Distributing Awards Efficiently: More on King Solomon's Problem

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Abstract

We consider a multi-awards generalization of King Solomon's problem: k identical and indivisible awards should be distributed among n agents, $k < n$, with the top k valuation agents receiving the awards. Agents have complete information about each others' valuations. Glazer and Ma (1989) analyzed the single-prize (i.e., $k = 1$) version of this problem. We show that in the 'more than two agents' problem the mechanism of Glazer and Ma admits inefficient equilibria and thus fails to solve Solomon's problem. So, first we modify their mechanism to rule out inefficient equilibria and implement efficient prize allocation in subgame perfect equilibrium when there are at least three agents. Then it is shown that a simple repeated application of our modified mechanism will distribute k (> 1) prizes efficiently in subgame perfect equilibria without any monetary transfers in equilibrium. Finally, in the multi-awards case we relax the complete information assumption and achieve implementation of efficient allocation by iterative elimination of weakly dominated strategies, using a generalization of Olszewski's (2003) mechanism. *JEL* Classification Number: D78.

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1 Introduction

Glazer and Ma (1989) had formally addressed King Solomon's dilemma – the problem of giving a baby to the baby's true mother with two women both claiming to be the true mother. This problem, as Glazer and Ma had noted, is generically equivalent to awarding an indivisible prize to one of several agents who valued the prize most. While the agents knew each others' valuations, the planner (or King Solomon) had no such information. Glazer and Ma constructed extensive form mechanisms implementing the award rule in subgame perfect equilibrium (SPE) without any monetary transfers in equilibrium.[1]

More recently Perry and Reny (1999) have relaxed the complete information assumption of Glazer and Ma: in the case of two contenders as in King Solomon's problem, each agent knows her own value and each agent knows which of them has the higher value; however, neither agent knows the precise value of the other agent.[2] The authors suggest a variant of the second-price sealed-bid all-pay auction that implements the efficient allocation in iteratively undominated strategies. Latest, Olszewski (2003) has constructed a 'simpler' mechanism that requires only two rounds of elimination (of weakly dominated strategies) as opposed to Perry and Reny's (1999) four rounds of elimination.

Our primary objective is to generalize King Solomon's problem in a different direction, assuming complete information. We consider the problem of distributing k identical and indivisible prizes among n agents where $k < n$.[3] The objective of the planner is to give the prizes (in equilibrium) to the *top k valuation agents* at zero monetary costs to the planner and the agents.[4] The top k valuation agents need not be unique: corresponding to a decreasing order valuations $u_{(1)} \geq u_{(2)} \geq \ldots \geq u_{(k)} \geq \ldots \geq u_{(n)}$, there can be more than one ordering of the agent indexes

[1] Moore (1992) suggested a mechanism implementing Solomon's choice function in undominated Nash equilibrium, but criticized the mechanism for its use of an 'integer game' construction. He also described an extensive form mechanism, similar to Glazer and Ma's construction, for implementation in SPE.

[2] In the case of more than two contenders, also considered by Perry and Reny, each agent knows her own value and the identity of the highest value agent, and the highest value agent knows that her value is strictly higher than all other agents' values. In fact, only the highest value agent might know that she has the highest value.

[3] $k = 1$ is the problem considered by Glazer and Ma.

[4] Bag (1996) considered a related prize distribution problem – the problem of dividing a given amount of *divisible* resources among n agents. In the current paper, the prizes are *indivisible*.

if there are ties in some of the valuations. Thus, the agents who receive the prizes must have valuations among the top k ranks in at least one of the corresponding ordering of agent indexes.

To generalize the result of Glazer and Ma (1989) in the case of multiple awards requires a further analysis of their single-prize mechanism for *three or more* agents; they formally prove the implementation result for the two-agents problem only and in the appendix they outline a more elaborate mechanism claiming that it implements efficient allocations when there are at least three agents. It turns out that this latter claim of Glazer and Ma is *not always* true. There are two problems with their mechanism, one trivial and a second problem more substantial due to ties in agent valuations. As a result, Glazer–Ma mechanism results in multiple equilibria involving inefficient allocations of the prize. So, first we modify Glazer–Ma mechanism to complete the task of solving King Solomon's problem for a single prize and arbitrary number of agents (Theorem 1), then generalize the modified mechanism for the multiple awards problem (Theorem 2).

We also adopt the informational generalizations of Perry and Reny and Olszewski in our multiple awards case. Using a generalized version of Olszewski's (2003) mechanism efficient prize allocation is implemented when no two agents' valuations are ever tied, by iterative elimination of weakly dominated strategies. Our single- and multi-prize mechanisms for the complete information case contrast with our multi-prize generalization of Olszewski's incomplete information mechanism. First, the solution concepts are different (SPE vs. iterative deletion). Moreover, in the first two mechanisms because agents move *sequentially* (and thus the game has perfect information) backwards induction makes the equilibrium solution (SPE) especially attractive, while there is no such nice feature in the incomplete information version as agents move simultaneously. Finally, the first two mechanisms have the advantage of dealing with ties in agent valuations, whereas the incomplete information mechanism does not deal with ties.[5]

Our generalized formulation of King Solomon's problem will have the following applications. The prizes can be a fixed number of research grants of equal worth to be distributed based on the applicants' productivity unknown to the grant authority. A local government may want to award a limited number of commercial licenses to enterprizing individuals in an attempt to give their careers a start; or it may

[5]In fact, when the valuations can be arbitrarily close, the mechanism suggested by Olszewski and our generalization involve randomization.

decide to distribute fixed grants for multipurpose development projects (children's park, schools, libraries, adult education etc.) among various councils who know each others' overall benefits that might result from the grants but which might be unknown to the grant awarding committee.

In the next two sections, we focus on the complete information version of King Solomon's problem and its multi-awards generalization. In section 4, we relax the informational restriction. The Appendix contains an equilibrium existence result.

2 Mechanisms and Results: Complete Information

The planner wants to distribute a total of $k \geq 1$ identical, indivisible prizes among a set of agents, $\mathcal{N}$, with cardinality n. Agents have *complete information* about each others' valuations. The planner does not know the agents' valuations. Each agent ℓ's valuation, u_ℓ, is from the interval $[0, d]$ with at least one agent drawing a positive valuation.[6] The (net) utility from the award to agent ℓ is given by $u_\ell - \chi$ (or simply $-\chi$ if no award is given) where χ is the agent's monetary payment to the planner. Our implementation solution concept is subgame perfect equilibrium (SPE), same as in Glazer and Ma (1989).

Initially, let us consider the mechanism proposed by Glazer and Ma for $k = 1$ and $n > 2$ (the single-prize mechanism for more than two contenders).

Glazer–Ma Mechanism. Fix any ordering of the agents and index the chosen order as $1, 2, \ldots, n$, which is common knowledge.

Stage 0: Each agent l, $l = 2, 3, \ldots, n$, announces a real number ϵ_l from the interval $[0, d]$. Let $\epsilon = \min\{\epsilon_l, l = 2, \ldots, n\}$. If $\epsilon = 0$, the prize is given to agent 1. Otherwise, proceed to Stage i, where $i = 1$.

Stage i: Agent i says the prize will be "mine" or "not mine." If she says "not mine," then proceed to Stage $i+1$, $i = 1, \ldots, n-2$. If she says "mine," then proceed to Stage $i.i+r$, where $r = 1$. If at Stage $n-1$, agent $n-1$ says "not mine," then agent n gets the prize.

[6]In the extreme case of all agents having zero valuations for the prize, our mechanisms will trivially implement efficient allocations both in the single-prize and multiple awards problem.

Stage $i.i+r$: Agent $i+r$ says "challenge" or "not challenge." If agent $i+r$ says "not challenge" and $i+r+1 \leq n$, then proceed to Stage $i.i+r+1$; if agent $i+r$ says "not challenge" and $i+r=n$, then agent i gets the prize. If agent $i+r$ says "challenge," agent i and agent $i+r$ each pays ϵ. Then they proceed to game $\gamma(i, i+r)$.

$\gamma(i, i+r)$: Agent i bids $\hat{u}_i$ from $[0, d]$ and pays $\hat{u}_i$. Then agent $i+r$ bids $\hat{u}_{i+r}$ from $[0, d]$ and pays $\hat{u}_{i+r}$. The agent with the higher bid gets the prize. If there is a tie, agent $i+r$ gets the prize. $\|$

There are two difficulties with Glazer–Ma mechanism. *First*, if *at least one agent* (barring agent 1) announces zero at Stage 0 so that $\min_{l \neq 1} \epsilon_l = 0$, positive announcements by individual agents become inconsequential as the prize is given to agent 1 without going through the challenge/no-challenge route. This can give rise to inefficient equilibria with the prize not being awarded to the highest valuation agent, as the following example shows.

Example 1. Suppose there are only three agents with ordering 1,2,3, valuations $v_1 < v_2 < v_3$ and a single prize to be allocated. Then announcements $\epsilon_2 = \epsilon_3 = 0$ will be an equilibrium, given any SPE in the continuation game for $\min_{l \neq 1} \epsilon_l > 0$, so that the prize goes to agent 1 who has the lowest valuation.

Second, in the case of ties in agent valuations, the agents in Glazer–Ma mechanism can get stuck with all announcing the same high 'epsilon' in equilibrium where the prize is inefficiently awarded, if lowering of 'epsilon' by one of the agents with a higher valuation (than the inefficient winner's valuation) triggers a subgame in which some other agent gets to win the prize, as in the following example.

Example 2. Consider four agents with ordering 1,2,3,4, valuations $v_2 < v_1 < v_3 = v_4$ and a single prize to be allocated. Suppose $\epsilon_2 = \epsilon_3 = \epsilon_4 \, (= \bar{\epsilon}) \geq v_4$ so that $\min_{l \neq 1} \epsilon_l = \bar{\epsilon}$, and agent 1 claims the prize. Clearly agents 2,3, or 4 will not challenge. Now suppose agent 3 deviates to announce some $\hat{\epsilon}_3$, $0 < \hat{\epsilon}_3 < v_4$; instead of $\min_{l \neq 1} \epsilon_l$, now $\hat{\epsilon}_3$ is the required initial payment for disagreement. Following agent 3's deviation there will be two equilibria (by case (3) in Glazer and Ma's proof of Theorem 2), one of which will result in agent 4 receiving the prize and the other resulting in agent 3 receiving the prize. Thus, depending on which continuation equilibrium is expected to be played, agent 3 might not have an incentive to deviate from $\bar{\epsilon}$. Similarly, agent 4 may not have an incentive to deviate from $\bar{\epsilon}$ either. This will sustain $\bar{\epsilon}$ announcements in an SPE that results in the prize going to agent 1.

Both examples 1 and 2 would hold whether agents announce 'epsilon' simultaneously or sequentially.

The difficulties highlighted in the two examples could not occur in the two-agents problem analyzed by Glazer and Ma, but become relevant when there are at least three agents. The first difficulty, a minor one, requires only a little modification in the mechanism. The second difficulty is more serious: multiple equilibria in continuation games due to ties among best two (or more) agents' valuations leave little incentives for any agent to deviate from an inefficient equilibrium. Some careful modifications to Glazer–Ma mechanism are required mainly to *break* this kind of multiplicity.

Our modification of Glazer–Ma mechanism is as follows.

The Mechanism Γ. Fix any ordering of the agents and index the chosen order as $1, 2, \ldots, n$, which is common knowledge.

Stage 0: Agents $1, 2, \ldots, n$ sequentially announce real numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ from the interval $[0, d]$. Let

$$\epsilon = \begin{cases} \min_\ell\{\epsilon_\ell | \epsilon_\ell > 0\}, & \text{if } \epsilon_\ell > 0 \text{ for some } \ell \in \{1, 2, \ldots, n\} \\ 0, & \text{otherwise.} \end{cases}$$

If $\epsilon = 0$, give the prize to agent 1. Otherwise, define

$$\tilde{\epsilon}_\ell = \begin{cases} \epsilon_\ell, & \text{if } \epsilon_\ell > 0 \\ d, & \text{if } \epsilon_\ell = 0, \end{cases}$$

and proceed to Stage 1.

For any $1 \leq i < n$ define the following:

Stage i: Agent i says the prize will be "mine" or "not mine." If she says "not mine," then proceed to Stage $i+1$, $i = 1, \ldots, n-2$. If she says "mine," then proceed to Stage $i.i+r$, where $r = 1$. If at Stage $n-1$ agent $n-1$ says "not mine," then agent n gets the prize.

Stage $i.i+r$: Agent $i+r$ says "challenge" or "not challenge." If agent $i+r$ says "not challenge" and $i+r < n$, then proceed to Stage $i.i+r+1$; if $i+r$ says "not challenge" and $i+r = n$, then agent i gets the prize. If agent $i+r$ says "challenge," agent i and agent $i+r$ each pays ϵ to the planner. Then they proceed to play the disagreement game $\gamma(i, i+r)$.

$\gamma(i, i+r)$: Agent i bids $\hat{u}_i \in [0, d]$ and pays $\hat{u}_i$ to the planner. Then agent $i+r$ bids $\hat{u}_{i+r} \in [0, d]$ and pays $\hat{u}_{i+r}$ to the planner. The agent with the higher bid gets the prize; if there is a tie, agent $i+r$ gets the prize. Finally, if agent i (agent $i+r$) is the winner, she pays $\tilde{\epsilon}_i$ (resp. $\tilde{\epsilon}_{i+r}$) to the planner and the loser pays nothing.[7] $\|$

The key points of difference between Glazer–Ma mechanism and our modification can now be summarized as follows:

• in our mechanism agents announce 'epsilon' sequentially, whereas in Glazer and Ma the nature of announcements is not clearly specified;[8]

• our definition of ϵ differs from that of Glazer and Ma;

• in the event of disagreement our mechanism requires the winner to pay additionally an $\tilde{\epsilon}_\ell$ (on top of ϵ), unlike in Glazer and Ma.

The mechanism for multiple prizes is a simple extension of the single-award mechanism and outlined next.

The Mechanism Γ^k. k identical prizes are distributed by at most k-rounds application of the mechanism Γ. At the beginning, draw an agent ordering once-for-all.

Round 1. Allocate the first unit of the prizes applying the mechanism Γ. The recipient of the prize, the challenger if there is a claim and a challenge, and any agent who does not claim the prize when it is *her* turn to claim – all leave(s) the game at the end of the first round. Proceed to the next round.

Round 2. Allocate the second unit of the prizes once again applying the mechanism Γ for the order, inherited from the first round, with respect to the agents who survived the first round. Similar to round 1, the recipient of the second prize, the challenger if there is a claim and a challenge, and any agent not claiming

[7] The assumption that agents' valuations are bounded is not essential in the construction of the mechanism Γ. The only role of the upper bound, d, is in defining $\tilde{\epsilon}_\ell$ when $\epsilon_\ell = 0$. If d is unbounded, keep the mechanism Γ same as above except that whenever $\epsilon_\ell = 0$ agent ℓ is not allowed to proceed beyond Stage 0 and the bids in the game, $\gamma(i, i+r)$, can be any non-negative real number.

[8] Since they do not formally analyze the case of many agents, the ambiguity about the nature of announcements remained.

the prize on her turn – all leave(s) the game at the end of the second round. Proceed to the next round.

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Follow this sequential-rounds procedure until all k units have been distributed. If in the process the number of remaining agents becomes equal to the number prizes left, give each agent a prize. ‖

The following definitions are in the context of any subgame at the beginning of a round with τ more prizes left, $1 \leq \tau \leq k$.

Definition 1 *An agent is* <u>*deserving*</u> *if she merits a place among top τ ranks for* <u>*at least one*</u> *ordering of the remaining agents' indexes arranged in a decreasing order of valuations.*

An agent is <u>*surely deserving*</u> *if she merits a place among top τ ranks for* <u>*all*</u> *ordering of the remaining agents' indexes arranged in a decreasing order of valuations.*

An agent is <u>*marginal*</u> *if she is deserving but not surely deserving.*

An agent is <u>*undeserving*</u> *if she is not deserving.*

Note that if $m = \tau$ where m is the number of deserving agents, then all the deserving agents are also surely deserving.

Theorem 1 (**Single-prize implementation**) *Suppose there are $n \geq 2$ number of contenders for a single prize. In every SPE of the mechanism Γ, the award goes to the deserving agent placed 'last' in the agent order* (*'last' among all deserving agents*)*. Moreover, no agent pays any monetary transfers to the planner.*

In the case of multiple awards problem, $\Gamma^{\mathcal{A},\tau}$ denotes any subgame starting in a round $k - \tau + 1$ with remaining agents $\mathcal{A} \subset \mathcal{N}$ and τ prizes to be distributed.[9] Let $\int_\tau$ $(0 \leq \int_\tau \leq \tau)$ be the number of surely deserving agents in this subgame.

Theorem 2 (**Multiple awards implementation**) *Consider SPE. In any subgame $\Gamma^{\mathcal{A},\tau}$ for any $\mathcal{A}$ and $1 \leq \tau \leq k$, τ prizes are distributed as follows:*

Each surely deserving agent will receive a prize.

[9] Γ^k is simply a subgame $\Gamma^{\mathcal{A},\tau}$, where $\mathcal{A}$ the set of all n agents and $\tau = k$.

If more than τ agents have positive valuations then any remaining prizes will be received by $\tau - \int_\tau$ marginal agents strictly in the reverse order from the last; if less than τ agents have positive valuations and thus the marginal agents have zero valuations, the remaining prizes will be received by some $\tau - \int_\tau$ marginal agents.[10]

Finally, no agent pays any monetary transfers.

In both Theorems 1 and 2, the recipient(s) of the prize(s) is (are) *uniquely* determined by the order of agents.[11] This implies, if the planner has some strict preference ordering over who should receive the prize(s) when there are ties in agent valuations, he would like to place the agents in an order exactly opposite to his preferred ranking.

Theorems 1 and 2 may hold vacuously. In the Appendix we outline strategies to ensure that an equilibrium exists.

3 Proofs of Theorems 1 and 2

For any strategy profile and any round, suppose ϵ_ℓ is announced in that round by agent ℓ. Denote agent ℓ's *virtual valuation* in that round (which is her net utility in the challenge game if she wins the prize) by

$$\vartheta_\ell = u_\ell - \tilde{\epsilon}_\ell.$$

By definition, ϑ_ℓ can be negative. Note that virtual valuation does not take into account the initial $\epsilon > 0$ that an agent will incur for entering into the challenge game.

Lemma 1 *Consider any SPE σ of Γ^k. For any subgame of Γ^k, the continuation strategy profile given by σ is such that a claim will be challenged by an agent in any round only if her virtual valuation in that round is positive and weakly exceeds the claimant's virtual valuation in that round.*

Proof. The challenger's virtual valuation must be positive for her to be willing to pay a positive ϵ for entering into the challenge game.

[10]For *any* $\tau - \int_\tau$ marginal agents, there will be at least one SPE such that these marginal agents each receive a prize.

[11]Except in the special situation of the multi-awards problem where less than k agents draw positive valuations.

To show the second result, suppose not. Suppose there is an equilibrium such that in the j-th round subgame, agent i with virtual valuation ϑ_i claims a prize and is challenged by some agent $i+r$ whose virtual valuation is ϑ_{i+r}, and $\vartheta_{i+r} < \vartheta_i$.

As per mechanism rule, both i and $i+r$ pay $\epsilon > 0$ each and proceed to play the game $\gamma(i, i+r)$. Now, by a similar argument as in Glazer and Ma (see case (1) in the proof of their Theorem 2) the claimant i will bid $\hat{u}_i = \vartheta_{i+r}$ and the challenger $i+r$ will bid $\hat{u}_{i+r} = 0$, and the prize will go to claimant i. But then agent $i+r$ would have been better off not to challenge agent i and save $\epsilon > 0$, a contradiction. **Q.E.D.**

Lemma 2 *Fix any SPE σ of Γ^k. Then there does not exist a subgame such that the continuation strategies of σ prescribe a claim to be followed by a challenge.*

Proof. Suppose not. Then there is some equilibrium σ of Γ^k such that it results in agent i claiming a prize to be later challenged by agent $i+r$ in some subgame. Then by Lemma 1, in that round $\vartheta_{i+r} \geq \vartheta_i$ and moreover $\vartheta_{i+r} > 0$. As agent $i+r$ challenges, both i and $i+r$ pay $\epsilon > 0$ each and proceed to play $\gamma(i, i+r)$.

Suppose $\vartheta_{i+r} > \vartheta_i$. But then by a similar argument as in case (2) of Theorem 2 of Glazer and Ma, in the disagreement subgame the claimant i would bid $\hat{u}_i = 0$, the challenger $i+r$ would bid $\hat{u}_{i+r} = 0$, and the prize will go to the challenger. But then agent i would have been better off not to claim the prize and instead save $\epsilon > 0$, a contradiction.

Suppose $\vartheta_{i+r} = \vartheta_i$. By a similar argument as in case (3) of Theorem 2 of Glazer and Ma, agent i will receive a payoff of zero in the bidding game $\gamma(i, i+r)$. So agent i will not make a claim to be challenged by agent $i+r$ and pay $\epsilon > 0$, a contradiction. **Q.E.D.**

Proof of Theorem 1. Consider any SPE strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$. Denote the deserving agent placed 'last' (among all deserving agents) by index j. By the assumption that at least one agent draws a positive valuation, $u_j > 0$. First, we claim that agent j will win the prize.

Suppose not, so that agent j's equilibrium payoff is non-positive. Denote the equilibrium announcements at Stage 0 by (ϵ_ℓ).

Let agent j deviate to another strategy σ'_j that is otherwise identical to the equilibrium strategy σ_j except that:

(1) announce $\hat{\epsilon}_j$ such that

(i) $0 < \hat{\epsilon}_j < \tilde{\epsilon}_\ell \quad \forall \ell < j$,

(ii) $\hat{\epsilon}_j < u_j - u_\ell \quad \forall \ell > j$,[12]

(iii) $2\hat{\epsilon}_j < u_j$;

(2) always challenge; and
(3) always claim.

Following agent j's deviation to σ'_j, let $\hat{\epsilon}$ be the modified ϵ taking into account any follow-up changes to ϵ_ℓ, $\ell > j$. Clearly, $0 < \hat{\epsilon} \leq \hat{\epsilon}_j$. Next we check agent j's payoffs if she deviates, taking others' strategies as given.

Since the mechanism Γ is same as Γ^1, Lemma 2 rules out claim by any agent $\ell < j$ followed by challenge by another agent $\ell' < j$. So if some agent ℓ claims before agent j, by challenging agent j will win in the bidding game $\gamma(\ell, j)$ because $\vartheta_j = u_j - \hat{\epsilon}_j > u_\ell - \tilde{\epsilon}_\ell = \vartheta_\ell$ (inequality follows from (1.i)) and case (2) of Theorem 2 of Glazer and Ma would apply. Agent j's overall payoff is then $u_j - \hat{\epsilon}_j - \hat{\epsilon} \geq u_j - 2\hat{\epsilon}_j > 0$ (the last inequality follows from (1.iii)).

Next, check agent j's payoff from claiming the prize (given that no agent $\ell < j$ claims). For $\ell > j$, condition (1.ii) implies $u_j - \hat{\epsilon}_j > u_\ell \geq 0$ so that j's virtual valuation is positive and exceeds the virtual valuation of every undeserving agent to follow. Thus, if agent j is challenged, she will win in the bidding game $\gamma(j, \ell)$ (by case (1), Theorem 2, Glazer and Ma) and her overall payoff will be $u_j - \hat{\epsilon}_j - \hat{\epsilon}$, which is already shown to be positive; otherwise agent j receives the prize unchallenged and gets a positive payoff.

Thus, by deviating agent j will always receive positive payoffs, contradicting that she will not win the prize.

Finally, since by Lemma 2 there can be no disagreement in equilibrium, neither agent j nor any other agent pays any monetary transfers to the planner. **Q.E.D.**

Proof of Theorem 2. Suppose in any final round subgame $\Gamma^{\tilde{\mathcal{A}},1}$ for any $\tilde{\mathcal{A}} \subset \mathcal{N}$ with only one prize left to be distributed, at least one agent in $\tilde{\mathcal{A}}$ has a positive valuation. By Theorem 1, in any equilibrium of $\Gamma^{\tilde{\mathcal{A}},1}$ the deserving agent placed 'last' (among all deserving agents) will win the prize without paying any monetary transfers.

[12]Clearly, $u_j - u_\ell > 0$.

Next, in any subgame $\Gamma^{\mathcal{A},\tau}$ for any $\mathcal{A} \subset \mathcal{N}$, denote by $\mathcal{W}(\mathcal{A},\tau)$ the union of $\int_\tau$ surely deserving agents and the 'last' $\tau - \int_\tau$ marginal agents in the agent order.

Now assume the following hypothesis is true: *Fix any* $1 \leq \tau \leq k-1$. *In any subgame* $\Gamma^{\mathcal{A},\tau}$ *for any* $\mathcal{A} \subset \mathcal{N}$, *of whom at least* τ *agents have positive valuations, every member of* $\mathcal{W}(\mathcal{A},\tau)$ *will win a prize without paying any monetary transfers.*

Then consider any subgame $\Gamma^{\mathcal{A}',\tau+1}$ for any $\mathcal{A}' \subset \mathcal{N}$, of whom at least $\tau+1$ agents have positive valuations. We will argue that every member of $\mathcal{W}(\mathcal{A}',\tau+1)$ will win a prize without paying any monetary transfers.

Suppose not. Since by Lemma 2 there can be no disagreement in equilibrium, our contraposition implies that for some $\mathcal{A}' \subset \mathcal{N}$ and some subgame $\Gamma^{\mathcal{A}',\tau+1}$, some agent $f \notin \mathcal{W}(\mathcal{A}',\tau+1)$ will win a prize. This, in turn, implies some agent $\rho \in \mathcal{W}(\mathcal{A}',\tau+1)$ is *not* going to win any prize in this subgame so that her payoff must be non-positive in the subgame. Consider the following strategy σ'_ρ by agent ρ in subgame $\Gamma^{\mathcal{A}',\tau+1}$:

- σ'_ρ is otherwise the same as her equilibrium strategy if the number of prizes left is less than or equal to τ; and
- in the subgame with $\tau+1$ prizes, agent ρ's strategy, σ'_ρ, is again otherwise the same as her equilibrium strategy except that:

(1) at Stage 0 of the first round of the subgame agent ρ announces $\hat{\epsilon}_\rho$ such that

(i) $0 < \hat{\epsilon}_\rho < \tilde{\epsilon}_\ell \;\; \forall \ell < \rho$,

(ii) $\hat{\epsilon}_\rho < u_\rho - u_\ell \quad \forall \ell > \rho,\; \ell \notin \mathcal{W}(\mathcal{A}',\tau+1)$,[13]

(iii) $2\hat{\epsilon}_\rho < u_\rho$;[14]

(2) challenge only claims by agents whose virtual valuations are strictly less than her new virtual valuation $u_\rho - \hat{\epsilon}_\rho$;
(3) always claim.

We now check agent ρ's payoffs when σ'_ρ is chosen in the subgame.

No agent $\ell < \rho$ with virtual valuations below that of agent ρ will claim in the initial round: agent ρ will challenge any such claim and win to receive a positive payoff, by condition (1.iii). Given condition (1.i), any agent $\ell < \rho$ who would

[13] Such an ℓ must be undeserving so that $u_\rho - u_\ell > 0$.

[14] $u_\rho > 0$ by the assumption that at least $\tau+1$ agents have positive valuations.

possibly claim is one with a higher valuation than that of agent ρ,[15] implying that ℓ is also surely deserving and therefore a member of $\mathcal{W}(\mathcal{A}', \tau+1)$. Since agent ρ is not going to challenge any such claim, she will proceed to some subgame $\Gamma^{\mathcal{A},\tau}$ where $\mathcal{A} \subset \mathcal{A}' \setminus \ell$ and be a member of $\mathcal{W}(\mathcal{A}, \tau)$. But then by our hypothesis agent ρ will win a prize without paying any monetary transfers.

No agent $\ell > \rho$ will challenge ρ's claim in the first round of the subgame. Why? If $\ell \notin \mathcal{W}(\mathcal{A}', \tau+1)$ then it must be that $u_\rho > u_\ell$, which, by condition (1.ii), implies that agent ℓ will lose in the bidding game to follow ($\vartheta_\rho > \vartheta_\ell$). Therefore agent ℓ will not challenge. On the other hand, if $\ell \in \mathcal{W}(\mathcal{A}', \tau+1)$ then agent ℓ knows that by not challenging agent ρ and thus proceeding to any subgame $\Gamma^{\mathcal{A},\tau}$, where $\mathcal{A} \subset \mathcal{A}' \setminus \rho$, she will be a member of $\mathcal{W}(\mathcal{A}, \tau)$ and therefore receive a prize without paying any monetary transfers (by hypothesis). So, by not challenging, agent ℓ additionally saves $\epsilon > 0$.

Thus, by deviating agent ρ will receive positive payoffs, contradicting that f will win a prize.

Our hypothesis is true for $\tau = 1$ and any $\tilde{\mathcal{A}}$ (with at least one of them having a positive valuation), so use induction to conclude that in any game $\Gamma^{\mathcal{A},\tau}$ with at least τ agents having positive valuations, every member of $\mathcal{W}(\mathcal{A}, \tau)$, i.e. $\int_\tau$ surely deserving agents and $\tau - \int_\tau$ marginal agents in the reverse order from the last, will win a prize without paying any monetary transfers. Since by Lemma 2 there can be no disagreement in equilibrium, agents outside $\mathcal{W}(\mathcal{A}, \tau)$ do not pay any monetary transfers. Thus, Theorem 2 is established in the special case when at least τ agents have positive valuations.

Next, consider for any $\mathcal{A}$ and any τ the case when less than τ agents have positive valuations. These agents are surely deserving, while rest of the agents who all have zero valuations are marginal agents. Now each surely deserving agent will win a prize by applying the same argument as in the previous case, by defining $\mathcal{W}(\mathcal{A}, \tau)$ (for all τ) to consist of only the surely deserving agents, and modifying the induction hypothesis as follows:[16] *Fix any $1 \leq \tau \leq k-1$. In any subgame $\Gamma^{\mathcal{A},\tau}$ for any $\mathcal{A} \subset \mathcal{N}$ with less than τ agents having positive valuations, every member of $\mathcal{W}(\mathcal{A}, \tau)$ will win a prize without paying any monetary transfers.* The rest of the

[15] Otherwise agent ρ's virtual valuation will be higher.

[16] If a surely deserving agent does not win a prize then equate her deviation strategy with the deviation strategy σ'_ρ of agent ρ, defined above, to obtain a contradiction.

prizes will be received by *some* $\tau - \int_\tau$ marginal agents. Finally, Lemma 2 ensures that no agent pays any monetary transfers to the planner. **Q.E.D.**

4 Relaxing Informational Restriction

In this section, we analyze the multiple awards problem but under less than complete information assumption: Each agent knows her own valuation plus the *identity* of the top k valuation agents but not their exact valuations (with the exception of her own valuation, if she happens to be one of them). As before, the planner wants to give k prizes to the top k valuation agents. This is a further generalization of the single-prize problem studied by Perry and Reny (1999), and Olszewski (2003). Our analysis below will be based on Olszewski's paper.

Let us start with the following preliminary assumption:

Assumption 1 *It is commonly known that there exists some* $\delta > 0$ *such that* $\forall i, j \in \{1, \ldots, n\}$, $i \neq j$, $|u_i - u_j| > \delta$.

Assumption 1 would be valid if, for example, each agent's valuation is drawn from a finite set and no two sets have nonempty intersection. Clearly, the assumption will hold in other situations as well.[17] Later on this assumption is relaxed.

The following mechanism, which is a generalization of Olszewski's single-prize mechanism, will distribute k prizes efficiently.

The Mechanism $\mathcal{M}$.

Each agent says "mine" or "not mine."

If exactly k agents say "mine" then each such agent is awarded a prize and the rest get zero payoffs.

If more than k agents say "mine," each agent gets a zero payoff.

If less than k agents say "mine" then all agents participate in a $(k+1)$th-price sealed-bid auction, defined as follows. Arrange the bids in a descending order: $b_{(1)} \geq \cdots \geq b_{(k)} \geq \cdots \geq b_{(n)}$. If top k bidders are unique according to any ordering, give each such bidder a prize for which they individually pay the $(k+1)$th highest bid, $b_{(k+1)}$, and $n-k$ losers pay nothing. If top k bidders are not unique, give each

[17] While there may still exist a commonly known (finite) upper bound, d, to agents' valuations, clearly the upper bound is never attained for more than one agent. This will be true even when Assumption 1 is replaced by Assumption 2.

agent bidding higher than $b_{(k+1)}$ a prize, no agent bidding lower than $b_{(k+1)}$ gets any prize, and the remaining prizes are randomly distributed among the agents who all bid the same value $b_{(k+1)}$; those receiving a prize all pay $b_{(k+1)}$, and the losers pay nothing. Finally, the planner makes extra payments to all agents as follows: each of the k winners receive $b_{(k+1)} - \delta$ while the remaining $n - k$ agents receive $b_{(k)} - \delta$ each. ||

By standard arguments, truthful bidding is a weakly dominant strategy for each bidder in the $(k+1)$th-price auction defined above. Moreover, by Assumption 1, any two agents' bids will differ by at least δ, thus there will be no ties. So the agents' payoffs can be described by the following matrices.

Any deserving agent ℓ's payoff (ℓ placed along the Row and 'other agents' placed along the Column) is:[18]

	k or more say mine	$k-1$ say mine	less than $k-1$ say mine
mine	0	u_ℓ	$u_\ell - \delta$
not mine	0	$u_\ell - \delta$	$u_\ell - \delta$

Any undeserving agent ℓ's payoff (ℓ placed along the Row and 'other agents' placed along the Column) is:

	k or more say mine	$k-1$ say mine	less than $k-1$ say mine
mine	0	u_ℓ	$u_{(k)} - \delta$
not mine	0	$u_{(k)} - \delta$	$u_{(k)} - \delta$

When ℓ is deserving (i.e., among top k valuation agents), clearly saying "mine" is a weakly dominant strategy. When ℓ is undeserving (i.e., not among top k valuation agents), saying "not mine" is a weakly dominant strategy because $u_{(k)} - \delta > u_\ell$ (by Assumption 1 and the fact that ℓ is undeserving). Thus, the top k valuation agents will say "mine," the rest say "not mine," and the prizes go to the top k valuation agents.

Similar to Olszewski we now relax the assumption of common knowledge lower bound on the difference in valuations, by using some real-valued random variable C such that the probability of $C > r$ is positive for every real number r.

[18] Any agent knows whether she is deserving or undeserving because she knows the identity of the top k valuation agents.

Assumption 2 *It is commonly known that* $\forall i, j \in \{1, \ldots, n\}$, $i \neq j$, $u_i \neq u_j$.

Modify the mechanism $\mathcal{M}$ by changing specifications only when *less than k agents say "mine"* (leaving the rest unchanged), as follows.

Agents take part in the following modified $(k+1)$th-price sealed-bid auction. Arrange the bids once again as $b_{(1)} \geq \cdots \geq b_{(k)} \geq \cdots \geq b_{(n)}$.

If $b_{(k)} > C$ then distribute the prizes as follows:

if top k bidders are unique according to any ordering, then each $\ell \leq k$ gets a prize and pays $\max\{b_{(k+1)}, C\}$, and no $\ell > k$ gets a prize nor pays anything;

if top k bidders are not unique, then each agent bidding higher than $b_{(k+1)}$ gets a prize, no agent bidding lower than $b_{(k+1)}$ gets any prize, and the remaining prizes are randomly distributed among the agents who all bid the same value $b_{(k+1)}$; those receiving a prize all pay $\max\{b_{(k+1)}, C\}$, and the losers pay nothing.

If $b_{(k)} < C$ then no one gets any prize.

Finally, irrespective of whether any prize is given or not, the planner pays $b_{(k+1)}$ to each $\ell \leq k$, and pays $b_{(k)}$ to each $\ell > k$. ||

Again, truthful bidding is a weakly dominant strategy for each bidder in the above modified $(k+1)$th-price auction. Below we show that the top k valuation agents will say "mine" and the rest of the agents will say "not mine".

Since agents bid truthfully, the actual bids can be ordered as $u_{(1)} > u_{(2)} > \cdots > u_{(k)} > u_{(k+1)} > \cdots > u_{(n)}$.[19]

Consider the payoffs to any deserving agent $\ell \leq k$:

If $C \leq u_{(k+1)}$ then ℓ gets a prize and her payoff is $u_\ell - u_{(k+1)} + u_{(k+1)} = u_\ell$;

If $u_{(k+1)} < C < u_{(k)}$ then ℓ gets a prize and her payoff is $u_\ell - C + u_{(k+1)} < u_\ell$;

If $u_{(k+1)} < u_{(k)} < C$ then ℓ does not get any prize and her payoff is $u_{(k+1)} < u_\ell$.

Denote the expected payoff to agent $\ell \leq k$ in the bidding game (plus the receipt from the planner) by $E_\ell(B)$. Clearly $E_\ell(B) < u_\ell$. Now we can summarize agent ℓ's payoff in the modified mechanism in the following matrix:

	k or more say mine	$k-1$ say mine	less than $k-1$ say mine
mine	0	u_ℓ	$E_\ell(B)$
not mine	0	$E_\ell(B)$	$E_\ell(B)$

[19]Note that because of Assumption 2 there will be no ties.

Next, consider any undeserving agent $\ell > k$. Agent ℓ will not win any prize in the bidding game, so her payoffs in the modified mechanism can be summarized as follows:

	k or more say mine	$k-1$ say mine	less than $k-1$ say mine
mine	0	u_ℓ	$u_{(k)}$
not mine	0	$u_{(k)}$	$u_{(k)}$

For $\ell \leq k$ (i.e., ℓ deserving), clearly saying "mine" is a weakly dominant strategy. On the other hand, for $\ell > k$ (i.e., ℓ undeserving), saying "not mine" is a weakly dominant strategy. Thus, the modified mechanism will implement efficient allocation of k prizes.

The implementation result in this section under weaker assumption about agents' information depends crucially on the basic assumption that there is no possibility of a tie in any two (or more) agents' valuations. The same is also true of Olszewski (2003), and Perry and Reny (1999). In contrast, our results in section 2 allow for possible ties.

5 Appendix

Theorems 1 and 2 are about equilibrium characterizations. Below we outline strategies to guarantee that an equilibrium exists.

Let $\Omega^\ell = \{l > \ell | u_l < u_\ell\}$.

Recall, $\Gamma^{\mathcal{A},\tau}$ is the subgame with only τ more prizes left, $1 \leq \tau \leq k$, for any set of remaining agents $\mathcal{A}$. Also, let $\mathcal{W}(\mathcal{A},\tau)$ be the union of $\int_\tau$ surely deserving agents and the 'last' $\tau - \int_\tau$ marginal agents in the agent order, as defined in the proof of Theorem 2.

• The strategy of any $\ell \in \mathcal{A}$ such that $u_\ell > 0$ in the first round of any subgame $\Gamma^{\mathcal{A},\tau}$ is as follows, where throughout $l \in \mathcal{A}$:

(A.1) announce ϵ_ℓ in that round to satisfy

(i) $0 < \epsilon_\ell < \tilde{\epsilon}_l \;\; \forall l < \ell$,

(ii) $\epsilon_\ell < u_\ell - u_l \quad \forall l \in \Omega^\ell$,

(iii) $2\epsilon_\ell < u_\ell$;

(A.2) if $\tau = 1$ then claim iff $\forall l > \ell$,

$$\text{either} \quad \vartheta_\ell > \vartheta_l \qquad \text{or} \quad \{\vartheta_\ell = \vartheta_l \text{ and } \vartheta_l - \epsilon < 0\}; \tag{1}$$

if $\tau > 1$ then claim iff $\forall l > \ell$, either $l \in \mathcal{W}(\mathcal{A} \setminus \ell, \tau - 1)$ or (1) holds;

(A.3) in the case $\tau = 1$, if some $l < \ell$ claims then challenge her iff

$$\vartheta_\ell \geq \vartheta_l \tag{2}$$

and

$$\vartheta_\ell - \epsilon \geq 0; \tag{3}$$

in the case $\tau > 1$, if some $l < \ell$ claims then challenge her iff $\ell \notin \mathcal{W}(\mathcal{A} \setminus l, \tau - 1)$, and (2) and (3) hold;

(A.4) in the challenge game $\gamma(i, i + r)$ where $\ell = i$ or $i + r$, if $\vartheta_i \neq \vartheta_{i+r}$, play the unique SPE defined in Glazer and Ma (for their single-prize, two-bidders challenge game) in their Theorem 2 proof;[20] if $\vartheta_i = \vartheta_{i+r}$, play the SPE of Glazer and Ma such that $i + r$ wins with probability 1;

• The strategy of any $\ell \in \mathcal{A}$ with $u_\ell = 0$, in any round is:

(B.1) announce arbitrary $\epsilon_\ell > 0$ in that round;

(B.2) never claim;

(B.3) never challenge any claim;

(B.4) play the challenge game as in (A.4).

This completes the description of an equilibrium. ||

Existence Proof. We provide a sketch of the proof that the above strategies constitute an SPE of Γ^k, by showing that for any $\ell \in \mathcal{A}$ such that $u_\ell > 0$ the choices specified in the first round of any subgame $\Gamma^{\mathcal{A},\tau}$ by the strategy described by (A.1)-(A.4) are optimal for ℓ, given that in the continuation game all players follow the strategies described by (A.1)–(A.4) or (B.1)–(B.4). (For the case of a player $\ell \in \mathcal{A}$ such that $u_\ell = 0$ the strategy described by (B.1)–(B.4) is clearly optimal in any subgame.)

[20] Except that now virtual valuations play the role of valuations. Even if an agent's virtual valuation can be negative, the fact that the only permissible bids in the challenge game are from the interval $[0, d]$ makes it straightforward to apply, with minor modification, Glazer and Ma's analysis of the challenge game (for non-negative valuations) to our setting.

We shall sketch this proof by considering the choices of ℓ in the first round of $\Gamma^{\mathcal{A},\tau}$ backwards starting with the choices in the challenge game of this round.

Before outlining the sketch, note that if the above strategies are implemented, then in any subgame $\Gamma^{\mathcal{A}',\tau'}$ the set of agents who receive the prizes is $\mathcal{W}(\mathcal{A}',\tau')$. In our arguments below, throughout $l \in \mathcal{A}$.

• In the challenge game, the specified strategies will be optimal as in Glazer and Ma.

• Let us check whether agent ℓ will enter the challenge game or not, assuming that in the continuation game everyone else follows the above strategies.

For $\tau = 1$, clearly the strategy of when to challenge is optimal. So consider $\tau > 1$. Suppose that some agent $l < \ell$ claims.

If either (2) or (3) does not hold then clearly it is optimal for agent ℓ not to challenge, and hence not to enter the challenge game. Next suppose $\ell \in \mathcal{W}(\mathcal{A} \setminus l, \tau - 1)$. Given the order of agents in $\mathcal{A}$, label the agents in $\mathcal{W}(\mathcal{A} \setminus l, \tau - 1)$ in reverse order $z_1, z_2, \ldots$ up to agent ℓ. The last agent, z_1, will not challenge l, because then either l remains unchallenged or is challenged by some $k > z_1$, $k \notin \mathcal{W}(\mathcal{A} \setminus l, \tau - 1)$ and in either case z_1 will receive a prize in a later round without paying monetary transfers (note that $\mathcal{W}(\mathcal{A} \setminus l, \tau - 1) = \mathcal{W}(\mathcal{A} \setminus \{l, k\}, \tau - 1)$). Agent z_2 will not challenge l either, because after her only a member outside $\mathcal{W}(\mathcal{A} \setminus l, \tau - 1)$ (such as k) might challenge l, and z_2 will then receive a prize without paying monetary transfers. Repeating this argument backwards, $\ell \in \mathcal{W}(\mathcal{A} \setminus l, \tau - 1)$ will not challenge l. Therefore ℓ will not enter the challenge game.

Next, suppose $\ell \notin \mathcal{W}(\mathcal{A} \setminus l, \tau - 1)$, and (2) and (3) hold. It is not difficult to see that $\ell \notin \mathcal{W}(\mathcal{A} \setminus \{l, k\}, \tau - 1)$ for any $k > \ell$ who challenges l in the continuation game, because then $k \notin \mathcal{W}(\mathcal{A} \setminus l, \tau - 1)$ so that $\mathcal{W}(\mathcal{A} \setminus l, \tau - 1) = \mathcal{W}(\mathcal{A} \setminus \{l, k\}, \tau - 1)$. But then because ℓ is not a member of $\mathcal{W}(\mathcal{A} \setminus l, \tau - 1)$, by letting go the opportunity to challenge the claim by l agent ℓ will receive zero payoff. But by conditions (2) and (3), agent ℓ can obtain a non-negative payoff by challenging l. So it is optimal for agent ℓ to challenge.

• Next we check whether it is optimal for agent ℓ to claim, assuming that in the continuation game everyone else follows the above strategies.

For $\tau = 1$, clearly the strategy of when to claim is optimal. So consider $\tau > 1$. Suppose for some $l > \ell$, $l \notin \mathcal{W}(\mathcal{A} \setminus \ell, \tau - 1)$, both $\vartheta_l \geq \vartheta_\ell$ and $\vartheta_l - \epsilon \geq 0$. By definition of (A.3), agent l will challenge a claim by agent ℓ and l will win in the challenge game. Therefore agent ℓ will not claim.

On the other hand, if $\forall l > \ell$, either $l \in \mathcal{W}(\mathcal{A} \setminus \ell, \tau - 1)$ or (1) holds then no such l will challenge (by definition of (A.3)). So it is optimal for agent ℓ to claim.

• The optimality of the announcement strategy (A.1) can be demonstrated using similar arguments to that in Theorem 2 proof. **Q.E.D.**

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