

On the thermodynamic boundary conditions of a solidifying mushy layer with outflow

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The free-boundary problem between a liquid region and a mushy layer (a reactive porous medium) must respect both thermodynamic and fluid dynamical considerations. We develop a steady, two-dimensional forced-flow configuration to investigate the thermodynamic condition of marginal equilibrium that applies to a solidifying mushy layer with outflow and requires that streamlines are tangent to isotherms at the interface. We show that a ‘two-domain’ approach in which the mushy layer and liquid region are distinct domains is consistent with marginal equilibrium by extending Stokes equations in a narrow transition region within the mushy layer. We show that the tangential fluid velocity changes rapidly in the transition region to satisfy marginal equilibrium. In convecting mushy layers with liquid channels, a buoyancy gradient can drive this tangential flow. We use asymptotic analysis in the limit of small Darcy number to derive a regime diagram for the existence of steady solutions. Thus we show that marginal equilibrium is a robust boundary condition and can be used without precise knowledge of the fluid flow near the interface.

Key words:

1. Introduction

The boundary conditions between a mushy layer (a reactive porous medium) and a liquid melt involve a complex interaction between thermodynamics and fluid flow (Beckermann & Wang 1995; Schulze & Worster 1999, 2005). A mushy layer comprising dendritic crystals and residual melt often forms when a two-component mixture solidifies. Pressure gradients, buoyancy gradients, external shear and internal phase change can all drive fluid flows through the porous medium. Boundaries between a mushy and a liquid region are a vital part of any description of a mushy layer. For example, at the wall of a dissolution channel in a solidifying, convecting mushy layer (Copley *et al.* 1970), there is an outflow into the channel, and appropriate boundary conditions are needed to determine the size of the channel. For this situation, motivated by the idea that a mushy layer grows to relieve supercooling of the liquid, Schulze & Worster (1999, 2005) derived a condition of marginal equilibrium, which requires that streamlines are tangent to isotherms at the boundary.

In this paper, we investigate a potential inconsistency of marginal equilibrium associated with a slip in the tangential velocity at a mush–liquid interface (Beavers & Joseph

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1967). We use the idea of a *transition region* inside the mushy layer that has a width of a few pore lengths where we apply Stokes equation for viscous flow, following Le Bars & Worster (2006). This allows us to investigate the effect of slip in the tangential velocity within a two-domain formulation of mushy-layer theory, while retaining a continuous velocity on the microscale, which is necessary for marginal equilibrium. We do not intend the transition region to constitute a complete description of the fluid flow near the interface. However, it is a self-consistent alternative to using the Darcy-Brinkman equation to describe the flow in both the mushy layer and liquid region. These alternatives were shown by Le Bars & Worster (2006) to give comparable results for a transition region of prescribed dimensionless width

$$\delta = c\mathcal{D}^{1/2}, \quad \text{where } c = O(1), \mathcal{D} = \Pi_0/H^2, \quad (1.1)$$

and the dimensionless Darcy number \mathcal{D} is the ratio of a characteristic permeability Π_0 to the square of a length scale H characteristic of the width of the mushy layer.

In sections 2 and 3 respectively we present interior equations and boundary conditions for an ideal mushy layer (Worster 1997, 2000; Schulze & Worster 2005). In section 4, we apply these equations to a simple ‘toy’ problem with a two-dimensional corner flow, using a configuration originally formulated by Conroy & Worster (2006) and Le Bars & Worster (2006), because the marginal equilibrium condition at an outflow boundary is fundamentally two-dimensional. In section 5 we present semi-analytical results that explain the regions of parameter space in which there is a steady mush–liquid interface, and we discuss these results in section 6.

2. Mushy-layer and liquid-region equations

We work with the equations for an ideal mushy layer describing conservation of mass, heat, solute and momentum, as well as a liquidus relationship between temperature T and interstitial (or liquid region) concentration C , respectively

$$\nabla \cdot \mathbf{q} = 0, \quad (2.1)$$

$$\frac{DT}{Dt} = \kappa \nabla^2 T + \frac{L}{c_p} \frac{D^s \phi}{Dt}, \quad (2.2)$$

$$\frac{DC}{Dt} - \phi \frac{D^s C}{Dt} = (C - C_S) \frac{D^s \phi}{Dt}, \quad (2.3)$$

$$\mu \mathbf{u} = -\Pi \nabla p, \quad (2.4)$$

$$T = T_L(C) \equiv T_m - m(C - C_S). \quad (2.5)$$

We use the material derivatives D^s/Dt moving with the solid phase of velocity \mathbf{v} , and D/Dt moving with the total flux $\mathbf{q} = \mathbf{u} + \mathbf{v}$, where \mathbf{u} is the Darcy velocity. In the idealization, it is assumed that the phases have the same material and thermal properties (particularly latent heat of solidification L and heat capacity c_p) and that the liquidus is linear with gradient $-m$. The concentration of the solid phase $C_S = 0$ and the melting temperature is T_m . The solute diffusivity, which is much less than thermal diffusivity κ , is neglected. The permeability of the porous medium is Π and the dynamic viscosity of the fluid is μ . We also neglect buoyancy-driven flow.

Equations (2.1–2.3) hold in the liquid region where \mathbf{q} is the fluid velocity and $\phi = 0$. In the liquid region, fluid flow is governed by Stokes equation

$$\mu \nabla^2 \mathbf{q} = \nabla p. \quad (2.6)$$

Neglect of solute diffusion in the liquid region implies that the concentration is constant

along streamlines there, which is an important consideration in the development of the interfacial conditions between mush and liquid (Schulze & Worster 1999).

An important feature of the model is that there is a transition region between the mushy layer and the liquid region in which the solid fraction is non-zero and the thermodynamics is described by the mushy-layer equations while the flow is described by Stokes equation.

3. Mushy-layer boundary conditions

At a mush–liquid interface, we apply continuity of temperature and heat flux

$$[T]_l^m = 0, \quad VL\phi_m = c_p\kappa[\mathbf{n} \cdot \nabla T]_l^m, \quad (3.1a, b)$$

where V is the velocity of the interface relative to the solid phase in the direction normal to the interface (from mush to liquid) and subscripts m and l denote quantities evaluated on the mush and liquid sides of the interface respectively. Solute conservation across the interface implies that

$$-[C]_l^m U = \phi_m(C_m - C_S)V, \quad (3.2)$$

where U is the velocity of the liquid phase relative to the mush–liquid interface. So if, as in the case of outflow, $[C]_l^m = 0$ (the concentration is continuous), then $\phi_m = 0$ at the interface, and vice versa.

Schulze & Worster (1999, 2005) have shown that the conditions of heat and salt conservation are insufficient to determine the location of the interface for a solidifying mushy layer, $V > 0$. Therefore, motivated by the fact that a mushy layer grows to alleviate constitutional supercooling, they derive a condition of marginal equilibrium. For inflow, $U < 0$, the marginal equilibrium condition extends the liquidus condition (2.5) into the liquid such that the temperature satisfies $\mathbf{n} \cdot \nabla T_l = \mathbf{n} \cdot \nabla T_L(C_l)$, as originally proposed by Worster (1986). For outflow, $U > 0$, the marginal equilibrium condition requires that

$$\frac{DT}{Dt} = 0. \quad (3.3)$$

4. Solidification with a corner flow

4.1. Problem formulation

We consider a semi-infinite rectangular channel of width H in a directional solidification configuration (with the pulling speed V in the negative z -direction, which is equivalent to a solidification rate V). This configuration permits steady solutions in which the mush–liquid interface is fixed, the solid phase has velocity $\mathbf{v} = -V\hat{\mathbf{z}}$, and corresponds to the experimental apparatus of Peppin *et al.* (2007). We impose boundary temperatures $T = T_m - (k_{1,2}/H)x$, $0 < x < \infty$, on two permeable walls, where x is the distance along the channel. We impose $k_1 > k_2$ such that the lower wall, which is adjacent to a mushy layer, is colder than the upper wall, which is adjacent to a liquid melt that occupies a fraction $1 - \eta$ of the channel, as shown in figure 1(a). To control the flow, we impose the total normal flux $\mathbf{q} \cdot \mathbf{n} = Q_{1,2}$ at the permeable walls. We also impose no slip at the upper wall in the figure.

Figure 1(b) shows the dimensionless version of this problem. We non-dimensionalize lengths with respect to the dimensional channel width H and velocities with respect to Q_1 . Henceforth, x and z denote dimensionless lengths. The ratio of imposed fluxes $r = Q_2/Q_1$ determines the direction of the flow, with $r < 1$ corresponding to flow towards the positive x -direction. We restrict attention to the case $r > 0$ to ensure outflow.

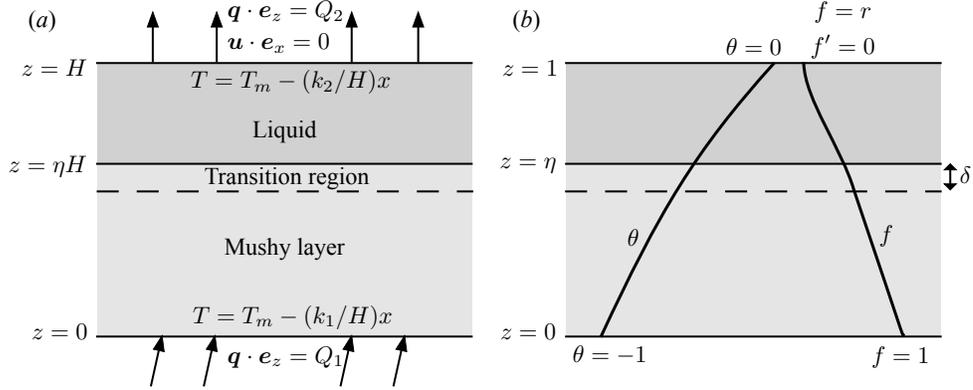


FIGURE 1. Problem formulation in (a) dimensional variables and (b) dimensionless variables. The dimensionless temperature $\theta(z)$ and flow function $f(z)$ are defined in equations (4.1, 4.2). The transition region has a prescribed dimensionless width $\delta = c\mathcal{D}^{1/2}$.

We seek a self-similar solution in which the dimensionless temperature and interstitial concentration can be written in the form

$$\mathcal{T} - \theta(z) = \frac{T_m - T}{(k_1 - k_2)x} = \frac{m(C - C_S)}{(k_1 - k_2)x}, \quad (4.1)$$

where $\mathcal{T} = k_2/(k_1 - k_2) > \theta$ gives a measure of the ratio of horizontal to vertical temperature gradients at the upper boundary.

Motivated by the separable solution for Stokes flow in a corner (Batchelor 1967), we seek a corresponding similarity solution for the dimensionless material flux in the form

$$\mathbf{q}/Q_1 = [-xf'(z), f(z)], \quad (4.2)$$

cf. Le Bars & Worster (2006). This mathematical framework allows for a two-dimensional flow to be analysed conveniently with one-dimensional equations in z . We note that this similarity solution is consistent with symmetry boundary conditions on the flow at $x = 0$, and $T(0, z) = T_m$.

4.2. Dimensionless fluid flow

In the liquid and transition regions, the flow satisfies the dimensionless Stokes equation $\nabla^2 \mathbf{q} = \nabla p$. Eliminating the pressure, we find that Stokes equation becomes $f^{(4)}(z) = 0$. In the mushy layer, the flow satisfies Darcy's law (2.4), which becomes $(f'/\Pi)' = 0$. In the simplest case of uniform permeability $\Pi = \Pi_0$, the solution that satisfies the boundary conditions at $z = 0$ and $z = 1$ is

$$f = C_0 z + 1 \quad 0 \leq z \leq \eta - \delta, \quad (4.3)$$

$$f = B_0(1 - z)^3 + A_0(1 - z)^2 + r \quad \eta - \delta \leq z \leq 1, \quad (4.4)$$

where A_0 , B_0 , and C_0 are constants that can be derived from mass conservation, no slip, and continuity of pressure at the boundary between Stokes and Darcy flow, respectively

$$[f] = 0, \quad [f'] = 0, \quad f''|^{+} = -f''|^{-}/\mathcal{D} \quad (z = \eta - \delta). \quad (4.5a, b, c)$$

We find that

$$A_0 = -3B_0 \left[\frac{\gamma}{2} + \frac{\mathcal{D}}{\gamma} \right], \quad B_0 = \frac{2(r-1)}{\gamma^3 + 6\mathcal{D}(2-\gamma)}, \quad C_0 = 6\mathcal{D}B_0, \quad (4.6a, b, c)$$

where $\gamma = 1 - \eta + \delta$. Note that the streamfunction f is continuously differentiable (although its second derivative is not continuous and we use superscripts in equation (4.6c) to denote relevant side of the Stokes–Darcy boundary). Thus there are well defined streamlines at the mush–liquid interface, essential both to deriving and also imposing the marginal equilibrium condition (Schulze & Worster 2005). We now solve the thermodynamic part of the problem.

4.3. Dimensionless interior equations

The ideal mushy-layer equations (2.2, 2.3) in the case of steady directional solidification give

$$f\theta' - f'(\theta - \mathcal{T}) = \theta''/Pe - \mathcal{S}\mathcal{V}\phi' \quad 0 \leq z \leq \eta, \quad (4.7)$$

$$f\theta' - f'(\theta - \mathcal{T}) = -\mathcal{V}[\phi(\theta - \mathcal{T})]' \quad 0 \leq z \leq \eta, \quad (4.8)$$

where $\mathcal{V} = V/Q_1$ represents the ratio of the solidification rate to the imposed flow and $Pe = Q_1H/\kappa$ is a Péclet number representing the ratio of advection by the imposed flow to diffusion of heat. The Stefan number $\mathcal{S} = L/(c_p(k_1 - k_2)x)$, so we can neglect latent heat release at sufficiently large x where $\mathcal{S} \ll 1$.

Equation (4.7) with $\phi = 0$ is also the heat equation for the liquid region $\eta \leq z \leq 1$. Salt conservation in the liquid region is governed by $\mathbf{q} \cdot \nabla C = 0$, so $C \propto xf(z)$.

4.4. Dimensionless boundary conditions

The dimensionless boundary conditions on temperature are

$$\theta(0) = -1, \quad \theta(1) = 0, \quad [\theta]_l^m = 0, \quad [\theta']_l^m = 0, \quad (4.9a - d)$$

provided $\mathcal{S} \ll 1$ so that latent heat release at the interface is insignificant.

For solidifying outflow ($V, U > 0$), conservation of solute (3.2) and marginal equilibrium (3.3) require that

$$[C]_m^l = 0 \Rightarrow \phi_m = 0, \quad (4.10)$$

$$\mathbf{q} \cdot \nabla \theta = 0 \Rightarrow f\theta' - f'(\theta - \mathcal{T}) = 0 \quad (z = \eta). \quad (4.11)$$

In general, we solve these equations numerically, iterating such that equation (4.11) is satisfied. We find the first integral of equation (4.8) analytically, using (4.7), to determine

$$\phi = \frac{1}{Pe_V} \frac{\theta' - \theta(\eta)}{\mathcal{T} - \theta} \quad 0 \leq z \leq \eta, \quad (4.12)$$

where $Pe_V = VH/\kappa = \mathcal{V}Pe$ is a Péclet number based on the solidification rate. Thus the solid fraction is inversely proportional to the solidification rate V . Figure 2 shows a typical example of a solution. Note that marginal equilibrium is satisfied, so streamlines are tangential to isotherms at the mush–liquid interface.

5. Results in the case of a solidifying mushy layer with outflow

Steady solutions only exist in a limited region of parameter space (figure 3) because of two crucial constraints. (I): If the outflow Q_2 becomes too strong relative to the inflow Q_1 , the isotherms become convex, which is inconsistent with the twin requirements of marginal equilibrium and a positive solid fraction, as shown in sketch **A** in figure 3. Physically, the mushy layer is dissolved away as there is not enough advection from the cold lower boundary. An important implication of this first constraint is that there are no solutions without a transition region. (II): If Q_2 is too weak, the direction of the

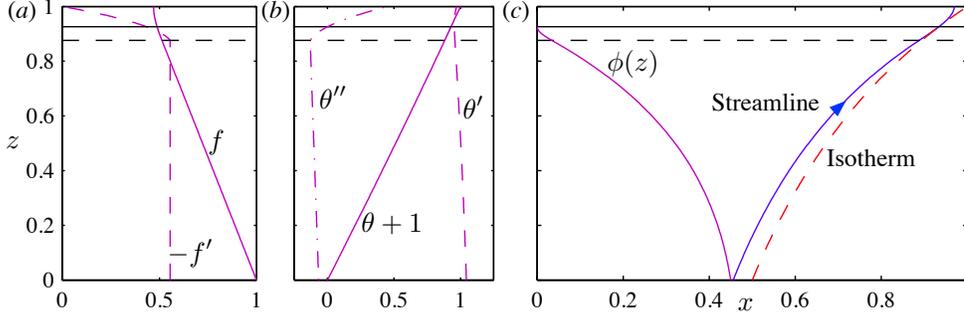


FIGURE 2. A typical solution for the dimensionless variables (magenta curves, labelled). The transition region between the dashed horizontal line and the solid horizontal line at the mush–liquid interface has width $\delta = 0.05$ ($c = 0.5$, $\mathcal{D} = 0.01$). Other parameters are $r = 0.47$, $Pe = \mathcal{T} = 1$. (a) Flow function f and the negative of its derivative, proportional to the vertical and horizontal velocities respectively. (b) Temperature θ and its derivatives. Note that θ'' becomes increasingly negative in the mushy layer and only starts to increase in the transition region, and $\theta''(\eta) = 0$. (c) the solid fraction ϕ , which increases away from the interface, and an illustration of the marginal equilibrium condition: a streamline (blue, direction of flow indicated by an arrow) and an isotherm (red, dashed) are tangential at the mush–liquid interface.

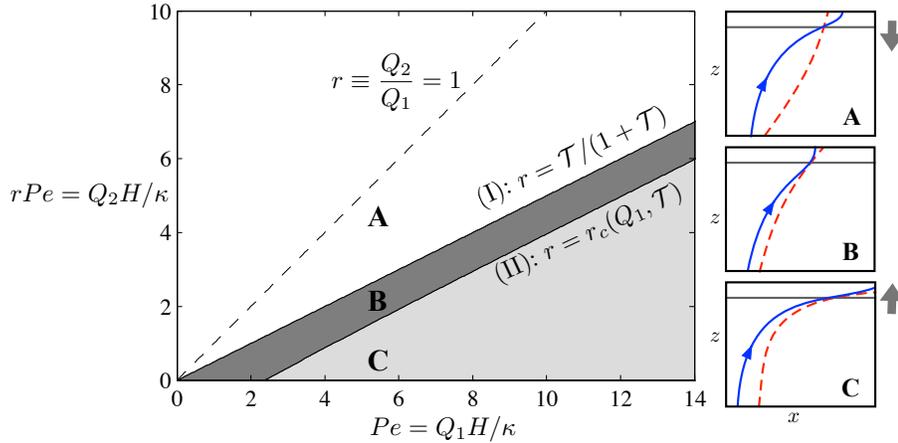


FIGURE 3. A regime diagram showing the dependence on the imposed flow. In region **A**, there are no steady solutions because the solid fraction is negative. This region is divided into two by the dashed line $r = 1$ ($Q_2 = Q_1$), which separates flow to the left from flow to the right. In region **B**, there is a unique steady solution. The width of the liquid region decreases towards region **C**, in which there are no steady solutions because the mushy layer occupies the whole channel. The boundaries are labelled (I) and (II) and are discussed in sections 5.1 and 5.2 respectively. The right column shows sketched streamlines (blue, direction of flow indicated with an arrow) and isotherms (dashed, red) and a vertical arrow showing the retreat (**A**), steady state (**B**) or advance (**C**) of the mush–liquid interface. The regime diagram is in the physically meaningful limit of small Darcy number $\mathcal{D} \rightarrow 0$, $\mathcal{T} = 1$, and it is independent of $c > 0$. Thus the regime diagram does not depend on the absolute size of the transition region, provided it is nonzero.

temperature gradient at the upper wall reverses, which is inconsistent with marginal equilibrium, as shown in sketch **C** in figure 3. Physically, the mushy layer advances and the width of the liquid region goes to zero.

5.1. *The concavity of isotherms (I)*

We take the asymptotic limit Darcy number $\mathcal{D} \rightarrow 0$, in which both the transition region and liquid region are narrow (see section 5.3). Mathematically, it is required to prove that $\theta''(z) \leq 0$ in the mushy layer given that $\phi \geq 0$. Informally, the argument is that equation (2.3) shows that $DC/Dt > 0$, since all the terms on the right hand side are positive, at least sufficiently close to the mush–liquid interface. Then the liquidus condition (2.5) shows that $DT/Dt < 0$, whence $\nabla^2 T < 0$ using (2.2), as required. More formally, we differentiate (4.7) with respect to z to find

$$\theta''' / Pe = f\theta'' - f''(\theta - \mathcal{T}). \quad (5.1)$$

In the mushy layer (excluding the narrow transition zone) $f'' = 0$ and $f > 0$, so θ'' has a definite sign. However, in the transition zone, $f'' \neq 0$ and θ'' changes rapidly (as $\mathcal{D}^{-1/2}$) to satisfy $\theta''(\eta) = 0$ (cf. figure 2b). Given that $\phi(\eta) = 0$, ϕ must increase as z decreases away from the interface $z = \eta$. From equations (4.7, 4.8), this requires $\theta''(\eta^-) < 0$, so $\theta''(0) < 0$, as claimed.

The physical reason why the transition region is crucial can also be inferred from equation (5.1). In the mushy layer, the horizontal velocity is uniform. However, in the transition region it decreases rapidly, cf. figure 2(a, c). This decrease allows horizontal and vertical advection of heat to balance at the mush–liquid interface, as required by the marginal equilibrium condition (4.11). (Note that only the horizontal velocity changes across the transition region to leading order as $\mathcal{D} \rightarrow 0$.)

Furthermore, our asymptotic analysis shows that f, f', θ, θ' and θ'' are all $O(1)$ in the Darcy number, so the full solution for θ and θ' can be found to leading order in the Darcy number by solving (4.7) with the linear flow function $f(z) = 1 + C_0 z$ because the mush occupies almost all the domain. Now $C_0 \sim (r - 1)$ from equation (4.6). For notational simplicity, we introduce a depth-dependent Péclet number $P(z) = Pe [1 - z(1 - r)]$ and difference $\Delta P = Pe(1 - r)$. Then we find the exact solution for the temperature field

$$\theta(z) = \mathcal{T} + K_1 P + K_2 \left[\sqrt{\frac{\pi}{2\Delta P}} P \operatorname{erf} \left(\frac{P}{\sqrt{2\Delta P}} \right) + \exp \left(\frac{-P^2}{2\Delta P} \right) \right], \quad (5.2)$$

where K_1 and K_2 are constants determined by the boundary conditions $\theta(0) = -1$ and $\theta(1) = 0$. Thus there is a unique solution that satisfies the boundary conditions, except in the singular, one-dimensional case of $\Delta P = 0$. Note that equation (5.2) holds if $\Delta P < 0$, and gives real solutions.

We determine the boundary (I) in figure 3 by considering the marginal case $\theta''(z) = 0$, which occurs if and only if

$$r = \mathcal{T} / (1 + \mathcal{T}). \quad (5.3)$$

A consequence is that $r < 1$ (greater inflow than outflow meaning that flow is from left to right and so from fresher to saltier).

 5.2. *The heat flux from the upper heat exchanger is positive (II)*

The marginal equilibrium condition (4.11) requires that horizontal advection of warm fluid from the left is balanced by the vertical advection of cold fluid from below. Thus there are no steady solutions unless $\theta'(1) > 0$.

We determine the boundary (II) in figure 3 by considering the marginal case $\theta'(1) = 0$. To fix ideas, consider the natural special case of no flux through the upper wall, $r = 0$. There is a critical $\mathcal{T} = \mathcal{T}_C$, above which there are no solutions, where

$$\mathcal{T}_C = \left[\sqrt{\pi Pe/2} \operatorname{erf} \left(\sqrt{Pe/2} \right) + \exp(-Pe/2) - 1 \right]^{-1}. \quad (5.4)$$

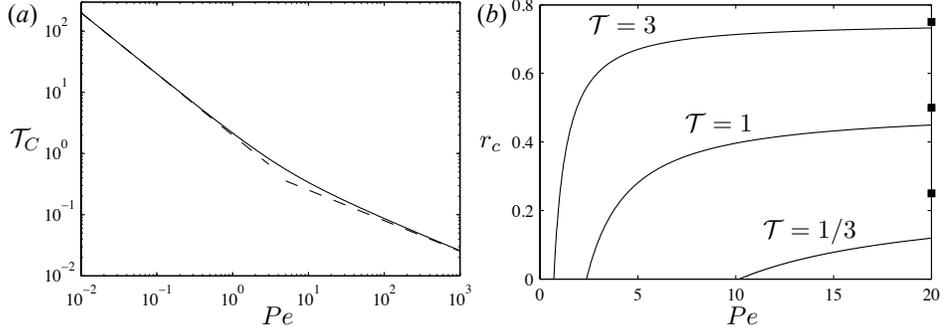


FIGURE 4. (a) The critical temperature gradient ratio \mathcal{T}_C in the special case $r = 0$. The dashed lines give the asymptotic scalings described in the main text. (b) The critical ratio of outflow to inflow r_c , above which there are steady solutions, for three given values of \mathcal{T} . Note that the $r_c = 0$ intercept is the critical Pe_C in (a). The solid black squares denote the asymptotic value $r_c \sim \mathcal{T}/(1 + \mathcal{T})$ (equation 5.6). Note that for small \mathcal{T} , at $Pe = 20$ the ratio is much less than the asymptotic value, indicating a wide region of solutions.

The inverse of this relationship gives a critical $Pe_C(\mathcal{T})$ (*i.e.* input flux Q_1). Note that $\mathcal{T}_C \sim 2/Pe$ as $Pe \rightarrow 0$ and $\mathcal{T}_C \sim \sqrt{2/\pi Pe}$ as $Pe \rightarrow \infty$, as shown in figure 4(a).

More generally, we find the boundary (II) – defined by, say, $r = r_c(Pe, \mathcal{T})$ – that satisfies a transcendental equation

$$(1 + \mathcal{T}) \exp\left(-\frac{r_c Pe^2}{2\Delta P}\right) - \mathcal{T} \exp\left(-\frac{Pe^2}{2\Delta P}\right) = \mathcal{T} \sqrt{\frac{\pi}{2\Delta P}} Pe \left[\operatorname{erf}\left(\frac{Pe}{\sqrt{2\Delta P}}\right) - \operatorname{erf}\left(\frac{r_c Pe}{\sqrt{2\Delta P}}\right) \right]. \quad (5.5)$$

It is not difficult to show that this equation has a solution for all $Pe > Pe_C$, which we plot in figure 4(b), and that, asymptotically for large Pe ,

$$r_c \sim \mathcal{T}/(1 + \mathcal{T}). \quad (5.6)$$

This ratio also defines the top of region **B**. Thus the region of parameter space where solutions exist becomes asymptotically narrow for large Pe . However, if \mathcal{T} is small, there is a relatively wide region of solutions for a given Pe . Thus if the horizontal temperature gradient is weak compared to the vertical gradient near the interface, marginal equilibrium is a much weaker constraint, because a smaller change in horizontal velocity is required to balance the horizontal and vertical transport of heat near the interface.

5.3. Liquid channel width and the necessity of a transition region

We now find a scaling for the liquid channel width in this configuration. Marginal equilibrium (4.11) gives

$$-f'(\eta) = f(\eta)\theta'(\eta)/(\mathcal{T} - \theta(\eta)) \sim r\theta'(1)/\mathcal{T}. \quad (5.7)$$

Note that $\theta'(1)$ is a function of (r, \mathcal{T}, Pe) that can be determined from equation (5.2). Now the conditions on the flow (4.5) give scalings

$$f_+''' = f_-'/\mathcal{D} \sim f_+'/\mathcal{D} \quad (z = \eta - \delta), \quad (5.8)$$

where subscript + denotes the quantity on the transition zone side of the mush–transition zone interface and subscript – the mush side. These scalings combine to give a liquid

region width

$$1 - \eta \sim a_0 \mathcal{D}^{1/2}. \quad (5.9)$$

An alternative way to interpret this is to note that equation (5.1) implies that $f''(\eta) \sim \delta^{-1} \sim \mathcal{D}^{-1/2}$, so the liquid region width must scale with $\mathcal{D}^{1/2}$ to balance viscous shear stresses. We determine the prefactor a_0 from the asymptotic solution for the flow $f(z)$ and equation (5.7). These yield the quadratic equation for a_0

$$a_0 \left(\frac{c}{2} + \frac{1}{a_0 + c} \right) = \frac{r\theta'(1)}{\mathcal{T}(1-r)}. \quad (5.10)$$

For any finite $c > 0$, this equation has exactly one positive solution a_0 . However, for $c = 0$ there are no solutions ($c \rightarrow 0$ is a singular limit). Therefore, the transition region is crucial for the the existence of steady solutions to the problem.

5.4. Solid fraction

Since $\theta'' < 0$ in the mushy layer, $\phi > 0$. Furthermore, equation (4.12) shows that

$$\phi = \frac{1}{Pe_V} \frac{\theta'(z) - \theta'(\eta)}{\mathcal{T} - \theta(z)}, \quad (5.11)$$

so ϕ increases as z decreases away from the mush–liquid interface. The physical requirement that $\phi \leq 1$ gives a critical Pe_V (or solidification rate V) above which solutions are physically meaningful. One important feature of the case of solidifying outflow is that both the size of the solid fraction in the mushy layer and also the concentration in the liquid region are determined everywhere by heat and salt conservation, and not by any imposed external boundary condition on ϕ . By contrast, in the case of a dissolving mushy layer, it is appropriate to impose ϕ at the lower heat exchanger.

6. Discussion

The thermodynamic marginal equilibrium condition determines the position of the interface between a solidifying mushy layer and a purely liquid region when there is a fluid flow from the mush to the liquid. A potential difficulty with this condition arises in cases where a slip in the tangential velocity occurs at the interface. We have shown that extending Stokes equations into the mushy layer in a transition region whose width scales with $\mathcal{D}^{1/2}$ is a viable way to resolve this difficulty. This avoids relying on a single-domain approach with a fluid dynamical equation (such as Darcy-Brinkman) applying in both the liquid and the mushy layer, and shows that marginal equilibrium is a robust boundary condition.

By considering the simple ‘toy’ problem of solidification with a corner flow, we have understood several important features of flow in a mushy layer using simple physical and geometrical arguments, as well as asymptotic analysis. For our external boundary conditions on the flow, the transition region is strictly necessary for the existence of steady solutions. The need for a transition region is physically associated with the need for a rapid change in the component of the velocity tangential to the mush–liquid interface. Similarly, the vertical velocity increases rapidly in what we call the active region near a vertical chimney in a mushy layer with convection due to baroclinic torque (Rees Jones & Worster 2013). This is another way to explain why a finite active region near the chimney is needed to sustain mushy-layer convection. In our particular pressure-driven flow configuration, the width of the liquid region and the transition region both scale with $\mathcal{D}^{1/2}$ (which dimensionally is a few pore lengths), which is somewhat problematic

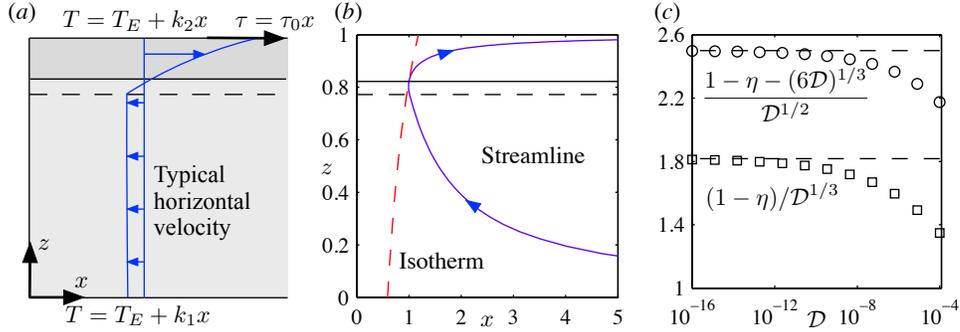


FIGURE 5. (a) shear-driven flow with temperature gradient from cold to warm as x, z increase. (b) typical isotherm (red dashed) and streamline (blue with arrows) for same parameters as figure 2. Note the the much wider liquid region. (c) asymptotic scalings (dashed lines) for the width of the liquid region, verifying the first (squares) and second (circles) terms in equation (6.1) for $\mathcal{T} = 1 \Rightarrow k_1 = 2k_2$, and $c = 1$.

within the formal continuum limit. Nevertheless, the physical mechanisms that give rise to the regime diagram (figure 3) still apply at moderate values of \mathcal{D} , and the diagram itself is independent of the prefactor c in the width of the transition region.

Furthermore, there are other configurations in which the liquid region is asymptotically large compared to the transition region. By way of example, we briefly consider a shear-driven flow. If we reverse the overall temperature gradient and impose a viscous shear at the top of the liquid region that increases linearly with x , the flow circulation that ensures positive solid fraction has the opposite configuration to the main pressure-driven flow part of this paper, as shown in figure 5. Note that the equation of salt conservation cannot formally be treated in the similarity framework because far along the channel the local temperature would be above the liquidus. Using similar asymptotic techniques to section 5.1, we find that the liquid region width

$$1 - \eta \sim (6\mathcal{D})^{1/3} - \left(\frac{c}{2} + \frac{1}{c} \frac{k_1}{k_2} \right) \mathcal{D}^{1/2}. \quad (6.1)$$

Again, there are no solutions without a transition region ($c = 0$), because the overall temperature gradient and marginal equilibrium require that $u > 0$ at the interface, but u is a constant negative number in the mush. The shear in the liquid region drives the increase in u . In this case, the width of the liquid region is set by marginal equilibrium, and the thermal properties only appear at higher order. The first and second order terms are both independent of the Péclet number, which should be defined in terms of the imposed shear. Finally, we note that the $\mathcal{D}^{1/3}$ scaling applies to chimneys in convecting mushy layers where shear is important (*e.g.* Rees Jones & Worster 2013).

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REFERENCES

- BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.
 BEAVERS, G. S. & JOSEPH, D. D. 1967 Boundary conditions at a naturally permeable wall. *J. Fluid Mech.* **30**, 197–207.

- BECKERMANN, C. & WANG, C.Y. 1995 Multiphase/-scale modeling of alloy solidification. In *Annual Reviews of Heat Transfer* (ed. C. L. Tien), vol. 6, pp. 115–198. Begell House.
- CONROY, D. & WORSTER, M. G. 2006 Mush–liquid interfaces with cross flow. Geophysical Fluid Dynamics Program report, Woods Hole Oceanographic Institute.
- COPLEY, S. M., GIAMEI, A. F., JOHNSON, S. M. & HORNBECKER, M. F. 1970 The origin of freckles in unidirectionally solidified castings. *Metall. Trans.* **1**, 2193–2204.
- LE BARS, M. & WORSTER, M. G. 2006 Interfacial conditions between a pure fluid and a porous medium: implications for binary alloy solidification. *J. Fluid Mech.* **550**, 149–173.
- PEPPIN, S. S. L., AUSSILLOUS, P., HUPPERT, H. E. & WORSTER, M. G. 2007 Steady-state mushy layers: experiments and theory. *J. Fluid Mech.* **570**, 69–77.
- REES JONES, D. W. & WORSTER, M. G. 2013 Fluxes through steady chimneys in a mushy layer during binary alloy solidification. *J. Fluid Mech.* **714**, 127–151.
- SCHULZE, T. P. & WORSTER, M. G. 1999 Weak convection, liquid inclusions and the formation of chimneys in mushy layers. *J. Fluid Mech.* **388**, 197–215.
- SCHULZE, T. P. & WORSTER, M. G. 2005 A time-dependent formulation of the mushy-zone free-boundary problem. *J. Fluid Mech.* **541**, 193–202.
- WORSTER, M. G. 1986 Solidification of an alloy from a cooled boundary. *J. Fluid Mech.* **167**, 481–501.
- WORSTER, M. G. 1997 Convection in mushy layers. *Ann. Rev. Fluid Mech.* **29** (1), 91–122.
- WORSTER, M. G. 2000 Solidification of fluids. In *Perspectives in Fluid Dynamics: a Collective Introduction to Current Research* (ed. G. K. Batchelor, H. K. Moffatt & M. G. Worster), pp. 393–446. Cambridge University Press.