Aristotle on the metaphysical status of mathematical entities

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Preface

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration. I state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution. It does not exceed the prescribed word limit for the Classics Degree Committee.

Στην μνήμη της γιαγιάς μου

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V.P.

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Synopsis

The purpose of this dissertation is to provide an account of the metaphysical status of mathematical entities in Aristotle. Aristotle endorses a form of realism about mathematical entities: for him as well as for Platonists, anti–realism, the view that mathematical objects do not exist, is not a viable option. The thesis consists of two main parts: a part dedicated to the objects of geometry, and a part dedicated to numbers. Furthermore, I have included an introductory chapter about a passage in the second chapter of Book B of the *Physics* (193b31-194a7) where Aristotle endorses a form of naïve realism with regard to mathematical entities.

Many of the passages that give us an insight into Aristotle's philosophy of mathematics are to be found in the third chapter of Book M of the *Metaphysics*. Aristotle's primary concern there, however, is not so much to present his own positive account as to provide answers to a series of (not so obvious) Platonic arguments. In the second chapter of my thesis, I discuss some of those arguments and highlight their role in Aristotle's own position about the metaphysical status of geometrical entities. In a passage that is of crucial importance to understand Aristotle's views regarding the mode of existence of the objects of mathematics (*Meta.* M.3, 1078a25-31), Aristotle allows for the potential existence of them. I argue that Aristotle's sketchy remarks in *Meta*. M.3 point towards a geometry based on the commonsensical notion of the solid. This account can be further developed if we take into consideration the purpose of the preceding chapter M.2: to refute Platonic arguments that attribute greater metaphysical status to 'limit entities' (entities bounding and within a physical body), that is, to points, lines, and surfaces. According to Aristotle, such 'limit entities' have only a potential existence–what does this claim amount to? To answer this question, I will explore a more traditional reading of this claim and I will also put forward a more radical one: from a contemporary perspective, this reading makes

Aristotelian geometry a distant cousin of modern Whiteheadian or Tarskian geometries.

Providing an account of the metaphysical status of number in Aristotle poses quite a few challenges. On the one hand, the scarcity of the evidence forces commentators to rely on a few scattered remarks (primarily from the *Physics*) and to extract Aristotle's own views from heavily polemical contexts (such as the convoluted arguments that occupy much of books M and N of the *Metaphysics*). On the other hand, the Fregean tradition casts a great shadow upon the majority of the interpretations; indeed, a great amount of the relevant scholarship is dominated by Fregean tendencies: it is, for example, widely held that numbers for Aristotle are not supposed to be properties of objects, much like colour, say, or shape, but second-order properties (properties–of–properties) of objects. The scope of the third chapter is to critically examine some of the Fregean-inspired arguments that have led to a thoroughly Fregean depiction of Aristotle, and to lay the foundations for an alternative reading of the crucial texts.

Chapter 1: Aristotle's realism about mathematicals in *Physics* B.2

[1.1] Introduction

In the following lines from the second chapter of Book B of the *Physics* Aristotle endorses a form of naïve realism with regard to mathematical entities:

> For natural bodies have planes, solids, lengths, and points, about which the mathematician carries out his investigations.[1] [*Physics* B.2, 193b23–25; Charlton's trans. mod.]

Why do I label Aristotle's realism as a 'naïve' one? The main reasons are two: One, the above lines do not tell us much about the metaphysical status of lower-dimensional entities, that is, of points, lengths, planes. Things get better a few lines below, where Aristotle says that 'the mathematician, too, deals with these things, but he does not consider each of them as *boundary of natural bodies*' (περὶ τούτων μὲν οὖν πραγματεύεται καὶ ὁ μαθηματικός, ἀλλ' οὐχ ᾗ φυσικοῦ σώματος πέρας ἕκαστον, 193b31-32; Chartlon's trans. mod., italics mine). It is safe, then, to assume that Aristotle is considering mathematical entities in this context (surfaces, lengths, and points) as *limit entities*, entities bounding or limiting natural bodies or some continuous magnitude. The solid extensions of bodies as well as *limit entities* (πέρατα)–points, surfaces, and lines–that bound or demarcate a continuous magnitude, constitute the subject matter of the mathematicians and are 'objectively there', a real feature of the natural world. There arises a certain need, then, to better understand the nature of such limit entities. According to the following passage limit entities do not enjoy separate existence from the bodies they bound; a limit must always be the limit of something:

> There are some people who think that there must be entities of this sort, because the point is the limit and extreme of the line, the line of the plane, and the plane of the solid. We must therefore have a look at this argument too, and see whether it is not extremely feeble. *Extremes are not real objects; they are all rather limits.* (Even walking, and movement in general, has a sort of limit; so this would be an individual and a real object which is

[1] καὶ γὰρ ἐπίπεδα καὶ στερεὰ ἔχει τὰ φυσικὰ σώματα καὶ μήκη καὶ στιγμάς, περὶ ὧν σκοπεῖ ὁ μαθηματικός.

absurd.) *But even if they are, they will all belong to the particular perceptible things* (it was to these that the argument applied); so why should they be separate?[2] [*Meta.* N.3, 1090b5–1090b13; Annas' trans.; italics mine]

Henry Mendell greatly expands on Aristotle's realism in *Physics* B.2:

> On this view mathematical theorems will be true, because they are true about solids, surfaces of solids, and edges of surfaces. Whatever the shapes of physical bodies may be, it is still true that they each have per se some shape. These need not even be shapes which are easy to analyse. The question, sometimes raised, whether Aristotle thinks there are geometricals is, from this perspective, easy to resolve. In a sense, there must be. That is, all bodies have volume with shape, surface with shape, and so forth. The objects of mathematics are in a way established. [Mendell (1986), p.78]

My second reason for labeling Aristotle's realism as 'naïve' has to do with the so-called *precision problem*: Are the shapes of the bodies we have around us the shapes that the geometers actually study–for example, are the bodies around us perfectly spherical or perfectly planar and so on? Aristotle does not fully address the precision problem in this passage of the *Physics*. Does he address it explicitly somewhere else? For a better understanding of the issue and its implications we will have to wait until we discuss the crucial passages in Book M of the *Metaphysics* in the second chapter of this work.

[2] εἰσὶ δέ τινες οἳ ἐκ τοῦ πέρατα εἶναι καὶ ἔσχατα τὴν στιγμὴν μὲν γραμμῆς, ταύτην δ' ἐπιπέδου, τοῦτο δὲ τοῦ στερεοῦ, οἴονται εἶναι ἀνάγκην τοιαύτας φύσεις εἶναι. δεῖ δὴ καὶ τοῦτον ὁρᾶν τὸν λόγον, μὴ λίαν ᾖ μαλακός. οὔτε γὰρ οὐσίαι εἰσὶ τὰ ἔσχατα ἀλλὰ μᾶλλον πάντα ταῦτα πέρατα (ἐπεὶ καὶ τῆς βαδίσεως καὶ ὅλως κινήσεως ἔστι τι πέρας· τοῦτ' οὖν ἔσται τόδε τι καὶ οὐσία τις· ἀλλ' ἄτοπον)· —οὐ μὴν ἀλλὰ εἰ καὶ εἰσί, τῶνδε τῶν αἰσθητῶν ἔσονται πάντα (ἐπὶ τούτων γὰρ ὁ λόγος εἴρηκεν)· διὰ τί οὖν χωριστὰ ἔσται;

[1.2] A discussion of the *Physics* B.2 passage

Nevertheless, the second chapter of Book B of the *Physics* gives us a first glimpse of Aristotle's position regarding the metaphysical status of mathematicals. Let us examine closely the following important passage from that chapter:

> Both the student of nature and the mathematician deal with these things; but the mathematician does not consider them as boundaries of natural bodies. Nor does he consider things which supervene as supervening on such bodies. That is why he separates them; for they are separable in thought from change, and it makes no difference; no error results. Those who talk about ideas do not notice that they too are doing this: they separate physical things though they are less separable than the objects of mathematics. That becomes clear if you try to define the objects and the things which supervene in each class. Odd and even, straight and curved, number, line, and shape, can be defined without change but flesh, bone, and man cannot. They are like snub nose, not like curved. The point is clear also from those branches of mathematics which come nearest to the study of nature, like optics, harmonics, and astronomy. They are in a way the reverse of geometry. Geometry considers natural lines, but not as natural; optics treats of mathematical lines, but considers them not as mathematical but as natural.[3] [*Physics* B.2, 193b31-194a7; Charlton's trans.]

Mathematicians study the very same natural points, lines, etc. which are the limits of natural bodies, yet they do not study them as the limits of natural bodies. Nor do they examine their attributes as attributes that belong to natural bodies (193b31–3). Rather, mathematical entities are regarded in a special way: as separate from matter or change: Aristotle is emphasising that the mathematician makes use of a special sort of *cognitive separation* in his study of

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[3] περὶ τούτων μὲν οὖν πραγματεύεται καὶ ὁ μαθηματικός, ἀλλ' οὐχ ᾗ φυσικοῦ σώματος πέρας ἕκαστον· οὐδὲ τὰ συμβεβηκότα θεωρεῖ ᾗ τοιούτοις οὖσι συμβέβηκεν· διὸ καὶ χωρίζει· χωριστὰ γὰρ τῇ νοήσει κινήσεώς ἐστι, καὶ οὐδὲν διαφέρει, οὐδὲ γίγνεται ψεῦδος χωριζόντων. λανθάνουσι δὲ τοῦτο ποιοῦντες καὶ οἱ τὰς ἰδέας λέγοντες· τὰ γὰρ φυσικὰ χωρίζουσιν ἧττον ὄντα χωριστὰ τῶν μαθηματικῶν. γίγνοιτο δ' ἂν τοῦτο δῆλον, εἴ τις ἑκατέρων πειρῷτο λέγειν τοὺς ὅρους, καὶ αὐτῶν καὶ τῶν συμβεβηκότων. τὸ μὲν γὰρ περιττὸν ἔσται καὶ τὸ ἄρτιον καὶ τὸ εὐθὺ καὶ τὸ καμπύλον, ἔτι δὲ ἀριθμὸς καὶ γραμμὴ καὶ σχῆμα, ἄνευ κινήσεως, σὰρξ δὲ καὶ ὀστοῦν καὶ ἄνθρωπος οὐκέτι, ἀλλὰ ταῦτα ὥσπερ ῥὶς σιμὴ ἀλλ' οὐχ ὡς τὸ καμπύλον λέγεται.

mathematical entities. It is not completely clear how one should conceive this cognitive separation: is it to be understood as a way of grasping what is already there? Or does it involve something more–perhaps some sort of mental construction? Aristotle is adamant that 'no error results' in employing it. It is reasonable to suppose that the cognitive separation Aristotle has in mind in this passage is what most commentators call 'abstraction'; simply on the basis of etymological considerations, there is more or less a consensus among commentators regarding the fundamental characteristics of abstraction, which is to be understood as

> … an elimination or prescinding from irrelevant features of the physical and sensible world…The resulting picture, I believe, is one in which the mathematical features 'remaining' after the abstraction process *were actually there in the sensible and physical realm all along*. Abstraction is simply a means of focusing one's attention, as it were, on those features by eliminating from consideration other features not germane to one's present mathematical investigations. [White (1993), p.176; my emphasis][4]

No error arises in one's mathematical reasoning because abstraction, understood in this way, does not involve any distortion or misrepresentation of how the physical world is. The following passage gives us an insight into this sort of abstraction:

> And since, as the mathematician investigates abstractions (for in his investigation he eliminates all the sensible qualities, e.g. weight and lightness, hardness and its contrary, and also heat and cold and the other sensible contrarieties, and leaves only the quantitative and continuous, sometimes in one, sometimes in two, sometimes in three dimensions, and the attributes of things *qua* quantitative and continuous, and does not consider them in any other respect, and examines the relative positions of some and the consequences of these, and the commensurability and incommensurability of others, and the ratios of others; but

[4] An extensive discussion of the passages in the Aristotelian corpus where the term 'abstraction' occurs can be found in [Cleary (1985)]. The gist of his analysis as well as the relation of abstraction with the *qua* locution are examined in the second chapter of my thesis, in the context of the *Meta*. M.3 discussion; it seems that Aristotle employs a more enhanced version of cognitive separation in M.3, one that is not merely a way of grasping what is already there, but involves some sort of mental construction.

> yet we say there is one and the same science of all these things—geometry), the same is true with regard to being.[5] [*Meta.* K.3, 1061a28-b4; Ross' trans.; his emphasis]

The mathematician theorises (τὴν θεωρίαν ποιεῖται, 1061a29; θεωρεῖ, 1061a30) about 'abstractions' (τὰ ἐξ ἀφαιρέσεως, 1061a29). When the mathematician abstracts, he omits (περιαιρέει, 1061a29) the features that are not relevant to his present concern (1061a29-32). He leaves only quantity and what is continuous (τὸ ποσὸν καὶ συνεχές, 1061a32), that is, he focuses his attention on the mathematical features of natural bodies: on their solid extensions, their surfaces, and delineations (τῶν μὲν ἐφ' ἓν τῶν δ' ἐπὶ δύο τῶν δ' ἐπὶ τρία). Besides 'isolating' mathematical entities that are found in the sensible world (solids, planes, lines, etc.), the mathematician also considers their properties (the ratios of some, the relative positions of others, etc.). The following passage is also pertinent to our discussion:

> The so-called 'abstract objects' <the mind thinks> in the following way: if <one> had thought of the snub as snub <one would have thought of it> not <abstractly>; whereas <if one> had thought of <the snub> qua concave one would have thought of it abstractly, <that is>, qua concave, <one> would have thought of it without the flesh; in this sense the mathematicals, though not separate, <the mind> considers them as separate when it thinks them. In general, the mind is identical with its objects.[6] [*De Anima* III.7, 431b12-19; my trans.]

Metaphysically, mathematicals cannot exist in separation (οὐ κεχωρισμένα, 431b16) from the objects of natural world; concavity is always the concavity of

[5] καθάπερ δ' ὁ μαθηματικὸς περὶ τὰ ἐξ ἀφαιρέσεως τὴν θεωρίαν ποιεῖται (περιελὼν γὰρ πάντα τὰ αἰσθητὰ θεωρεῖ, οἷον βάρος καὶ κουφότητα καὶ σκληρότητα καὶ τοὐναντίον, ἔτι δὲ καὶ θερμότητα καὶ ψυχρότητα καὶ τὰς ἄλλας αἰσθητὰς ἐναντιώσεις, μόνον δὲ καταλείπει τὸ ποσὸν καὶ συνεχές, τῶν μὲν ἐφ' ἓν τῶν δ' ἐπὶ δύο τῶν δ' ἐπὶ τρία, καὶ τὰ πάθη τὰ τούτων ᾗ ποσά ἐστι καὶ συνεχῆ, καὶ οὐ καθ' ἕτερόν τι θεωρεῖ, καὶ τῶν μὲν τὰς πρὸς ἄλληλα θέσεις σκοπεῖ καὶ τὰ ταύταις ὑπάρχοντα, τῶν δὲ τὰς συμμετρίας καὶ ἀσυμμετρίας, τῶν δὲ τοὺς λόγους, ἀλλ' ὅμως μίαν πάντων καὶ τὴν αὐτὴν τίθεμεν ἐπιστήμην τὴν γεωμετρικήν), τὸν αὐτὸν δὴ τρόπον ἔχει καὶ περὶ τὸ ὄν.

[6] Translating the following version of the (tortuous) text: τὰ δὲ ἐν ἀφαιρέσει λεγόμενα <νοεῖ> ὥσπερ, εἴ <τις> τὸ σιμὸν ᾗ μὲν σιμὸν οὔ, κεχωρισμένως δὲ ᾗ κοῖλον <εἴ τις> ἐνόει, ἄνευ τῆς σαρκὸς ἂν ἐνόει ἐν ᾗ τὸ κοῖλον—οὕτω τὰ μαθηματικά, οὐ κεχωρισμένα <ὄντα>, ὡς κεχωρισμένα νοεῖ, ὅταν νοῇ <ᾗ> ἐκεῖνα. ὅλως δὲ ὁ νοῦς ἐστιν, ὁ κατ' ἐνέργειαν, τὰ πράγματα.

some thing. The mathematician, however, is able to isolate mathematical features (in this case, concavity) in thought; interested as he is only in the essential characteristics of concavity (since it is not part of the essence of concavity whether it is the concavity of this or that nose), he proceeds to examine concavity as such. On the other hand, as the *Physics* passage has made clear, if one were to consider the snub qua snub, one would not think of it irrespectively of matter, since to be snub is to be concave in a special way, to be nasally concave. A similar discussion takes place at the end of *De Anima* I.1, where Aristotle emphasises the way in which the affections of the soul are inseparable:

> We have said that the affections of the soul are not separate from the physical matter of living beings in the way in which anger and fear are not separate, and not in the way in which line and plane are.[7] [*De Anima* I.1, 403bl7-19; mod. Charles' translation]

How are we to compare the affections of the soul (τὰ πάθη τῆς ψυχῆς, like anger and fear) with lines and planes? In what way the former are not like the latter? To answer this question we have to go back a few lines:

> <The attributes> which are not separate, and which are not treated as attributes of such and such a body but in abstraction, <are studied by> the mathematician.[8] [*De Anima* I.1, 403b14-15; my trans.]

Mathematicals are not separate from physical matter but they can be treated in abstraction from it; affections of the soul, are by contrast, neither separable (in thought or in definition) from matter nor separate in existence from such matter. As D. Charles points out: 'if one does not think of fear and anger as enmattered in certain types of perceptual matter, one will (in his view) make mistakes in one's reasoning about those affections. One will fail, for example, to know when and why they occur.'[9] In lines 403a25-27, Aristotle offers an example of how an affection of the soul (like anger) ought to be properly defined: 'To be angry is a

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[7] ἐλέγομεν δὴ ὅτι τὰ πάθη τῆς ψυχῆς οὕτως ἀχώριστα τῆς φυσικῆς ὕλης τῶν ζῴων, ᾗ γε τοιαῦθ' ὑπάρχει <οἷα> θυμὸς καὶ φόβος, καὶ οὐχ ὥσπερ γραμμὴ καὶ ἐπίπεδον.

[8] τῶν δὲ μὴ χωριστῶν μέν, ᾗ δὲ μὴ τοιούτου σώματος πάθη καὶ ἐξ ἀφαιρέσεως, ὁ μαθηματικός.

[9] In [Charles (2009), p.5].

process of this type of body or part or capacity of such a body caused in this way for the sake of such and such a goal. (Charles' trans.: τὸ ὀργίζεσθαι κίνησίς τις τοῦ τοιουδὶ σώματος ἢ μέρους ἢ δυνάμεως ὑπὸ τοῦδε ἕνεκα τοῦδε).

Further justification for understanding 'separation' as 'abstraction' may be found in Philoponus' and Simplicius' commentaries of the *Physics* passage:

> The mathematician discusses the shapes and their accompanying features without further thinking of whatever sort of matter these belong in, but separating them in thought from all matter he studies in this way their accompanying features; the natural scientist, however, thinking of the shape and the rest of the attributes studies them as being in matter.[10] [Philoponus: *On Aristotle 'Physics' 2*, 219.28-33; Lacey's trans. mod.]

> The mathematician differs from the natural scientist in the first instance in that the natural scientist talks not only about the properties of natural bodies but also about their matter, while the mathematician is in no way concerned with the matter.[11] [Simplicius: *On Aristotle 'Physics' 2*, 290.27-29; Fleet's trans.]

Mathematical entities then, though inseparable in reality from physical matter, they can be separated in thought, and the mathematicians are doing nothing wrong in separating away perceptible matter from them and defining them (λέγειν τοὺς ὅρους) without mentioning any such matter. To illustrate his point, Aristotle draws a comparison between mathematical entities and their attributes–which can be correctly defined without appealing to matter or change–and natural forms–which cannot be defined in that way (194a2-6).[12] We can point to Simplicius' commentary of the passage, where the latter suggests that the *Physics* B.2 passage offers a general criterion to determine what can and what cannot be mentally separated:

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[10] Ὁ μὲν μαθηματικὸς διαλέγεται περὶ τῶν σχημάτων καὶ τῶν συμβαινόντων αὐτοῖς μηδὲν προσεπινοῶν ἐν ὁποιᾳδηποτοῦν ὕλῃ ταῦτα ὑπάρχει, ἀλλὰ χωρίσας αὐτὰ πάσης ὕλης τῇ διανοίᾳ οὕτω τὰ συμβαίνοντα αὐτοῖς θεωρεῖ, ὁ μέντοι φυσικὸς ἐπινοῶν τὸ σχῆμα καὶ τὰ λοιπὰ τῶν παθῶν, ὡς ἐν ὕλῃ αὐτὰ θεωρεῖ.

[11] Διαφέρει δὲ ὁ μαθηματικὸς τοῦ φυσικοῦ πρῶτον μὲν ὅτι ὁ φυσικὸς οὐ περὶ τῶν συμβεβηκότων μόνον τοῖς φυσικοῖς σώμασι λέγει, ἀλλὰ καὶ περὶ τῆς ὕλης, τοῦ μαθηματικοῦ μηδὲν περὶ ὕλης πολυπραγμονοῦντος.

[12] Following [Peramatzis (2011), p.75].

> He offers a general rule to determine what can and what cannot be mentally separated. When, in defining what we are separating, we do not include in the definition the entity from which we are separating it, and do not carry it along in our conception, but instead define and conceive it as something per se, it is then that we say such a thing is separable in definition and thought (for example, when defining the mathematical body we talk about that which has three dimensions without in any way carrying along the matter or the movement of the natural body; in defining the plane surface we talk about that which has nothing more than length and breadth; and it is the same in the case of numbers). But when the original entities appear inevitably as part and parcel of the definition which we seek to give, together with the properties which we are separating, which cannot even be thought of without them, then we say that such entities cannot be separated even in concept and thought. Such entities are flesh, bone and man.[13] [Simplicius: *On Aristotle 'Physics' 2*, 293.29-294.5; Fleet's trans.]

The criterion is to examine whether any error occurs in the mathematician's reasoning if mathematicals are to be thought of independently of physical matter.[14] The application of this criterion in the case of mathematical and natural forms allows Aristotle to show the error in the ways of the Platonists. As Simplicius argues, no error arises if one thinks of the solid body as 'that which has three dimensions without in any way carrying along the matter or the movement of the natural body' (293.32-34). In this, mathematical entities differ from entities such as flesh, bone, and man; for not only is each of these natural and essentially enmattered, but each cannot be thought of without matter (294.3-5). At the end of the *Physics* passage, Aristotle compares mathematics

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[13] καὶ παραδίδωσι κανόνα τῶν τε τῇ ἐπινοίᾳ δυναμένων χωρίζεσθαι καὶ τῶν μή. καὶ γὰρ ὅταν μὲν ταῦτα ἃ χωρίζομεν ὁριζόμενοι μὴ παραλαμβάνωμεν ἐν τῷ ὁρισμῷ ἐκεῖνα, ὧν χωρίζομεν αὐτά, μηδὲ τῇ ἐννοίᾳ συναναφέρωμεν, ἀλλ' αὐτὰ καθ' αὑτὰ ὁριζώμεθα καὶ ἐννοῶμεν (ὡς τὸ μαθηματικὸν σῶμα ὁριζόμενοι λέγομεν τὸ τὰς τρεῖς ἔχον διαστάσεις οὐδαμοῦ τὴν ὕλην ἢ τὴν κίνησιν τοῦ φυσικοῦ σώματος συναναφέροντες καὶ τὸ ἐπίπεδον τὸ μῆκος καὶ πλάτος μόνον ἔχον καὶ ἐπὶ ἀριθμῶν ὁμοίως), τότε χωριστὰ λόγῳ καὶ ἐπινοίᾳ τὰ τοιαῦτα λέγομεν εἶναι. ὅταν δὲ βουλομένοις ὁρίσασθαι τὰ χωριζόμενα συνεμφαίνηται πάντως ἐκεῖνα, ὧν χωρίζεται, καὶ μηδὲ δύνηται χωρὶς ἐκείνων νοεῖσθαι, τότε καὶ τῇ νοήσει καὶ ἐπινοίᾳ τὰ τοιαῦτα ἀχώριστα λέγομεν. τοιαῦτα δὲ σὰρξ καὶ ὀστοῦν καὶ ἄνθρωπος.

[14] So Peramatzis (2011), p.72. Consult [Charles (2009), p.5] and especially [Peramatzis (2011), pp.71-73] for a discussion of the criterion.

with 'those branches of mathematics which come nearest to the study of nature' (τὰ φυσικώτερα τῶν μαθημάτων):

> The point is clear also from those branches of mathematics which come nearest to the study of nature, like optics, harmonics, and astronomy. They are in a way the reverse of geometry. Geometry considers natural lines, but not as natural; optics treats of mathematical lines, but considers them not as mathematical but as natural. [*Physics* B.2, 194a7-12; Charlton's trans.]

The geometer is investigating natural lines 'not as natural' (οὐχ ᾗ φυσική). Thus, the mathematician studies the essence of concavity or curvature and its properties: the mathematician can mentally separate/cognitively isolate curvature from its instantiations and study its properties without any error arising from this process. On the other hand, the subject matter of astronomy is not merely sphericity and its properties, but the sphericity of planets, and other celestial bodies. But the astronomer does not study just that; he also takes into account the motion of the celestial bodies, that is, he studies those bodies as moving spheres. It would be a mistake for the astronomer to consider the sphericity of the planets in isolation from their motion. For that would reduce astronomy to a geometry of spheres; and how could astronomy then explain the apparent motions of the stars and other celestial bodies? The point that Aristotle is trying to make here is that an applied mathematician such as an astronomer merely studies a conjunction of properties (such as the sphericity of the celestial bodies and their motions); in other words astronomy studies the heavenly bodies qua (moving and having magnitude). Aristotle's claim that astronomy is in a way the reverse of geometry can be understood as implying that the former involves an addition, since it takes the shapes from the superordinate science and studies them in conjunction with the motion of the celestial bodies, whereas the latter involves an abstraction (in the sense of subtraction) in that it studies only shapes themselves and their essential properties.[15]

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[15] Cf. the discussion in *Post. An.* A.13 (78b36–79a11), where Aristotle argues that subordinate sciences 'make use of <mathematical> forms' (κέχρηται τοῖς εἴδεσιν, 79a7). I do not mean, of course, that Aristotle postulates *conjunctive* properties in addition to the *conjuncts*: if a celestial body has the property of being spherical (S) and the property of being in motion (M), then it has

Especially illuminating about the contrast between astronomy and geometry is the commentary of Philoponus (220.1-5): On the one hand, Philoponus cites Theodosius (a mathematician and astronomer of the second century BC) and his work *On Spheres*, as a paradigmatic case of a geometer that studies the attributes of the sphere, without taking into account any matter; instead, he focuses his attention on the spherical shape and whatever holds true of spheres, e.g. that if a sphere is cut by a plane a circle comes about and so on.[16] Philoponus contrasts that case with the case of Autolycus (an astronomer of the fourth century BC) and his work *On Moving Sphere*, where the latter writes about moving spheres and what holds true of them, and thus he is a concerned with a more particular kind of subject than Theodosius, and he is closer to the natural philosopher in that he also takes into account motion (he examines a combination of shape and motion).[17] (220.4-9) This helpful comment can be examined alongside Aristotle's

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the conjunctive property of being spherical and being in motion (S&M); this is not to say that it has three distinct properties S, M and S&M. (Following [Armstrong (1978), p.30ff]). What does it mean that the optician studies the mathematical lines not qua mathematical but qua physical? Drawing an analogue with the investigations of the astronomer (where we saw that the subject matter of astronomers is not merely geometrical aspects of the heavenly bodies (e.g. their shape) but also properties such as being in motion), we can argue similarly for the subject matter of optics: it is not merely the geometrical aspects of visual rays (properties such as being straight or circular and so on) but also other, physical properties, the nature of which has to be determined. Now McKirahan in his article 'Aristotle's Subordinate Sciences' points to Euclid's *Optics* as the paradigmatic case that will offer some illumination on the subject matter of optics. This approach is partially correct: Euclid's *Optics* studies only the geometrical aspect of visual rays and does not make any reference to further physical properties that those lines might have; it does not study properties such as their being in motion (since those rays travel outwards from the eye/the light object), as well as their strength and weakness. (Burnyeat also makes this point in [Burnyeat (2005), pp.36-37]). One need only look at Aristotle's *Meteorologica* and the explanation of phenomena such as the reflection of the visual rays, to get an understanding of the importance of properties such as the weakness of the visual rays in our analysis of them. (For extensive discussion consult [Wilson (2013)] esp. ch.12).

[16] ὁ γοῦν Θεοδόσιος ἐν τοῖς Σφαιρικοῖς διδάσκων τὰ συμβαίνοντα πάθη τῇ σφαίρᾳ οὐδὲν προσλογίζεται ὕλην, ἀλλὰ χωρίσας πάσης οὐσίας τὸ σφαιρικὸν σχῆμα οὕτω τὰ συμβαίνοντα αὐτῷ ἐπισκέπτεται, ὅτι ἐὰν σφαῖρα ἐπιπέδῳ τμηθῇ κύκλον ποιεῖ, καὶ ὅσα ἄλλα.

[17] ὁ δὲ Αὐτόλυκος Περὶ κινουμένης σφαίρας γράψας καὶ ὅσα συμβαίνει τῇ κινουμένῃ σφαίρᾳ, μερικώτερός ἐστι τοῦ Θεοδοσίου καὶ μᾶλλον τῷ φυσικῷ προσεγγίζων (ἡ γὰρ κίνησις ἐγγύς

own remarks about astronomy in *Meta*. M.3, 1077b22-30, where he claims that the subject matter of the latter is just moving solids etc. and not some peculiar intermediate entities as some Platonists suppose.

[1.3] The question of idealisation

All the above discussion, however, presupposes one thing: that there are perfect instantiations of the various geometrical entities in the physical world. Provided that those entities are exemplified in the physical world, no misrepresentation of the reality around us results when the mathematicians cognitively isolate those entities from all those irrelevant, extra-mathematical elements that do not pertain to their investigations. For, it is easy to see that when astronomers study the celestial bodies qua spheres no misrepresentation of their reality results *given that they are perfect spheres*; or consider Simplicius' example of the solid body:

> For not even is the solid assumed by him to be natural <body>, but only something with three dimensions as if such things existed per se; for the mathematician concerns himself with the features that can be mentally separated.[18] [Simplicius: *On Aristotle 'Physics' 2*, 290.34-291.2; Fleet's trans.]

One might claim, however, that in the case of geometry, there is a margin for approximation: the surface of my desk is unlikely to be perfectly planar and my basketball is not a perfect sphere. One then might argue that just because some of the minutial aberrations and irregularities (such as the unevenness on the surface of my desk) are omitted this does not mean that our definitions are not representative of reality. We cannot, of course, talk of abstraction in the sense described previously: this is not a mere elimination from consideration anymore but something more, namely idealisation.[19] According to this alternative reading,

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πως ἐστι τῆς οὐσίας)· εἰ γὰρ καὶ μὴ ἐπινοεῖ οὐσίαν τινὰ ἐν τῇ κινουμένῃ σφαίρᾳ, ἀλλ' οὖν σύνθεσίν τινα λαμβάνει τοῦ σχήματος καὶ τῆς κινήσεως, καὶ ταύτῃ ἐγγύς πως ἐστι τῆς οὐσίας.

[18] οὐδὲ γὰρ τὸ στερεὸν αὐτῷ φυσικὸν ὑπόκειται, ἀλλ' αὐτὸ τοῦτο μόνον τὸ τριχῇ διεστώς. ὡς εἰ καὶ καθ'ἑαυτὰ ἦν τοιαῦτα· περὶ γὰρ τὰ τῇ νοήσει χωριστὰ καταγίνονται.

[19] In a similar discussion, Mendell calls it 'ideal abstraction' in [Mendell (1986), pp.73-75].

there is no error when the geometers are thinking of their objects as separate from change/perceptible matter because those objects are the idealised versions of everyday ones. One may retort that any talk of idealisation is dangerous, inasmuch as it leads us away from the realism Aristotle espouses in that passage. A first response would be to limit the scope of the idealisation: not every geometrical object needs to be considered as an idealised version of a sensible one; furthermore, the issue of idealisation is limited in geometry –one does not talk of idealisation in the case of arithmetic. Does Aristotle allow for idealisation in his discussion of the metaphysical status of mathematicals in *Metaphysics* M and N? And is the picture there consistent with his naïve realism in the *Physics* passage? Answers to questions like the above will have to wait until a proper examination of the relevant passages.

Chapter 2: Aristotle on the metaphysical status of geometricals

[2.1] (Not really) a tetrachotomy

The following passage from the first chapter of Book M of the *Metaphysics* lists several options for the *mode of existence* of mathematicals:

> If the objects of mathematics exist, then (i) they must exist either in sensible objects, as some say, or (ii) separate from sensible objects (and this also is said by some), or if they exist in neither of these ways, either (iii) they do not exist, or (iv) they exist in some other way. So that the subject of our discussion will be not whether they exist but how they exist.[20] [*Meta.* M.1, 1076a32-37; Ross' trans.]

Aristotle's primary interest, as the last sentence of this passage makes clear (ὥσθ' ἡ ἀμφισβήτησις ἡμῖν ἔσται οὐ περὶ τοῦ εἶναι ἀλλὰ περὶ τοῦ τρόπου, 1076a36-37), lies in the specific mode of existence of the objects of mathematics. The third option, however, the option of their non-existence, is *not* stated here merely for the sake of completeness. For Aristotle as well as for Platonists, *anti-realism*, the view that mathematical objects do not exist, does not merit serious consideration. If the propositions of mathematics are true, then they are true of things that exist; Aristotle and the Platonists surely think that the propositions of mathematics are true. One then should not overemphasise the extent of disagreement between Aristotle and the Platonists with regard to their respective philosophies of mathematics as, for example, Julia Annas does.[21] As Myles Burnyeat explains, any discussion about the metaphysical status of mathematical entities that purports to be an accurate reflection of the Greek philosophy of mathematics has to presuppose (or at least to be largely based on) a *realist conception of mathematical truth*:

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[20] ἀνάγκη δ', εἴπερ ἔστι τὰ μαθηματικά, (i) ἢ ἐν τοῖς αἰσθητοῖς εἶναι αὐτὰ καθάπερ λέγουσί τινες, (ii) ἢ κεχωρισμένα τῶν αἰσθητῶν (λέγουσι δὲ καὶ οὕτω τινές)· ἢ εἰ μηδετέρως, (iii) ἢ οὐκ εἰσὶν ἢ (iv) ἄλλον τρόπον εἰσίν· ὥσθ' ἡ ἀμφισβήτησις ἡμῖν ἔσται οὐ περὶ τοῦ εἶναι ἀλλὰ περὶ τοῦ τρόπου.

[21] Annas claims that what distinguishes Aristotle from Plato is 'the question whether mathematical objects exist'. In [Annas (1976), pp. 26-27].

> No-one in this debate thinks to explain mathematical truth as *theorem-hood* (derivability from the axioms). No one has the idea that mathematical truth could be internal to mathematical statements in the manner of analytic statements like 'Bachelors are unmarried.' No one suggests it would be enough to regard mathematical theorems as approximately true of the physical world . . . Here, as elsewhere in Greek philosophy, the discussion is constrained by a heavily *realist concept of truth*. [Burnyeat (1987), p.224; italics mine]

Burnyeat rightly cautions against ascribing to Aristotle a notion of mathematical truth as *theorem-hood* (=derivability from the axioms, also known as *if-thenism*), for the simple reason that this notion implies that mathematical statements can be *devoid of content*:[22]

> Now the entities referred to in a given science are entities whose existence is necessary for the theorems of the science to be true. That is why option (iii) in the tetrachotomy of M.1 has no takers and receives no discussion. *It would mean that mathematics was not true*. All parties to the debate agree that mathematics is true. All parties are therefore committed to

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[22] Hilary Putnam in his article 'The Thesis that Mathematics is Logic' attributes to Russell the following philosophy of mathematics: 'Mathematicians are in the business of showing that *if* there is any structure which satisfies such-and-such axioms (e.g., the axioms of group theory), *then* that structure satisfies such-and-such further statements (some theorems of group theory or other).' [Putnam (1967), p.281; italics mine]. Putnam himself also adopted this philosophy of mathematics in this same article. There is, however, a danger of vacuity that lurks underneath; Dale Jacquette offers the following critique of *if-thenism*: 'The inferences invoked in Putnam's if-thenism . . . when their content is universally reduced to conditional deductive form, *appear to be altogether vacuous of specific mathematical content*.' [Jacquette (2004), p.320; italics mine]. *If-thenism*, however, has roots that go back at least to the Ockhamist tradition and the debate between indivisibilists and anti-indivisbilists; the following is a passage from an anonymous disciple of Ockham who gives us a glimpse of Ockham's interpretation of Aristotle's philosophy of mathematics: 'According to the preceding principle he [i.e. Ockham] posits that <u>one must not admit indivisibles such as those commonly conceded, such as points, lines, surfaces, and things of that kind</u>. In fact, neither reason, nor experience, nor authority prohibit us from doing so. <u>He states that the texts authorised by Aristotle should be interpreted conditionally</u>. When Aristotle asserts, for example, that the circle is a shape such that the lines from its center to its circumference are all equal, he states that it must be understood thus: <u>the circle is a shape such that, if a point existed, the lines from this point to its circumference will be equal</u>. That is how one ought to explain all the postulates and conclusions relative to indivisibles.' (Contained in [Duhem (1985), p.21]; underlining mine).

> accepting that mathematicals exist. The dispute, as M.1 was bound to conclude (1076a36–37), is about their manner of existence.[23] [Burnyeat (1987), p.221; italics mine]

The discussion that follows Aristotle's listing of the possible views regarding the mode of existence for mathematicals can be summarised as follows: Aristotle's first target is the people who endorse a view of immanent Platonism (i.e. that mathematicals somehow exist *in* the sensibles) (M.2, 1076a38-b11); he then argues against a more traditional view of Platonism (i.e. against the view that mathematicals enjoy *separate* existence from the sensibles) (M.2, 1076b11-1077b14). He concludes the discussion by asking whether mathematicals do not exist at all, or exist in some other way (M.2, 1077b14-17). Then, in M.3 (1077b17-1078a31) he explains (albeit in a cryptic way) the special mode of existence that mathematical objects do have.

[23] As it can be seen from the first of Burnyeat's passages, he makes the stronger claim that we *cannot* even regard mathematical theorems as *approximately true* of the physical world. I am not so sure that Aristotle would agree with such a claim. In any case Burnyeat's argument in the passage quoted could be substantially strengthened if we invoke, as Cleary rightfully does, *the argument from the sciences*. As Cleary notes, the argument is based on the 'fundamental assumption is that any genuine science must have a real or existent object. Since Aristotle shares [with the Platonists] that epistemological assumption, he cannot accept the non-existence of the objects of mathematics as that would leave his paradigmatic sciences without foundations.' In [Cleary (1995), p.280].

[2.2] The notion of priority and its role in the M.2 discussion

[2.2.1] The multifaceted notion of priority

Aristotle's general strategy consists in showing that the Platonists' arguments can, at best, account for the priority *in account* or *definition* of mathematicals to sensible objects–those arguments cannot establish the priority in *substance* of mathematicals to sensible objects. Much of the discussion is framed around a conception of mathematicals as *boundaries* and their priorities to the solids they bound.[24] Aristotle's criticism of the view that mathematicals exist in separation from the sensibles begins as follows:

> But, again, it is not possible that such entities should exist separately. For if besides the sensible solids there are to be other solids which are separate from them and prior to the sensible solids, it is plain that besides the planes also there must be other and separate planes and points and lines; for consistency requires this.[25] [*Meta*. M.2., 1076b11-16; Ross' trans.]

One of the main tenets of (what can be called the orthodox) Platonic philosophy of mathematics is that mathematical entities enjoy *separate* existence over and above sensible substances; the above passage highlights a further aspect of this philosophy: that such mathematicals are somehow *prior* (πρότερα) to sensible substances. The following brief excursus intends to shed some light on the multifaceted notion of priority, a concept that plays such an important role in Aristotle's discussion of the metaphysical status of mathematical objects.

Aristotle's notion of separation is sometimes tied to the notion of *priority in substance*: in *Meta.* Δ.11, 1019a1-4, for instance, Aristotle says that things are called prior (πρότερα) in virtue of their nature and substance (κατὰ φύσιν καὶ οὐσίαν) when it is possible for them to exist without other things (ἐνδέχεται εἶναι

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[24] Stephen Menn is one of the few scholars who have highlighted the fact that Aristotle actually spends a large part of the second chapter of Book M of the *Metaphysics* responding to Platonist arguments framed in terms of *priority* rather than *separation*. In [Menn, 'Ιγ3', p.19].

[25] ἀλλὰ μὴν οὐδὲ κεχωρισμένας γ' εἶναι φύσεις τοιαύτας δυνατόν. εἰ γὰρ ἔσται στερεὰ παρὰ τὰ αἰσθητὰ κεχωρισμένα τούτων ἕτερα καὶ πρότερα τῶν αἰσθητῶν, δῆλον ὅτι καὶ παρὰ τὰ ἐπίπεδα ἕτερα ἀναγκαῖον εἶναι ἐπίπεδα κεχωρισμένα καὶ στιγμὰς καὶ γραμμάς (τοῦ γὰρ αὐτοῦ λόγου)·

ἄνευ ἄλλων), whereas those other things cannot exist without them; this division, according to Aristotle, was used by Plato (τὰ μὲν δὴ οὕτω λέγεται πρότερα καὶ ὕστερα, τὰ δὲ κατὰ φύσιν καὶ οὐσίαν, ὅσα ἐνδέχεται εἶναι ἄνευ ἄλλων, ἐκεῖνα δὲ ἄνευ ἐκείνων μή· ᾗ διαιρέσει ἐχρήσατο Πλάτων, 1019a1-4). Furthermore, in *Meta*. Z.1, 1028a31-b2, Aristotle says that substance is prior (πρῶτον) in three ways, in definition, in knowledge and in nature (λόγῳ καὶ γνώσει καὶ φύσει). That substance is prior in nature is then explained in terms of *separation*: none of the other predicates is separate but only substance (τῶν μὲν γὰρ ἄλλων κατηγορημάτων οὐθὲν χωριστόν, αὕτη δὲ μόνη):

> Now we speak of what is primary in many ways, but substance is primary in every way-in definition, in knowledge and in nature. For none of the other predicates is separate but this alone; and in definition too this is primary, since in the definition of everything there must occur the definition of a substance; and we think we know a thing most fully when we know what the man is, or the fire, rather than when we know its quality or quantity or place-since it is also true that each of these themselves we know only when we know what that quantity or quality is. [*Meta*. Z.1, 1028a31-b2; Bostock's trans. mod.]

From the *Meta*. Δ.11 passage we may infer that A is separate from B iff A can exist without B (ὅσα ἐνδέχεται εἶναι ἄνευ ἄλλων), or, equivalently, iff A exists independently of B. We may also infer an association of separation with priority: if A can exist without B, but B cannot exist without A (ἐκεῖνα δὲ ἄνευ ἐκείνων), then A is not only separate from but also prior in substance to B.[26] Priority in substance, then, is understood as follows: A is prior in substance to B if A can exist apart from B, but B cannot exist apart from A.

Another sense of separation is separation in account or in definition. An item, A, is separate in definition (or in account) from another item, B, if and only if A is (or can be) defined without reference to B. For example, white can be defined apart from man but female cannot be defined apart from animal (ὥσπερ τὸ λευκὸν ἄνευ τοῦ ἀνθρώπου ἐνδέχεται ἀλλ' οὐ τὸ θῆλυ ἄνευ τοῦ ζῴου, 1030b25-26).

[26] For extensive discussion on the matter see [Fine (1984)].

However, instead of definitional separation, Aristotle often speaks of *definitional priority*. Peramatzis [27] lists several passages where Aristotle discusses definitional priority:

> 1) For with respect to the account the former are defined in terms of the latter, and the latter are prior in that they are without the former. [*Meta.* Z.10, 1034b30-2; Bostock's trans.]
>
> 2) The parts of the formula, into which the formula is divided, are prior – some of them or all of them; and the formula of the right angle is not divided into the formula of the acute, but that of the acute into that of the right; for one who defines the acute uses the right angle; for the acute is less than the right. And, similarly, in the case of the circle and the semicircle; for the semicircle is defined through the circle and the finger through the whole; for finger is such-and-such a part of man. [*Meta.* Z.10, 1035b4–11; Bostock's trans.]
>
> 3) <Things are prior> in definition to those things whose definitions are compounded from definitions of them. [*Meta.* M.2, 1077b3–4; Annas' trans.]
>
> 4) And in definition, too, this [i.e. substance] is primary (for it is necessary that in each thing's definition there should occur the definition of a substance). [*Meta*. Z.1, 1028a34–6; Bostock's trans.]

It seems then that definitional priority involves as a necessary part the concept of separation in definition.[28] Consider the second passage listed above (*Meta.* Z.10, 1035b4–11): the account of the right angle does not include the account of the acute but the converse is true (ὁ δὲ τῆς ὀρθῆς λόγος οὐ διαιρεῖται εἰς ὀξείας λόγον, ἀλλ' <ὁ> τῆς ὀξείας εἰς ὀρθήν, 1035b6–7). One defines the acute angle in terms of the right angle but not conversely (χρῆται γὰρ ὁ ὁριζόμενος τὴν ὀξεῖαν τῇ ὀρθῇ· "ἐλάττων" γὰρ "ὀρθῆς" ἡ ὀξεῖα, 1036b7–8). Priority in account then is understood as follows: A is prior in account to B if, in giving an account of B, we formulate it in terms of A, whereas in giving an account of A we need not appeal to B.

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[27] See [Peramatzis (2011), p.24 ff.] for extensive discussion.

[28] So [Peramatzis (2011), p.25].

The following passage from the *Metaphysics* Z.13 introduces yet another sense of priority:

> Further, it is absurd and impossible that a this and a substance, if it is composed of anything, should be composed not of substances, nor of a this, but of a quality. For then the quality, which is not a substance, will be prior to substance and the this. And this is impossible; for attributes cannot be prior to substance either in formula or in time or in generation, since if they were they would also be separable. [*Meta.* Z.13, 1038b23-29; Bostock's trans.]

The passage introduces priority in generation or 'coming-to-be': A is prior in generation to B if B is further ahead in a process of generation than A. Priority plays also an important role in *Metaphysics* Θ, where Aristotle discusses actuality and potentiality:

> Since it has been determined in how many ways prior is said, it is evident that actuality is prior to potentiality. And I mean by potentiality not only that defined kind which is called an origin of change in something else or in a thing *qua* something else, but generally all origins of change or remaining static. For nature too is in the same class as potentiality; for it is an origin of change, though not in something else but in a thing itself *qua* itself. Then actuality is prior to all potentiality of this sort both in account and in substance; and in time in one way it is and in another way it is not. [*Meta.* Θ.8, 1049b4-12; Makin's trans.]

For Aristotle, actuality is prior to potentiality 1) in account (an account of a potential thing A will necessarily be formulated in terms of an actual thing B) and 2) in substance (an actual thing A can exist without a potential thing B, but the reverse does not hold). In time, however, actuality is in one sense prior but in another sense it is not: one can claim, for instance, that the chicken temporally precedes the egg; but one might also claim that an actually-existing egg does temporally precede the potentially existing chicken. Aristotle also holds that what is posterior in generation is prior in substance and in form (τῇ γενέσει ὕστερα τῷ εἴδει καὶ τῇ οὐσίᾳ πρότερα):

> But indeed actuality is prior in substance too, first because things posterior in coming to be are prior in form and in substance (for example, adult to boy and man to seed; for the one

> already has the form, the other does not).[29] [*Meta*. Θ.8, 1050a4-7; Makin's trans.]

This point will be of crucial importance in Aristotle's criticisms of Platonic views in *Metaphysics* M.2 as we shall see shortly.

[2.2.2] Aristotle's arguments against the metaphysical priority of lower-dimensional entities

The context of aporia #12 in Book B of the *Metaphysics* is one of crucial importance for our understanding of the M.2 discussion.[30] The aporia is stated as follows:

> A question connected with these is whether numbers and bodies and planes and points are substances or not. If they are not, it baffles us to say what being is and what the substances of things are.[31] [*Meta*. B.5, 1001b26-29; Ross' trans.]

In the discussion that ensues Aristotle considers a Platonist argument that ascribes greater metaphysical status not to three-dimensional bodies, but rather to the lower-dimensional quantities that determine or bound them:

> And as to the things which might seem most of all to indicate substance, water and earth and fire and air, of which composite bodies consist, heat and cold and the like are modifications of these, not substances, and the body which is thus modified alone persists as something real and as a substance. But, on the other hand, a body is surely less of a substance than a surface, and a surface less than a line, and a line less than a unit and a point. For a body is bounded by these; and they are thought to be capable of existing without body, but a body cannot exist without these.[32] [*Meta*. B.5, 1001b26-1002a8; Ross'

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[29] Ἀλλὰ μὴν καὶ οὐσίᾳ γε, πρῶτον μὲν ὅτι τὰ τῇ γενέσει ὕστερα τῷ εἴδει καὶ τῇ οὐσίᾳ πρότερα (οἷον ἀνὴρ παιδὸς καὶ ἄνθρωπος σπέρματος· τὸ μὲν γὰρ ἤδη ἔχει τὸ εἶδος τὸ δ' οὔ).

[30] Following the division of aporiae in [Crubellier & Laks (2009), pp.1-2].

[31] Τούτων δ' ἐχομένη ἀπορία πότερον οἱ ἀριθμοὶ καὶ τὰ σώματα καὶ τὰ ἐπίπεδα καὶ αἱ στιγμαὶ οὐσίαι τινές εἰσιν ἢ οὔ. εἰ μὲν γὰρ μή εἰσιν, διαφεύγει τί τὸ ὂν καὶ τίνες αἱ οὐσίαι τῶν ὄντων·

[32] ἃ δὲ μάλιστ' ἂν δόξειε σημαίνειν οὐσίαν, ὕδωρ καὶ γῆ καὶ πῦρ καὶ ἀήρ, ἐξ ὧν τὰ σύνθετα σώματα συνέστηκε, τούτων θερμότητες μὲν καὶ ψυχρότητες καὶ τὰ τοιαῦτα πάθη, οὐκ οὐσίαι, τὸ δὲ σῶμα τὸ ταῦτα πεπονθὸς μόνον ὑπομένει ὡς ὄν τι καὶ οὐσία τις οὖσα. ἀλλὰ μὴν τό γε σῶμα ἧττον οὐσία τῆς ἐπιφανείας, καὶ αὕτη τῆς γραμμῆς, καὶ αὕτη τῆς μονάδος καὶ τῆς στιγμῆς· τούτοις γὰρ ὥρισται τὸ σῶμα, καὶ τὰ μὲν ἄνευ σώματος ἐνδέχεσθαι δοκεῖ εἶναι τὸ δὲ σῶμα ἄνευ τούτων ἀδύνατον.

trans.]

According to a certain Platonist line of thought, the fundamental elements of the world (earth, air, fire and water) have a better claim to be substances than the things composed out of them.[33] Furthermore, the Platonists seem to argue for the substantial priority of the boundaries of things to the things they bound; more specifically, Platonists argue not only that surfaces are prior to bodies, but also that lines are prior to surfaces, points to lines and that units are prior to points. But why do Platonists suppose that lower-dimensional entities are prior in substance to higher-dimensional ones? One answer is provided by the passage above: if B is delimited (or defined, ὥρισται) by A, then A is more substantial than B; for example, a triangle is bounded by three straight lines, therefore, the lines are more substantial than the triangle itself.[34] This conforms well with Alexander's commentary of the passage:

> The things by which something is defined and given its form are substance to a higher degree than that which is defined by them. . . For it is not possible to conceive of body without a surface, or of surface without a line, or of line without a point - for these items are included in the definitions of those things: body is said to be that which has length, breadth and depth; surface that which has length and breadth; and line that which has length without breadth, and points as limits - but a point is conceived of even apart from a line, and a line apart from a plane, and a plane without body.[35] [Alexander: *Comm. on*

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[33] A parallel passage can be found in *Meta.* Z.2 where Aristotle reports that 'some think that the limits of a body–i.e. surfaces, lines, points, and units–are substances, and more so than the body and the solid.' In [*Meta*. Z.2, 1028b16–18; Bostock's trans.].

[34] Stephen Menn offers an extensive discussion of aporia #12 in [Menn, 'Iβ3', pp.29-32]. Menn traces the origins of such a Platonist argument to the *Timaeus*: 'Aristotle is thinking here of the kind of argument that the *Timaeus* makes, after reducing the nature of things to body and thus to 'depth': 'depth is always necessarily circumscribed by surface, and the plane base-surface is constituted out of triangles' (53c6-8), and so on. . . . Aristotle is calling attention to one important feature of Plato's strategy of argument, namely that it argues that the boundaries of things are prior to the things.' In [Menn, 'Iβ3', p.29].

[35] οἷς ὁρίζεταί τι καὶ εἰδοποιεῖται, ἐκεῖνα τοῦ ὁριζομένου ὑπ' αὐτῶν μᾶλλον οὐσία· . . . σῶμα μὲν γὰρ ἄνευ ἐπιφανείας οὐχ οἷόν τε νοηθῆναι, οὐδ' αὖ ἐπιφάνειαν ἄνευ γραμμῆς, οὐδὲ ταύτην χωρὶς σημείου. ἐν γὰρ τοῖς ὁρισμοῖς αὐτῶν συμπαραλαμβάνεται κἀκεῖνα· σῶμα μὲν γὰρ λέγεται εἶναι τὸ μῆκος καὶ

Meta. Book B, 229.22-230.7; Madigan's trans.]

Yet another reason the Platonists attribute greater metaphysical status to lower-dimensional entities is supplied from the following passages from *Metaphysics* M.2:

> And, in general, conclusions contrary alike to the truth and to the usual views follow, if one supposes the objects of mathematics to exist thus as separate entities. For if they exist thus they must be prior to sensible spatial magnitudes, but in truth they must be posterior; for the incomplete spatial magnitude is in the order of generation prior, but in the order of substance posterior, as the lifeless is to the living.[36] [*Meta*. M.2, 1077a14-21; Ross' trans.]

> Again, the modes of generation of the objects of mathematics show that we are right. For the dimension first generated is length, then comes breadth, lastly depth, and the process is complete. If, then, that which is posterior in the order of generation is prior in the order of substance, body will be prior to the plane and the line. And in *this* way also it is more complete and more whole, because it can become animate. How, on the other hand, could a line or a plane be animate? The supposition passes the power of our senses.[37] [*Meta*. M.2,1077a24-31; Ross' trans.]

The examination of the passages reveals that that Platonists invoke a certain process of generation for geometrical entities. A (moving) point generates a line, a (moving) line generates a plane, and a (moving) plane generates a solid.[38]

πλάτος καὶ βάθος ἔχον, ἐπιφάνεια δὲ ὃ μῆκος καὶ πλάτος ἔχει, γραμμὴ δὲ μῆκος ἀπλατές, ἧς πέρατα σημεῖα· σημεῖον δὲ νοεῖται καὶ χωρὶς γραμμῆς, καὶ αὕτη χωρὶς ἐπιπέδου, καὶ ἐπίπεδον ἄνευ σώματος.

[36] ὅλως δὲ τοὐναντίον συμβαίνει καὶ τοῦ ἀληθοῦς καὶ τοῦ εἰωθότος ὑπολαμβάνεσθαι, εἴ τις θήσει οὕτως εἶναι τὰ μαθηματικὰ ὡς κεχωρισμένας τινὰς φύσεις. ἀνάγκη γὰρ διὰ τὸ μὲν οὕτως εἶναι αὐτὰς προτέρας εἶναι τῶν αἰσθητῶν μεγεθῶν, κατὰ τὸ ἀληθὲς δὲ ὑστέρας· τὸ γὰρ ἀτελὲς μέγεθος γενέσει μὲν πρότερόν ἐστι, τῇ οὐσίᾳ δ' ὕστερον, οἷον ἄψυχον ἐμψύχου.

[37] ἔτι αἱ γενέσεις δηλοῦσιν. πρῶτον μὲν γὰρ ἐπὶ μῆκος γίγνεται, εἶτα ἐπὶ πλάτος, τελευταῖον δ' εἰς βάθος, καὶ τέλος ἔσχεν. εἰ οὖν τὸ τῇ γενέσει ὕστερον τῇ οὐσίᾳ πρότερον, τὸ σῶμα πρότερον ἂν εἴη ἐπιπέδου καὶ μήκους· καὶ ταύτῃ καὶ τέλειον καὶ ὅλον μᾶλλον, ὅτι ἔμψυχον γίγνεται· γραμμὴ δὲ ἔμψυχος ἢ ἐπίπεδον πῶς ἂν εἴη; ὑπὲρ γὰρ τὰς αἰσθήσεις τὰς ἡμετέρας ἂν εἴη τὸ ἀξίωμα.

[38] Cf. *De Anima* I.4, 409a3-5: ἔτι δ'ἐπεί φασι κινηθεῖσαν γραμμὴν ἐπίπεδον ποιεῖν, στιγμὴν δὲ γραμμήν… Cherniss tentatively attributes this view of generation of mathematicals from lower-dimensional ones to Speusippus. See [Cherniss (1944), pp.396-397]. For the view that a line is generated by a flowing point see also Sextus, *Against the Mathematicians*, Book 9, section 380. For a detailed discussion of Sextus' arguments consult [Betegh (2015), esp. pp.154-165].

When this process of generation has reached the solid it has reached its end (τέλος, 1077a26). Proclus gives us a more detailed insight of this process of generation when commenting on Euclid's definition of the line ('a line is breadthless length'):

> The line has also been defined in other ways. Some define it as the 'flowing of a point', others as 'magnitude extended in one direction'. The latter definition indicates perfectly the nature of the line, but that which calls it the flowing of a point appears to explain it in terms of its generative cause and sets before us not line in general, but the immaterial line. This line owns its being to the point, which, though without parts, is the cause of the existence of all divisible things; and the 'flowing' indicates the forthgoing of the point and its generative power that extends to every dimension without diminution and, remaining itself the same, provides existence to all divisible things.[39] [Proclus: *A Commentary on The First Book of Euclid's Elements*, 97.6-17; Morrow's trans. mod.]

It seems then that Platonists ague that if A is prior in generation to B, then A is prior in substance to B. The result is that points are prior in substance to lines, lines are prior in substance to surfaces, surfaces are prior in substance to solids. Aristotle's response in lines 1077a26-27 makes use of his own principle, established in *Meta*. Θ.8, that what is posterior in generation is substantially prior (εἰ οὖν τὸ τῇ γενέσει ὕστερον τῇ οὐσίᾳ πρότερον). Even if one grants the Platonists that solids are somehow generated from lower-dimensional geometricals, solids are more complete (καὶ ταύτῃ καὶ τέλειον καὶ ὅλον μᾶλλον, 1077a28) than what lies at the beginning of the process of generation: the line and the plane, for example, may be prior in generation to the solid but the solid body (τὸ σῶμα) is prior in substance to them. In lines 1077a28-29 Aristotle invokes the capacity of the solid body for becoming animate (ἔμψυχον) as the reason that makes it more substantial than lower-dimensional entities; it seems

[39] Ἀφορίζονται δὲ αὐτὴν καὶ κατ' ἄλλας μεθόδους, οἱ μὲν ῥύσιν σημείου λέγοντες, οἱ δὲ μέγεθος ἐφ' ἓν διαστατόν. ἀλλ' οὗτος μὲν ὁ ὅρος τέλειός ἐστιν τὴν οὐσίαν σημαίνων τῆς γραμμῆς, ὁ δὲ σημείου ῥύσιν εἰπὼν ἔοικεν ἀπὸ τῆς αἰτίας αὐτὴν τῆς γεννητικῆς δηλοῦν καὶ οὐ πᾶσαν γραμμὴν ἀλλὰ τὴν ἄυλον παρίστησι· ταύτην γὰρ ὑφίστησι τὸ σημεῖον ἀμερὲς ὑπάρχον, ὑπάρξεως δὲ τοῖς μεριστοῖς αἴτιον ὄν. ἡ δὲ ῥύσις τὴν πρόοδον ἐνδείκνυται καὶ τὴν γόνιμον δύναμιν, τὴν ἐπὶ πᾶσαν διάστασιν φθάνουσαν καὶ οὐκ ἐλαττουμένην, τὴν αὐτὴν μὲν ἑστῶσαν, πᾶσι δὲ τοῖς μεριστοῖς τὴν οὐσίαν παρεχομένην.

rather bizarre to suggest that a line or a plane can become animate. At this point Julia Annas claims that Aristotle is guilty of confusing mathematical solids with physical bodies:

> …Aristotle is thinking of a physical object as a solid object made up of planes, lines, etc., so the latter are 'incomplete' in that although there have to be planes, etc. to make up a solid, the solid is that via which the planes must be identified and not vice versa. The relation of planes, etc. to solids is compared with that of the earth that becomes a man. But if this is Aristotle's arguments he is confusing a mathematician's solid with a physical object; the latter is not made up of planes in the way the former is. [Annas (1976), p.145]

Is Aristotle guilty of such charge? Annas claims that a physical object is not made up of planes in the way a mathematical solid is. Annas does not elaborate on the supposed difference in composition between a mathematical solid and a sensible one. In what way does Socrates' composition of planes differ from a cube's composition of squares? Does a cube consist of infitely many squares in a way that Socrates does not consist of infinitely many planes? When Aristotle poses (1077a33-34) the question about how lines can be substances, one of the options provided is that they might be substances as forms/shapes or in a matter–like way; he rejects both options:

> Again, body is a sort of substance; for it already has in a sense completeness. But how can lines be substances? Neither as a form or shape, as the soul perhaps is, nor as matter, like body; for we have no experience of anything that can be put together out of lines or planes or points, while if these had been a sort of material substance, we should have observed things which could be put together out of them.[40] [*Meta*. M.2, 1077a31-36; Ross' trans.]

Aristotle thinks that it is not very plausible to ascribe a process of generation for solids beginning from lower-dimensional entities, since nothing seems to be capable of being assembled from lines and planes and points. Crucially, however, he denies that lower-dimensional entities can even exist as forms or shapes of the things they bound (contrary, for example, to the naïve realism espoused in

[40] ἔτι τὸ μὲν σῶμα οὐσία τις (ἤδη γὰρ ἔχει πως τὸ τέλειον), αἱ δὲ γραμμαὶ πῶς οὐσίαι; οὔτε γὰρ ὡς εἶδος καὶ μορφή τις, οἷον εἰ ἄρα ἡ ψυχὴ τοιοῦτον, οὔτε ὡς ἡ ὕλη, οἷον τὸ σῶμα· οὐθὲν γὰρ ἐκ γραμμῶν οὐδ' ἐπιπέδων οὐδὲ στιγμῶν φαίνεται συνίστασθαι δυνάμενον, εἰ δ' ἦν οὐσία τις ὑλική, τοῦτ' ἂν ἐφαίνετο δυνάμενα πάσχειν.

Physics B.2). What does this claim amount to? What is the metaphysical status of lower-dimensional entities? An answer to those question will have to wait the examination of the M.3 chapter of the *Metaphysics*.[41]

[2.2.3] Mathematicals as prior in definition

What sense of priority does Aristotle allow for mathematicals? The following passage from *Metaphysics* M.2 might shed some light to this question:

> Let it be granted that <the mathematical objects> are prior in formula. Yet not everything which is prior in formula is also prior in substance. Things are prior in substance if more able to go on existing when separated from the latter, and prior in definition to things whose definitions are compounded from definitions of them. These do not apply together. For if there are no attributes distinct from real objects (e.g. a moving or a white) then white is prior in definition to white man, but not in substance, since it cannot exist separately but only together with the compound (by compound I mean the white man). So clearly the result of abstraction is not prior, nor the result of addition subsequent, for the expression 'white man' is the result of adding a determinant to 'white'.[42] [*Meta*. M.2, 1077a36-b11, Ross' trans. mod.]

This passage describes two notions of priority discussed earlier. We learn that A is prior in substance (τῇ οὐσίᾳ) to B if A 'is more able to go on existing' when

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[41] There is also a passage in *De Caelo* III.1, where Aristotle is objecting that those who generate bodies out of planes contradict the laws of mathematics:

> But this last theory which composes every body of planes is, as is seen at a glance, in many respects in plain contradiction with mathematics. It is, however, wrong to remove the foundations of a science unless you can replace them with others more convincing. And, secondly, the same theory which composes solids of planes clearly composes planes of lines and lines of points, so that a part of a line need not be a line. [*De Caelo* III.1, 299a2-9; Stock's trans.]

[42] τῷ μὲν οὖν λόγῳ ἔστω πρότερα, ἀλλ' οὐ πάντα ὅσα τῷ λόγῳ πρότερα καὶ τῇ οὐσίᾳ πρότερα. τῇ μὲν γὰρ οὐσίᾳ πρότερα ὅσα χωριζόμενα τῷ εἶναι ὑπερβάλλει, τῷ λόγῳ δὲ ὅσων οἱ λόγοι ἐκ τῶν λόγων· ταῦτα δὲ οὐχ ἅμα ὑπάρχει. εἰ γὰρ μὴ ἔστι τὰ πάθη παρὰ τὰς οὐσίας, οἷον κινούμενόν τι ἢ λευκόν, τοῦ λευκοῦ ἀνθρώπου τὸ λευκὸν πρότερον κατὰ τὸν λόγον ἀλλ' οὐ κατὰ τὴν οὐσίαν· οὐ γὰρ ἐνδέχεται εἶναι κεχωρισμένον ἀλλ' ἀεὶ ἅμα τῷ συνόλῳ ἐστίν (σύνολον δὲ λέγω τὸν ἄνθρωπον τὸν λευκόν), ὥστε φανερὸν ὅτι οὔτε τὸ ἐξ ἀφαιρέσεως πρότερον οὔτε τὸ ἐκ προσθέσεως ὕστερον· ἐκ προσθέσεως γὰρ τῷ λευκῷ ὁ λευκὸς ἄνθρωπος λέγεται.

separated from B, and that A is prior in definition (τῷ λόγῳ) to B when the definition of the latter contains the definition of A (ὅσων οἱ λόγοι ἐκ τῶν λόγων). A parallel for priority in account/definition can be found in *Meta*. Δ.11, 1018b34ff. where the musical is said to be prior to the musical man in account, 'for the account cannot exist as a whole without the part' (Ross' trans) (καὶ κατὰ τὸν λόγον δὲ τὸ συμβεβηκὸς τοῦ ὅλου πρότερον, οἷον τὸ μουσικὸν τοῦ μουσικοῦ ἀνθρώπου· οὐ γὰρ ἔσται ὁ λόγος ὅλος ἄνευ τοῦ μέρους· 1018b34-36). Just like the M.2 passage, Aristotle cautions against thinking that this entails priority in substance since it is not possible for the musical to exist unless there is someone who is musical (καίτοι οὐκ ἐνδέχεται μουσικὸν εἶναι μὴ ὄντος μουσικοῦ τινός, 1018b36-37). The term 'compound' (σύνολον) in the M.2 passage seems to refer to the combination of an accident like *white* with a substantial subject like *man*.[43] White, he says, is definitionally prior (πρότερον κατὰ τὸν λόγον) to white man, since the definition of the former does not include that of the latter, though the converse is not true. But white is not prior in substance to white man (οὐ κατὰ τὴν οὐσίαν) since it cannot exist without it (οὐ γὰρ ἐνδέχεται εἶναι κεχωρισμένον ἀλλ' ἀεὶ ἅμα τῷ συνόλῳ ἐστίν). Presumably Aristotle does not wish to confine himself to white men but he thinks that white cannot exist without some white things (not necessarily men). Likewise, Aristotle says, mathematical objects are prior in definition to sensible bodies (τῷ μὲν οὖν λόγῳ ἔστω πρότερα) but this priority does not entail priority in substance.[44] The argument ought to be applied to three-dimensional bodies and to the lower–dimensional quantities that determine or bound them. Aristotle sums up his refutation of the Platonists as follows:

> It has, then, been sufficiently pointed out that the objects of mathematics are not substances in a higher sense than bodies are, and that they are not prior to sensibles in

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[43] So Cleary (1995), p.303.

[44] I, thus, find myself in disagreement with Ian Mueller who claims that the context of the B.12 aporia is not in any way connected with the discussion in M.2 ('it is unlikely that Aristotle is conscious of a direct connection of M.2 and 3 with aporia 12', in [Crubellier & Laks (2009), p.190]). That Aristotle grants the Platonists that mathematicals are only prior in account and not in substance is precisely his response to Platonic principles such as the one in that aporia: that if A is prior in account to B, then A is prior in substance to B.

> being, but only in formula, and that they cannot in any way exist separately. But since they could not exist *in* sensibles either, it is plain that they either do not exist at all or exist in a special way and therefore do not exist without qualification. For 'exist' has many senses.[45] [*Meta.* M.2, 1077b12-17; Ross' trans.]

The results of the discussion so far: mathematical entities (and by that Aristotle has in mind lower-dimensional or limit entities) are not more substantial than three-dimensional bodies; they are only prior in formula and they cannot exist in separation anywhere. We still need, however, an account of what is their mode of existence. A careful reader will notice that Aristotle has already defused the Protagorean objection that mathematical statements do not refer to any objects that exist. For, as the previous discussion has made clear, Aristotle already points to a philosophy of mathematics based on the readily available notion of the solid; his further investigations will amount to a clarification (if any) of the metaphysical status of lower-dimensional entities (points, lines, and planes).

[45] Ὅτι μὲν οὖν οὔτε οὐσίαι μᾶλλον τῶν σωμάτων εἰσὶν οὔτε πρότερα τῷ εἶναι τῶν αἰσθητῶν ἀλλὰ τῷ λόγῳ μόνον, οὔτε κεχωρισμένα που εἶναι δυνατόν, εἴρηται ἱκανῶς· ἐπεὶ δ' οὐδ'ἐν τοῖς αἰσθητοῖς ἐνεδέχετο αὐτὰ εἶναι, φανερὸν ὅτι ἢ ὅλως οὐκ ἔστιν ἢ τρόπον τινὰ ἔστι καὶ διὰ τοῦτο οὐχ ἁπλῶς ἔστιν· πολλαχῶς γὰρ τὸ εἶναι λέγομεν.

[2.3] Mathematicals in sensible things. A fanciful doctrine?

[2.3.1] Arguments against the existence of intermediate mathematicals in the sensibles

Let us closely examine Aristotle's argument against the existence of mathematicals in sensible things:

> That it is impossible for mathematical objects to exist in sensible things and at the same time that the doctrine in question is a fabricated one, has been said already in our discussion of difficulties, –the reasons being that it is impossible for two solids to be in the same place, and that according to the same argument all the other powers and characteristics also should exist in sensible things, none of them existing separately. This we have said already. But, further, it is obvious that on this theory it is impossible for any body whatever to be divided; for it would have to be divided at a plane, and the plane at a line, and the line at a point, so that if the point cannot be divided, neither can the line, and if the line cannot, neither can the plane nor the solid. What difference then does it make whether sensible things are of this kind, or, without being so themselves, have such things in them? The result will be the same; if the sensible things are divided the others will be divided too, or else not even the sensible things can be divided.[46] [*Meta.* M.2, 1076a38-b11; Ross' trans.]

At the outset of his discussion, Aristotle summarily dismisses the doctrine of mathematicals existing in the sensibles, pointing the reader to an earlier treatment of the issue in Book B of the *Metaphysics* (εἴρηται μὲν καὶ ἐν τοῖς διαπορήμασιν); the relative passage is located in *Meta*. B.2:

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[46] Ὅτι μὲν τοίνυν ἔν γε τοῖς αἰσθητοῖς ἀδύνατον εἶναι καὶ ἅμα πλασματίας ὁ λόγος, εἴρηται μὲν καὶ ἐν τοῖς διαπορήμασιν ὅτι δύο ἅμα στερεὰ εἶναι ἀδύνατον, ἔτι δὲ καὶ ὅτι τοῦ αὐτοῦ λόγου καὶ τὰς ἄλλας δυνάμεις καὶ φύσεις ἐν τοῖς αἰσθητοῖς εἶναι καὶ μηδεμίαν κεχωρισμένην· —ταῦτα μὲν οὖν εἴρηται πρότερον, ἀλλὰ πρὸς τούτοις φανερὸν ὅτι ἀδύνατον διαιρεθῆναι ὁτιοῦν σῶμα· κατ' ἐπίπεδον γὰρ διαιρεθήσεται, καὶ τοῦτο κατὰ γραμμὴν καὶ αὕτη κατὰ στιγμήν, ὥστ' εἰ τὴν στιγμὴν διελεῖν ἀδύνατον, καὶ τὴν γραμμήν, εἰ δὲ ταύτην, καὶ τἆλλα. τί οὖν διαφέρει ἢ ταύτας εἶναι τοιαύτας φύσεις, ἢ αὐτὰς μὲν μή, εἶναι δ' ἐν αὐταῖς τοιαύτας φύσεις; τὸ αὐτὸ γὰρ συμβήσεται· διαιρουμένων γὰρ τῶν αἰσθητῶν διαιρεθήσονται, ἢ οὐδὲ αἱ αἰσθηταί.

> Now there are some who say that these so-called intermediates between the Forms and the perceptible things exist, not apart from the perceptible things, however, but in these; the impossible results of this view would take too long to enumerate, but it is enough to consider such points as the following:—It is not reasonable that this should be so only in the case of these intermediates, but clearly the Forms also might be in the perceptible things; for the same account applies to both. Further, it follows from this theory that there are two solids in the same place, and that the intermediates are not immovable, since they are in the moving perceptible things. And in general to what purpose would one suppose them to exist, but to exist in perceptible things? For the same paradoxical results will follow which we have already mentioned; there will be a heaven besides the heaven, only it will be not apart but in the same place; which is still more impossible.[47] [*Meta.* B.2, 998a7-19; Ross' trans.]

Aristotle in *Meta.* B.2 is considering a doctrine according to which mathematicals are metaphysically between (μεταξύ) Platonic Forms and sensibles; according to a particular version of that doctrine (the one discussed in *Meta*. B.2, 998a7-19), intermediate mathematicals exist not apart from but *in* sensibles (οὐ μὴν χωρίς γε τῶν αἰσθητῶν ἀλλ' ἐν τούτοις). The following brief excursus is intended to shed some light on the nature of intermediates with the specific intention of highlighting their essential to our discussion features.[48] Aristotle explicitly attributes to Plato and his followers the doctrine that there are three fundamental types of entities, the Forms, the intermediate objects of mathematics and the sensible things, in two places. The first, most informative, passage is to be found in Aristotle's account of Plato's philosophy in *Metaphysics* A.6:

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[47] εἰσὶ δέ τινες οἵ φασιν εἶναι μὲν τὰ μεταξὺ ταῦτα λεγόμενα τῶν τε εἰδῶν καὶ τῶν αἰσθητῶν, οὐ μὴν χωρίς γε τῶν αἰσθητῶν ἀλλ' ἐν τούτοις· οἷς τὰ συμβαίνοντα ἀδύνατα πάντα μὲν πλείονος λόγου διελθεῖν, ἱκανὸν δὲ καὶ τὰ τοιαῦτα θεωρῆσαι. οὔτε γὰρ ἐπὶ τούτων εὔλογον ἔχειν οὕτω μόνον, ἀλλὰ δῆλον ὅτι καὶ τὰ εἴδη ἐνδέχοιτ' ἂν ἐν τοῖς αἰσθητοῖς εἶναι (τοῦ γὰρ αὐτοῦ λόγου ἀμφότερα ταῦτά ἐστιν), ἔτι δὲ δύο στερεὰ ἐν τῷ αὐτῷ ἀναγκαῖον εἶναι τόπῳ, καὶ μὴ εἶναι ἀκίνητα ἐν κινουμένοις γε ὄντα τοῖς αἰσθητοῖς. ὅλως δὲ τίνος ἕνεκ' ἄν τις θείη εἶναι μὲν αὐτά, εἶναι δ' ἐν τοῖς αἰσθητοῖς; ταὐτὰ γὰρ συμβήσεται ἄτοπα τοῖς προειρημένοις· ἔσται γὰρ οὐρανός τις παρὰ τὸν οὐρανόν, πλήν γ' οὐ χωρὶς ἀλλ' ἐν τῷ αὐτῷ τόπῳ· ὅπερ ἐστὶν ἀδυνατώτερον.

[48] The discussion is not meant to be exhaustive. For a fuller treatment consult Wedberg (1955), Brentlinger (1963), Annas (1975).

> Further, besides sensible things and Forms he <i.e. Plato> says that there are the objects of mathematics, which occupy an intermediate position, differing from sensible things in being eternal and unchangeable, from Forms in that there are many alike, while the Form itself is in each case unique.[49] [*Meta*. A.6, 987b14-18; Ross' trans.]

The second one is taken from *Meta*. Z.2:

> Thus Plato held that the Forms and the objects of mathematics were two kinds of substance, perceptible bodies being the third kind.[50] [*Meta*. Z.2, 1028b19-21; Bostock's trans.]

In what sense mathematical objects are between the sensibles and the Forms? The term μεταξύ serves as a straightforward indicator of their *intermediate* ontological status with respect to those fundamental Platonic categories. On the one hand, intermediates differ from physical objects in being *eternal* and *unchangeable* (διαφέροντα τῶν μὲν αἰσθητῶν τῷ ἀΐδια καὶ ἀκίνητα εἶναι, 987b16-17), just like Forms are; one the other hand, they are dissimilar to Forms in that there are *many of the same kind*, while each Form is *unique* ([διαφέροντα] τῶν δ' εἰδῶν τῷ τὰ μὲν πόλλ' ἄττα ὅμοια εἶναι τὸ δὲ εἶδος αὐτὸ ἓν ἕκαστον μόνον, 987b17-18). Commentators more or less agree that the intermediates serve the following primary purposes: 1) They address the mathematician's need for *a plurality of entities* in their statements, hence providing a direct solution to a specific problem that plagues the mathematical Forms, the *uniqueness problem*. Thus, Annas:

> A Form has to be unique of its kind, whereas mathematical statements seem to refer to a plurality of entities, and these cannot be identified either with Forms or with physical objects. Hence intermediates are posited to be the objects of such statements. [Annas (1975), p.151]

2) Apart from being a straightforward solution to the uniqueness problem, the

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[49] ἔτι δὲ παρὰ τὰ αἰσθητὰ καὶ τὰ εἴδη τὰ μαθηματικὰ τῶν πραγμάτων εἶναί φησι μεταξύ, διαφέροντα τῶν μὲν αἰσθητῶν τῷ ἀΐδια καὶ ἀκίνητα εἶναι, τῶν δ' εἰδῶν τῷ τὰ μὲν πόλλ' ἄττα ὅμοια εἶναι τὸ δὲ εἶδος αὐτὸ ἓν ἕκαστον μόνον.

[50] ὥσπερ Πλάτων τά τε εἴδη καὶ τὰ μαθηματικὰ δύο οὐσίας, τρίτην δὲ τὴν τῶν αἰσθητῶν σωμάτων οὐσίαν.

intermediates serve as a solution to the *perfection problem*; their existence seems indispensable to those who do advocate a realist conception of mathematical truth, while at the same time endorsing a skepticism about the adequacy of sensible objects as the subject matter of mathematics.

Aristotle is vehemently opposed to the theory of intermediates *in* the sensibles. In the *Meta*. B.2 passage he speaks about the 'impossible consequences' of this theory, an exhaustive analysis of which would require a large account (οἷς τὰ συμβαίνοντα ἀδύνατα πάντα μὲν πλείονος λόγου διελθεῖν, 998a9-10); in the *Meta*. M.2 passage he speaks about this doctrine in an apparently scornful manner, as an utterly 'fictitious' and 'impossible' one (Ὅτι μὲν τοίνυν ἔν γε τοῖς αἰσθητοῖς ἀδύνατον εἶναι καὶ ἅμα πλασματίας ὁ λόγος, 1077a38-39). In both passages, he does not wish to provide an exhaustive list of all the absurdities that stem from this doctrine, focusing his attention on certain major ones. The very first point that he makes in the B.2 passage is that–by parity of reasoning–the Forms too could be posited as present in the sensibles (οὔτε γὰρ ἐπὶ τούτων εὔλογον ἔχειν οὕτω μόνον, ἀλλὰ δῆλον ὅτι καὶ τὰ εἴδη ἐνδέχοιτ' ἂν ἐν τοῖς αἰσθητοῖς εἶναι (τοῦ γὰρ αὐτοῦ λόγου ἀμφότερα ταῦτά ἐστιν, 998a11-13). Aristotle's point is that the people who claim that mathematicals are in the sensibles, also posit the existence of a separate realm for Forms *not in the sensibles*; given that they do not provide sufficient justification for such a metaphysical distinction, why cannot one claim that not only mathematicals but also Forms are present in the sensibles?[51]

This sense of 'in–ness' has to be explained. For Aristotle, something is said to be in another in many ways:

> Next we must find in how many ways one thing is said to be in another. (1) In one way, as the finger is in the hand, and, generally, the part in the whole. (2) In another, as the whole is in the parts—the whole does not exist apart from the parts. (3) In another, as man is in

[51] This is how Alexander understands Aristotle's claim here (201.13-18). Whose view is this? In 201.18-20 Alexander attributes a version of this view to Eudoxus (this is probably a reference to the discussion in *Meta*. A.9 991a13-18) and he also points to the discussion in *Meta*. Book N (where the reference is probably to the Pythagorean doctrine of things composed out of numbers discussed in 1090a20-1090b5).

> animal and, generally, species in genus. (4) In another, as the genus is in the species and, generally, the part of the species in the definition. (5) In another, as health is in hot and cold things, and, generally, as the form is in the matter. (6) In another, as the affairs of Greece are in the king [of Persia], and, generally, as things are in the first thing productive of change. (7) In another, as a thing is in its good, and, generally, in its end (that is, the that-for-the-sake-of-which). (8) And—most properly of all so called—as a thing is in a vessel, and, generally, in a place.[52] [*Physics* Δ.3, 210a14-24; Morison's trans.]

The passage lists eight ways something is said to be in another; however, I think it is the first one that is pertinent to our discussion about the intermediates in the sensibles: one way (τρόπος) in which something is said to be *in* something else is 'as a finger is in the hand and generally the part in the whole' (ὡς ὁ δάκτυλος ἐν τῇ χειρὶ καὶ ὅλως ἐν τῷ ὅλῳ). Aristotle also tells us that one of the ways in which *being in* is said, is κυριώτατον ('most properly so called'), namely the eighth way, being in as *in a place* (πάντων δὲ κυριώτατον τὸ ὡς ἐν ἀγγείῳ καὶ ὅλως ἐν τόπῳ).[53] Aristotle, in both passages (*Meta.* 998a13-14, 1076b1), raises the following–seemingly straightforward–objection against metaphysically locating mathematicals in the sensibles: if we take the intermediate solid to be *literally in* the sensible one, *they would both occupy the same place (at the same*

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[52] Μετὰ δὲ ταῦτα ληπτέον ποσαχῶς ἄλλο ἐν ἄλλῳ λέγεται. (1) ἕνα μὲν δὴ τρόπον ὡς ὁ δάκτυλος ἐν τῇ χειρὶ καὶ ὅλως τὸ μέρος ἐν τῷ ὅλῳ. (2) ἄλλον δὲ ὡς τὸ ὅλον ἐν τοῖς μέρεσιν· οὐ γάρ ἐστι παρὰ τὰ μέρη τὸ ὅλον. (3) ἄλλον δὲ τρόπον ὡς ὁ ἄνθρωπος ἐν ζῴῳ καὶ ὅλως εἶδος ἐν γένει. (4) ἄλλον δὲ ὡς τὸ γένος ἐν τῷ εἴδει καὶ ὅλως τὸ μέρος τοῦ εἴδους ἐν τῷ λόγῳ. (5) ἔτι ὡς ἡ ὑγίεια ἐν θερμοῖς καὶ ψυχροῖς καὶ ὅλως τὸ εἶδος ἐν τῇ ὕλῃ. (6) ἔτι ὡς ἐν βασιλεῖ τὰ τῶν Ἑλλήνων καὶ ὅλως ἐν τῷ πρώτῳ κινητικῷ. (7) ἔτι ὡς ἐν τῷ ἀγαθῷ καὶ ὅλως ἐν τῷ τέλει· τοῦτο δ' ἐστὶ τὸ οὗ ἕνεκα. (8) πάντων δὲ κυριώτατον τὸ ὡς ἐν ἀγγείῳ καὶ ὅλως ἐν τόπῳ.

[53] Morison is correct in acknowledging a *close connection* between the *locative* sense of being in and being in in the sense of *parthood*, though he does not elaborate on what this closeness amounts to:

> Aristotle says that the locative sense of 'in' is the primary one. Clearly, there is a close link between this way of being in and the way in which a part is in the whole (the first way)… [Morison (2002), p.74]

Morison gives the following necessary and sufficient conditions under which parthood and locative 'in-ness' occur:

> (Parthood): x is in y as a part is in its whole iff x is a part of y and y is a whole.
>
> (Locative 'in-ness'): x is in y as something is in its place iff y is a place of x. See [Morison (2002), pp.73-74].

time), something absurd. An escape option for the Platonist would be to attribute a *difference in size* between a bronze sphere, say, and a mathematical one contained in the former; let us say, for instance, that a bronze sphere contains a smaller mathematical sphere. However, the *Meta.* B.2 passage provides us with a ready–made objection: how can something capable of change (the bronze sphere) contain something that is incapable of change (a mathematical sphere)? For, if the bronze sphere is susceptible to change then it follows that every part of it is susceptible to change; hence, the smaller internal sphere will also be susceptible to change. It follows that the intermediates will not be immovable.[54]

[2.3.2] Arguments against a conception of lower-dimensional mathematicals as constitutive parts of bodies

The absence of a characterisation of mathematicals in the *Meta.* M2 passage as 'intermediates' –there is no occurrence of the term τὰ μεταξὺ in the passage– might strike the reader as something peculiar, given that there is a straightforward reference to a parallel passage in the second book of the *Metaphysics* and many of the difficulties of the doctrine presented there are also part of the M.2 analysis. A plausible answer to this oddity may be provided after an examination of the last part of the passage in question:

> But, further, it is obvious that on this theory it is impossible for any body whatever to be divided; for it would have to be divided at a plane, and the plane at a line, and the line at a

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[54] As Arthur Madigan points out, Aristotle's objection can be formulated in terms of a two-level paradox, i.e. 'a contradiction between a predicate that belongs to an intermediate because it is an intermediate, and a predicate that belongs to it because it is the particular intermediate it is.'[54] In [Madigan (1986), fn.13, p.154]. One may argue like this: 1) An intermediate sphere is not susceptible to change. This is something that can be inferred from the essential nature of intermediates as eternal, unchangeable entities. 2) But, the intermediate sphere is part of a sensible one (according to this particular doctrine of intermediates). Furthermore, 3) Sensible spheres are susceptible to change, and 4) if something is susceptible to change, any part of it is susceptible to change. Hence, 5) the intermediate sphere is susceptible to change (contradicts premise 1). For the positions of Syrianus and Asclepius on the view of intermediates in the sensibles one can consult [Madigan (1986), pp.165-169].

> point, so that if the point cannot be divided, neither can the line, and if the line cannot, neither can the plane nor the solid. What difference then does it make whether sensible things are of this kind, or, without being so themselves, have such things in them? The result will be the same; if the sensible things are divided the others will be divided too, or else not even the sensible things can be divided. [*Meta.* M.2, 1076b4–11; Ross' trans.]

Julia Annas labels Aristotle's argumentative strategy a 'bad' one:

> This is not a good argument. Aristotle only obtains his conclusions by foisting implausibly crude conceptions on to his opponent, making him think of mathematical operations as if they were precisely analogous to physical operations, the sole difference being that they are performed on a more rarefied subject matter. [Annas (1976), p.139][55]

What does Annas mean by 'implausibly crude conceptions'? And what of the disparity between mathematical operations and physical ones? Ian Mueller offers a first response to Annas' criticism:

> It is certainly true that Aristotle's argument looks very crude in the light of relatively modern ideas about continuity and divisibility, but the literature that has come down to us suggests that Aristotle himself was the first person to work out detailed ideas on these notions. And it is quite clear that Aristotle's ideas involved assigning a special sense in which points are in lines, lines in planes, and planes in bodies by saying that one of these things is only potentially rather than actually in another. It is not unreasonable for him to insist that a person who lacks the potentiality–actuality distinction must think of, e.g., points as actually in lines.[56]

Annas correctly acknowledges the wider scope of Aristotle's argument. She seems, however, to have misunderstood the context of Aristotle's argument when she claims that it 'is not limited to intermediates but applies to any type of ideal mathematical object'.[57] For, as commentators have argued, Aristotle's intended target is the people who think of mathematical objects not simply as *ideal* (i.e. objects that perfectly satisfy the mathematician's definitions) parts of

[55] Ross also expresses some skepticism: '<Aristotle> treats the divisibility of the line at a point as implying the division of the point, and one might be disposed to question this'. [Ross (1924), vol.II, p.412]

[56] In [Crubellier & Laks (2009), pp.199].

[57] In [Annas (1976), pp.138-139].

physical objects–illustrated by my example of a mathematical sphere within a sensible one–but as *constitutive* parts of those, i.e. those who conceive a continuous magnitude of dimension n as being constituted out of lower dimensional entities of dimension $n–1$, e.g. a line out of points, a surface out of lines, a solid out of planes.[58] As Michael White helpfully remarks,

> The new argument strikes against any geometrically reasonable conception of surfaces/planes, lines, and points that posits them as actually metaphysically constitutive of or present in physical bodies because of the simple fact that any geometrically reasonable conception of a point must hold that it is indivisible. [White (1993), p.171][59]

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[58] See [Menn, 'Iγ3', fn.59, p.21] and [White (1993), pp.171-172].

[59] White, however, misses the opportunity to designate the Atomists as (at least part of) the intended audience of Aristotle's argument. Regarding the Atomists' doctrine of the composition of bodies out of quanta of some sort, the following passage is illuminating:

> Democritus and Leucippus say that there are indivisible bodies out of which everything else is composed, infinite both in number and in variety of shape; and that compounds differ from each other in respect of these components, and in respect of the position and the arrangement of these components. [*De Gen. et Cor*. I.1, 314a20-24; Williams' trans.]

White, however, is mistaken in his reading of the sentence τί οὖν διαφέρει ἢ ταύτας εἶναι τοιαύτας φύσεις, ἢ αὐτὰς μὲν μή, εἶναι δ' ἐν αὐταῖς τοιαύτας φύσεις. He takes the ταύτας as referring to the limit entities within sensible bodies, and his reading of the first disjunct ταύτας εἶναι τοιαύτας φύσεις is, 'those mathematical features are of such nature <that is, of such constitutive nature>'. He then reads the second disjunct αὐτὰς μὲν μή, εἶναι δ' ἐν αὐταῖς τοιαύτας φύσεις as 'even though the <limit entities> are not <of such constitutive nature>, they nonetheless exist within such bodies':

> Two initial observations seem to be in order. The first pertains to Aristotle's comment, immediately following the argument, that it does not make any difference whether these mathematical features (in null, one, two, and three dimensions - that is, the στιγμαί, γραμμαί, ἐπίπεδα, and στερεά) are (constituents of) sensible, physical bodies or whether, although they are not thought of as constituting physical bodies, they exist in (ἐν) such bodies. This comment suggests, I believe, that Aristotle considers this argument not to be directed exclusively at 'partial platonism', to use Annas' term; that is, he does not consider it to be exclusively directed at a conception of μαθηματικά as platonic forms or form-like οὐσίαι immanent in sensible, physical entities. [White (1993), p.171]

But then what happened to the second τοιαύτας φύσεις? Surely it must have the same sense as the previous one. Would that sense be, according to White, 'of constitutive nature'? Then the sentence becomes nonsensical: 'There is no difference whether those mathematicals are

Let us now return to the *Meta*. M.2 passage under examination. It is reasonable to assume that sensible, physical things can be subjected to change; a stone, for instance, can be cut into two parts. Now, let us further assume, as the upholder of this theory does, that mathematicals are *constitutive parts* of physical objects. To escape the unpleasant consequences of his analysis being paradoxical, assuming he subscribes to the principle that any part of something that is susceptible to change is also susceptible to change, one has to maintain that mathematicals *too* are divisible into parts.[60] But what will happen when we reach the ultimate constitutive parts of physical objects (the points, say), ultimate in the sense of being incapable of further division?

We may represent Aristotle's reasoning, semi-formally, as follows:

1) B -> (B_S & S), if a physical body is divisible, then the body is divisible along a surface and the surface is itself divisible;
2) S -> (S_L & L), if a surface is divisible, then the surface is divisible along a line and the line is itself divisible;
3) L -> (L_P & P), if a line is divisible, then the line is divisible along a point and the point is itself divisible;

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constitutive entities, or whether, even though they are not constitutive of sensible bodies, they exist within physical bodies as constitutive entities'. I think it makes much more sense to take the ταύτας, αὐτὰς, αὐταῖς, as referring to the same thing (as Ross does in his translation), i.e. to physical bodies (τὰ αἰσθητά) and to understand the τοιαύτας φύσεις as 'of such nature, that is, not susceptible to change'. The sentence then will be translated as 'What difference then does it make whether sensible things are unchangeable, or, without being so themselves, have such unchangeable things (e.g. surfaces, lines, points) in them?' This translation can make sense of the next sentences τὸ αὐτὸ γὰρ συμβήσεται· διαιρουμένων γὰρ τῶν αἰσθητῶν διαιρεθήσονται, ἢ οὐδὲ αἱ αἰσθηταί. 'The result will be the same; if the sensible things are divided the others will be divided too, or else not even the sensible things can be divided.' As Aristotle's example shows, even if one were to claim that sensible, changeable, things have undivided things in themselves such as points, the result will be that the sensible things will be unchangeable themselves.

[60] In a way Annas is right about complaining; this is not a convincing argument. Mr Denyer has pointed out to me that even if we assume that every part of a changeable thing must itself be changeable, this still wouldn't imply that the part must be changeable in the same way as the whole; Mr Denyer's example: 'I can become Prime Minister, my toe cannot.' Yet another example due to my partner, Stefania Mataragka: 'A woman can become pregnant, her hand cannot.'

But

4) ~P, for a point is not divisible, hence

5) (~L_P or ~P), by the usual rules for introducing a disjunction;

6) ~(P & L_P), by the de morgan laws; Thus,

7) ~L, from 3, 6 by modus tollens;

8) (~L or ~ S_L), by the usual rules for introducing a disjunction;

9) ~(L & S_L), by the de morgan laws; Thus,

10) ~S, from 2, 10 by modus tollens;

11) (~S or ~B_S), by the usual rules for introducing a disjunction;

12) ~(B_S & S), by the de morgan laws; Hence,

13) ~B, from 1,12 by modus tollens.

Aristotle's treatment of this version of immanent mathematicals shows that the adherer of this view cannot produce a paradox-free version of this theory. For, either he has to claim that physical objects cannot be susceptible to change – cannot be divided, for example – or he has to claim that there are ultimate indivisible immanent mathematicals such as points, leading him also to the same paradoxical result that *a physical body cannot be subjected to change*.

[2.4] *Metaphysics* M.3 analysis and related excursus

Some of the most interesting passages that pertain to Aristotle's philosophy of mathematics are located in the third chapter of Book M of the *Metaphysics*. Aristotle's primary concern in that chapter is to provide answers to a series of Platonic[61] arguments. In what follows I will proceed to discuss some of those arguments and highlight their role in Aristotle's overall position about the metaphysical status of mathematicals.

[2.4.1] The analogy from the universal propositions in mathematics and the related discussion–part one

The major part of *Meta*. M.3 (1077b17-1078a5) is reserved for the discussion of three analogies that Aristotle uses to highlight the close ties between mathematical objects and the actual world.[62] The first analogy is the following:

> For just as universal propositions in mathematics are not about objects which exist separately from magnitudes and numbers, but are about these, only not *as* having magnitude or being divisible, clearly it is also possible for there to be statements and proofs about perceptible magnitudes, but not *as* perceptible but *as* being of a certain kind.[63] [*Meta*. M.3, 1077b17-22; Lear's trans. mod.]

What exactly is the parallel that Aristotle draws here? To answer this question properly let us first fix our attention to the things that Aristotle labels as 'the universal propositions in mathematics' (τὰ καθόλου ἐν τοῖς μαθήμασιν).[64] Now τὰ καθόλου ἐν τοῖς μαθήμασιν could indicate either *general* mathematical principles that apply to both numbers and spatial magnitudes such as the principle that *if we take equals from equals then equals should remain*, principles gathered elsewhere under the label τὰ κοινά (cf. κοινὰ δὲ οἷον τὸ ἴσα ἀπὸ ἴσων ἂν ἀφέλῃ, ὅτι

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[61] Not necessarily Plato's own but of the wider Academy.

[62] See [Hussey (2011), p.108].

[63] ὥσπερ γὰρ καὶ τὰ καθόλου ἐν τοῖς μαθήμασιν οὐ περὶ κεχωρισμένων ἐστὶ παρὰ τὰ μεγέθη καὶ τοὺς ἀριθμοὺς ἀλλὰ περὶ τούτων μέν, οὐχ ᾗ δὲ τοιαῦτα οἷα ἔχειν μέγεθος ἢ εἶναι διαιρετά, δῆλον ὅτι ἐνδέχεται καὶ περὶ τῶν αἰσθητῶν μεγεθῶν εἶναι καὶ λόγους καὶ ἀποδείξεις, μὴ ᾗ δὲ αἰσθητὰ ἀλλ' ᾗ τοιαδί.

[64] Universal mathematics is mentioned also in *Meta*. E.1, 1026a25-27 and in K.7, 1064b8-9.

ἴσα τὰ λοιπά, *Post. An.,* 76a41), or, as Jonathan Lear suggests, *theorems* applicable to both numbers and spatial magnitudes such as the theorem that *proportions alternate* (i.e., that if a:b::c:d, then a:c::b:d).[65]

In a related passage from *Post. An.* A.5, Aristotle reports that the theorem that proportions alternate used to be proved separately (ἐδείκνυτό ποτε χωρίς) for numbers, lines, solids, and times; he cites as reason for this the fact that there was no name comprehending all these things as one (διὰ τὸ μὴ εἶναι ὠνομασμένον τι ταῦτα πάντα ἕν), things which differ in species from one another. However, now–Aristotle reports–the theorem is proved universally (νῦν δὲ καθόλου δείκνυται), given that numbers, lines, solids, and times, presumably share a common character (ὃ καθόλου ὑποτίθενται ὑπάρχειν): they are all *quantities*. The τοδί in the text points to a certain universal aspect of numbers, lines, planes, etc., about which the theorem that proportions alternate is now proved:[66]

> And <it might seem> that proportion alternates for things as numbers and as lines and as solids and as times-as once it used to be proved separately, though it is possible for be to be proved of all cases by a single demonstration. But because all these things–numbers, lengths, times, solids–do not constitute a single named item and differ in sort from one another, it used to be taken separately. But now it is proved universally; for it did not belong to things as lines or as numbers, but as *this* which they suppose to belong universally.[67] [*Post. An.* A.5, 74a17-25; Barnes' trans.]

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[65] In this Lear follows Ross in [Ross (1924), vol.II, p.413]. Ross considers Eudoxus' theory of proportion as the most characteristic example of 'the universal propositions in mathematics'. Ps.-Alexander (729.23-25) provides an example from the general theory of proportion and another from the general principles of equality: οἷον διὰ τοῦ ἀπὸ ἴσων ἴσα ἂν ἀφέλῃς, τὰ καταλειπόμενα ἴσα ἐστί, καὶ διὰ τοῦ ἂν τέσσαρά τινα ᾖ ἀνάλογον, τὸ ὑπὸ τῶν ἄκρων ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων, καὶ ἄλλων πολλῶν τοιούτων. Syrianus (89.32-34) also offers the same examples: οἷον διὰ τοῦ 'ἐὰν ἀπὸ ἴσων ἴσα ἀφέλῃς, τὰ καταλειπόμενά ἐστιν ἴσα' καὶ διὰ τοῦ 'ἐὰν τέτταρα ᾖ ἀνάλογον, τὸ ὑπὸ τῶν ἄκρων ἴσον ἐστὶ τῷ ὑπὸ τῶν μέσων' καὶ ἄλλων πολλῶν τοιούτων.

[66] So [Cleary (1995), p.311].

[67] καὶ τὸ ἀνάλογον ὅτι καὶ ἐναλλάξ, ᾗ ἀριθμοὶ καὶ ᾗ γραμμαὶ καὶ ᾗ στερεὰ καὶ ᾗ χρόνοι, ὥσπερ ἐδείκνυτό ποτε χωρίς, ἐνδεχόμενόν γε κατὰ πάντων μιᾷ ἀποδείξει δειχθῆναι· ἀλλὰ διὰ τὸ μὴ εἶναι ὠνομασμένον τι ταῦτα πάντα ἕν, ἀριθμοί μήκη χρόνοι στερεά, καὶ εἴδει διαφέρειν ἀλλήλων, χωρὶς ἐλαμβάνετο. νῦν δὲ καθόλου δείκνυται· οὐ γὰρ ᾗ γραμμαὶ ἢ ᾗ ἀριθμοὶ ὑπῆρχεν, ἀλλ' ᾗ τοδί, ὃ καθόλου ὑποτίθενται ὑπάρχειν.

The above passage from the *Analytics* is part of a series of Aristotle's examples in which he illustrates three cases where one might incorrectly think that has proved of *A* that belongs *primitively* and *universally* to *B* (74a4 ff.). Aristotle says that we may fail to find this (most general) *B* in the following cases: a) the first case is where *it is not possible to grasp* a more general kind *B* above *C* to which *A* belongs, so that we think that *C* is the most general one; e.g. if the only triangles we had met with were isosceles, we might think that *having two right angles* belonged universally to the *isosceles* triangle: 'we make this error when either we cannot grasp anything higher apart from the particular' (Barnes' trans.); (ἀπατώμεθα δὲ ταύτην τὴν ἀπάτην, ὅταν ἢ μηδὲν ᾖ λαβεῖν ἀνώτερον παρὰ τὸ καθ' ἕκαστον, 74a7-8, with the aforementioned example at 74a16-17: καὶ εἰ τρίγωνον μὴ ἦν ἄλλο ἢ ἰσοσκελές, ᾗ ἰσοσκελὲς ἂν ἐδόκει ὑπάρχειν); b) the second case is when we can grasp something higher above different species of things but *there is no name for it* ('or we can < grasp something higher apart from the particular> but it is nameless for objects different in sort' (Barnes' trans.); (ἢ ᾖ μέν, ἀλλ' ἀνώνυμον ᾖ ἐπὶ διαφόροις εἴδει πράγμασιν, 74a8-9); c) the third case is when that of which we prove A is 'in fact a partial whole' or as Barnes explains: '*C*, of which *A* is proved, is actually a species of *B*, to which *A* belongs universally';[68] (ἢ τυγχάνῃ ὂν ὡς ἐν μέρει ὅλον ἐφ' ᾧ δείκνυται, 74a9-10), with an example[69] at 74a13-16: 'now if someone would prove that right <angles> do not meet, the demonstration would seem to hold of this because of its holding of all right <angles>' (Barnes' trans.); (εἰ οὖν τις δείξειεν ὅτι αἱ ὀρθαὶ οὐ συμπίπτουσι, δόξειεν ἂν τούτου εἶναι ἡ ἀπόδειξις διὰ τὸ ἐπὶ πασῶν εἶναι τῶν ὀρθῶν).[70]

Aristotle says that the theorem that proportions alternate used to be proved separately (ἐδείκνυτό ποτε χωρίς) for numbers, lines, solids, and times because

[68] In [Barnes (1975), p.122].

[69] The reference is to Euclid's *Elem*. I.28, that, 'if a straight line intersecting two straight lines makes the exterior angle equal to the interior and opposite angle falling on the same side of it . . . the two straight lines will be parallel' (Ross' trans.). In [Ross (1949), p.525].

[70] As Ross explains, 'the error lies in supposing that the parallelness of the lines follows from the fact that the exterior and the interior and opposite angle are equal by being both of them right angles, instead of following merely from their equality.' In [Ross (1949), p.525].

there was no name comprehending all these things as one (διὰ τὸ μὴ εἶναι ὠνομασμένον τι ταῦτα πάντα ἓν).[71] Thus, the *Post. An.* A.5 74a17-25 passage is used by Aristotle to illustrate the *second* case above: There was no name for something over lines, numbers, etc. which differ in species. So we missed the fact that those theorems can be proved for these universally.[72] The phrase νῦν δὲ

[71] Henry Mendell offers a helpful synopsis of the *Post. An.* A.5 discussion and an extensive discussion of *the problem of universal science* in Aristotle, i.e. whether the latter held a science of universal mathematicals, a 'posology' to use Mendell's own coinage. Whereas Mendell concedes that the universal propositions in mathematics are seemingly suitable candidates for the constitution of a universal mathematical science, he expresses his doubts as to whether Aristotle really accepted such a science. Mendell highlights the fact that in the *Post. An*. passage (and in the passages that deal with universal mathematicals in *Meta*. M.2) Aristotle *does not* employ his own term ποσόν to name this more general subject. In [Mendell(1985), pp.229-250]. John Cleary (who also offers a helpful commentary on the *Analytics* passage and its relation with the M.3 parallel) suggests that this is due to Academic infulence: points, lines, planes, and solids, constitute a series whose members are related as *prior* to *posterior*. He points to a certain passage from the *Nicomachean Ethics* where Aristotle tells us (1096al7-19) that the Platonists refused to posit Forms for any series of *prior* and *posterior* elements. This Platonic position is consistent, Cleary claims, with 'Aristotle's report (*Post. An*. A.5) that there was no general name for quantity, and that proportions were proved separately for each kind of quantity before Eudoxus.' In [Cleary (1995), fn. 97; pp.310-311]; for some useful discussion see also [Cleary(1995), pp.290-292 & 307-312].

[72] Cf. Proclus, 392.22-27 (Morrow's trans.): 'A man may mistakenly suppose, Aristotle says, that he is proving something universally when he is not, because the common subject to which the character primarily belongs has no name. For instance, it is not possible to say what the common element is in numbers, magnitudes, motions, and sounds, to all of which the rule of alternate proportion applies.' (λανθάνει δέ, φησὶν Ἀριστοτέλης, τὸ μὴ καθόλου δεικνύς τις ὡς καθόλου διὰ τὸ εἶναι ἀνώνυμον τὸ κοινόν, ᾧ πρώτως ὑπάρχει τὸ σύμπτωμα. τί γὰρ κοινὸν ἀριθμοῖς καὶ μεγέθεσι καὶ κινήσεσι καὶ φθόγγοις, οἷς ἅπασιν ὑπάρχει τὸ ἐναλλάξ, οὐκ ἔστιν εἰπεῖν.); Philoponus (77.6-9, McKirahan's trans.) however claims that the passage falls under case a) above; for he says that the reason that the theorem that proportions alternate was proved separately for each case was that we were not aware that there is a more general subject to which the various species of quantity belong: 'He means that it was demonstrated rather roughly in each case because we do not know what is the one thing predicated in common in all these cases, whether it is quantity, for example, or something else, in virtue of which numbers, magnitudes and times are one in their common genus.' (ἀπεδείκνυτο οὖν, φησίν, ὁλοσχερέστερον ἐφ' ἑκάστου διὰ τὸ μὴ εἰδέναι ἡμᾶς τί ἐστι τὸ ἐπὶ πάντων τούτων ἓν κοινῶς κατηγορούμενον, οἷον εἴτε τὸ ποσὸν εἴτε ὁτιοῦν ἄλλο, καθὸ ἀριθμοί τε καὶ μεγέθη καὶ χρόνοι ἕν εἰσι τῷ κοινῷ αὐτῶν γένει.)

καθόλου δείκνυται, as Heath suggests,[73] is a reference to the proof forming part of Eudoxus' new theory of proportion. According to Heath, the accuracy of Aristotle's remark may be verified if we take a look at Euclid's *Elements* Books V and VII: regarding the former it is generally accepted that contains the Eudoxean theory of proportion which is applicable to all magnitudes alike (notice the use of the general term 'magnitude' (μέγεθος) throughout this book, see for instance V.16: Ἐὰν τέσσαρα μεγέθη ἀνάλογον ᾖ, καὶ ἐναλλὰξ ἀνάλογον ἔσται); the latter Book contains what is considered to be an older theory of proportion applicable only to numbers and apparently of Pythagorean origin.[74]

This first parallel of *Meta*. M.3 cautions against the postulation of extra, separately existing objects (οὐ περὶ κεχωρισμένων ἐστὶ παρὰ τὰ μεγέθη καὶ τοὺς ἀριθμούς, 1077b18-19) that satisfy those universal propositions. Such a Platonic move is more *explicitly* suggested in a passage from the previous chapter of Book M of the *Metaphysics*; the context is again about τὰ καθόλου ἐν τοῖς μαθήμασιν:

> Besides, there are some universal mathematical propositions, whose application extends beyond these substances. Here then we shall have another substance between, and separate from, the Ideas and the intermediates,—a substance which is neither number nor points nor spatial magnitude nor time. And if this is impossible, plainly it is also impossible that the *former* should exist in separation from sensible things.[75] [*Meta*. M.2, 1077a9-14; Ross' trans. mod.]

The above passage forms part of a bigger section of M.2 where Aristotle argues against the view that mathematicals exist separately from the sensibles (M.2, 1076b11-1077b14)[76]. As we have already seen, in mathematics we have certain universal propositions which are not specifically about magnitudes or numbers

[73] In [Heath (1949), p.223].

[74] For this suggestion and some further discussion see [Heath(1949), pp.43-44].

[75] ἔτι γράφεται ἔνια καθόλου ὑπὸ τῶν μαθηματικῶν παρὰ ταύτας τὰς οὐσίας. ἔσται οὖν καὶ αὕτη τις ἄλλη οὐσία μεταξὺ κεχωρισμένη τῶν τ' ἰδεῶν καὶ τῶν μεταξύ, ἣ οὔτε ἀριθμός ἐστιν οὔτε στιγμαὶ οὔτε μέγεθος οὔτε χρόνος. εἰ δὲ τοῦτο ἀδύνατον, δῆλον ὅτι κἀκεῖνα ἀδύνατον εἶναι κεχωρισμένα τῶν αἰσθητῶν.

[76] This is the orthodox Platonic view; Ross attributes this view to Plato and Speusippus. In [*Ross* (1924), vol.II, p.412].

(which, as the passage indicates, the Platonists regard as separate substances). It seems then that they have to postulate some other substance which is 'neither number nor points nor spatial magnitude nor time' and is separate from Forms and intermediates (to make matters worse this substance must also be *between* Forms and intermediates). If this is impossible, then it is also impossible that numbers, lines, points, planes, etc. should exist in separation from the sensibles. But what are the Platonist's reasons behind the postulation of a class of separate *universal mathematicals* that satisfy those propositions? (cf. ps.–Alex. : ἕτερον γὰρ τούτων ἁπάντων ἔσται καθολικώτερον ὂν καὶ τῶν γραμμῶν καὶ τῶν ἐπιπέδων καὶ χρόνων καὶ στερεῶν καὶ τῶν ἄλλων ἁπάντων, 729.28-30). Now, it is not immediately obvious why one would proceed to postulate separately existing universal mathematical objects or why one would identify them with either the Forms or the Intermediates or, as in the case of M.2, 1077a9-14, add the extra claim that those mathematicals are between Forms and Intermediates (ἔσται οὖν καὶ αὕτη τις ἄλλη οὐσία μεταξὺ κεχωρισμένη τῶν τ' ἰδεῶν καὶ τῶν μεταξύ).

Part 1077b17-20 of the first parallel pertains to the postulation of separate universal mathematicals, beginning from a universal treatment of different species of quantity. Let us try to uncover the Platonist arguments in it; I believe that this part should be understood primarily within the context of the *one over many argument.*[77] In *Meta*. A.9, Aristotle mentions five arguments for the existence of Platonic Forms:

> Further, of the ways in which we prove that the Forms exist, none is convincing; for from some no inference necessarily follows, and from some it follows that there are Forms of things of which we think there are no Forms. For according to the arguments from the sciences there will be Forms of all things of which there are sciences, and according to the one over many there will be Forms even of negations, and according to <the argument> that there is an object for thought even when the thing has perished, <there will be Forms> of perishable things; for there is an image of these. Further, of the more accurate arguments, some lead to Ideas of relations, of which we say there is no independent class,

[77] Stephen Menn also remarks in passing that Aristotle is trying to respond here to Platonically–inspired *one over many* arguments. See [Menn, 'Iγ3', p.22].

> and others involve the difficulty of the 'third man'.[78] [*Meta*. A.9, 990b8 − 17 ; Ross' trans. mod.; the passage is almost identical to that in M.4, 1079a4–13]

The five arguments listed in the passage: the *arguments from the sciences*, the *one over many argument*, the *object of thought argument*, the *argument from relatives*, and the *argument that introduces the third man*. Those arguments were discussed in detail in the first book of Aristotle's work *On Ideas*, extensive excerpts of which are preserved by Alexander in his commentary on *Meta.* A.9.[79] Gail Fine offers the following description for the role of the one over many argument: 'According to the *one over many agument* there are separated, everlasting Forms corresponding to every general term true of groups of things':[80]

> They use also the following argument to establish that there are ideas: If each of the many men is a man, and if each of the many animals is an animal, and the same applies in the other cases; and if in the case of each of these it is not that something is predicated of itself

[78] ἔτι δὲ καθ' οὓς τρόπους δείκνυμεν ὅτι ἔστι τὰ εἴδη, κατ' οὐθένα φαίνεται τούτων· ἐξ ἐνίων μὲν γὰρ οὐκ ἀνάγκη γίγνεσθαι συλλογισμόν, ἐξ ἐνίων δὲ καὶ οὐχ ὧν οἰόμεθα τούτων εἴδη γίγνεται. κατά τε γὰρ τοὺς λόγους τοὺς ἐκ τῶν ἐπιστημῶν εἴδη ἔσται πάντων ὅσων ἐπιστῆμαι εἰσί, καὶ κατὰ τὸ ἓν ἐπὶ πολλῶν καὶ τῶν ἀποφάσεων, κατὰ δὲ τὸ νοεῖν τι φθαρέντος τῶν φθαρτῶν· φάντασμα γάρ τι τούτων ἔστιν. ἔτι δὲ οἱ ἀκριβέστεροι τῶν λόγων οἱ μὲν τῶν πρός τι ποιοῦσιν ἰδέας, ὧν οὔ φαμεν εἶναι καθ'αὑτὸ γένος, οἱ δὲ τὸν τρίτον ἄνθρωπον λέγουσιν.

[79] The *locus classicus* for Aristotle's *On Ideas* is G.E.L. Owen's paper 'A Proof in the *Peri Idewn*', where he discusses mainly the argument from relatives. Other significant scholarly contributions are those of Daniel H. Frank (in his book *The Arguments 'From the Sciences' in Aristotle's Peri Idewn* he discusses the omonymous arguments) and of Gail Fine (her work *On Ideas: Aristotle's Criticism of Plato's Theory of Forms* offers a detailed analysis of the majority of the arguments; her analysis relies heavily on modern philosophers such as D.M. Armstrong and his conception of realism). Aristotle's *On Ideas* was also subjected to extensive analysis in the May Week Seminar (Cambridge, Summer 2018).

[80] In [Fine(1993), p.103]. I will not say much about Plato's *own* version of the *one over many* argument. A passage commonly asscociated with Plato's one over many argument is that of *Rep*. Book X, 596a6-7: 'Do you want us to begin our inquiry with the following point then, in accordance with our usual method? I mean, as you know, we usually posit some one particular Form in connection with each set of many things to which we apply the same name'(Reeve's trans.). Another passage that apparently conveys an over many argument is that of *Parmenides* 132a1-4.

> but that there is something which is predicated of all of them and which is not the same as any of them, then this is some being besides the particular beings which is separated from them and everlasting. For it is in every case predicated in the same way of all the numerically succesive <particulars>. And what is a one in addition to many, separated from them, and everlasting is an idea. Therefore there are ideas.[81] [Alexander: *Comm. on Meta.*, 80.8-15, Fine's trans.]

According to Fine's analysis[82] the argument begins from the following premise: Whenever many particular thing*s* are *F*, they are *F* in virtue of having some one thing, the *F*, predicated of them. What is the nature of the *F* that is predicated of *Fs*? Fine maintains that the two most plausible candidates are *linguistic predicates* and *properties*.[83] If F were a *linguistic predicate*, the following reading of the aforementioned premise occurs: Whenever a group of particulars are *F*, they are *F* in virtue of having some one predicate, '*F*', predicated of them. But this would mean-as Fine correctly points out-that a linguistic predicate such as 'man' would be identified with the Form of man; something absurd.[84] Thus, Fine suggests an alternative interpretation: that *F* is a *property*. The premise becomes then: Whenever a group of particulars are *F*, they are *F* because they share the property of being *F*.[85] Now let us recall the first part of the parallel (1077b17-20)

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[81] Χρῶνται καὶ τοιούτῳ λόγῳ εἰς κατασκευὴν τῶν ἰδεῶν. εἰ ἕκαστος τῶν πολλῶν ἀνθρώπων ἄνθρωπός ἐστι καὶ τῶν ζῴων ζῷον καὶ ἐπὶ τῶν ἄλλων ὁμοίως, καὶ οὐκ ἔστιν ἐφ' ἑκάστου αὐτῶν αὐτὸ αὑτοῦ τι κατηγορούμενον, ἀλλ' ἔστι τι ὃ καὶ πάντων αὐτῶν κατηγορεῖται οὐδενὶ αὐτῶν ταὐτὸν ὄν, εἴη ἄν τι τούτων παρὰ τὰ καθ' ἕκαστα ὄντα ὂν κεχωρισμένον αὐτῶν ἀίδιον· ἀεὶ γὰρ ὁμοίως κατηγορεῖται πάντων τῶν κατ' ἀριθμὸν ἀλλασσομένων. ὃ δὲ ἕν ἐστιν ἐπὶ πολλοῖς κεχωρισμένον τε αὐτῶν καὶ ἀίδιον, τοῦτ' ἔστιν ἰδέα· εἰσὶν ἄρα ἰδέαι.

[82] Fine offers the following reconstruction of the argument: '(1) Whenever many (πολλά) *Fs* are *F*, they are *F* in virtue of having some one thing, the *F*, predicated of them. (2) No particular (καθ' ἕκαστον) *F* is *F* in virtue of itself. (3) The *F* is in every case (ἀεὶ) predicated in the same way of all the numerically successive *Fs* (τῶν κατ' ἀριθμὸν ἀλλασσομένων). (4) Therefore the *F* is something besides (παρὰ) particular *Fs*. (5) Therefore the *F* is separated from (κεχωρισμένον) particular *Fs* and is everlasting (ἀίδιον). (6) Whatever is a one over many, separated, and everlasting is a Form. (7) Therefore the *F* is a Form.' In [Fine(1993), p.104].

[83] In [Fine, op. cit., p.106].

[84] ibid.

[85] op.cit., pp.106-108. Fine's properties are not meant to be distinguished from species and types: 'I use 'property' more broadly, so that it includes all these types of entities. [...]The crucial point

and the related discussion in *Post. An.* A.5, 74a17-25. The subject matter of universal mathematical propositions are numbers, lines, solids, etc., i.e. different kinds of quantity. Does this mean that we have to postulate a peculiar entity (call it 'Quantity'), to which they are all related in the same way? It seems that a consistent application of the *one over many* argument requires the Platonists to do so. In the original *one over many argument* Fine takes καθ' ἕκαστα to refer to particulars.[86] It is true that when Aristotle speaks of καθ' ἕκαστα he often refers to particulars (Socrates as contrasted to man); however, this not always the case, as he frequently refers instead to kinds or species which may be said to be particular in relation to something more universal (man in contrast to animal).[87] In the *Post. An.* discussion that pertains to the first part of our parallel καθ' ἕκαστα clearly stand for (low-level) kinds; as Barnes comments, in a different case the first error (i.e. the fact that is not possible to grasp a more general kind B above C to which attribute A belongs, so that we think that C is the most general subject) would remain 'unillustrated':[88]

> It must not escape our notice that it often happens that we make mistakes and that what is being proved does not belong primitively and universally in the way in which it seems to be being proved universally and primitively. We make this error when either we cannot grasp anything higher apart from *the particular*, or we can but it is nameless for objects

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for our purposes is that on the realist conception universals are explanatory entities of roughly the sort that properties conceived in realist fashion have been taken to be. I shall use 'property', 'explanatory property', and 'genuine property' interchangeably.' In [Fine (1993), p.247].

86 In [Fine, op. cit., p.104].

87 John Cooper in his book *Reason and Human Good in Aistotle* offers an extensive list of passages that correspond to those two readings of καθ' ἕκαστα: for καθ' ἕκαστα as referring to particulars he cites the following places: *Meta*. B.4 999b33; Z.15 1039b28-31; M.9 1086a32-34; Λ.5 1071a27-29, *Cat*. 2b3, *De Int.* 18a33, *Pr. An.* A.27 43a27, *Post. An.* B.19 100al6-18, *De Gen. An.* I.3 768al-2, *N.E.* III.3 1112b33-l 113a2. For καθ' ἕκαστα as referring to low-level types or species he mentions *Topics* I.12 105a13-14; *Cat.* 15b1-2, H*ist. An.* V.1 539bl5, *De Gen. An.* III.11 763bl5, *Post. An.* A.13 79a4-6, *Post. An.* B.13 97b28-31, *De Part. An.* I.4 644a28-33, b6-7. See [Cooper(1975), pp.28-29].

88 In [Barnes (1975), p.122].

> different in sort or that of which it is proved is in fact a partial whole.[89] [*Post. An.* A.5, 74a4-10, Barnes' trans.]

There is however a 'more accurate' argument that can accommodate a more general reading of καθ' ἕκαστα. In *Met.* A. 9 (990b15–17) where Aristotle lists the arguments for the existence of the Forms, he tells us that of the more accurate arguments (οἱ ἀκριβέστεροι τῶν λόγων) 'some lead to Ideas of relations and others involve the difficulty of *the third man*' (οἱ μὲν τῶν πρός τι ποιοῦσιν ἰδέας, ὧν οὔ φαμεν εἶναι καθ'αὑτὸ γένος, οἱ δὲ τὸν τρίτον ἄνθρωπον λέγουσιν). Fine argues that this *one over many* argument is *different* from the one presented previously; whereas in the previous one over many argument the particulars in question were sensible ones, this argument deals with a plurality of things that are not necessarily sensible particulars (πλείονά τινα).[90] But why is this a more accurate argument? The reason is, according to Fine, that this argument focuses on the *similarity* of the things of which *F* is predicated:

> In Aristotle's view, that is, Plato uses a one over many argument not to explain particularity, but to explain similarity. The accurate one over many argument brings this out by focusing on the *F*–ness of *F* things, without restricting the relevant things to sensible particulars. Given Plato's belief that the form of *F* is also an *F* thing, though not an *F* sensible particular, this will turn out to be important <for the third man argument>. [Fine (1993), p.201]

Fine traces this 'more accurate' *one over many* argument within the following lines from Alexander's commentary:

> If what is predicated truly of some plurality of things is also <some> other thing besides the things of which it is predicated, being separated from them (for this is what those who posit the ideas think they prove; for this is why, according to them, there is such a thing as man-itself, because the man is predicated truly of the particular men, these being a

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[89] Δεῖ δὲ μὴ λανθάνειν ὅτι πολλάκις συμβαίνει διαμαρτάνειν καὶ μὴ ὑπάρχειν τὸ δεικνύμενον πρῶτον καθόλου, ᾗ δοκεῖ δείκνυσθαι καθόλου πρῶτον. ἀπατώμεθα δὲ ταύτην τὴν ἀπάτην, ὅταν ἢ μηδὲν ᾖ λαβεῖν ἀνώτερον παρὰ τὸ καθ' ἕκαστον, ἢ ᾖ μέν, ἀλλ' ἀνώνυμον ᾖ ἐπὶ διαφόροις εἴδει πράγμασιν, ἢ τυγχάνῃ ὂν ὡς ἐν μέρει ὅλον ἐφ' ᾧ δείκνυται·

[90] In [Fine, op. cit. p.199].

> plurality, and it is other than the particular men)—but if this is so, there will be a third man.[91] [Alex. 84.22–7; Fine's trans.]

Thus a Platonist ought to formulate the following argument for the existence of the Form of Quantity: We have lines, planes, solids, numbers, and so on, which are all quantities. From this we infer that there is a separate Form of Quantity by virtue of which they all are quantities. How do we infer this? We are relying on an *one over many* principle that generates a *separate* Form for a collection of things that all have something in common.

We saw then how one could proceed to postulate the existence of the Form of Quantity based on a version of the *one over many* argument. According to Alexander, Aristotle's response to the one over many argument is as follows:

> It is clear that this argument too does not validly deduce that there are ideas; rather, it too tends to prove that what is predicated in common is something other than the particulars of which it is predicated. [92] [Alex. 81.7-11, Fine's trans.]

It seems that the error in the arguments for the existence of the Forms lies in the *separation inference*. Why is this problematic? Well, it may be the case that the Form ought to be *different* from what it is predicated of but this need not imply separation. What can be said about the notion of separation that Aristotle associates with Platonic Forms? As I have already pointed out in this chapter, in *Meta.* Δ. 11, 1019a1-4, for instance, Aristotle says that things are called prior (πρότερα) in virtue of their nature and substance (κατὰ φύσιν καὶ οὐσίαν) when it is possible for them to exist without other things (ἐνδέχεται εἶναι ἄνευ ἄλλω*ν*), whereas those other things cannot exist without them; this division was used by

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[91] εἰ τὸ κατηγορούμενόν τινων πλειόνων ἀληθῶς καὶ ἔστιν ἄλλο παρὰ τὰ ὧν κατηγορεῖται, κεχωρισμένον αὐτῶν (τοῦτο γὰρ ἡγοῦνται δεικνύναι οἱ τὰς ἰδέας τιθέμενοι· διὰ τοῦτο γάρ ἐστί τι αὐτοάνθρωπος κατ' αὐτούς, ὅτι ὁ ἄνθρωπος κατὰ τῶν καθ' ἕκαστα ἀνθρώπων πλειόνων ὄντων ἀληθῶς κατηγορεῖται καὶ ἄλλος τῶν καθ' ἕκαστα ἀνθρώπων ἐστίν)—ἀλλ' εἰ τοῦτο, ἔσται τις τρίτος ἄνθρωπος.

[92] δῆλον δὲ ὅτι οὐδὲ οὗτος ὁ λόγος ἰδέας εἶναι συλλογίζεται, ἀλλὰ δεικνύναι βούλεται καὶ αὐτὸς ἄλλο εἶναι τὸ κοινῶς κατηγορούμενον τῶν καθ' ἕκαστα ὧν κατηγορεῖται.

Plato (τὰ μὲν δὴ οὕτω λέγεται πρότερα καὶ ὕστερα, τὰ δὲ κατὰ φύσιν καὶ οὐσίαν, ὅσα ἐνδέχεται εἶναι ἄνευ ἄλλων, ἐκεῖνα δὲ ἄνευ ἐκείνων μή· ᾗ διαιρέσει ἐχρήσατο Πλάτων.). Therefore, from the Δ.11 passage we may infer A is separate from B iff A can exist without B (ὅσα ἐνδέχεται εἶναι ἄνευ ἄλλων), or, equivalently, iff A exists independently of B. Fine speaks of 'capacity for independent existence' for this particularly important notion of separation.[93] Separation defined that way is always separation from something; in the case of the one over many argument it is separation from particulars[94] (παρὰ τὰ καθ' ἕκαστα ὄντα ὂν κεχωρισμένον αὐτῶν, Alex., 80.8-15). Thus, the Platonic Forms are separate from particulars in the sense that they can exist whether or not particulars exist (Fine actually puts this in terms of 'instantiation' – the Platonic Forms (unlike their Aristotelian counterparts) can exist without their instances).

In the following passages from the *Metaphysics* Aristotle describes the origins of Plato's theory of Forms and how it differs from the Socratic theory:

> The belief about the forms occurred to those who asserted it because they were convinced of the truth of the Heracleitean arguments that all sensibles are always flowing, so that if knowledge and wisdom are to be about anything, there must be some different and enduring natures, besides the sensible ones, for there is no knowledge of flowing things. Now Socrates was concerned with the moral virtues, and he was the first to seek universal definitions in connection with them . . . It was reasonable for Socrates to try to find what a thing is, because he was seeking to argue deductively, and the starting-point of deductions is what a thing is . . . For there are just two things one might fairly ascribe to Socrates—inductive arguments and universal definitions, both of which are concerned with the starting-point of knowledge. But Socrates did not make universals or definitions separate, but they <the Platonists> separated them, and they called these sorts of beings 'ideas'.[95] [*Meta*. M.4, 1078b12–32; Fine's trans.]

[93] See Fine(1984) for extensive discussion.

[94] ibid.

[95] συνέβη δ' ἡ περὶ τῶν εἰδῶν δόξα τοῖς εἰποῦσι διὰ τὸ πεισθῆναι περὶ τῆς ἀληθείας τοῖς Ἡρακλειτείοις λόγοις ὡς πάντων τῶν αἰσθητῶν ἀεὶ ῥεόντων, ὥστ' εἴπερ ἐπιστήμη τινὸς ἔσται καὶ φρόνησις, ἑτέρας δεῖν τινὰς φύσεις εἶναι παρὰ τὰς αἰσθητὰς μενούσας· οὐ γὰρ εἶναι τῶν ῥεόντων ἐπιστήμην. Σωκράτους δὲ περὶ τὰς ἠθικὰς ἀρετὰς πραγματευομένου καὶ περὶ τούτων ὁρίζεσθαι καθόλου ζητοῦντος πρώτου … ἐκεῖνος δ' εὐλόγως ἐζήτει τὸ τί ἐστιν· συλλογίζεσθαι γὰρ ἐζήτει, ἀρχὴ δὲ τῶν

In the second passage, which contains a more detailed account of the origin of separate Forms, Aristotle claims that the Platonists not only make Forms entirely distinct from particulars, but they also consider them to be separate substances. Their error, however, lies in that no universal can be a separate substance:

> For they treat ideas both as universals and again, at the same time, as separate and as particulars. But it has been argued before that this is impossible. Those who said that the substances were universals combined these things <universality and particularity> in the same thing because they did not make them <the substances> the same as sensibles. They thought that the particulars in sensibles were flowing and that none of them endured, but that the universal is besides these things and is something different from them. Socrates motivated this <view>, as we were saying before, through definitions; but he did not separate <universals> from particulars. And he was right not to separate them. This is clear from the results. For it is not possible to acquire knowledge without the universal; but separating is the cause of the difficulties arising about the ideas. But they, on the assumption that any substances besides the sensible and flowing ones had to be separate, had no others, and so they set apart the substances spoken of universally, so that it followed that universal and particular <natures> were virtually the same natures. This in itself, then, would be one difficulty for the view discussed.[96] [*Meta*. M.9, 1086a32–b13; Fine's trans.]

συλλογισμῶν τὸ τί ἐστιν … δύο γάρ ἐστιν ἅ τις ἂν ἀποδοίη Σωκράτει δικαίως, τούς τ' ἐπακτικοὺς λόγους καὶ τὸ ὁρίζεσθαι καθόλου· ταῦτα γάρ ἐστιν ἄμφω περὶ ἀρχὴν ἐπιστήμης)· —ἀλλ' ὁ μὲν Σωκράτης τὰ καθόλου οὐ χωριστὰ ἐποίει οὐδὲ τοὺς ὁρισμούς· οἱ δ' ἐχώρισαν, καὶ τὰ τοιαῦτα τῶν ὄντων ἰδέας προσηγόρευσαν.

[96] ἅμα γὰρ καθόλου τε ποιοῦσι τὰς ἰδέας καὶ πάλιν ὡς χωριστὰς καὶ τῶν καθ' ἕκαστον. ταῦτα δ' ὅτι οὐκ ἐνδέχεται διηπόρηται πρότερον. αἴτιον δὲ τοῦ συνάψαι ταῦτα εἰς ταὐτὸν τοῖς λέγουσι τὰς οὐσίας καθόλου, ὅτι τοῖς αἰσθητοῖς οὐ τὰς αὐτὰς [οὐσίας] ἐποίουν· τὰ μὲν οὖν ἐν τοῖς αἰσθητοῖς καθ' ἕκαστα ῥεῖν ἐνόμιζον καὶ μένειν οὐθὲν αὐτῶν, τὸ δὲ καθόλου παρὰ ταῦτα εἶναί τε καὶ ἕτερόν τι εἶναι. τοῦτο δ', ὥσπερ ἐν τοῖς ἔμπροσθεν ἐλέγομεν, ἐκίνησε μὲν Σωκράτης διὰ τοὺς ὁρισμούς, οὐ μὴν ἐχώρισέ γε τῶν καθ' ἕκαστον· καὶ τοῦτο ὀρθῶς ἐνόησεν οὐ χωρίσας. δηλοῖ δὲ ἐκ τῶν ἔργων· ἄνευ μὲν γὰρ τοῦ καθόλου οὐκ ἔστιν ἐπιστήμην λαβεῖν, τὸ δὲ χωρίζειν αἴτιον τῶν συμβαινόντων δυσχερῶν περὶ τὰς ἰδέας ἐστίν. οἱ δ' ὡς ἀναγκαῖον, εἴπερ ἔσονταί τινες οὐσίαι παρὰ τὰς αἰσθητὰς καὶ ῥεούσας, χωριστὰς εἶναι, ἄλλας μὲν οὐκ εἶχον ταύτας δὲ τὰς καθόλου λεγομένας ἐξέθεσαν, ὥστε συμβαίνειν σχεδὸν τὰς αὐτὰς φύσεις εἶναι τὰς καθόλου καὶ τὰς καθ' ἕκαστον. αὕτη μὲν οὖν αὐτὴ καθ' αὑτὴν εἴη τις ἂν δυσχέρεια τῶν εἰρημένων.

In both passages Aristotle is accusing Platonists of separating the universal. More specifically, in the second passage above, Aristotle accuses the Platonists of a serious *category mistake*: that Platonists make forms *both* universals *and* particulars. That sensibles are in flux (ῥεῖν) is the reason why Plato inferred that there must be forms conceived as the basic objects of knowledge and definition; as we can see in the previously cited passage from *Meta.* M.4, knowledge and definition require the existence of things that are not in flux (ὥστ' εἴπερ ἐπιστήμη τινὸς ἔσται καὶ φρόνησις, ἑτέρας δεῖν τινὰς φύσεις εἶναι παρὰ τὰς αἰσθητὰς μενούσας· οὐ γὰρ εἶναι τῶν ῥεόντων ἐπιστήμη). It seems that Aristotle is in agreement with what motivates the theory of forms, namely that knowledge and definition has to be of something different than the ever–changing particulars.[97] But why the confusion of universals with particulars? The reasons may be traced in the last lines of the second passage: Platonists' answer to the inadequacy of the sensibles is that there must be some non–sensible substances besides the sensible ones; but they did not have any other non-sensible substances apart from their own Forms (ἄλλας μὲν οὐκ εἶχον); so they posited separate Forms; (καὶ πάλιν ὡς χωριστὰς καὶ τῶν καθ' ἕκαστον, 1086a33-34, χωριστὰς εἶναι, ἄλλας μὲν οὐκ εἶχον ταύτας δὲ τὰς καθόλου λεγομένας ἐξέθεσαν, ὥστε συμβαίνειν σχεδὸν τὰς αὐτὰς φύσεις εἶναι τὰς καθόλου καὶ τὰς καθ' ἕκαστον, 1086b9-11), so Forms are particulars (as well as universals). But–Aristotle complains–how can something be a substance particular and a universal? According to Fine, the crucial assumption is that separation implies particularity. Fine offers an account of Aristotle's reasoning in terms of instantiation:

> But as I (following Aristotle) understand separation, the claim that forms—universals—are separate is simply the claim that they can exist whether or not any corresponding sensible particulars exist. Why does Aristotle take this to show that forms are particulars? The answer is that he believes that universals exist when and only when they are instantiated; in his view, only substance particulars are separate (see e.g. *Meta.* 1028a33–4). So he claims that if forms are separate they are (substance) particulars because *he* accepts the controversial view that universals cannot exist uninstantiated. He is

[97] The theme that knowledge is of the universal whereas perception is of the particulars can be found elsewhere in Aristotle (e.g. in *Meta*. B.6, 999a26-b3).

> therefore not convicting Plato of internal inconsistency: he means that Plato's views do not square with the truth. He sees that Plato introduces forms simply to be universals; that they are particulars results only if we accept the controversial Aristotelian assumption, which Aristotle takes Plato to reject, that universals cannot exist uninstantiated. [Fine(1993), p.61]

The first part of our parallel, then, can be viewed as a very condensed criticism of Aristotle to a (more accurate) one-over-many argument that would lead to the postulation of a Form of Quantity that enjoys separate existence from numbers and magnitudes.

[2.4.2] The analogy from the universal propositions in mathematics and the related discussion– part two

The very best that can be extracted from the first part of our parallel is this: what is of importance in the universal propositions of mathematics is that the objects that satisfy them share a common feature. But in M.2, 1077a9-14 (where the context again is about τὰ καθόλου ἐν τοῖς μαθήμασιν) the text reads:

> Besides, there are some universal mathematical propositions, whose application extends beyond these substances. Here then we shall have another substance between, and separate from, the Ideas and the intermediates,—a substance which is neither number nor points nor spatial magnitude nor time. And if this is impossible, plainly it is also impossible that the *former* should exist in separation from sensible things.[98] [*Meta*. M.2, 1077a9-14; Ross' trans. mod.]

As we have said previously, the passage is part of Aristotle's arguments against mathematicals as separately existing entities. A few lines before (1077a9-14) Aristotle makes a reference to Book B of the *Metaphysics*: ἔτι ἅπερ καὶ ἐν τοῖς ἀπορήμασιν… (1076b39-1077a1); this is a reference back to the fifth aporia in that Book. A formulation of the fifth aporia may be found in lines 997a34-b3:

> Further, must we say that sensible substances alone exist, or that there are others besides

[98] ἔτι γράφεται ἔνια καθόλου ὑπὸ τῶν μαθηματικῶν παρὰ ταύτας τὰς οὐσίας. ἔσται οὖν καὶ αὕτη τις ἄλλη οὐσία μεταξὺ κεχωρισμένη τῶν τ' ἰδεῶν καὶ τῶν μεταξύ, ἣ οὔτε ἀριθμός ἐστιν οὔτε στιγμαὶ οὔτε μέγεθος οὔτε χρόνος. εἰ δὲ τοῦτο ἀδύνατον, δῆλον ὅτι κἀκεῖνα ἀδύνατον εἶναι κεχωρισμένα τῶν αἰσθητῶν.

> these? And are substances of one kind or are there several kinds of substances, as those say who assert the existence both of the Forms and of the intermediates with which they say the mathematical sciences deal?[99] [*Meta*. B.2, 997a34-b3; Ross' trans.]

Aristotle in the beginning of the first chapter of Book M of the *Metaphysics* gives us the context of the inquiry that will follow:

> Now since our inquiry is whether there is or is not besides the sensible substances any which is immovable and eternal, and, if there is, what it is, …[100] [*Meta*. M.1, 1076a10-12; Ross' trans.]

But why cannot sensible substances be the subject matter of the sciences? Among the five arguments for the existence of the Forms listed previously, there are some that pertain to *sciences*. That sensible things are somehow *inadequate* as proper objects of the sciences serves as a key premise of the so-called 'arguments from sciences' in Aristotle's work *On Ideas*. There are three such arguments in the *On Ideas*; let us examine the first two:

> 1) If every science performs its own function by referring to some one and the same thing, and not to any of the particulars, there would be, for each science, some other thing besides the sensibles, which is eternal and a model of the things that come in each science. And such is the Form.[101] [Alex. 79.5-8; Frank's trans. slightly mod.]
>
> 2) Moreover, the things about which the sciences are concerned are. But the sciences are concerned with some other things besides the particulars; for <the particulars> are indefinite and indeterminate, whereas the things about which the sciences are concerned are determinate. Therefore, there are some things besides the particulars, and these things

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[99] ἔτι δὲ πότερον τὰς αἰσθητὰς οὐσίας μόνας εἶναι φατέον ἢ καὶ παρὰ ταύτας ἄλλας, καὶ πότερον μοναχῶς ἢ πλείω γένη τετύχηκεν ὄντα τῶν οὐσιῶν, οἷον οἱ λέγοντες τά τε εἴδη καὶ τὰ μεταξύ, περὶ ἃ τὰς μαθηματικὰς εἶναί φασιν ἐπιστήμας;

[100] ἐπεὶ δ' ἡ σκέψις ἐστὶ πότερον ἔστι τις παρὰ τὰς αἰσθητὰς οὐσίας ἀκίνητος καὶ ἀΐδιος ἢ οὐκ ἔστι, καὶ εἰ ἔστι τίς ἐστι …

[101] εἰ πᾶσα ἐπιστήμη πρὸς ἕν τι καὶ τὸ αὐτὸ ἐπαναφέρουσα ποιεῖ τὸ αὑτῆς ἔργον καὶ πρὸς οὐδὲν τῶν καθ' ἕκαστον, εἴη ἄν τι ἄλλο καθ' ἑκάστην παρὰ τὰ αἰσθητὰ ἀίδιον καὶ παράδειγμα τῶν καθ' ἑκάστην ἐπιστήμην γινομένων. τοιοῦτον δὲ ἡ ἰδέα.

are the Forms.[102] [Alex. 79.8-11; Frank's trans. slightly mod.]

In the first argument we have the specification of the conditions which a proper object of a science must fulfill.[103] Thus if every science 'functions with reference to some one and the same thing' (πρὸς ἕν τι καὶ τὸ αὐτὸ ἐπαναφέρουσα)[104] and concerns itself with 'none of the particular things' (πρὸς οὐδὲν τῶν καθ' ἕκαστον), then there would be for each science some other thing besides the particulars, an entity which is 'everlasting and a paradigm' (ἀίδιον καὶ παράδειγμα) of the things that come to be within that science. From the second argument we further infer that sensible things (τὰ αἰσθητὰ) are *inadequate* as objects of the sciences because they are somehow *indefinite* (ἄπειρα) and *indeterminate* (ἀόριστα), whereas the objects of the sciences should be *determinate* (ὡρισμένα). Both arguments conclude that, for each science, there should be a proper object (eventually identified as a Form) which is different from the sensibles. It seems that the καθ' ἕκαστα here are sensible particulars and not low-level types or properties.[105] Terms such as ἀόριστον and ἄπειρον connote two kinds of indeterminacy, *qualitative* and *quantitative* one:

> The indeterminacy can be quantitative or qualitative. That is it can be indeterminate how many of something are dealing with – for example how many particulars instantiate a given universal, or how many properties a given thing has. (This second sort of quantitative indeterminacy shows that a single thing can be quantitatively indeterminate.) Or there can be some indeterminacy in the nature of, or in a given description of, some or all of a thing's properties. If, for example, we say only that something is hot, or hotter than something else, it is indeterminate what degree of heat it has. [Fine (1993), p.71]

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[102] ἔτι ὧν ἐπιστῆμαί εἰσι, ταῦτα ἔστιν· ἄλλων δέ τινων παρὰ τὰ καθ' ἕκαστά εἰσιν αἱ ἐπιστῆμαι· ταῦτα γὰρ ἄπειρά τε καὶ ἀόριστα, αἱ δὲ ἐπιστῆμαι ὡρισμένων· ἔστιν ἄρα τινὰ παρὰ τὰ καθ' ἕκαστα, ταῦτα δὲ αἱ ἰδέαι

[103] See [Cleary (2013), p.337].

[104] Frank remarks: 'this does not mean that for each science there is numerically just one object which acts as the object of the scientist's concern. <...> In the third argument of sciences we learn that the geometer concerns himself with both *equal* simpliciter and *commensurable* simpliciter.' In [Frank(1984), p.128].

[105] For this view see also [Cleary, op. cit., p.338]. Frank argues that the καθ' ἕκαστα *are* low-level types or properties [Frank(1984), pp.22–23]. Fine argues that we should opt for a more general reading of the καθ' ἕκαστα that includes both senses [Fine(1993), p.79].

One may understand qualitative indeterminacy with regard to the objects of the mathematical sciences, as for example Burnyeat does, that 'when mathematical properties are attributed to sensible things, the result both is and is not the case': 'a tabletop or a diagram both is and is not square, a cow or a line drawn to represent unity both is and is not one.'[106] One could also illuminate the quantitative indeterminacy of the sensibles by pointing to *Parmenides* (129c-d), where Plato argues that Socrates is 'one', because he is one man among a company of seven men, and we can equally say that he is 'many' in virtue of his upper and lower, front and back, and left and right parts. Or, we can go to the *Republic* Book VII where Plato argues that the 'ones' and the 'numbers' grasped by the senses are not truly ones and numbers, since 'we do see the same thing as one and as an unlimited number at the same time' (525a4-5).

Given the inadequacy of sensible objects then, one might be tempted to posit Forms as the proper objects of the sciences. A more suitable move, however, would be to posit Intermediates as proper objects: we have already seen that they differ from sensible things in being eternal and unchangeable (ἀΐδια καὶ ἀκίνητα), and from Forms in that there are many alike (πόλλ' ἄττα ὅμοια), while the Form itself is in each case unique (*Meta*. A.6, 987b14-18). Given that mathematical statements require things that are many–per–type (the so-called *uniqueness problem*), one may modify the original *argument from sciences* so as to postulate Intermediates instead of Forms:

1)The theorems of mathematics are true. 2)The theorems of mathematics are not true about sensible things for the latter are indeterminate. 3)The theorems of

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[106] In [Burnyeat (1987), pp.225-226]. Compare also the following passage from *Meta*. Z.15:

> For this reason, also, there is neither definition nor demonstration of sensible individual substances, because they have matter whose nature is such that *they are capable both of being and of not being*; for which reason all the individual instances of them are destructible. If then demonstration is of necessary truths and definition involves knowledge, and if, just as knowledge cannot be sometimes knowledge and sometimes ignorance, but the state which varies thus is opinion, so too demonstration and definition can- not vary thus, but it is opinion that deals with that which can be otherwise than as it is, clearly there can neither be definition nor demonstration of sensible individuals. [*Meta*. Z.15, 1039b27-1042a2; Ross' trans.; italics mine]

mathematics are true of intermediate mathematicals.

The argument is a modification of similar arguments in [Burnyeat (1987), pp.221-222] and in [Menn,'Iγ3', p.20] and is based on the following argument from *Meta*. N.3:

> But those who make <number> separate assume that it exists and is separate because the axioms would not be true of sensible things, while the statements <of mathematics> *are* true and delight the soul; and similarly with the magnitudes of mathematics.[107] [*Meta*. N.3 1090a35-b1; Ross' trans. mod]

The passage is about *anyone* who accepts that numbers exist in separation from the sensibles not merely about people who posit numbers as Forms (see 1090a15ff.). Premise 2) in this Platonist argument illustrates the so-called precision problem, the fact physical objects might *fail* to have the mathematical properties we study. A formulation of the precision problem may be found in the following passage:

> For neither are perceptible lines such lines as the geometer speaks of (for no perceptible thing is straight or curved in this way; for a hoop touches a straight edge not at a point, but as Protagoras said it did, in his refutation of the geometers), nor are the movements and complex orbits in the heavens like those of which astronomy treats,[108] nor have

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[107] οἱ δὲ χωριστὸν ποιοῦντες, ὅτι ἐπὶ τῶν αἰσθητῶν οὐκ ἔσται τὰ ἀξιώματα, ἀληθῆ δὲ τὰ λεγόμενα καὶ σαίνει τὴν ψυχήν, εἶναί τε ὑπολαμβάνουσι καὶ χωριστὰ εἶναι· ὁμοίως δὲ καὶ τὰ μεγέθη τὰ μαθηματικά.

[108] Cf. passage 529c-d from *Republic* Book VII where Socrates acknowledges that the motions of the heavenly bodies are the most beautiful and most exact among those of sensible things; he insists, however, that those motions are not the subject matter of the astronomers because they fall short of the true motions which only can be grasped by reason and thought and not by the senses:

> [Socrates]:… these ornaments in the heavens, since they are ornaments in something visible, may certainly be regarded as having the most beautiful and most exact motions that such things can have. But these fall short of the true ones – those motions in which things that are really fast or really slow, as measured in true numbers and as forming all the true geometrical figures, are moved relative to one another, and that move the things that are in them. And these, of course, must be grasped by reason and thought, not by sight. Don't you agree? [Glaucon]: Of course. [529c7-d6; Reeve's trans.]

> geometrical points the same nature as the actual stars.[109] [*Meta.* B.2, 997b34-998a6; Ross' trans.]

Lear cautions us not to disregard the *aporetic* context of the above passage:

> One should not consider this passage in isolation from the context in which it occurs. *Metaphysics* B.2 is a catalogue of philosophical problems (aporiai) presented from various points of view. None of it should be thought of as a presentation of Aristotle's considered view on the subject. It is rather a list of problems in response to which he will form his philosophical position. Immediately before the quoted passage Aristotle is putting forward the problem for the Platonists that the belief in Form-like intermediates involves many difficulties (997bl2-34). The quoted passage can thus be read as an imagined Platonist's response: 'Yes, the belief in intermediates is problematic, but, on the other hand, giving them up involves difficulties, too.' Here it is an imagined Platonist speaking, and not Aristotle. So Aristotle is not endorsing Protagoras' view; he is presenting it as one horn of a dilemma that must be resolved. [Lear (1982), p.176][110]

The dilemma the Platonist presents us with is, according to Lear, the following: either we have to endorse a theory of intermediate mathematicals or abandon them altogether and face the problem of precision that tantalises the sensible objects. Lear proposes the following resolution of the dilemma:

> We have already seen Aristotle's proposed resolution; and it is one that involves asserting that some physical objects perfectly possess geometrical properties. [Lear (1982), p.176]

I agree with Lear on this point. My disagreement has to do with the somewhat wider scope of mathematical features Lear claims to be 'exactly' (i.e. mathematically precisely) physically instantiated. The question that naturally

[109] (οὔτε γὰρ αἱ αἰσθηταὶ γραμμαὶ τοιαῦταί εἰσιν οἵας λέγει ὁ γεωμέτρης (οὐθὲν γὰρ εὐθὺ τῶν αἰσθητῶν οὕτως οὐδὲ στρογγύλον· ἅπτεται γὰρ τοῦ κανόνος οὐ κατὰ στιγμὴν ὁ κύκλος ἀλλ' ὥσπερ Πρωταγόρας ἔλεγεν ἐλέγχων τοὺς γεωμέτρας), οὔθ' αἱ κινήσεις καὶ ἕλικες τοῦ οὐρανοῦ ὅμοιαι περὶ ὧν ἡ ἀστρολογία ποιεῖται τοὺς λόγους, οὔτε τὰ σημεῖα τοῖς ἄστροις τὴν αὐτὴν ἔχει φύσιν.

[110] In this Lear probably follows Syrianus' commentary of the passage: 'The argument is not directed against those who bring in several kinds of substance; it is rather in agreement with them' [Syrianus: *Comm. in Meta.*, 27.8-9; Madigan's trans.]. Consult [Madigan (1986), pp.162-165] for a valuable discussion on the commentaries of Syrianus and Asclepius in this passage.

arises, then, is the following: Are any mathematical features 'exactly' physically instantiated? And, if so, which features are so instantiated? I will return to the problem of precision later in my essay.

Returning to the M.2, 1077a9-14 passage, why would certain objects lie between (μεταξύ) Intermediates and Forms? It is not immediately clear; ps–Alex. complains that Aristotle is being a little vague there: ἔχει δέ τινα βραχεῖαν ἀσάφειαν ἡ λέξις ἡ ἔσται οὖν καὶ αὕτη τις ἄλλη οὐσία μεταξὺ κεχωρισμένη τῶν τε ἰδεῶν καὶ τῶν μεταξύ· λέγει δὲ ἰδέας μὲν τὸ αὐτοποσὸν καὶ τὰ τοιαῦτα, μεταξὺ δὲ τὰ μαθηματικά· μεταξὺ γὰρ τῶν τε ἰδεῶν καὶ τῶν αἰσθητῶν ἐτίθεντο τὰ μαθηματικά. λέγει οὖν ὅτι ἔσται τις ἄλλη φύσις καθ' αὑτὴν οὖσα μεταξὺ τῶν τε ἰδεῶν καὶ τῶν μεταξύ, τουτέστι τῶν μαθηματικῶν 729.34-730.10). A solution might be this: We have seen that, given the *uniqueness* problem, one may modify the original *argument from sciences* so as to postulate Intermediates instead of Forms. Then one could perhaps formulate an *one over many* argument to show that there is a Form of Quantity. And one could invoke a *self-predication* principle and a *non-identity* one to formulate a *third man* argument: We have a set of intermediate lines, planes, solids, numbers, and so on which are all quantities. From this we infer that there is a separate Form of Quantity by virtue of which they all are quantities. How do we infer this? We are relying on a *one over many* principle that postulates a Form for a set of things that all have something in common. But we can proceed even further: We now consider together the items discussed in the first step (that is, all the intermediate quantities) and Quantity, the Form by virtue of which they all are quantities. In order to do so, we need a principle that tells us that the Form of Quantity is itself a *quantity*: a Form ought to be subject to *self-predication*. From this via the one over many principle we infer that there is a separate Form of Quantity by virtue of which they (the members of the previous set, the intermediate quantities and the Form of Quantity) all are quantities. The Form introduced in this step is a second Form (Quantity$_2$), distinct from the original Form (Quantity$_1$) introduced earlier. In order to infer this we need a *non-identity* principle: The Form by virtue of which a set of things are all quantities is not itself a member of that set. And we can continue the process ad infinitum.[111] It seems then that Aristotle must have something like that on his mind when he claims that there ought to be some other substance between the ideas and the mathematicals.

Let us conclude the analysis of the first parallel (1077bl7-22) by offering a more expanded version of it: Just as the universal propositions of mathematics are about kinds of quantity (lines, numbers, etc.) insofar as they are quantities and not insofar as they are continuous or (in)divisible (οὐχ ᾗ δὲ τοιαῦτα οἷα ἔχειν μέγεθος ἢ εἶναι διαιρετά), and not about some special, separate entity called 'quantity', similarly geometry is about sensible magnitudes, not qua sensible but qua such and such (ᾗ τοιαδί); geometry is about sensible magnitudes qua solids or qua planes or qua lines, and not (we may add) about separately existing solids, planes, and lines. The very least that can be extracted from this first parallel is that the link between mathematics and the actual world should not be severed by the postulation of mathematicals that exist in separation from the sensibles.

[2.4.3] The analogy from physics/astronomy

The second parallel (1077b22-30) is now brought forward to further support or explain the first one (cf. ps–Alex.: πρὸς τούτῳ καὶ δι' ἄλλου ὑποδείγματος σαφηνίζει τὸ λεγόμενον λέγων ὥσπερ γὰρ καὶ ᾗ κινούμενα μόνον πολλοὶ λόγοι εἰσί, 734.33-35):

> For just as there are many statements about things merely as moving, apart from the nature of each such thing and their incidental properties (and this does not mean that there has to be either some moving object separate from the perceptible objects, or some such entity marked off in them), so in the case of moving things there will be statements and branches of knowledge them about them, not as moving but merely as bodies, and again merely as planes and merely as lengths, as divisible, and indivisible but with position, and merely as indivisible.[112] [*Meta*. M.3, 1077b22-30; Annas' trans.]

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[111] In formulating the argument I am relying on Cohen's analysis in [Cohen(1971)] and on Fine's one in [Fine (1993), ch.15, esp.pp.210-211].

[112] ὥσπερ γὰρ καὶ ᾗ κινούμενα μόνον πολλοὶ λόγοι εἰσί, χωρὶς τοῦ τί ἕκαστόν ἐστι τῶν τοιούτων καὶ τῶν συμβεβηκότων αὐτοῖς, καὶ οὐκ ἀνάγκη διὰ ταῦτα ἢ κεχωρισμένον τι εἶναι κινούμενον τῶν

In this passage Aristotle states that there are many statements (πολλοὶ λόγοι) about things only insofar as they are moving (ᾗ κινούμενα μόνον), without any reference to the essential nature of such things or to their attributes (χωρὶς τοῦ τί ἕκαστόν ἐστι τῶν τοιούτων καὶ τῶν συμβεβηκότων αὐτοῖς). Now those statements could be either the propositions that physicists study (οἱ φυσικοὶ: ps–Alex., 734.33ff.) or those of astronomers or more general motion principles; an example would be: *things that move at an equal speed cover the same distance in an equal time* (ps–Alex., 734.36-37, τὰ ἰσοταχῶς κινούμενα ἐν ἴσῳ χρόνῳ τὸ αὐτὸ διάστημα διέξεισιν, and also Syrianus, 95.19-20). In any case, although the ᾗ κινούμενα locution implies that the subject–matter of those statements are not sensible things themselves, but rather sensible things qua moving, it is not necessary to postulate special 'moving objects' as 1) *separate* from the sensibles or 2) as located *in* the sesibles (ἢ κεχωρισμένον τι εἶναι κινούμενον τῶν αἰσθητῶν ἢ ἐν τούτοις τινὰ φύσιν εἶναι ἀφωρισμένην). The aforementioned disjunction refers to the two possible modes of existence for mathematical objects, initially articulated in 1076a32-34 (ἀνάγκη δ', εἴπερ ἔστι τὰ μαθηματικά, ἢ ἐν τοῖς αἰσθητοῖς εἶναι αὐτὰ καθάπερ λέγουσί τινες, ἢ κεχωρισμένα τῶν αἰσθητῶν), and refuted in M.2. The locution χωρὶς τοῦ τί ἕκαστόν ἐστι τῶν τοιούτων καὶ τῶν συμβεβηκότων αὐτοῖς explains what happens when someone studies sensibles qua moving; the commentary of ps–Alex. (734.37-735.7) is especially helpful: if we consider the motion principle stated above, *things that move at an equal speed cover the same distance in an equal* time (τὰ ἰσοταχῶς κινούμενα ἐν ἴσῳ χρόνῳ τὸ αὐτὸ διάστημα διέξεισιν), it is clear that nothing *specific* is required about the nature of the things that satisfy this principle, but what matters is their motion and its essential attributes (οὐδὲν ἁπτόμενοι τῶν ὑποκειμένων πραγμάτων, ἀλλὰ περὶ τῆς κινήσεως αὐτῶν μόνης διαλεγόμεθα). Thus, as ps–Alex. comments, physicists/astronomers that study thigs qua moving do not consider whether the things that are in motion are men or the heavens and they do not study men or the heavens as such, but instead they study the very nature of motion and what holds true of men and the heavens qua being in motion (οἱ φυσικοὶ δεικνύουσι

αἰσθητῶν ἢ ἐν τούτοις τινὰ φύσιν εἶναι ἀφωρισμένην, οὕτω καὶ ἐπὶ τῶν κινουμένων ἔσονται λόγοι καὶ ἐπιστῆμαι, οὐχ ᾗ κινούμενα δὲ ἀλλ' ᾗ σώματα μόνον, καὶ πάλιν ᾗ ἐπίπεδα μόνον καὶ ᾗ μήκη μόνον, καὶ ᾗ διαιρετὰ καὶ ᾗ ἀδιαίρετα ἔχοντα δὲ θέσιν καὶ ᾗ ἀδιαίρετα μόνον.

πάντα τὰ καθ' αὐτὰ ταῖς κινήσεσι μὴ θεωροῦντες μηδὲ πολυπραγμονοῦντες τὰ ὑποκείμενα τίνα ἐστί, πότερον ἄνθρωποι ἢ οὐρανός, ἀλλ' οὐδὲ τί συμβέβηκε τῷ ἀνθρώπῳ ἢ τῷ οὐρανῷ θεωροῦσιν, ἀλλ' αὐτὴν καθ' αὑτὴν τὴν τῆς κινήσεως φύσιν καὶ τὰ καθ' αὑτὰ ὑπάρχοντα αὐταῖς ἀνιχνεύουσί τε καὶ ἀποδεικνύουσι, ps–Alex. 734.37-735.7).

Aristotle then, following a line of thought similar to the first parallel where the context was the universal propositions of mathematics, reiterates the invalidity of any Platonic argument that establishes either the separate existence of some Form of Motion or the existence of Intermediate moving things. Aristotle in this second parallel seems to be particularly concerned with the objects of astronomical study and it would be rather useful to offer a brief analysis of his discussion regarding the absurdities that stem from positing astronomical Intermediates in *Meta*. 997b12-20:

> Further, if we are to posit besides the Forms and the sensibles the intermediates between them, we shall have many difficulties. For clearly on the same principle there will be lines besides the lines-in-themselves and the sensible lines, and so with each of the other classes of things; so that since astronomy is one of these mathematical sciences there will also be a heaven besides the sensible heaven, and a sun and a moon (and so with the other heavenly bodies) besides the sensible ones. Yet how are we to believe these things? It is not reasonable even to suppose these bodies immovable, but to suppose their *moving* is quite impossible.[113] [*Meta*. B.2, 997b12-20; Ross' trans.]

Aristotle begins by asserting that those who posit Intermediates will run into many absurdities; for he explains this means positing Intermediate lines in addition to the Form line and to the sensible ones; and likewise for every other kind of thing: if astronomy is a science of Intermediates there will be a heaven, a sun and moon and heavenly bodies over and above the sensible heaven. Following Alexander, I understand that the γενῶν in ln. 997b15 refers to the

[113] ἔτι δὲ εἴ τις παρὰ τὰ εἴδη καὶ τὰ αἰσθητὰ τὰ μεταξὺ θήσεται, πολλὰς ἀπορίας ἕξει· δῆλον γὰρ ὡς ὁμοίως γραμμαί τε παρά τ' αὐτὰς καὶ τὰς αἰσθητὰς ἔσονται καὶ ἕκαστον τῶν ἄλλων γενῶν· ὥστ' ἐπείπερ ἡ ἀστρολογία μία τούτων ἐστίν, ἔσται τις καὶ οὐρανὸς παρὰ τὸν αἰσθητὸν οὐρανὸν καὶ ἥλιός τε καὶ σελήνη καὶ τἆλλα ὁμοίως τὰ κατὰ τὸν οὐρανόν. καίτοι πῶς δεῖ πιστεῦσαι τούτοις; οὐδὲ γὰρ ἀκίνητον εὔλογον εἶναι, κινούμενον δὲ καὶ παντελῶς ἀδύνατον.

subject genera of the mathematical sciences which include astronomy[114] (cf. Alex., 197.35-198.3: ὡς γὰρ ἐπὶ γραμμῆς ἔχει (ἔστι γάρ τις γραμμὴ κατ' αὐτοὺς παρά τε τὴν αἰσθητὴν καὶ τὴν ἰδέαν ἡ μαθηματική, περὶ ἣν ἡ γεωμετρία πραγματεύεται), οὕτως ἕξει καὶ ἐπὶ τῶν ἄλλων, δηλονότι τῶν μαθηματικῶν). Thus, Alexander points out that the τούτων in the next sentence could refer to those mathematical sciences of which astronomy is part (cf. Alex. 198.4-5: ἐπεὶ οὖν ἐστιν ἐν ταῖς μαθηματικαῖς καὶ ἡ ἀστρολογία). It is not plausible for the heavens to be considered as immovable mathematical objects (οὐδὲ γὰρ ἀκίνητον εὔλογον εἶναι) because it is their very nature to be movable (cf. Alex. 198.8-11: πῶς δὲ οἷόν τε εἶναι ἥλιόν τινα μαθηματικὸν μὴ κινούμενον, ἢ κόσμον ἢ ἄλλο τι τῶν ἄστρων; ἡ γὰρ οὐσία καὶ ἡ φύσις τούτων μετὰ τῆς τοιᾶσδε κινήσεως). So a Platonist who studies the sensible heavens has two options: either to regard them as immovable mathematical objects of intermediate nature, thereby disregarding certain important aspects of them, such as their motion (this treatment of the heavens would probably classify one as a geometer rather than as an astronomer) or to regard them as movable mathematical objects again of intermediate nature, something impossible for objects non-material and non-sensible by nature such as the Intermediates (as Alex. explains in 198. 11-14: πολὺ δ' ἔτι ἀλογώτερον τοῦ λέγειν κόσμον καὶ ἥλιον ἀκίνητα τὸ λέγειν εἶναι μὲν αὐτὰ κινούμενα, οὐκ αἰσθητὰ δὲ ἀλλὰ μαθηματικά· ἀδύνατον γὰρ κινεῖσθαι τὸ μὴ ὂν ἔνυλον καὶ τῇ αὑτοῦ φύσει αἰσθητόν).[115]

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[114] This is also Cleary's reading in [Cleary (1995), p.250].

[115] As Madigan points out, Asclepius interprets Aristotle's argument not as a dilemma but as a *two-level paradox*, i.e. 'a contradiction between a predicate that belongs to an intermediate because it is an intermediate and a predicate that belongs to it because it is the particular intermediate it is'. In [Madigan (1986), fn.13, p.154]. Insofar as the intermediate heaven is a heaven it must be in motion; insofar as it is an intermediate object it must be unmoved; hence it will be both in motion and unmoved which is impossible (καθὸ μὲν γάρ ἐστιν οὐρανός, δεῖ αὐτὸν κινεῖσθαι, καθὸ δὲ διανοητός, ἔσται ἀκίνητος· ὥστε ὁ αὐτὸς καὶ κινούμενος καὶ ἀκίνητος, ὅπερ ἐστὶν ἀδύνατον, 168.28-31). Madigan informs us that Asclepius deals with the paradox by distinguising (168.31-169.1) in a neo-Platonic fashion the sensible heaven which is in motion (αἰσθητός ἐστιν οὐρανὸς ὁ φαινόμενος), the intelligible heaven in the Demiurge (νοητὸς οὐρανὸς ὁ λόγος ὁ ἐν τῷ δημιουργῷ, presumably the Form-heaven), and the intermediate heaven that is the object of reason which exists in our soul and it is unmoved (διανοητὸς ὁ λόγος τοῦ οὐρανοῦ ὁ ἐν τῇ ψυχῇ

The absurdities that stem from the postulation of intermediate heavens is part of *a more general* Aristotelian argumentative strategy that can be outlined as follows: If the Platonists try to respond to problems such as that of the indeterminacy of the sensibles by postulating separately existing mathematicals, then the same problems would require them to posit entities such as the intermediate heavens and so on.[116] The case becomes especially problematic for the more 'applied' mathematical sciences (e.g. optics) since it leads to peculiar entities such as intermediate sensations and similarly for the other sciences. This seems to be what Aristotle has in mind in the following passage:

> And similarly with the things of which optics and mathematical harmonics treat. For these also cannot exist apart from the sensible things, for the same reasons. For if there are sensible things and sensations intermediate between Form and individual, evidently there will also be animals intermediate between animals-in-themselves and the perishable animals. [*Meta*. B.2, 997b20-24; Ross' trans.]

But let us return to the discussion of this second parallel. Just as there are theorems that are just about sensible bodies, not qua sensible bodies but only qua moving, so likewise, there can be theorems that are just about sensible moving bodies (οὕτω καὶ ἐπὶ τῶν κινουμένων), not qua moving but only qua bodies (οὐχ ᾗ κινούμενα δὲ ἀλλ' ᾗ σώματα μόνον), and again only qua surfaces (πάλιν ᾗ ἐπίπεδα μόνον), and only qua lengths (ᾗ μήκη μόνον); and qua divisibles (i.e. just as continuous quantities, ᾗ διαιρετὰ); and qua indivisibles having position (i.e. just as points, ᾗ ἀδιαίρετα ἔχοντα δὲ θέσιν); and qua indivisibles alone (i.e. just as units, ᾗ ἀδιαίρετα μόνον). Aristotle's seemingly straightforward conclusion needs further analysis if we scrutinise it through the prism of the precision problem: the fact that astronomy studies celestial bodies qua points does not entail that those bodies are actually points, only that they are regarded as being indivisible and having position; and what happens when I examine my

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ἡμῶν ὑπάρχων· οὗτος δὲ ἀκίνητός ἐστι λόγος ὑπάρχων.). The intermediate heaven then is not both moved and unmoved but simply unmoved, and thus the paradox is solved. In [Madigan(1986), pp.154-155].

[116]This is also what Stephen Menn argues in [Menn, 'Iγ3', p.22]. This also seems to be Cleary's conclusion. See [Cleary(1995), pp. 250-259] for extensive discussion.

desk qua plane? Does this mean that my desk has a perfectly planar surface? These are also objections that Syrianus raises against Aristotle. Syrianus objects that Aristotle's analogy does not work properly; for there is no problem of precision with regard to the precise (uniform circular) motion of the celestial bodies–but one might wonder whether the same thing can be said about the shapes of the objects around us; whether there are such things as perfect spheres, cubes and so on: 'For he who does not wish that there be motion outside of sensible objects does not conceive of some more exact form of motion in the immobile realm, whereas the geometer does conceive of other shapes more exact than perceptible ones.'[117] In [Com. on *Meta*. M-N, 95.22-26, trans. by Dillon and O'Meara].

[2.4.4] Conclusions and further discussion

A conclusion then is drawn in lines 1077b31-4 based on the above two parallels is: just as it is true to say ἁπλῶς that the moving things which are the subject matter of physics/astronomy exist, so it is true to say ἁπλῶς that mathematical objects exist:

> So, since it is true to say without qualification not only that separate things exist but also non-separate things exist (e.g. that moving-things exist), it is also true to say, without qualification, that mathematical objects exist, and are as they are said to be.[118] [*Meta*. M.3, 1077b31-4; Annas' trans. mod.]

Aristotle by beginning with 'therefore, since…' (ὥστ' ἐπεί) states the conclusion of the previous argument, which is that non-separate objects in general, and mathematical objects in particular, do exist.[119] The expression ἁπλῶς modifies λέγειν and εἰπεῖν and not εἶναι.[120] For, in *Metaphysics* M.2 Aristotle explicitly

[117] ῥητέον οὖν ὅτι πρῶτον μὲν οὐχ ὅμοιόν ἐστι τὸ περὶ τὴν κίνησιν τοῖς σχήμασιν· ὁ μὲν γὰρ μὴ βουλόμ ενος εἶναι κίνησιν ἔξω τῶν αἰσθητῶν οὐκ ἐννοεῖ τὴν ἐν τοῖς ἀκινήτοις ἀκριβεστέραν, ὁ δὲ γεωμέτρης ἐ πινοεῖ τῶν φαινομένων σχημάτων ἀκριβέστερα·

[118] ὥστ' ἐπεὶ ἁπλῶς λέγειν ἀληθὲς μὴ μόνον τὰ χωριστὰ εἶναι ἀλλὰ καὶ τὰ μὴ χωριστά (οἷον κινούμενα εἶναι), καὶ τὰ μαθηματικὰ ὅτι ἔστιν ἁπλῶς ἀληθὲς εἰπεῖν, καὶ τοιαῦτά γε οἷα λέγουσιν.

[119] In [Lear (1982), p.170].

[120] In agreement with [Lear, ibid.] and [Cleary (1995), p.319].

states that mathematical objects do not exist without qualification (οὐχ ἁπλῶς ἔστιν, 1077b16). It seems that Cleary makes a plausible suggestion when he points out that the distinction that Aristotle makes between separate and non-separate things corresponds to the distinction between *independent substances* and *dependent attributes* (he highlights the use of the term κινούμενα as Aristotle's example of such non-separate entities).[121] Thus Cleary offers the following interpretation of the passage:

> Since attributes can be truly said to exist (when one speaks simply or without qualification), the same is true of mathematical objects which 'are such as they are said to be' (1077b33-34). [Cleary (1995), p.319]

One may supplement Cleary's account by reminding the reader that when Aristotle talks about τὰ μαθηματικά he frequently means *lower-dimensional* entities such as surfaces, lines and points, which according to him are limit entites and do not enjoy separate existence from the things they are limits of. In any case, the role of this passage is indicative of Aristotle's realism towards mathematicals: they do exist, albeit in a dependent manner.

In the third analogy (1077b34-1078a5) a parallelism is established between the science of medicine and geometry:

> It is true to say of other branches of knowledge, without qualification, that they are about this or that-not what is incidental (e.g. not the white, even if the branch of knowledge deals with the healthy, and the healthy is white) but what each branch of knowledge is about, the healthy if <it studies its subject> as healthy, man if <it studies it> as man. And likewise with geometry.[122] [*Meta*. M.3, 1077b34-1078a2; Annas' trans.]

Since it is true to say without qualification (ἁπλῶς ἀληθὲς εἰπεῖν) that the science of medicine is about 'the healthy' (ὑγιεινόν) and not about something incidental (οὐχὶ τοῦ συμβεβηκότος) to it (e.g. 'the white', οἷον ὅτι λευκοῦ, εἰ τὸ ὑγιεινὸν λευκόν, ἡ δ' ἔστιν ὑγιεινοῦ), so too with geometry. It may be the case that every

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[121] Cleary, ibid.

[122] καὶ ὥσπερ καὶ τὰς ἄλλας ἐπιστήμας ἁπλῶς ἀληθὲς εἰπεῖν τούτου εἶναι, οὐχὶ τοῦ συμβεβηκότος (οἷον ὅτι λευκοῦ, εἰ τὸ ὑγιεινὸν λευκόν, ἡ δ' ἔστιν ὑγιεινοῦ) ἀλλ' ἐκείνου οὗ ἐστὶν ἑκάστη, εἰ <ᾗ> ὑγιεινὸν ὑγιεινοῦ, εἰ δ' ᾗ ἄνθρωπος ἀνθρώπου, οὕτω καὶ τὴν γεωμετρίαν·

object that has geometrical properties is a sensible one; it does not follow that the proper objects of geometry are sensible objects as such, but sensible objects qua lines, planes, solids, and so on. Aristotle once again is quick to point out that these proper objects do not exist in separation from the sensible ones.[123] Thus the analogy ends as follows:

> The mathematical branches of knowledge will not be about perceptible objects just because their objects happen to be perceptible, though not <studied> as perceptible; but nor will they be about separate objects over and above these.[124] [*Meta*. M.3, 1078a2-5; Annas' trans.]

Concluding the presenation of these analogies, we may say that via them, Aristotle wishes to highlight the fact that mathematics have close ties to the natural world. And this is only natural, for otherwise they wouldn't be applicable to the world.[125] However, despite these close ties, they do not treat specifically with perceptible objects, but have some other proper object, its existence has to be accommodated within those ties. It is not the case that mathematics and physics, for example, deal with the same objects but in a different manner, as J. Annas repeatedly claims in her commentary: 'mathematics is distinguished from science not by its subject-matter but by its method'.[126] Mathematics have a proper object, namely numbers and magnitudes; Aristotle acknowledges that in the beginning of chapter M as we have already seen: οἷον ἀριθμοὺς καὶ γραμμὰς

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[123] Aristotle's discussion mirrors the *third argument from the sciences*:

> Further, if medicine is the science not of this health but of health without qualification, there will be some health itself. And if geometry is the science not of this equal and of this commensurate but of equal without qualification and of commensurate without qualification, there will be some equal itself and some commensurate itself. And these things are the ideas. [Alex. in *Meta*., 79.3-15; Fine's trans.]

Aristotle would agree, I think, with the Platonists that medicine is not the science of this instance of health (τῆσδε τῆς ὑγιείας) nor is geometry the science of this instance of equality (τοῦδε τοῦ ἴσου). Rather, medicine is about 'the healthy' and geometry is about 'the equal'; and these do not exist *without qualification* as the Platonists assert, but in some οὐχ ἁπλῶς sense.

[124] οὐκ εἰ συμβέβηκεν αἰσθητὰ εἶναι ὧν ἐστί, μὴ ἔστι δὲ ᾗ αἰσθητά, οὐ τῶν αἰσθητῶν ἔσονται αἱ μαθηματικαὶ ἐπιστῆμαι, οὐ μέντοι οὐδὲ παρὰ ταῦτα ἄλλων κεχωρισμένων.

[125] As Lear also points out in [Lear(1982), p.81].

[126] In [Annas (1976), pp.29 & 148].

καὶ τὰ συγγενῆ τούτοις, 1076a18-19. He is not contesting that these objects exist but the manner in which they exist (ὥσθ’ ἡ ἀμφισβήτησις ἡμῖν ἔσται οὐ περὶ τοῦ εἶναι ἀλλὰ περὶ τοῦ τρόπου, 1076a36-37).

Aristotle continues by arguing that many things that hold of sensibles *qua* planes and lengths:

> Many properties hold true of things in their own right as being, each of them, of a certain type – for instance there are attributes peculiar to animals as being male or as being female (yet there is no female or male separate from animals). So there are properties holding true of things merely as lengths or as planes.[127] [*Meta*. M.3, 1078a5-9; Annas’ trans.]

Aristotle appeals to πάθη that are ἴδια to animals, not qua animals, but qua male or female: the male animal is γεννητικόν and the female is ὑποδεκτικὸν τῶν σπερμάτων.[128] When considered qua male, the male animal possesses his γεννητικόν καθ’ αὑτόν, since to be γεννητικόν in this way is essential to his being a male. But it is not essential to his being an animal as such. We can then establish an analogy with mathematicals as follows: just as there are many things true of animals qua being male or female, so too in the case of geometry there are many things true of things qua lines or planes.[129]

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[127] πολλὰ δὲ συμβέβηκε καθ’ αὑτὰ τοῖς πράγμασιν ᾗ ἕκαστον ὑπάρχει τῶν τοιούτων, ἐπεὶ καὶ ᾗ θῆλυ τὸ ζῷον καὶ ᾗ ἄρρεν, ἴδια πάθη ἔστιν (καίτοι οὐκ ἔστι τι θῆλυ οὐδ’ ἄρρεν κεχωρισμένον τῶν ζῴων)· ὥστε καὶ ᾗ μήκη μόνον καὶ ᾗ ἐπίπεδα.

[128] The examples are taken from ps-Alex., 737.9-10.

[129] This is in line with Syrianus’ reading of the passage: Ἐπειδὴ δὲ περὶ τὰ καθ’ αὑτὰ τοῖς μεγέθεσιν ὑπάρχοντα πραγματευομένης τῆς γεωμετρίας φησὶν ὅτι δύναται, καὶ εἰ μὴ χωριστὰ εἴη τῶν αἰσθητῶν τὰ μεγέθη, περὶ τὰ καθ’ αὑτὰ ὑπάρχοντα τοῖς ἀχωρίστοις διατρίβειν (καὶ γὰρ ἄλλοις ἀχωρίστοις, φησί, καθ’ αὑτά τινα ὑπάρχει· τὸ γὰρ θῆλυ καὶ τὸ ἄρρεν ἀχώριστα μὲν τῶν ζῴων, ἴδια δέ τινα ἔχει πάθη, οἷον τὸ μέν ἐστι γεννητικόν, τὸ δὲ θρεπτικὸν καὶ ὑποδεκτικὸν τῶν σπερματικῶν λόγων). ‘But since, accepting that geometry deals with essential properties of magnitudes, he says that it is possible, even if the magnitudes are not separate from sensible objects, for it to focus on essential properties of inseparable entities (for after all, he says, things pertain essentially to other inseparable entities: ‘female’ and ‘male’, for instance, are inseparable from living things, but yet have distinctive attributes, e.g. that the latter is generative, while the former is nutritive and receptive of seminal reason-principles).’ [Syrianus: *Comm. in Meta. M-N*, 97.18–24; trans. by Dillon and O’Meara]

The previous passage prompts us to examine a bit closer the role of the qua locution that dominates much of the M.3 discussion. J.J. Cleary understands the use of the qua locution as inextricably connected with a logical process of abstraction (or subtraction) developed by Aristotle mainly in *Post. An.* A.5.[130] In a difficult passage, Aristotle, as Barnes says, 'offers a recipe for discovering to what subject a given predicate belongs primitively':[131]

> So when do you not know universally, and when do you know simpliciter? Well, clearly <you could know simpliciter> if it were the same thing to be a triangle and to be equilateral (either for each or for all). But if it is not the same but different, and it belongs as triangle, you do not know. Does it belong as triangle or as isosceles? And when does it belong in virtue of this as primitive? And of what does the demonstration hold universally? Clearly whenever after abstraction it belongs primitively-e.g. two right angles will belong to bronze isosceles triangle, but also when being bronze and being isosceles have been abstracted. But not when figure or limit have been. But they are not the first. Then what is first? If triangle, it is in virtue of this that it also belongs to the others, and it is of this that the demonstration holds universally. [*Post. An.* A.5, 74a32-b4; Barnes' trans.]

Abstraction (or substraction) is then for Cleary a logical procedure which allows one to isolate the primary subject of a given attribute. Cleary outlines the procedure as follows:

> Finding an answer through subtraction seems to presuppose a certain order in which aspects are taken away; e.g. the bronze aspect of the sensible triangle is subtracted before the isosceles aspect. After each step in the procedure, one can ask whether the attribute in question has been eliminated along with the particular aspect that has been conceptually subtracted. Attributes like 'heavy' and 'light', for instance, disappear along with the bronze aspect which is therefore their primary subject. Similarly, the attribute of having the sum of its internal angles equal to two right angles is eliminated when one subtracts the aspect of triangularity from this sensible figure. [Cleary (1995), p.313]

Similarly, Lear understands the 'qua' locution as indicating a predicate-filter in the following manner: If, for example, *b* is a bronze isosceles triangle, then to

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130 For discussion see [Cleary (1995), pp.312-318].

131 In [Barnes(1975), p.123].

consider *b* qua triangle is to apply a predicate filter that filters out the predicates like bronze and isosceles that happen to be true of *b*, but are irrelevant to the purposes of the geometer:[132]

> Generalizing, one might say that Aristotle is introducing an *as*-operator, or *qua*-operator, which works as follows. Let *b* be an Aristotelian substance and let "*b qua F*" signify that *b* is being considered *as an F*. Then a property is said to be true of *b qua F* if and only if *b* is an *F* and its having that property follows of necessity from its being an *F*: $G(b \text{ } qua \text{ } F) \leftrightarrow F(b) \,\&\, (F(x) \vdash G(x))$. Thus to use the *qua*-operator is to place ourselves behind a veil of ignorance: we allow ourselves to know only that *b* is *F* and then determine on the basis of that knowledge alone what other properties must hold of it. If, for example, *b* is a bronze isosceles triangle–$Br(b)$ & $Is(b)$ & $Tr(b)$ – then to consider *b* as a triangle–*b qua Tr*–is to apply a predicate filter: it filters out the predicates like *Br* and *Is* that happen to be true of *b*, but are irrelevant to our current concern. [Lear (1982), p.168]

Lear's understanding of the qua-operator is especially restrictive.[133] The biscuit box in my desk qua solid has the shape of a cube. Having a cubical shape is not a property that follows necessarily from my biscuit box's being a solid. Hence we should perhaps weaken Lear's claim; as we shall see Aristotle tells us that the geometer studies the man qua solid: thus it might be better to argue that the geometer studies whatever holds true for Socrates qua solid. Yet another reason to weaken Lear's claim is that it is not always the case that something is precisely F. Suppose that I am treating my desk qua a planar surface; this does not mean that my desk is a planar surface. Further examples can be brought forward by the applied mathematical sciences: the astronomer treats the planets qua points for the purposes of his enquiry. This does not mean that the celectial bodies are actually points. Compare Alexander's commentary on *Meta.* B.2, 997b34-998a6:

> For the astronomer assumes lines, the spiral and the circle,which are lengths without breadth, and posits that motions occur in accord with these; but among sensible things there is no length without breadth. Further, they assume that the stars are certain points,

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[132] Lear's interpretation is more general in that it does not presuppose a certain order of filtered–out predicates.

[133] I would like to thank Mr Denyer for pointing this out to me. A similar complaint can be found in [Mendell (1986), pp.47-49].

and have the status of points in the heaven; but a point is something which has no parts, and none of the stars is such a thing.[134] [Alexander: *Comm. on Aristotle's Meta*. 200.23-28; Madigan's trans.]

[134] γραμμὰς γὰρ λαμβάνει ὁ ἀστρολόγος καὶ τὴν ἕλικα καὶ τὸν κύκλον, ἅ ἐστι μήκη ἀπλατῆ, καὶ ἐπὶ τούτων τὰς κινήσεις ὑποτίθεται γίγνεσθαι· οὐδὲν δέ ἐστιν ἐν τοῖς αἰσθητοῖς μῆκος ἄνευ πλάτους. ἔτι οἱ μὲν λαμβάνουσι τὰ ἄστρα σημεῖά τινα καὶ σημείων ἐπέχειν λόγον ἐν τῷ οὐρανῷ, τὸ δὲ σημεῖόν ἐστιν οὗ μέρος οὐδέν, οὐδὲν δὲ τῶν ἄστρων τοιοῦτον.

[2.5] The crucial M.3 passage (1078a17-31)

> So if one posits objects separate from what is incidental to them and studies them as such, one will not because of this speak falsely, any more than when one draws a line on the ground and calls it a foot long when it is not; for the error is not included in the premises. The best way to investigate each thing would be this: to separate and posit what is not separate, as the arithmetician does and the geometer. For man is, qua man, one and indivisible. But the <arithmetician> first posits an indivisible one, and then studies whether anything follows, qua indivisible, for man. While the geometer does not <study> qua man or qua indivisible but qua solid. For these which would belong to him even if in some way he was not indivisible - it is clear that they may also belong to him without them <= without the presuppositions 'man', 'indivisible'>. So that, because of this, the geometers speak correctly, and they speak about beings, which really are; for being is double: 'entelechy'; and 'as matter'. [135] [*Meta*. M.3, 1078a17-31; Lear/Netz trans.]

[2.5.1] Untangling a highly compressed text

Let us examine the first part of this passage:

> So if one posits objects separate from what is incidental to them and studies them as such, one will not because of this speak falsely, any more than when one draws a line on the ground and calls it a foot long when it is not; for the error is not included in the premises. The best way to investigate each thing would be this: to separate and posit what is not separate, as the arithmetician does and the geometer. [*Meta*. M.3, 1078a17-23, Lear's trans.]

It seems that in this passage Aristotle is primarily concerned with the way the mathematicians deal with propositions that are about a geometrical *individual*.

[135] ὥστ' εἴ τις θέμενος κεχωρισμένα τῶν συμβεβηκότων σκοπεῖ τι περὶ τούτων ᾗ τοιαῦτα, οὐθὲν διὰ τοῦτο ψεῦδος ψεύσεται, ὥσπερ οὐδ' ὅταν ἐν τῇ γῇ γράφῃ καὶ ποδιαίαν φῇ τὴν μὴ ποδιαίαν· οὐ γὰρ ἐν ταῖς προτάσεσι τὸ ψεῦδος. ἄριστα δ' ἂν οὕτω θεωρηθείη ἕκαστον, εἴ τις τὸ μὴ κεχωρισμένον θείη χωρίσας, ὅπερ ὁ ἀριθμητικὸς ποιεῖ καὶ ὁ γεωμέτρης. ἓν μὲν γὰρ καὶ ἀδιαίρετον ὁ ἄνθρωπος ᾗ ἄνθρωπος· ὁ δ' ἔθετο ἓν ἀδιαίρετον, εἶτ' ἐθεώρησεν εἴ τι τῷ ἀνθρώπῳ συμβέβηκεν ᾗ ἀδιαίρετος. ὁ δὲ γεωμέτρης οὔθ' ᾗ ἄνθρωπος οὔθ' ᾗ ἀδιαίρετος ἀλλ' ᾗ στερεόν. ἃ γὰρ κἂν εἰ μή που ἦν ἀδιαίρετος ὑπῆρχεν αὐτῷ, δῆλον ὅτι καὶ ἄνευ τούτων ἐνδέχεται αὐτῷ ὑπάρχειν <τὸ δυνατόν>, ὥστε διὰ τοῦτο ὀρθῶς οἱ γεωμέτραι λέγουσι, καὶ περὶ ὄντων διαλέγονται, καὶ ὄντα ἐστίν· διττὸν γὰρ τὸ ὄν, τὸ μὲν ἐντελεχείᾳ τὸ δ' ὑλικῶς.

That much can be extracted from Aristotle's examples: we have a line considered as one foot long, a man considered as a unit for counting or as a solid for the geometer's purposes. Similar discussion can be found in the following passages (two from the *Analytics* and one from the book N of the *Metaphysics*):

> One should not think that any absurdity results from setting something out. For we do not make use of it insofar as it is a particular thing; instead, it is like the geometer who calls this a foot-long line, this a straight line, and says that they are breadthless, though they are not, but does not use these things as though he were deducing from them.[136] [*Pr. An.* A.41, 49b33-37, Smith's trans.]

> Nor does the geometer supposes falsehoods, as some have said, stating that one should not use a falsehood but that the geometer speaks falsely when he says that the <line> which is not a foot long is a foot long or that the drawn <line> which is not straight is straight. But the geometer does not conclude anything from there being this line which he himself has described, but <from> what is made clear through them.[137] [*Post. An.* A.10, 76b39-77a3; Barnes' trans.]

> That is why it used to be said that you have to assume something false, like geometers when they assume a line to be a foot long when it is not a foot long. But this cannot be right. Geometers do not make any false assumptions (it is not a premise in their reasoning).[138] [*Meta*. N.2, 1089a21-25; Annas' trans.]

Now what Aristotle seems to have in mind is something taken from the actual practice of mathematicians, namely the *ekthesis* part of a typical geometrical proof, the part that has to do with the consideration of an *arbitrary instance*. Consider, for example, the first proposition of Euclid's *Elements* together with the

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[136] Οὐ δεῖ δ' οἴεσθαι παρὰ τὸ ἐκτίθεσθαί τι συμβαίνειν ἄτοπον· οὐδὲν γὰρ προσχρώμεθα τῷ τόδε τι εἶναι, ἀλλ'ὥσπερ ὁ γεωμέτρης τὴν ποδιαίαν καὶ εὐθεῖαν τήνδε καὶ ἀπλατῆ εἶναι λέγει οὐκ οὔσας, ἀλλ' οὐχ οὕτως χρῆται ὡς ἐκ τούτων συλλογιζόμενος.

[137] οὐδ' ὁ γεωμέτρης ψευδῆ ὑποτίθεται, ὥσπερ τινὲς ἔφασαν, λέγοντες ὡς οὐ δεῖ τῷ ψεύδει χρῆσθαι, τὸν δὲ γεωμέτρην ψεύδεσθαι λέγοντα ποδιαίαν τὴν οὐ ποδιαίαν ἢ εὐθεῖαν τὴν γεγραμμένην οὐκ εὐθεῖαν οὖσαν. ὁ δὲ γεωμέτρης οὐδὲν συμπεραίνεται τῷ τήνδε εἶναι γραμμὴν ἣν αὐτὸς ἔφθεγκται, ἀλλὰ τὰ διὰ τούτων δηλούμενα.

[138] διὸ καὶ ἐλέγετο ὅτι δεῖ ψεῦδός τι ὑποθέσθαι, ὥσπερ καὶ οἱ γεωμέτραι τὸ ποδιαίαν εἶναι τὴν μὴ ποδιαίαν· ἀδύνατον δὲ ταῦθ' οὕτως ἔχειν, οὔτε γὰρ οἱ γεωμέτραι ψεῦδος οὐθὲν ὑποτίθενται (οὐ γὰρ ἐν τῷ συλλογισμῷ ἡ πρότασις).

customary Greek divisions of a proposition as provided by Ian Mueller:

> *protasis:* On a given finite straight line to construct an equilateral triangle. *ekthesis:* Let *AB* be the given finite straight line. *diorismos*: Thus it is required to construct an equilateral triangle on the straight line *AB*. *kataskeue:* With center *A* and distance *AB* let the circle *BCD* be described; again, with center *B* and distance *BA* let the circle *ACE* be described; and from the point *C*, in which the circles cut one another, to the points *A*, *B* let the straight lines *CA*, *CB* have been joined. *apodeixis:* Now, since the point *A* is the center of the circle *CDB*, *AC* is equal to *AB*. Again, since the point *B* is the center of the circle *CAE*, *BC* is equal to *BA*. But *CA* was also proved equal to *AB*; therefore each of the straight lines *CA*, *CB* is equal to *AB*. And things which are equal to the same thing are also equal to one another; therefore *CA* is also equal to *CB*. Therefore the three straight lines CA, *AB*, *BC* are equal to one another. *sumperasma:* Therefore the triangle *ABC* is equilateral; and it has been constructed on the given finite straight line *AB*. Which was required to be done.[139] [Mueller (1981), p.11]

In the above example, the *ekthesis* part is when Euclid tells us 'let *AB* be the given finite straight line'. Furthermore, a figure is normally assumed to be drawn in connection with the Euclidean ekthesis.[140] Now one can easily see that the individual letters pick out geometrical points and that pairs of letters pick out the lines bounded by the two points which the letters pick out. So in the

139 Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι. Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ. Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι. Κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ, καὶ πάλιν κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΑ κύκλος γεγράφθω ὁ ΑΓΕ, καὶ ἀπὸ τοῦ Γ σημείου, καθ' ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπὶ τὰ Α, Β σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ. Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστὶν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστὶν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν. Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον, καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς ΑΒ. [Ἐπὶ τῆς δοθείσης ἄρα εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συνέσταται]· ὅπερ ἔδει ποιῆσαι.

140 Hintikka makes a connection between ekthesis and the modern principle of instantiation: 'The prominent role of ekthesis in the geometrical proofs of ancient mathematicians may indeed be considered as an acknowledgement of the importance of instantiation for the kind of logic that is needed in elementary geometry – which is mainly first-order logic (quantification theory).' In [Hintikka(2004), p.147].

Euclidean proof, '*AB*' designates an individual line. But then one may ask which line does it designate? One option is that '*AB*' picks out the line in the geometer's diagram. But there are certain reasons not to think so; apart from the obvious one that pertains to the imprecision problem that tantalises the perceptibles there are also other reasons that have to do with the role of the diagrams in Greek mathematical thought. As Netz in his groundbreaking book *The Shaping of Deduction in Greek Mathematics: a Study in Cognitive History* argues, Greek diagrams were schematic and non-representational.[141] Yet another option is for '*AB*' to pick out a Platonic ideal line. It seems that Aristotle in the M.3 passage *concedes* that the mathematician may have to resort to separate mathematicals, merely for heuristic purposes however. Aristotle claims that 'the best way to investigate each thing would be this: to separate and posit what is not separate' (ἄριστα δ' ἂν οὕτω θεωρηθείη ἕκαστον, εἴ τις τὸ μὴ κεχωρισμένον θείη χωρίσας). What does he mean? Perhaps something along these lines: it is indeed *mostly convenient* to postulate mathematicals as separately existing entities–even though in reality they are not separate (thus the ἄριστα here seems to be a heuristic 'best'). Let us try to elucidate the sense of separation in this passage. One possible interpretation is to take 'separation' to mean 'separation in definition'. However, I do not think this is a plausible claim for this passage; there is no attempt here to formulate a definition for mathematical objects nor is there any discussion about priority in definition. Rather, as E. Hussey points out,

> It seems that the threefold repetition of forms of the verb *tithenai* in the sense of 'posit' seems to show sufficiently (at least for a working hypothesis) that what happens in separation is this: one assumes the separated existence of something that does not in fact exist in separation. The account of 1078al7-28 shows that the arithmetician's separation consists of assuming the existence in separation of, for example, an indivisible unit which is not a man or any other sensible substance but a unit.' [Hussey (2011), pp.116-117]

Accordingly, the geometer's separation consists in assuming the existence in separation of a solid. Since separation in definition seems irrelevant in this context, most commentators understand that this separation involves a *fiction*:

1) <Separation> purports to create or reveal a new type of object, the 'separated

[141] See [Netz (2003), esp. pp.54-57].

> mathematical object', but this is in fact a fiction. [Hussey(2011), p.117]
>
> 2) For Aristotle, abstraction amounts to no more than the separation of one predicate that belongs to an object and the postulation of an object that satisfies this predicate alone. [Lear (1982), p.186]

The similarities between Hussey's analysis of separation and Lear's one terminate here, however. Hussey is quick to adopt a more *radical* approach in his interpretation. According to his analysis of the M.3 discussion, mathematical objects for Aristotle are special sort of objects (what he calls 'representative objects') which are *distinct* from any particular object; in advancing his interpretation, Hussey draws inspiration from Kit Fine's *arbitrary objects*. Fine provides the following characterisation of arbitrary objects:

> In addition to individual objects, there are arbitrary objects: in addition to individual numbers, arbitrary number; in addition to individual men, arbitrary men. With each arbitrary object is asscociated an appropriate range of individual objects, its values: with each arbitrary number, the range of individual numbers; with each arbitrary man, the range of individual men. An arbitrary object has those properties common to the individual objects in its range. So an arbitrary number is odd or even, an arbitrary man is mortal, since each individual number is odd or even, each individual man is mortal. On the other hand, an arbitrary number fails to be prime, an arbitrary man fails to be a philosopher, since some individual number is not prime, some individual man is not a philosopher.[142] [Fine (1985), p.5]

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[142] In this Kit Fine draws inspiration from Locke and other older theories. Cellucci offers the following helpful synopsis of Locke's theory:

> According to Locke, Euclid's proofs of Proposition I.32 is carried out not on an individual triangle but on the 'general triangle', that is, "the general Idea of a Triangle", which "must be neither Oblique, nor Rectangle, neither Equilateral, Equicrural, nor Scalenon; but all and none of these at once" [Locke (1975), p.596]. Once established that, in the general triangle, the three interior angles of the triangle are equal to two right angles, one may conclude that this holds for any triangle, since the properties of the general triangle are common to all triangles, so "he that hath got the" general "Idea of a Triangle" is "certain that its three Angles are equal to two right ones" (ibid., p. 651). General Ideas are obtained from particular objects "leaving out but those particulars wherein they differ, and retaining only those wherein they agree" (ibid., p. 412). This "is called Abstraction, whereby Ideas taken from particular Beings, become general Representatives of all of the same

Thus, for example, when we stipulate that *AB* be an arbitrary triangle, we fix the reference of '*AB*' to an arbitrary triangle, which is an entity *distinct from* any of the familiar particular triangles. Hussey, then, proceeds to advance the following interpretation of Aristotle's philosophy of mathematics: mathematical objects for Aristotle are unseparated arbitrary objects, in the sense outlined in the above passage.[143] The geometer proceeds to separate such objects in thought in order to facilitate his proofs. Is this really a viable interpretation? Apart from the lack of textual evidence–something that Hussey himself acknowledges,[144] there is also the fact that those objects are susceptible to serious logical objections:[145] For example, take a representative number. Then it is odd or even, since every individual number is odd or even. But it is not odd, since some individual number is not odd; and it is not even, since some individual number is not even. Therefore, a representative number has inconsistent properties: it is both odd or even and neither odd nor even.[146] Or take a representative triangle. Then it is isosceles or not isosceles. But it isn't isosceles, since some individual triangle is not isosceles; and it isn't not-isosceles, since some individual triangle is isosceles. Therefore it is isosceles or not-isosceles, and neither isosceles nor not-isosceles. Several other problems (besides the actual textual evidence) remain. As Fine points out, it would be a mistake to think that arbitrary objects are *on par* with particulars, for then 'one can say the same sort of things about each. So one is led to the absurd conlusion that one might count with arbitrary objects or have tea with an arbitrary man'.[147] Consider an arbitrary natural number. The arbitrary number is not an individual number although it should be by definition given

kind" (ibid., p. 159). In [Cellucci (2009), p.4].

For a brief overview of similar theories consult [Santambrogio(1988), pp.630-631].

143 '…*unseparated* representative objects, which according to this interpretation are the proper objects of mathematics…' in [Hussey(2011), p.128].

144 'Neither here <i.e. in M.3> nor elsewhere does Aristotle state expressly that he takes mathematical objects to be representative objects.' In [Hussey(2011), p.120].

145 Something that Hussey also acknowledges; ibid., p.119.

146 This is an objection that goes back to Berkeley as Kit Fine himself acknowledges. In [Fine (1985), p.9].

147 In [Fine(1985), p.8].

that all numbers share the property of 'being an individual number'.[148] Fine solves this problem by distinguishing between *generic* properties (such as 'being odd' or 'being even') and *classical* ones (such as 'being an individual number'). 'Being an individual number' is thus not attributed to an arbitrary number, because it is not a generic property. Could such an ad–hoc distinction be attributed to Aristotle? Hussey, boldly claims yes. He also seems to understand that arbitrary (or 'representative', in his terminology) objects are *on par* with ordinary particulars:

> Every actual triad is a triad of distinct actual individuals. And, again, that property seems to be essential to its being a triad. So then also must the representative triad be a triad of distinct 'representative' individuals, which are its 'matter'. [Hussey(2011), p.130]

It seems then that the property 'consists of distinct individuals' is according to Hussey a generic property and thus it is properly attributed to the arbitrary triad. Consider now a triangle. In every triangle the sum of its interior angles is 180^{o}. This is an essential property of triangles. If we follow Hussey's reasoning then one should claim that the arbitrary triangle must have some arbitrary interior angles whose sum is 180^{o}. But, as John Macnamara objects, 'what would ground the belief that the sum of the three arbitrary angles is 180?'[149]

Hussey's primary reason for the postulation of arbitrary objects is the same one Fine uses to develop his theory of arbitrary objects; namely that stipulations such as 'let *ABC* be a triangle', or 'let *n* be a number' and so on, are involved in the application of the rule of *universal generalisation*. As Fine says, 'we may establish that all objects of a certain kind have a given property by showing that an arbitrary object of that kind has that property.'[150] Consider Euclid's proof of Proposition I.32: In any triangle, the interior angles are equal to two right angles. The 'ektheis' part is the following: Let *ABC* be a triangle. Then Euclid proceeds to show that the interior angles of *ABC* are equal to two right angles. He can then conclude that, since *ABC* is an arbitrary triangle, all triangles have their interior

[148] op. cit., p.12.

[149] In [Macnamara(1988), p.305].

[150] See [Fine(1985), p.1]. See also [Hussey(2011), pp.126-127].

angles equal to two right angles. There is however the following problem related to this rule, the so-called 'problem of *universal generalisation*. Celluci offers the following formulation of the problem: 'Generally, what entitles one to conclude that a property, established for an individual object, holds for any individual object of the same kind?'[151] Burnyeat explains the situation for Greek logic:

> Greek logic does not command an explicit formulation of the rule of universal generalisation: from 'f*a*', where '*a*' denotes any arbitrarily selected individual, infer '(*x*)f*x*'. We may say that Euclid is relying implicitly on the rule when, after proving that Pythagoras' theorem is true of the triangle *ABC*, he infers that it is a general truth. But this, with its *caveat* 'implicitly', does little more than record our conviction, in our own terms, that the inference is valid. Euclid simply does not tell us what he thinks it is about the triangle *ABC* which entitles the mathematician to his general conclusion. [Burnyeat (1987), pp.230-231]

Burnyeat invites us to consider the following passage from Proclus in which he seemingly describes such rule of universal generalisation:

> Furthermore, mathematicians are accustomed to draw what is in a way a double conclusion. For when they have shown something to be true of the given figure, they infer that it is true in general, going from the particular to the universal conclusion. Because they do not make use of the particular qualities of the subjects but draw the angle or the straight line in order to place what is given before our eyes, they consider that what they infer about the given angle or straight line can be identically asserted for every similar case. They pass therefore to the universal conclusion in order that we may not suppose that the result is confined to the particular instance. This procedure is justified, since for the demonstration they use the objects set out in the diagram not as these particular figures, but as figures resembling others of the same sort. It is not as having such-and-such a size that the angle before me is bisected, but as being rectilinear and nothing more. Its particular size is a character of the given angle, but its having rectilinear sides is a common feature of all rectilinear angles. Suppose the given angle is a right angle. If I used its rightness for my demonstration, I should not be able to infer anything about the whole class of rectilinear angles; but if I make no use of its rightness and consider only its rectilinear character, the proposition will apply equally to all angles with rectilinear sides. [*In Euc. Elem. I* 207.4–25; trans. Morrow]

[151] In [Cellucci(2009), p.4].

The terms 'set out' in this passage are individual geometrical objects.[152] How could this passage make sense in conjuction with what Aristotle tells us about the need for separate mathematicals in mathematical practice? Lear tells us that the fiction of separate mathematicals allows us to attain *generality*:

> The postulation of separated geometrical objects enables us to attain knowledge that is more general. And it is through this general knowledge that one can discover the explanation (aitia) of why something is the case. For by abstracting one can see that the full explanation of a triangle's having the 2R property is that it is a triangle and not, say, that it is bronze or isosceles (cf. *Post. An*. A.5). In a limited sense, though, the abstract proof is unnecessary. For of any particular physical triangle *d* we can prove that it has interior angles equal to two right angles without first proving this <for a separate triangle> *c*: we could prove that *d* has the property directly. The proof that a physical object possesses a geometrical property via a proof that a pure geometrical object possesses that property is a useful but unnecessary, detour. However, if we want to know why the object possesses the property, the abstract proof is of crucial importance. [Lear (1982), pp.174-175; underlining mine]

> The reason that the 'crossing' is valuable though is that one thereby proves a general theorem applicable to all triangles rather than simply proving that a certain property holds of a particular triangle. [Lear(1982), pp.187-188; underlining mine]

It seems then that Hussey's view is not supported by the available evidence on the problem of universal generalisation. But let us return to the discussion of the M.3 passage. One the one hand, Aristotle says, a man qua man is one and indivisible. But the arithmetician first posits an indivisible one, and then studies whether anything follows, qua indivisible, for man. What does this mean? The arithmetician simply posits an indivisible unit (or better, indivisible units) and he examines the attributes of those units (say, being an odd number of units) which are the same as the attributes of men, say, qua indivisible units. Aristotle is being more explicit with the geometrical example:

> For these which would belong to him even if in some way he was not indivisible - it is clear that they <=divisibility properties> may also belong to him without them <= without the presuppositions 'man', 'indivisible'>. [*Meta*. M.3, 1078a26-28; Netz trans.]

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[152] See [Barnes (2007), p.349].

Aristotle tries to express himself via a counterfactual conditional. The most detailed exposition of Aristotle's complex argument is provided by Netz:

> 1)We are given man as it is: indivisible. 2) We envisage another, counterfactual, not-indivisible-man. 3) We derive a set of '*a* (counterfactually) *not-indivisible-man*'s properties'. 4) We assume that this set is a subset of 'actual man properties'. (This is valid on these assumptions: that counterfactually not-indivisible-man differs from actual man by the *lack* of a defining property, and that sets of properties are a direct, monotonic projection from sets of defining properties: add defining properties and you add some extra, derived properties, remove defining properties and you remove some extra, derived properties.) 5) We now go back to actual man. We know that we have a subset of his properties, discovered through the counterfactual route: we may simply attach to him this subset, and equate this with the subset of properties which belong to man qua not indivisible. [Netz (2006), p.34; his italics]

Netz concludes:

> In other words: there is no distinction between (i) studying a counterfactual divisible man, and (ii) studying man as he actually is, but without the assumption of divisibility. The two routes converge: there is no duty on counterfactuality. [Netz(2006), p.29]

Netz in a way expands on Lear's analysis: Consider a spherical body; it is always enmattered. Aristotle argues that whatever holds true of the counterfactual, non-enamattered sphere, the same holds true of the sphere qua not being enmattered. Let us recap the discussion of the M.3 passage: Aristotle acknowledges that in a geometrical proof we find ourselves uttering sentences that seem to be committed to separate mathematicals; sentences that could not be true unless separate mathematicals existed. But Aristotle does not believe that mathematical proofs are really about entities that enjoy separate existence from the sensibles. Aristotle presumably thinks that those sentences are advanced in a fictional spirit. We make *as if* there are separate mathematicals that satisfy those sentences so as to draw some useful results. The geometrical propositions are *really* about actual/potential mathematicals, mathematicals that do not enjoy separate existence from the sensibles. Netz encapsulates brilliantly the above points:

> There seem to be, for him, three kinds of objects in a mathematical proof: 1. The token

> sign (A line in the diagram). 2. The token signified (A Platonic, ideal line). 3. The type of things for which the conclusion holds true. . . . *neither* of the tokens 1 and 2 belong to the type 3. The token sign does not belong to the type of things for which the conclusion holds, and this can be proved, I think, on the basis of a reconstruction of the nature of Greek diagrams, which were schematic and non-representational. As for the ideal Platonic object, we just saw that this is not one of the things of which the conclusion holds, for the reason that it does not exist and is merely a virtual object set up as a heuristic tool. [Netz(2006), pp.25-26]

[2.5.2] The metaphysical priority of bodies

Aristotle's argument against mathematical objects in the sensibles need not be understood as an argument against the *potential* existence of mathematical objects within sensible ones. Let us consider again the following crucial passage of the M.3:

> While the geometer does not <study> qua man or qua indivisible but qua <solid>. For these which would belong to him even if in some way he was not indivisible - it is clear that they may also belong to him *without* them <= without the presuppositions 'man', 'indivisible'>. So that, because of this, the geometers speak correctly, and they speak about beings, which really are; for being is double: 'entelechy'; and 'as matter'. [*Meta*. M.3, 1078a25-31; Netz' trans.]

In the geometrical case, I can examine Plato *only in virtue of being solid*. For this is a property that he actually holds (1078a25-26). But consider what can follow from this examination: Plato qua solid is divisible into two parts. But then, it is not clear that Plato qua solid is the subject matter of geometry (for it is true of him that he is a solid and divisibility is an essential property of solids) or of arithmetic (since one might say that what follows from his being solid is that he is divisible into two half-parts, or into three third-parts, etc.). But I do believe this is precisely the point Aristotle wishes to make. In some sense, the continuity of the solid, grounds the *potential existence* both of other geometricals and of numbers.

It is not bizarre to suggest that Aristotle is laying in M.3 the foundations for a geometry of solids, of three-dimensional bodies. The discussion of the relevant

M.2 passages earlier in this essay revealed that Aristotle considers bodies to have much greater metaphysical priority when compared to lower-dimensional entities such as surfaces and points. In this section we will examine a certain passage from the first chapter of the first book of *De Caelo* that contains a rather detailed analysis of the notion of body. The focal point for our analysis will be a very helpful paper entitled 'The Perfection of Bodies: Aristotle's *De Caelo* I.1' by Gábor Betegh, Francesca Pedriali, and Christian Pfeiffer (henceforth Betegh et al.). In this article, the authors argue, Aristotle demonstrates 'that bodies are complete and perfect in virtue of being extended in three dimensions'.[153] For the purposes of my inquiry I will follow closely the authors' argument as well as their division of the chapter. In the first few lines of the chapter (268a1–6) Aristotle tells us that the subject matter of physical science includes bodies, their attributes and their principles (cf. *Physics* B.2, 193b22-194a15). In the next part of the chapter (268a6–268b5) Aristotle proceeds to discuss the notion of body (focusing specifically on the perfection or completeness of bodies).[154] The most relevant bit for my purposes is the following, where Aristotle gives us a definition of body:

> Hence continuous is that which is divisible into ever divisible parts, body is that which is divisible in every way. Of magnitude, that which is extended in one dimension is a line, that which is extended in two is a surface and that which is extended in three dimensions is a body. There is no other magnitude beyond these, since the three is all and the thrice is in every way.[155] [*De Caelo* I.1, 268a6–10; trans. by Betegh et al.]

Aristotle's definition of 'body' is based on the notions of continuity and divisibility.[156] More specifically, Aristotle first defines the continuous (συνεχές) in terms of which body and the other lower-dimensional magnitudes will be defined; the continuous is that which is divisible into ever divisible parts (συνεχὲς μὲν οὖν ἐστι τὸ διαιρετὸν εἰς ἀεὶ διαιρετά); the line is that which is

[153] In [Betegh et al. (2013), p.30].

[154] See [Betegh et al., op. cit., pp.31-32].

[155] Συνεχὲς μὲν οὖν ἐστι τὸ διαιρετὸν εἰς ἀεὶ διαιρετά, σῶμα δὲ τὸ πάντῃ διαιρετόν. Μεγέθους δὲ τὸ μὲν ἐφ' ἓν γραμμή, τὸ δ' ἐπὶ δύο ἐπίπεδον, τὸ δ' ἐπὶ τρία σῶμα· καὶ παρὰ ταῦτα οὐκ ἔστιν ἄλλο μέγεθος διὰ τὸ τὰ τρία πάντα εἶναι καὶ τὸ τρὶς πάντῃ.

[156] In [Betegh et al., op. cit., pp.38].

continuous in one dimension, a plane that which is continuous in two dimensions, a body that which is continuous in all (three) dimensions.[157] The passage concludes with the enigmatic claim that there is no other magnitude beyond these because 'the three is all and the thrice is in every way' (τὸ τὰ τρία πάντα εἶναι καὶ τὸ τρὶς πάντῃ). According to Betegh et al., Aristotle's elliptic remark ought to be understood as 'being extended in three dimensions and being divisible in three ways is being extended in *all* dimensions and being divisible in *all* ways'.[158] To justify their claim, the authors point to lines 268a10-20 where Aristotle makes a connection between 'three' and 'all' by employing peculiar means; as the authors note, 'Aristotle apparently cannot do better but appeal to an alleged Pythagorean doctrine, to a set of cult practices, and to Greek linguistic usage.'[159] What is of crucial importance for my interpretation is that after this connection has been established Aristotle invokes, as the authors point out, a hierarchy among the types of magnitude, with 'body' being on top: it is the only type of magnitude which can be considered 'perfect'. We will come to this passage after an examination of Aristotle's term τέλειον .

The entry for the term τέλειον in Aristotle's philosophical lexicon is the following:

> We call complete: 1) in one sense, that outside which not even one part is to be found, as for instance the complete time of each thing is that outside which there is not time to be found which is part of that time. 2) Also, that which in respect of excellence and goodness

[157] Betegh et al. note that certain commentators (the authors mention Wildberg (1988) as a prominent example) have expressed their sheer puzzlement over Aristotle's focus on mathematical characteristics of bodies such as three-dimensionality and divisibility. According to those commentators, Aristotle somehow vacillates between a conception of bodies as physical entities and a mere mathematical one (physical bodies/geometrical solids ?). Betegh et al., correctly–in my opinion–resist such interpretations: they point out that Aristotle is not switching from physical project to a mathematical one but that he is merely highlighting certain characteristics of perceptible magnitudes that both the physicist and the mathematician study (continuity in three-dimensions, divisibility). See [Betegh et al. (2013), pp. 35-36] for a discussion.

[158] In [Betegh et al., *op. cit.*, p.37].

[159] See [Betegh et al., *op. cit.*, p.39 and pp.39-44] for an extensive discussion.

> cannot be surpassed relative to its genus, as for instance a doctor is complete and a flautist is complete when they are without deficiency in respect of the form of their own proper excellence. It is in this way that, transferring it to the case of bad things, we speak of a complete slanderer and a complete thief–as indeed we even call them good: a good thief and a good slanderer. And excellence is a kind of completion, for each thing is complete and every substance is complete when in respect of the form of its own proper excellence no portion of its natural magnitude is deficient. 3) Again, things which have reached their fulfillment, when it is worth while, are called complete, for they are complete by virtue of having attained their fulfillment; so that, since a fulfillment is something ultimate, we also say, transferring it to the case of worthless things, that a thing has been completely spoilt and completely destroyed when there is no deficiency in its destruction and badness but it has reached the ultimate. (That is why even life's end is metaphorically called a fulfillment, because both are ultimate. A <thing's> fulfillment, i.e. what is it for, is ultimate.) [*Meta*. Δ.16, 1021b12-30; Kirwan's trans. mod.]

Aristotle says, first, that in one sense a thing is said to be complete outside of which it is impossible to find any of its parts. For example, a period of time (a day, say) is said to be complete when none of its parts can be found outside of it. As Betegh et al. note, their translation of τέλειον as 'complete' corresponds to the first sense of the term in *Metaphysics* Δ.16: 'insofar as body is divisible and extended in *all* the dimensions in which a magnitude can be extended and divided, body is a complete magnitude.'[160] There is also a second sense of τέλειον, with an axiological undertone. According to this second sense, perfect is said that 'which which in respect of excellence and goodness cannot be surpassed relative to its genus' (καὶ τὸ κατ' ἀρετὴν καὶ τὸ εὖ μὴ ἔχον ὑπερβολὴν πρὸς τὸ γένος, 1021b14-15). Thus a man is said to be a perfect doctor or a perfect flautist when he lacks nothing pertaining to the particular ability in virtue of which he is said to be a good physician or a good flute player. As Betegh et al. note, 'insofar as no further magnitude can surpass body, body is the perfect magnitude according to the second meaning of τέλειον'.[161] Yet another sense of τέλειον pertains to the fulfilment of a goal. A special case pertains to a goal which is 'worth seeking' (σπουδαῖον). Since this third sense of τέλειον concerns things that have reached a

[160] In [Betegh et al., *op. cit.*, p.44].

[161] ibid.

final state, it can also be applied in a secondary manner to things that have been completely spoiled or destroyed. For example, a thing is said to be perfectly spoiled or corrupted when 'there is no deficiency in its destruction and badness but it has reached the ultimate' (ὅταν μηδὲν ἐλλείπῃ τῆς φθορᾶς καὶ τοῦ κακοῦ ἀλλ' ἐπὶ τῷ ἐσχάτῳ ᾖ, 1021b27-28). And this is the reason why death is, somewhat metaphorically, called an end, because it is something final.[162]

In *De Caelo* I.1, 268a20-28 Aristotle argues that bodies possess a privileged position when compared to lower-dimensional entities: it is the only type of magnitude which can be considered 'perfect':

> Therefore, since 'every' and 'all' and 'complete' do not differ from one another in respect of form, but only, if at all, in their matter and in that to which they are applied, body alone among magnitudes can be complete. For it alone is determined by the three dimensions, that is, is an 'all'. But if it is divisible in three dimensions it is every way divisible, while the other magnitudes are divisible in one dimension or in two; for the divisibility and continuity of magnitudes depend upon the number of the dimensions, one sort being continuous in one direction, another in two, another in all.[163] [*De Caelo*, I.1, 268a20-28; Stock's trans.]

Betegh et al. offer the following reconstruction of Aristotle's argument:

> (1) Magnitudes are defined by the number of the dimensions in which they are extended and divisible. (2) Body is defined by 'three'. (3) 'Three' implies 'all'. (4) 'All' implies 'complete and perfect'. (5) Hence, body is complete and perfect. (6) The other magnitudes, in contrast to body, are not complete and perfect, since they are defined by 'one' (line) or 'two' (surface) respectively.[164]

And based on the M.3 and the *De Caelo* passages we can, as Betegh et al. do,

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[162] If we invoke the Platonic generation of solids in *Meta*. M.2, 1077a24-31, we may also call bodies 'complete' in this third sense. They have also attained a goal 'worth seeking' in that they can become animate (1077a28-29).

[163] Ὥστ' ἐπεὶ τὰ πάντα καὶ τὸ πᾶν καὶ τὸ τέλειον οὐ κατὰ τὴν ἰδέαν διαφέρουσιν ἀλλήλων, ἀλλ' εἴπερ, ἐν τῇ ὕλῃ καὶ ἐφ' ὧν λέγονται, τὸ σῶμα μόνον ἂν εἴη τῶν μεγεθῶν τέλειον· μόνον γὰρ ὥρισται τοῖς τρισίν, τοῦτο δ' ἐστὶ πᾶν. Τριχῇ δὲ ὂν διαιρετὸν πάντῃ διαιρετόν ἐστιν· τῶν δ'ἄλλων τὸ μὲν ἐφ' ἓν τὸ δ' ἐπὶ δύο· ὡς γὰρ τοῦ ἀριθμοῦ τετυχήκασιν, οὕτω καὶ τῆς διαιρέσεως καὶ τοῦ συνεχοῦς· τὸ μὲν γὰρ ἐφ' ἓν συνεχές, τὸ δ' ἐπὶ δύο, τὸ δὲ πάντῃ τοιοῦτον.

[164] In [Betegh et al., *op. cit.*, p.48].

attribute to him a mathematical realism of the following kind:

> The basic properties of being <three-dimensionally> extended and being infinitely divisible are properties that physical bodies have in a precise and realistic way.[165]

[2.5.3] Ian Mueller's interpretation of the M.3 passage and its background

The interpretation I would like to offer for the crucial *Metaphysics* M.3 passage allows for the potential existence (ὑλικῶς has the meaning of δυνάμει) of mathematical entities, *on the basis of the fundamental concept of the solid*. Before examining the implications of this interpretation, it is important to discuss certain other views regarding Aristotle's philosophy of mathematics. A very influential interpretation with many followers is that of Ian Mueller;[166] in his article 'Aristotle's doctrine of abstraction in the commentators' he summarises his position as follows:

> Mathematical objects are embodied in pure extension underlying physical objects; the geometer's abstraction of non-geometric properties enables him to apprehend these things which satisfy the mathematician's definitions. [Mueller (1990), pp.464-465]

According to Mueller, Aristotle seems to endorse a conception of mathematical entities, 'not as matter-less properties, but as substance-like individuals with a special matter-intelligible matter'.[167] Mueller attributes the former view, that for Aristotle mathematical entities are forms that can be separated in thought from matter, to the interpretations of the *Physics* B.2 passage advanced by Philoponus and Simplicius.[168] To properly understand Mueller's claim that 'mathematical objects are embodied in pure extension underlying physical objects' we need to keep in mind that he identifies this 'pure extension' with intelligible matter. According to his interpretation, the matter of geometrical objects is the matter of

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165 In [Betegh et al., *op. cit.*, p.61].

166 Mueller's influence is prominent in [Modrak (1989)] and [Menn, 'Iγ3']. Annas is initially sympathetic to Mueller's account; she dismisses it, however, rather briefly as a 'reconstruction' that 'depends fairly heavily on the later Greek commentators. See [Annas (1975), pp.30-31].

167 See [Mueller (1970), p.164].

168 In [Mueller, *op. cit.*, p.162].

physical objects with some of its properties abstracted away and only extension left.[169] Intelligible matter is, in his view, pure and indeterminate extension, 'potentially' existent in physical objects, and can be abstracted from any sensible body. It is this extension that constitutes the substratum for the various specific mathematicals:

> For by abstraction one eliminates all sensible characteristics and arrives at the idea of pure extension. Pure extension does not seem to be sensible in the way that triangularity is, nor is it completely undifferentiated or purely potential in the way that prime matter seems to be. We cannot see a thing as just extended but only as extended so and so much with a certain shape. Simple extendedness we must grasp rationally. Geometric properties are imposed on this intelligible matter, but these properties are not the approximate properties of sensible substances precisely because they are *imposed* upon intelligible matter. The resultant objects are still intelligible rather than sensible. [Mueller (1970), pp.168-169; italics mine]

One might follow Bostock in formulating the following, in my opinion devastating, objection to Mueller's view: if this so-called 'pure extension' is 'supposed to be *perfectly* square, spherical, and so on, in what sense does it 'underlie' the material objects that are only *imperfect* examples of these properties?'[170] And if a perfect three-dimensional sphere somehow underlies a non-perfect sensible (a bronze, say) one, isn't this susceptible to Aristotle's criticisms that no two solids can occupy the same place at the same time? What is particularly problematic in Mueller's account is his talk of the *imposition* of some determinate mathematical property on this indeterminate 'pure extension':

> Thus it becomes necessary to distinguish two kinds of geometric object in Aristotle. First, there are the basic objects: points, lines, planes, solids. The last three are conceived of as indeterminate extension and, therefore, as matter on which geometric properties are *imposed*. The *imposition* of these properties produces the ordinary geometric figures, straight or curved lines, triangles, cubes; etc. The definition of such a figure will include

[169] I follow Menn's charitable understanding of Mueller's account, in [Menn, 'Iγ3', fn.76, pp. 28-29].

[170] In [Bostock (2012), p.477].

> both the form, the properties imposed, and the matter; but in the definition this matter will also play the role of genus. A circle is a plane figure. [Mueller (1970), pp.167]

It seems to me that Mueller's talk of 'imposition' implies that mathematical entities are somehow *mental entities*, a conception which is hard to reconcile with Aristotle's realistic tendencies expressed in the crucial M.3 passage. What does Mueller mean when he claims that the geometer 'imposes' geometric properties on some indeterminate extension (what Mueller calls 'basic objects', i.e. points, lines, planes, solids)? How is this *different* from bringing that entity to actuality by the mind? Serious epistemic objections also arise: imposition presupposes knowledge of what is to be imposed, that is, of the specific geometrical properties. Mueller in his article does not elaborate on how we come to have this knowledge: Is it perhaps the case that we possess some previous understanding of mathematical principles and concepts which then we project on this 'indeterminate extension'? If so, then–as commentators have argued–it seems that Mueller does not offer a genuine alternative to the Platonic account of mathematical knowledge:

> Aristotle does accept Plato's mathematical epistemology: mathematicians treat objects which are different from all sensible things, perfectly fulfill given conditions, and are apprehensible by pure thought. [Mueller (1970), p.157][171]

For the purposes of my interpretation I simply note that I espouse Lear's treatment of intelligible matter, namely that perceptible objects

> …have intelligible matter insofar as they can be objects of thought rather than perception: that is, it is the object one is thinking about that has intelligible matter. The evidence for this is Aristotle's claim that intelligible matter is 'the matter which exists in perceptible objects but not as perceptible, for example, mathematical objects' (1036a11-12). [Lear (1982), pp.182-183]

Perhaps the invocation of intelligible matter as the matter of mathematicals has

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[171] Jonathan Lear offers the following scathing criticism of Mueller's account: 'This interpretation must view Aristotle as caught in the middle of a conjuring trick: trying to offer an apparently Platonic account of mathematical knowledge while refusing to allow the objects that the knowledge is knowledge of.' In [Lear (1982), p.161].

its origins in certain passages from *De Anima* Book III where Aristotle says that the forms of sensible things are only potentially intelligible (ch.4, 430a6-7; ch.8, 431b24-28).[172] However, what Aristotle says there need not be interpreted as a claim about the metaphysical status of mathematical entities. One can follow Mignucci in arguing that

> the forms of sensible things are potentially intelligible in the sense that they cannot be thought of if their representations are not brought to actuality. <u>The process of actualisation therefore concerns the condition of forms with respect to a mind which thinks of them, but not their ontological status.</u> To think of a dog does not imply that the form of the dog is contained only potentially in the dog and that it is brought to actuality by the thinker. The form of the dog is actually in the individual dog. [Mignucci (1987), p.183; underlining mine]

Or we can, perhaps, trace the origins of Mueller's interpretation to Syrianus' commentary of Aristotle's *Metaphysics* M and N. Regarding the views of Syrianus Stephen Menn says the following:

> <There is a certain> interpretive tradition on M.3 1078a26-31, going back at least to Syrianus, and apparently to a text of the authentic Alexander that Syrianus is using, that takes Aristotle to mean that mathematical objects (whether geometrical or arithmetical) *are potentially in sensible things, and are made actual by the intellect's act of contemplating them*. [Menn, 'Iγ3', fn.76, p.29, italics mine]

The key texts from Syrianus' commentary are the following:

> 1. <The mathematical objects> acquire whatever existence they possess in some other way – that is, they are generated in us by abstraction, which is in fact his own <i.e. Aristotle's> view.[173] [Syrianus: *On Aristotle Metaphysics M-N*, 84.12-14; Dillon and O'Meara trans.]
>
> 2. Having said in what way he does not think that the objects of mathematics exist, now he undertakes to tell us what sort of existence one might suppose them to have. His preferred view is that mathematical magnitudes and figures neither exist on their own nor in sense-objects while being distinct from sense objects, but that they are derived conceptually

[172] See [Mignucci (1987), p.183].

[173] ἢ ἄλλῳ τρόπῳ τὴν οἵαν ποτὲ ὑπόστασιν αὐτὰ λαμβάνειν, τουτέστιν ἐξ ἀφαιρέσεως ἐν ἡμῖν γεννᾶσθαι, καθάπερ αὐτῷ καὶ δοκεῖ.

from sense objects by abstraction.[174] [*On Aristotle Metaphysics M-N*, 94.30-34; Dillon and O'Meara trans.]

3. <Aristotle's> reply is that they <the mathematicians> are dealing with things that do not exist in actuality, but potentially.[175] [*On Aristotle Metaphysics M-N*, 99.21-22; Dillon and O'Meara trans.]

Syrianus seems to be opting for a mentalist reading of the crucial M.3 passage: the mathematical objects are the product of our abstractive intellectual power (84.12-13: ἐξ ἀφαιρέσεως ἐν ἡμῖν γεννᾶσθαι), derived from the objects of sense via abstraction (94.33-34: ἐκ τῶν αἰσθητῶν κατὰ ἀφαίρεσιν ἐπινοεῖσθαι). Menn's helpful remarks shed some light on the Alexandrian origins of Syrianus' interpretation:

This tradition is connected with Alexander's theory of the νοῦς ποιητικός as abstracting from matter, and making what is potentially intelligible in a sensible thing (or in a phantasma) actually intelligible, whether as a universal form or as a mathematical, each of which would exist in the potential intellect; universals and mathematicals would thus have a foundation in bodies, but formally exist and be completed only in a soul. [Menn, 'Iγ3', fn.76, p.29, italics mine]

Whether this anti-realistic interpretation of mathematical entities (and universals in general) can be rightly attributed to Alexander or not,[176] it is a position that Syrianus raises a barrage of objections against:

Our reply to this must be, first of all, to ask what it is that brings figure and magnitude from potentiality to actuality. For the geometrician does not cognise the potential by keeping it potential, but by making it actual; and if this is so, he does so by giving it shape and making it more exact and perfect. How, then, could he do this if he did not possess actualised entities within himself? For it is your principle, Aristotle, that the potential is only brought to perfection and actuality by the actual. And then again, geometry cannot take all its data from sensible objects; for it deals with many shapes and attributes of

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[174] Εἰπὼν πῶς οὐκ οἴεται εἶναι τὰ μαθηματικά, νῦν πειρᾶται λέγειν ποίαν ἄν τις αὐτὰ νομίσειεν ἔχειν ὑπόστασιν. ἀρέσκει οὖν αὐτῷ μήτε καθ' αὑτὰ ὑφεστάναι τὰ μαθηματικὰ μεγέθη καὶ σχήματα μήτε ἐν τοῖς αἰσθητοῖς εἶναι ἄλλα ὄντα παρὰ τὰ αἰσθητά, ἀλλ' ἐκ τῶν αἰσθητῶν κατὰ ἀφαίρεσιν ἐπινοεῖσθαι·

[175] ὁ δέ φησι περὶ μὴ ὄντων μὲν ἐνεργείᾳ, δυνάμει δ' ὄντων.

[176] I am hesitant to ascribe this anti-realistic view of mathematical objects to Alexander but I cannot develop further that thought here.

> shapes which are not to be found in the sensible world. And again, if these things exist in actuality in the sensible realm (for it is in this sense that the Aphrodisian interprets the text here), while being studied in themselves only potentially, how can what is potential be more exact than what is actual?[177] [*On Aristotle Metaphysics M-N*, 99.31-100.5; Dillon and O'Meara trans.]

Syrianus reasonably complains that Aristotle does not elaborate on the potentiality he has in mind. If this is a potentiality akin to the potentiality of a statue of Hermes within a block of wood, a potentiality ultimately analysable in terms of actual existence, then the role of the geometer is akin to the that of the sculptor who can carve the wood and produce a separate statue of Hermes. If Aristotle's position regarding mathematical entities is that they exist only potentially in physical objects and are made actual by the mathematician's mind then he has to provide an account of what is the function of the mind in this process, an account that will presumably explain the source of the 'perfection' of those objects. Could they be those entities that an Ideal Geometer could construct? Syrianus seems particularly troubled by the fact that sensible objects may fail to have the required precision, i.e. they may fail to completely satisfy the geometer's definitions (100.1-3: ἔπειτα δὲ οὐδὲ πάντα ἀπὸ τῶν αἰσθητῶν δύναται λαμβάνειν ἡ γεωμετρία· πολλὰ γὰρ σχήματα καὶ πάθη θεωρεῖ σχημάτων, ἃ ὁ αἰσθητὸς κόσμος οὐχ ὑποδέδεκται.). Syrianus complains that one cannot confer precision to mathematicals enjoying potential existence in the sensibles, unless one already possesses an adequate understanding of what is for a mathematical object to be precise (99.34-35: πῶς ἂν οὖν δύναιτο ταῦτα ποιεῖν μὴ ἔχων τὰ ἐνεργείᾳ ἐν ἑαυτῷ;). This is made clearer in the following passage:

> In general, in response to <Aristotle's> overall view it must be said that we also do not observe all shapes or all numbers as being inherent in sensible objects, that is to say, all

[177] πρὸς ἃ ῥητέον, ὅτι πρῶτον μὲν τί τὸ ἀπὸ τοῦ δυνάμει εἰς ἐνέργειαν ἄγον τὸ σχῆμα καὶ τὸ μέγεθος; οὐ γὰρ δὴ τὸ δυνάμει φυλάξας δυνάμει νοεῖ ὁ γεωμέτρης, ἀλλ' ἐνεργείᾳ αὐτὸ ποιήσας· εἰ δὲ τοῦτο, μορφοῖ αὐτὸ καὶ ἀκριβέστερον ποιεῖ καὶ τελειοῖ. πῶς ἂν οὖν δύναιτο ταῦτα ποιεῖν μὴ ἔχων τὰ ἐνεργείᾳ ἐν ἑαυτῷ; σὸν γάρ ἐστιν, ὦ Ἀριστότελες, ὅτι ὑπὸ μόνου τοῦ ἐνεργείᾳ τὸ δυνάμει τελειοῦται καὶ εἰς ἐνέργειαν ἄγεται. ἔπειτα δὲ οὐδὲ πάντα ἀπὸ τῶν αἰσθητῶν δύναται λαμβάνειν ἡ γεωμετρία· πολλὰ γὰρ σχήματα καὶ πάθη θεωρεῖ σχημάτων, ἃ ὁ αἰσθητὸς κόσμος οὐχ ὑποδέδεκται. εἶτα <εἰ> ἐνεργείᾳ μέν ἐστιν ἐν τοῖς αἰσθητοῖς ταῦτα (οὕτω γὰρ ὁ Ἀφροδισιεὺς τοῦτο τὸ ῥητὸν ἐξηγεῖται), δυνάμει δὲ θεωρεῖται καθ' ἑαυτά, πῶς ἀκριβέστερόν ἐστι τὸ δυνάμει τοῦ ἐνεργείᾳ;

> those with which the mathematical sciences concern themselves, nor is it possible that things that derive from sense-objects should enjoy such precision. And if he were to explain that we ourselves add to them what is lacking and thus make them more exact and then contemplate them as such, he will have to tell us first of all whence we are able to confer perfection on these; for we would not find any other truer cause of this than that propounded by the ancients, that the soul in its essence has prior possession of the reason-principles of all things.[178] [*On Aristotle Metaphysics M-N*, 95.29-36; Dillon and O'Meara trans.]

The last sentence of the previous passage reveals Syrianus' neo-Platonic agenda. Christian Widlberg has offered the following concise synopsis of Syrianus' position regarding the metaphysical status of mathematicals:

> What in fact happens, according to Syrianus, is that the human intellect possesses an innate understanding of mathematical principles and concepts which it projects onto the plane of our imagination from above in order to grasp them rationally as the substances that they are: "… geometry aims to contemplate the soul's partless reason-principles (*logoi*) themselves but, being too feeble to employ these intellections, which are free of images, it extends these principles into imagined and extended shapes and magnitudes, and thus contemplates the former in the latter" (*In Metaph. 13-14,* 91.31–34; trans. Dillon/O'Meara, modified). The place of mathematical objects is in our imagination, Syrianus suggests (*In Metaph.* 186.17–23), and the case is comparable to matter receiving form, except that matter "does not know what it is receiving, nor can it hold on to it," whereas the imagination, when it receives the mathematical blueprint from above, holds on to it to some extent and acquires an understanding of it.[179]

What would be an alternative to Mueller's interpretation that mathematical entities exist only potentially in sensible things and are brought to actuality by the mind? Instead of offering an interpretation that closely resembles a neo-Platonic (or a Kantian) one, Mueller could have adopted a more empiricst one by

[178] Ἁπλῶς δὲ πρὸς ἅπασαν αὐτοῦ τὴν δόξαν τοῦτο ῥητέον, ὅτι μήτε τεθεάμεθα πάντα τὰ σχήματα ἢ πάντας τοὺς ἀριθμοὺς ἐν τοῖς αἰσθητοῖς, περὶ ὅσα καὶ ὅσους αἱ μαθηματικαὶ διατρίβουσι, μήτε δυνατόν ἐστιν ἀκριβείᾳ τοσαύτῃ χρῆσθαι τὰ ἐκ τῶν αἰσθητῶν εἰλημμένα. εἰ δὲ ὅτι ἡμεῖς αὐτοῖς προστίθεμεν τὸ ἐνδέον καὶ ἀκριβέστερα ποιοῦμεν καὶ οὕτω θεωροῦμεν, ἀποφαίνοιτο, πρῶτον μὲν πόθεν δυνάμεθα αὐτὰ τελειοῦν, ἀναγκαῖον εἰπεῖν· οὐ γὰρ ἂν ἄλλην αἰτίαν ἀληθεστέραν εὕροιμεν τῆς ὑπὸ τῶν παλαιῶν εἰρημένης, ὅτι κατ' οὐσίαν ἡ ψυχὴ προείληφε πάντων τοὺς λόγους.

[179] Wildberg (2009).

arguing that ideal geometrical objects can be thought of as the 'constructive output of an ideal subject', to use Philip Kitcher's terms. In his book *The Nature of Mathematical Knowledge*, Philip Kitcher argues that mathematics is grounded in our manipulations of physical reality:

> I construe arithmetic as an *idealising theory:* the relation between arithmetic and the actual operations of human agents parallels that between the laws of ideal gases and the actual gases which exist in our world. We may personify the idealisation, by thinking of arithmetic as describing the constructive output of an ideal subject, whose status as an ideal subject resides in her freedom from certain accidental limitations imposed on us. [Kitcher (1984), p.109; his italics]

He cautions us however not to regard his approach as an orthodox constructivist approach according to which we already possess knowledge of mathematical properties; it is not the case that mathematical statements are about private mental entities (*pace* Mueller):

> *To say that arithmetic in particular, or mathematics in general, is true in virtue of the constructive output of an ideal subject, does not commit me to the thesis that we can have intuitive knowledge of mathematical truths* or to the thesis that there are (real or apparent) violations of the law of the excluded middle. I suggest that we have no way of knowing in advance what powers should be attributed to our ideal subject. Rather the description of that ideal subject and the conditions of her performance must be tested against our actual manipulations of reality. From Kant on, constructivist philosophies of mathematics have supposed that we can know a priori what constructions we can and cannot perform, or, to put it another way, what powers should be given to the ideal constructive subject. But there is no reason to bind this epistemological claim to the basic ontological thesis of constructivism. *Instead, we can adopt a more pragmatic attitude to the question of which mathematical operations are possible or what powers the ideal subject has, adjusting our treatment of these issues to the manipulations of the world which we actually perform*. [Kitcher (1984), pp.109-110; italics mine]

He also cautions us not to read his account as one that commits us to the existence of an Ideal Geometer:

> At this point, it is important to forestall a possible misunderstanding. In regarding mathematics as an idealising theory of our actual operations, I shall sometimes talk about the ideal operations of an ideal subject. That is not to suppose that there *is* a mysterious

> being with superhuman powers. Rather, as I shall explain in the next section, mathematical truths are true in virtue of stipulations which we set down, specifying conditions on the extensions of predicates *which actually are satisfied by nothing at all but are approximately satisfied by operations we perform (including physical operations)*. [Kitcher (1984), p.110; his italics]

We may reapproach the precision problem about geometricals as follows: a statement of stereometry cannot be referring to sublunary spheres or cubes for those objects are *approximately* spherical or cubical. We might claim, however, that such a statement applies to ideal spheres, cubes and so on. According to a Kitcherian modification to Mueller's interpretation we may say that a perfect sphere exists potentially within a slab of marble in that it can be brought to actuality by the acts of an Ideal Geometer, much like the Hermes inside the marble slab can be brought to actuality by the sculptor. The acts of the Ideal Geometer are merely the idealisation of the acts of the human geometers. As we shall see, however, the nature of the sublunary matter is such that even if there was an Ideal Geometer, then no matter how spherical he could make the sphere, it could always be made into something a bit more spherical. Does Aristotle really address the problem of precision that occupies such a central place in Syrianus' commentary? It is time to offer some answers on Aristotle's behalf.

[2.6] The potential being of geometricals

[2.6.1] The meaning of ὑλικῶς

Jonathan Barnes claims that when Aristotle says that mathematical objects exist ὑλικῶς, and not in actuality, he means not that they exist δυνάμει, but that

> … squares (say) exist in the same way that bronze (say) exists: bronze exists insofar as there are bronze statues, squares exist insofar as there are square areas. Bronze, evidently, is not an abstract stuff whose existence depends in some way on the mental exertions of the bronze-smith. Nor surely is there any implication in Aristotle's text that squares depend on geometers, or upon the mental activities of geometers, for their existence. [Barnes

(1985), p. 110][180]

Although I agree with Barnes' comments against mentalist readings of ὑλικῶς, I do side with Menn when the latter complains that his interpretation is especially restrictive:

> [Barnes] has done nothing to get rid of the obvious implication of the text that when something is said to exist ὑλικῶς, as opposed to ἐντελεχείᾳ this means that it exists only δυνάμει. It is true that for Aristotle bronze, like whiteness, exists only in dependence on the things which are named paronymously from it but that gives no warrant at all for saying that these things do not exist ἐντελεχείᾳ: three-dimensional extension, and the particular shape that Socrates has at the present moment, exist actually and not merely potentially. They exist, of course, only as particular attributes of actual substances which may be considered apart from those substances and the other attributes of those substances-the same status that whiteness has in the white man, and that unity and indivisibility have as attributes of Socrates. [Menn, 'Iγ3', fn.76, p.29]

When Aristotle says that mathematical objects exist ὑλικῶς, he is conceding that, in a sense, there are more mathematical objects than those that are actually embodied. Perhaps he means that there are such things in the sense that they have a *potential existence*: it is possible for them to exist actually, (perhaps) to exist in actual physical objects. David Bostock offers the following interpretation of the M.3 passage in terms of the above notion of potentiality:

> One presumes that his thought here is that these objects, when considered as existing separately, can be said to exist potentially because it is possible for them to exist actually, i.e. to exist in actual physical objects. Thus a circle exists actually in a circular table-top, and the number 7 exists actually wherever there are (say) 7 cows. Then the idea would be that some rather complex geometrical figures, e.g. a regular icosahedron, may not actually exist anywhere in the physical world, but this figure still has a potential existence because it could do so. The same would apply to a very large number, too large to be exemplified. [Bostock (2009), p.21]

Before we proceed any further, it would be useful to expand Bostock's thought by introducing a distinction between those mathematical entities that exist

180 A similar interpretation can be found in [Mignucci (1987), pp.183-184].

actually, and those that do not. In the case of arithmetical entities it is reasonable to assume that some of them have concrete (i.e. spatiotemporally–located) instances, for example, the number *three* (e.g. in a team of three horses in a field). Let us call those mathematical entities that exist actually, *basic mathematical entities*. The same can be said for geometrical entities, for example *solids* (recall Aristotle's example in *Meta.* M.3, of a man qua solid). Then there are some that do not. A regular icosahedron is, perhaps, an example of a mathematical entity that falls into the second category. Let us call mathematical entities that do not exist actually but potentially, *secondary entities*. The aforementioned distinction between basic and secondary entities pertains to the metaphysical status of mathematicals and it is crucial, I believe, for our understanding of the M.3 passage, a place where Aristotle distinguishes between mathematical entities existing ἐντελεχείᾳ and those that exist ὑλικῶς. Where should one draw the line between basic and secondary entities? A *broad* conception of basic mathematical entities could be used, covering all such entities that are actually part of the physical world: that conception is endorsed by Lear, who claims that things such as perfect spheres, perfect circles and perfect straight edges exist actually in the world. [181] Alternatively a more *narrow* understanding of 'basic' could be employed, which pushes more entities into the secondary category. I believe that a narrow conception of basic mathematical entities is closer to Aristotle's own views than a broader one.

[2.6.2] A normal understanding of potentiality

The sense of potentiality that is ascribed to certain mathematicals has to be sufficiently explained. Normally when we say something is potentially Φ we imply that it is possible that it should be actually Φ. A potentiality so understood, as a possibility analysable in terms of actual occurrence is supported by the following passage from *Meta.* Θ6:

> So then: energeia is a thing's being around not in the way we say <something is around> in capacity. We say, for instance, that a herm in wood <is around> in capacity and a half line within a whole line <is around in capacity>, because it could be cut off, and even that

[181] In [Lear (1982) pp.180-181].

> someone who is not contemplating is <in capacity> a knower, if he is able to contemplate. And <all these things are around> in energeia too.[182] [*Meta.* Θ.6, 1048a30–5; Beere's trans.]

A statue of Hermes in a block of wood has potential existence in the sense that the sculptor can carve the wood and produce a separate Hermes, and, similarly, the half-line has potential existence in that it could be separated out of the whole line. In both cases there are two items, A (Hermes, half line), and B (wood, whole line); A is in B, and A could be separated out from B.[183] Given this proper sense of potentiality, one might say that a slab of marble can be carved into a perfect cube (i.e. into a cube that perfectly satisfies the geometer's definition); the latter enjoys potential existence within the slab. We can also claim something similar about lower-dimensional or limit entities.

A major deficit of Annas' commentary of the *Metaphysics* M.2-3 is that she eschews any reference to Aristotle's *own* understanding of limit entities within a continuous body or continuous stretch of magnitude; such entities at most they exist within a body potentially, as loci where the body/stretch of magnitude could actually be divided.[184] Thus, Aristotle's argument against mathematical objects in the sensibles need not be understood as an argument *against* the potential existence of mathematical objects within sensible ones. In the case of limit entities such as points, one might claim that it is possible for a point–previously existing only potentially within a line segment–to become actual, perhaps as the terminus of one of the two resulting line segments after a division has occurred. This is indeed the view espoused by Richard Pettigrew in his article 'Aristotle on the subject matter of geometry':

> Thus, when the solid is divided, it will not be divided at a point that actually exists. Rather, the point that is identified, and thus brought to actuality, by the act of dividing *will belong to one of the two lines that result from the division*. [Pettigrew (2009), p.252; italics mine.]

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[182] ἔστι δὴ ἐνέργεια τὸ ὑπάρχειν τὸ πρᾶγμα μὴ οὕτως ὥσπερ λέγομεν δυνάμει· λέγομεν δὲ δυνάμει οἷον ἐν τῷ ξύλῳ Ἑρμῆν καὶ ἐν τῇ ὅλῃ τὴν ἡμίσειαν, ὅτι ἀφαιρεθείη ἄν, καὶ ἐπιστήμονα καὶ τὸν μὴ θεωροῦντα, ἂν δυνατὸς ᾖ θεωρῆσαι· τὸ δὲ ἐνεργείᾳ.

[183] See [Makin (2006), p.136].

[184] See, e.g., [White (1992), p.204]. Cf. *Meta.* B.5, 1002a18-b11; *Phys.* Θ.8, 263a23-b9, 262a21-26.

Although this claim makes sense for someone with sufficient knowledge of *modern topology*, it cannot be attributed to Aristotle, I am afraid, on the pain of being overly anachronistic. From a contemporary perspective, there is indeed a sense in which a line is divided at a point: One simply assigns the point to one or the other of the line segments resulting from the division, thus leaving the other line segment 'open' at the end of division, that is, without a terminal point.[185] According to Pettigrew's interpretation, if we employ the Aristotelian potentiality–actuality distinction, we might claim that the point, previously enjoying potential existence within the line, is actualised as the terminus of one of the two line segments. Modern views of topology, however, involve a point-set ontology of the continuous. A continuous magnitude of n dimension is conceived as a set of n-1 dimensional elements; thus a line is conceived as a set of points, actually infinite in number (non-denumerably infinite), linearly ordered, dense (between two points there is always another point) and continuous in the sense of Dedekind continuity: a line is Dedekind continuous iff for every 'cut' of the line into parts either the first linear segment has a last point and the second one does not have a first one, or the first linear segment does not have a last point but the second one has a first one. Aristotle of course does not hold that a line is conceived as a set of points; he argues against conceiving a continuous magnitude of dimension n as being *actually* constituted out of geometrically conceived limit entities of dimension n–1, e.g. a line out of points, a surface out of lines, a solid out of planes in *De gen. et. cor.* I.2 and *Phys.* Z.1.[186] Furthermore, his notion of density is fundamentally different from the modern (Dedekind) one as M. White has argued:

> There is a sense in which Aristotle is quite willing to admit the 'density' of points in a (one-dimensional) continuous magnitude. Since a 'line is always between points' (*Phys*. Z.1, 231b9), it is always theoretically *possible* to divide this line and mark a third point between the two. I refer to this notion as '*distributive density*'. Note that distributive

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[185] Cf. Pettigrew's confident remark: 'Indeed, it is in exactly this way that the division of a line is modelled in modern analytic geometry.' In [Pettigrew (2009), pp.252-253].

[186] E.g. in *Physics* Z.1, 231a24-26: ἀδύνατον ἐξ ἀδιαιρέτων εἶναί τι συνεχές, οἷον γραμμὴν ἐκ στιγμῶν, εἴπερ ἡ γραμμὴ μὲν συνεχές, ἡ στιγμὴ δὲ ἀδιαίρετον. For this remark regarding Aristotle's topology of the continuous see also [White (1993), pp.171-172].

> density, as I am using the concept, does not postulate an 'actually infinite' collection of discrete elements, linearly ordered in such a way that between any two there is a third, distinct element (and hence an infinite number of distinct elements). I refer to a linear array of an actually infinite collection of intuitively 'discrete' elements, such that between any two elements there is a third element, as '*collective density*'. There is no indication that Aristotle ever conceives a (one-dimensional) continuous magnitude as constituted by a collectively dense array of points. [White (1992), p.22; his italics]

It is now time to address Pettigrew's interpretation; let us consider a linear segment AB: if we divide it in half, say at point C, then it is hard to see how could Aristotle have claimed that the resulting partitions are of the kind AC, CB where the first partition includes the point C but the second not (or vice-versa).[187] The following account from *Meta.* B.5 seems to be Aristotle's first stab at explaining the division of a continuously extended body at a surface (with one added caveat: the context is an intensely dialectical discussion of whether limit entities can be substantial entities):

> But points and lines and surfaces cannot be in process of becoming nor of perishing, though they at one time exist and at another do not. For when bodies come into contact or are separated, their boundaries instantaneously become one at one time—when they touch, and two at another time—when they are separated; so that when they have been put together one boundary does not exist but has perished, and when they have been separated the boundaries exist which before did not exist. For it cannot be said that the point (which is indivisible) was divided into two. And if the boundaries come into being and cease to

[187]A point also made in [White (1992), p.20]. It is, perhaps, noteworthy to quote Franz Brentano's objections to an earlier version of the modern conception, that of Bernard Bolzano: 'According to the doctrine here considered, in contrast, the divisions of the line would not occur in points, but in some absurd way behind a point and before all others of which however none would stand closest to the cut. One of the two lines into which the line would be split upon division would therefore have an end point, but the other no beginning point. This inference has been quite correctly drawn by Bolzano, who was led thereby to his *monstrous doctrine* that there would exist bodies with and without surfaces, the one class containing just so many as the other, because contact would be possible only between a body with a surface and another without.' [Brentano (2010), p.105, italics mine]. Bolzano holds a conception of boundaries as 'the aggregate of all the extreme ether-atoms which still belong to it' and according to his conception of contact two bodies can only be in contact when one is bounded and the other one is not. In [Bolzano (2014), §66, pp.167-168].

> be, from what do they come into being?[188] [*Meta.* B.5, 1002a32-b4; Ross' trans.]

This aporetic passage contains what some scholars have described as Aristotle's 'constructivist' conception of a point, 'constructivist' in the sense that points and other limit entities are always conceived in terms of other continuous magnitudes rather than vice versa. To use Fred Miller's description: the point is 'an accidental feature of magnitudes undergoing operations' (p.100)[189]. When bodies come into contact, their limits which were two before contact, become one at contact. Regarding division, as White puts it 'where there was a single potential surface or plane at what was to be the locus of division, there are now two distinct planes or surfaces of the resulting parts of the body.'[190] Aristotle is, perhaps, arguing that there can be no *complete* process of generation for such entities. One is bound to wonder, however, whether there can be an account that treats such limit entities as progressively actualised without ever reaching a state of complete being–I will examine this fascinating alternative shortly.

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[188] τὰς δὲ στιγμὰς καὶ τὰς γραμμὰς καὶ τὰς ἐπιφανείας οὐκ ἐνδέχεται οὔτε γίγνεσθαι οὔτε φθείρεσθαι, ὁτὲ μὲν οὔσας ὁτὲ δὲ οὐκ οὔσας. ὅταν γὰρ ἅπτηται ἢ διαιρῆται τὰ σώματα, ἅμα ὁτὲ μὲν μία ἁπτομένων ὁτὲ δὲ δύο διαιρουμένων γίγνονται· ὥστ' οὔτε συγκειμένων ἔστιν ἀλλ' ἔφθαρται, διῃρημένων τε εἰσὶν αἱ πρότερον οὐκ οὖσαι (οὐ γὰρ δὴ ἥ γ' ἀδιαίρετος στιγμὴ διῃρέθη εἰς δύο), εἴ τε γίγνονται καὶ φθείρονται, ἐκ τίνος γίγνονται;

[189] In [Miller (1982), p.100]; this interpretation can also be found in [White (1992), pp.15-16].

[190] In [White (1993), p.174].

[2.7] Towards a new interpretation of the M.3 passage

[2.7.1] A radical interpretation

Aristotle's schematic account for a *geometry of solids* in *Metaphysics* M.3 can be further developed if we take into consideration his primary purpose in the preceding chapters: to refute Platonist–based arguments regarding the prior metaphysical status of the so–called 'lower–dimensional' or 'limit entities', i.e. of points, lines, and surfaces. One of the passages where Aristotle is more informative about the nature of such limit entities is the following:

> But if we are to suppose lines or what comes after these (I mean the primary surfaces) to be principles, these at least are not separable substances, but sections and divisions – the former of surfaces, the latter of bodies (while points are sections and divisions of lines); and *further they are limits of these same things; and all these are in other things and none is separable*.[191] [*Meta.* K.2, 1060b12-17, Ross' trans.; italics mine]

We have already offered an account that is based on what commentators have called 'the constructive conception of a point': according to that account whereas before the division of a line there was one *potential* point, after the division we get two points *actually* existing as limits of lines. However, I would like to focus on a more radical interpretation according to which limit entities whether bounding a body or lying within a body enjoy the potentiality of the infinite/void, that is the potentiality of something that can come progressively close to being actualised, without ever reaching complete actuality. This radical interpretation will make heavy use of the potentiality of the infinite and what Michael White has called 'Aristotle's intuitive concept of a limit' (see sections 2.7.2 and 2.7.3 for some discussion): we can make sense, I suggest, of Aristotle's peculiar term ὑλικῶς by making a connection with the way the infinite is ὑλικῶς: Aristotle in his

[191] εἴ γε μὴν γραμμὰς ἢ τὰ τούτων ἐχόμενα (λέγω δὲ ἐπιφανείας τὰς πρώτας) θήσει τις ἀρχάς, ταῦτά γ' οὐκ εἰσὶν οὐσίαι χωρισταί, τομαὶ δὲ καὶ διαιρέσεις αἱ μὲν ἐπιφανειῶν αἱ δὲ σωμάτων (αἱ δὲ στιγμαὶ γραμμῶν), ἔτι δὲ πέρατα τῶν αὐτῶν τούτων· πάντα δὲ ταῦτα ἐν ἄλλοις ὑπάρχει καὶ χωριστὸν οὐδέν ἐστιν.

discussion of the infinite in *Physics* Γ.6 says that 'the infinite is potentially as matter is' (καὶ δυνάμει οὕτως ὡς ἡ ὕλη, 206b14-15): insofar as limit entities enjoy potential being in a matter-like way, they are incomplete just as the infinite is (206b33-2017a15).

[2.7.2] A more 'exotic' potentiality

Let us now take into serious consideration *another kind of potentiality*, that of the void and the infinite (henceforth, I will label this kind of potentiality 'potentiality$_2$'). That this potentiality has to be distinguished from the usual sense of potentiality (enjoyed, for example, by the herm in the wood), is something that can be seen in the following passages:

> 'Being potentially' should not be understood in such a way that (just as if it is possible for this to be a statue, it will be a statue) so also there is an infinite that will be actually.[192] [*Physics* Γ.6, 206a18–21; Coope's trans.]

> And the infinite and the void, and other such like things, are said to be potentially and actually in another way from many other things, for example what sees and what walks and what is seen. For these things can sometimes be truly said without qualification as well (for what is seen is on the one hand so called because it is seen, and on the other because it is capable of being seen); but the infinite is not potentially in this way, namely that it will be actually separate, but by coming into being. For it is the division's not coming to an end which makes it the case that this actuality is potentially, and not the infinite being separated.[193] [*Meta.* Θ.6, 1048b9-17; Makin's trans.]

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[192] οὐ δεῖ δὲ τὸ δυνάμει ὂν λαμβάνειν, ὥσπερ εἰ δυνατὸν τοῦτ' ἀνδριάντα εἶναι, ὡς καὶ ἔσται τοῦτ' ἀνδριάς, οὕτω καὶ ἄπειρον ὃ ἔσται ἐνεργείᾳ·

[193] ἄλλως δὲ καὶ τὸ ἄπειρον καὶ τὸ κενόν, καὶ ὅσα τοιαῦτα, λέγεται δυνάμει καὶ ἐνεργείᾳ <ἢ> πολλοῖς τῶν ὄντων, οἷον τῷ ὁρῶντι καὶ βαδίζοντι καὶ ὁρωμένῳ. ταῦτα μὲν γὰρ ἐνδέχεται καὶ ἁπλῶς ἀληθεύεσθαί ποτε (τὸ μὲν γὰρ ὁρώμενον ὅτι ὁρᾶται, τὸ δὲ ὅτι ὁρᾶσθαι δυνατόν)· τὸ δ' ἄπειρον οὐχ οὕτω δυνάμει ἔστιν ὡς ἐνεργείᾳ ἐσόμενον χωριστόν, ἀλλὰ γνώσει. τὸ γὰρ μὴ ὑπολείπειν τὴν διαίρεσιν ἀποδίδωσι τὸ εἶναι δυνάμει ταύτην τὴν ἐνέργειαν, τὸ δὲ χωρίζεσθαι οὔ. Regarding Aristotle's claim in lines 1048b14-15 that the infinite exists potentially γνώσει, I have adopted Burnyeat's interpretation according to which Aristotle's point is that 'we know that further division will always be possible; not that we know that 'there is a possibility of any number of divisions' but,

Aristotle claims that there is some sense in which the infinite and the void is potentially but not actually. What does he mean? The following discussion follows closely Ursula Coope's excellent article 'Aristotle on the Infinite'. Commentators agree that Aristotle's claim seems a bit peculiar because, as Coope acknowledges, 'it is hard to see what can be meant by saying that something is potentially F, if this does not imply that that thing *could* be actually F'.[194] Things get more complicated because Aristotle after saying that the infinite is 'in no other way than this: potentially' (ἄλλως μὲν οὖν οὐκ ἔστιν, οὕτως δ' ἔστι τὸ ἄπειρον, δυνάμει, 206b12-13), he adds that 'it is actually too, in the way that we say that the day and the contest are' (καὶ ἐντελεχείᾳ δὲ ἔστιν, ὡς τὴν ἡμέραν εἶναι λέγομεν καὶ τὸν ἀγῶνα, 206b13–14). Why does Aristotle compare the way in which the infinite is to the way in which the day or the contest is? The following passage elucidates a bit the way the day and the contest are:

> But since being is in many ways, just as the day or the contest is by the constant occurring of other and other, in this way too the infinite is. (For in these cases also there is 'potentially' and 'actually'. The Olympic games *are* both in virtue of the contest's being able to occur and in virtue of the contest's occurring).[195] [*Physics* Γ.6, 206a21–25; Coope's trans.]

In her article, Coope offers a helpful comparative analysis of two major–albeit contrasting–interpretations of Aristotle's notion of infinity, that of Jaako Hintikka and that of Jonathan Lear. Whereas Hintikka's interpretation relies heavily on lines 206b13-14 to argue that the infinite is something enjoying actual existence (its being is like the being of a day or a contest), Lear argues instead that the infinite is only potential, by emphasising that the infinite, unlike a day or a contest, is always *incomplete*, something without limit or boundary. Coope,

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rather that we know that 'for any number of divisions, there is always a possibility of more' – and presumably for better reasons than that we have always found so in practise up to the point we have now reached in the dividing process.' See [Burnyeat (1984), p.127].

[194] For Coope's comment, see [Coope (2012), p.271]. A similar claim is made in [Bostock (1972), p.117].

[195] ἀλλ' ἐπεὶ πολλαχῶς τὸ εἶναι, ὥσπερ ἡ ἡμέρα ἔστι καὶ ὁ ἀγὼν τῷ ἀεὶ ἄλλο καὶ ἄλλο γίγνεσθαι, οὕτω καὶ τὸ ἄπειρον (καὶ γὰρ ἐπὶ τούτων ἔστι καὶ δυνάμει καὶ ἐνεργείᾳ· Ὀλύμπια γὰρ ἔστι καὶ τῷ δύνασθαι τὸν ἀγῶνα γίγνεσθαι καὶ τῷ γίγνεσθαι)·

however, complains that neither of these two interpretations can explain the crucial fact that Aristotle 'says all of these things together'.[196] Thus, she provides her own–in my opinion very persuasive–interpretation which will form the bedrock for much of my argument regarding the potential status of lower-dimensional geometricals.

Jaakko Hintikka is one of the most prominent supporters of the view that every (genuine) possibility is at some time actualised (the so-called *principle of plenitude*). Hintikka invokes this principle with respect to infinity as follows: magnitudes have a potential for undergoing an infinite process (a process of being infinitely divided) and this potential can be actualised or, in Coope's words, 'according to his interpretation the actuality of the infinite consists in the actuality of an infinite process of division'.[197] As Coope notes, this interpretation can make sense of Aristotle's claim that the infinite is actual in the way in which a day or a contest is, that is, 'the sense in which the infinite is actual is that it is going on'.[198] It cannot account, she notes, for the fact that the infinite is *unlike* a whole day or contest and the fact that Aristotle goes to great lengths to emphasise that there is a sense in which the infinite is *only potentially*.[199] She cites David Bostock's solution, who argues that Aristotle's point is that the process of dividing a line into infinitely many parts is 'one that cannot be *completed*.'[200] Nevertheless, she argues, this explanation is inadequate, because it does not account for what she calls 'the connection between the uncompleteability of this process and the existence of an unfufillable potential'.[201]

Let us examine the two crucial passages where Aristotle distinguishes the potentiality ascribed to infinity from the usual sense of potentiality more closely. It seems that, according to Aristotle, not every potentiality has to be

[196] In [Coope (2012), p.274].

[197] See [Hintikka (1966)] and [Coope, *op.cit.*, p.276].

[198] In [Coope, *op. cit.*, p.275].

[199] ibid.

[200] ibid. She refers to [Bostock (1972), p.39].

[201] ibid.

accompanied by the corresponding actuality.[202] Things that enjoy potential existence, such as the Hermes in the wood can (assuming normal conditions) eventually be actual. By contrast, infinite things are not potential in this sense; it is not the case that there will ever be an actual, separate, infinite thing. In *Physics* 206a18-21 Aristotle contrasts the potentiality of the infinite with the potentiality (as discussed in *Metaphysics* Θ.6) of a statue: in the latter case, once the sculptor's work is complete, the statue will enjoy actual existence. In the former case, the unlimited divisibility of a magnitude, in which its infinity consists, does not mean that there will ever be an end to this process, something that will enjoy actual existence, an infinite in actuality. Thus, Jonathan Lear's objection to Hintikka's interpretation (that the actuality of the infinite consists in the actuality of an infinite process of division) is that 'there is no process which could correctly be considered the actualisation of an infinite division of a line'.[203] The simple reason is that 'any such process will terminate after finitely many divisions':[204]

> To see this more clearly, what sort of process might be considered the actualisation of an infinite division. It could not be a physical process of actually cutting a finite physical magnitude, for, obviously, any physical cut we make in such a magnitude will have finite size and thus the magnitude will be completely destroyed after only finitely many cuts. Nor could it be a process of theoretical division: i.e. a mental operation which distinguishes parts of the magnitude. For no mortal could carry out more than a finite number of theoretical divisions.[205]

Coope remarks that Lear's account does indeed explain the way the infinite is potentially in a way that it is *not* actually: 'For a magnitude to be infinitely divisible is for it to have potentials that could never be fully actualised by any process of dividing.'[206] However, she complains, 'Lear's interpretation leaves it

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[202] Richard Sorabji has pointed out that Aristotle reserves the principle of plenitude for special cases, such as the everlasting properties of everlasting objects. See [Sorabji (1980), pp.128–35].

[203] See [Lear (1982), p.190]. Extensive bits from Lear's paper are discussed in [Coope, *op.cit*, p.276].

[204] In [Lear, *op.cit.*, p.190].

[205] In [Lear, ibid].

[206] In [Coope*, op.cit.*, p.276].

mysterious how Aristotle can claim that the infinite is actually 'in the way that a day or a contest is'[207]. Coope refines Lear's reading by distinguishing between two senses according to which it is impossible for there to be an infinite process of division for a continuous stretch of magnitude: i) such a process *cannot occur*, ii) such a process *cannot be occurring*.[208] She argues that Lear's interpretation employs the first sense, because of certain considerations that pertain to Aristotle's conception of *processes*:

> … a process *occurs* only if the whole of it occurs, but in the case of an infinite process there is no such thing as the whole of it. A finite process, such as a contest, occurs over a certain length of time; what this means is that the whole process is spread out over a certain length of time. But this is not true of an infinite process. On Aristotle's view, the whole of such a process could not occur even over infinitely much time, for as we have seen, he holds that being infinite implies not being whole (206b33ff.)[209]

She thinks however that these considerations are not sufficient to claim that a process of division ad infinitum *cannot be occurring*. This allows her to offer a defence of Hintikka's interpretation against Lear:

> Hintikka should say that infinite divisibility is the potential for a certain process to be occurring. This potential is completely fulfilled when a magnitude is *being divided* ad infinitum. Of course, the whole of such a process can never occur, but that (he might say) does not imply that the magnitude has some potential that cannot be fulfilled. The magnitude cannot undergo a whole process of division ad infinitum, but then it does not have the potential to undergo such a process; on the other hand, the magnitude does have the potential to be undergoing a process of division ad infinitum, and this potential can be fulfilled.[210]

But still this does not 'make sense of Aristotle's claim that the infinite is potentially in a way in which it is not actually. Coope will go on to investigate the way in which a process is the actualisation of a potential. Coope's main argument relies on a distinction between an activity (*energeia*) and a process (*kinêsis*)

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[207] *ibid.*

[208] In [Coope, *op.cit*, p.277].

[209] *ibid.*

[210] In [Coope, *op. cit.*, p.278].

based on *Metaphysics* Θ.6 1048b18–35. The gist of her analysis is contained in the following passage:

> Undergoing a process essentially involves having some potential that is not fully realised, whereas engaging in an activity does not essentially involve having an unrealised potential. Of course, when I am engaged in an activity (such as seeing), I will have all sorts of unrealised potentials. (Perhaps I am not using my sense of smell.) The point is that having such unrealised potentials is not essential to what it is to be engaged in the activity of seeing, whereas there will always be a certain unrealised potential that is essential to the undergoing of a process.[211]

Thus, according to Coope's account, one cannot claim –alongside Hintikka- that a magnitude has the potential to be undergoing the process of being infinitely divided which is a potential that can be completely fulfilled 'at any moment when the magnitude is undergoing such a process'.[212] Rather, the potential in question must be a potential which is 'only incompletely fulfilled while the magnitude is undergoing that process'.[213] More specifically, it is a potential that 'has no complete fulfilment'.[214] Coope acknowledges that this is a deviation from the standard reading of potentiality:

> If a potential has no complete fulfilment, then how can we specify what potential it is? The answer is that we have to define this potential in a non-standard way: we have to specify what would count as fulfilling it *as completely as possible*. When we say that a magnitude is infinitely divisible, the potential we attribute to it should be defined as follows. It is a potential that *has no complete fulfilment* but that is fulfilled *as completely as it can be* in the process by which the line is *being divided* ad infinitum. There is thus a sense in which it, like other potentials, is defined by its maximal fulfilment, but the maximal fulfilment that defines it is not a complete fulfilment, merely a fulfilment that is *as complete as possible.*[215]

[211] In [Coope, *op. cit.*, p.279].

[212] In [Coope, *op. cit.*, p.280].

[213] In [Coope, *op. cit.*, p.280].

[214] In [Coope, *op. cit.*, p.281].

[215] ibid.

Coope's account provides us with a brilliant way to reconcile the two peculiar claims by Aristotle: that 1) the infinite is in no other way than potentially and 2) that it is actually too in the way that the day and the games are:

> The potential that we ascribe to something when we say that it is infinitely divisible is a potential that can be fulfilled in a way: it can be *incompletely* fulfilled. It is incompletely fulfilled while the magnitude is being divided ad infinitum, just as the potential for a day to occur is incompletely fulfilled while the day is going on, or the potential for a game to occur is incompletely fulfilled while the game is taking place. The difference is that in the case of these potentials (for the day or the game to occur), there is a corresponding complete fulfilment (the occurrence of the day or of the game), whereas the potential we ascribe to something when we say it is infinitely divisible is a potential that has no complete fulfilment. It is thus 'only potential', in that it has no complete fulfilment, but also 'actual' in a way, in that it does (like the potential involved in the day or the games) have an incomplete fulfilment.[216]

[2.7.3] Aristotle's 'intuitive concept of a limit'

Coope's account can be further enhanced, I suggest, by examining more closely Aristotle's two notions of potential infinity, the infinity of magnitudes 'by division' and that 'by addition'. In *Phys.* Γ.6 Aristotle discusses those two senses of potential infinity for magnitudes:

> The infinite by addition, too, is potentially in this way; this infinite, we say, is in a way the same as the infinite by division. For it will always be possible to obtain something beyond but it will not exceed any magnitude, as it does in the division where it will always be less than any assigned magnitude.[217] [*Phys*. Γ.6, 206b16-20; Hussey's trans. mod.]

Infinity by addition and infinity by division seem to be closely interrelated, Aristotle, however, opts for a detailed exposition of the former:

[216] In [Coope, *op. cit.*, p.282].

[217] καὶ κατὰ πρόσθεσιν δὴ οὕτως ἄπειρον δυνάμει ἔστιν, ὃ ταὐτὸ λέγομεν τρόπον τινὰ εἶναι τῷ κατὰ διαίρεσιν· ἀεὶ μὲν γάρ τι ἔξω ἔσται λαμβάνειν, οὐ μέντοι ὑπερβαλεῖ παντὸς μεγέθους, ὥσπερ ἐπὶ τὴν διαίρεσιν ὑπερβάλλει παντὸς ὡρισμένου καὶ ἀεὶ ἔσται ἔλαττον.

> The infinite by way of addition is in a sense the same as the infinite by way of division. Within a finite magnitude the infinite by way of addition occurs in a way inverse to that by division; for, as we see the magnitude being divided to infinity, the sum of the parts taken appears to tend toward something definite. For if, in a finite magnitude, one takes a definite part and then from what remains keeps on taking a part, not equal to the first part but always using the same ratio, he will not traverse the original finite magnitude; but if he is to so increase the ratio that the parts taken are always equal, he will traverse it, because every finite magnitude is exhausted by any definite magnitude. Thus it is in this and not in any other way that the infinite exists, namely, potentially and by way of diminution.[218] [*Physics* Γ.6, 206b3–13; Apostle's trans. mod.]

Aristotle's understanding of the infinity of addition seems to require a prior understanding of the (more familiar perhaps) notion of the infinity by division. He claims that the notion of infinity by addition for magnitudes involves a process which is 'in a way' (πως) the inverse of the process of division (ἀντεστραμμένως; *cf.* τὴν ἀντεστραμμένην πρόσθεσιν, 207a23). Suppose we have a magnitude AB and we divide it by means of a geometrical progression, as for example is prescribed in the Zenonian Dichotomy: we first take away its half, then the half of that, and so on. We then have the sequence: $\frac{1}{2}$AB, $\frac{1}{4}$ AB, $\frac{1}{8}$AB . . . The Zenonian Dichotomy is an instant of the more general case of division described in lines 206b7-12. One can then use this division to form the inverse process of addition $\frac{1}{2}$ AB+$\frac{1}{4}$ AB+$\frac{1}{8}$AB . . . Such a sum tends toward 'something definite' (πρὸς τὸ ὡρισμένον, 206b6), namely the whole AB. Thus, whereas in the process of division we move towards the complete 'exhaustion' of a magnitude, in the process of addition we move towards reaching a finite magnitude. That there is always one bit of magnitude that can be added or taken away underlines

[218] τὸ δὲ κατὰ πρόσθεσιν τὸ αὐτό ἐστί πως καὶ τὸ κατὰ διαίρεσιν· ἐν γὰρ τῷ πεπερασμένῳ κατὰ πρόσθεσιν γίγνεται ἀντεστραμμένως· ᾗ γὰρ διαιρούμενον ὁρᾶται εἰς ἄπειρον, ταύτῃ προστιθέμενον φανεῖται πρὸς τὸ ὡρισμένον. ἐν γὰρ τῷ πεπερασμένῳ μεγέθει ἂν λαβών τις ὡρισμένον προσλαμβάνῃ τῷ αὐτῷ λόγῳ, μὴ τὸ αὐτό τι τοῦ ὅλου μέγεθος περιλαμβάνων, οὐ διέξεισι τὸ πεπερασμένον· ἐὰν δ' οὕτως αὔξῃ τὸν λόγον ὥστε ἀεί τι τὸ αὐτὸ περιλαμβάνειν μέγεθος, διέξεισι, διὰ τὸ πᾶν πεπερασμένον ἀναιρεῖσθαι ὁτῳοῦν ὡρισμένῳ. ἄλλως μὲν οὖν οὐκ ἔστιν, οὕτως δ' ἔστι τὸ ἄπειρον, δυνάμει τε καὶ ἐπὶ καθαιρέσει.

the fact that the infinite by addition and division are two instances of potential infinity; the processes themselves are inverse.

Michael White offers a helpful comparative analysis of Aristotle's notion of potential infinity and modern treatments of infinity. Using modern algebra one *identifies* the sum S of a denumerably infinite series of addenda (i.e. a series the members of which can be put into one-to-one correspondence with the natural numbers), $s_1 + s_2 + \ldots + s_n + \ldots$, with the limit of the denumerably infinite sequence $\{t_n\}$, where each term t_n in the sequence is the sum of the first n s_i s. To take the example of the linear magnitude AB above, the sum S = ½ AB + ¼ AB + 1/8 AB…will be identified with the limit of the infinite sequence $\{(2^n - 1)/2^n \text{ AB}\}$ = $\frac{1}{2}$ AB, $\frac{3}{4}$ AB, $\frac{7}{8}$ AB, $\frac{15}{16}$ AB,…, which is AB.[219] But does Aristotle allow this identification? According to Aristotle the sum will approach a definite limit, namely AB. As White notes, Aristotle allows the processes of division and addition be 'infinitely extendable' in the sense that 'one can always take the division and correlative summation a step beyond any finite division/summation'.[220] Thus, Aristotle, as White points out,

> does have an 'intuitive concept' of a limit of a process of summation-a process that can be *indefinitely* extended in the Aristotelian sense of being extendable beyond *any* finite number of stages. We here mean by 'limit' a least finite magnitude (i) which the process of summation never (i.e. at any finite stage) exceeds and (ii) to which the process of summation approaches closer at each successive stage but never (i.e. at any finite stage) reaches. There is no evidence, however, that Aristotle moves from this conception of limit (which obviously grounds the mathematical notion of a limit) to what I shall call the *mathematical sum/union* notion of a limit, which *by definition* serves as the sum or union of an infinite series. [White (1992), p.141][221]

Aristotle's core conception of infinity as 'that which with respect to quantity, it is always possible to take something beyond what has been taken' (*Phys.* Γ.6.

[219] In [White (1992), p.9].

[220] In [White, *op. cit.*, p.11].

[221] In [White, *op. cit.*, p.141].

207a7–8), prevents him from identifying the sum of an indefinitely extendable series of addenda with the limit that such a series converge to:

> Aristotle's notion of the infinite is tied to a process that can be thought of as consisting in a series of discrete 'stages' or 'steps'—either a process of addition (*prosthesis*) of a unit or quantity to other such units/quantities or a process of the division (*diairesis*) of some original, fixed quantity into subquantities. Such a process can, in principle, always be carried a step beyond any determinate (i.e. finite) number of steps. 'The infinite', in this sense, Aristotle contrasts with what is 'complete and whole' (*teleion kai holon*) (207a10). [White (1992), p.11]

How should one then understand the corresponding claim about the potentiality of the void in *Metaphysics* Θ.6, 1048b9-17? Picking up a suggestion made by Burnyeat, we may say that 'we can always (at least in theory) reduce the amount of air in a container beyond the point to which we have already reduced it, but never produce a perfect void.' [222]

[2.7.4] Menn's interpretation and its problems

The potentiality of the infinite may also be applied to more complex or derivative geometrical objects (triangles, straight lines, tetrahedra, etc.) as follows. We may claim, for example, that a perfect marble sphere exists potentially within a marble block, without this potentiality ever being actualised in the sense of resulting in a separate, perfectly spherical, marble object. This is not due to our human limitations but due to the nature of the sublunary matter. The nature of the sublunary matter is such that even if there were an Ideal Geometer who was unhampered by human physical and mental limitations, then no matter how spherical he could make the artifact, it could always be made into something a bit more spherical. The sphere however can come progressively closer to being actualised, beyond any given limit, and this is enough to claim that is has potential existence. Stephen Menn attributes this kind of potentiality to such complex geometrical objects:

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[222] Burnyeat (1984), p.127.

> If no actual bodies are bounded by perfectly flat planes, then the potentiality for being divided along a perfectly flat plane will never be actualised, but this does not disqualify it from counting as a potentiality: it will be like the potentialities for the void and for the infinite which Aristotle discusses at Θ.6 1048b9-17, which can never be entirely actualised, but which can come progressively closer to being actualised, beyond any given limit, and this is enough to say that 'this ἐνέργεια [sc. the void or the infinite] exists δυνάμει, but is not separated'. . . [Geometrical objects], existing potentially within the matter of sensible things not qua sensible, are, as potentialities, as eternal and unchanging as a Platonist could wish, without any need for separate existence. [Menn, 'Iγ3', p.28]

And again he makes his thesis more explicit when he claims:

> I take Aristotle to think, not that geometrical objects are physical objects with some of their properties abstracted away (since at least sublunar physical objects are not perfectly straight, circular etc.), but rather that the matter of geometrical objects is the matter of physical objects with some of its properties abstracted away (and only extension left), and that geometrical objects exist potentially in that matter. I suppose it is not possible for a physical object ever to become perfectly straight, but the straightness is still potentially in the object, in the same way that infinity is potentially in the objects–it can be asymptotically approached. These potentialities will be actualised (so far as they ever are) either by human acts of thought or by artificial acts of construction (drawing approximately straight lines etc.) caused by those acts of thought. [Menn, 'IIIα3', p.26]

Menn's focus is on sublunary geometrical objects; his view is that we have sublunary physical objects that do not perfectly satisfy the geometers' definitions (for example, a marble sphere that has some indentations, an approximately straight drawn line that has some curviness, a drawn triangle with jagged sides) but we can 'asymptotically approach' the more perfect versions of these (a marble sphere with no indentations, a perfectly straight line and a perfect triangle). The following passage from *De Caelo* may be cited in support of Menn's claim about the nature of the sublunary matter:

> It is plain from the foregoing that the universe is spherical. It is plain, further, that it is so accurately turned that no manufactured thing nor anything else within the range of our observation can even approach it. For the matter of which these are composed does not admit of anything like the same regularity and finish as the substance of the enveloping

body.[223] [*De Caelo* II.4, 287b14-287b20; mod. Stocks' trans.]

Unlike the heavenly sphere (κόσμος), which is perfectly spherical, no artifact (χειρόκμητον) or sensible object (τῶν ἡμῖν ἐν ὀφθαλμοῖς φαινομένων) enjoys such kind of (geometrical) perfection. For the matter of those artifacts does not admit of anything like the evenness (ὁμαλότητα) or the precision (ἀκρίβειαν) as the nature of the encompassing body. As Theokritos Kouremenos notes, lines 287b14-18 can be understood in two ways: 1) either that, unlike the heavenly sphere, *no sphere* made or perceived by us enjoys such kind of precision, or 2) no artifact or perceivable object enjoys *such kind of precision* as the heavenly sphere.[224] The second construal entails the stronger claim that sphericity is not the only geometric property that sublunary objects fail to instantiate perfectly.[225] One, however, need not follow Kouremenos in attributing to Aristotle the utterly skeptical view that physical objects do not perfectly instantiate any geometrical property.[226] For, Aristotle can ground his realism about geometricals (as well as about arithmeticals) on the commonsensical notion of the solid, perfectly exemplified in physical objects around us. Thus, if we endorse the second reading of lines 287b14-18, we may simply infer that there are no perfect instantiations of the various *specific* geometrical objects (spheres, straight lines, circles, etc.) in the sublunary world.

But why does Menn place such a great emphasis on the objects of the sublunary world? I am not so sure that Aristotle means to exclude them from his discussion in the M.3 passage. After all, in the analogy from astronomy (*Meta*. M.3, 1077b22-3) he tells about theorems that are just about moving bodies (οὕτω καὶ ἐπὶ τῶν κινουμένων), not qua moving but only qua bodies (οὐχ ᾗ κινούμενα δὲ ἀλλ' ᾗ

[223] Ὅτι μὲν οὖν σφαιροειδής ἐστιν ὁ κόσμος, δῆλον ἐκ τούτων, καὶ ὅτι κατ' ἀκρίβειαν ἔντορνος οὕτως ὥστε μηθὲν μήτε χειρόκμητον ἔχειν παραπλησίως μήτ' ἄλλο μηθὲν τῶν ἡμῖν ἐν ὀφθαλμοῖς φαινομένων. Ἐξ ὧν γὰρ τὴν σύστασιν εἴληφεν, οὐδὲν οὕτω δυνατὸν ὁμαλότητα δέξασθαι καὶ ἀκρίβειαν ὡς ἡ τοῦ πέριξ σώματος φύσις·

[224] In [Kouremenos (2003), fn.28, p.476].

[225] ibid.

[226] A claim constantly made throughout his article. See esp. the abstract in [Kouremenos (2003), p.463].

σώματα μόνον), and again only qua surfaces (πάλιν ᾗ ἐπίπεδα μόνον), and only qua lengths (ᾗ μήκη μόνον); and qua divisibles (i.e. just as continuous quantities, ᾗ διαιρετά); and qua indivisibles having position (i.e. just as points, ᾗ ἀδιαίρετα ἔχοντα δὲ θέσιν); and qua indivisibles alone (i.e. just as units, ᾗ ἀδιαίρετα μόνον). As the analogy makes clear, there can be statements about celestial bodies regarding their volume, their shape, their delineations, and so on, without having to postulate separately existing planes, lines, and solids. Furthermore, it is clear from the *De Caelo* that the celestial bodies do possess perfect sphericity. The ideal spheres are simply the spheres of the superlunary world. As for the other solids, they may be thought of as existing potentially within those spheres *in the normal sense of potentiality* (see discussion in section 2.6.2). Menn points to the right direction when he claims that in the M.3 passage,

> Aristotle does not introduce a potentiality-actuality opposition in talking about the white or about unity, but only in talking about geometricals such as tetrahedron, which are not attributes of any actual substance, since no actual substance is perfectly tetrahedral. [Menn, 'Iγ3', p.27]

According to his interpretation, however, geometricals <u>are constituted of intelligible matter</u>; he claims that the ὑλικῶς in the M.3 passage refers to intelligible matter:

> The 'matter' of which Aristotle speaks here is not Socrates' flesh and blood, but what he in some texts calls 'intelligible matter', the matter of geometrical things ('some matter is sensible, some intelligible, sensible like bronze and wood and whatever matter is movable, intelligible what is present in the sensibles not qua sensibles, like the mathematicals', Z.10, 1036a9-12). This will be bare extension--in the present case, three-dimensional extension--existing not separately from the sensibles, like the matter of Platonic or Speusippean mathematicals, but 'in the sensibles not qua sensibles'; and this matter (unlike the matter of Platonic or Speusippean mathematicals, which is not potentially anything) is potentially divided along planes, spherical surfaces, or any other possible bounding surfaces. [Menn, 'Iγ3', pp.27-28]

I do not think this is right. The M.3 passage does not warrant any such reference to intelligible matter. How should one then understand the role of intelligible matter that Menn is so willing to identify as the substratum of mathematicals? One ought to distinguish between the mode of existence of mathematicals and the way the mathematicians study them. On the one hand we have a claim about the metaphysical status of the mathematicals: they do not enjoy independent existence over and above their (actual or possible) material exemplifications; on the other hand we have a cognitive one: the mathematicians can acquire a conception of mathematicals without invoking the matter in which they are (or can be) exemplified. The latter is a claim about how the mind thinks of the mathematical entities and *not* about their metaphysical status; when they become the objects of a mathematician's consideration they do have intelligible matter.[227] When I claim that a perfect tetrahedron exists potentially within a perfect celestial sphere, I am not making a claim about intelligible matter. The matter of the sphere in question is superlunary matter. The potentially existing tetrahedron can simply be considered qua tetrahedron, qua a solid determinant. When the potentially existing tetrahedron is considered in this way it is an object of the geometrician's consideration, and we may say it has intelligible matter.[228]

[2.7.5] My interpretation and a modern analogue

Let me summarise my interpretation thus far: the physical world is essentially a world of solids. Aristotle's realism about geometrical entities is essentially a realism about solids: either the complex solids around us, the not-so-easy-to-analyse solids of the sublunary world, or the solids of the supelunary world, the perfect celestial spheres or the potentially existing solids in them. I see no reason to deny that the latter, potentially in the proper sense of potentiality, do not exhibit the requisite precision: that a tetrahedron, for example, existing potentially within a superlunary sphere is not perfectly tetrahedral; that it does not have perfectly planar surfaces or that its edges are not perfectly straight. Aristotle's realism with regard to solids is the answer to someone who argues that there are no geometrical objects that perfectly satisfy the geometers' definitions. Certain questions arise, however, when one focuses on the

[227] A point also made in [Mignucci (1987), p.183].

[228] For this understanding of the role of intelligible matter see also [Lear (1982), p.182].

metaphysical status of limit entities that bound or lie within a body. We have already discussed White's account according to which limit entities *within* a continuous body enjoy some kind of potential existence, as loci where the body/stretch of magnitude could be divided.[229] White, however, distinguishes between the aforementioned limit entities, and those that bound a continuous body. What can be said about the latter? Do they enjoy actual or potential existence? Michael J. White thinks the former:

> On the other hand, the Aristotelian conception would seem to commit us to the *actual (as opposed to the merely potential) existence of some limit entities, e.g. the surface of a body, the terminus of a line inscribed in a wax tablet*. There is no indication that Aristotle conceives of such entities as somehow spatially indeterminate or extended: they are true limits in the sense that they possess zero measure (are not extended) in at least one dimension; but there is an obvious sense in which they are 'objectively out there', a real feature of the physical world. Aristotelian entities of this sort are assumable, I believe, to what Avrum Stroll, in a recent book on surfaces calls the 'DS' conception of surfaces. DS surfaces are geometrical or 'abstractions' in the sense that they possess zero measure or are unextended in-at least-one dimension. In other words, they are not corporeal in so far as 'corporeality' connotes possession of three dimensions. Yet DS surfaces are regarded as belonging to (and indeed as circumscribing and hence helping to define) the bodies or continuous stretches of magnitudes that they demarcate. [White (1992), pp.204-205; italics mine]

I am not sure that White succeds in illuminating his position by invoking Avrum Stroll's account. The key passage from Stroll's book is the following:

> The intuitive idea is that one wishes to say that a marble has a surface or that a lake has a surface and yet that this surface is not a physical part of the marble or the lake. *We arrive at such a conception by a process that consists of the progressive thinning out of a physical surface until we are left with a kind of logical limit or conceptual limit to the object*. But it is still the object's surface that we are speaking about, not an interface that doesn't belong to the object. [Stroll (1988), pp.46-47; italics mine]

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[229] See, e.g., [White (1992), p.204].

Stroll's 'DS'[230] conception of surfaces is an attempt for reconciling a purely geometric conception (two-dimensional surfaces) with physical reality: he provides us with an account of how we can get from three–dimensional bodies found in nature to geometric skins, or limit entities. It seems that Stroll's conception of surfaces is consistent with the commonsensical claim that surfaces are 'parts of', or 'belong to' the things of which they are surfaces of. In the above passage, Stroll seems to allude to a *Whiteheadian* solution of the problem, by invoking the idea of ever thinner layers of a physical object 'until we are left with a kind of logical limit or conceptual limit to the object'.[231] A detailed exposition of the method–termed 'extensive abstraction' by Whitehead–is provided in the latter's 'La théorie relationniste de l'espace', an essay that includes an application of extensive abstraction to three-dimensional physical objects at various distances from one another.[232] The general idea is to identify points, lines, and planes in a continuum with 'abstractive sets', sets containing infinitely many converging, nested, extended entities. Consider for example a cone–shaped physical object. Then, in Whiteheadian fashion, its tip is

> … identifiable for all practical or theoretical purposes with an *abstractive set* of extended parts of the cone which form an infinite nested series honing in on the place where the cone ends - and likewise for every other inner or outer boundary of a part of the cone, or of

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[230] Where 'DS' according to Stroll stands 'for the conception that holds that surfaces belong to their corresponding bodies'. See [Stroll (1988), pp.50-51]

[231] ibid.

[232] For a succinct exposition of Whitehead's method and its place within the history of topology consult [Zimmerman (1996)]. In this article, W.D. Zimmerman notes that Whitehead's method was 'very well-received, quickly adopted by the likes of Bertrand Russell, Jean Nicod, and Alfred Tarski' [Zimmerman (1966), p.162]. Interestingly enough, Zimmerman cites a passage from [Tarski (1956)] where Tarski seems to suggest that Lesniewski had at least been thinking of something along the lines of Whitehead's method independently: 'Some years ago Lesniewski suggested the problem of establishing the foundations of a geometry of solids, understanding by this term a system of geometry destitute of such geometrical figures as points, lines, and surfaces, and admitting as figures only solids...' (In [Tarski (1956), p.24], included in [Zimmerman (1996, p.174], underlining mine).

> any other extended body. [Zimmerman (1996), p.162][233]

In such a Whiteheadian account, however, the abstractive set is an unending sequence of solids. As Laguna explains, 'there can not be a smallest solid of the set; because, if there were, any such solid which it contained would be contained by all the solids of the set.'[234] Thus, I think White is mistaken in explaining the actual existence of the limit entities that bound or demarcate a physical object–such as the tip of the cone–by invoking Stroll's (or rather Whitehead's account). On Stroll's (or better Whitehead's) account it seems that those limit entities enjoy the kind of potentiality that Aristotle attributes to infinity and the void,[235]

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[233] The example can be found in [Zimmerman (1996), p.162]. According to Jean Nicod an abstractive class is 'defined by a class of volumes such that any one of two of its members is either included in the other or includes it, and no volume is included in all its members.' [Nicod (1930), p.40]. As Nicod cautions, however, 'these conditions are not sufficient to guarantee that all volumes of an abstractive class would have only one point in common. They might have as their common nucleus not a volume but a line or a surface. Thus an abstractive class formed from discs of a constant diameter and diminishing thickness converges to a circle; an abstractive class generated by a series of cylinders of constant height and decreasing diameter would reduced it to a line segment (the altitude).' In [Nicod (1930), pp.40-41].

[234] In [Laguna (1922), p.453]. Laguna provides a much clearer exposition of Whitehead's method of extensive abstraction in [Laguna (1922)].

[235] It seems that an account of the metaphysical status of limit entities bounding a physical body or a continuous stretch of magnitude can be reformulated as an account of limit entities within a magnitude, if we take into consideration two things: 1) the fact that for Aristotle the world is a massive continuous magnitude and 2) Aristotle's notion of potentiality pertaining to the infinite and the void. In fact, even though White attributes actual being to lower-dimensional entities bounding a body, he–somewhat surprisingly–offers an account of such entities that can better be understood along the lines of my interpretation, namely as entities enjoying potential$_2$ being within a continuous stretch of magnitude: 'Consider a three-dimensional body *A*, surrounded by its spatial environment *B*. We consider a three-dimensional region *i* of spatial extension that clearly contains some of *A* and some of *B* and increasingly small regions *i'* nested in *i*. According to the doctrine of geometrical realism advanced by Aristotle, there can be constructed a monotonic non-increasing sequence of such regions which converges to a two-dimensional limit entity, i.e. the (geometrical) surface of *A* or interface between *A* and its surrounding spatial environment.' In [White (1992), p.287]. If, however, White is alluding to Stroll's conception of surfaces, then there cannot be a smallest region such that it is identical to the interface between A and its surroundings. At the very best, this two–dimensional entity enjoys the potentiality

that is, they can progressively come close to being actualised, beyond any given limit.

This account can also make sense of the claim in *Meta*. M.2. that lower–dimensional entities ought not to be understood as (the actual) form or shape of bodies:

> Again, body is a sort of substance; for it already has in a sense completeness. But how can lines be substances? Neither as a form or shape, as the soul perhaps is, nor as matter, like body; for we have no experience of anything that can be put together out of lines or planes or points, while if these had been a sort of material substance, we should have observed things which could be put together out of them. [*Meta*. M.2, 1077a31-36; Ross' trans.]

A similar account can be provided for the metaphysical status of limit entities within a body: one can, for example, begin from the notion of a sphere and replace the notion of an actually existing point in the centre of that sphere with a sequence of enclosed spheres that gradually converge to that point.

If we are to consider the M.3 passage as Aristotle's final word on the mode of existence of mathematicals, then his curious example about the geometer studying a man qua solid, points towards a conception of geometry that is based on the commonsensical and readily available notion of the solid. Such a geometry, combined with the claim that Aristotle attributes some kind of potential (in terms of potentiality$_2$) existence to the limit entities bounding and within a physical body, combined with 'his intuitive sense of a limit'[236] has a closer affinity to modern Whiteheadian or Tarskian geometries than it has to Euclidean ones. In a sense, this peculiar kind of potentiality ascribed to lower–dimensional entities has not so much eliminated them but analysed them in terms of more fundamental geometrical entities, namely (sets of) solids. The fact that no human possesses any of the many determinate solid shapes need not be a problem for the modern geometer with Aristotelian inspirations. Theodore de Laguna developed such a geometry, beginning with 'solid' and the relation 'can

Aristotle ascribes to the infinite and the void: it can become as 'thin' as possible without ever becoming an actually existing surface.

[236] See discussion in section [2.7.3].

connect' as the fundamental notions and utilising (a modified form of) Whitehead's method of 'extensive abstraction' to give an account of points in terms of *an unending sequence of solids*. The key idea that lies in utilising Aristotle's notion of potentiality$_2$ with respect to lower-dimensional geometrical objects is that the geometrical formulas about them are descriptive not of actually existing 2-dimensional, 1-dimensional, or 0-dimensional entities, but of sets or sequences of 3-dimensional objects that converge to the aforementioned entities asymptotically.

[2.7.6] Further suggestions

Although the above interpretation of Aristotelian limit entities was described as a Whiteheadian one, it has roots that go back at least to the scholastic tradition and the discussions regarding the problem of contact between a sphere and a plane (which in turn is based on Aristotle's comment that a bronze sphere touches a line at a point in *De Anima* I.1, 403a10-16). The complicated discussions that arose around this problem took either a straightforwardly realist stance towards limit entities (because they were deemed necessary to explain contact) or a more anti–realist approach (based on Ockham's account of contact). If one were to save White's interpretation of limit entities as entities that actually determine and help to define more complex ones, then one should look perhaps to the positions of Francisco Suarez and Franz Brentano who rely extensively on Aristotle's 'constructivist conception of point' in *Meta*. B.5. As Zimmerman notes, they both advocate the following:

> (i) Extended objects have indivisible parts, (ii) every extended object (including each of the infinitely many proper parts of a solid body) is surrounded by a 'skin' of point-sized parts which constitutes its two-dimensional surface, (iii) distinct extended objects touch when two such indivisible boundaries coincide, and (iv) the three-dimensionally extended parts of a thing are not made up out of indivisibles alone but also contain some 'atomless gunk', a substance all of whose parts have proper parts.' [Zimmerman (1996), p.158]

According to this reading, continuous stretches of magnitudes are surrounded by determinate lower–dimensional entities which enjoy actual existence: a three–dimensional sphere, for example, is surrounded by an actually existing two–

dimensional surface, an one–dimensional line by two actually exisitng zero–dimensional points, and so on. This interpretation has to accommodate, however, Aristotle's claim in *Meta* M.2 that mathematicals cannot actually exist as forms or shapes of substances: is that Brentanian view compatible with Aristotle's claim that lower-dimensional entities cannot exist in the sensibles? For, I understand that the claim that every body is surrounded by an actual two-dimensional surface is equivalent to the claim that this surface is in the body as some sort of an actual constituent of it (that is, in the first sense of 'in-ness' in *Physics* Δ.3, 210a14-24, where 'in-ness' is understood as parthood).[237] But even if one does accommodate this clain one has to provide the reader with a satisfactory analysis of Aristotle's notion of contact.[238] According to my–Ockhamist if you like–preferences, contact does not require the actual existence of limit entities. I cannot argue for this claim here but I can point to a tradition of followers of Ockham such as Adam Wodeham and John Buridan, both of whom relied on methods of proportional division *ad infinitum* to explain how the parts of divisible bodies can be said to touch each other.[239]

Moreover, one can argue that the interpretation I am advancing was conceptually available to Aristotle not only because it makes heavy use of his

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[237] See, for example Chisholm's understanding of Brentanian boundaries in [Chisholm (1983), pp.90-91].

[238] I find the contemporary attempts to explain Aristotle's understanding of contact extremely disappointing, not least because they do not take into account the vast scholastic tradition on the matter. For a contemporary contribution that is based on Bolzano's 'monstrous neighbors' but is nevertheless adequate enough as an introduction to the problem of contact in Aristotle, consult [Bartha (2001)].

[239] Cf. the following passage from Wodeham:

> For example, <a sphere would touch a plane> by means of its <lower> half, constructed traversely; and by means of a half of that same <half> constructed in parallel - <that is>, the lower half similarly reaching the plane, and so on *ad infinitum*, as can be proven by argument and also using the examples introduced above here. [Wodeham: *De indiv.* 2.3.14; Wood's trans.]

For a helpful discussion on the Ockhamist tradition on contact consult [Zupko (1993)], where the above and other passages are discussed extensively.

notion of potential infinity[240] but also because Aristotle was acquainted with Eudoxus' astronomical model of *concentic spheres*. According to Eudoxus the complex, apparent paths of the various celestial bodies are the result of the circular motions of a certain number of concentric spheres: the spheres are of different size, one inside another, they move about a diameter as axis in different though uniform speeds with the earth at rest at the common centre of those spheres. Aristotle transformed this purely geometrical theory into a more mechanical one, by arguing that the spheres need to be in contact with one another so that they would consitute a continuous system of spheres.[241] As a concluding remark, there is a suggestion by Sambursky according to which Aristotle's account of the potential existence of limit entities bounding a continuous stretch of magnitude might have been picked up by the Stoics, in their attempt to remove limit entities from reality.[242]

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[240] Which, as White points out, was merely a layman's guide to Eudoxus' method of exhuastion. [White (1992), p.144]

[241] For a detailed discussion of the theory of concentric spheres in Eudoxus and Aristotle one can consult [Heath (2013)]. Aristotle describes the Eudoxean system of concetric spheres in *Meta*. Λ.8 1073b17-1074a14; a more detailed source of information is the commentaty of Simplicius on Book II of *De Caelo*.

[242] Aristotle's conception of limit entities had a major influence on the Stoic conception of spatial magnitude. Both Aristotle and the Stoics shared a view of spatial magnitude as something infinitely divisible (see [White (1992)] for a detailed discussion, esp. ch.7). According to Sambursky, the Stoics 'discarded the conception of the discrete surface of a body, or generally a distinct boundary of *n-1* dimensions forming the surface of a figure of *n* dimensions (n=1,2,3), and replaced it by an infinite sequence of boundaries defining the surfaces of inscribed and circumscribed figures which converge from both sides to the figure in question and thus define it as a dynamic entity.' In [Sambursky (1959), p.96]. White rightfully complains that Sambursky's replacing of the 'distinct boundary' of a body by an 'infinite sequence of boundaries converging to that boundary' seems entirely Pickwickian; however, Sambursky's account can be saved if instead of sequence of boundaries we speak of sequences of three-dimensional parts that converge to the boundary in question (à la Whitehead).

Chapter 3: Aristotle on the metaphysical status of numbers: An exploration

Number is the easiest of all things to remember. (*Rhetoric* Γ.9, 1409b5-6)

[3.1] Introduction: A Fregean Aristotle?

It seems that in ordinary language, number-words function primarily as adjectives.[243] We say that my desk has wooden legs, but equally well that it has four legs. Frege acknowledges this as he sets out to introduce his own view of number:

> In language, numbers most commonly appear in adjectival form and attributive construction in the same sort of way as the words hard or heavy or red, which have for their meanings properties of external things. It is natural to ask whether we must think of the individual numbers too as such properties, and whether, accordingly, the concept of Number can be classed along with that, say, of colour. [*Foundations*, §21, p.27][244]

If number-words play an adjectival role in ascriptions of number, it is quite natural to follow Frege in asking the following question: Should we think of numbers as *properties of things*?[245] The adjectival function of number-words encourages a view of number-words as first-level predicates, and hence as standing for properties of things. However, this account of number has been neglected (at best) in view of the supposedly decisive arguments against it formulated by Frege in the *Foundations of Arithmetic*. Most of Frege's arguments are designed to show that attributions of number cannot have the same logical form as, for example, the attributions of colour. The following passage from the *Foundations* encapsulates Frege's position:

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[243] I say 'primarily' because number-words occur in two forms: 1) as adjectives, as in ascriptions of numbers (sentences that begin with 'There are' followed by a number-adjective, e.g. 'There are four books on my desk'), and 2) as nouns, as in most number-theoretic propositions (e.g. '2+2=4').

[244] The failure to distinguish between use and mention appears in the original. All excerpts are from Austin's translation of Frege's *Die Grundlagen der Arithmetik*.

[245] Notice that Frege speaks of 'external' (=perceptible?) things.

> [T]he content of a statement of number is an assertion about a concept. This is perhaps clearest with the number 0. If I say 'Venus has 0 moons', there simply does not exist any moon or agglomeration of moons for anything to be asserted of; but what happens is that a property is assigned to the concept 'moon of Venus', namely that of including nothing under it. If I say 'the King's carriage is drawn by four horses', then I assign the number four to the concept 'horse that draws the King's carriage'. [*Foundations*, §46, p.59; underlining mine]

The Fregean rejection of numbers as first-order properties of things is taken for granted without much justification by two of the foremost commentators of Aristotle's philosophy of number, Mignucci and Lear.[246] Mignucci in particular, who is responsible for the most detailed account of Aristotle's philosophy of number to be found in the scholarship, is clearly inspired by Fregean-style arguments in his analysis when he writes:

> Even if one limits oneself to natural numbers – as Aristotle does – it is difficult, since Frege's *Foundations of Arithmetic* to think, that numbers are affections of objects. What is the object to which 12 applies? Of course, we can say that the Apostles were wise and that the Apostles were 12. But it is only a likeness of surface grammar[247] that could lead one to think that 12 is a predicate of the Apostles in the same way as wise is. From the fact that the Apostles were wise we can infer that John was wise if we know that John was an Apostle. The parallel inference of John was 12 from the Apostles were 12 is nonsense. Shall we dismiss Aristotle's view without further ado? [Mignucci (1987), p.188; underlining mine]

Mignucci's interpretation can be summarised as follows: numbers for Aristotle are not first-level, but second-level concepts,[248] since they are supposed to be

[246] In [Lear (1982)] and [Mignucci (1987)]. Lear writes: 'The main obstacle from giving a successful account of arithmetic is that number is not a property of an object.' He immediately points to Frege's *Foundations of Arithmetic* in an attempt to substantiate his claim. See [Lear (1982), p.183].

[247] Mr Denyer has pointed out to me that the expression 'the Apostles are twelve' is not idiomatic English. So there might be no likeness of surface grammar after all!

[248] The term 'concept' is a convenient term that covers both *properties* and *relations.*

involved in expressing a property of a sortal concept,[249] the property of *having so many instances*. What goes on when we ascribe a number to something - as when, for example, we say 'the Apostles are 12'? According to Mignucci's Fregean analysis, this is an assertion of second-level, ascribing a property to a first-level concept. The subject of the assertion is the concept *is an Apostle*, and the predicate is, in effect, *is a concept with 12 instances*. In what follows, I would proceed to unearth the Fregean presuppositions that guide Mignucci's analysis and try to show two things: first, that not all Fregean arguments against numbers as first-order predicates are sound, and, second, that there are serious textual as well as contextual reasons against the view that Aristotle held a Fregean account of number.

[249] In speaking about sortals I follow P.F. Strawson; in his work *Individuals: An Essay in Descriptive Metaphysics,* Strawson espouses a view of sortals as *universals*: 'A sortal universal supplies a principle for distinguishing and counting individual particulars which it collects.' In [Strawson (1964), p.168]. Some familiar examples of sortals: *horse, apple* and *man.*

[3.2] Frege's arguments against numbers as properties of perceptibles

[3.2.1] A first argument

In the first of these passages, Frege argues against the classification of numbers alongside colours as attributes of perceptible things,[250] because numerical predication is different from colour predication:

> Is it not in a totally different sense that we speak of a tree as having 1000 leaves and again as having green leaves? The green colour we ascribe to each leaf, but not the number 1000. If we call all the leaves of a tree taken together its foliage, then the foliage too is green, but it is not 1000. To what then does the property 1000 belong? It almost looks as though it belongs neither to any single one of the leaves, nor to the totality of them all; is it possible that it does not really belong to things in the external world at all? [*Foundations*, §22, p.28]

To understand this argument it is important to distinguish between two kinds of predicates, *distributive* ones and *collective* ones. A predicate like '…is green' is distributive in that it holds of some objects only if it holds individually for each one of them. In a sentence like 'The leaves are green', the predicate 'green' applies distributively, that is, any leave is green. Non-distributive or collective predicates, on the other hand, are possessed by two or more objects together, but by none of them individually: e.g. Russell and Whitehead wrote *Principia Mathematica* (together); Castor and Pollux were twins. Consider the sentence 'The leaves of the tree are 1000'. Although we ascribe the green colour to each single leaf this is not the case, according to Frege, with the number 1000. If '1000' were a proper predicate, then it would function similarly to other predicates, that is distributively. Since the predicate '1000' displays quite different behaviour from ordinary predicates, like colour ones, it cannot be a predicate of 'things in the external world', Frege tells us. As many scholars have

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[250] We also count non-perceptible things such as ideas and unicorns.

observed,[251] 'Frege's error here is that he glosses over the familiar distinction between distributive and collective predicates. But why is it a mistake if one takes number-terms to function as collective predicates? And we can push this question a bit further: Is it true that there can be *no account whatsoever* of numerical predicates which analyses them as ordinary (level-1) collective predicates, that is as being true of objects? If there is such an account, then Frege is not justified in asserting that numerical predicates cannot be of level-1, and that the sentence 'The leaves of the tree are 1000' must be analysed as a level-2 predication, that is, as asserting of the concept '…is a leaf of the tree' that it has 1000 instances.[252] As we shall see, such an account is provided by Socrates in the *Hippias Major*.

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[251] For a more detailed discussion one can consult the following works: [Lambros (1976), pp.381-383], [Oliver&Smiley (2013), pp.71-72], [Ganeri (1996), pp.114-115].

[252] As, for example, Jonardon Ganeri remarks in [Ganeri (1996), p.115]. According to a certain reading of Frege's semantics of plural terms advanced by Dummett, the latter function predicatively. Thus, Dummett says: 'A plural noun phrase, even when preceded by the definite article, cannot be functioning analogously to a singular term. There are, of course, complex objects; but their continued existence depends on the maintenance of some relation between their components [....] But a plural subject of predication or ascription cannot stand for any such composite object, both because it presupposes no relation between the objects alluded to, and because it determines which those objects are, in a way in which no composite object is uniquely articulable into components. There is no such thing as a 'plurality', which is the misbegotten invention of a faulty logic: it is only as referring to a concept that a plural phrase can be understood, because only a concept-word admits a plural. But to say that it refers to a concept is to say that, under a correct analysis, the phrase is seen to figure predicatively. Thus 'All whales are mammals', correctly analysed, has the form 'If anything *is a whale*, it is a mammal', and 'The Kaiser's carriage is drawn by four horses' the form 'There are four objects each of which *is a horse that draws the Kaiser's carriage*'.' In [Dummett (1991), p.93; his italics]. But there is some tension in Frege's semantics of plural terms that has been brought into light in a paper by Alex Oliver. Oliver–while discussing Dummett's analysis of Frege's semantics of plural terms–brings to our attention Fregean passages like the following: 'on the other hand, the phrase "the Romans" in the sentence "the Romans conquered Gaul" is to be regarded as a proper name, for here we are not saying of each Roman that he has conquered Gaul; we are speaking of the Roman people, which is to be regarded logically as an object' [Oliver (1994), pp.75-76]. It seems that in passages like the above Frege's thesis is that when a plural description is combined with a collective predicate like 'conquered Gaul' the description is a proper name standing for a whole! [op.cit., p.76]. Thus, applied to the above example, the phrase 'the leaves of the tree' is a proper name

[3.2.2] A second argument

Another well known argument of Frege is the following:

> If I give someone a stone with the words: Find the weight of this, I have given him precisely the object he is to investigate. But if I place a pile of playing cards in his hands with the words: Find the number of these, this does not tell him whether I wish to know the number of cards, or of complete packs of cards, or even say of honour cards at skat. To have given him the pile in his hands is not yet to have given him completely the object he is to investigate; I must add some further word - cards, or packs, or honours. Nor can we say that in this case the number words exist in the same thing side by side, as different colours do. I can point to the patch of each individual colour without saying a word, but I cannot in the same way point to the individual members. If I can call the same object red and green with equal right, it is a sure sign that the object named is not what really has the green colour; for that we must first get a surface which is green only. Similarly, an object to which I can ascribe different numbers with equal right is not what really has a number. [*Foundations*, §22, pp.28-29]

Frege's argument seems to be the following: from the fact that an ascription of number 'involves' a concept, it follows that an ascription of number is 'about' a concept – or in other words, the concept itself is the subject of numerical predication. Is Frege right? It seems that Frege's conclusion, namely that the bearers of numerical properties cannot be objects, does not follow. I do not disagree that we often have to supply a concept to *clarify* which entities we are counting; in many cases using a bare demonstrative will not suffice. Thus, Kris McDaniel offers a first response to Frege's argument above:

> If I hand you some cards and merely ask you, 'how many of them are there?', you will probably say 'fifty-four' since the natural interpretation of 'them' is one in which 'them' refers to the cards. But if I tell you then that the answer is not 'fifty-four' you will struggle to answer correctly unless I clue you in on what the 'them' referred to in the context of my

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standing for a whole when it is combined with the numerical collective predicate '...is 1000'; on a certain reading then, Frege's thesis is dangerously close to that of Mill, who asserts that numbers are properties of complex wholes. [op.cit., p.77]. I say more on Mill's view of number later in this chapter.

> utterance. (Perhaps ‘them’ referred in that context to the suits, in which case the correct answer was ‘four’.) [McDaniel (2013), p.218]

Kris McDaniel quickly points out that the case is similar with colours. If I ask you ‘What colour is this?’, you will not be in a position to answer my question unless you know which object I am pointing at. I can give you a hint by telling you that ‘this’ referred’ to my sweater. Shall we infer from this that colours should be attributed not to sweaters but to concepts of sweaters?[253]

[253]See [McDaniel (2013), p.219].

[3.3] A Fregean reading of *Phys*. Δ.12, 220b10-12

Let us consider the following passage from the *Physics*:

> The number of a hundred horses and that of a hundred men is one and the same, but the things of which it is the number are different-the horses are different from the men.[254] [*Phys*. Δ.12, 220b10-12; Hussey's trans.]

What goes on when we ascribe a number to something - as when, for example, we say 'there are 100 horses in the field'? As we have already seen, according to Mignucci's Fregean analysis, this is an assertion of second-level, ascribing a property to a first-level concept. The subject of the assertion is the concept *a horse in the field*, and the predicate is, *is a concept with 100 instances*:

> If a Fregean terminology were allowed, Aristotle's view could be clarified by saying that mathematical numbers are not first-level, but second-level concepts, since they are supposed to be involved in expressing a property of a sortal concept. If 'a hundred horses' means that the concept horse has a hundred instances, the mathematical number 100 contributes essentially to express such a property. [Mignucci (1987), p.198]

To properly understand Mignucci's interpretation of the passage, the reader needs to follow closely Frege's arguments such as the ones discussed previously in this chapter. Mignucci perhaps wishes to say something along these lines: no individual horse (an entity of level-0) can be said to be 100; each horse is a single thing. Or he might be opting for the stronger claim, that there cannot be *any* entity of level-0 to which the numerical property 100 can plausibly be ascribed.[255] Since there are no suitable bearers of numerical properties at level-0, Mignucci insists that we have to go a level up in the type hierarchy. In this he follows Frege; for the latter–as Bell comments–offers either i) a single entity of level-0, or ii) a single entity of level-1 (a concept) as the only possible bearer of a

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[254] ἔστι δὲ ὁ ἀριθμὸς εἷς μὲν καὶ ὁ αὐτὸς ὁ τῶν ἑκατὸν ἵππων καὶ ὁ τῶν ἑκατὸν ἀνθρώπων, ὧν δ' ἀριθμός, ἕτερα, οἱ ἵπποι τῶν ἀνθρώπων.

[255] That Mignucci seems to opt for the stronger claim can be seen later in his article, when he argues against the Millian view of numbers as properties of complex objects (or aggregates). See section 3.8.1 in this chapter.

number property, and by 'suggesting that because there cannot be any appropriate candidates of the first sort, an ascription of number must be an assertion about an item of the second kind - a concept'.[256]

[256] In [Bell (1990), p.66].

[3.4] Two senses of number

Passages that shed some light on the metaphysical status of numbers in Aristotle are rare; some of the most important of them are to be found in his account of time. There, Aristotle constantly emphasises that number is said in two ways:

> Hence time is a kind of number. But number is so called in two ways: we call number both that which is counted and countable, and that by which we count. Time is that which is counted and not that by which we count.[257] [*Phys*. Δ.11, 219b5-8; Hussey's trans.]

The distinction is also made in the following passage:

> Time is not the number by which we count but the number which is counted, and this number turns out to be always different before and after, because the nows are different. *The number of a hundred horses and that of a hundred men is one and the same, but the things of which it is the number are different-the horses are different from the men.*[258] [*Phys*. Δ.12, 220b8-12; Hussey's trans., italics mine]

Let us try to explicate these two senses of number:

[3.4.1] A first sense of number

The first sense of number is number as 'that which is counted and countable' (τὸ ἀριθμούμενον καὶ τὸ ἀριθμητόν), i.e. number as a plurality of units. This is the standard sense of number as employed by the mathematicians of the time.[259] Across the Aristotelian corpus, we find various definitions of number all of which come to the same thing: number is 'a plurality of indivisibles' (πλῆθος ἀδιαιρέτων, *Meta.* M.9, 1085b22), or 'a plurality of units' (πλῆθος μονάδων, *Meta.* I.1,

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[257] ἐπεὶ δ' ἀριθμός ἐστι διχῶς (καὶ γὰρ τὸ ἀριθμούμενον καὶ τὸ ἀριθμητὸν ἀριθμὸν λέγομεν, καὶ ᾧ ἀριθ μοῦμεν), ὁ δὴ χρόνος ἐστὶν τὸ ἀριθμούμενον καὶ οὐχ ᾧ ἀριθμοῦμεν.

[258] ὁ δὲ χρόνος ἀριθμός ἐστιν οὐχ ᾧ ἀριθμοῦμεν ἀλλ' ὁ ἀριθμούμενος, οὗτος δὲ συμβαίνει πρότερον καὶ ὕστερον ἀεὶ ἕτερος· τὰ γὰρ νῦν ἕτερα. ἔστι δὲ ὁ ἀριθμὸς εἷς μὲν καὶ ὁ αὐτὸς ὁ τῶν ἑκατὸν ἵππων καὶ ὁ τῶν ἑκατὸν ἀνθρώπων, ὧν δ' ἀριθμός, ἕτερα, οἱ ἵπποι τῶν ἀνθρώπων.

[259] For a detailed list of pre-Euclidean passages regarding the concept of *arithmos* consult [Pritchard (1995), pp.27-30].

1053a30).[260] The origins of these definitions are to be found perhaps in the wider Pythagorean tradition, where units are represented by pebbles and each number has some characteristic shape (e.g. triangular, square, etc.). Neo-Pythagoreans, such as Iamblichus and Nicomachus, define number variously as a 'compound of units' or as a 'definite plurality'. Nicomachus, specifically, combines several definitions into one when he says that 'number is a definite plurality or a compound of units or a flow of quantity composed of units' (Ἀριθμός ἐστι πλῆθος ὡρισμένον ἢ μονάδων σύστημα ἢ ποσότητος χύμα ἐκ μονάδων συγκείμενον, *Introd. Arithm*. Book 1, ch.7, 1.1-2). Commenting on this, Iamblichus ascribes to Thales the description of number as a 'compound of units' (μονάδων σύστημα), who 'follows the Egyptian view' (κατὰ τὸ Αἰγυπτιακὸν ἀρέσκον). He ascribes to 'Eudoxus the Pythagorean' the definition 'definite multitude'(πλῆθος ὡρισμένον).[261] As for the expression 'flow of quantity composed of units', Pritchard notes that the first part is rather opaque, while the last three words are taken from the Euclidean definition of number: Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος (*Elements*, Book VII, def. 2).[262]

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[260] What is for something to be indivisible? What is for something to be a unit? Aristotle's report of a certain Pythagorean (according to Proclus' comm. on the First Book of Euclid's *Elements*: 95.21-22) definition of the unit is not really helpful; for the latter essentially depends on the notion of the point and its negative character does not really help us to grasp what is to be a unit: 'that which is indivisible with respect to quantity in all dimensions and has no position is called 'unit'' (τὸ μὲν οὖν κατὰ τὸ ποσὸν ἀδιαίρετον, τὸ μὲν πάντη καὶ ἄθετον λέγεται μονάς, *Meta*. Δ.6, 1016b24-25); 'a unit is a position-less substance' (μονὰς οὐσία ἄθετος, *An. Post.* A.27, 87a36; also *An. Post.* A.32, 88a33-34); 'a unit is a point without position' (ἡ γὰρ μονὰς στιγμὴ ἄθετός ἐστιν, *Meta.* M.8, 1084b26-27). In some cases Aristotle considers the term 'indivisible' in the sense of 'cannot be divided' to be equivalent to 'one': 'The one and the many are opposed in several ways, of which one is the opposition of the one and plurality as indivisible and divisible; for that which is either divided or divisible is called a plurality, and that which is indivisible or not divided is called one.' (Ἀντίκειται δὲ τὸ ἓν καὶ τὰ πολλὰ κατὰ πλείους τρόπους, ὧν ἕνα τὸ ἓν καὶ τὸ πλῆθος ὡς ἀδιαίρετον καὶ διαιρετόν· τὸ μὲν γὰρ ἢ διῃρημένον ἢ διαιρετὸν πλῆθός τι λέγεται, τὸ δὲ ἀδιαίρετον ἢ μὴ διῃρημένον ἕν, *Meta*. I.3, 1054a20-23, cf. I.6, 1057a12-17). Recall also the discussion in *Republic* Book VII where Plato argues that the 'ones' and the 'numbers' grasped by the senses are not truly ones and numbers, since 'we do see the same thing as one and as an unlimited number at the same time' (525a4-5).

[261] In *Nicom. arithm. introduc*., p.10, ln.8-10 and ln.17-20, respectively.

[262] In [Pritchard (1995), pp.25-26].

[3.4.2] An Aristotelian response to Frege

Consider the following passage from *Metaphysics* N.1 passage 1087b33-1088a14:

> The measure must always be some one and the same thing applying to all cases; for example, if there are horses the measure is horse, if men it is man. If there are a man, a horse, and a god, the measure will perhaps be living thing, and their number will be a number of living things. If there are a man, white, and walking, they will hardly have a number, because they all belong to the same thing which is numerically one. Still, they will have a number of categories or some such term.[263] [*Meta*. N.1, 1088a8-1088a14; Annas' trans.]

The pluralities that Aristotle mentions in this passage can be classified into the following three categories: a) Pluralities that are composed of members of the same kind, i.e. horses, men (1088a9); b) Pluralities composed of members of different kinds but of the same genus, i.e. a man, a horse, and a god (1088a10); c) Pluralities composed of members of different Aristotelian categories, i.e. a man, white, walking (1088a11-12). If for cases a) and b) Aristotle accepts that we can count them without particular effort, case c) is more problematic. Aristotle is circumspect: 'If there are a man, white, and walking, they will hardly have a number, because they all belong to the same thing which is numerically one; still they will have a number of categories or some such term' (1088a11-14). Aristotle in effects tells us that were one to ask 'what is the number of them' the most natural response would be 'three categories' because the 'them' refers to the categories or to something similar. What the above passage makes clear, is that counting presupposes agreement upon some unit concept. This, however, does not necessarily mean that number is predicated of the concept (à la Frege). Certain pluralities of the passage (such as the plurality whose members are a man, a horse, and a god) can be specified either by listing their members one by

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[263] δεῖ δὲ ἀεὶ τὸ αὐτό τι ὑπάρχειν πᾶσι τὸ μέτρον, οἷον εἰ ἵπποι, τὸ μέτρον ἵππος, καὶ εἰ ἄνθρωποι, ἄνθρωπος. εἰ δ' ἄνθρωπος καὶ ἵππος καὶ θεός, ζῷον ἴσως, καὶ ὁ ἀριθμὸς αὐτῶν ἔσται ζῷα. εἰ δ' ἄνθρωπος καὶ λευκὸν καὶ βαδίζον, ἥκιστα μὲν ἀριθμὸς τούτων διὰ τὸ ταὐτῷ πάντα ὑπάρχειν καὶ ἑνὶ κατὰ ἀριθμόν, ὅμως δὲ γενῶν ἔσται ὁ ἀριθμὸς ὁ τούτων, ἤ τινος ἄλλης τοιαύτης προσηγορίας.

one *or* by identifying them as instances of a certain concept (the concept 'living thing'). Aristotle with his notion of measure explicates the sense of number as *that which is countable*, i.e. a plurality of objects of a certain kind. The members of this plurality are also in a trivial sense indivisible (in the sense of undivided). What is one man is not also divisible into many men, what is one horse, is not also divisible into many horses:[264]

> Reasonable, too, it is that while in number there is a limit at the minimum, but in the direction of 'more' number always exceeds any multitude…The reason for this is that one is indivisible, whatever may be one (e.g. a man is one man and not many), but number is a plurality of ones, a certain 'many' of them. So there must be a halt at the indivisible. [*Phys.* Γ.7, 207b1-8; Hussey's trans.]

[3.4.3] Aristotle's criticisms of Platonic Form numbers in *Meta*. M.6-8

This is not however the only sense of number that we find in Aristotle. Mignucci invites us to consider the following difficulty that stems from the view that numbers are certain collections of objects: If numbers have to be identified with collections of objects (3, say, with a collection of three men), then collections of objects which differ simply because they have different members must be *different* numbers:

> Not only do three men differ from four cats, they also differ from three dogs, as everyone will admit. But if three men is a number, say 3, then three dogs is a different number 3; it is a different number, i.e. a different 3. *According to this view there can be as many 3s as there are possible collections of these objects*. It is not difficult to recognise that Aristotle could be charged with such a criticism. [Mignucci (1987), p.196; italics mine]

Is Mignucci right? Could Aristotle be charged with such a criticism? Or is this an objection that Aristotle raises *against* a certain view of numbers? And if so why is this view problematic? To answer these questions, let us turn our attention to the *constitutive* units of the numbers. One might, perhaps, claim that the units of

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[264] See also [Wedberg (1955), pp.72-74]. We leave aside here problematic cases such as the raindrops: raindrops can coalesce, resulting in claims such as 1 + 1 = 1. I owe this remark to Mr Denyer.

the number 3 (the three men), cannot be added to the units of the 3* (the three dogs), given that they differ in kind. Or, if number 3 is to be identified arbitrarily with a collection of three men, say, and number 4 with a collection of four cats, say, then we cannot claim that 3 is part of 4. Thus, numbers in this sense fail to account for (at least some of) the operations and relations of arithmetic.[265] Mignucci, however, does not provide much justification for the claims he makes here; in order to address them properly we need to turn our attention to *Metaphysics* M.6-8 and Aristotle's discussion of the various Academic theories of number there.

Much of Aristotle's argumentation in *Metaphysics* M.6-8 seems to be centered around the fact that Platonic Form numbers cannot account for some of the more mundane operations of arithmetic. In those passages Aristotle assumes from beginning to end that Platonic Form numbers are collections of units, the Platonic number Four, for example, is a collection of four units. There seems to be quite a disagreement among scholars about whether Plato held a different conception of Form numbers, a conception which was not fully understood by his followers in the Academy, and which was therefore wrongly criticised by

[265] Mignucci's objections remind me of Benacerraf's ones against numbers conceived as sets. In his paper 'What Numbers Could Not Be', Benacerraf argues that numbers cannot be sets. The essence of his argument is that there are many different ways to represent numbers as sets, and that we cannot give proper reasons for preferring one way over the others. Thus, Benacerraf: 'If numbers are sets, then they must be particular sets, for each set is some particular set. But if the number 3 is really one set rather than another, it must be possible to give some cogent reason for thinking so; for the position that this is an unknowable truth is hardly tenable.' [Benacerraf (1965),p.62]. Benacerraf concludes that numbers cannot be sets. Thus Mignucci seems to adopt the following Benacerraf-inspired argument: 1)There are many different ways to represent 3 as collections of objects, e.g. 3 as a collection of three men or 3 as a collection of 3 cats. 2)Either none of these accounts is correct, or one of them is, or both are correct. 3)If both accounts are collect the set of three men is identical to the set of three dogs, which is clearly wrong. Thus the accounts cannot both be correct. 4)If one of them is correct, say 3 is three men, then why is it so? We ought to give some proper reason to explain our choice. It seems, however, that our choice is completely arbitrary. Thus, 5) none of the accounts is the correct one, so that numbers cannot be collections of objects. (Following the exposition of Bencarraf's argument in [Wetzel (1989), pp.273-274].)

Aristotle. Scholars such as Cherniss and Tarán rely primarily on an influential article by Cook Wilson in order to reject the criticisms offered by Aristotle as based largely on an incorrect interpretation of Platonic Form numbers,[266] whereas others such as Burnyeat and Pritchard think that Aristotle has correctly understood the Platonic theory of Form Numbers.[267] But what is this different conception? According to the 'Cook Wilson camp', Form numbers are not collections of units but they are–just like other Platonic Forms–fundamentally unique and part-less. As Tarán puts it: 'These numbers, however, are *not* congeries of units, as Aristotle thinks they are, but merely the hypostatisation of the universals which constitute the series of natural numbers.'[268] I will not discuss the dispute in detail but, in the course of this essay, I will point to several places in Plato's dialogues that support both readings of Form numbers. What is important for my essay is that Aristotle *interprets* Platonic Form numbers (Two, Three, Four,…) as collections of two units, three units, four units, and so on.[269]

Platonic Form numbers, according to Aristotle, enjoy substantial existence (1080a15-16), they are ordered, and they differ in kind from one another (1080a17-18).[270] In the following lines (1080a18-30) Aristotle tells us that there are the following possibilities for the units of which they are composed: (i) the units are also different in kind so that the units are all incomparable with each other; (ii) the units do not differ in kind and are all comparable with each other, as is the case with the units of the mathematical number; (iii) each unit is comparable with the other units within each number, but incomparable with the units of the other numbers (for example, each unit in the triad is comparable

[266] In [Wilson (1904)], [Tarán (1978)], and [Cherniss (1944), pp.513-524].

[267] In [Burnyeat (1987) pp.234-235] and [Pritchard(1995) pp.33-38].

[268] Tarán (1978), p.83. I am not entirely sure as to what Tarán means when he writes about 'the hypostatisation of the universals which constitute the series of natural numbers'; does he refer to the natural number series: 1,2,3,…?

[269] Given, however, that the Forms are self-predicable, then, for example, the Form of three would have to be unqualifiedly three, which means it must consist of three units.

[270] εἴπερ ἐστὶν ὁ ἀριθμὸς φύσις τις καὶ μὴ ἄλλη τίς ἐστιν αὐτοῦ ἡ οὐσία ἀλλὰ τοῦτ' αὐτό, ὥσπερ φασί τινες, ἤτοι εἶναι τὸ μὲν πρῶτόν τι αὐτοῦ τὸ δ' ἐχόμενον, ἕτερον ὂν τῷ εἴδει ἕκαστον. (*Meta*. M.6, 1080a15-18).

with the other unit in the triad but not with any of the units in the pentad) . The term συμβλητός is usually translated 'comparable'; according to Ross things are comparable if and only if they are members of the same kind (he points to *Phys*. 248b8, 249a3, *Top*. 107b17, *Meta*. I. 1055a6). Its negation, άσύμβλητος, is equivalent in meaning with 'specifically different' (ἕτερον ὂν τῷ εἴδει) and elsewhere συμβλητός is taken as equivalent in meaning with 'undifferentiated' (ἀδιάφορος, 1081a5-6).[271] How are we supposed to understand mathematical numbers which consist of 'undifferentiated' units (option (ii) above)? Perhaps like this: we are given an (infinite) pool of ideal units, all of them are of the same type, and each mathematical number is a collection of such units; since any two units make a two there will be many two's and the Form number Two will no longer be unique; so on this option numbers cannot be Forms (1081a5-17). Regarding option (i) above, it seems that there exist many (actually, infinitely many) distinct units a,b,c,d,..., each unit unique in kind, and each number is a collection of such units: take 2 to be (b,c) and 4 to be (d,e,f,g); one cannot then argue that 2 is part of 4. Option (iii) above is introduced in the following passage:[272]

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[271] Ross (1924), vol. II, p.427.

[272] Commentators have struggled connecting Aristotle's classification of the units that constitute each number in lines 1080a17-35 and his classification of number in 1080a35-37. To the best of my knowledge Tarán offers the best explanation of what is happening there; there seem to be the following three kinds of numbers: (a) incomparable numbers with units all incomparable, (b) mathematical numbers with units all comparable, and (c) incomparable numbers with the units of each number comparable with each other but incomparable with those of other numbers. What has then become of the view of the units that are all comparable with each other, as is the case with the units of the mathematical number introduced in (ii) above? Aristotle seems to argue as follows: if the units in each Form number are fully comparable with the units of another Form number then those numbers are not one-per-type anymore but they *become* mathematical numbers, i.e. collections of units which are many-per-type (cf. 1081a5-6: εἰ μὲν οὖν πᾶσαι συμβληταὶ καὶ ἀδιάφοροι αἱ μονάδες, ὁ μαθηματικὸς γίγνεται ἀριθμὸς καὶ εἷς μόνος) [Tarán (1978), p.87]. Aristotle tells us that no one has ever held view (a) (*cf. Metaph.* 1080b8-9 and 1081a35-36), some have held view (b) (these are Speusippus and the Pythagoreans, *cf.* 1080bl4-21; the difference being that the Pythagoreans endorse this view of number without further supposing that it is separate from perceptible objects, but that perceptible objects are composed out of such numbers, cf. 1080b16-20), someone held view (c) (this is the anonymous Platonist of *Metaph.*

> Or some units must be comparable and some not, e.g. Two is first after One, and then comes Three and then the other numbers, and the units in each number are comparable, e.g. those in the first Two with one another, and those in the first Three with one another, and so with the other numbers; but the units in the Two itself are not comparable with those in the Three itself; and similarly in the case of the other successive numbers. Therefore while mathematical number is counted thus—after one, two (which consists of another one besides the former one), and three (which consists of another one besides these two), and the other numbers similarly, ideal number is counted thus—after One, a distinct Two which does not include the first One, and a Three which does not include the Two, and the other numbers similarly.[273] [*Meta*. M.6, 1080a23-35; Ross' trans. mod.]

This view of Form numbers is examined in more detail in lines 1081b35-1083a17; one of the problems that stem from this view is discussed in the following passage:

> For example, in the original Ten there are ten units, and Ten is composed both of these and of two Fives. Since the original Ten is not just any number and is not composed of just any Fives (or just any units) the units in this Ten must differ. If they do not differ, the Fives of which the Ten consists will not differ either, but since they do differ, the units will differ too. But if they do differ, will there be no other Fives in <the Ten> but only these two, or will there be? It is absurd if there are not; but if there are, what kind of Ten will be composed of them? There is no other Ten in the Ten over and above itself.[274] [*Meta*. M.7,

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1080b21-22), others (b) and (c) (this is the view Aristotle ascribes to Plato, *cf. Metaph.* 1080b11-14 with 987bl4-18), and certain others have identified (b) and (c) (this view is implicitly attributed to Xenocrates; on Xenocrates' identification of the ideas with mathematical numbers *cf. Metaph.* 1080b22-23 and 28-30, 1028b24-27, 1069a35, and 1076a20-21). The main source for the attributions above is [Tarán (1978)]; consult also Menn['Iγ3'] for extensive discussion.

273 ἢ τὰς μὲν συμβλητὰς τὰς δὲ μή (οἷον εἰ ἔστι μετὰ τὸ ἓν πρώτη ἡ δυάς, ἔπειτα ἡ τριὰς καὶ οὕτω δὴ ὁ ἄλλος ἀριθμός, εἰσὶ δὲ συμβληταὶ αἱ ἐν ἑκάστῳ ἀριθμῷ μονάδες, οἷον αἱ ἐν τῇ δυάδι τῇ πρώτῃ αὑταῖς, καὶ αἱ ἐν τῇ τριάδι τῇ πρώτῃ αὑταῖς, καὶ οὕτω δὴ ἐπὶ τῶν ἄλλων ἀριθμῶν· αἱ δ' ἐν τῇ δυάδι αὐτῇ πρὸς τ ὰς ἐν τῇ τριάδι αὐτῇ ἀσύμβλητοι, ὁμοίως δὲ καὶ ἐπὶ τῶν ἄλλων τῶν ἐφεξῆς ἀριθμῶν· διὸ καὶ ὁ μὲν μαθ ηματικὸς ἀριθμεῖται μετὰ τὸ ἓν δύο, πρὸς τῷ ἔμπροσθεν ἑνὶ ἄλλο ἕν, καὶ τὰ τρία πρὸς τοῖς δυσὶ τούτοις ἄλλο ἕν, καὶ ὁ λοιπὸς δὲ ὡσαύτως· οὗτος δὲ μετὰ τὸ ἓν δύο ἕτερα ἄνευ τοῦ ἑνὸς τοῦ πρώτου, καὶ ἡ τριὰς ἄνευ τῆς δυάδος, ὁμοίως δὲ καὶ ὁ ἄλλος ἀριθμός)·

274 οἷον γὰρ ἐν τῇ δεκάδι αὐτῇ ἔνεισι δέκα μονάδες, σύγκειται δὲ καὶ ἐκ τούτων καὶ ἐκ δύο πεντάδων ἡ δεκάς. ἐπεὶ δ' οὐχ ὁ τυχὼν ἀριθμὸς αὐτὴ ἡ δεκὰς οὐδὲ σύγκειται ἐκ τῶν τυχουσῶν πεντάδων, ὥσπερ οὐδὲ μονάδων, ἀνάγκη διαφέρειν τὰς μονάδας τὰς ἐν τῇ δεκάδι ταύτῃ. ἂν γὰρ μὴ διαφέρωσιν, οὐδ' αἱ πεντάδες διοίσουσιν ἐξ ὧν ἐστιν ἡ δεκάς·ἐπεὶ δὲ διαφέρουσι, καὶ αἱ μονάδες διοίσουσιν. εἰ δὲ

1082a1-11, Annas' trans. mod.]

Aristotle's objections highlight the difficulties that arise from the counter-intuitive result that the ordinary operations of arithmetic (such as the addition) cannot be applied consistently to this particular version of Form numbers. One wishes to say that Form number Ten is made up of ten units and also of two Fives (σύγκειται δὲ καὶ ἐκ τούτων καὶ ἐκ δύο πεντάδων ἡ δεκάς). If the first Five consists of units of a certain kind then the second Five, being a different one, will necessarily consist of units of another kind (call the second one, 'Five*'). Hence the Ten consists of Five and Five*, that is, the units in the Ten do not all belong to the same kind, which runs contrary to the Platonist belief that units in the same Number are undifferentiated from one another (αἱ δ' ἐν τῷ αὐτῷ ἀριθμῷ ἀδιάφοροι ἀλλήλαις, 1081b35-37).

Another argument is the following:

> Besides, if every unit and another unit make two there will be a two made up of a unit from the original Two and another from the original Three, which will thus be made of differentiated units. Also will it be before or after it? It seems rather as if it must be before, since one of the units comes about together with three, and the other together with two. We suppose that in general one and one make two, whether they are equal or unequal-good and bad, for instance, or man and horse; but people with these views suppose that not every two units make two.[275] [*Meta.* M.7, 1082b11-19, Annas' trans. mod.]

In this passage, Aristotle complains that on this view of Form numbers one cannot add two units (however heterogeneous these units are), something that offends common sense. He asks: Can you make a two from one unit from the original Two and one from the original Three? Recall that the units in the Two

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διαφέρουσι, πότερον οὐκ ἐνέσονται πεντάδες ἄλλαι ἀλλὰ μόνον αὗται αἱ δύο, ἢ ἔσονται; εἴτε δὲ μὴ ἐνέσονται, ἄτοπον· εἴτ' ἐνέσονται, ποία ἔσται δεκὰς ἐξ ἐκείνων; οὐ γὰρ ἔστιν ἑτέρα δεκὰς ἐν τῇ δεκάδι παρ' αὑτήν.

[275] ἔτι εἰ ἅπασα μονὰς καὶ μονὰς ἄλλη δύο, ἡ ἐκ τῆς δυάδος αὐτῆς μονὰς καὶ ἡ ἐκ τῆς τριάδος αὐτῆς δυὰς ἔσται ἐκ διαφερουσῶν τε, καὶ πότερον προτέρα τῆς τριάδος ἢ ὑστέρα; μᾶλλον γὰρ ἔοικε προτέραν ἀναγκαῖον εἶναι· ἡ μὲν γὰρ ἅμα τῇ τριάδι ἡ δ' ἅμα τῇ δυάδι τῶν μονάδων. καὶ ἡμεῖς μὲν ὑπολαμβάνομεν ὅλως ἓν καὶ ἕν, καὶ ἐὰν ᾖ ἴσα ἢ ἄνισα, δύο εἶναι, οἷον τὸ ἀγαθὸν καὶ τὸ κακόν, καὶ ἄνθρωπον καὶ ἵππον· οἱ δ' οὕτως λέγοντες οὐδὲ τὰς μονάδας.

are of one kind and the units in the Three are of a different one. The resulting two is obviously different from the original Two, since its members are different; call it 'Two*'. Following this line of thought and taking into account the serial ordering of Form numbers (1080a17-18) Aristotle asks: what would be the place of this Two* in the order? It seems that it should be before Three since it consists of a unit of Two and a unit of Three. But this is absurd because there are no numbers between Two and Three. Hence on this view we cannot say that 'every two units make two'.

It is time, however, we turned our attention to some of the more positive evidence Aristotle has to offer us.

[3.4.4] A second sense of number

Number as 'that by which we count'. The *Physics* passages previously cited indicate that there is a sense of number different from that of number as a plurality of units/measures. Three men, four horses, five cows are countable numbers, *arithmoi*. They are collections of objects, from which the numbers we count with have to be distinguished. The latter are normally identified in the scholarship with *abstract or abstracted* numbers.[276] What does this claim amount to? Could we, perhaps, tentatively suggest that an abstract number is what all the collections with the same amount of instances have in common? To answer this question let us return to Aristotle's account of time; many commentators have complained that there are certain problems with it. What is particularly problematic is that Aristotle does not seem to remain consistent to the view that time is 'that which is counted and not that by which we count'.

It is not, of course, my intention to offer here a full explication of Aristotle's conception of time and to discuss alternative interpretations. Rather, I wish to focus on Bostock's analysis, which, in my opinion, is one of the few that takes into account that time is something that is predicated of the various movements that have some time:

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[37] In [Gaukroger (1982), pp.312-313]; [Mignucci (1987), pp.197-198], [Hussey (1983), p.161]. For an alternative interpretation see [Coope (2005)].

> But one of the striking features of Aristotle's discussion of time is his *failure* to mention that the word 'time' has many senses. Some of these are well illustrated by the way the phrase 'a particular time' (χρόνος τις or χρόνος τις ὡρισμένος) is quite ambiguous between a date pure and simple (yesterday noon), a dated temporal stretch (from noon yesterday to noon today), and a quantity of temporal stretch (24 hours). There are also other senses of this phrase. For example, when one says 'dinner is always at 7.30 sharp' one may properly be said to be giving a particular time as the time of dinner, but this time is what one might call a *recurring* date (and similarly with recurring periods). If Aristotle had paused to point out these ambiguities explicitly he would have saved himself from the appearance of outright contradiction on several occasions. (For example, 'earlier and later times are always different' (220b9–10), and only three lines later 'one and the same time may occur again and again, e.g. a year' (220b13–l4).)[277] [Bostock (2006), p.143; his italics]

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[277] Bostock adds that when Aristotle provides an explanation for his point that simultaneous movements have exactly the same time he once again considers time as a number: 'Of equal and simultaneous movements', he says, 'the number is one and the same, wherever they may occur' (223b11–12), and he adds in comparison that 7 dogs and 7 horses have the same number. On this occasion, however, it seems that time answers to the number 7 'with which we number'. It seems to me that Bostock is correct in complaining that 'earlier he has used the point that 100 horses is not the same thing as 100 men in order to justify saying that non-simultaneous occurrences do not have the same time, explicitly comparing the times to the different things numbered.' In [Bostock (2006), p.142; his translations]. Bostock helpfully lists other problematic places such as lines 220b4-5, where Aristotle is trying to explain why time cannot be quick or slow, and says simply that no 'number with which we number' is quick or slow, evidently implying that time is a number with which we number. [ibid, p.143]. According to Coope's analysis, to say that the 7 dogs and the 7 horses are the same number is not to imply that there is some one thing, some abstract number, that is the number of each [Coope (2005), p.120]. What Coope perhaps means is that the fact that the 7 horses are the same in number with the 7 dogs ought not to be interpreted as an identity statement between two abstract particulars, i.e. 7=7. Rather, we ought to say that '=' is not a symbol of identity but of equinomerosity between countable numbers. She is surely right in that. But she still does not take into account that those arithmoi (of horses and dogs) not only are equinumerous but that they also belong to the same species of number, that they are each, a 7. She is then hard-pressed to explain that 'time is a number of continuous change simpliciter, not of one particular type' (223a33– 223b1; her trans.). It seems that time, understood in a more universal manner, corresponds to the common kind to which 7 dogs and 7 horses belong, namely to the kind of 7-membered pluralities, or to number 7 simpliciter.

David Bostock offers the following plausible explanation as to why Aristotle does not seem to remain consistent to the view that time is ‘that which is counted and not that by which we count’:

> The source of the trouble is that Aristotle has two quite different reasons for calling time a number, the first is that a time always has an amount of duration, i.e. it is a subject of which amounts of duration are predicated, and the second is that a time is itself predicated of the various movements that have that time. The second reason really amounts to no more than the point that a time is a universal, and is a very thin ground for regarding it as some kind of ‘number’. [Bostock (2006), p.142]

Especially illuminating about this second sense of number is Aristotle’s treatment of the now in the following passage:

> In so far as the now is a limit it is not time, but belongs to it. But in so far as it numbers, it is a number. For limits pertain only to that of which they are limits, whereas the number of these horses—ten—may hold elsewhere too.[278] [*Phys.* Δ.11, 220a21–4; Bostock’s trans.]

What this passage shows is that there is a certain contrast between the limiting function of the now and its numbering function. The difference is this: ‘limits pertain only to that of which they are limits, whereas the number of these horses— ten—may hold elsewhere too’. It seems then that the crucial feature of a number is just that it is universal, and the now ‘in so far as it numbers’ may be treated as universal as well, namely as a date holding of all the momentary events which have that date.[279] As Bostock explains:

> It is this which entitles us to call the now a number (and perhaps also what entitles us to regard it as a time). The boundary of a particular movement is not a number, since it is the

[278] ᾗ μὲν οὖν πέρας τὸ νῦν, οὐ χρόνος, ἀλλὰ συμβέβηκεν· ᾗ δ’ ἀριθμεῖ, ἀριθμός· τὰ μὲν γὰρ πέρατα ἐκείνου μόνον ἐστὶν οὗ ἐστιν πέρατα, ὁ δ’ ἀριθμὸς ὁ τῶνδε τῶν ἵππων, ἡ δεκάς, καὶ ἄλλοθι.

[279] It is this passage where Coope seems particularly troubled to accommodate it in her interpretation: ‘My interpretation does not solve all the difficulties about this passage. It is odd to find Aristotle saying that the now (as opposed to a series of counted nows) is a number. Moreover, it is not clear quite what he means when he says that the ten of these horses is ‘also elsewhere’. On my interpretation, what one would expect him to say is that the ten that is the number we use to count these horses could also be used to count ten things of some other kind (for instance, ten dogs).’ In [Coope (2005), fn.18 in p.124].

boundary merely of that movement and not of anything else. The boundary even of a stretch of time is not a number simply by being a boundary. What makes it a number is that the boundary of a stretch of time is also the boundary of all movements that have that time, i.e. it is a universal. [Bostock (2006), p.149]

[3.5] The beginnings of a non-Fregean account

Is it somehow possible to retain the intuitive idea that number-predicates are first-level predicates, applicable to things of the world? Consider the following passage:

> Reasonable, too, it is that while in number there is a limit at the minimum, but in the direction of 'more' number always exceeds any multitude, yet in the case of magnitudes, on the contrary, they exceed any magnitude in the direction of 'less' but in that of 'more' there is no infinite magnitude. The reason for this is that one is indivisible, whatever may be one (e.g. a man is one man and not many), but number is a plurality of ones, a certain 'many' of them. So there must be a halt at the indivisible. ('*Three' and 'two' are paronymous names, and similarly each of the other numbers.*)[280] [*Phys.* Γ.7, 207b1-10; Hussey's trans. mod., italics mine]

Let us have a closer look at the last line from the *Physics* passage above, where Aristotle claims that says that 'three and two are paronymous names – and so too is each of the other numbers' (τὸ γὰρ τρία καὶ δύο παρώνυμα ὀνόματά ἐστιν, ὁμοίως δὲ καὶ τῶν ἄλλων ἀριθμῶν ἕκαστος, *Physics*, Γ.7, 207b8-10). Perhaps, he means that number nouns are in some way derivative from the number adjectives. According to Mignucci from this last sentence we may even extract some useful information about numbers in the second sense:

> … the substantives 'two', 'three', and so on are said to be paronymous nouns in the sense that they are derivative from the corresponding adjectives. This does not mean that the nouns 'two', 'three', and so on are etymologically derived from the corresponding adjectives *but that they have their ontological ground in the adjectives*. Now the numbers which are said to be pluralities of ones are obviously countable numbers, while the nouns

[280] εὐλόγως δὲ καὶ τὸ ἐν μὲν τῷ ἀριθμῷ εἶναι ἐπὶ μὲν τὸ ἐλάχιστον πέρας ἐπὶ δὲ τὸ πλεῖον ἀεὶ παντὸς ὑπερβάλλειν πλήθους, ἐπὶ δὲ τῶν μεγεθῶν τοὐναντίον ἐπὶ μὲν τὸ ἔλαττον παντὸς ὑπερβάλλειν μεγέθους ἐπὶ δὲ τὸ μεῖζον μὴ εἶναι μέγεθος ἄπειρον. αἴτιον δ' ὅτι τὸ ἕν ἐστιν ἀδιαίρετον, ὅ τι περ ἂν ἓν ᾖ (οἷον ἄνθρωπος εἷς ἄνθρωπος καὶ οὐ πολλοί), ὁ δ' ἀριθμός ἐστιν ἕνα πλείω καὶ πόσ' ἄττα, ὥστ' ἀνάγκη στῆναι ἐπὶ τὸ ἀδιαίρετον (τὸ γὰρ τρία καὶ δύο παρώνυμα ὀνόματά ἐστιν, ὁμοίως δὲ καὶ τῶν ἄλλων ἀριθμῶν ἕκαστος).

> 'two', 'three' and so on stand for mathematical numbers. Therefore the point that Aristotle makes here is probably that *mathematical numbers are considered properties of groups of objects*.' [Mignucci (1987), p.199; italics mine]

Why is this important? Well, number-words occur in two forms: 1) as adjectives, as in ascriptions of numbers (sentences that begin with 'There are' followed by a number-adjective, e.g. 'There are four cows in the field'), and 2) as nouns, as in most number-theoretic propositions (e.g. '2+2=4'). However, as Dummett in his commentary of Frege's *Grundlagen* notes, it is crucial that any analysis must display some sort of connection between the two uses of number-terms mentioned previously. It is not sufficient to give separate explanations of number-adjectives and of number-nouns, without providing for an explicit relation between them: 'otherwise, we should be unable to appeal to the equation '5+2+0=7' to justify inferring that there are were seven animals in the field from the fact that there were five sheep, two cows and no other animals there.'[281] Dummett points to some of the available strategies: 1) We may first explain the adjectival use of number, and then explain the corresponding numerical terms by reference to it – Dummett calls this 'the adjectival strategy'. 2) Conversely, we may explain the use of numerals as singular terms, and then explain the corresponding number-adjectives by reference to it – 'the substantival strategy'.[282] Mignucci in effect claims that Aristotle here opts for the first strategy, the adjectival one. However, while much of what Mignucci says here seems to me correct, later in his article he shifts from a straightforward reading of numbers as *properties of objects* to numbers as *properties of concepts*, by presupposing a *Fregean analysis of number*.

A much more informative account of number is given in the following passage from the *Physics*:

> It is correct, too, to say that the number of the sheep and of the dogs is the same, if each number is equal, but that the ten is not the same <ten> nor are they the same ten, just as the equilateral and the scalene are not the same triangles, though they are the same figure, since both are triangles. Things are said to be the same X if they do not differ by the

[281] In [Dummett (1991), p.99].

[282] ibid.

> difference of an X, but <not the same X> if they do, for example, a triangle differs from a triangle by the difference of a triangle, and therefore they are different triangles; but they do not <differ by the difference> of a figure, but are in one and the same division [of figure]. For one kind of figure is a circle, another a triangle, and one kind of triangle is equilateral, another is scalene. So they are the same figure, that is a triangle, but not the same triangle; and so it is the same number, since the number of them does not differ by the difference of a number, but not the same ten, since the things it is said of are different: dogs in the one case, horses in the other.[283] [*Phys*. Δ.14, 224a2-15; Hussey's trans. mod.]

In which sense the number of the ten sheep and the number of ten dogs is the same? To properly answer this question, Aristotle supplies us with the following principle: two things are the same if they do not differ by the 'difference of an X' . 'To differ by the difference of an X' means 'to differ in respect of a differentia falling immediately under X'. Scalene and equilateral triangles are the same figure because they both belong in the species *triangle* of the genus *figure*; they are figures that 'do not differ by the difference of a figure'. They are not the same triangle, however, because they do not belong in the same species of the genus *triangle*; 'they differ by the difference of a triangle'. Similarly with ten sheep and ten dogs.

It seems that in Aristotle's view number is like a genus. We may divide this genus into the species two, three, four, and so on. Ten sheep and ten dogs are the same number in that they both fall under the same species of the genus number, *ten*; 'they do not differ by the difference of number'. They are the same kind of number, but they are not the same ten; 'they differ by the difference of a ten'. The analogy between triangles and numbers suggests a notion of number as genus, with species two, three, etc..., each species having as its particular members

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[283] λέγεται δὲ ὀρθῶς καὶ ὅτι ἀριθμὸς μὲν ὁ αὐτὸς ὁ τῶν προβάτων καὶ τῶν κυνῶν, εἰ ἴσος ἑκάτερος, δεκὰς δὲ οὐχ ἡ αὐτὴ οὐδὲ δέκα τὰ αὐτά, ὥσπερ οὐδὲ τρίγωνα τὰ αὐτὰ τὸ ἰσόπλευρον καὶ τὸ σκαληνές, καίτοι σχῆμά γε ταὐτό, ὅτι τρίγωνα ἄμφω· ταὐτὸ γὰρ λέγεται οὗ μὴ διαφέρει διαφορᾷ, ἀλλ' οὐχὶ οὗ διαφέρει, οἷον τρίγωνον τριγώνου <τριγώνου> διαφορᾷ διαφέρει· τοιγαροῦν ἕτερα τρίγωνα· σχήματος δὲ οὔ,ἀλλ' ἐν τῇ αὐτῇ διαιρέσει καὶ μιᾷ. σχῆμα γὰρ τὸ μὲν τοιόνδε κύκλος, τὸ δὲ τοιόνδε τρίγωνον, τούτου δὲ τὸ μὲν τοιόνδε ἰσόπλευρον, τὸ δὲ τοιόνδε σκαληνές. σχῆμα μὲν οὖν τὸ αὐτό, καὶ τοῦτο τρίγωνον, τρίγωνον δ' οὐ τὸ αὐτό. καὶ ἀριθμὸς δὴ ὁ αὐτός (οὐ γὰρ διαφέρει ἀριθμοῦ διαφορᾷ ὁ ἀριθμὸς αὐτῶν), δεκὰς δ' οὐχ ἡ αὐτή· ἐφ' ὧν γὰρ λέγεται, διαφέρει· τὰ μὲν γὰρ κύνες, τὰ δ' ἵπποι.

(e.g.) these two books, these two horses, … etc.[284] We might claim then that numbers in this sense are universals—kinds—rather than particulars: indeed, that they are kinds of collections, that is, that they are kinds whose instances are collections of objects. For example, the number 2 is, by this account, the kind of two-membered collections. Hence, on this view, numbers are certainly not themselves collections, any more than the kind triangle is itself a triangle.[285]

This passage can form the basis of an Aristotelian account of numerical predication. As Laura Castelli points out, it seems that Aristotle is claiming that when we say 'ten dogs' and when we say 'ten sheep' the numerical predicate 'ten' means the same in both cases, even though it is ascribed to two different collections.[286] It might be of some help to compare Aristotle's view with other non-Fregean views such as G.E. Moore's conception of number. In his work *Some Main Problems of Philosophy,* G.E. Moore endorses a conception of numbers as universals. He argues that, for instance, any pair of things, irrespectively of the nature of its members, possesses the property of *being two*: 'Every pair or couple of things, no matter what the things may be, obviously has some property which belongs to all other pairs or couples and to nothing else - *the* property which we express by saying that each of them is a pair or a couple.'[287] And similarly for all the other numbers.[288] Thus, Moore writes, 'the number two, therefore, does seem to me as good an instance as can be given of a universal'.[289]

If our interpretation of Aristotle's numbers as universals (kinds) is sound, then those numbers are certainly different from the Platonic Form numbers he

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[284] See also [Hussey (1983), p.161] for this reading of the text.

[285] For this last remark, see, e.g. [Lowe (2001), p.220]. It may be of some value to compare Aristotle's view of numbers as kinds with Cantor's own conception of numbers: as Stefania Centrone informs us, Cantor wrote the following in a letter to Giuseppe Peano: 'I conceive of numbers as 'forms' or 'species' (general concepts) of sets.' In [Centrone (2010), p.12; her trans.]. E.J. Lowe has also presented similar views about numbers as kinds of sets in his [Lowe (2001)] and in a series of articles. See also the treatment in [Mayberry (2000), esp. ch.2].

[286] See [Castelli (2010), p.201].

[287] In [Moore (1953), p.366; his italics].

[288] Op. cit., p.368.

[289] Op. cit., p.366. Moore's passages are discussed extensively in [O'Connor (1982), pp.151-154].

criticises in much of *Metaphysics* M and N; the latter are certain collections of ideal units. This view of Platonic Form numbers is perhaps not unexpected: it is what someone gets by combining the theory of Form numbers as universals with the idea that universals are self-predicable. Let us leave aside the problematic principle of self-predication and try to trace the origins of Aristotle's theory of numbers as universals.

[3.6] A surprising Platonic account

In Plato's dialogue *Hippias Major,* the sophist Hippias, in his exchange with Socrates, acknowledges only one legitimate kind of predication, namely the *distributive* one:[290]

> [Hip.:] If both of us were just, wouldn't each of us be too? Or if each of us were unjust, wouldn't both of us? Or if we were healthy, wouldn't each be? Or if each of us had some sickness or were wounded or stricken or had any other tribulation, again, wouldn't both of us have that attribute? Similarly, if we happened to be gold or silver or ivory, or, if you like, noble or wise or honoured or even old or young or anything you like that goes with human beings, isn't it really necessary that each of us be that as well? [Soc.:] Of course. [*Hippias Major* 300e8-301b1; Scaltsas' trans.]

For Hippias the only legitimate kind of predication is distributive predication: things are F if and only if each one of them is F. Socrates agrees that there are cases like the ones that Hippias mentions. But, additionally, Socrates puts forward counterexamples to Hippias's theory; in certain cases what we say of things is true if the things jointly satisfy the predicate:

> [Soc.:] We were so foolish, my friend, before you [Hippias] said what you did, that we had an opinion about me and you that each of us is one, but that we would not both be one (which is what each of us would be) because we are not one but two. But now, we have been instructed by you that if two is what we both are, two is what each of us must be as well; and if each is one, then both must be one as well.…

[290] Much like Frege does.

> Then it's not entirely necessary, as you [Hipppias] said it was a moment ago, that whatever is true of both is also true of each, and that whatever is true of each is also true of both. [*Hippias Major*, 301d5- 302b3; Scaltsas' trans.]

Socrates' counterexample to Hippias's assumption about distributive predication that pertains to our discussion is an instant of *numerical predication*. Each of Socrates and Hippias is one, while *they*, collectively, are two. The attribute of 'being two' belongs to them, but not to each of them; it is instantiated only in Socrates and Hippias together. Plato presents us here with an ingenious metaphysical account of plural predication, an account that allows the predicated attribute–the property 2–to belong to *all the subjects together*.[291]

According to the Platonic view of plural predication advanced in the *Hippias Major*, a single instance of a property is jointly owned by several subjects. Scaltsas invites us to consider the example of a book being commonly owned by two siblings: 'the book is not divided between the two siblings so that the one of them owns the first half of the book and the other the second half. Rather, each of the siblings owns the whole book together with the other sibling; but neither of them owns the book fully by himself or herself.'[292] Does it make sense to claim that the two siblings become something one, metaphysically, when they co-own the book? Should we perhaps look for a new object (an abstract entity like the *set* of the siblings, or even a concrete one like the *mereological fusion* of them) that will act as a *unitary bearer* for this property? I do not think so. One need not pursuit the metaphysically extravagant unification of the subjects into one entity in such cases.[293] Socrates then does not seem to endorse Mignucci's (or Frege's) insistence on single bearers for number properties (whether these are Fregean

[291] See Scaltsas (2017) for an extensive discussion.

[292] In [Scaltsas (2017), p.12].

[293] *Cf.* Scaltsas, ibid: 'The unification of the subjects into one entity is precisely what does not occur in the case of plural belonging of the collective type. The subjects do not become one in order to manage co-possession, any more than the two pillars become one when they hold a statue, or the wheels of a car become one when they sustain the car. When we say that the siblings are two, it would undermine the truth conditions of this statement if the attribution of twoness to the siblings turned them into one entity, or even one subject.'

concepts, sets, mereological fusions). Socrates espouses a view of numbers as non-distributive properties of multitudes.[294]

[3.7] Platonic complications and interim conclusions

A similar treatment of numbers as attributes of objects can be found in certain passages in the *Phaedo*. Consider, for example, the following passage:

> And again, wouldn't you beware of saying that when one is added to one, the addition is the reason for their coming to be two, or when one is divided, that division is the reason? You'd shout loudly that you know no other way in which each thing comes to be, except by participating in the peculiar Being of any given thing in which it does participate; and in these cases you own no other reason for their coming to be two, save participation in twoness: things that are going to be must participate in that, and whatever is going to be one must participate in oneness. [*Phaedo*, 101c1-7; Gallop's trans. mod.]

Plato uses a series of words ending in -ας for Form numbers in this passage: μονάς, δυάς. These are translated in English as 'Oneness', and 'Twoness', respectively. While rejecting addition and division as reasons for the things'

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[294] 'Multitude' here is a technical term borrowed from Simons (1982): it is a collective noun much like 'audience' and 'congregation'. In his later article 'On Multitudes' Simons distinguishes between multitudes, sets, and mereological sums or fusions. According to Simons, the identity of a multitude is determined solely and completely by what members it has: multitudes with the same members are identical. Just like multitudes, sets with the same members are identical. But he also points to some crucial differences: a set is a single thing whereas a multitude is essentially many things. Furthermore he maintains that, while a multitude of concrete individuals such as the books on my table is concrete (in the sense that it occupies a volume which is the sum of the volumes occupied by the individual members), a set is something rather abstract, outside of space and time. Whereas a multitude is essentially its members, a set something additional, a new individual. Lastly, while there can be no empty multitude, there is an empty set. Mereological sums or fusions are complex individuals and just like multitudes they can be said to be concrete if their members are concrete; a fusion is nothing over and above its parts. In [Simons (2011) , pp.4-6].

coming to be one and two, Plato offers the alternative that they participate in the Forms Oneness and Twoness, respectively. But why does Socrates reject addition (πρόσθεσις) and division (σχίσις) as reasons for things' coming to be two or one? An answer may be found earlier in the text. In lines (97a1-b3) it seems that Socrates understands 'addition' as 'juxtaposition' (ἡ σύνοδος τοῦ πλησίον ἀλλήλων) and 'division' as 'dispersion' (ἀπάγεται καὶ χωρίζεται ἕτερον ἀφ' ἑτέρου), and then he claims that it is not the juxtaposition of two items that constitutes their being two, but something else. His reason is that juxtaposition and dispersion are opposites to one another, and it cannot be right to say that two opposite causes may equally be the reason for the same result.[295] Or Socrates may acknowledge that it is not necessary for two things to be close to each other for them to be two. We may draw a useful comparison, as Gallop does, with what Frege says about predications of number: 'Must we literally hold a rally of blind in Germany before we can attach any sense to the expression "the number of blind in Germany?"' (*The Foundations of Arithmetic*, p.30) [296]

Number Forms, much like other Forms in the *Phaedo*, have to be single and partless. In one of his attempts in that dialogue to prove the immortality of the soul Socrates claims that if anything at all is going to be constant and unchanging then incomposite things will be:

> Now is it not that which is compounded and composite naturally liable to be decomposed, in the same way in which it was compounded? And if anything is uncompounded, is not that, if anything, naturally unlikely to be decomposed?...Then it is most probable that things which are always the same and unchanging are the uncompounded things and the things that are changing and never the same are the composite things? [*Phaedo*, 78c1-8; Gallop's trans. mod.]

The above passage is part of the Affinity Argument (78c-79e). The argument provides us with some more information about the nature of forms: they are incomposite and one-per-type (μονοειδές, 78d5), and for this reason they are unchanging and eternal. It seems that in lines 101c1-7 Plato advocates a

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[295] In [Bostock (1986), p.137].

[296] In [Gallop (1975), p.173].

conception of numbers as Forms, attributes or properties of things, e.g. the Form of F seems to be just the attribute F itself, so that to 'participate' (μετέχειν) in F is like having that attribute. We predicate 'three' of various triplets of things in a way similar to that in which we predicate 'man' of various men; thus, it is only natural to assume that the number 3 is a Form on par with the Form of man. The Platonic picture presented above shares certain features with the *Hippias Major* analysis, namely the treatment of number Forms as properties.

Plato maintains a distinction between numbers as collections of objects and Form numbers in the Final Argument for the immortality of the soul. Plato, from a metaphysical point of view, seems to be concerned with two levels, namely Forms and the exemplification of Forms. Let us have a closer look at the numerical example in *Phaedo's* Final Argument:

> 'Take a good look then at what I want to show. It's this. Apparently not only do the opposites we spoke of not accept each other. In addition, whatever things are not opposite to each other but always have the opposites, these too it seems will not accept the character, whatever it may be, that is opposite to the character that is in them. When this opposite character advances towards them, they either perish or get out of the way. (1) We will in fact admit, won't we, that three will sooner perish, sooner put up with anything, than stay behind and while it is still three become even ?' (104c1-3) 'Yes, indeed,' said Cebes. 'Further, twoness is not opposite to threeness.' 'No, it isn't.' 'Not only therefore will the opposite forms not stay behind when one of them advances upon the other. In addition there are certain other things that will not stay behind at the advance of an opposite.' 'Yes, you're quite right.' 'Will you agree then to our defining, if we can, what sort of things these are ?' 'By all means.' (2) 'Is this what they would be then: they are what compel whatever thing they possess to have not only their own character, but the character of some opposite as well, as a character which will belong to it for good?' (104d1-3) 'What do you mean quite?' 'I mean just what we said before. (3) You appreciate presumably that whatever the form of three possesses must of necessity be not only three but odd as well.' (104d5-7) 'Quite.' 'Well, we maintain that the form opposite to the character which brings this about could never come to such an object.' 'No, it couldn't.' 'And what brought it about was in this case the form odd? ' 'Yes.' 'Opposite to this is the form of even ?' 'Yes.' 'So the form of even will never come to three?' 'No, it won't.' (4) 'Three then has no share in the even?' (104e3) 'No, it hasn't.' 'Threeness therefore is not-even?' 'Yes.' 'So what I was suggesting we define, what sort of things are not opposite to something and yet will not accept it, the

> opposite—the example we have had just now is threeness which is not opposite to the even and yet all the same will not accept it, for the reason that threeness always brings along the opposite of the even, and in the same way twoness brings along the opposite of the odd, and fire brings along the opposite of the cold, and so on and so forth—: well see whether you would define them in this way. Not only will the opposite not accept its opposite. There is in addition that which brings along a certain opposite into whatever object it comes to. The thing that brings along the opposite will never accept the opposite of what is brought along.' [*Phaedo*, 104b6-105a5; O'Brien's trans; underlining mine]

The passage is extremely complex and admits of various interpretations but for my purposes I have numbered and highlighted certain places where one might recognise a distinction between Form numbers and the things that are 'occupied' by them: a characteristic example is the distinction between 'the Form of Three' (ἡ τῶν τριῶν ἰδέα), and a particular three (τὰ τρία). O'Brien (1967) notes that this distinction is kept up throughout the passage but I am making the weaker claim that there is such distinction in the passage: At lines 104c1-3 (see place (1) in the text) the three (τὰ τρία) that will 'sooner perish' must be particular three, since only the (sensible) particulars can perish, not the Form. Furthermore, Socrates' preliminary definition at 104d1-3 (place (2) in the text) is cast in terms of Form and particular. It will be Forms which 'occupy' particulars and impress their character (ἰδέαν), upon them; the definition is applied to 'the Form of Three' (ἡ τῶν τριῶν ἰδέα), which occupies a (particular) three (τρισίν) (104d5-7, place (3) in the text).[297] At 104e3 (place (4) in the text) the three (τὰ τρία), that 'has no share in even', 104e3, must be a particular three. Thus the claim is that whatever the Form of Three occupies must be odd (d5-8, place (3) in the text); hence any triad will not have part in the Form opposite to Odd, the Form Even (d9-e4, place (4) in the text). Although the Platonic picture presented above shares certain features with the *Hippias Major* analysis, namely the treatment of Form numbers as properties, Plato often speaks of the Form as an archetype (or ideal standard)

[297] What is it that is supposed to be possessed by the Form of Three and compelled not to be three but also odd (104d5-7)? O'Brien (p.212 and 217) strangely claims that it is the number three (three not as set, but rather as a mereological simple?), but I agree with Gallop (p.206) that it can very well be a numbered collection, of books say.

which is copied or imitated by the participants.[298]

It seems to me that we need to look no further beyond *Hippias Major* and the discussion in the *Phaedo*, if we really wish to understand Aristotle's conception of number as species that is presented in the *Physics* passage.[299] Certain provision must be made to strip the philosophy of number presented there of any extravagant Platonic features, first and foremost, the fact that Platonic Form numbers are 'over and above' the things that have those numbers. Aristotle in *Metaphysics* M.6-8 argues against a certain view of Platonic numbers, that conceives them as certain sets of ideal units. It is important to notice that the main objection Aristotle raises against a possible identification of numbers with such collections of units has to do with the inability of such collections to account for many arithmetical operations. Let us now revisit passage 220b8-12 of the *Physics*: What goes on when we ascribe a number to something - as when, for example, we say 'The horses in the field are 5'? It is certainly true, of course, that no one thing can as such be that which is 5. Aristotle points to a reasonable answer, however: the sentence is equivalent to 'There is a pentad of horses in the field' or 'the horses in the field are a five'. In the number-as-species interpretation that means that the horses in the field (a countable

[298] In much of the discussion in the Recollection Argument and the Final Argument Plato compares the Forms to sensible particulars which are F, and he notes that those particulars always combine being F with being not-F, while the Form of F-ness is unqualifiedly F. It seems that in the *Phaedo* we have two quite different views of what the forms are: on the one hand they are perfect examples of properties, and on the other hand they are the properties themselves. See [Bostock (1986), pp.198-201].

[299] In fact, *Hippias Major* may be considered *the* proper dialogue that supports both Platonic and Aristotelian readings. Paul Woodruff in his article 'Socrates and Ontology: The Evidence of the *Hippias Major*' has argued–in my opinion convincingly–that early dialogues such as the *Hippias Major* are 'ontologically neutral in that there is no particular ontology that they require, and, though they tempt one to provide them an ontology, the proof of their neutrality is that Plato and Aristotle respond differently, but with equal respect, to the temptation. Socrates' exercises in definition are at the same time a rich breeding ground for Plato's lavish ontology and assimilable gracefully to Aristotle's more austere one'. [Woodruff (1978), p.102] Aristotle tells us that Socrates takes the objects of definition to be universals and he constantly complains that Socrates did not 'separate' the universals as the Platonist did (cf. *Meta*. M.4, 1078b12-32, *Meta*. M.9, 1086a32-b13).

number/arithmos–or number in the first sense) belongs to the number species (or number in the second sense) *five*; we may say that the *five* is the kind of five-membered collections. How can we analyse a simple arithmetical statement of addition—such as '2+2=4' according to this interpretation? Perhaps like this: for any collections of objects x, y, z, if x is a 2, y is a 2, and z is a 4, and if x and y are disjoint (i.e. they have no members in common), then the union of x and y is equinumerous to z.[300]

[3.8] The thesis that composition is identity

[3.8.1] Frege against Mill

Part of Mignucci's Fregean baggage is his hostile attitude to Mill and his philosophy of number:

> If our interpretation is sound, Aristotle is far from a view such as Mill's, according to which a number is a physical property of an agglomerate of things and expresses the characteristic manner in which the agglomerate is made up. Without a loss of realism, Aristotle is not committed to such an empiricist position. [Mignucci (1987), p.201]

It does seem a bit odd to deny Aristotle a place in the empiricist tradition. But why does Mignucci so straightforwardly deny any similarity between Aristotle's and Mill's views on number? After all, for J. S. Mill (as well as for Aristotle), all numbers must be numbers of something:

> All numbers must be numbers of something: there are no such things as numbers in the abstract. *Ten* must mean ten bodies, or ten sounds, or ten beatings of the pulse. But though numbers must be numbers of something, they may be numbers of anything. Propositions, therefore, concerning numbers, have the remarkable peculiarity that they are propositions concerning all things whatever, all objects, all existences of every kind, known to our experience. [*System of Logic*, Book 2, ch.6, §2]

A similar view to Mill can be traced in the Aristotelian corpus; for example in *Meta.* 1092b19-20 Aristotle states clearly: 'A number, whatever it is, is always a

[300] A similar analysis can be found in [Lowe (2001), pp.224-225].

number of certain things, of fire or of earth or of units' (ἀεὶ ὁ ἀριθμὸς ὃς ἂν ᾖ τινῶν ἐστιν, ἢ πύρινος ἢ γήϊνος ἢ μοναδικός).[301] According to Mill, number terms denote agglomerations of things (complex wholes) and connote properties of those agglomerations. Thus, he writes:

> Each of the numbers two, three, four, etc., denotes physical phenomena, and connotes a physical property of those phenomena. Two, for instance, denotes all pairs of things, and twelve all dozens of things, connoting what makes them pairs, or dozens…What, then, is that which is connoted by a name of number? Of course, some property belonging to the agglomeration of things which we call by the name; and that the property is, the characteristic manner in which the agglomeration is made up of, and may be separated into, parts. [*System of Logic*, Book 3, ch.24, §5]

Frege quotes the last sentence[302] and objects to it on a number of grounds. His criticism is that an agglomeration may be separated into parts in various ways and thus we cannot talk about the number of parts in an agglomeration. Let us examine more carefully the relevant passage:

> And it is quite true that, while I am not in a position, simply by thinking of it differently, to alter the colour or hardness of a thing in the slightest, I am able to think of the Iliad, either as one poem, or as 24 books, or as some large number of verses. […] Nor can we say in this case that the different numbers exist in the same thing side by side, as different colours do. I can point to the patch of each individual colour without saying a word, but I cannot in the same way point to the individual numbers. If I can call the same object red and green with equal right, it is a sure sign that the object named is not what really has the green colour; for that we must first get a surface which is green only. Similarly an object to which I can ascribe different numbers with equal right is not what really has a number. [*Foundations*, §22, pp.28-29]

Frege seems to argue that there is no unique way by which a whole (such as the *Iliad*) can be divided into parts: the *Iliad* may be considered as one poem, or as

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[301] Furthermore, Aristotle sometimes places number in the category of *relatives*, because number is always a number of things (cf. the discussion in *Meta*. I.6, 1056b8-1057a7). Number is a relative (for every number is a number of something) and the defining mark of a relative is that its linguistic expression requires completion by a genitive; it is always 'of' or 'than' something else (cf. *Cat.* 6a36-b11; *Meta*. Δ.15).

[302] In [*Foundations*, §23, pp.29-30].

24 books, or as some large number of verses. If the *Iliad* was uniquely divided into parts (say, into 24 books) then one could not claim that it is also 15,693 lines. It seems then that there are multiple ways of dividing a whole into parts; but that, as Frege argues, would entail that incompatible numerical properties are attributed to the same thing: much like the colours red and green cannot both be truly ascribed of a single thing at the same time, the *Iliad* cannot both be 24 and 15,693. It seems then that the object at hand, the *Iliad* for example, is not that which 'really has a number'. Thus Frege concludes: 'If I can call the same object red and green with equal right, it is a sure sign that the object named is not what really has the green colour; for that we must first get a surface which is green only. Similarly an object to which I can ascribe different numbers with equal right is not what really has a number.'[303]

What is exactly is Frege's argument here? Is his position that numerical attributions are subjective, that the object in question does not in any way determine whether or not certain numbers are being ascribed to it? Dummett understands Frege in this way:

> When we regarded it [i.e. number] as ascribed to a complex, an aggregate, it seemed that the number to be ascribed depended on our subjective way of regarding it; as one copse, or as five trees; as four companies, or as five hundred men. But there is nothing subjective about it: it is the concept *copse* or *tree*, *company* or *man* which we invoke in the ascription

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[303] Frege argues similarly elsewhere in the *Foundations*: a pile of playing cards can considered as a number of cards, or as a number of complete packs of cards (*Foundations*, §22, pp.28-29); a bundle of straw may be thought of as 100 straws or as some huge number of cells or molecules (§23, p.30); a copse of trees may be considered as a single thing or as five trees (§46, p.59); four companies may be thought of as 500 men (§46, p.59). The reader who wishes for a more extensive discussion of Frege's arguments against Mill's view of number properties can consult Andrew D. Irvine's article 'Frege on Number Properties'. E. J. Lowe remarks that Frege's arguments are part of a wider *reductio ad absurdum* of the view that numbers are attributes of objects [Lowe (2005), p.84]. He points to what Frege says later in the *Foundations*: 'Several examples given earlier gave the false impression that different numbers may belong to the same thing. This is to be explained by the fact that we were there taking objects to be what has number. As soon as we restore possession to the rightful owner, the concept, numbers reveal themselves as no less mutually exclusive in their own sphere than colours are in theirs.' [*Foundations*, §48, p.61].

> of number, that determines objectively which number it must be. [Dummett (1991), p.88; his italics]

One could respond to Frege that the number of parts in the *Iliad* is fixed independently of the concepts one employs to count them. A whole such as the *Iliad* may turn out to be *composed* of a plethora of different parts, each of which have different numbers associated with them. Let us try to summarise our response to Frege: the *Iliad* is one thing and no other numerical predicate should be applied to the *Iliad* as such. It is true, however, that the *Iliad* is composed of 24 books and that those books contain many pages of verses. We should not infer from the fact that those books are 24 that the *Iliad* is 24 in number. Thus we can always distinguish between *composition* and *identity*.[304] No contradiction arises when one says of the books that compose the *Iliad* that they are twenty-four, even though the *Iliad* is one; the relation between the *Iliad* and the books is one of composition, not one of identity.

Is Mill's view susceptible to Frege's objections? It seems that Mill can escape Frege's objections; for he proceeds to explain what he means as follows:

> What we call a collection of objects *two*, *three*, or *four*, they are not two, three, or four, in the abstract; they are two, three, or four things of some particular kind; pebbles, horses, inches, pounds weight. What the name of the number connotes is, the manner in which single objects of the given kind must be put together, in order to produce that particular aggregate. [*System of Logic*, Book 3, ch.24, §5]

As Glenn Kessler has argued, what Mill says here is that when we say that an aggregate has the number 2, what we really mean is that 'an aggregate composed in a certain way from a certain kind of part has the number 2. Any aggregate that

[304] See Kris McDaniel (2013) for an extensive argument. McDaniel also points out that even if composition is identity, the answer to the puzzles raised by Frege is that numerical properties are not contraries: 'In short, something can be both one and many. And if this is so, why couldn't a thing be both one in number and twenty-four in number? Just as the friend of the view that composition is identity gives up the intuition that one thing cannot be many things, so too she should abandon the intuition that something cannot be both one in number and twenty-four in number.' In [McDaniel (2013), pp.217-218].

differs in number from this aggregate will, of necessity, be composed of different parts, and hence, will be a different object.'[305] And we can perhaps attribute a similar view to Aristotle: recall that for Aristotle a number in the first sense is something composed of things of a certain kind; an *arithmos* of syllables (number in the first sense) has a determinate number, or, equivalently, belongs to a certain number species (number in the second sense). In an *arithmos* in Aristotle's first sense the units are already determined: an arithmos of syllables is not also an *arithmos* of letters.

Frege raises a much more interesting objection against Mill's view that numbers are properties of 'external objects'. Frege argues that number 'is applicable over a far wider range'. [*Foundations*, §24, p.30]. Frege is surely right in this: as well as being able to count dogs and apples, we can also count the figures of the syllogism, the Muses and other non-sensible things.[306] Aristotle would agree that number ascriptions need not involve only perceptibles. An objection that Aristotle raises in *Meta*. N.3 against Academic theories of number, is that those theories cannot account for the application of numbers to perceptible things (see esp.1090a30-1090b5); he does not say that numbers are properties of perceptibles. Were Aristotle to uphold such a Millian position his conception of

[305] See [Kessler(1980), p.67]. As Kessler explains: 'If we want to put this in a more modern setting we could do so by noting that Frege-style aggregates are individuated in terms of their 'atomic parts'. That is, they are individuated in terms of those parts which themselves have no proper parts. … On the alternative reading I have just considered, a given aggregate can have as parts only certain kinds of things. Anything that is not of the appropriate kind will not be a part of the aggregate and will therefore not figure into the identity conditions of the aggregate. In a sense, it is still true that aggregates are individuated in terms of their parts. However, the parts in terms of which the aggregate is individuated may themselves have (proper) parts.' In [Kessler(1980), pp.67-68].

[306] Bell points out that Frege's objection can be directed towards his conception of number: 'If, with Frege, we take the analysis of ascriptions of number to involve objectual quantification . . . then it becomes impossible to provide a plausible analysis of many actual ascriptions of number that we make, apparently unproblematically, in everyday life. For instance, we have no difficulty in understanding, and if we have the appropriate knowledge we have no difficulty in answering, questions like the following: 'How many years was Robinson Crusoe marooned on his island?' . . . 'How many daughters had King Lear?' . . . On Frege's theory the only true answer to these questions is: zero. And intuitively that seems to be the wrong answer. See [Bell (1990), p.76].

arithmetic would be more or less identical to logistiké (λογιστική), the art of calculation, considered as a subsidiary of Arithmetic by most mathematicians and philosophers in antiquity. The distinction between ἀριθμητική (the theory of number) and λογιστική (the art of calculation) was of some importance in Greek mathematics. A passage from Proclus' commentary on the first book of Euclid's *Elements* sheds some light on the subject matter of logistiké:

> Nor does the student of calculation consider the properties of number as such, but of numbers as present in sensible objects; and hence he gives them names from the things being numbered, calling them sheep numbers or cup numbers.[307] [Comm. on the First Book of Euclid's *Elements*, 40.2-5; Morrow's trans.]

It is quite odd that Aristotle not only fails to identify anywhere arithmetic with *logistikê*, but also he does not even discuss the latter even though the distinction was of particular importance to Plato.[308]

[3.8.2] Frege and Plato

Frege of course postulates concepts as those things that do not admit of multiple numbers. It is perhaps worthwhile comparing his arguments against Mill with those of Plato: much like Frege, Plato opts for things other than perceptibles, so as to avoid what he understands to be their major deficiency, their *quantitative indeterminacy*. Thus, in *Republic* Book VII, Plato seems to argue that perceptible objects are not really suitable candidates for ascriptions of number, since 'we do see the same thing as one and as an unlimited number at the same time' (525a4-5). Plato's argument is strikingly similar to that of Frege above. We can supplement *Republic's* account with excerpts from the *Phaedo*, where Plato expounds his position through the example of the concept of equality. In order to understand what it is for something to 'be equal' one cannot rely on perceptible things. Equal perceptible things like sticks and stones are not equal in the same

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[307] οἶδ' αὖ ὁ λογιστικὸς αὐτὰ καθ' ἑαυτὰ θεωρεῖ τὰ πάθη τῶν ἀριθμῶν, ἀλλ' ἐπὶ τῶν αἰσθητῶν, ὅθεν καὶ τὴν ἐπωνυμίαν αὐτοῖς ἀπὸ τῶν μετρουμένων τίθεται, μηλίτας καλῶν τινας καὶ φιαλίτας.

[308] In the *Republic* Book VII, 522c, 525a-c, 526b; *Gorgias* 451b-c and *Theatet*. 198c. For a discussion of the importance of the distinction see [Heath (1921), pp.13-16]. Far more authoritative is the account given in Klein (1968).

way as the Form of Equality is (*Phaedo*, 74d-e, 75a), they are deficiently so, whereas the Form is paradigmatically so (74e). Also in the *Parmenides*, Plato argues that Socrates 'is one', e.g., because he is one man among a company of seven men, and we can equally say that he is 'many' in virtue of his upper and lower, front and back, and left and right parts (*Parm*. 129c–d). There is, however, a proper way in which something can be one or many, which, according to Plato, is grasped by reasoning (129d–e), thus independently of sense perception.

The things that do for Plato the job that concepts do for Frege, i.e the entities to which numbers apply without any others' applying, are mathematical numbers. As we saw in the *Hippias Major* and the *Phaedo*, there are Forms of number, viz. the Forms of Oneness, Twoness, Threeness and so on. The statements of arithmetic, however, are not about them but about mathematical numbers. Mathematical numbers are intermediate between the Form numbers and collections of sensible things. The Academics' reasons for postulating such entities are two: 1) The problem of precision, namely the fact that physical objects might *fail* to have the mathematical properties we study, and 2) the uniqueness problem, the fact that mathematical statements need more than one objects. Aristotle tells us that the intermediates differ from sensible things in being eternal and unchangeable (ἀΐδια καὶ ἀκίνητα), and from Forms in that there are many alike (πόλλ' ἄττα ὅμοια), while the Form itself is in each case unique (ἔτι δὲ παρὰ τὰ αἰσθητὰ καὶ τὰ εἴδη τὰ μαθηματικὰ τῶν πραγμάτων εἶναί φησι μεταξύ, διαφέροντα τῶν μὲν αἰσθητῶν τῷ ἀΐδια καὶ ἀκίνητα εἶναι, τῶν δ' εἰδῶν τῷ τὰ μὲν πόλλ' ἄττα ὅμοια εἶναι τὸ δὲ εἶδος αὐτὸ ἓν ἕκαστον μόνον, *Meta*. A.6, 987b14-18).[309]

[3.8.3] Plato and Aristotle on composition as identity

Plato in Socrates' Dream in the *Theatetus* (201e-206b) discusses the view that composition is identity, i.e. that a whole is merely the sum of its parts. The context of the discussion is the third definition of knowledge as true judgement with an account. Socrates reports a dream which involves the crucial premise that there is a certain epistemological asymmetry between parts and complex

[309] Cf. *Meta*. N.3 1090a35-b1.

wholes, namely parts are unknowable, whereas complex wholes are knowable (by the account of their parts).[310] Letters and syllables are used as examples of parts and wholes, respectively. Socrates proceeds to give two different refutations of this epistemological asymmetry. The first refutation results in epistemological symmetry between parts and wholes (it takes the form of a dilemma in which Socrates argues that parts and wholes are either just as knowable as each other or just as unknowable as each other):[311]

> [Soc.:] Well now, if the complex is both many elements and a whole, with them as its parts, then complexes and elements are equally capable of being known and expressed, since all the parts turn out to be the same thing as the whole. [Theaet.:] Yes, surely. [Soc.:] But if, on the other hand, the complex is single and without parts, then complexes and elements are equally unaccountable and unknowable—both of them for the same reason. [*Theat*., 205d7-e4; Levett's trans.]

Socrates begins his enquiry as follows:

> [Soc.:] Look here, what do we mean by "the syllable"? The two letters (or if there are more, all the letters)? Or do we mean some single form produced by their combination? [*Theat*., 203c4–6; Levett's trans.]

He then invites us to consider the first horn of the above question. If we assume that a syllable is all its letters and we know the syllable 'SO', then, since 'SO' is the same as the two letters 'S' and 'O', we know the letters also. Thus, the letters are just as knowable as the syllable, something that refutes the epistemological asymmetry hypothesis:

> [Soc.:] Then take the case of the two letters, S and O; these two are the first syllable of my name. If a man knows the syllable, he must know both the letters? [Theaet.:] Of course. [Soc.:] So he knows S and O. [Theat., 203c8-d2; Levett's trans.]

The crucial premise that Socrates' argumentative strategy depends on is that <u>a whole is identical with its parts</u>. However, this should not be taken as an

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310 Verity Harte calls this 'the Asymmetry Thesis'. In [Harte (2002), p.33].

311 Ibid.

indication that Plato endorses such a view.[312]

As Harte helpfully comments, 'since the identification of a whole with its parts is the (sole) shared premiss on which both horns of the dilemma depend, if there is something at fault in the arguments of the dilemma, which Plato intends to highlight, this identification is the most likely candidate. [...] Reasons to be suspicious of this identification are also provided by the lengths to which Socrates goes to defend it and the lengths to which Theaetetus goes to try to resist it.'[313] In the following passage Socrates gives us an argument for the thesis that composition is identity:

> [Soc.:] Well now, is there any difference between all of them and all of it? For instance, when we say 'one, two, three, four, five, six'; or, 'twice three', or 'three times two', 'four and two', 'three and two and one'; are we speaking of the same thing in all these cases or different things? [Theaet.:] The same thing. [Soc.:] That is, six? [Theaet.:] Precisely. [Soc.:] Then with each expression have we not spoken of all the six? [Theat.:] Yes. [Soc.:] And when we speak of all of them, aren't we speaking of all of it? [Theat.:] We must be. [Soc.:] That is, six? [Theat.:] Precisely. [Soc.:] Then in all things made up of number, at any rate, by 'the sum' and 'all of them' we mean the same thing? [Theat.] So it seems. [*Theat*., 204b10-d3; Levett's trans.]

The passage is of particular importance since it was perhaps the one that prodded Aristotle to inquire about the unity of number and to accuse the Platonists of not providing a principle of unity for number, as we shall see later in this chapter. We have become accustomed to think of the numbers as those unique abstract particulars 1,2,3,4, . . . This conception of number, however, stands in stark contrast to the ancient one. As I have said many times in this

[312] This is essentially Scaltsas' reading of the Platonic position. [Scaltsas (1994), pp.59-61] According to Burnyeat, Plato rejects the thesis that composition is identity and subscribes to the Aristotelian position of substantial or substantial-like composition (even in the case of numbers). [Burnyeat (1990), pp.206-209]. According to Verity Harte in certain passages in the *Theaetetus*, the *Parmenides*, and the *Sophist*, Plato discusses a notion of composition as identity without endorsing such a view, whereas in other texts from the *Parmenides*, the *Sophist*, the *Philebus* and the *Timaeus*, he endorses a view of wholes as things genuinely unified, as an alternative to the rejected composition-as-identity view [Harte (2002), pp.2-3].

[313] See [Harte (2002), p.39].

chapter, an arithmos (number in the first sense) is a collection of units, something that corresponds also to Euclid's definition: 'number is a collection composed of units (Ἀριθμὸς δὲ τὸ ἐκ μονάδων συγκείμενον πλῆθος, *Elem.*, Book VII, def. 2). Thus, it is better to understand number three, for instance, as a trio of units. And one is not an arithmos, by definition. Harte, on the contrary, does not seem to employ *that* conception of number. Rather, she understands that an arithmos is a collection in the *plural*, i.e. something that lacks *unity*, or, to use Aristotle's term, something that is like a 'heap'. As she writes: 'The term 'collection', of course, is grammatically singular. However, here and in what follows, I use the term 'collection' as a convenient way in which to refer, in the plural, to many things. A collection is a plurality, or, better, many things, plurally quantified.'[314] For Harte, number terms like '3' are plural terms denoting what I have called elsewhere multitudes.[315] She also claims that this conception is 'the conception of ordinary Greek mathematics' . I am not so sure about this. Euclid's definition of number point to number as some *one* thing composed out of parts. According to Burnyeat, Plato rejects the thesis that composition is identity in the case of numbers and subscribes to the Aristotelian position of substantial or substantial-like composition.[316] I will discuss Aristotle's position on the matter shortly. Let us return to the above passage. How should we read the arithmetical propositions '6=4+2' and '6=3+2+1'? Burnyeat points out that someone who has been influenced by Frege might invoke a distinction between sense/reference to better understand the expressions 'twice three', 'three times two', 'four and two', 'three and two and one', (204c): perhaps all those expressions designate the same thing, the number 6, but they differ in their sense. Thus a Fregean-inspired reader would take '=' as an expression of identity. However, this view is mistaken. Given the ancient conception of arithmos, the symbol '=', as Burnyeat correctly points out, should be taken as a symbol of *equinomerosity*. Thus, instead of self-identity when we write '3=3' (the number 3 is identical to itself), we mean that a triplet has the same number of units as another a triplet, i.e. any triplet is equinumerous with any other. Then '3+2=5', as Burnyeat explains,

[314] Harte, op.cit., p.27.

[315] See fn.295.

[316] In [Burnyeat (1990), pp.207-208].

means not that the number 5 is identical with the number which is the sum of 3 and 2, but that there are exactly as many units in a pentad as in a triplet together with a pair.[317]

Aristotle discusses the problem of the unity of substance and its relation to its parts in *Meta.* Z.17 (1041b11-31). According to Aristotle a whole is something more than the sum of its parts:

> As regards that which is compounded out of something so that the whole is one — not like a heap, however, but like a syllable, — the syllable is not its elements, ''ba'' is not the same as ''b'' and ''a'', nor is flesh fire and earth; for when they are dissolved the wholes, i.e., the flesh and the syllable, no longer exist, but the elements of the syllable exist, and so do fire and earth. The syllable, then, is something—not only its elements (the vowel and the consonant) but also something else, and the flesh is not only fire and earth or the hot and the cold, but also something else. Since, then, that something must be either an element or composed of elements, (1) if it is an element the same argument will again apply; for flesh will consist of this and fire and earth and something still further, so that the process will go on to infinity; while (2) if it is a compound, clearly it will be a compound not of one but of many (or else it will itself be that one), so that again in this case we can use the same argument as in the case of flesh or of the syllable. But it would seem that this is something, and not an element, and that is the *cause* which makes *this* thing flesh and *that* a syllable. And similarly in all other cases. And this is the *substance* of each thing; for this is the primary cause of its being; and since, while some things are not substances, as many as are substances are formed naturally and by nature, their substance would seem to be this nature, which is not an element but a principle. An *element* is that into which a thing is divided and which is present in it as matter; e.g. ''a'' and ''b'' are the elements of the syllable.[318] [*Met.* Z.17, 1041b11–33; Ross' trans.; his italics]

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[317] op. cit., pp.205-207.

[318] ἐπεὶ δὲ τὸ ἔκ τινος σύνθετον οὕτως ὥστε ἓν εἶναι τὸ πᾶν, μὴ ὡς σωρὸς ἀλλ' ὡς ἡ συλλαβή—ἡ δὲ συλλαβὴ οὐκ ἔστι τὰ στοιχεῖα, οὐδὲ τῷ βα ταὐτὸ τὸ β καὶ α, οὐδ' ἡ σὰρξ πῦρ καὶ γῆ (διαλυθέντων γὰρ τὰ μὲν οὐκέτι ἔστιν, οἷον ἡ σὰρξ καὶ ἡ συλλαβή, τὰ δὲ στοιχεῖα ἔστι, καὶ τὸ πῦρ καὶ ἡ γῆ)· ἔστιν ἄρα τι ἡ συλλαβή, οὐ μόνον τὰ στοιχεῖα τὸ φωνῆεν καὶ ἄφωνον ἀλλὰ καὶ ἕτερόν τι, καὶ ἡ σὰρξ οὐ μόνον πῦρ καὶ γῆ ἢ τὸ θερμὸν καὶ ψυχρὸν ἀλλὰ καὶ ἕτερόν τι—εἰ τοίνυν ἀνάγκη κἀκεῖνο ἢ στοιχεῖον ἢ ἐκ στοιχείων εἶναι, εἰ μὲν στοιχεῖον, πάλιν ὁ αὐτὸς ἔσται λόγος (ἐκ τούτου γὰρ καὶ πυρὸς καὶ γῆς ἔσται ἡ σὰρξ καὶ ἔτι ἄλλου, ὥστ' εἰς ἄπειρον βαδιεῖται)· εἰ δὲ ἐκ στοιχείου, δῆλον ὅτι οὐχ ἑνὸς ἀλλὰ πλειόνων, ἢ ἐκεῖνο αὐτὸ ἔσται, ὥστε πάλιν ἐπὶ τούτου τὸν αὐτὸν ἐροῦμεν λόγον καὶ ἐπὶ τῆς σαρκὸς ἢ συλλαβῆς.

Aristotle invites us to consider a complex whole, a syllable, and its elements, its letters. He then tells us that the syllable is something more than the sum of its letters. The reason, Aristotle tells us, is that if the syllable is dissolved, we still have the letters but not the whole any more. This shows that the syllable is something more than its letters. Perhaps there is an extra *element*, X, in the syllable in addition to its letters. But then one might argue that all we have now is a new sum of the letters plus X. So either we should concede that the syllable is after all the sum of its elements (letters+X) or the same argument applies: if the syllable is dissolved we still have the elements (letters, X) but not the whole anymore. In the latter case regress threatens: the syllable is something more than (letters+X) by virtue of a new element, Y, and so on. Thus Aristotle argues: 'It would seem that this is something and not an element, and that is the cause which makes this thing flesh and that a syllable. And similarly in all other cases. And this is the substance of each thing; for this is the primary cause of its being … which is not an element but a principle.' (1041b25-31). Just like the arrangement of the letters in the syllable (something which is not a letter nor composed of letters) is the solution of the problem of the unity of the letters into a single whole, the *ousia* of a particular substance is the solution of the problem of the unity of the elements of the substance into a single whole.[319]

δόξειε δ' ἂν εἶναι τὶ τοῦτο καὶ οὐ στοιχεῖον, καὶ αἴτιόν γε τοῦ εἶναι τοδὶ μὲν σάρκα τοδὶ δὲ συλλαβήν· ὁμοίως δὲ καὶ ἐπὶ τῶν ἄλλων. οὐσία δὲ ἑκάστου μὲν τοῦτο (τοῦτο γὰρ αἴτιον πρῶτον τοῦ εἶναι)—ἐπεὶ δ' ἔνια οὐκ οὐσίαι τῶν πραγμάτων, ἀλλ' ὅσαι οὐσίαι, κατὰ φύσιν καὶ φύσει συνεστήκασι, φανείη ἂν [καὶ] αὕτη ἡ φύσις οὐσία, ἥ ἐστιν οὐ στοιχεῖον ἀλλ' ἀρχή—· στοιχεῖον δ' ἐστὶν εἰς ὃ διαιρεῖται ἐνυπάρχον ὡς ὕλην, οἷον τῆς συλλαβῆς τὸ α καὶ τὸ β.

[319] I am thus in agreement with Scaltsas' reading of the argument. See [Scaltsas (1994), pp.64-65].

[3.9] Aristotelian complications

[3.9.1] A hylomorphic account of number?

In certain places in the *Metaphysics* Aristotle advocates, rather surprisingly, a *hylomorphic* account of number. In those places (most notably in chapters H.3 and H.6) Aristotle asks what is it that makes number one:

> Let us now consider the problem we have already mentioned concerning both definitions and numbers, namely: what is the cause of their unity?[320] [*Meta*. H.6, 1045a7-8; Bostock's trans.]

In trying to determine Aristotle's own solution to the problem about the unity of number Cleary helpfully presents us with the following possibilities: (a) that there is some internal form that unifies the matter of number, which consists of indivisible units; (b) that some external form is imposed on these units by the mathematician; (c) that number does not have any unifying form but is merely a heap or multitude of units.[321] The available edvidence, however, is not adequate enough to supply us with a definite answer. Cleary himself opts for the 3rd option: 'Aristotle seems to hold that number is not something unified like a substance but rather more like a heap (σωρός), since number consists of units that differ from each other, and each number is counted simply by adding units.'[322] In this Cleary seems to follow Jacob Klein's influential view. Klein claims that on the Aristotelian account 'number is simply not *one* thing but a 'heap' of things or monads.' He continues: ''Being a number' is not a *koinon* to be taken as a 'whole' *above* and *alongside*, as it were, the parts of the 'heap''(his italics).[323] Klein points to *Meta.* M.7, 1082a22-24 for justification: 'two men are not some

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[320] Περὶ δὲ τῆς ἀπορίας τῆς εἰρημένης περί τε τοὺς ὁρισμοὺς καὶ περὶ τοὺς ἀριθμούς, τί αἴτιον τοῦ ἓν εἶναι;

[321] See [Cleary (2013), p.431].

[322] ibid. p.437. In his earlier [Cleary (1995)] Cleary endorses option b), where an ordinal form is imposed by the counter to the units he counts. See [Cleary (1995), pp.373-375].

[323] Both in Klein [(1968), p.220].

one thing over and above both of them, and this must be so with units too' (ἀλλ' ὥσπερ οἱ δύο ἄνθρωποι οὐχ ἕν τι παρ' ἀμφοτέρους, οὕτως ἀνάγκη καὶ τὰς μονάδας). From this, it would seem that Aristotle attributes no unity to number whatsoever. However, Klein's claim that number is a heap need not represent Aristotle's *own* view on the matter. Rather, Aristotle's understanding of number as a heap is the conclusion that follows naturally where one to admit that there is not any principle of unity by which a number is made to be something more than its parts. For Aristotle formulates the matter as follows: 'For either it is not <a unity>, but is like a heap, or it is, and then it should be explained what it is that makes it one out of many' (ἢ γὰρ οὐκ ἔστιν ἀλλ' οἷον σωρός, ἢ εἴπερ ἐστί, λεκτέον τί τὸ ποιοῦν ἓν ἐκ πολλῶν, *Meta.* H.3,1044a4-5). The absence of such a principle of unity is attributed to the Platonists in *Meta*. M.7 1082a15–22. I discuss the passage later in this chapter.

We can, however, develop a bit further the view of number as something that lacks unity. What happens then when we ascribe a number to something–as when, for example, we say 'The horses are 5'? If one were to endorse that view, then something like the Socratic position in the *Hippias Major* seems like a promising start: One could say of the horses (where 'horses' is a plural term referring to an irreducibly plural entity, namely the horses) that they are, collectively, 5. In other words, one needs to substantiate this claim by providing an account of *plural predication* in Aristotle.

There is some evidence that Aristotle was aware of the distributive-collective distinction:

1)There is a passage in the *Politics* (Book II, chapter 3) where he points out that 'all' may be used in two senses— collectively as in 'all together' and distributively as in 'each separately'.[324] As Oliver and Smiley remark: 'Aristotle

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[324] The passage is the following:

> Again, even if it is best that the association should as far as possible be one, this does not seem to have been shown to be so by the argument, 'if all say "mine" and "not mine" at the same time' (Socrates thinks this is an indication of the state's being completely one)-because 'all' is used in two senses. If all individually is meant, then this may perhaps be nearer to what Socrates wants to bring about; for each man will always refer to the same person as his son, and to the same woman as his wife; and he will speak in the same way of his

makes use of this distinction against Plato 'by detecting an elementary fallacy in the political vision of the *Republic*. [. . .] A Platonic commune of wives and children is one where all together say 'mine', not each separately. And he argues against Socrates that while the 'each separately' sense may be desirable, the 'all together' sense is not.'[325]

2) Aristotle draws a distinction between distributive and collective predication in order to resolve the fallacies of composition and division. Aristotle presents the fallacy of division after the fallacy of combination has been discussed, and provides us with an example. In the *Sophistical Refutations* 4, 166a33–35, he says: 'Upon division depend the fallacies that two and three are five, and even and odd, and the greater is equal (for it is that amount and more besides).' According to Schiaparelli's analysis of the argument, there are two sentences in this passage ('two and three are five, and even and odd') that are in need of explanation and an obviously false sentence ('the greater is equal') that should be read as a conclusion of the previous sentences. The sentence 'two and three are five' can be understood as 'two is five' and 'three is five'. These premises together with the background assumptions 'two is even' and 'three is odd' lead to the conclusions 'five is even' and 'five is odd', which is absurd. The absurd conclusion is a result of a distributive application of the numerical predicate 'five'.[326]

The above passages could, perhaps, form part of a bigger account where one could argue that Aristotle understands numerical predication as an instance of

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> possessions, and each thing that befalls him. But that is not in fact how people will speak who hold wives and children in common. They will *all* speak, but not individually, and the same with regard to possessions: all, but not individually. So then, 'all say' is clearly some sort of fallacy; for 'all' and 'both', and 'odd' and 'even', owing to their double senses, generate contentious syllogisms even in discussion. So, while in one way it is admirable, but impossible, that all should say the same thing, in another way it is not at all conducive to concord. [*Pol.* II.3, 1261b16-32; Saunders' trans.; his italics]

[325] Oliver and Smiley (2013), p.17.

[326] Translations are Schiaparelli's. As for the second argument, Schiaparelli argues that an analogous reading is possible: the sentence 'two and three are five' is taken as if there were two sentences, i.e. 'two is five' and 'three is five', whence one concludes that five is both equal and greater then two and three, i.e. 'the greater is equal', something absurd. Consult [Schiaparelli (2003), pp.123-125] for a detailed reconstruction of the argument.

plural predication.

[3.9.2] Discussing *Meta*. H.3

A suitable answer to the problem of the unity of numbers would require a close investigation of the discussion in H.3. In this chapter Aristotle offers another argument along the lines of the argument in Z.17 discussed in the previous section; there is something in a substance that is not one of its parts, but a different sort of entity, which is the cause of being and the *ousia* of a particular substance:

> Nor then is man an animal *and* two-footed. If these are matter, then there must also be something over and above them, something which is not an element and not composed of elements but is the substance; and this they eliminate when they state only the matter. So if this is the cause of man's being, and this is the substance, they will be failing to state the substance itself![327] [*Meta*. H.3, 1043b11-14; Bostock's trans.]

Aristotle begins his argument by claiming that a syllable is not just its letters *plus* an arrangement:

> Now, on investigation it is evident that a syllable is not composed of the letters and their combination, and a house is not bricks and a combination.[328] [*Meta*. H.3, 1043b4-6; Bostock's trans.]

Aristotle's examples serve to remind us the Z.17 lessons: that the substance does not depend on the form in the way it depends on its material parts: a substance is not a mere aggregate of matter and form, much like a syllable is not just its letters plus combination, and a house is not merely bricks plus combination. In H.3 Aristotle tell us that 'if substances are in a certain way numbers, it is in this way, not as some say as collections of units' (φανερὸν δὲ καὶ διότι, εἴπερ εἰσί πως ἀριθμοὶ αἱ οὐσίαι, οὕτως εἰσὶ καὶ οὐχ ὥς τινες λέγουσι μονάδων·, 1043b32-34). What is the distinction at hand? One suggestion is that the distinction is between

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327 οὐδὲ δὴ ὁ ἄνθρωπός ἐστι τὸ ζῷον καὶ δίπουν, ἀλλά τι δεῖ εἶναι ὃ παρὰ ταῦτά ἐστιν, εἰ ταῦθ' ὕλη, οὔτε δὲ στοιχεῖον οὔτ' ἐκ στοιχείου, ἀλλ' ἡ οὐσία· ὃ ἐξαιροῦντες τὴν ὕλην λέγουσιν. εἰ οὖν τοῦτ' αἴτιον τοῦ εἶναι, καὶ οὐσία τοῦτο, αὐτὴν ἂν τὴν οὐσίαν οὐ λέγοιεν.

328 οὐ φαίνεται δὴ ζητοῦσιν ἡ συλλαβὴ ἐκ τῶν στοιχείων οὖσα καὶ συνθέσεως, οὐδ' ἡ οἰκία πλίνθοι τε καὶ σύνθεσις.

a number in Aristotle's first sense, i.e. a collection composed of units, such as a trio of horses, and a number of abstract units, each exactly like one another in every respect.[329] This understanding of the distinction points to the right direction, since by ἀριθμὸς μονάδων Aristotle probably has in mind a Platonic notion of number, which has no principle of unity and is more like a *heap* (cf. the discussion in *Meta*. M.7 1082a15–22). Bostock offers yet another suggestion: the distinction is between a number that is akin to a syllable (recall that the letters have to be arranged in a certain manner to make the syllable) and a number of abstract units. I will discuss his suggestion in detail shortly. Beyond those two suggestions it seems that Aristotle understands the former kind of number as some kind of whole, a compound, perhaps, of form and matter. In 1043b34-1044a1 Aristotle tells us that both numbers and definitions are similar in the following ways: (1) both are divisible into indivisibles (the 'units' of the definition are those that are 'indivisible' in the sense of being 'indefinable');[330] (2) neither the definition nor the number will survive addition/subtraction without losing their identity.[331] The discussion continues as follows:

> Further, a number must be something in virtue of which it is a unity, though people cannot now say what it is that makes it so, if indeed it is. (For either it is not, but is like a heap, or it is, and then it should be explained what it is that makes it one out of many.) Similarly, a definition is a unity, and again people cannot explain this either. Nor is this surprising, for the explanation is in each case the same; substances are one in this way, not by being a kind of unit or point, but because each substance is an actuality and a certain nature.[332] [*Meta*. H.3,1044a2-9; Bostock's trans.]

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[329] A suggestion made by Burnyeat in [Burnyeat et al. (1984), p.21].

[330] As Bostock helpfully remarks in [Bostock (1994), p.268].

[331] ὅ τε γὰρ ὁρισμὸς ἀριθμός τις· διαιρετός τε γὰρ καὶ εἰς ἀδιαίρετα (οὐ γὰρ ἄπειροι οἱ λόγοι), καὶ ὁ ἀριθμὸς δὲ τοιοῦτον. καὶ ὥσπερ οὐδ' ἀπ' ἀριθμοῦ ἀφαιρεθέντος τινὸς ἢ προστεθέντος ἐξ ὧν ὁ ἀριθμός ἐστιν, οὐκέτι ὁ αὐτὸς ἀριθμός ἐστιν ἀλλ' ἕτερος, κἂν τοὐλάχιστον ἀφαιρεθῇ ἢ προστεθῇ, οὕτως οὐδὲ ὁ ὁρισμὸς οὐδὲ τὸ τί ἦν εἶναι οὐκέτι ἔσται ἀφαιρεθέντος τινὸς ἢ προστεθέντος.

[332] καὶ τὸν ἀριθμὸν δεῖ εἶναί τι ᾧ εἷς, ὃ νῦν οὐκ ἔχουσι λέγειν τίνι εἷς, εἴπερ ἐστὶν εἷς (ἢ γὰρ οὐκ ἔστιν ἀλλ' οἷον σωρός, ἢ εἴπερ ἐστί, λεκτέον τί τὸ ποιοῦν ἓν ἐκ πολλῶν)· καὶ ὁ ὁρισμὸς εἷς ἐστίν, ὁμοίως δὲ οὐδὲ τοῦτον ἔχουσι λέγειν. καὶ τοῦτο εἰκότως συμβαίνει· τοῦ αὐτοῦ γὰρ λόγου, καὶ ἡ οὐσία ἓν οὕτως, ἀλλ' οὐχ ὡς λέγουσί τινες οἷον μονάς τις οὖσα ἢ στιγμή, ἀλλ' ἐντελέχεια καὶ φύσις τις ἑκάστη.

Aristotle in this passage claims that a number is a unity, a definition is a unity, and that the explanation in each case is the same; a substance is a unity because it is 'an actuality and a certain nature' (1044a9). The term 'actuality' (ἐντελέχεια) indicates form (*cf.* Z.13, 1038b6) as does the expression 'a certain nature' (φύσις τις) (the form of a natural object was identified with its 'nature' at Z.17, 1041b28-30).[333] Aristotle's substance, then, is a unity because it consists of certain materials with a certain form. In the case of paradigmatic substances such as Socrates, the form is its soul (cf. the discussion in Z.16, 1040b5-16); in the case of the syllable, the form is the arrangement (σύνθεσις) of the material components (the letters) which prevents the syllable from being considered a mere heap (Z.17, 1041b11-17). But what about the unity of number? Drawing some analogies with the example of the syllable, Bostock tentatively suggests that Aristotle thinks of number as an arrangement, structure or pattern of units: for instance, three is a triangular number, four is a square number, and so on. We may even generalise this idea as Bostock does: 'If we tone down this suggestion a bit, but still without losing its basic idea, we may perhaps think of Aristotle as holding that we have three horses only where the three form a group, not necessarily in any particular pattern, but at least so situated that they are all <u>close to one another</u>. For this too is an arrangement of a kind though not such a specific kind as suggested previously.'[334] (underlining mine). Despite the initial plausibility of the idea which stems from the Pythagorean tradition of figured numbers, one cannot be utterly satisfied with it; as Bostock himself acknowledges: '…of course such view is mistaken, for three horses remain three however they may be scattered, but it would not be too surprising if Aristotle had failed to grasp this point.'[335] Now it seems rather implausible to me that Aristotle would have failed to grasp the crucial lesson of the *Phaedo,* namely that the units of a number need not be juxtaposed or otherwise physically related in order to form a number.

Yet another passage that points towards an interpretation of number as some

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[333] In [Bostock (1994), p.268].

[334] ibid.

[335] ibid.

sort of composite is the following:

> The cause of the mistake they made was the fact that they were making their search at one and the same time from the side of mathematics and from that of definitions of universals. From the former side they regarded one, their principle, as a point. (A unit is a point without position. So they put things together from minimum parts, as others have done, and the unit becomes the matter of numbers, and at the same time prior to Two-though also subsequent, in fact, because two is a whole and a unity and form.) But because they were looking for the universal they treated the one that is predicated as also being a part even so. But it is impossible for both of these things to apply simultaneously to the same thing.[336] [*Meta*. M.8, 1084b23-32; Annas' trans.]

Aristotle complains that the Platonists conflate the mathematical and dialectical modes of inquiry. In the passage above (as well as in the preceding discussion) Aristotle endorses a *hylomorphic* account of number to explain the error in the Platonists' thinking:

> Insofar as number is composite, it is one that comes first, but insofar as the universal and form are prior, it is number that comes first; each of the units is part of number as matter, but the number is their form.[337] [*Meta*. M.8,1084b4-6; Annas' trans.]

On the one hand the Platonists as mathematicians gave certain priority to the elements of the (composite) number, i.e. to the units, thereby subscribing, we may add, to the thesis that composition is identity: a whole is merely the sum of its parts. This means that number three, for instance, is merely three units, or in Aristotle's terminology a 'heap' of three units. On the other hand the Platonists as philosophers were concerned with the formal unity of number, thereby acknowledging that number is something more than the sum of its elements, its units, by something extra, the form of number. However–and this takes us back

[336] αἴτιον δὲ τῆς συμβαινούσης ἁμαρτίας ὅτι ἅμα ἐκ τῶν μαθημάτων ἐθήρευον καὶ ἐκ τῶν λόγων τῶν καθόλου, ὥστ' ἐξ ἐκείνων μὲν ὡς στιγμὴν τὸ ἓν καὶ τὴν ἀρχὴν ἔθηκαν (ἡ γὰρ μονὰς στιγμὴ ἄθετός ἐστιν· καθάπερ οὖν καὶ ἕτεροί τινες ἐκ τοῦ ἐλαχίστου τὰ ὄντα συνετίθεσαν, καὶ οὗτοι, ὥστε γίγνεται ἡ μονὰς ὕλη τῶν ἀριθμῶν, καὶ ἅμα προτέρα τῆς δυάδος, πάλιν δ' ὑστέρα ὡς ὅλου τινὸς καὶ ἑνὸς καὶ εἴδους τῆς δυάδος οὔσης)· διὰ δὲ τὸ καθόλου ζητεῖν τὸ κατηγορούμενον ἓν καὶ οὕτως ὡς μέρος ἔλεγον. ταῦτα δ' ἅμα τῷ αὐτῷ ἀδύνατον ὑπάρχειν.

[337] ᾗ μὲν δὴ σύνθετος ὁ ἀριθμός, τὸ ἕν, ᾗ δὲ τὸ καθόλου πρότερον καὶ τὸ εἶδος, ὁ ἀριθμός· ἑκάστη γὰρ τῶν μονάδων μόριον τοῦ ἀριθμοῦ ὡς ὕλη, ὁ δ' ὡς εἶδος.

to the Z.17 lesson–they treated the form as part of this composite number, on equal footing with the material part, i.e. the units (ὡς μέρος ἔλεγον, 1084b31-32). This, Aristotle claims, does not solve the problem of unity for this composite number, since it leads to regress. It is pretty obvious that one cannot apply both the principle that composition-is-identity and its negation to the same thing, i.e. to number as something composite (1084b32). Now, I take it, there is a sense in which number has some principle of unity, a formal aspect, which is not also a part of number, unlike its units. A certain account that sheds some light on the matter is Syrianus' one.

[3.9.3] Syrianus' helpful account

A much more detailed insight into such a hylomorphic account of number is given by Syrianus in his commentary of Aristotle's *Metaphysics* M and N. It is worthy of special remark, since Syrianus addresses the problem regarding the unity of number by adopting a hylomorphic approach:

> So then, neither is it the case that five is constituted from substance and accident, as with 'white man', nor yet from genus and differentia, as is 'man' from 'animal' and 'two-footed', nor by five units being in contact with each other, as in the case of a bundle of sticks, nor by being mixed together, like honey-wine, nor by having a certain placing, as in the case of stones going to make up a house. However, it is not so, as in the case of countable objects, that there is nothing over and above the individual objects; for let us grant him for the moment that the conjunction of two men is nothing over and above each of them (although it is in fact Plato's view that all these combinations themselves receive the different numbers by virtue of participation in some Form, as is written in the *Phaedo*; but let this not be attributed to countable objects just for the moment); but it is not because numbers are composed of indivisible units that they have something other than those units (for the many points are indivisible, but nonetheless they are not considered to make up something else besides themselves as subjects), but because there is something in them analogous respectively to matter and form. For instance, when we add three to four and make seven, we express what we are doing in these terms, but our statement actually is not true; for the units when joined together with the other units make up the substratum of the number seven, but the actual seven is made up of this number of units and the Form of

Seven.[338] [*On Aristotle Metaphysics M-N*,132.29-133.7; Dillon and O'Meara trans. mod.]

What is the context of Syrianus' remarks? In the above passage Syrianus expresses his own position when commenting on the Aristotelian passage 1082a20–26. In this passage Aristotle asks 'How is it possible that a number like two be a unity?'. He states that some things are one by contact, others by mixture, and others by position, but none of these alternatives can possibly apply to the units of which two and three consist.

> Besides, some things are one by contact, some by mixture, some by position but none of these can apply to the units of which Two and Three are made up. Two men are not some one thing over and above both of them, and this must be so with units too. Their being indivisible will make no difference; points are indivisible too, but still two of them do not make anything over and above the two.[339] [*Meta*. M.7, 1082a20-26; Annas' trans. mod.]

If one were to raise the issue of the unity of number Two, say, one cannot claim, for example, that the units are next to each other in any way (1082a20-22). As Bostock remarks, 'the only thing that the Platonist can say to distinguish these units from any other pair of units is that they are the ones that make up *the* number two.' (his italics) But, of course, he continues, 'this reply is evidently

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[338] οὔτ' οὖν ἐξ οὐσίας καὶ συμβεβηκότος ὑφέστηκεν ὁ πέντε, ὡς ὁ λευκὸς ἄνθρωπος, οὔτε ἐκ γένους καὶ διαφορᾶς, ὡς ὁ ἄνθρωπος ἐκ ζῴου καὶ δίποδος, οὔτε τῶν πέντε μονάδων ἁπτομένων ἀλλήλων, ὡς ἡ δέσμη τῶν ξύλων, οὔτε μιγνυμένων, ὡς τὸ οἰνόμελι, οὔτε θέσιν ὑπομεινασῶν, ὡς οἱ λίθοι τὴν οἰκίαν ἀποτελοῦσιν. οὐ μὴν οὐδ' ὡς τὰ ἀριθμητὰ οὐδέν ἐστι παρὰ τὰ καθ' ἕκαστα· συγκεχωρήσθω γὰρ αὐτῷ πρὸς τὸ παρὸν τοὺς δύο ἀνθρώπους μηδὲν εἶναι παρ' ἑκάτερον (καίτοι Πλάτωνι πάντα ταῦτα μεθέξει τινὸς εἴδους δοκεῖ τοὺς διαφόρους ἀριθμοὺς καὶ αὐτὰ δέχεσθαι, ὡς ἐν τῷ Φαίδωνι γέγραπται· ἀλλ' οὖν μὴ ὑπαρχέτω τοῦτο τοῖς ἀριθμητοῖς πρὸς τὸ παρόν)· ἀλλ' οἵ γε ἀριθμοὶ οὐχ ὅτι ἐξ ἀδιαιρέτων σύγκεινται τῶν μονάδων, διὰ τοῦτο ἕτερόν τι ἔχουσι παρὰ τὰς μονάδας (καὶ γὰρ τὰ πολλὰ σημεῖα ἀδιαίρετα, ἀλλ' ὅμως οὐ δοκεῖ τι αὐτά γε συμπληροῦν ἄλλο παρ' αὐτὰ τὰ ὑποκείμενα), ἀλλ' ὅτι ἔστι τι ἐν αὐτοῖς τὸ μὲν ὕλῃ τὸ δὲ εἴδει ἀναλογοῦν. ἀμέλει ὅταν τὸν τρία τῷ τέσσαρα συντιθῶμεν καὶ ποιῶμεν τὸν ἑπτά, λέγομεν μὲν οὕτως, οὐ μήν ἐστι τὸ λεγόμενον ἀληθές· ἀλλὰ γὰρ αἱ μονάδες ταῖς μονάσι συμπλακεῖσαι τὸ ὑποκείμενον ποιοῦσι τοῦ ἑπτὰ ἀριθμοῦ, γίγνεται δὲ ὁ ἑπτὰ ἐκ μονάδων τοσῶνδε καὶ τῆς ἑπτάδος.

[339] ἔτι τὰ μὲν ἁφῇ ἐστὶν ἓν τὰ δὲ μίξει τὰ δὲ θέσει· ὧν οὐδὲν ἐνδέχεται ὑπάρχειν ταῖς μονάσιν ἐξ ὧν ἡ δυὰς καὶ ἡ τριάς· ἀλλ' ὥσπερ οἱ δύο ἄνθρωποι οὐχ ἕν τι παρ' ἀμφοτέρους, οὕτως ἀνάγκη καὶ τὰς μονάδας. καὶ οὐχ ὅτι ἀδιαίρετοι, διοίσουσι διὰ τοῦτο· καὶ γὰρ αἱ στιγμαὶ ἀδιαίρετοι, ἀλλ' ὅμως παρὰ τὰς δύο οὐθὲν ἕτερον ἡ δυὰς αὐτῶν.

circular, and the truth is that if there were such abstract units then none of them would be distinguished from any others, so there could be no saying what it is that 'unifies' or 'holds together' those particular units that are supposed to constitute the number two, i.e. what it is that distinguishes them from other units.'[340]

It seems that the Platonists cannot offer an appropriate principle of unity for number such that its parts (i.e. the units) actually constitute something of a whole, and not simply a heap. Aristotle in the passage above offers several principles of unity and his strategy consists in showing that Platonic number cannot share in any of them and so cannot be anything more than a heap, an irreducibly plural entity. The types of unity offered are the following: (a) unity by contact (ἁφῇ); (b) unity by mixture (μίξει); (c) unity by position (θέσει). A more detailed account of those types of unity is offered in *Metaphysics* Δ.6. Aristotle there offers as an example of unity of contact, a collection of planks glued or tied together (1015b36ff.). Unity by contact is not, of course, appropriate for numbers: for some horses to be three it is not necessary that they are in touch with one another (a generalisation of this observation is that the horses need not even be close to each other). Finally, unity by blending is not of relevance here since it belongs to such things as the mixture of honey with wine that constitutes honey-wine (οἰνόμελι in Syrianus' account). Unity by position pertains to some slabs that are positioned in a certain way to form a threshold, or to the letters that make up a syllable. Therefore Aristotle reaches the conclusion that none of *these* types of unity can account for the unity of Form numbers. Thus, in Form Number two, for instance, there seems to be no unity apart from the two units. Furthermore, the indivisibility of the units is not sufficient to explain the unity of Form numbers: drawing on a parallel with points, he argues that they are also indivisible, yet a pair of them does not constitute a separate entity beyond them (1082a23-26). Aristotle in effect claims that in a number of two points the fact that the units are of the same type–i.e. that they are points–is not sufficient to establish the unity of the number in question: the common type does not unify the points into a whole (much like the fact that the elements of the syllable are

[340] In [Bostock (1994), p.269].

both of the same type–letters–is not enough for them to constitute a single whole).[341]

According to Syrianus, number five is a unified whole, but the unity of five is not due to the conditions Aristotle accepts for the unity of a thing in the passage 1082a20–26: Five is not constituted by five units being in contact with each other, nor by being mixed together, nor by having a certain placing. His answer is that they are somehow composites of form and matter (132.7-8).[342] According to Syrianus, when we add three to four and make seven, we express what we are doing in these terms, but our statement is actually not true. It is not that the seven is identical to the sum of three and four, but that the seven consists of seven units, which are equinumerous with the totality of units in the three together with the units of the four. Syrianus' interpretation falls under option b) in Cleary's list of possible solutions regarding the unity of number; we have some external form that is imposed on these units by the mathematician:

> What is it, then, that applies the Form of Seven to the units? What is it, after all, that applies the Form of Bed to such and such a combination of pieces of wood? Surely it is plain that it is the soul of the carpenter that, in virtue of possessing the appropriate art, imposes forms on bits of wood for the making of a bed; and it is the soul of the mathematician that, by possessing within itself the originative Monad, imposes form upon, and generates all numbers.[343] [*On Aristotle Metaphysics M-N*,133.8-12; Dillon and O'Meara trans.]

Syrianus' answer is based on a parallel between the soul of the carpenter and the soul of the mathematician: just as the soul of the carpenter applies the Form of Bed to such and such a combination of pieces of wood because he possesses the appropriate art, the soul of the mathematician by possessing the appropriate

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[341] This is a point that has not been generally appreciated by commentators. An exception is Edward Halper in [Halper (1989), p.259].

[342] [Mouzala (2015)] contains a useful discussion of Syrianus' arguments.

[343] τίς οὖν ὁ τὴν ἑπτάδα ταῖς μονάσιν ἐπιφέρων; τίς δὲ ὁ τὸ εἶδος τῆς κλίνης τῇ τοιᾷδε συνθέσει τῶν ξύλων; ἢ δῆλον ὅτι ψυχὴ μὲν τεκτονικὴ τῷ ἔχειν τὴν τέχνην εἰδοποιεῖ τὰ ξύλα πρὸς κλίνης ἀπογέννησιν· ψυχὴ δὲ ἀριθμητικὴ τῷ ἔχειν ἐν ἑαυτῇ τὴν ἀρχηγικὴν μονάδα πάντας εἰδοποιεῖ καὶ ὑφίστησι τοὺς ἀριθμούς.

knowledge of the Numbers imposes numerical form upon the units.[344]

[3.9.4] Further refinements

Although the discussion in H.3 draws certain parallels between substances and numbers, it would be of some significance to highlight the differences between number forms and substantial forms. Recall that the syllable, Aristotle tells, is not just its elements (its letters) but it is the letters in a certain arrangement. Much like the arrangement is something that is not an element nor composed of elements, the substance (form) is of *a different type* than the (material) elements it unifies, and is responsible for the being of a substance by combining these elements into a unified whole. Scaltsas cautions us not to read Aristotle's position about the unifying role of the substantial form over the material elements as a claim that the substantial form is some kind of *relation*:

> But now, his claim that the substantial form unites the various elements (out of which the substance is made up) into a single whole seems to clash with his claim that the substantial form is not a relation. It would appear that what the substantial form needs to do is precisely to interconnect the various elements of a substance to each other so as to make up a single whole. Hence, it must be a relation. … The difference is that a substantial form unites elements into a whole by tying their identity to the identity of the whole, while a relation leaves the identity of the relata intact. Thus, given a related whole of ten juxtaposed books, each of the books is identifiable independently of its relation to the other books. The identity of the relata is not determined by the relation they bear to each other. But in the case of a substance, something comes to be a component of the substance by being identified in terms of the relation it bears to the whole, for example the skin, liver, or brain, of a human being. The unity that is achieved in a substance is achieved by the identity-interdependence of all the constituents in it, as determined by the substantial form. The constituents that emerge from the incorporation of the elements into the whole are what they are because of their role in the whole. [Scaltsas (1990), p.588]

Apart from the example of the ten juxtaposed books, Scaltsas notes that Aristotle's example of the syllable, is not a substantial whole. His argument is that each of the letters is identifiable independently of its relation to the other.

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[344] See also [Mouzala (2015), p.180]. For a similar to Syrianus' interpretation according to which the counter imposes the ordinal form to the units he counts see [Cleary (1995), pp.373-375].

This shows that the syllable is not an Aristotelian substance but rather, a related whole.[345] We may say something similar for the composite numbers. If, e.g., a triplet of birds were a substance then the incorporation of the birds into the whole would involve the reidentification of them. However, the identity of the birds is not determined by the relation they hold to each other.

Although Aristotle asks about the cause of unity in numbers at the beginning of H.6, the issue is not further addressed in this chapter. However, it is possible to draw some useful conclusions from the discussion there. Aristotle's primary example in this chapter is a composite, a bronze sphere, the discussion of the unity of which takes up almost all of lines 1045a25-35. For the purposes of my interpretation, I endorse Verity Harte's reading of the H.6 discussion, according to which what makes a bronze sphere one is that it is a unified realisation of a form, the form of sphericity. Thus, according to Harte's account, 'a composite is one because there is a unitary form which it exemplifies' or in other words, 'a composite is one because it is one something: that is, the unity of a composite is parasitic on the unity of the something it is, its form.'[346] What could be the composite in the case of number? One proposal is that the composite is a particular collection of objects, a pentad of horses, a trio of musicians, the twelve gods of Olympus. If we take into account Syrianus' hylomorphic analysis of number as well as Aristotle's comments in H.3 and in H.6 and elsewhere, then we can say that any such collection is a composite of matter (units) in a certain form (cardinal structure). Perhaps looking into the philosophy of mathematics known as *structuralism* might supplement this picture; Shapiro provides us with the following examples of cardinal structures:

> For each natural number *n*, there is a structure exemplified by all systems that consist of exactly *n* objects. For example, the *4 pattern* is the structure common to all collections of four objects. The 4 pattern is exemplified by the starting infielders on a baseball team (not counting the battery), the corners of my desk, and two pairs of shoes. We define the 2 pattern, 3 pattern, and so on, similarly. Let us call these 'cardinal structures', or 'finite cardinal structures'. [Shapiro (1997), p.115; his italics]

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[345] In [Scaltsas (1990), p.590, fn.33].

[346] In [Harte (1996), pp.292-293].

He defines *system* as a related whole and *structure* as the 'abstract form of the system':

> I define a *system* to be a collection of objects with certain relations. An extended family is a system of people with blood and marital relationships, a chess configuration is a system of pieces under spatial and 'possible-move' relationships, a symphony is a system of tones under temporal and harmonic relationships, and a baseball defense is a collection of people with on-field spatial and 'defensive-role' relations. A *structure* is the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system. [Shapiro (1997), pp.73-74]

I am not sure that Shapiro's conception of cardinal structures gives us lots of information about the nature of the relations in that structure. Most of his examples take the relations between the objects of the system to be of a spatio-temporal character, something that could invoke a criticism akin to Frege's against Mill regarding the applicability of number to non-perceptible things. A much more promising account is provided by Stanley Jevons in his work *The Principles of Sciences* where he tells us that 'number is but another name for *diversity*'.[347] A bit later he expands on what he understands as the 'abstract number': 'if they are really three men and not one and the same ... in speaking of them I imply the existence of the requisite differences. Abstract number then is the *empty form of difference*.'[348] A modern advocate of Jevon's thought is John Bigelow. Bigelow invites us to consider the equivalence between (a) The number of F's is at least three, and (b) $\exists x \exists y \exists z. (x \neq y \& x \neq z \& y \neq z \& Fx \& Fy \& Fz)$. We can observe that (b) does contain the following open sentence, with three variables x, y, and z: $(x \neq y \& x \neq z \& y \neq z)$. Any open sentence like this, with three free variable will be true of various triples of things. This open sentence, in particular, will be true of any triple of things that are distinct from one another. It seems then that there is indeed something, a universal, which is instantiated by each triple of numerically distinct things. Bigelow calls it 'the relation of threefold mutual distinctness', or 'the form of threeness'. The source of numbers,

[347] See [Jevons (1913), p.156; his italics].

[348] In [Jevons (1913), p.158; his italics].

then, is the relation of non-identity, or the form of twoness. On this account, natural numbers begin at two: the number two being simple the dyad, the relation of mutual distinctness expressed by the open sentence $(x \neq y)$.[349]

Yet another suggestion for a composite number stems from the fact that number is *something essentially ordered*. For this understanding of number it may be helpful to invoke Aristotle's treatise on time at *Physics* Δ.l0-14 where he defines it as kind of number, 'a number of change with respect to the before and after' (τοῦτο γάρ ἐστιν ὁ χρόνος, ἀριθμὸς κινήσεως κατὰ τὸ πρότερον καὶ ὕστερον, 219b1-2). Number in the sense of an ordered group (a row of houses, a row of dots, a stack of coins, etc.) can be very well considered as a composite of form and matter. As in the case of a particular syllable, Aristotle's point is, perhaps, that we have a row of three houses when the houses are considered in a particular order, namely a first house followed by a second one and the latter by a third one; otherwise the houses are a mere heap, something with no unity whatsoever. According to this interpretation, numbers in the sense of linearly ordered groups are compounds of matter (units of a certain kind) and form (ordinal patterns, for example, the ordinal 3 pattern, the structure of any group of three objects *considered in a particular order*—a first, a second, and a third).[350] What are the (material) parts or members of this composite? The houses apparently. What is it, then, that provides the unity for the composite? Is the composite one in virtue of some one common kind to which each of the houses belongs? One might respond that the common kind provides some kind of unity but it cannot account for the unity in question. It is true that the things numbered are all houses, but there is no particular reason why there are three houses and not five houses as far as the nature of house goes. Thus, it cannot be the common measure that accounts for the unity of the composite. Rather, what accounts for the unity of this compound is the *form of threeness*, the specific type of ordinal structure the compound is a realisation of.[351]

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[349] In [Bigelow (1988), pp.48-54].

[350] Shapiro also talks about ordinal structures. In [Shapiro (1997), pp.115-116]

[351] A much more satisfactory analysis of form number (something that Aristotle unfortunately does not provide us with) would include an epistemic account of *how do we get* from a sequence

Several problems remain: more specifically, what can be said about infinite sequences? Aristotle argues in the *Physics* 204a8-206a8 that the infinite cannot be *actually*, and that it must exist potentially if it exists at all. However, the problem is that it seems that something can only have potential existence if it is possible for it to exist actually, like in the case of a statue in a marble slab. The infinite, however, does not have potential existence in that way; there can be no infinitely extended magnitude or an infinite aggregate of things. The infinite is not 'whole' or 'complete'. Let us now have a closer look at Aristotle's concept of *potential infinity* that pertains to numbers.

> But in the direction of more it is always possible to conceive of <more>-since the halvings of magnitude are infinite. Hence [the infinite in number] is potentially, but not in actual operation, though what is taken always exceeds any definite multitude. But this number is not separable, and the infinity does not stay still but comes to be, in the same way as time

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of dots, a stack of coins, etc. to the formal structure of such a sequence. In his paper 'Mathematical Knowledge and Pattern Cognition' Resnik gives us the archetype of such an account; Resnik invites us to consider this sequence of dots . . . . and try to understand it from a mathematical perspective:

> If we were impressed by the immediate succession of one dot after another we might make statements such as 1) no dot has more than one immediate successor; 2) if one dot succeeds another then the latter does not succeed the former; 3) there is a dot which succeeds no dot and every dot but it succeeds a dot; 4) there is no dot between a dot and its successor. Or if we were impressed by the ordering of the dots we might come up with these other statements: 5) if one dot comes before a second and the second before a third then the first comes before the third; 6) if one dot comes before another then the second does not come before the first; 7) given two distinct dots one comes before the other; 8) given any sub-sequence of dots there will be one in the sub-sequence which comes before the others. … *What is the epistemology of this situation*? How do we arrive at these beliefs and what justification do we have for making these claims? I think that the claims are simply obvious to anyone who has sufficient mathematical experience to understand them and who attends to the diagram. *I think that neither deduction nor introspection is needed to verify these claims; they are in a sense read off the drawing. So long as we are taking our perceptual faculties for granted, they need no further justification*. …[N]otice that it is also evident that (1) - (8) continue to hold when the talk of dots is replaced by talk of a sequence of squares, stars, a row of houses, a stack of coins, etc. and, furthermore, the claims remain valid if several dots are taken away from or added to the original sequence. These additional assertions are as evident or almost as evident as the original ones. *We have thus arrived at knowledge of an abstract pattern or structure*. [Resnik (1975), pp. 33-34; italics mine]

and the number of time [*Physics* Γ.7, 207b10-15; Hussey's trans. mod.]

Aristotle asserts that it is always possible to think of a larger number because the divisions of a length are infinite (207b10-11); Aristotle explicitly states later that it is the structure of the magnitude (i.e. its continuity) that entails the infinite divisibility of it ('it is clear that everything that is continuous is divisible to what is itself always divisible', *Phys*. Z.1, 231b15-16). If numbers are to be conceived as sequences of things, then what could be said about the notion of an *infinite sequence*? According to Aristotle one cannot claim that an infinite sequence of divisions enjoys potential being in that it is possible for it to be actual. Aristotle explicitly rejects this sort of potentiality for the infinite in 206a18-21. Instead it seems that he opts for the weaker statement to the effect that any finite sequence of things can be extended.[352]

[3.10] Conclusion

In this chapter I offered an exploration of Aristotle's philosophy of number that is fundamentally different from the standard Fregean interpretation. Beginning from the standard Greek notion of number as a collection of units, I tried to

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[352] Thus Aristotle's view is *fundamentally different* from Hellman's modal treatment of sequences. One of the most influential forms of structuralism is the modal structuralism developed in Geoffrey Hellman's *Mathematics Without Numbers*. According to Berry's definition, modal structuralism 'is a nominalist philosophy of mathematics which maintains that mathematicians can systematically express truths even if there are no mathematical objects, by interpreting statements about mathematical objects as modal claims about what is logically possible.' [Berry (2018), p.1]. A useful summary of Hellman's modal structuralism is provided by Owen Griffiths: 'Consider some arithmetic sentence S; then according to Hellman one might paraphrase S as follows: necessarily, if there is an ω-sequence, then S is true in that sequence. He calls this the *hypothetical* component of his view. However, this paraphrase faces an immediate vacuity problem: our universe might not contain that many objects so there might not be any ω-sequences after all. For this reason, Hellman introduces the *categorical* component: it is possible that an ω-sequence exists. The categorical component guarantees that, if the hypothetical component is true, it is non-vacuously true.' [Griffiths(2015)]. However, as Owen Griffiths has noticed, Hellman's *categorical* component seems to state the possibility of an actual infinity (since both components are expressed in second-order S5 with the Barcan Formula) [Owen Griffiths (2015)].

explicate the second sense of number that occurs in the *Physics* texts. While it seems that Aristotle understands numbers in this second sense as species of collections, things get a lot more complicated when the discussion comes to the issue of the unity of number. Aristotle's discussion of the issue leaves much to be desired; however, some light may be shed if we also take into account the *Theatetus* and invoke the Euclidean definition of number as a composition of units. My analysis of the issue of the unity of this composite number pointed to a conception of number as a related whole, with two sub-conceptions emerging: a) a cardinal conception of composite number, where number is understood as a related whole of mutually distinct units, and, b) an ordinal conception, where number is understood a sequence of units. I have also briefly explored an account that treats numerical predication as an instance of plural predication, where no such unity is required. One might, however, be disappointed in that Aristotle does not seem to provide a clear answer to the question 'How are these conceptions of number interrelated?' But if there is any consolation to the reader, he was not alone in this; as Dummett complains, even Frege failed to understand the importance of ordinal numbers and to provide an account of the relation between the ordinal and the cardinal conception:

> <Frege's> definition of the natural numbers did not achieve the generality for which he aimed. He assumed, as virtually everyone else at the time would have done, that the most general application of the natural numbers is to give the cardinality of finite sets. The procedure of counting does not merely establish the cardinality of the set counted: it imposes a particular ordering on it. It is natural to think this ordering irrelevant, since any two orderings of a finite set will have the same order type; but, if Frege had paid more attention to Cantor's work, he would have understood what it revealed, that the notion of ordinal number is more fundamental than that of cardinal number. This is true even in the finite case; after all, when we count the strokes of a clock, we are assigning an ordinal number rather than a cardinal. If Frege had understood this, he would therefore have characterised the natural numbers as finite ordinals rather than a cardinal. He was well aware that Cantor was concerned with ordinal rather than cardinal numbers; but . . . he dismissed the difference as a mere divergence of interest, and never perceived its significance. [Dummett (1991), p.293]

Bibliography

Alexander of Aphrodisias, *On Aristotle Metaphysics 2 and 3*, trans. by W. E. Dooley, S. J., and A. Madigan, S. J., Duckworth, London, 1992.

Annas, J., *Aristotle's Metaphysics: Books M and N*, Clarendon Press, Oxford, 1976.

— 'On the "Intermediates"', *Archiv für Geschichte der Philosophie*, 57(2), 1975, pp.146-166.

— 'Aristotle, Number and Time', *Philosophical Quarterly*, 25(99), 1975, pp.97–113.

Apostle, H.G. (trans. & ed.), *Aristotle's Physics*, Bloomington, IN, 1969.

Aristotle, *The Complete Works of Aristotle: The Revised Oxford Translation*, Jonathan Barnes (ed.), Princeton University Press, 1984.

— *Prior Analytics*, Robin Smith (ed., trans.), Hackett Publishing Company, Indianapolis (IN), 1989.

Armstrong, D.M., *A theory of universals, Volume 2: Universals and Scientific Realism*, Cambridge University Press,1978.

Barnes, J., *Truth, etc.: Six Lectures on Ancient Logic*, Oxford University Press, 2007.

— 'Aristotle's Arithmetic', *Revue de Philosophie Ancienne*, 3(1), 1985, pp. 97-133.

—*Aristotle's Posterior Analytics*, Clarendon Press, 1975.

Bartha, P., 'Monstrous Neighbors or Curious Coincidence: Aristotle on Boundaries and Contact', *History of Philosophy Quarterly*, 18(1), 2001, pp.1-16.

Beere, J., *Doing and Being: An Interpretation of Aristotle's 'Metaphysics' Theta*, Oxford University Press, 2009.

Bell, D., *Husserl*, Routledge, London, 1990.

Benacerraf, P., 'What numbers could not be', *Philosophical Review*, 74(1), 1965, pp.47-73.

Berry, S., 'Modal structuralism simplified', *Canadian Journal of Philosophy*, 48(2), 2018, pp.200-222.

Betegh, G., 'Body: M 9.359–440', in K. Algra and K. Ierodiakonou (ed.), *Sextus Empiricus and Ancient Physics*, Cambridge University Press, 2015, pp. 130–183.

Betegh, G., Pedriali, F. & Pfeiffer, C., 'The Perfection of Bodies: Aristotle's De Caelo I.1', *Rhizomata*, 1(1), 2013, pp. 30-62.

Bigelow, J., *The Reality of Numbers: A Physicalist's Philosophy of Mathematics*, Oxford University Press, 1988.

Bolzano, B., *Paradoxes of the Infinite*, Routledge, London, 2014.

Bostock, D., 'Aristotle's philosophy of mathematics' in C. Shields (ed.), *The Oxford Handbook on Aristotle*, Oxford University Press, 2012, p.465-492.

–– *Philosophy of Mathematics: An Introduction*, Wiley-Blackwell, Oxford, 2009.

–– *Space, time, matter and form : essays on Aristotle's Physics*, Clarendon Press, Oxford, 2006.

–– *Aristotle: 'Metaphysics' Books Z and H*, Oxford University Press, 1994.

–– *Plato's Phaedo*, Oxford University Press, 1986.

–– 'Zeno and the potential infinite', *Proceedings of the Aristotelian Society,* 73, 1972, pp. 37–51.

Brentano, F., *Philosophical Investigations of Space, Time and the Continuum*, B. Smith (trans.), Routledge, New York, 2010.

Brentlinger, J. A., 'The Divided Line and Plato's Theory of Intermediates', *Phronesis*, 8(2), 1963, pp. 146-166.

Burkert, W., *Lore and Science in Ancient Pythagoreanism*, E. Minar (tr.), Harvard University Press, 1972.

Burnyeat, M., 'Archytas and Optics', *Science in Context,* 18(1), 2005, pp.35-53.

–– *The 'Theaetetus' of Plato*, with trans. by M. J. Levett, Hackett, Indianapolis, 1990.

–– 'Platonism and Mathematics. A Prelude to Discussion' in Graeser, A. (ed.), *Mathematics and Metaphysics in Aristotle: Akten des X. Symposium Aristotelicum*, Haupt, Bern, 1987, pp.213–240.

Burnyeat, M. F., et al., *Notes on Eta and Theta*, unpublished ms, Faculty of Philosophy, University of Oxford, 1984.

Cantor, G., *Briefe*, Meschkowski H. & Nilson W. (eds.), Springer, Berlin, 1991.

Castelli, L., *Problems and Paradigms of Unity: Aristotle's Accounts of the One*, Academia Verlag, Sankt Augustin, 2010.

Celucci, C., 'The universal generalisation problem', *Logique & Analyse*, 52(205), 2009, pp.3-20.

Centrone, S., *Logic and philosophy of mathematics in the early Husserl*, Springer, Dordrecht, 2010.

Charles, D., 'Aristotle's Psychological Theory', *Proceedings of the Boston Area Colloquium of Ancient Philosophy*, 24 (1), 2009, pp.1-49.

Charlton, W., *Aristotle's: Physics I, II*, Oxford University Press, 1970.

Cherniss, H. F., *The Riddle of the Early Academy*, Berkeley, 1945.

— *Aristotle's Criticism of Plato and the Academy*, vol.1, The Johns Hopkins Press Baltimore, 1944.

Chisholm, R., 'Boundaries as dependent particulars', *Grazer Philosophische Studien*, 20(1), 1983, pp.87-95.

Cleary, J., *Studies on Plato, Aristotle, and Proclus: collected essays on ancient philosophy of John Cleary*, John Dillon & Brendan O'Byrne & Fran O'Rourke (eds.), Brill, Leiden, 2013.

— *Aristotle and mathematics : Aporetic Method in Cosmology and Metaphysics*, Brill, Leiden, 1995.

— 'On the Terminology of 'Abstraction' in Aristotle', *Phronesis*, 30(1), 1985, pp. 13–45.

Cohen, S. M., 'The Logic of the Third Man', *Philosophical Review*, 80(4), 1971, pp.448-475.

Coope, U., 'Aristotle on the Infinite' in the *Oxford Handbook of Aristotle*, C. Shields (ed.), Oxford, 2012, pp.267-286.

— *Time for Aristotle: 'Physics' IV.10-14*, Oxford University Press, 2005.

Cooper, J., *Reason and Human Good in Aristotle*, Harvard University Press, 1975.

Crubellier, M. and Laks, A. (eds.), *Aristotle: Metaphysics Beta Symposium Aristotelicum*, Oxford University Press, 2009.

Duhem, P., *Medieval Cosmology*, Roger Ariew (trans.), University of Chicago Press, Chicago, 1985.

Dummett, M., Frege: *Philosophy of Mathematics*, Harvard University Press, 1991.

— *Frege: Philosophy of Language*, Duckworth, London, 1973.

Fine, G., *On Ideas-Aristotle's Criticism of Plato's Theory of Forms*, Oxford University Press, 1993.

— 'Separation', *Oxford Studies in Ancient Philosophy*, 2, 1984, pp.31–87.

Fine, K., *Reasoning with Arbitrary Objects*, Basil Blackwell, Oxford, 1985.

Frank, D.H., *The Arguments 'From the Sciences' in Aristotle's Peri Ideon*, Peter Lang, New York, 1984.

Frege, G., *The Foundations of Arithmetic*, trans. by J.L. Austin, 2nd rev. ed., Harper, New York, 1953.

Gallop, D., *Plato's Phaedo*, Oxford University Press, 1975.

Ganeri, J., 'Numbers as Properties of Objects: Frege and the Nyāya', *Studies in Humanities and Social Sciences*, 3(2), 1996, pp.111-121.

Gaukroger, S., 'The one and the many: Aristotle on the individuation of numbers', *The Classical Quarterly*, 32(2), 1982, pp.312-322.

Griffiths, O., 'Philosophy of Mathematics: Nominalism', online notes: https://www.phil.cam.ac.uk/curr-students/II/II-lecture-notes/pom-7.pdf/at_download/file

Halper, E., 'Some Problems in Aristotle's Mathematical Ontology', *Proceedings of the Boston Area Colloquium in Ancient Philosophy*, 5(1), 1989, pp.247-276.

Harte, V., *Plato on parts and wholes: The metaphysics of structure*, Oxford University Press, 2002.

— 'Aristotle Metaphysics H.6: a dialectic with Platonism', *Phronesis*, 41(3), 1996, pp.276-304.

Heath, T.L., 'THE THEORY OF CONCENTRIC SPHERES–EUDOXUS, CALLIPPUS, AND ARISTOTLE' in *Aristarchus of Samos, the Ancient Copernicus: A History of Greek Astronomy to Aristarchus, Together with Aristarchus's Treatise on the Sizes and Distances of the Sun and Moon*, Cambridge University Press, 2013, pp.190-224.

— *Mathematics in Aristotle*, Oxford University Press, 1949.

— *A History of Greek Mathematics*, vol.1, Clarendon Press, Oxford, 1921.

Hellman, G., *Mathematics without numbers: Towards a modal-structural interpretation*, Clarendon Press, Oxford, 1989.

Hintikka, J., *Analyses of Aristotle*, New York, Kluwer, 2004.

— 'Aristotelian infinity', *Philosophical Review*, 75(2), 1966, pp.197–218.

Hussey , E., 'Aristotle on Mathematical Objects', *Apeiron*, 24(4), 2011, pp. 105-134.

— *Aristotle's 'Physics': books III and IV*, Oxford University Press, 1983.

Irvine, A., 'Frege on Number Properties', *Studia Logica: An International Journal for Symbolic Logic*, 96(2), 2010, pp.239-260.

Jacquette, D., ‘Mathematical Fiction and Structuralism in Chihara's Constructibility Theory’, *History and Philosophy of Logic*, 25(4), 2004, pp.319-324.

Jevons, S., *The Principles of Science*, Macmillan and Co., London, 1913.

Kessler, G., ‘Frege, Mill, and the foundations of arithmetic’, *The Journal of Philosophy*, 77(2), 1980, pp.65-79.

Kirwan, C., *Aristotle Metaphysics Books Γ, Δ, and E*, Oxford University Press, 1993.

Kitcher, P., *The Nature of Mathematical Knowledge*, Oxford University Press, 1984.

Klein, J., *Greek Mathematical Thought and the Origin of Algebra*, E. Brann (trans.), M.I.T. Press, Cambridge (MA), 1968.

Kouremenos, T., ‘Aristotle on geometric perfection in the physical world’, *Mnemosyne*, 56(4), 2003, pp.463-479.

Kretzmann, N.(ed.), *Infinity and Continuity in Ancient and Medieval Thought*, Cornell University Press, Ithaca (NY), 1982.

Laguna, T., ‘Point, Line, and Surface, as Sets of Solids’, *The Journal of Philosophy*, 19(17), 1922, pp.449-461.

Lambros, C.H., ‘Are Numbers Properties of Objects?’, *Philos. Stud.*, 29, 1976, pp.381-389.

Lear, J., ‘Aristotle's Philosophy of Mathematics’, *Philosophical Review*, 91(2), 1982, pp.161-92.

— ‘Aristotelian Infinity’, *Proceedings of the Aristotelian Society*, 80, 1980, pp. 187-210.

Locke, J., *An essay concerning human understanding*, ed. P.H. Nidditch. Oxford, Oxford University Press, 1975.

Long, A.A, and Sedley, N.D., (eds.), *The Hellenistic Philosophers* (vol.1), Cambridge University Press, 1986.

Lowe, E. J., *Locke*, Routledge, London, 2005.

— *The Possibility of Metaphysics*, Oxford University Press, 2001.

— ‘Are the natural numbers individuals or sorts?’, *Analysis*, 53(3), 1993, pp.142-146.

Macnamara, J., ‘Review of Reasoning with Arbitrary Objects by Kit Fine’, *The Journal of Symbolic Logic*, 53(1), 1988, pp. 305-306.

Madigan, A., ‘Syrianus and Asclepius on Forms and Intermediates in Plato and Aristotle’, *Journal of the History of Philosophy*, 24(2), 1986, pp.149-171.

Makin, S., *Aristotle: Metaphysics Book Θ*, Oxford University Press, 2006.

Mayberry, J.P., *The Foundations of Mathematics in the Theory of Sets*, Cambridge University Press, 2000.

McDaniel, K., ‘Existence and Number’, *Analytic Philosophy*, 54(2), 2013 pp. 209–228.

McKirahan, R., *Philoponus: On Aristotle Posterior Analytics 1.1-8*, Bloomsbury, 2008.

— ‘Aristotle’s Subordinate Sciences’, *British Journal for the History of Science*, 11(3), 1978, pp. 197–220.

Mendell, H., *Aristotle and the Mathematicians, Some Cross Currents in the Fourth Century*, Ph.D. Dissertation, Stanford University, 1986.

Menn, S., ‘Iγ3:The argument-structure of MN and their place in the *Metaphysics*’ [Iγ3], in *The Aim and the Argument of Aristotle's Metaphysics*, online draft.

— ‘Iβ3: The "substantive" aporiai and the quarrel of the disciplines’ [Iβ3], in *The Aim and the Argument of Aristotle's Metaphysics*, online draft.

— ‘IIIα3: Θ7-9: conclusions for the ἀρχαί and the priority of ἐνέργεια.’ [IIIα3], in *The Aim and the Argument of Aristotle's Metaphysics*, online draft.

Mignucci, M., ‘Aristotle’s Arithmetic’ in Graeser, A. (ed.), *Mathematics and Metaphysics in Aristotle: Akten des X. Symposium Aristotelicum*, Haupt, Bern, 1987, pp.175-211.

Mill, J. S., *System of Logic*, Harper, New York, 1874.

Miller, F. D. Jr.,’Aristotle against the Atomists’, in Kretzmann (ed.), *Infinity and Continuity*, Cornell University Press, Ithaca (NY), 1982, pp.87-111.

Modrak, D. K. W., ‘Aristotle on the Difference between Mathematics and Physics and First Philosophy’, *Apeiron*, 22(4), 1989, pp.121-139.

Moore, G. E., *Some Main Problems of Philosophy*, Macmillan, London, 1953.

Morison, B., *On location: Aristotle's concept of place*, Clarendon Press, Oxford, 2002.

Morrow, G. R., *Proclus: A Commentary on the First Book of Euclid’s Elements*, Princeton University Press, 1970.

Mouzala, M., ‘Syrianus on the Platonic Tradition of the Separate Existence of Numbers’, *Peitho/Examina Antiqua*, 1(6), 2015, pp.167-194.

Mueller, I., ‘Aristotle’s doctrine of abstraction in the commentators’, in R. Sorabji (ed.), *Aristotle Transformed: The ancient commentators and their influence*, London, Duckworth, 1990, pp.464-465.

— *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, MIT Press, 1981.

— ‘Aristotle on Geometrical Objects’, *Archiv für Geschichte der Philosophie*, 52(2), 1970, p.164.

Netz, R., ‘Aristotle’s *Metaphysics* M.3: realism and the philosophy of QUA’, *Stanford/Princeton Working Papers in Classics*, 2006.

— *The shaping of deduction in Greek mathematics: A study in cognitive history*, Cambridge University Press, 2003.

Nicod, J., *Foundations of Geometry and Induction*, Routledge and Kegan Paul, London, 1930.

O'Brien, D., ‘The Last Argument of Plato's Phaedo’, *The Classical Quarterly*, 17(2), 1967, pp.198-231.

O’Connor, D., *The Metaphysics of G.E. Moore*, D.Reidel Publishing Company, Dordrecht, 1982.

Oliver, A., ‘Frege and Dummett are two’, *The Philosophical Quarterly*, 44(174), 1994, pp.74-82.

Oliver A. and Smiley T., *Plural Logic*, Oxford University Press, 2013.

Owen, G.E.L., ‘A Proof in the Peri Idewn’, *The Journal of Hellenic Studies*, 77(1), 1957, pp.103-111.

Peramatzis, M., *Priority in Aristotle’s Metaphysics*, Oxford University Press, 2011.

Pettigrew, R., ‘Aristotle on the Subject Matter of Geometry’, *Phronesis*, 54(3), 2009, pp. 239-260.

Philoponus, *On Aristotle ‘Physics 2’*, trans. by A. R. Lacey, Duckworth, London, 1993.

Pritchard, P., *Plato's Philosophy of Mathematics*, Academia Verlag, Sankt Augustin, 1995.

Putnam, H., ‘The thesis that mathematics is logic’ in R. Schoenman (ed.), *Bertrand Russell, Philosopher of the Century*, Allen & Unwin, London, 1967, pp.273–303.

Reeve, C. D. C., *Plato's Republic*, Hackett Pub. Co, Indianapolis, 2004.

Resnik, M., 'Mathematical Knowledge and Pattern Cognition', *Canadian Journal of Philosophy*, 5(1), 1975, pp. 25-39.

Ross, W. D., *Aristotle's 'Prior' and 'Posterior Analytics'*, Oxford University Press, 1949.

— *Aristotle's 'Metaphysics'*, 2 vols., Oxford University Press, 1924.

Russell, B., *The Analysis of Matter*, George Allen and Unwin, London, 1927.

Sambursky, S, *Physics of the Stoics*, Routledge and Kegan Paul, London, 1959.

Santambrogio, M., 'Review of Reasoning with Arbitrary Objects by K. Fine', *Noûs*, 22(4), 1988, pp. 630-635.

Scaltsas, T., 'Sharing a Property', *Philosophical Enquiry*, 41(2-3), 2017, pp. 3-16.

— *Substances and universals in Aristotle's Metaphysics*, Cornell University Press, Ithaca (NY), 1994.

— 'Is a Whole Identical to its Parts?', *Mind*, 99(396), 1990, pp.583-598.

Schiaparelli, A, 'Aristotle on the fallacies of combination and division in *Sophistici Elenchi* 4', *History and Philosophy of Logic*, 24(2), 2003, pp.111-129.

Shapiro, S., *Philosophy of Mathematics: Structure and Ontology*, Oxford University Press, 1997.

Simons, P. M.,'On Multitudes', Lecture at Opole University, 2011.

— 'Number and Manifolds', in B. Smith (ed.), *Parts and Moments: Studies in Logic and Formal Ontology*, Philosophia Verlag, München, 1982, pp.160–98.

Simplicius, *On Aristotle Physics 2*, trans. by B. Fleet, Duckworth, London, 1997.

Sorabji, R., *Necessity, Cause, and Blame: Perspectives on Aristotle's Theory*, Duckworth, London, 1980.

Strawson, P.F., *Individuals*, Methuen, London, 1964.

Stroll, A., *Surfaces*, University of Minnesota Press, Minneapolis, 1988.

Syrianus, *On Aristotle Metaphysics 13-14*, translations by J. Dillon and D. O'Meara, Duckworth, London, 2006.

Tarán, L., 'Aristotle's Classification of Number in *Metaphysics* M.6, 1086a15–37', *Greek, Roman, and Byzantine Studies*, 19, 1978, pp.83–90.

Tarski, A., 'Foundations of the Geometry of Solids' (trans. by J. H. Woodger), in A. Tarski, *Logics, Semantics, Metamathematics, Papers from 1923-1938*, Clarendon Press, Oxford, 1956.

Wedberg, A., *Plato's Philosophy of Mathematics*, Almquist and Wiksell, Stockholm, 1955.

Wetzel, L., 'That Numbers Could Be Objects', *Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition*, 56(3), 1989, pp. 273-292.

White, M., 'The Metaphysical Location of Aristotle's Μαθηματικά', *Phronesis*, 38(2), 1993, pp.166-182.

—*The continuous and the discrete: Ancient Physical Theories from a Contemporary Perspective*, Clarendon Press, Oxford 1992.

Whitehead, A.N., 'La théorie relationniste de l'espace', *Revue de Métaphysique et de Morale* 23, pp.423-454, (trans. in P.J. Hurley, "Whitehead's 'Relational Theory of Space': Text, Translation and Commentary", *Philosophy Research Archives* 5, 1979).

Wiggins, D., *Identity and Spatio-Temporal Continuity*, Basil Blackwell, Oxford, 1967.

Wildberg, C., 'Syrianus', in Edward N. Zalta (ed.),*The Stanford Encyclopedia of Philosophy*, 2009 Edition,

URL = <https://plato.stanford.edu/archives/fall2009/entries/syrianus/>.

— *John Philoponus' Criticism of Aristotle's Theory of Aether*, Berlin, Walter de Gruyter, 1988.

Williams, C. J. F., *Aristotle's De Generatione et Corruptione*, Oxford University Press, 1982.

Wilson, J. C., 'On the Platonist Doctrine of the assumbletoi arithmoi', *The Classical Review*, 18(5), 1904, pp.247–60.

Wilson, M., *Structure and Method in Aristotle's Meteorologica: A More Disorderly Nature*, Cambridge University Press, 2013.

Wood, R. (ed.&trans.), *Adam Wodeham's Tractatus de indivisibilis*, Kluwer, Dordrecht, 1988.

Woodruff, P., 'Socrates and Ontology: The Evidence of the *Hippias Major*', *Phronesis*, 23(2), 1978, pp.101-117.

Zimmerman, W.D., 'Indivisible Parts and Extended Objects: Some Philosophical Episodes from Topology's Prehistory', *The Monist*, 79(1), 1996, pp. 148- 180.

Zupko, J., ‘Nominalism Meets Indivisibilism’, *Medieval Philosophy & Theology,* 3, 1993, pp.158-185.