Discrete approximate subgroups of Lie groups

Simon Machado

Clare College
University of Cambridge
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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. I further state that no substantial part of my thesis has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee

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Abstract

We study generalisations of a theorem of Yves Meyer concerning the structure of approximate lattices. An approximate lattice is a discrete approximate subgroup of a locally compact group - i.e. a subset that is closed under multiplication up to an error controlled by a finite set - that has finite co-volume. We study, successively, generalisations of Meyer's theorem in soluble Lie groups, in amenable locally compact groups and in higher-rank semi-simple algebraic groups.

Along the way, we investigate properties of closed and discrete approximate subgroups of locally compact groups in general.

Contents

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Chapter 1

Introduction

1.1 Discrete approximate subgroups in Lie groups

The main goal of this thesis is to study a fascinating family of aperiodically ordered sets of Euclidean spaces. The so-called *Meyer sets* were introduced by Yves Meyer in his pioneering book [60]. They are defined as those subsets $\Lambda \subset \mathbb{R}^n$ that satisfy:

(I) Λ is an (R, r)-*Delone set*:

1. Λ is *r-uniformly discrete*: $\inf_{\lambda_1 \neq \lambda_2 \in \Lambda} |\lambda_1 - \lambda_2| \geq r > 0$;
2. Λ is *R-relatively dense*: $\sup_{x \in \mathbb{R}^n} |x - \Lambda| \leq R$;

(II) there exists a finite subset F such that

$$\Lambda - \Lambda := \{\lambda_1 - \lambda_2 : \lambda_1, \lambda_2 \in \Lambda\} \subset F + \Lambda.$$

Meyer sets appear in a wide variety of situations: number theory with the Pisot numbers ([60]), aperiodic tilings of Euclidean spaces with the Penrose tiling (P3) ([29, 28]), mathematical models of quasi-crystals ([70]), Fourier-analytic considerations ([60]). See the survey [62] for this and more.

The ubiquity of Meyer sets makes Meyer's main theorem even more striking. He proved that, even though they are highly aperiodic objects, Meyer sets originate in a precise way from higher-dimensional lattices. His theorem can be stated as follows.

Theorem 1.1.1 (Meyer, Theorem IV, [60]). *Let $\Lambda \subset \mathbb{R}^n$ be a Meyer set. There is an integer $m \geq 0$ and a lattice $\Gamma \subset \mathbb{R}^n \times \mathbb{R}^m$ such that*

$$\Lambda \subset p_{\mathbb{R}^n}\left(\Gamma \cap (\mathbb{R}^n \times [-1; 1]^m)\right)$$

where $p_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denotes the natural projection.

Subsets of the form $p_{\mathbb{R}^n}\left(\Gamma \cap (\mathbb{R}^n \times [-1; 1]^m)\right)$ - the *model sets* - are Meyer sets of a very interesting kind. Often, investigating properties of model sets

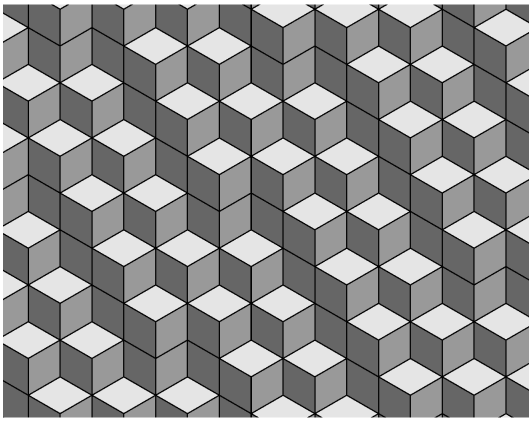

Figure 1.1: Vertices are a model set

reduces to investigating the same properties for the higher-dimensional lattice Γ it comes from. Therefore, Theorem 1.1.1 has had numerous applications (see [60, 70, 62, 4]) and, in particular, reshaped the way we think about quasicrystals ([3]).

Figure 1.1 below provides a good illustration of Theorem 1.1.1. There the model set corresponds to the set of vertices of the tiles. To obtain this kind of picture, one considers an irrational plane H in the three dimensional Euclidean space $\mathbb{R}^3$, and projects - via the orthogonal projection - all the faces of unit squares with integer coordinate corners that intersect H in their interior.

In general, model sets are built in a similar way. The triple $(\mathbb{R}^n, \mathbb{R}^m, \Gamma)$ is called the *cut-and-project scheme*. And a model set is obtained by first "cutting a strip" of points of Γ along $\mathbb{R}^n$ - the subset $\Gamma \cap \mathbb{R}^n \times [-1;1]^m \subset \mathbb{R}^n \times \mathbb{R}^m$ - and taking the projection of that strip to $\mathbb{R}^n$ parallel to $\mathbb{R}^m$ seen as subsets of $\mathbb{R}^n \times \mathbb{R}^m$.

Condition (II) asserts that Meyer sets belong to a larger class of objects, the so-called *approximate subgroups*. A K-approximate subgroup, as defined by Tao [73, Definition 3.8], is a subset Λ of a group G such that $e \in \Lambda$, $\Lambda = \Lambda^{-1}$ and there exists a subset F in G of size at most K with

$$\Lambda^2 := \{\lambda_1\lambda_2 : \lambda_1, \lambda_2 \in \Lambda\} \subset F\Lambda.$$

Loosely speaking, Λ satisfies all the conditions to be a subgroup - up to a multiplicative error controlled by a subset of size K.

The theory of approximate subgroups, with an emphasis on *finite* approximate subgroups, has seen dramatic developments occur over the past two decades. The starting point of these investigations most probably lies, however, in the striking theorem of Freiman [37] - proved decades before the term "approximate subgroups" was coined by Tao - that classifies finite approximate subgroups of $\mathbb{Z}^n$. It asserts roughly that given a finite K-approximate subgroup Λ of $\mathbb{Z}^n$ there are non-negative integers $N_1, \dots, N_r$ and integer coordinate vectors $q_1, \dots, q_r$ such that

$$\Lambda \subset Q := \{l_1 q_1 + \ldots + l_r q_r : \forall i, |l_i| \leq N_i\},$$

with both $|Q|/|\Lambda|$ and r bounded above by constants depending on K only.

Meyer's theorem and Freiman's theorem can be summed up in very similar terms. Meyer's theorem asserts that Meyer sets - that is, essentially, the class of Delone approximate subgroups of Euclidean spaces - behave, with respect to the additive law, as boxes of the form $[-1;1]^m$ in some Euclidean space; Freiman's theorem asserts that finite K-approximate subgroups of $\mathbb{Z}^n$ behave like boxes $\prod_{i=1}^r \{-N_i, \dots, N_i\}$ in some $\mathbb{Z}^r$. This resemblance can, in fact, be made very precise in at least two ways. First of all, an *ad hoc* way that entails noticing that Meyer sets are inductive limits of finite K-approximate subgroups for some fixed K (see the work of Konieczny [50]). This idea immediately provides new results about Meyer sets, and shows that Meyer's theorem can be seen as a consequence of Freiman's theorem. There is also a more abstract way, that shows that both theorems essentially reduce to finding the right *model* of an approximate subgroup (see Definition 1.1.9). We investigate this at length in Chapter 4.

The theory of finite approximate subgroups in non-commutative groups goes far beyond Freiman's theorem. Possibly the most complete result is the Breuillard–Green–Tao structure theorem [21] that asserts that a K-approximate subgroup of a group G is the extension of a subgroup by a nilprogression (a nilpotent generalisation of arithmetic progressions). A core idea in [21] was to exploit a seminal result due to Hrushovski [46] that allowed one to find the right way to model a pseudo-finite approximate subgroup. This enabled Breuillard, Green and Tao to study first order properties of families of finite approximate subgroups by means of the structure theory of locally compact groups due to Gleason and Yamabe [81] and Montgomery–Zippin [61]. For approximate subgroups of $\mathrm{GL}_n(K)$, the product theorem due to Helfgott [43], later generalised by Breuillard–Green–Tao [20] and Pyber–Szabo [64], provides finer information. For instance, fascinating applications to spectral gaps in $\mathrm{SL}_2(\mathbb{F}_p)$ and $\mathrm{SU}_n(\mathbb{C})$ were made possible through the Bourgain–Gamburd machine [18].

For infinite approximate subgroups, while less is known, some substantial results are available. The aforementioned work of Gleason and Yamabe [81] and Montgomery and Zippin [61] can be regarded as the first study of infinite approximate subgroups of a special kind: the compact neighbourhoods of the identity of locally compact groups. Since, Kreitlon-Carolino generalised the Breuillard–Green–Tao theorem to provide a structure theorem for open and relatively compact approximate subgroups in locally compact groups [24] - effectively improving on the results of Gleason–Yamabe and Montgomery–Zippin. In the same line of ideas is Weil's work about building topologies for measurable groups (see [41, §62]) whose influence reached the work of, among others, Hrushovski [46], Massicot–Wagner [59], Sanders [68], Croot–Sisask [27]. More recently, de Saxcé proved a product theorem in compact simple Lie groups [6] which was extended to perfect compact Lie groups by de Saxcé–He [42]. De Saxcé also used these ideas to classify approximate subgroups of simple Lie groups with intermediate Hausdorff dimension [31]. Hrushovski established in [46] a structure theorem for infinite approximate subgroups satisfying model-theoretic assumptions. In his ground-breaking work [47], he recently extended this result in a weakened form but valid for *all* approximate subgroups.

Another direction is the study of discrete approximate subgroups of Lie groups. In the abelian set-up, some of the early results come from the aforementioned work of Meyer [60] or the work of Lagarias on quasi-regular sets and their link with Meyer sets [51]. The systematic study of discrete approximate subgroups of non-commutative locally compact groups was initiated by Björklund, Hartnick and Pogorzelski ([12]) and Björklund and Hartnick ([9]). In [9], they studied discrete approximate subgroups of locally compact groups with a particular focus on *approximate lattices*: objects that simultaneously generalise lattices of locally compact groups and Meyer sets (note that certain discrete approximate subgroups of non-commutative Lie groups had however been studied before, for instance in the work of Chifan and Ioana [25]). In a series of papers starting with [9], they showed that approximate lattices share many properties with lattices. For instance, envelopes of approximate lattices are unimodular ([9, Theorem 1.1], see Theorem 2.2.22), the Borel density theorem generalises to approximate lattices (with co-author Stulemeijer [13]), and certain approximate lattices satisfy heredity properties with respect to Kazhdan-type and Haagerup-type properties of their ambient group ([11]). Furthermore, a more geometric study of discrete approximate subgroup was recently initiated by Cordes, Hartnick and Tonić [26].

In this thesis, our main motivation will be to further the study of approximate lattices and provide partial answers to a question Björklund and Hartnick raised in ([9, Problem 1]):

To which locally compact groups is it possible to extend Meyer's theorem?

To do so formally, Björklund–Hartnick defined generalisations of model sets as well as generalisations of Meyer sets - the aforementioned *approximate lattices.*

Model sets are straightforward to generalise [9, §2.3]. A triple (G, H, Γ) is called a *cut-and-project scheme* if G, H are locally compact groups and Γ is a lattice in $G \times H$ that projects injectively to G and densely to H. Take now a relatively compact symmetric neighbourhood of the identity $W_0 \subset H$. Then the set

$$p_G\left(\Gamma \cap (G \times W_0)\right)$$

where $p_G : G \times H \to G$ denotes the natural projection is called a *model set.*

In stark contrast, there are several paths one can take to generalise approximate lattices [9, 47, 54]. Broadly speaking, an approximate lattice in a locally compact group G is any uniformly discrete approximate subgroup that has "finite co-volume". We will mainly be interested in three types of approximate lattices in this thesis: *uniform approximate lattices* that are the closest in spirit to Meyer sets and were defined by Björklund and Hartnick ([9, Definition 2.6]), *approximate lattices* in the sense of Hrushovski [47, Definition A.1] and $\star$-*approximate lattices* (see Definition 2.2.15).

On the one hand, a uniformly discrete approximate subgroup Λ of a locally compact group G is an *approximate lattice* if there is a Borel subset $\mathcal{F} \subset G$ of finite volume such that $\Lambda\mathcal{F} = G$. If $\mathcal{F}$ is moreover relatively compact, then Λ is a *uniform approximate lattice*. On the other hand, $\star$-approximate lattices are defined by the existence of a proper invariant probability measure on a dynamical system associated to Λ that generalises the notion of quotient (G/Λ) when Λ is not a subgroup.

Let us provide a brief, but precise, description of what $\star$-approximate lattices are. Note that $\star$-approximate lattices were defined in [54] but this definition is largely inspired from the definition of *strong approximate lattices* due to Björklund and Hartnick (see [9]). Define the Chabauty space of G as the set $\mathcal{C}(G)$ of closed subsets of G. The space $\mathcal{C}(G)$ is naturally endowed with the Chabauty topology generated by the subsets $U^W := \{Y \in \mathcal{C}(G) : Y \cap W \neq \emptyset\}$ for W open and $U_K := \{Y \in \mathcal{C}(G) : Y \cap K = \emptyset\}$ for K compact. The space $\mathcal{C}(G)$ is then a compact space and the natural action $g, Y \mapsto gY$ is continuous. Given $X \in \mathcal{C}(G)$ we write Ω_X the closure $\overline{G \cdot X}$ of the orbit of X. The set Ω_X can be seen as a set of left-cosets of sorts of X - in fact, Ω_X is the set of those subsets that cannot be distinguished locally from a left-coset of X:

$$\Omega_X := \{Y \in \mathcal{C}(G) : \forall K \subset G \text{ compact}, \exists g \in G \text{ such that } Y \cap K = gX \cap K\}.$$

In particular, Ω_Γ is the one-point compactification of G/Γ when Γ is a subgroup - the extra point at infinity, when needed, being $\emptyset$ ([13, Lemma 2.3]). We will say that a uniformly discrete approximate subgroup is a $\star$-*approximate lattice* if there is $X \subset \Lambda$ such that there is a G-invariant Borel probability measure μ on Ω_X with $\mu(\{\emptyset\}) = 0$.

Out of the three, the notion of approximate lattices in the sense of Hrushovski is the more general. In what follows, when we simply say approximate lattices, we will therefore refer to approximate lattices in the sense of Hrushovski. The relation between these various notions is better summarised in the following:

Theorem 1.1.2 (Hierarchy of approximate lattices)**.** *We have:*

$$\begin{array}{ccccccc} \textit{Strong app. lattice} & \Rightarrow & \star - \textit{App. lattice} & \Rightarrow & \textit{App. lattice} & \Rightarrow & \textit{BH-app. lattice} \\ & & & & \Uparrow & & \\ & & & & \textit{Uniform app. lattice} & & \end{array}$$

See Section 2.2 for the definitions of *strong approximate lattices* and *approximate lattices in the sense of Björklund–Hartnick*.

Björklund, Hartnick and Pogorzelski showed that model sets also naturally fit in the above diagram:

Theorem 1.1.3 (Björklund–Hartnick–Pogorzelski, Theorem 1.1 [12])**.** *Let (G, H, Γ) be a cut-and-project scheme. Let W_0 be a symmetric relatively compact neighbourhood of the identity such that the boundary ∂W_0 is Haar-null. Then the model set $M := p_G(G \times W_0 \cap \Gamma)$ is a strong approximate lattice - where $p_G : G \times H \to G$ denotes the natural projection.*

The question raised by Björklund and Hartnick therefore reduces to proving or disproving a converse to Theorem 1.1.3. It becomes:

Question 1 (Björklund–Hartnick, Problem 1, [9])*.* Let Λ be an approximate lattice (resp. a uniform approximate lattice/$\star$-approximate lattice) in a locally compact group G. Is Λ contained in a model set?

Stating that Λ is simply *contained* in a model set understates the real phenomenon. This assertion is in fact equivalent to Λ being commensurable with a model set. We will say that two subsets X, Y of some group are *commensurable* if there is a finite set F such that $X \subset FY \cap YF$ and $Y \subset FX \cap XF$. Following [9] and [47], if Λ is commensurable with a model set, we will say that Λ is a *Meyer set*.

We point out that the wording here differs from the Euclidean space case. Indeed, in Euclidean spaces Meyer sets simply correspond to uniform approximate lattices. In contrast, in [9], [47] and in this dissertation, Meyer sets in general locally compact groups are defined as subsets that arise from the cut-and-project construction. In loose terms, by Meyer sets we will mean the sets that satisfy a version of a generalisation of Meyer's theorem.

In the amenable set-up, a complete answer to Question 1 stems from the simple, yet key, observation that links the theory of discrete approximate subgroups and finite approximate subgroups (see Theorem 1.1.10).

Theorem 1.1.4. *Let Λ be an approximate lattice in a locally compact amenable group G. Then Λ is a Meyer set.*

A natural counterpart to the case of amenable locally compact groups is given by semi-simple groups. We know thanks to the groundbreaking work of Hrushovski that Meyer's theorem can be extended to semi-simple algebraic groups.

Theorem 1.1.5 (Hrushovski, Corollary 6.11, [47])**.** *Let Λ be an approximate lattice in a group of points of a simple algebraic group over some local field, or a finite product of such groups. Then Λ is a Meyer set.*

The tools in the proof of Theorem 1.1.5 are model-theoretic and far from the combinatorial and ergodic-theoretic tools we will use and develop in this thesis.

Shortly before Hrushovski's theorem, we proved using tools developed in the proof of Margulis' arithmeticity theorem [58] a generalisation of Meyer's theorem to $\star$-approximate lattices in higher-rank semi-simple algebraic groups.

Theorem 1.1.6. *Let Λ be a $\star$-approximate lattice in a finite product G of simple algebraic groups over characteristic* 0 *local fields G. Suppose that G has no rank-*1 *factor. Then Λ is a Meyer set.*

To prove Theorem 1.1.6 we follow Margulis' original strategy. Our first step towards Theorem 1.1.6 is thus to prove a superrigidity statement with the help of Zimmer's cocycle superrigidity ([82]) applied to a cocycle naturally associated to Λ and first studied by Björklund and Hartnick in [11].

The Levi decomposition (see e.g. [45, VIII Theorem 4.3]) shows that affine algebraic groups over characteristic 0 fields are semi-direct products of soluble groups with semi-simple groups. In view of the above theorems, it is natural to wonder how approximate lattices behave with respect to the Levi decomposition. We provide perspectives on this question:

Theorem 1.1.7. *Let G be a group of points of an affine algebraic group over a characteristic* 0 *field, or a finite product of such groups. Let $\Lambda \subset G$ be an approximate lattice in G. Suppose that G has no compact factor and let R denote its radical. Then $\Lambda^2 \cap R$ is a uniform approximate lattice in R. In addition, the projection of Λ to G/R is an approximate lattice in G/R.*

The group G/R is known to be a product of semi-simple algebraic groups. Theorem 1.1.7 in combination with the positive answers of Question 1 in amenable and semi-simple Lie groups (Theorems 1.1.4 and 1.1.6) therefore gives hope towards a more general extension of Meyer's theorem.

In stark contrast with this intuition, we mention another result of Hrushovski that proves that Meyer's theorem cannot be extended to all locally compact groups.

Theorem 1.1.8 (Hrushovski, §7.9, [47])**.** *There is an approximate lattice in an extension of $\mathbb{Q}_p$ by $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$ that is* not *commensurable with a model set.*

Along the proofs of these three theorems (Theorems 1.1.4, 1.1.6 and 1.1.7), we show many other results of interest related to Meyer sets and their generalisations, as well as closed approximate subgroups of locally compact groups. Let us now describe in more details the material of this thesis.

Chapter 2: A primer on approximate subgroups, approximate lattices and good models

The purpose of Chapter 2 is threefold. First of all, we introduce several well-known and elementary results about apporximate subgroups such as Ruzsa's covering lemma (Lemma 2.1.6). Our main sources are a survey on growth in groups due to Helfgott [44], an article by Tao [73] and Tointon's book on approximate subgroups [74]. Note, however, that in the literature such results are often stated for *finite* approximate subgroups. We provide here statements fit for our purpose, along with proofs.

We then introduce approximate lattices and their most elementary properties. We write down definitions for the following types of approximate lattices:

1. *Uniform approximate lattices* that were defined by Björklund–Hartnick in [9];

2. *Strong approximate lattices* - also defined in [9] - with an ergodic-theoretic flavour;

3. $\star$-*approximate lattices* defined by the author in [54] (see Chapter 5) that can be seen as a more flexible version of the strong approximate lattices;

4. the object we will refer to as *approximate lattices* - first systematically studied by Hrushovski in [47];

5. the *approximate lattices* in the sense of Björklund and Hartnick [9].

In this thesis however, we will mainly be concerned with studying (1), (3) and (4). After defining these five notions of approximate lattices, we prove Theorem 1.1.2 about the hierarchy that exists among them.

We then turn to the last objective of this chapter. We wish to build a bridge between three fields: aperiodic order i.e. the classical theory of Meyer sets, the theory of finite approximate subgroups, and the theory of lattices in locally compact groups. Part of this goal will be achieved through the study of *good models.*

Definition 1.1.9. Let Λ be an approximate subgroup of a group Γ. A group homomorphism $f : \Gamma \to H$ with target a locally compact group H is called a *good model (of* (Λ, Γ)*)* if:

1. $f(\Lambda)$ is relatively compact;

2. there is $U \subset H$ a neighbourhood of the identity such that $f^{-1}(U) \subset \Lambda$.

If Γ is generated by Λ, then we say that f is a good model of Λ. If Λ' is an approximate subgroup commensurable with Λ, we will sometimes say that Λ' is a *Meyer subset.*

Definition 1.1.9 is a tweaked version of a notion defined by Breuillard, Green and Tao ([21, Definition 3.5]) that already appeared implicitly in earlier works of, for instance, Hrushovski (see e.g. [46, Theorem 4.2]). A key result of this dissertation asserts that good models of approximate lattices and model sets are closely linked. More precisely:

Theorem 1.1.10. *Let* Λ *be an approximate lattice in some locally compact group. Then* Λ *has a good model if and only if* Λ *contains a good model.*

While its proof is elementary, this observation is central to our work. Indeed, it suggests that one may be able to transfer ideas from the theory of finite approximate subgroups to the theory of approximate lattices.

Even though it was not formalised as such, the idea of good models is implicit in many works on Meyer sets. The good model that arises from a cut-and-project scheme is often called the $\star$-map (see e.g. [4, Chapter 7]) and was studied as the *address map* by Lagarias [51, p.370]. In addition, the strategy of Meyer's original proof of Theorem 1.1.1 really aims at building a good model, rather than a cut-and-project scheme.

Chapter 3: Infinite approximate subgroups in soluble linear groups

The main goal of chapter 3 is to provide a first approach to questions regarding the structure of discrete approximate subgroups. We will study here the case of approximate subgroups of soluble Lie groups.

Our main result is a generalisation of a theorem due to Schreiber:

Theorem 1.1.11 (Schreiber, Proposition 2, [69])**.** *Let V be a real vector space and let $\Lambda \subset V$ be an approximate subgroup. There exists a vector subspace $W \subset V$ and a compact subset $K \subset V$ such that:*

$$\Lambda \subset W + K \text{ and } W \subset \Lambda + K.$$

Theorem 1.1.11 was recently given a new proof and an extension to discrete approximate subgroups of the Heisenberg group by Alexander Fish in [35, Theorem 2.2]. We further extend Schreiber's theorem to all approximate subgroups of soluble affine algebraic groups over $\mathbb{R}$:

Theorem 1.1.12 (Coarse structure of soluble real linear approximate groups)**.** *Let $\Lambda \subset \mathrm{GL}_n(\mathbb{R})$ be an approximate subgroup generating a soluble subgroup. Then Λ is compactly commensurable with a Zariski-closed soluble subgroup H of $\mathrm{GL}_n(\mathbb{R})$ normal in the Zariski-closure of $\langle\Lambda\rangle$.*

Here, we say that subsets $\Lambda, \Xi \subset G$ are *compactly commensurable* if there is a compact subset $K \subset G$ with $\Lambda \subset K\Xi$ and $\Xi \subset K\Lambda$. Compact commensurability is an equivalence relation.

These considerations are not unrelated to Question 1 and generalisations of Meyer's theorem. Indeed, we show that one can use Theorem 1.1.12 to prove our first partial answer to Question 1:

Theorem 1.1.13 (Meyer-type theorem for soluble Lie groups)**.** *Let $\Lambda \subset G$ be a uniform approximate lattice in a connected soluble Lie group. Then Λ is a Meyer set.*

Even when G is abelian, our proof provides a new strategy to show Meyer's theorem. This chapter is based on the article [56].

Chapter 4: Good models, closed approximate subgroups and amenable groups

The main object of Chapter 4 is to continue the study of good models started in Chapter 2. We will also be interested in applications of good models to the study of closed approximate subgroups and approximate subgroups of amenable groups.

We start off by proving that non-trivial quasi-morphisms *à la* Brooks [23] yield constructions of approximate groups that are not Meyer subsets:

Theorem 1.1.14. *Let F_2 be the free group over two generators a and b. For any two reduced words $w, x \in F_2$ define $o(x, w)$ as the number of occurrences of w in x. Then for any $w \in F_2 \setminus \{a, b, a^{-1}, b^{-1}, e\}$ of length l the set*

$$\left\{g \in F_2 \,\middle|\, |o(g, w) - o(g, w^{-1})| \leq 3l \right\}$$

is an approximate subgroup but not a Meyer subset.

As the proof will show, the above construction works much more generally:

Theorem 1.1.15 (Quasi-kernels are not Meyer sets)**.** *Let G be a group and let $f : G \to \mathbb{R}$ be a quasimorphism. Then $f^{-1}([-R; R])$ is a 4-approximate subgroup of G for $R \geq 0$ sufficiently large. Moreover, $f^{-1}([-R; R])$ is a Meyer set if and only if f is within bounded distance of a group homomorphism $G \to \mathbb{R}$.*

This would enable one to produce approximate subgroups that are not Meyer subsets in a great variety of groups: free groups [23], non-elementary hyperbolic groups [32] and mapping class groups of closed hyperbolic surfaces [8].

We consider next two interesting classes of approximate subgroups: compact approximate subgroups and amenable approximate subgroups (Definition 4.3.1). We will see that these types of approximate subgroups always have good models, and thus are particularly regular types of approximate subgroups.

Concerning closed approximate subgroups, the authors of [13] noticed a striking result about approximate subgroups that are closed in the Zariski-topology:

Theorem 1.1.16 (Björklund–Hartnick–Stulemeijer, Theorem 17, [13])**.** *Let k be a field and let $\mathbb{G}$ be an affine algebraic group over k. If $\Lambda \subset \mathbb{G}(k)$ is an approximate subgroup,then there exists a connected k-algebraic subgroup $\mathbb{H}$ of $\mathbb{G}$, and element $g \in \mathbb{G}(k)$ normalising $\mathbb{H}(k)$ and a finite subset $F \subset \mathbb{G}(k)$ such that*

$$g\mathbb{H}(k) \subset \overline{\Lambda} \subset F\mathbb{H}(k) \cap \mathbb{H}(k)F$$

where $\overline{\Lambda}$ denotes the Zariski-closure of Λ.

Studying compact approximate subgroups and their good models leads to a closed-subgroup theorem for approximate subgroups which provides a topological counter-part to the algebraic Theorem 1.1.16.

Theorem 1.1.17 (Closed-approximate-subgroup theorem)**.** *Let Λ be a closed approximate subgroup of a locally compact group G. There is an injective locally compact group homomorphism $\phi : H \to G$ and an open approximate subgroup Ξ of H such that $\phi_{|\phi^{-1}(\Lambda)}$ is proper and $\Lambda \subset \phi(\Xi) \subset \Lambda^3$. Furthermore, if G is a Lie group, then H is a Lie group.*

Here, a good model of some compact approximate subgroup contained in Λ^2 appears implicitly as the inverse of the map ϕ. Theorem 1.1.17 combined with Schreiber's theorem (Theorem 1.1.11) show that, modulo a compact error term, the structure of closed approximate subgroups of Euclidean spaces is akin to the structure of closed subgroups.

Proposition 1.1.18. *Let Λ be a closed approximate subgroup in $\mathbb{R}^d$. Then we can find a vector subspace $V_o \subset \mathbb{R}^d$, as well as a uniformly discrete approximate subgroup Λ_d and a compact approximate subgroup K_e both in a supplementary subspace V_d of V_o such that Λ is commensurable with $V_o + \Lambda_d + K_e$.*

Likewise, we are able to prove that amenability assumptions force approximate subgroups to have good models. It is well known that in many situations existence of invariant finitely additive measures implies existence of a good model (see e.g. [46, 48, 68, 27]). In our language this becomes:

Theorem 1.1.19 (Hrushovski, Theorem 4.2, [46])**.** *Let Λ be an amenable approximate subgroup of some group. Then Λ^4 has a good model.*

The definition of amenable approximate subgroups can be found in Section 4.3. Note that the above result is not the full strength of Hrushovski's theorem as [46, Theorem 4.2] holds with much weaker assumptions.

We show that closed approximate subgroup close to an amenable subgroup are amenable themselves. Thus proving:

Theorem 1.1.20. *Let G be a locally compact second countable group, $H \subset G$ be a normal amenable closed subgroup and $K \subset G$ be a compact subset. If Λ is a closed approximate subgroup contained in KH then the closure of Λ^4 has a good model.*

As a corollary, we have:

Theorem 1.1.21 (Meyer theorem for amenable groups)**.** *If Λ is an approximate lattice in an amenable second countable locally compact group G, then Λ is Meyer set.*

Theorem 1.1.20 also allows us to study approximate subgroups in a neighbourhood of the amenable radical of a Lie group. This strategy yields the generalisation of theorems due to Auslander [1, Theorem 1] and Mostow [63, Lemma 3.9] mentioned above (Theorem 1.1.7). In fact, Theorem 1.1.7 is a consequence of the more general Theorem 1.1.22 below combined with a generalisation of the Borel density theorem for approximate lattices of Björklund–Hartnick–Stulemeijer [13].

Theorem 1.1.22. *Let Λ be an approximate subgroup in G a group of points of an affine algebraic group over local fields of characteristic 0 or a finite product of such groups. Let A be its amenable radical and let R be its soluble radical. If the projections of $\langle\Lambda\rangle$ to all non-compact simple factors are Zariski-dense, then the approximate subgroup $\Lambda^2 \cap A$ is a uniform approximate lattice in A. If, moreover, the projections of $\langle\Lambda\rangle$ to all compact simple factors are dense, then $\Lambda^2 \cap R$ is a uniform approximate lattice in R.*

Chapter 5: $\star$-Approximate lattices in higher-rank semi-simple algebraic groups

In chapter 5 our main goal is to prove our generalisation of Meyer's theorem to semi-simple algebraic groups (Theorem 1.1.6).

As mentioned above, our strategy will follow the proof strategy of Margulis' superrigidity and arithmeticity theorems. We state now these two cornerstone results of the theory of lattices:

Theorem 1.1.23 (Margulis' superrigidity, Theorem 2, [58])**.** *Let $G = \prod_{\alpha\in A} G_\alpha$ be a finite product where G_α is the group of k_α-points of some simple k_α-group and k_α is a local field for all $\alpha \in A$. Suppose that Γ is irreducible i.e. the projection of Γ to $\prod_{\alpha\in B} G_\alpha$ is dense for any $B \subsetneq A$. Suppose moreover that G has rank at least 2. Let $\tau : \Gamma \to \mathbb{H}(k)$ be a group homomorphism with k a local field, $\mathbb{H}$ a simple k-group and $\tau(\Gamma)$ Zariski-dense. Then:*

1. *either $\tau(\Gamma)$ is bounded;*
2. *or there is a continuous group homomorphism $\pi : G \to \mathbb{H}(k)$ that factors through the natural projection $G \to G_\alpha$ for some α and that extends τ.*

For the sake of simplicity we only state a Lie theoretic version of Margulis' arithmeticity theorem:

Theorem 1.1.24 (Margulis' arithmeticity, Theorem 1', [58])**.** *Let $G = \prod_{\alpha\in A} G_\alpha$ be a finite product where G_α is the group of $\mathbb{R}$-points of some simple $\mathbb{R}$-group. Suppose that G has $\mathbb{R}$-rank at least 2 and let $\Gamma \subset G$ be an irreducible lattice. Then there are a $\mathbb{Q}$-group $\mathbb{H}$ and a continuous group homomorphism $\pi : \mathbb{H}(\mathbb{R}) \to G$ with finite index image and compact kernel such that $\pi(\mathbb{H}(\mathbb{Z}))$ is commensurable with Γ.*

We refer to [58] for more details, relevant definitions and general statements. Margulis' strategy (see [58, §IX.2]) to prove his arithmeticity theorem consists in:

- first, proving superrigidity;

- then proving that lattices are finitely generated;
- finally, using superrigidity to study the natural action of field automorphisms on the group G seen as a group of matrices.

We will follow precisely this approach to show the following detailed version of Theorem 1.1.6:

Theorem 1.1.25. *Consider a finite set A, and for all $\alpha \in A$ a characteristic 0 local field k_α and a simple k_α-group $\mathbb{G}_\alpha$ of k_α-rank ≥ 2. Let Λ be a $\star$-approximate lattice in $G_A := \prod_{\alpha \in A} \mathbb{G}_\alpha(k_\alpha)$. Then there are a finite set B and :*

1. *a characteristic 0 local field k_β for every $\beta \in B$;*
2. *a simple k_β-group $\mathbb{H}_\beta$ for every $\beta \in B$;*
3. *a lattice $\Gamma \subset G_A \times \prod_{\beta \in B} \mathbb{H}_\beta(k_\beta)$ that projects injectively to G_A;*

such that $\langle \Lambda \rangle = p_A(\Gamma)$ - where p_A is the natural projection to G_A - and Λ is commensurable with a model set associated to the cut-and-project scheme $(G_A, \prod_{\beta \in B} \mathbb{H}_\beta(k_\beta), \Gamma)$.

Note that Theorem 1.1.25 states that both Λ and the group $\langle \Lambda \rangle$ it generates have an arithmetic origin.

To prove both the superrigidity result and the finite generation result we will need, we study a generalisation of the cocycles considered by Zimmer ([82, p.98]) in his proof of the superrigidity theorem. These cocycles were first studied in the context of approximate lattices by Björklund and Hartnick ([11]) in order to define an induction process for approximate lattices and extend properties such as Kazhdan's property (T) and the Haagerup property to uniform model sets.

The cocycle we will consider is defined as follows: given a strong approximate lattice Λ and a Borel section $s : \Omega_\Lambda \to G$ such that $s(X) \in X$ we define the natural cocycle α_s on Ω_Λ by $\alpha_s(g, X) := s(gX)^{-1} g s(X)$. In particular, $\alpha(\cdot, X)$ takes values in $X^{-1}X$ and, hence, in $\Lambda^{-1}\Lambda \subset \langle \Lambda \rangle$.

A crucial ingredient in Zimmer's proof of Margulis' superrigidity is a result of Mackey [57] describing the structure of cocycles arising from a transitive action. This fails in our situation. Indeed, Mackey's proof crucially uses the existence of large stabilisers, whereas Ω_Λ typically has trivial stabilisers. Circumventing this difficulty is the central point of our work. Given a group homomorphism $T : \langle \Lambda \rangle \to H$ with target a topological group we first relate the range of the cocycle $T \circ \alpha_s$ to $T(\Lambda)$ as follows.

Theorem 1.1.26. *Suppose that $T \circ \alpha_s$ is cohomologous to a cocycle taking values in a closed subgroup $L \subset H$. Then there is $h \in H$ such that for all neighbourhoods of the identity $\mathcal{V} \subset H$ there is $\Lambda' \subset \Lambda^2$ commensurable with Λ such that*

$$T(\Lambda') \subset \mathcal{V} h L h^{-1} \mathcal{V}^{-1}$$

Theorem 1.1.26 gives more than enough information when applied to group homomorphisms taking values in the group of isometries of a CAT(0) space (Propositions 5.1.13 and 5.1.14). In particular, this puts us in a position to apply Zimmer's cocycle superrigidity ([82]) and obtain the following:

Theorem 1.1.27 (Superrigidity for $\star$-approximate lattices)**.** *Let Λ, G_A be as in Theorem 1.1.25. Take a local field k and a simple k-group $\mathbb{H}$. Let $T : \langle\Lambda\rangle \to \mathbb{H}(k)$ be a group homomorphism. Then one of the following is true:*

1. *there is a continuous group homomorphism $\pi : G_A \to \mathbb{H}(k)$ extending T;*

2. *$T(\Lambda)$ is bounded.*

This result directly extends Margulis' superrigidity from lattices to $\star$-approximate lattices. We prove in fact a special case first: Theorem 1.1.26 together with cocycle superrigidity yield the case when $\dim \mathbb{H} \leq \dim \mathbb{G}$. We use it in combination with a finiteness statement (Proposition 5.3.2) which is itself a consequence of the closed-approximate-subgroup theorem (Theorem 1.1.17). This special case is central in the proof of Theorem 1.1.25, which in turn yields the general case of Theorem 1.1.27.

If we assume instead that T is a unitary representation of $\langle\Lambda\rangle$ on a Hilbert space $\mathcal{H}$, we can introduce - as in Björjkund and Hartnick's [11] - a unitary representation of G on $L^2(\Omega_\Lambda, \mathcal{H})$ by

$$\pi(g)(f)(X) := T \circ \alpha(g^{-1}, X)^{-1} \left(f(g^{-1}X)\right),$$

for all g and almost all X. Theorem 1.1.26 enables us to transfer information from the induced representation of G to the representation of Λ: if π fixes a unit vector, then there is a unit vector x such that $T(\Lambda)(x)$ is totally bounded (see Proposition 5.2.4 below). We build upon this remark to define a notion of *property (T) for approximate subgroups*:

Definition 1.1.28 (Property (T) for approximate subgroups)**.** An approximate subgroup Λ has *property (T)* if for any unitary representation π of $\langle\Lambda\rangle$ almost with invariant vectors there is a sub-representation σ with $\sigma(\Lambda)$ totally bounded in the strong topology.

See Definition 5.2.1 below for precisions. A notion of property (T) for approximate subgroups - in fact of the equivalent property (FH) - had been previously defined by Björklund–Hartnick (see [11] for this and definitions of several strengthening such as property (TT) and (TTT) for approximate subgroups). They say an approximate subgroup has "property (T)" if the pair $(\langle\Lambda\rangle, \Lambda)$ has the relative Kazhdan property as defined by Cornulier [30]. They go on to show, among other results, that uniform model sets of property (T) groups have property (T) ([30]). For general model sets this is in fact a consequence of a general result of Margulis:

Theorem 1.1.29 (Margulis, Theorem 6.3 [58])**.** *Let Γ be a lattice in a product $G \times H$ of locally compact groups. Take $W \subset H$ a compact neighbourhood of the identity and write $\Lambda_W := G \times W \cap \Gamma$. Suppose that G has property (T) and that the projection of Γ to H is dense. Let $(\pi, \mathcal{H})$ be any unitary representation of Γ almost with invariant vectors. Then there is a subrepresentation σ of π such that $\sigma(\Lambda_W)$ is relatively compact in the strong topology.*

Margulis' theorem shows that model sets in property (T) groups have property (T) in the sense of Björklund–Hartnick as well as in the sense of this dissertation.

In fact Property (T) for approximate subgroups implies property (T) in the sense of Björklund–Hartnick and it reduces to the usual property (T) when Λ is a subgroup (Subsection 5.2.2). Moreover, a $\star$-approximate lattice has property (T) if and only if the ambient group has property (T) (Proposition 5.2.4). This fact is key and leads to the last ingredient in our proof:

Theorem 1.1.30. *Let Λ be a $\star$-approximate lattice in a locally compact group with property (T). Then $\langle\Lambda\rangle$ is finitely generated.*

A brief comparison with Hrushovski's Meyer-type theorem for semi-simple groups

Now that we have given an overview of the techniques employed to prove Theorem 1.1.6, let us briefly discuss the difference with Hrushovski's Theorem 1.1.5. At the heart of the method presented in the previous discussion is the idea of *induction*. We take an action of the approximate lattice Λ - embodied here by the group homomorphism T - and transform it into an action of the ambient group G_A. One advantage of this method is that it enables us to use the rich theory of actions of semi-simple algebraic groups (e.g. Zimmer's superrigidity [82]) to draw information on the action of G_A, and, thus, on the action of Λ. One drawback of this method however is that if the group G_A does not satisfy sufficiently strong hypotheses, then we are not able to use this theory. In particular, this makes the case of lattices in $\mathrm{SL}_2(\mathbb{R})$ *a priori* unreachable with our method. A more technical drawback is that to apply Zimmer's superrigidity one needs to understand the *algebraic hull* of the cocycle we use. With our current knowledge, we are therefore bound to property (T) ambient groups. We refer to the proof of Proposition 5.3.1 and more generally, to the work of Fisher and Margulis on cocycle superrigidity results without algebraic hull assumptions [36].

In stark contrast, Hrushovski's strategy is closer to the strategy used in the proof of Theorem 1.1.4. In [47], Hrushovski first shows that any approximate subgroup - irrelevant of its ambient group - admits a weakened version of a good model:

Theorem 1.1.31 (Hrushovski, Theorem 4.2, [47])**.** *Let Λ be an approximate subgroup. There is a locally compact group H, a compact subset $S \subset H$ and a map $f : \langle\Lambda\rangle \to H$ such that:*

1. *for all* $\gamma_1, \gamma_2 \in \langle\Lambda\rangle$, $f(\gamma_1\gamma_2)f(\gamma_2)^{-1}f(\gamma_1)^{-1} \in S$;
2. *for all* $h \in H$, $hSh^{-1} = S$;
3. *for every neighbourhood of the identity* $W \subset H$, $f^{-1}(WS)$ *is an approximate subgroup commensurable with* Λ.

Hrushovski then notices that when $S = \{e\}$ the approximate subgroup Λ must be a Meyer subset. He then shows that S must be trivial whenever Λ is an approximate lattice in a semi-simple algebraic group, as otherwise one would find a non-trivial normal abelian subgroup of the semi-simple ambient group. In summary, first studying Λ in and of itself provides Hrushovski with additional information that allows him to show that the only assumption needed on the ambient group to prove that Λ has a good model is the absence of non-trivial normal abelian subgroups i.e. semi-simplicity. This way, Hrushovski avoids the shortcomings of higher-rank assumptions.

Unfortunately, we will not be able to explain the groundbreaking model-theoretic methods that enabled Hurshovski to prove Theorem 1.1.31 as this would go far beyond the scope of this dissertation.

1.1.1 Two questions

We conclude this introduction with two questions that we think are the natural continuation of this work.

Another cornerstone of the theory of Meyer sets in Euclidean spaces is the following theorem of Lagarias':

Theorem 1.1.32 (Lagarias, [51])**.** *Let* Λ *be a subset of* $\mathbb{R}^n$. *Suppose that* Λ *is relatively dense and that* $\Lambda - \Lambda$ *is uniformly discrete. Then* Λ *is a Meyer set.*

In the following question we wonder to what extent Lagarias' theorem can be extended to non-commutative groups.

Question 2. Let X be a subset of a locally compact group G such that $X^{-1}X$ is uniformly discrete and $\Omega X = G$ for some Borel subset Ω of finite Haar measure. Is X an approximate lattice ? What if $Xx^{-1}X$ is uniformly discrete for all $x \in X$? Same questions assuming that Ω_X admits a proper invariant probability measure.

Note that, while the two "finite co-volume" assumptions discussed in Question 2 are clearly related, none implies the other. This naturally leads to our next question:

Question 3. Let Λ be an approximate lattice in a locally compact group. Is there an integer $m \geq 0$ such that Λ^m is a $\star$-approximate lattice/ a strong approximate lattice?

In amenable groups, Question 3 is already known to be true. And, in fact, m can always be taken to be 1. Beyond amenable groups Question 3 is linked with measure rigidity ideas. Conversely, any *regular* model set is a strong approximate lattice, so Hrushovski's result from [47] asserts that if G is a semi-simple algebraic group, then an approximate lattice is always commensurable with a strong approximate lattice.

1.1.2 Source of the chapters

Chapter 3 was published as an independent paper in [56] (open access copyright license).

Chapter 4 and Chapter 5 are available as pre-publications on the arXiv (resp. [55] and [54]). And an article about Meyer's theorem in nilpotent Lie groups [53] (partly completed during the first months of the PhD course of the author) was not included.

1.1.3 Notations

Given two subsets $X, Y \subset G$ of a group we will write $XY := \{xy : x \in X, y \in Y\}$, $X^n := \{x_1 \cdots x_n : x_1, \ldots, x_n \in X\}$ for all $n \geq 1$, $X^0 = \{e\}$ and $X^{-1} := \{x^{-1} : x \in X\}$. When the ambient group is abelian we will also sometimes use additive notations. The group generated by X will be denoted $\langle \Lambda \rangle$. In Chapters 3 and 4, and when Λ denotes a symmetric subset, we will sometimes write Λ^∞ instead of $\langle \Lambda \rangle$.

If X is a subset of a group G, we will write $\mathrm{Comm}_G(X)$ the sets of elements $g \in G$ such that gXg^{-1} is commensurable with X. We say that $g \in G$ *commensurates* X if $g \in \mathrm{Comm}_G(X)$.

When X is a homogeneous space, a Haar-measure on X will often be denoted by μ_X or m_X.

Given a Cartesian product of groups $G \times H$ we will identify G and $G \times \{e\}$, as well as H and $\{e\} \times H$.

Chapter 2

A primer on approximate subgroups, approximate lattices and good models

2.1 Approximate subgroups and commensurability

We collect here well-known facts about approximate subgroups and commensurability in a form and with hypotheses suitable to our discussion. Our main sources are a survey on growth in groups due to Helfgott [44], an article by Tao [73] and Tointon's book on approximate subgroups [74] (see also [73, 21, 46] for more background material). Let us first recall the definition of approximate subgroups.

Definition 2.1.1 (Tao, Definition 3.8, [73])**.** A subset Λ of some group G is a K-*approximate subgroup* if $e \in \Lambda$, $\Lambda^{-1} = \Lambda$ and there is $F \subset G$ of size K such that $\Lambda^2 \subset F\Lambda$.

As explained in the introduction, we will often consider approximate subgroups and subsets of groups *up to a finite error*. Commensurability (that, to the author's knowledge, appears first in the introduction of Hrushovski's [46]) formalises that notion:

Definition 2.1.2. Let $X, Y \subset G$ be two subsets of a group. We say that:

1. X is *covered by finitely many left (resp. right) translates of* Y if there is $F \subset G$ finite such that $X \subset FY$ (resp. $X \subset YF$);

2. X is (K-)commensurable with Y if there is $F \subset G$ finite ($|F| \leq K$) such that $X \subset FY \cap YF$ and $Y \subset FX \cap XF$.

Let us make a series of useful remarks:

(a) We will sometimes omit to write the prefix left-, as we will mainly use notions pertaining to multiplication on the left.

(b) It is often useful to think of the condition $X \subset FY \cap YF$ as a conjonction of conditions $X \subset FY$ and $X \subset YF$.

(c) Commensurability is an equivalence relation. Reflexivity and symmetry are straightforward. The transitivity of commensurability relies on the following observation. If $X \subset FY \cap YF$ and $Y \subset F'Z \cap ZF'$, then $X \subset FY$ and $Y \subset F'Z$. So $X \subset FF'Z$. Symmetrically, $X \subset ZF'F$. Therefore, $X \subset FF'Z \cap ZF'F$.

(d) In what follows, we will most often be interested in symmetric subsets. When X and Y are symmetric subsets, then they are commensurable if and only if there is $F \subset G$ finite such that $X \subset FY$ and $Y \subset FX$. Indeed, since $X \subset FY$ and $X \subset YF^{-1}$, then $X \subset \tilde{F}Y \cap Y\tilde{F}$ where $\tilde{F} = F \cup F^{-1}$.

Lemma 2.1.3. *Let X, Y, Z be subsets of a group G. If $X \subset YZ$, then $X \subset \left(Y \cap XZ^{-1}\right) Z$.*

If $X \subset YZ \cap ZY$, then $X \subset \left(Y \cap XZ^{-1}\right) Z \cap Z \left(Y \cap Z^{-1}X\right)$. In particular, if X, Y are commensurable subsets of G, they are already commensurable subsets of the group generated by X and Y. In other words, the fact that two subsets are commensurable does not depend on the ambient group.

Approximate subgroups in fact provide a natural family of pairwise commensurable subsets.

Lemma 2.1.4. *Let Λ be a K-approximate subgroup of some group. Then Λ is K^{k-1}-commensurable with Λ^k for all $k \geq 1$.*

Proof. Since Λ is an approximate subgroup, there is a finite subset F of size at most K such that $\Lambda^2 \subset F\Lambda$. For all $k \geq 0$, we thus have $\Lambda^{k+1} \subset F^k\Lambda$. Since, conversely, $\Lambda \subset \Lambda^{k+1}$, we have that Λ^{k+1} and Λ are K^k-commensurable. □

The following is a partial converse:

Lemma 2.1.5. *Let Λ be an approximate subgroup of some group. Let $\Lambda' \subset \langle \Lambda \rangle$ be commensurable with Λ. Then there is $n \geq 0$ such that $\Lambda' \subset \Lambda^n$. In particular, $\Lambda' \cup \Lambda'^{-1} \cup \{e\}$ is an approximate subgroup commensurable with Λ.*

Proof. Let F be a finite subset such that $\Lambda' \subset F\Lambda$. Since $\Lambda' \subset \langle \Lambda \rangle$, we may take $F \subset \langle \Lambda \rangle$ (Lemma 2.1.3). But F is finite and $\langle \Lambda \rangle = \bigcup_{n \geq 0} \Lambda^n$. So there is $n \geq 0$ such that $F \subset \Lambda^n$. Hence, $\Lambda' \subset \Lambda^{n+1}$. □

Seeing Lemma 2.1.5, one may wonder if a symmetric subset commensurable with an approximate subgroup is always an approximate subgroup. Unfortunately, such a statement fails in general. Indeed, suppose that H is a subgroup of some group G and that $g \in G$ does not commensurate H (i.e. H and gHg^{-1} are not commensurable). Then the symmetric subset $X := \{g, g^{-1}\} \cup H$ is commensurable with the approximate subgroup (in fact the subgroup) H, but X^3 contains gHg^{-1}. So it is not commensurable with H nor with X.

We prove now a well-known and elementary criterion for some subset to be covered by finitely many translates of another.

Lemma 2.1.6 (Ruzsa's covering lemma, Ruzsa, [67])**.** *Take X, Y subsets of some group and let $F \subset X$ be maximal such that the subsets fY for $f \in F$ are pairwise disjoint. Then $X \subset FYY^{-1}$.*

Although proved and used in [67], Ruzsa's covering lemma is not explicitly stated there.

Proof. Take $x \in X$. Then there is $f \in F$ such that $xY \cap fY \neq \emptyset$. So $x \in fYY^{-1} \subset FYY^{-1}$. □

We will now prove three key lemmas about approximate subgroups and commensurability. A reassuring consequence of the following is the fact that two subgroups Γ_1 and Γ_2 are commensurable in the sense of Definition 2.1.2 if and only if they are commensurable in the usual group theoretic sense i.e. $\Gamma_1 \cap \Gamma_2$ is a finite-index subgroup of both Γ_1 and Γ_2. For references concerning such results, see e.g. [74, Proposition 2.6.5] or [44, Lemma 4.2].

Lemma 2.1.7. *Take $X, Y_1, \ldots Y_n$ subsets of a group G. Assume that there exist $F_1, \ldots, F_n \subset G$ finite such that $X \subset F_iY_i$ for all $i \in \{1, \ldots, n\}$. Then there is $F' \subset X$ with $|F'| \leq |F_1| \cdots |F_n|$ such that*

$$X \subset F' \left(Y_1^{-1}Y_1 \cap \cdots \cap Y_n^{-1}Y_n\right).$$

Proof. Write $F := \{(f_1, \ldots, f_n) \in F_1 \times \cdots \times F_n | X \cap f_1Y_1 \cap \cdots \cap f_nY_n \neq \emptyset\}$ and for every $f = (f_1, \ldots, f_n) \in F$ take $y_f \in X \cap f_1Y_1 \cap \cdots \cap f_nY_n$. Then

$$X \subset \bigcup_{(f_1,\ldots,f_n) \in F} f_1Y_1 \cap \ldots \cap f_nY_n \subset \bigcup_{f \in F} y_f \left(Y_1^{-1}Y_1 \cap \ldots \cap Y_n^{-1}Y_n\right).$$

So $F' := \{y_f | f \in F\}$ is the set we are looking for. □

Lemma 2.1.8. *Let $\Lambda_1, \ldots, \Lambda_n$ be $K_1, \ldots, K_n$-approximate subgroups of some group. Then:*

1. *if $k_1, \ldots, k_n \geq 2$ are integers, then there is F with $|F| \leq K_1^{k_1-1} \cdots K_n^{k_n-1}$ such that*
$$\Lambda_1^{k_1} \cap \cdots \cap \Lambda_n^{k_n} \subset F \left(\Lambda_1^2 \cap \cdots \cap \Lambda_n^2\right);$$
2. *if $k_1, \ldots, k_n \geq 2$ are integers, then $\Lambda_1^{k_1} \cap \cdots \cap \Lambda_n^{k_n}$ is a $K_1^{2k_1-1} \cdots K_n^{2k_n-1}$-approximate subgroup;*
3. *if $\Lambda_1', \ldots, \Lambda_n'$ is a family of approximate subgroups such that Λ_i' is commensurable with Λ_i for all $1 \leq i \leq n$, then $\Lambda_1'^2 \cap \cdots \cap \Lambda_n'^2$ is commensurable with $\Lambda_1^2 \cap \cdots \cap \Lambda_n^2$.*

Proof. For all $i \in \{1, \dots, n\}$ we have

$$\Lambda_1^{k_1} \cap \cdots \cap \Lambda_n^{k_n} \subset \Lambda_i^{k_i} \subset F_i \Lambda_i$$

where F_i has size at most $K_i^{k_i-1}$. According to Lemma 2.1.7, there is F of size at most $K_1^{k_1-1} \cdots K_n^{k_n-1}$ such that

$$\Lambda_1^{k_1} \cap \cdots \cap \Lambda_n^{k_n} \subset F\left(\Lambda_1^2 \cap \cdots \cap \Lambda_n^2\right).$$

This proves part (1). To prove part (2) notice that we have the inclusions

$$\left(\Lambda_1^{k_1} \cap \cdots \cap \Lambda_n^{k_2}\right)^2 \subset \Lambda_1^{2k_1} \cap \cdots \cap \Lambda_n^{2k_n}$$

and

$$\Lambda_1^2 \cap \cdots \cap \Lambda_n^2 \subset \Lambda_1^{k_1} \cap \cdots \cap \Lambda_n^{k_n}.$$

Then apply part (1). Part (3) is a consequence of Lemma 2.1.7 applied to both $\Lambda_1'^2 \cap \cdots \cap \Lambda_n'^2$ together with $\Lambda_1, \dots, \Lambda_n$, and $\Lambda_1^2 \cap \cdots \cap \Lambda_n^2$ together with $\Lambda_1', \dots, \Lambda_n'$. □

The condition $k_1, \dots, k_n \geq 2$ in Lemma 2.1.8 is necessary. Indeed, consider the set $X := \{\pm 2^n : n \in \mathbb{N}\} \cup \{0\}$. One may check that X is *not* an approximate subgroup. However, $Y := X \cup (2\mathbb{Z}+1)$ is symmetric and commensurable with and contained in $\mathbb{Z}$. Therefore, Y is an approximate subgroup. However, $2\mathbb{Z} \cap Y = X$ is *not* an approximate subgroup.

Just like we can take the intersection of *powers* of approximate subgroups and obtain an approximate subgroup, it is possible to take push-forwards and pull-backs of squares of approximate subgroups by group homomorphisms. The case of push-forwards is clear, and for the pull-backs we have:

Lemma 2.1.9. *Let Λ_1 and Λ_2 be two commensurable approximate subgroups of a group G. Let $\phi : H \to G$ be a group homomorphism. Then $\phi^{-1}(\Lambda_1^2)$ and $\phi^{-1}(\Lambda_2^2)$ are commensurable approximate subgroups of H.*

Proof. By Lemma 2.1.8 the subsets $\phi(H) \cap \Lambda_1^2$ and $\phi(H) \cap \Lambda_2^2$ are commensurable approximate subgroups. Take $\{i, j\} \subset \{1, 2\}$, we can find a finite subset $F_{ij} \subset H$ such that $(\phi(H) \cap \Lambda_i^2)^2 \subset \phi(F_{ij})\left(\phi(H) \cap \Lambda_j^2\right)$. In other words,

$$\phi^{-1}(\Lambda_i^2)^2 \subset F_{ij}\phi^{-1}(\Lambda_j^2).$$

So $\phi^{-1}(\Lambda_1^2)$ and $\phi^{-1}(\Lambda_2^2)$ are commensurable approximate subgroups. □

2.2 The definitions of approximate lattices

2.2.1 Uniform discreteness

Our first task is to introduce useful notions of discreteness.

Definition 2.2.1. A subset X of locally compact group G is *left-uniformly discrete* (resp. *right-uniformly discrete*) if there is an open set $U \subset G$ such that $X^{-1}X \cap U = \{e\}$ (resp. $XX^{-1} \cap U = \{e\}$). We say that X is *locally finite* if for all compact $K \subset G$, $|X \cap K| < \infty$.

When *left-* or *right-* is not specified we will always mean the left- version of uniform discreteness.

Notice the apparent discrepancy between the definition of uniform discreteness in Definition 2.2.1 and the definition from the introduction. In fact, the definition from the introduction can easily be seen to fit within the more general framework introduced here. Indeed, if X is a subset of (G, d) a metric group together with a left-invariant distance, then

$$\inf_{x_1 \neq x_2 \in X} d(x_1, x_2) \geq r > 0$$

is equivalent to $X^{-1}X \cap B_d(e, r) = \{e\}$ where $B_d(e, r)$ denotes the open ball of radius r centered at e.

We mention now another characterisation of uniform discreteness.

Lemma 2.2.2. *Let X be a subset of locally compact group. For $W \subset G$ neighbourhood of the identity, the multiplication map $X \times W \to G$ is injective if and only if $X^{-1}X \cap WW^{-1} = \{e\}$.*

Proof. Suppose first that $X^{-1}X \cap WW^{-1} = \{e\}$. If $x_1, x_2 \in X$ and $w_1, w_2 \in W$ are such that $x_1w_1 = x_2w_2$, then $x_2^{-1}x_1 = w_2w_1^{-1}$. So $x_1 = x_2$ and $w_1 = w_2$.

Conversely, suppose that the multiplication map $X \times W \to G$ is injective for some open neighbourhood of the identity W. Then for all $x_1, x_2 \in X$ and for all $w_1, w_2 \in W$ with $x_1 \neq x_2$ we have $x_1w_1 \neq x_2w_2$ i.e. $x_2^{-1}x_1 \neq w_2w_1^{-1}$. So $x_2^{-1}x_1 \notin WW^{-1}$ and $X^{-1}X \cap WW^{-1} = \{e\}$. □

Finally, we prove that for approximate subgroups the notion of uniform discreteness persists even after taking high powers.

Lemma 2.2.3. *Let Λ be an approximate subgroup of a locally compact group G. If Λ is uniformly discrete, then Λ^n is uniformly discrete for all integers $n \geq 0$.*

Proof. Let K be a compact subset of G. Since Λ is uniformly discrete there is V a neighbourhood of the identity such that $\Lambda \cap U^{-1}U = \{e\}$. There is now F_1 finite such that $K \subset F_1V$. For $f \in F_1$, we find that if $\lambda_1, \lambda_2 \in \Lambda \cap fV$, then $\lambda_1 = \lambda_2$. Therefore, $|\Lambda \cap K| \leq |F_1| < \infty$.

Take now an integer $n \geq 1$. There is $F_2 \subset G$ finite such that $\Lambda^{2n} \subset \Lambda F_2$. So for any subset K compact we have

$$|\Lambda^{2n} \cap K| \leq |\Lambda F_2 \cap K| \leq |F_2||\Lambda \cap KF_2^{-1}| < \infty.$$

Choose a compact neighbourhood of the identity V. Then $V \cap \Lambda^{2n}$ is finite, *a fortiori* discrete, and contains $\{e\}$. So there is a further neighbourhood of the identity $V' \subset V$ such that $\Lambda^{2n} \cap V' = \{e\}$. In other words, Λ^n is uniformly discrete. □

2.2.2 Approximate lattices and uniform approximate lattices

We have already introduced approximate lattices and uniform approximate lattices earlier in the introduction. Nevertheless, we provide a definition.

Definition 2.2.4. An approximate subgroup Λ of a locally compact group G is :

- (Hrushovski, [47, Definition A.1]) an *approximate lattice* if Λ is uniformly discrete and there exists a Borel subset $\mathcal{F} \subset G$ of finite Haar measure such that $\Lambda\mathcal{F} = G$;
- (Björklund–Hartnick, [9, Definition 2.6]) a *uniform approximate lattice* if in addition $\mathcal{F}$ can be chosen relatively compact.

We first make an easy observation about powers of approximate lattices.

Lemma 2.2.5. *Let Λ be an approximate subgroup of a locally compact group G. If Λ is an approximate lattice (resp. a uniform approximate lattice), then Λ^n is an approximate lattice (resp. a uniform approximate lattice) for every integer $n \geq 1$. Conversely, if there is an integer $n \geq 1$ such that Λ^n is an approximate lattice (resp. a uniform approximate lattice), then Λ is an approximate lattice (resp. a uniform approximate lattice).*

Proof. Suppose that Λ is an approximate lattice. Suppose that $\mathcal{F}$ is a Borel subset of finite Haar measure such that $\Lambda\mathcal{F} = G$. Then for every $n \geq 1$, $\Lambda^n\mathcal{F} = G$. Moreover, Λ^n is a uniformly discrete approximate subgroup according to Lemma 2.2.3. So Λ^n is an approximate lattice.

Conversely, suppose that Λ^n is an approximate lattice for some $n \geq 1$. Then Λ is uniformly discrete since $\Lambda \subset \Lambda^n$. Since Λ is an approximate subgroup, there is a finite subset F such that $\Lambda^n \subset \Lambda F$. Since, moreover, Λ^n is an approximate lattice, there is a Borel subset $\mathcal{F}$ of finite Haar measure such that $\Lambda^n\mathcal{F} = G$. Then $F\mathcal{F}$ is Borel and of finite Haar measure as well. Furthermore, $\Lambda F\mathcal{F} \supset \Lambda^n\mathcal{F} = G$. So Λ is an approximate lattice.

The same proof works for the case of uniform lattices. $\square$

Hrushovski proved a precision on the type of subset $\mathcal{F}$ one can hope to find:

Lemma 2.2.6 (Hrushovski, Proposition A.2, [47])**.** *Let Λ be an approximate lattice in a second countable locally compact group G. Let μ_G denote a (left-)Haar measure on G. There is $\mathcal{F} \subset G$ Borel such that:*

1. *$\mu_G(\mathcal{F}) < \infty$;*
2. *$\Lambda^2\mathcal{F} = G$;*
3. *the multiplication map $\Lambda \times \mathcal{F} \to G$ is one-to-one.*

The proof of Lemma 2.2.6 is based on the following observation:

Lemma 2.2.7. *Let X be a countable subset of a locally compact group G. Let μ_G be a (left-)Haar measure. Let $B_1, B_2 \subset G$ be two Borel subsets such that $XB_1 = G$ and the multiplication map $X^{-1} \times B_2 \to G$ is one-to-one. Then $\mu_G(B_2) \leq \mu_G(B_1)$.*

Proof. Since $XB_1 = G$ we have

$$\mu_G(B_2) \leq \sum_{x\in X} \mu_G(B_2 \cap xB_1) = \sum_{x \in X} \mu_G(x^{-1}B_2 \cap B_1).$$

But the subsets $x^{-1}B_2$ for $x \in X$ are pairwise disjoint. So we get

$$\mu_G(B_2) \leq \sum_{x\in X} \mu_G(x^{-1}B_2 \cap B_1) = \mu_G(B_1 \cap \bigcup_{x\in X} x^{-1}B_2) \leq \mu_G(B_1).$$

□

Proof of Lemma 2.2.6. Let V be a compact neighbourhood of the identity such that $VV^{-1} \cap \Lambda^2 = \{e\}$ and let $(g_n)_{n\geq 0}$ be a sequence of elements of G such that $G = \bigcup_{n\geq 0} Vg_n$. Define inductively $B_n := Vg_n \setminus \bigcup_{m<n} \Lambda^2 B_m$ and $B := \bigcup_{n\geq 0} B_n$. Since $\Lambda B_i \cap \Lambda B_j = \emptyset$ for $i \neq j$ and the multiplication map $\Lambda \times V \to G$ is one-to-one, we have that the multiplication map $\Lambda \times B \to G$ is one-to-one. Moreover, we have that $Vg_n \subset B_n \cup \bigcup_{m<n} \Lambda^2 B_m$ for all $n \geq 0$. So $\Lambda^2 B = G$. Since Λ is an approximate lattice and according to Lemma 2.2.7, we have that $\mu_G(B)$ is finite. □

2.2.3 Strong approximate lattices

We will now turn to the definitions of strong approximate lattices. To do so, we introduce the *Chabauty space* of G. Let G be a locally compact second countable group and let $\mathcal{C}(G)$ be the set of closed subsets of G. The *Chabauty-Fell* topology on $\mathcal{C}(G)$ is defined by the subbase of open subsets:

$$U^V = \{F \in \mathcal{C}(G) | F \cap V \neq \emptyset\}$$

and,

$$U_K = \{F \in \mathcal{C}(G) | F \cap K = \emptyset\}$$

for all $V \subset G$ open and $K \subset G$ compact. The Chabauty space of G is defined as $\mathcal{C}(G)$ endowed with the Chabauty–Fell topology. One can check that the map

$$\begin{aligned} G \times \mathcal{C}(G) &\to \mathcal{C}(G) \\ (g, F) &\mapsto gF \end{aligned}$$

defines a continuous action of the group G on $\mathcal{C}(G)$ and that $\mathcal{C}(G)$ is a compact metrizable set (see [34]). Convergence in the Chabauty space can also be characterised as follows (see [13, Section 2.2]). A sequence $(F_i)_{i\geq 0}$ converges to $F \in \mathcal{C}(G)$ if and only if:

1. for every $x \in F$ there are $x_i \in F_i$ for all $i \in \mathbb{N}$ such that $x_i \to x$ as $i \to \infty$;

2. If $x_i \in F_i$ for all $i \in \mathbb{N}$ then every accumulation point of $(x_i)_{i\geq 0}$ lies in F.

Definition 2.2.8. Let $F \in \mathcal{C}(G)$. The *invariant hull* is

$$\Omega_F := \overline{\{gF : g \in G\}}$$

the closure of the G-orbit of F.

The structure of the space Ω_F is closely linked to the structure of F. For instance:

Proposition 2.2.9 (Björklund–Hartnick, Proposition 4.4, [9])**.** $\emptyset \in \Omega_F$ *if and only if* F *is relatively dense.*

Lemma 2.2.10 (Björklund–Hartnick–Stulemeijer, Lemma 2.3, [13])**.** *If* H *is a closed subgroup, then* Ω_H *is isomorphic as a compact* G*-space to the one-point compactification of* G/H*.*

Recall that a lattice Γ of a locally compact G is a discrete subgroup such that G/Γ admits a G-invariant Borel probability measure. In other words, Γ is a lattice if and only if Ω_Γ admits a *proper* G-invariant (for the left-action) Borel probability measure in the following sense:

Definition 2.2.11. We say that a Borel measure ν on $\mathcal{C}(G)$ is *proper* if $\nu(\{\emptyset\}) = 0$.

This now justifies the definition of a strong approximate lattice.

Definition 2.2.12 (Björklund–Hartnick, Definition 4.9, [9])**.** Let Λ be an approximate subgroup of a locally compact second countable group G. We say that Λ is a *strong approximate lattice* if:

1. Λ is uniformly discrete;

2. there is a proper G-invariant Borel probability measure ν on Ω_Λ.

In particular, a subgroup is a lattice if and only if it is a strong approximate lattice.

2.2.4 $\star$-Approximate lattices

We begin by defining a generalisation of the invariant hull.

Definition 2.2.13. Let X_0 be a closed subset of a locally compact second countable group G. The *extended invariant hull* $\Omega^{ext}_{X_0,G}$ of X_0 in G is the subset of $\mathcal{C}(G)$ defined by

$$\{X \in \mathcal{C}(G) : X^{-1}X \subset \overline{X_0^{-1}X_0}\}.$$

When the group G considered is clear from context we will simply write $\Omega^{ext}_{X_0}$.

The extended invariant hull is a natural generalisation of the invariant hull defined above:

Lemma 2.2.14. *Let X_0 be a closed subset of a locally compact second countable group G. Then $\Omega^{ext}_{X_0,G}$ is a closed subset of $\mathcal{C}(G)$ stable under the G-action. Thus, the set $\Omega^{ext}_{X_0,G}$ is a metrizable compact continuous G-space. Moreover, the invariant hull Ω_{X_0} is contained in $\Omega^{ext}_{X_0,G}$.*

Proof. Stability under the G-action is clear. That $\Omega^{ext}_{X_0,G}$ is closed is a consequence of [9, Lemma 4.6]. Let us recall the argument. Consider a sequence $(X_n)_{n\geq 0}$ of elements of $\Omega^{ext}_{X_0,G}$ that converges towards $X \in \mathcal{C}(G)$ and take $x, y \in X$. There are $x_n, y_n \in X_n$ for all integers $n \geq 0$ such that $x_n \to x$ and $y_n \to y$. Note that for all integers $n \geq 0$ we have $x_n^{-1}y_n \in \overline{X_0^{-1}X_0}$. So we have $x^{-1}y = \lim_{n\geq 0} x_n^{-1}y_n \in \overline{X_0^{-1}X_0}$. Hence, the closed subset X belongs to $\Omega^{ext}_{X_0,G}$. Finally, we have the $X_0 \in \Omega^{ext}_{X_0,G}$ so $\Omega_{X_0} \subset \Omega^{ext}_{X_0,G}$. □

We can therefore mimic the definition of a strong approximate lattice in order to define $\star$-approximate lattices.

Definition 2.2.15. Let Λ be an approximate subgroup of a locally compact second countable group G. We say that Λ is a $\star$-*approximate lattice* if:

1. Λ is uniformly discrete;
2. there is a proper G-invariant Borel probability measure ν on $\Omega^{ext}_{\Lambda,G}$.

Definition 2.2.15 differs slightly from the one presented in the introduction. We explain in the following why the two definitions are equivalent:

Proposition 2.2.16. *Let Λ be a uniformly discrete approximate subgroup of a locally compact second countable group G. The following are equivalent:*

1. *the approximate subgroup Λ is a* $\star$-approximate lattice*;*
2. *there is $X_0 \subset \Lambda^2$ with $X_0^{-1}X_0 \subset \Lambda^2$ such that there is a proper G-invariant ergodic Borel probability measure ν on Ω_{X_0} the invariant hull of X_0 in G.*

Proof. Let us start by showing (1) $\Rightarrow$ (2). Since Λ is a $\star$-approximate lattice there exists a proper G-invariant ergodic Borel probability measure on the metrizable compact G-space $\Omega^{ext}_{\Lambda,G}$ (e.g. [82, 3.2.21]). Let S denote the support of ν. Since ν is ergodic there is an element $X \in S$ whose orbit is dense in S (e.g. [82, 2.1.7]). We thus have $S = \overline{G \cdot X} = \Omega_X$. But $X \neq \emptyset$ since ν is proper. So ν gives rise to a proper G-invariant Borel probability measure on Ω_X. Define now $X_0 := x^{-1}X$ for some $x \in X$. We have $X_0 = x^{-1}X \subset \Lambda^2$, $X_0^{-1}X_0 = X^{-1}X \subset \Lambda^2$ and $\Omega_X = \Omega_{X_0}$ admits a proper G-invariant ergodic Borel probability measure.

Conversely, the inclusion $i : \Omega_{X_0} \to \Omega^{ext}_{\Lambda,G}$ is injective, continuous and satisfies $i^{-1}(\{\emptyset\}) \subset \{\emptyset\}$. So $i_*\nu$ is a proper G-invariant ergodic Borel probability measure on $\Omega^{ext}_{\Lambda,G}$. □

2.2.5 Approximate lattices in the sense of Björklund–Hartnick

We will now briefly define approximate lattices in the sense of Björklund–Hartnick - BH-approximate lattices for short. To do so, we will consider so-called *admissible* measures of a locally compact group G. A Borel probability measure μ on G is admissible if it is absolutely continuous with respect to the Haar measure and if its support generates G as a semi-group. A Borel probability measure ν on Ω_Λ is called μ-stationary if $\mu * \nu = \nu$ (see [9, §4.2]).

Definition 2.2.17 (Björklund–Hartnick, Definition 4.12, [9])**.** A uniformly discrete approximate subgroup of a second countable locally compact group G is a *BH-approximate lattice* if for all admissible measures μ on G there is a proper μ-stationary measure ν on Ω_Λ.

Except for the above definition, we will not be studying BH-approximate lattices in this dissertation. We refer to [9, §4] for discussions and results concerning these objects.

2.2.6 The periodization map

Given a locally compact group G and a closed subgroup H one may use the quotient map I that sends a continuous function $f : G \to \mathbb{R}$ with compact support to the function $I(f) : G/H \to \mathbb{R}$ defined by $I(f)(gH) := \int_H f(gh)d\mu_H(h)$ to naturally relate the Haar measure on G and the Haar measure on G/H (see e.g. [65, Section I]).

In our situation as well, we can define a map called the *periodization* map.

Definition 2.2.18 (Björklund–Hartnick, §5.1, [9])**.** Let X be a uniformly discrete subset of a second countable locally compact group G. For ϕ is a continuous function with compact support defined on G, its *periodization* is defined for all $Y \in \Omega_X$ as

$$\mathcal{P}\phi(Y) := \sum_{y \in Y} \phi(y).$$

The periodization map was first used to study approximate lattices by Björklund and Hartnick in order to prove that envelopes of approximate lattices are unimodular (see [9, §5]). The periodization map of Björklund–Hartnick in fact originates in the celebrated work of Siegel on the so-called *Siegel transform* [71].

The periodization map will allow us to exploit the strong connections between measures on the extended hull $\Omega_{X_0}^{ext}$ and measures on G. Note that since we are dealing with the extended hull instead of the invariant hull, the definition is different from, though extremely similar to, Definition 2.2.18.

Definition 2.2.19. Let X_0 be a uniformly discrete subset of a locally compact second countable group G. Then the *periodization* map defined by

$$\begin{aligned} \mathcal{P}_{X_0} :\ C_c^0(G) &\longrightarrow C^0(\Omega_{X_0}^{ext}) \\ f &\longmapsto \left(X \mapsto \sum_{x\in X} f(x)\right) \end{aligned}$$

is well-defined, continuous and left-equivariant. Here, $C_c^0(G)$ denotes the set of continuous functions on G with compact support and $C^0(\Omega_{X_0}^{ext})$ is the set of continuous functions on $\Omega_{X_0}^{ext}$. Moreover, any function in the image of $\mathcal{P}_{X_0}$ has support contained in $\Omega_{X_0}^{ext} \setminus \{\emptyset\}$.

Our first crucial observation is that the periodization maps show how to relate the measure of basic open subsets of the Chabauty-Fell topology in the extended invariant hull with the measure of open and compact subsets of G through a simple formula:

Lemma 2.2.20. *Let X_0 be a uniformly discrete subset of a locally compact second countable group G and let ν be a Borel probability measure on $\Omega_{X_0}^{ext}$. Take an open subset $V \subset G$ and a compact subset $K \subset G$. Then*

$$(\mathcal{P}_{X_0})^* \nu(V) = \int_{\Omega_{X_0}^{ext}} |X \cap V| d\nu(X) \leq |V^{-1}V \cap X_0^{-1}X_0| \nu(U^V),$$

and

$$(\mathcal{P}_{X_0})^* \nu(K) = \int_{\Omega_{X_0}^{ext}} |X \cap K| d\nu(X) \leq |K^{-1}K \cap X_0^{-1}X_0|(1 - \nu(U_K)).$$

Proof. Choose a sequence of non-negative continuous functions with compact support $(\phi_n)_{n\geq 0}$ that point-wise converges increasingly to $\mathbb{1}_V$. For all integers $n \geq 0$ and all $X \in \Omega_{X_0}$ we have

$$\mathcal{P}_{X_0}(\phi_n)(X) = \sum_{x\in X} \phi_n(x) = \sum_{x \in X\cap V} \phi_n(x).$$

We have

$$\lim_{n\to\infty} \mathcal{P}_{X_0}(\phi_n)(X) = \sum_{x\in X\cap V} \mathbb{1}_V(x) = |X \cap V|.$$

In addition, the sequence $(\mathcal{P}_{X_0}(\phi_n))_{n\geq 0}$ is increasing. Hence, the map $X \mapsto |X \cap V|$ is measurable and, according to the monotone convergence theorem, we have

$$(\mathcal{P}_{X_0})^* \nu(V) = \int_{\Omega_{X_0}^{ext}} |X \cap V| d\nu(X).$$

Similarly, choose a sequence of non-negative continuous functions with compact support $(\psi_n)_{n\geq 0}$ that point-wise converges decreasingly to $\mathbb{1}_K$. Again

$$\lim_{n\to\infty} \mathcal{P}_{X_0}(\psi_n)(X) = \sum_{x\in X\cap K} \mathbb{1}_K(x) = |X \cap K|.$$

So as above

$$(\mathcal{P}_{X_0})^* \nu(K) = \int_{\Omega_{X_0}^{ext}} |X \cap K| d\nu(X).$$

Finally, the right-hand inequalities follow from the facts that for any $X \in \Omega_{X_0}^{ext}$ and any subset $Y \subset G$ we have

$$|X \cap Y| \leq |X^{-1}X \cap Y^{-1}Y| \leq |X_0^{-1}X_0 \cap Y^{-1}Y|.$$

□

This formula is of particular importance when applied to a proper G-invariant Borel probability measure as, in this situation, the pull-back by the periodization map is always a Haar measure. Similar observations were also exploited in [9, §5] and [71].

Corollary 2.2.21. *Let X_0 be a uniformly discrete subset of a locally compact second countable group G and let ν be a proper G-invariant Borel probability measure on $\Omega_{X_0}^{ext}$. Then $(\mathcal{P}_{X_0})^*\nu$ is a Haar-measure on G.*

2.2.7 First properties of approximate lattices

We gather here some elementary results concerning approximate lattices that will be useful throughout this dissertation.

Unimodularity

One of the first results on strong approximate lattices and uniform approximate lattices is the unimodularity of their envelope.

In [9, §5], the periodization maps were used to prove a unimodularity statement.

Theorem 2.2.22 (Björklund–Hartnick, Theorem 1.11 [9])**.** *If G is a locally compact second countable group that contains a strong approximate lattice or a uniform approximate lattice, then G is unimodular i.e. the Haar measure on G is both left- and right-invariant.*

Similarly, Hrushovski proved a unimodularity theorem for approximate lattices.

Proposition 2.2.23 (Hrushovski, Appendix A, [47])**.** *Let G be a locally compact second countable group and suppose that G contains an approximate lattice Λ. Then G is unimodular.*

Proof. Let $\mathcal{F}$ be a Borel subset of finite Haar measure such that $\Lambda\mathcal{F} = G$. Fix a left-Haar measure μ_G on G and let Δ_G denote the modular function of G. Choose a neighbourhood of the identity $V \subset G$ such that the multiplication map $\Lambda \times V \to G$ is one-to-one. Then for every $g \in G$ the multiplication map $\Lambda \times Vg \to G$ is also one-to-one. Therefore, $\Delta_G(g)\mu_G(V) = \mu_G(Vg) \leq \mu_G(\mathcal{F})$ by Lemma 2.2.7. So Δ_G is bounded i.e. it is trivial. □

Finally, we prove an extension of 2.2.22 to $\star$-approximate lattices:

Proposition 2.2.24. *Let G be a locally compact second countable group. Suppose that G contains a $\star$-approximate lattice. Then G is unimodular.*

We will require an additional result:

Proposition 2.2.25 (Björklund–Hartnick, Corollary 4.22, [9])**.** *Let Λ be a $\star$-approximate lattice in a locally compact second countable group G and $V \subset G$ be any neighbourhood of the identity. Then there is a compact subset $K \subset G$ such that $V\Lambda K = G$.*

Proof. This is precisely [9, Corollary 4.22] applied to Λ in combination with the equivalent definition of $\star$-approximate lattices (Proposition 2.2.16). □

Proof. Fix a proper G-invariant Borel probability measure ν on Ω_Λ. Let μ_G be the left-Haar measure given by $(\mathcal{P}_\Lambda)^*\nu$ and let Δ_G be the modular function of G. Take V a neighbourhood of the identity such that $\Lambda^4 \cap V^{-1}V = \{e\}$. For all $\lambda \in \Lambda$ we have $\Lambda^2 \cap \lambda^{-1}V^{-1}V\lambda = \{e\}$. By Lemma 2.2.20, for all $\Lambda \in \Lambda$ we thus have

$$\Delta_G(\lambda)\mu_G(V) = \mu_G(V\lambda) \leq 1.$$

So Δ_G is bounded by $\mu_G(V)^{-1}$ on Λ. By Proposition 2.2.25 this means that Δ_G is bounded. Hence, G is unimodular. □

These two unimodularity theorems will be necessary below to establish that $\star$-approximate lattices are approximate lattices (Theorem 1.1.2).

Finite co-volume and commensurability

We express now a simple and useful commensurability criterion in terms of uniform discreteness and finite co-volume.

Proposition 2.2.26 (Hrushovski, Appendix A, [47])**.** *Let X_0 be a uniformly discrete subset of a locally compact second countable group G and let Y be a subset such that $X_0^{-1}Y^{-1}$ is uniformly discrete. Fix a Haar measure μ_G on G. Suppose that there is a Borel subset $\mathcal{F}$ of finite Haar measure such that $X_0\mathcal{F} = G$. Then Y is covered by finitely many left translates of $X_0X_0^{-1}$.*

Proof. Let $F \subset Y$ be such that the subsets $(X_0^{-1}f^{-1})_{f\in F}$ are disjoint. Since $X_0^{-1}F^{-1} \subset X_0^{-1}Y^{-1}$ and $X_0^{-1}Y^{-1}$ is uniformly discrete there is a neighbourhood $V \subset G$ of the identity such that the subsets $(X_0^{-1}f^{-1}v)_{f\in F, v\in V}$ are disjoint. By Lemma 2.2.7 we thus have

$$|F|\mu_G(V) = \mu_G(F^{-1}V) \leq \mu_G(\mathcal{F}).$$

So the subset F is finite and according to Ruzsa's covering lemma Y is covered by finitely many left-translates of $X_0X_0^{-1}$. □

Proposition 2.2.26 is not contained explicitly in [47]. It was however implicitly used in the proof of the following:

Corollary 2.2.27 (Hrushovski, Lemma A.4, [47])**.** *If $\Lambda_1 \subset \Lambda_2$ are two approximate lattices in a locally compact second countable group G, then Λ_1 and Λ_2 are commensurable.*

Proof. The subset $\Lambda_1\Lambda_2$ is contained in Λ_2^2. Since the latter is uniformly discrete and Λ_1 is an approximate lattice, we may apply Proposition 2.2.26. So Λ_2 is covered by finitely many translates of Λ_1^2. Hence, Λ_2 is commensurable with Λ_1. □

In fact, [47, Lemma A.4] tells us more than simply Corollary 2.2.27. We discuss it in the next section.

Some characterisations of approximate lattices

A simple, yet remarkable, observation due to Meyer asserts that uniform approximate lattices of Euclidean spaces arise naturally as a combination of co-compactness and a stronger discreteness assumption.

Lemma 2.2.28 (Meyer, [60])**.** *Let $X \subset \mathbb{R}^n$ be a relatively dense subset such that $X - X - X$ is uniformly discrete. Then there is a finite subset $F \subset \mathbb{R}^n$ such that $X - X \subset F + X$.*

Björklund and Hartnick noticed that Lemma 2.2.28 extends in a natural way to non-commutative locally compact groups.

Lemma 2.2.29 (Björklund–Hartnick, Proposition 2.9, [9])**.** *Suppose that Λ is a subset of a locally compact group G such that:*

1. *$e \in \Lambda = \Lambda^{-1}$;*
2. *there is $\mathcal{F}$ compact such that $\Lambda\mathcal{F} = G$;*
3. *Λ^3 is locally finite.*

Then Λ is a uniform approximate lattice.

Proof. Take $\lambda_1, \lambda_2 \in \Lambda$. According to (2) there is $\lambda_3 \in \Lambda$ such that $\lambda_3^{-1}\lambda_1\lambda_2 \in \mathcal{F}$. So $\lambda_3^{-1}\lambda_1\lambda_2 \in \mathcal{F} \cap \Lambda^3$. But $\mathcal{F} \cap \Lambda^3$ is finite. Thus, $\Lambda^2 \subset \Lambda\left(\mathcal{F} \cap \Lambda^3\right)$ i.e. Λ is an approximate subgroup. □

A result in the same spirit - but with slightly larger exponents involved - holds for non-uniform approximate lattices.

Lemma 2.2.30 (Hrushovski, Lemma A.4, [47])**.** *Suppose that Λ is a subset of a second countable locally compact group G such that:*

1. *$e \in \Lambda = \Lambda^{-1}$;*

2. *there is a Borel subset $\mathcal{F} \subset G$ of finite Haar measure such that $\Lambda\mathcal{F} = G$;*

3. *Λ^4 is uniformly discrete.*

Then Λ^2 is an approximate lattice.

Proof. Proposition 2.2.26 shows that there is a finite subset $F \subset G$ such that $\Lambda^3 \subset F\Lambda^2$. So Λ^2 is an approximate subgroup. Hence, Λ^2 is an approximate lattice. □

Measurability of maps on the hull

Questions of continuity and measurability of maps between hulls will be central. We conclude this section with a simple result that enables us to prove that some maps are Borel.

Lemma 2.2.31. *Let X be a topological space, let G be a locally compact second countable group and let $\Phi : X \to \mathcal{C}(G)$ be a map. If $\Phi^{-1}(U_K)$ is a Borel subset for every compact subset $K \subset G$, then Φ is Borel measurable.*

Proof. Let $V \subset G$ be any open subset. Then V is σ-compact since G is second countable locally compact. So let $(K_n)_{n\geq 0}$ be any sequence of compact subsets such that $K_0 \subset K_1 \subset \dots$ and $\bigcup_{n\geq 0} K_n = V$. Then one checks that $U^V = \bigcup_{n\geq 0} G \setminus U_{K_n}$ so $\Phi^{-1}(U^V)$ is Borel. We conclude that Φ is Borel. □

2.2.8 Link between the definitions

We will now establish:

Theorem (Hierarchy of approximate lattices, Theorem 1.1.2)**.** *We have:*

$$\text{Strong app. lattice} \Rightarrow \star - \text{App. lattice} \Rightarrow \text{App. lattice} \Rightarrow \text{BH-app. lattice}$$
$$\Uparrow$$
$$\text{Uniform app. lattice}$$

In general, these implications are not equivalences. When the ambient group is amenable however, we have:

Lemma 2.2.32 (Björklund–Hartnick, Rem. 4.14.(1), [9])**.** *Suppose that G is an amenable locally compact group. Then any approximate lattice in the sense of Blörklund–Hartnick is a strong approximate lattice.*

Proof. Since G is amenable, there is an admissible Borel probability measure μ on G such that any μ-stationary measure is G-invariant according to a result of Kaimanovich–Vershik [49] and independently Rosenblatt [66]. But, if Λ is an approximate lattice in the sense of Blörklund–Hartnick, then there is a proper μ-stationary Borel probability measure on Ω_Λ i.e. a proper G-invariant Borel probability measure on Ω_Λ. □

We start with two technical results that we deem interesting in their own right. We first consider the link between $\star$-approximate lattices and approximate lattices.

Proposition 2.2.33. *Let Λ be a $\star$-approximate lattice in a locally compact second countable group G. Then there is a Borel subset $B \subset G$ with finite (left-)Haar measure such that $B\Lambda^2 = G$ and $B^{-1}B \cap \Lambda^2 = \{e\}$.*

Proof. Let ν be a proper G-invariant Borel probability measure on Ω_Λ and let μ_G denote the Haar-measure $(\mathcal{P}_\Lambda)^*\nu$. Let V be a compact neighbourhood of the identity such that $V^{-1}V \cap \Lambda^2 = \{e\}$ and let $(g_n)_{n\geq 0}$ be a sequence of elements of G such that $G = \bigcup_{n\geq 0} g_n V$. Define inductively $B_n := g_n V \setminus \bigcup_{m<n} B_m\Lambda^2$ and $B := \bigcup_{n\geq 0} B_n$. Since $B_i\Lambda \cap B_j\Lambda = \emptyset$ for $i \neq j$ and the multiplication map $V \times \Lambda \to G$ is one-to-one, we have that the multiplication map $B \times \Lambda \to G$ is one-to-one. Moreover, we have that $gV_n \subset B_n \cup \bigcup_{m<n} B_m\Lambda^2$ for all $n \geq 0$. So $B\Lambda^2 = G$. Now the Haar measure μ_G is inner regular so there is a sequence of compact subsets $K_n \subset B$ with $\sup_{n\geq 0} \mu_G(K_n) = \mu_G(B)$. Since $K_n \subset B$, we have that $\mathcal{P}_\Lambda(\mathbb{1}_{K_n}) \leq 1$. Hence, $\mu_G(K_n) \leq 1$ for all integers $n \geq 0$ by Lemma 2.2.20. So $\mu_G(B) \leq 1$. □

Note that Proposition 2.2.33 does not say that Λ is an approximate lattice directly as B appears on the "wrong side".

In order to prove now that approximate lattices are BH-approximate lattices, we first have to reformulate the definition of approximate lattices in terms of measures on the hull.

Proposition 2.2.34. *Let Λ be an approximate lattice in a locally compact group G. Fix a Haar measure μ_G on G. Then there exist a finite measure ν on Ω_Λ and a constant $C \geq 1$ such that*

$$\frac{1}{C}\mu_G \leq \mathcal{P}_\Lambda^*\mu \leq C\mu_G.$$

To prove Proposition 2.2.34 we first have to show a result about elements of the invariant hull of an approximate lattice.

Lemma 2.2.35. *Let Λ be an approximate lattice. Let F be a finite subset such that $\Lambda^2 \subset \Lambda F$. If $\mathcal{F} \subset G$ is such that $\Lambda\mathcal{F} = G$, then for all $X \in \Omega_\Lambda \setminus \{\emptyset\}$, $XF\mathcal{F} = G$.*

Proof. Take $X \in \Omega_\Lambda \setminus \{\emptyset\}$. There is a sequence $(g_n)_{n\geq 0}$ such that $g_n\Lambda \to X$ as $n \to \infty$. Choose $x \in X$, then there is a sequence $(\lambda_n)_{n\geq 0}$ of elements of Λ such that $g_n\lambda_n \to x$ i.e. $g_n =: x_n\lambda_n^{-1}$ with $x_n \to x$. But now

$$g_n\Lambda F \supset g_n\Lambda^2 = x_n\lambda_n^{-1}\Lambda^2 \supset x_n\Lambda.$$

Taking limits on both sides we get $XF \supset x\Lambda$. Therefore, $XF\mathcal{F} \supset x\Lambda\mathcal{F} = G$. □

Proof of Proposition 2.2.34. Suppose first that Λ is an approximate lattice. Recall that G is unimodular by Proposition 2.2.23. There is $\mathcal{F} \subset G$ a Borel subset of finite Haar measure such that $\Lambda^2\mathcal{F} = G$ and the multiplication map $\Lambda \times \mathcal{F} \to G$ is one-to-one (Lemma 2.2.6). Let $F \subset \Lambda^3$ be a finite subset such that $\Lambda^2 \subset F\Lambda$. Take also F' such that $\Lambda F \subset \Lambda^4 \subset F'\Lambda$. Now the map $\Lambda \times F\mathcal{F} \to G$ is surjective.

Let us compute the size of the fibres of this map. Take $\lambda_1, \lambda_2 \in G$, $f_1, f_2 \in F$ and $v_1, v_2 \in \mathcal{F}$ such that $\lambda_1 f_1 v_1 = \lambda_2 f_2 v_2$. Choose $f_1', f_2' \in F'$ and $\lambda_1', \lambda_2' \in \Lambda$ such that $\lambda_1 f_1 = f_1'\lambda_1'$ and $\lambda_2 f_2 = f_2'\lambda_2'$. If $f_1' = f_2'$ now, we find that $\lambda_1' v_1 = \lambda_2' v_2$. Because $\Lambda \times \mathcal{F} \to G$ is one-to-one, this means that $v_1 = v_2$. In turn, we have $\lambda_1 f_1 = \lambda_2 f_2$. So if $f_1 = f_2$ we have $\lambda_1 = \lambda_2$. In other words, given a choice of f_1' and f_1, there is at most one solution. So the fibres are of size at most $|F||F'|$.

Write $\mathcal{F}_1 = F\mathcal{F}$. For every continuous function with compact support $\phi \in C_c^0(G)$ we can define the quantity

$$M(\phi) := \int_G \mathbb{1}_{\mathcal{F}_1^{-1}}(g) \sum_{\lambda \in \Lambda} \phi(g\lambda)\mu_G(g) = \int_G \mathbb{1}_{\mathcal{F}_1^{-1}}(g)\mathcal{P}_\Lambda(\phi)(g\Lambda)\mu_G(g).$$

Since G is unimodular we have moreover

$$M(\phi) = \int_G \left(\sum_{\lambda\in\Lambda} \mathbb{1}_{\mathcal{F}_1^{-1}}(g\lambda)\right) \phi(g)\mu_G(g).$$

Therefore, if ϕ is moreover non-negative,

$$\mu_G(\phi) \leq M(\phi) \leq |F||F'|\mu_G(\phi).$$

Therefore, M defines a positive linear functional on $\mathcal{P}_\Lambda\left(C_c^0(G)\right)$ seen as a subspace of $C_c^0(\Omega_\Lambda \setminus \{\emptyset\})$. We wish now to apply the Hahn–Banach theorem - in the version of [17, §II.3, Proposition 1] - to extend M as a positive linear functional of $C_c^0(\Omega_\Lambda \setminus \{\emptyset\})$.

We will show that this is possible thanks to Lemma 2.2.35. Take $(V_n)_{n\geq 0}$ an increasing sequence of relatively compact open subsets of G such that:

$$\mathcal{F}_1^{-1} \subset \textstyle\bigcup_{n\geq 0} V_n \text{ and } \forall n \geq 0,\ \mu_G(V_n) \leq 2\mu_G(\mathcal{F}_1^{-1}).$$

Consider now $(\phi_n)_{n\geq 0}$ a sequence of elements of $C_c^0(G)$ taking non-negative values such that for all n:

$$\mu_G(\phi_n) \leq 2\mu_G(V_n) \text{ and } \forall g \in V_n,\ \phi_n(g) = 1.$$

We have

$$\mathcal{P}_\Lambda(\phi_n)(X) \geq |X \cap V_n|.$$

So $\mathcal{P}_\Lambda(\phi_n)(X) \geq 1$ for all $X \in U^{V_n}$. But, according to Lemma 2.2.35,

$$\bigcup_{n\geq 0} U^{V_n} \supset \{X \in \Omega_\Lambda : X \cap \mathcal{F}_1^{-1} \neq \emptyset\} = \Omega_\Lambda \setminus \{\emptyset\}.$$

Since the subsets U^{V_n} are open, we have that for all $\psi \in C_c^0(\Omega_\Lambda \setminus \{\emptyset\})$ there is $n \geq 0$ such that $\mathcal{P}(\phi_n) \geq \psi$. Therefore, we can apply the Hahn–Banach theorem [17, §II.3, Proposition 1].

Hence, we find a positive linear functional ν on the set $C_c^0(\Omega_\Lambda \setminus \{\emptyset\})$ such that $\nu(\mathcal{P}_\Lambda(\phi)) = M(\phi)$ for all $\phi \in C_c^0(G)$. According to the Riesz–Markov–Kakutani representation theorem, ν corresponds to integration against a Radon measure - that we denote by ν as well - on $\Omega_\Lambda \setminus \{\emptyset\}$. We extend ν to Ω_Λ by assigning $\nu(\{\emptyset\}) = 0$.

It remains only to prove that ν is a finite measure. But

$$\begin{aligned}\nu(\Omega_\Lambda) = \nu\left(\bigcup_{n\geq 0} U^{V_n}\right) &\leq \liminf_n \nu(\mathcal{P}_\Lambda(\phi_n)) \\ &\leq \liminf_n |F'||F|\mu_G(\phi_n) \\ &\leq 4|F'||F|\mu_G(\mathcal{F}_1) < \infty\end{aligned}$$

where the first inequality is a consequence of Fatou's lemma. □

We can now prove Theorem 1.1.2.

Proof of Theorem 1.1.2. First of all, it is clear that uniform approximate lattices are types of approximate lattices. Now, the fact that a strong approximate lattice is a $\star$-approximate lattice is a consequence of Lemma 2.2.14.The $\star$-approximate lattices are approximate lattices as a consequence of Proposition 2.2.33 combined with Proposition 2.2.24. It remains to prove that approximate lattices are approximate lattices in the sense of Björklund–Hartnick. Take Λ an approximate lattice, μ a Borel probability measure on G, and ν, C and μ_G as in Proposition 2.2.34. Then for all $\phi \in C_c^0(G)$ taking non-negative values

$$\mu * \nu(\mathcal{P}_\Lambda(\phi)) = \int_G \nu(\mathcal{P}_\Lambda(\phi_g))d\mu(g)$$

where ϕ_g denotes the map $h \in G \mapsto \phi(gh)$. Therefore, using the bounds from Proposition 2.2.34 and left-invariance of the Haar measure, we have

$$\frac{1}{C}\mu_G(\phi) \leq \mu * \nu(\mathcal{P}_\Lambda(\phi)) \leq C\mu_G(\phi).$$

Since this is true for any such μ, we find that any cluster point ν' of the sequence $\left(n^{-1}\sum_{i=1}^n \mu^{*i} * \nu\right)$ is μ-stationary and satisfies

$$\frac{1}{C}\mu_G(\phi) \leq \nu'(\mathcal{P}_\Lambda(\phi)) \leq C\mu_G(\phi).$$

So ν' is not concentrated on $\emptyset$. So Λ is a BH-approximate lattice. □

2.2.9 Cut-and-project schemes and model sets

We recall now the definition of cut-and-project schemes and model sets.

Definition 2.2.36. A triple (G, H, Γ) is a *cut-and-project scheme* if:

1. G and H are locally compact groups;
2. $\Gamma \subset G \times H$ is a lattice;
3. Γ projects injectively to G and densely to H.

Cut-and-project schemes enable us to efficiently build interesting families of approximate lattices:

Definition 2.2.37. Let (G, H, Γ) be a cut-and-project scheme. Take $W_0 \subset H$ a symmetric relatively compact neighbourhood of the identity and let $p_G : G \times H \to G$ denote the natural projection. We call the subset

$$M := p_G\left((G \times W_0) \cap \Gamma\right)$$

a *model set* (associated to (G, H, Γ)). Following [9] we will sometimes write $M =: P_0(G, H, \Gamma, W_0)$.

The following result explains why model sets are interesting.

Proposition 2.2.38 (Björklund–Hartnick, [9]; Björklund–Hartnick–Pogorzelski, [12])**.** *Let (G, H, Γ) be a cut-and-project scheme, let $W_0 \subset H$ be a symmetric relatively compact neighbourhood of the identity and let $M := P_0(G, H, \Gamma, W_0)$. Then:*

1. *if ∂W_0 is Haar null, then M is a strong approximate lattice;*
2. *M is a uniform approximate lattice if and only if $\Gamma \subset G \times H$ is a uniform lattice;*
3. *if M' is another model set associated to (G, H, Γ), then M and M' are commensurable.*

We provide a short proof of a weaker statement.

Lemma 2.2.39. *Let (G, H, Γ) be a cut-and-project scheme and let M be a model set associated to (G, H, Γ). Then M is a $\star$-approximate lattice.*

Proof. Write $p : G \times H \to G$ the projection. Let $W_0 \subset H$ be the symmetric relatively compact neighbourhood associated to M. Take W_1 a compact symmetric neighbourhood of the identity such that $W_1^2 \subset W_0$. Define the map

$$\begin{aligned} \Phi : (G \times H)/\Gamma &\longrightarrow \Omega_M^{ext} \\ (g,h)\Gamma &\longmapsto (g,h)\Gamma \cap (G \times W_1). \end{aligned}$$

The map Φ is clearly G-equivariant and if $(g,h) \in G \times H$, then

$$((g,h)\Gamma \cap (G \times W_1))^{-1} ((g,h)\Gamma \cap (G \times W_1)) \subset \Gamma \cap \left(G \times W_1^2\right) \subset M.$$

So Φ is well-defined. Moreover, for every compact subset $K \subset G$, we have $\Phi((g,h)\Gamma) \in U_K$ if and only if $(g,h)\Gamma \cap K \times W_1 = \emptyset$. In particular, $\Phi^{-1}(U_K)$ is an open subset of $(G \times H)/\Gamma$. So Φ is Borel measurable by Lemma 2.2.31. Let μ denote a Haar measure on $(G \times H)/\Gamma$ normalised so that μ is a probability measure. The push-forward ν of μ via Φ is a G-invariant Borel probability measure on Ω_M. It remains only to check that it is proper. If $\Phi((g,h)\Gamma) = \emptyset$, then $(g,h)\Gamma \cap G \times W_1 = \emptyset$ i.e. $\Gamma \cap G \times h^{-1}W_1 = \emptyset$. But Γ projects densely to H and $h^{-1}W_1$ has non-empty interior. Hence, there are no such (g,h). So ν is indeed proper. □

2.3 Good models

2.3.1 Definition and first properties

Let us recall the definition of good models:

Definition (Definition 1.1.9)**.** Let Λ be an approximate subgroup of a group Γ. A group homomorphism $f : \Gamma \to H$ with target a locally compact group H is called a *good model (of* (Λ, Γ)*)* if:

1. $f(\Lambda)$ is relatively compact;
2. there is $U \subset H$ a neighbourhood of the identity such that $f^{-1}(U) \subset \Lambda$.

Remark 2.3.1.
1. if (Λ, Γ) has a good model, then $\Gamma \subset \mathrm{Comm}_\Gamma(\Lambda)$;
2. restricting the range of the good model f we can always assume that f has dense image.

Definition 1.1.9 involves both the choice of a map f and an open subset U. However, up to commensurability, the choice of U does not matter as the following shows:

Lemma 2.3.1. *Let H be a locally compact group, Γ be a discrete group, V_1 and V_2 be symmetric relatively compact neighbourhoods of the identity in H and $f : \Gamma \to H$ be a group homomorphism. The subsets $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are commensurable approximate subgroups.*

Proof. Take $i, j \in \{1, 2\}$. The identity belongs to the interior $\mathrm{int}(V_j)$ of V_j so

$$V_i^2 \subset \bigcup_{h \in V_i^2} h\,\mathrm{int}(V_j).$$

But $\mathrm{int}(V_j)$ is open and V_i^2 is relatively compact, and, thus, there is a finite subset $F_{ij} \subset V_i^2$ such that $V_i^2 \subset F_{ij}U$. Since V_1 and V_2 are moreover symmetric subsets, we have that V_1 and V_2 are commensurable approximate subgroups. Choose now a symmetric open neighbourhood of the identity W such that W^2 is contained in V_1 and V_2. Then $f^{-1}(W^2), f^{-1}(V_1^2)$ and $f^{-1}(V_2^2)$ are commensurable approximate subgroups by Lemma 2.1.8. But for $i = 1, 2$ we have $f^{-1}(W^2) \subset f^{-1}(V_i) \subset f^{-1}(V_i^2)$. So $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are commensurable approximate subgroups. □

Corollary 2.3.2. *Let Λ be an approximate subgroup of a group Γ and $f : \Gamma \to H$ be a good model of (Λ, Γ). If $U \subset H$ is a symmetric neighbourhood of the identity such that $f^{-1}(U) \subset \Lambda$, then $f^{-1}(U)$ is an approximate subgroup commensurable to Λ.*

What may come as a surprise at first is the fact that the property *to have a good model* is stable under group homomorphisms.

Lemma 2.3.3. *Let Λ be an approximate subgroup of a group Γ. Suppose that (Λ, Γ) has a good model. We have :*

1. *if $\phi_1 : \Gamma_1 \to \Gamma$ is a group homomorphism, then $\phi_1^{-1}(\Lambda)$ is an approximate subgroup and $(\phi_1^{-1}(\Lambda), \Gamma_1)$ has a good model;*
2. *if $\phi_2 : \Gamma \to \Gamma_2$ is a group homomorphism, then $(\phi_2(\Lambda), \phi_2(\Gamma))$ has a good model.*

Proof. Let $f : \Gamma \to H$ be a good model of (Λ, Γ) and let $U \subset H$ be an open subset as in Definition 1.1.9. Set furthermore $\Lambda_1 := \phi_1^{-1}(\Lambda)$ and $f_1 := f \circ \phi_1$. Then $f_1(\Lambda_1) = f(\Lambda)$ is relatively compact and $f_1^{-1}(U) \subset \Lambda_1$. Hence, Λ_1 is an approximate subgroup by Lemma 2.3.1 and f_1 is a good model of (Λ_1, Γ_1). Let us now prove (2). Take a good model $f : \Gamma \to H$ of (Λ, Γ) with dense image and $U \subset H$ a symmetric neighbourhood of the identity such that $f^{-1}(U^2) \subset \Lambda$. Define $N := \overline{f(\ker(\phi_2))}$ which is a normal subgroup since $f(\Gamma)$ is dense. Now

$$(p_{H/N} \circ f)^{-1}(p_{H/N}(U)) \subset f^{-1}(U^2 f(\ker(\phi_2))) \subset f^{-1}(U^2)\ker(\phi_2) \subset \Lambda \ker(\phi_2)$$

where $p_{H/N} : H \to H/N$ denotes the natural projection. Therefore, the obvious map $\phi_2(\Gamma) \to H/N$ is a good model of $(\phi_2(\Lambda), \phi_2(\Gamma))$. □

2.3.2 Group-theoretic characterisation of good models

We will prove now a simple and detailed characterisation of good models. The following lemma is well-known, albeit in a different language, to model theorists (see e.g. [46, 59]) and in fact, as we argue in the proof below, reduces to old results of Weil on general topology and uniform structures ([77]).

Lemma 2.3.4. *Let Λ be an approximate subgroup of a group Γ. The following are equivalent:*

1. *there is a good model $f : \Gamma \to H$ of (Λ, Γ);*
2. *there exists a sequence $(\Lambda_n)_{n \geq 0}$ of approximate subgroups such that:*

(a) $\Lambda_0 = \Lambda$;

(b) for all integers $n \geq 0$ *and all* $\gamma \in \Gamma$, *the approximate subgroups* $\gamma\Lambda_n\gamma^{-1}$ *and* Λ *are commensurable;*

(c) for all integers $n \geq 0$, *we have* $\Lambda_{n+1}^2 \subset \Lambda_n$;

3. there exists a family of subsets $\mathcal{B}$ *such that:*

(a) there is $\Xi \in \mathcal{B}$ *with* $\Xi \subset \Lambda$;

(b) all elements of $\mathcal{B}$ *contain* e *and are commensurable to* Λ;

(c) for all $\Lambda_1 \in \mathcal{B}$ *and* $\gamma \in \Gamma$, *there is* $\Lambda_2 \in \mathcal{B}$ *with* $\gamma\Lambda_2^{-1}\Lambda_2\gamma^{-1} \subset \Lambda_1$.

Moreover, when any of the three statements above is satisfied:

(4) with $\mathcal{B}$ *as in (3), we can choose a good model* $f : \Gamma \to H$ *such that* f *has dense image and* $\mathcal{B}$ *is a neighbourhood basis for the identity with respect to the initial topology on* Γ *given by* f;

(5) there is a good model $f_0 : \Gamma \to H_0$ *of* (Λ, Γ) *such that for any other good model* $f : \Gamma \to H$ *of* (Λ, Γ) *we have a continuous group homomorphism* $\phi : H_0 \to H$ *with compact kernel such that* $f = \phi \circ f_0$;

(6) if Λ *is a* K*-approximate subgroup, then there exists a sequence* $(\Lambda_n)_{n\geq 0}$ *with* $\Lambda_0 = \Lambda^8$ *and as in (2) such that* Λ *is covered by* $C_{K,n}$ *left-translates of* Λ_n *for all* $n \geq 0$, *where* $C_{K,n}$ *is an integer that depends on* K *and* n *only.*

Parts (4) and (5) tell about the compatibility of good models with other structures on the approximate subgroup Λ (see the proofs of Theorem 4.2.1). They can be considered as part of a larger theme, through the lens of model theory (see for instance [48]). We keep here, however, with an elementary point-set topology approach as we will not need more.

Proof. (1) $\Rightarrow$ (2):

Choose a neighbourhood of the identity $U \subset H$ such that $f^{-1}(U) \subset \Lambda$. Because of the continuity of the multiplication map at (e, e), for all neighbourhood of the identity W there is a symmetric neighbourhood of the identity W' such that $(W')^2 \subset W$. We can thus find, by induction, a sequence $(U_n)_{n>0}$ of relatively compact symmetric neighbourhoods of the identity in H such that $U_0 = U$ and $U_{n+1}^2 \subset U_n$ for all integers $n \geq 0$. Define now $(\Lambda_n)_{n\geq 0}$ by $\Lambda_0 = \Lambda$ and $\Lambda_n = f^{-1}(U_n)$. We readily check that for all integers $n \geq 0$ we have $\Lambda_{n+1}^2 \subset \Lambda_n$. Furthermore, for all $\gamma \in \Gamma$, $\gamma\Lambda_n\gamma^{-1}$ is an approximate subgroup commensurable with Λ by Corollary 2.3.2. So (1) $\Rightarrow$ (2) is proved.

(2) $\Rightarrow$ (3):

Let $(\Lambda_n)_{n\geq 0}$ be as in (2). For any two subsets $X, Y \subset G$ define

$$X^Y := \bigcap_{y \in Y} yXy^{-1}.$$

Define now $\mathcal{B}$ by

$$\mathcal{B} := \left\{ \left(\Lambda_n^2\right)^F \mid \forall n \in \mathbb{N}, \forall F \subset \Gamma, |F| < \infty \right\}.$$

We know that $\Lambda_1^2 \subset \Lambda$ and that for all $\Xi \in \mathcal{B}$ we have $e \in \Xi$ and Ξ is an approximate subgroup commensurable to Λ (Lemma 2.1.8). Now, for all $n \in \mathbb{N}$ and $F \subset \Gamma$ finite we have

$$\left(\left(\Lambda_{n+1}^2\right)^F\right)^2 \subset \left(\Lambda_{n+1}^4\right)^F \subset \left(\Lambda_n^2\right)^F$$

and for $\gamma \in \Gamma$ we find

$$\gamma \left(\Lambda_n^2\right)^F \gamma^{-1} \subset \left(\Lambda_n^2\right)^{\gamma F}.$$

So $\mathcal{B}$ satisfies (3).

(3) $\Rightarrow$ (1):

Equip the group $\langle \Lambda \rangle$ with the topology defined by

$$\mathcal{T} = \{U \subset \Gamma | \forall \gamma \in U, \exists \Xi \in \mathcal{B}, \gamma\Xi \subset U\}.$$

By [15, Chapter III, §1.2, Proposition 1] the topology $\mathcal{T}$ is the unique topology that makes G into a topological group and such that $\mathcal{B}$ is a neighbourhood basis for e. Now, the closure $\overline{\{e\}}$ of the identity is a closed normal subgroup and the group $\Gamma/\overline{\{e\}}$ equipped with the quotient topology is the maximal Hausdorff factor of Γ. Let $p : \Gamma \to \Gamma/\overline{\{e\}}$ be the natural map. Then $\{p(\Xi) | \Xi \in \mathcal{B}\}$ is a neighbourhood basis for the identity in $\Gamma/\overline{\{e\}}$. But the subsets that belong to $\mathcal{B}$ are pairwise commensurable, so the neighbourhoods $\{p(\Xi) | \Xi \in \mathcal{B}\}$ are pre-compact. Hence, the topological group $\Gamma/\overline{\{e\}}$ has a completion by [77, Theorem X]. In other words, there is a locally compact group H and a group homomorphism

$$i : \Gamma/\overline{\{e\}} \to H$$

such that i has dense image and is a homeomorphism onto its image. Define the map $f := i \circ p$. We will show that f is a good model. The group H is a complete space and Λ is pre-compact in the topology $\mathcal{T}$ according to assumption (b). So $f(\Lambda)$ is a relatively compact subset of H. Recall that i is a homeomorphism onto its image, the map p is open and $\mathcal{B}$ is a neighbourhood basis for the identity. There is thus a neighbourhood of the identity $U \subset H$ such that $f^{-1}(U) \subset \Lambda$ according to assumption (a).

Statement (4) is straightforward from the proof of (3) $\Rightarrow$ (1). Let us prove (5). Let $\mathcal{T}_0$ denote the initial topology on Γ with respect to the class of all good models $f : \Gamma \to H$ of (Λ, Γ). In other words, the topology $\mathcal{T}_0$ is generated by the family of subsets $\{f^{-1}(U)\}_{f,U}$ where $f : \Gamma \to H$ and U run through all good models of (Λ, Γ) and all open subsets $U \subset H$. Define $\mathcal{B}_0$ as $\{U \in \mathcal{T}_0 | e \in U \subset \Lambda\}$. Since Λ is assumed to have a good model we know that $\mathcal{B}_0$ is not empty. So take $\Xi \in \mathcal{B}_0$. Then there are good models

$(f_i : \Gamma \to H_i)_{1 \le i \le r}$ of (Λ, Γ) and open relatively compact neighbourhoods of the identity $U_i \subset H_i$ for all $1 \le i \le r$ such that

$$\bigcap_{1 \le i \le r} f_i^{-1}(U_i) \subset \Xi \subset \Lambda.$$

But according to Corollary 2.3.2 and Lemma 2.1.8 we know that Λ is commensurable to $\bigcap_{1 \le i \le r} f_i^{-1}(U_i)$, hence to Ξ. So $\mathcal{B}_0$ satisfies conditions (a) and (b) of (3). But conditions (c) and (d) of (3) are also satisfied since $(G, \mathcal{T}_0)$ is a topological group. Let now $f_0 : \Gamma \to H_0$ be as in the proof of (3) $\Rightarrow$ (1). Then every good model $f : \Gamma \to H$ of (Λ, Γ) is continuous with respect to $\mathcal{T}_0$. According to the universal properties of quotients and completions one can therefore find a continuous group homomorphism $\phi : H_0 \to H$ such that $\phi \circ f_0 = f$.

Take a good model $f : \Gamma \to H$ of (Λ, Γ) with dense image. We can find a relatively compact open symmetric neighbourhood of the identity $U \subset H$ such that $\Lambda \subset f^{-1}(U) \subset \Lambda^2$. So $f^{-1}(U)$ is a K^3-approximate subgroup, and, hence, U is a K^3-approximate subgroup as well. But by [24, Cor. 5.6 and Lem. 5.4] (or Lemma 4.3.4 below) we can find an symmetric open neighbourhood of the identity $S \subset U^4$ such that S^{16} is contained in U^4 and C_K left-translates of S cover U for some constant C_K that depends on K only. We thus define $\Lambda_1 = f^{-1}(S)$ and $C_{K,1} = C_K$. A proof by induction on n then shows (6). □

As a consequence of Lemma 2.3.4 we show that Meyer subsets almost have a good model:

Proposition 2.3.5. *Let Λ be an approximate subgroup of some group. If Λ is a Meyer subset then there is a positive integer such that Λ^n has a good model.*

Proof. Let Λ_0 be an approximate subgroup commensurable with Λ that has a good model. By Lemma 2.3.4 there is a sequence $(\Lambda_n)_{n \ge 0}$ of approximate subgroups commensurable with Λ such that $\Lambda_{n+1}^2 \subset \Lambda_n$ for all integers $n \ge 0$. Lemma 2.3.4 again implies that the approximate subgroup $\Lambda \cup \Lambda_0$ has a good model, and so has $(\Lambda \cup \Lambda_0) \cap \langle \Lambda \rangle$ by Lemma 2.3.3. But $(\Lambda \cup \Lambda_0) \cap \langle \Lambda \rangle$ is commensurable to Λ (Lemma 2.1.7) and $\langle \Lambda \rangle$ is generated by Λ. So there is a positive integer n such that $(\Lambda \cup \Lambda_0) \cap \langle \Lambda \rangle$ is contained in Λ^n. □

2.3.3 Model sets and good models

We will now explain the relationship between the language of good models and the language of model sets from [60, 9, 62]. Our main result is the following:

Proposition 2.3.6. *Let Λ be an approximate lattice in a locally compact group G. Then Λ has a good model if and only if Λ contains a model set.*

When G is abelian, this equivalence boils down to the construction of *star-map* (see for instance [4, §7.2]). At the end of this section, we will also prove a result that will be instrumental in the later parts of this dissertation. Let us now prove two elementary lemma towards Proposition 2.3.6.

Lemma 2.3.7. *Let Λ be an approximate subgroup in a locally compact group G and let Γ be a subgroup with $\Lambda \subset \Gamma \subset Comm_G(\Lambda)$. Let $f : \Gamma \to H$ be a good model of (Λ, Γ). The graph $\Gamma_f \subset G \times H$ of f is discrete if and only if Λ is discrete.*

Proof. Note here that the assumption that $\Lambda \subset \Gamma \subset Comm_G(\Lambda)$ is redundant. It is implied by the existence of a good model of (Λ, Γ) (see Definition 1.1.9 and the remark following it). We nevertheless include it to put an emphasis on this property.

Since f is a good model there is a neighbourhood of the identity $U_H \subset H$ such that $f^{-1}(U_H) \subset \Lambda$. Since, moreover, Λ is discrete, there is a neighbourhood of the identity $U_G \subset G$ such that $\Lambda \cap U_G = \{e\}$. We thus have $\Gamma_f \cap (U_G \times U_H) = \{e\}$. Which proves the if part.

Conversely, write $K := \overline{f(\Lambda^2)}$ and note that $\Gamma_f \cap \{e_G\} \times K = \{e\}$. By compactness of K there is a neighbourhood of $\{e_G\} \times K$ of the form $U_G \times U_H$ such that $\Gamma_f \cap (U_G \times U_H) = \{e\}$. So $U_G \subset G$ is a neighbourhood of G and $\Lambda^2 \cap U_G = \{e\}$. □

Lemma 2.3.8. *Let Λ be an approximate subgroup of a group G and take $\Lambda \subset \Gamma \subset \mathrm{Comm}_G(\Lambda)$. Let $f : \Gamma \to H$ be a good model of (Λ, Γ) with dense image. Suppose that $\mathcal{F} \subset G$ satisfies $\Lambda\mathcal{F} = G$ and $f(\Lambda) \subset U$ with U an open subset of G. Then $\Gamma_f \left(\mathcal{F} \times U^{-1}U\right) = G \times H$.*

Proof. Here again the assumption that $\Lambda \subset \Gamma \subset Comm_G(\Lambda)$ is redundant.

Take $(g, h) \in G \times H$. Choose first γ such that $f(\gamma)h \in U$. Choose now $\lambda \in \Lambda$ such that $\lambda^{-1}g \in \mathcal{F}$. We find $(\lambda^{-1}\gamma, f(\lambda^{-1}\gamma))(g, h) \in \mathcal{F} \times U^{-1}U$. □

Proof of Proposition 2.3.6. Suppose that Λ has a good model $f : \langle\Lambda\rangle \to H$ and take one with dense image. According to Lemma 2.3.7 the graph Γ_f of f is a discrete subgroup of $G\times H$. Now, write U a relatively compact neighbourhood of $\overline{f(\Lambda)}$ and take $\mathcal{F} \subset G$ of finite volume such that $\Lambda\mathcal{F} = G$. We have that $\Gamma_f \left(\mathcal{F} \times U^{-1}U\right) = G \times H$ (Lemma 2.3.8). So Γ_f is a lattice. Hence, if V is a relatively compact neighbourhood of the identity in H such that $f^{-1}(V) \subset \Lambda$, then the model set $M = P_0(G, H, \Gamma_f, V)$ associated to (G, H, Γ_f) and V is contained in Λ.

Conversely, consider (G, H, Γ) a cut-and-project associated to a model set M contained in Λ. Note that the restriction to Γ of the natural projection $p_G : G \times H \to G$ is injective. Define the map $f : \langle M\rangle \to H$ by $f(\gamma) := p_H \circ (p_G)^{-1}_{|\Gamma}(\gamma)$. Then f is a good model of M. By Lemma 2.3.4, Λ must have a good model too. □

We show finally a key lemma that we will use in the later parts of the proof of Theorem 1.1.25. Lemma 2.3.9 was first observed, independently, by Hrushovski ([47]) and the author ([54]).

Lemma 2.3.9. *Let Λ be an approximate lattice of a locally compact group G. Suppose that $f : \langle\Lambda\rangle \to H$ is a group homomorphism towards a locally compact* *group with $f(\Lambda)$ relatively compact. Suppose furthermore that the graph Γ_f is discrete. Then f is a good model of some power Λ^m of Λ.*

Proof. Take a compact neighbourhood of the identity $W \subset H$ such that $f(\Lambda) \subset W$ and note that $\Lambda' := f^{-1}(W)$ is a discrete approximate subgroup containing Λ. Indeed, according to Lemma 2.1.7 $\Gamma_f \cap (G \times W)$ is an approximate subgroup and, by assumption, it is uniformly discrete. Moreover, since the restriction $(p_G)_{|G\times W}$ of the natural map $p_G : G \times H \to G$ is proper, we have that $\Lambda' = p_G(\Gamma_f \cap (G \times W))$ is a uniformly discrete approximate subgroup as well. But $\Lambda \subset \Lambda'$ and Λ is an approximate lattice. By Corollary 2.2.27, Λ and Λ' are commensurable. Since $\Lambda' \subset \langle\Lambda\rangle$ there is a positive integer m such that $\Lambda' \subset \Lambda^m$. Therefore, f is a good model of Λ^m. □

Chapter 3

Infinite approximate subgroups in soluble linear groups

3.1 Approximate subgroups in soluble linear groups

In this section we prove Theorem 1.1.12 which extends Schreiber's theorem (Theorem 1.1.11) to soluble real algebraic groups. In the proof of the following proposition we rely on the theory of algebraic groups. See [80, §2 and 3] for a survey of the specific tools we will use and [72] for a general introduction to affine algebraic groups. Note that Proposition 3.1.1 below is in fact the main result of this chapter. We will call upon Proposition 3.1.1 rather than Theorem 1.1.12 in the rest of the chapter.

Proposition 3.1.1. *Let $\Lambda \subset \mathbb{G}(\mathbb{R})$ be an approximate subgroup in the group of $\mathbb{R}$-points of a Zariski-connected soluble real algebraic group such that Λ^{∞} is Zariski-dense. Then there is a closed connected normal subgroup $H \lhd \mathbb{G}(\mathbb{R})$ such that Λ is compactly commensurable with H.*

The subgroup H is the connected component of the identity of the group of $\mathbb{R}$-points of some normal algebraic subgroup $\mathbb{H}$ of $\mathbb{G}$. Moreover, there exists a unique minimal *such $\mathbb{H}$ i.e. contained in every other normal algebraic subgroup $\mathbb{L}$ whose group of $\mathbb{R}$-points is compactly commensurable with Λ.*

Proof. We start with the uniqueness part. It will be an easy consequence of the following: take two normal algebraic subgroups $\mathbb{H}_1, \mathbb{H}_2 \subset \mathbb{G}$ such that the projection of Λ is relatively compact in $\mathbb{G}(\mathbb{R})/\mathbb{H}_i(\mathbb{R})$ for $i = 1, 2$. Then the projection of Λ through the diagonal group homomorphism $\Delta : \mathbb{G}(\mathbb{R}) \to (\mathbb{G}/\mathbb{H}_1)(\mathbb{R}) \times (\mathbb{G}/\mathbb{H}_1)(\mathbb{R})$ is relatively compact. But Δ is algebraic so there is a compact subset $K \subset \mathbb{G}(\mathbb{R})$ such that Λ is contained in $K\left(\mathbb{H}_1(\mathbb{R}) \cap \mathbb{H}_2(\mathbb{R})\right)$ (as a consequence of [80, 3.16]). In particular, if Λ is compactly commensurable with $\mathbb{H}_1(\mathbb{R})$ and $\mathbb{H}_2(\mathbb{R})$, then Λ is compactly commensurable with $\mathbb{H}_1(\mathbb{R}) \cap \mathbb{H}_2(\mathbb{R})$. Now, by the descending chain condition for Zariski-closed subsets of $\mathbb{G}(\mathbb{R})$

there is a Zariski-closed normal subgroup compactly commensurable with Λ that does not contain any Zariski-closed normal proper subgroup compactly commensurable with Λ. So by the discussion it must be contained in every other Zariski-closed normal subgroup compactly commensurable with Λ.

Let us now move on to proving existence. As Λ^∞ is Zariski-dense, we know that $\Lambda^\infty \cap [\mathbb{G}(\mathbb{R}), \mathbb{G}(\mathbb{R})]$ is Zariski-dense in $[\mathbb{G}(\mathbb{R}), \mathbb{G}(\mathbb{R})]$. Moreover, $[\mathbb{G}(\mathbb{R}), \mathbb{G}(\mathbb{R})]$ is a connected simply connected nilpotent Lie group ([80, Lemma 3.4]) so $\overline{\Lambda^\infty \cap [\mathbb{G}(\mathbb{R}), \mathbb{G}(\mathbb{R})]}$ is co-compact by [65, Theorem 2.1]. As a consequence, there is $k \in \mathbb{N}$ greater than 2 such that $\Lambda' := \Lambda^k \cap [\mathbb{G}(\mathbb{R}), \mathbb{G}(\mathbb{R})]$ with Λ'^∞ Zariski-dense in $[\mathbb{G}(\mathbb{R}), \mathbb{G}(\mathbb{R})]$. By Lemma 2.1.8 we know that Λ' is an approximate subgroup. According now to the induction hypothesis there is a closed connected subgroup $H_1 \triangleleft [\mathbb{G}(\mathbb{R}), \mathbb{G}(\mathbb{R})]$ compactly commensurable with Λ'. In addition, for all $\lambda \in \Lambda$, we have $\lambda\,(\Lambda')\,\lambda^{-1} \subset \Lambda^{k+2} \cap [\mathbb{G}(\mathbb{R}), \mathbb{G}(\mathbb{R})]$. But, according to Lemma 2.1.8 approximate subgroups $\Lambda^{k+2} \cap [\mathbb{G}(\mathbb{R}), \mathbb{G}(\mathbb{R})]$ and Λ' are commensurable. Therefore, H_1 and $\lambda H_1 \lambda^{-1}$ are compactly commensurable. We thus have $H_1 = \lambda H_1 \lambda^{-1}$ (see Lemma 3.1.2 below).

Now H_1 is connected, so its normaliser $N(H_1)$ is equal to the stabiliser of its Lie algebra. But then $N(H_1)$ is Zariski-closed and contains Λ which is Zariski dense. So $N(H_1) = \mathbb{G}(\mathbb{R})$ and H_1 is normal. Since H_1 is connected in a unipotent subgroup, H_1 is the group of $\mathbb{R}$-points of an algebraic subgroup $\mathbb{H}_1$ normal in $\mathbb{G}$ (e.g. [80, Lemma 3.20]). Therefore, the natural map $\mathbb{G}(\mathbb{R})/\mathbb{H}_1(\mathbb{R}) \to (\mathbb{G}/\mathbb{H}_1)(\mathbb{R})$ is an embedding and its image contains the connected component of the identity in $(\mathbb{G}/\mathbb{H}_1)(\mathbb{R})$. Moreover, the Zariski-closure of the image of Λ^∞ contains the image of $\mathbb{G}(\mathbb{R})$.

Set $p : \mathbb{G}(\mathbb{R}) \to (\mathbb{G}/\mathbb{H}_1)(\mathbb{R})$ the canonical projection, $\tilde{\Lambda} := p(\Lambda^2)$ and $\tilde{G}$ the Zariski-closure of $\tilde{\Lambda}$.

Claim 3.1.1. *The approximate subgroup $\tilde{\Lambda}$ is compactly commensurable with a Zariski-closed subgroup of the centre $Z(\tilde{G})$ of $\tilde{G}$.*

Let us first show how Proposition 3.1.1 follows from this claim. There is a Zariski-closed subgroup $V \subset Z(\tilde{G})$ such that $\tilde{\Lambda}$ is compactly commensurable with V. So we can find a compact subset $K_1 \subset \mathbb{G}(\mathbb{R})$ such that

$$\tilde{\Lambda} \subset p(K_1)V \text{ and } V \subset p(K_1)\tilde{\Lambda}.$$

According to the first inclusion we have $\Lambda \subset K_1 p^{-1}(V)$. On the other hand, there is $K_2 \subset \mathbb{G}(\mathbb{R})$ compact such that $H_1 \subset K_2\Lambda$, where H_1 is the subgroup defined above. Finally,

$$p^{-1}(V) \subset K_1 \Lambda H_1 = K_1 H_1 \Lambda \subset K_1 K_2 \Lambda^2.$$

So the subgroup $p^{-1}(V)$ is compactly commensurable with Λ. But $p^{-1}(V)$ is equal to the $\mathbb{R}$-points of an algebraic subgroup $\mathbb{H} \subset \mathbb{G}$ since both V and H_1 are Zariski-closed subgroups. The subgroup V being central implies furthermore that $\mathbb{H}$ is normal. Finally, we know that $\mathbb{H}(\mathbb{R})$ has finitely many connected components ([78]), so denoting by H the connected component of the identity in $\mathbb{H}(\mathbb{R})$ we indeed find a subgroup satisfying the conclusions of Proposition 3.1.1.

Now let us move to the proof of the claim. We will use the fact that the set of commutators of elements of $\tilde{\Lambda}$ is relatively compact to show that $\tilde{\Lambda}$ is contained in a 'neighbourhood' of the centre. The group $\tilde{\Lambda}^\infty \cap [\tilde{G},\tilde{G}]$ is co-compact in $[\tilde{G},\tilde{G}]$. So there is $l \in \mathbb{N}$ such that

$$\mathrm{span}_{\mathbb{R}}(\log(\tilde{\Lambda}^l \cap [\tilde{G},\tilde{G}])) = \mathfrak{Lie}([\tilde{G},\tilde{G}]),$$

where log is the logarithm map from $[\tilde{G},\tilde{G}]$ to its Lie algebra. In addition,

$$\bigcup_{\lambda\in\tilde{\Lambda}} \lambda \left(\tilde{\Lambda}^l \cap [\tilde{G},\tilde{G}]\right) \lambda^{-1} \subset \tilde{\Lambda}^{l+2} \cap [\tilde{G},\tilde{G}].$$

But by definition of H_1 we know that $\tilde{\Lambda} \cap [\tilde{G},\tilde{G}]$ is relatively compact. And thanks to Lemma 2.1.8 and since $\tilde{\Lambda} = p(\Lambda)^2$ the subset $\tilde{\Lambda}^r \cap [\tilde{G},\tilde{G}]$ is an approximate subgroup commensurable with $\tilde{\Lambda} \cap [\tilde{G},\tilde{G}]$ for all $r \in \mathbb{N}$. The right-hand side of the above inclusion is therefore a relatively compact subset. Hence, the family of linear operators $(\mathrm{Ad}(\lambda)_{|\mathfrak{Lie}([\tilde{G},\tilde{G}])})_{\lambda\in\Lambda}$ is uniformly bounded. Since

$$\begin{aligned}\rho : (\mathbb{G}/\mathbb{H})(\mathbb{R}) &\to \mathrm{GL}(\mathfrak{Lie}([\tilde{G},\tilde{G}]))\\ g &\mapsto Ad(g)_{|\mathfrak{Lie}([\tilde{G},\tilde{G}])}\end{aligned}$$

is an algebraic group homomorphism, there is a compact set $K_3 \subset \tilde{G}$ such that

$$\tilde{\Lambda} \subset \ker(\rho)K_3.$$

In the following we will denote $\ker(\rho)$ by Z. Note that since $[\tilde{G},\tilde{G}]$ is connected, Z is in fact the centraliser of $[\tilde{G},\tilde{G}]$.

Now for any $g \in \tilde{G}$ define the map

$$\begin{aligned}\theta_g : &Z \to \tilde{G}\\ &h \mapsto [g,h],\end{aligned}$$

where $[g,h]$ denotes $ghg^{-1}h^{-1}$. For $h_1, h_2 \in Z$ we have

$$\begin{aligned}\theta_g(h_1)\theta_g(h_2) &= gh_1g^{-1}h_1^{-1}gh_2g^{-1}h_2^{-1}\\ &= gh_1g^{-1}gh_2g^{-1}h_2^{-1}h_1^{-1} \qquad \text{as } h_1 \in Z\\ &= g(h_1h_2)g^{-1}(h_1h_2)^{-1}\\ &= \theta_g(h_1h_2).\end{aligned}$$

So θ_g is an algebraic group homomorphism.

For all $\lambda \in \tilde{\Lambda}$ let $f(\lambda)$ denote an element of Z such that $\lambda f(\lambda)^{-1} \in K_3$. Write also $\delta(\lambda) := \lambda f(\lambda)^{-1}$. Now, for $\gamma \in \tilde{\Lambda}$ and $\lambda \in \tilde{\Lambda}$ we have

$$\begin{aligned}\theta_\gamma(f(\lambda)) &= \gamma f(\lambda)\gamma^{-1}f(\lambda)^{-1}\\ &= \gamma\delta(\lambda)^{-1}\lambda\gamma^{-1}\left(\delta(\lambda)^{-1}\lambda\right)^{-1}\\ &= \left(\gamma\delta(\lambda)^{-1}\gamma^{-1}\right)\gamma\lambda\gamma^{-1}\lambda^{-1}\left(\delta(\lambda)\right).\end{aligned}$$

In particular
$$\theta_\gamma(f(\tilde{\Lambda})) \subset \gamma K_3^{-1}\gamma^{-1}(\tilde{\Lambda}^4 \cap [\tilde{G},\tilde{G}])K_3.$$
So $\theta_\gamma(f(\tilde{\Lambda}))$ is a relatively compact set.

Set now
$$\begin{aligned}\theta_\Upsilon : Z &\to (Z)^n\\ g &\mapsto (\theta_{\gamma_1}(g),\dots,\theta_{\gamma_n}(g))\end{aligned}$$
where Υ is any finite family $\{\gamma_1,\dots,\gamma_n\}$ of elements of $\tilde{\Lambda}$. We readily see that
$$\ker(\theta_\Upsilon) = \bigcap_{1\le i\le n} Z_{\tilde{G}}(\gamma_i) \cap Z.$$
We know that Z is an algebraic subgroup and θ_Υ is an algebraic group morphism. Moreover, since $\theta_\Upsilon(f(\tilde{\Lambda}))$ is relatively compact as a subset of $\tilde{G}^n$ and $\theta_\Upsilon(Z)$ is closed, it is relatively compact as a subset of $\theta_\Upsilon(Z)$. Thus, there is a compact set K_4 such that $f(\tilde{\Lambda}) \subset K_4\ker(\theta_\Upsilon)$. Setting $K_\Upsilon := K_3K_4$, we get $\tilde{\Lambda} \subset K_\Upsilon \ker(\theta_\Upsilon)$.

Since $\tilde{\Lambda}^\infty$ is Zariski-dense, the centraliser of elements are Zariski-closed and Zariski-closed subsets satisfy the descending chain condition, we can find $\Upsilon \subset \tilde{\Lambda}^\infty$ finite such that the centraliser of Υ is equal to the centre $Z(\tilde{G})$ of $\tilde{G}$. Hence,
$$\tilde{\Lambda} \subset K_\Upsilon \ker(\theta_\Upsilon) = K_\Upsilon Z(\tilde{G}).$$
But $Z(\tilde{G})$ is a Zariski-closed subgroup, so it has a finite number of connected components ([78]). Besides, the connected component of the identity is isomorphic to $\mathbb{R}^k \times \mathbb{T}^l$ for some $k,l \in \mathbb{N}$. There is therefore a central subgroup $W \subset \tilde{G}$ and a compact subset $K_5 \subset \tilde{G}$ such that $\tilde{\Lambda} \subset K_5W$ and $W \simeq \mathbb{R}^k$

Finally, choose a function $g : \tilde{\Lambda} \to W$ such that for all $\lambda \in \tilde{\Lambda}$, $g(\lambda^{-1}) = g(\lambda)^{-1}$ and $b(\lambda) := g(\lambda)\lambda^{-1} \in K_5$. There is a finite subset $F \subset \tilde{G}$ such that for all $\lambda_1,\lambda_2 \in \tilde{\Lambda}$ there is $\lambda \in \tilde{\Lambda}$ satisfying $\lambda\lambda_1^{-1}\lambda_2^{-1} \in F$. Relying on the centrality of W we compute
$$\begin{aligned} g(\lambda)g(\lambda_1)^{-1}g(\lambda_2)^{-1} &= b(\lambda)\lambda\lambda_1^{-1}b(\lambda_1)^{-1}g(\lambda_2)^{-1}\\ &= \lambda\lambda_1^{-1}b(\lambda_1)^{-1}g(\lambda_2)^{-1}b(\lambda)\\ &= \lambda\lambda_1^{-1}g(\lambda_2)^{-1}b(\lambda_1)^{-1}b(\lambda)\\ &= \lambda\lambda_1^{-1}\lambda_2^{-1}b(\lambda_2)^{-1}b(\lambda_1)^{-1}b(\lambda).\end{aligned}$$
Above, we used the fact that elements of W are fixed under conjugation by $b(\lambda)$ to go from the first line to the second. And that $g(\lambda_2)$ is central to go from the second to the third. We therefore find
$$g(\lambda)g(\lambda_1)^{-1}g(\lambda_2)^{-1} \in FK_5^{-2}K_5.$$
We have thus proved that the subset $g(\tilde{\Lambda})$ is such that $g(\tilde{\Lambda})+g(\tilde{\Lambda})$ is compactly commensurable with $g(\tilde{\Lambda})$. We then have that $g(\tilde{\Lambda})$ is compactly commensurable with an approximate subgroup (see Lemma 3.1.3 below). By Schreiber's

theorem (Theorem 1.1.11) there is therefore a closed connected subgroup V_1 compactly commensurable with $g(\tilde{\Lambda})$ - hence to $\tilde{\Lambda}$. But $V_1 \subset Z(\tilde{G})$ is compactly commensurable with its Zariski-closure V_2 (see [79, Cororollary 4.1]). So V_2 is compactly commensurable with $\tilde{\Lambda}$. $\square$

We prove now the two technical lemmas used in the proof of Proposition 3.1.1.

Lemma 3.1.2. *Let H_1, H_2 be two connected subgroups of a simply connected nilpotent group N. Then H_1 and H_2 are compactly commensurable if and only if they are equal.*

Proof. Let $K \subset N$ be a compact subset such that $H_2 \subset H_1K$. We proceed by induction on the length of the upper central series. If $N \simeq \mathbb{R}^n$ for some $n \in \mathbb{N}$, the result is obvious. Otherwise let Z be the centre of N, by induction hypothesis the projections of H_1 and H_2 to N/Z are equal. So given $g \in H_2$, there is $z \in Z$ such that $gz \in H_1$. Moreover for all $n \in \mathbb{N}$ there are $h_n \in H_1$ and $k_n \in K$ such that $g^n = h_nk_n$. Hence,

$$\forall n \in \mathbb{N},\ z^nk_n = z^nh_n^{-1}g^n = h_n^{-1}(gz)^n \in H_1.$$

So,

$$\forall n \in \mathbb{N}, \log(z) + \frac{\log(k_n)}{n} \in \log(H_1),$$

where log is the logarithm map from N to its Lie alebra. But $\log(H_1)$ is closed (since it is a vector subspace of the Lie algebra) so we find $z \in H_1$ and $g \in H_1$. $\square$

Lemma 3.1.3. *Let V be a locally compact abelian group and $\Lambda \subset V$ a symmetric subset such that $\Lambda + \Lambda$ is compactly commensurable with Λ. Then Λ is compactly commensurable with an approximate subgroup.*

Proof. Let $U \subset V$ be a symmetric compact neighbourhood of 0, then $\Lambda + U$ is a symmetric set compactly commensurable with Λ. Moreover, let $K \subset V$ be a compact subset such that $\Lambda + \Lambda \subset \Lambda + K$ and $F \subset V$ be a finite subset such that $K + U + U \subset U + F$. Then we have,

$$(\Lambda + U) + (\Lambda + U) \subset \Lambda + K + U + U \subset (\Lambda + U) + F.$$

$\square$

Proof of Theorem 1.1.12. Let G be the Zariski-closure of Λ^∞. Then G is the group of $\mathbb{R}$-points of a soluble algebraic group. Let $\tilde{G}$ denote the group of $\mathbb{R}$-points of its Zariski-connected component of the identity. The subgroup Λ^∞ is Zariski-dense in G so we can find a finite subset $X \subset \Lambda^\infty$ such that $X\tilde{G} = G$. There is therefore an integer $n \geq 1$ such that $X \subset \Lambda^n$; a classical argument - found e.g. in [52] - about generation in finite index subgroups now shows that $\tilde{G} \cap \Lambda^\infty$ is generated by $\Lambda^{2n+1} \cap \tilde{G}$. By Lemma 2.1.8, the set $\Lambda^{2n+1} \cap \tilde{G}$ is an approximate subgroup. Applying Proposition 3.1.1 to $\Lambda^{2n+1} \cap \tilde{G}$ we get a minimal Zariski-closed normal subgroup H of $\tilde{G}$ that is compactly commensurable with $\Lambda^{2n+1} \cap \tilde{G}$. But $\Lambda^{2n+1} \cap \tilde{G}$ is commensurable with $\Lambda^2 \cap \tilde{G}$ according to Lemma 2.1.8 and $\Lambda^2 \cap \tilde{G}$ is commensurable with Λ by Lemma 2.1.8. The subgroup H is thus compactly commensurable with Λ. Now for any $\lambda \in \Lambda$ notice that $\lambda \left(\Lambda^{2n+1} \cap \tilde{G}\right) \lambda^{-1} = \lambda \left(\Lambda^{2n+1}\right) \lambda^{-1} \cap \tilde{G}$ is commensurable with $\Lambda^{2n+1} \cap \tilde{G}$ according to Lemma 2.1.8. So $\lambda H \lambda^{-1}$ is a normal Zariski-closed subgroup of $\tilde{G}$ and is compactly commensurable with $\lambda \left(\Lambda^{2n+1} \cap \tilde{G}\right) \lambda^{-1}$ - hence to $\Lambda^{2n+1} \cap \tilde{G}$. By minimality of H we find $H \subset \lambda H \lambda^{-1}$. The inclusion being true for all elements in Λ, we have that H is normalised by Λ^∞ and by its Zariski-closure. □

3.2 Consequences

In [38] Fried and Goldman proved that every soluble subgroup Γ of $\mathrm{GL}_n(\mathbb{R})$ admits a *syndetic hull*. More precisely:

Proposition 3.2.1 (Theorem 1.6, [38])**.** *Let $\mathbb{G}$ be a soluble real algebraic group and $\Gamma \subset \mathbb{G}(\mathbb{R})$ a subgroup. Then there is $H < \mathbb{G}(\mathbb{R})$ such that H is a closed connected subgroup (in the Euclidean topology), $\Gamma \cap H$ has finite index in Γ and Γ and H are compactly commensurable.*

We show how this theorem is in fact a consequence of Theorem 1.1.12.

Proof. Without loss of generality we can assume that Γ is Zariski-dense. Let $\mathbb{G}^0$ be the Zariski-connected component of the identity of $\mathbb{G}$. The subgroup $\Gamma \cap \mathbb{G}^0(\mathbb{R})$ has finite index in Γ and is Zariski-dense in $\mathbb{G}^0(\mathbb{R})$. Then applying Proposition 3.1.1 to $\Gamma \cap \mathbb{G}^0(\mathbb{R})$ we get a closed connected normal subgroup $H \subset \mathbb{G}^0(\mathbb{R})$ such that Γ is compactly commensurable with H. So the image of $\Gamma \cap \mathbb{G}^0(\mathbb{R})$ in $\mathbb{G}^0(\mathbb{R})/H$ via $p : \mathbb{G}^0(\mathbb{R}) \to \mathbb{G}^0(\mathbb{R})/H$ is contained in a compact subgroup K. Let K_0 be the connected component of the identity of K in the Euclidean topology, then set $\tilde{\Gamma} = \Gamma \cap p^{-1}(K_0)$. The subgroup $\tilde{\Gamma}$ has finite index in $\Gamma \cap \mathbb{G}^0(\mathbb{R})$, hence in Γ, and is co-compact in the closed connected subgroup $p^{-1}(K_0)$. □

We also get a generalisation of the well-known fact that closed soluble subgroups of $\mathrm{GL}_n(\mathbb{R})$ are compactly generated. Note that in Proposition 3.2.2 below we do not assume the approximate subgroup Λ considered to be closed. Specialised to (not necessarily closed) soluble subgroups of $\mathrm{GL}_n(\mathbb{R})$ our conclusion is that any soluble subgroup is generated by a subset that is relatively compact in $\mathrm{GL}_n(\mathbb{R})$.

Proposition 3.2.2. *Let G be the $\mathbb{R}$-points of a soluble real algebraic group and $\Lambda \subset G$ an approximate subgroup. Then there is a compact subset K such that $\Lambda^2 \cap K$ generates Λ^∞.*

Proof. Let H be the Zariski-closure of Λ^∞. Proceeding as in the proof of Theorem 1.1.12 we find an integer $k \geq 2$ such that $\Lambda' := \Lambda^k \cap H^0$ generates a Zariski-dense subgroup of the Zariski-connected component of the identity H^0 of H. Note moreover that Λ' is an approximate subgroup commensurable with Λ (Lemma 2.1.8). Applying Proposition 3.1.1 to Λ' we find a connected subgroup $H \leq G$ such that G and Λ' are compactly commensurable. Since Λ and Λ' are commensurable, there is a compact symmetric subset $K \subset G$ such that $\Lambda \subset KH$ and $H \subset K\Lambda$. Choose also V a compact neighbourhood of the identity in H. As H is connected for the Euclidean topology, V generates H. Now, for any $\lambda \in \Lambda$ choose $h \in H$ such that $\lambda h^{-1} \in K$. Since V generates H, we can find a sequence $(h_i)_{0 \leq i \leq r}$ of elements of H such that $h_0 = e, h_r = h$ and $h_{i+1}h_i^{-1} \in V$. In addition, we can find a sequence $(\lambda_i)_{0 \leq i \leq n}$ of elements of Λ such that $\lambda_0 = e, \lambda_r = \lambda$ and for all $0 \leq i \leq r$, $\lambda_i h_i^{-1} \in K^{-1} = K$. Thus, $\lambda_{i+1}\lambda_i^{-1} \in KV^{-1}K^{-1}$. Finally, Λ^∞ is generated by $\Lambda^2 \cap KV^{-1}K^{-1}$. □

Corollary 3.2.3. *If Λ is discrete, then Λ^∞ is finitely generated.*

3.3 Uniform approximate lattices in abelian groups

We will investigate morphisms commensurating approximate subgroups in $\mathbb{R}^n$. This will turn out to be useful in the proof of Theorem 1.1.13. Our goal is to understand homomorphisms that commensurate a uniform approximate lattice. Let us start with a result concerning lattices.

Proposition 3.3.1. *Let $\Lambda \subset \mathrm{GL}_n(\mathbb{R})$ be an approximate subgroup and suppose there are $\Gamma_1 \subset \Gamma_2$ lattices in $\mathbb{R}^n$ such that $\lambda(\Gamma_1) \subset \Gamma_2$ for all $\lambda \in \Lambda$. Then there is an approximate subgroup $\Xi \subset \Lambda^4$ commensurable with Λ such that $\Xi \subset \mathrm{Aut}(\Gamma_1)$.*

Proof. We can assume that $\Gamma_1 = \mathbb{Z}^n$. Let m be the order of Γ_2/Γ_1 and $p_1, \ldots, p_r$ the prime factors of m. Then any matrix in Λ has entries lying in $\frac{1}{m}\mathbb{Z}$.

Set

$$\begin{aligned} \phi : \mathrm{GL}_n(\mathbb{R}) &\to \mathbb{R}^*_+ \\ M &\mapsto |\det(M)| \end{aligned}$$

then ϕ is a group homomorphism and $\phi(\Lambda) \subset \frac{1}{m^n}\mathbb{Z}$ is a discrete approximate subgroup bounded away from 0 so $\phi(\Lambda)$ is finite. As a consequence,

$$\tilde{\Lambda} = \phi^{-1}(\{1\}) \cap \Lambda^2$$

is an approximate subgroup commensurable with Λ by Lemma 2.1.8.

Consider the diagonal embedding

$$\iota : \mathrm{SL}_n\left(\mathbb{Z}\left[\frac{1}{m}\right]\right) \hookrightarrow \prod_{i=1}^{r} \mathrm{SL}_n(\mathbb{Q}_{p_i}).$$

Now $\iota(\tilde{\Lambda})$ is relatively compact and $\prod_{i=1}^{r} \mathrm{SL}_n(\mathbb{Z}_{p_i})$ is open. Therefore, there are $\lambda_1, \ldots, \lambda_s \in \tilde{\Lambda}$ such that

$$\iota(\tilde{\Lambda}) \subset \bigcup_{i=1}^{s} \iota(\lambda_s)(\prod_{i=1}^{r} \mathrm{SL}_n(\mathbb{Z}_{p_i})).$$

So

$$\tilde{\Lambda}^2 \cap \iota^{-1}\left(\prod_{i=1}^{r} \mathrm{SL}_n(\mathbb{Z}_{p_i})\right)$$

is an approximate subgroup commensurable with $\tilde{\Lambda}$ by Lemma 2.1.8. Since

$$\iota^{-1}(\prod_{i=1}^{r} \mathrm{SL}_n(\mathbb{Z}_{p_i}) = \mathrm{SL}_n(\mathbb{Z}),$$

we can set $\Xi := \tilde{\Lambda}^2 \cap \mathrm{SL}_n(\mathbb{Z})$. □

Now, we can deduce

Proposition 3.3.2. *Let $\Lambda \subset \mathrm{GL}_n(\mathbb{R})$ be an approximate subgroup and suppose there are $\Lambda_1 \subset \Lambda_2$ approximate lattices in $\mathbb{R}^n$ such that $\lambda(\Lambda_1) \subset \Lambda_2$ for all $\lambda \in \Lambda$. Then there are $\Xi \subset \Lambda^4$ commensurable with Λ and an injective group homomorphism $\Xi^\infty \to \mathrm{SL}_m(\mathbb{Z})$ for some $m \geq n$. .*

This result is not needed in the sequel, however it gives a good insight into the remaining part of the proof of Theorem 1.1.13. Indeed, a similar argument will be used to prove Proposition 3.4.2.

Proof. According to Corollary 3.2.3 any approximate subgroup commensurable with Λ_1 generates a subgroup with finite rank. Take Λ_0 commensurable with Λ_1 generating a group of minimal rank. By Lemma 2.1.8 the approximate subgroup $\Lambda_0^2 \cap \Lambda_1^2$ is commensurable with Λ_1. But $\mathrm{rank}((\Lambda_0^2 \cap \Lambda_1^2)^\infty) \leq \mathrm{rank}(\Lambda_0^\infty)$, so there is equality of ranks. We will therefore assume that $\Lambda_0 \subset \Lambda_1^2$. We also know that for all $\lambda \in \Lambda$ the approximate group $\lambda(\Lambda_0)$ is commensurable with $\lambda(\Lambda_1)$ which in turn is commensurable with Λ_2. So Λ_0 and $\lambda(\Lambda_0)$ are commensurable. Hence, Λ_0 is commensurable with $\Lambda_0^2 \cap \lambda(\Lambda_0^2)$ (Lemma 2.1.8) . Also, minimality of $\mathrm{rank}(\Lambda_0^\infty)$ yields

$$\mathrm{rank}((\Lambda_0^2 \cap \lambda(\Lambda_0^2))^\infty) = \mathrm{rank}(\Lambda_0^\infty) = \mathrm{rank}(\lambda(\Lambda_0^\infty)).$$

So $\Lambda_0^2 \cap \lambda(\Lambda_0^2)$ generates a finite index subgroup of Λ_0^∞.

Therefore, λ is an isomorphism of the $\mathbb{Q}$-span $\mathrm{span}_{\mathbb{Q}}(\Lambda_0^\infty)$ of Λ_0^∞. Choosing a basis of $\mathrm{span}_{\mathbb{Q}}(\Lambda_0^\infty)$ adapted to the subgroup Λ_0^∞ we thus have a group homomorphism $\phi : \Lambda^\infty \to \mathrm{GL}_m(\mathbb{Q})$ where $m = \mathrm{rank}(\Lambda_0^\infty)$. Since $\mathrm{span}_{\mathbb{R}}(\Lambda_0^\infty) = \mathbb{R}^n$ this group homomorphism is injective. Now, for all $\lambda \in \Lambda$ we have $\lambda(\Lambda_0^\infty) \subset \Lambda_2^\infty \cap \mathrm{span}_{\mathbb{Q}}(\Lambda_0^\infty)$. But

$$\begin{aligned}\operatorname{rank}(\Lambda_0^\infty) &\leq \operatorname{rank}\left(\Lambda_2^\infty \cap \operatorname{span}_{\mathbb{Q}}(\Lambda_0^\infty)\right)\\ &\leq \dim_{\mathbb{Q}}(\operatorname{span}_{\mathbb{Q}}(\Lambda_0^\infty))\\ &= \operatorname{rank}(\Lambda_0^\infty).\end{aligned}$$

So Λ_0^∞ has finite index in $\Lambda_2^\infty \cap \operatorname{span}_{\mathbb{Q}}(\Lambda_0^\infty)$. So Proposition 3.3.1 applied to Λ_0^∞, $\Lambda_2^\infty \cap \operatorname{span}_{\mathbb{Q}}(\Lambda_0^\infty)$ and Λ gives an approximate subgroup Ξ commensurable with Λ that consists of automorphisms of Λ_0^∞. Rewording the last statement using ϕ we find

$$\phi(\Xi) \subset \mathrm{SL}_m(\mathbb{Z}).$$

□

Remark 3.3.1. From the proof of Proposition 3.3.2, we have that for any discrete approximate lattice $\Lambda \subset \mathbb{R}^n$ the subgroup of those elements $g \in \mathrm{GL}_n(\mathbb{R})$ such that $g(\Lambda)$ is commensurable with Λ is isomorphic to a subgroup of $\mathrm{GL}_m(\mathbb{Q})$ where m is the minimal rank of an approximate subgroup commensurable with Ξ.

3.4 Meyer's Theorem for soluble Lie groups

We will now turn to the proof of Theorem 1.1.13. Our first step towards this goal is to prove a version under an additional assumption.

Proposition 3.4.1. *Let $\Lambda \subset G$ be a uniform approximate lattice in a connected soluble Lie group. If Λ^∞ is polycyclic then Λ is a Meyer subset.*

Proof. According to a theorem of Auslander (see [2] or the proof of [65, Theorem 4.28]), Λ^∞ admits an embedding as a Zariski-dense lattice in R the group of $\mathbb{R}$-points of a soluble algebraic group. In the following we will consider Λ^∞ as a subgroup of R. Moreover, we can assume without loss of generality that R is Zariski-connected. Indeed, there is a finite index subgroup Γ of Λ^∞ such that the Zariski closure of Γ is Zariski-connected. Furthermore, proceeding as in the proof of Theorem 1.1.12 we can find $n \in \mathbb{N}$ such that $\Lambda^n \cap \Gamma$ is an approximate subgroup commensurable with Λ (Lemma 2.1.8) and generates Γ.

Now according to Proposition 3.1.1 there is a closed connected normal subgroup $N \triangleleft R$ such that Λ is compactly commensurable with N. Let $p : R \to R/N$ denote the natural projection. We know that $p(\Lambda)$ is relatively compact, so we can choose a symmetric compact neighbourhood W_0 of $p(\Lambda)$. Now $p^{-1}(W_0)$ is an approximate subgroup compactly commensurable with N. But $\Lambda \subset \Lambda^\infty \cap p^{-1}(W_0)$ so both subsets are compactly commensurable with the subgroup N. So Λ and $\Lambda^\infty \cap p^{-1}(W_0)$ are compactly commensurable i.e. there is a compact subset $K \subset R$ such that

$$\Lambda^\infty \cap p^{-1}(W_0) \subset K\Lambda.$$

Since Λ and $\Lambda^\infty \cap p^{-1}(W_0)$ are moreover contained in Λ^∞ we have

$$\Lambda^\infty \cap p^{-1}(W_0) \subset (K \cap \Lambda^\infty)\Lambda.$$

But Λ^∞ is a discrete subgroup of R so $K \cap \Lambda^\infty$ is finite - meaning that Λ is commensurable with $\Lambda^\infty \cap p^{-1}(W_0)$. Finally, $p_{|(\Lambda^\infty \cap p^{-1}(W_0))^\infty}$ is a good model of $\Lambda^\infty \cap p^{-1}(W_0)$. Hence, Λ is a Meyer set (Proposition 2.3.6) . □

Proposition 3.4.2. *Let $\Lambda \subset G$ be a uniform approximate lattice in a connected soluble Lie group. Then there is a uniform approximate lattice Λ' commensurable with Λ such that $(\Lambda')^\infty$ is polycyclic.*

Proof. Let us first show that we can assume G to be simply connected. Indeed, if G is not simply connected we proceed as follows. Let $p : \tilde{G} \to G$ be a universal cover, then $p^{-1}(\Lambda)$ is a uniform approximate lattice in $\tilde{G}$. Suppose $p^{-1}(\Lambda)$ is commensurable with an approximate subgroup Λ' such that Λ' generates a polycyclic group. Then $p(\Lambda')$ is commensurable with Λ and $p(\Lambda')$ generates a polycyclic group as well.

From now on G is supposed simply connected. Let N denote the nilpotent radical of G, $k \in \mathbb{N}$ and $\Xi \subset \Lambda^k \cap N$ be an approximate subgroup. First of all, let us show that Ξ^∞ is finitely generated. Since G is a simply connected soluble Lie group, G does not contain any non-trivial compact subgroup ([80, Corollary 2.18]). So N does not contain any non-trivial compact subgroup, and thus N is simply connected. Now N is a connected simply connected nilpotent Lie group so it is the group of $\mathbb{R}$-points of a unipotent algebraic group (see [65, Theorem 4.1]) and Ξ is a discrete approximate subgroup. Hence, Ξ^∞ is finitely generated by Corollary 3.2.3.

The rest of the proof will rely on the following lemma that links finitely generated subgroups of connected simply connected nilpotent Lie groups to finite dimensional $\mathbb{Q}$-Lie algebras.

Lemma 3.4.3. *[65, Chapter IV] Let $\Gamma \subset N$ be a finitely generated group in a connected simply connected nilpotent Lie group. Then Γ is torsion-free nilpotent, $\mathbb{Q}\log(\Gamma)$ is a finite dimensional $\mathbb{Q}$ Lie algebra and $\dim_{\mathbb{Q}}(\mathbb{Q}\log(\Gamma)) = \mathrm{rank}(\Gamma)$.*

Here the rank of Γ is the dimension of its Malcev completion, i.e. the unique connected simply connected nilpotent Lie group that admits a lattice isomorphic to Γ, and log denotes the logarithm map from N to its Lie algebra. Lemma 3.4.3 is a consequence of [65, Theorems 2.18, 2.12, 2.10 and 2.11].

The group Ξ^∞ is finitely generated, torsion-free and nilpotent so it has finite rank. Among all approximate subgroups Ξ commensurable with $\Lambda^2 \cap N$ - note that $\Lambda^2 \cap N$ is an approximate subgroup by Lemma 2.1.8 - such that there is $k \in \mathbb{N}$ satisfying $\Xi \subset \Lambda^k \cap N$, choose one with minimal rank. Let Ξ denote this approximate subgroup and let k be such that $\Xi \subset \Lambda^k \cap N$.

Take $\lambda \in \Lambda$. Note that by Lemma 2.1.8 and since

$$\Lambda^k \cap N \subset \lambda \left(\Lambda^k \cap N\right) \lambda^{-1} \subset \Lambda^{k+2} \cap N$$

the subsets $\Lambda^k \cap N$, $\lambda\left(\Lambda^k \cap N\right)\lambda^{-1}$ and $\Lambda^{k+2}\cap N$ are pairwise commensurable approximate subgroups. Therefore, Ξ and $\lambda\Xi\lambda^{-1}$ are contained in and commensurable with $\Lambda^{k+2}\cap N$. So $\Xi^2 \cap \lambda\Xi^2\lambda^{-1}$ is an approximate subgroup commensurable with Ξ according to Lemma 2.1.8. But $\left(\Xi^2 \cap \lambda\Xi^2\lambda^{-1}\right)^\infty \subset \Xi^\infty$ so these two subgroups have the same rank. As a consequence, $\left(\Xi^2 \cap \lambda\Xi^2\lambda^{-1}\right)^\infty$ has finite index in both Ξ^∞ and $\lambda\Xi^\infty\lambda^{-1}$. So the subgroups Ξ^∞ and $\lambda\Xi^\infty\lambda^{-1}$ are commensurable. There is thus $n \in \mathbb{N}$ such that for all $\gamma \in \Xi^\infty$, we have $\gamma^n \in \Xi^\infty \cap \lambda\Xi^\infty\lambda^{-1}$. Therefore,

$$n\log(\Xi^\infty) \subset \log(\lambda\Xi^\infty\lambda^{-1}).$$

Hence,

$$\mathbb{Q}\log(\Xi^\infty) = \mathbb{Q}\log(\lambda\Xi^\infty\lambda^{-1}).$$

We have thus proved that $\exp(\mathbb{Q}\log(\Xi^\infty))$ is stable under conjugation by elements of Λ^∞. Moreover, $\exp(\mathbb{Q}\log(\Xi^\infty))$ is a group by the Baker–Campbell–Hausdorff formula (see [65, Chapter IV]) and any finitely generated subgroup in it has rank less than or equal to $\dim_{\mathbb{Q}}(\mathbb{Q}\log(\Xi^\infty)) = \mathrm{rank}(\Xi^\infty)$ according to Lemma 3.4.3. Let Γ denote the subgroup generated by $\Lambda^{k+2}\cap\exp(\mathbb{Q}\log(\Xi^\infty))$. Since $(\Lambda^{k+2}\cap N)^\infty$ is nilpotent and finitely generated according to a combination of Lemma 2.1.8 and Corollary 3.2.3, the subgroup Γ is finitely generated as well. In addition, Γ contains Ξ^∞. So $\mathrm{rank}(\Gamma) = \mathrm{rank}(\Xi^\infty)$ and Ξ^∞ has finite index in Γ.

We know by [65, Theorem 2.12] that there exist free abelian subgroups $\Gamma_1, \Gamma_2 \subset \mathbb{Q}\log(\Xi^\infty)$ of rank $\dim_{\mathbb{Q}}(\mathbb{Q}\log(\Xi^\infty))$ such that

$$\Gamma_1 \subset \log(\Xi^\infty) \subset \log(\Gamma) \subset \Gamma_2.$$

This implies in particular that for all $\lambda \in \Lambda$ we have $\mathrm{Ad}(\lambda)(\Gamma_1) \subset \Gamma_2$ since $\lambda\Xi^\infty\lambda^{-1} \subset \Gamma$. According now to Proposition 3.3.1, there is an approximate subgroup $X \subset \mathrm{Ad}(\Lambda)^4$ commensurable with $\mathrm{Ad}(\Lambda)$ such that $\phi(\Gamma_1) = \Gamma_1$ for all $\phi \in X$. Define $\tilde{\Lambda} := \mathrm{Ad}^{-1}(X)^2 \cap \Lambda^2$. By Lemma 2.1.8 we find that $\tilde{\Lambda}$ is an approximate subgroup commensurable with Λ. And by construction $\mathrm{Ad}(\lambda)(\Gamma_1) = \Gamma_1$ for all $\lambda \in \tilde{\Lambda}$. Therefore, the subgroup H of Ξ^∞ generated by $\exp(\Gamma_1)$ has finite index in Ξ^∞ and $H \cap \tilde{\Lambda}^\infty$ is normalised by $\tilde{\Lambda}$.

Consider $p : \tilde{\Lambda}^\infty \to \tilde{\Lambda}^\infty/(H\cap\tilde{\Lambda}^\infty)$ the canonical projection. We claim that $p(\tilde{\Lambda}^2) \cap Z(\tilde{\Lambda}^\infty/(H \cap \tilde{\Lambda}^\infty))$ is commensurable with $p(\tilde{\Lambda})$. First, note that Ξ is commensurable with $\Xi^2 \cap H$ by Lemma 2.1.8, $\Lambda^8 \cap N$ is commensurable with Ξ by Lemma 2.1.8 and $\tilde{\Lambda} \subset \Lambda^2$. Note also that N contains all commutators of elements of G. So the set of commutators of elements of $p(\tilde{\Lambda})$ is contained in $p(\Lambda^8\cap N)$. But $p(\Lambda^8\cap N)$ is commensurable with $p(\Xi^2\cap H) = \{e\}$, and, hence, is finite. But $p(\tilde{\Lambda}^\infty)$ is finitely generated since $\tilde{\Lambda}$ is uniform approximate lattice in a connected Lie group (see the proof of Proposition 3.2.2 or [9, Theorem 1.13]), and $p(\tilde{\Lambda})$ generates $p(\tilde{\Lambda}^\infty)$. We can therefore find a finite generating family $\Upsilon := \{\gamma_1, \dots, \gamma_n\} \subset p(\tilde{\Lambda})$. Define now

$$\theta_\Upsilon : p(\tilde{\Lambda}^\infty) \to p(\tilde{\Lambda}^\infty)^n$$
$$\gamma \mapsto ([\gamma_1, \gamma], \dots, [\gamma_n, \gamma]),$$

where $[\gamma_i, \gamma] := \gamma_i \gamma \gamma_i^{-1} \gamma^{-1}$. Note now that for all γ, γ' in $p(\tilde{\Lambda})$ if $\theta_\Upsilon(\gamma) = \theta_\Upsilon(\gamma')$ then

$$\gamma Z(\tilde{\Lambda}^\infty/(H \cap \tilde{\Lambda}^\infty)) = \gamma' Z(\tilde{\Lambda}^\infty/(H \cap \tilde{\Lambda}^\infty)).$$

Indeed, for $i \in \{1, \dots, n\}$ we have

$$\gamma_i \gamma \gamma_i^{-1} \gamma^{-1} = [\gamma_i, \gamma] = [\gamma_i, \gamma'] = \gamma_i \gamma' \gamma_i^{-1} \gamma'^{-1}$$

which implies

$$\left(\gamma_i \gamma^{-1} \gamma_i^{-1}\right) \gamma_i \gamma' \gamma_i^{-1} \gamma'^{-1} (\gamma) = e.$$

But

$$\left(\gamma_i \gamma^{-1} \gamma_i^{-1}\right) \gamma_i \gamma' \gamma_i^{-1} \gamma'^{-1} (\gamma) = [\gamma_i, \gamma^{-1}\gamma'].$$

So $\gamma^{-1}\gamma'$ belongs to the centraliser of Υ i.e. belongs to the centre $Z(\tilde{\Lambda}^\infty/(H \cap \tilde{\Lambda}^\infty))$ of $\tilde{\Lambda}^\infty/(H \cap \tilde{\Lambda}^\infty)$ because Υ is a generating family.

But $\theta_\Upsilon(p(\tilde{\Lambda}))$ is finite so there are $\gamma_1', \dots, \gamma_r' \in \tilde{\Lambda}^\infty/(H \cap \tilde{\Lambda}^\infty)$ such that $p(\tilde{\Lambda}) \subset \bigcup \gamma_i' Z(\tilde{\Lambda}^\infty/(H \cap \tilde{\Lambda}^\infty))$. We thus have that $\tilde{\Lambda}$ is covered by finitely many left-cosets of $p^{-1}(Z(\tilde{\Lambda}^\infty/(H \cap \tilde{\Lambda}^\infty)))$. According to Lemma 2.1.8 now, $\Lambda' := \tilde{\Lambda}^2 \cap p^{-1}(Z(\tilde{\Lambda}^\infty/(H \cap \tilde{\Lambda}^\infty)))$ is an approximate subgroup commensurable with $\tilde{\Lambda}$ and Λ.

Finally, Λ' is a uniform approximate lattice in G as it is commensurable with Λ. So it generates a finitely generated subgroup (as above see the proof of Proposition 3.2.2 or [9, Theorem 1.13]). Moreover, $H \cap \tilde{\Lambda}^\infty \subset \Lambda'^\infty$ is a finitely generated torsion-free nilpotent normal subgroup such that $\Lambda'^\infty/(H \cap \tilde{\Lambda}^\infty)$ is abelian and finitely generated. Hence, Λ'^∞ is polycyclic. □

Proof of Theorem 1.1.13. Let $\Lambda \subset G$ be a uniform approximate lattice in a connected soluble Lie group. According to Proposition 3.4.2 Λ is commensurable with a uniform approximate lattice Λ' with Λ' polycyclic. By Proposition 3.4.1 the uniform approximate lattice Λ' is therefore a Meyer set. So Λ is a Meyer set. □

Chapter 4

Good models, closed approximate subgroups and amenable groups

4.1 An approximate subgroup without a good model

Recall that a *quasi-morphism* of a group G is a map $f : G \to \mathbb{R}$ such that

$$C(f) := \sup_{g_1,g_2\in G} |f(g_1g_2) - f(g_1) - f(g_2)| < \infty.$$

We say that f is *symmetric* if for all $g \in G$ we have $f(g^{-1}) = -f(g)$ and that it is *homogeneous* if for all $n \in \mathbb{Z}$ and $g \in G$ we have $f(g^n) = nf(g)$. Just like group homomorphisms quasi-morphisms give rise to families of approximate subgroups. The approximate subgroups produced that way are often called *quasi-kernels.*

Lemma 4.1.1. *Let G be a group and $f : G \to \mathbb{R}$ be a symmetric quasi-morphism. Then for all $R > C(f)$ the set $f^{-1}([-R;R])$ is a $2\frac{2R+C(f)}{R-C(f)+2}$ -approximate subgroup.*

Proof. Let Λ denote the set $f^{-1}([-R;R])$. Then Λ is symmetric since f is symmetric and the set $f(\Lambda^2)$ is contained in $[-2R - C(f); 2R + C(f)]$. Set $\delta := R - C(f) > 0$ and choose a finite subset $F \subset \Lambda^2$ such that $f(F)$ is a maximal $\delta/2$-separated subset of $f(\Lambda^2)$. We know that $|F| < 2\frac{2R+C(f)}{R-C(f)+2}$ and in addition we have

$$f(\Lambda^2) \subset \bigcup_{\gamma\in F} f(\gamma) + [-\delta;\delta].$$

Take $\lambda \in \Lambda^2$ and $\gamma \in F$ such that $|f(\lambda) - f(\gamma)| \leq \delta$. We have

$$|f(\gamma^{-1}\lambda) - (f(\lambda) - f(\gamma))| \leq C(f),$$

so

$$|f(\gamma^{-1}\lambda)| \leq C(f) + \delta = R.$$

Hence, $\gamma^{-1}\lambda \in \Lambda$ and $\Lambda^2 \subset F\Lambda$. □

Since all bounded maps are quasi-morphisms, quasi-morphisms are often studied up to a bounded error. This gives an equivalence relation between quasi-morphisms that can be translated as a commensurability condition on quasi-kernels.

Lemma 4.1.2. *Let G be a group and $f_1, f_2 : G \to \mathbb{R}$ be two symmetric quasi-morphisms. Suppose that $\sup_{g\in G} |f_1(g) - f_2(g)| < \infty$. Then for all $R_1 > C(f_1)$ and $R_2 > C(f_2)$ the approximate subgroups $\Lambda_1 := f_1^{-1}([-R_1; R_1])$ and $\Lambda_2 := f_2^{-1}([-R_2; R_2])$ are commensurable. More precisely, there is $F \subset G$ with $|F| \leq \max(\frac{2(R_1+\eta)}{R_2-C(f_2)+2}, \frac{2(R_2+\eta)}{R_1-C(f_1)+2})$ such that $\Lambda_1 \subset F\Lambda_2$ and $\Lambda_2 \subset F\Lambda_1$.*

Proof. Write $\delta_1 := R_1 - C(f_1)$ and $\eta := \sup_{g\in G} |f_1(g) - f_2(g)|$. Choose a maximal $\delta_1/2$-separated subset F_1 of $f_1(\Lambda_2)$. Since $f_1(\Lambda_2) \subset [-(R_2+\eta); R_2+\eta]$ we know that $|F_1| \leq \frac{2(R_2+\eta)}{\delta_1+2}$. As in the proof of Lemma 4.1.1 we find that $\Lambda_2 \subset F_1\Lambda_1$. By symmetry there is $F_2 \subset G$ with $|F_2| \leq \frac{2(R_1+\eta)}{R_2-C(f_2)+2}$ such that $\Lambda_1 \subset F_2\Lambda_2$. □

Our main result links properties of quasi-morphisms to whether the quasi-kernel is a Meyer subset or not. Proposition 4.1.3 is a precise version of Theorem 1.1.15 from the introduction.

Proposition 4.1.3. *Let G be a finitely generated group and let $f : G \to \mathbb{R}$ be a homogeneous quasi-morphism. Choose a real number $R > C(f)$. If the approximate subgroup $f^{-1}([-R; R])$ is a Meyer subset then f is a group homomorphism.*

Proof. If f is bounded, then $f = 0$. So assume that f is unbounded. Take $R' > C(f)$ such that $f^{-1}([-R'; R'])$ generates G. We know that $f^{-1}([-R'; R'])$ is an approximate subgroup (Lemma 4.1.1) and a Meyer subset (Lemma 4.1.2). By Proposition 2.3.5 there is an integer $n \geq 1$ such that there are a good model $f_0 : G \to H$ of some power, say n, of $f^{-1}([-R'; R'])$ with dense image. In particular, f_0 is a good model of $\Lambda := f^{-1}([-n(R' + C(f)); n(R' + C(f))])$ according to Lemma 4.1.2. Since $f_0(G)$ is dense in H we have that $\overline{f_0(\Lambda)}$ is a neighbourhood of the identity. So the subgroup generated by the compact set $\overline{f_0(\Lambda)}$ is open and contains $f_0(G)$, hence equals H. The group H is thus compactly generated. We will now show that $f = cf_0$ for some real number $c > 0$. We start with two claims:

Claim 4.1.1. *The set of commutators $\{h_1h_2h_1^{-1}h_2^{-1} | h_1, h_2 \in H\}$ is relatively compact.*

By Lemma 4.1.2 the approximate subgroup $f_0(f^{-1}([-3C(f); 3C(f)]))$ is commensurable to $f_0(\Lambda)$. So $K := \overline{f_0(f^{-1}([-3C(f); 3C(f)]))}$ is compact. Take now $\gamma_1, \gamma_2 \in G$. We have $|f(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1})| \leq 3C(f)$, hence

$$f_0(\gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}) \in K.$$

But $f_0(G)$ is dense in H so we find $\{h_1h_2h_1^{-1}h_2^{-1} | h_1, h_2 \in H\} \subset K$.

Claim 4.1.2. *Let $\gamma_0 \in G$ be such that $f(\gamma_0) > 0$. Then there is a compact subset $K \subset H$ such that for all $\gamma \in G$ we have $f_0(\gamma) \subset \langle f_0(\gamma_0)\rangle K$ where $\langle f_0(\gamma_0)\rangle$ is the subgroup generated by $f_0(\gamma_0)$.*

Write $R_0 := f(\gamma_0)$, then the subset $f^{-1}([-R_0 - C(f); R_0 + C(f)])$ is commensurable to Λ by Lemma 4.1.2. As above, we find that the subset

$$K := \overline{f_0(f^{-1}([-R_0 - C(f); R_0 + C(f)]))}$$

is compact. But for any $\gamma \in G$ there is an integer l such that $|f(\gamma) - lf(\gamma_0)| \leq R_0$ so $\gamma_0^{-l}\gamma \in K$.

Now all conjugacy classes of H are relatively compact according to Claim 4.1.1. So we can find a compact normal subgroup $K \subset H$ and non-negative integers k, l such that $H/K \simeq \mathbb{R}^k \times \mathbb{Z}^l$ (see [75]). We will show that $k + l \leq 1$. Let $p : H \to H/K$ denote the natural projection. Then the group homomorphism $p \circ f_0$ has dense image and $p \circ f_0(\Lambda)$ is relatively compact. If H/K is compact then $H/K \simeq \{e\}$. Otherwise we can find $\gamma_0 \in G \setminus \Lambda$ so $f(\gamma_0) \geq R > 0$. According to Claim 4.1.2 every $\gamma \in G$ satisfies $p \circ f_0(\gamma) \in \langle p \circ f_0(\gamma_0)\rangle L$ where L is some compact subset of H/K. Therefore $\langle p \circ f_0(\gamma_0)\rangle$ is an infinite cyclic co-compact subgroup, and, hence, $k + l \leq 1$. Choose a neighbourhood $U \subset H$ of the identity such that $f_0^{-1}(U) \subset \Lambda$. Since K is a compact subgroup of H and the subgroup $f_0(G)$ is dense we can find an integer $m \geq 0$ such that $K \subset f(\Lambda^m)U$. Then $V := p(U) \subset H/K$ is such that

$$f_0^{-1}(p^{-1}(V)) \subset f_0^{-1}(UK) \subset \Lambda^{m+2}.$$

So $p \circ f_0$ is a good model of Λ^{m+2} with image dense in $\mathbb{R}$ or $\mathbb{Z}$. Since f is unbounded and $f(\Lambda^{m+2})$ is bounded, we can find $\gamma \in G \setminus \Lambda^{m+2}$, implying $f(\gamma) \neq 0$ and $p \circ f_0(\gamma) \neq 0$. The map

$$\hat{f} : g \mapsto f(\gamma)p \circ f_0(g) - p \circ f_0(\gamma)f(g)$$

is therefore well-defined and a homogeneous quasi-morphism with $\hat{f}(\gamma) = 0$. Moreover, any set commensurable to Λ has bounded image by $\hat{f}$. Take $g \in G$, there is $n \in \mathbb{Z}$ such that $|f(g) - nf(\gamma)| \leq f(\gamma)$. This implies that $G = \langle\gamma\rangle f^{-1}([-C(f) - f(\gamma); f(\gamma) + C(f)])$. But $f^{-1}([-C(f) - f(\gamma); f(\gamma) + C(f)])$ is commensurable to Λ according to Lemma 4.1.2, and, hence, is mapped to a bounded set by $\hat{f}$. So $\hat{f}$ must have bounded image and therefore must be trivial. In other words $f = \frac{f(\gamma)}{p \circ f_0(\gamma)} p \circ f_0$. □

Proof of Theorem 1.1.14. Recall that w is a non-trivial reduced word of length l in F_2 the free group over $\{a, b\}$ and suppose that w has at least two letters. According to [23, §3.(a)] the map

$$\begin{aligned} f_w : &F_2 \longrightarrow \mathbb{R} \\ &g \longmapsto o(g, w) - o(g, w^{-1}) \end{aligned}$$

is a symmetric quasi-morphism with $C(f_w) \leq 3(l-1)$. Moreover, f_w is within bounded distance of a unique homogeneous quasi-morphism, $\tilde{f}_w$ say, that is not a group homomorphism. But according to Lemma 4.1.1 the set

$$\left\{g \in F_2 \,\middle|\, |o(g,w) - o(g,w^{-1})| \leq 3l \right\} = f_w^{-1}([-3l;3l)])$$

is an approximate subgroup. Moreover, by Lemma 4.1.2 it is commensurable to $\tilde{f}_w^{-1}([-C(\tilde{f}_w)-1;C(\tilde{f}_w)+1])$. So if $\left\{g \in F_2 \,\middle|\, |o(g,w) - o(g,w^{-1})| \leq 3l \right\}$ is a Meyer subset, then $\tilde{f}_w^{-1}([-C(\tilde{f}_w)-1;C(\tilde{f}_w)+1])$ is a Meyer subset, and, hence, $\tilde{f}_w$ is a group homomorphism according to Proposition 4.1.3. A contradiction. □

Proposition 4.1.3 combined with [23, §3.(a)] in fact proves that any ultrapower of the approximate subgroup

$$\left\{g \in F_2 \,\middle|\, |o(g,w) - o(g,w^{-1})| \leq 3l \right\},$$

with w that has at least two letter, is not a Meyer subset. Which means that it is not commensurable to an approximate subgroup that has a good model. In particular, this seems to contradict that "even without the definable amenability assumption a suitable Lie model exists" conjectured in [59, p. 57].

4.2 A closed-approximate-subgroup theorem

We give in this section a proof of Theorem 1.1.17 and investigate some applications.

4.2.1 Globalisation in Hausdorff Topological Groups

We start by proving a general form of Theorem 1.1.17.

Theorem 4.2.1. *Let Λ be a compact approximate subgroup of a Hausdorff topological group G. Let Γ be a subgroup of G such that $\Lambda \subset \Gamma \subset \mathrm{Comm}_G(\Lambda)$. There is a locally compact group H, an injective continuous group homomorphism $\phi : H \to G$ and a compact symmetric neighbourhood of the identity V that generates H such that $\phi(V) = \Lambda^2$.*

The key observation needed to prove Theorem 4.2.1 is the fact that locally a closed approximate subgroup behaves like a group.

Lemma 4.2.2. *Let Λ be a closed approximate subgroup of a locally compact group G and let Ξ be a subset covered by finitely many left-translates of Λ. There is a neighbourhood of the identity $U(\Xi) \subset G$ such that :*

$$\Xi \cap U(\Xi) \subset \Lambda^2 \cap U(\Xi).$$

Proof. Choose a finite subset $F \subset G$ such that $\Xi \subset F\Lambda$. Define the open subset

$$U := G \setminus \left(\bigcup_{f \in F, f \notin \Lambda} f\Lambda \right).$$

Since $e \in f\Lambda$ implies $f \in \Lambda^{-1} = \Lambda$, the subset U contains the identity. We thus have

$$\begin{aligned} U \cap \Xi &\subset U \cap F\Lambda \\ &\subset \bigcup_{f \in F, f \in \Lambda} f\Lambda \\ &\subset \Lambda^2. \end{aligned}$$

□

Remark 4.2.1. The content of Lemma 4.2.2 is particularly well illustrated in totally disconnected groups. If Λ is a closed approximate subgroup of a totally disconnected group G, then Lemma 4.2.2 in combination with the Van Dantzig theorem provides a compact open subgroup O of G such that $\Lambda^2 \cap O$ is also a subgroup. Therefore, approximate subgroups and subgroups seem to behave similarly, at least locally. Theorem 4.2.1 confirms this guess.

Proof of Theorem 4.2.1. For all $\gamma \in \Gamma$, since γ commensurates Λ - hence commensurates Λ^4 - there is a neighbourhood of the identity $U(\gamma)$ such that $\gamma\Lambda^4\gamma^{-1} \cap U(\gamma) \subset \Lambda^2$ (Lemma 4.2.2). Choose a neighbourhood basis for the identity $\mathcal{B}$ made of closed subsets in G and define $\mathcal{B}_\Lambda$ as the family of subsets $\{\Lambda^2 \cap U^{-1}U | U \in \mathcal{B}\}$. The subsets in $\mathcal{B}_\Lambda$ are all contained in and commensurable to Λ^2 by Lemma 2.1.7. Take any $U \in \mathcal{B}$ and any $\gamma \in \Gamma$ and choose $V \in \mathcal{B}$ such that $\gamma(V^{-1}V)^2\gamma^{-1} \subset U(\gamma) \cap U$. Then

$$\begin{aligned} \gamma\left(V^{-1}V \cap \Lambda^2\right)^2\gamma^{-1} &\subset \gamma(V^{-1}V)^2\gamma^{-1} \cap \gamma\Lambda^4\gamma^{-1} \\ &\subset U(\gamma) \cap U \cap \gamma\Lambda^4\gamma^{-1} \\ &\subset U \cap \Lambda^2. \end{aligned}$$

So $\mathcal{B}_\Lambda$ checks all conditions of (3) of Lemma 2.3.4.

Choose a good model $f : \Gamma \to H$ of (Λ^2, Γ) with dense image and such that $\mathcal{B}_\Lambda$ is a neighbourhood basis for the identity in the initial topology ((4) of Theorem 2.3.4). We know the family $\{\overline{f(\Xi)} | \Xi \in \mathcal{B}_\Lambda\}$ is a neighbourhood basis for the identity in H and $\ker(f) = \bigcap_{\Xi \in \mathcal{B}_\Lambda} \Xi \subset \bigcap_{U \in \mathcal{B}} U^{-1}U = \{e\}$. Take $\lambda \in \Lambda^2$ and take any $U \in \mathcal{B}$ with $U^{-1}U \subset U(e)$. Then

$$\Lambda^2 \cap \lambda U^{-1}U = \lambda(\lambda^{-1}\Lambda^2 \cap U^{-1}U) \subset \lambda(\Lambda^4 \cap U^{-1}U) = \lambda(\Lambda^2 \cap U^{-1}U),$$

so the restriction of f to Λ^2 is a continuous map. Hence, $\{f(\Xi) | \Xi \in \mathcal{B}_\Lambda\} = \{\overline{f(\Xi)} | \Xi \in \mathcal{B}_\Lambda\}$ is a neighbourhood basis for the identity in H. Since $f(\Lambda^2)$ has non-empty interior and $f(\Gamma)$ is dense in H, we find $f(\Gamma) = f(\Gamma)f(\Lambda^2) = H$.

So f is bijective. But for all $\Xi \in \mathcal{B}_\Lambda$ we have $f^{-1}(f(\Xi)) = \Xi$. So $f^{-1} : H \to G$ is a continuous one-to-one group homomorphism and Λ^2 is the image of the compact neighbourhood of the identity $f(\Lambda^2)$. □

We can now turn to the proof of Theorem 1.1.17. This result is akin to Cartan's closed-subgroup theorem as it shows that closed approximate subgroups of Lie groups have a Lie group structure, at least locally. In particular, it enables one to define - unambiguously - the Lie algebra associated to a closed approximate subgroup of a Lie group. Note that this last fact could also be proved as a consequence of Lemma 4.2.2 and [16, Chapter III, §8, Prop. 2].

Proof of Theorem 1.1.17. Apply Theorem 4.2.1 to $(\Lambda^2 \cap V^2, \Lambda^\infty)$ where V is any symmetric compact neighbourhood of the identity in G. This yields an injective continuous group homomorphism $\phi : H \to G$ with image Λ^∞ and such that $\phi^{-1}(\Lambda^2 \cap V^2)$ is a compact neighbourhood of the identity. By the Baire category theorem $\phi^{-1}(\Lambda)$ has non-empty interior. The approximate subgroup $\phi^{-1}(\Lambda)$ is therefore contained in the interior of $\phi^{-1}(\Lambda^3)$ and commensurable to the approximate subgroup defined as the interior of $\phi^{-1}(\Lambda^2)$. We have moreover that for any $K \subset G$ compact the subset $K \cap \Lambda$ is covered by finitely many left-translates of $\Lambda^2 \cap V^2$. So $\phi^{-1}(K \cap \Lambda)$ is compact and ϕ is proper.

If moreover G is a Lie group, then H is a Lie group as a consequence of [16, Chapter III, §8, Corollary 1]. □

We will use Lemma 4.2.2 once more later on in Section 4.3 to define - at least locally - the quotient of an ambient group by a closed approximate subgroup. This will then enable us to build local Borel sections of closed approximate subgroups (see the proof of Proposition 4.3.6).

4.2.2 Closed approximate subgroups of Euclidean spaces

As a first consequence we investigate the structure of closed approximate subgroups of Euclidean spaces:

Proposition 4.2.3. *Let Λ be a closed approximate subgroup of $\mathbb{R}^n$. There are two vector subspaces V_o and V_d of $\mathbb{R}^n$, a uniformly discrete approximate subgroup $\Lambda_d \subset V_d$ and a compact approximate subgroup $K \subset V_d$ such that $V_o \oplus V_d = \mathbb{R}^n$ and Λ is commensurable to $V_o + \Lambda_d + K$. Furthermore:*

1. *there is a vector subspace $V_e \subset V_d$ such that we can choose K to be any compact open neighbourhood of the identity in V_e and $V_e \cap \Lambda_d^2 = \{0\}$;*

2. *there are a non-negative integer m, a linear map $\phi : \mathbb{R}^m \to V_d$, a subspace $V_d' \subset \mathbb{R}^m$ with $V_d' \cap \ker(\phi) = \{e\}$ such that Λ_d is contained in and commensurable to $\phi\left(\mathbb{Z}^m \cap (V_d' + [-1; 1]^m)\right)$.*

Proof. In the following we will use additive notations for sums of sets. If X, Y are subsets of a vector space, then $X + Y := \{x + y | x \in X, y \in Y\}$. Moreover, $kX := \{x_1 + \cdots + x_k | x_1, \ldots, x_k \in X\}$ and $-X := \{-x | x \in X\}$.

The main ingredients of the proof will be Schreiber's theorem (Theorem 1.1.11) as well as the close-approximate-subgroup theorem (Theorem 1.1.17). Recall that a consequence of Schreiber's theorem asserts that there exists a relatively compact subset $K \subset 4\Xi$ that generates $\langle \Xi \rangle$ (Proposition 3.2.2).

We start with three minor cases. Note that the first part of (1) is a consequence of Theorem 1.1.17 and the second part will follow naturally from our choice of V_o. Suppose now that Λ is a uniformly discrete approximate subgroup. Then $\langle \Lambda \rangle$ is finitely generated (Proposition 3.2.2 again), so $\langle \Lambda \rangle \simeq \mathbb{Z}^d$. Take a linear map $\phi : \mathbb{R}^d \to \mathbb{R}^n$ such that the restriction of ϕ to $\mathbb{Z}^d$ yields an isomorphism $\langle \Lambda \rangle \simeq \mathbb{Z}^d$. Then (2) is a consequence of Schreiber's theorem in $\mathbb{R}^d$ (Theorem 1.1.11). Suppose finally that Λ has non-empty interior. Then the interior of 2Λ is symmetric, contains the identity and is commensurable to Λ. Hence, it is an open approximate subgroup commensurable to Λ. Take V' and K as in Schreiber's theorem and let W be any bounded neighbourhood of the identity. We know that $K + K + W$ is covered by finitely many left-translates of 2Λ. So $V' + K \subset \Lambda + K + K + W$ is covered by finitely many left-translates of Λ. But $\Lambda \subset V' + K$ and K is covered by finitely many left-translates of W, so Λ and $V' + W$ are commensurable.

Let us go back to the general case. Notice that if V_1, V_2 are two vector subspaces covered by finitely many left-translates of Λ, then $V_1 + V_2$ is covered by finitely many left-translates of Λ. Indeed, take $F_1, F_2 \subset \mathbb{R}^n$ finite such that $V_1 \subset F_1 + \Lambda$ and $V_2 \subset F_2 + \Lambda$. Then $V_1 + V_2 \subset F_1 + F_2 + \Lambda + \Lambda$. Therefore, there is a maximal vector subspace V_o that is covered by finitely many left-translates of Λ. Take V_d any supplementary subspace of V_o. Then $V_o \cap 2\Lambda$ is commensurable to V_o by Lemma 2.1.8. So Λ is commensurable to $V_o + V_d \cap 2\Lambda$ according to Lemma 2.1.8 again. Let $i : L \to \mathbb{R}^n$ be the injective Lie group homomorphism given by Theorem 4.2.1 applied to $V_d \cap 2\Lambda$ and let Λ' denote the inverse image of $V_d \cap 2\Lambda$. Let L^0 denote the connected component of L, then $L^0 \simeq \mathbb{R}^k$. We have that $2\Lambda' \cap L^0$ is an approximate subgroup with non-empty interior so it is commensurable to $V' + W$ where $V' \subset L^0$ is a vector subspace and W is a compact neighbourhood of the identity in L^0. By construction of V_d we know that $V' = \{0\}$. So $2\Lambda' \cap L^0$ is a compact neighbourhood of the identity in L^0. But L is a torsion-free abelian Lie group and is moreover compactly generated according to the above discussion. Thus, $L \simeq \mathbb{R}^k \times \mathbb{Z}^l$ for some non-negative integers k, l. So we can identify L with a closed subgroup of $\mathbb{R}^k \times \mathbb{R}^l$ with $L^0 = \mathbb{R}^k \times \{0\}$. According to Schreiber's theorem there is vector subspace $V \subset \mathbb{R}^k \times \mathbb{R}^l$ and a compact neighbourhood of the identity $K \subset \mathbb{R}^k \times \mathbb{R}^l$ such that $\Lambda' \subset V + K$ and $V \subset \Lambda' + K$. But one can check that $L^0 \cap V = \{0\}$. So choose a vector subspace V' such that $L^0 \oplus V \oplus V' = \mathbb{R}^k \times \mathbb{R}^l$. The projection of L to $V \oplus V'$ parallel to L^0 is then a discrete subgroup $\Gamma \subset V \oplus V'$ and we find $L = L^0 \oplus \Gamma$. Moreover, the projection of Λ' to L^0 is a bounded subset with non-empty interior. So we readily check that Λ' is commensurable to $2\Lambda' \cap \Gamma + 2\Lambda' \cap L^0$. Since $i_{|\Lambda'}$ is proper, this yields the desired result. □

The situation becomes even more striking when Λ is a closed approximate subgroup in a one dimensional Euclidean space:

Corollary 4.2.4. *Let Λ be a closed approximate subgroup of $\mathbb{R}$. Then one and only one of the following is true:*

1. *Λ is finite;*
2. *there are real numbers $0 < a < b < \infty$ such that $[-a; a] \subset 2\Lambda \subset [-b; b]$;*
3. *Λ is a uniform approximate lattice i.e. uniformly discrete and relatively dense;*
4. *there is $n \in \mathbb{N}$ such that $n\Lambda = \mathbb{R}$.*

In particular, Λ is uniformly discrete or Λ has non-empty interior.

4.3 Amenable approximate subgroups

4.3.1 Definition

We start by defining a general notion of amenable approximate subgroups in locally compact groups. Our definition is close to the one given by Massicot–Wagner in [59, p.57]. And indeed their argument carries over completely to our context. The two notions are however not equivalent - ours being less flexible, but tailored for the study of closed approximate subgroups.

Definition 4.3.1. Let Λ be a closed approximate subgroup of a locally compact group G. Let $\mathcal{B}(\Lambda)$ be the set of Borel subsets of G that are contained in finitely many translates of Λ. We say that Λ is *amenable* if there is a finitely additive probability measure m defined on $\mathcal{B}(\Lambda)$ such that:

1. $m(\Lambda) = 1$;
2. $\forall g \in G, X \in \mathcal{B}(\Lambda), m(gX) = m(X)$.

We note that it is equivalent to ask for a left-invariant measure defined on the subsets of Λ only.

Lemma 4.3.2. *Let m be a finitely additive measure defined on the Borel subsets of Λ and such that:*

1. *$m(\Lambda) = 1$;*
2. *$m(gX) = m(X)$ whenever $g \in G, X \subset \Lambda$ and $gX \subset \Lambda$.*

Then Λ is amenable and m can be extended to a finitely additive measure as in Definition 4.3.1.

Proof. Consider $X \in \mathcal{B}(\Lambda)$ and $X_1, \dots, X_r$ is a Borel partition of X such that there are $f_1, \dots, f_r \in G$ with $f_i X_i \subset \Lambda$. We will prove that the quantity $\sum_{i=1}^r m(f_i X_i)$ depends only on X. Defining $\tilde{m}(X) = \sum_{i=1}^r m(f_i X_i)$ then yields the extension we are looking for. Take $Y_1, \dots, Y_s$ a second partition with $g_1, \dots, g_s \in G$ as above. We have

$$\begin{aligned}
\sum_{i=1}^r m(f_i X_i) &= \sum_{i=1}^r m(f_i X_i) \\
&= \sum_{i=1}^r \sum_{j=1}^s m(f_i(X_i \cap Y_j)) \\
&= \sum_{j=1}^s \sum_{i=1}^r m(f_i(X_i \cap Y_j)) \\
&= \sum_{j=1}^s \sum_{i=1}^r m(f_i g_j^{-1} g_j(X_i \cap Y_j)) \\
&= \sum_{j=1}^s \sum_{i=1}^r m(g_j(X_i \cap Y_j)) \\
&= \sum_{i=1}^s m(g_j Y_j)
\end{aligned}$$

where we have used left-invariance - i.e. part (2) - to go from the fourth to the fifth line. □

Note that a careful study of elementary properties of invariant finitely additive measures was carried out in [48].

4.3.2 Good models of amenable approximate subgroups

A striking fact about amenable approximate subgroups, first proved under very general assumptions by Hrushovski in [46], is the existence of a good model.

Theorem 4.3.3 (Hrushovski, Theorem 4.2, [46]). *Let Λ be an amenable closed approximate subgroup of a locally compact group G. Then $\overline{\Lambda^4}$ has a continuous good model.*

We will provide an elementary argument for this fact, based on the argument of Massicot–Wagner ([59]) inspired by Sanders [68], and Croot and Sisask [27], who proved it for abelian finite approximate groups. The core of the proof is the:

Lemma 4.3.4. *Let Λ be an amenable closed approximate subgroup of a σ-compact locally compact group G and let m be a positive integer. There is an approximate subgroup $S \subset \Lambda^2$ commensurable to Λ such that $S^m \subset \Lambda^4$.*

Proof. For any Haar measurable subset $B \subset \overline{\Lambda^8}$ define $\mu(B) := m(\mathbb{1}_B)$ where $\mathbb{1}_B$ denotes the indicator function. Let $\Xi \subset \Lambda$ be Haar measurable such that $\mu(\Xi) \geq t\mu(\Lambda)$ for some $t \in (0;1]$. Set

$$X(\Xi) := \{g \in \Lambda^2 | \mu(g\Xi \cap \Xi) \geq st\mu(\Lambda)\}$$

where $s = \frac{t}{2K}$. The approximate subgroup Λ is covered by at most $N := \lfloor \frac{1}{s} \rfloor$ left translates of $X(\Xi)$. Otherwise we could build inductively a sequence $(g_i)_{0 \leq i \leq N}$ such that $g_i \in \Lambda \setminus \bigcup_{j<i} g_j X(\Xi)$. This would mean

$$\mu(g_i\Xi \cap g_j\Xi) < st\mu(\Lambda)$$

for all $0 \leq j < i \leq N$. And therefore,

$$\begin{aligned}
K\mu(\Lambda) &\geq \mu(\bigcup_{0\leq i \leq N} g_i\Xi) \\
&\geq (N+1)\mu(\Xi) - \sum_{0 \leq i < j \leq N} \mu(g_i\Xi \cap g_j\Xi) \\
&> (N+1)t\mu(\Lambda) - \frac{N(N+1)}{2} st\mu(\Lambda) \\
&\geq (1 - s\frac{N}{2})(N+1)t\mu(\Lambda) \\
&\geq K\mu(\Lambda).
\end{aligned}$$

A contradiction. Set

$$f(t) := \inf \left\{ \frac{\mu(\Xi\Lambda)}{\mu(\Lambda)} \middle| \Xi \subset \Lambda \text{ closed},\ \mu(\Xi) \geq t\mu(\Lambda) \right\}.$$

Note that f is well defined since the product of two σ-compact subsets is a σ-compact subset, hence Haar measurable. We also know that $f(t) \in [1;K]$ for all $t \leq 1$. Take $t \geq c_{K,m}$ such that $f(\frac{t^2}{2K}) \geq (1 - \frac{1}{4m})f(t)$ (where we can choose $c_{K,m} = \frac{1}{(2K)^{2^n-1}}$ with $n = \left\lceil \frac{\log(K)}{\log((1-\frac{1}{4m})^{-1})} \right\rceil$ see e.g. [59, Lem. 11]) and choose $\Xi \subset \Lambda$ closed such that $\mu(\Xi) \geq t\mu(\Lambda)$ and $\frac{\mu(\Lambda\Xi)}{\mu(\Lambda)} \geq (1 + \frac{1}{4m})f(t)$.

If $g \in X(\Xi)$ we have:

$$\begin{aligned}
\mu(g\Xi\Lambda \cap \Xi\Lambda) &\geq \mu((g\Xi \cap \Xi)\Lambda) \\
&\geq f(\frac{t^2}{2K})\mu(\Lambda) \\
&\geq (1 - \frac{1}{4m})f(t)\mu(\Lambda) \\
&\geq \frac{1 - \frac{1}{4m}}{1 + \frac{1}{4m}}\mu(\Xi\Lambda).
\end{aligned}$$

Hence,

$$\mu((g\Xi\Lambda)\,\Delta\,(\Xi\Lambda)) \leq 2\left(1-\frac{1-\frac{1}{4m}}{1+\frac{1}{4m}}\right)\mu(\Xi\Lambda)$$
$$\leq \frac{1}{m}\mu(\Xi\Lambda).$$

Thus, by an easy induction we have for all $k \leq m$:

$$\mu((g_1\cdots g_k\Xi\Lambda)\,\Delta\,(\Xi\Lambda)) \leq \frac{k}{m}\mu(\Xi\Lambda). \tag{$*$}$$

As a consequence, $X(\Xi)^m \subset \Lambda^4$ and $\lfloor \frac{2K}{t} \rfloor \leq \frac{2K}{c_{K,m}} + 1 \approx (2K)^{2^{4\log(K)m}}$ translates of $X(\Xi)$ cover Λ. □

Proof of Theorem 4.3.3. By Lemma 4.3.4 there is therefore a closed approximate subgroup $\Lambda_1 \subset \overline{\Lambda^2}$ commensurable with Λ and such that $\overline{\Lambda_1^8} \subset \overline{\Lambda^4}$. Since Λ_1 is commensurable with Λ_2, it is easy to see that the closed approximate subgroup Λ_1 is amenable. We can thus build inductively a sequence of closed approximate subgroups $(\Lambda_n)_{n\geq 0}$ commensurable with Λ such that $\Lambda_0 = \Lambda$ and $(\overline{\Lambda_{n+1}^4})^2 \subset \overline{\Lambda_{n+1}^8} \subset \overline{\Lambda_n^4}$ for all integers $n \geq 0$. By Lemma 2.3.4 applied to the sequence $(\overline{\Lambda_n^4})_{n\geq 0}$ we obtain that $\overline{\Lambda^4}$ has a good model. □

4.3.3 Approximate subgroups in amenable groups

The proof of Theorem 1.1.20 consists of two steps. We first show that any neighbourhood of a normal amenable subgroup is an amenable approximate subgroup (Proposition 4.3.5). We then prove heredity of amenability for closed approximate subgroups (Proposition 4.3.6).

Proposition 4.3.5. *Let G be a locally compact group, let H be a closed amenable normal subgroup and let $W \subset G$ be a compact symmetric neighbourhood of the identity. Then the approximate subgroup WH of G is amenable.*

Most of the proof of Proposition 4.3.5 consists in adapting classical material from the theory of amenable locally compact groups (as found in [40]) to our setting. For the sake of completeness, and since Proposition 4.3.5 is crucial in the following, we write out the proof completely.

Proof. Let us first recall some notations and definitions (see [40] for more details). Fix a left-Haar measure μ_G on G. Define the left- and right-translates of a function $f : G \to \mathbb{R}$ by $g \in G$ as the maps ${}_gf : x \mapsto f(g^{-1}x)$ and $f_g : x \mapsto f(xg)$ respectively. A function $f : G \to \mathbb{R}$ is *right-uniformly continuous* if for any real number $\epsilon > 0$ there is a neighbourhood $U(\epsilon) \subset G$ of the identity such that for all $g \in U(\epsilon)$ and $x \in G$ we have $|f(x) - f(gx)| < \epsilon$. The set of right-uniformly continuous bounded functions on G will be denoted by $C^0_{b,ru}(G)$. Likewise the set of continuous bounded functions (resp. continuous functions with compact support) on G will be denoted by $C_b^0(G)$ (resp. $C_c^0(G)$) . One readily checks that G acts continuously by left-translations on the normed vector space $C_{b,ru}^0(G)$ equipped with the norm $||\cdot||_\infty$. The *convolution* $\phi * f$ of $\phi \in L^1(G)$ and $f \in L^\infty(G)$ is defined by

$$\begin{aligned} \phi * f : G &\longrightarrow \mathbb{R} \\ x &\longmapsto \int_G \phi(t) f(t^{-1}x) d\mu_G(t). \end{aligned}$$

We have $||\phi * f||_\infty \leq ||\phi||_1 ||f||_\infty$ and $\phi * f \in C_{b,ru}^0(G)$. If moreover $f \in L^1(G)$ then by Fubini $||\phi * f||_1 \leq ||\phi||_1 ||f||_1$. In addition, for any $g \in G$ we have $\phi * ({}_g f) = \Delta(g^{-1}) \phi_{g^{-1}} * f$ (where $\Delta : G \to \mathbb{R}$ is the modular function) and ${}_g(\phi * f) = ({}_g \phi) * f$. A linear map $F : X \to \mathbb{R}$ is said *left-invariant* if the subspace $X \subset C_{b,ru}^0(G)$ is stable by the G-action and if for every $g \in G$ and $f \in X$ we have $F({}_g f) = F(f)$. It is said *positive* is for all $f \in X$ with $f \geq 0$ we have $F(f) \geq 0$.

We can now proceed to the first step of the proof. The vector subspace of $C_{b,ru}^0(G)$ we want to consider is

$$X := \{f \in C_{b,ru}^0(G) | \ p(\mathrm{supp}(f)) \text{ is relatively compact.}\}$$

where $p : G \to G/H$ is the natural projection. We will prove the following claim.

Claim 4.3.1. *There exists a non-trivial left-invariant positive linear map* $m : X \to \mathbb{R}$.

Fix $\mu_{G/H}$ a right-Haar measure on G/H. First of all, note that X is stable under the action of G. Since H is an amenable locally compact group there is a left-invariant mean $m_H : C_b^0(H) \to \mathbb{R}$ according to [40, Theorem 2.2.1]. This means that m_H is a left-invariant positive linear functional such that for any $f \in C_b^0(H)$ we have $m_H(f) \leq ||f||_\infty$ and $m_H(\mathbb{1}_H) = 1$. Take $f \in X$ and consider the map

$$\begin{aligned} \tilde{f} : G &\to \mathbb{R} \\ x &\mapsto m_H\left(({}_x f)_{|H}\right). \end{aligned}$$

We will show that $\tilde{f}$ is continuous and invariant under left-translation by elements of H. Indeed, if $h, x \in H$ and $g \in G$ then ${}_{hg} f(x) = ({}_g f)(h^{-1}x)$. But $h^{-1}x \in H$ if and only if $x \in H$. So ${}_{hg} f_{|H} = {}_h\left({}_g f_{|H}\right)$, and, hence, for $x \in G$ we have

$${}_h\tilde{f}(x) = \tilde{f}(h^{-1}x) = m_H\left({}_{h^{-1}x} f_{|H}\right) = m_H\left({}_{h^{-1}}\left({}_x f_{|H}\right)\right) = m_H\left({}_x f_{|H}\right) = \tilde{f}(x).$$

Moreover, for any $x_1, x_2 \in G$ we have

$$\begin{aligned} |\tilde{f}(x_1) - \tilde{f}(x_2)| &= |m_H({}_{x_1} f_{|H}) - m_H({}_{x_2} f_{|H})| \\ &\leq ||{}_{x_1} f - {}_{x_2} f||_\infty. \end{aligned}$$

But f is right-uniformly continuous, so $\tilde{f}$ is continuous. Therefore, there exists a unique continuous function $f_{G/H} : G/H \to \mathbb{R}$ such that

$$\left(f_{G/H} \circ p\right)(x) = m_H({}_xf_{|H})$$

where $p : G \to G/H$ is the natural projection. One readily checks that the map $f \mapsto f_{G/H}$ is linear, sends non-negative functions to non-negative functions and $||f_{G/H}||_\infty \leq ||f||_\infty$ for all $f \in C^0_{b,ru}(G)$. Furthermore, we have $\text{supp}(f_{G/H}) \subset \overline{p(\text{supp}(f))}$, so $f_{G/H}$ is a continuous function with compact support. We are thus able to define

$$\begin{aligned} m : X &\longrightarrow \mathbb{R} \\ f &\longmapsto \int_{G/H} f_{G/H}(t) d\mu_{G/H}(t). \end{aligned}$$

The map m is a positive linear map. Choose a compact neighbourhood of the identity $U \subset G$ and $f \in X$ such that $f(x) = 1$ for all $x \in UH$. Then for all $x \in UH$ we have ${}_xf_{|H} = 1$, so $f_{G/H}(p(x)) = 1$. This implies $m(f) \geq \mu_{G/H}(p(U)) > 0$ so m is non-trivial. It only remains to check that m is left-invariant. Take $g, x \in G$ and $f \in X$. Then

$$\begin{aligned} \left(({}_gf)_{G/H} \circ p\right)(x) &= m_H(({}_x({}_gf))_{|H}) \\ &= m_H(({}_{xg}f)_{|H}) \\ &= (f_{G/H} \circ p)(xg) \\ &= f_{G/H}(p(x)p(g)). \end{aligned}$$

Therefore, $({}_gf)_{G/H} = (f_{G/H})_{p(g)}$. But $\mu_{G/H}$ is right-invariant so

$$\begin{aligned} m({}_gf) &= \int_{G/H} ({}_gf)_{G/H}(t) d\mu_{G/H}(t) \\ &= \int_{G/H} \left(f_{G/H}\right)_{p(g)}(t) d\mu_{G/H}(t) \\ &= \int_{G/H} f_{G/H}(t) d\mu_{G/H}(t) \\ &= m(f). \end{aligned}$$

So Claim 4.3.1 is proved.

Let us move on to the second step. First define

$$P(G) := \{f \in \mathcal{C}^0_c(G) | \forall x \in G, f(x) \geq 0, ||f||_1 = 1\}.$$

We will prove the following.

Claim 4.3.2. *Set $Y \subset L^\infty(G)$ be the subspace of functions f supported on some Haar measurable subset B with $p(B)$ is relatively compact. Then for any $\phi \in P(G)$ the map*

$$\begin{aligned} m_\phi : Y &\longrightarrow \mathbb{R} \\ f &\longmapsto m(\phi * f), \end{aligned}$$

is independent of ϕ and m_ϕ is non-trivial, left-invariant and positive.

Note first of all that for all $f \in Y$ and $\phi \in C_c^0(G)$ we have $\phi * f \in X$. So m_ϕ is well-defined. It is moreover a positive linear functional since ϕ is assumed to take non-negative values. Take now $f \in X$ that takes non-negative values and consider the linear functional

$$\begin{aligned} F : C_c^0(G) &\longrightarrow \mathbb{R} \\ \phi &\longmapsto m(\phi * f). \end{aligned}$$

Since f takes non-negative values and m is positive, the linear functional F is positive. By the Riesz–Markov–Kakutani representation theorem F is given by a regular Borel measure. We have moreover that for all $g \in G$,

$$F({}_g\phi) = m(({}_g\phi) * f) = m({}_g(\phi * f)) = m(\phi * f) = F(\phi).$$

So according to the Haar theorem there exists a constant $k(f) \in \mathbb{R}$ such that $F(\phi) = k(f) \int_G \phi(t) d\mu_G(t)$ for all non-negative functions $\phi \in C_c^0(G)$ where μ_G is a left-Haar measure. In particular, if $\phi_1, \phi_2 \in P(G)$ then $m_{\phi_1}(f) = F(\phi_1) = k(f) = F(\phi_2) = m_{\phi_2}(f)$. So m_ϕ is independent of ϕ. Now take $g \in G$. Then

$$m_\phi({}_g f) = m(\phi *_g f) = m(\Delta(g^{-1})\phi_{g^{-1}} * f).$$

But $\Delta(g^{-1})\phi_{g^{-1}} \in C_c^0(G)$ is non-negative and

$$\int_G \Delta(g^{-1})\phi_{g^{-1}}(t) d\mu_G(t) = \int_G \phi(t) d\mu_G(t) = 1,$$

so $\Delta(g^{-1})\phi_{g^{-1}} \in P(G)$. As a consequence,

$$m_\phi({}_g f) = m_{\Delta(g^{-1})\phi_{g^{-1}}}(f) = m_\phi(f).$$

So it remains to prove that m_ϕ is non-trivial. Let K be a compact subset that supports ϕ and W be any compact neighbourhood of the identity. Then $\phi * \mathbb{1}_{K^{-1}WH} \geq \mathbb{1}_{WH}$, hence $m_\phi(\mathbb{1}_{K^{-1}WH}) > 0$.

We can now conclude the proof of Proposition 4.3.5. Let W be any compact neighbourhood and $\phi \in P(G)$. The map

$$X \in \mathcal{B}(WH) \longmapsto \frac{1}{m_\phi(\mathbb{1}_{WH})} m_\phi(\phi * \mathbb{1}_X)$$

is a finitely additive measure as in Definition 4.3.1. So WH is amenable. $\square$

We will now prove heredity:

Proposition 4.3.6. *Let $\Lambda_1 \subset \Lambda_2$ be closed approximate subgroups of some second countable locally compact group. If Λ_2 is amenable, then Λ_1 is amenable.*

Proof. By Theorem 4.2.1 we can assume that Λ_2 has non-empty interior. Suppose we have found a Borel subset $B \subset \Lambda_2^3$ such that the multiplication map $\Lambda_1^5 \times B \to G$ is one-to-one, and such that $F\Lambda_1 B \supset \Lambda_2$ for some finite subset

F. Then for all $X \subset \Lambda_1^5$ the subset $XB \subset \Lambda_2^{15}$ is Borel. So we can define $m'(X) = m(XB)$. Now m' is clearly left-invariant since m is. In addition, since for $X_1, X_2 \subset \Lambda_1^5$ we have that $X_1 \cap X_2 = \emptyset \Rightarrow X_1 B \cap X_2 B$, m' is finitely additive. It remains to check that $0 < m'(\Lambda_1) < \infty$. We readily see that $m'(\Lambda_1) \leq m(\Lambda_2^4) < \infty$. By left-invariance

$$m'(\Lambda_1) = m(\Lambda_1 B) \geq \frac{1}{|F|} m(F\Lambda_1 B) \geq \frac{1}{|F|} m(\Lambda_2) > 0.$$

It thus remains to find such a Borel set B. We will proceed in two steps, first find $B' \subset \Lambda_2^2$ Borel such that the multiplication map $\Lambda_1 \times B' \to G$ is one-to-one and has image with non-empty interior. Then we will complete B' to a subset as described above.

Let us proceed with the first step. Notice that Λ_2^2 is a neighbourhood of the identity. Take W a symmetric compact neighbourhood of the identity such that $W^4 \subset \Lambda^2 \cap U(\Lambda_1^{10})$ - where $U(\Lambda_1^{10})$ is as described in Lemma 4.2.2. Then we can define an equivalence relation on W by setting $w_1 \sim w_2$ if and only if $w_1 w_2^{-1} \in \Lambda_1^2$. We have that $\sim$ is clearly symmetric and reflexive as Λ_1^2 is a symmetric subset that contains the identity. If now $w_1 \sim w_2$ and $w_2 \sim w_3$ where $w_1, w_2, w_3 \in W$, then $w_1 w_3^{-1} = \left(w_1 w_2^{-1}\right)\left(w_2 w_3^{-1}\right) \in \Lambda_1^4 \cap W^4 = \Lambda_1^2 \cap W^4$ since $W^4 \subset U_{\Lambda^{10}}$. So $w_1 \sim w_3$. Since equivalence classes are of the form $(\Lambda_1^2 w) \cap W$, the quotient topology makes $W/\sim$ into a compact space and the natural projection $W \to W/\sim$ is open and proper. According to a theorem of Federer and Morse [33] we can thus find a Borel subset $B' \subset W$ such that $W \subset \Lambda_1^2 B'$ and the map $\Lambda_1^2 \times B' \to G$ is one-to-one. Since $B'B'^{-1} \subset W^2 \subset U(\Lambda_1^{10})$ we have in fact that $\Lambda_1^5 \times B' \to G$ is one-to-one.

Take a dense sequence $(g_n)_{n\geq 1}$ of Λ_2. Then $\Lambda_2 \subset \bigcup_{n\geq 0} \Lambda_1^2 B g_n$. Define inductively B_m by $B_0 := B'$, $B_m := B' g_m \setminus \left(\Lambda_1^{10}(\bigcup_{0\leq k<m} B_k)\right)$ for all $m \geq 1$, and set $B_\infty := \bigcup_{m\geq 0} B_m$. For all $m \geq 0$, $\Lambda_1^5 B_m \cap \Lambda_1^5 \left(\bigcup_{0\leq k<m} B_k\right) = \emptyset$. We have also $B_m \subset B' g_m$ so the multiplication map $\Lambda_1^5 \times B_m \to G$ is one-to-one by construction of B'. Therefore, the multiplication map $\Lambda_1^5 \times B_\infty \to G$ is one-to-one. Finally, for every m we have $B' g_m \subset B_m \cup \Lambda_1^{10}(\bigcup_{0\leq k<m} B_k) \subset \Lambda_1^{10} B_\infty$. Hence, $\Lambda_2 \subset \Lambda_1^{10} B_\infty$. So B_∞ is the subset we are looking for. □

Recall the statement of Theorem 1.1.20:

Theorem. *Let G be a locally compact second countable group, $H \subset G$ be a normal amenable closed subgroup and $K \subset G$ be a compact subset. If Λ is a closed approximate subgroup contained in KH then the closure of Λ^4 has a good model.*

Proof of Theorem 1.1.20. We have that Λ^2 is contained in $W^2 H$. But $W^2 H$ is amenable by Proposition 4.3.5. So Λ is amenable by Proposition 4.3.6. Hence, $\overline{\Lambda}^4$ has a good model according to Theorem 4.3.3. □

Corollary 4.3.7. *Let G be an amenable second countable locally compact group. If $\Lambda \subset G$ is a closed approximate subgroup, then Λ is amenable.*

Proof. Corollary 4.3.7 is a consequence of Theorem 1.1.20 applied to G and $H = G$. □

Corollary 4.3.8. *If Λ is an approximate subgroup of a countable discrete amenable group G, then Λ is amenable.*

Proof. The group G equipped with the discrete topology is an amenable locally compact group. Moreover, Λ is a uniformly discrete approximate subgroup in this topology. So Corollary 4.3.8 is a consequence of Corollary 4.3.7. □

Proof of Theorem 1.1.21. Since Λ is uniformly discrete it is amenable by Corollary 4.3.7 and Λ^4 is uniformly discrete as well, hence closed. So the approximate subgroup $\overline{\Lambda^4} = \Lambda^4$ has a good model, f say, by Theorem 4.3.3. Thus, the approximate subgroup Λ^4 contains a model set by Proposition 2.3.6. □

4.3.4 Consequences and questions

We gather in this subsection various consequences of the above results.

Proposition 4.3.9. *Let Λ be an approximate lattice in a connected amenable Lie group. Then there is $\Lambda' \subset \Lambda^4$ commensurable to Λ that has a good model $f : \Lambda^\infty \to H$ with target a connected amenable Lie group.*

Proof. According to Corollary 4.3.7 and Theorem 4.3.3 there is a good model of Λ^4. In addition, there is an approximate subgroup $\Lambda' \subset \Lambda^4$ commensurable with Λ such that Λ' has a good model $f : (\Lambda')^\infty \to H$ with dense image and target a connected Lie group according to Lemma 2.3.3. By Proposition 2.3.6 the graph of f, denoted by Γ_f, is a lattice in $G \times H$. Let A denote the amenable radical (i.e. the maximal normal amenable closed subgroup) of H. Then $G \times A$ is the amenable radical of $G \times H$. Let $p : G \times H \to H/A$ be the natural map. By [39, Corollary 1.4] the image $p(\Gamma_f)$ is a discrete subgroup of H/A. We can therefore choose a compact neighbourhood of the identity $U \subset H$ with $f^{-1}(U) \subset \Lambda'$ and sufficiently small so that $\Gamma_f \cap (G \times U) \subset G \times A$. But $\overline{f(f^{-1}(U))}$ contains the interior of U since f has dense image. So the interior of U is contained in A which implies that $A = H$ because H is connected. Thus, the map $f : (\Lambda')^\infty \to H$ is a good model of Λ' with dense image and target a connected amenable Lie group. □

We pursue this strategy in the context of S-adic Lie groups (i.e. locally a finite product of real and p-adic Lie groups, see e.g. [7]) as this shows that approximate lattices are uniform:

Proposition 4.3.10. *Let Λ be an approximate lattice in an amenable S-adic Lie group G. Then Λ is a uniform approximate lattice.*

Proof. There are Ξ commensurable to Λ and $f : \Xi^\infty \to H$ a good model of Ξ with dense image and target a connected Lie group without non-trivial normal compact subgroups by Theorem 1.1.20. Since G is amenable, we know that

Λ is a strong approximate lattice ([9, Rem. 4.14.(1)]). So the graph $\Gamma_f \subset G \times H$ of f is a lattice by Proposition 2.3.6 and there is a symmetric compact neighbourhood of the identity $W_0 \subset H$ such that $\Xi := P_0(G, H, \Gamma_f, W_0)$. Since G is normal, closed and amenable we know by [7, Th. 6.6] that there is closed normal subgroup $H' \subset H$ such that $\Gamma' := \Gamma_f \cap G \times H'$ is a uniform lattice in $G \times H'$. So Ξ contains the uniform approximate lattice $P_0(G, H', \Gamma', W_0 \cap H')$. Hence, Λ is uniform. □

As another consequence, we prove finite generation of discrete approximate subgroups of soluble Lie groups that generalises Corollary 3.2.3.

Proposition 4.3.11. *Let R be a connected soluble Lie group. If $\Lambda \subset R$ is a uniformly discrete approximate subgroup, then Λ^∞ is finitely generated.*

Proof. According to Corollary 4.3.8 and Theorem 4.3.3 the approximate subgroup Λ^4 has a good model. According to Lemma 2.3.3 there is an approximate subgroup $\Lambda' \subset \Lambda^4$ that has a good model $f : (\Lambda')^\infty \to H$ with dense image and target a connected Lie group. Since $(\Lambda')^\infty$ is soluble we obtain that H is soluble. Now, the graph of f, denoted by Γ_f, is a discrete subgroup of the connected soluble Lie group $R \times H$ and therefore Γ_f is finitely generated by [65, Proposition 3.8]. Let $F_1 \subset \Lambda'$ be a finite set of generators of $(\Lambda')^\infty$ and $F_2 \subset \Lambda^\infty$ be a finite subset such that $\Lambda \subset F_2\Lambda'$. Then $F_1 \cup F_2$ is a finite set that generates Λ^∞. □

4.4 Generalisation of theorems of Mostow and Auslander

4.4.1 Intersections of approximate lattices and closed subgroups

We will show a general theorem about intersections of approximate lattices and closed subgroups. Proposition 4.4.1 is close in spirit to a classical fact about lattices (see for instance [65, Theorem 1.13]). See also [10] for other results around this topic in the framework of strong (and) uniform approximate lattices.

Results in this section are mainly concerned with uniform approximate lattices and, indirectly, $\star$-approximate lattices.

Proposition 4.4.1. *Let Λ be a uniformly discrete approximate subgroup of a locally compact group G. Assume that H is a closed subgroup of G such that $p(\Lambda)$ is uniformly discrete where $p : G \to G/H$ is the natural map. We have:*

1. *if Λ is a uniform approximate lattice, then $\Lambda^2 \cap H$ is a uniform approximate lattice in H;*

2. *if Λ is a $\star$-approximate lattice in G second countable and H is normal, then $\Lambda^2 \cap H$ is a $\star$-approximate lattice in H and $p(\Lambda)$ is a $\star$-approximate lattice in G/H;*

3. *if* Λ *is an approximate lattice in* G *second countable and* H *is normal, then* $\Lambda^2 \cap H$ *is an approximate lattice in* H *and* $p(\Lambda)$ *is an approximate lattice in* G/H.

Proof. We will first prove (1). We know that $\Lambda^2 \cap H$ is an approximate subgroup according to Lemma 2.1.8. Moreover, since Λ is uniformly discrete so is $\Lambda^2 \cap H$. We must prove that $\Lambda^2 \cap H$ is relatively dense in H. Let $K \subset G$ be a compact subset such that $K\Lambda = G$. Since $p(\Lambda)$ is locally finite there is $F \subset \Lambda$ finite such that $K^{-1}H \cap \Lambda \subset FH$. Take any $h \in H$ then there are $\lambda \in \Lambda$ and $k \in K$ such that $k\lambda = h$. Which implies $\lambda \in K^{-1}H \cap \Lambda$ and we can find $f \in F$ such that $f^{-1}\lambda \in H \cap \Lambda^2$. Therefore, $h \in KF\left(H \cap \Lambda^2\right)$.

Let us move on to the proof of (2). Again, note that $\Lambda^2 \cap H$ is uniformly discrete and is an approximate subgroup according to Lemma 2.1.8. Fix a proper G-invariant Borel probability measure ν_0 on Ω_Λ^{ext}. Let $\mathcal{P}(\Omega_\Lambda^{ext}, H)$ denote the set of H-invariant Borel probability measures. Since $\nu_0 \in \mathcal{P}(\Omega_\Lambda^{ext}, H)$ and is proper, we have by the usual Krein–Millman argument that there exists $\nu_1 \in \mathcal{P}(\Omega_\Lambda^{ext}, H)$ which is ergodic and proper. Then ν_1 is a Borel probability measure on the compact metrizable space Ω_Λ^{ext}, so ν_1 has a well-defined support K. The compact subset K is H-invariant with $\nu_1(K) = 1$ and for any open subset $U \subset \Omega_\Lambda^{ext}$ we have $\nu_1(U \cap K) = 0$ if and only if $U \cap K = \emptyset$. Thus, according to e.g. [82, Proposition 2.1.7], there is $P_1 \in K$ such that $K = \overline{H \cdot P_1}$. Furthermore, $P_1 \neq \emptyset$ since $\overline{H \cdot \emptyset} = \{\emptyset\}$ and $\nu_1(\{\emptyset\}) = 0$. Choose now $p_1 \in P_1$. Then $e \in p_1^{-1}P_1 \subset \Lambda^2$ by [9, Lemma 4.6]. Moreover, the subgroup H is normal so $\nu_2 := \left(p_1^{-1}\right)_* \nu_1$ is an H-invariant ergodic Borel probability measure with $\nu_2(\{\emptyset\}) = 0$ and support $p_1^{-1}K = \overline{H \cdot P_2}$ where $P_2 = p_1^{-1}P_1$. Define the map

$$\begin{aligned} \pi : \overline{H \cdot P_2} &\longrightarrow \mathcal{C}(H) \\ P &\longmapsto P \cap H. \end{aligned}$$

We see that π is H-equivariant and takes values in $\Omega_{\Lambda^2 \cap H}^{ext}$. We claim moreover that π is continuous. Since $p(\Lambda^2)$ is locally finite we know that there is an open subset $U \subset G$ such that $\Lambda^2 H \cap U = H$. Note in addition that $p(P) \subset p(\Lambda^2)$ for all $P \in \overline{H \cdot P_2}$. So for any open subset $V \in G$ we have

$$\begin{aligned} \pi^{-1}(U^V) &= \pi^{-1}(\{X \in \mathcal{C}(H) | X \cap V \neq \emptyset\}) \\ &= \{P \in \overline{H \cdot P_2} | P \cap (U \cap W) \neq \emptyset\} \\ &= U^{U \cap W} \cap \overline{H \cdot P_2}, \end{aligned}$$

where $W \subset G$ is any open subset such that $H \cap W = V$. Likewise for any compact subset $L \subset H$ we have

$$\begin{aligned} \pi^{-1}(U_L) &= \pi^{-1}(\{X \in \mathcal{C}(H) | X \cap L = \emptyset\}) \\ &= \{P \in \overline{H \cdot P_2} | P \cap L = \emptyset\} \\ &= U_L \cap \overline{H \cdot P_2}, \end{aligned}$$

where we consider $L \subset H \subset G$. So π is indeed a continuous map. Thus, $\pi(\overline{H \cdot P_2})$ is a compact subset of $\mathcal{C}(H)$ and $\pi(P_2) = P_2 \cap H$ has dense orbit in $\pi(\overline{H \cdot P_2})$. So $\pi(\overline{H \cdot P_2}) = \Omega_{P_2 \cap H}$. Set $\nu_3 := \pi_* \left((\nu_2)_{|\overline{H \cdot P_2}} \right)$ where $(\nu_2)_{|\overline{H \cdot P_2}}$ is the restriction of the measure ν_2 to its support $\overline{H \cdot P_2}$ which is a well defined H-invariant ergodic Borel probability measure since Ω_Λ is metric compact. Then ν_3 is a H-invariant ergodic Borel probability measure on $\Omega_{P_2 \cap H}$. Suppose now that $\nu_3(\{\emptyset\}) > 0$, then $\nu_3(\{\emptyset\}) = 1$ by ergodicity. Thus, $\pi^{-1}(\{\emptyset\})$ is an H-invariant compact co-null subset of $\overline{H \cdot P_2}$. Which means $\pi^{-1}(\{\emptyset\}) = \overline{H \cdot P_2}$ because $\overline{H \cdot P_2}$ is the support of ν_2. Therefore $\pi(P_2) = \emptyset$. A contradiction. Hence, $\nu_3(\{\emptyset\}) = 0$ so ν_3 is a proper H-invariant Borel probability measure on $\Omega^{ext}_{\Lambda^2 \cap H}$.

We prove now that $p(\Lambda)$ is a $\star$-approximate lattice in G/H. Since $p(\Lambda)$ is uniformly discrete, the map

$$\begin{aligned} P_{G/H} : \Omega^{ext}_{\Lambda,G} &\to \Omega^{ext}_{p(\Lambda),G/H} \\ X &\mapsto p(X) \end{aligned}$$

is well-defined and easily seen to be Borel (e.g. by Lemma 2.2.31). One thus readily checks that the push-forward of any proper G-invariant Borel probability measure on $\Omega^{ext}_{\Lambda,G}$ is a proper G/H-invariant Borel probability measure on $\Omega^{ext}_{p(\Lambda),G/H}$.

Let us show (3). By assumption we may choose $F_{G/H}$ a measurable subset of positive Haar measure (possibly infinite) such that the multiplication map $p(\Lambda) \times F_{G/H} \to G/H$ is one-to-one. Take a measurable section of $F_{G/H}$ in G i.e. a Borel subset $\tilde{F}_{G/H} \subset G$ such that the projection from $\tilde{F}_{G/H} \subset G$ to $F_{G/H}$ is bijective. Let now $F_H \subset H$ be any Borel subset of positive measure such that the multiplication $\Lambda^2 \cap H \times F_H \to H$ is one-to-one. We first notice that the multiplication map $\Lambda \times F_H \tilde{F}_{G/H} \to G$ is one-to-one. Indeed, take $\lambda_1, \lambda_2 \in \Lambda, \tilde{f}_1, \tilde{f}_2 \in \tilde{F}_{G/H}$ and $f_1, f_2 \in F_N$ such that $\lambda_1 f_1 \tilde{f}_1 = \lambda_2 f_2 \tilde{f}_2$. Projecting to G/H we see that $\tilde{f}_1 = \tilde{f}_2$ and $p(\lambda_1) = p(\lambda_2)$. So $\lambda_2^{-1}\lambda_1 f_1 = f_2$. But $\lambda_2^{-1}\lambda_1 \in \Lambda^2 \cap H$ so $f_1 = f_2$ and $\lambda_1 = \lambda_2$. By Lemma 2.2.7 we see that $\tilde{F}_{G/H} F_H$ has finite Haar measure. Thus, by standard computations, F_H and $F_{G/H}$ have finite Haar measure in H and G/H respectively. So by [47, A.2] again $\Lambda^2 \cap H$ and $p(\Lambda)$ are approximate lattices. □

We prove next a converse of sorts of Proposition 4.4.1:

Lemma 4.4.2. *Let Λ be a uniformly discrete approximate subgroup of a locally compact group G. Assume that H is a closed subgroup of G such that $\Lambda^2 \cap H$ is a uniform approximate lattice in H. Then $p(\Lambda)$ is a locally finite subset of G/H where $p : G \to G/H$ is the natural map.*

Proof. Let $K \subset G/H$ be a compact subset. Then there is a compact subset $L \subset G$ such that $p(L) = K$. Since $\Lambda^2 \cap H$ is relatively dense in H there is a compact subset $L' \subset G$ such that $LH \subset L' \left(\Lambda^2 \cap H\right)$. Take $\lambda \in \Lambda \cap LH$ then

$$\lambda \in \left(L' \left(\Lambda^2 \cap H\right)\right) \subset \left(\left(L' \cap \Lambda^3\right) \left(\Lambda^2 \cap H\right)\right),$$

so $p(\Lambda) \cap K \subset p(L' \cap \Lambda^3)$ that is indeed finite. □

As a first application we investigate the intersections of uniform approximate lattices with centralisers:

Corollary 4.4.3. *Let Λ be a uniform approximate lattice in a locally compact group G. Then for all $\gamma \in \Lambda^\infty$ the approximate subgroup $\Lambda^2 \cap C(\gamma)$ is a uniform approximate lattice in $C(\gamma)$ the centraliser of γ. Moreover, if G is a Lie group and Λ^∞ is dense in G, then $\Lambda^2 \cap Z(G)$ is a uniform approximate lattice in $Z(G)$ the centre of G.*

Proof. Let $n \geq 0$ be an integer such that $\gamma \in \Lambda^n$ and consider the map

$$\begin{aligned} \varphi : G &\longrightarrow G \\ g &\longmapsto g\gamma g^{-1}. \end{aligned}$$

Then φ factors as $\varphi = \psi \circ p$ where $\psi : G/C(\gamma) \to G$ is a continuous injective map and $p : G \to G/C(\gamma)$ is the natural map. But $\varphi(\Lambda) \subset \Lambda^{n+2}$ so is locally finite. Hence, $p(\Lambda)$ is locally finite as well. By part (1) of Proposition 4.4.1 we deduce that $\Lambda^2 \cap C(\gamma)$ is a uniform approximate lattice in $C(\gamma)$.

Now if G is a Lie group and Λ^∞ is dense in G, then $Z(G) = \bigcap_{\gamma \in \Lambda^\infty} C(\gamma)$. But $Z(G)$ is a Lie group and so are the $C(\gamma)$'s. Thus, there are $\gamma_1, \ldots, \gamma_r \in \Lambda^\infty$ such that $\dim(Z(G)) = \dim(\bigcap_i C(\gamma_i))$. Consider now the map

$$\begin{aligned} \varphi : G &\longrightarrow G^r \\ g &\longmapsto (g\gamma_1 g^{-1}, \ldots, g\gamma_r g^{-1}). \end{aligned}$$

As above φ factors as $\varphi = \psi \circ p$ with $\psi : G/\left(\bigcap_i C(\gamma_i)\right) \to G^r$ an injective and continuous map and $p : G \to G/\left(\bigcap_i C(\gamma_i)\right)$ the natural map. But $\varphi(\Lambda) \subset \prod_{1 \leq i \leq r} \Lambda^{n+2}$ where n is a positive integer such that $\{\gamma_1, \ldots, \gamma_r\} \subset \Lambda^n$. Thus, $\varphi(\Lambda)$ is locally finite and so is $p(\Lambda)$. By part (1) of Proposition 4.4.1 we deduce that $\Lambda^2 \cap \bigcap_i C(\gamma_i)$ is a uniform approximate lattice in $\bigcap_i C(\gamma_i)$. But $Z(G)$ is an open subgroup of $\bigcap_i C(\gamma_i)$ so $p'(\Lambda^2 \cap \bigcap_i C(\gamma_i))$ is obviously locally finite where $p' : \bigcap_i C(\gamma_i) \to \left(\bigcap_i C(\gamma_i)\right)/Z(G)$ is the natural map. By part (1) of Proposition 4.4.1 once again we have that

$$\left(\Lambda^2 \cap \bigcap_i C(\gamma_i)\right)^2 \cap Z(G) \subset \Lambda^4 \cap Z(G)$$

is a uniform approximate lattice in $Z(G)$. By Lemma 2.1.8 we find that $\Lambda^2 \cap Z(G)$ is a uniform approximate lattice in $Z(G)$. □

4.4.2 Borel density for approximate lattices

The Borel density theorem asserts that lattices in simple algebraic groups are Zariski-dense. In [13], Björklund, Hartnick and Stulemeijer proved an extension of this theorem valid for strong approximate lattices as well:

Theorem 4.4.4 (Björklund–Hartnick–Stulemeijer, [13])**.** *Let k be a local field and let $\mathbb{G}$ be a semi-simple algebraic group defined over k. Let Λ be either a strong approximate lattice or a uniform approximate lattice in $\mathbb{G}(k)$. Then Λ is Zariski-dense.*

This theorem will not be enough for our purpose. We will in fact use a strengthening of this theorem. The historical route to show Borel density-type theorems for groups with finite co-volume is to start by proving that the subgroup considered has *property (S)* (see e.g. [14]).

Definition 4.4.5 (Definition 1.1, [14])**.** Let G be a locally compact group. A closed subset $X \subset G$ has *property (S)* if for all neighbourhoods $\Omega \subset G$ of the identity and all $g \in G$ there is $n \in \mathbb{N}$ such that $g^n \in \Omega X \Omega$.

It so happens that approximate lattices have property (S). Hence, they exhibit similar density properties.

Proposition 4.4.6 (Hrushovski, [47])**.** *Let Λ be an approximate lattice in a locally compact second countable group G. Then Λ^2 has property (S).*

We produce here a proof that parallels the proof of property (S) for lattices. Note that the original proof from [47] is almost identical - with the exception that it does not use the language of hulls.

Proof. By assumption there is a proper G-invariant Borel probability measure ν on Ω_X as in Proposition 2.2.34. Let $C > 0$ be the constant coming from Proposition 2.2.34. If Ω is any symmetric neighbourhood of the identity, then the open subset U^{Ω} satisfies $\nu\left(U^{\Omega}\right) > 0$. Therefore, for any $g \in G$ there is an integer $1 \leq n < C\left(\nu\left(U_{\Omega}\right)\right)^{-1}$ such that $\nu(U_{\Omega} \cap g^n U_{\Omega}) > 0$. So we can find $P \in U_{\Omega} \cap g^n U_{\Omega}$. Thus, $P \cap \Omega \neq \emptyset$ and $P \cap g^n\Omega \neq \emptyset$. That implies that $P^{-1}P \cap \Omega g^n \Omega \neq \emptyset$. But $P^{-1}P \subset \Lambda^2$ so $g^n \in \Omega\Lambda^2\Omega$. □

In the next section we will use the following density theorem due to Borel:

Proposition 4.4.7 (Borel, [14])**.** *Let G be the k-points of an almost simple algebraic group over a local field k. Suppose that G is not compact and let Γ be a group with property (S). Then Γ is Zariski-dense.*

4.4.3 Proof of Theorem 1.1.22

We will use results from Section 4.3 to show that the strategy due to Auslander in [1] carries over to the approximate subgroup setting. Hence, we will first show a generalisation of the Zassenhaus lemma. We define inductively the set of commutators of a subset X by $X^{(1)} := [X, X]$ and $X^{(n+1)} := [X, X^{(n)}]$.

Lemma 4.4.8 (Zassenhaus-type lemma)**.** *Let G be a product of groups of points of affine algebraic groups over characteristic 0 local fields and let A be* *a normal abelian subgroup. Suppose that there is a distance d_A defined on A that is left-invariant and such that the map*

$$\begin{aligned} s : G &\longrightarrow \mathbb{R} \\ g &\longmapsto \sup_{a \in A \setminus \{e\}} \frac{d(gag^{-1}, a)}{d(a, e)} \end{aligned}$$

is well-defined and continuous at e. Then there is a neighbourhood of the identity $W \subset G$ such that for all compact subsets K of WA and any neighbourhood of the identity $V \subset G$ there is an integer $n_0 \geq 0$ with $K^{(n)} \subset V$ for all $n \geq n_0$.

Proof. Take ϵ_1 and $\epsilon_2 > 0$ and choose neighbourhoods $W_1, W_2 \subset G$ such that for all $g \in W_i$ we have $s(g) \leq \epsilon_i$. Now for $w_1, w_2 \in W_2$ and $a_1, a_2 \in A$ we compute that

$$[w_1 a_1, w_2 a_2] = [w_1, w_2] w_2 w_1 ([w_2^{-1}, a_1][w_1^{-1}, a_2])(w_2 w_1)^{-1}.$$

So

$$[w_1 B_d(r_1), w_2 B_d(r_2)] \subset [w_1, w_2] B_d \left((1 + \epsilon_1)(1 + \epsilon_2)(\epsilon_2 r_1 + \epsilon_1 r_2)\right) \qquad (*)$$

where $B_d(r)$ is the ball of radius r centred at the identity and $r_1, r_2 \geq 0$.

But by [65, Cor. 8.15] (the proof holds verbatim over local fields of characteristic 0) we can find a neighbourhood of the identity $W \subset G$ such that $s(g) \leq 1/4$ for all $g \in W$, $W^{(n+1)} \subset W^{(n)}$ for all $n \geq 0$ and for any neighbourhood of the identity $V \subset G$ there is $n_0 \geq 0$ such that $W^{(n_0)} \subset V$. We then compute using $(*)$ that for all $r \geq 0$ and for all neighbourhoods of the identity $V' \subset G$ there is $n_0 \geq 0$ such that $(WB_d(r))^{(n)} \subset V'$ for every $n \geq n_0$. □

Lemma 4.4.9. *Let Λ be an approximate subgroup in a group G and let Γ be a subgroup of G that commensurates and contains Λ. Assume furthermore that Λ generates a soluble subgroup and that one of the following holds:*

1. *G is a real Lie group and Λ is relatively compact;*
2. *$G = G_1 \times \cdots \times G_n$ is a finite product of groups of k_i-points of algebraic groups over fields k_i of characteristic 0 with $i = 1, \ldots, n$.*

Then Λ is covered by finitely many left-translates of a normal soluble subgroup of Γ.

Proof. Suppose (1). Then write $\overline{\Lambda}$ the closure of Λ and Γ' the subgroup generated by Γ and $\overline{\Lambda}$. Let $\phi : L \to G$ be the Lie group homomorphism given by the closed-approximate-subgroup theorem (Theorem 1.1.17). Let L^0 be the connected component of the identity in L. Since $\pi^{-1}(\overline{\Lambda})$ has non-empty interior $\Gamma \cap \phi(L^0)$ is contained in $\overline{\Lambda}^{\infty}$. Whence L^0 is soluble. But $\phi(L^0)$ is normal in Γ', and finitely many left-translates of $\phi(L^0)$ cover $\overline{\Lambda}$. So $\Gamma \cap \phi(L^0)$ works.

Suppose now (2). Write $G = G_1 \times \cdots \times G_n$. Suppose first that $n = 1$. Take any two subgroup H_1, H_2 of G such that Λ is covered by finitely many left-translates of H_1 and in finitely many left-translates of H_2. Then Λ is contained in finitely many left-translates of $H_1 \cap H_2$ (e.g. Lemma 2.1.7). Since G with the Zariski-topology is a Noetherian topological space, there is a minimal Zariski-closed subgroup $H \subset G$ such that Λ is covered by finitely many left-translates of H. Now, for any $\gamma \in \Gamma$, $\gamma\Lambda\gamma^{-1}$ is covered by finitely many left-translates of $\gamma H \gamma^{-1}$. Hence, Λ is covered by finitely many left-translates of $\gamma H \gamma^{-1}$ and by minimality $\gamma H \gamma^{-1} = H$. If now $n \geq 1$, apply the above reasoning in each factor to find soluble subgroups $H_i \subset G_i$ normalised by Λ such that Λ is covered by finitely many left-translates of $H := H_1 \times \cdots H_n$. Then $H \cap \Gamma$ works. □

The next result is reminiscent of a lemma due to Wang ([65, Theorem 8.24]):

Lemma 4.4.10. *Let G be a product of finitely many groups of points of affine algebraic groups over local fields of characteristic 0, let R be a closed normal subgroup of G and Λ be a uniformly discrete approximate subgroup. Suppose moreover that there is a normal series $R = R_0 \supset R_1 \supset \ldots \supset R_n = \{e\}$ such that for all $i \in \{1, \ldots, n\}$, R_i is a closed normal subgroup of G and R_{i-1}/R_i satisfies the conditions of Lemma 4.4.8 as a subgroup of G/R_i. Then there is a symmetric compact neighbourhood of the identity $W \subset G$ such that $\Lambda^2 \cap WR$ is contained in a soluble normal subgroup of Λ^∞.*

Proof. Take V a symmetric compact neighbourhood of the identity. The subset $\Lambda^2 \cap V^2R$ is an approximate subgroup by Lemma 2.1.8. Since Λ is uniformly discrete we know that $\Lambda^2 \cap V^2R$ is amenable by Theorem 1.1.20 and so $(\Lambda^2 \cap V^2R)^4 \subset \Lambda^8 \cap V^8R$ has a good model by Theorem 4.3.3. Now there is $\Xi_1 \subset \Lambda^8 \cap V^8R$ that has a good Lie model by Lemma 2.3.3. There is therefore $\Xi_2 \subset \Xi_1$ such that $[\Xi_2^2, \Xi_2^2] \subset \Xi_2^2$ by [65, Cor. 8.15].

Let us proceed by induction on n the length of the normal series. If $n = 0$, take W such that $\Xi_2^2 \cap W = \{e\}$. Otherwise $n \geq 1$, assume by induction that we can find a symmetric compact neighbourhood of the identity W_1 in G such that $\Lambda^8 \cap W_1R_1$ is contained in a soluble normal subgroup of Λ^∞. By Lemma 4.4.8 there is $W_0 \subset G$ such that for all compact subsets $K \subset W_0R_0$ there is a positive integer n such that $K^{(n)} \subset W_1R_1$. For any finite subset $F \subset \Xi_2^2 \cap W_2^2R$ we can thus find an integer n such that $F^{(n)} \in W_1R_1$. Hence, $F^{(n)} \in \Xi_2^2 \cap W_1R_1$. So by [65, Lem. 8.17] and the induction hypothesis F generates a soluble subgroup. Now by [65, Cor. 8.4] the approximate subgroup $\Xi_2^2 \cap W_2^2R$ generates a soluble subgroup (for, if G is the group of rational points of an algebraic group over a local field of characteristic 0 we can embed it in $\mathrm{GL}_n(\mathbb{C})$ for some integer n). Let Γ be the projection to G/R of the subgroup given by Lemma 4.4.9 applied to $\Xi_2^2 \cap W_2^2R$. In both cases Γ is soluble and normalised by the projection of Λ^∞. Since the projection of $\Lambda^2 \cap V^2R$ is contained in finitely many left-translates of Γ (Lemma 2.1.8), there exists a symmetric compact neighbourhood of the identity W such that the projection of $\Lambda^2 \cap WR$ is contained in the closure of Γ. □

Proof of Theorem 1.1.22. Let A be the amenable radical (i.e. the maximal normal amenable closed subgroup) and R be the soluble radical (i.e. the maximal normal soluble closed subgroup) of G. Note that R has a series $R = R_0 \supset R_1 \supset \ldots \supset R_n = \{e\}$ of closed normal subgroups of G such that R_i is normal in G and R_{i-1}/R_i is isomorphic to one of the following: a finitely generated abelian group, a p-adic torus, a real or p-adic finite dimensional vector space or a compact abelian (real or p-adic) Lie group (see e.g. [72, §14.1.1]). So one checks that R satisfies the assumptions of Lemma 4.4.10 using absolute values. Therefore, there is a soluble normal subgroup Γ of Λ^∞ and some neighbourhood of the identity W in G such that $\Lambda^2 \cap WR$ is contained in Γ. Let S be any non-compact simple factor of G and let $p_S : G \to S$ be the natural map. Then $p_S(\Gamma)$ is solvable normalised by $p_S(\Lambda^\infty)$ which is Zariski-dense by assumption. So $p_S(\Gamma)$ is is finite. Hence, the subset $p_S(\Lambda^2 \cap WR)$ is finite. We thus know that the projection of $\Lambda^2 \cap WR$ to G/A is finite. Since R is co-compact in A and $\Lambda^2 \cap WR$ is commensurable to $\Lambda^2 \cap VR$ for any other symmetric compact neighbourhood of the identity V (Lemma 2.1.8) we find that the projection of Λ to G/A is uniformly discrete.

Let S' be a compact almost simple factor of G and let $p_{S'} : G \to S'$ be the natural map. If $p_{S'}(\Lambda^\infty)$ is dense, then $p_{S'}(\Gamma)$ must be finite as it is soluble and normalised by $p_{S'}(\Lambda^\infty)$. So the Levi decomposition shows that the projection of $\Lambda^2 \cap W^2R$ to G/R is finite, hence the projection of Λ to G/R is uniformly discrete. □

Proof of Theorem 1.1.7. According to Proposition 4.4.6 we know that Λ has property (S). If S is any simple factor of G, then the projection of Λ to S also has property (S). By the Borel density theorem ([14]) the group it generates - the projection of Λ^∞ - is Zariski-dense. So Λ satisfies the conditions of Theorem 1.1.22. By Proposition 4.4.1 the approximate subgroup $\Lambda^2 \cap R$ is an approximate lattice in R. And we have that $\Lambda^2 \cap A$ is uniform by Proposition 4.3.10. □

Question 4. With the notations of Theorem 1.1.22. Let $p : G \to G/A$ denote the natural projection. Suppose that both $p(\Lambda)$ and $\Lambda^2 \cap A$ are contained in model sets. Under which conditions is Λ contained in a model set?

Chapter 5

$\star$-Approximate lattices in higher-rank semi-simple algebraic groups

5.1 Cocycles and $\star$-approximate lattices

5.1.1 Cocycles associated to sections of extended hulls

We will now define certain cocycles that will mimic properties of cocycles on transitive spaces as studied by Mackey [57] and Zimmer [82]. Our construction will essentially follow the construction of Björklund and Hartnick ([11]). In [11] they used the same construction to establish a tentative notion of induction for *uniform* strong approximate lattices. The two results below are essentially Proposition 2.9 and Lemma 2.10 from [11] without the assumption that X_0 is a *strong uniform* approximate lattice. We write down their proof to show that this stronger assumption is indeed not needed.

We start by defining a suitable notion of sections of the set $\mathcal{C}(G)$.

Definition 5.1.1. Let G be a locally compact group and $\mathcal{B}$ be a Borel subset of $\mathcal{C}(G)$. A *Borel section* of $\mathcal{B}$ is a Borel map $s : \mathcal{B} \to G$ such that for any $X \in \mathcal{B}$ we have $s(X) \in X$.

Precisely, the cocycles we are interested in are:

Lemma 5.1.2. *Let X_0 be a uniformly discrete subset of a locally compact group G. Let $s : \Omega_{X_0}^{ext} \setminus \{\emptyset\} \to G$ be a Borel section. Then the map*

$$\begin{aligned} \alpha_s : G \times (\Omega_{X_0}^{ext} \setminus \{\emptyset\}) &\longrightarrow G \\ (g, X) &\longmapsto s(gX)^{-1} g s(X) \end{aligned}$$

is a strict Borel cocycle that takes values in $\overline{X_0^{-1}X_0}$.

Proof. The map α_s is Borel since s is Borel. For all $g, h \in G$ and $X \in \Omega^{ext}_{X_0} \setminus \{\emptyset\}$ we have

$$\alpha_s(g, hX)\alpha_s(h, X) = s(ghX)^{-1}gs(hX)s(hX)^{-1}hs(X) = s(ghX)^{-1}ghs(X) = \alpha_s(gh, X).$$

So α_s is a strict cocycle. Moreover, we have

$$\alpha_s(g, X) = s(gX)^{-1}gs(X) \in (gX)^{-1}gX \subset X^{-1}X \subset \overline{X_0^{-1}X_0}.$$

□

It remains only to prove the existence *Borel sections* of extended invariant hulls.

Proposition 5.1.3. *Let X_0 be a uniformly discrete subset of a locally compact second countable group G. Then there is a Borel section $s : \Omega^{ext}_{X_0} \setminus \{\emptyset\} \to G$.*

Proof. Since X_0 is uniformly discrete there is an open neighbourhood V of the identity in G such that $X_0^{-1}X_0 \cap V^{-1}V = \{e\}$. Thus, we have $|X \cap gV| \leq 1$ for any $X \in \Omega^{ext}_{X_0}$ and any $g \in G$. If $X \in U^{gV}$, then define $s_g(X)$ as the unique element of the subset $X \cap gV$. The maps $s_g : U^{gV} \to G$ are well-defined. Moreover, if $W \subset G$ is any open subset, then $s_g^{-1}(W) = U^{gV \cap W}$. So the maps s_g are continuous. Take a sequence $(g_i)_{i \geq 0}$ such that $\bigcup_{i \geq 0} g_i V = G$. Then $\bigcup_{i \geq 0} U^{g_i V} = \Omega^{ext}_{X_0} \setminus \{\emptyset\}$. Now define $B_i := U^{g_i V} \setminus \left(\bigcup_{0 \leq j < i} U^{g_j V}\right)$ for all integers $i \geq 0$ and $s : \Omega^{ext}_{X_0} \setminus \{\emptyset\} \to G$ as the unique map such that $s_{|B_i} = (s_{g_i})_{|B_i}$. Since the subsets B_i are Borel and the maps s_g are continuous, the map s is Borel. Moreover, for all $X \in \Omega^{ext}_{X_0} \setminus \{\emptyset\}$ we indeed have $s(X) \in X$. □

5.1.2 A reduction lemma

Before we move on with our study of the cocycles defined in the previous section, we make an elementary observation about the range of Borel maps that satisfy some functional equation with respect to a cocycle. To this end we first recall a well known fact about ergodicity and dense subgroups:

Lemma 5.1.4. *Let G be a locally compact group, $D \subset G$ be a dense subgroup and X be a compact G-space. If ν is an ergodic G-invariant Borel probability measure, then the D-action on (X, ν) is also ergodic.*

We now prove the fact that will be used as the starting point of the proof of Proposition 5.1.7 in Subsection 5.1.3.

Lemma 5.1.5. *Let G and H be two locally compact second countable groups. Let X be a compact metric G-space and ν be a G-invariant ergodic Borel probability measure on X. Let $\alpha : G \times X \to H$ be a Borel cocycle that take values in a subset $A \subset H$ and let $B \subset H$ be another subset. Suppose that there is a Borel map $\phi : X \to H$ such that for all $g \in G$ and ν-almost every $x \in X$ we have*

$$\phi(gx) = \alpha(g, x)\phi(x)B.$$

Then there is $h \in H$ such that for every neighbourhood of the identity V in H we have

$$\nu - a.e.\ x \in X, \phi(x) \in AVhB^{-1}.$$

Proof. Consider the measure $\phi_*\nu$. It is a Borel probability measure on the locally compact second countable group H. So $\phi_*\nu$ has a well-defined non-trivial support $S \subset H$ and we can choose $h \in S$. Then for any open neighbourhood V of e we have

$$\phi_*\nu(Vh) = \nu(\phi^{-1}(Vh)) > 0.$$

So for all $g \in G$ and ν-almost every $x \in \phi^{-1}(Vh)$ we have

$$\phi(gx) \in AVhB^{-1}.$$

But for any countable dense subgroup D of G there is $Y \subset \phi^{-1}(Vh)$ with $\nu(Y) = \nu(\phi^{-1}(Vh)) > 0$ such that for all $x \in Y$ and all $d \in D$ we have $\phi(dx) \in AVhB^{-1}$. So $DY \subset \phi^{-1}\left(AVhB^{-1}\right)$. And we know that $\nu(DY) = 1$ by Lemma 5.1.4. □

Likewise, we prove a result for cocycles taking values in unitary groups that will be at the heart of our investigation of property (T) for ⋆-approximate lattices.

Lemma 5.1.6. *Let G be a locally compact group that acts continuously on a compact metric space and let ν be a G-invariant ergodic Borel probability measure on X. Let $\mathcal{H}$ be a separable Hilbert space and let $\alpha : X \times G \to U(\mathcal{H})$ be a Borel cocycle with target the unitary group of $\mathcal{H}$. Suppose that there exists a Borel map $\phi : X \to \mathcal{H}$ such that for all $g \in G$ and ν-almost every $x \in X$ we have*

$$\phi(gx) = \alpha(g, x)\phi(x).$$

Then there is $\xi \in \mathcal{H}$ such that for every $\epsilon > 0$ we have

$$\nu - a.e.\ x \in X, \phi(x) \in A(B(\xi, \epsilon))$$

where $B(\xi, \epsilon)$ denotes the ball of centre ξ and radius ϵ.

Proof. Consider the measure $\phi_*\nu$. It is a Borel probability measure on the separable Hilbert space $\mathcal{H}$ whose (strong) topology is second countable. So $\phi_*\nu$ has a well-defined non-trivial support $S \subset H$ and we can choose $\xi \in S$. Then for any $\epsilon > 0$ we have $\nu(\phi^{-1}(B(\xi, \epsilon))) > 0$. For all $g \in G$ and ν-almost every $x \in \phi^{-1}(B(\xi, \epsilon))$ we have

$$\phi(gx) = \alpha(g, x)\phi(x) \in A(B(\xi, \epsilon)).$$

And we conclude as above. □

5.1.3 Range of cocycles and compact cocycles

Throughout this section we fix a $\star$-approximate lattice Λ in a locally compact group G and a proper G-invariant ergodic Borel probability measure ν_0 on Ω_Λ^{ext}. Let $s : \Omega_\Lambda^{ext} \setminus \{\emptyset\} \to G$ be a Borel section given by Proposition 5.1.3 and let $\alpha_s : G \times \Omega_\Lambda^{ext} \setminus \{\emptyset\} \to G$ be the strict Borel cocycle associated to s (Lemma 5.1.2). Since $\nu_0(\{\emptyset\}) = 0$ the cocycle α_s gives rise to a Borel cocycle, that we denote by α_s as well, over $(\Omega_\Lambda^{ext}, \nu_0)$. Finally, we fix an abstract group homomorphism $T : \langle \Lambda \rangle \to H$ with target a locally compact second countable group. Our goal is to prove the following result linking the set $T(\Lambda)$ to the range of the cocycle $T \circ \alpha_s$.

Proposition 5.1.7. *Suppose that the Borel cocycle $T \circ \alpha_s$ is cohomologous to a cocycle that takes values in a subset $L \subset H$. Then there is $h_0 \in H$ such that for every neighbourhood of the identity V in H there is a compact subset $K \subset H$ with*

$$T(\Lambda) \subset V h_0 (L^{-1}L)^2 h_0^{-1} K.$$

The proof reduces to two steps. First, we build a cocycle α' cohomologous to α_s, taking values in some power of Λ (here Λ^6) but such that $T \circ \alpha'$ still takes values in some thickening of $L^{-1}L$.

Lemma 5.1.8. *There is $h_0 \in H$ such that for every neighbourhood of the identity $V \subset H$ we can find Borel maps $f : \Omega_\Lambda^{ext} \to G$ and $h : \Omega_\Lambda^{ext} \to H$ defined ν_0-almost everywhere such that for all $g \in G$ and ν_0-almost every $X \in \Omega_\Lambda^{ext}$ we have*

$$\alpha_f(g, X) := f(gX)^{-1} g f(X) \in \Lambda^6 \text{ and } T(\alpha_f(g, X)) \in V h L^{-1} L h(X)^{-1}.$$

Proof. First of all, note that α_s takes values in Λ^2 by Lemma 5.1.2 and that the subset Λ^2 is countable. So $T \circ \alpha_s$ is a well-defined Borel cocycle that takes values in $T(\Lambda^2)$. Choose a measurable map $\phi : \Omega_\Lambda^{ext} \to H$ such that for all $g \in G$ and ν_0-almost every $X \in \Omega_\Lambda^{ext}$ we have

$$T \circ \alpha_s(g, X) = \phi(gX) \beta(g, X) \phi(X)^{-1}.$$

By Lemma 5.1.5 there is $h_0 \in H$ such that for every neighbourhood V of the identity in H and ν_0-almost every $X \in \Omega_\Lambda^{ext}$ we have

$$\phi(X) \in T(\Lambda^2) V h_0 L^{-1}.$$

Consider an enumeration $(\lambda_n)_{n \geq 0}$ of Λ^2 and define

$$\begin{aligned} p_V : T(\Lambda^2) V h L^{-1} &\longrightarrow \Lambda^2 \\ g &\longmapsto \lambda_{\min\{n \in \mathbb{N} | g \in T(\lambda_n) V h_0 L^{-1}\}}. \end{aligned}$$

Since $p_V^{-1}(\{\lambda_0, \dots, \lambda_n\}) = \bigcup_{0 \leq i \leq n} T(\lambda_n) V h_0 L^{-1}$ the map p_V is Borel measurable. We can thus define the map $h : \Omega_\Lambda^{ext} \to V h_0 L^{-1}$ for ν_0-almost every

$X \in \Omega_\Lambda^{ext}$ by $\phi(X) = T(p_V(\phi(X)))h(X)$. We have for all $g \in G$ and ν_0-almost every $X \in \Omega_\Lambda^{ext}$,

$$T \circ \alpha_s(g, X) = T\left(p_V(\phi(gX))\right) h(gX)\beta(g, X)h(X)^{-1}T(p_V(\phi(X))^{-1}).$$

So

$$T(f(gX)^{-1}gf(X)) = h(gX)\beta(g, X)h(X)^{-1},$$

where f is defined for ν_0-almost every $X \in \Omega_\Lambda^{ext}$ by $f(X) := s(X)p_V(\phi(X))^{-1}$. Then one checks that f and h work. □

We then show that the range of such a cocycle α_f must be large in some power of Λ, thus proving that the set of elements $\lambda \in \Lambda$ such that $T(\lambda)$ belongs to some thickening of $L^{-1}L$ is large. However, the range of α_f as defined by the map $\Omega_{\Lambda,G}^{ext} \to \mathcal{C}(G)$ given by $X \mapsto \{\alpha_f(g, X) | g \in G\}$ might not be well-behaved (e.g. non-Borel) so we proceed more carefully.

Proposition 5.1.9. *Let $f : \Omega_{\Lambda,G}^{ext} \to G$ be a Borel measurable map and let $\alpha_f : G \times \Omega_{\Lambda,G}^{ext} \to G$ denote the cocycle defined by $\alpha_f(g, X) = f(gX)^{-1}gf(X)$ for all $g \in G$ and ν_0-almost all $x \in \Omega_{\Lambda,G}^{ext}$. Suppose that $\Delta : \Omega_{\Lambda,G}^{ext} \to \mathcal{C}(G)$ is a Borel measurable map defined ν_0-almost everywhere such that :*

1. *α_f takes values in Λ^n for some integer $n \geq 0$;*
2. *for all $g \in G$ and ν_0-almost all $X \in \Omega_{\Lambda,G}^{ext}$ we have $\alpha_f(g, X) \in \Delta(X)$.*

Then there is a finite subset $F \subset \Lambda$ such that

$$\Lambda \subset \bigcup_{X \in \Omega_{\Lambda,G}^{ext}} \Delta(X)\Delta(X)^{-1}F$$

where we set $\Delta(X)\Delta(X)^{-1} = \emptyset$ when $\Delta(X)$ is not defined.

We will also make an independent use of this proposition in our study of property (T).

Proof. We first show that we can suppose that Δ takes values in $\Omega_{\Lambda^n,G}^{ext}$. The map $\Delta' : \Omega_{\Lambda,G}^{ext} \to \mathcal{C}(G)$ defined by $X \to \Delta(X) \cap \Lambda^n$ satisfies $(\Delta')^{-1}(U_K) = \Delta^{-1}(\Lambda^n \cap K)$ for all compact subsets $K \subset G$. So Δ' is Borel by Lemma 2.2.31. Since in addition $\alpha_f(g, X) \in \Delta(X) \cap \Lambda^n$ for all $g \in G$ and ν_0-almost all $X \in \Omega_{\Lambda,G}^{ext}$ we get the desired result. So suppose from now on that Δ takes values in $\Omega_{\Lambda^n,G}^{ext}$. Let $\Phi : \Omega_\Lambda^{ext} \longrightarrow \Omega_{\Lambda^n}^{ext}$ be the map defined by $X \longmapsto f(X)\Delta(X)^{-1}$. We know that Φ is Borel and well-defined ν_0-almost everywhere since both f and Δ are. The crux of the proof is to deduce a simple equation involving f, Δ (or equivalently Φ) and intertwining the G-actions on $\Omega_{\Lambda,G}^{ext}$ and $\Omega_{\Lambda^n,G}^{ext}$. But we know that for all $g \in G$ and ν_0-almost all $X \in \Omega_{\Lambda,G}^{ext}$ we have

$$\alpha_f(g, X) \in \Delta(X).$$

So we obtain
$$f(gX) \in gf(X)\Delta(X)^{-1} = g\Phi(X). \tag{$*$}$$
We will now use $(*)$ to show the existence of a real number $\epsilon > 0$ such that $(\mathcal{P}_{\Lambda^n})^* (\Phi_* \nu_0) \geq \epsilon \mu_G$ where μ_G is a Haar-measure on G (fixed from now on) and $\mathcal{P}_{\Lambda^n}$ denotes the periodization map introduced in Definition 2.2.19. So let ν_1 denote the push-forward of ν_0 by Φ. For all open subsets $W \subset G$ and all elements $g \in G$ we have

$$\begin{aligned}
\nu_1 \left(gU^W\right) &= \int_{\Omega_\Lambda^{ext}} \mathbb{1}_{U^W} \left(g^{-1}\Phi(X)\right) d\nu_0(X) \\
&\geq \int_{\Omega_\Lambda^{ext}} \mathbb{1}_W(f(g^{-1}X))d\nu_0(X) \\
&= \int_{\Omega_\Lambda^{ext}} \mathbb{1}_W(f(X))d\nu_0(X) = f_*\nu_0(W).
\end{aligned}$$

where the second-to-last line is implied by $(*)$ and the last one is obtained by G-invariance of ν_0. Since $gU^W = U^{gW}$, the above inequalities imply that for all $h \in G$ and all open subsets $W \subset G$ we have $\nu_1(U^W) \geq \mu_0(hW)$ where $\mu_0 := f_*\nu_0$. Define $\mu_1 := (\mathcal{P}_{\Lambda^n})^*\nu_1$ where $\mathcal{P}_{\Lambda^n}$ is the periodization map. Since Λ^n is uniformly discrete there is an open neighbourhood of the identity $W \subset G$ such that $W^{-1}W \cap \Lambda^{2n} = \{e\}$. By Lemma 2.2.20 for all $g, h \in G$ we have

$$\mu_1(gW) = \nu_1(U^{gW}) \geq \mu_0(hW).$$

Furthermore, given any Borel probability measure μ on G we have

$$\mu * \mu_0(hW) \leq \mu_1(gW).$$

Now take μ that has density a continuous compactly supported function $\delta : G \to \mathbb{R}$ with respect to μ_G on G. We know that $\mu * \mu_0$ is absolutely continuous with respect to μ_G and that $\mu * \mu * \mu_0$ is a Borel probability measure on G that has continuous density, ρ say, with respect to μ_G. Since ρ is a non-trivial non-negative continuous function we can find $h \in G$, a neighbourhood of the identity $W' \subset W$ and a real number $\epsilon > 0$ such that $f_{|hW'} \geq \epsilon > 0$. Hence, for all $g \in G$ and all Borel subsets $B \subset gW'$ we have

$$\mu_1(B) \geq \mu * \mu * \mu_0(hg^{-1}B) \geq \epsilon\mu_G(hg^{-1}B) = \epsilon\mu_G(B).$$

But G is second countable so we can find a a sequence $(g_i)_{i\geq 0}$ of elements of G and a countable Borel partition $(W_i)_{i\geq 0}$ of G such that $W_i \subset g_iW'$ for all integers $i \geq 0$. Thus, for all Borel subsets $B \subset G$ we have

$$\mu_1(B) = \sum_{i\geq 0} \mu_1 \left(B \cap W_i\right) \geq \epsilon \sum_{i\geq 0} \mu_G \left(B \cap W_i\right) = \epsilon\mu_G(B).$$

We will now prove Proposition 5.1.9 as a consequence of a version of Ruzsa's covering lemma. Let $\mathcal{B} \subset \Omega_\Lambda^{ext}$ be a co-null Borel subset such that Φ is well-defined for every $X \in \mathcal{B}$. Let $F \subset \Lambda$ be such that for every $X \in \mathcal{B}$ the subsets

$(\Phi(X)f)_{f\in F}$ are pairwise disjoint. Choose moreover a symmetric neighbourhood of the identity $W \subset G$ such that $W^{-1}W \cap \Lambda^{2n+2} = \{e\}$. We know that for $X \in \mathcal{B}$ we have

$$F^{-1}\left(\Phi(X)\right)^{-1}\left(\Phi(X)\right)F \subset \Lambda^{2n+2},$$

so

$$W^{-1}W \cap F^{-1}\Phi(X)^{-1}\Phi(X)F = \{e\}.$$

Therefore,

$$FW^{-1}WF^{-1} \cap \Phi(X)^{-1}\Phi(X) = FF^{-1} \cap \Phi(X)^{-1}\Phi(X).$$

But the subsets $(\Phi(X)f)_{f\in F}$ are pairwise disjoint so

$$FW^{-1}WF^{-1} \cap \Phi(X)^{-1}\Phi(X) = \{e\}.$$

Recall that we defined $\mu_1 = (\mathcal{P}_{\Lambda^n})^*\nu_1$ so that, by Lemma 2.2.20 and the above discussion, we have

$$1 \geq \nu_1(U^{WF^{-1}}) = \mu_1(WF^{-1}) \geq \epsilon\mu_G(WF^{-1}).$$

But G is unimodular and $F^{-1} \subset \Lambda \subset \Lambda^{n+1}$, so

$$1 \geq \epsilon\mu_G(WF^{-1}) = \sum_{f\in F}\epsilon\mu(Wf^{-1}) = \epsilon\,|F|\,\mu_G(W).$$

We thus have

$$|F| \leq \frac{1}{\epsilon\mu_G(W)} < \infty.$$

So take one such F maximal for the inclusion. The subset F is finite and for all $\lambda \in \Lambda$ there are $f \in F$ and $X \in \mathcal{B}$ such that $\Phi(X)\lambda \cap \Phi(X)f \neq \emptyset$ i.e. $\lambda \in \Phi(X)^{-1}\Phi(X)f$. But $\Phi(X) = f(X)\Delta(X)^{-1}$ so we have the inclusion $\lambda \in \Delta(X)\Delta(X)^{-1}F$. □

Proof of Proposition 5.1.7. Let h_0, f and h be as in Lemma 5.1.8 and define the map

$$\begin{aligned}\Delta : \Omega_\Lambda^{ext} &\longrightarrow \Omega_{\Lambda^6}^{ext}\\ X &\longmapsto \Lambda^6 \cap T^{-1}(Vh_0L^{-1}Lh(X))\end{aligned}$$

which is well-defined ν_0-almost everywhere and such that for all $g \in G$ and ν_0-almost every $X \in \Omega_\Lambda^{ext}$ we have

$$f(gX)^{-1}gf(X) \in \Delta(X).$$

For any compact subset $K \subset G$ we have

$$\begin{aligned}\Delta^{-1}\left(U_K\right) &= \{X \in \Omega_\Lambda^{ext} | \Delta(X) \cap K = \emptyset\}\\ &= \{X \in \Omega_\Lambda^{ext} | T(\Lambda^6 \cap K) \cap h(X)^{-1}L^{-1}Lh_0^{-1}V^{-1} = \emptyset\}\\ &= \Omega_\Lambda^{ext} \setminus h^{-1}\left(L^{-1}Lh_0^{-1}V^{-1}\left(T(\Lambda^6 \cap W)\right)^{-1}\right).\end{aligned}$$

So Δ is Borel measurable according to Lemma 2.2.31. We may now conclude the proof of Proposition 5.1.7. Indeed, by Proposition 5.1.9 we have $F \subset \Lambda$ finite

$$\begin{aligned} T(\Lambda) &\subset T\left(\left(\bigcup_{X\in\mathcal{B}} \Delta(X)\Delta(X)^{-1}\right) F\right) \\ &\subset \left(\bigcup_{X\in\mathcal{B}} T\left(\Delta(X)\Delta(X)^{-1}\right)\right) T(F) \\ &\subset Vh_0(L^{-1}L)^2 h_0^{-1} V^{-1} T(F). \end{aligned}$$

□

Corollary 5.1.10. *Suppose that the Borel cocycle $T \circ \alpha_s$ is cohomologous to a cocycle that takes values in a compact subgroup $L \subset H$. Then $T(\Lambda)$ is a relatively compact subset of H.*

Proof. Take a compact neighbourhood of the identity $V \subset H$. There are $h_0 \in H$ and $K \subset H$ given by Proposition 5.1.7 such that $T(\Lambda) \subset Vh_0Lh_0^{-1}K$. But $Vh_0Lh_0^{-1}K$ is compact. □

5.1.4 Constant cocycles

We end this section with an application of Proposition 5.1.7. Along with Corollary 5.1.10 these will enable us to use the strength of Zimmer's cocycle superrigidity.

Let Λ, G, H, T and s be as in Subsection 5.1.3. Assume that the cocycle $T \circ \alpha_s$ is cohomologous to a constant cocycle associated to a continuous group homomorphism $\pi : G \to H$. We start by showing that in general a conjugate of π extends T modulo a two-sided error.

Proposition 5.1.11. *There is $h \in H$ such that for all $n \in \mathbb{N}$ and for all neighbourhoods of the identity $V \subset G \times H$ there exists a compact set $K_V \subset G \times H$ with*

$$\forall \lambda \in \Lambda^n,\ T(\lambda) \in V\pi^h(\lambda)K_V$$

where $\pi^h(g) := h\pi(g)h^{-1}$.

Proof. Let us first prove the case $n = 1$. Let $i : \langle\Lambda\rangle \to G$ denote the inclusion map and consider the diagonal map $i \times T : \langle\Lambda\rangle \to G \times H$. The cocycle $(i \times T) \circ \alpha_s : G \times \Omega_\Lambda^{ext} \to G \times H$ is cohomologous to the constant cocycle associated to the continuous group homomorphism $\mathrm{id} \times \pi : G \to G \times H$. By Proposition 5.1.7 there is $(g, h) \in G \times H$ such that for all neighbourhoods of the identity $V \subset G \times H$ there is a compact subset $K \subset G \times H$ such that

$$i \times T(\Lambda) \subset V(g,h)\Gamma_\pi(g^{-1}, h^{-1})K = V\Gamma_{\pi^{h\pi(g)^{-1}}}K.$$

So a quick computation shows that for all neighbourhoods of the identity $W \subset H$ there is a compact subset $K_W \subset H$ such that for all $\lambda \in \Lambda$ we have

$$T(\lambda) \in W\pi^{h\pi(g)^{-1}}(\lambda)K_W.$$

The case $n > 1$ follows readily noticing that there is a finite subset $F \subset \langle\Lambda\rangle$ such that $\Lambda^n \subset \Lambda F$. □

Following [58, Def. V.3.5] we will say that a continuous group homomorphism $\pi : G \to H$ *almost extends* T if $T(\gamma)\pi(\gamma)^{-1}$ is centralised by $\pi(G)$ for all $\gamma \in \langle\Lambda\rangle$. When H is abelian the error term from Proposition 5.1.11 becomes simpler. So we find that π^h almost extends T and the error term is bounded on Λ (in the spirit of the extension results from [62] and [53]). To deal with the non-commutative case we first draw another formula from Proposition 5.1.11:

Corollary 5.1.12. *With h given by Proposition 5.1.11. Choose $\gamma \in \langle\Lambda\rangle$ and let $\delta(\gamma) := T(\gamma)^{-1}\pi^h(\gamma)$ then for all neighbourhoods of the identity $V \subset H$ there is a compact subset K such that*

$$\forall g \in G, \delta(\gamma)\pi^h(g) \in V\pi^h(g)K.$$

Proof. Take $\lambda \in \Lambda$. By Proposition 5.1.11 we know that for all neighbourhoods of the identity $W \subset H$ there is a compact subset $K_W \subset H$ such that

$$\pi^h(\gamma\lambda) \in W^{-1}T(\gamma\lambda)K_W^{-1} \subset W^{-1}T(\gamma)W\pi^h(\lambda)K_W K_W^{-1}.$$

So for all neighbourhoods of the identity $W' \subset H$ there is a compact subset $K' \subset H$ such that for all $\lambda \in \Lambda$ we have

$$\delta(\gamma)\pi^h(\lambda) \in W'\pi^h(\lambda)K'.$$

One can now deduce Corollary 5.1.12 because ⋆-approximate lattices are bi-syndetic (Proposition 2.2.25). □

We now wish to utilise the "two-sided" error from Proposition 5.1.11 to prove that π^h extends T. As we will see below, Corollary 5.1.12 allows us to reformulate this in more geometric terms. We will use the non-commutativity to consider the two error terms in two different ways. Our intuition will be to consider multiplication on the left as a rotation and multiplication on the right as a translation.

To do so we will study what Corollary 5.1.12 asserts along a ray i.e. along a set of the form $\{g, g^2, g^3, \ldots\}$. The conclusion of Corollary 5.1.12 becomes that the element $\delta(\gamma)$ may displace (multiplication on the right) the elements of the ray $\{\pi^h(g), \pi^h(g^2), \ldots\}$ a lot - this is controlled here by the potentially large set K - but should not make the ray rotate too much (multiplication on the left) - which is controlled here by the small neighbourhood V. Since $\delta(\gamma)$ acts on the left of the ray, and, thus, rotates the ray, this should mean that $\delta(\gamma)$ itself is small.

It will be possible to make this intuition work when the target group H satisfies sufficiently strong geometric assumptions. Fortunately, we will be interested in semi-simple algebraic groups. These groups are known to come with associated CAT(0) space on which they act with compact stabilisers and co-compact image: the symmetric spaces when the field of definition is Archimedean and the Bruhat–Tits building otherwise. The CAT(0) property tells us that, loosely speaking, a space has non-negative curvature (see e.g. [22] for definitions and background material). In our situation, considering CAT(0) spaces will ensure that our intuition about rays, rotations and displacement discussed above can be made precise by means of the visual boundary.

Proposition 5.1.13. *Let X be a complete CAT(0) space and suppose that H acts continuously by isometries on X. Take $x_0 \in X$ and let Y be the closure of $\pi^h(G) \cdot x_0$ in $X \cup \partial X$ equipped with the cone topology. Then $\delta(\gamma)$ stabilises $Y \cap \partial X$ point-wise.*

Proof. Take $\xi \in Y \cap \partial X$ and let $(g_n)_{n \geq 0}$ be a sequence of elements of G such that $\pi^h(g_n) \cdot x_0 \to \xi$ as n goes to ∞. Choose a compact neighbourhood of the identity $V \subset H$ and a compact subset $K \subset H$ given by Corollary 5.1.12. For all $n \geq 0$ we can find $v_n \in V$ and $k_n \in H$ such that $\delta(\gamma)\pi^h(g_n) = v_n\pi^h(g_n)k_n$. Note that upon considering sub-sequences we can assume that $v_n \to v \in \overline{V}$. We now have

$$d_X(\delta(\gamma)\pi^h(g_n) \cdot x_0, v_n\pi^h(g_n) \cdot x_0) = d(x_0, k_n \cdot x_0)$$

where d_X denotes the distance on X. But $\pi^h(g_n) \cdot x_0 \to \xi \in \partial X$ and $k_n \in K$ compact so $\delta(\gamma) \cdot \xi = v \cdot \xi$. As this holds true for any V we find $\delta(\gamma) \cdot \xi = \xi$. □

Applying the above Proposition 5.1.13 to symmetric spaces and Bruhat–Tits buildings we are able to prove the main result of this section:

Proposition 5.1.14. *Let l be a local field and assume that H is the group of l-points of a connected almost simple algebraic l-group $\mathbb{H}$, and that π has unbounded Zariski-dense image. Then π^h almost extends T.*

Proof. Fix $\gamma \in \langle \Lambda \rangle$. Let X be: the symmetric space associated to $\mathbb{H}$ if l is Archimedean, or the Bruhat–Tits building associated to $\mathbb{H}$ otherwise. Since $\pi(G)$ is unbounded we know by Proposition 5.1.14 that $\delta(\gamma)$ fixes point-wise a non-empty closed subset $Y \subset \partial X$ stable under the action of $\pi^h(G)$. But the point-wise stabiliser of Y, being the intersection of parabolic subgroups, is a Zariski-closed proper subgroup of H and it is normalised by $\pi^h(G)$. It is thus central, and so $\delta(\gamma)$ is central as well. Since this is true for all $\gamma \in \langle \Lambda \rangle$ we find that π^h almost extends T. □

5.2 Property (T) for approximate subgroups

We will now give a tentative definition of Property (T) for approximate subgroups. Our main goal in doing so is to prove that $\star$-approximate lattices in Kazhdan groups generate a finitely generated subgroup.

5.2.1 Definition and first properties

Recall that given a locally compact group G, a unitary representation $(\pi, \mathcal{H})$ and a subset $Q \subset G$, a (Q, ϵ)-*invariant vector* for some $\epsilon > 0$ is a unit vector $\xi \in \mathcal{H}$ such that $||\pi(g)\xi - \xi|| < \epsilon$ for all $g \in Q$. If there are (Q, ϵ)-invariant vector for all compact subsets Q and all $\epsilon > 0$ we say that π *almost has invariant vectors.*

Definition 5.2.1 (Property (T) for approximate subgroups)**.** Let Λ be an approximate subgroup of some group and $\langle\Lambda\rangle$ the group it generates. We say that Λ has *property (T)* if there are a finite subset $Q \subset \Lambda$ and $\epsilon > 0$ such that for any unitary representation $(\pi, \mathcal{H}_\pi)$ of $\langle\Lambda\rangle$ that has (Q, ϵ)-invariant vectors there is a unit vector $\xi \in \mathcal{H}_\pi$ such that $\pi(\Lambda)(\xi)$ is totally bounded.

Equivalence between Definition 5.2.1 and the shorter definition given in the introduction is a clear consequence of the following lemma:

Lemma 5.2.2. *Let Λ be an approximate subgroup in some group, let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of the group $\langle\Lambda\rangle$ it generates and let $\xi \subset \mathcal{H}_\pi$ be any vector. The following are equivalent:*

1. *for all $\delta > 0$ there is an approximate subgroup $\Lambda(\delta, \xi)$ contained in and commensurable with Λ^2 such that for all $\lambda \in \Lambda(\delta, \xi)$ we have $||\pi(\lambda)(\xi) - \xi|| < \delta$;*

2. *the subset $\pi(\Lambda)(\xi)$ is totally bounded;*

3. *there is a sub-representation $(\sigma, \mathcal{H}_\sigma)$ with $\xi \in \mathcal{H}_\sigma$, and such that $\sigma(\Lambda)$ is totally bounded in the strong topology.*

Proof. Let us start with (1) $\Rightarrow$ (2). Take $\delta > 0$, the subset $\pi(\Lambda(\delta, \xi))(\xi)$ is contained in $B_{\mathcal{H}_\pi}(\xi, \delta)$ the ball of centre ξ and radius δ. But there is a finite subset $F_\delta \subset \Lambda^3$ such that $\Lambda \subset F_\delta\Lambda(\delta, \xi)$ so $\pi(\Lambda)(\xi) \subset \pi(F_\delta)(B_{\mathcal{H}_\pi}(\xi, \delta)) \subset \bigcup_{f \in F_\delta} B_{\mathcal{H}_\pi}(\pi(f)(\xi), \delta)$. Therefore, $\pi(\Lambda)(\xi)$ is covered by finitely many balls of radius δ. The set $\pi(\Lambda)(\xi)$ is thus totally bounded.

Notice now that for any $\gamma \in \langle\Lambda\rangle$ there is $F_\gamma \subset \langle\Lambda\rangle$ finite such that $\Lambda\gamma \subset F_\gamma\Lambda$. So $\pi(\Lambda)(\pi(\gamma)(\xi))$ is relatively compact as well. Let $(\sigma, \mathcal{H}_\sigma)$ denote the sub-representation of π generated by ξ i.e. $\mathcal{H}_\sigma$ is the closure of the linear span of $\pi(\langle\Lambda\rangle)(\xi)$. We know that $\pi(\Lambda)(\xi)$ contains a countable dense subset (it is totally bounded in a metric space), so we readily check that $\mathcal{H}_\sigma$ contains a countable dense subset $(\xi_n)_{n\geq 0}$ such that $\sigma(\Lambda)(\xi_n)$ is relatively compact for all $n \geq 0$. By the Arzela-Ascoli theorem, since $\sigma(\Lambda)$ is obviously a set of equicontinuous operators, we know that $\sigma(\Lambda)$ is relatively compact in the point-wise topology. So (2) $\Rightarrow$ (3).

Finally (3) $\Rightarrow$ (1): choose $\delta > 0$ and let $V_\delta(\xi)$ be the open subset of $U(\mathcal{H}_\sigma)$ defined by $\{T \in U(\mathcal{H}_\sigma) \mid ||T(\xi) - \xi|| < \delta\}$. Then $V_\delta(\xi)$ is symmetric and $V_\delta(\xi)^2 \subset V_{2\delta}(\xi)$. But $\sigma(\Lambda)$ is covered by finitely many left-translates of $V_\delta(\xi)$. Hence, $\Lambda^2 \cap V_{2\delta}(\xi)$ is an approximate subgroup commensurable with Λ (Lemma 2.1.8). So $\Lambda(2\delta, \xi) := \Lambda^2 \cap V_{2\delta}(\xi)$ works. □

Remark 5.2.1. As the proof shows, or since compact metric subsets are separable, we can always find σ as in (2) such that $\mathcal{H}_\sigma$ is separable.

5.2.2 Links with relative property (T)

For illustration purposes, we relate Definition 5.2.1 to the notion of property (T) for approximate subgroups due to Björklund–Hartnick i.e. relative property (T) of the pair $(\langle\Lambda\rangle, \Lambda)$ (see [11, 30]). Note that we assume here that Λ is countable, but we will later see that this is always true (Proposition 5.2.5).

Proposition 5.2.3. *Let Λ be a countable approximate subgroup of some group. We have:*

1. *if Λ has property (T), then Λ has property (T) in the sense of Björklund–Hartnick i.e. $(\langle\Lambda\rangle, \Lambda)$ has the relative property (T);*

2. *conversely, if Λ generates a lattice in an algebraic group over a local field and $(\langle\Lambda\rangle, \Lambda)$ has the relative property (T), then Λ has property (T);*

3. *if G is a locally compact group, N is a closed normal subgroup, (G, N) has relative property (T), and $\Gamma \subset G$ is a lattice, then there is a compact subset $K \subset G$ such that for any compact neighbourhood of the identity W containing K the approximate subgroup $\Gamma \cap NW$ has property (T).*

The proof follows from the results found in [30, 5]. We do not include a proof since we do not use these facts in the following. Note moreover that it would be interesting to know if a general converse to (1) is true.

5.2.3 Heredity

We now turn to the heart of Section 5.2: we will prove a certain heredity result about property (T) for $\star$-approximate lattices.

Proposition 5.2.4. *Let Λ be a $\star$-approximate lattice in a locally compact second countable group G. The following are equivalent:*

1. *G has property (T);*

2. *Λ has property (T) for approximate subgroups.*

Proof. We begin with (2) $\Rightarrow$ (1) as the proof will use the ideas of Subsection 5.2.1. Let (Q,ϵ) be a Kazhdan pair for Λ and let $(\pi,\mathcal{H}_\pi)$ be a unitary representation of G that admits (Q,ϵ)-invariant vectors. Let $\xi\in\mathcal{H}$ and $(\Lambda(\xi,\delta))_{\delta\geq 0}$ be as in Definition 5.2.1 and take $\delta>0$. Take $V\subset G$ open such that $\pi(V)\subset V_{\delta/3}:=\{T\in U(\mathcal{H}_\sigma)\,|\,||T(\xi)-\xi||<\delta/3\}$. By Proposition 2.2.25 a finite number of left-translates of the subset $X(\delta,\xi):=V\Lambda(\delta/3,\xi)V$ cover G. But then for all $g\in X(\delta,\xi)$ we have $||\pi(g)\xi-\xi||\leq\delta$. So there is a sub-representation $(\sigma,\mathcal{H}_\sigma)$ such that $\sigma(G)$ is relatively compact by Lemma 5.2.2. According to the Peter–Weyl theorem σ, and hence π, has a finite dimensional sub-representation. But that means that G has property (T) according to the characterisation of property (T) from [5, Th.2.12.4].

Conversely, let ν be a proper G-invariant ergodic measure on $\Omega^{ext}_{\Lambda,G}$, let (Q,ϵ) be a Kazhdan pair for G and let $(\pi,\mathcal{H}_\pi)$ be a unitary representation of $\langle\Lambda\rangle$. Note that since G is σ-compact ([5, Th.1.3.1]), $\langle\Lambda\rangle$ is countable. So we can assume that $\mathcal{H}_\pi$ is separable. Take a Borel section $s:\Omega^{ext}_{\Lambda,G}\to G$ as in Definition 5.1.1 and let $\alpha_s: G\times\Omega^{ext}_{\Lambda,G}\to\langle\Lambda\rangle$ be the Borel cocycle given by Lemma 5.1.2. Thanks to the cocycle identity $\alpha(gh,X)=\alpha(g,hX)\alpha(h,X)$ we can define a unitary representation σ of G on $\mathcal{H}_\sigma:=L^2(\Omega^{ext}_{\Lambda,G},\mathcal{H}_\pi;\nu)$ by $\sigma(g)(f): X\to\pi(\alpha(g^{-1},X)^{-1})(f(g^{-1}X))$ (see e.g. [82] for this and more). Arguing exactly as in [82, Th. 9.1.1] we show that there are a finite subset $Q'\subset\Lambda$ and $\epsilon'>0$ such that if $(\pi,\mathcal{H}_\pi)$ has a (Q',ϵ')-invariant vector, then $(\sigma,\mathcal{H}_\sigma)$ has a (Q,ϵ)-invariant vector. So suppose from now on that $(\pi,\mathcal{H}_\pi)$ has a (Q',ϵ')-invariant vector. Since (Q,ϵ) is a Kazhdan pair for G there is $\phi\in L^2(\Omega^{ext}_{\Lambda,G},\mathcal{H}_\pi;\nu)$ with norm 1 such that $\sigma(g)(\phi)=\phi$ for all $g\in G$. Therefore, for all $g\in G$ and ν-almost all $X\in\Omega^{ext}_{\Lambda,G}$ we have

$$\pi(\alpha(g^{-1},X)^{-1})(\phi(g^{-1}X))=\phi(X).$$

Note first that by ergodicity of the action of G on $\Omega^{ext}_{\Lambda,G}$ and since ϕ has norm 1 we have that $\phi(X)$ has norm 1 in $\mathcal{H}_\pi$ for ν-almost all $X\in\Omega^{ext}_{\Lambda,G}$. We will now proceed as in Section 5.1 to produce a unit vector $\xi\in\mathcal{H}_\pi$ and a sequence of approximate subgroups $(\Lambda(\xi,\delta))_{\delta>0}$ commensurable with Λ such that $\pi(\Lambda(\xi,\delta))(\xi)\subset B_{\mathcal{H}_\pi}(\xi,\delta)$ where $B_{\mathcal{H}_\pi}(\xi,\delta)$ is the ball of centre ξ and radius δ in $\mathcal{H}_\pi$. By Lemma 5.1.6 and since α_s takes values in Λ^2 there is $\xi\in\mathcal{H}_\pi$ with norm 1 such that for all $\delta>0$ and ν-almost all $X\in\Omega^{ext}_{\Lambda,G}$ we have $\phi(X)\in\pi(\Lambda^2)(B_{\mathcal{H}_\pi}(\xi,\delta))$. As in Lemma 5.1.8 we can build $f_\delta:\Omega^{ext}_{\Lambda,G}\to G$ and α_{f_δ} defined for all $g\in G$ and almost all $X\in\Omega^{ext}_{\Lambda,G}$ by $\alpha_{f_\delta}(g,X):=f_\delta(gX)^{-1}gf_\delta(X)$ such that f_δ is Borel, α_{f_δ} takes values in Λ^6 and $\pi\circ\alpha_{f_\delta}$ takes values in $V_\delta(\xi):=\{T\in U(\mathcal{H}_\pi)\,|\,||T(\xi)-\xi||<\delta\}$. According to Proposition 5.1.9 applied to α_{f_δ} and $\Delta: X\mapsto V_\delta(\xi)$ we find that there is $F\subset\Lambda$ finite such that $\pi(\Lambda)\subset\pi(F)V_{2\delta}(\xi)$. So $\Lambda(2\delta,\xi):=\Lambda^2\cap\pi^{-1}(V_{2\delta}(\xi))$ and ξ are as in Definition 5.2.1. □

5.2.4 Finite generation

It is well-known that property (T) for groups implies finite generation (see e.g. [5]). We will show in the same spirit that:

Proposition 5.2.5. *If an approximate subgroup Λ of some group has property (T) then the subgroup $\langle\Lambda\rangle$ it generates is finitely generated. More precisely, if (Q,ϵ) is any Kazhdan pair, then Λ is covered by finitely many left translates of the subgroup Δ generated by Q.*

Proof. Let (Q,ϵ) be a Kazhdan pair and let Δ denote the subgroup generated by Q. Then the indicator function $\mathbb{1}_\Delta$ is a (Q,ϵ)-invariant vector of the quasi-regular representation $(\pi, L^2(\langle\Lambda\rangle/\Delta))$. So we can find $\phi \in L^2(\langle\Lambda\rangle/\Delta)$ and $(\Lambda(\delta,\phi))_{\delta>0}$ a family of approximate subgroups contained in Λ^2 and commensurable with Λ such that $||\pi(\lambda)(\phi)-\phi|| < \delta$ for all $\delta > 0$ and $\lambda \in \Lambda(\delta,\phi)$. Now let $p : \langle\Lambda\rangle \to \langle\Lambda\rangle/\Delta$ denote the natural projection. Take $\gamma \in \langle\Lambda\rangle$ such that $\phi(p(\gamma)) = \alpha > 0$. So for all $\lambda \in \Lambda(\alpha/2,\phi)$ we have $|\phi(p(\lambda^{-1}\gamma)) - \phi(p(\gamma))| \leq ||\pi(\lambda)(\phi)-\phi|| < \alpha/2$, meaning $\lambda^{-1}\gamma \in \phi^{-1}([\alpha/2;+\infty))$. Since $\phi^{-1}([\alpha/2;+\infty))$ is finite, we can find a finite set F of representatives of $\phi^{-1}([\alpha/2;+\infty))$ in $\langle\Lambda\rangle$. Then $\lambda^{-1}\gamma\Delta \cap F\Delta \neq \emptyset$ and $\Lambda(\alpha/2,\phi)$ is contained in $F\Delta\gamma$. But there is a finite subset $F' \subset \langle\Lambda\rangle$ such that $\Lambda \subset F'\Lambda(1/3,\phi)\gamma^{-1} \subset F'F\Delta$. Since $F' \cup F \cup \Delta \subset \langle\Lambda\rangle$, $F' \cup F \cup \Delta$ generates $\langle\Lambda\rangle$. But Δ is finitely generated. So $\langle\Lambda\rangle$ is finitely generated. □

As a corollary, we prove Theorem 1.1.30:

Proof of Theorem 1.1.30. According to Proposition 5.2.4 we know that Λ has property (T). So $\langle\Lambda\rangle$ is finitely generated as a consequence of Proposition 5.2.5. □

5.3 Superrigidity and arithmeticity

From now on let A be a finite set, let $(k_\alpha)_{\alpha\in A}$ be a family of local fields of characteristic 0 and let $(\mathbb{G}_\alpha)_{\alpha\in A}$ be a family of connected absolutely simple k_α-groups with k_α-rank ≥ 2. For any subset $B \subset A$ set $G_B := \prod_{\alpha\in B}\mathbb{G}_\alpha(k_\alpha)$ and let $p_B : G_A \to G_B$ denote the natural map. Moreover, for all $\alpha \in A$ let G_α denote $\mathbb{G}_\alpha(k_\alpha)$ and let p_α denote $p_{\{\alpha\}}$.

5.3.1 Superrigidity in bounded dimension

For any local field l of characteristic 0 we let $p(l)$ denote the unique element of $\mathcal{P} \cup \{\infty\}$ (the set of prime numbers together with $\{\infty\}$) such that l is a finite extension of the $p(l)$-adic field $\mathbb{Q}_{p(l)}$ (where $\mathbb{Q}_\infty := \mathbb{R}$).

Proposition 5.3.1. *Let Λ be a $\star$-approximate lattice in G_A and $T : \langle\Lambda\rangle \to \mathbb{H}(l)$ be a group homomorphism towards the l-points of an affine l-group $\mathbb{H}$. We have:*

1. *if* $p(k_\alpha) \neq p(l)$ *for all* $\alpha \in A$*, then* $T(\Lambda)$ *is a relatively compact subset of* $\mathbb{H}(l)$*;*

2. *if for every* $\alpha \in A$ *such that* $p(k_\alpha) = p(l)$*,* $\mathbb{G}_\alpha$ *is absolutely simple, we know that* $\mathbb{H}$ *is connected,* $\dim(\mathbb{G}_\alpha) \geq \dim(\mathbb{H})$ *and* $T(\Lambda)$ *is not relatively compact in* $\mathbb{H}(l)$*, then there is a continuous group homomorphism* $\pi : G_A \to \mathbb{H}(l)$ *that almost extends* T*.*

Proof. Let ν be an ergodic proper G-invariant Borel probability measure on Ω_Λ^{ext}. Let s be a Borel section of Ω_Λ^{ext} and consider the Borel cocycle $T \circ \alpha_s : G_A \times \Omega_\Lambda^{ext} \to \mathbb{H}(l)$. Let $\mathbb{L}$ be the algebraic hull of $T \circ \alpha_s$ ([82, Prop. 9.2.1]) and let $\beta : G_A \times \Omega_\Lambda^{ext} \to \mathbb{H}(l)$ be a Borel cocycle cohomologous to $T \circ \alpha_s$ that takes values in $\mathbb{L}(l)$. Let $\mathbb{F} \ltimes \mathbb{U}$ be a Levi decomposition of $\mathbb{L}$ with $\mathbb{F}$ reductive and $\mathbb{U}$ unipotent. Let $p : \mathbb{L}(l) \to \mathbb{F}(l)$ be the natural map. Then the algebraic hull of $p \circ \beta : G_A \times \Omega_\Lambda^{ext} \to \mathbb{F}(l)$ is $\mathbb{F}$. Indeed, otherwise there would exist a proper l-subgroup $\mathbb{F}'$ and a Borel map $\psi : \Omega_\Lambda^{ext} \to \mathbb{F}(l)$ such that for all $g \in G_A$ and ν-almost every $X \in \Omega_\Lambda^{ext}$ we have $\psi(gX)\,(p \circ \beta)\,(g, X)\psi(X)^{-1} \in \mathbb{F}'(l)$. Taking a Borel map $\tilde{\psi} : \Omega_\Lambda^{ext} \to \mathbb{L}(l)$ such that $p \circ \tilde{\psi} = \psi$ we would have for all $g \in G_A$ and ν-almost every $X \in \Omega_\Lambda^{ext}$ that

$$\tilde{\psi}(gX)\beta(g, X)\tilde{\psi}(X)^{-1} \in \left(\mathbb{F}' \ltimes \mathbb{U}\right)(l) \subsetneq \mathbb{L}(l).$$

A contradiction. Thus, the cocycle $p \circ \beta$ has a reductive algebraic hull. By [36, Th. 3.16] there are a continuous group homomorphism $\pi : G_A \to \mathbb{F}(l)$ and a cocycle $z : G_A \times \Omega_\Lambda^{ext} \to \mathbb{F}(l)$ that takes values in a compact subgroup centralising $\pi(G_A)$ such that $p \circ \beta$ is cohomologous to the cocycle defined ν-almost everywhere by $g, X \mapsto \pi(g)z(g, X)$.

Suppose first that $p(k_\alpha) \neq p(l)$ for all $\alpha \in A$. Then π is trivial according to [58, I.2.6.1, (i)]. So $p \circ \beta$ is cohomologous to a cocycle that takes values in a compact subgroup of $\mathbb{F}(l)$. Reasoning as above we see that β is cohomologous to a cocycle that takes values in an amenable subgroup. By [82, Th. 9.1.1] we thus have that β is cohomologous to a cocycle that takes values in a compact subgroup of $\mathbb{H}(l)$. Whence $T \circ \alpha_s$ is cohomologous to a cocycle that takes values in a compact subgroup of $\mathbb{H}(l)$. By Corollary 5.1.10 the subset $T(\Lambda)$ is relatively compact in $\mathbb{H}(l)$.

Suppose now that the assumptions of (2) are satisfied. If π is trivial, then as above we conclude that $T(\Lambda)$ is relatively compact in $\mathbb{H}(l)$. Otherwise according to [58, I.2.6.2] the Zariski closure of $\pi(G_A)$ is semi-simple. Moreover, one can see applying [58, I.2.6.1,(iii)] again that $\dim(\mathbb{G}) \leq \dim(\mathbb{F})$ for some $\alpha \in A$ with $p(k_\alpha) = p(l)$. As a consequence, we have the equality $\dim(\mathbb{F}) = \dim(\mathbb{H})$ and this yields $\mathbb{H} = \mathbb{L} = \mathbb{F}$ since $\mathbb{H}$ is connected. So $p = \mathrm{id}$ and $p \circ \beta = \beta$. So π almost extends T according to Proposition 5.1.14. □

5.3.2 Compact finiteness

We prove now a general finiteness property of compact images of approximate groups.

Proposition 5.3.2. *Let Λ be an approximate subgroup of some group. Take a family $\mathcal{F}$ of group homomorphisms $\tau : \langle\Lambda\rangle \to H_\tau$ with $\tau(\langle\Lambda\rangle)$ dense and $\overline{\tau(\Lambda)}$ compact with non-empty interior. Suppose moreover that H_τ contains a non-discrete non-compact topologically simple group S_τ contained in all normal subgroups. Then there is $X \subset \mathcal{F}$ finite such that:*

1. *for any group homomorphism $\phi : \langle\Lambda\rangle \to G \in \mathcal{F}$ there is $\tau : \langle\Lambda\rangle \to H \in X$ and a continuous group homomorphism $\pi : H \to G$ such that $\pi \circ \tau = \phi$;*

2. *the intersection $\overline{\tau_X(\Lambda^2)} \cap \prod_{\tau\in X} S_\tau$ is open in $\prod_{\tau\in X} S_\tau$ where $\tau_X := \prod_{\tau\in X} \tau$.*

Proof. For any $X \subset \mathcal{F}$ write H_X the target of τ_X and $p_X : H_{\mathcal{F}} \to H_X$ the natural projection. Since $\overline{\tau_{\mathcal{F}}(\Lambda)}$ is compact, there is a topology on the subgroup L it generates finer than the induced topology, with L locally compact, and $\overline{\tau_{\mathcal{F}}(\Lambda^2)} =: V$ a neighbourhood of the identity (Theorem 4.2.1). For every $\tau \in \mathcal{F}$ one sees that L projects surjectively to H_τ. Write $N_\tau \leq L$ the kernel of $L \to H_\tau$. Since $\bigcap_{\tau\in\mathcal{F}} N_\tau = \{e\}$ and V is compact, there is $X \subset \mathcal{F}$ finite such that $\bigcap_{\tau\in X} N_\tau \cap V = \{e\}$. So $\bigcap_{\tau\in X} N_\tau$ is a discrete subgroup of a compactly generated locally compact group. The projection of $\bigcap_{\tau\in X} N_\tau$ to any H_σ is thus a normal countable subgroup, hence trivial. So $\bigcap_{\tau\in X} N_\tau = \{e\}$. Take one such $X \subset \mathcal{F}$ of minimal cardinality and write $L_X := p_X(L)$. Note that $L_X \simeq L$. By minimality we know that $L_X \cap H_\tau$ is non-trivial for all $\tau \in X$. So it contains S_τ since L projects surjectively to H_τ. In particular $\prod_{\tau\in X} S_\tau \subset L_X$ so X satisfies (2). Take $\sigma \notin X$, by Goursat's lemma $p_{X\cup\{\sigma\}}(L)$ must be the graph of a continuous group homomorphism $\phi : L_X \to H_\sigma$. But there must be exactly one $\tau \in X$ such that $\phi(S_\tau)$ is non-trivial. Thus, ϕ yields a continuous group isomorphism $\psi : H_\tau \to H_\sigma$ such that $\sigma = \psi \circ \tau$. □

Note that when H is a group of rational points of a simple algebraic group over a local field of characteristic 0, $\tau : \langle\Lambda\rangle \to H$ satisfies the hypothesis as soon as $\tau(\Lambda)$ is relatively compact and $\tau(\langle\Lambda\rangle)$ is dense in a finite index open subgroup. This can be easily seen by considering the Lie algebra of $\overline{\tau(\Lambda)}$ (Theorem 4.2.1).

5.3.3 Arithmeticity

We will proceed by induction on the cardinality of A. To do so we will need the following lemma about reduction of $\star$-approximate lattices:

Lemma 5.3.3. *Let Λ be a $\star$-approximate lattice in G_A:*

1. *take $\gamma \in \langle\Lambda\rangle$ and $B_\gamma := \{\alpha \in A : p_\alpha(\gamma) \neq e\}$, then $p_{B_\gamma}(\Lambda)$ is a $\star$-approximate lattice;*

2. *take $B \subset A$, if $p_B(\Lambda)$ is a $\star$-approximate lattice, then $p_{A\setminus B}(\Lambda)$ is a $\star$-approximate lattice.*

Proof. First of all, for $g \in G_A$ we have $g\gamma g^{-1} = p_{B_\gamma}(g)\gamma p_{B_\gamma}(g)^{-1}$. Let N be the normal subgroup of $\langle\Lambda\rangle$ generated by γ. Since $\langle\Lambda\rangle$ has property (S) (Proposition 4.4.6), the Borel density theorem (see Proposition 4.4.7) yields $C_{G_A}(N) = G_{A\setminus B}$ where $C_{G_A}(\cdot)$ is the centraliser in G_A. There is thus $F \subset N$ finite such that $C_{G_A}(F) = G_{A\setminus B}$. Take $n \geq 0$ such that $F \subset \Lambda^n$, there is a neighbourhood of the identity $U \subset G_{B_\gamma}$ such that $\Lambda^{n+4} \cap UfU^{-1} = \{f\}$ for all $f \in F$. So $\lambda f \lambda^{-1} = f$ if $f \in F$ and $\lambda \in \Lambda^2 \cap p_{B_\gamma}^{-1}(U)$. Hence, $p_{B_\gamma}(\Lambda^2) \cap U = \{e\}$ so $p_{B_\gamma}(\Lambda)$ is uniformly discrete. Now if $p_B(\Lambda)$ is locally finite, then $\Lambda^2 \cap G_{A\setminus B}$ is a $\star$-approximate lattice in $G_{A\setminus B}$ by Proposition 4.4.1. We can thus find $\gamma \in \Lambda^2 \cap G_{A\setminus B}$ such that $B_\gamma = A \setminus B$ (e.g. Proposition 2.2.25). So (2) follows from (1) and Proposition 4.4.1. □

Proof of Theorem 1.1.25. The crux of the proof is to establish the following claim:

Claim 5.3.1. *There is H a finite product of groups of rational points of simple algebraic groups over local fields of characteristic* 0 *and a group homomorphism $\tau : \langle\Lambda\rangle \to H$ such that $\tau(\Lambda)$ is relatively compact and topologically generates an open finite index subgroup, and the image of the diagonal map $\langle\Lambda\rangle \to G_A \times H$ is discrete.*

Let us first explain why the claim is sufficient. Since Λ is a $\star$-approximate lattice and is contained in the projection to G of $\mathrm{id} \times \tau(\langle\Lambda\rangle) \cap G \times W_0$ for some compact neighbourhood of the identity $W_0 \subset H$, the subgroup $\Gamma := \mathrm{id} \times \tau(\langle\Lambda\rangle)$ is a lattice (Lemma 2.3.9 and Proposition 2.3.6). So Λ is contained in and commensurable with a model set coming from the cut-and-project (G, H, Γ).

We will prove the claim by induction on $|A|$. If $|A| = 0$ the result is obvious. Suppose now that $|A| \geq 1$. If there is a proper non-trivial subset $B \subset A$ such that $p_B(\Lambda)$ is a $\star$-approximate lattice, then $p_{A\setminus B}(\Lambda)$ is a $\star$-approximate lattice as well by Lemma 5.3.3. Applying Claim 5.3.1 to both $p_B(\Lambda)$ and $p_{A\setminus B}(\Lambda)$ gives H and τ as in Claim 5.3.1 except that $\tau(\Lambda)$ might not topologically generate a finite index open subgroup of $H := \prod_{i\in I} H_i$ where the H_i's are the simple factors of H. However, at the very least the projection of $\tau(\Lambda)$ to any simple factor H_i topologically generates an open finite index subgroup of H_i according to the induction hypothesis. So take $J \subset I$ and p_J as in part (2) of Proposition 5.3.2. Then $p_J \circ \tau$ satisfies all the conditions of Claim 5.3.1.

Suppose otherwise that there is no such B. Note that if $\langle\Lambda\rangle \subset G_A$ is discrete, then the claim is immediate. We will suppose from now on that it is not discrete. According to part (1) of Lemma 5.3.3 the projection of $\langle\Lambda\rangle$ to any factor is injective. And it is Zariski-dense according to Proposition 4.4.6 and the Borel density theorem (Proposition 4.4.7). Choose α with $\dim(G_\alpha)$ minimal. According to Theorem 1.1.30 the group $\langle\Lambda\rangle$ is finitely generated. Therefore the set $\{\mathrm{Tr}\,\mathrm{Ad}\, p_\alpha\gamma : \gamma \in \langle\Lambda\rangle\}$ generates a finitely generated field K. According to [76] we can identity G_α with the k_α-points of a K-simple group $\mathbb{H}$ such that $p_\alpha(\langle\Lambda\rangle) \subset \mathbb{H}(K)$. Let $\mathcal{F}$ be the family $\{\widehat{\sigma} : \langle\Lambda\rangle \to \mathbb{H}^\sigma(k)\}_\sigma$ where $\sigma : K \to k$ runs through the isomorphism classes of field embeddings with k

local, $\sigma(K)$ dense and such that the natural map $\widehat{\sigma} : \langle\Lambda\rangle \to \mathbb{H}^\sigma(k)$ has non-compact image but sends Λ to a relatively compact set. Note that if $\widehat{\sigma}(\langle\Lambda\rangle)$ is discrete, then $\widehat{\sigma}(\Lambda)$ must be finite. So there is $\alpha \in A$ such that $p_\alpha(\Lambda)$ is finite - contradicting Proposition 2.2.25. Now $\mathbb{G}_\alpha$ is assumed absolutely simple, so a classical argument shows that $\widehat{\sigma}(\langle\Lambda\rangle)$ must be dense in a finite index open subgroup. Hence, we can apply Proposition 5.3.2 and we obtain a finite subset $\mathcal{F}' \subset \mathcal{F}$. Write $H := \prod_{\langle\Lambda\rangle\to\mathbb{H}^\sigma(k)\in\mathcal{F}'} \mathbb{H}^\sigma(k)$ and let $\tau : \langle\Lambda\rangle \to H$ be the natural map. Suppose that $\mathrm{id} \times\tau(\langle\Lambda\rangle) \subset G_A \times H$ is not discrete. There is an infinite subset $X \subset \langle\Lambda\rangle$ such that $\mathrm{id} \times\tau(X)$ is bounded in $G_A \times H$. But by [19, Lem. 2.1] there is $\sigma : K \to k$ with k local, $\sigma(K)$ dense and $\widehat{\sigma}(X)$ unbounded. Now, either $\widehat{\sigma}(\Lambda)$ is unbounded and, by Proposition 5.3.1, $\widehat{\sigma}$ extends to a continuous group homomorphism $G_A \to H^\sigma(k)$, or $\widehat{\sigma}(\Lambda)$ is bounded and, by Proposition 5.3.2, we can find a continuous group homomorphism $\pi : H \to \mathbb{H}^\sigma(k)$ such that $(\widehat{\sigma})_{|\langle\Lambda\rangle} = \pi \circ \tau$. But in both cases this means that $\widehat{\sigma}(X)$ is relatively compact. A contradiction. □

Finally, we deduce *a posteriori* a superrigidity theorem without assumptions on the dimension of the target group.

Proof of Theorem 1.1.27. Let $T : \langle\Lambda\rangle \to \mathbb{L}(k)$. Take B finite, characteristic 0 local fields $(k_\beta)_{\beta\in B}$, simple k_β-groups $(\mathbb{H}_\beta)_{\beta\in B}$, a lattice $\Gamma \subset G_A \times \prod_{\beta\in B} \mathbb{H}_\beta(k_\beta)$ given by Theorem 1.1.25. Identify $\langle\Lambda\rangle$ with Γ. According to Margulis' superrigidity there is a continuous group homomorphism $\pi : G_A \times \prod_{\beta\in B} \mathbb{H}_\beta(k_\beta) \to \mathbb{L}(k)$ that extends T. Moreover, we know that π factors though the natural projection to one of the simple factors of $G_A \times \prod_{\beta\in B} \mathbb{H}_\beta(k_\beta)$. But $\pi(\Lambda) = T(\Lambda)$ is unbounded, so π factors through the natural projection $p_\alpha : G_A\times\prod_{\beta\in B} \mathbb{H}_\beta(k_\beta) \to \mathbb{G}_\alpha(k_\alpha)$ for some $\alpha \in A$. In particular, π factors through the natural projection $p_A : G_A \times \prod_{\beta\in B} \mathbb{H}_\beta(k_\beta) \to G_A$. We thus have a continuous group homomorphism $G_A \to \mathbb{L}(k)$ that extends T. □

Bibliography

[1] Louis Auslander. On radicals of discrete subgroups of Lie groups. *Amer. J. Math.*, 85:145–150, 1963.

[2] Louis Auslander. On a Problem of Philip Hall. *Annals of Mathematics*, 86(1):112–116, 1967.

[3] Françoise Axel and Denis Gratias, editors. *Beyond quasicrystals.* Springer-Verlag, Berlin; Les Éditions de Physique, Les Ulis, 1995. Papers from the Winter School held in Les Houches, March 7–18, 1994.

[4] Michael Baake and Uwe Grimm. *Aperiodic order. Vol. 1*, volume 149 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2013. A mathematical invitation, With a foreword by Roger Penrose.

[5] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs.* Cambridge University Press, Cambridge, 2008.

[6] Yves Benoist and Nicolas de Saxcé. A spectral gap theorem in simple Lie groups. *Invent. Math.*, 205(2):337–361, 2016.

[7] Yves Benoist and Jean-François Quint. Lattices in S-adic Lie groups. *J. Lie Theory*, 24(1):179–197, 2014.

[8] Mladen Bestvina and Koji Fujiwara. Bounded cohomology of subgroups of mapping class groups. *Geom. Topol.*, 6:69–89, 2002.

[9] Michael Björklund and Tobias Hartnick. Approximate lattices. *Duke Math. J.*, 167(15):2903–2964, 2018.

[10] Michael Björklund and Tobias Hartnick. Spectral theory of approximate lattices in nilpotent Lie groups. *arXiv preprint arXiv:1811.06563*, 2018.

[11] Michael Björklund and Tobias Hartnick. Analytic properties of approximate lattices. *Ann. Inst. Fourier (Grenoble)*, 70(5):1903–1950, 2020.

[12] Michael Björklund, Tobias Hartnick, and Felix Pogorzelski. Aperiodic order and spherical diffraction, I: auto-correlation of regular model sets. *Proc. Lond. Math. Soc. (3)*, 116(4):957–996, 2018.

[13] Michael Björklund, Tobias Hartnick, and Thierry Stulemeijer. Borel density for approximate lattices. *Forum Math. Sigma*, 7:Paper No. e40, 27, 2019.

[14] Armand Borel. Density properties for certain subgroups of semi-simple groups without compact components. *Annals of Mathematics*, 72(1):179–188, 1960.

[15] N. Bourbaki. *Éléments de mathématique. Topologie générale. Chapitres 1 à 4.* Hermann, Paris, 1971.

[16] Nicolas Bourbaki. *Lie groups and Lie algebras / Nicolas Bourbaki. Part I, chapters 1-3.* Actualites scientifiques et industrielles. H ; Addison-Wesley, Paris : Reading, Mass, 1975.

[17] Nicolas Bourbaki. *Espaces vectoriels topologiques. Chapitres 1 à 5.* Masson, Paris, new edition, 1981. Éléments de mathématique. [Elements of mathematics].

[18] Jean Bourgain and Alex Gamburd. Uniform expansion bounds for Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$. *Ann. of Math. (2)*, 167(2):625–642, 2008.

[19] E. Breuillard and T. Gelander. A topological Tits alternative. *Ann. of Math. (2)*, 166(2):427–474, 2007.

[20] Emmanuel Breuillard, Ben Green, and Terence Tao. Approximate subgroups of linear groups. *Geom. Funct. Anal.*, 21(4):774–819, 2011.

[21] Emmanuel Breuillard, Ben Green, and Terence Tao. The structure of approximate groups. *Publ. Math. Inst. Hautes Études Sci.*, 116:115–221, 2012.

[22] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1999.

[23] Robert Brooks. Some remarks on bounded cohomology. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference*, volume 97, pages 53–63, 1981.

[24] Pietro Kreitlon Carolino. *The Structure of Locally Compact Approximate Groups.* ProQuest LLC, Ann Arbor, MI, 2015. Thesis (Ph.D.)–University of California, Los Angeles.

[25] Ionut Chifan and Adrian Ioana. On relative property (T) and Haagerup's property. *Trans. Amer. Math. Soc.*, 363(12):6407–6420, 2011.

[26] Matthew Cordes, Tobias Hartnick, and Vera Tonić. Foundations of geometric approximate group theory. *arXiv preprint arXiv:2012.15303*, 2020.

[27] Ernie Croot and Olof Sisask. A probabilistic technique for finding almost-periods of convolutions. *Geom. Funct. Anal.*, 20(6):1367–1396, 2010.

[28] Nicolaas Govert de Bruijn. Algebraic theory of Penrose's non-periodic tilings of the plane. II. In *Indagationes Mathematicae (Proceedings)*, volume 84, pages 53–66. Elsevier, 1981.

[29] Nicolaas Govert de Bruijn. Algebraic theory of Penrose's non-periodic tilings of the plane. *Kon. Nederl. Akad. Wetensch. Proc. Ser. A*, 43(84):1–7, 1981.

[30] Yves de Cornulier. Relative Kazhdan property. *Ann. Sci. École Norm. Sup. (4)*, 39(2):301–333, 2006.

[31] Nicolas de Saxcé. Borelian subgroups of simple Lie groups. *Duke Math. J.*, 166(3):573–604, 2017.

[32] David B. A. Epstein and Koji Fujiwara. The second bounded cohomology of word-hyperbolic groups. *Topology*, 36(6):1275–1289, 1997.

[33] H. Federer and A. P. Morse. Some properties of measurable functions. *Bull. Amer. Math. Soc.*, 49:270–277, 1943.

[34] J. M. G. Fell. A Hausdorff topology for the closed subsets of a locally compact non-Hausdorff space. *Proc. Amer. Math. Soc.*, 13:472–476, 1962.

[35] Alexander Fish. Extensions of Schreiber's theorem on discrete approximate subgroups in $\mathbb{R}^d$. *J. Éc. polytech. Math.*, 6:149–162, 2019.

[36] David Fisher and G. A. Margulis. Local rigidity for cocycles. In *Surveys in differential geometry, Vol. VIII (Boston, MA, 2002)*, volume 8 of *Surv. Differ. Geom.*, pages 191–234. Int. Press, Somerville, MA, 2003.

[37] G. A. Freĭman. On the addition of finite sets. *Dokl. Akad. Nauk SSSR*, 158:1038–1041, 1964.

[38] David Fried and William M. Goldman. Three-dimensional affine crystallographic groups. *Adv. in Math.*, 47(1):1–49, 1983.

[39] Andrew Geng. When are radicals of Lie groups lattice-hereditary? *New York J. Math.*, 21:321–331, 2015.

[40] Frederick P. Greenleaf. *Invariant means on topological groups and their applications.* Van Nostrand Mathematical Studies, No. 16. Van Nostrand Reinhold Co., New York-Toronto, Ont.-London, 1969.

[41] Paul R. Halmos. *Measure Theory.* D. Van Nostrand Co., Inc., New York, N. Y., 1950.

[42] Weikun He and Nicolas de Saxcé. Sum-product for real Lie groups. *J. Eur. Math. Soc. (JEMS)*, 23(6):2127–2151, 2021.

[43] H. A. Helfgott. Growth and generation in $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. *Ann. of Math. (2)*, 167(2):601–623, 2008.

[44] Harald A. Helfgott. Growth in groups: ideas and perspectives. *Bull. Amer. Math. Soc. (N.S.)*, 52(3):357–413, 2015.

[45] Gerhard P. Hochschild. *Basic theory of algebraic groups and Lie algebras*, volume 75 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1981.

[46] Ehud Hrushovski. Stable group theory and approximate subgroups. *J. Amer. Math. Soc.*, 25(1):189–243, 2012.

[47] Ehud Hrushovski. Beyond the lascar group. *arXiv preprint arXiv:2011.12009*, 2020.

[48] Ehud Hrushovski, Krzysztof Krupinski, and Anand Pillay. Amenability and definability, 2019.

[49] V. A. Kaĭmanovich and A. M. Vershik. Random walks on discrete groups: boundary and entropy. *Ann. Probab.*, 11(3):457–490, 1983.

[50] Jakub Konieczny. Characterisation of meyer sets via the freiman–ruzsa theorem. *arXiv preprint arXiv:2103.02289*, 2021.

[51] J. C. Lagarias. Meyer's concept of quasicrystal and quasiregular sets. *Comm. Math. Phys.*, 179(2):365–376, 1996.

[52] A. M. Macbeath and S. Świerczkowski. On the set of generators of a subgroup. *Nederl. Akad. Wetensch. Proc. Ser. A 62 = Indag. Math.*, 21:280–281, 1959.

[53] Simon Machado. Approximate lattices and Meyer sets in nilpotent Lie groups. *Discrete Analysis*, 1:18 pp., 2020.

[54] Simon Machado. Approximate lattices in higher-rank semi-simple groups. *arXiv preprint arXiv:2011.01835*, 2020.

[55] Simon Machado. Good models, infinite approximate subgroups and approximate lattices. *arXiv preprint arXiv:2011.01829*, 2020.

[56] Simon Machado. Infinite approximate subgroups of soluble Lie groups. *Math. Ann.*, 382(1-2):285–301, 2022.

[57] George W. Mackey. Induced representations of locally compact groups. I. *Ann. of Math. (2)*, 55:101–139, 1952.

[58] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[59] Jean-Cyrille Massicot and Frank O. Wagner. Approximate subgroups. *J. Éc. polytech. Math.*, 2:55–64, 2015.

[60] Yves Meyer. *Algebraic numbers and harmonic analysis*, volume 2. Elsevier, 1972.

[61] Deane Montgomery and Leo Zippin. Small subgroups of finite-dimensional groups. *Ann. of Math. (2)*, 56:213–241, 1952.

[62] Robert V Moody. Meyer sets and their duals. *NATO ASI Series C Mathematical and Physical Sciences-Advanced Study Institute*, 489:403–442, 1997.

[63] G. D. Mostow. Arithmetic subgroups of groups with radical. *Ann. of Math. (2)*, 93:409–438, 1971.

[64] László Pyber and Endre Szabó. Growth in finite simple groups of Lie type. *J. Amer. Math. Soc.*, 29(1):95–146, 2016.

[65] M. S. Raghunathan. Discrete subgroups of Lie groups. pages ix+227, 1972.

[66] Joseph Rosenblatt. Ergodic and mixing random walks on locally compact groups. *Math. Ann.*, 257(1):31–42, 1981.

[67] Imre Z. Ruzsa. An analog of Freiman's theorem in groups. Number 258, pages xv, 323–326. 1999. Structure theory of set addition.

[68] Tom Sanders. Approximate groups and doubling metrics. *Math. Proc. Cambridge Philos. Soc.*, 152(3):385–404, 2012.

[69] Jean-Pierre Schreiber. Approximations diophantiennes et problemes additifs dans les groupes abéliens localement compacts. *Bull. Soc. Math. France*, 101:297–332, 1973.

[70] Marjorie Senechal. *Quasicrystals and geometry.* Cambridge University Press, Cambridge, 1995.

[71] Carl Ludwig Siegel. A mean value theorem in geometry of numbers. *Annals of Mathematics*, 46(2):340–347, 1945.

[72] Tonny Albert Springer. *Linear algebraic groups.* Springer Science & Business Media, 2010.

[73] Terence Tao. Product set estimates for non-commutative groups. *Combinatorica*, 28(5):547–594, 2008.

[74] Matthew C. H. Tointon. *Introduction to Approximate Groups.* London Mathematical Society Student Texts. Cambridge University Press, 2019.

[75] V. I. Ušakov. Topological $\overline{FC}$-groups. *Sibirsk. Mat. Ž.*, 4:1162–1174, 1963.

[76] È. B. Vinberg. Rings of definition of dense subgroups of semisimple linear groups. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:45–55, 1971.

[77] André Weil. *Sur les espaces à structure uniforme et sur la topologie générale.* Paris, 1938.

[78] Hassler Whitney. Elementary structure of real algebraic varieties. *Ann. of Math. (2)*, 66:545–556, 1957.

[79] Dave Witte. Zero-entropy affine maps on homogeneous spaces. *Amer. J. Math.*, 109(5):927–961, 1987.

[80] Dave Witte. Superrigidity of lattices in solvable Lie groups. *Invent. Math.*, 122(1):147–193, 1995.

[81] Hidehiko Yamabe. A Generalization of A Theorem of Gleason. *Annals of Mathematics*, 58(2):351–365, 1953.

[82] Robert J Zimmer. *Ergodic theory and semisimple groups*, volume 81. Springer Science & Business Media, 2013.