Some Numerical Considerations in $\mathcal{H}_\infty$ Control (special issue JCW)

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Abstract

Certain $\mathcal{H}_\infty$-control problems are considered and potential numerical difficulties. An alternative test for checking that the solution to an algebraic Riccati equation is positive semi-definite which is robust to rounding error even when the matrix is singular is presented. Finally the $\mathcal{H}_\infty$ loop-shaping method is summarised and its numerical properties are shown to be satisfactory.

This paper is dedicated to our teacher, friend and colleague Jan Willems.

Keywords: H-infinity control, numerical methods, loop-shaping control

1. Introduction

This short paper is concerned with $\mathcal{H}_\infty$-control problems and the numerical behaviour of solutions to the corresponding algebraic Riccati equations (ARE's). The theory of ARE's has a long and distinguished history with the 1971 paper of Jan Willems [17] being fundamental. Numerical software for solving ARE's has also received much attention with methods endeavouring to be as comprehensive as possible and using well-established methods from numerical linear algebra (e.g. [1]). Special purpose methods have also been derived that exploit Hamiltonian structures [2]. How such numerical software is used in practise is of interest. At one extreme it could be to design a control law that would then be implemented, in which case issues such as the problem being well-posed and the solution insensitive to small perturbations are important. However other legitimate questions can be posed without the expectation of controller implementation. For example 'what-if' questions could be asked such as how does the minimal closed-loop gain depend of system parameters and specification. The designer could then determine the critical parameters on which to concentrate before performing a final design.

In the context of $\mathcal{H}_\infty$-control theoretical results are preceded by a list of assumptions which correspond to rank tests. A user would not typically check these conditions and it is of interest to determine the behaviour when the conditions are not satisfied or nearly not satisfied. In the solution one of the conditions that needs to be checked is whether the solution, X_∞ of an ARE is positive semidefinite when it may be singular. This might appear to require some test dependent on the machine precision. We will show in Theorem 2 that there is a more robust alternative to this test. In section 4 we consider the $\mathcal{H}_\infty$ loop-shaping method and demonstrate that analogous numerical issues are even less problematic and additional improvements in computational approaches are possible.

2. Notation

In this note we will use standard notation as for example in [18]. $\mathbb{R}$ and $\mathbb{C}$ will denote the real complex numbers (resp.). $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) denotes the open right half plane (resp. open left half plane). Let C^* denote the Hermitian transpose of $C \in \mathbb{C}^{m\times n}$. For $D \in \mathbb{C}^{m\times m}$, $D \geq 0$ indicates that D is positive semi-definite. $\text{diag}(x_1, x_2, \ldots, x_m)$ will denote the diagonal matrix whose diagonal entries are $x_1, x_2, \ldots, x_m$. For $A \in \mathbb{R}^{n\times n}$, $\sigma(A)$ denotes the spectrum of A i.e. the set of the eigen values of A.

For $P(s)$ a rational transfer function, bounded at infinity, we denote the system with input u, state x and output y with state-space realisation: $\dot{x}(t) = x(t) + Bu(t); y(t) = Cx(t) + Du(t)$, as

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{1}$$

and for systems with inputs, $w \in \mathbb{R}^{m_1}$ and $u \in \mathbb{R}^{m_2}$ and outputs $z \in \mathbb{R}^{p_1}$ and $y \in \mathbb{R}^{p_2}$

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|c:c} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \hdashline C_2 & D_{21} & D_{22} \end{array}\right] \tag{2}$$

The closed-loop system given in Fig. 1 is denoted as $z = \mathcal{F}_\ell(P, K)w$ where

$$\mathcal{F}_\ell(P,K) = P_{11}(s) + P_{12}(s)K(s)\left(I - P_{22}(s)K(s)\right)^{-1} P_{21}(s) \tag{3}$$

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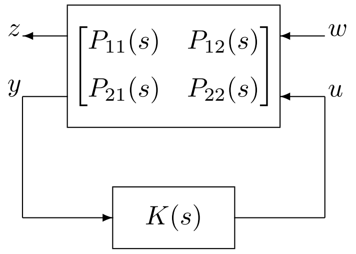

Figure 1: Linear Fractional Transformation

3. The $\mathcal{H}_\infty$ Control Problem

3.1. Background

In this section we will consider the $\mathcal{H}_\infty$-control problem to minimise $\|T_{z\leftarrow w}\|_\infty$ in Fig. 1, or equivalently for a given value of γ find a stabilising $K(s)$ such that $\|\mathcal{F}_\ell(P,K)\|_\infty < \gamma$. The solution is well-known (see e.g. [5], [18]). We will make the following standard assumptions for the state-space model in (2):

A1. (A, B_2) is stabilizable and (A, C_2) is detectable;

A2. $D_{12} = \begin{bmatrix} 0 \\ I_{m_2} \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I_{p_2} \end{bmatrix}$; (denote $D_{12}^\perp = \begin{bmatrix} I_{(p_1-m_2)} \\ 0 \end{bmatrix}$ and $D_{21}^\perp = \begin{bmatrix} I_{(m_1-p_2)} & 0 \end{bmatrix}$.)

A3. rank $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all ω;

A4. rank $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all ω;

A5. $D_{11} = 0$ and $D_{22} = 0$. (only included for simplicity of formulae).

Now define the Hamiltonian matrices:

$$H_\infty = \begin{bmatrix} A - B_2 D_{12}^* C_1 & \gamma^{-2} B_1 B_1^* - B_2 B_2^* \\ -C_1^* D_{12}^\perp D_{12}^{\perp *} C_1 & -(A - B_2 D_{12}^* C_1)^* \end{bmatrix} \quad (4)$$

$$J_\infty = \begin{bmatrix} (A - B_1 D_{21}^* C_2)^* & \gamma^{-2} C_1^* C_1 - C_2^* C_2 \\ -B_1 D_{21}^{\perp *} D_{21}^\perp B_1^* & -(A - B_1 D_{21}^* C_2) \end{bmatrix} \quad (5)$$

and the standard results is:

Theorem 1. *[5] For the system given by (2) satisfying assumptions A1–5, then there exists K such that Fig. 1 is internally stable and $\|\mathcal{F}_\ell(P,K)\|_\infty < \gamma$ if and only if*

C1. $\exists\ X_\infty = X_\infty^*$ *such that* $\begin{bmatrix} X_\infty & -I \end{bmatrix} H_\infty \begin{bmatrix} I \\ X_\infty \end{bmatrix} = 0$ *and* $\sigma\left\{ \begin{bmatrix} I & 0 \end{bmatrix} H_\infty \begin{bmatrix} I \\ X_\infty \end{bmatrix} \right\} \subset \mathbb{C}_-$*;*

C2. $\exists\ Y_\infty = Y_\infty^*$ *such that* $\begin{bmatrix} Y_\infty & -I \end{bmatrix} J_\infty \begin{bmatrix} I \\ Y_\infty \end{bmatrix} = 0$ *and* $\sigma\left\{ \begin{bmatrix} I & 0 \end{bmatrix} J_\infty \begin{bmatrix} I \\ Y_\infty \end{bmatrix} \right\} \subset \mathbb{C}_-$*;*

C3. $X_\infty \geq 0$, $Y_\infty \geq 0$*; and*

C4. the spectral radius, $\rho(X_\infty Y_\infty) < \gamma^2$.

3.2. Example

In this subsection we give an example that can be solved symbolically to illustrate some potential numerical difficulties with $\mathcal{H}_\infty$ controller synthesis. Let

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+1)} & \frac{(s-\epsilon)}{(s+2)} \\ \frac{(s-1)}{(s+1)} & 0 \end{bmatrix} \quad (6)$$

$$= \left[\begin{array}{cc|c:c} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -(2+\epsilon) \\ \hline 1 & 1 & 0 & 1 \\ \hdashline -2 & 0 & 1 & 0 \end{array}\right] \quad (7)$$

and consider the problem,

$$E_K(s) := \mathcal{F}_\ell(P,K) = \frac{1}{(s+1)} + \frac{(s-1)(s-\epsilon)}{(s+1)(s+2)} K(s)$$
$$\gamma_o = \inf_{K \in \mathcal{H}_\infty} \|E_K\|_\infty \quad (8)$$

Note that for $\epsilon = 0$ assumption A3 is not satisfied at $\omega = 0$.

This is a standard model matching problem with the interpolation constraints (the Nevanlinna-Pick problem):

$$\epsilon < 0 \qquad E_K(1) = 0.5 \quad (9)$$
$$\epsilon = 0 \qquad E_K(1) = 0.5, E_K(0) = 1 \quad (10)$$
$$\epsilon > 0 \qquad E_K(1) = 0.5, E_K(\epsilon) = \frac{1}{(1+\epsilon)} \quad (11)$$

whose solution (see e.g. [4], pp. 154-160) gives,

$$\gamma_o = \begin{cases} \frac{1}{2} =: \gamma_o^- & \text{for } \epsilon < 0 \\ 1 =: \gamma_o^0 & \text{for } \epsilon = 0 \\ \frac{1}{4}\left(1 + \sqrt{1 + 8/(1+\epsilon)}\right) =: \gamma_o^+ & \text{for } \epsilon > 0 \end{cases} \quad (12)$$

For $\epsilon < 0$ the optimal K, $K_o^-(s) = \frac{(s+2)}{2(s-\epsilon)}$, and $E_{K_o^-} = \frac{1}{2}$. For $\epsilon = 0$ there is a family of K giving $\|E_{K_o^0}\|_\infty = 1$, e.g. $K_o^0 = 0$ or $K_o^0 = -\frac{(s+2)}{(s+3)}$.

For $\epsilon > 0$ some manipulation gives $K_o^+ = -\gamma_o^+ \frac{(s+2)}{(s+\alpha)}$ where $\alpha = 1 + 2(1+\epsilon)\gamma_o^+$, and $E_{K_o^+} = -\gamma_o^+ \frac{(s-\alpha)}{(s+\alpha)}$.

The behaviour as $\epsilon \uparrow 0$ or $\epsilon \downarrow 0$ will now be considered. For $\epsilon < 0$, $K_o^-(s)$ cancels the zero at ϵ and if this is only done approximately by a term $(s-\hat{\epsilon})$, then $E_K(0) = 1 - \frac{\epsilon}{2\hat{\epsilon}}$, giving an unreliable result (for a small frequency range). For $\epsilon \geq 0$ there is no similar sensitivity. There is clearly a discontinuity at $\epsilon = 0$ in γ_o and K_o and if the near violation of assumption A3 is buried by numerical precision unreliable results are likely.

The standard approach to this problem would be to perform a search over γ to find bounds on γ_o by for each γ-value asking the question: does there exist $K \in \mathcal{H}_\infty$ such that $\|E_K\|_\infty < \gamma$?

The Hamiltonian matrices will be

$$H_\infty = \begin{bmatrix} -1 & 0 & \gamma^{-2} & 0 \\ 2+\epsilon & \epsilon & 0 & -(2+\epsilon)^2 \\ 0 & 0 & 1 & -(2+\epsilon) \\ 0 & 0 & 0 & -\epsilon \end{bmatrix}$$

$$J_\infty = \begin{bmatrix} 1 & 0 & \gamma^{-2}-4 & \gamma^{-2}-4 \\ 0 & -2 & \gamma^{-2}-4 & \gamma^{-2}-4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

For $\epsilon < 0$, X_∞ and Y_∞ are given by (resp.)

$$X_\infty^- = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; Y_\infty^- = \begin{bmatrix} \frac{2}{(4-\gamma^{-2})} & 0 \\ 0 & 0 \end{bmatrix},$$

giving $Y_\infty^- \geq 0 \Leftrightarrow \gamma > \frac{1}{2} = \gamma_o^-$ and

$$K_\gamma^- = \frac{\gamma^{-2}(s+2)}{(s-\epsilon)((4-\gamma^{-2})s + (4+\gamma^{-2}))}$$

For $\epsilon > 0$, $Y_\infty = Y_\infty^+ = Y_\infty^-$ and X_∞ is given by:

$$X_\infty^+ = \frac{2\epsilon}{(1+\epsilon)^2 - \gamma^{-2}} \begin{bmatrix} 1 \\ \frac{(1+\epsilon)}{(2+\epsilon)} \end{bmatrix} \begin{bmatrix} 1 & \frac{(1+\epsilon)}{(2+\epsilon)} \end{bmatrix} \tag{13}$$

The spectral radius condition becomes:

$$\rho(X_\infty^+ Y_\infty^+) = \frac{4\epsilon}{((1+\epsilon)^2 - \gamma^{-2})(4-\gamma^{-2})} < \gamma^2 \Leftrightarrow \gamma > \gamma_o^+.$$

In determining γ_o^+ for small $\epsilon > 0$, since $\gamma_o^+ \approx 1$ there will be an ϵ cancellation in the term $\frac{2\epsilon}{(1+\epsilon)^2-\gamma^{-2}}$ leading to potentially unreliable results. Also note that both X_∞^+ and Y_∞^+ have rank 1 for all γ and $\epsilon > 0$, so that determining whether they are positive semidefinite might require an accuracy tolerance term which is a potential problem.

This simple example has been included to illustrate how sensitive the solution can be when, in this case, assumption A3 is nearly violated, and also that testing for semi-definiteness can be an issue. The former problem could be interpreted as a problem with the problem formulation, since it allows a pole-zero cancellation near $s = 0$ for $\epsilon < 0$, however for $\epsilon > 0$, $K = 0$ gives $\|E_K\|_\infty = 1$ which is nearly optimal since $\gamma_o^+ \approx 1 - \epsilon/3$, so that an 'acceptable' solution is available.

3.3. Solving the Algebraic Riccati Equations

Solving for X_∞ in C1 corresponds to solving an Algebraic Riccati equation (ARE) with an indefinite quadratic term $X_\infty(\gamma^{-2}B_1B_1^* - B_2B_2^*)X_\infty$. For $\gamma > \gamma_o$ H_∞ will have no eigen-values on the imaginary axis and one solution method is to find a basis for the stable invariant subspace of H_∞, i.e. find $X_1, X_2 \in \mathbb{R}^{n\times n}$ such that (see e.g. [1], [11])

$$H_\infty \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} T_X, \quad \text{with } \sigma(T_X) \in \mathbb{C}_- \tag{14}$$

For Hamiltonian matrices (and exact arithmetic) it can be shown that $X_1^*X_2 = X_2^*X_1$ so that $X_\infty = X_2X_1^{-1} = X_1^{*-1}X_2^* = X_\infty^*$. Using the Q-Z algorithm here does not preserve the Hamiltonian structure and methods that do preserve this structure have been developed (see [2] and the references therein for example [3]).

It is normally recommended that iterative refinement is used in solving ARE's. If $\hat{X}_i$ is an approximate solution to C1 then Newton's method will give the refined solution, $\hat{X}_{i+1}$ satisfying the Lyapunov equation,

$$\begin{aligned} &\hat{X}_{i+1}\hat{A}_i + \hat{A}_i\hat{X}_{i+1} - \hat{X}_i(\gamma^{-2}B_1B_1^* - B_2B_2^*)\hat{X}_i \\ &\qquad + C_1^*D_{12}^\perp D_{12}^{\perp *}C_1 = 0 \\ &\text{where} \quad \hat{A}_i = A - B_2D_{12}^*C_1 + (\gamma^{-2}B_1B_1^* - B_2B_2^*)\hat{X}_i \end{aligned} \tag{15}$$

In the $\mathcal{H}_2$ problem ($\gamma = \infty$) this refinement has excellent properties with $\hat{X}_{i+1} \geq \hat{X}_i$ and convergence guaranteed as long as the initial approximate solution, $\hat{X}_0$ makes $\hat{A}_0$ stable [10]. There is also the opportunity to calculate a square root of the solution. Unfortunately when $\gamma < \infty$ the monotonicity property does not hold and square root algorithms do not appear to exist. However if $\hat{X}_0$ is close enough to X_∞ then convergence of $\hat{X}_i$ to X_∞ should be assured.

3.4. Testing Condition C3

The condition in C3 of Theorem 1 that $X_\infty \geq 0$ is potentially problematic since X_∞ being singular is not pathological and can arise naturally, e.g. X_∞ will be singular if sensor dynamics are included but the sensor output does not affect z. Similarly Y_∞ will be singular if actuator dynamics are not affected by input disturbances. The rank of X_∞ and Y_∞ is constant for all $\gamma > \gamma_o$ [8]. If $\mathrm{rank}(X_\infty) < n$ then numerical imprecision in calculating X_∞ may suggest that $X_\infty \not\geq 0$. However the following result (c.f. Lemma 6 in [5]) gives an alternative equivalent condition without this potential difficulty.

Theorem 2.

(a) *Given X_∞ satisfying C1 in Theorem 1 then $X_\infty \geq 0$ if and only if $\sigma(A - B_2D_{12}^*C_1 - B_2B_2^*X_\infty) \subset \mathbb{C}_-$.*

(b) *Given Y_∞ satisfying C2 in Theorem 1 then $Y_\infty \geq 0$ if and only if $\sigma(A - B_1D_{21}^*C_2 - Y_\infty C_2^*C_2) \subset \mathbb{C}_-$.*

Proof. (a) X_∞ satisfying condition C1 is equivalent to,

$$\begin{aligned} &X_\infty(A - B_2D_{12}^*C_1) + (A - B_2D_{12}^*C_1)^*X_\infty \\ &+ X_\infty(\gamma^{-2}B_1B_1^* - B_2B_2^*)X_\infty + C_1^*D_{12}^\perp D_{12}^{\perp *}C_1 = 0 \end{aligned} \tag{16}$$

and $\sigma(A - B_2D_{12}^*C_1 + (\gamma^{-2}B_1B_1^* - B_2B_2^*)X_\infty) \subset \mathbb{C}_-$. Defining $\tilde{A} = A - B_2D_{12}^*C_1 - B_2B_2^*X_\infty$, and $\tilde{C} = \begin{bmatrix} \gamma^{-1}B_1^*X_\infty \\ B_2^*X_\infty \\ D_{12}^{\perp *}C_1 \end{bmatrix}$, gives

$$X_\infty\tilde{A} + \tilde{A}^*X_\infty + \tilde{C}^*\tilde{C} = 0 \tag{17}$$

and $(\tilde{C}, \tilde{A})$ is detectable since $\sigma(\tilde{A}+\begin{bmatrix}\gamma^{-1}X_\infty B_1 & 0 & 0\end{bmatrix}\tilde{C}) \subset \mathbb{C}_-$ by assumption. Hence $X_\infty \geq 0$ implies $\sigma(\tilde{A}) \subset \mathbb{C}_-$ by Lemma 3.19(iii) in [18]. Conversely if $\sigma(\tilde{A}) \subset \mathbb{C}_-$, then $X_\infty \geq 0$ by Lemma 3.18(ii) in [18]. The proof for part (b) is the dual argument. □

Remark 1. Although this result is not new its use as an alternative to checking $X_\infty \geq 0$ in numerical procedures is perhaps novel.

Remark 2. This condition can be applied to the example in §3.2 when for $\epsilon < 0$ the eigen values of $\tilde{A}$ in condition (a) of Theorem 2 are $\{-1, \epsilon\}$ and for condition (b) $\{-\frac{(4+\gamma^{-2})}{(4-\gamma^{-2})}, -2\}$. For $\epsilon > 0$ these become respectively: $\{-1, -2\frac{((1+\epsilon)^3-\gamma^{-2})}{((1+\epsilon)^2-\gamma^{-2})}\}$ and $\{-\frac{(4+\gamma^{-2})}{(4-\gamma^{-2})}, -2\}$. These are all consistent with the results in the example. Clearly, for any γ these values are all negative if and only if the (rank deficient) solutions X_∞ and Y_∞ are in fact positive semi-definite.

3.5. Assumptions A1, A3 and A4

Although the theoretical results come with assumptions such as A1, A3 and A4 a typical user would not independently check these are satisfied. (Note that assumptions A2 can be satisfied by suitable scaling of u and y and orthogonal transformations of w and z so these assumptions are without loss of generality as long as the weight on u is positive definite and that w can effect all the measurements independently). If assumptions A1, A3 or A4 are not satisfied then a small perturbation of the data will typically make them satisfied. It is helpful to know how the algorithms will fail if these assumptions are violated or nearly violated.

Suppose (A, B_2) is not stabilizable so that there exists $x \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}_+$ such that $x^*A = \lambda x^*$ and $x^*B_2 = 0$. For $\gamma = \infty$, $H_\infty \begin{bmatrix}0\\x\end{bmatrix} = -\lambda^* \begin{bmatrix}0\\x\end{bmatrix}$ and hence X_1 in (14) will necessarily be singular and the $\mathcal{H}_2$ problem is not solvable. For $\gamma < \infty$ since $\nexists\, F \in \mathbb{R}^{m_2\times n}$ such that $\sigma(A - B_2F) \subset \mathbb{C}_-$, Theorem 2 gives that $X_\infty \ngeq 0$. Similarly if (C_2, A) is not detectable any Y_∞ satisfying C2 is not positive semi-definite. Hence if A1 is violated then the problem is not solvable and the algorithm will fail in a predictable way.

If assumption A3 is violated at $\omega = \omega_o$ then $\exists\, x, u$ such that ,

$$\begin{bmatrix}A - j\omega_o I & B_2\\ C_1 & D_{12}\end{bmatrix}\begin{bmatrix}x\\u\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad (18)$$

$$\implies H_\infty \begin{bmatrix}x\\0\end{bmatrix} = j\omega_o \begin{bmatrix}x\\0\end{bmatrix} \quad \forall\,\gamma \quad (19)$$

and this will imply that the algorithm should fail. However numerical inaccuracy might make the calculated eigen values off the imaginary axis. As seen in the example in 3.2 there might be significant sensitivity in the solution in spite of the problem seeming to have a sensible solution.

Indeed if for some value of γ either H_∞ or J_∞ have an eigen value at $j\omega_o$ then the largest singular value of $\mathcal{F}_\ell(P(j\omega_o), K(j\omega_o))$ is necessarily $\geq \gamma$ for any K, independent on any closed-loop stability requirement, as demonstrated in the following Lemma (whose simple proof is in the Appendix).

Lemma 3. *If* $\exists\, \omega_o \in \mathbb{R}$ *and* $\xi_1, \xi_2 \in \mathbb{C}^n$ *such that,* $\det(j\omega_o I - A + B_2D_{12}^*C_1) \neq 0$ *and*

$$H_\infty \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} = j\omega_o \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix},\ \begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} \neq 0 \quad (20)$$

then $\exists\, w_o \in \mathbb{C}^{m_1}$ *such that for* $z_o := P_{11}(j\omega_o)w_o + P_{12}(j\omega_o)u$, $z_o^*z_o \geq \gamma^2 w_o^*w_o$ *for all* $u \in \mathbb{C}^{m_2}$.

3.6. A Descriptor Form for the $\mathcal{H}_\infty$ controller

In Theorem 1 a realisation of the central controller is given by (e.g. Theorem 17.1 in [18])

$$K = \left[\begin{array}{c|c}\hat{A} & \hat{B}\\ \hline \hat{C} & 0\end{array}\right]$$

where

$$\begin{aligned}\hat{A} =&\ A - B_2D_{12}^*C_1 + \gamma^{-2}B_1B_1^*X_\infty - B_2B_2^*X_\infty\\ &- \hat{B}(C_2 + \gamma^{-2}D_{21}B_1^*X_\infty)\\ \hat{B} =&\ Z_\infty(B_1D_{21}^* + Y_\infty C_2^*)\\ \hat{C} =&\ -(D_{12}^*C_1 + B_2^*X_\infty)\\ Z_\infty =&\ (I - \gamma^{-2}Y_\infty X_\infty)^{-1}\end{aligned}$$

It is noted that from (14) forming $X_\infty = X_2X_1^{-1}$, $Y_\infty = Y_2Y_1^{-1}$ and Z_∞ requires the inverses of matrices that might not be well-conditioned especially when $\gamma \approx \gamma_o$. If a descriptor form for K is acceptable then the following formulae are easily derived and contain no matrix inverses (see also [13] [6])

$$K = \hat{C}_d\left(s\hat{E}_d - \hat{A}_d\right)^{-1}\hat{B}_d \quad (21)$$

where

$$\hat{E}_d = Y_1^*X_1 - \gamma^{-2}Y_2^*X_2 \quad (22)$$

$$\hat{C}_d = -(D_{12}^*C_1X_1 + B_2^*X_2) \quad (23)$$

$$\hat{B}_d = Y_1^*B_1D_{21}^* + Y_2^*C_2^* \quad (24)$$

$$\hat{A}_d = \hat{E}_dT_X - \hat{B}_d(C_2X_1 + \gamma^{-2}D_{21}B_1^*X_2) \quad (25)$$

where X_1, X_2 and T_X are defined in (14) and Y_1 and Y_2 from the corresponding definition for J_∞. Also an alternative condition for $X_\infty \geq 0$ and $Y_\infty \geq 0$, as in Theorem 2 is the generalised eigen value problem condition: only $\lambda \in \mathbb{C}_-$ satisfy

$$\begin{aligned}\det(\lambda X_1 - (A - B_2D_{12}^*C_1)X_1 + B_2B_2^*X_2) &= 0\\ \det(\lambda Y_1 - ((A - B_1D_{21}^*C_2)^*Y_1 + C_2^*C_2Y_2) &= 0\end{aligned}$$

which can be checked via the Q-Z algorithm. The spectral radius condition, C4, can be formulated as the generalised

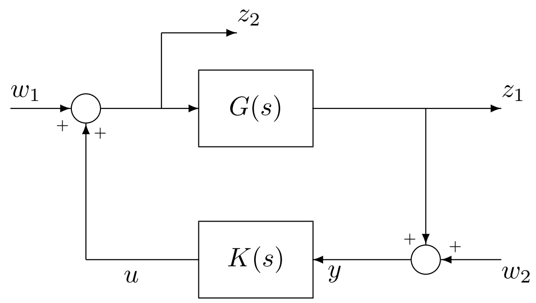

Figure 2: $\mathcal{H}_\infty$-Loop-shaping block diagram

eigen value problem (without requiring inverses of X_1 or Y_1): that all λ such that $\det(\lambda X_1^* Y_1 - X_2^* Y_2) = 0$ satisfy $\lambda < \gamma^2$.

Remark 3. As γ decreases towards γ_o, X_∞ and Y_∞ both increase monotonically [8]. The ARE's will then fail to have the desired solutions when X_1 or Y_1 become singular or when H_∞ or J_∞ have eigen values on the imaginary axis. With the optimal value of γ, $\hat{E}_d$ may become singular but the descriptor formulae may nevertheless give an optimal controller which can then have a proper transfer function rather than a strictly proper transfer function.

One potential disadvantage of this approach is that it is not apparent how iterative refinement as outlined in §3.3 could be adapted.

4. $\mathcal{H}_\infty$ Loop-shaping

The $\mathcal{H}_\infty$ Loop-shaping control system design method has been shown to have many appealing features from both mathematical and design perspectives (see e.g. [16] [14]). In this section we will discuss some numerical aspects and will not review the design rationale. We will show that the computations can be very straightforward and reliable.

The central computational problem is to solve the $\mathcal{H}_\infty$-control problem in Fig. 2 : for a given $\gamma \in \mathbb{R}_+$ find an internally stabilizing $K(s)$ such that the closed-loop transfer function from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ has $\mathcal{H}_\infty$-norm $< \gamma$. i.e.

$$\|\mathcal{F}_\ell(P,K)\|_\infty < \gamma, \text{ where } P = \begin{bmatrix} G \\ I \end{bmatrix} (I - KG)^{-1} \begin{bmatrix} I & K \end{bmatrix} \tag{26}$$

A main result from [12][18] is that

Theorem 4. *In Fig. 2 let* $G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ *in which* (A,B) *is stabilizable and* (C,A) *is detectable, then* $\exists K$ *such that Fig. 2 is internally stable and* $\|\mathcal{F}_\ell(P,K)\|_\infty < \gamma$ *if and only if*

D1. $\exists\ X = X^*$ *such that* $\sigma(A - BB^*X) \subset \mathbb{C}_-$ *and*

$$A^*X + XA + C^*C - XBB^*X = 0 \tag{27}$$

D2. $\exists\ Z = Z^*$ *such that* $\sigma(A - ZC^*C) \subset \mathbb{C}_-$ *and*

$$AZ + ZA^* + BB^* - ZC^*CZ = 0 \tag{28}$$

D3. $X \geq 0$, $Z \geq 0$*; and*

D4. $1 + \rho(XZ) < \gamma^2$.

If conditions D1–D4 are satisfied then two state-space realisations of the central (maximum entropy) controller are given by

$$K = \left[\begin{array}{c|c} A + BF_\infty + L_2 C & -L_2 \\ \hline F_\infty & 0 \end{array}\right] \tag{29}$$

$$= \left[\begin{array}{c|c} A + BF_2 + L_\infty C & -L_\infty \\ \hline F_2 & 0 \end{array}\right] \tag{30}$$

where (31)

$$F_2 = -B^*X,\ L_2 = -ZC^*,\ F_\infty = F_2 W^{-1},\ L_\infty = W^{-1} L_2 \tag{32}$$

$$W = I - \gamma^{-2}(I + ZX). \tag{33}$$

Remark 4. The state-space realisation in (29) can be obtained from that in (30) by the state transformation W^{-1}. Both these realisations are in 'observer form' and hence the closed-loop poles will be $\sigma(A + L_2 C)$ and $\sigma(A + BF_\infty)$ which are the same as $\sigma(A + BF_2)$ and $\sigma(A + L_\infty C)$ respectively. Note that the observer poles and controller poles are switched between the two realisations. The observer in (30) gives the Kalman filter state estimate if the elements of w_1 and w_2 are assumed to be independent white noise processes with equal spectral densities. However in a loop-shaping problem set-up such white noise assumptions are unlikely to be valid and hence such a state estimate should be treated with due caution.

Remark 5. The ARE's in (27) and (28) are very standard and could be solved by a variety of methods (see [1]), including iterative refinement and square root algorithms. If a Schur-type method is used a modest efficiency is possible as follows. The corresponding Hamiltonian matrices for (27) and (28) are,

$$H_2 = \begin{bmatrix} A & -BB^* \\ -C^*C & -A^* \end{bmatrix},\ J_2 = \begin{bmatrix} A^* & -C^*C \\ -BB^* & -A \end{bmatrix}, \tag{34}$$

From the Schur form for H_2 two ordered Schur forms can be determined to give,

$$H_2 \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} T_X,\ H_2 \begin{bmatrix} -Z_2 \\ Z_1 \end{bmatrix} = \begin{bmatrix} -Z_2 \\ Z_1 \end{bmatrix} (-T_Z) \tag{35}$$

$$\Rightarrow J_2 \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} T_Z \tag{36}$$

$$X = X_2 X_1^{-1},\ Z = Z_2 Z_1^{-1} \tag{37}$$

where $\sigma(T_X) \subset \mathbb{C}_-$ and $\sigma(T_Z) \subset \mathbb{C}_-$. The refinement step can then be solved by the square-root solution method to the Lyapunov equation given in [7], giving the observability Gramian of $\left(A - BB^*\hat{X}_i, \begin{bmatrix} B^*\hat{X}_i \\ C \end{bmatrix}\right)$ as $\hat{X}_{i+1} = R^*R$. Similarly for $\hat{Z}_{i+1} = S^*S$. Let the svd of $SR^* = U\Sigma V^*$ with $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$, $\sigma_i \geq \sigma_{i+1}, \forall i$. Then condition D4 gives $1 + \sigma_1^2 < \gamma^2$ and $\gamma_{\mathrm{opt}} = \sqrt{1 + \sigma_1^2}$. These σ_i were introduced as the "LQG-characteristic values" in [9]

The algorithm in [15] can now be used to give a change in state coordinates such that $X = Z = \Sigma$ in the new state coordinates. In this method the terms $\sigma_i^{-\frac{1}{2}}$ occur so there are potential problems if some $\sigma_i \ll 1$ and [15] suggested that such states are truncated with a minimal model reduction error. In the present context this will imply that if $\hat{G}$ is the model truncated to k states then the ν-gap metric, $\delta_\nu(G, \hat{G}) < 2\sum_{i=k+1}^{n} \sigma_i$ (see [16]). E.g. for $\sigma_{k+i} < 10^{-3}$ the approximation error will be negligible and for $\sigma_i > 10^{-3}$ the terms $\sigma_i^{-\frac{1}{2}}$ should not present numerical problems.

The above illustrates some advantages of this method since robustness to uncertainty in the gap metric and approximation in the gap metric are both relative to unity. If the resulting robustness in the gap metric, $\sim 1/\gamma$, turns out to be smaller than say 0.2, then if such a controller is implemented its closed-loop behaviour is most likely to be poor, and for example the controller should be re-designed with a less ambitious loop-shape.

Remark 6. For the balanced coordinates given above the normalised left and right coprime factorisations of G are given by

$$\begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|c} A + L_2C & \begin{bmatrix} B & L_2 \end{bmatrix} \\ \hline C & \begin{bmatrix} 0 & I \end{bmatrix} \end{array}\right] \tag{38}$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \left[\begin{array}{c|c} A + BF_2 & B \\ \hline \begin{bmatrix} C \\ F_2 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \end{array}\right] \tag{39}$$

with controllability Gramians are Σ and $\Sigma\left(I + \Sigma^2\right)^{-1}$ resp. and observability Gramians $\Sigma\left(I + \Sigma^2\right)^{-1}$ and Σ resp.. Hence the Hankel singular values of these coprime factorisations are $\sigma_i/\sqrt{1 + \sigma_i^2}$. Here σ_1 determines the optimal performance and the small σ_i give the opportunities for model reduction, with bounds on the resulting performance and robustness guaranteed via results on the ν-gap metric as in [16].

Remark 7. In this section we have demonstrated that the $\mathcal{H}_\infty$-loop shaping problem is well-behaved especially when compared to some of the potential issues mentioned in §3. An area where numerical difficulties might arise is when the internal description of G is poorly scaled and initial diagonal scaling can help avoid this ([1]).

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Appendix: Proof of Lemma 3.

Proof. A sinusoidal analysis at frequency ω_o, together with the change of variable, $u = v - D_{12}^*C_1x$, gives

$$\begin{aligned} j\omega_o x &= (A - B_2D_{12}^*C_1)x + B_1w + B_2v \\ z &= D_{12}^{\perp}D_{12}^{\perp *}C_1x + D_{12}v \end{aligned}$$

and defining

$$\begin{aligned} L &:= j\omega_o I - A + B_2 D_{12}^* C_1;\ M := D_{12}^{\perp *} C_1 L^{-1}; \\ F_1 &:= M B_1;\ F_2 := M B_2;\ N := (I + F_2^* F_2)^{-1} F_2^* F_1 \end{aligned}$$

gives $x = L^{-1}(B_1 w + B_2 v)$, and

$$\begin{aligned} & z^* z - \gamma^2 w^* w = x^* C_1^* D_{12}^{\perp} D_{12}^{\perp *} C_1 x + v^* v - \gamma^2 w^* w \\ & = (F_1 w + F_2 v)^* (F_1 w + F_2 v) + v^* v - \gamma^2 w^* w \\ & = (v + Nw)^* \left(I + F_2^* F_2\right)(v + Nw) \\ & \qquad + w^* (-\gamma^2 I + F_1^* (I + F_2 F_2^*)^{-1} F_1) w \\ & \geq w^* (-\gamma^2 I + F_1^* (I + F_2 F_2^*)^{-1} F_1) w \ \ \forall \ \ v \end{aligned} \tag{.1}$$

Now (20) gives

$$L\xi_1 = (\gamma^{-2} B_1 B_1^* - B_2 B_2^*)\xi_2;\ L^* \xi_2 = C_1^* D_{12}^{\perp} D_{12}^{\perp *} C_1 \xi_1$$

giving

$$\xi_1 = L^{-1}(\gamma^{-2} B_1 B_1^* - B_2 B_2^*) L^{*-1} C_1^* D_{12}^{\perp} D_{12}^{\perp *} C_1 \xi_1$$

and defining $q := D_{12}^{\perp *} C_1 \xi_1$ gives

$$\gamma^2 q = (I + F_2 F_2^*)^{-1} F_1 F_1^* q \tag{.2}$$

Now let $w = w_o := F_1^* q$ when (.2) implies

$$q^* F_1 (-\gamma^2 I + F_1^* (I + F_2 F_2^*)^{-1} F_1)) F_1^* q = 0$$

and then (.1) gives $z_o^* z_o \geq \gamma^2 w_o^* w_o$ as required. $\square$