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Reciprocal constructions using conic sections & Poncelet duality

Marina KONSTANTATOU*, Allan McROBIE[a]

*,[a] University of Cambridge, Department of Engineering
Trumpington Street, CB2 1PZ, Cambridge, UK
mk822@cam.ac.uk

Abstract

Graphic Statics offers a geometrical framework for the design & analysis of structural systems with reciprocity between the 'form' and 'force' diagrams being a fundamental notion. In this paper we provide a detailed illustration and explanation of Maxwell's reciprocal construction for 2-dimensional trusses through a 3-dimensional paraboloid of revolution. Even though Maxwell briefly mentions this method there is no record of the detailed construction. The main focus of this research is placed on the synthesis of Poncelet duality in projective geometry and of reciprocity in graphic statics resulting in the definition of a generalised dual statement for the construction of reciprocal diagrams (Maxwell 2D & Rankine 3D). This is based on the construction of their higher dimensional polyhedral Airy stress functions, using conic sections. We generalise the construction for other conics, such as the sphere and ellipsoid, and we extend the definition for 3-dimensional trusses and corresponding higher-dimensional conic sections.

Keywords: structural design, graphic statics, reciprocal diagrams, projective geometry. Airy stress function, static equilibrium, Maxwell, Rankine, Poncelet duality, truss

1. Introduction

In the context of structural design using graphic statics the traditional approach has been that of deriving reciprocal figures for 2D trusses by means of parallel drawing; i.e. by geometrical transformations on the plane (Cremona [10], Wolfe [21], Allen and Zalewski [2]) and for 3D trusses by means of geometrical transformations in space (Rankine [19], Akbarzadeh [1]). In the literature of graphic statics we have found no mention of higher dimensional reciprocal constructions based on conic sections. Drawing analogues from fields such as rigidity theory and scene analysis which have independently researched Maxwell's and Rankine's reciprocal constructions (Crapo and Whiteley [6]) we derive and visualise a reciprocal construction where the predominant geometrical objects are polyhedral Airy stress functions. In this approach planar trusses (form diagrams) and their reciprocal force diagrams, which can interchange roles, are conceived as the corresponding 2D projections of these reciprocal polyhedra. This purely geometrical methodology is largely based on the explanation and illustration of the reciprocal construction with the paraboloid of revolution 'in the ordinary sense' as briefly mentioned by Maxwell in his 1870 paper [13]; a construction based on projective geometry which was common knowledge at the time but is defunct in the 20th and 21st century. Maxwell's 'ordinary' procedure of analysing trusses was not mentioned, explained, or used in subsequent graphic statics literature.

We will place this construction in a 'Poncelet duality' and projective geometry framework and discuss how this contributes to a general paradigm shift in geometry-based structural design & analysis, *viz*, from thinking of the structure as a set of points and lines to conceiving it as one of the possible projections of a set of points and spaces between structural members. Even though projective geometry has been mentioned in the literature in the context of graphic statics, until now there has not been a geometrical construction explicitly based and formalized on Poncelet duality. We clarify pole & polar plane mappings and relate them with the different kinds of reciprocals. Even though the literature contains matrix-based constructions for deriving 2D and 3D reciprocals (e.g. Micheletti [16]) here we restrict our attention to geometry-based constructions only, deriving and using their algebraic equivalents when necessary. Moreover, the geometrical constructions introduced here are applied to 2D and 3D trusses - projections of spherical plane-faced polyhedra and 4-polytopes respectively.

1.1. Objectives

The aim of this research is firstly to interpret and visualize Maxwell's arcane reciprocal construction using conic sections. This will be based on the overarching idea of duality in projective geometry (which we will refer to as 'Poncelet duality'). Subsequently, we will extend and develop this reciprocal construction taking as our main geometrical object the higher dimensional Airy stress function. This leads to an elegant method which gives insight to the axial loads and can produce, by orthographic projection, a number of structures. Lastly we introduce 'generalised' reciprocal diagrams using conic sections other than the paraboloid of revolution and discuss their potential.

1.2. Contents

Firstly we review briefly fundamentals such as graphic statics, reciprocity between form and force diagrams, Airy stress functions, projective geometry, and conics. We then define and visualise our geometrical constructions and the polarity on which they are based, explaining the distinction between Maxwell and Cremona reciprocals. We then implement the geometrical algorithms in a CAD environment, giving a number of examples in 2D and 3D. Lastly, we discuss the potential of this approach.

2. Review

Graphic statics is a 19th century geometrical design & analysis method based on the construction of reciprocal 'form' and 'force' diagrams. This method was developed at the time from natural scientists such as Maxwell, Rankine, Culmann, Varignon, Bow, and Cremona among others (see for example Charlton [8]). Today, renewed interest in graphic statics is underpinned by the widespread use of Computer Aided Design (CAD) which holds the promise of overcoming the 19th century limitations, and results in a generalized, intuitive, visual design & analysis method that can lead to materially efficient and aesthetically elegant structures. It was not until recently that Maxwell's fundamental observation that self-stressed 2D trusses are projections of 3D plane-faced polyhedral, which in turn are essentially piecewise linear versions of Airy stress functions, was reintroduced into the field of graphic statics (Baker *et al.*, [3]; Mitchell *et al.*, [17]; McRobie *et al.*, [14]). However, currently there is no generalized methodology for reciprocal (or dual) structures in 3-dimensions although case-specific progress has been made (Akbarzadeh [1], Micheletti [16], McRobie [15]). To the knowledge of the authors there is no methodology of drawing n-dimensional reciprocal diagrams for n-dimensional trusses based on their dual (n+1)-dimensional polyhedral Airy stress functions via a conic polarity mapping. However, a reference can be found in the field of rigidity theory (Crapo and Whiteley [7]).

2.1. Reciprocal diagrams & Airy stress functions

The development of the geometrical theory of reciprocal diagrams and subsequently of graphic statics is attributed to Maxwell (Cremona [10]) and Rankine (Maxwell [14]). It is believed that Maxwell was thinking of reciprocity in the context of projective geometry and duality, which was introduced by Poncelet (Cremona [10], Poncelet [18]), in order to employ a correspondence (or mapping or 'polarity') between the dual form and force diagrams (Harman [12]). Maxwell underlines that planar reciprocal diagrams in equilibrium are (orthographic or perspective) projections of dual plane faced polyhedra

(Cremona [9]) and these dual polyhedra are constructed based on the theory of pole & polar plane using a conic, which naturally implies a 'polarity'.

We shall distinguish between 4 kinds of reciprocal diagrams; Maxwell 2D reciprocals where (form) edges correspond to perpendicular (force) edges; Cremona 2D reciprocals with the same correspondence but with parallel edges; Rankine 3D reciprocals where (form) edges correspond to perpendicular (force) faces; and 3D Cremona reciprocals where spatial (form) edges correspond to parallel spatial (force) edges. The above reciprocals can be created by using higher dimensional polarities. We should note that the form and force diagrams have interchangeable roles and we do not distinguish between structural members and lines of action of the forces. As a consequence, if a 2D truss without external loading can be perceived as a projection of a polyhedron then it is capable of being self-stressed (Baker *et al*., [3]).

Following recent publications on the topic (Baker et al., [3]; Mitchell et al., [17]; McRobie et al., [14]) the reciprocal construction followed the logic: 2D truss – lift to 3D polyhedral Airy stress function – construct 2D reciprocal truss – lift to 3D reciprocal polyhedral Airy stress function. As a result, from an initial Airy stress function we derived a family of reciprocal Airy with a possibly high number of freedoms depending on the initial 2D structure. With the current construction we define in a non-ambiguous way the two dual, reciprocal Airy polyhedra which we can manipulate and project obtaining reciprocal 2D diagrams. That is, for 2D trusses, the construction sequence was 2D to 3D to 2D to 3D, whereas the procedure presented here allows direct transition from 3D to 3D between the dual Airy polyhedra. For 3D trusses, the Rankine construction was previously 3D to 4D to 3D to 4D, but the construction here allows direct transition from 4D to 4D between the dual Airy polytopes.

2.2. Simply-connected polyhedra

Reciprocal diagrams can be constructed for 2D and 3D trusses if they have an underlying planar graph; which means they can be drawn topologically with no edge or face crossing respectively (Crapo [6]). This implies that the polyhedron (or 4-polytope in the case of a spatial truss) of which the truss is a projection is topologically spherical (Crapo [6]). For a topologically spherical polyhedron with vertices V, edges E, and faces F: $P(V, E, F)$ we have Euler's formula: $V-E+F=2$. However, it is not enough for a 2D truss to have an underlying planar graph in order to be a projection of a plane-faced polyhedron; apart from the topological requirements there are geometrical requirements as well. Further, the planar graphs should be 3-vertex connected (if we remove any two vertices and adjacent edges the graph does not separate into two subgraphs) and have more than three vertices (Whiteley [20]).

2.3. Projective geometry

Projective geometry is the system of propositions remaining after we have removed the notions of circles, distances, angles, intermediacy, and parallelism from our familiar Euclidean geometry; on the projective plane two lines always meet. Even though this sounds like a system too simple to be interesting it has yielded surprising, elegant, and rich results (Coxeter [4]). Projective geometry can be derived from affine geometry which in turn can be derived from Euclidean geometry. Projective geometry as a system of propositions has as primitive concepts solely the line, the point, and the incidence. The point at infinity was conceived independently by the German astronomer Kepler and the French architect Desargues. Consequently, two parallel lines not intersecting on the Euclidean plane, will now intersect at a point at infinity on the Projective plane. Similarly, two parallel planes, not intersecting in Euclidean space, will now intersect at a line at infinity in the Projective space (Coxeter [4]). These concepts of Projective plane & space were formalised and established by the French mathematician Poncelet while in exile after the defeat of the Grande Armée from Kutuzov's troops in Krasnoi in 1812. It is important to stress that these 'points at infinity' are not special cases but ordinary points of the Projective plane. We should note that projective geometry was not only a very active and promising mathematical field, but its propositions were common knowledge for 19^{th} century natural scientists. This is not the case today, when it is considered to be a rather inaccessible field for the modern engineer and designer.

2.3.1. Poncelet duality

As a consequence of treating the point at infinity as an ordinary point on the projective plane any two lines meet at a point and in turn any line can be defined by two points. We observe that in projective geometry any theorem and statement which holds for lines and points can be 'dualised' in terms of points and lines and hold as well. This 'principle of duality' which is attributed to Poncelet, yields elegant and powerful results.

We define 2-dimensional (2D) geometry to be the geometry of points and lines on a plane, 3-dimensional (3D) geometry to be the geometry of points, lines, and planes in space, and 4-dimensional (4D) geometry to be the geometry of points, lines, planes, and spaces in 4-dimensional space. The principle of duality can be applied to the above geometries by interchanging their primitive elements (fig.1) and where necessary pairs of words (collinear-concurrent, join-meet). By arranging definitions, theorems, and true statements in a 'parallel column' configuration where the geometrical elements and any pairs of words have been interchanged we derive their duals and we observe that these are also true (Cremona [10]). For example the statement: 'Two points (*a.b*) and (*c.d*) can be joined by a line *l*' can be dualised as follows: 'Two lines (*AB*) and (*CD*) meet at a point *L*'. As a result, we assert that in Projective geometry all theorems, axioms, statements, and definitions imply their duals which we can confidently use.

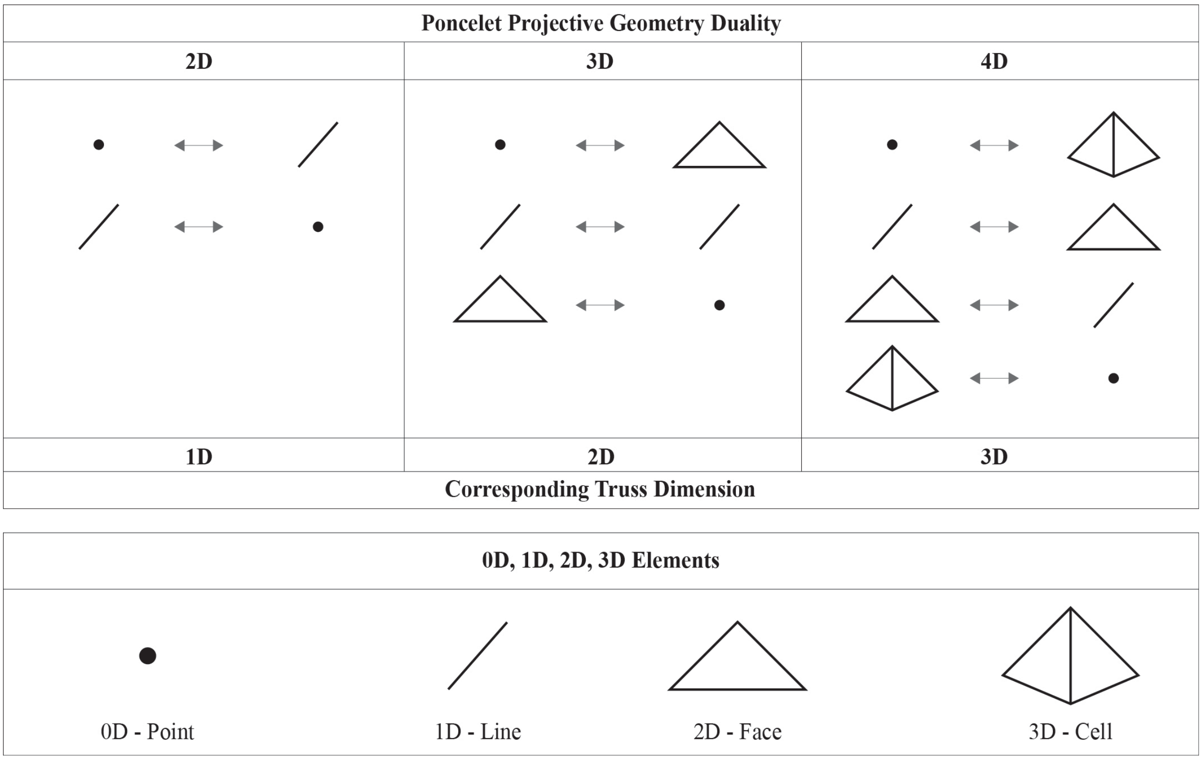

Figure 1: 2D, 3D, and 4D principle of duality

2.3.2. Conics

The familiar distinction between conics (ellipse, parabola, hyperbola) does not hold on the projective plane where only *the conic* lives. However, we can still apply these notions. In particular, we can relate the position of the conic with respect to the line at infinity $l\infty$. If the conic has no common points with the line at infinity $l\infty$ then its centre P is a point interior to the conic (what we see as an ellipse on the Euclidean plane); if the conic has one common point with $l\infty$ (which as a result is a tangent) then the centre of the 'parabola' P is self-conjugate lying on $l\infty$; if the conic has two common points with $l\infty$ then the centre P is a point exterior to the conic (what we see as a hyperbola in Euclidean plane).

3. Constructions

We can place our reciprocal construction completely on the projective plane/ space since we can even define a conic only with lines, points and incidence. We can induce a conic polarity to derive the reciprocal polyhedra which can then be embedded in the Euclidean space at the last step in order to obtain the projections which will be the Maxwell 2D reciprocals in the case of the paraboloid of revolution.

3.1. Pole & polar plane (conic polarity)

Polarity on the plane is a transformation taking points to lines and dually lines to points. A polarity preserves incidence and has degree 2. For a point *P* (that we name the pole) a conic polarity transforms it to its image which is a line *p* (that we name the polar) as follows: from *P* we draw the two tangents to the conic, which touch it in the points *Q, R*. If we now connect points *Q, R* with a line *p* we obtain the polar line of the pole *P*. A Self-conjugate point *Q* is incident with its polar *q*; that is *Q* lies on *q* (fig.2, Left).

More generally, we can apply this type of polarity in 3-space using a conic of revolution (e.g. sphere, ellipsoid, paraboloid) as the locus of self-conjugate points. The property of the self-conjugate points to lie on the conic and their respective polar lines (or planes in the spatial case) to be tangent to the conic enables us to define the following general geometrical construction: for a point (pole) *P* outside the conic we can construct its polar plane *p* by drawing the tangent cone from *P* to the conic and joining any three of the resulting coplanar intersection points *Pi* (fig.2, Right). Conversely, for a plane intersecting the conic in a curve *c* we can take tangent planes on any three points on *c*, the intersection of which will gives us the pole *P*.

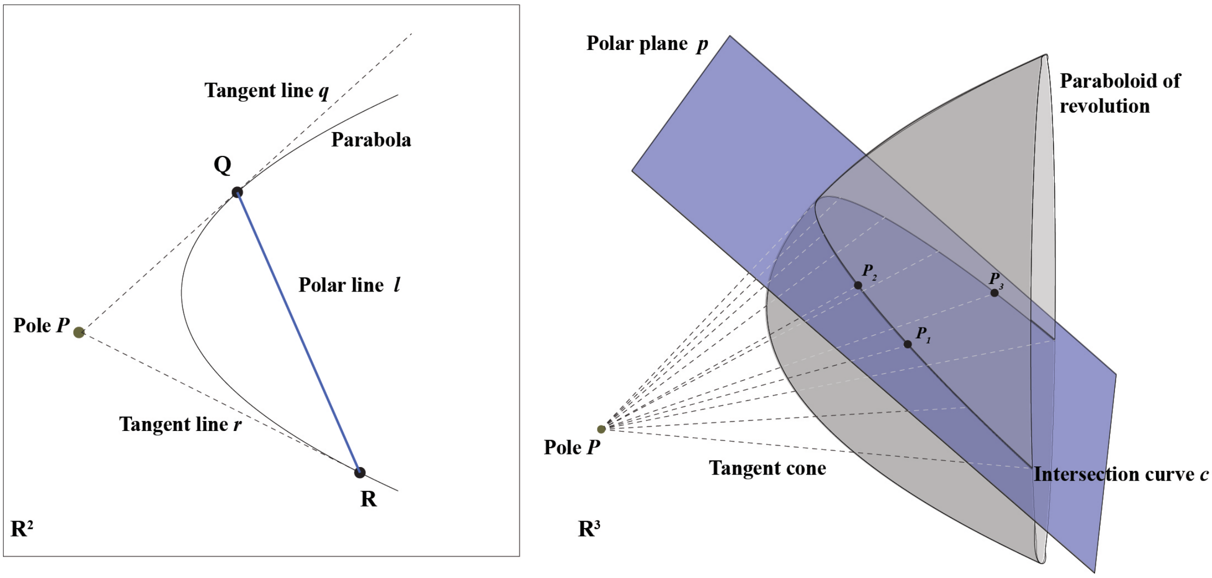

Figure 2: Left: A conic polarity mapping points to lines and vice versa; points lying on the conic are self-conjugate, Right: Polarity induced by a paraboloid of revolution, mapping planes to points and vice versa

Naturally, we can now derive the general construction of reciprocal polyhedra *P(V, E, F), P'(V', E', F')* in a 3D Poncelet duality fashion (points map to faces) using a conic polarity as follows: for a given set of vertices *V* we obtain the reciprocal polar planes on which the faces *F'* of the reciprocal polyhedron *P'* lie on. The intersections of these polar planes give us the edges *E'* and vertices *V'* of *P'*. We have thus arrived in a geometrical construction of dual polyhedral Airy stress functions using the paraboloid of revolution.

We should note that since this polarity has a period 2 we can also start from an initial set of faces and derive the same two reciprocal polyhedra. Also if a vertex does not lie outside the conic, or a face inside the conic we can still easily construct their reciprocals; for a point *P* inside the conic we take three arbitrary planes *Pi*, these intersect the conic in three intersection curves *ci*, from these we define the corresponding tangent cones which in turn define three points *Pi*, these three points suffice to define the polar plane *p*. Dually we can work for a plane *p* outside the conic.

We formally define this type of conic polarity in R^3 (named 'Maxwell polarity' in Crapo and Whiteley [7]) as a pair of transformations L, L^{-1}; mapping points, defined as triples (x, y, z), to planes, defined as quadruples $(A, B, 1, C)$ as follows: for a given plane *p* described from the equation $z=Ax+By+C$ we define L: $L(A, B, 1, C)=(cA, cB, -C)$ which gives us the triple of coordinates for the reciprocal point (pole) *P*. Conversely, for a given point *P* described from the coordinates (ξ, η, ζ) we define L^{-1}: $L^{-1}(\xi, \eta, \zeta) =(\xi/c, \eta/c, 1, -\zeta)$ which gives us the quadruple defining the equation of the reciprocal polar plane *p*. We should note that *c* is an arbitrary number which we define in the equation of the paraboloid of revolution in R^3 : $2cz=x^2+y^2$. Furthermore, the transformation *L* preserves incidence; if a point *Q* lies on a plane *q* then the plane $L(Q)$ will be incident with the point $L^{-1}(q)$.

3.2. Dual Airy stress functions using conic polarity

3.2.1. 3D reciprocity

We visualise this reciprocal construction for the simplest case of a convex polyhedron, a tetrahedron (fig.3). We note that the resulting reciprocal polyhedra have indeed a period 2 and preserve incidence; a face from *F* maps to a vertex in *V'* which then maps to the same face; three faces in *F* defining a vertex in *V* correspond to thee vertices in *V'* which lie on the same face in *F'*. Also corresponding edges of the projected 'form' and 'force' diagrams are perpendicular indicating a Maxwell 2D reciprocal configuration.

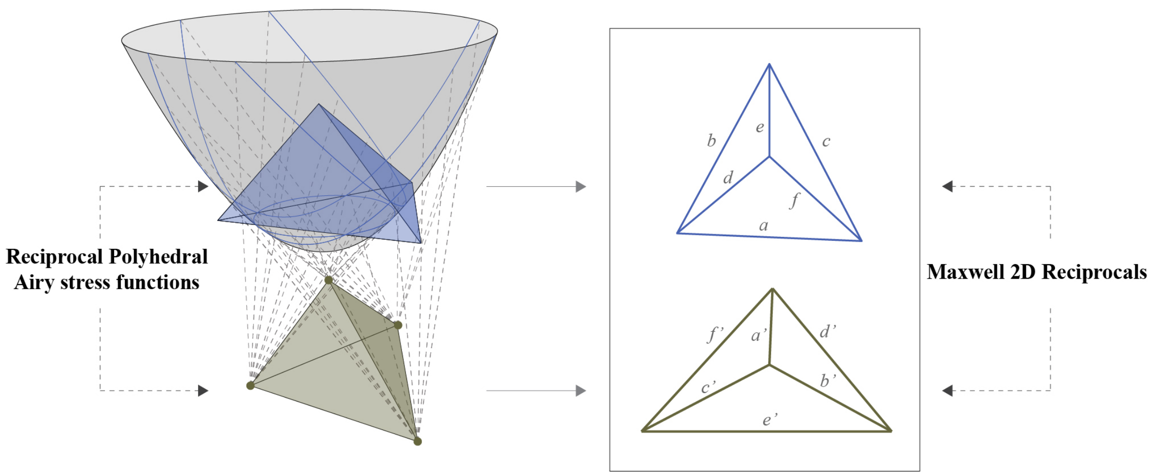

Figure 3: Reciprocal polyhedral Airy stress functions and reciprocal 2D figures for a tetrahedron

3.2.2. 4D reciprocity

More generally for a convex 4-polytope $P(V, E, F, C)$ (which is a set of Vertices, Faces, Edges, and Cells (volumes) in 4-space) we can apply our construction using a 4D Poncelet duality correspondence (mapping points to cells, and edges to faces) as follows: for a cell in *C* we take the hyperplane (generalisation of a plane; like the plane is a 2D sub-space of 3-space, a hyperplane is a 3D sub-space of 4-space) *P* on which it lies (let us assume for convenience that this lies inside the conic, but this construction can be readily generalized as in 3.1) and we then take its intersection with the hyper-paraboloid of revolution in 4-space. From this intersection surface we choose 4 arbitrary points and draw tangent hyperplanes to the conic which will intersect in a point *p* which is the pole of the given polar hyperplane *P* and the reciprocal vertex in *V'*. Consequently, by generalizing in 4-space the pole & polar construction from *P(V, E, F, C)* we can obtain the reciprocal 4-polytope *P'(V', E', F', C')*. As above we formalise this polarity by the pair of transformations L, L^{-1}; mapping points *(x, y, z, w)*, to planes *(A, B, C, 1, D)* as follows: for a given plane *p* described from the equation $w=Ax+By+Cz+D$ we define L: $L(A, B, C, 1, D)=(cA, cB, cC, -D)$ which gives us the coordinates for the reciprocal point (pole) *P*. Conversely, for a given point P $(\xi, \eta, \zeta, \theta)$ we define L^{-1}: $L^{-1}(\xi, \eta, \zeta, \theta) =(\xi/c, \eta/c, \zeta/c, 1, -\theta)$ with the equation of the paraboloid of revolution in R^4 : $2cw=x^2+y^2+z^2$.

We visualise this construction for the simplest convex 4-polytope, the 4-simplex, whose 3D projection is a tetrahedron enclosing 4 tetrahedra (fig.4). Interestingly, Maxwell refers to this in the end of his 1864 paper [12] as the simplest 3D truss that has a reciprocal. Moreover, revisiting Maxwell's construction of 3D reciprocals in [13] after the exegesis we present here it is clear that he is suggesting a reciprocal construction based on the duality of 4-polytopes. As we mentioned in 2.2 we can now readily infer that this 3D truss is self-stressed since it is a projection of a 4-polytope. The correspondence between edges in *E* and perpendicular faces in *F'* indicate a Rankine 3D reciprocal configuration. We have thus arrived in a reciprocal construction of Rankine 3D reciprocals based on the higher-paraboloid of revolution and Poncelet duality.

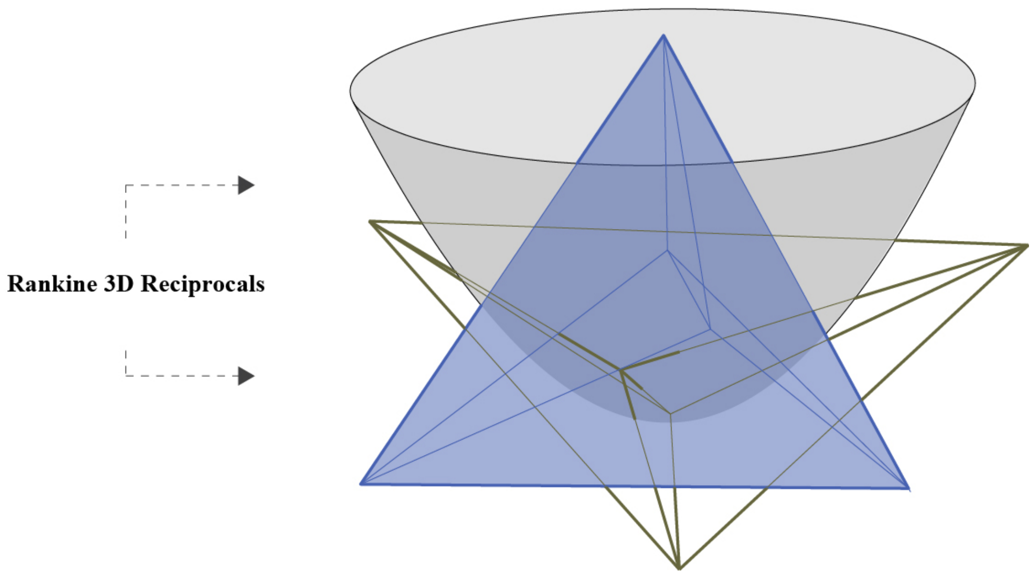

Figure 4: Reciprocal 4-polytopic Airy stress functions and reciprocal Rankine 3D figures for a 4-simplex (for visualisation purposes only the edges of the reciprocal are shown, the conic is the 3D projection of the 4D-hyperboloid of revolution)

3.3. Conic polarity & Cremona reciprocals

Cremona 2D reciprocals differ to Maxwell 2D reciprocals in that corresponding edges of the 'form' and 'force' diagrams are parallel rather than perpendicular. Both planar reciprocal figures (Cremona 2D & Maxwell 2D) can be derived by central or parallel projection on a plane of their respective plane-faced polyhedral Airy stress functions. We outlined above the construction of the Maxwell 2D reciprocals in terms of a conic polarity which maps points (poles) to faces (polar planes) and vice versa (fig.2). We examine here the feasibility of a construction of Cremona 2D reciprocals using reciprocal polyhedral Airy stress functions with respect to a conic. The difference in polarity between the two lies in the fact that for Cremona we need a 'skew polarity' (Cremona [9], Crapo [6]) which means that the poles *P* (vertices in *V*) need to be incident, and thus self-conjugate, with their polar planes *p* (faces in *F'*) and vice versa. In our previous conic construction the only such self-conjugate points are the ones lying on the conic.

In particular, if we need the poles (vertices in *V*) to lie on the corresponding reciprocal polar planes (faces in *F'*) and mutually the reciprocal poles in *V'* to lie on the corresponding polar planes in *F* then we have the case that three vertices (in *V*) should lie on the conic and the plane of the face *f* (in *F*) which they define should be tangent to the conic, but since *f* is also self-conjugate it should intersect the conic only in one point (its pole in *V'*), but by definition the three points which defined *f* lie on it and also on the conic. So *f* has 4 discrete intersection points with the conic. *Reductio ad absurdum*. We cannot create Cremona 2D reciprocals by a conic polarity.

4. Implementation & results

We implement the constructions in the Rhino CAD environment using the Grasshopper parametric geometry platform and Python scripting. The constructions were validated by three different ways which coincide: geometric constructions developed for this research; corresponding algebraic constructions; and Maxwell's equations [13].

4.1. 2D trusses on Euclidean plane

As a starting point of this research we assumed that Maxwell had in mind the construction described in 3.1. Generating in this fashion the reciprocal 2D diagrams, the following figures validate that we obtain exactly the same results as the diagrams of the Maxwell [12] paper. As suggested by the produced figures, corresponding edges are indeed perpendicular when we use a conic polarity of a paraboloid of revolution to map dual polyhedral Airy stress functions (fig.5).

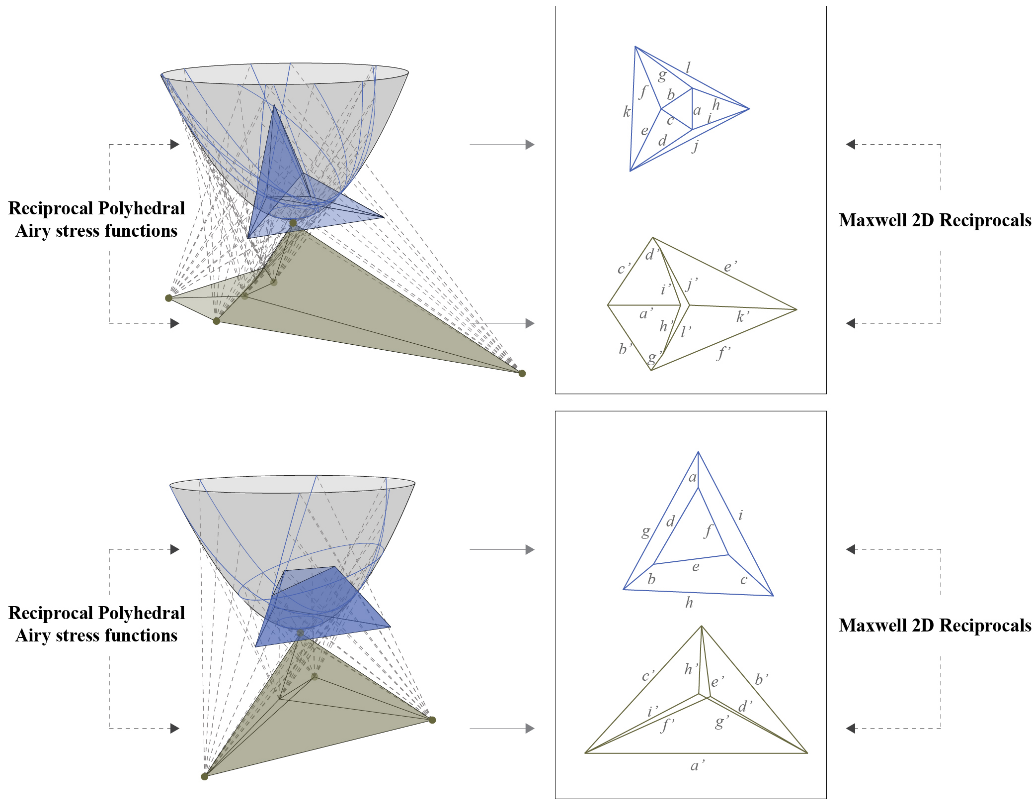

Figure 5: Reciprocal polyhedra and reciprocal 2D figures for Maxwell figures IV-4 & V-5

Moreover, if we induce a conic polarity using a sphere and an ellipsoid to create reciprocal polyhedra, we observe that the edges are not perpendicular any more (fig.6). These 'generic' dual Airy can be produced using any conic section of revolution other than the paraboloid (which is the limiting case where the corresponding edges are perpendicular).

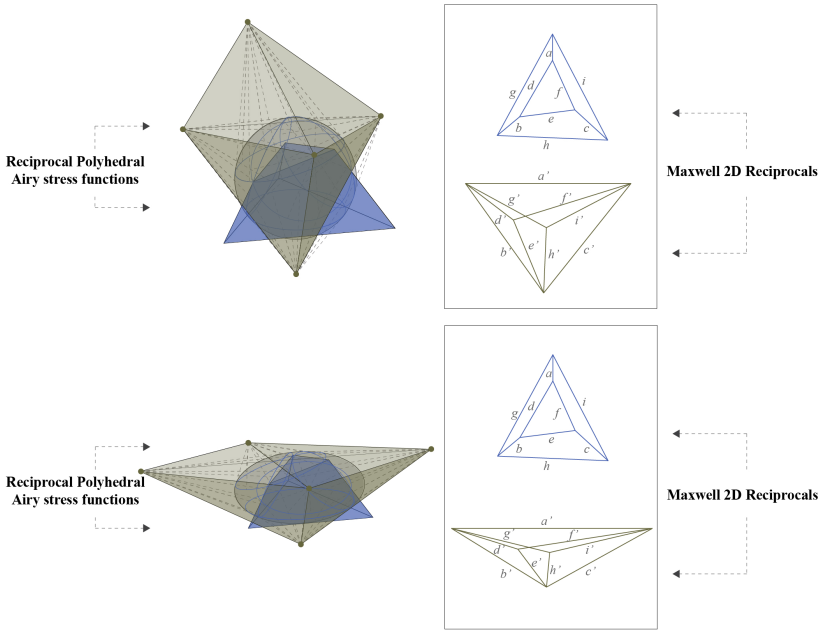

Figure 6: Reciprocal polyhedra and reciprocal 2D figures using conic polarities induced by sphere & ellipsoid

4.2. 3D trusses in Euclidean space

We construct in the above way the Rankine 3D reciprocal of a 20hedral tensegrity by 'coning' the single cell of the tensegrity, which under our polarity would map to a single point since it is a single space (McRobie [15]). By coning it with a point *p* in (0, 0, 0, 0) we create 20 extra cells apart from the external one which is the structural perimeter (and is not lifted in the fourth dimension). We then lift *p* to the 4th dimension by assigning a value to the 4th co-ordinate *w*. In this way our 20 internal cells can now define 20 polar hyperplanes (in *C*) and a 4-polytopic Airy stress function which we will map through our conic polarity to the poles in *V'* thus constructing the reciprocal 4-polytope. When projected back to 3-space, this gives us the Rankine 3D reciprocal of the 20hedral tensegrity (fig.7).

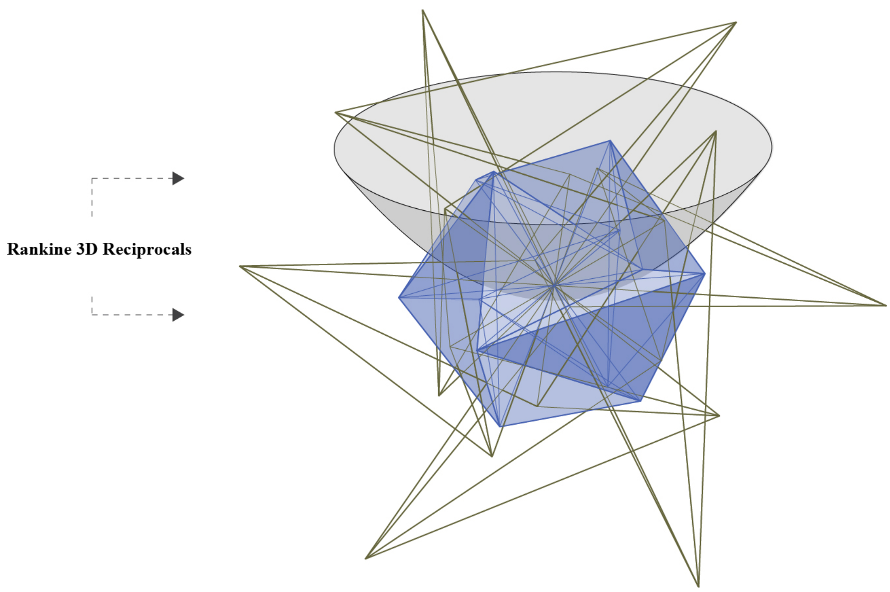

Figure 7: Rankine 3D reciprocal for the 20hedral tensegrity via higher dimensional conic polarity, in blue the coned tensegrity in green the edges of the Rankine 3D reciprocal

5. Concluding remarks & future work

Light was shed on Maxwell's arcane construction which we generalised and developed for 2D and 3D trusses - projections of convex polyhedra and 4-polytopes respectively. We contributed to a fundamentally novel way of thinking of trusses in terms of their underlying higher dimensional Airy stress function – which when projected can create numerous structures. We defined a reciprocal construction using Poncelet duality which can work for any conic of revolution resulting in Maxwell 2D, Rankine 3D, or generalised reciprocals depending on the dimension and category of input conic. The ultimate aim of our research is to create an intuitive geometry-based design and analysis tool which will have Airy stress functions as fundamental geometrical objects and will take into consideration trusses, frames, equilibrium, and stability.

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