Existence of Equilibria and Core Convergence in Economies with Bads

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February 2004

CWPE 0413

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Abstract

We consider an exchange economy in which there are infinitely many consumers and some commodities are bads, that is, cause disutility to consumers. We give an example of such an economy for which there is no competitive equilibrium or its variants (quasi- or pseudo-equilibrium). We also give examples of the failure of the so called uniform integrability condition of equilibrium allocations of increasingly populous finite economies, and also the failure of the core convergence property.

Keywords: General equilibrium, bads, existence, core convergence.
JEL Classification Codes: C62, C71, D41, D51, D61, D63, Q21.

Existence of Equilibria and Core Convergence in Economies with Bads

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1 Introduction

1.1 Setup

This paper is concerned with exchange economies in which some commodities are bads, that is, cause disutility to consumers. To simplify the analysis, we concentrate on the case where there are only two types of commodities, one of which is a good and the other is a bad. The good can be considered just as any consumption good, while the bad should be considered as garbage or toxic wastes. The exchange economy is populated with infinitely many consumers, each of whom is negligible in size relative to the entire economy. As will be seen in the subsequent analysis, what turns out to be crucial is not the infinity of the population of consumers *per se* but the infinity of types of consumers.

The resource feasibility constraint in all the notions of Pareto-efficiency, competitive equilibria, and the core, of this paper requires the demand to be exactly equal to supply for each commodity. An alternative resource feasibility constraint would allow the demand to be less than the supply;

[1] This paper is a spin-off of the collaboration with Akira Yamazaki, to whom I am most grateful. Robert Anderson gave me many helpful and insightful comments. Communications with Atsushi Kajii and conversations with Mamoru Kaneko were stimulating. I also thank workshop participants at Kyoto and seminar participants at Otaru University of Commerce, Hokkaido University, Kyoto University, and Hitotsubashi University, especially Tomoki Inoue, Tomoichi Shinozuka, and Kotaro Suzumura. I gratefully acknowledge the hospitality of Institute of Economic Research at Hitotsubashi University. Parts of the paper, such as the proof methods for Lemmas 7 and 8 and Propositions 14 and 16, are contained in the Kyoto workshop proceeding (Hara (2003)).

and the corresponding competitive equilibrium concept is called a free-disposal equilibrium. We do not use this weaker constraint, because that would allow the bad to be freely disposed of and hence trivialize the problem of the efficient allocation of bads. While a free-disposal equilibrium involves only non-negative prices, an competitive equilibrium satisfying the exact resource feasibility constraint must involve negative prices for bads. This paper explores some consequences of allowing for negative prices.

In most of this paper, the consumption set is assumed to be the non-negative orthant R^2_+. This is mostly for simplicity, except that there is no upper bound on the possible consumption levels for the bad, which we will see turns out to be an important property. In our examples, every consumer's initial endowment for each commodity is strictly positive.

1.2 Our Results and the Relationship with the Literature

There have of course been many contributions in general equilibrium theory in which bads and non-monotone preference relations are incorporated. While Arrow and Debreu (1954) used the free-disposal equilibrium as the equilibrium concept, the equilibrium concept of McKenzie (1959, 1981) requires the exact equality of supply and demand with possibly negative prices. His work had later been followed by Bergstrom (1976), Kuhn and Hart (1975), Polemarchakis and Siconolfi (1993), and others. In the models of these papers, there assumed to be only finitely consumers.

Aumann (1964) introduced a general equilibrium model of an exchange economy with infinitely many consumers to establish the core equivalence theorem. The existence of a competitive equilibrium in such a economy was subsequently established by Aumann (1966), Schmeidler (1969), and Hildenbrand (1970). Aumann (1966) and Schmeidler (1969) assumed that every consumer has a monotone preference relation. Hildenbrand (1970) considered a production economy, in which consumers' preference relations need not be monotone but the production technology can dispose of any commodity without incurring any costs in terms of other commodities. In the context of an exchange economy, therefore, his theorem would only establish the existence of a free-disposal equilibrium. Cornet, Topuzu, and Yildiz (2003) proved the existence of a free-disposal equilibrium in an extended sense, in that the cone in the commodity space representing the feasible directions of disposal may be narrower than the non-positive orthant. Yet their result falls short of the existence of an equilibrium when the free disposal is completely impossible. To summarize, unlike the case of economies consisting of finitely many consumers, any equilibrium existence theorem without monotone preference relations and the free disposability has been provided. Neither has an example of the non-existence. This paper presents such an example (Example 10).

Our example is similar to the example of the non-existence of an equilibrium in an exchange economy of Araujo (1985), in that there is neither an efficient and individually rational allocation nor an efficient and envy-free allocation in both our and his examples. The difference is, roughly, that the role of consumers and commodities are swapped in his example, so that there are only two consumers but infinitely many commodities.

Given that an economy of infinitely many negligible consumers can be approximated by a large finite economy with respect to the weak topology of probability measures, and that every finite economy has an equilibrium, one might wonder what would happen to the limit of the sequence of equilibria of finite economies that converges to the infinite economy having no equilibrium. We will see (Example 22) that the sequence of equilibrium allocations of finite economies may not satisfy the so-called *uniform integrability* condition, so that, as the free-disposal is impossible, the sequence does not possess any limit that would correspond to an equilibrium allocation of the infinite economy.[2] This is an instance of the divergence, pointed out by Anderson (1992), in equilibrium outcomes between infinite and large finite economies due to the assumptions implicit in the formulation of an infinite economy. Another instance of the same nature is the failure of the core convergence without monotone preference relations, as exemplified by Manelli (1991a). We give an example (Example 24) of the failure of the core convergence, which is simpler than his. Both of the two examples illustrate a potential source of market power arising from preference for (or, the willingness to accept) bads, not from initial endowment allocations.

1.3 Significance of the Results

How to allocate bads efficiently is an important economic issue. A general equilibrium model with bads is an appropriate theoretical framework to tackle this question, since bads are often generated in conjunction with the production or consumption activities of goods, and a general equilibrium model would then be necessary to assess the welfare consequences of such joint productions. The model of this paper is a static exchange economy with only two commodities, one good and one bad. It is an appropriate benchmark case, just like the Edgeworth box economy and the Robison-Crusoe economy.

We assume that there are complete markets for goods and bads. In particular, no consumer can escape from consuming bads without paying

[2] One could construct a sequence of equilibrium allocations of finite economies on the same space of consumers, which have an almost everywhere limit. Given the failure of uniform integrability, however, the conclusion of Fatou's lemma holds with strict inequalities, leading the almost everywhere limit to violate the exact resouce-feasibilty constraint.

prices.[3] We also assume that bads generate no externalities.[4] While some important examples of bads, such as pollution, do have externalities, it is a good theoretical exercise to take up a model with no externalities to study efficient allocation of bads.

In this framework, the first and second welfare theorems still hold, as long as prices may be negative, so that an allocation is efficient if and only if it is a competitive equilibrium (or quasi-equilibrium) allocation once some appropriate transfers are made. The natural starting point of our analysis is thus to see whether any of the Pareto-efficient allocations can be attained through the market mechanism with negative prices. Underlying the two welfare theorems is the hypothesis of the price-taking behavior. It is for this reasons that we look into a model with infinitely many consumers, each of whom is negligible in size relative to the entire economy. Indeed, Aumann (1964) introduced such a model to establish the core equivalence theorem, which justifies the hypothesis of the price-taking behavior. It has been later extended by Hildenbrand (1968) to incorporate bads.

The example (Example 10) of non-existence of a competitive equilibrium should be taken as a seriously disturbing fact, as it shows that even the simplest model of bads cannot pass the most basic internal consistency test for economic models. The example (Example 24) of the failure of core convergence undermines the relevance of the core equivalence theorem to large finite economies. Furthermore, the example (Example 22) of the failure of the uniform integrability condition shows that in a large finite economy, an almost negligibly small group of consumers may end up consuming almost all of bads in the economy. Such an equilibrium

[3] Alternatively, Shapley and Shubik (1969) proposed a co-operative game in which consumers can do so.

[4] In fact, Arrow (1969) shows that, with appropriate modifications of the commodity space and utility functions, an economy with externalities can be made a special case of economies with incomplete markets.

allocation cast serious doubts on the plausibility of the price-taking behavior in finite economies, however large it may be.

1.4 Organization of the Paper

The formal model is presented in the next section. Section 2 presents the model and basic concepts of this paper. Section 3 presents the leading example of the non-existence of a competitive equilibrium. Section 4 shows that in addition to the non-existence of a competitive equilibrium, there is neither an efficient and individually rational allocation nor an efficient and envy-free allocation. Section 5 shows that the non-existence survives various modifications of the example. Section 6 investigates the limit behavior of core and equilibrium allocation of finite economies approximating the infinite economy for which there is no competitive equilibrium. Section 7 concludes.

2 Model

The space of (names of) consumers is given by a complete measure space $(A, \mathcal{A}, \mu)$ with $0 < \mu(A) < \infty$. Denote by $\mathcal{U}$ the set of all real-valued functions defined on the two-dimensional non-negative orthant R^2_+, denoted by X, that are continuous, quasi-concave, strictly increasing in the first coordinate, and strictly decreasing in the second coordinate, endowed with the C^0 compact open topology. This is the space of utility functions we shall consider in this paper. The interpretation is that there are two commodities; X is the consumption set for every consumer; the first commodity is a good; and the second commodity is a bad. An (private ownership) *economy* is characterized as a pair of an $(\mathcal{A} \otimes \mathcal{B}(\mathcal{U}))$-measurable mapping $u : A \to \mathcal{U}$ and an $(\mathcal{A} \otimes \mathcal{B}(R^2))$-measurable and integrable mapping $e : A \to R^2$, where $\mathcal{B}$ stands for the Borel σ-field. The interpretation is that the utility function $u(a)$, which we also write

u_a, represents consumer a's preference relation, and $e(a)$ is his initial endowment vector.

Definition 1 Let $B \in \mathcal{A}$ and $f : B \to X$, then f is an *allocation within* B if it is integrable and satisfies $\int_B f = \int_B e$. An allocation within A is also simply called an allocation.

Note that an allocation, by definition, satisfies the resource feasibility constraint, which is met with the strict equality rather than the weak inequality $\int_B f \leq \int_B e$, to eliminate the possibility of free disposal.

Definition 2 Let f and g be two allocations, then g is a *weak improvement* on f if $u_a(g(a)) \geq u_a(f(a))$ for almost every $a \in A$, with a strict inequality for every a in some measurable subset of positive measure. It is a *strong improvement* on f if $u_a(g(a)) > u_a(f(a))$ for almost every $a \in A$.

Definition 3 An allocation of an economy is *strongly efficient* if there is no weak improvement on it; it is *weakly efficient* if there is no strong improvement on it.

A strong improvement is thus a weak improvement but the converse need not be true. Yet if preference relations are strongly monotone, then there is a strong improvement whenever there is a weak one, and the two notions of efficiency coincide with each other. We will see towards the end of this section that this fact remains to be true in an economy in which there are one good and one bad.

A *price vector* is, by definition, a non-zero vector of R^2.

Definition 4 An allocation f is *strongly supportable* if there exists a price vector p such that for almost every $a \in A$ and every $x \in X$, $p \cdot x > p \cdot f(a)$ whenever $u_a(x) > u_a(f(a))$. It is *weakly supportable* if there exists a price vector p such that for almost every $a \in A$ and every $x \in X$, $p \cdot x \geq p \cdot f(a)$ whenever $u_a(x) > u_a(f(a))$.

For each consumer a, we shall refer to the utility maximization condition for strong supportability as the *strong utility maximization condition* and the utility maximization condition for weak supportability as the *weak utility maximization condition*. The above notion of weak supportability has been considered in Hildenbrand (1968), Mas-Colell, Whinston, and Green (1995), Hurwicz and Richter (2001), and many others. As they have pointed out, with the locally non-satiation assumption on utility functions, then the weak utility maximization condition is equivalent to the cost minimization condition, that is, for almost every $a \in A$ and every $x \in X$, $p \cdot x \geq p \cdot f(a)$ whenever $u_a(x) \geq u_a(f(a))$. Given this, we see that the weak utility maximization condition is equivalent to the strong utility maximization condition if $p \cdot f(a) > \inf\{p \cdot x \mid x \in X\}$.

Definition 5 A pair (p, f) of a price vector p and an allocation f is a *strong equilibrium* if f is strongly supported by p and $p \cdot f(a) \leq p \cdot e(a)$ for almost every $a \in A$. It is a *weak equilibrium* if f is weakly supported by p and $p \cdot f(a) \leq p \cdot e(a)$ for almost every $a \in A$.

The strong equilibrium is commonly known as a Walrasian or competitive equilibrium, but we opt for adding the adjective "strong" to distinguish it from a weak equilibrium. The weak equilibrium concept coincides with the pseudo-equilibrium and quasi-equilibrium, but we call it an "weak" equilibrium for simplicity. Even if the weak inequality $\leq$ is replaced by $=$, the condition would still be equivalent in our setup, because $\int_A f = \int_A e$. Hence a weak equilibrium is a strong equilibrium if $p \cdot e(a) > \inf\{p \cdot x \mid x \in X\}$ for almost every $a \in A$, that is, the so-called *minimum income condition* is met. This condition is met if $e_1(a) > 0$ for almost every $a \in A$ (since $p_1 > 0$ at every weak equilibrium) or if $p_2 < 0$

(since, then, $\inf\{p \cdot x \mid x \in X\} = -\infty$).

We now introduce the useful notion of linked and non-linked allocations. It is taken from Proposition 7.2.7 of Mas-Colell (1985).[5]

Definition 6 Let $B \in \mathcal{A}$ and f be an allocation within B, then f is *linked* if $f(a) \in \operatorname{int} X$ for every a in some measurable subset of B of positive measure. Otherwise, it is *non-linked*.

According to this definition, at a non-linked allocation, almost no consumer consumes both of the two commodities. The following lemma on non-linked allocations is technical but underlies some of the results of this paper.

Lemma 7 *Let $B \in \mathcal{A}$ and f be an allocation within B. If g is an non-linked allocation within B and $u_a(g(a)) \geq u_a(f(a))$ for almost every $a \in B$, then $g(a) = f(a)$ for almost every $a \in B$.*

Proof of i nce the measure space $(A, \mathcal{A}, \mu)$ is complete, we can assume without loss of generality that $g(a) \notin \operatorname{int} X$ and $u_a(g(a)) \geq u_a(f(a))$ for *every $a \in B$.*

Since g is an non-linked allocation within B, for every $a \in B$, if $g_2(a) > 0$, then $g_1(a) = 0 \leq f_1(a)$. Since $u_a(g(a)) \geq u_a(f(a))$, this implies that $g_2(a) \leq f_2(a)$. This of course holds when $g_2(a) = 0$. Thus $g_2(a) \leq f_2(a)$ for every $a \in B$. Since $\int_B g_2 = \int_B e_2 = \int_B f_2$, this implies that $g_2(a) = f_2(a)$ for every $a \in B$. Since $u_a(g(a)) \geq u_a(f(a))$, this implies that $f_1(a) \geq g_1(a)$ for every $a \in B$. As before, then, $g_1(a) = f_1(a)$ for every $a \in B$. ///

[5] There are apparently many predecessors of this definition. Also, Definition 4.3.5 of Mas-Colell (1985) is more explicit but less suited to our analysis because he assumed differentiability of utility functions while we do not do so at this point. A more detailed account on this property is contained in a workshop proceeding (Hara (2003)).

Lemma 8 *An allocation is strongly efficient if and only if it is weakly efficient.*

Lemma 8 implies that there is no need to distinguish strong and weak efficiency. We shall therefore refer to them simply as efficiency. Its proof is somewhat intricate and thus relegated to the appendix.[6] The following is a combination of the first and second welfare theorems. We skip its easy proof.

Theorem 9 *An allocation is efficient if and only if it is weakly supportable.*

3 Leading Example

The following example is our leading example. We shall explore various properties of this example and also check the robustness of these properties when the example is modified.

Example 10 Let A be the open interval $(0, 1)$, $\mathcal{A}$ be the set of Lebesgue measurable subsets of A, and μ be the Lebesgue measure restricted on A. For each $a \in A$, let

$$u_a(x) = x_1 - a\,(x_2)^2$$

and $e(a) = (2, 1)$.

In this example, for every $a \in A$, $e(a) \in \operatorname{int} X$ and u_a is smooth, strictly differentiably quasi-concave, and can be extended to the entire R^2.

Proposition 11 *There is no weak (and hence strong) equilibrium in Example 10.*

[6] I am grateful to Tomoki Inoue for pointing out the need to check measurability in the proof.

Proof of i nce $e(a) \in \operatorname{int} X$ for every $a \in A$, it suffices to show that there is no strong equilibrium. To do so by a contradiction argument, suppose that there is a strong equilibrium (p, f).

Note that $p_1 > 0$ by the strong utility maximization condition. It also implies that $p_2 < 0$ because, otherwise, $f_2(a) = 0$ for almost every $a \in A$, implying that $\int_A f_2 = 0$, but this would contradict $\int_A f_2 = \int_A e_2 = 1$.

Since $\nabla u_a(e(a)) = (1, -2a)$ and $2a < 2$ for every $a \in A$, if $|p_2| \geq 2$, then $f_2(a) > 1$ for almost every $a \in A$ and hence $\int_A f_2 > 1$. But this is a contradiction to $\int_A f_2 = \int_A e_2 = 1$. Thus $|p_2| < 2$.

Since $|p_2| < 2$, $p \cdot e(a) \geq 2 - |p_2| > 0$ and hence the budget line $\{x \in X \mid p \cdot x = p \cdot e(a)\}$ must intersect with the horizontal, but not the vertical, axis. Hence if $f(a)$ is on the boundary of X, then $f_2(a) = 0$. However, since $p_2 < 0$ and $\dfrac{\partial u_a(f(a))}{\partial x_2} = 2af_2(a) = 0$, the first-order condition for the strong utility maximization would contradict $f_2(a) = 0$. Thus $f(a) \in \operatorname{int} X$ for almost every $a \in A$. Again by the first-order condition, therefore, $f_2(a) = \dfrac{|p_2|}{2a}$ for almost every $a \in A$. But then f_2 would not be integrable because the real-valued function $a \mapsto 1/a$ on A is not integrable either. This is a contradiction. Hence there is no strong equilibrium. ///

The non-integrability of the real-valued function $a \mapsto 1/a$ on A is the crucial property for the above non-existence result. It will appear repeatedly in the subsequent analysis.

4 More on Non-Existence

A strong equilibrium allocation is efficient and individually rational, and envy-free in terms of net demands. In this section, we show that there are stronger non-existence results in Example 10. First, there is no allocation that is both efficient and individually rational. Second, there is no allocation that is both efficient and envy-free in terms of net demands. The following equivalence between efficiency and unlinkedness is crucial for both results.

Lemma 12 *In Example 10, an allocation is efficient if and only if it is non-linked.*

Proof of f an allocation is non-linked, then it is weakly supportable by $p = (1, 0)$ and hence efficient.

To prove the converse by a contradiction argument, let f is an efficient and linked allocation. Then f is weakly supportable, by a price vector p. Then, as we saw in the proof of Proposition 11, $p_1 > 0$ and hence we can assume that $p_1 = 1$. Let $B \in \mathcal{A}$ be such that $\mu(B)$ and $f(a) \in \operatorname{int} X$ for every $a \in B$. By the first-order condition of efficient interior allocations, $2af_2(a) = -p_2$ for almost every $a \in B$. Thus $p_2 < 0$ and the minimum income condition is met for every consumer. Hence the strong utility maximization condition is met for every consumer as well. The first-order condition, allowing for the possibility of $f_1(a) = 0$, is then that $2af_2(a) \geq |p_2|$, that is,

$$f_2(a) \geq \frac{|p_2|}{2a}.$$

This weak inequality therefore holds for almost every $a \in A$. But his is a contradiction because f_2 is integrable but the real-valued function $a \mapsto 1/a$ on A is not. ///

Note that to show that every non-linked allocation is efficient, we did not use the specification of u and e in Example 10. Hence the property is true in every economy with one good and one bad.

4.1 Individual Rationality

The following definition is standard.

Definition 13 An allocation f is *individually rational* if $u_a(f(a)) \geq u_a(e(a))$ for almost every $a \in A$.

Proposition 14 *In Example 10, there is no efficient and individually rational allocation.*

Proof of u ppose that there is an efficient and individually rational allocation f. By Lemma 12, f is non-linked. By Lemma 7, $f(a) = e(a)$ for almost every $a \in A$. But this contradicts $e(a) = (2, 1)$ for every $a \in A$. ///

4.2 Envy-Freeness

We consider the envy-free property with respect to net demands.

Definition 15 An allocation f is *envy-free* if there exists a $B \in \mathcal{A}$ such that $\mu(A \setminus B) = 0$ and for every $a \in B$ and every $b \in B$, if $e(a) + (f(b) - e(b)) \in X$, then $u_a(e(a) + (f(b) - e(b))) \leq u_a(f(a))$.

The definition states that at an envy-free allocation, almost no consumer can get strictly better off by receiving the net demands that another consumer receives.

Proposition 16 *In Example 10, there exists no efficient and envy-free allocation.*

Proof of u ppose that there is an efficient and envy-free allocation f. Define

$$A_1 = \{a \in A \mid f_1(a) > e_1(a)\},$$
$$A_2 = \{a \in A \mid f_2(a) > e_2(a)\}.$$

Then $\mu(A_1 \cap A_2) = 0$ because f is non-linked by Lemma 12. Suppose that $\mu(A_1) > 0$ and $\mu(A_2) > 0$, then

$$\mu(A_1 \cap (A \setminus A_2)) = \mu(A_1 \setminus (A_1 \cap A_2)) = \mu(A_1) > 0,$$
$$\mu((A \setminus A_1) \cap A_2) = \mu(A_2 \setminus (A_1 \cap A_2)) = \mu(A_2) > 0.$$

But the consumers $a \in (A \setminus A_1) \cap A_2$ would envy those $a \in A_1 \setminus (A_1 \cap A_2)$ and hence f could not be envy-free. We must thus have either $\mu(A_1) = 0$ and $\mu(A_2) = 0$. If $\mu(A_1) = 0$, then $f_1(a) \leq e_1(a)$ for almost every $a \in A$ and, since $\int_A f_1 = \int_A e_1$, $f_1(a) = e_1(a)$ for almost every $a \in A$. Then, by the envy-free property, $f_2(a) = e_2(a)$ for almost every $a \in A$. Thus $f(a) = e(a)$ for almost every $a \in A$. We can analogously show that the same equality is obtained also when $\mu(A_2) = 0$. But this contradicts $e(a) = (2, 1)$ for every $a \in A$. ///

5 Robustness of the Leading Example

In this section we argue that the non-existence results of our leading example (Example 10) is robust in many directions of modification. We will omit detailed proofs for most of the propositions below, as they would be straightforward modifications of the preceding proofs. In all of the modifications below, just as in Example 10, A be the open interval $(0, 1)$, $\mathcal{A}$ be the set of Lebesgue measurable subsets of A, and μ be the Lebesgue measure restricted on A.

5.1 Consumption Sets

It has so far been assumed that the consumption set X equals the non-negative orthant R^2_+, but this assumption may seem to be implausible, because it implies that a consumer can survive only with bads, not consuming the good at all.[7] The next proposition shows that a weak equilibrium may still not exist even if X is a proper subset of R^2_+, to incorporate the situation where, for example, a consumer needs to consume more of the good for survival when he consumes more of the bad.

Proposition 17 *Let u and e be as in Example 10 for every $a \in A$, and let $X_a \supseteq \{x \in X \mid u_a(x) \geq u_a(e(a))\}$ for every $a \in A$, then there is no weak equilibrium for the economy with initial endowments e and utility functions u_a restricted on X_a for every $a \in A$*

This proposition can be proved by noting that the set of individually rational consumption vectors remains the same as for Example 10 and thus the demand function is the same as well. Since the set $\{x \in X \mid u_a(x) \geq u_a(e(a))\}$ of individually rational consumption vectors has no upper bound on the possible consumption levels for the bad, neither does X_a. Indeed, one can show that if there were an upper bound (with other things being equal), then there would exist a strong equilibrium.

As explained in textbooks such as Kolstad (1999, Section 4.III.C), a standard technique to transform an economy with bads into an economy without bads is to incorporate, say, "garbage disposal" as a commodity in place of "garbage". Since garbage disposal makes a good, one would often conclude that the standard results on equilibria, including their existence, in economies without bads are all applicable to those with bads. This argument, however, is flawed, as can be seen from the argument in the previous paragraph: Since there is no upper bound on garbage

[7] Kotaro Suzumura and Tomoichi Shinozuka pointed out this to me.

consumptions, the garbage disposal must be measured by negative numbers, where the more negative the garbage disposal level is, the more the garbage itself is to be consumed; and there is no lower bound on the possible consumption levels for garbage disposal in the transformed consumption set. However, since the consumption sets are assumed to be bounded from below in the existence theorems for infinite economies (Aumann (1966), Schmeidler (1969), and Hildenbrand (1970)), these existence theorems, consistent with Example 10, do not imply existence of equilibria in economies with bads.

5.2 Goods for Low Levels of Consumption

While, in the leading example, the second commodity is a bad for every consumer at every consumption level, this aspect of preferences is not crucial for an equilibrium not to exist. The following proposition shows that even if the commodity is a good for low levels of consumptions, an equilibrium may not exist as long as the satiation levels are sufficiently low.

Proposition 18 *Let e be as in Example 10 and $r : A \to R_+$ be integrable. Define $u_a(x) = x_1 - a(x_2 - r(a))^2$, then there is a strong (and hence weak) equilibrium if and only if $\int_A r \geq 1$.*

Proof of e note by $g(a, p_2)$ consumer a's demand for the second commodity under the price vector $p = (1, p_2)$. This is well defined for every $a \in A$ and every $p_2 \in R$, and p is a strong equilibrium price vector if and only if $g(\cdot, p_2)$ is integrable and $\int_A g(\cdot, p_2) = 1$. Note that $g(a, p_2)$ is nonnegative and continuous in p_2, and satisfies $g(a, p_2) \lesseqgtr r(a)$ if and only if $p_2 \gtreqless 0$. Hence, in particular, if $p_2 \geq 0$, then $g(\cdot, p_2)$ is integrable and, by the bounded convergence theorem, the function $p_2 \mapsto \int_A g(\cdot, p_2)$ is continuous in $p_2 \geq 0$. Moreover, $g(a, p_2) \to 0$ as $p_2 \to \infty$ for every $a \in A$. Hence, again by the bounded convergence theorem, $\int_A g(\cdot, p_2) \to 0$ as $p_2 \to \infty$.

Now, if $\int_A r \geq 1$, then $\int_A g(\cdot, 0) = \int_A r \geq 1$. Hence, by the intermediate value theorem, there exists a $p_2^* \geq 0$ such that $\int_A g(\cdot, p_2^*) = 1$. Then $p^* = (1, p_2^*)$ is a strong (and hence weak) equilibrium price vector.

On the other hand, if $\int_A r < 1$, then $\int_A g(\cdot, p_2) \leq \int_A r < 1$ for every $p_2 \geq 0$. We must thus have $p_2 < 0$ at equilibrium. Then $g(a, p_2) \geq r(a) + \frac{|p_2|}{2a}$ by the first-order condition for a maximum, holding with an equality whenever $g(a, p_2) > 0$. But the function $a \mapsto \frac{|p_2|}{2a}$ is not integrable. Thus g is not integrable either and there is no strong (and hence weak) equilibrium. ///

5.3 Initial Endowments

While all consumers have the same initial endowments $(2, 1)$ in Example 10, to establish the non-existence results, it is not necessary for all consumers' initial endowments to be equal.

Proposition 19 *Let u be as in Example 10 and $e : A \to X$ be integrable.*

1. *There is a strong equilibrium if and only if $e_2(a) = 0$ for almost every $a \in A$.*
2. *There exists no weak equilibrium if and only if $e_1(a) > 0$ for almost every $a \in A$.*

Proof of . If $e_2(a) = 0$ for almost every $a \in A$, then $p = (1, 0)$ and e constitute a strong equilibrium. If not, and if $p = (p_1, p_2)$ were an equilibrium price vector, then $p_1 > 0$ and $p_2 < 0$. By the first-order condition admitting the boundary consumptions, if $g(a, p_2)$ is consumer a's demand for the bad under the price vector $p = (1, p_2)$ with $p_2 < 0$, then, for every $a \in A$,

$$g(a, p_2) = \begin{cases} \dfrac{|p_2|}{2a} & \text{if } e_1(a) - |p_2|e_2(a) + \dfrac{|p_2|^2}{2a} > 0, \\ e_2(a) - \dfrac{e_1(a)}{|p_2|} & \text{otherwise.} \end{cases}$$

This implies that $g(a, p_2) \geq \dfrac{|p_2|}{2a}$. Since $a \mapsto \dfrac{|p_2|}{2a}$ is not integrable, there is no strong equilibrium.

2. Since a weak equilibrium would also be a strong equilibrium if $e_1(a) > 0$ for almost every $a \in A$, part 1 implies that there is no weak equilibrium if $e_1(a) > 0$ for almost every $a \in A$. On the other hand, if there exists a $B \in \mathcal{A}$ such that $\mu(B) > 0$ and $e_1(a) = 0$ for every $a \in B$, then the price vector $p = (1, 0)$ and every non-linked allocation satisfying $f_1(a) = g_1(a)$ for almost every $a \in A$ constitute a weak equilibrium. ///

5.4 Distribution of the Intensity of Disutility

In Example 10, the utility functions are given by $u_a(x) = x_1 - a(x_2)^2$. Taking the good as the numeraire, we can say that a represents the intensity of disutility from consuming the bad, and that the intensity is uniformly distributed over the unit interval $(0, 1)$. As has been argued earlier, the crucial property of the uniform distribution is that the real valued function $a \mapsto 1/a$ defined on $(0, 1)$ is not integrable. The following proposition substantiates this claim.

Proposition 20 *Let e be as in Example 10 and $r : A \longrightarrow A$ be measurable. Define $u_a(x) = x_1 - r(a)(x_2)^2$, then there exists a strong (and hence weak) equilibrium if and only if $a \mapsto 1/r(a)$ is integrable.*

Since the distribution of the intensity of disutility on $A = (0, 1)$ is $\mu \circ r^{-1}$, the proposition says that there is a strong (and hence weak) equilibrium if and only if there are relatively few consumers having very low intensities of disutility. It can be proved by the same argument as for Proposition 11 and, in particular, implies that the non-existence result can be obtained even when there are only countably many types. When $a \mapsto 1/r(a)$ is integrable, the equilibrium price vector is given by $p = \left(1, -\left(\int r^{-1}\right)^{-1}\right)$.

5.5 Range of Marginal Disutility

In the leading example (Example 10), the marginal disutility $\dfrac{\partial u_a(x)}{\partial x_2} = 2ax_2$ ranges from zero to infinity as the consumption level x_2 of the bad goes to infinity. This fact might be considered as an indispensable feature of the example for two reasons. First, it implies that every efficient allocation is supportable only by the zero price for the bad, and a weak equilibrium is not a strong equilibrium only if the price for the bad equals zero. Second, since the marginal utility diverges to infinity, the bad can potentially be cause an arbitrarily large marginal disutility. Indeed, unlike the u_a of Example 10, if there is a bound on marginal utility, as assumed under the names of *C-monotone preferences* in Grodal, Trockel, and Weber (1984), and *proper preferences* in Manelli (1991a, 1991b), one may obtain a core convergence theorem, which is somewhat suggestive of positive existence results for the limit, infinite economy. The following proposition shows that our non-existence result still holds even when the marginal utility can be bounded from above and below by any non-negative numbers. It therefore implies that neither the zero marginal utility at zero consumption nor its divergence to infinity when the consumption level goes to infinity is a crucial property of our leading example.

Proposition 21 *Let* $\underline{q}$ *and* $\overline{q}$ *be such that* $0 \leq \underline{q} < \overline{q} \leq \infty$*. Then, there exists a* $u : A \to \mathcal{U}$ *and* $w \in \operatorname{int} X$ *such that*

1. u_a *can be written in the form of*

$$u_a(x) = x_1 - s_a(x_2),$$

where, for every $a \in A$*,* $s_a : R_+ \to R_+$ *is a twice continuously differentiable function such that* $\underline{q} < s'_a(x_2) < \overline{q}$ *and* $s''_a(x_2) > 0$ *for every* $x_2 \in R_{++}$*.*

2. *There is no weak (and hence strong) equilibrium of the economy defined by* u *and the constant mapping* $e : A \to X$ *with* $e(a) = w$ *for every* $a \in A$*,*

Proof of e prove the proposition first for the case of $\overline{q} = \infty$ and then for the case of $\overline{q} < \infty$.

Suppose first that $\overline{q} = \infty$. Then define

$$s_a(x_2) = a(x_2)^2 + \underline{q}x_2,$$
$$w = (2 + \underline{q}, 1).$$

Then, just as in the proof of Proposition 11, we can show that we can take $p_1 = 1$ without loss of generality; that $p_2 < -\underline{q}$ because otherwise $\int_A f_2 = 0$; that $|p_2| < 2 + \underline{q}$ because otherwise $\int_A f_2 > 1$; but that if $\underline{q} < |p_2| < 2 + \underline{q}$, then $f_2(a) = \dfrac{|p_2| - \underline{q}}{2a}$, contradicting the integrability of f_2.

Suppose next that $\overline{q} < \infty$. Define $r : (0, 1] \to R_{++}$ by

$$r(a) = \frac{\overline{q} - \underline{q}}{4a}.$$

Then, for each $a \in A$, there exist an $\alpha(a) \in R$ and a $\beta(a) \in R_{++}$ such that the function $q_a : A \to R_+$ defined by

$$q_a(x_2) = \begin{cases} \underline{q} + 2ax_2 & \text{for } x_2 \leq r(a), \\ \overline{q} - \dfrac{\alpha(a)}{\beta(a)} \exp\left(-\beta(a)x_2\right) & \text{for } x_2 > r(a) \end{cases} \tag{1}$$

is continuously differentiable. Then $q_a(r(a)) = \widehat{q}$, where $\widehat{q} = \dfrac{\overline{q} + \underline{q}}{2}$, and $\underline{q} < q_a(x_2) < \overline{q}$ and $q_a'(x_2) > 0$ for every $x_2 \in R_{++}$. Then define $s_a : R_+ \to R_+$ by

$$s_a(x_2) = \int_0^{x_2} q_a(t)\, dt,$$

then s_a is twice continuously differentiable. Moreover, $s_a'(r(a)) = \widehat{q}$, and $\underline{q} < s_a'(x_2) < \overline{q}$ and $s_a''(x_2) > 0$ for every $x_2 \in R_{++}$. Define $u : A \to \mathcal{U}$ by

$$u_a(x) = x_1 - s_a\left(x_2\right)$$

then u is measurable.

Define $e : A \to \operatorname{int} X$ by

$$e(a) = (\widehat{q}r(1), r(1))$$

for every $a \in A$, then e is of course integrable. Since $e(a) \in \operatorname{int} X$, it is sufficient to prove that there is no strong equilibrium for the economy defined by u and e. To do so by a contradiction argument, suppose that (p, f) is a strong equilibrium. Then $p_1 > 0$ and hence we can assume that $p_1 = 1$.

If $p_2 \geq -\underline{q}$, then $p \cdot e(a) \geq (\widehat{q} - \underline{q})r(1) > 0$ and hence the budget line $\{x \in X \mid p \cdot x = p \cdot e(a)\}$ must intersect with the horizontal, but not the vertical, axis. Thus the utility maximization condition implies that $f_2(a) = 0$ for almost every $a \in A$, which contradicts $\int_A f = \int_A e$. Thus $p_2 < -\underline{q} \leq 0$.

Since $e_2(a) = r(1) < r(a)$,

$$\left|\frac{\partial u_a(e(a))}{\partial x_2}\right| = s_a'(e_2(a)) < s_a'(r(a)) = \widehat{q}$$

for every $a \in A$. Thus, if $|p_2| \geq \widehat{q}$, then $f_2(a) > e_2(a)$ for almost every $a \in A$ and hence $\int_A f_2 > \int_A e_2$, which is a contradiction. Thus $|p_2| < \widehat{q}$.

Hence $p \cdot e(a) > 0$ and the budget line $\{x \in X \mid p \cdot x = p \cdot e(a)\}$ must intersect with the horizontal, but not the vertical, axis. Hence if $f(a)$ is on the boundary of X, then $f_2(a) = 0$. However, since

$$|p_2| > \underline{q} = s_a'(0) = q_a(0) = \left|\frac{\partial u_a(f(a))}{\partial x_2}\right|,$$

the first-order condition for the strong utility maximization, even allowing for the boundary consumption, could not be met. Hence $f_2(a) > 0$. Thus, by the first-order condition for an interior consumption, $s_a'(f_2(a)) = q_a(f_2(a)) = |p_2|$. Since $|p_2| < \widehat{q}$, this implies that $f_2(a) < r(a)$ and hence $\underline{q} + 2af_2(a) = |p_2|$. That is,

$$f_2(a) = \frac{|p_2| - \underline{q}}{2a}$$

for almost every $a \in A$. But then f_2 would not be integrable because the real-valued function $a \mapsto 1/a$ on A is not integrable either. This is a contradiction. Hence there is no strong equilibrium. ///

6 Large Finite Economies

The purpose of this section is to clarify the nature of the non-existence of equilibria of the leading example (Example 10) by looking into a sequence of equilibria of finite economies that converges, with respect to the weak topology of probability measures, to the infinite economy of the leading example. We will see in Proposition 23 that as the economy becomes large, an arbitrary small group of consumers may end up consuming almost all of the bad in the economy. Such a sequence of allocations has no limit corresponding to an allocation of the infinite economy because, by definition, every allocation in the infinite economy must assign zero consumption to every group of consumers of measure zero. We shall also give an example of the failure of the limit theorem of the core of finite economies. Such an example was already given by Manelli (1991), but our example is similar to Example 10 and admits an easier economic interpretation.

6.1 Equilibria

Example 22 For each positive integer n, define a probability measure space $(A^n, \mathcal{A}^n, \mu^n)$ by letting $A^n = \{1, 2, \ldots, n\}$, $\mathcal{A}^n$ be the power set of A^n, and μ^n be the uniform probability distribution on A^n. Define $u^n : A^n \to \mathcal{U}$ by

$$u_a^n(x) = x_1 - \frac{a}{n}(x_2)^2$$

and $e^n : A^n \to X$ by $e^n(a) = (2, 1)$ for every $a \in A^n$.

Proposition 23 *1. Write* $S^n = \sum_{a=1}^{n} \frac{1}{a}$ *and define*

$$p^n = \left(1, -\frac{2}{S^n}\right),$$

$$f^n(a) = \left(2 + \frac{2}{S^n}\left(\frac{n}{S^n a} - 1\right), \frac{n}{S^n a}\right).$$

Then, for every n, (p^n, f^n) *is the unique strong (and hence weak) equilibrium of the economy* (u^n, e^n).

2. The sequence of induced probability measures $\mu^n \circ (u^n \times e^n)^{-1}$ *on* $\mathcal{U} \times X$ *converges weakly to* $\mu \circ (u \times e)^{-1}$*, where* u *and* e *are defined as in Example 10.*

3. There exists a sequence $(a^n)_n$ of positive integers such that

$$a^n \leq n \text{ for every } n, \tag{2}$$

$$\frac{a^n}{n} \to 0 \text{ as } n \to \infty, \tag{3}$$

$$\frac{1}{n}\sum_{a=1}^{a^n} f_2^n(a) \to 1 \text{ as } n \to \infty. \tag{4}$$

This proposition says that the group of consumers, $\{1, \ldots, a^n\} \subset A^n$, in the n-th finite economy (u^n, e^n), tends to occupy an arbitrarily small proportion in population (3) but almost the entire share in the bads consumption (4).

Proof of . This is routine.

2. Just as in Example 2.2 of Billingsley (1999), we can show that the distribution assigning probability $1/n$ to points $1/n, 2/n, \ldots, n/n$ converges weakly to the uniform distribution on the interval $(0, 1)$. Since $e^n(a) = (2, 1)$ for every n and $a \in A^n$ and $e(a) = (2, 1)$ for every $a \in A$, and also since $a \mapsto u_a$ of A into $\mathcal{U}$ is continuous, this implies that $\mu^n \circ (u^n \times e^n)^{-1}$ converges weakly to $\mu \circ (u \times e)^{-1}$.

3. For each n, let a^n be the positive integer such that

$$n^{1-(\log n)^{-1/2}} \leq a^n < n^{1-(\log n)^{-1/2}} + 1.$$

To prove (3), note first that

$$\log \frac{n^{1-(\log n)^{-1/2}}}{n} = -(\log n)^{-1/2} \log n = -(\log n)^{1/2} \to -\infty$$

as $n \to \infty$. Thus

$$\frac{n^{1-(\log n)^{-1/2}}}{n} \to 0$$

and hence (3) is proved. As for (4), since $\log n < S^n < 1 + \log n$ for every n,

$$\begin{aligned}\frac{1}{n}\sum_{a=1}^{a^n} f_2^n(a) &= \frac{1}{n}\sum_{a=1}^{a^n} \frac{n}{S^n a} = \frac{S^{a^n}}{S^n} \\ &> \frac{\log a^n}{1+\log n} \geq \frac{\log(n^{1-(\log n)^{-1/2}})}{\log n} \frac{\log n}{1+\log n} \\ &= \left(1 - (\log n)^{-1/2}\right) \frac{\log n}{1+\log n}.\end{aligned}$$

Since the far right hand side converges to 1, (4) is proved. ///

6.2 Failure of the Core Convergence

Example 24 For each positive integer n, define a probability measure space $(A^n, \mathcal{A}^n, \mu^n)$ by letting $A^n = \{0, 1, \ldots, n\}$, $\mathcal{A}^n$ be the power set of A^n, and μ^n be the uniform probability distribution on A^n. Define $u^n : A^n \to \mathcal{U}$ by

$$u_a^n(x) = \begin{cases} x_1 - \dfrac{1}{n+2}(x_2)^2 & \text{for } a = 0, \\ x_1 - (x_2)^2 & \text{for } a \geq 1 \end{cases}$$

and define $e^n : A^n \to X$ by $e^n(a) = (2, 1)$ for every $a \in A^n$.

The sequence of type distributions of finite economies, $\mu^n \circ (u^n \times e^n)^{-1}$, on $\mathcal{U} \times X$ converges weakly to the degenerated probability measure on $\mathcal{U} \times X$ that puts probability one on the pair of utility function $x \mapsto x_1 - (x_2)^2$ and initial endowment vector $(2, 1)$. Hence the type of consumer $a = 0$, who cares little about the bad, disappears at the limit.[8]

[8] In fact, if the space $\mathcal{U}$ of utility functions were to be extended to accommodate the possibility that the second commodity may be *neutral*, then the sequence of supports $\operatorname{supp} \mu^n \circ (u^n \times e^n)^{-1}$ would not converge to this single type with respect to the closed convergence topology.

Proposition 25 *In Example 24, for each n, the unique strong equilibrium is given by*

$$p^n = (1, -1),$$
$$f^n(a) = \begin{cases} \left(\frac{1}{2}n + 2, \frac{1}{2}n + 1\right) & \text{for } a = 0, \\ \left(\frac{3}{2}, \frac{1}{2}\right) & \text{for } a \geq 1. \end{cases}$$

The proof is straightforward, so we omit it. Note that neither the equilibrium price vector p^n nor the consumption vector $f^n(a)$ for consumer $a \geq 1$ depends on n. More importantly, the single consumer $a = 0$ always consumes more than half of the aggregate endowment of both commodities, however large n may be. On the other hand, the equilibrium of the limit economy with respect to the weak convergence is $p = (1, -2)$ and $f(a) = (2, 1)$ for almost every a. Hence there is a discontinuity in the equilibrium correspondence with respect to the weak convergence topology. The sequence of distributions of the equilibrium allocations f^n converges weakly to the degenerated probability measure putting probability one on $(3/2, 1/2)$. This does not correspond to any consumption allocation of the limit economy, because half of both commodities are disposed of.

We show that that the sequence of finite economies in Example 24 does not have the core convergence property, by measuring the gap from budget feasibility and utility maximization in money metric. To be more precise, take $P = \left\{p \in R^2 \mid p_1 = 1\right\}$ to be the price space.[9] For each n, define $\psi^n : X \times P \times A^n \to R_+$ by

$$\psi^n(x, p, a) = \max\{p \cdot (x - e(a)), 0\} + \sup\{p \cdot (x - y) \mid y \in X \text{ and } u_a^n(y) \geq u_a^n(x)\}.$$

Thus $\psi^n(x, p, a)$ measures the gap between the given consumption vector $x \in X$ and the demand of consumer $a \in A^n$ under the price vec-

[9] Once could of course take, say, $P = \left\{p \in R^2 \mid |p_1| + |p_2| = 1\right\}$ as the space of normalized price vectors, but the result obtained below needs no change.

tor $p \in P$, where the first term measures the value of x in excess of his wealth under p, and the second term measures the deviation from cost minimization of x under p. With our specification of the utility functions and endowments in Example 24, cost minimization, weak utility maximization, and strong utility maximization are equivalent. Hence, for every price vector p and an allocation f for the economy $((A^n, \mathcal{A}^n, \mu^n), u^n, e^n)$, $\psi^n(f(a), p, a) = 0$ for every $a \in A^n$ if and only if (p, f) is a strong equilibrium. The above definition of ψ^n is a modification of the gap measure in Anderson (1978), which is, according to our notation, $|p \cdot (x - e(a))| + |\inf\{p \cdot (y - e(a)) \mid y \in X \text{ and } u_a(y) \geq u_a(x)\}|$.[10]

The notion of the core convergence property we employ requires that the sequence $(\psi^n)_n$ converges in measure to zero. More precisely, a sequence of finite economies, $((A^n, \mathcal{A}^n, \mu^n), u^n, e^n)_n$, has the *core convergence property* if for every sequence $(g^n)_n$ of core allocation of the economies $((A^n, \mathcal{A}^n, \mu^n), u^n, e^n)$ and for every $\varepsilon > 0$, there exist a sequence $(p^n)_n$ in P and a positive integer N such that for every $n > N$,

$$\mu^n\left(\{a \in A^n \mid \psi^n(g^n(a), p^n, a) > \varepsilon\}\right) < \varepsilon.$$

The following is an example of the violation of the two core convergence properties.

Proposition 26 *For each n, define an allocation $g^n : A^n \to X$ of the economy $((A^n, \mathcal{A}^n, \mu^n), u^n, e^n)$ by*

$$g^n(a) = \begin{cases} \left(\frac{5}{8}n + 2, \frac{1}{2}n + 1\right) & \text{for } a = 0, \\ \left(\frac{11}{8}, \frac{1}{2}\right) & \text{for } a \geq 1. \end{cases}$$

Then, for every n, g^n belongs to the core of $((A^n, \mathcal{A}^n, \mu^n), u^n, e^n)$, but there exists an $\varepsilon > 0$ such that for every sequence $(p^n)_n$ of price vectors

[10] The failure of the core convergence property by the example can be established for his gap measure as well.

in P,

$$\mu^n \left(\{a \in A^n \mid \psi^n(g^n(a), p^n, a) \geq \varepsilon\}\right) \to 1. \tag{5}$$

To explain intuitively why the sequence of the g^n does not have the core convergence property, recall that at the equilibrium allocations f^n, the single consumer 0 consumes more than half of them, however large n may be. It is thus reasonable to guess that he must retain some monopoly power all along the sequence of finite economies, by being less unwilling to accept the bad than any other consumer. In fact, what we see in the above proposition is that even if we modify the equilibrium allocations f^n by transferring some amounts of the good from all consumers $a \geq 1$ to $a = 0$, the resulting allocation will also belong to the core. This is what was done to construct the allocations g^n.

The choice of the level of transfer, $\frac{1}{8}$, is not completely arbitrary. To see this, note that

$$u_a\left(\frac{5}{4}, \frac{1}{2}\right) = 1 = u_a(e(a))$$

for every $a \geq 1$. That is, the consumption vector $\left(\frac{5}{4}, \frac{1}{2}\right)$ just satisfies the individual rationality constraint for $a \geq 1$, and we can take at most $\frac{1}{4} = \frac{3}{2} - \frac{5}{4}$ units of the first commodity from type $a = 0$ at the equilibrium allocation f^n without violating the individual rationality constraint. So we transfer just a half of it, $\frac{1}{8} = \frac{1}{2}\frac{1}{4}$, so that his individual rationality constraint is still met with strict inequality. This implies that there is no objecting coalition consisting of the $a \geq 1$. It thus remains to show that there is no objection involving consumer 0. This task will turn out to be easy, thanks to the quasi-linearity of utility functions with respect to the good and by the strong supportability of the f^n.

Proof of e first prove that for every n, g^n belongs to the core of (u^n, e^n). Indeed, no coalition consisting only of consumers $a \geq 1$ can object to g^n; if there were such a coalition, then, since all members have the identical, strict quasi-concave utility function, there would be another objection at which every member receives the same consumption vector. But this consumption vector must be $e(a) = (2,1)$. This is a contradiction to $u_a^n\left(g^n(a)\right) > u_a^n(e(a))$. Hence there is no objection consisting only of consumers $a \geq 1$.

To show that there is no objection involving the consumer $a = 0$, note first that since

$$\frac{\partial u_a(g^n(a))}{\partial x_2} = -1$$

for every $a \geq 0$, g^n is strongly supportable by $p = (1, -1) \in P$.

Now, by means of a contradiction argument, suppose that there is a weak objection $(\{0\} \cup C, h)$ to some g^n, where C is a subset of $\{1, \dots, n\}$ and may be empty or equal to $\{1, \dots, n\}$. Then

$$p \cdot h(a) \geq p \cdot g^n(a)$$

for every $a \in C$,

$$p \cdot h(0) \geq p \cdot g^n\left(0\right),$$

and at least one of these $|C| + 1$ weak inequalities must be strict. Thus, by taking summation over $a \in \{0\} \cup C$, we obtain

$$p \cdot \left(\sum_{a \in \{0\} \cup C} h(a) \right) > p \cdot \left(\sum_{a \in \{0\} \cup C} g^n(a) \right).$$

The left hand side is equal to $(1 + |C|)\, p \cdot w$, where $w = (2, 1) \in X$. On the other hand, since $p \cdot g^n(0) = p \cdot w + \frac{n}{8}$ and $p \cdot g^n\left(a\right) = p \cdot w - \frac{1}{8}$ for every $a \in C$, the right hand side is equal to

$$(1 + |C|)\, p \cdot w + \frac{n - |C|}{8}.$$

Since $n - |C| \geq 0$, this is a contradiction. Hence there is no objection involving the consumer $a = 0$.

We now move on to prove that there exists a positive number $\varepsilon > 0$ for which (5) holds. Since all $a \geq 1$ have the same utility function and initial endowments, and receive the same consumption vector $g^n(a) = \left(\frac{11}{8}, \frac{1}{2}\right)$ for every n, both of the two terms of the measure of deviation for $a \geq 1$,

$$\begin{aligned}\psi^n(g^n(a), p, a) &= \max\{p \cdot (g^n(a) - e(a)), 0\} + \sup\{p \cdot (g^n(a) - y) \mid y \in X, \\ \text{and } u_a(y) &\geq u_a(g^n(a))\}\end{aligned}$$

depends only on p. We thus denote them by $\psi_1(p)$ and $\psi_2(p)$. Since $\left(\frac{11}{8}, \frac{1}{2}\right)$ is not the demand, $\psi_1(p) + \psi_2(p) > 0$ for every $p \in P$. The first term $\psi_1(p)$ is a constant function of p_2 for $p_2 < -\frac{3}{4}$ and a strictly increasing function for $p_2 > -\frac{3}{4}$. The second term $\psi_2(p)$ is equal to

$$p \cdot \left(\frac{11}{8}, \frac{1}{2}\right) - \inf\left\{p \cdot y \mid y \in X \text{ and } u_a(y) \geq u_a\left(\frac{11}{8}, \frac{1}{2}\right)\right\}.$$

Hence it is a convex function of p_2. The value of the function is non-negative everywhere and equal to zero only at $p_2 = -1$. Hence $\psi_2(p)$ is a decreasing function of p_2 for $p_2 < -1$ and an increasing function of for $p_2 > -1$. Therefore, $\psi(g^n(a), p, a)$ is a strictly decreasing function of p_2 for $p_2 < -1$ and a strictly increasing function for $p_2 > -\frac{3}{4}$. Thus the minimum,

$$\min\left\{\psi_1(p) + \psi_2(a) \mid p \in P \text{ and } -1 \leq p_2 \leq -\frac{3}{4}\right\}, \tag{6}$$

is equal to

$$\inf\{\psi(g^n(a), p, a) \mid p \in P\}$$

for every positive integer n and $a \geq 1$. If we denote the minimum (6) by ε, then

$$\mu^n(\{a \in A^n \mid \psi^n(g^n(a), p^n, a) \geq \varepsilon\}) \geq \frac{n}{n+1}$$

for every sequence $(p^n)_n$, and the right hand side converges to one as $n \to \infty$. ///

We should point out that it is possible to construct an example of the failure of core convergence with the modifications of Proposition 21. Indeed, for the case of $\overline{q} < \infty$, if we define $(A^n, \mathcal{A}^n, \mu)$ as in Example 24 and (u^n, e^n) by

$$u_a^n(x) = \begin{cases} x_1 - s_{1/(n+2)}(x_2) & \text{for } a = 0, \\ x_1 - s_1(x_2) & \text{for } a \geq 1, \end{cases}$$
$$e^n(a) = w,$$

where $s_{1/(n+2)} : R_+ \to R$, $s_1 : R_+ \to R$, and $w \in \operatorname{int} X$ were defined as in the proof of Proposition 21, then the strong equilibrium (p^n, f^n) is given by

$$p^n = \left(1, -\frac{\overline{q} + 3\underline{q}}{4}\right),$$
$$f_2^n(a) = \begin{cases} \dfrac{n+2}{8}(\overline{q} - \underline{q}) & \text{for } a = 0, \\ \dfrac{1}{8}(\overline{q} - \underline{q}) & \text{for } a \geq 1. \end{cases}$$

Again, a transfer of a sufficiently small amount of the the good from each $a \geq 1$ to $a = 0$ gives rise to a sequence of core allocations violating the core convergence property. Since the upper bound $\overline{q}$ and lower bound $\underline{q}$ of marginal disutility can be made at any levels, we have both proper preferences and the failure of core convergence as in Manelli (1991).

7 Conclusion

We have explored some problems arising from the presence of bads in economies with infinitely many consumers. The most significant result was a work-out example (Example 10) of the non-existence of equilibria. We have also shown (Section 4) that in this example, there is even neither an efficient and individually rational allocation nor an efficient and envy-free allocation; and (Section 5) that the non-existence result survives various types of modifications of the example. Two examples were presented (Section 6), one to show that the limit of the sequence of equilibrium allocations of increasingly populous finite economies may not even be a resource-feasible allocation, and the other to show that the limit theorem of the core fails even in a simple setting of economies.

The most important future research topic is perhaps to give a set of sufficient conditions for the existence of equilibria. What we can see from Proposition 21 is that the existence of equilibria cannot be guaranteed by imposing bounds on the size of marginal disutility from bads. For any of the utility functions (equality (1)) constructed in its proof, however, one can show that at any given point, the Gaussian curvatures of indifference curves of the u_a converges to zero as $a \to 0$ and that the equi-convexity condition of Anderson (1981) is not satisfied. These conditions have turned out to be crucial for the limit theorem for the core, and may well be so too in our non-existence result.

A Proof of Lemma 8

It is sufficient to prove that if there exists a weak improvement g on an allocation f, then there also exists a strong improvement on f. Define

$$B = \{a \in A \mid u_a(g(a)) > u_a(f(a))\},$$
$$C = \{a \in A \mid g(a) \in \operatorname{int} X\}$$

Then $C \in \mathcal{A}$. We can prove that $B \in \mathcal{A}$ as follows: The mapping $(v, x) \mapsto v(x)$ of $\mathcal{U} \times X$ into R is continuous and hence $(\mathcal{B}(\mathcal{U}) \otimes \mathcal{B}(X), \mathcal{B}(R))$-measurable. Since both $a \mapsto u_a$ and f are measurable, by D.I.(4) of Hildenbrand (1974), the mapping $a \mapsto (u_a, f(a))$ of A into $\mathcal{U} \times X$ is $(\mathcal{A}, \mathcal{B}(\mathcal{U}) \otimes \mathcal{B}(X))$-measurable. Since the mapping $a \mapsto u_a(f(a))$ of A into R is the composite of these two mappings, it is $(\mathcal{A}, \mathcal{B}(R))$-measurable. We can similarly show that $a \mapsto u_a(g(a))$ is $(\mathcal{A}, \mathcal{B}(R))$-measurable. Thus

$B \in \mathcal{A}$.

If $\mu(A \setminus B) = 0$, then g would itself be a strong improvement on g. In the rest of the proof, therefore, we assume that $\mu(A \setminus B) > 0$.

Case 1 $\mu(B \cap C) > 0$.

For each positive integer n, define

$$D_n = \left\{ a \in B \cap C \mid g_1(a) > \frac{1}{n} \text{ and } u_a\left(g(a) - \frac{1}{n}y\right) > u_a(f(a)) \right\},$$

where $y = (1, 0) \in X$, then $D_n \in \mathcal{A}$, $D_n \subseteq D_{n+1}$ for every n, and $\bigcup_n D_n = B \cup C$ by the continuity of the u_a. Hence there exists an n such that $\mu(D_n) > 0$. Define then $h : A \to X$ by

$$h(a) = \begin{cases} g(a) - \dfrac{1}{n}y & \text{if } a \in D_n, \\ g(a) + \dfrac{\mu(D_n)}{n\mu(A \setminus B)}y & \text{if } a \in A \setminus B, \\ g(a) & \text{otherwise.} \end{cases}$$

Since the u_a are strictly increasing in the first commodity, h is a strong improvement on f.

Case 2 $\mu(B \cap C) = 0$.

In this case, it is sufficient to construct another weak improvement g' on f that falls into Case 1. For each positive integer n, define

$$\Gamma_n = \{(a, x) \in B \times X \mid u_a(x) < u_a(f(a))\} \bigcap \left\{ (a, x) \in B \times X \mid \|x - g(a)\| \leq \frac{1}{n} \right\},$$

where $\|\cdot\|$ denotes the Euclidean norm. Since both sets on the right hand side belongs to $\mathcal{A} \otimes \mathcal{B}(X)$, so does Γ_n. Since the measure space $(A, \mathcal{A}, \mu)$ (and hence its subspace B) is complete and X is a closed subset of R^2, by D.II.(11) of Hildenbrand (1974), the projection of Γ_n onto B, that is,

$$\left\{a \in B \mid \text{there exists an } x \in X \text{ such that } u_a(x) < u_a(f(a)) \text{ and } \|x - g(a)\| \leq \frac{1}{n}\right\},$$

belongs to $\mathcal{A}$. Denote this set by B_n, then $B_n \supseteq B_{n+1}$ for every n and $\bigcap_n B_n = \varnothing$ by continuity. Thus there exists an n such that $\mu(B \setminus B_n) > 0$.

By Lemma 7, $\mu(C) > 0$ and hence $\mu((A \setminus B) \cap C) > 0$. Then there exist a $\delta > 0$ and a $D \in \mathcal{A}$ such that $D \subseteq (A \setminus B) \cap C$, $\mu(D) > 0$, and $\min\{g_1(a), g_2(a)\} > \delta$ for every $a \in D$. We can assume without loss of generality that $\delta < \dfrac{\mu(B \setminus B_n)}{n\mu(D)}$.

Since each $v \in \mathcal{U}$ is continuous, strictly increasing in the first commodity, and strictly decreasing in the second, for every $x \in X$ with $\min\{x_1, x_2\} > \delta$, there exists a unique $y \in \operatorname{int} X$ such that $\|y\| = \delta$ and $v(x - y) = v(x)$. It is easy to show that the mapping $\kappa : (v, x) \mapsto y$ is continuous. Thus the mapping $k : D \to \operatorname{int} X$ defined by $k(a) = \kappa(u_a, g(a))$ is $(\mathcal{A}, \mathcal{B}(X))$-measurable. Then $\|k(a)\| = \delta$, $g(a) - k(a) \in \operatorname{int} X$, and $u_a(g(a) - k(a)) = u_a(g(a))$. Now define $g' : A \to X$ by

$$g'(a) = \begin{cases} g(a) - k(a) & \text{if } a \in D, \\ g(a) + \dfrac{1}{\mu(B \setminus B_n)} \displaystyle\int_D k & \text{if } a \in B \setminus B_n, \\ g(a) & \text{otherwise.} \end{cases}$$

Then g' is an allocation. Moreover, $\displaystyle\int_D k \in \operatorname{int} X$ and

$$\left\| \frac{1}{\mu(B \setminus B_n)} \int_D k \right\| \leq \frac{1}{\mu(B \setminus B_n)} \int_D \|k\| = \frac{\mu(D)\delta}{\mu(B \setminus B_n)} < \frac{1}{n}.$$

Hence, by the choice of B_n, $u_a(g'(a)) > u_a(f(a))$ for every $a \in B \setminus B_n$. Thus, if B' and C' are defined for g' just as B and C were defined for g, then $\mu(B' \cap C') \geq \mu(B \setminus B_n) > 0$. Thus g' is a weak improvement on f that falls into Case 1.

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