is given by the hook length formula, Theorem 2.2.6.

Example 2.2.8. *Let* $\lambda = (3,2)$ *and* $t = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}$. *Then*

$$C(t) = \{(1), (14), (25), (14)(25)\},$$

and

$$e_t = \begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} - \begin{array}{ccc} \hline 2 & 3 & 4 \\ \hline 1 & 5 \\ \cline{1-2} \end{array} - \begin{array}{ccc} \hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2} \end{array} + \begin{array}{ccc} \hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2} \end{array}.$$

Alternatively the Specht module S^λ can be characterised as follows [Jam77]:

Theorem 2.2.9 (Kernel Intersection Theorem)**.**

$$S^\lambda = \bigcap_{i=1}^{r-1} \bigcap_{v=0}^{\lambda_{i+1}-1} Ker(\psi_{i,v}) \subseteq M^\lambda.$$

We shall now list some other important properties of the Specht modules.

Theorem 2.2.10. *[Jam78][Theorem 4.12] If* k *is a field of characteristic* 0*, then*

$$\{\mathcal{S}^\lambda : \lambda \vdash n\}$$

is a complete set of (non-isomorphic) simple $k\mathcal{S}_n$*-modules.*

Theorem 2.2.11. *[Jam78][Theorem 8.11] Let* $\mathbb{F}_p$ *denote the field of* p *elements. Then,* $S^\lambda_{\mathbb{F}_p}$ *is the modular representation of* $\mathcal{S}_n$ *obtained from* $S^\lambda_{\mathbb{Q}}$ *by modular reduction.*

If λ is a p-regular partition, and $\mathbb{F}$ is a field of characteristic p, then the module $D^\lambda_{\mathbb{F}} := \frac{S^\lambda_{\mathbb{F}}}{S^\lambda_{\mathbb{F}} \cap (S^\lambda_{\mathbb{F}})^\perp}$ where λ is p-regular, and orthogonality is with respect to the inner product, is simple.

Theorem 2.2.12. *[Jam78][Theorem 11.5] If* k *is a field of characteristic* p*, then*

$$\{\mathcal{D}^\lambda : \lambda \vdash n, \lambda\ \ p - \text{regular}\}$$

is a complete set of (non-isomorphic) simple $k\mathcal{S}_n$*-modules.*

2.2.4 Specht-shadow modules

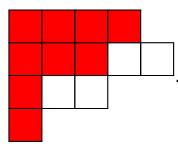
Specht modules may be seen as a special case of a more general submodule of the permutation module M^λ, which we call a *Specht shadow module*:

Definition 2.2.13. *Let $\lambda \vDash n$ and let $\mu \vdash m$ be such that λ has at least as many parts as μ and that $\mu_i \leq \lambda_i$ for all i; that is, μ is a partition and λ a composition with $Y(\mu) \subseteq Y(\lambda)$. Let t be a λ-tableau. We define the μ-*column stabiliser *of t, denoted $C^\mu(t)$, to be the subset of the column stabiliser of t which fixes the entries outside μ. The μ-*column symmetriser *is the element of the group algebra, $R\mathcal{S}_n$, given by*

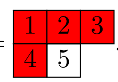
$$\kappa_t^\mu = \sum_{\sigma \in C^\mu(t)} (-1)^\sigma \sigma,$$

*and define the μ-*polytabloid *$e_t^\mu = \kappa_t^\mu \{t\}$. The* Specht shadow module *$S_R^{\mu,\lambda} \subseteq M_R^\lambda$ is the R span of the μ-polytabloids. We call (μ, λ) a* pair of partitions for n, *although in general λ may be a composition and not a partition.*

We will often draw the pair (μ, λ) by highlighting $Y(\mu)$ as a subset of $Y(\lambda)$, for example, if $\lambda = (4, 5, 3, 1)$ and $\mu = (4, 3, 1, 1)$, then we draw



We may always assume that $\mu_1 = \lambda_1$, as adding boxes to the first row of μ does not change the μ-column stabiliser, and so we identify the pairs of partitions (μ, λ) and $(\bar{\mu}, \lambda)$, where $\bar{\mu}_1 = \lambda_1$ and $\bar{\mu}_i = \mu_i$ for all other i. If $\mu = (0)$, or equivalently $\mu = (\lambda_1)$, then $S_R^{\mu,\lambda} = M_R^\lambda$, while if λ is a partition and $\mu = \lambda$ then $S_R^{\mu,\lambda} = S_R^\lambda$.

Example 2.2.14. *Let $\lambda = (3, 2)$, $\mu = (3, 1)$ and $t =$* 

. Then

$$C^\mu(t) = \{(1), (14)\},$$

and

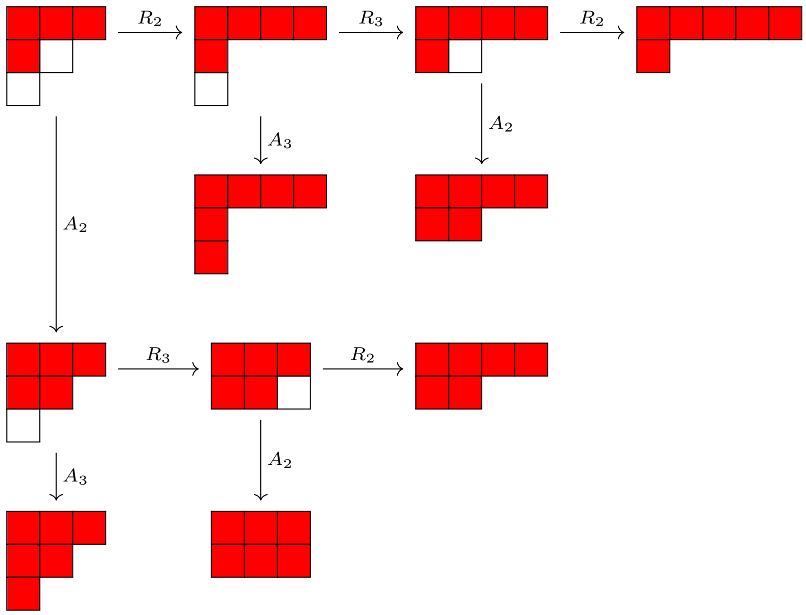
$$e_t = \frac{\overline{\quad 1 \;\; 2 \;\; 3 \quad}}{\underline{4 \;\; 5}} - \frac{\overline{\quad 2 \;\; 3 \;\; 4 \quad}}{\underline{1 \;\; 5}}$$

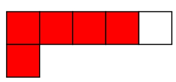
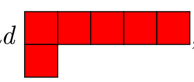
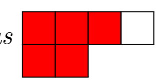
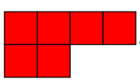
Given a pair of partitions (μ, λ) with $\mu \neq \lambda$, we shall define two families of operators, "raise" and "add" which give rise to new pairs of partitions.

Definition 2.2.15. *Let (μ, λ) be pair of partitions with $\mu \neq \lambda$ and let $c > 1$ be such that $\mu_{c-1} = \lambda_{c-1}$ and $\mu_c \neq \lambda_c$.*

- *If $\mu_{c-1} > \mu_c$, then $A_c(\mu, \lambda)$ is the pair of partitions obtained from (μ, λ) by changing μ_c to μ_{c+1}, while if $\mu_{c-1} = \mu_c$ then $A_c(\mu, \lambda)$ is the pair $(\emptyset, \emptyset)$*
- *$R_c(\mu, \lambda)$ is the pair of partitions obtained from (μ, λ) by changing λ_c to μ_c and λ_{c-1} to $\lambda c - 1 + \lambda_c - \mu_c$.*

Example 2.2.16.



Notice that we have identified  *and* *, as well as*  *and* .

James has investigated the relationship between these operators and the maps $\psi_{i,v}$:

Proposition 2.2.17. *[Jam77][Theorem 4.11]*

- $\psi_{c-1,\mu_c}(S^{\mu,\lambda}) = S^{R_c(\mu,\lambda)}$ *and* $S^{\mu,\lambda} \cap ker(\psi_{c-1,\mu_c}) = S^{A_c(\mu,\lambda)}$
- $(S^{\mu,\lambda})/(S^{A_c(\mu,\lambda)}) \cong S^{R_c(\mu,\lambda)}$

It follows that $S^{\mu,\lambda}$, and in particular M^λ, has a filtration by Specht modules. The factors and their order of appearance is independent of the ground field and can be computed by successive applications of raising and adding operators to the pair (μ, λ).

Example 2.2.18. *Let* $\lambda = (3, 2, 1)$ *and* $\mu = (3, 1)$ *as in Example 2.2.16. Then we see* *that*

$$S^{\mu,\lambda} \sim \begin{matrix} S^{(5,1)} \\ S^{(4,2)} \\ S^{(4,1^2)} \\ S^{(4,2)} \\ S^{(3,3)} \\ S^{(3,2,1)} \end{matrix}$$

We have an alternative characterisation of the Specht-shadow module, which is a generalisation of James' Kernel Intersection Theorem [Jam78][§17]:

Proposition 2.2.19. *Let (μ, λ) be be a pair of partitions, then*

$$S^{\mu,\lambda} = \bigcap_{i=1}^{r-1} \bigcap_{v=0}^{\mu_{i+1}-1} Ker(\psi_{i,v}) \subseteq M^{\lambda}.$$

2.2.5 Decomposition numbers

An important problem in the representation theory of the symmetric group is calculation of the decomposition numbers, $d_{\mu,\lambda} := [S_{\mathbb{F}}^{\lambda} : D_{\mathbb{F}}^{\mu}]$, which record the composition multiplicity of the irreducible module $D_{\mathbb{F}}^{\mu}$ in the Specht module $S_{\mathbb{F}}^{\lambda}$. If μ is p-regular, the decomposition number $d_{\mu,\lambda}$ is the same for any field of characteristic p, and thus the notation makes no reference to the field. Of course the decomposition numbers depend on the characteristic of the field, we shall occasionally refer to p-decomposition numbers if necessary to avoid confusion. There is currently no known algorithm for calculating these numbers, but they are known in a number of special cases. In addition to these special cases, there are some ad hoc results which allow decomposition numbers to be calculated. The most powerful of these techniques is the study of Schaper filtration, together with its sum formula, which is described in detail in Chapter 3. In the remainder of this section we shall collate a number of other tools and techniques for calculating decomposition numbers.

The decomposition numbers for $\mathcal{S}_n$ are often displayed in the *decomposition matrix*, whose rows are indexed by partitions $\lambda \vdash n$, and whose columns are indexed by p-regular partitions of n.

Proposition 2.2.20. *[Jam78][Corollary 12.2] If λ is p-regular, S^{λ} has a unique top composition factor D^{λ}. All other composition factors are isomorphic to some D^{μ} with $\mu \rhd \lambda$. If λ is p-singular, all the composition factors of S^{λ} are isomorphic to some D^{μ} with $\mu \rhd \lambda$.*

James proved a generalisation of this result:

Proposition 2.2.21. *[Jam76b][Theorem A] Lat $\lambda \vdash n$ and denote by λ^R the p-regularisation of n. Let $\mu \vdash n$ be a p-regular partition such that $d_{\mu,\lambda} \neq 0$, then $\mu \trianglerighteq \lambda^R$. Moreover, $d_{\lambda^R,\lambda} = 1$.*

We have a beautiful combinatorial description of the blocks for the symmetric group, known as Nakayama's conjecture. Although still referred to as a conjecture it was proven by Brauer and Robinson in 1947 [BR47] [Rob47], seven years after it was first conjectured by Nakayama. A short proof, due to Meier and Tappe [MT76], can be found in James and Kerber's book [JK81][§6.1].

Proposition 2.2.22 (Nakayama's conjecture)**.** *The blocks of $\mathcal{S}_n$ are labelled by p-core partitions γ, such that $(n- \mid \gamma \mid)$ is divisible by p. The Specht module S^λ lies in the block γ if and only if $C_p(\lambda) = \gamma$.*

This allows us to fill more entries of the decomposition matrix with 0's.

Corollary 2.2.23. *Let λ and μ be partitions of n and suppose that μ is p-regular. If $C_p(\lambda) \neq C_p(\mu)$ then $d_{\mu,\lambda} = 0$.*

The decomposition numbers are also known in a few special cases. In particular, James determined the decomposition numbers when λ is a two part partition $\lambda = (n-k, k)$ [Jam76c][Jam76a], and Peel calculated the decomposition numbers for hook partitions $\lambda = (n-k, 1^k)$ [Pee71].

Proposition 2.2.24. *[Jam76c][Jam76a] Let $\lambda = (n-k, k)$ and $\mu = (n-j, j)$ be partitions of n with $k \leq j$. Then*

$$d_{\mu,\lambda} = \begin{cases} 1 & \textit{if } \binom{n-2j+1}{k-j} \equiv 1 (\textit{mod } p) \\ 0 & \textit{otherwise} \end{cases}.$$

Proposition 2.2.25. *[Pee71] Let p be odd and $\lambda = (n-k, 1^k)$ with $k < n$.*

1. *If $p \nmid n$ then S^λ is irreducible.*
2. *If $p \mid n$ then, S^λ is irreducible if and only if $k = 0$ or $k = n-1$. Otherwise S^λ has two composition factors, and its socle is isomorphic to the cosocle of $S^{(n-k-1,1^{k+1})}$.*

This result determines the number of composition factors, exactly what the composition factors are can be determined by James' p-regularisation theorem, Proposition 2.2.21.

The final results in this section are the principles of row and column removal, which allow us to relate some of the decomposition numbers of $\mathcal{S}_n$ to decomposition numbers for a smaller symmetric group. James principle of row removal states that if $\lambda_1 = \mu_1$ then the decomposition numbers $d_{\lambda,\mu}$ and $d_{\hat{\lambda},\hat{\mu}}$ are equal, where $\hat{\lambda} = (\lambda_2, \lambda_3, \dots)$ and

$\hat{\mu} = (\mu_2, \mu_3, \dots)$. The principle of column removal, if $\lambda'_1 = \mu'_1$, is similar. We shall state a generalisation of each due to Donkin [Don85].

Proposition 2.2.26 (Generalised principle of row removal). *Let λ and μ be partitions of n with μ p-regular and $\lambda \rhd \mu$. Suppose that for some j we have $\sum_{i=1}^{j} \lambda_i = \sum_{i=1}^{j} \mu_i$, then*

$$d_{\lambda,\mu} = d_{\lambda^t,\mu^t} d_{\lambda^b,\mu^b},$$

where

$$\lambda^t = (\lambda_1, \dots, \lambda_j), \qquad \mu^t = (\mu_1, \dots, \mu_j),$$
$$\lambda^b = (\lambda_{j+1}, \dots), \text{ and} \qquad \mu^b = (\mu_{j+1}, \dots)$$

Proposition 2.2.27 (Generalised principle of column removal). *Let λ and μ be partitions of n with μ p-regular and $\lambda \rhd \mu$. Suppose that for some j we have $\sum_{i=1}^{j} \lambda'_i = \sum_{i=1}^{j} \mu'_i$, then*

$$d_{\lambda,\mu} = d_{\lambda^l,\mu^l} d_{\lambda^r,\mu^r},$$

where

$$\lambda^l = ((\lambda')^t)', \qquad \mu^l = ((\mu')^t)',$$
$$\lambda^r = ((\lambda')^b)', \text{ and} \qquad \mu^r = ((\mu')^b)'.$$

2.3 Representation theory of GL_n

The approach to the representation theory of the symmetric group outlined in the previous section is highly combinatorial, and is largely the approach which is taken in this thesis. There is another approach which relates the representation theory of the general linear group to the representation theory of the symmetric group, via the Schur functor, which allows powerful tools from the theory of algebraic groups to be used. Although this approach is not used often in this thesis, a brief overview shall be given. More details can be found, for example, in the book of Green [Gre06].

2.3.1 Polynomial representations of GL_n

If k is any field and n a positive integer, then GL_n denotes the group of all non-singular $n \times n$ matrices with entries in k. For a pair of elements $i, j \in [n]$, denote by $c_{i,j}$ the map,

$$\begin{aligned} c_{i,j} : GL_n &\to k \\ g &\mapsto g_{i,j}, \end{aligned}$$

which takes the matrix g to its i,jth entry. Denote by $A_k(n)$ the k-algebra generated by the functions $c_{i,j}$. If k is an infinite field then these functions are algebraically independent, and thus $A_k(n)$ can be regarded as the algebra of all polynomials in the *indeterminates* $c_{i,j}$. For $r \geq 0$ we denote the subspace of $A_k(n)$ consisting of homogeneous polynomials of degree r in the $c_{i,j}$ by $A_k(n,r)$. This gives $A_k(n)$ a grading:

$$A_k(n) = \bigoplus_{r \geq 0} A_k(n,r).$$

We restrict our study of the representation theory of $kGL_n(k)$ to the $A_k(n)$-*representation theory*; the study of $kGL_n(k)$ modules whose coefficient space is a subset of $A_k(n)$. The category of all finite dimensional $kGL_n(k)$-modules is denoted $\mod(kGL_n(k))$, while the full subcategory corresponding to the $A_k(n)$ and $A_k(n,r)$ representation theory will be denoted simply by $M_k(n)$ and $M_k(n,r)$, respectively. We shall call a module, $V \in M_k(n)$, *homogeneous* if $V \in M_k(n,r)$ for some r.

We then have the following:

Theorem 2.3.1. *[Gre06][2.2c] Let k be an infinite field and let $V \in M_k(n)$. Then V has a direct sum decomposition;*

$$V = \bigoplus_{r \geq 0} V_r,$$

where for each $r \geq 0$ the $V_r \in M_k(n,r)$.

This means that any indecomposable module $V \in M_k(n)$ is homogeneous, so we focus our attention to $M_k(n,r)$.

For integers $n, r \geq 1$ we write $I(n,r)$ for the set of all functions $\underline{i} : [r] \to [n]$, which we shall write as a vector $\underline{i} = (i_1, i_2, \ldots, i_r)$ with each $i_j \in [n]$. If we define the monomials $c_{\underline{i},\underline{j}} := c_{i_1,j_1} c_{i_2,j_2} \cdots c_{i_r,j_r}$, then we observe that $A_k(n,r)$ is spanned by $\{c_{\underline{i},\underline{j}} : \underline{i}\underline{j} \in I(n,r)\}$. This set, however, does not form a basis as the indeterminates commute. To obtain a basis we first define an equivalence relation which captures this fact.

The symmetric group $\mathcal{S}_r$ acts naturally on $I(n,r)$ (on the right) by *place permutation*; if $\underline{i} = (i_1, i_2, \ldots, i_r)$ and $\sigma \in \mathcal{S}_r$, then

$$\underline{i} \cdot \sigma = (i_{\sigma(1)}, \ldots, i_{\sigma(r)}).$$

Similarly, $\mathcal{S}_r$ acts on $I(n,r)\times I(n,r)$ by $(\underline{i},\underline{j})\cdot\sigma = (\underline{i}\cdot\sigma, \underline{j}\cdot\sigma)$. We write $\underline{i}\sim\underline{j}$ to indicate that $\underline{i}$ and $\underline{j}$ are in the same $\mathcal{S}_r$ orbit of $I(n,r)$, and $(\underline{i},\underline{j})\sim(\underline{k},\underline{l})$ if $(\underline{i},\underline{j})$ and $(\underline{k},\underline{l})$ are in the same orbit of $I(n,r)\times I(n,r)$; that is there is some $\sigma\in\mathcal{S}_r$ such that $\underline{k}=\underline{i}\cdot\sigma$ and $\underline{l}=\underline{j}\cdot\sigma$. Clearly $c_{\underline{i},\underline{j}} = c_{\underline{k},\underline{l}}$ if and only if $(\underline{i},\underline{j})\sim(\underline{k},\underline{l})$.

Proposition 2.3.2. *[Gre06][§2.1] $A_k(n,r)$ has a k-basis the set of distinct monomials $c_{\underline{i},\underline{j}}$, which are in bijection with the $\mathcal{S}_r$ orbits of $I(n,r)\times I(n,r)$. In particular the dimension of $A_k(n,r)$ is*

$$\binom{n^2+r+1}{r}.$$

2.3.2 The Schur algebra

The Schur algebra has many equivalent characterisations, but we shall continue following the approach of Green [Gre06]. Fix $r\geq 0$. We define the Schur algebra $S_k(n,r)$ to be the dual space of $A_k(n,r)$:

$$S_k(n,r) = A_k(n,r)^* = \mathrm{Hom}_k(A_k(n,r),k).$$

As a k-space, $S_k(n,r)$ is spanned by $\{\xi_{\underline{i},\underline{j}} : \underline{i},\underline{j}\in I(n,r)\}$, where

$$\xi_{\underline{i},\underline{j}}(c_{\underline{p},\underline{q}}) = \begin{cases} 1 & \text{if } (\underline{i},\underline{j})\sim(\underline{p},\underline{q}) \\ 0 & \text{otherwise} \end{cases}.$$

Given $\underline{i},\underline{j},\underline{k},\underline{l},\underline{p},\underline{q}\in I(n,r)$, define $Z(\underline{i},\underline{j},\underline{k},\underline{l},\underline{p},\underline{q})$ to be the number of $\underline{s}\in I(n,r)$ such that

$$(\underline{i},\underline{j})\sim(\underline{p},\underline{s}) \qquad \text{and} \qquad (\underline{s},\underline{q})\sim(\underline{k},\underline{l}).$$

We define a multiplication rule for the basis elements of $S_k(n,r)$ by;

$$\xi_{\underline{i},\underline{j}}\xi_{\underline{k},\underline{l}} = \sum_{(\underline{p},\underline{q})} Z(\underline{i},\underline{j},\underline{k},\underline{l},\underline{p},\underline{q})\xi_{\underline{p},\underline{q}},$$

where the sum is over a set of representatives $(\underline{p},\underline{q})$ of $\mathcal{S}_r$ orbits on $I(n,r)\times I(n,r)$. This multiplication makes the Schur algebra an associative k-algebra. The importance of the Schur algebra is the following result:

Theorem 2.3.3. *[Gre06][§2.4] Let k be an infinite field. Then the category $M_k(n,r)$, of finite-dimensional r-homogeneous polynomial representations of GL_n, and the category* mod $S_k(n,r)$, *of finite-dimensional left $S_k(n,r)$ modules, are equivalent.*

This allows us to focus entirely on the representation theory of the Schur algebras.

2.3.3 Weyl modules

We now define a class of modules for the Schur algebra, which are analogous to the Specht modules for the symmetric groups. Again, this will be done using the combinatorics of tableaux, but in this section only we shall relax our condition that t is bijective. For the purpose of defining Weyl modules we have:

Definition 2.3.4. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \vdash r$. A* λ-tableau *is a map*

$$t : Y(\lambda) \to [r].$$

The tableau obtained by writing the entries $1, \ldots, r$ from left to right shall be called the basic $\lambda-$*tableau, and denoted T. The position (i,j) in the Young diagram $Y(\lambda)$ shall be referred to as the $T((i,j))$*th place. *If $\underline{i} \in I(n,r)$ then we shall denote by $T_{\underline{i}}$ the tableau*

$$\underline{i} \circ T : Y(\lambda) \to [n].$$

Note that the basic tableau is necessarily bijective, while $T_{\underline{i}}$ may not be.

Let E be an n dimensional vector space with basis $\{e_1, \ldots, e_n\}$, and let $E^{\otimes r}$ be the r-fold tensor product, with basis

$$\{e_{\underline{i}} = e_{i_1} \otimes \cdots \otimes e_{i_r} : \underline{i} \in I(n,r)\}.$$

This is naturally an $S(n,r)$-module, with

$$\xi_{\underline{i},\underline{j}} \cdot e_{\underline{k}} = \sum_{\underline{l}} e_{\underline{l}},$$

where the sum is over all $\underline{l} \in I(n,r)$ with $(\underline{i},\underline{j}) \sim (\underline{l},\underline{k})$. There is a natural action of $\mathcal{S}_r$ on $E^{\otimes r}$, by permuting the order of the factors. This action commutes with the action of the Schur algebra and may be written on the right, giving $E^{\otimes r}$ a bimodule structure. Let $\underline{l} \in I(n,r)$ be such that $T_{\underline{l}}((i,j)) = i$; that is the tableaux $T_{\underline{l}}$ has all entries in the ith row equal to i. Define

$$f_{\underline{l}} = \sum_{\sigma \in C(T)} (-1)^{\sigma} e_{\underline{l}} \cdot \sigma.$$

The Weyl module V_λ is defined to be the $S(n,r)$-submodule of $E^{\otimes r}$ generated by $f_{\underline{l}}$. Similar to the basis theorem for Specht modules, we have:

Proposition 2.3.5. *[Car74][Theorem 3.5] The set*

$$\{b_{\underline{i}} = \xi_{\underline{i},\underline{l}} f_{\underline{l}} : \underline{i} \in I(n,r), T_{\underline{i}} \in std(\lambda)\}$$

is a k-basis of V_λ.

We list some more results analogous to those in Section 2.2:

Theorem 2.3.6. *If k is a field of characteristic 0, then the modules V_λ are simple. Moreover, the set of V_λ as λ ranges over the set of partitions of r with at most n parts are a complete set of simple modules for $S_k(n,r)$.*

Theorem 2.3.7. *Let k be an infinite field of characteristic $p > 0$. Then V_λ has a simple cosocle $L_\lambda := \mathrm{cosoc}(V_\lambda)$. Moreover, the set of L_λ as λ ranges over the set of partitions of r with at most n parts are a complete set of irreducible modules for $S_k(n,r)$.*

Theorem 2.3.8. *Let k be an infinite field of characteristic $p > 0$. Then the composition factors of $\mathrm{rad}(V_\lambda)$ are all of the form L_μ with $\lambda \rhd \mu$.*

2.3.4 The Schur functor

In this subsection we make more precise the earlier remark that Weyl modules are analogous to Specht modules by defining the Schur functor. Fix an infinite field k and integers $n \geq r$. Let $\underline{i} = (1, 2, \ldots, r) \in I(n,r)$. Observe the element $\xi_{\underline{i},\underline{i}} \in S_k(n,r)$ is idempotent and denote this element by e. Let $f : \mathrm{mod}(S_k(n,r)) \to \mathrm{mod}(eS_k(n,r)e)$ be the restriction functor, which we call the *Schur functor*.

Theorem 2.3.9. *[Gre06][§6]*

- *$eS_k(n,r)e$ is isomorphic to $k\mathcal{S}_r$.*
- *For any partition $\lambda \vdash r$, $f(V_\lambda)$ is isomorphic to $S^{\lambda'}$.*
- *If k has characteristic $p > 0$ and $\mu \vdash r$, then $f(L_\mu)$ is isomorphic to $D^{\mu'}$ if μ' is p-regular, and 0 otherwise.*
- *If k has characteristic $p > 0$ and $\lambda, \mu \vdash n$ with μ' is p-regular. Then*

$$[V_\lambda : L_\mu] = [S^{\lambda'} : D^{\mu'}].$$

Amongst other things this gives us another tool for calculating decomposition numbers. In fact, no proof of Propositions 2.2.26 and 2.2.27 which remain in the setting of the symmetric group are known. These results are obtained by first showing the equivalent fact about Weyl modules and applying the Schur functor.

This point of view allows many powerful tools from the representation theory of algebraic groups to be used, however it can sometimes obfuscate the role of the symmetric group. For this reason, we prefer to rely on proofs which remain in the setting of the symmetric group, which use largely combinatorial tools, where possible.

Chapter 3

Schaper's Filtration

One of the few tools we have for studying the structure of Specht modules is the study of its Schaper filtration. This filtration, which is defined in terms of the inner product on the permutation module, is analogous to the Jantzen filtration in the setting of algebraic groups. It's main applications are due to the accompanying sum formula, which first appeared in Schaper's thesis (in German). This sum formula gives upper bounds on the decomposition numbers for the symmetric group. In this chapter we shall first introduce the filtration and then show how the utility of this formula can be improved. This motivates the study of Schaper numbers, which are studied in the second section of this chapter. Many of the results in this chapter appear in [JM21] which was written in collaboration with Dr Martin.

3.1 The Schaper filtration

For a given Specht module $S^{\lambda}_{\mathbb{Z}}$, prime p and integer $i \geq 0$ we define the submodule

$$S^{\lambda}_{i} = \{x \in S^{\lambda}_{\mathbb{Z}} : p^i \mid \langle x, y \rangle \forall y \in S^{\lambda}_{\mathbb{Z}}\}$$

and denote by $\bar{S^{\lambda}_{i}}$ its reduction mod p to obtain the *Schaper filtration*:

$$S^{\lambda}_{\mathbb{F}_p} = \bar{S^{\lambda}_{0}} \geq \bar{S^{\lambda}_{1}} \geq \bar{S^{\lambda}_{2}} \geq \cdots .$$

All composition factors of $S^{\lambda}_{\mathbb{F}_p}$ must appear in the quotients of this filtration and hence studying this filtration would reveal the decomposition numbers for the symmetric groups. Unfortunately, the layers of this filtration are not known in general, but despite this we are able to use combinatorial tools to calculate an upper bound for the decomposition number $[S^\lambda_{\mathbb{F}_p} : D^\mu_{\mathbb{F}_p}]$. From now on we shall omit the subscript indicating the ring over which the module is defined as we shall always work over a field of characteristic p and the decomposition numbers depend only on the characteristic of the field, and not on the field itself.

Let $\lambda \vdash n$ and define $H(\lambda)$ be the set of triples (g, h, ν) where $\nu \trianglerighteq \lambda$ is a partition of n and g and h are hooks of $Y(\lambda)$ and $Y(\nu)$ respectively, such that removing the corresponding rim-hooks leaves the same partition $Y(\lambda\backslash g) = Y(\nu\backslash h)$.

Theorem 3.1.1 (Schaper's Sum Formula). *Let μ be a p-regular partition not equal to λ, then*

$$\sum_{i\geq 1}[\bar{S^\lambda_i} : D^\mu] = \sum_{(g,h,\nu)\in H(\lambda)} \alpha_{(g,h,\nu)}[S^\nu : D^\mu],$$

where the coefficients are given by $\alpha_{(g,h,\nu)} = \mathrm{val}_p(\mid g \mid)(-1)^{l(g)+l(h)+1}$.

The top factor of this filtration $\bar{S^\lambda_0}/\bar{S^\lambda_1}$ is D^λ if the partition λ is p-regular, otherwise is zero and hence the sum formula gives an upper bound on $[S^\lambda : D^\mu]$ for $\mu \triangleright \lambda$, as any composition factor isomorphic to D^μ must appear in a quotient further along the filtration.

Definition 3.1.2. *The ith* Schaper layer *of the Specht module S^λ is*

$$L_i = \bar{S^\lambda_i}/\bar{S^\lambda_{i+1}}.$$

We say that the kth Schaper layer of S^λ is the top layer *if k is the least such integer such that $L_k \neq 0$. The integer k will be denoted by $\nu_p(\lambda)$ and shall be called the* (p) Schaper number *of λ.*

It is common to use also ν_p to denote the p-valuation of an integer; that is the highest power of p that divides that integer, so to avoid confusion we shall use ν_p exclusively for Schaper numbers, and use val_p for the p-valuation of an integer. Indeed, this is the motivation for this notation to be used for Schaper numbers, as $\nu_p(\lambda)$ is the highest power of p which divides all of the integers $\langle x, y\rangle$ for $x, y \in S^\lambda$. Of course, as polytabloids span the Specht module $\nu_p(\lambda)$ is the highest power of p dividing $\langle x, y\rangle$ where x and y are polytabloids. That is $\nu_p(\lambda) = \mathrm{val}_p(g^\lambda)$, where

$$g^\lambda = \text{g.c.d}\{\langle x, y\rangle : x, y \in S^\lambda \text{are polytabloids}\},$$

as defined in [Jam78, Definition 10.3].

An irreducible module appearing in the ith layer is counted by the formula i times as it is a composition factor of $\bar{S^\lambda_j}$ for all $j \leq i$, and hence knowing which layer is the top layer allows us to improve the upper bound for the decomposition numbers obtained from Schaper's sum formula.

Corollary 3.1.3.

$$[S^\lambda : D^\mu] \leq \frac{\sum_{i\geq 1}[\bar{S_i^\lambda} : D^\mu]}{\nu_p(\lambda)}.$$

Fayers showed that Schaper numbers of partitions are superadditive in the following sense:

Proposition 3.1.4. *[Fay03, Theorem 3.5] Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be partitions of n and m respectively. Then $\nu_p(\lambda \star \mu) \geq \nu_p(\lambda) + \nu_p(\mu)$, where $\lambda \star \mu$ is the partition of $n+m$ which has rows of lengths $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_{s-1}$ and μ_s.*

This result is reminiscent of Donkin's generalisation [Don85] of the principle of row removal [Jam81] and is useful in determining lower bounds on the Schaper number of a partition.

3.2 The Schaper Number of λ

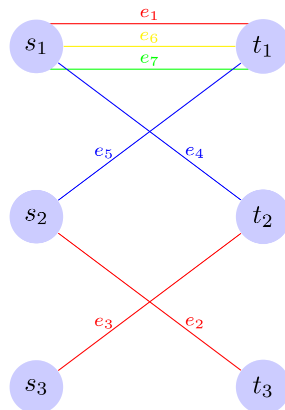
In this section we shall turn to characterising partitions λ with a certain Schaper number. We use a number of results and techniques due to Fayers [Fay03], which are stated here. A corollary of the following theorem of James tells us that $\nu_p(\lambda) \geq 1$ if and only if λ is p-singular:

Theorem 3.2.1. *[Jam78, Theorem 10.4] Suppose λ has z_j parts equal to j for each j. Then*

$$\mathrm{val}_p(\prod_{j=1}^{\infty} z_j!) \leq \nu_p(\lambda) \leq \mathrm{val}_p(\prod_{j=1}^{\infty} (z_j!)^j).$$

We shall use the graph-theoretic approach introduced by Fayers [Fay03]. Recall if s and t are row equivalent λ-tableaux we define the graph $G = G(s,t)$ as follows: the vertex set of G is $\{s_1, s_2, \ldots, s_{\lambda_1}, t_1, t_2, \ldots, t_{\lambda_1}\}$ and the edge set is $\{e_1, \ldots, e_n\}$ and the edge e_k goes from s_i to t_j if k appears in column i of s and column j of t. The graph below is the graph $G(s,t)$ for the $(3,2,1,1)$-tableaux

$$s = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array} \quad \text{and } t = \begin{array}{|c|c|c|}\hline 1 & 3 & 2 \\ \hline 5 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}.$$



We consider colourings of $G(s,t)$ with colours $c_1,\ldots,c_{\lambda'_1}$ and we call such a colouring admissible if for each $l \leq \lambda'_1$ there is precisely one edge of colour c_l incident on each of the vertices $s_1,\ldots,s_{\lambda'_l},t_1,\ldots,t_{\lambda'_l}$. The set of all admissible colourings of G will be denoted $A(G)$. Observe there is a bijection between the admissible colourings of G and pairs (u,v) of λ-tableaux with $s \sim_{\text{col}} u \sim_{\text{row}} v \sim_{\text{col}} t$. This correspondence is given by colouring the edge e_i with colour i if it appears in row i of u, or equivalently row i of v. For example, in the colouring above c_1, c_2, c_3 and c_4 are the colours red, blue, yellow and green, respectively. The only other possible admissible colourings are obtained by permuting the colours assigned to e_1, e_6 and e_7, just as the only tableaux u and v with $s \sim_{\text{col}} u \sim_{\text{row}} v \sim_{\text{col}} t$ are obtained from s and t respectively by permuting the positions of the entries $1, 6$ and 7. Given a graph G and a set of distinguished edges $\mathcal{E}$, we shall call an admissible colouring $C \in A(G)$ *respectable* (with respect to $\mathcal{E}$) if it assigns a different colour to each edge in $\mathcal{E}$. For example; the graph above is respectable with respect to the set $\{e_1, e_6, e_7\}$, but not with respect to $\{e_4, e_5\}$.

Observe each admissible colouring induces a permutation of $\{1, 2, \ldots, \lambda'_l\}$ for each l by sending i to j if there is an edge from s_i to t_j of colour l. If u and v are the corresponding tableaux then this permutation, π_{uv}, takes the lth row of u to the lth row of v. Define the product of all of the signatures of these permutations for all l to be the signature of the colouring, $(-1)^C$, and observe that as $(-1)^C = (-1)^{\pi_{uv}} = (-1)^{\pi_{st}}(-1)^{\pi_{us}}(-1)^{\pi_{tv}}$ we get the following result:

Proposition 3.2.2. *[Fay03, Proposition 3.6]*

$$\sum_{C \in A(G)} (-1)^C = (-1)^{\pi_{st}} \langle e_s, e_t \rangle.$$

Fayers uses this approach to prove a result reminiscent of principle of column removal [Jam81]:

Proposition 3.2.3. *[Fay03, Theorem 3.7] Let $\hat{\lambda}$ be the partition whose Young diagram is obtained by removing the first column of the Young diagram for λ. Then $\nu_p(\lambda) \geq \nu_p(\hat{\lambda})$.*

An important consequence of the proof of Proposition 3.2.3 is the following:

Proposition 3.2.4. *Let s and t be λ-tableaux. If there are m edges from s_1 to t_1 in $G(s,t)$ then $\langle e_s, e_t \rangle$ is divisible by $m!p^{\nu_p(\hat{\lambda})}$, where $\hat{\lambda}$ is the partition whose Young diagram is obtained by removing the first column of the Young diagram for λ.*

This graph theoretic approach allows Fayers to go further than James, and characterise all of Specht modules whose Schaper number is at least two:

Theorem 3.2.5. *[Fay03, Theorem 3.8] Let $\lambda \vdash n$. Then $\nu_p(\lambda) \geq 2$ if and only if one of the following hold:*

(i) λ is doubly p-singular; that is there exists i, j with $i \geq j + p$ and $\lambda_i = \lambda_{i+p-1}$ and $\lambda_j = \lambda_{j+p-1}$.

(ii) There exist i such that $\lambda_i \leq \lambda_{i+2p-2} + 1$ and $\lambda_{i+p-1} \geq 2$.

This result, together with Proposition 3.1.4 immediately gives the corollary below. The reader can see the obvious extension of this and should now be able to construct partitions with arbitrarily large Schaper numbers.

Corollary 3.2.6. *Let $\lambda \vdash n$. Then $\nu_p(\lambda) \geq 3$ if one of the following hold:*

(i) λ is triply p-singular; that is it contains three disjoint p-singularities.

(ii) There exist i, j with $\{i, \ldots, i+2p-2\} \cap \{j, \ldots, j+p-1\} = \emptyset$ such that $\lambda_i \leq \lambda_{i+2p-2}+1$ and $\lambda_{i+p-1} \geq 2$ and $\lambda_j = \lambda_{j+p-1}$.

Before continuing we shall give an example of how decomposition numbers can be calculated using Theorem 3.2.5:

Example 3.2.7. *Let $p = 2$ and consider the block of $\mathbb{F}_2\mathcal{S}_{13}$ containing all Specht modules S^{λ} where λ has 2-core $(2, 1)$. Assume that the decomposition numbers are known for $\mathcal{S}_n$ where $n < 13$. Using column elimination [Don85] and by observing the linear relations between the ordinary characters of $\mathcal{S}_{13}$ on 2-regular classes we can compute the first part of the first column of the decomposition matrix below:*

	$(12,1)$
$(12,1)$	1
$(10,3)$	0
$(10,1^3)$	1
$(8,5)$	1
$(8,3,2)$	x
$(8,3,1^2)$	$x+1$
$(8,2^2,1)$	x

Schaper's sum formula, Theorem 3.1.1, tells us

$$[S^{(8,3,2)} : D^{(12,1)}] \leq [S^{(12,1)} : D^{(12,1)}] + [S^{(8,5)} : D^{(12,1)}]$$

and

$$\begin{aligned}[S^{(8,2,2,1)} : D^{(12,1)}] \leq & - 2[S^{(12,1)} : D^{(12,1)}] + [S^{(10,1^3)} : D^{(12,1)}] \\ & - 2[S^{(8,5)} : D^{(12,1)}] + 2[S^{(8,3,2)} : D^{(12,1)}] \\ & + [S^{(8,3,1,1)} : D^{(12,1)}].\end{aligned}$$

That is $x \leq 2$ and $x \leq 3x-2$. Theorem 3.2.5 allows us to improve the second inequality, as we know that the Schaper number of $(8,2,2,1)$ is at least two. Thus, using Corollary 3.1.3, the second inequality becomes $x \leq \frac{3x-2}{2}$ and we conclude that $x = 2$.

Of course this decomposition number can be calculated using other techniques, but this calculation demonstrates how a better understanding of Schaper numbers may lead to new decomposition numbers for the symmetric group.

The following lemma gives us another way of constructing partitions of large Schaper number.

Lemma 3.2.8. *Let $\lambda = ((x+1)^a, x^b, (x-1)^c) \vdash n$. Then*

$$\nu_p(\lambda) \geq \nu_p((x^{a+b+c})) - \mathrm{val}_p(\binom{a+b+c}{a,b,c}) - \mathrm{val}_p(c!).$$

We shall prove this by induction on a.

Lemma 3.2.9 (Base case)**.** *Let $\lambda = (x^b, (x-1)^c) \vdash n$. Then*

$$\nu_p(\lambda) \geq \nu_p((x^{b+c})) - \mathrm{val}_p(\binom{b+c}{b}) - \mathrm{val}_p(c!).$$

Proof. We want to calculate $\mathrm{val}_p(\langle e_s, e_t \rangle)$, where s and t are λ-tableaux. We may assume that s and t are row equivalent by acting by the column stabiliser of s on e_s, which will only (possibly) change the sign of $\langle e_s, e_t \rangle$. If no element of the column stabiliser of s makes $s \sim_{row} t$ then $\langle e_s, e_t \rangle$. Let $G = G(s, t)$, so

$$\sum_{C \in A(G)} (-1)^C = (-1)^{\pi_{st}} \langle e_s, e_t \rangle.$$

Form the graph G' from G by adding c edges from s_x to t_x. We shall now define a correspondence between $C \in A(G)$ and $C' \in A(G')$ for which there are precisely $\binom{b+c}{c} \cdot c!$ admissible colourings of G' for each $C \in A(G)$.

Fix some ordering $c_1 < \cdots < c_{b+c}$ on the colours and take $C' \in A(G')$. Let $c_{i_1} < \cdots < c_{i_b}$ be the colours assigned to the edges incident on s_x which are also edges in G and $c_{j_1} < \cdots < c_{j_c}$ be the remaining colours. We get an admissible colouring of G by colouring each edge in G with the colour c_k if it has colour c_{i_k} in C' and colour c_{b+k} if it has colour c_{j_k} in C'. Clearly there are $\binom{b+c}{c} \cdot c!$ admissible colourings of G' that get sent to each $C \in A(G)$ as this construction only depends on the relative positions of the colours chosen for the edges which appear in G, and is not affected by permuting the colours assigned to the c edges between s_x and t_x which do not appear in G.

This correspondence is sign preserving, as edges with the same colour in C' get assigned the same colour in C, so

$$\binom{b+c}{c} \cdot c! \sum_{C \in A(G)} (-1)^C = \sum_{C' \in A(G')} (-1)^C,$$

and the result follows. □

Remark. *Whenever this graph theoretic approach is used we can explicitly reconstruct the proof and calculate inner products directly, however care must be taken to keep track of signs. Although the proof may be more complicated if we try to keep track of the tableaux, the correspondence between $C \in A(G)$ and $C' \in A(G')$ can usually be more easily understood from this point of view. Tableaux corresponding to G' contain c entries which do not appear in tableaux corresponding to G, all in the final column. In any pair of tableaux (u', v') corresponding to an admissible colouring on G' we simply delete the nodes containing the extra entries and slide the corresponding rows to the bottom, without changing their relative order.*

We shall now prove Lemma 3.2.8.

Proof of Lemma 3.2.8. Suppose $\lambda = ((x+1)^a, x^b, (x-1)^c)$ with $a > 0$. Let s and t be two row equivalent λ-tableaux and construct the graph $G = G(s, t)$. If there is an edge from s_{x+1} to t_{x+1} then there is a faithful and signature preserving action of $\mathcal{S}_a$ on the admissible colourings of G, by permuting colours $c_1, \ldots, c_a$. Summing the signatures of all admissible colourings of G in which e has colour c_a we get $\frac{\langle e_s, e_t \rangle}{a}$, which is divisible by $p^{\mathrm{val}_p(\langle e_s, e_t \rangle) - \mathrm{val}_p(a)}$. Deleting the edge e results in a graph $G(s', t')$ for some $((x+1)^{a-1}, x^{b+1}, (x-1)^c)$-tableaux s' and t'. There is an obvious correspondence between admissible colourings of $G(s', t')$ and those admissible colourings of G where e has the colour c_a. This corresponds to the correspondence between pairs of tableaux (u, v) with $s \sim_{\mathrm{col}} u \sim_{\mathrm{row}} v \sim_{\mathrm{col}} t$ and the entry corresponding to edge e appearing in row a, and pairs of tableaux (u', v') with $s' \sim_{\mathrm{col}} u' \sim_{\mathrm{row}} v' \sim_{\mathrm{col}} t'$. Observe that, as this correspondence preserves the signature, the sum of all admissible colourings of G in which the edge e has colour a is $\langle e_{s'}, e_{t'} \rangle$. Thus

$$\begin{aligned}
\mathrm{val}_p(\langle e_s, e_t \rangle) &= \mathrm{val}_p(a \langle e_{s'}, e_{t'} \rangle) \\
&\geq \mathrm{val}_p(a) + \nu_p(((x+1)^{a-1}, x^{b+1}, (x-1)^c)) \\
&\geq \mathrm{val}_p(a) + \nu_p((x^{a+b+c})) - \mathrm{val}_p(\binom{a+b+c}{a-1, b+1, c}) - \mathrm{val}_p(c!) \\
&\geq \nu_p((x^{a+b+c})) - \mathrm{val}_p(\binom{a+b+c}{a, b, c}) - \mathrm{val}_p(c!)
\end{aligned}$$

as required.

Now suppose there is no edge from s_{x+1} to t_{x+1}. Let $e_{i_1}, e_{i_2}, \ldots, e_{i_a}$ be the edges which meet s_{x+1}, and $e_{j_1}, e_{j_2}, \ldots, e_{j_a}$ be the edges which meet t_{x+1}. Suppose also that e_{i_k} meets $t_{f(k)}$ and e_{j_k} meets $s_{g(k)}$. For each $\sigma \in \mathcal{S}_a$ define a graph G_σ as follows: delete the vertices s_{λ_1} and t_{λ_1} from the graph G and then add edges $e'_1, \ldots, e'_a$ and $E_1, \ldots, E_c$ such that e'_k is incident on $s_{g(k)}$ and $t_{f(\sigma_k)}$, and each E_k goes from $s_{\lambda_1 - 1}$ to $t_{\lambda_1 - 1}$. Let $\mathcal{E} = \{e'_1, \ldots, e'_a, E_1, \ldots, E_c\}$ and denote the set of respectable colourings of G_σ with respect to $\mathcal{E}$ by $R(G_\sigma)$.

Each admissible colouring $C \in A(G)$ determines a $\sigma \in \mathcal{S}_a$ by drawing edges so that the edges e_{i_k} and $e_{j_{(\sigma k)}}$ have the same colour in C. The colouring C also gives rise to $c!$ respectable colourings $C' \in R(G_\sigma)$ for this permutation σ with $e'_1, \ldots, e'_a$ having colours $c_1, \ldots, c_a$ in some order, while $E_1, \ldots, E_c$ have the colours $c_{a+b+1}, \ldots, c_{a+b+c}$ in some order. The edges of G_σ which appear in G are given the same colour as in C, the edges e'_k are given the same colour as e_{i_k} and the edges $E_1, \ldots, E_c$ are given the colours $c_{a+b+1}, \ldots, c_{a+b+c}$ in some order. By examining the permutations induced by the colourings we see that

$$(-1)^C = (-1)^a (-1)^{C'}.$$

Conversely, a respectable colouring $C' \in R(G_\sigma)$ where $e'_1, \ldots, e'_a$ have colours $c_1, \ldots, c_a$ and the edges $E_1, \ldots, E_c$ have the colours $c_{a+b+1}, \ldots, c_{a+b+c}$ gives rise to an admissible colouring $C \in A(G)$ by giving all the edges which appear in both G and G_σ the same colour in C as in C', and by giving each of e_{i_k} and e_{j_k} the same colour as e'_k. Again we see that

$$(-1)^C = (-1)^a (-1)^{C'},$$

and we also observe that these two operations are mutually inverse, thus

$$(-1)^{\pi_{st}} \langle e_s, e_t \rangle = \frac{(-1)^a}{c!} \sum_{\sigma \in \mathcal{S}_a} \sum_{C} (-1)^C,$$

where the sum is over all respectable colourings of G_σ where the edges $e'_1, \dots, e'_a$ have colours $c_1, \dots, c_a$ and $E_1, \dots, E_c$ have the colours $c_{a+b+1}, \dots, c_{a+b+c}$. There is a faithful signature preserving action of $\mathcal{S}_m$ on $R(G_\sigma)$ by permuting all the colours, so we get

$$(-1)^{\pi_{st}} \langle e_s, e_t \rangle = \frac{(-1)^a}{c!} \frac{1}{\binom{a+b+c}{a,b,c}} \sum_{\sigma \in \mathcal{S}_a} \sum_{C \in R(G_\sigma)} (-1)^C.$$

We will now show that we may replace the sum over $R(G_\sigma)$ by one over $A(G_\sigma)$.

For an admissible colouring $C \in A(G_\sigma)$ we define

$$C_l = | \{ k : e'_k \text{ has colour } c_l \} |,$$

and observe that C is respectable if and only if each $C_l = 1$. For integers $d_1, \dots, d_{\lambda'_1}$ we define $\mathbf{C}(d_1, \dots, d_{\lambda'_1})$ to be the set of pairs (σ, C'), where $\sigma \in \mathcal{S}_p$ and $C' \in A(G_\sigma)$ with $C_l = d_l$ for all l. The group $\mathcal{S}_{d_1} \times \mathcal{S}_{d_2} \times \cdots \times \mathcal{S}_{d_{\lambda'_1}}$ acts, with signature, on $\mathbf{C}(d_1, \dots, d_{\lambda'_1})$ by permuting the endpoints of the edges (i.e. the elements of $\{t_{f(1)}, \dots, t_{f(m)}\}$) of the same colour. If any of the d_l are greater than one, then there is some permutation $\rho \in \mathcal{S}_{d_1} \times \mathcal{S}_{d_2} \times \cdots \times \mathcal{S}_{d_{\lambda'_1}}$ with negative signature. If $\rho C = D$ then

$$(-1)^D = -(-1)^C.$$

Summing over all pairs $(\sigma, C') \in \mathbf{C}(d_1, \dots, d_{\lambda'_1})$, we obtain

$$\sum_{(\sigma, C') \in \mathbf{C}(d_1, \dots, d_{\lambda'_1})} (-1)^C = - \sum_{(\sigma, C') \in \mathbf{C}(d_1, \dots, d_{\lambda'_1})} (-1)^C,$$

and so is zero, hence

$$\sum_{\sigma \in \mathcal{S}_m} \sum_{C \in A(G_\sigma)} (-1)^C = \sum_{\sigma \in \mathcal{S}_m} \sum_{C \in R(G_\sigma)} (-1)^C.$$

Thus

$$(-1)^{\pi_{st}}\langle e_s, e_t\rangle = \frac{(-1)^a}{c!}\frac{1}{\binom{a+b+c}{a,b,c}}\sum_{\sigma\in\mathcal{S}_a}(-1)^{\pi_{s_\sigma t_\sigma}}\langle e_{s_\sigma}, e_{t_\sigma}\rangle,$$

and hence $\nu_p(\lambda) \geq \nu_p((x^{a+b+c})) - \mathrm{val}_p(\binom{a+b+c}{a,b,c}) - \mathrm{val}_p(c!)$. □

3.3 Schaper Numbers for $p = 2$

We shall now investigate which other partitions have high Schaper number for $p = 2$.

Lemma 3.3.1. *Let $\lambda \vdash n$ and suppose there exists an i such that $\lambda_i \leq \lambda_{i+2} + 1$ and $\lambda_{i+1} \geq 3$, then $\nu_2(\lambda) \geq 3$.*

Proof. By Propositions 3.1.4 and 3.2.3 and Lemma 3.2.8 it suffices to show that $\nu_2((3^3)) \geq 3$. We observe that this calculation has being carried out by Lübeck [Lue], but we shall include it here for completeness. Let s and t be row equivalent (3^3)-tableaux and let $G = G(s,t)$. Suppose there is a pair of edges between any two vertices; without loss of generality let these vertices be s_1 and t_1. We have already seen (Theorem 3.2.5) that $\nu_2((2^3)) \geq 2$, and so, by Proposition 3.2.4, we conclude that $8 \mid \langle e_s, e_t \rangle$. If there are no pairs of edges then, possibly after relabelling and reordering,

$$s = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \text{ and } t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 6 & 4 & 5 \\ \hline 8 & 9 & 7 \\ \hline \end{array},$$

and the polytabloids e_s and e_t are orthogonal. □

Lemma 3.3.2. *Let $\lambda \vdash n$ and suppose there exist i and j such that $\lambda_i = \lambda_{i+1} = \lambda_j + 2 = \lambda_{j+1} + 2 \geq 4$, then $\nu_2(\lambda) \geq 3$.*

Proof. By Propositions 3.1.4 and 3.2.3 it suffices to show that $\nu_2((4,4,2,2)) \geq 3$. Using Lemma 3.2.8 it suffices to show that $\nu_2((3^4)) \geq 5$, which again has been verified by Lübeck [Lue]. It also follows from Propositions 3.2.4 and 4.1.29 by observing first $\nu_2((1^4)) = 3$, and then that any graph $G = G(s,t)$ where s, t are (2^4)-tableaux necessarily contains a pair of edges between two vertices which, without loss of generality, we may assume to be s_1 and t_1 and so $\nu_2((1^4)) \geq 4$. Similarly any (3^4)-tableaux necessarily contains a pair of edges between two vertices which again we may assume to be s_1 and t_1, and thus $\nu_2((3^4)) \geq 5$. □

We are now ready to state the main results of this chapter for $p = 2$.

Theorem 3.3.3. *Let $\lambda \vdash n$ and $p = 2$. Then $\nu_2(\lambda) \geq 3$ if and only if one of the following hold:*

(i) *λ is triply 2-singular.*

(ii) *There exist i, j with $\{i, i+1, i+2\} \cap \{j, j+1\} = \emptyset$ such that $\lambda_i \leq \lambda_{i+2} + 1$ and $\lambda_{i+1} \geq 2$ and $\lambda_j = \lambda_{j+1}$.*

(iii) *λ is 4-singular.*

(iv) There exist i, j such that $\lambda_i = \lambda_{i+1} = \lambda_j + 2 = \lambda_{j+1} + 2 \geq 4$.

(v) There exist i such that $\lambda_i \leq \lambda_{i+2} + 1$ and $\lambda_{i+1} \geq 3$.

Proof. The 'if' direction is Corollary 3.2.6, Proposition 4.1.29 and Lemmas 3.3.1 and 3.3.2. To prove the 'only if' direction we must show that if λ satisfies one of the properties of Theorem 3.2.5 but none of the properties in the statement then $\nu_2(\lambda) = 2$. First, suppose λ is doubly 2-singular and let $\lambda_i = \lambda_{i+1}$ and $\lambda_j = \lambda_{j+1}$ be the two disjoint singularities. As λ is not 4-singular and does not satisfy condition *(iv)* or *(v)* from the statement, we may assume that $\lambda_i \geq \lambda_j + 3$ and also that there are no other rows of length λ_i or λ_j, nor are there rows of lengths $\lambda_i \pm 1$ or $\lambda_j \pm 1$. In this case λ^r, the 2-regularisation, of λ, is

$$\lambda^r{}_k = \begin{cases} \lambda_k & k \notin \{i, i+1, j, j+1\} \\ \lambda_k + 1 & k \in \{i, j\} \\ \lambda_k - 1 & k \in \{i+1, j+1\} \end{cases}.$$

We shall show that D^{λ^r} is in the second Schaper layer, and thus the Schaper number of λ is two. As $[S^\lambda : D^{\lambda^r}] = 1$, the value of $\sum_{i=1}[S^\lambda_{(i)} : D^{\lambda^r}]$ is the number of the layer in which D^{λ^r} appears. By Schaper's formula,

$$\sum_{i=1}[S^\lambda_i : D^{\lambda^r}] = \sum_\nu a_\nu [S^\nu : D^{\lambda^r}],$$

for $\nu \rhd \lambda$. As $[S^\nu : D^{\lambda^r}] = 0$ for all $\nu \ntrianglelefteq \lambda^r$, the sum is over all ν such that $\lambda \lhd \nu \unlhd \lambda^r$, and thus any ν contributing to the sum must have $\nu_k = \lambda_k$ for all $k \notin \{i, i+1, j, j+1\}$. Also, a_ν is zero unless there are rim-hooks g and h of $Y(\lambda)$ and $Y(\nu)$ respectively such that $\mathrm{val}_p(\mid g \mid) \neq 0$ and $Y(\lambda \backslash g) = Y(\nu \backslash h)$. The only contributing terms are when $\nu \in \{\lambda', \lambda''\}$ where

$$\lambda'_k = \begin{cases} \lambda_k & k \notin \{i, i+1\} \\ \lambda_k + 1 & k = i \\ \lambda_k - 1 & k = i+1 \end{cases} \quad \text{and } \lambda''_k = \begin{cases} \lambda_k & k \notin \{j, j+1\} \\ \lambda_k + 1 & k = j \\ \lambda_k - 1 & k = j+1 \end{cases},$$

with $a_{\lambda'} = a_{\lambda''} = 1$. By row and column removal [Jam81], or by observing that each of these partitions have λ^r as their 2-regularisations, we see that $[S^\nu : D^{\lambda^r}] = 1$ for $\nu \in \{\lambda', \lambda''\}$, and thus $\sum_{i=1}[S^\lambda_{(i)} : D^{\lambda^r}] = 2$ as required. If λ satisfies property *(ii)* of Theorem 3.2.5, but none of the conditions of the statement, the only 2-singularity in λ is a pair of rows of length 2 and we conclude $\nu_2(\lambda) = 2$ by Proposition 4.1.29. □

As before Proposition 3.1.4 allows us to get some conditions for which $\nu_2(\lambda) \geq 4$, which are the first six conditions below.

Theorem 3.3.4. *Let $\lambda \vdash n$ and $p = 2$. Then $\nu_2(\lambda) \geq 4$ if and only if one of the following hold:*

(i) *λ is quadruply 2-singular; that is there are i, j, k and l such that $\lambda_i = \lambda_{i+1}$, $\lambda_j = \lambda_{j+1}$, $\lambda_k = \lambda_{k+1}$ and $\lambda_l = \lambda_{l+1}$ with $\{i, i+1\}$ $\{j, j+1\}$, $\{k, k+1\}$ and $\{l, l+1\}$ pairwise disjoint.*

(ii) *There exists i such that $\lambda_i = \lambda_{i+3} = 1$ and $j \notin \{i, \ldots, i+3\}$ with $\lambda_j = \lambda_{j+1}$.*

(iii) *There exist i, j with $\{i, i+1, i+2\} \cap \{j, j+1\} = \emptyset$ such that $\lambda_i \leq \lambda_{i+2} + 1$ and $\lambda_{i+1} \geq 3$ and $\lambda_j = \lambda_{j+1}$.*

(iv) *There exists i, j, k with $i \geq j + 2 \geq k + 2$ and $\lambda_i = \lambda_{i+1}$ and $\lambda_j = \lambda_{j+1}$ and $\lambda_k \leq \lambda_{i+2} + 1$ and $\lambda_{i+1} = 2$.*

(v) *There exist i, j, k with $\{i, i+1\} \cap \{j, j+1\} \cap \{k, k+1\} = \emptyset$ such that $\lambda_i = \lambda_{i+1} = \lambda_j + 2 = \lambda_{j+1} + 2 \geq 4$ and $\lambda_k = \lambda_{k+1}$.*

(vi) *There exists i such that $\lambda_i = \lambda_{i+3} > 1$.*

(vii) *There exists i such that $\lambda_i \leq \lambda_{i+3} + 2$ with $\lambda_{i+1} \geq 4$, $\lambda_{i+2} \geq 3$ and $\lambda_{i+3} \geq 2$.*

Proof. Observe that the 'if' direction follows from Theorem 3.3.3, Theorem 3.2.5, Proposition 3.1.4 and Proposition 4.1.29 for conditions *(i)-(v)*. We observed that $\nu_2((2^4)) \geq 4$ in the proof of Lemma 3.3.2. Also in that proof we show that $\nu_2((3^4)) \geq 5$ and hence $\nu_2(\lambda) \geq 4$ for

$$\lambda \in \{(4,4,3,2), (4,4,4,2), (4,4,3,3), (4,4,4,3), (5,4,4,3)\}.$$

To see that a partition satisfying *(vii)* has $\nu_2(\lambda) \geq 4$ it remains to check this for $\lambda \in \{(5,4,4,4), (6,5,4,4)\}$. This follows from the fact that $\nu_2((4^4)) \geq 6$, which can be checked by computing the inner products of polytabloids e_s and e_t for all s and t where $G(s,t)$ contains no pairs of edges.

To prove the 'only if' direction we will show that if λ satisfies one of the conditions from Theorem 3.3.3, but none of the conditions in the statement, then the Schaper number of λ is three. If λ is triply 2-singular, with $\lambda_i = \lambda_{i+1}$, $\lambda_j = \lambda_{j+1}$ and $\lambda_k = \lambda_{k+1}$, then similarly to before these lengths all differ by at least 3 and all other rows have lengths that differ by at least 2 from λ_i, λ_j and λ_k. The only contributing terms in the sum $\sum_\nu a_\nu [S^\nu : D^{\lambda^r}]$ are those for which $\nu \in \{\lambda'\lambda'', \lambda'''\}$, where $\lambda'_l = \begin{cases} \lambda_l & l \notin \{i, i+1\} \\ \lambda_l + 1 & l = i \\ \lambda_l - 1 & l = i+1 \end{cases}$,

$$\lambda''_l = \begin{cases} \lambda_l & l \notin \{j, j+1\} \\ \lambda_l + 1 & l = j \\ \lambda_l - 1 & l = j+1 \end{cases} \quad \text{and } \lambda'''_l = \begin{cases} \lambda_l & l \notin \{k, k+1\} \\ \lambda_l + 1 & l = k \\ \lambda_l - 1 & k = j+1 \end{cases},$$

which all appear with coefficient $a_{\lambda'} = a_{\lambda''} = 1$. As before $[S^\nu : D^{\lambda^r}] = 1$ if ν is any of the above, as the 2-regularisation of each of these ν is λ^r, and thus $\nu_2(\lambda) = 3$.

If λ is 4-singular, but does not satisfy any of the conditions in the statement, then the rows of the same length are of length 1 and λ is not 6-singular so, by Proposition 4.1.29, $\nu_2(\lambda) = 3$.

Let λ satisfy property *(v)* of Theorem 3.3.3 but none of the conditions in the statement. If there are two rows of length 3, then by Proposition 4.1.29, $\nu_2(\lambda) = 3$, so we may assume

$$\lambda \in \{\eta\#(k+2, k, k, k-1, k-3)\#\xi, \eta\#(k+3, k+1, k, k, k-2)\#\xi,$$
$$\eta\#(k+3, k, k, k, k-3)\#\xi \text{ for 2-regular partitions } \eta, \xi \text{ and } k \geq 4\},$$

where # denotes concatenation of partitions. In all three cases, just as before, we shall show that the simple module corresponding to the p-regularisation of λ lies in the 3rd, and therefore top, Schaper layer.

Let λ be of the form $\eta\#(k+2, k, k, k-1, k-3)\#\xi$, then $\lambda^r = \eta\#(k+2, k+1, k, k-2, k-3)\#\xi$. The only ν contributing to the sum $\sum_\nu a_\nu[S^\nu : D^{\lambda^r}]$ are $\eta\#(k+2, k+1, k-1, k-1, k-3)\#\xi$ which appears with coefficient 1, and λ^r itself, which appears with coefficient 2. Both of these have $[S^\nu : D^{\lambda^r}] = 1$, as the 2-regularisation of both ν and λ^r is λ^r, and hence $\sum_\nu a_\nu[S^\nu : D^{\lambda^r}] = 3 = \nu_2(\lambda)$

Now consider a partition of the form $\lambda = \eta\#(, k+3, k, k, k, k-3)\#\xi$. The p-regularisation is $\lambda^r = \eta\#(k+3, k+2, k, k-2, k-3)\#\xi$. The only ν contributing to the sum $\sum_\nu a_\nu[S^\nu : D^{\lambda^r}]$ are $\eta\#(k+3, k+2, k-1, k-1, k-3)\#\xi$, $\eta\#(k+3, k+1, k+1, k-2, k-3)\#\xi$ and λ^r itself, which all appear with coefficient 1 and have $[S^\nu : D^{\lambda^r}] = 1$, as before, so $\sum_\nu a_\nu[S^\nu : D^{\lambda^r}] = 3 = \nu_2(\lambda)$.

Finally, if λ is of the form $\eta\#(k+3, k+1, k, k, k-2)\#\xi$, then $\lambda^r = \eta\#(k+3, k+2, k, k-1, k-3)\#\xi$. The only ν contributing are $\eta\#(k+3, k+1, k+1, k-1, k-3)\#\xi$, with coefficient 1, and λ^r itself, with coefficient 2. Again both have $[S^\nu : D^{\lambda^r}] = 1$ so $\sum_\nu a_\nu[S^\nu : D^{\lambda^r}] = 3 = \nu_2(\lambda)$.

Let λ satisfy property *(iv)* of Theorem 3.3.3 but no conditions of the statement. Then we may assume $\lambda = \eta\#(k+2, k, k, k, k-2, k-2, k-4)\#\xi$ for 2-regular partitions η, ξ and $k \geq 4$. The p-regularisation of λ is $\lambda^r = \eta\#(k+2, k+1, k, k, k-2, k-3, k-4)\#\xi$. As before the only ν contributing to the sum $\sum_\nu a_\nu[S^\nu : D^{\lambda^r}]$ are $\eta\#(k+2, k+3, k-1, k-2, k-2, k-4)\#\xi$, $\eta\#(k+4, k, k, k-1, k-3, k-4)\#\xi$ and λ^r itself, which all appear with coefficient 1 and have $[S^\nu : D^{\lambda^r}] = 1$.Thus $\sum_\nu a_\nu[S^\nu : D^{\lambda^r}] = 3$ and hence D^{λ^r} lies in the 3rd, and therefore top, Schaper layer.

If λ satisfies property *(ii)* of Theorem 3.3.3 but none of the conditions in the statement then $\lambda_i = 2$ and $\lambda_j \notin \{2, 3\}$. If $\lambda_j = 1$ then $\nu_2(\lambda) = 3$ by Proposition 4.1.29, so we may assume $\lambda = (r, r-1, \ldots, m+2, m, m, m-2, \ldots, 3, 2, 2, 1)$. Suppose further that $m \neq 4$. We shall construct row equivalent λ-tableaux t and u such that $16 \nmid \langle e_t, e_u \rangle$. We shall choose t to be the initial tableaux, that is the tableaux whose entries are, from left to right and top to bottom, $1, 2, 3, \ldots$. We then choose u to be the unique tableaux which is row equivalent to t and whose rows of unique length have entries in descending order from left to right and whose rows of length m are obtained from t by permuting the other rows that occur as a pair as described below: If the pair of rows of length m appearing in t is

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	$\ldots$	a_{m-1}	a_m
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	$\ldots$	b_{m-1}	b_m

,

then set the corresponding rows of u to be

a_m	a_{m-2}	a_{m-1}	$\ldots$	a_8	a_5	a_6	a_3	a_4	a_1	a_2
b_{m-1}	b_m	b_{m-3}	$\ldots$	b_6	b_7	b_4	b_5	b_2	b_3	b_1

,

and set the last rows of u to be

$n-3$	$n-4$
$n-2$	$n-1$
n	

.

It is easy to see that any tabloid $\{v\}$ common to e_t and e_u must have $R_i(\{v\}) = R_i(t)$ for any row i of unique length with $\mid R_i(t) \mid \neq 1$. For example, the elements occurring first in each row of t occur last in the rows in u except the row of length m where it is the second to last entry. Apart from in this row, these entries can not appear lower in $\{v\}$ than they do in t and so they must appear in the same row. Similarly we see that if $\lambda_l = \lambda_{l+1}$ then $R_l(\{v\}) \cup R_{l+1}(\{v\}) = R_l(t) \cup R_{l+1}(t)$ and thus $\langle e_t, e_u \rangle = \langle e_{t'}, e_{u'} \rangle \cdot \langle e_{t''}, e_{u''} \rangle$, where t' is the tableau consisting of only the pair of rows in t of length m and t'' is the tableau consisting of last three rows of t, with u' and u'' defined similarly. It is easy to see that $\langle e_{t'}, e_{u'} \rangle = 2$ and $\langle e_{t''}, e_{u''} \rangle = 12$, although we shall sketch proofs to give demonstrate to the reader how to obtain this, as these techniques are used throughout. It then follows that $\langle e_t, e_u \rangle = 24$, which is not divisible by 16.

Claim. $\langle e_{t'}, e_{u'} \rangle = 2$

Proof of claim. Observe that the graph $G(s,t)$, where

$s =$

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	$\dots$	a_{m-1}	a_m
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	$\dots$	b_{m-1}	b_m

and

$t =$

a_m	a_{m-2}	a_{m-1}	$\dots$	a_8	a_5	a_6	a_3	a_4	a_1	a_2
b_{m-1}	b_m	b_{m-3}	$\dots$	b_6	b_7	b_4	b_5	b_2	b_3	b_1

,

has only two admissible colourings, as once either c_1 or c_2 is chosen for some edge e the colour of the other edges is determined. ✓

Claim. $\langle e_{t''}, e_{u''} \rangle = 12$

Proof of claim. Let

$s =$

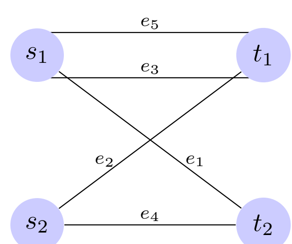
1	2
3	4
5	

and $t =$

2	1
3	4
5	

.

Then the graph $G(s,t) =$



.

The claim then follows from analysing the admissible colourings of the graph $G(s,t)$.

✓

We may do a similar thing if $m = 4$, in which case we may assume $\lambda = (r, r-1, \cdots, 7, 6, 4, 4, 2, 2, 1)$. If we set t to be the initial tableau and set u to be the row equivalent tableau with entries in descending order in all rows except the rows of length

4 which we set to

$$\begin{array}{|c|c|c|c|}\hline a_3 & a_4 & a_1 & a_2 \\ \hline b_4 & b_2 & b_3 & b_1 \\ \hline \end{array},$$

as before. In this case we see that $\langle e_t, e_u \rangle = \langle e_{t'}, e_{u'} \rangle$, where

$$t' = \begin{array}{|c|c|cc}\hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline 5 & 6 & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{8} \\ \hline 9 & 10 & & \\ \cline{1-2} 11 & 12 & & \\ \cline{1-2} \multicolumn{1}{|c|}{13} & & & \\ \cline{1-1} \end{array}$$

and

$$u' = \begin{array}{|c|c|cc}\hline 3 & 4 & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} \\ \hline 8 & 6 & \multicolumn{1}{c|}{7} & \multicolumn{1}{c|}{5} \\ \hline 10 & 9 & & \\ \cline{1-2} 12 & 11 & & \\ \cline{1-2} \multicolumn{1}{|c|}{13} & & & \\ \cline{1-1} \end{array}.$$

Similar calculations show that this inner product is 8, and hence not divisible by 16, so the Schaper number of λ is at most three.

Now suppose λ satisfies the final property of Theorem 3.3.3. Recall the case where λ has two rows of length 2 and one of length 1 was dealt with earlier, so we may assume $\lambda = (r, r-1 \cdots, k+3, k+1, k+1, k-1, k-1, k-3, \cdots, 2, 1)$ and thus $\lambda^r = (r, r-1, \cdots, k+3, k+2, k+1, k-1, k-2, k-3, \cdots, 2, 1)$. The contributing terms are $(r, r-1, \cdots, k+3, k+2, k, k-1, k-1, k-3, \cdots, 2, 1)$, $(r, r-1, \cdots, k+3, k+1, k+1, k, k-2, k-3, \cdots, 2, 1)$ and λ^r, all with coefficient 1. All of these have λ^r as their p-regularisation so $[S^\nu : D^{\lambda^r}] = 1$ and therefore $\sum_\nu a_\nu [S^\nu : D^{\lambda^r}] = 3 = \nu_2(\lambda)$, completing the proof. $\square$

Armed with our new knowledge of Schaper numbers we are able to go further than Example 3.2.7 and determine new decomposition numbers. The next example is a calculation of a decomposition number which does not appear in the literature.

Example 3.3.5. *Let $p = 2$ and consider the block of $\mathbb{F}_2\mathcal{S}_{20}$ containing all Specht modules S^λ where λ has 2-core $(3,2,1)$. The decomposition numbers are known for $\mathcal{S}_n$ where $n < 20$ and can be found in the GAP character table library. There are 15 2-regular partitions in this block, and standard techniques, such as r-induction [JK81, Section 6.3]* *allow the decomposition numbers in the last nine (in the dominance order) columns, and the columns labelled by* $(11,8,1)$ *and* $(13,6,1)$ *to be calculated. Simply by using column elimination [Don85] we get the following part of the decomposition matrix:*

	$(17,2,1)$	$(15,4,1)$	$(13,6,1)$	$(13,4,3)$	$(11,8,1)$	$(11,6,3)$
$(17,2,1)$	1					
$(15,4,1)$	0	1				
$(13,6,1)$	0	0	1			
$(13,4,3)$	1	0	1	1		
$(11,8,1)$	0	1	0	0	1	
$(11,6,3)$	1	0	1	1	0	1

The next row we seek to calculate is the row corresponding to the partition $(11,4,3,2)$. *Of course* $[S^{(11,4,3,2)} : D^{(11,4,3,2)}] = 1$ *and the composition multiplicities of* $D^{(11,6,3)}$ *and* $D^{(11,8,1)}$ *in* $S^{(11,4,3,2)}$ *are the same as the composition multiplicities in* $S^{(4,3,2)}$ *of* $D^{(6,3)}$ *and* $D^{(8,1)}$*, respectively. Three decomposition numbers remain to be calculated, namely;* $[S^{(11,4,3,2)} : D^{(17,2,1)}]$, $[S^{(11,4,3,2)} : D^{(15,4,1)}]$, *and* $[S^{(11,4,3,2)} : D^{(13,4,3)}]$.

Schaper's sum formula, Theorem 3.1.1, tells us

$$[S^{(11,4,3,2)} : D^{\mu}] \leq [S^{(15,4,1)} : D^{\mu}] + [S^{(11,8,1)} : D^{\mu}].$$

Neither $S^{(15,4,1)}$ *or* $S^{(11,8,1)}$ *have* $D^{(17,2,1)}$ *or* $D^{(13,4,3)}$ *as a composition factor, but both have* $D^{(15,4,1)}$ *as a composition factor with multiplicity 1. Thus the row of the decomposition matrix corresponding to the partition* $(11,4,3,2)$ *is:*

	$(17,2,1)$	$(15,4,1)$	$(13,6,1)$	$(13,4,3)$	$(11,8,1)$	$(11,6,3)$	$(11,4,3,2)$
$(11,4,3,2)$	0	x	0	0	1	0	1,

where $1 \leq x \leq 2$. *We now consider the row corresponding to the partition* $(11,3,3,2,1)$, *and note that the Schaper number of this partition is at least* 3, *by Theorem 3.3.3 (in fact, that the Schaper number is exactly 3 follows from Theorem 3.3.4). Linear relations* *between the characters allow us to determine that this row is*

	(17, 2, 1)	(15, 4, 1)	(13, 6, 1)	(13, 4, 3)	(11, 8, 1)	(11, 6, 3)	(11, 4, 3, 2)
(11, 3, 3, 2, 1)	1	x	1	1	1	0	1.

The sum of modules appearing in Schaper's sum formula is:

$$\sum_{i\geq 1} S_i^{(11,3,3,2,1)} = S^{(17,2,1)} - 2S^{(15,2,1^3)} + S^{(13,2^3,1)}$$
$$+S^{(11,8,1)} - S^{(11,6,3)} - 2S^{(11,6,1^3)}$$
$$+S^{(11,4,3,2)} + 2S^{(11,4,3,1^2)} + S^{(11,4,2^2,1)}.$$

Writing the characters of Specht modules corresponding to 2*-singular partitions in terms of* 2*-regular partitions we observe that* $\sum_{i\geq 1} S_i^{(11,3,3,2,1)}$ *has the same composition factors as*

$$-S^{(17,2,1)} - S^{(15,4,1)} - S^{(13,6,1)} + 4S^{(13,4,3)} - S^{(11,8,1)} + 4S^{(11,4,3,2)}.$$

Thus $x \leq \frac{4x-2}{3}$*, which rules out the possibility that* $x = 1$*. We conclude, then, that* $[S^{(11,4,3,2)} : D^{(15,4,1)}] = 2$ *and the first seven (2-regular) rows of the decomposition matrix of this block are:*

	(17, 2, 1)	(15, 4, 1)	(13, 6, 1)	(13, 4, 3)	(11, 8, 1)	(11, 6, 3)	(11, 4, 3, 2)
(17, 2, 1)	1						
(15, 4, 1)	0	1					
(13, 6, 1)	0	0	1				
(13, 4, 3)	1	0	1	1			
(11, 8, 1)	0	1	0	0	1		
(11, 6, 3)	1	0	1	1	0	1	
(11, 4, 3, 2)	0	2	0	0	1	0	1.

We may continue as in the above example to attempt to calculate all of the decomposition numbers in this block. Applying this method will give a number of inequalities in terms of the unknown decomposition numbers. Again, like all other known methods for determining decomposition numbers, these are not guaranteed to give all of the decomposition numbers. However, with our new knowledge of Schaper numbers we are able to go further than has been done before and to provide improved bounds for decomposition numbers.

The explicit methods of this chapter are sufficient to continue classifying higher and higher Schaper numbers, although it will become computationally heavy. We seek a more general understanding. We shall modify the proof of the *only if* direction of Theorem 3.3.4 to determine the Schaper number of partitions whose Schaper number only has contributions from pairs of rows, or rows of length 1 or 2.

Proposition 3.3.6. *Let $\lambda = \mu\#(2^a, 1^b)$ be a partition of n with where μ is a partition whose rows are all at least 4. Suppose further that μ does not satisfy property* (ii) *of Theorem 3.2.5; that is that there is no i such that $\lambda_i \leq \lambda_{i+2}+1$, and that if $\lambda_i = \lambda_{i+1}$ and $\lambda_j = \lambda_{j+1}$ for $i > j$ then $\lambda_i - \lambda_j > 2$, or $\lambda_i \leq 2$.*

Let m be the number of 2-singularities in μ

$$\nu_2(\lambda) = m + min_{x \leq a}\{\mathrm{val}_2(a!(a-x)!(b+x)!\}$$

Proof. We shall first show that $\nu_2((2^a, 1^b)) = \min_{x\leq a}\{\mathrm{val}_2(a!(a-x)!(b+x)!\}$. Let t and u be row equivalent $(2^a, 1^b)$ tableaux. Then u is obtained from t by swapping the order in which the entries in $x \leq a$ of the rows of length 2 appear. Then the inner product $\langle e_t, e_u\rangle = (a!(a-x)!(b+x)!$ and thus $\nu_2((2^a, 1^b)) = \min_{x\leq a}\{\mathrm{val}_2(a!(a-x)!(b+x)!\}$. This was observed by Fayers in [Fay03], where the Schaper filtration is explicitly calculated. It follows that $\nu_2(\lambda) \geq m + \min_{x\leq a}\{\mathrm{val}_2(a!(a-x)!(b+x)!\}$.

We shall construct row equivalent λ-tableaux t and u such that the highest power of 2 dividing $\langle e_t, e_u\rangle$ is $m+\min_{x\leq a}\{\mathrm{val}_2(a!(a-x)!(b+x)!\}$. We shall choose t to be the initial tableaux, and u to be the tableaux which is row equivalent to t obtained as follows: if i is the only row of length λ_i then reverse the order of the entries, reverse the order of the entries of the first x rows of length 2, and if rows i and $i+1$ are of the same length $\lambda_i = k_j \geq 4$, then permute the entries as described below: If the pair of rows of length m appearing in t is

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	$\dots$	a_{k-1}	a_k
b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	$\dots$	b_{k-1}	b_k

,

then set the corresponding rows of u to be

a_k	a_{k-2}	a_{k-1}	$\dots$	a_8	a_5	a_6	a_3	a_4	a_1	a_2
b_{k-1}	b_k	b_{k-3}	$\dots$	b_6	b_7	b_4	b_5	b_2	b_3	b_1

It is easy to see that any tabloid $\{v\}$ common to e_t and e_u must have $R_i(\{v\}) = R_i(t)$ for any row i of unique length with $\mid R_i(t) \mid \geq 4$. It must also have $R_i(\{v\}) \cup R_{i+1}(\{v\}) = R_i(t) \cup R_{i+1}(t)$ for rows i and $i+1$ of length $k \geq 4$ and $\cup_{i \geq j} R_i(\{v\}) = \cup_{i \geq j} R_i(t)$ where $j = \mu'_1 + 1$. The last equality is the set of entries which appear in rows of length 1 or 2. Thus we see that the inner product

$$\langle e_t, e_u \rangle = \langle e_{t'}, e_{u'} \rangle \Pi_{i=1}^m \langle e_{t_i}, e_{u_i} \rangle,$$

where t' and u' are obtained by restricting t and u to the partition $(2^a, 1^b)$ and t_i and u_i are obtained by restricting t and u to the ith 2-singularity of μ. Clearly $\langle e_t, e_u \rangle = (a!(a-x)!(b+x)!) \cdot 2^m$, and the result follows. □

Similar ideas can be used to determine the Schaper number of similar partitions. We conclude this section by making a number of conjectures. It is expected that these may need to be modified as they are intended only to serve as a starting point for further work on 2-Schaper numbers.

Conjecture 3.3.7. *Let $\lambda \vdash n$ and $p = 2$. Then $\nu_2(\lambda) \geq 5$ if and only if one of the following hold:*

- *(i) $\lambda = \mu \star \eta$ with $\nu_2(\mu) + \nu_2(\eta) \geq 5$*
- *(ii) There exists i such that $\lambda_i = \lambda_{i+3} \geq 3$.*
- *(iii) There exists i such that $\lambda_i \leq \lambda_{i+4} + 1$ and at least 4 of these 5 rows are of the same length and $\lambda_{i+1} \geq 2$.*
- *(iv) There exists i such that $\lambda_i = \lambda_{i+7} \geq 1$.*
- *(v) There exists i such that $\lambda_i = \lambda_{i+1} = \lambda_{i+2} + 1 = \lambda_{i+3} + 1 \geq 5$.*
- *(vi) There exists i such that $\lambda_i \leq \lambda_{i+4} + 3$ with $\lambda_{i+2} \geq 4$, $\lambda_{i+3} \geq 3$ and $\lambda_{i+4} \geq 2$ and some $j \in \{i, i+1, i+2, i+3, i+4\}$ with $\lambda_j = \lambda_{j+1} \geq 5$.*

Of course, a partition satisfying the first of these conditions has 2-Schaper number at least 5, by Proposition 3.1.4. It was observed in Lemma 3.3.2 that $\nu_2((3^4)) \geq 5$. The fact that conditions *(iii)* and *(iv)* both imply $\nu_2(\lambda) \geq 5$ follow from Proposition 3.3.6 and Proposition 4.1.29, respectively. The partition $\lambda = (5, 5, 4, 4)$ has 2-Schaper number at least 5, due to Lemma 3.2.8 and the proof of Theorem 3.3.3 and thus any partition satisfying *(v)* has 2-Schaper number at least 5. The last condition is not known to force a partition to have 2-Schaper number at least 5. It is part of a larger family of partitions which may have large Schaper number:

Definition 3.3.8. *Let $\lambda \vdash n$ and $k \geq 3$. We say λ has a k-*staircase *if there are $i, \dots, i+k-1$ such that $\lambda_i \leq \lambda_{i+k-1} + k - 2$ with $\lambda_{i+j} \geq k+1-j$ for $1 \leq j \leq k-1$ and some $1 \leq j \leq k-2$ with $\lambda_{i+j} = \lambda_{i+j+1} \geq k$. That is; if there are $k-1$ consecutive rows, who differ in length by at most $k-2$ boxes, which contain a pair of rows of length at least k, and whose ith row (from the bottom) contains at least $i+1$ boxes.*

Conjecture 3.3.9. *Let $\lambda \vdash n$ and suppose λ has a k-staircase. Then $\nu_p(\lambda) \geq k$.*

This conjecture is seen to be hold for $k = 3$ and $k = 4$ by Theorem 3.3.3 and Theorem 3.3.4 respectively. We see in these results, and Conjecture 3.3.7 that a k-staircase is not the only way for a partition to have a high 2-Schaper number, and that a partition with a k-staircase may in fact have a 2-Schaper number greater than k.

3.4 Schaper Numbers for Odd Primes

The problem of characterising partitions with high Schaper number for odd primes is more difficult. Unlike in Theorem 3.2.5, where there is a nice characterisation for all primes, small primes must be treated separately when characterising partitions with higher Schaper numbers. In this section we will give a necessary list of conditions for partitions to have Schaper number at least three for odd primes. Throughout this section p is assumed to be odd.

Theorem 3.4.1. *Let $\lambda \vdash n$, p be an odd prime and $\nu_p(\lambda) \geq 3$. Then one of the following conditions holds:*

(i) *λ is triply p-singular.*

(ii) *There exist i, j with $\{i, \dots, i+2p-2\} \cap \{j, \dots, j+p-1\} = \emptyset$ such that $\lambda_i \leq \lambda_{i+2p-2}+1$ and $\lambda_{i+p-1} \geq 2$ and $\lambda_j = \lambda_{j+p-1}$.*

(iii) *There exists i with $\lambda_i = \lambda_{i+2p-1} \geq 2$.*

(iv) *There exists i such that $\lambda_i = \lambda_{i+p-1} = \lambda_{i+p} + 1 = \lambda_{i+2p-1} + 1 \geq 3$.*

(v) *There exist i such that $\lambda_i \leq \lambda_{i+3p-3} + 2$ with $\lambda_{i+2p-2} \geq 2$ and some p consecutive rows between λ_i and λ_{i+3p-3} have length $k \geq 3$.*

Proof. We shall show that if λ satisfies one of the conditions of Theorem 3.2.5, but not any of the conditions in the statement then $\nu_p(\lambda) = 2$.

First suppose λ is doubly p-singular. If the p-singularities are of the same length then this length must be 1, and so we are done by Proposition 4.1.29. If they differ in length by 1 then $\lambda = (\cdots, 4^{i_4}, 3^{i_3}, 2^{p+i_2}, 1^{p+i_1})$ with $i_j \leq p-1$ for all j, $i_1 + i_2 \leq p-2$ and $i_2 + i_3 \leq p-2$. Similar to the proof of Theorem 3.3.4, if we let t be the initial λ-tableau and u be the tableau obtained from t by reversing the entries in all rows except $i_2 + 1$ of the rows of length 2, then we have constructed tableaux such that $p^3 \nmid \langle e_t, e_u \rangle$. We can see this by observing that the entries appearing in some row of length $\lambda_i \geq 3$ in a tableaux v such that $t \sim_{\text{col}} v \sim_{\text{row}} v' \sim_{\text{col}} u$ must appear in a row of that same length in u. Then note $\langle e_t, e_u \rangle = \langle e_{t'}, e_{u'} \rangle \langle e_{t''}, e_{u''} \rangle$ where t' and u' are tableaux containing all rows of length greater than two and t'' and u'' contain the remaining rows. Clearly no power of p divides $\langle e_{t'}, e_{u'} \rangle$, and considering the graph $G(t'', u'')$ we see that $p^3 \nmid \langle e_{t''}, e_{u''} \rangle$.

Now suppose the lengths of these two singularities differ by 2 or more and that neither of them are of length 1, We will now show that the module D^{λ^r} appears in the second Schaper layer of S^λ. Observe that when we take the p-regularisation of such a partition boxes can only move into the next position in the p ladder; that is to say a box is either fixed or it moves up p rows and into the column to its right. This is because if it were able to move further then we must have $2p-1$ rows who differ by 2, or $3p-2$ rows who differ by 3.

Again, $[S^\lambda : D^{\lambda^r}] = 1$ so the value of $\sum_{i=1}[S^\lambda_{(i)} : D^{\lambda^r}] = \sum_\nu a_\nu [S^\nu : D^{\lambda^r}]$ is the number of the Schaper layer in which D^{λ^r} appears. The term $[S^\nu : D^{\lambda^r}]$ can only contribute if ν is obtained from λ by unwrapping a single mp-hook and wrapping it further up the Young diagram, and if $\lambda \lhd \nu \unlhd \lambda^r$.

Any hook which contains boxes not in one of the two singularities would result in a ν which is not dominated by λ^r so the only options are the two p-hooks which have their foot in the removable box of a p-singularity. Such a hook must then be wrapped in a way so that all of its boxes are placed in the same column that they appear in $Y(\lambda)$, or the column immediately to the right. The leg length of the hook as it appears in $Y(\lambda)$ is p and in $Y(\lambda^r)$ it is $p-1$, so the coefficient $a_\nu = +1$. Also, as $\nu^r = \lambda^r$ we have $[S^\nu : D^{\lambda^r}] = 1$, and hence $\sum_{i=1}[S^\lambda_{(i)} : D^{\lambda^r}] = \sum_\nu a_\nu [S^\nu : D^{\lambda^r}] = 2$, as claimed.

Now suppose that the lengths of these two singularities differ by 2 or more and that there is a p-singularity of length 1. We shall construct λ-tableaux t and u such that the inner product between the polytabloids e_t and e_u is divisible by p^2 but not p^3. We may assume that

$$\lambda = (\cdots, (k+1)^{i_{k+1}}, k^{p+i_k}, (k-1)^{i_{k-1}}, \cdots, 2^{i_2}, 1^{p+i_1}),$$

with $i_{k+1} + i_k + i_{k-1} < p-2$ and $i_j < p$ for all j. As before we choose t to be the initial

λ-tableau and u to be the tableau row equivalent to t which is obtained by reversing the order of entries in all of the rows except for $p+i_k-\max\{i_{k+1},i_{k-1}\}$ of the rows of length k. Of these remaining rows, we set $p-\max\{i_{k+1},i_{k-1}\}-1$ of these to

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & a_5 & a_6 & a_3 & a_4 & a_1 & a_2 \\ \hline \end{array}$$

and the other i_k+1 rows to

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \cdots & a_7 & a_4 & a_5 & a_2 & a_3 & a_1 \\ \hline \end{array},$$

where

$$\begin{array}{|c|c|c|c|c|c|c|}\hline a_1 & a_2 & a_3 & a_4 & \cdots & \cdots & a_k \\ \hline \end{array}$$

is the corresponding row of t.

First observe that any entry that appears in a row of length i for $i \notin \{k-1,k,k+1\}$ of a tabloid common to e_t and e_u must also appear in a row of that length in t and u. This allows us to deduce that $\langle e_t, e_u\rangle = \langle e_{t'}, e_{u'}\rangle\langle e_{t''}, e_{u''}\rangle$ where t' and u' are the tableau whose rows are the same as the rows of t and u whose length is not $k-1$, k or $k+1$, and t'' and u'' are the $((k+1)^{i_{k+1}}, k^{p+i_k}, (k-1)^{i_{k-1}})$-tableaux whose rows are the same as the corresponding rows of t and u respectively. Observe also that $\mathrm{val}_p(\langle e_{t'}, e_{u'}\rangle) = 1$ so to complete the proof it remains to prove that $p^2 \nmid \langle e_{t''}, e_{u''}\rangle$.

To see this consider the tableaux

$$t'' = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \tilde{a_1} & \tilde{a_2} & \tilde{a_3} & \cdots & \cdots & \cdots & \cdots & \tilde{a_{k+1}} \\ \hline \tilde{x_1} & \tilde{x_2} & \tilde{x_3} & \cdots & \cdots & \cdots & \tilde{x_k} \\ \cline{1-7} \tilde{y_1} & \tilde{y_2} & \tilde{y_3} & \cdots & \cdots & \cdots & \tilde{y_k} \\ \cline{1-7} \tilde{z_1} & \tilde{z_2} & \tilde{z_3} & \cdots & \cdots & \cdots & \tilde{z_k} \\ \cline{1-7} \tilde{c_1} & \tilde{c_2} & \tilde{c_3} & \cdots & \cdots & \tilde{c_{k-1}} \\ \cline{1-6} \end{array}$$

and

$$u'' = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \tilde{a_{k+1}} & \tilde{a_k} & \cdots & \cdots & \cdots & \cdots & \tilde{a_2} & \tilde{a_1} \\ \hline \cdots & \cdots & \cdots & \tilde{x_3} & \tilde{x_4} & \tilde{x_1} & \tilde{x_2} \\ \cline{1-7} \cdots & \cdots & \cdots & \tilde{y_5} & \tilde{y_2} & \tilde{y_3} & \tilde{y_1} \\ \cline{1-7} \tilde{z_k} & \cdots & \cdots & \tilde{z_4} & \tilde{z_3} & \tilde{z_2} & \tilde{z_1} \\ \cline{1-7} \tilde{c_{k-1}} & \cdots & \cdots & \tilde{c_3} & \tilde{c_2} & \tilde{c_1} \\ \cline{1-6} \end{array},$$

where the $\tilde{a}_i, \tilde{x}_i, \tilde{y}_i, \tilde{z}_i$ and $\tilde{c}_i$ represent columns of length $i_{k+1}, p+i_k-\max\{i_{k+1}, i_{k-1}\}, i_k+1, \max\{i_{k+1}, i_{k-1}\}$ and i_{k-1} respectively. Observe that for any tabloid $\{T\}$ common to $e_{t''}$ and $e_{u''}$, the permutations required to make t'' and u'' row equivalent to T have the same number of transpositions and therefore the same sign. This means that $\langle e_{t''}, e_{u''}\rangle$ is the number of tabloids $\{T\}$ common to $e_{t''}$ and $e_{u''}$. We shall count such tabloids by constructing tableau U which are column equivalent to u'' and row equivalent to T. Observe that once we have chosen which $p - \max\{i_{k+1}, i_{k-1}\}$ of the rows of length k in U have entries in their last box which come from the second column of t'' (of which there are $\binom{p+i_k}{p-\max\{i_{k+1},i_{k-1}\}}$ possible choices, a number divisible by p) then U is chosen by choosing the order in which entries in the other columns appear. By considering that U must be row equivalent to some tableau which is column equivalent to t'' we observe that we are only choosing the order of either $i_{k+1} + p - \max\{i_{k+1}, i_{k-1}\} - 1$, $i_{k+1} + i_k + 1$, $p - \max\{i_{k+1}, i_{k-1}\} - 1 + \max\{i_{k+1}, i_{k-1}\}$, $p - \max\{i_{k+1}, i_{k-1}\} - 1 + i_{k-1}$,$i_k + 1 + \max\{i_{k+1}, i_{k-1}\}$ or $i_k + 1 + i_{k-1}$ elements. The number of possible choices here is the product of the factorials of these numbers, which is not divisible by p. We conclude that $p^2 \nmid \langle e_{t''}, e_{u''}\rangle$ and thus $\mathrm{val}_p(\langle e_t, e_u\rangle) = 2$, as required.

Now suppose that λ satisfies the other condition of Theorem 3.2.5, but not any of the conditions in the statement, that is there exist i such that $\lambda_i \leq \lambda_{i+2p-2} + 1$ and $\lambda_{i+p-1} \geq 2$. As before we shall show that the D^{λ^r} appears in the second layer, as in the proof of Theorem 3.3.3.

We may assume that

$$\lambda = (\cdots, (k+1)^{i_{k+1}}, k^{p+i_k}, (k-1)^{i_{k-1}}, (k-2)^{i_{k-2}}, \cdots, 2^{i_2}, 1^{i_1}),$$

with $i_{k+1} + i_k < p - 1$, $i_k + i_{k-1} \geq p - 1$, and not satisfying any of the conditions of Theorem 3.4.1, or that

$$\lambda = (\cdots, (k+1)^{i_{k+1}}, k^{i_k}, (k-1)^{p+i_{k-1}}, k - 2^{i_{k-2}}, \cdots, 2^{i_2}, 1^{i_1})$$

with $i_k + i_{k-1} \geq p - 1$, and not satisfying any of the conditions of Theorem 3.4.1. In the first case the p-regularisation of λ is

$$\lambda^r = (\cdots, (k+1)^{i_{k+1}+i_k+1}, k^{i_{k-1}}, (k-1)^{2p-i_k-i_{k+1}-3}, \\ (k-2)^{i_{k-2}+i_{k+1}+i_k+2-p}, (k-3)^{i_{k-3}}, \cdots, 2^{i_2}, 1^{i_1}),$$

while in the second it is

$$\lambda^r = (\cdots, (k+1)^{i_{k+1}+i_k+i_{k-1}+2-p}, k^{2p-i_k-i_{k-1}-3},$$
$$(k-1)^{i_k+i_{k-1}+i_{k-2}+2-p}, (k-2)^{2p-3-i_{k-1}-i_{k-2}}, (k-3)^{i_{k-1}+i_{k-2}+i_{k-3}}, \cdots),$$

if $i_{k-1} + i_{k-2} \geq p-1$ and

$$\lambda^r = (\cdots, (k+1)^{i_{k+1}+i_k+i_{k-1}+2-p}, k^{2p-i_k-i_{k-1}-3},$$
$$(k-1)^{i_{k-1}}, (k-2)^{i_{k-1}+i_{k-2}+1}, (k-3)^{i_{k-3}}, \cdots)$$

otherwise. Observe that in each of these cases the only μ that can contribute to the sum in Theorem 3.1.1 are those μ which are obtained from λ by unwrapping an mp hook and wrapping it back on higher up the diagram in such a way that $\lambda \lhd \mu \unlhd \lambda^r$. Observe that there are only two such mp hooks. One is the p-hook whose foot is in the row of the same length as the p-singularity, and the other is a $2p$ hook. There is a unique way that each of these can be wrapped and each of these has p-regularisation λ^r, hence each will contribute one to the sum, and thus $\sum_{i=1}[S^\lambda_{(i)} : D^{\lambda^r}] = 2$ and D^{λ^r} appears in the second layer. □

We shall now investigate which of these conditions are sufficient for $\nu_p(\lambda) \geq 3$, for which we make the following conjecture:

Conjecture 3.4.2. *Let $\lambda \vdash n$ satisfy one of the conditions of Theorem 3.4.1, then $\nu_p(\lambda) \geq 3$.*

If the conjecture is true then we have a complete characterisation of partitions with Schaper number at least three. In the remainder of this paper we shall make progress towards the conjecture by dealing separately with each of the conditions in Theorem 3.4.1. Of these conditions, only the last remains open, although some progress is made towards this case in Lemma 3.4.5.

Lemma 3.4.3. *Let $\lambda \vdash n$ and suppose there exists an i with $\lambda_i = \lambda_{i+2p-1} \geq 2$, then $\nu_p(\lambda) \geq 3$.*

Proof. By Propositions 3.1.4 and 3.2.3 is suffices to show $\nu_p(\lambda) \geq 3$ for $\lambda = (2^{2p})$. Let s and t be row equivalent (2^{2p})-tableaux and let $G = G(s,t)$ as above. Observe that, without loss of generality, there are $a \geq p$ edges from s_1 to t_1, so $p^3 \mid \langle e_s, e_t \rangle$ by Proposition 3.2.4 □

Lemma 3.4.4. *Let $\lambda \vdash n$ and suppose there exists i such that $\lambda_i = \lambda_{i+p-1} = \lambda_{i+p} + 1 = \lambda_{i+2p-1} + 1 \geq 3$. Then $\nu_p(\lambda) \geq 3$.*

Proof. Again, by Propositions 3.1.4 and 3.2.3 we are reduced to showing $\nu_p((3^p, 2^p)) \geq 3$. Let $\lambda = (3^p, 2^p)$ let s and t be row equivalent λ-tableaux. Consider the graph $G = G(s,t)$. If this graph contains no edges from s_3 to t_3 then by deleting these two vertices we obtain the graph G_σ for some s_σ, t_σ row equivalent (2^{2p})-tableaux. There is a one-to-one correspondence between admissible colourings C of G and pairs (σ, C') where $\sigma \in \mathcal{S}_p$ and C' is an admissible colouring of G_σ where the edges $e'_1, \ldots, e'_p$ have colours $c_1, \ldots, c_p$ in some order. Examining permutations induced by the colourings shows

$$(-1)^C = (-1)^p (-1)^{C'},$$

thus

$$(-1)^{\pi_{st}} \langle e_s, e_t \rangle = (-1)^p \sum_{\sigma \in \mathcal{S}_r} \sum_{C} (-1)^C,$$

where the second sum is over the admissible colourings of G_σ where the edges $e'_1, \ldots, e'_p$ have colours $c_1, \ldots, c_p$. Let $\mathcal{E} = \{e'_1, \ldots, e'_p\}$ and denote by $R(G_\sigma)$ the set of respectable colourings of G_σ, with respect to $\mathcal{E}$. Recall this is the set of admissible colourings of G_σ in which the edges e'_i all have different colours. There is a faithful signature preserving action of $\mathcal{S}_{2p}$ on $R(G_\sigma)$ given by permuting the colours $c_1, \ldots, c_{2p}$, so

$$(-1)^{\pi_{st}} \langle e_s, e_t \rangle = \frac{1}{\binom{2p}{p}} (-1)^p \sum_{\sigma \in \mathcal{S}_r} \sum_{C \in R(g_\sigma)} (-1)^C.$$

As in the proof of Lemma 3.2.9 we may replace the sum over $R(G_\sigma)$ by one over $A(G_\sigma)$, thus

$$(-1)^{\pi_{st}} \langle e_s, e_t \rangle = \frac{1}{\binom{2p}{p}} (-1)^p \sum_{\sigma \in \mathcal{S}_r} \sum_{C \in A(G_\sigma)} (-1)^{\pi_{s_\sigma t_\sigma}} \langle e_{s_\sigma}, e_{t_\sigma} \rangle.$$

As the binomial coefficient $\binom{2p}{p}$ is not divisible by p and as p^3 divides $\langle e_{s_\sigma}, e_{t_\sigma} \rangle$, by Lemma 3.4.3 we conclude that p^3 divides $\langle e_s, e_t \rangle$.

On the other hand, if there is an edge from s_3 to t_3, then there is a faithful and signature preserving action of $\mathcal{S}_p$ on the admissible colourings of G, by permuting the colours $c_1, \ldots, c_p$. Summing the signatures of all admissible colourings of G in which e has colour c_p we get $\frac{\langle e_s, e_t \rangle}{p}$, which is divisible by p^2 if and only if $\langle e_s, e_t \rangle$ is divisible by p^3. Deleting the edge e gives the graph $G(s', t')$ for $(3^{p-1}, 2^{p+1})$-tableaux s' and t'. There is a one-to-one correspondence between the admissible colourings of $G(s', t')$ and colourings of G in which e has colour c_p. This correspondence is signature preserving and so the sum of $(-1)^C$ over all admissible colourings C of G in which e has colour c_p is $\langle e_{s'}, e_{t'} \rangle = \frac{\langle e_s, e_t \rangle}{p}$ which is divisible by p^2, by Theorem 3.2.5. ☐

Lemma 3.4.5. *Let* $\lambda \vdash n$ *and suppose there exist* i *such that* $\lambda_i \leq \lambda_{i+3p-3} + 2$ *with*

$\lambda_{i+2p-2} \geq 2$. Suppose further that the p-singularity between rows i and $i+3p-2$ has length $\lambda_j \neq \lambda_i - 2$ and $\lambda_j \geq 3$.

Proof. By Propositions 3.1.4 and 3.2.3 it suffices to show the following partitions have Schaper number greater than or equal to three:

- $(3^a, 2^b, 1^c)$ with $a \geq p$, $a+b \geq 2p-1$ and $a+b+c = 3p-2$,
- $(4^a, 3^b, 2^c)$ with $a+b+c = 3p-2$, and one of a or $b \geq p$

which, by Lemma 3.2.8 follows from:

Claim. $\nu_p((3^{3p-2})) = 4$.

Let s and t be row equivalent (3^{3p-2})-tableaux and let $G = G(s,t)$. Each vertex in G has degree $3p-2$ and so we may assume, without loss of generality, there are at least p edges from s_1 to t_1. Lemma 3.4.3 gives that $\nu_p((2^{3p-2})) \geq 3$ and so we deduce, by Proposition 3.2.4, that $\nu_p((3^{3p-2})) = 4$, thus proving the claim. □

Remark. *Observe that Proposition 3.1.4, Theorem 3.2.5, and Lemmas 3.4.3 to 3.4.5 show that all the conditions in Theorem 3.4.1 are sufficient, except possibly the last, for which only the case $\lambda = (5^a, 4^b, 3^c)$ where $a+b+c = 3p-2$ and $a, b < p$ remains. To prove the conjecture it only remains to show that such a partition also has $\nu_p(\lambda) \geq 3$.*

Chapter 4

Combinatorial Designs

The study of combinatorial designs is perhaps one of the oldest branches of pure mathematics. Designs are widely studied and although the name 'design' may have different meanings in different contexts, the notion that design theory captures is that of 'balance'. The earliest examples of combinatorial designs appear in ancient China, with the construction of the 3 magic square, but the modern study of the subject was motivated by famous questions of Euler[Eul23], Kirkman[Kir53], and Steiner[Ste53], on the possibility of arranging sets of finite sets with certain rules. These motivating problems remained open for many years, and generalisations of them remain open today [BS59] [Kee14] [RCW71]. Combinatorial design theory has a number of applications to a range of fields, such as in experiment design or finite geometry. In this chapter we shall briefly survey the theory of integral designs, and then introduce a generalisation of these due to Wilson known as p-ary designs. The main result of this chapter is the classification of universal p-ary designs, p-ary designs which are t-designs for all t. We shall see some applications of the theory of p-ary designs to the representation theory of the symmetric group in a later chapter. Many of the results of this chapter appear in [Jol21]

4.1 Designs

The most general object studied in design theory is an incidence structure.

Definition 4.1.1. *An incidence structure is a triple $S = (\mathcal{P}, \mathcal{B}, \boldsymbol{I})$ consisting of three finite sets, with $\mathcal{P} \cap \mathcal{B} = \emptyset$ and $\boldsymbol{I} \subseteq \mathcal{P} \times \mathcal{B}$. The elements of $\mathcal{P}, \mathcal{B}$ and $\boldsymbol{I}$ are called* points, blocks, *and* flags *respectively.*

If $(p, B) \in \boldsymbol{I}$ is a flag then we say that p and B are *incident*. We may identify the blocks B with subsets of $\mathcal{P}$, but take care that it is possible that two blocks may be incident on the same set of points. If this does not happen, we call the incidence structure *simple*, otherwise we talk about the *multiplicity* of a block, the number of times a block containing the same set of points appears in $\mathcal{B}$. This allows us to use more geometric language and notation, replacing $(p, B) \in \boldsymbol{I}$ with $p \in B$ and saying that p is in (or on) B or that B passes through p. It also allows the set $\boldsymbol{I}$ to be omitted from the triple, as all of the information is encoded in the pair $(\mathcal{P}, B)$. The motivation for this language is clear from the natural example of an incidence structure, where $\mathcal{P}$ is a finite set of points in Euclidean space and $\mathcal{B}$ is the set of straight lines between them. By imposing certain regularity conditions on an incidence structure we obtain a combinatorial design.

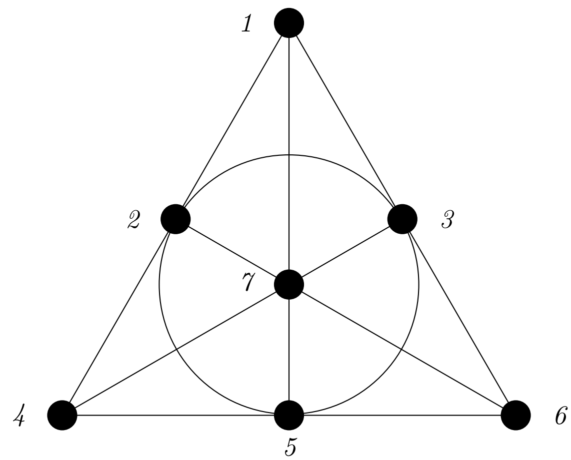
Definition 4.1.2. *Let $v \geq b \geq t \geq 0$ and $\mu \geq 0$ be integers. Denote by $[v]$ the set $\{1, 2, \ldots, v\}$. A* (integral) t-design of blocksize b on v points with coefficient μ, *a $t-(v, b, \mu)$-design is an incidence structure $\mathcal{D} = ([v], \mathcal{B})$ where each $B \in \mathcal{B}$ has $\mid B \mid= b$ and every set T of t points is contained in exactly μ blocks. If the coefficient of the design, μ, is zero then we shall say that $\mathcal{D}$ is a* null *design. We often omit the explicit reference to v, b, and μ and simply refer to $\mathcal{D}$ as a t-design.*

Example 4.1.3 (Constant design)**.** *The most obvious example of a $t-(v, b, \mu)$ design is the* constant design $\mathcal{C}_{[v],b}$*, a simple design whose blocks are all of the subsets of $[v]$ of size b. Each set of size t is contained in exactly $\binom{v-t}{b-t}$ blocks. A design whose blocks are all of the subsets of $[v]$ of size b, each appearing with the same multiplicity, c, may also be referred to as a constant design.*

Example 4.1.4 (Fano plane)**.** *Let $v = 7$ and let*

$$\mathcal{B} = \{\{1, 2, 4\}, \{1, 3, 6\}, \{1, 5, 7\}, \{4, 5, 6\}, \{2, 3, 5\}, \{3, 4, 7\}, \{2, 6, 7\}\}.$$

If we represent the points as nodes and the blocks as lines (including the circle) then this incidence system can be represented in the diagram below.



Observe that every subset of $[v]$ *of size* 2 *is contained in a unique block, and thus* $\mathcal{F} = ([v], \mathcal{B})$ *is a* $2-(7,3,1)$ *design. This design is known as the* Fano plane*.*

The Fano plane is an example of a projective plane, which in turn are examples of combinatorial 2-designs.

Definition 4.1.5. *A* projective plane *is an incidence structure satisfying:*

- *Every two blocks contain exactly one common point,*
- *Every two points lie in exactly one common block,*
- *There are four points such that no block contains more than two of them.*

The blocks of a projective plane are called lines*.*

The first two conditions are *dual* to each other, in the sense that if we interchanging the roles played by points and lines, then these conditions are interchanged. The final condition, which asserts the existence of a *quadrangle*, rules out the so-called *degenerate* projective planes, such as; the empty set, a single line with no points, or a single point which is incident on many lines, for example. It can be shown that if $(\mathcal{P}, \mathcal{B})$ is a projective plane then all of the blocks are of the same size. If all blocks are of size $n+1$, then we say that $(\mathcal{P}, \mathcal{B})$ is a projective plane of order n. The Fano plane is the unique projective plane of order 2, and there are unique projective planes of orders $3, 4, 5, 7$ and 8. The classification of projective planes is not known, and the only general restriction is that if $n = 1, 2(\mod 4)$ and n is not the sum of two squares then no projective plane of order n exists [BR49]. This rules out the existence of a projective plane of order 6, which also follows from the fact that Euler's 36 officer problem has no solution [BS59]. The non-existence of a projective plane of order 10 has been verified by computer calculation [Lam91].

Observe that the Fano plane $\mathcal{F}$ is also a 1-design, as all points are contained in exactly 3 blocks. Trivially $\mathcal{F}$ is a 0-design (with coefficient 7).

If a design, $\mathcal{D}$ is a j-design for all $j \leq t$ we will call it a $(v, b, \mu_0, \mu_1, \ldots, \mu_t)$-design, where μ_j the coefficient of $\mathcal{D}$ as a j-design. We call a $(v, b, \mu_0, \mu_1, \ldots, \mu_t)$-design a *null* if all of the coefficients μ_j are zero, and if $t = b - 1$ we call it a *universal* design. In particular, $\mathcal{F}$ is a $(7, 3, 7, 3, 1)$-design, and so is universal. Observe that the constant design is also a universal design. If a universal design has all of its coefficients 0 then we say that it is *universally null*, or when the context is clear *null*.

It is often convenient to have an algebraic method for recording a design. If $\mathcal{D} = ([v], \mathcal{B})$ is a design of block size b, then we shall represent $\mathcal{D}$ as a vector of length $\binom{v}{b}$ whose rows are indexed by the subsets of $[v]$ of size b. The entry in the row corresponding to a set A is the number of times A appears as a block in $\mathcal{B}$.

Example 4.1.6. *The Fano plane with the labelling above is represented by the following vector:*

$$\begin{matrix} \{1,2,4\} \\ \{1,3,6\} \\ \{1,5,7\} \\ \{4,5,6\} \\ \{2,3,5\} \\ \{3,4,7\} \\ \{2,6,7\} \\ \{1,2,3\} \\ \{1,2,4\} \\ \vdots \\ \{5,6,7\} \end{matrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

while the constant design is represented by $c\mathbf{1}_t$*, where* $\mathbf{1}_t$ *is the vector consisting of* $\binom{v}{t}$ *ones.*

This gives an algebraic interpretation of a t design. Denote by $A_t^b(v)$ the $\binom{v}{t} \times \binom{v}{b}$ matrix whose rows are indexed by the subsets of $[v]$ of size t and whose columns are indexed by subsets of $[v]$ of size b and where the entry

$$A_t^b(v)_{X,Y} = \begin{cases} 1 & \text{if } X \subseteq Y \\ 0 & \text{otherwise} \end{cases}.$$

We call this the *inclusion matrix*. Observe that if $\boldsymbol{d}$ is a vector corresponding to a

$t-(v,b,\mu)$ design then

$$A_t^b(v)\boldsymbol{d} = \mu\mathbf{1}_t. \tag{4.1}$$

This is as each entry counts the number of blocks containing a set X. This gives an algebraic interpretation of a design $t-(v,b,\mu)$ design as a non-negative integer solution of Equation (4.1). We shall abuse notation slightly here and refer to any integer solution as a design. This point of view allows some algebraic structure to be given to the set of all t-designs: the set of designs becomes a $\mathbb{Z}$-module, with addition and scalar multiplication naturally defined. We shall refer to this addition of designs as *superposition*. It is clear that the set of null designs is a submodule of the module of all designs. Moreover, the set of all $(v,b,\mu_0,\mu_1,\ldots,\mu_t)$ designs forms a $\mathbb{Z}$-module, which we will denote by $D_{t,b}$ and we shall denote by $N_{t,b} \subseteq D_{t,b}$ the submodule consisting of all null $(v,b,\mu_0,\mu_1,\ldots,\mu_t)$ designs.

From now on we will often identify a $t-(v,b,\mu)$ design $\mathcal{D}$ with the corresponding $\binom{v}{b}$ vector, $\boldsymbol{d}$.

Observe that

$$A_j^t(v)A_t^b(v) = \binom{b-j}{t-j}A_j^b(v),$$

If $\boldsymbol{d}$ is a $t-(v,b,\mu)$ design then

$$\begin{aligned} A_j^b(v)\boldsymbol{d} &= \frac{1}{\binom{b-j}{t-j}}A_j^t(v)A_t^b(v)\boldsymbol{d} \\ &= \frac{\mu_t}{\binom{b-j}{t-j}}A_j^t(v)\mathbf{1}_t \\ &= \frac{\binom{v-j}{t-j}}{\binom{b-j}{t-j}}\mu\mathbf{1}_j, \end{aligned}$$

and thus $\boldsymbol{d}$ is also a j-design. This observation forces a relationship between the coefficients of a universal design. Graver and Jurkat showed that this is the only obstruction to the existence of universal designs.

Theorem 4.1.7. *[GJ73] Let $b \le v$ be integers. There exists a $(v,b,\mu_0,\mu_1,\ldots,\mu_{b-1})$-design if and only if $\mu_{j+1} = \frac{s-j}{v-j}\mu_j$ for $0 \le j < b-1$.*

The necessity of these conditions has already been established. To prove the sufficiency we need the following result.

Theorem 4.1.8. *[GJ73, Section 4] Let $0 \le t < b \le v-t$. Then $A_{t+1}^b(v)N_{t,b} = N_{t,t+1}$.*

Remark. *It is clear that if $\boldsymbol{d} \in N_{t,b}$ then $A_{t+1}^b(v)\boldsymbol{d} \in N_{t,t+1}$ as*

$$A_t^{t+1}A_{t+1}^b(v) = (b-t)A_t^b.$$

Proof of Theorem 4.1.7. We have already seen that the conditions are necessary. We shall prove sufficiency of the conditions by induction on t, noting that if $t = 0$ then the design $\mathcal{D} = ([v], \mathcal{B})$, where $\mathcal{B}$ consists of μ_0 copies of the set $[b]$ is of the form we seek. Now assume that these conditions are sufficient for $t \geq 0$, and that $\mu_0, \mu_1, \mu_2, \ldots, \mu_{t+1}$ satisfy these conditions. Then there is some $(v, d, \mu_0, \mu_1, \mu_2, \ldots, \mu_t)$-design, $\boldsymbol{c'}$, of block size b. We shall construct $\boldsymbol{c}$ a $(v, d, \mu_0, \mu_1, \mu_2, \ldots, \mu_{t+1})$-design, of block size b. If $b \geq v - t$ then A_t^b is of full column rank and thus the only designs are multiples of the constant design. In this case $\boldsymbol{c'} = \alpha \mathbf{1}_b$ is also a $(v, d, \mu_0, \mu_1, \mu_2, \ldots, \mu_t, \mu'_{t+1})$-design. The relationship between the coefficients of the design established previously ensure that $\mu_{t+1} = \mu'_{t+1}$.

We now consider the case where $b < v - t$. Observe,

$$\begin{aligned} A_t^{t+1} A_{t+1}^b \boldsymbol{c'} &= (l-t) A_t^b \boldsymbol{c'} \\ &= (l-t)\mu_t \mathbf{1}_t \\ &= A_t^{t+1} \frac{l-t}{v-t} \mu_t \mathbf{1}_{t+1} \\ &= A_t^{t+1} \mu_{t+1} \mathbf{1}_{t+1}, \end{aligned}$$

thus $\boldsymbol{d'} := A_{t+1}^b \boldsymbol{c'} - \mu_{t+1}\mathbf{1}_{t+1} \in N_{t,t+1}$. By Theorem 4.1.8 there is a $\boldsymbol{d} \in N_{t,l}$ such that $A_{t+1}^l \boldsymbol{d} = \boldsymbol{d'}$. Setting $\boldsymbol{c} = \boldsymbol{c'} - \boldsymbol{d}$ we see that

$$A_{t+1}^l \boldsymbol{c} = A_{t+1}^l \boldsymbol{c'} - \boldsymbol{d'} = \mu_{t+1} \mathbf{1}_{t+1},$$

and the relationship between coefficients ensures this is a $(v, b, \mu_0, \mu_1, \mu_2, \ldots, \mu_t, \mu'_{t+1})$-design, as required. □

4.1.1 p-ary designs

Viewing designs as the solutions to Eq. (4.1) means it is natural to ask for designs over other rings. This section will be dedicated to designs which are solutions to Eq. (4.1) over the ring $\mathbb{Z}/p\mathbb{Z}$ where p is prime, also denoted $\mathbb{F}_p$. These were first studied by Wilson, [Wil09], who called them *p-ary designs*. Of course we may define these combinatorially, analogously to Definition 4.1.2:

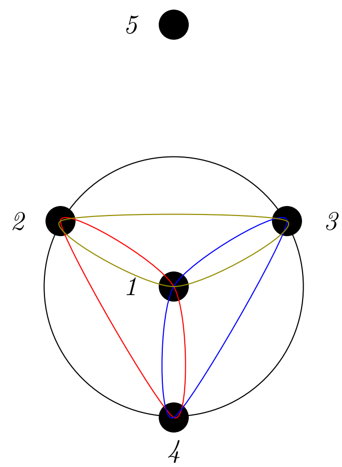
Definition 4.1.9. *Let $v \geq b \geq t \geq 0$ and $\mu \geq 0$ be integers. A* p-ary t-design of blocksize b on v points *is an incidence structure $\mathcal{D} = ([v], \mathcal{B})$ where each $B \in \mathcal{B}$ has $\mid B \mid= b$ and every set T of t points is contained in μ (mod p) blocks. We shall call μ the coefficient of the design, and if $\mu = 0$ then we shall say that $\mathcal{D}$ is a* null *design.*

Note that in this situation we may always choose $\mathcal{B}$ so that the multiplicities of the blocks are non-negative. Again we will usually identify a p-ary design $\mathcal{D}$ with the corresponding vector in $\mathbb{F}_p^{\binom{v}{b}}$. Of course, any integral t-design is necessarily a p-ary t-design for any prime p, although the converse is not true.

Example 4.1.10. *Let $v = 5$ and let*

$$\mathcal{B} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}.$$

If we represent the points as nodes and the blocks as curves, then this incidence system can be represented in the diagram below.



Observe that every subset of $[v]$ of size 2 containing 5 appears in no blocks, while each other pair is in exactly 2 blocks. Thus $\mathcal{D} = ([v], \mathcal{B})$ is not an integral t-design but it is a (null) 2-ary 2-design. The points $1, 2, 3,$ and 4 all appear in 3 (congruent to 1 $\mod 2$) blocks, while 5 appears in none, so $\mathcal{D}$ is not a 1-design, and therefore is not universal.

Proposition 4.1.11. *Let $\boldsymbol{d}$ be a $t-(v, b, \mu_t)$-design and let $j \leq b$ be such that $\binom{b-j}{t-j} \not\equiv 0\,(mod\ p)$. Then $\boldsymbol{d}$ is also a j-design, with coefficient*

$$\mu_j = \frac{\binom{v-j}{t-j}}{\binom{b-j}{t-j}}\mu_t.$$

Proof.

$$\begin{aligned}\binom{b-j}{t-j}A_j^b(v)\boldsymbol{d} &= A_j^t(v)A_t^b(v)\boldsymbol{d}\\ &= A_j^t(v)\mu_t\mathbf{1_t}\\ &= \binom{v-j}{t-j}\mu_t\mathbf{1_t}.\end{aligned}$$

The result follows since $\binom{b-j}{t-j}$ is invertible in $\mathbb{F}_p$. □

This shows that all integral $(b-1)$-(v, b, μ) designs are necessarily universal. Example 4.1.10 shows that this is not necessarily true for p-ary designs. In fact, Wilson [Wil09] showed that there are examples of t-designs which are not j designs whenever $\binom{b-j}{t-j} \equiv 0 \pmod p$.

We now investigate when universal p-ary designs can exist to seek to describe these designs.

4.1.2 Uniqueness of designs

Null universal designs are well understood via a James' kernel intersection theorem (Theorem 2.2.9) in the special case of two part partitions, which was proved in the context of the representation theory of the symmetric group, but is re-stated below in the language of designs.

Theorem 4.1.12 (James' Kernel Intersection Theorem)**.** *Let $X, Y \subseteq [v]$ with $|X| = |Y| = b$ and $X \cap Y = \emptyset$. Let $f : X \to Y$ be a bijection. Define*

$$\mathcal{D} = ([v], \mathcal{B})$$

by,

$$Z \in \mathcal{B} \text{ if } (Z \subseteq X \cup Y) \wedge (\forall x \in X)(x \in Z \implies f(x) \notin Z)$$

and the multiplicity of such a Z in $\mathcal{B}$ is $(-1)^{|Z \cap Y|}$.

Then $\mathcal{D}$ is a universally null p-ary design of blocksize b on $[v]$. Moreover any universally null p-ary design of blocksize b on $[v]$ is a linear combination of designs of this form.

Remark. *Theorem 4.1.12 gives us a basis for the module $N_{b-1,b}$. We observe here that the sum of the coefficients of each basis element given by Theorem 4.1.12 is zero, and thus there are no universally null integral designs whose blocks all appear with non-negative coefficient, other than the trivial design $([v], \emptyset)$.*

Example 4.1.13. *All universally null designs of blocksize 2 on [5] are linear combinations of the following designs.*

- $\mathcal{D}_{4,5} = ([5], \{\{1,2\}, \{4,5\}, -\{1,5\}, -\{2,4\}\})$
- $\mathcal{D}_{3,5} = ([5], \{\{1,2\}, \{3,5\}, -\{1,5\}, -\{2,3\}\})$
- $\mathcal{D}_{3,4} = ([5], \{\{1,2\}, \{3,4\}, -\{1,4\}, -\{2,3\}\})$

- $\mathcal{D}_{2,5} = ([5], \{\{1,3\}, \{2,5\}, -\{1,5\}, -\{2,3\}\})$
- $\mathcal{D}_{2,4} = ([5], \{\{1,3\}, \{2,4\}, -\{1,4\}, -\{2,3\}\})$

It remains to understand non-null universal designs; that is universal designs for which some coefficient is non-zero.

Definition 4.1.14. *Let $\boldsymbol{u}$ and $\boldsymbol{u}'$ be universal p-ary designs with coefficients μ_j and γ_j respectively . We say $\boldsymbol{u}$ and $\boldsymbol{u}'$ are similar if there is some $k \in \mathbb{F}_p$ such that $\mu_j = k\gamma_j$.*

If $\boldsymbol{u}$ and $\boldsymbol{u}'$ are similar, then the difference $\boldsymbol{u}' - k \cdot \boldsymbol{u}$ is a universally null design, and is of the form described in Theorem 4.1.12. Thus, to describe all universal designs it is suffices to describe universal designs up to similarity.

The existence, or otherwise, of non-null universal designs depends on number theoretic conditions on the integers v and b. Many of the results in the theory of p-ary designs involve determining whether certain binomial coefficients are 0 (mod p) or not, so we shall begin this section by stating some facts on divisibility of binomial coefficients, and making a number of definitions which give us the terminology to discuss the relationship between v and b.

Let $a = \sum_{i=0}^{\alpha} a_i p^i$ be the base p expansion of a; that is $0 \leq a_i \leq p-1$ and $a_\alpha \neq 0$. The p-adic valuation $\mathrm{val}_p(a)$ is the least i such that a_i is non-zero, we call α the p-adic length of a and write $l_p(a) = \alpha$.

Definition 4.1.15. *Let (a, b) be a two part partition, that is $a \geq b > 0$. We call a partition James if $val_p(a+1) > l_p(b)$, while if $b = p^\beta + \hat{b}$ and $\hat{b} < p^{val_p(a+1)} < p^\beta$ we call (a, b) pointed.*

So, for example, when $p = 5$ the partition $(4, 1)$ is James, the partition $(29, 26)$ is pointed, and the partition $(5, 1)$ is neither James nor pointed.

Lemma 4.1.16 (Kummer's Theorem)**.** *Let p be a prime and $a, b \in \mathbb{N}$, then $val_p(\binom{a+b}{b})$, the highest power of p that divides $\binom{a+b}{b}$, is the number of carries that occurs when a and b are added in their base p expansions.*

Lemma 4.1.17 (Lucas's Theorem)**.** *Let $a = \sum_{i=0}^{r} a_i p^i$ and $b = \sum_{i=0}^{r} b_i p^i$, with $0 \leq a_i, b_i \leq p-1$. Then*

$$\binom{a}{b} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\cdots\binom{a_r}{b_r} \; (mod\ p).$$

In particular, $\binom{a}{b} \equiv 0 \; (mod\ p)$ if and only if some $a_i < b_i$.

Corollary 4.1.18. *Let $a, b \in \mathbb{N}$. The binomial coefficients $\binom{a+1}{1}, \binom{a+2}{2}, \ldots \binom{a+b}{b}$ are all divisible by p if and only if $a \equiv -1 \; (mod\ p^{l_p(b)})$.*

Remark. *This gives an alternative characterisation of a James partition, in particular* $\lambda = (a, b)$ *is James if and only if* $a \equiv -1\,(mod\ p^{l_p(b)})$, *or equivalently* $l_p(b) < val_p(a+1)$.

The relationship in Proposition 4.1.11 puts conditions on the coefficients of a design. In light of this, to check a design is universal it suffices to check that it is a $(b-p^l)$-design for all $l \leq l_p(b)$.

Proposition 4.1.19. *Let* $v \geq b \geq 0$. *A design of blocksize* b *on* $[v]$ *is universal if and only if it is a* $(b-p^l)$*-design for all* $l \leq l_p(b)$.

Proof. Of course a universal design is a $(b-p^l)$-design. A $(b-p^l)$-design, is also a j design for all $j < b-p^l$ with $\binom{b-j}{b-p^l-j} \neq 0$; that is, for any j such that the sum $(b-j-p^l)+p^l$ has no carries in p-ary notation, by Lemma 4.1.16. This is precisely those j for which the coefficient of p^l in the p-ary expansion of $b-j$, which we shall denote $(b-j)_l$, is non zero. If $j < b$, then some $(b-j)_l \neq 0$, and as $\boldsymbol{u}$ is a $(b-p^l)$-design $\boldsymbol{u}$ is also a j-design by Proposition 4.1.11. □

Wilson has determined when non-null p-ary t-designs exist when $b \leq v-t$.

Theorem 4.1.20. *[Wil09] Let* $t \leq b \leq v-t$. *Then there is a non-null* p*-ary* t*-design of block size* b *if and only if*

$$\binom{b-i}{t-i} \equiv 0\,(mod\ p) \quad implies \quad \binom{v-i}{t-i} \equiv 0\,(mod\ p)$$

for all $i \leq t$.

Corollary 4.1.21. *There are non-null* p*-ary* $(b-p^l)$*-designs of blocksize* b *on* $[a+b]$ *if and only if* $a_l \not\equiv -1\,(mod\ p)$ *or* $b \leq p^{l+1}$.

Proof. By Theorem 4.1.20 a non-null $(b-p^l)$-design exists if

$$\binom{b-j}{p^l} \equiv 0\,(\text{mod } p) \quad \text{implies} \quad \binom{a+b-j}{a+p^l} \equiv 0\,(\text{mod } p).$$

If $a_l \equiv -1\,(\text{mod } p)$ and $b > p^{l+1}$ then setting $j = b-p^l$ we see that non-null designs can not exist. On the other hand if $b \leq p^{l+1}$ then $\binom{b-j}{p^l} \not\equiv 0\,(\text{mod } p)$ for all $j < b-p^l$ so there are non-null $(b-p^l)$-designs. Finally, if $a \not\equiv -1\,(\text{mod } p)$ then $\binom{b-j}{p^l} \equiv 0\,(\text{mod } p)$ whenever $(b-j)_l = 0$. If $(b-j)_l = 0$ then the sum $(a+p^l)+(b-j-p^l)$ necessarily has a carry in p-ary notation, so $\binom{a+b-j}{a+p^l} \equiv 0\,(\text{mod } p)$ by Lemma 4.1.16. □

Combining this with the relationship between coefficients established in Proposition 4.1.11, we obtain more integers j for which a universal design for (a, b) is null.

Proposition 4.1.22. *Let $a \geq b$. If a universal design of blocksize b on $[a+b]$, $\boldsymbol{u}$, is non-null as a j-design, then $(b-j)_m + a_m < p$ for all $m < l_p(b)$.*

Proof. Suppose $\boldsymbol{u}$ is non-null as a j-design with coefficient μ_j, and let $m < l_p(b)$ be such that $(b-j)_m \neq 0$. As $\boldsymbol{u}$ is non-null for j, we must have $\boldsymbol{u}$ is non-null for $b - p^m$, by Proposition 4.1.11, as

$$\mu_j = \frac{\binom{a+b-j}{b-p^m-j}}{\binom{b-j}{b-p^m-j}} \mu_{b-p^m}.$$

For $\boldsymbol{u}$ to be non-null as a j-design, we must have $\binom{a+b-j}{b-p^m-j} \neq 0$. Corollary 4.1.21 ensures that $a_m \not\equiv -1 \pmod{p}$ and thus $(a+p^m) + (b-j-p^m)$ having no carries is equivalent to $(a) + (b-j)$ having no carries. Using Lemma 4.1.16 we see that if $\boldsymbol{u}$ is non-null then $(a) + (b-j)$ has no carries, and therefore $(b-j)_m + a_m < p$ for all $m < l_p(b)$. □

Our next goal is to determine what the relationship is between the non-zero coefficients of a universal design. Let $\boldsymbol{u}$ be a universal design on $[a+b]$ with blocksize b, and let X be the set of all j with $(b-j)_m + a_m < p$ for all $m < l_p(b)$. Observe if $j \notin X$ then $\boldsymbol{u}$ must be a null j-design, and so X contains all j such that $\boldsymbol{u}$ is a non-null j-design. We shall define a partial ordering on X by setting $i \geq_X j$ if $i > j$ and $\binom{b-j}{i-j} \not\equiv 0 \pmod{p}$. If $i \geq_X j$ and μ_i and μ_j are the coefficients of $\boldsymbol{u}$ corresponding to i and j respectively, then $\mu_i = \frac{\binom{a+b-i}{j-i}}{\binom{b-i}{j-i}} \mu_j$. Following [Sch03] we define a connected component of a poset to be a connected component of the underlying graph, whose vertices are the elements of the poset, and whose (undirected) edges indicate that two elements are related. If two elements, i and j, are in the same connected component of X there is a relationship between the coefficients μ_i and μ_j obtained be following a path in the graph between i and j and repeated application of Proposition 4.1.11. The coefficients of a universal design are determined by the coefficients on each connected component, so the structure of X gives restrictions on the possible coefficients of a universal design.

Proposition 4.1.23. *If $\lambda = (a,b)$ is James, then X has a single connected component.*

Proof. Recall if λ is James then $b < p^{\mathrm{val}_p(a+1)}$, and $a_m \equiv -1 \pmod{p}$ for all $m < l_p(b)$. Write $b = \alpha p^\beta + \hat{b}$, where $\beta = l_p(b)$ and $\hat{b} < p^\beta$, and observe, by Proposition 4.1.22, that $X = \{\hat{b}, p^\beta + \hat{b}, \ldots, (\alpha-1)p^\beta + \hat{b}\}$, which, by Lemma 4.1.17, is a single connected component. □

Proposition 4.1.24. *If $\lambda = (a,b)$ is not James, and $b = \alpha p^\beta + \hat{b}$ then X has a single connected component, unless λ is pointed, in which case X has two connected components, one of which consists only of the element $\hat{b}$.*

Proof. Observe that $i, j \in X$ are comparable if and only if $(b-i)_m \leq (b-j)_m$ for all $m \leq l_p(b)$, or $(b-j)_m \leq (b-i)_m$ for all $m \leq l_p(b)$. Observe also that $(b-i)_m = 0$ for all $m < l_p(b)$ for which $a_m \equiv -1 \mod p$. The join of $i, j \in X$, if it exists, is the element $i \vee j = x$ such that $(b-x)_m = \max\{(b-i)_m, (b-j)_m\}$, the meet, $y = i \wedge j$, is the element y such that $(b-y)_m = \min\{(b-i)_m, (b-j)_m\}$. These may fail to be in X as it may be that $(b-x) > b$ or $b - y = 0$. If, however, $(b-i)_m$ and $(b-j)_m$ are both non-zero for some m then $i \wedge j \in X$.

Let x be such that $(b-x)_m = p - 1 - a_m$ for $m < \beta$ and $(b-x)_\beta = \alpha - 1$, and observe that $x \in X$ by Proposition 4.1.22. Clearly $j \in X$ with $j > \hat{b}$ is comparable to x. If $j < \hat{b} \in X$, or if $j = \hat{b}$ and $\alpha \neq 1$ then $x \wedge j \in X$.

It only remains to consider the case where $j = \hat{b}$ and $\alpha = 1$, which, if $\hat{b} > p^{\mathrm{val}_p(a+1)}$ is clearly comparable to $\hat{b} - p^{\mathrm{val}_p(a+1)}$, which is in the same component as x. It follows that if λ is not pointed then there is only one connected component of X.

On the other hand, when λ is pointed $\hat{b}$ is not comparable to any other element and thus is in a connected component of its own. This is as no $j < \hat{b}$ is in X as no $j < \hat{b}$ has $(b-j)_m = 0$ for all $m < l_p(b)$ where $a_m \equiv -1 \mod p$. Similarly no $j > \hat{b}$ has $(b-j)_\beta \geq 1$, so j and $\hat{b}$ are incomparable . □

We can now state the main result of this chapter, an existence and uniqueness theorem for universal p-ary designs.

Theorem 4.1.25. *Let $a \geq b \geq 0$ and let $\boldsymbol{u}$ be a non-null universal p-ary design of blocksize b on $[a+b]$. If (a,b) is neither pointed or James, then $\boldsymbol{u}$ is similar to the constant design. If (a,b) is James then $\boldsymbol{u}$ is unique up to similarity, while if (a,b) is pointed, and $b = p^\beta + \hat{b}$ then $\boldsymbol{u}$ can be written as the sum of two universal designs, $\boldsymbol{u} = \boldsymbol{u'} + \boldsymbol{c}$, where $\boldsymbol{u'}$ is non-null only as a $\hat{b}$ -design, while $\mathbf{c}$ is similar to the constant design.*

Proof of uniqueness in Theorem 4.1.25. If $\boldsymbol{u}$ is a universal p-ary design of blocksize b on $[a+b]$, then its coefficients are entirely determined by the connected components of X, thus an understanding of this poset allows us to determine the possible coefficients of designs. If (a,b) is not pointed, then non-null universal designs, if they exist, are unique up to similarity, while if (a,b) is pointed, then any design must be the sum of two designs, uniquely determined by its coefficients on each of the two connected components of X. □

4.1.3 Existence of designs

In the previous section we have seen a complete characterisation of universal null p-ary designs of blocksize b and described, up to similarity, the uniqueness of non-null universal designs. We now move to considering the existence of non-null universal p-ary designs of blocksize b on $[a+b]$. We first consider p-ary designs which come from the modulo p

reduction of integral designs. Clearly the constant design, $\mathcal{C}_{[a+b],b}$, is an integral design, with coefficients $\mu_i = \binom{a+b-i}{b-i}$, and therefore is null if and only if (a, b) is James.

Proposition 4.1.26. *Let $a \geq b$ be such that the partition (a, b) is neither James nor pointed. Then the constant design $\mathcal{C}_{[a+b],b}$, is the unique, up to similarity, universal p-ary design on $[a + b]$ of blocksize b.*

We shall begin this section by giving examples of non-null universal designs when (a, b) is James and when (a, b) is pointed. These will illustrate the more general constructions to follow.

Example 4.1.27. *Let $p = 2$ $a = 3$ and $b = 2$. Then the partition $(3, 2)$ is James. Let*

$$\mathcal{B} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

Each singleton appears as a subset of either 2, or 0 blocks, so $\mathcal{D} = ([5], \mathcal{B})$ is null as a p-ary 1-design. The empty set is a subset of all 3 blocks, so $\mathcal{D}$ is a non-null universal 2-ary design.

Example 4.1.28. *Let $a = b = 3$. When $p = 3$ the partition $(3, 3)$ is pointed. Let*

$$\mathcal{B} = \{B \subseteq [6] \mid (1 \notin B) \wedge (\mid B \mid = 3)\}.$$

First, observe that $\mathcal{D} = ([6], \mathcal{B})$ is a null 3-ary 2-design, as any pair containing 1 is in no blocks and every other pair is in exactly 3 blocks. Similarly, $\mathcal{D} = ([6], \mathcal{B})$ is a null 3-ary 1-design, as there are no blocks containing the element 1, and exactly 6 blocks containing any other point. The total number of blocks is 10, which is congruent to 1 mod p, thus $\mathcal{D}$ is a 3-ary $(6, 3, 1, 0, 0)$ design.

Proposition 4.1.29. *Let $a \geq b$, then there exists an integral design which is not similar to the constant design if and only if (a, b) is James.*

Proof. Theorem 4.1.7 ensures there is an integral universal design with coefficients μ_j as long as

$$\mu_j = \frac{\binom{a+b-j}{a}}{\binom{a+b}{a}} \mu_0.$$

Such a design restricts to a universal p-ary design which is not universally null if and only if some $\mu_i \not\equiv 0 \pmod{p}$. To ensure this we must take $\mu_s = k\frac{\binom{a+b-s}{a}}{p^d}$ where $k \in \mathbb{F}_p$ is non-zero and d is the least power of p dividing some $\binom{a+b-s}{a}$ for $s \in \{0, 1, \ldots, b - 1\}$. That is, $d = \min_{s<b}\{\mathrm{val}_p\binom{a+b-s}{b}\}$.

Observe that

$$A_j^b \boldsymbol{c}_{[a+b],b} = \binom{a+b-j}{b-j} \cdot \mathbf{1}_j$$
$$= k^{-1} p^d \mu_j \cdot \mathbf{1},$$

so the coefficients of the constant design are $k^{-1}p^d\mu_j$. If p^d is a unit in $\mathbb{F}_p$, that is if $d = 0$, then $\boldsymbol{u}$ is similar to the constant design. On the other hand if $d \geq 1$ then the constant design is universally null as a p-ary design and is not similar to $\boldsymbol{u}$. This means u is similar to the constant design if and only if $p \mid \binom{a+b-j}{a}$ for all $j \in \{0, 1, \ldots, b-1\}$, which by Corollary 4.1.18 is if and only if λ is James. □

Theorem 4.1.30. *The unique, up to similarity, universal p-ary design of blocksize b on $[a+b]$ when (a, b) is a James partition is the modulo p reduction of the integral design with coefficients $\mu_s = \frac{\binom{a+b-s}{a}}{p^d}$ where $d = min_{s<b}\{val_p\binom{a+b-s}{b}\}$.*

We have seen that if (a, b) is pointed then the constant design $\boldsymbol{c}_{[a+b],b}$ is non-null. We shall now construct another non-null design for (a, b) which is not similar to the constant design, completing the classification.

Proposition 4.1.31. *Let $a \geq b$ be such that $b = p^\beta$ and $val_p(a+1) < \beta$. Then there exists a universal p-ary design of blocksize b on $[a+b]$ which is null as a t-design for all $t > 0$ and non-null as a 0-design.*

Proof. Let $m = a - b + 1$ and define

$$\mathcal{B} = \{Y \mid (Y \cap [m] = \emptyset) \wedge (\mid Y \mid = b)\}.$$

Then any $Z \subseteq [a+b]$ with $|Z| = s$ is contained in exactly $\binom{a+b-m-s}{b-s}$ blocks of $\mathcal{B}$ if $Z \cap [m] = \emptyset$ and in no blocks otherwise. By our choice of m the coefficients are $\binom{a-m+1}{1}, \binom{a-m+2}{2}, \ldots, \binom{a-m+b-1}{b-1}$ are all divisible by p, by Corollary 4.1.18. Then $u = ([a+b], \mathcal{B})$ is a universal design which is non-null only as a 0-design. □

Let $u = ([a+p^\beta], \mathcal{B})$ be the design of blocksize p^β on $[a+p^\beta]$ constructed above. We shall modify u to construct a design for a pointed partition $(a, p^\beta + \hat{b})$, where $\hat{b} < p^{\mathrm{val}_p(a+1)} < p^\beta$, which is non-zero only as a $\hat{b}$-design.

Let $Y = \{a + p^\beta + 1, \ldots, a + b\}$, then Y is a set of size $\hat{b}$. Let $b = p^\beta + \hat{b}$ and $v = a + b$.

Define $\mathcal{B}_Y$ by

$$\{Z \backslash Y \mid (Z \in \mathcal{B}) \wedge (Y \subseteq Z)\},$$

and $\mathcal{B}^Y$ by

$$\mathcal{B}^Y = \{Z \mid Y \cap Z = \emptyset\},$$

and let $u^Y = ([v], \mathcal{B}^Y)$ and $u_Y = ([v], \mathcal{B}_Y)$

Given a subset $X \subset [v]$ we denote by δ_X the incidence system $\delta_X = ([v], X)$. Of course, this incidence system may not be a design, but we may treat them as vectors of length $\binom{v}{|X|}$. Consider $A^b_j u_Y$, by grouping terms by the size of their intersection with Y. First, consider the case where $\hat{b} < j < b$:

$$A^b_j(v)u_Y = (A^b_{j-\hat{b}}(v)u)_Y + \sum_{y \in Y}(A^b_{j-\hat{b}-1}(v)u)^{\{y\}}_{Y\backslash\{y\}} + \cdots + (A^b_j(v)u)^Y.$$

Each of these terms is 0, by our choice of u, so $A^{p^\beta+\hat{b}}_j(v)u_Y$ is 0.

Similarly for $j \leq \hat{b}$

$$\begin{aligned} A^b_j(v)u_Y &= \sum_{i=0}^{j} \sum_{|Y' \cap Y| = i} A^b_{j-i}(v)(u)_{Y'} \\ &= \sum_{|Y' \cap Y| = j} (A^b_0(v)u)_{Y'} \\ &= \mu_0 \cdot A^b_j(v)\delta_Y, \end{aligned}$$

where $\mu_0 \neq 0$ is the coefficient of u as a 0-design.

Observe that if Y is any subset of $[v]$, not necessarily $\{a + p^\beta + 1, \ldots, a + b\}$, then we may define u_Y as before, by first defining u on subsets of $[a + b]\backslash Y$ of size p^β.

Let $X \subseteq [a + b]$ of size $b - 1 = p^\beta + \hat{b} - 1$. Define $u_{\overline{X}} := \sum_{Y \subseteq X} u_Y$. Then

$$A^b_j(v)u_{\overline{X}} = \sum_{Y \subseteq X} A^b_j(v)u_Y,$$

which is 0 when restricted to sets of size j if $\hat{b} < j < b$. When $j \leq \hat{b}$,

$$\begin{aligned} A^b_j(v)u_{\overline{X}} &= \sum_{Y \subseteq X} A^b_j(v)\widehat{u_Y} \\ &= \sum_{Y \subseteq X} \mu_0 \cdot A^b_j(v)\delta_Y \\ &= \binom{p^\beta - 1 + \hat{b} - j}{\hat{b} - j}\mu_0 \cdot A^b_j(v)\delta_X, \end{aligned}$$

which is 0 if $j \neq \hat{b}$. So

$$A^b_{\hat{b}}u_{\overline{X}} = \mu_0 \cdot A^b_{\hat{b}}\delta_X.$$

If $\mathcal{U}$ is a non null p-ary $\hat{b}$-design of block size $b-1$ and coefficient α then setting

$$u_{\mathcal{U}} := \sum_X \mathcal{U}(X) u_{\overline{X}},$$

where the sum is over all sets X of size $b-1$ and $\mathcal{U}(X)$ is the multiplicity of$X$ as a block in the $\hat{b}$-design $\mathcal{U}$, we see

$$\begin{aligned} A^b_{\hat{b}} u_{\mathcal{U}} &= \sum_X \mathcal{U}(X) A^b_{\hat{b}} u_{\overline{X}} \\ &= \sum_X \mathcal{U}(X)\mu_0 \cdot A^b_{\hat{b}} \delta_X \\ &= \alpha\mu_0 \cdot A^b_{\hat{b}} \delta_X, \end{aligned}$$

and of course

$$A^b_j u_{\mathcal{U}} = 0$$

for all $j \neq \hat{b}$.

Theorem 4.1.32. *Let $a \geq b$ be such that $b = p^\beta + \hat{b}$ and $\hat{b} < p^{val_p(a+1)} < b$. Then there is a universal design of blocksize b on $[a+b]$ which is non-null only as a $\hat{b}$-design.*

Proof. An incidence system of the form $u_{\mathcal{U}}$ as described above is such a design, it remains to prove such an element exists; that is that there is a non null p-ary $\hat{b}$-design of block size $b-1$. By Theorem 4.1.20, we may construct such a design if (and only if) $\binom{a+b-1-i}{\hat{b}-i} \equiv 0 \,(\mathrm{mod}\ p)$ whenever $\binom{b-1-i}{\hat{b}-i} \equiv 0 \,(\mathrm{mod}\ p)$. Of course $\binom{b-1-i}{\hat{b}-i} = \binom{p^\beta+\hat{b}-1-i}{\hat{b}-i} \equiv 0 \,(\mathrm{mod}\ p)$ for all $i < \hat{b}$, so it remains to see that $\binom{a+b-1-i}{\hat{b}-i} \equiv 0 \,(\mathrm{mod}\ p)$ for all $i < \hat{b}$; that is, that $\binom{a+p^\beta+j}{j} \equiv 0 \,(\mathrm{mod}\ p)$ for all $j < \hat{b}$. This follows from Corollary 4.1.18, as $a + p^\beta \equiv -1 \,(mod\ p^{l_p(\hat{b})})$. ☐

Proof of existence in Theorem 4.1.25. By Proposition 4.1.29, if (a, b) is James, then we can construct an integral design of blocksize b on $[a + b]$ whose coefficients are not all 0 (mod p) following Theorem 4.1.7 [GJ73]. The reduction mod p of this design is a non-null universal p-ary design which is not similar to the constant design (which is universally null in this case). If (a, b) is not James then the constant design is an example of a non-null universal design. Finally, if (a, b) is pointed then Theorem 4.1.32 gives a non-null universal design, completing the proof of Theorem 4.1.25. ☐

This result demonstrates one of the key differences between the theory of integral designs and the theory of p-ary designs. While integral $(b-1)$-designs of blocksize b are necessarily universal and always exist, they are always similar to the constant design. Universal designs over $\mathbb{F}_p$, however, are not all similar to the constant design.

The theory of p-ary designs has been little studied, other than the work of Wilson, and the work in this chapter. The main motivation for this work was the relationship between universal designs and certain extensions of the trivial $k\mathcal{S}_n$-module by a Specht module over a fields of positive characteristic, as described in a later chapter, hence the interest in designs over fields of positive characteristic, and in particular over $\mathbb{Z}/p\mathbb{Z}$. From this point of view the problem is entirely settled, however, there are a number of design theoretic problems which remain open as any question that can be asked about integral designs has a p-ary design analogue. For example one may wish to find simple universal p-ary block designs, those where we restrict the design to taking the values 0 or 1, although the main result of this paper reduces this to the computation task of finding 0-1 vectors amongst the vectors which correspond to universal p-designs, a problem which is NP-complete. Similarly one may wish to find designs of smallest support, that is those designs who take non-zero values on the fewest sets. Wilson [Wil09], subject to some additional hypothesis, gives a lower bound for the size of the support of a p-ary t-design, which we would like to see extended to universal designs. Again, the results of this paper reduce this problem to a computational problem- that of finding vectors of smallest support within a vector space- however a precise description would be of interest.

The most obvious question one would like to answer is how to extend this work to designs modulo n, without the restriction that n is prime. Designs over $\mathbb{Z}/n\mathbb{Z}$, or n-ary designs, are yet to receive any attention, however many of the techniques used in this chapter can be used in their study. The goal should be to develop a theory of n-ary designs, analogous to the theory of integral designs, with an existence and uniqueness theorem extending the main result of this chapter.

Chapter 5

Cohomology

In this chapter we will investigate the cohomology of Specht modules for the symmetric group, and we assume the reader is familiar with the concepts of cohomology. We refer the reader to the books of Benson [Ben98] [Ben91], which contain far more than we need for this chapter. The zeroth cohomology is well understood due to James [Jam77], and recent work of Donkin and Geranios determined the dimension of the first degree cohomology [DG19]. The work of Donkin and Geranios was done in the setting of algebraic groups. We seek to understand first degree cohomology by studying only the representation theory of the symmetric group. This will see an application of the theory of combinatorial designs, introduced in Chapter 3, to the representation theory of the symmetric group. This chapter begins with a brief review of cohomology and then will see the methods of Donkin and Geranios. We then re-derive many of their results, using only combinatorial methods. In particular we give an explicit description of the extensions of Specht modules by the trivial module in the case of two part partitions, and are able to give bounds on the dimension of the first cohomology, including a number of far reaching conditions for this to br trivial. Many of the results in this chapter also appear in [Jol20a].

5.1 Cohomology

Recall from the previous chapter on designs, that a two part partition $\lambda = (a, b)$ is called James if $\mathrm{val}_p(a+1) > l_p(b)$. We extend this definition to general partitions and say that $\lambda = (\lambda_1, \ldots, \lambda_r)$ is James if $(\lambda_i, \lambda_{i+1})$ is James for all $1 \leq i < r$. For example the partition $\lambda = (24, 9, 4, 3)$ is James for $p = 5$, but is not James for any other prime. It follows from Lucas's Theorem, Lemma 4.1.17, that $\lambda = (\lambda_1, \ldots, \lambda_r)$ is James if and only if $\lambda_i \equiv -1($ mod $p^{l_p(\lambda_{i+1})})$ for all $1 \leq i < r$.

Given a finite group G and a kG-module M the 0th cohomology $H^0_k(G, M)$ may be identified with the group of G-invariant elements of M, denoted M^G. This means that in order to calculate the cohomology $H_k^0(\mathcal{S}_n, S^\lambda)$ it suffices to determine which elements of the Specht module are invariant under the action of $\mathcal{S}_n$. Denote by $f_\lambda \in M^\lambda$ the sum of all tabloids. Observe that f_λ is invariant under the action of $\mathcal{S}_n$. In fact, f_λ and its scalar multiples are the only $\mathcal{S}_n$-invariant elements of the permutation module M^λ:

$$(M^\lambda)^{\mathcal{S}_n} = \langle f^\lambda \rangle.$$

As the Specht module S^λ is a submodule of M^λ, calculating $(S^\lambda)^{\mathcal{S}_n}$ is as simple as determining whether or not $f_\lambda \in S^\lambda$:

$$(S^\lambda)^{\mathcal{S}_n} = \begin{cases} \langle f^\lambda \rangle & \text{if } f_\lambda \in S^\lambda \\ 0 & \text{otherwise} \end{cases}.$$

James' Kernel Intersection Theorem, Theorem 4.1.12 gives a combinatorial way of testing membership of Specht modules. It follows that

Theorem 5.1.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$. We have*

$$(S^\lambda)^{\mathcal{S}_n} = \begin{cases} \langle f^\lambda \rangle & \textit{if } \lambda \textit{ is James,} \\ 0 & \textit{otherwise} \end{cases}.$$

If we restrict our attention to two part partitions, $\lambda = (a, b)$, then identifying the tabloid

$$\{t\} = \begin{array}{cccc} \hline x_1 & \cdots & \cdots & x_a \\ \hline y_1 & \cdots & y_b & \\ \hline \end{array}$$

with the set $Y := \{y_1, \cdots, y_b\}$ allows us to restate this theorem in the language of designs from Chapter 4:

Corollary 5.1.2. *Let $\lambda = (a, b)$ and let $\mathcal{C}_{[a+b],b}$ be the constant design.*

$$(S^\lambda)^{\mathcal{S}_n} = \begin{cases} \langle \mathcal{C}_{[a+b],b} \rangle & \textit{if } \mathcal{C}_{[a+b],b} \textit{ is a universally null design,} \\ 0 & \textit{otherwise} \end{cases}.$$

Recall, in this setting the Specht module $S^{(a,b)}$ can be identified with the set of universally null designs (of blocksize b on $[a + b]$).

We shall now present two methods for calculating the first cohomology $H_k^1(\mathcal{S}_n, S^\lambda) = \mathrm{Ext}^1_{k\mathcal{S}_n}(S^\lambda, k)$.

5.2 An algebraic approach

The approach outlined in this section is that taken in a recent paper of Donkin and Geranios [DG19], who related the cohomology of the Specht modules to the cohomology of the general linear group. The purpose of this section is to give an idea of the algebraic approach in order to contrast it with the combinatorial approach that will be taken in the next section. We refer the reader to the paper of Donkin and Geranios for more details. Throughout this section k is an algebraically closed field of characteristic $p > 0$ and GL_n denotes the general linear group $GL_n(k)$.

Let $B_n \leq GL_n$ be the Borel subgroup of GL_n consisting of lower triangular matrices, and let T_n be the subgroup consisting of diagonal matrices. If $\lambda \vdash r$ is a partition with at most n parts then we denote by k_λ the one-dimensional B_n-module of weight λ and $\nabla(\lambda) = \mathrm{Ind}_{B_n}^{GL_n} k_\lambda$. Extensions of these modules are related to extensions by Specht modules by the following:

Theorem 5.2.1. *Let $n \geq r$. If $i = 0$ and $p \geq 3$, or $i = 1$ and $p \geq 5$, then*

$$\mathrm{Ext}^i_{GL_n}(\nabla(\mu), \nabla(\lambda)) = \mathrm{Ext}^i_{\mathcal{S}_r}(S^\mu, S^\lambda).$$

The natural GL_n module will be denoted by E and $S^r E$ is its rth symmetric power, or equivalently $S^r E = \nabla((n))$. When considering this module Donkin and Geranios are able to go slightly further, and show:

Theorem 5.2.2. *Let $\lambda \vdash r$ have no more than n parts, then*

$$\dim(H^1(\mathcal{S}_r, S^\lambda)) \geq \dim(\mathrm{Ext}^1_{GL_n}(S^r E, \nabla(\lambda))),$$

with equality when $p \geq 3$.

In particular, they reduced calculating the first cohomology of Specht modules to determining extensions of symmetric powers of the natural module. This can be reduced further, to a similar question about B_n-modules as follows:

Theorem 5.2.3. *[CPSK77][II, 4.7, Corollary a] For $i \geq 0$ we have*

$$\mathrm{Ext}^i_{GL_n}(S^r E, \nabla(\lambda)) = \mathrm{Ext}^i_{B_n}(S^r E, k_\lambda).$$

The paper is then dedicated to determining the extensions $\mathrm{Ext}^i_{B_n}(S^r E, k_\lambda)$ which they are able to do for all partitions.

5.3 A combinatorial approach

A purely combinatorial method for computing the first degree cohomology for a Specht module, similar to the approach taken by James in computing the zeroth degree cohomology (Theorem 5.1.1), was suggested by Hemmer [Hem12]. It is based on the observation that, like the zeroth cohomology, the first cohomology of a Specht module S^λ is controlled by M^λ.

Theorem 5.3.1. *[Hem12, Theorem 3.2] Let p be an odd prime and suppose U is a non-split extension of S^λ, then U is isomorphic to a submodule of M^λ.*

Proof. Applying $Hom_{k\mathcal{S}_n}(-, M^\lambda)$ the short exact sequence

$$0 \to S^\lambda \xrightarrow{f} U \to k \to 0,$$

we obtain the long exact sequence

$$0 \to \mathrm{Hom}_{k\mathcal{S}_n}(k, M^\lambda) \to \mathrm{Hom}_{k\mathcal{S}_n}(U, M^\lambda) \xrightarrow{\tilde{f}} \mathrm{Hom}_{k\mathcal{S}_n}(S^\lambda, M^\lambda) \to \mathrm{Ext}^1_{k\mathcal{S}_n}(k, M^\lambda) \to \cdots .$$

Observe that since $p \neq 2$ there are no non-split extensions of the trivial $\mathcal{S}_n$-module k by itself. It follows from the Eckmann-Shapiro lemma, Theorem 2.1.2, that $\mathrm{Ext}^1_{k\mathcal{S}_n}(k, M^\lambda) = 0$. Similarly, by Frobenius reciprocity, $\mathrm{Hom}_{k\mathcal{S}_n}(k, M^\lambda) = \mathrm{Hom}_{k\mathcal{S}_\lambda}(k, k) \cong k$. A result of James, [Jam78, Corollary 13.17], shows that $\mathrm{Hom}_{k\mathcal{S}_n}(S^\lambda, M^\lambda) \cong k$.

Thus we have a short exact sequence

$$0 \to k \to \mathrm{Hom}_{k\mathcal{S}_n}(U, M^\lambda) \xrightarrow{\tilde{f}} \mathrm{Hom}_{k\mathcal{S}_n}(S^\lambda, M^\lambda) \cong k \to 0.$$

As $\tilde{f}$ is surjective, the unique inclusion $\iota : S^\lambda \to M^\lambda$ lifts to a map $g \in \mathrm{Hom}_{k\mathcal{S}_n}(U, M^\lambda)$ satisfying $\iota = g \circ f$. The restriction of g to S^λ is surjective. Observe that the image of g contains S^λ, and thus if g is not injective then $g \circ f$ is the identity on S^λ and thus

$$0 \to S^\lambda \xrightarrow{f} U \to k \to 0$$

is split, which is a contradiction. □

Remark. *We do need the assumption that $p = 2$ here, as the fact there is a non-split extension of the trivial module $k = S^{(n)} = M^{(n)}$ by itself is both a contradiction to this result, and the failure of a crucial step in the proof.*

In light of this result, when attempting to construct extensions of trivial module by S^λ it suffices to restrict our attention to submodules of M^λ. Constructing such a non-split extension U is equivalent to finding a vector $u \in M^\lambda \backslash S^\lambda$ such that the subspace $\langle S^\lambda, u \rangle$ is the required module. Hemmer [Hem12] gave combinatorial conditions that determine when such a u can exist:

Theorem 5.3.2. *[Hem12, Theorem 3.6] Let $p \geq 3$ and $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, then $Ext^1(k, S^\lambda) \neq 0$ if and only if there is an element $u \in M^\lambda$ with the following properties:*

1. *For each $1 \leq i < r$ and $0 \leq v < \lambda_i$, $\psi_{i,v}(u) \in M^{\lambda'}$ is a multiple of $f_{\lambda'}$, at least one of which is a non-zero multiple.*
2. *There does not exist a scalar $c \in k$ such that all the $\psi_{i,v}(c \cdot f_\lambda - u) = 0$.*

If such a u exists then the subspace $\langle S^\lambda, u \rangle \subseteq M^\lambda$ spanned by S^λ and u is a non-split extension of the Specht module S^λ by the trivial module.

We will call an element $u \in M^\lambda$ satisfying the above conditions *Hemmer*.

Proof. Suppose that $\text{Ext}^1(k, S^\lambda) \neq 0$. By Theorem 5.3.1 we can find a submodule of M^λ which is extension of S^λ by k. This module must be of the form $\langle S^\lambda, u \rangle$ for some $u \in M^\lambda$. For any $\sigma \in \mathcal{S}_n$, we have $\sigma \cdot u = u + v$ for some $v \in S^\lambda$, and as the Specht module is in the kernel of all of the maps $\psi_{i,v}$, by the kernel intersection theorem Theorem 2.2.9

$$\sigma \cdot \psi_{i,v}(u) = \psi_{i,v}(u).$$

Thus $\psi_{i,v}(u)$ is a fixed point of $\mathcal{S}_n$ and therefore is a multiple of $f_{\lambda'}$. As $u \notin S^\lambda$ at least one of these $\psi_{i,v}(u)$ is non-zero, and thus (1) holds. Suppose that some $c \in k$ was such that $\psi_{i,v}(c \cdot f_\lambda - u) = 0$ for all $1 \leq i < r$ and $0 \leq v < \lambda_i$. Then, by the kernel intersection theorem $c \cdot f_\lambda - u \in S^\lambda$. It cannot be that $c \cdot f_\lambda \in S^\lambda$, as otherwise $u \in S^\lambda$, so $f_\lambda \in U$. Of course, f_λ spans a fixed subspace of M^λ (and hence of U) and thus U is split, a contradiction.

Conversely, suppose there is a u which satisfies (1) and (2). Condition (1) ensures that $U = \langle S^\lambda, u \rangle \subseteq M^\lambda$ is a submodule of M^λ with $U/S^\lambda \cong k$:

$$\psi_{i,v}(\sigma \cdot u - u) = \sigma \cdot \psi_{i,v}(u) - \psi_{i,v}(u) = 0,$$

thus $\sigma \cdot u - u \in S^\lambda$.

Condition (2) ensures that U is non-split. □

The second condition is automatic when $H^0(\mathcal{S}_n, S^\lambda) \neq 0$, by Theorem 5.1.1. In [Web13], Weber uses this method to give a far reaching combinatorial condition which sufficient for first degree cohomology to be trivial. He remarks that "the strength of Hemmer's method does not lie in proving non-trivial but trivial first cohomology" and laments the difficulty of constructing Hemmer elements. Throughout the remainder of this chapter we shall construct Hemmer elements in some special cases and calculate upper bounds on the dimension of the cohomology. This allows us to take this remark further and say that the strength of Hemmer's method is actually in determining an upper bound for the dimension of the first cohomology (and in particular determining when it is trivial).

5.3.1 Two-part partitions

We shall first restrict ourselves to Specht modules indexed by two part partitions. Note that in this case we may identify a tabloid with the set of entries in its second row. Condition (1) of Theorem 5.3.2 in this context is equivalent to the condition that u is a non-null universal p-ary design of blocksize b on $[a+b]$, while condition (2) ensures that it is not similar to the constant design. Thus, finding a Hemmer element is equivalent to finding non-null universal p-ary designs which are not similar to the constant design, which was done in the previous chapter, Chapter 4.

Recall that we call a partition (a, b) James if $\mathrm{val}_p(a+1) > l_p(b)$ and pointed if $b = p^\beta = \hat{b}$ with $\beta > \mathrm{val}_p(a+1) > l_p(b)$. We saw in Theorem 4.1.25 that existence, or otherwise of universal designs depends on the shape of the partition (a, b). We restate this result below:

Theorem 5.3.3. *Let $a, b \in \mathbb{N}$, with $a \geq b$ and let u be a non-null universal p-ary design of blocksize b on $[a+b]$. If (a, b) is neither pointed or James, then u is similar to the constant design. If (a, b) is James then u is unique up to similarity, while if $(a, p^\beta + \hat{b})$ is pointed then $u = u' + c$ where u' is non-null only as a $\hat{b}$-design, while c is similar to the constant design.*

Corollary 5.3.4. *Let $\lambda = (a, b) \vdash n$, and $p \geq 3$ then*

$$\dim(H^1(\mathcal{S}_n, S^\lambda)) = \begin{cases} 1 & \text{if } \lambda \text{ is James or pointed,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. A non-split extension of S^λ corresponds to a Hemmer element in M^λ. If u and v are similar Hemmer elements, then there is some α such that $\psi_{1,j}(u - \alpha v) = 0$ for all j. Then $u - \alpha v \in S^\lambda$ by Theorem 2.2.9, and thus the extensions they define are the same and $\dim(H^1(\mathcal{S}_n, S^\lambda)) = 1$. Similarly, in the case where λ is pointed, we may have Hemmer elements u and v, which are not similar. Without loss of generality we may assume that $u = v + f_\lambda$ by subtracting off some $v' \in S^\lambda$, in which case the extensions $\langle S^\lambda, u\rangle$ and $\langle S^\lambda, v\rangle$ are equivalent and $\dim(H^1(\mathcal{S}_n, S^\lambda)) = 1$. If λ is neither pointed or James, then there are no Hemmer elements in M^λ, by Theorem 4.1.25, and thus $\dim(H^1(\mathcal{S}_n, S^\lambda)) = 0$. $\square$

Observe that the above result recovers the results of Hemmer [Hem09], and Donkin and Geranios for the case of two part partitions [DG19], however our proof is entirely in the setting of the symmetric group. Following the results of Chapter 4 we are able to explicitly construct Hemmer elements.

Theorem 5.3.5. *Let $\lambda = (a, b)$ be a partition, and suppose $\langle S^\lambda, x\rangle$ is a non-split extension of the Specht module by the trivial module. Then either*

- *λ is pointed and $\langle S^\lambda, x\rangle$ is equivalent to $\langle S^\lambda, u\rangle$, where u is the Hemmer element $u_{\mathbf{U}}$ described in Chapter 4, or*
- *λ is James and $\langle S^\lambda, x\rangle$ is equivalent to $\langle S^\lambda, u\rangle$, where u is the Hemmer element described in Chapter 4.*

5.4 General Partitions

The results in the previous section utilise the correspondence between Hemmer elements for two part partitions and combinatorial designs. The combinatorial objects which correspond to Hemmer elements for general partitions are much more complicated than designs, although they are built up from designs in some sense. If we take a Hemmer element for a partition λ and restrict our attention to a pair of adjacent rows in λ then we obtain a p-ary design. This allows us to use the results of the previous section to determine bounds on the degree of the cohomology.

Lemma 5.4.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and suppose u is a Hemmer element in M^λ with $\psi_{i,v}(u) = c \cdot f_{\lambda'}$ for some $c \neq 0$, then $(\lambda_i, \lambda_{i+1})$ is James or pointed. If $j \leq i-2$ or $j \geq i+2$, then $(\lambda_j, \lambda_{j+1})$ is a James partition.*

Proof. Observe that we may group the terms appearing in u, which are tabloids, by the union of the entries appearing in the ith and $(i+1)$th rows. Restricting our attention to any one of these groupings we see that we have an element u' with $\psi_{i,l}(u') = c_l \cdot f_{\lambda^l}$ and $c_v = c \neq 0$. In particular, u' is a universal p-ary design. It can not be the constant design as then u would not satisfy condition 2 of Theorem 5.3.2. We conclude that u' is Hemmer, and thus the partition $(\lambda_i, \lambda_{i+1})$ is James or pointed.

Now suppose that j is such that $\mid i-j \mid\geq 2$. Then $\psi_{j,l}(u)$ is some scalar multiple of $f_{\lambda'}$, but agrees with u on rows i and $i+1$. As u is not the constant design on restriction to these two rows we must have $\psi_{j,l}(u) = 0$. Denote by $\hat{\lambda}$ the partition obtained by deleting the rows i and $i+1$ from λ. Observe that as $\psi_{i,v}(u) = c \cdot f_{\lambda'}$ is non zero, every possible $\hat{\lambda}$ tabloid must appear in u with the same (non-zero) multiplicity. Thus the coefficient of any tabloid appearing in $\psi_{j,l}(u)$ is the product of some $c \neq 0$ and $\binom{\lambda_j+\lambda_{j+1}-s}{\lambda_{j+1}-s}$. As $\psi_{j,l}(u) = 0$ we conclude that $\binom{\lambda_j+\lambda_{j+1}-s}{\lambda_{j+1}-s} = 0$ for all s, and hence $(\lambda_j, \lambda_{j+1})$ is James. □

Corollary 5.4.2. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$. Then dim$(H^1(\mathcal{S}_n, S^\lambda)) \leq l$, where l is the number of rows i for which $(\lambda_i, \lambda_{i+1})$ is James or pointed, and $(\lambda_j, \lambda_{j+1})$ is James for all $j \in [r-1]$ with $|i-j| \geq 2$.*

Proof. If u is a Hemmer element in M^λ then u restricted to each pair of rows i and $i+1$ is either Hemmer, similar to the constant design, or is mapped to zero by each $\psi_{i,s}$. By Corollary 5.3.4, u is uniquely determined, up to similarity and constants, and hence the extension $\langle S^\lambda, u\rangle$ is determined up to equivalence, by the rows i and $i+1$ for which u is Hemmer. If u is Hemmer when restricted to rows i and $i+1$ then $(\lambda_i, \lambda_{i+1})$ is James or pointed, and $(\lambda_j, \lambda_{j+1})$ is James for all $j \in [r-1]$ with $|i-j| \geq 2$. □

Corollary 5.4.3. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ with $r \geq 5$, and let $2 < i+2 < j < r$. If $(\lambda_i, \lambda_{i+1})$ and $(\lambda_j, \lambda_{j+1})$ are both not James, then $H^1(\mathcal{S}_n, S^\lambda) = 0$.*

Remark. *This is the main result of [Web13], and gives a large class of partitions for which the first degree cohomology is trivial.*

For the remainder of this section we shall carefully analyse Hemmer elements for general partitions, and will improve Corollary 5.4.2. We remark here that Donkin and Geranios have determined exactly the dimension of $H^1(\mathcal{S}_n, S^\lambda)$ for all partitions, and this is the bound we obtain from our combinatorial methods. We shall not explicitly construct Hemmer elements for general partitions, and so our result only gives an upper bound on the dimension of $H^1(\mathcal{S}_n, S^\lambda)$. Future work which constructs Hemmer elements would complete the entirely combinatorial proof of two of the three main theorems of [DG19], namely Theorem 12.29 and Theorem 12.30. The last of the main results of [DG19], Theorem 12.31, can not be obtained by the methods in this paper, as over fields of characteristic 2 non-split extensions of the trivial module k by S^λ do not necessarily have to be isomorphic to submodules of M^λ.

We have already observed in Corollary 5.4.2 that dim$H^1(\mathcal{S}_n, S^\lambda)$ is bounded above by the number of pairs of consecutive rows as, for a Hemmer element u and a fixed i, the coefficients in $\psi_{i,l}(u)$ are related. In general, however, there may be relationships between the coefficients in $\psi_{i,l}(u)$ and $\psi_{j,m}(u)$, in which case we will say the pairs of rows $(i, i+1)$ and $(j, j+1)$ are *dependent*. This dependence is clearly an equivalence relation, and then an improved bound is that dim$H^1(\mathcal{S}_n, S^\lambda)$ is at most the number of equivalence classes of dependent pairs of rows. We will now investigate when pairs of rows are dependent.

Let $\lambda = (a, b, c)$ and suppose that the pairs (a, b) and (b, c) are independent. Then there must be Hemmer elements $u, v \in M^\lambda$ with $\psi_{1,i}(u) = 0$ and $\psi_{2,j}(v) = 0$ for all $1 \leq i \leq b$ and $1 \leq j \leq c$. In particular,

$$u \in \bigcap_{i=1}^{b} \ker(\psi_{1,i}) = S^{(a,b),(a,b,c)}$$

and

$$v \in \bigcap_{j=1}^{c} \ker(\psi_{2,j}) = S^{(b,c),(a,b,c)}.$$

As the Hemmer elements u and v are such that $\psi_{1,l}(u) \neq 0$ and $\psi_{1,k}(v) \neq 0$ for some l, k, the partitions (a, b) and (b, c) are either James or pointed.

The Specht-shadow module $S^{(a,b),(a,b,c)}$ has a filtration by Specht modules with factors $S^{(a+x,b+y,c-x-y)}$ with $x + y \leq c$ and $b + y \leq a$. As $u \in S^{(a,b),(a,b,c)}$ the extension $\langle S^\lambda, u\rangle$ is a submodule of $S^{(a,b),(a,b,c)}$. Also, since u is Hemmer, this is an extension of $S^{(a,b,c)}$ by the trivial module, and thus some factor of $S^{(a,b),(a,b,c)}$, not including the bottom factor S^λ, must have the trivial module as a submodule. This only occurs when one of these factors is James, by Theorem 5.1.1, and so u can only exist if there is some x and y with $x + y \leq c$ and $b + y \leq a$ for which $(a + x, b + y, c - x - y)$ is a James partition.

Similarly, the Specht-shadow module $S^{(b,c),(a,b,c)}$ has a filtration by Specht modules with factors $S^{(a+x+y,b-x,c-y)}$ with $y \leq c \leq b - x$. If there is a Hemmer element $v \in S^{(b,c),(a,b,c)}$, then some factor of $S^{(b,c),(a,b,c)}$, not including the bottom factor $S^{(a,b,c)}$, must have the trivial module as a submodule. Thus v can only exist if there is some x and y with $y \leq c \leq b - x$ for which $(a + x + y, b - x, c - y)$ is a James partition.

Lemma 5.4.4. *The pairs (a, b) and (b, c) are dependent if (a, b, c) is James and $b = p^{val_p(a+1)} - 1$.*

Proof. We shall prove that there can be no Hemmer element $u \in S^{(a,b),(a,b,c)}$. As (a, b, c) is a James partition, $\text{val}_p(a + 1) > l_p(b)$ and $\text{val}_p(b + 1) > l_p(c)$. The partition $(a + x, b + y, c - x - y)$ is James if $(a + x, b + y)$ and $(b + y, c - x - y)$ are both James. It is clear that $l_p(b + y)$ is either $l_p(b)$ or $l_p(b) + 1$, and thus $(a + x, b + y)$ can only be James if $\text{val}_p(x) > l_p(b)$. Of course $x \leq c$ and so we must have $x = 0$. The only factors of $S^{(a,b),(a,b,c)}$ indexed by James partitions are those of the form $S^{(a,b+y,c-y)}$ with $y \leq \min\{c, a - b\}$. The partition $(b + y, c - y)$ is only James if $y \equiv c \pmod{p^l}$ for some l (recall $\text{val}_p(b + 1) > l_p(c)$). Finally observe that $b + y \leq a$ as long as $b \neq p^{\text{val}_p(a+1)} - 1$, and if $b = p^{\text{val}_p(a+1)} - 1$ then no such u exists. □

Remark. *We have actually shown something stronger: If (a, b, c) is James and $b \neq p^{val_p(a+1)} - 1$ then $S^{(a,b),(a,b,c)}$ has a Specht module factor, other than $S^{(a,b,c)}$, with a trivial submodule. This tells us exactly where to look to find the Hemmer element u described above.*

Lemma 5.4.5. *Let (a, b) be pointed and (b, c) either pointed or James. Then the pairs (a, b) and (b, c) are dependent.*

Proof. As before, we shall prove that there can be no Hemmer element $u \in S^{(a,b),(a,b,c)}$. Let $b = p^\beta + \hat{b}$ with $\hat{b} < p^{\text{val}_p(a+1)} < p^\beta$. For the partition $(a + x, b + y, c - x - y)$ to be

James, we must have $\text{val}_p(a + x + 1) > l_p(b + y) \geq \beta$. As the partition (b, c) is either James or pointed and $x \leq c$, we have that $\text{val}_p(a + x + 1) \leq \text{val}_p(a + 1) < \beta$, so this partition is never James. $\square$

Lemma 5.4.6. *Let (a, b) be James and (b, c) be pointed. Then the pairs (a, b) and (b, c) are dependent.*

Proof. This time we shall prove that there can be no Hemmer element $v \in S^{(b,c),(a,b,c)}$. Let $c = p^\gamma + \hat{c}$ with $\hat{c} < p^{\text{val}_p(b+1)} < p^\gamma$ and observe $\text{val}_p(a + 1) > l_p(b)$. There is no Hemmer element v as long as no partition $(a+x+y, b-x, c-y)$ is James for $y \leq c \leq b-x$. Observe that $\text{val}_p(a + x + y + 1) = \text{val}_p(x + y)$, and so if $(a + x + y, b - x)$ is James then $y = 0$. The partition $(b - x, c)$ is James if and only if $\text{val}_p(b - x + 1) > \gamma$, but then $(a + x, b - x)$ is not James, and so v can not exist. $\square$

Lemma 5.4.7. *Let (a, b, c) be James and suppose $l_p(b) = l_p(c)$. Then the pairs (a, b) and (b, c) are dependent.*

Proof. We shall actually show that if (a, b, c) is James then $S^{(b,c),(a,b,c)}$ has a Specht module factor, other than $S^{(a,b,c)}$, with a trivial submodule if and only if $l_p(b) \neq l_p(c)$. As (a, b, c) is James, $\text{val}_p(a+1) > l_p(b)$ and $\text{val}_p(b+1) > l_p(c)$. Again, $\text{val}_p(a+x+y+1) = \text{val}_p(x+y)$, and so if $(a+x+y, b-x)$ is James then $y = 0$. Subject to the condition that $x \geq b - c$ and that $(b - x, c)$ is James, the partition $(a + x, b - x)$ is James if and only if $x \equiv b \pmod{p^l}$ for some $l \geq l_p(c)$. This can be satisfied (for non-zero x) if and only if $l_p(b) \neq l_p(c)$. $\square$

In order to state the main results of this chapter, we must make some definitions to capture when pairs of rows in a partition are dependent. We shall follow [DG19] in making the following definitions. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a James partition, the *segments* of λ are the equivalence classes on $\{1, \ldots, n\}$ generated by the equivalence relation $r \sim s$ if $l_p(\lambda_r) = l_p(\lambda_s)$. We shall call two integers $1 \leq r, s \leq n$ *adjacent* if they are in the same segment, or if $1 < r < n$, $s = r + 1$ and s is the only element in its segment, and $\lambda_r = p^{\text{val}_p(\lambda_{r-1}+1)} - 1$. The *p-segments* of λ are the equivalence classes on $\{1, \ldots, n\}$ generated by adjacency.

Theorem 5.4.8. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a James partition of length $n \geq 2$. Let l denote the number of p-segments of $\{1, \ldots, n\}$.*

1. *If $l_p(\lambda_1) = l_p(\lambda_2)$, then $dim(H^1(\mathcal{S}_n, S^\lambda)) \leq l$.*
2. *If $l_p(\lambda_1) > l_p(\lambda_2)$, then $dim(H^1(\mathcal{S}_n, S^\lambda)) \leq l - 1$.*

Proof. Observe that if i and $i+1$ in the same segment then $(\lambda_{i-1}, \lambda_i)$ and $\lambda_i, \lambda_{i+1})$ are dependent, by Lemma 5.4.7. If i and $i+1$ are in the same p-segment, but not the same segment, then $\lambda_i = p^{\mathrm{val}_p(\lambda_{i-1}+1)} - 1$ and $l_p(\lambda_{i-1}) > l_p(\lambda_i) > l_p(\lambda_{i+1})$. In this situation, then by Lemma 5.4.4, $(\lambda_{i-1}, \lambda_i)$ and $(\lambda_i, \lambda_{i+1})$ are dependent. Thus for any i and j lying in the same p-segment $(\lambda_{i-1}, \lambda_i)$ and $(\lambda_{j-1}, \lambda_j)$ are dependent. If $l_p(\lambda_1) = l_p(\lambda_2)$, then 1 and 2 are in the same p-segment and the number of equivalence classes of dependent pairs of rows is k, while if $l_p(\lambda_1) > l_p(\lambda_2)$ then 1 and 2 are in different p-segments, and as there are no pairs of rows corresponding to the p-segment $\{1\}$, the number of equivalence classes of dependent pairs of rows is $l-1$. □

We have already seen that if λ is not James, then Hemmer elements can not exist unless the only pairs of non-James rows are close together (Corollary 5.4.3). We shall now prove something even stronger, namely that if λ is a non-James partition and $u \in M^\lambda$ is a Hemmer element, then u is unique, up to *similarity*, where we say Hemmer elements u and v are similar if there exists a $c \in k$ such that $\phi_{i,j}(u) = c \cdot \phi_{i,j}(v)$ for all appropriate i, j. Note that this generalises the notion of similarity for designs. Equivalently:

Theorem 5.4.9. *Let λ be a non-James partition. Then dim$(H^1(\mathcal{S}_n, S^\lambda)) \leq 1$.*

Proof. If $u \in M^\lambda$ is Hemmer and $(\lambda_j, \lambda_{j+1})$ is non-James, then $\psi_{i,l}(u) = 0$ for all i such that $\mid i - j \mid \geq 2$. Thus $\psi_{i,l}(u)$ is only possibly non-zero for $i \in \{j-1, j, j+1\}$. It follows from Lemma 5.4.5 that $(\lambda_{j-1}, \lambda_j)$ and $(\lambda_j, \lambda_{j+1})$ are dependent and from Lemma 5.4.6 that $(\lambda_j, \lambda_{j+1})$ and $(\lambda_{j+1}, \lambda_{j+2})$ are dependent. This implies that Hemmer elements can not be chosen independently, as far from the pointed pair we have $\psi_{i,l}(u) = 0$, while the value of $\psi_{i,l}(u)$ close to the pointed pair is determined by the coefficient of $\psi_{j,l}(u) = c \cdot f_{\lambda'}$ for some l such that $c \neq 0$. □

Observe that the upper bounds on the dimension of H^1 given in this section are known, by the work of Donkin and Geranios, to agree with the dimension. To complete an entirely combinatorial proof of their results it remains to construct Hemmer elements in these cases.

5.4.1 General Hemmer elements

The previous section gives an upper bound on the number of Hemmer elements which can exist, but in practice finding any is difficult. In this section we shall give a description of where these Hemmer elements can be found. We shall describe some Hemmer elements in some special cases for some three part partitions.

James partitions

To illustrate how complicated Hemmer elements are in general we shall give an example in the smallest possible case. If $p = 3$, then the smallest James partition with three parts is the partition $\lambda = (2, 2, 1)$. By Theorem 5.4.8, if a non-split extension of S^λ by the trivial module exists, then it is unique. This can also be seen by observing the Specht filtration for M^λ, given by Proposition 2.2.17:

$$M^{(2,2,1)} \sim \begin{array}{l} S^{(5)} \\ S^{(4,1)} \\ S^{(4,1)} \\ S^{(3,2)} \\ S^{(3,1,1)} \\ S^{(3,2)} \\ S^{(2,2,1)} \end{array} .$$

Any non-split extension of S^λ by the trivial module is a submodule of M^λ, by Theorem 5.3.1, and thus must correspond to a trivial submodule of one of the factors in the Specht filtration. The only factor with a trivial submodule, other than S^λ itself, is $S^{(5)}$, and thus this is the only candidate for a non-split extension of S^λ. If we can find a Hemmer element in M^λ then we can conclude that $\mathrm{Ext}^1_{k\mathcal{S}_n}(S^{(2,2,1)}, k) = 1$.

Consider the element $u \in M^{(2,2,1)}$ given by,

$$\begin{aligned} u = {} & \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 1 & 3 \\ \hline 4 & 5 \\ \hline 2 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 1 & 5 \\ \hline 3 & 4 \\ \hline 2 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline 1 & \\ \cline{1-1}\end{array} \\ & + \begin{array}{cc}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline 5 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 3 & 5 \\ \hline 1 & 2 \\ \hline 4 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 3 & 5 \\ \hline 2 & 4 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 4 & 5 \\ \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} \\ & + \begin{array}{cc}\hline 4 & 5 \\ \hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{cc}\hline 3 & 4 \\ \hline 1 & 5 \\ \hline 2 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & \\ \cline{1-1}\end{array} - \begin{array}{cc}\hline 1 & 4 \\ \hline 3 & 5 \\ \hline 2 & \\ \cline{1-1}\end{array} \end{aligned}$$

$$- \begin{array}{cc}\hline 2 & 5 \\ \hline 1 & 4 \\ \hline 3 & \\ \hline \end{array} - \begin{array}{cc}\hline 2 & 5 \\ \hline 3 & 4 \\ \hline 1 & \\ \hline \end{array} - \begin{array}{cc}\hline 3 & 4 \\ \hline 2 & 5 \\ \hline 1 & \\ \hline \end{array} - \begin{array}{cc}\hline 3 & 5 \\ \hline 1 & 4 \\ \hline 2 & \\ \hline \end{array} - \begin{array}{cc}\hline 4 & 5 \\ \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} .$$

Observe that

$$\psi_{2,0}(u) = f_{(2,3)},$$

$$\psi_{1,0}(u) = 2f_{(4,0,1)},$$

and

$$\psi_{1,1}(u) = f_{(3,1,1)},$$

Of course, as $(2,2,1)$ is a James partition $f_{(2,2,1)} \in S^{(2,2,1)}$ and therefore the image of $f_{(2,2,1)}$ under each of the maps $\psi_{2,0}, \psi_{1,0}$ and $\psi_{1,1}$ is 0. Thus u is a Hemmer element and we conclude:

Proposition 5.4.10. *Let k be a field of characteristic 3. The Specht module $S^{(2,2,1)}$ has a unique non-split extension by the trivial module, spanned by $S^{(2,2,1)}$ and the element u described above.*

Although in principle the Specht filtration tells us where to look to find a Hemmer element, even in the smallest possible case this Hemmer element is difficult to construct; in this case there are 20 tabloids involved in u. This Hemmer element is not unique, however, as for any $v \in S^{(2,2,1)}$, the element $u + v$ is also a Hemmer element.

Having seen a concrete example, we now move on to describing where to find Hemmer elements for James partitions in general.

Let $\lambda \vdash n$. If $\{1\}$ is a p-segment of λ then label the p-segments of λ by $\{1\} = X_0, X_1, \ldots, X_l$, otherwise label the p-segments $X_1, \ldots, X_l$. Then there are at most l independent Hemmer elements, by Theorem 5.4.8. If there are l independent Hemmer elements, then they can be chosen so each is non-null only on the rows corresponding to some p-segment X_i. Let $1 \leq i \leq l$ and write $\lambda = \mu \star \eta$, where η is a partition with $\mid X_i \mid +1$ parts corresponding to the p-segment X_i. That is, the parts of η are $\{\lambda_j : j \in X_i \text{ or } j+1 \in X_i\}$. A Hemmer element which is non-null only on rows corresponding to X_i is necessarily in the Specht-shadow module $S^{\mu,\mu\#\eta}$. Observe that $\mu\#\eta$ is the concatenation of μ and η and is a composition, but not necessarily a partition. If no Hemmer element exists in $S^{\mu,\mu\#\eta}$ then the bound in Theorem 5.4.8 is not obtained, otherwise these Specht-shadow modules are exactly where one must look to find the l Hemmer elements.

Special case: $\lambda = (a, b, 1)$

To illustrate the comments in the previous section we shall look for Hemmer elements in the case that $\lambda = (a, b, 1)$ is a James partition with $l(a) = l(b)$, $b \neq p^{\mathrm{val}_p(a+1)} - 1$, and $l(b) \neq 0$. In this case the partition has p-segments $\{1, 2\}$ and $\{3\}$. We aim to find Hemmer elements $u \in S^{(a,b),(a,b,1)}$ and $v \in S^{(b,1),(b,1,a)}$. First, observe that

$$S^{(a,b),(a,b,1)} \sim \begin{array}{l} S^{(a+1,b)} \\ S^{(a,b+1)} \\ S^{(a,b,1)} \end{array} .$$

Any non-split extension of $S^{(a,b,1)}$ contained in $S^{(a,b),(a,b,1)}$ must correspond to a trivial submodule of one of the factors in the Specht filtration. As $(a, b+1)$ is James and $(a+1, b)$ is not, the extension, if it exists, must be a submodule of

$$\begin{array}{l} S^{(a,b+1)} \\ S^{(a,b,1)} \end{array} ,$$

which is $S^{(a,b),(a,b,1)} \cap \ker\psi_{2,0}$, by Proposition 2.2.17. The module $S^{(a,b+1)}$ has a trivial submodule. Let $u \in S^{(a,b),(a,b,1)} \cap \ker\psi_{2,0}$ be such that u, together with $S^{(a,b,1)}$ spans $S^{(a,b,1)}$ and this trivial submodule. Then u satisfies condition 1 of Theorem 5.3.2. As $u \notin S^{(a,b,1)}$ and $(a, b, 1)$ is James, then u must satisfy condition 2 of Theorem 5.3.2. In particular u is Hemmer and the module $\langle S^{(a,b,1)}, u\rangle$ is a non-split extension of $S^{(a,b,1)}$.

If another independent Hemmer element exists then without loss of generality it is contained in $S^{(b,1),(b,1,a)}$. Observe.

$$S^{(b,1),(b,1,a)} \sim \begin{array}{l} S^{(a+b,1)} \\ S^{(a+b-1,2)} \\ \vdots \\ S^{(a+1,b)} \\ S^{(a+b-1,1,1)} \\ S^{(a+b-2,2,1)} \\ \vdots \\ S^{(a,b,1)} \end{array} .$$

The only factors with trivial submodules appearing in this filtration are of the form $S^{(a+x,b-x,1)}$ where $x = b - (b \bmod p^{\alpha})$ for some α. That is; x is obtained from b by writing b in its base p expansion and then replacing the last α entries with 0 for some α. This tells us in which layers of the filtration a Hemmer element possibly occurs. As it can not occur in any of the layers indexed by two part partitions, nor the layer $S^{(a+b-x,x,1)}$

if $x < p-1$, we can restrict our attention to:

$$S^{(b,p-1,1),(b,a,1)} \sim \begin{array}{l} S^{(a+b+1-p,p-1,1)} \\ S^{(a+b-p,p,1)} \\ \vdots \\ S^{(a,b,1)} \end{array} .$$

If, in one of the layers indexed by a James partition, there is some v such that $\langle S^{(a,b,1)}, v\rangle$ is a $k\mathcal{S}_n$-module then this v is a Hemmer element and $\dim(H^1(\mathcal{S}_n, S^{a,b,c})) = 2$.

Example 5.4.11. *The smallest partition which is James and has more than 1 p-segment (other than $\{1\}$) is the partition $\lambda = (8,5,1)$ when $p = 3$. There is a Hemmer element*

$$u \in S^{(8,5),(8,5,1)} \sim \begin{array}{l} S^{(9,5)} \\ S^{(8,6)} \\ S^{(8,5,1)} \end{array} ,$$

as described above. If there is another Hemmer element then it is in the $k\mathcal{S}_n$-module

$$S^{(5,2,1),(5,8,1)} \sim \begin{array}{l} S^{(11,2,1)} \\ S^{(10,3,1)} \\ S^{(9,4,1)} \\ S^{(8,5,1)} \end{array} .$$

The only layer which is indexed by a James partition is $S^{(11,2,1)}$. This has a trivial submodule which either extends $S^{(10,3,1)}$, $S^{(9,4,1)}$, or $S^{(8,5,1)}$. It can not span a non-split extension of $S^{(8,5,1)}$ as the $(8,5,1)$ is James and thus $f_{(8,5,1)} \in S^{(8,5,1)}$.

Non-James partitions

If λ is a non-James partition, then it follows from Theorem 5.4.9 that there is at most one non-split extension of S^λ by the trivial module. These extensions do not exist, however, if there are two pairs of rows which are non-James that are far apart. Let j be the least j such that $(\lambda_j, \lambda_{j+1})$ is non-James and write $\lambda = \mu \star (\lambda_{j-1}, \lambda_j, \lambda_{j+1})$. It follows from Theorem 5.4.9 that if there is a Hemmer element $u \in M^\lambda$, then $u \in S^{\mu,\mu\star(\lambda_{j-1},\lambda_j,\lambda_{j+1})}$. To find such an element one must first find a Specht filtration of $S^{\mu,\mu\star(\lambda_{j-1},\lambda_j,\lambda_{j+1})}$ and determine which, if any, of the factors have a trivial submodule. If no such factor exists then we can conclude that $\dim(H^1(\mathcal{S}_n, S^\lambda)) = 0$.

Example 5.4.12. *Let $p \geq 5$ and let $\lambda = (\lambda_1, \ldots, \lambda_{r-3}, 1, 1, 1)$ be such that $(\lambda_1, \ldots, \lambda_{r-3}, 1)$ is James. Any Hemmer element in M^λ must be in $S^{(\lambda_1,\ldots,\lambda_{r-3}),\lambda}$. This has a filtration* *by Specht modules indexed by partitions μ where μ is obtained from λ by moving the three boxes in the last three rows. As $p \geq 5$ this can never be done in such a way to ensure that μ is James, and so there is no factor in the filtration of $S^{(\lambda_1,\ldots,\lambda_{r-3}),\lambda}$ with a trivial submodule.*

The example above shows that:

Proposition 5.4.13. *Let $p \geq 5$ and let $\lambda = (\lambda_1, \ldots, \lambda_{r-3}, 1, 1, 1)$ be such that $(\lambda_1, \ldots, \lambda_{r-3}, 1)$ is James. Then*

$$dim(H^1(\mathcal{S}_n, S^{\lambda})) = 0.$$

This technique can be used to determine a range of other families of Specht modules who have no non-split extensions by the trivial module. Despite these being already known by the work of Donkin and Gernios [DG19], this is, as Weber remarks [Web13], the true strength of the combinatorial approach. The construction of general Hemmer elements, where they exist, remains elusive.

Chapter 6

FI modules

In this chapter we shall study **FI**-modules, which contain information about sequences of $k\mathcal{S}_n$-modules. In particular, in the second section of this chapter we shall study representable **FI**-modules and will describe a number of their submodules. One submodule of interest corresponds to a sequence of $k\mathcal{S}_n$-modules whose dimensions are given by the rank of the inclusion matrix, which was defined in Chapter 4 but will be defined again. This gives both a new representation theoretic interpretation of the well known rank formula for inclusion matrices and a purely representation theoretic proof of this fact, which makes up the first part of this chapter, and also appears in [Jol20b].

6.1 Rank formula for the inclusion matrix

Recall the *inclusion matrix,* $A_i^n(m)$, where $i \leq n \leq m$, is the $\binom{m}{i} \times \binom{m}{n}$ matrix whose rows are indexed by subsets of $[m] := \{1, 2, \ldots, m\}$ of size i and whose columns are indexed by subsets of $[m]$ of size n. The entry corresponding to position X, Y is 1 if $X \subseteq Y$ and 0 otherwise. For example:

$$A_2^3(4) = \begin{array}{c} \\ \{1,2\} \\ \{1,3\} \\ \{1,4\} \\ \{2,3\} \\ \{2,4\} \\ \{3,4\} \end{array} \begin{array}{c} \begin{array}{cccc} \{1,2,3\} & \{1,2,4\} & \{1,3,4\} & \{2,3,4\} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}.$$

Observe that this matrix is of full rank over $\mathbb{R}$, however is not of full rank over a field of characteristic 2, where the last column is the sum of the first three columns.

The inclusion matrix arises in a number of combinatorial investigations. Gottlieb proved that over a field of characteristic 0 this matrix has full rank [Got66]. Linial and Rothschild then determined a formula for the rank of this matrix over the field of two elements, as well the special case when $n = i + 1$ over the field of three elements [LR81]. Wilson solved the problem over any field by proving the following [Wil90]:

Theorem 6.1.1. *Let k be a field of characteristic p and suppose $i \leq \min\{n, m - n\}$. Then*

$$rank_k(A_i^n(m)) = \sum_{p \nmid \binom{n-j}{i-j}} \binom{m}{j} - \binom{m}{j-1},$$

where $\binom{m}{-1}$ is interpreted as 0.

We shall give a new proof of Theorem 6.1.1 by constructing a $k\mathcal{S}_m$-module spanned by the columns of $A_i^n(m)$ which, of course, has dimension $\mathrm{rank}_k(A_i^n(m))$. This proof shall make use of the representation theory of the symmetric group as described in chapter two. In particular we need the Specht modules, permutation modules and Specht-shadow modules indexed by two part partitions. When λ is a two part partition the notation can be slightly simplified, as the family of maps $\psi_{i,v}$ are only a one parameter family because the row is fixed; that is $i = 1$. We shall review the concepts from chapter two in this context in order to simplify much of the notation.

6.1.1 Two part partitions

When $\lambda = (\lambda_1, \lambda_2)$ is a two part partition, we can identify the λ-tabloid $\{t\}$ with the set of elements appearing in the second row of $\{t\}$. We then see that $M^{(m-i,i)}$ has a basis indexed by the i subsets $X \subseteq_i [m]$; that is, the subsets of size i. We shall frequently alternate between these two notations depending on notational convenience.

Let t be a $(m - i, i)$-tableau and let $0 \leq j \leq i$. The *j-column stabiliser* of t, denoted $C_j(t)$ is the set of permutations that fix all but the first j columns of t and only permute elements that appear in the same column of t. In particular, $C_j(t)$ is generated by the j transpositions which swap an element appearing in the first j entries in the first row of t with the element appearing below it.

The *j-column symmetriser* is the element of the group algebra

$$\kappa_j(t) := \sum_{\sigma \in C_j(t)} (-1)^\sigma \sigma,$$

and the *j-polytabloid* is

$$e_t^j = \kappa_j(t)\{t\} \in M^{(m-i,i)},$$

where $(-1)^\sigma$ is the sign of the permutation σ. Note in particular that

$$e_t^0 = \{t\}.$$

Recall the Specht-shadow module $S^{(m-i,j)(m-i,i)} \subseteq M^{(m-i,i)}$ is the submodule of the permutation module spanned by the j-polytabloids. The Specht module $S^{(m-i,i)}$ is just the Specht-shadow module $S^{(m-i,i)(m-i,i)}$. This gives us a chain of submodules

$$M^{(m-i,i)} \supseteq S^{(m-i,1)(m-i,i)} \supseteq \cdots S^{(m-i,i-1)(m-i,i)} \supseteq S^{(m-i,i)} \supseteq 0.$$

It can be shown that the successive quotients $S^{(m-i,j)(m-i,i)}/S^{(m-i,j+1)(m-i,i)}$ are isomorphic as $k\mathcal{S}_n$-modules to the Specht module $S^{(m-j,j)}$, and so

$$M^{(m-i,i)} \sim \begin{array}{c} S^{(m)} \\ S^{(m-1,1)} \\ \vdots \\ S^{(m-i,i)} \end{array}.$$

In the notation of sets we define the $k\mathcal{S}_n$-homomorphism

$$\psi_j : M^{(m-i,i)} \to M^{(m-j,j)}$$

by

$$\psi_j(X) = \sum_{Z \subseteq_j X} Z,$$

for $X \subseteq_i [m]$. Observe this is the map $\psi_{1,j}$ defined in Chapter 2.

Proposition 6.1.2. *Let t be an $(m-i,i)$-tableau and e_t^j its j-polytabloid. Let $k < j \leq i$ then*

$$\psi_k(e_t^j) = 0,$$

while

$$\psi_j(e_t^j) = e_{t'}^j,$$

where t' is the $(m-j,j)$-tableau obtained from t by moving the last $i-j$ entries in the bottom row to the end of the top row.

Proof.

$$\begin{aligned}\psi_k(e_t^j) &= \psi_k(\kappa_j(t)\{t\}) \\ &= \psi_k(\sum_{\sigma \in C_j(t)} (-1)^\sigma \sigma\{t\}) \\ &= \sum_{\sigma \in C_j(t)} (-1)^\sigma \sigma \psi_k(\{t\}) \\ &= \sum_{\sigma \in C_j(t)} (-1)^\sigma \sigma \sum_{\{s\}} \{s\},\end{aligned}$$

where the second sum is over all $(m-k,k)$-tabloids, $\{s\}$, whose second row is a subset of the second row of $\{t\}$. As $k < j$, some element in the first j entries of the second row of t must lie in the top row of $\{s\}$, and there must be a transposition $\sigma \in C_j(t)$ fixing $\{s\}$. The terms involving this transposition and the terms not involving this transposition have opposite signs and cancel, thus

$$\psi_k(e_t^j) = 0.$$

Similarly

$$\psi_j(e_t^j) = \sum_{\sigma \in C_j(t)} (-1)^\sigma \sigma \sum_{\{s\}} \{s\},$$

with cancellation for any $\{s\}$ which contain, in their first row, an element which appears as one of the first j elements of the second row of t. Therefore

$$\begin{aligned}\psi_j(e_t^j) &= \sum_{\sigma \in C_j(t)} (-1)^\sigma \sigma\{t'\} \\ &= \sum_{\sigma \in C_j(t')} (-1)^\sigma \sigma\{t'\} \\ &= e_{t'}^j.\end{aligned}$$

□

So, when restricted to $S^{(m-i,j)(m-i,i)} \subseteq M^{(m-i,i)}$ the image of ψ_j is isomorphic to $S^{(m-j,j)}$ and its kernel is $S^{(m-i,j)(m-i,i)}$. This gives an alternative characterisation of $S^{(m-i,j)(m-i,i)}$ as:

$$S^{(m-i,j)(m-i,i)} = \cap_{k=0}^{j-1}(\ker(\psi_k : M^{(m-i,i)} \to M^{(m-k,k)})).$$

To prove Theorem 6.1.1 in the next section we will identify a submodule of $M^{(m-i,i)}$ and study its images under this map. We shall conclude this section by stating a special case of the hook length formula Theorem 2.2.6, which gives the dimension of a Specht module.

Theorem 6.1.3. *Let $j, m \in \mathbb{N}$ be such that $m - j \geq j$. Then*

$$\dim S^{(m-j,j)} = \binom{m}{j} - \binom{m}{j-1},$$

where $\binom{m}{-1} = 0$.

The astute reader will have noticed that is the term which appears in the formula in Theorem 6.1.1.

6.1.2 Proof of rank formula

The rows of the inclusion matrix $A_i^n(m)$ are indexed by the i subsets of $[m]$, which is naturally the basis for the $k\mathcal{S}_m$-module $M^{(m-i,i)}$. The columns of $A_i^n(m)$ span a submodule of $M^{(m-i,i)}$ which has dimension $\mathrm{rank}_k(A_i^n(m))$. We shall denote this submodule by $P_i^n(m)$. Our analysis of $M^{(m-i,i)}$ in the previous section gives rise to a chain of submodules

$$P_i^n(m) \supseteq P_i^n(m)_1 \supseteq P_i^n(m)_2 \supseteq \cdots \supset P_i^n(m)_i \supseteq 0,$$

where $P_i^n(m)_j := P_i^n(m) \cap S^{(m-i,j)(m-i,i)}$. This shows that

$$P_i^n(m) \sim \begin{array}{c} L^{(m)} \\ L^{(m-1,1)} \\ \vdots \\ L^{(m-i,i)} \end{array},$$

where each $L^{(m-j,j)}$ is some submodule of $S^{(m-j,j)}$. In particular, $L^{(m-j,j)}$ is the image of $P_i^n(m)_j$ under the map $\psi_j : M^{(m-i,i)} \to M^{(m-j,j)}$.

Observe that the columns of the matrix $A_i^n(m)$ correspond to the images of the n-subsets of $[m]$ under the homomorphism $\psi_i : M^{(m-n,n)} \to M^{(m-i,i)}$. The module $P_i^n(m)$ is thus the image of $\psi_i : M^{(m-n,n)} \to M^{(m-i,i)}$. Observe that although $(m-n,n)$ may not be a partition, we can still define the permutation module $M^{(m-n,n)}$, and we can also define j-polytabloids for any $j < \min\{m-n, n\}$. Denote by $x \in P_i^n(m)$ the image of the j-polytabloid corresponding to a $(m-n,n)$-tableau t, for $j \leq i$. Our assumption that

$i < \min\{m-n, n\}$ ensures that this j-polytabloid is well defined.

$$\begin{aligned} x &:= \psi_i(e_t^j) \\ &= \sum_{\sigma \in C_j(t)} (-1)^\sigma \sigma \sum_{\{s\}} \{s\} \end{aligned}$$

where the second sum is over all $(m-i, i)$-tabloids, $\{s\}$, whose second row is a subset of the second row of $\{t\}$. Of course we have cancellation of any terms for which the first j entries from the second row of t do not appear in the second row of $\{s\}$, so the sum is over all $(m-i, i)$-tabloids whose second row is a subset of the second row of $\{t\}$ of size i containing these first j entries. Observe then that

$$x = \sum_s e_s^j,$$

where the sum is over all $(m-i, i)$-tableaux obtained by moving $n-i$ of the last $n-j$ entries of the second row of t to the top row. As x is a sum of j-polytabloids, $x \in S^{(n-i,j)(m-i,i)}$ and thus $x \in P_i^n(m)_j$, and so its image under ψ_j is in $L^{(m-j,j)}$.

Proposition 6.1.4. *If* $p \nmid \binom{n-j}{i-j}$ *then* $L^{(m-j,j)} = S^{(m-j,j)}$.

Proof. Let $x = \sum_s e_s^j$ as above. Then, by Proposition 6.1.2,

$$\begin{aligned} \psi_j(x) &= \psi_j(\sum_s e_s^j) \\ &= \sum_s e_{s'}^j \end{aligned}$$

where s' is the $(m-j, j)$-tableau obtained from s by moving the last $i-j$ entries of the bottom row to the top row. Each term is equal, and and the sum is over all $(m-i, i)$-tableaux obtained by moving $n-i$ of the last $n-j$ entries of the second row of t to the top row, of which there are $\binom{n-j}{i-j}$. Thus

$$\psi_j(x) = \binom{n-j}{i-j} e_{s'}^j,$$

and hence $e_{s'}^j \in L^{(m-j,j)}$. In fact, this shows that any j-polytabloid is in $L^{(m-j,j)}$ and thus $L^{(m-j,j)} = S^{(m-j,j)}$. □

Proof of Theorem 6.1.1. Consider the image of $P_i^n(m)_j$ under the map ψ_j, which is a submodule $L^{(m-j,j)} \subseteq S^{(m-j,j)}$. By Proposition 6.1.4, if $p \nmid \binom{n-j}{i-j}$ then $L^{(m-j,j)} =$

$S^{(m-j,j)}$. On the other hand, if $p \mid \binom{n-j}{i-j}$ then for any column $y = \psi_i(Y)$ of $A_i^n(m)$,

$$\begin{aligned}\psi_j(y) &= \sum_{X \subset_i Y} \psi_j(X) \\ &= \sum_{X \subseteq_i Y} \sum_{Z \subseteq_j X} Z \\ &= \sum_{Z \subseteq_j Y} \binom{n-j}{i-j} Z \\ &= 0.\end{aligned}$$

This means that ψ_j is the zero map on $P_i^n(m)$, thus $L^{(m-j,j)} = 0$. We conclude that

$$P_i^n(m) \sim \begin{array}{c} L^{(m)} \\ L^{(m-1,1)} \\ \vdots \\ L^{(m-i,i)} \end{array},$$

with

$$L^{(m-j,j)} = \begin{cases} 0 & \text{if } p \mid \binom{n-j}{i-j} \\ S^{(m-j,j)} & \text{if } p \nmid \binom{n-j}{i-j} \end{cases}.$$

The dimension of this module is then:

$$\begin{aligned}\dim_k(P_i^n(m)) &= \sum_{j=0}^{m} \dim_k(L^{(m-j,j)})) \\ &= \sum_{p \nmid \binom{n-j}{i-j}} \dim_k(S^{(m-j,j)})) \\ &= \sum_{p \nmid \binom{n-j}{i-j}} \binom{m}{j} - \binom{m}{j-1},\end{aligned}$$

where the last equality is due to Theorem 6.1.3. The rank of the inclusion matrix $A_i^n(m)$ over k is the dimension of the $k\mathcal{S}_n$-module $P_i^n(m)$, thus proving the result. □

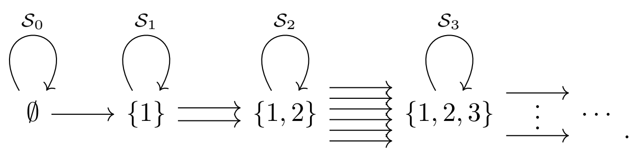
This gives a representation theoretic interpretation of this well-known rank formula as the dimension of a $k\mathcal{S}_n$-module spanned by the columns of the inclusion matrix. This module came up in an investigation of submodules of the representable **FI**-modules, which are discussed in the final section of this dissertation.

6.2 FI-modules

FI-modules were introduced by Church, Ellenberg and Farb and appear naturally in many areas of mathematics [CEF15]. They encode the structure of a sequence of representations of the symmetric group, and there is some hope that studying these objects may lead to new information on the representation theory of the symmetric group. The acronym **FI** comes from **F**inite sets and **I**njective maps, which form the objects and morphisms, respectively, in the category **FI**. We shall abuse notation and refer to a skeleton of this category as the category **FI**.

Definition 6.2.1. *The category* **FI** *has as its objects the sets* $[n] := \{1, 2, \ldots, n\}$ *for* $n \in \mathbb{N}_0$*, where* $[0] := \emptyset$ *and as its morphisms are all injections between these sets.*

Clearly the set of morphisms from an object $[n]$ to itself, together with their composition, forms a group isomorphic to the symmetric group on n elements. Also, all injections from $[n]$ to $[m]$ can be realised as the composition of an injection from $[n]$ to $[n+1]$, an injection from $[n+1]$ to $[n+2]$, and so on, up to an injection from $[m-1]$ to $[m]$. In particular, we may visualise the category **FI** as below:



Definition 6.2.2. *An* **FI**-module *over a commutative ring R is a functor from the category* ***FI*** *to the category of R-modules.*

We will assume throughout that the ring R is actually a field. Observe that the **FI**-module V assigns, to each object in **FI**, an $R\mathcal{S}_n$-module, which we denote by $V(n) = V_n$ and will refer to as the nth *layer* of V. We will often write an **FI**-module V by listing its layers in a column as shown:

$$V = \begin{matrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ \vdots \end{matrix} .$$

We shall denote the image of an injection $\alpha : [m] \to [n]$ under V by α_*, or, by abuse of notation, when the context is clear, simply by α. Such a map α_* is $\mathcal{S}_m$-equivariant with respect to the action of $\mathcal{S}_m$ on V_m and of $\mathcal{S}_m \subseteq \mathcal{S}_n$ on V_n. The action of $\mathcal{S}_n$ on V_n means that we only need to define the image of the natural embedding $\iota_{n,n+1} : [n] \to [n+1]$.

This means that an **FI**-module is a sequence of Sn-representations, together with $\mathcal{S}_n$-equivariant maps $(\iota_{n,n+1})_* : V_n \to V_{n+1}$.

Example 6.2.3. • *The most obvious example of an* **FI***-module is the 0 module*

$$0 = \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{matrix}$$

where the $\mathcal{S}_n$-equvariant maps $(\iota_{n,n+1})_* : V_n \to V_{n+1}$ are the maps which send 0 to 0.

- *Slightly more interesting is the* **FI***-module*

$$V = \begin{matrix} V_0 \\ V_1 \\ V_2 \\ V_3 \\ \vdots \end{matrix}$$

where V_n is any $k\mathcal{S}_n$-module, and all maps are the 0 map.

- *We may also define the* **FI***-module*

$$\mathbf{R} = \begin{matrix} R \\ R \\ R \\ R \\ \vdots \end{matrix}$$

where all maps are isomorphisms.

Not all sequences of $k\mathcal{S}_n$-modules together with $\mathcal{S}_n$-equivariant maps give rise to an **FI**-module. The following result tells us which sequences occur:

Theorem 6.2.4 (The **FI**-module criterion)**.** *[Wil18] Suppose that $\{W_n\}$ is a sequence of $\mathcal{S}_n$-representations with $\mathcal{S}_n$-equivariant maps $\phi_n : W_n \to W_{n+1}$. Let $G \cong \mathcal{S}_{n-m}$ be the stabiliser of $\iota_{m,n}$, the natural embedding, under post-composition. Then W_n with maps*

$(\iota_{n,n+1})_* = \phi_n$ *is an **FI**-module if and only if for all* $m < n$, $\sigma \in G$ *and* $v \in im((\iota_{n,m})_*)$,

$$\sigma \cdot v = v.$$

Definition 6.2.5. *A* submodule U *of an **FI**-module* V *is a sequence of* $\mathcal{S}_n$*-sub-representations* $U_n \subseteq V_n$ *that is closed under the action of the **FI** morphisms.*

Given an **FI**-module V, we can always form an family of submodules by truncation at a layer N, by taking the submodule $U_n \subseteq V_n$ to be the 0 module for $n < N$, and setting $U_n = V_n$ otherwise. This is closed under **FI** morphisms because we only have **FI** morphism $\alpha_* : V_m \to V_n$ when $m \leq n$. We denote the module obtained from V by truncation at layer N by $V^{\geq N}$.

$$\mathbf{V}^{\geq \mathbf{N}} = \begin{matrix} \vdots \\ 0 \\ V_N \\ V_{N+1} \\ \vdots \end{matrix}$$

We say an **FI**-module V is *generated* by a set $S \subseteq \bigsqcup_{n \geq 0} V_n$ if V is the smallest **FI** module containing S. We say that V is *finitely generated* if it is generated by some finite set S, and say that V is *generated in degree* d if d is the smallest integer d such that V is generated by $S = \bigsqcup_{n=0}^{d} V_d$. Observe that if V is finitely generated then it necessarily has *finite generation degree*, however V may be of finite generation degree, but not finitely generated if some V_n is not finitely generated as an $\mathcal{S}_n$-representation. This gives us another way to construct submodules of V, namely by taking the submodule generated by a subset $S \subseteq \bigsqcup_{n \geq 0} V_n$. We shall dedicate the the rest of this dissertation to finding submodules of a particular class of **FI**-modules, namely the representable **FI**-modules:

Definition 6.2.6. *Let* $i \in \mathbb{N}$. *Define the **FI**-module* P_i *by*

$$P_i(n) := R \cdot \mathrm{Hom}_{\mathbf{FI}}([i], [n]),$$

*with the **FI** morphisms acting by post-composition. We call an **FI**-module of this form* representable.

Observe that if $i < n$ then $P_i(n) = 0$, while if $i \geq n$ then $P_i(n) = M^{(n-i,1^i)}$, so each layer of a representable module is a Young Permutation Module M^λ. We will identify the element $\alpha \in Hom_{\boldsymbol{FI}}([i], [n])$ with the $(n-i, 1^i)$-tabloid whose $j+1$th row is the singleton $\{\alpha(j)\}$. The inclusion map $(\iota_{n,n+1})_* : M^{(n-i,1^i)} \to M^{(n+1-i,1^i)}$ sends the $(n-i,1^i)$-tabloid whose jth row is the singleton $\{t_j\}$, to the $(n+1-i,1^i)$-tabloid whose jth row is the singleton $\{t_j\}$; that is, $(\iota_{n,n+1})$ extends the first row of a tabloid $\{t\}$ by adding the entry $n+1$. This is an important class of **FI**-modules because they play the role of "free" **FI**-modules, in the sense that finitely generated **FI**-modules appear as quotients of representable **FI**-modules.

$$P_i(n) = \begin{cases} 0 & \text{if } n < i \\ M^{(n-i,1^i)} & \text{if } n \geq i \end{cases},$$

6.2.1 Submodules of P_i

Of course we immediately see an infinite family of submodules $P_i^{\geq N}$ obtained by truncation, so we now turn to see what other submodules of P_i we can find. Our general strategy for finding submodules of an **FI**-module V is to take a set $S \subseteq \bigsqcup_{n\geq 0} V_n$ and to find the module that is generated by this set. Of course this must will give rise to an $\mathcal{S}_n$-submodule of V_n at each layer, so we may assume that S consists of submodules at each layer, for example truncation at N is achieved by taking $S = \bigsqcup_{n\geq N} V_n$. Apart from truncation, we shall only consider submodules generated by a set S from a single layer of V. In particular, S will be a $k\mathcal{S}_n$-submodule of some V_n, as other submodules are obtained from taking the sums of the the submodules generated by a submodule at a single layer.

As seen above the representable **FI**-module P_i has a Young permutation module M^λ in each layer. Of course each M^λ has many submodules. We shall describe the **FI**-submodule generated by a number of these submodules.

0 Module

The most obvious and least interesting submodule of P_i is that generated by the 0 module $0 \subseteq P_i(n)$. Of course this is just the 0-**FI**-module, which has the 0 module at every layer.

Trivial Module

Each non-zero layer has a submodule isomorphic to the trivial module, which is spanned by the sum of all $(n-i,1^i)$-tabloids. We will denote the **FI**-module generated by the trivial submodule of $P_i(n)$ by P_i^n. The image of the trivial module $P_i^n(n)$ under the natural inclusion $(\iota_{n,m})_*$ is the sum of tabloids with $\{n+1, n+2, \ldots, m\}$ in their first row, or equivalently, the sum of all $(m-i,1^i)$-tabloids whose last i rows contain entries from $[n] \subseteq [m]$. Any subset of $[n]$ of size i appears in the last i rows of a such a tabloid in each possible permutation, so we may identify this with a sum of $(m-i,i)$-tabloids. This shows that $P_i^n(m)$ is a submodule of $M^{(m-i,i)}$, namely the one generated by the elements $\sum_{X\subseteq_i Y}\{t_X\}$ where Y is a subset of $[m]$ of size n, the tabloid $\{t_X\}$ is the $M^{(m-i,i)}$-tabloid whose second row is X and the sum is over all subsets $X \subseteq Y$ of size i. The module $P_i^n(m)$ is the module spanned by the columns of the inclusion matrix $A_i^n(m)$, which was studied by in the previous section.

Proposition 6.2.7. *Let k be a field of characteristic 0 or $p > n$, then*

$$P_i^n(m) = \begin{cases} 0 & \text{if } m < n \\ M^{(n,m-n)} & \text{if } n \le m \le n+i \\ M^{(m-i,i)} & \text{if } m > n+i \end{cases}.$$

Observe that we can obtain more submodules of P_i by truncation of this family. Of course truncating P_i^n at $m \ge n+i$ will result in the module $P_n^{\ge m}$, while truncating P_i^n at any $n < m < n+i$ results in a module we have not already seen.

Proposition 6.2.8. *Let k be a field of characteristic $p \le n$, then*

1. *$P_i^n(m) = 0$ if $m < n$*

2. *$P_i^n(m) \subseteq M^{(m-i,i)}$ has dimension $\sum_{p \nmid \binom{m-i-j}{m-n-j}} \binom{m}{j} - \binom{m}{j-1}$, if $n \le m \le n+i$*

3. *$P_i^n(m)$ has a composition series whose factors are the Specht modules $S^{(m-j,j)}$ if $p \nmid \binom{n-j}{i-j}$ if $m > n-i$.*

In particular, if $m > n-i$ then

$$P_i^n(m) \sim \begin{matrix} L^{(m)} \\ L^{(m-1,1)} \\ \vdots \\ L^{(m-i,i)} \end{matrix},$$

with

$$L^{(m-j,j)} = \begin{cases} 0 & \text{if } p \mid \binom{n-j}{i-j} \\ S^{(m-j,j)} & \text{if } p \nmid \binom{n-j}{i-j} \end{cases}.$$

Proof. The first part is trivial, the second follows from the observation that $A_i^n(m)^T = A_{m-n}^{m-i}(m)$ and Theorem 6.1.1, while the final part follows from the proof of Theorem 6.1.1. □

Specht module

The Specht Module S^λ is the submodule of M^λ spanned by the polytabloids. The image of a polytabloid e_t under the map $(\iota_{n,m})_*$ is simply the polytabloid $e_{t'}$, where t' is the tableau obtained from t by adding $m-n$ boxes to the first row of t and filling them with the entries $n+1, n+2, \ldots, m$, in any order, and so each layer of the submodule generated by a Specht module is a Specht module. This allows us to define the following family of **FI**-modules:

Definition 6.2.9. *Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$. Define an FI module S^λ_{FI} by*

$$S^\lambda_{FI}(m) = \begin{cases} 0 & \text{if } m < n \\ S^{(\lambda_1+m-n,\lambda_2,\ldots,\lambda_r)} & \text{if } m \geq n \end{cases}$$

with the maps

$$\iota : S^\lambda_{FI}(m) \to S^\lambda_{FI}(m+1)$$

sending the polytabloid $e(t)$ to the polytabloid $e(t')$, where t' is the $(\lambda_1+m-n+1, \lambda_2, \ldots, \lambda_r)$ tableau obtained by adding a node containing $(n+1)$ to the first row of t.

The fact that these are indeed **FI**-modules follows from Theorem 6.2.4. At any layer $n > i$ the submodule generated by the Specht module $S^{(n-i,1^i)}$ is $S^{(n-i,1^i)}_{FI}(m)$. Observe that this is simply the submodule $S^{(1^{i+1})}_{FI}(m)$ truncated at n.

We do, however, get a different example of an **FI**-submodule of the representable **FI**-module P_i if we take the submodule generated by $S^{(1^i)}$ at layer i. This gives the module $S^{(1^i)}_{FI}(m)$ can be truncated at any layer $n > i$ to obtain a submodule different to any of those previously described.

Remark. *Over fields of positive characteristic $S^{(1^{i+1})}_{FI}$ contains a proper submodule, V, which is not obtained by truncation for each $mp \geq n$ as follows:*

$$V(k) = \begin{cases} 0 & \text{if } k < mp \\ D^{(k-i+1,1^{i-1})^r} & \text{if } k = mp \\ S^{(k-i,1^i)} & \text{if } k > mp, \end{cases}$$

where λ^r denotes the p-regularisation of λ. Similarly, $S^{(1^i)}_{FI}$ contains the proper submodules,

$$V^m(k) = \begin{cases} 0 & \text{if } k < mp \\ D^{(k-i,1^i)^r} & \text{if } k = mp \\ S^{(k-i,1^{i-1})} & \text{if } k > mp \end{cases}.$$

Specht-shadow modules

The previous example can be seen as a special case of the **FI**-submodule generated by the Specht-shadow module $S^{\mu\#,\mu}$ at layer $|\,\mu\,|$. Recall that the Specht-shadow module $S^{\mu\#,\mu}$ is spanned by the $\mu\#$-polytabloids. Similar to above, the image of a $\mu\#$-polytabloid under the map $(\iota_{n,m})_*$ is simply the $\mu\#$-polytabloid obtained from t by adding $m-n$ boxes to the first row and filling them with the entries $n+1, n+2, \ldots, m$, in any order. This means that if we define the **FI**-module $S_{FI}^{(n-i,1^j)\#,(n-i,1^i)}$ as follows, then $S_{FI}^{(n-i,1^j)\#,(n-i,1^i)}$ is a submodule of P_i for all $n \geq i$:

Definition 6.2.10. *Let $j \leq i$, then $(n-i,1^j)\#,(n-i,1^i)$ is a pair of partitions as in [Jam78][15.5]. Define an FI module $S_{FI}^{(n-i,1^j)\#,(n-i,1^i)}$ by*

$$S_{FI}^{(n-i,1^j)\#,(n-i,1^i)}(m) = \begin{cases} 0 & \text{if } m < n \\ S^{(\lambda_1+m-n,1^j)(\lambda_1+m-n,1^i)} & \text{if } m \geq n \end{cases}$$

with the maps

$$\iota : S_{FI}^{(n-i,1^j)\#,(n-i,1^i)}(m) \to S_{FI}^{(n-i,1^j)\#,(n-i,1^i)}(m+1)$$

sending the j-polytabloid $e(t)$ to the polytabloid $e(t')$, where t' is the $(\lambda_1+m-n+1,1^i)$ tableau obtained by adding a node containing $(n+1)$ to the first row of t.

Other Permutation Modules

We can find copies of many other permutation modules M^μ as a submodule of $M^{(n-i,1^i)}$. In fact we can obtain M^μ for any $\mu \vdash n$ with $\mu_1 = n-i$:

$M^{(n-i,1^i)}$ has a basis consisting of all $(n-i,1^i)$-tabloids, which we can identify with all ordered i-tuples. If we define an equivalence relation on this basis by sating two tabloids $\{s\}$ and $\{t\}$ are μ-equivalent if the entries in row 1 of $\{s\}$ and $\{t\}$ are the same, and the entries appearing in the next μ_2 rows of $\{s\}$ and $\{t\}$ the same, and the next μ_3 rows and so on. For example:

$$\begin{array}{l} \hline 1\ 2\ 3\ 4 \\ \hline 5 \\ \hline 6 \\ \hline 7 \\ \hline 8 \\ \hline 9 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{l} \hline 1\ 2\ 3\ 4 \\ \hline 6 \\ \hline 5 \\ \hline 8 \\ \hline 7 \\ \hline 9 \\ \hline \end{array}$$

are $(4,2,2,1)$-equivalent. The submodule spanned by the sums of the μ-equivalence classes is isomorphic to M^μ. For example, we identify the sum

$$\begin{array}{l}\hline 1\ 2\ 3\ 4\\ \hline 5\\ \hline 6\\ \hline 7\\ \hline 8\\ \hline 9\\ \hline\end{array} + \begin{array}{l}\hline 1\ 2\ 3\ 4\\ \hline 5\\ \hline 6\\ \hline 8\\ \hline 7\\ \hline 9\\ \hline\end{array} + \begin{array}{l}\hline 1\ 2\ 3\ 4\\ \hline 6\\ \hline 5\\ \hline 7\\ \hline 8\\ \hline 9\\ \hline\end{array} + \begin{array}{l}\hline 1\ 2\ 3\ 4\\ \hline 6\\ \hline 5\\ \hline 8\\ \hline 7\\ \hline 9\\ \hline\end{array},$$

with the $(4,2,2,1)$-tabloid

$$\begin{array}{l}\hline 1\ 2\ 3\ 4\\ \hline 5\ 6\\ \hline 7\ 8\\ \hline 9\\ \hline\end{array}.$$

Clearly the image of a $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ class sum of $(n-i, 1^i)$-tabloids is a $(\mu_1 + 1, \mu_2, \ldots, \mu_r)$ class sum of $(n-i+1, 1^i)$-tabloids. It follows that:

Proposition 6.2.11. *Let $\mu \vdash n$ with $\mu_1 = n - i > 0$, and define*

$$M^\mu_{FI}(m) = \begin{cases} 0 & \text{if } m < n \\ M^{(\mu_1+m-n,\mu_2,\ldots,\mu_r)} & \text{if } m \geq n \end{cases}$$

with the maps

$$\iota : M^\mu_{FI}(m) \to M^\mu_{FI}(m+1)$$

sending the class sum $\{t\}$ to the class sum $\{t'\}$, where $\{t'\}$ is the $(\mu_1+m+1-n, \mu_2, \ldots, \mu_r)$ tabloid obtained by appending $(n+1)$ to the first row of $\{t\}$. Then, $M^\mu_{FI}(m) \subseteq P_i$.

Of course the permutation module M^μ contains a Specht module S^μ, which can be realised inside $M^{(n-i,1^i)}$ as the span of appropriate signed sums of sums of μ-equivalence classes. These signed sums play the role of polytabloids, and it follows that the representable **FI**-module P_i has the Specht **FI**-module $S^\mu_{\mathbf{FI}}$ as an **FI**-submodule for all $\mu \vdash m+i$ with $\mu_1 = m$.

Some submodules of P_5

We conclude this chapter by giving a number of examples of submodules of the representable **FI**-module P_5 over a field of characteristic 0. We will only write the layers, as the maps are given by restriction of the maps in P_5.

P_5	0	P_5^5	P_5^6	P_5^7	$S_{\mathbf{FI}}^{(1^5)}$	$S_{\mathbf{FI}}^{(1^6)}$	$S_{\mathbf{FI}}^{(3,1^3)\#(3,1^5)}$	$M_{\mathbf{FI}}^{(4,3,2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	0	0	0	0	0	0	0
$M^{(1^5)}$	0	k	0	0	$S^{(1^5)}$	0	0	0
$M^{(1^6)}$	0	$M^{(5,1)}$	k	0	$S^{(2,1^4)}$	$S^{(1^6)}$	0	0
$M^{(2,1^5)}$	0	$M^{(5,2)}$	$M^{(6,1)}$	k	$S^{(3,1^4)}$	$S^{(2,1^5)}$	0	0
$M^{(3,1^5)}$	0	$M^{(5,3)}$	$M^{(6,2)}$	$M^{(7,1)}$	$S^{(4,1^4)}$	$S^{(3,1^5)}$	$S^{(3,1^3)\#(3,1^5)}$	0
$M^{(4,1^5)}$	0	$M^{(5,4)}$	$M^{(6,3)}$	$M^{(7,2)}$	$S^{(5,1^4)}$	$S^{(4,1^5)}$	$S^{(4,1^3)\#(4,1^5)}$	$M^{(4,3,2)}$
$M^{(5,1^5)}$	0	$M^{(5,5)}$	$M^{(6,4)}$	$M^{(7,3)}$	$S^{(6,1^4)}$	$S^{(5,1^5)}$	$S^{(5,1^3)\#(5,1^5)}$	$M^{(5,3,2)}$
$M^{(6,1^5)}$	0	$M^{(6,5)}$	$M^{(6,5)}$	$M^{(7,4)}$	$S^{(7,1^4)}$	$S^{(6,1^5)}$	$S^{(6,1^3)\#(6,1^5)}$	$M^{(6,3,2)}$
$M^{(7,1^5)}$	0	$M^{(7,5)}$	$M^{(7,5)}$	$M^{(7,5)}$	$S^{(8,1^4)}$	$S^{(7,1^5)}$	$S^{(7,1^3)\#(7,1^5)}$	$M^{(7,3,2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The leftmost column is the representable **FI**-module P_5 and the second column is the trivial example of the 0 submodule. The next three columns are examples of the modules generated by the trivial modules at layers 5, 6, and 7, respectively. The layers of these three **FI**-submodules have dimensions given by the rank of inclusion matrices. The remaining columns are examples of **FI**-submodules generated by various permutation, Specht and Specht-shadow modules at different layers. Truncating any of these modules at any layer also gives an **FI**-submodule, thus the examples given in this section give rise to a large number of **FI**-submodules of the representable **FI**-modules.

Chapter 7

Conclusions

Throughout this dissertation we have developed and applied a number of combinatorial techniques for use in the study of the representation theory of the symmetric group. This has allowed new decomposition numbers of Specht modules to be calculated and new extensions of the trivial $k\mathcal{S}_n$-module by Specht modules to be constructed. While the main motivation of this dissertation is to use combinatorial techniques to solve representation theoretic questions, and thus remaining in the setting of the symmetric group and avoiding the use of the theory of algebraic groups and the Schur functor, these combinatorial tools are of independent interest. In particular, this dissertation lays the foundation for the further study of p-ary designs. In the final chapter of this dissertation a new proof of the rank formula is obtained demonstrating the applications of the representation theory of the symmetric group to solving combinatorial problems.

In Chapter 4 we developed the theory of p-ary designs, which generalise the much more widely studied integral designs seen in [GJ73], for example. This modular theory is more complicated as t-designs over $\mathbb{Z}_p$ are more common than t-designs over $\mathbb{Z}$. Although in this new theory not all designs are universal, we see that universal designs are also more common over $\mathbb{Z}_p$. The main achievement of this chapter is an existence and uniqueness style theorem for universal p-ary designs. We construct, up to similarity, all universal designs of constant block-size on a finite set. This brings our knowledge of the theory of p-ary designs up to point seen in [GJ73] for the theory of integral designs. Many more design theoretic questions remain open in the modular case so this will become a rich field of study. We also anticipate many more applications of the theory of p-ary designs, beyond those discussed later in this dissertation in Chapter 5.

Schaper's filtration of the Specht modules, studied in the fourth chapter, provides a tool to determine decomposition numbers for the symmetric group. The Schaper number of a partition counts the number of empty layers at the top of the Schaper filtration of the Specht module indexed by that partition and knowledge of this number improves the upper bound on the decomposition numbers given by Schaper's sum formula. In some cases this improved bound will allow new decomposition numbers to be calculated exactly. We see in this chapter that a classification of partitions whose Schaper number is at least 3 must separate the case $p = 2$ from the odd prime cases, and we complete this classification when $p = 2$, as well as the classification of partitions whose Schaper number is at least 4. The odd characteristic case is dealt with by Theorem 3.4.1 and Conjecture 3.4.2 and evidence for this conjecture is presented in this chapter. We also demonstrate how this can be used to calculate new decomposition numbers for the symmetric group, calculating a new decomposition number for $\mathcal{S}_{20}$ when $p = 2$. This is the smallest symmetric group for which the decomposition matrix is unknown for $p = 2$, and future work utilising the results of this chapter will allow more decomposition numbers to be determined.

As well as dealing with the decomposition numbers for the symmetric group, this dissertation deals also with extensions of the trivial $k\mathcal{S}_n$-module by the Specht modules. In Chapter 5 we see how the combinatorial tools developed in Chapter 4 allow the extensions by Specht modules indexed by two part partitions to be calculated. Extensions of the trivial $k\mathcal{S}_n$-module by these Specht modules are in bijection with universal p-ary designs which are not similar to the constant design, so our work on design theory gives a complete characterisation of such extensions. For partitions with more than two parts we seek to find Hemmer elements, which generalise partitions and describe the extensions of the trivial $k\mathcal{S}_n$-module by these Specht modules. We see a number of families of partitions for which no Hemmer elements, and therefore no non-split extensions by the corresponding Specht module, exist. We also give restrictions on the number of non-split extensions of the trivial module by a given Specht module, up to equivalence. In particular, we observed that there is at most one non-split extension if the partition is not James. A number of remarks and examples demonstrate how to find Hemmer elements, or extensions, in general. If these Hemmer elements could be classified then we would obtain a purely combinatorial understanding of the first degree cohomology of the Specht modules and an explicit description of the extensions of the trivial $k\mathcal{S}_n$-module by the Specht modules.

In the final chapter of this thesis we saw an introduction into the theory of **FI**-modules, which encode information about a sequence of representations for the symmetric group. In particular we study representable **FI**-modules and describe a number of families of submodules of **FI**-modules, namely those generated by $k\mathcal{S}_n$-submodules of interest. The **FI**-submodule generated by a trivial $k\mathcal{S}_n$-module gives rise to a $k\mathcal{S}_m$-module whose dimension is given by the rank of the inclusion matrix. This gives rise to a new proof of the rank of this matrix, which was known due to Richard Wilson, as well as a representation theoretic interpretation of this fact.

Combinatorial tools have proven to be effective at dealing with a number of problems in the representation theory of the symmetric group, especially those dealing with the Specht modules, which have a combinatorial definition. This allows us to employ tools such as Schaper's filtration when attempting to understand the structure of Specht modules, and the theory of combinatorial designs when studying their extensions. On the other hand, the representation theory of the symmetric group provides motivation for the study of combinatorial designs over fields of characteristic p and gives an algebraic interpretation of the inclusion matrix. The interaction of these two subjects, combinatorics and representation theory of the symmetric group, gives rise to much rich and interesting mathematics which this dissertation gives a sample of. Future work on the representation theory of the symmetric group should be focused on the combinatorics which underpin this theory. This combinatorics is interesting in its own right and may have many other applications, but strong motivation for this study is to develop an approach to the representation theory of the symmetric group which remains purely in the setting of the symmetric group.

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