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The Ruelle zeta function at zero for nearly hyperbolic 3-manifolds

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Abstract We show that for a generic conformal metric perturbation of a compact hyperbolic 3-manifold Σ with Betti number b_1, the order of vanishing of the Ruelle zeta function at zero equals $4 - b_1$, while in the hyperbolic case it is equal to $4 - 2b_1$. This is in contrast to the 2-dimensional case where the order of vanishing is a topological invariant. The proof uses the microlocal

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approach to dynamical zeta functions, giving a geometric description of generalized Pollicott–Ruelle resonant differential forms at 0 in the hyperbolic case and using first variation for the perturbation. To show that the first variation is generically nonzero we introduce a new identity relating pushforwards of products of resonant and coresonant 2-forms on the sphere bundle $S\Sigma$ with harmonic 1-forms on Σ.

Let (Σ, g) be a compact connected oriented 3-dimensional Riemannian manifold of negative sectional curvature. The Ruelle zeta function

$$\zeta_{\mathrm{R}}(\lambda) = \prod_{\gamma} \left(1 - e^{i\lambda T_\gamma}\right), \quad \operatorname{Im}\lambda \gg 1 \tag{1.1}$$

is a converging product for $\operatorname{Im}\lambda$ large enough and continues meromorphically to $\lambda \in \mathbb{C}$ as proved by Giulietti–Liverani–Pollicott [34] and Dyatlov–Zworski [20]. Here the product is taken over all primitive closed geodesics γ on (Σ, g) and T_γ is the length of γ.

In this paper we study the order of vanishing of ζ_{R} at $\lambda = 0$, defined as the unique integer $m_{\mathrm{R}}(0)$ such that $\lambda^{-m_{\mathrm{R}}(0)}\zeta_{\mathrm{R}}(\lambda)$ is holomorphic and nonzero at 0. Our main result is

Theorem 1 *Let (Σ, g_H) be a compact connected oriented hyperbolic 3-manifold and $b_1(\Sigma)$ be the first Betti number of Σ. Then:*

1. *For (Σ, g_H) we have $m_{\mathrm{R}}(0) = 4 - 2b_1(\Sigma)$.*
2. *There exists an open and dense set $\mathscr{O} \subset C^\infty(\Sigma; \mathbb{R})$ such that for any $\mathbf{b} \in \mathscr{O}$, there exists $\varepsilon > 0$ such that for any $\tau \in (-\varepsilon, \varepsilon)\backslash\{0\}$ and $g_\tau := e^{-2\tau\mathbf{b}} g_H$, the manifold (Σ, g_τ) has $m_{\mathrm{R}}(0) = 4 - b_1(\Sigma)$.*

Part 1 of Theorem 1 was proved by Fried [25, Theorem 3] using the Selberg trace formula. The novelty is part 2, which says that *for generic conformal perturbations of the hyperbolic metric the order of vanishing of ζ_{R} equals* $4 - b_1(\Sigma)$. In particular, when $b_1(\Sigma) > 0$ (fulfilled in many cases, in particular for mapping tori over pseudo-Anosov maps [24, Theorem 13.4]), $m_{\mathrm{R}}(0)$ is not topologically invariant. Theorem 1 is the first result on instability of the order of vanishing of ζ_{R} at 0 for Riemannian metrics. It is in contrast to the 2-dimensional case, where Dyatlov–Zworski [21] showed that $m_{\mathrm{R}}(0) = b_1(\Sigma) - 2$ for *any* compact connected oriented negatively curved surface (Σ, g), and is complementary to a recent breakthrough on the (acyclic) *Fried conjecture* by Dang–Guillarmou–Rivière–Shen [16], see §1.3 below.

A result similar to Theorem 1 holds for contact perturbations of $S\Sigma$, see Theorem 4 in §4.

1.1 Outline of the proof

We now outline the proof of Theorem 1. We use the microlocal approach to Pollicott–Ruelle resonances and dynamical zeta functions, which we review here – see §2 for details and §1.3 for a historical overview. Let $M = S\Sigma$ be the sphere bundle of (Σ, g) and $X \in C^\infty(M; TM)$ be the generator of the geodesic flow. The geodesic flow is a *contact flow*, i.e. there exists a 1-form $\alpha \in C^\infty(M; T^*M)$ such that $\iota_X\alpha = 1$, $\iota_X d\alpha = 0$, and $\alpha \wedge d\alpha \wedge d\alpha$ is a nonvanishing volume form. When g has negative curvature, the geodesic flow is *Anosov*, i.e. the tangent spaces $T_\rho M$ decompose into a direct sum of the flow, unstable, and stable subspaces. Denote by E_u^*, E_s^* the dual unstable/stable subbundles of the cotangent bundle T^*M, that is, E_u^*, E_s^* are the annihilators of unstable/stable plus flow directions; these define closed conic subsets of T^*M.

Define the spaces of *resonant k-forms at 0*

$$\operatorname{Res}_0^k := \{u \in \mathcal{D}'(M; \Omega^k) \mid \iota_X u = 0,\ \mathcal{L}_X u = 0,\ \operatorname{WF}(u) \subset E_u^*\}. \quad (1.2)$$

Here Ω^k is the (complexified) bundle of k-forms, $\mathcal{L}_X = d\iota_X + \iota_X d$ is the Lie derivative with respect to X, and for any distribution $u \in \mathcal{D}'(M; \Omega^k)$ we denote by $\operatorname{WF}(u) \subset T^*M\setminus 0$ the *wavefront set* of u, see for instance [38, Chapter 8]. The wavefront set condition makes Res_0^k into a finite dimensional space, which is a consequence of the interpretation of Res_0^k as the eigenspace at 0 of the operator $P_{k,0} := -i\mathcal{L}_X$ acting on certain anisotropic Sobolev spaces tailored to the flow (see [29, Theorem 1.7] and [21, Lemma 2.2]). We similarly define the spaces of *generalized resonant k-forms at 0*

$$\begin{aligned}
\operatorname{Res}_0^{k,\ell} &:= \{u \in \mathcal{D}'(M; \Omega^k) \mid \iota_X u = 0,\ \mathcal{L}_X^\ell u = 0,\ \operatorname{WF}(u) \subset E_u^*\},\\
\operatorname{Res}_0^{k,\infty} &:= \bigcup_{\ell\geq 1} \operatorname{Res}_0^{k,\ell}.
\end{aligned}$$

The *semisimplicity* condition for k-forms states that $\operatorname{Res}_0^{k,\infty} = \operatorname{Res}_0^k$, which means that the operator $P_{k,0}$ has no nontrivial Jordan blocks at 0. We also have the dual spaces of *generalized coresonant k-forms at 0*, replacing E_u^* with E_s^* in the wavefront set condition:

$$\operatorname{Res}_{0*}^{k,\ell} := \{u_* \in \mathcal{D}'(M; \Omega^k) \mid \iota_X u_* = 0,\ \mathcal{L}_X^\ell u_* = 0,\ \operatorname{WF}(u) \subset E_s^*\}.$$

Since $E_u^* \cap E_s^* = \{0\}$, wavefront set calculus makes it possible to define $u \wedge u_*$ as a distributional differential form as long as $\operatorname{WF}(u) \subset E_u^*$, $\operatorname{WF}(u_*) \subset E_s^*$.

The order of vanishing of the Ruelle zeta function at 0 can be expressed as the alternating sum of the dimensions of the spaces of generalized resonant k-forms, see (2.59):

$$m_{\mathrm{R}}(0)=\sum_{k=0}^{4}(-1)^k\dim\mathrm{Res}_0^{k,\infty}.$$

Thus the problem reduces to understanding the spaces $\mathrm{Res}_0^{k,\infty}$ for $k = 0,1,2,3,4$. The proof of Theorem 1 computes their dimensions, listed in the table below, from which the formulas for $m_{\mathrm{R}}(0)$ follow immediately. See Theorem 2 in §3 for the hyperbolic case and Theorem 3 in §4, as well as §4.4, for the case of generic perturbations.

Dimension of	**Hyperbolic**	**Perturbation**
$\mathrm{Res}_0^0=\mathrm{Res}_0^{0,\infty}$	1	1
$\mathrm{Res}_0^1=\mathrm{Res}_0^{1,\infty}$	$2b_1(\Sigma)$	$b_1(\Sigma)$
Res_0^2	$b_1(\Sigma)+2$	$b_1(\Sigma)+2$
$\mathrm{Res}_0^{2,2}=\mathrm{Res}_0^{2,\infty}$	$2b_1(\Sigma)+2$	$b_1(\Sigma)+2$
$\mathrm{Res}_0^3=\mathrm{Res}_0^{3,\infty}$	$2b_1(\Sigma)$	$b_1(\Sigma)$
$\mathrm{Res}_0^4=\mathrm{Res}_0^{4,\infty}$	1	1

Note that the semisimplicity condition holds for $k = 0,1,3,4$ in both the hyperbolic case and for generic perturbations. However, semisimplicity fails for $k = 2$ in the hyperbolic case (assuming $b_1(\Sigma) > 0$), and it is restored for generic perturbations. Also, since $b_2(M) = b_1(\Sigma) + 1$ (see (2.28)), we may interpret the dimension of Res_0^2 in the perturbed case as the 'topological part' coming from the bijection with the de Rham cohomology group $H^2(M;C)$ and the extra invariant form $d\alpha$.

The cases $k = 0,4$ of the above table are well-known: the semisimplicity condition holds and Res_0^0, Res_0^4 are spanned by 1, $d\alpha\wedge d\alpha$, see Lemma 2.4. One can also see that the map $u\mapsto d\alpha\wedge u$ gives an isomorphism from $\mathrm{Res}_0^{1,\ell}$ to $\mathrm{Res}_0^{3,\ell}$. Thus it remains to understand the spaces $\mathrm{Res}_0^{k,\infty}$ for $k = 1,2$ and this is where the situation gets more complicated.

The spaces $\mathrm{Res}_0^k\cap\ker d$ of resonant states that are closed forms play a distinguished role in our argument. Similarly to [21] we introduce linear maps π_k from $\mathrm{Res}_0^k\cap\ker d$ to the de Rham cohomology groups $H^k(M;\mathbb{C})$, see (2.61). We show that the map π_1 is an isomorphism, see Lemma 2.8. This gives the dimension of the space of *closed* forms in Res_0^1: since $b_1(M)=b_1(\Sigma)$,

$$\dim(\mathrm{Res}_0^1\cap\ker d)=b_1(\Sigma).$$

In the hyperbolic case, the other $b_1(\Sigma)$-dimensional space of *non-closed* forms in Res_0^1 is obtained by rotating the closed forms by $\pi/2$ in the dual unstable space, see §3.3. This rotation commutes with the geodesic flow because the geodesic flow is conformal on the stable/unstable spaces, see (3.7). This additional symmetry, which is only present in the hyperbolic case, is related to the presence of a closed 2-form $\psi \in C^\infty(M; \Omega^2)$ which is invariant under the geodesic flow and is not a multiple of $d\alpha$, see §3.2.3. The space Res_0^2 is spanned by $d\alpha$, ψ, and the differentials du where u are the non-closed forms in Res_0^1, see §3.4. We also show in §3.4 that each $du \in d(\mathrm{Res}_0^1)$ lies in the range of $\mathcal{L}_X$, producing $b_1(\Sigma)$ Jordan blocks for the operator $P_{2,0}$.

In the case of the perturbation $g_\tau = e^{-2\tau\mathbf{b}} g_H$, we use first variation techniques and make the following *nondegeneracy assumption* (see §4.4): for the spaces Res_0^1, Res_{0*}^1 and the contact form α defined using the hyperbolic metric g_H, and denoting by $\pi_\Sigma : M = S\Sigma \to \Sigma$ the projection map, we assume that

$$(du, du_*) \mapsto \int_M (\pi_\Sigma^* \mathbf{b})\alpha \wedge du \wedge du_* \quad \text{defines a nondegenerate pairing on} \quad d(\mathrm{Res}_0^1) \times d(\mathrm{Res}_{0*}^1). \tag{1.3}$$

Under the assumption (1.3), we show that the non-closed 1-forms in Res_0^1 move away once τ becomes nonzero (i.e. they turn into generalized resonant states for nonzero Pollicott–Ruelle resonances), see §4.1. Thus for $0 < |\tau| < \varepsilon$ all the resonant 1-forms are closed and we get $\dim \mathrm{Res}_0^1 = b_1(\Sigma)$. Further analysis shows that semisimplicity is restored for $k = 2$ and $\dim \mathrm{Res}_0^2 = b_1(\Sigma) + 2$.

It remains to show that the nondegeneracy assumption (1.3) holds for a generic choice of the conformal factor $\mathbf{b} \in C^\infty(\Sigma; \mathbb{R})$. The difficulty here is that $\mathbf{b}$ can only depend on the point in Σ and not on elements of $S\Sigma$ which is where $\alpha \wedge du \wedge du_*$ lives. We reduce (1.3) to the following statement on nontriviality of pushforwards (see Proposition 4.10): for each real-valued resonant 1-form for the hyperbolic metric $u \in \mathrm{Res}_0^1$ we have

$$du \neq 0 \quad \Longrightarrow \quad \pi_{\Sigma*}(\alpha \wedge du \wedge \mathcal{J}^*(du)) \neq 0. \tag{1.4}$$

Here $\mathcal{J} : (x, v) \mapsto (x, -v)$ is the antipodal map on $M = S\Sigma$ and $\pi_{\Sigma*}$ is the pushforward of differential k-forms on M to $(k-2)$-forms on Σ obtained by integrating along the fibers, see (2.19).

The statement (1.4) concerns resonant 1-forms for the hyperbolic metric $g = g_H$, which are relatively well-understood. However, it is complicated by the fact that $\pi_{\Sigma*}(\alpha \wedge du \wedge \mathcal{J}^*(du))$ is merely a distribution, so we cannot hope to show it is nonzero by evaluating its value at some point. Instead we pair it with functions in $C^\infty(\Sigma)$ which have to be chosen carefully so that we can compute the pairing. More precisely, we prove the following identity (Theorem 5 in §5):

$$Q_4 F = -\tfrac{1}{6}\Delta_g |\sigma|_g^2 \quad \text{where} \quad \pi_{\Sigma *}(\alpha \wedge du \wedge \mathcal{J}^*(du)) = F\, d\,\mathrm{vol}_g\,. \quad (1.5)$$

Here $d\,\mathrm{vol}_g$ is the volume form on (Σ, g), Δ_g is the Laplace–Beltrami operator, $Q_4 : \mathcal{D}'(\Sigma) \to C^\infty(\Sigma)$ is a naturally defined smoothing operator, and

$$\sigma := \pi_{\Sigma *}(d\alpha \wedge u) \in C^\infty(\Sigma; T^*\Sigma)$$

is proved to be a nonzero harmonic 1-form on (Σ, g). The identity (1.5) implies the nontriviality statement (1.4): if $F = 0$ then $|\sigma|_g^2$ is constant, but hyperbolic 3-manifolds do not admit harmonic 1-forms of nonzero constant length as shown in Appendix A. This finishes the proof of Theorem 1.

If one is interested instead in conformal perturbations of the contact form α, then one needs to show that $\alpha \wedge du \wedge du_*$ is not identically 0 assuming that $u \in \mathrm{Res}_0^1$, $u_* \in \mathrm{Res}_{0*}^1$ and $du \neq 0$, $du_* \neq 0$. The latter follows from the full support property for Pollicott–Ruelle resonant states proved by Weich [54]. See Theorem 4 in §4 for details.

We finally note that it would have been possible to introduce a flat unitary twist in our discussion. Namely, we can consider a Hermitian vector bundle over Σ endowed with a unitary flat connection A. Resonant spaces can be defined using the operator d_A and the holonomy of A provides a way to twist the Ruelle zeta function as well, we refer to [12] for details. We do not pursue this extension here in order to simplify the presentation.

1.2 A conjecture

Theorem 1 can be interpreted as follows: the hyperbolic metric has non-closed resonant states due to the extra symmetries, and by destroying these symmetries we make all resonant states closed. We thus make the following conjecture about generic contact Anosov flows:

Conjecture 1 *Let M be a compact $2n+1$ dimensional manifold and α a contact 1-form on M such that the corresponding flow is Anosov with orientable stable/unstable bundles. Define the spaces Res_0^k, $0 \le k \le 2n$, by (1.2) and let $\pi_k : \mathrm{Res}_0^k \cap \ker d \to H^k(M; \mathbb{C})$ be defined by (2.61). Then for a generic choice of α we have:*

(1) the semisimplicity condition holds in all degrees $k = 0, \dots, 2n$;
(2) $d(\mathrm{Res}_0^k) = 0$ for all $k = 0, \dots, 2n$;
(3) for $k = 0, \dots, n$ the map π_k is onto, $\ker \pi_k = d\alpha \wedge \mathrm{Res}_0^{k-2}$, and $\dim \ker \pi_k = \dim \mathrm{Res}_0^{k-2}$.

Denoting by $b_k(M)$ the k-th Betti number of M, we then have

$$\dim \mathrm{Res}_0^k = \sum_{j=0}^{\lfloor k/2 \rfloor} b_{k-2j}(M), \quad 0 \le k \le n; \quad \dim \mathrm{Res}_0^{2n-k} = \dim \mathrm{Res}_0^k \tag{1.6}$$

and the order of vanishing of the Ruelle zeta function at 0 is given by (see [20, (2.5)])

$$m_{\mathrm{R}}(0) = \sum_{k=0}^{2n} (-1)^{k+n} \dim \mathrm{Res}_0^k = \sum_{k=0}^{n} (-1)^{k+n} (n+1-k) b_k(M). \tag{1.7}$$

The proof of part 2 of Theorem 1 (see Theorem 3 in §4, as well as §4.4) shows that Conjecture 1 holds for $n = 2$ and geodesic flows of generic nearly hyperbolic metrics (while the conjecture is stated for generic metrics that do not have to be nearly hyperbolic). Moreover, [21] shows that Conjecture 1 holds for $n = 1$ and any contact Anosov flow.

Note that the conditions (1) and (2) of Conjecture 1 imply (3). Indeed, by the work of Dang–Rivière [18, Theorem 2.1] the cohomology of the complex $(\mathrm{Res}^{k,\infty}, d)$, with $\mathrm{Res}^{k,\infty}$ defined in (2.38) below with $\lambda_0 := 0$, is isomorphic to the de Rham cohomology of M (with the isomorphism mapping each closed form in $\mathrm{Res}^{k,\infty}$ to its cohomology class). By (2.43) and the semisimplicity condition (1), we have $\mathrm{Res}^{k,\infty} = \mathrm{Res}_0^k \oplus (\alpha \wedge \mathrm{Res}_0^{k-1})$. By condition (2), we have $d(u + \alpha \wedge v) = d\alpha \wedge v$ for all $u \in \mathrm{Res}_0^k$, $v \in \mathrm{Res}_0^{k-1}$. If $k \le n$, then $d\alpha \wedge : \mathrm{Res}_0^{k-1} \to \mathrm{Res}_0^{k+1}$ is injective, so $\mathrm{Res}^{k,\infty} \cap \ker d = \mathrm{Res}_0^k$ and $d(\mathrm{Res}^{k-1,\infty}) = d\alpha \wedge \mathrm{Res}_0^{k-2}$. This gives condition (3).

Note also that for $n = 2$ the set of contact forms satisfying Conjecture 1 is open in $C^\infty(M; T^*M)$. Indeed, by the perturbation theory discussed in §4.1, more specifically (4.18), if we take a sufficiently small perturbation of a contact form satisfying Conjecture 1, then $\dim \mathrm{Res}_0^{1,\infty} \le b_1(M)$ and $\dim \mathrm{Res}_0^{2,\infty} \le b_2(M) + 1$. By Lemma 2.8 we see that semisimplicity holds for $k = 1$ and $d(\mathrm{Res}_0^1) = 0$. Then Lemma 2.11 together with Lemma 2.4 give all the conclusions of Conjecture 1. A similar argument might work in the case of higher n. Thus the main task in proving the conjecture is to show that (1) and (2) hold on a dense set of contact forms.

One can make a similar conjecture for geodesic flows of generic negatively curved compact orientable $n+1$-dimensional Riemannian manifolds (Σ, g), with $M = S\Sigma$. In particular, if $n = 2m$ is even, then Σ is odd dimensional and thus has Euler characteristic 0. By the Gysin exact sequence we have $b_k(M) = b_k(\Sigma)$ for $0 \le k < n$ and $b_n(M) = b_n(\Sigma) + b_0(\Sigma)$. Moreover, by

Poincaré duality we have $b_k(\Sigma) = b_{n+1-k}(\Sigma)$. Thus (1.7) becomes

$$m_{\mathrm{R}}(0) = b_0(\Sigma) + \sum_{k=0}^{m} (-1)^k (2m + 1 - 2k) b_k(\Sigma).$$

This is in contrast to the hyperbolic case, where by [25, Theorem 3]

$$m_{\mathrm{R}}(0) = \sum_{k=0}^{m} (-1)^k (2m + 2 - 2k) b_k(\Sigma).$$

Note that we only expect Conjecture 1 to hold for generic flows/metrics rather than, say, all non-hyperbolic metrics: for $n = 2$ the proof of Theorem 1 uses first variation which by the Implicit Function Theorem suggests that there is a 'singular submanifold' of metrics passing through the hyperbolic metric on which Conjecture 1 fails.

1.3 Previous work

The treatment of Pollicott–Ruelle resonances of an Anosov flow as eigenvalues of the generator of the flow on anisotropic Banach and Hilbert spaces has been developed by many authors, including Baladi [3], Baladi–Tsujii [9], Blank–Keller–Liverani [5], Butterley–Liverani [6], Gouëzel–Liverani [33], and Liverani [46,47] (some of the above papers considered the related setting of Anosov maps). In this paper we use the microlocal approach to dynamical resonances, introduced by Faure–Sjöstrand [29] and developed further by Dyatlov–Zworski [20]; see also Faure–Roy–Sjöstrand [28], Dyatlov–Guillarmou [15], as well as Dang–Rivière [17] and Meddane [48] for the treatment of Morse–Smale and Axiom A flows.

The study of the relation of the vanishing order $m_{\mathrm{R}}(0)$ to the topology of the underlying manifold M has a long history, going back to the works of Fried [25,26] for geodesic flows on hyperbolic manifolds. The paper [25] also related the leading coefficient of ζ_{R} at 0 to *Reidemeister torsion*, which is a topological invariant of M. It considered the more general setting of a twisted zeta function corresponding to a unitary representation. One advantage of such twists is that one can choose the representation so that the twisted de Rham complex is *acyclic*, i.e. has no cohomology, and then one expects ζ_{R} to be holomorphic and nonvanishing at 0.

In [27, p. 66] Fried conjectured a formula relating the Reidemeister torsion with the value $\zeta_{\mathrm{R}}(0)$ for geodesic flows on all compact locally homogeneous manifolds with acyclic representations. Fried's conjecture was proved by Shen [53] for compact locally symmetric reductive manifolds, following earlier contributions by Bismut [4] and Moscovici–Stanton [49]. The abovementioned works [4,25,26,49,53] used representation theory and Selberg trace formulas, which do not extend beyond the class of locally symmetric manifolds.

In recent years much progress has been made on understanding the relation between the behavior of ζ_R at 0, as well as the dimensions of $\mathrm{Res}_0^{k,\ell}$, with topological invariants for general (not locally symmetric) negatively curved Riemannian manifolds and Anosov flows:

- Dyatlov–Zworski [21] computed $m_R(0)$ for any contact Anosov flow in dimension 3 with orientable stable/unstable bundles, including geodesic flows on compact oriented negatively curved surfaces;
- Dang–Rivière [18, Theorem 2.1] showed that the chain complex $(\mathrm{Res}^{\bullet,\infty}, d)$, where $\mathrm{Res}^{k,\infty} = \mathrm{Res}^{k,\infty}(0)$ is defined in (2.39) below, is homotopy equivalent to the usual de Rham complex and hence their cohomologies agree. One can see that Conjecture 1 is compatible with this result, using (2.43) and the fact that $(d\alpha\wedge)^k : \Omega_0^{n-k} \to \Omega_0^{n+k}$ is a bundle isomorphism for $0 \le k \le n$;
- Hadfield [35] showed a result similar to [21] for geodesic flows on negatively curved surfaces with boundary;
- Dang–Guillarmou–Rivière–Shen [16] computed $\dim \mathrm{Res}_0^{k,\infty}$ for hyperbolic 3-manifolds and proved Fried's formula relating $\zeta_R(0)$ to Reidemeister torsion for nearly hyperbolic 3-manifolds in the acyclic case; see also Chaubet–Dang [11];
- Küster–Weich [44] obtained several results on geodesic flows on compact hyperbolic manifolds and their perturbations, in particular showing that $\dim \mathrm{Res}_0^1 = b_1(\Sigma)$ when $\dim \Sigma \neq 3$;
- Cekić–Paternain [12] studied volume preserving Anosov flows in dimension 3, giving the first example of a situation where $m_R(0)$ jumps under perturbations of the flow and thus is not topologically invariant;
- Borns-Weil–Shen [10] proved a result similar to [21] for nonorientable stable/unstable bundles.

Our Theorem 1 gives a jump in $m_R(0)$ for geodesic flows on 3-manifolds and indicates that the situation for the hyperbolic case is different from that in the case of generic metrics. We stress that it is more difficult to obtain results for generic metric perturbations (such as Theorem 1) than for generic perturbations of contact forms (such as Theorem 4 in §4) due to the more restricted nature of metric perturbations.

One of our main technical results (Theorem 5) bears (limited) similarities to known pairing formulas for Patterson–Sullivan distributions such as those established by Anantharaman–Zelditch [2], Hansen–Hilgert–Schröder [37], Dyatlov–Faure–Guillarmou [14], and Guillarmou–Hilgert–Weich [32]. We briefly discuss this in the Remark after Theorem 5.

1.4 Structure of the paper

- §2 discusses contact Anosov flows on 5-manifolds and sets up the scene for the rest of the paper. In particular, it introduces Pollicott–Ruelle resonances, (co-)resonant states, dynamical zeta functions, de Rham cohomology, and geodesic flows. It also proves various general lemmas about the maps π_k and semisimplicity.
- §3 gives a complete description of generalized resonant states at 0 for hyperbolic 3-manifolds, proving part 1 of Theorem 1. The approach in this section is geometric, as opposed to the algebraic route taken in [25] and [16].
- §4 discusses contact perturbations of geodesic flows on hyperbolic 3-manifolds. It proves Theorem 3 which is a general perturbation statement using the nondegeneracy condition (1.3), as well as Theorem 4 on generic contact perturbations. It also gives the proof of part 2 of Theorem 1, relying on the key identity (1.5).
- §5 contains the proof of the identity (1.5) (stated in Theorem 5), using a change of variables, a regularization procedure, and the results of §3.
- Finally, Appendix A gives a proof of the fact that hyperbolic 3-manifolds have no nonzero harmonic 1-forms of constant length.

2 Contact 5-dimensional flows

In this section we study general contact Anosov flows on 5-dimensional manifolds. Some of the statements below apply to non-contact Anosov flows and to other dimensions, however we use the setting of 5-dimensional contact flows for uniformity of presentation.

2.1 Contact Anosov flows

Assume that M is a compact connected 5-dimensional C^∞ manifold and $\alpha \in C^\infty(M; T^*M)$ is a contact 1-form on M, namely

$$d\,\mathrm{vol}_\alpha := \alpha \wedge d\alpha \wedge d\alpha \neq 0 \quad \text{everywhere.}$$

We fix the orientation on M by requiring that $d\,\mathrm{vol}_\alpha$ be positively oriented. Let $X \in C^\infty(M; TM)$ be the associated Reeb field, that is the unique vector field satisfying

$$\iota_X \alpha = 1, \quad \iota_X d\alpha = 0. \tag{2.1}$$

Note that this immediately implies (where $\mathcal{L}_X$ denotes the Lie derivative)

$$\mathcal{L}_X\alpha = d\iota_X\alpha + \iota_X d\alpha = 0.$$

We assume that the flow generated by X,

$$\varphi_t := e^{tX} : M \to M,$$

is an *Anosov flow*, namely there exists a continuous flow/unstable/stable decomposition of the tangent spaces to M,

$$T_\rho M = E_0(\rho) \oplus E_u(\rho) \oplus E_s(\rho), \quad \rho \in M, \quad E_0(\rho) := \mathbb{R}X(\rho) \tag{2.2}$$

and there exist constants $C, \theta > 0$ and a smooth norm $|\bullet|$ on the fibers of TM such that for all $\rho \in M$, $\xi \in T_\rho M$, and t

$$|d\varphi_t(\rho)\xi| \le Ce^{-\theta|t|} \cdot |\xi| \quad \text{if} \quad \begin{cases} t \le 0, & \xi \in E_u(\rho) \quad \text{or} \\ t \ge 0, & \xi \in E_s(\rho). \end{cases} \tag{2.3}$$

The flow/unstable/stable decomposition gives rise to the dual decomposition of the cotangent spaces to M,

$$\begin{aligned} T_\rho^* M = E_0^*(\rho) \oplus E_u^*(\rho) \oplus E_s^*(\rho), \quad E_0^* &:= (E_u \oplus E_s)^\perp, \\ E_u^* := (E_0 \oplus E_u)^\perp, \quad E_s^* &:= (E_0 \oplus E_s)^\perp. \end{aligned} \tag{2.4}$$

Since $\mathcal{L}_X\alpha = 0$, we see from (2.3) that $\alpha|_{E_u \oplus E_s} = 0$ and thus

$$E_0^* = \mathbb{R}\alpha.$$

Since α is a contact form and $d\alpha$ vanishes on $E_u \times E_u$ and on $E_s \times E_s$ (as follows from (2.3) and the fact that $\mathcal{L}_X d\alpha = 0$), we have $\dim E_u = \dim E_s = 2$.

2.1.1 Bundles of differential forms

We define the vector bundles over M

$$\Omega^k := \wedge^k(T^*M), \quad \Omega_0^k := \{\omega \in \Omega^k \mid \iota_X\omega = 0\} \simeq \wedge^k(E_u^* \oplus E_s^*). \tag{2.5}$$

Note that smooth sections of Ω^k are differential k-forms on M.

We use the de Rham cohomology groups

$$H^k(M; \mathbb{C}) := \frac{\{u \in C^\infty(M; \Omega^k) \mid du = 0\}}{\{dv \mid v \in C^\infty(M; \Omega^{k-1})\}}. \tag{2.6}$$

Unless otherwise stated, we will always take Ω^k to be complexified. We define the Betti numbers

$$b_k(M) := \dim H^k(M; \mathbb{C}).$$

Since M is connected and by Poincaré duality we have

$$b_0(M) = 1, \quad b_k(M) = b_{5-k}(M).$$

The bundles Ω^k and Ω_0^k are related as follows:

$$\Omega^k \simeq \Omega_0^k \oplus \Omega_0^{k-1}$$

with the canonical isomorphism and its inverse given by

$$u \mapsto (u - \alpha \wedge \iota_X u, \iota_X u), \quad (v, w) \mapsto v + \alpha \wedge w. \tag{2.7}$$

Denote by $d\alpha\wedge$ the map $u \mapsto d\alpha \wedge u$ and by $d\alpha\wedge^2$ the map $u \mapsto d\alpha \wedge d\alpha \wedge u$, then we have linear isomorphisms (as both maps are injective and image and domain have the same dimension)

$$d\alpha\wedge : \Omega_0^1 \to \Omega_0^3, \quad d\alpha\wedge^2 : \Omega_0^0 \to \Omega_0^4. \tag{2.8}$$

We also have a nondegenerate bilinear pairing between sections of Ω_0^k and Ω_0^{4-k} given by

$$u \in C^\infty(M; \Omega_0^k),\ u_* \in C^\infty(M; \Omega_0^{4-k}) \ \mapsto\ \langle\langle u, u_* \rangle\rangle := \int_M \alpha \wedge u \wedge u_* \tag{2.9}$$

which in particular identifies the dual space to $L^2(M; \Omega_0^k)$ with $L^2(M; \Omega_0^{4-k})$. If $A : C^\infty(M; \Omega_0^k) \to \mathcal{D}'(M; \Omega_0^k)$ is a continuous operator, where $\mathcal{D}'$ denotes the space of distributions, then its *transpose operator* is the unique operator $A^T : C^\infty(M; \Omega_0^{4-k}) \to \mathcal{D}'(M; \Omega_0^{4-k})$ satisfying

$$\langle\langle Au, u_* \rangle\rangle = \langle\langle u, A^T u_* \rangle\rangle \quad \text{for all} \quad u \in C^\infty(M; \Omega_0^k),\ u_* \in C^\infty(M; \Omega_0^{4-k}). \tag{2.10}$$

2.2 Geodesic flows

A large class of examples of contact Anosov flows is given by geodesic flows on negatively curved manifolds, which is the setting of the main results of this paper. More precisely, assume that (Σ, g) is a compact connected oriented 3-dimensional Riemannian manifold. Define M to be the sphere bundle of Σ and let π_Σ be the canonical projection:

$$M := S\Sigma = \{(x, v) \in T\Sigma\colon |v|_g = 1\}, \quad \pi_\Sigma : M \to \Sigma.$$

Define the *canonical*, or *tautological*, 1-form α on M as follows: for all $\xi \in T_{(x,v)}M$,

$$\langle \alpha(x, v), \xi \rangle = \langle v, d\pi_\Sigma(x, v)\xi \rangle_g. \tag{2.11}$$

Then α is a contact form, the corresponding flow φ_t is the geodesic flow, and $d\,\mathrm{vol}_\alpha$ is the standard Liouville volume form up to a constant, see for instance [52, §1.3.3]. If the metric g has negative sectional curvature, then the flow φ_t is Anosov, see for instance [42, Theorem 3.9.1].

We have the time reversal involution

$$\mathcal{J} : M \to M, \quad \mathcal{J}(x, v) = (x, -v) \tag{2.12}$$

which is an orientation reversing diffeomorphism satisfying

$$\mathcal{J}^*\alpha = -\alpha, \quad \mathcal{J}^*X = -X, \quad \varphi_t \circ \mathcal{J} = \mathcal{J} \circ \varphi_{-t} \tag{2.13}$$

and the differential of $\mathcal{J}$ maps E_0, E_u, E_s into E_0, E_s, E_u.

2.2.1 Horizontal and vertical spaces

Recall from (2.2) that an Anosov flow induces a splitting of the tangent bundle TM into the flow, unstable, and stable subbundles. For geodesic flows there is another splitting, into *horizontal and vertical subbundles*, which we briefly review here. See [52, §1.3.1] for more details.

Let $(x, v) \in M = S\Sigma$. The vertical space at (x, v) is the tangent space to the fiber $S_x\Sigma$:

$$\mathbf{V}(x, v) := \ker d\pi_\Sigma(x, v) \subset T_{(x,v)}M.$$

To define a complementary horizontal subspace of $T_{(x,v)}M$, we use the metric. The *connection map* of the metric is the unique bundle homomorphism $\mathcal{K}$: $TM \to T\Sigma$ covering the map π_Σ such that for any curve on M written as

$$\rho(t) = (x(t), v(t)), \quad x(t) \in \Sigma, \quad v(t) \in S_{x(t)}\Sigma$$

we have

$$\mathcal{K}(\rho(t))\dot{\rho}(t) = \mathbf{D}_t v(t) \ \in \ T_{x(t)}\Sigma, \tag{2.14}$$

where $\mathbf{D}_t v(t)$ denotes the Levi–Civita covariant derivative of the vector field $v(t)$ along the curve $x(t)$ (see e.g. [13, Proposition 2.2] for a precise definition). Note that since $d_t\langle v(t), v(t)\rangle_g = 0$, the range of $\mathcal{K}(x, v)$ is g-orthogonal to v.

We now define the horizontal space as

$$\mathbf{H}(x, v) := \ker \mathcal{K}(x, v) \ \subset \ T_{(x,v)}M.$$

We have the splitting

$$T_{(x,v)}M = \mathbf{H}(x, v) \oplus \mathbf{V}(x, v), \quad \dim \mathbf{H}(x, v) = 3, \quad \dim \mathbf{V}(x, v) = 2$$

and the isomorphisms (here $\{v\}^{\perp}$ is the g-orthogonal complement of v in $T_x\Sigma$)

$$d\pi_{\Sigma}(x, v) : \mathbf{H}(x, v) \to T_x\Sigma, \quad \mathcal{K}(x, v) : \mathbf{V}(x, v) \to \{v\}^{\perp}$$

which together give the following isomorphism $T_{(x,v)}M \to T_x\Sigma \oplus \{v\}^{\perp}$:

$$\xi \mapsto (\xi_H, \xi_V), \quad \xi_H = d\pi_{\Sigma}(x, v)\xi, \quad \xi_V = \mathcal{K}(x, v)\xi. \tag{2.15}$$

We use the map (2.15) to identify $T_{(x,v)}M$ with $T_x\Sigma \oplus \{v\}^{\perp}$.

Under the identification (2.15), the contact form α and its differential satisfy (see [52, Proposition 1.24])

$$\begin{aligned} \alpha(x, v)(\xi) &= \langle \xi_H, v\rangle_g, \\ d\alpha(x, v)(\xi, \eta) &= \langle \xi_V, \eta_H\rangle_g - \langle \xi_H, \eta_V\rangle_g. \end{aligned} \tag{2.16}$$

Using the splitting (2.15), we define the *Sasaki metric* $\langle \bullet, \bullet\rangle_S$ on M as follows:

$$\langle \xi, \eta\rangle_S := \langle \xi_H, \eta_H\rangle_g + \langle \xi_V, \eta_V\rangle_g. \tag{2.17}$$

We finally remark that the generator X of the geodesic flow has the following form under the isomorphism (2.15):

$$X(x, v)_H = v, \quad X(x, v)_V = 0. \tag{2.18}$$

2.2.2 De Rham cohomology of the sphere bundle

We now describe the de Rham cohomology of $M = S\Sigma$ in terms of the cohomology of Σ. To relate the two, we use the pullback operators

$$\pi_\Sigma^* : C^\infty(\Sigma; \Omega^k) \to C^\infty(M; \Omega^k), \quad 0 \le k \le 3$$

and the pushforward operators defined by integrating along the fibers of $S\Sigma$

$$\pi_{\Sigma *} : C^\infty(M; \Omega^k) \to C^\infty(\Sigma; \Omega^{k-2}), \quad 2 \le k \le 5. \tag{2.19}$$

Here the orientation on each fiber $S_x\Sigma$ is induced by the orientation on Σ: if v, v_1, v_2 is a positively oriented orthonormal basis of $T_x\Sigma$, then the vertical vectors corresponding to v_1, v_2 form a positively oriented basis of $T_v(S_x\Sigma)$. The pushforward operation can be characterized as follows: if $X_1, \dots, X_{k-2}$ are vector fields on Σ and $\widetilde{X}_1, \dots, \widetilde{X}_{k-2}$ are vector fields on M projecting to $X_1, \dots, X_{k-2}$ under $d\pi_\Sigma$, then for any $\omega \in C^\infty(M; \Omega^k)$ and $x \in \Sigma$

$$\pi_{\Sigma *}\omega(x)(X_1, \dots, X_{k-2}) = \int_{S_x\Sigma} \iota_{\widetilde{X}_{k-2}} \cdots \iota_{\widetilde{X}_1}\omega.$$

Another characterization of $\pi_{\Sigma *}$ is that for any $\omega \in C^\infty(M; \Omega^k)$ and any compact $k-2$ dimensional oriented submanifold with boundary $Y \subset \Sigma$, we have

$$\int_{\pi_\Sigma^{-1}(Y)} \omega = \int_Y \pi_{\Sigma *}\omega. \tag{2.20}$$

Here the orientation on $\pi_\Sigma^{-1}(Y)$ is induced by the orientation on Y. If $Y = \Sigma$ is the entire base manifold, then the orientation on $\pi_\Sigma^{-1}(\Sigma) = S\Sigma$ featured in (2.20) is *opposite* to the usual orientation on $M = S\Sigma$, induced by $d\,\mathrm{vol}_\alpha = \alpha \wedge d\alpha \wedge d\alpha$. In fact, using (2.16) we can compute that

$$\pi_{\Sigma *} d\,\mathrm{vol}_\alpha = -8\pi\, d\,\mathrm{vol}_g, \tag{2.21}$$

where $d\,\mathrm{vol}_g$ is the volume form on Σ induced by g and the choice of orientation, by applying $d\,\mathrm{vol}_\alpha$ to the vectors $X = (v, 0), (v_1, 0), (v_2, 0), (0, v_1), (0, v_2)$ written using the horizontal/vertical decomposition (2.15), where v, v_1, v_2 is a positively oriented g-orthonormal basis on Σ.

The pushforward map has the following properties (see for instance [8, Propositions 6.14.1 and 6.15] for the related case of vector bundles):

$$d\pi_{\Sigma *} = \pi_{\Sigma *} d, \tag{2.22}$$

$$\pi_{\Sigma *}\big(\omega_1 \wedge (\pi_\Sigma^* \omega_2)\big) = (\pi_{\Sigma *}\omega_1) \wedge \omega_2. \tag{2.23}$$

Note that the maps $\pi_{\Sigma *}$, π_Σ^* can also be defined on distributional forms. For $\pi_{\Sigma *}$ this follows from the fact that pushforward is always well-defined on distributions as long as the fibers are compact and for the pullback π_Σ^* this follows from the fact that π_Σ is a submersion [38, Theorem 6.1.2].

Since the map $\mathcal{J}$ defined in (2.12) is an orientation reversing diffeomorphism of the fibers of $S\Sigma$, we also have

$$\pi_{\Sigma *}(\mathcal{J}^* \omega) = -\pi_{\Sigma *}\omega. \tag{2.24}$$

Since pullbacks commute with the differential d, and by (2.22), the operations π_Σ^*, $\pi_{\Sigma *}$ induce maps on de Rham cohomology, which we denote by the same letters:

$$\pi_\Sigma^* : H^k(\Sigma; \mathbb{C}) \to H^k(M; \mathbb{C}), \quad \pi_{\Sigma *} : H^k(M; \mathbb{C}) \to H^{k-2}(\Sigma; \mathbb{C}).$$

From the Gysin exact sequence (see for instance [8, Proposition 14.33], where the Euler class is zero since Σ is three-dimensional; alternatively one can use Künneth formulas and the fact that every compact orientable 3-manifold is parallelizable) we have isomorphisms

$$\pi_\Sigma^* : H^1(\Sigma; \mathbb{C}) \to H^1(M; \mathbb{C}), \quad \pi_{\Sigma *} : H^4(M; \mathbb{C}) \to H^2(\Sigma; \mathbb{C}) \tag{2.25}$$

and the exact sequences

$$0 \to H^2(\Sigma; \mathbb{C}) \xrightarrow{\pi_\Sigma^*} H^2(M; \mathbb{C}) \xrightarrow{\pi_{\Sigma *}} H^0(\Sigma; \mathbb{C}) \to 0, \tag{2.26}$$

$$0 \to H^3(\Sigma; \mathbb{C}) \xrightarrow{\pi_\Sigma^*} H^3(M; \mathbb{C}) \xrightarrow{\pi_{\Sigma *}} H^1(\Sigma; \mathbb{C}) \to 0. \tag{2.27}$$

In particular, we get formulas for the Betti numbers of the sphere bundle M:

$$\begin{aligned} b_0(M) = b_5(M) = 1, \quad b_1(M) = b_4(M) = b_1(\Sigma),& \\ b_2(M) = b_3(M) = b_1(\Sigma) + 1.& \end{aligned} \tag{2.28}$$

2.3 Pollicott–Ruelle resonances

We now review the theory of Pollicott–Ruelle resonances in the present setting. Define the first order differential operators

$$\begin{aligned} P_k &:= -i\mathcal{L}_X : C^\infty(M;\Omega^k) \to C^\infty(M;\Omega^k), \\ P_{k,0} &:= -i\mathcal{L}_X : C^\infty(M;\Omega_0^k) \to C^\infty(M;\Omega_0^k). \end{aligned}$$

Note that $P_{k,0}$ is the restriction of P_k to $C^\infty(M;\Omega_0^k)$ which is the space of all $u \in C^\infty(M;\Omega^k)$ which satisfy $\iota_X u = 0$.

For $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda$ large enough, the integral

$$R_k(\lambda) := i\int_0^\infty e^{i\lambda t} e^{-itP_k}\,dt : L^2(M;\Omega^k) \to L^2(M;\Omega^k) \qquad (2.29)$$

converges and defines a bounded operator on L^2 which is holomorphic in λ. Here the evolution group e^{-itP_k} is given by $e^{-itP_k}u = \varphi_{-t}^* u$. It is straightforward to check that $R_k(\lambda)$ is the L^2-resolvent of P_k:

$$R_k(\lambda) = (P_k - \lambda)^{-1} : L^2(M;\Omega^k) \to L^2(M;\Omega^k), \quad \operatorname{Im}\lambda \gg 1, \quad (2.30)$$

where we treat P_k as an unbounded operator on L^2 with domain $\{u \in L^2(M;\Omega^k) \mid P_k u \in L^2(M;\Omega^k)\}$ and $P_k u$ is defined in the sense of distributions.

2.3.1 Meromorphic continuation

Since φ_t is an Anosov flow, the resolvent $R_k(\lambda)$ admits a meromorphic continuation

$$R_k(\lambda) : C^\infty(M;\Omega^k) \to \mathcal{D}'(M;\Omega^k), \quad \lambda \in \mathbb{C},$$

see for instance [20, §3.2] and [29, Theorems 1.4, 1.5]. The proof of this continuation shows that $R_k(\lambda)$ acts on certain anisotropic Sobolev spaces adapted to the stable/unstable decompositions, see e.g. [20, §3.1]; this makes it possible to compose the operator $R_k(\lambda)$ with itself. Instead of introducing these spaces here, we use the spaces of distributions

$$\mathcal{D}'_\Gamma(M;\Omega^k) := \{u \in \mathcal{D}'(M;\Omega^k) \mid \mathrm{WF}(u) \subset \Gamma\}, \qquad (2.31)$$

where $\Gamma \subset T^*M\setminus 0$ is a closed conic set and $\mathrm{WF}(u)$ denotes the wavefront set of a distribution u. These spaces come with a natural sequential topology, see [38, Definition 8.2.2].

We have the wavefront set property of $R_k(\lambda)$ proved in [20, (3.7)]:

$$\mathrm{WF}'(R_k(\lambda)) \subset \mathscr{W} := \Delta(T^*M) \cup \Upsilon_+ \cup (E_u^* \times E_s^*), \tag{2.32}$$

where $\Delta(T^*M) \subset T^*M \times T^*M$ is the diagonal and $\Upsilon_+ = \{(\varphi_t(x), d\varphi_t(x)^{-T}\xi, x, \xi) \mid t \geq 0, \xi(X(x)) = 0\}$; for an operator $B : C^\infty(M) \to \mathcal{D}'(M)$ with Schwartz kernel $K_B \in \mathcal{D}'(M \times M)$, we denote $\mathrm{WF}'(B) = \{(x, \xi, y, -\eta) \mid (x, \xi, y, \eta) \in \mathrm{WF}(K_B)\} \subset T^*(M \times M)$. The Schwartz kernel of $R_k(\lambda)$ is meromorphic in λ with values in $\mathcal{D}'_{\mathscr{W}'}$ where $\mathscr{W}' := \{(x, \xi, y, -\eta) \mid (x, \xi, y, \eta) \in \mathscr{W}\}$. By the wavefront set calculus [38, Theorem 8.2.13] and since $E_u^* \cap E_s^* = 0$, $R_k(\lambda)$ defines a meromorphic family of continuous operators

$$R_k(\lambda) : \mathcal{D}'_{E_u^*}(M; \Omega^k) \to \mathcal{D}'_{E_u^*}(M; \Omega^k), \tag{2.33}$$

where we view $E_u^* \subset T^*M$ as a closed conic subset and define $\mathcal{D}'_{E_u^*}$ by (2.31).

Note that differential operators (in particular, $d, \iota_X, \mathcal{L}_X$) define continuous maps on the regularity classes $\mathcal{D}'_{E_u^*}$. We have

$$R_k(\lambda)(P_k - \lambda)u = (P_k - \lambda)R_k(\lambda)u = u \quad \text{for all } u \in \mathcal{D}'_{E_u^*}(M; \Omega^k). \tag{2.34}$$

For $\operatorname{Im}\lambda \gg 1$ and $u \in C^\infty(M; \Omega^k)$ this follows from (2.30); the general case follows from here by analytic continuation and since C^∞ is dense in $\mathcal{D}'_{E_u^*}$.

We also have the commutation relations

$$dR_k(\lambda)u = R_{k+1}(\lambda)du, \quad \iota_X R_k(\lambda)u = R_{k-1}(\lambda)\iota_X u \quad \text{for all} \quad u \in \mathcal{D}'_{E_u^*}(M; \Omega^k). \tag{2.35}$$

As with (2.34) it suffices to consider the case $\operatorname{Im}\lambda \gg 1$ and $u \in C^\infty(M; \Omega^k)$, in which (2.35) follows from (2.29) and the fact that d and ι_X commute with φ_{-t}^*.

The poles of the family of operators $R_k(\lambda)$ are called *Pollicott–Ruelle resonances* on k-forms. At each pole $\lambda_0 \in \mathbb{C}$ we have an expansion (see for instance [20, (3.6)])

$$R_k(\lambda) = R_k^H(\lambda; \lambda_0) - \sum_{j=1}^{J_k(\lambda_0)} \frac{(P_k - \lambda_0)^{j-1}\Pi_k(\lambda_0)}{(\lambda - \lambda_0)^j}, \tag{2.36}$$

where $R_k^H(\lambda; \lambda_0) : \mathcal{D}'_{E_u^*}(M; \Omega^k) \to \mathcal{D}'_{E_u^*}(M; \Omega^k)$ is a family of operators holomorphic in a neighborhood of λ_0, $J_k(\lambda_0) \geq 1$ is an integer, and $\Pi_k(\lambda_0)$:

$\mathcal{D}'_{E_u^*}(M;\Omega^k) \to \mathcal{D}'_{E_u^*}(M;\Omega^k)$ is a finite rank operator commuting with P_k and such that $(P_k-\lambda_0)^{J_k(\lambda_0)}\Pi_k(\lambda_0)=0$.

Taking the expansions of (2.35) at λ_0 we see that

$$d\Pi_k(\lambda_0)=\Pi_{k+1}(\lambda_0)d, \quad \iota_X\Pi_k(\lambda_0)=\Pi_{k-1}(\lambda_0)\iota_X. \tag{2.37}$$

2.3.2 Resonant states

The range of the operator $\Pi_k(\lambda_0)$ is equal to the space of *generalised resonant states* (see for instance [20, Proposition 3.3])

$$\mathrm{Res}^{k,\infty}(\lambda_0):=\bigcup_{\ell\ge 1}\mathrm{Res}^{k,\ell}(\lambda_0), \tag{2.38}$$

where we define

$$\mathrm{Res}^{k,\ell}(\lambda_0):=\{u\in\mathcal{D}'_{E_u^*}(M;\Omega^k)\mid (P_k-\lambda_0)^\ell u=0\}. \tag{2.39}$$

We define the *algebraic multiplicity* of λ_0 as a resonance on k-forms by

$$m_k(\lambda_0):=\operatorname{rank}\Pi_k(\lambda_0)=\dim\mathrm{Res}^{k,\infty}(\lambda_0). \tag{2.40}$$

The *geometric multiplicity* is the dimension of the space of *resonant states*

$$\mathrm{Res}^k(\lambda_0):=\mathrm{Res}^{k,1}(\lambda_0)=\{u\in\mathcal{D}'_{E_u^*}(M;\Omega^k)\mid (P_k-\lambda_0)u=0\}.$$

We say a resonance λ_0 of P_k is *semisimple* if the algebraic and geometric multiplicities coincide, that is $\mathrm{Res}^{k,\infty}(\lambda_0)=\mathrm{Res}^k(\lambda_0)$. This is equivalent to saying that $J_k(\lambda_0)=1$ in (2.36). Another equivalent definition of semisimplicity is

$$u\in\mathcal{D}'_{E_u^*}(M;\Omega^k),\ (P_k-\lambda_0)^2u=0 \implies (P_k-\lambda_0)u=0. \tag{2.41}$$

We note that the operators $\Pi_k(\lambda_0)$ are idempotent. In fact, applying the Laurent expansion (2.36) at λ_0 to $u\in\mathrm{Res}^{k,\ell}(\lambda_1)$ and using the identity $R_k(\lambda)u=-\sum_{j=0}^{\ell-1}(\lambda-\lambda_1)^{-j-1}(P_k-\lambda_1)^j u$ we see that

$$\Pi_k(\lambda_0)\Pi_k(\lambda_1)=\begin{cases}\Pi_k(\lambda_0) & \text{if } \lambda_1=\lambda_0,\\ 0 & \text{if } \lambda_1\neq\lambda_0.\end{cases} \tag{2.42}$$

2.3.3 Operators on the bundles Ω_0^k

The above constructions apply equally as well to the operators $P_{k,0}$ (except that the operator d does not preserve sections of Ω_0^k, so the first commutation relation in (2.37) does not hold, and the second one is trivial); we denote the resulting objects by

$$R_{k,0}(\lambda),\ J_{k,0}(\lambda_0),\ R^H_{k,0}(\lambda;\lambda_0),\ \Pi_{k,0}(\lambda_0),\ \mathrm{Res}_0^{k,\ell}(\lambda_0),\ m_{k,0}(\lambda_0).$$

Under the isomorphism (2.7) the operator P_k is conjugated to $P_{k,0} \oplus P_{k-1,0}$. Therefore (2.7) gives an isomorphism

$$\mathrm{Res}^{k,\ell}(\lambda_0) \simeq \mathrm{Res}_0^{k,\ell}(\lambda_0) \oplus \mathrm{Res}_0^{k-1,\ell}(\lambda_0). \tag{2.43}$$

Moreover, we get for all $u \in \mathcal{D}'_{E_u^*}(M;\Omega^k)$

$$\Pi_k(\lambda_0)u = \Pi_{k,0}(\lambda_0)(u - \alpha \wedge \iota_X u) + \alpha \wedge \Pi_{k-1,0}(\lambda_0)\iota_X u. \tag{2.44}$$

Since $\mathcal{L}_X d\alpha = 0$, the operations (2.8) give rise to linear isomorphisms

$$d\alpha\wedge : \mathrm{Res}_0^{1,\ell}(\lambda_0) \to \mathrm{Res}_0^{3,\ell}(\lambda_0), \quad d\alpha\wedge^2 : \mathrm{Res}_0^{0,\ell}(\lambda_0) \to \mathrm{Res}_0^{4,\ell}(\lambda_0) \tag{2.45}$$

which in particular give the equalities

$$m_{1,0}(\lambda_0) = m_{3,0}(\lambda_0), \quad m_{0,0}(\lambda_0) = m_{4,0}(\lambda_0). \tag{2.46}$$

2.3.4 Transposes and coresonant states

Since $\mathcal{L}_X\alpha = 0$ and $\int_M \mathcal{L}_X\omega = 0$ for any 5-form ω, we have

$$(P_{k,0})^T = -P_{4-k,0}, \quad k = 0,1,2,3,4, \tag{2.47}$$

where the transpose is defined using the pairing $\langle\!\langle \bullet, \bullet \rangle\!\rangle$, see (2.10). Thus the transpose of the resolvent $(R_{k,0}(\lambda))^T$ is the meromorphic continuation of the resolvent corresponding to the vector field $-X$; the latter generates an Anosov flow with the unstable and stable spaces switching roles compared to the ones for X. Similarly to (2.33) we have

$$(R_{k,0}(\lambda))^T : \mathcal{D}'_{E_s^*}(M;\Omega_0^{4-k}) \to \mathcal{D}'_{E_s^*}(M;\Omega_0^{4-k}), \tag{2.48}$$

where $\mathcal{D}'_{E_s^*}$ is the space of distributional sections with wavefront set contained in E_s^*. Same applies to the transposes of the operators $R^H_{k,0}(\lambda;\lambda_0)$ and $\Pi_{k,0}(\lambda_0)$

appearing in (2.36). The range of $(\Pi_{k,0}(\lambda_0))^T$ is the space of *generalised coresonant states* $\mathrm{Res}_{0*}^{4-k,\infty}(\lambda_0)$ where

$$\begin{aligned}\mathrm{Res}_{0*}^{k,\infty}(\lambda_0) &:= \bigcup_{\ell\geq 1} \mathrm{Res}_{0*}^{k,\ell}(\lambda_0),\\ \mathrm{Res}_{0*}^{k,\ell}(\lambda_0) &:= \{u_* \in \mathcal{D}'_{E_s^*}(M;\Omega_0^k) \mid (P_{k,0}+\lambda_0)^\ell u_* = 0\}.\end{aligned}$$

The space of *coresonant states* is defined as

$$\mathrm{Res}_{0*}^{k}(\lambda_0) := \mathrm{Res}_{0*}^{k,1}(\lambda_0) = \{u_* \in \mathcal{D}'_{E_s^*}(M;\Omega_0^k) \mid (P_{k,0}+\lambda_0) u_* = 0\}.$$

Similarly to (2.45) we have the isomorphisms

$$d\alpha\wedge : \mathrm{Res}_{0*}^{1,\ell}(\lambda_0) \to \mathrm{Res}_{0*}^{3,\ell}(\lambda_0), \quad d\alpha\wedge^2 : \mathrm{Res}_{0*}^{0,\ell}(\lambda_0) \to \mathrm{Res}_{0*}^{4,\ell}(\lambda_0). \tag{2.49}$$

In the special case when φ_t is a geodesic flow with the time reversal map $\mathcal{J}$ defined in (2.12), the pullback operator $\mathcal{J}^*$ gives an isomorphism between $\mathcal{D}'_{E_u^*}(M;\Omega_0^k)$ and $\mathcal{D}'_{E_s^*}(M;\Omega_0^k)$. Moreover, $\mathcal{J}^* P_{k,0} = -P_{k,0}\mathcal{J}^*$. This gives rise to isomorphisms between the spaces of generalised resonant and coresonant states

$$\mathcal{J}^* : \mathrm{Res}_{0}^{k,\ell}(\lambda_0) \to \mathrm{Res}_{0*}^{k,\ell}(\lambda_0). \tag{2.50}$$

2.3.5 Coresonant states and pairing

Since E_u^* and E_s^* intersect only at the zero section, we can define the product $u \wedge u_* \in \mathcal{D}'(M;\Omega_0^4)$ and thus the pairing $\langle\!\langle u, u_*\rangle\!\rangle$ for any $u \in \mathcal{D}'_{E_u^*}(M;\Omega_0^k)$, $u_* \in \mathcal{D}'_{E_s^*}(M;\Omega_0^{4-k})$, see [38, Theorem 8.2.10]. Note that this pairing is nondegenerate since both $\mathcal{D}'_{E_u^*}$ and $\mathcal{D}'_{E_s^*}$ contain C^∞, and the transpose formula (2.10) still holds since C^∞ is dense in $\mathcal{D}'_{E_u^*}$ and in $\mathcal{D}'_{E_s^*}$. In particular, we have a pairing

$$u \in \mathrm{Res}_{0}^{k,\infty}(\lambda_0),\ u_* \in \mathrm{Res}_{0*}^{4-k,\infty}(\lambda_0) \quad\mapsto\quad \langle\!\langle u, u_*\rangle\!\rangle \in \mathbb{C}. \tag{2.51}$$

This pairing is nondegenerate. Indeed, assume that $u \in \mathrm{Res}_0^{k,\infty}(\lambda_0)$ and $\langle\!\langle u, u_*\rangle\!\rangle = 0$ for all $u_* \in \mathrm{Res}_{0*}^{4-k,\infty}(\lambda_0)$. Since $\mathrm{Res}_{0*}^{4-k,\infty}(\lambda_0)$ is the range of $(\Pi_{k,0}(\lambda_0))^T$, we have

$$\begin{aligned}0 = \langle\!\langle u, (\Pi_{k,0}(\lambda_0))^T\varphi\rangle\!\rangle &= \langle\!\langle \Pi_{k,0}(\lambda_0)u, \varphi\rangle\!\rangle\\ &= \langle\!\langle u, \varphi\rangle\!\rangle \quad \text{for all } \varphi \in C^\infty(M;\Omega_0^{4-k}),\end{aligned}$$

where the last equality follows from the fact that $\Pi_{k,0}(\lambda_0)^2 = \Pi_{k,0}(\lambda_0)$ and u is in the range of $\Pi_{k,0}(\lambda_0)$. It follows that $u = 0$. Similarly one can show that if $\langle\langle u, u_* \rangle\rangle = 0$ for some $u_* \in \mathrm{Res}_{0*}^{4-k,\infty}(\lambda_0)$ and all $u \in \mathrm{Res}_0^{k,\infty}(\lambda_0)$, then $u_* = 0$.

Consider the operators on finite dimensional spaces

$$P_{k,0} - \lambda_0 : \mathrm{Res}_0^{k,\infty}(\lambda_0) \to \mathrm{Res}_0^{k,\infty}(\lambda_0), \tag{2.52}$$

$$-P_{4-k,0} - \lambda_0 : \mathrm{Res}_{0*}^{4-k,\infty}(\lambda_0) \to \mathrm{Res}_{0*}^{4-k,\infty}(\lambda_0), \tag{2.53}$$

which are transposes of each other with respect to the pairing (2.51). The kernels of ℓ-th powers of these operators are $\mathrm{Res}_0^{k,\ell}(\lambda_0)$ and $\mathrm{Res}_{0*}^{4-k,\ell}(\lambda_0)$, thus (using the isomorphisms (2.49))

$$\dim \mathrm{Res}_0^{k,\ell}(\lambda_0) = \dim \mathrm{Res}_{0*}^{4-k,\ell}(\lambda_0) = \dim \mathrm{Res}_{0*}^{k,\ell}(\lambda_0). \tag{2.54}$$

We now give a solvability result for the operators $P_{k,0}$. It follows from the Fredholm property of these operators on anisotropic Sobolev spaces but we present instead a proof using the Laurent expansion (2.36).

Lemma 2.1 *Assume that $w \in \mathcal{D}'_{E_u^*}(M; \Omega_0^k)$. Then the equation*

$$(P_{k,0} - \lambda_0)u = w, \quad u \in \mathcal{D}'_{E_u^*}(M; \Omega_0^k) \tag{2.55}$$

has a solution if and only if w satisfies the condition

$$\langle\langle w, u_* \rangle\rangle = 0 \quad \textit{for all} \quad u_* \in \mathrm{Res}_{0*}^{4-k}(\lambda_0). \tag{2.56}$$

Proof First of all, if (2.55) has a solution u, then for each $u_* \in \mathrm{Res}_{0*}^{4-k}(\lambda_0)$ we have

$$\langle\langle w, u_* \rangle\rangle = \langle\langle (P_{k,0} - \lambda_0)u, u_* \rangle\rangle = -\langle\langle u, (P_{4-k,0} + \lambda_0)u_* \rangle\rangle = 0,$$

that is the condition (2.56) is satisfied.

Now, assume that w satisfies the condition (2.56); we show that (2.55) has a solution. We start with the special case when $w \in \mathrm{Res}_0^{k,\infty}(\lambda_0)$. We use the pairing (2.51) to identify the dual space to $\mathrm{Res}_0^{k,\infty}(\lambda_0)$ with $\mathrm{Res}_{0*}^{4-k,\infty}(\lambda_0)$. By (2.56), w is annihilated by the kernel of the operator (2.53). Therefore w is in the range of the operator (2.52), that is (2.55) has a solution $u \in \mathrm{Res}_0^{k,\infty}(\lambda_0)$.

We now consider the case of general w satisfying (2.56). Taking the constant term in the Laurent expansion of the identity (2.34) at $\lambda = \lambda_0$, we obtain

$$(P_{k,0} - \lambda_0) R_{k,0}^H(\lambda_0; \lambda_0) w = w - \Pi_{k,0}(\lambda_0) w. \tag{2.57}$$

We have $\Pi_{k,0}(\lambda_0)w \in \mathrm{Res}_0^{k,\infty}(\lambda_0)$ and it satisfies (2.56), thus (2.55) has a solution with this right-hand side. Writing $w = \Pi_{k,0}(\lambda_0)w + \big(\mathrm{Id} - \Pi_{k,0}(\lambda_0)\big)w$, we may take as u the sum of this solution and $R^H_{k,0}(\lambda_0; \lambda_0)w$. □

Lemma 2.1 implies the following criterion for semisimplicity:

Lemma 2.2 *The semisimplicity condition* (2.41) *holds for the operator* $P_{k,0}$ *if and only if the restriction of the pairing* (2.51) *to* $\mathrm{Res}_0^k(\lambda_0) \times \mathrm{Res}_{0*}^{4-k}(\lambda_0)$ *is nondegenerate.*

Proof The condition (2.41) is equivalent to saying that the intersection of $\mathrm{Res}_0^k(\lambda_0)$ with the range of the operator $P_{k,0} - \lambda_0 : \mathcal{D}'_{E_u^*}(M; \Omega_0^k) \to \mathcal{D}'_{E_u^*}(M; \Omega_0^k)$ is trivial; that is, for each $w \in \mathrm{Res}_0^k(\lambda_0) \setminus \{0\}$ the equation (2.55) has no solution. By Lemma 2.1, this is equivalent to saying that w does not satisfy the condition (2.56), i.e. there exists $v \in \mathrm{Res}_{0*}^{4-k}(\lambda_0)$ such that $\langle\langle w, v \rangle\rangle \neq 0$. This is equivalent to the nondegeneracy condition of the present lemma. □

2.3.6 Zeta functions

We now discuss dynamical zeta functions. We assume that the unstable/stable bundles E_u, E_s are orientable (the non-orientable case is covered by [10]); this is true for the case of geodesic flows on orientable manifolds as follows from the fact that the vertical bundle trivially intersects the weak unstable bundle $\mathbb{R}X \oplus E_u$ (see [34, Lemma B.1]).

We say $\gamma : [0, T_\gamma] \to M$ is a *closed trajectory* of the flow φ_t of period $T_\gamma > 0$ if $\gamma(t) = \varphi_t(\gamma(0))$ and $\gamma(T_\gamma) = \gamma(0)$. We identify closed trajectories obtained by shifting t. The *primitive period* of a closed trajectory, denoted by $T_\gamma^\sharp$, is the smallest positive $t > 0$ such that $\gamma(t) = \gamma(0)$. We say γ is a *primitive* closed trajectory if $T_\gamma = T_\gamma^\sharp$.

Define the *linearised Poincaré map* $\mathcal{P}_\gamma := d\varphi_{-T_\gamma}(\gamma(0))|_{E_u \oplus E_s}$. We have $\det \mathcal{P}_\gamma = 1$ since the restriction of $d\alpha \wedge d\alpha$ to $E_u \oplus E_s$ is a φ_t-invariant nonvanishing 4-form. Since φ_t is an Anosov flow, the map $I - \mathcal{P}_\gamma$ is invertible (in fact $\mathcal{P}_\gamma$ has no eigenvalues on the unit circle).

For $0 \leq k \leq 4$, define the zeta function

$$\zeta_k(\lambda) := \exp\left(- \sum_\gamma \frac{T_\gamma^\sharp \, \mathrm{tr}(\wedge^k \mathcal{P}_\gamma) e^{i\lambda T_\gamma}}{T_\gamma \det(I - \mathcal{P}_\gamma)} \right), \quad \mathrm{Im}\,\lambda \gg 1, \tag{2.58}$$

where the sum is over all the closed trajectories γ. The series in (2.58) converges for sufficiently large $\mathrm{Im}\,\lambda$, see e.g. [20, §2.2].

The zeta function ζ_k continues holomorphically to $\lambda \in \mathbb{C}$ and for each $\lambda_0 \in \mathbb{C}$, the multiplicity of λ_0 as a zero of ζ_k is equal to $m_{k,0}(\lambda_0)$, the algebraic multiplicity of λ_0 as a resonance of the operator $P_{k,0}$ defined similarly to (2.40) – see [20, §4] for the proof.

By Ruelle's identity (see e.g. [20, (2.5)]) the Ruelle zeta function defined in (1.1) factorizes as follows:

$$\zeta_{\mathrm{R}}(\lambda) = \frac{\zeta_0(\lambda)\zeta_2(\lambda)\zeta_4(\lambda)}{\zeta_1(\lambda)\zeta_3(\lambda)}.$$

Using (2.46) we see that the order of vanishing of the function ζ_{R} at λ_0 is equal to

$$m_{\mathrm{R}}(\lambda_0) = \sum_{k=0}^{4}(-1)^k m_{k,0}(\lambda_0) = 2m_{0,0}(\lambda_0) - 2m_{1,0}(\lambda_0) + m_{2,0}(\lambda_0). \tag{2.59}$$

2.4 Resonance at 0

This paper focuses on the resonance at 0, which is why we henceforth put $\lambda_0 := 0$ unless stated otherwise. For instance we write

$$R^H_{k,0}(\lambda) := R^H_{k,0}(\lambda; 0), \quad \Pi_{k,0} := \Pi_{k,0}(0), \quad \mathrm{Res}_0^{k,\ell} := \mathrm{Res}_0^{k,\ell}(0).$$

Our main goal is to study the order of vanishing of the Ruelle zeta function at 0, which by (2.59) is equal to

$$m_{\mathrm{R}}(0) = 2m_{0,0}(0) - 2m_{1,0}(0) + m_{2,0}(0), \quad m_{k,0}(0) = \dim \mathrm{Res}_0^{k,\infty}.$$

Since $\mathcal{L}_X = d\iota_X + \iota_X d$, the space of resonant states at 0 for the operator $P_{k,0}$ is

$$\mathrm{Res}_0^k = \{u \in \mathcal{D}'_{E_u^*}(M; \Omega^k) \mid \iota_X u = 0,\ \iota_X du = 0\}. \tag{2.60}$$

In particular, the exterior derivative defines an operator $d : \mathrm{Res}_0^k \to \mathrm{Res}_0^{k+1}$. (Unfortunately this is no longer true for the spaces of generalised resonant states $\mathrm{Res}_0^{k,\ell}$ with $\ell \geq 2$, since d does not necessarily map these to the kernel of ι_X.)

2.4.1 0-Forms and 4-forms

We first analyze the resonance at 0 for the operators $P_{0,0}$ and $P_{4,0}$. The following regularity result is a special case of [21, Lemma 2.3] (see also [28, Lemma 4] for a similar statement in the case of Anosov maps):

Lemma 2.3 *Assume that*

$$u \in \mathcal{D}'_{E_u^*}(M;\mathbb{C}), \quad Xu \in C^\infty(M;\mathbb{C}), \quad \operatorname{Re}\langle Xu, u\rangle_{L^2(M;d\,\mathrm{vol}_\alpha)} \le 0.$$

Then $u \in C^\infty(M;\mathbb{C})$.

Using Lemma 2.3 we show the following statement similar to [21, Lemma 3.2] (we note that it straightforwardly generalizes to other dimensions, which was known already to [46, Corollary 2.11]):

Lemma 2.4 *The semisimplicity condition* (2.41) *holds at $\lambda_0 = 0$ for the operators $P_{0,0}$, $P_{4,0}$ and*

$$m_{0,0}(0) = m_{4,0}(0) = 1.$$

Moreover, $\operatorname{Res}_0^0 = \operatorname{Res}_{0*}^0$ *is spanned by the constant function 1 and* $\operatorname{Res}_0^4 = \operatorname{Res}_{0*}^4$ *is spanned by the form $d\alpha \wedge d\alpha$.*

Proof We only give the proof for 0-forms (i.e. functions); the case of 4-forms follows from here using the isomorphisms (2.45), (2.49).

Assume that $u \in \operatorname{Res}_0^0$. Then $Xu = 0$, so Lemma 2.3 implies that $u \in C^\infty(M;\mathbb{C})$. Thus the differential $du \in C^\infty(M;\Omega^1)$ is invariant under the flow φ_t; the stable/unstable decomposition (2.4) gives that $du \in E_0^*$ at every point. Together with the equation $Xu = 0$, this implies that $du = 0$ and thus (since M is connected) u is constant. We have shown that Res_0^0 is spanned by the function 1; applying the above argument to $-X$ we see that $\operatorname{Res}_{0*}^0$ is spanned by 1 as well.

To show the semisimplicity condition (2.41), assume that $u \in \mathcal{D}'_{E_u^*}(M;\mathbb{C})$ satisfies $X^2u = 0$. Then $Xu \in \operatorname{Res}_0^0$, so Xu is constant. Together with the identity $\int_M (Xu)\,d\,\mathrm{vol}_\alpha = 0$ this gives $Xu = 0$ as needed. □

2.4.2 Closed forms

We now study resonant states which are closed, that is elements of the space

$$\operatorname{Res}_0^k \cap \ker d = \{u \in \mathcal{D}'_{E_u^*}(M;\Omega^k) \mid \iota_X u = 0,\ du = 0\}.$$

We use a special case of [21, Lemma 2.1] which shows that de Rham cohomology in the spaces $\mathcal{D}'_{E_u^*}(M;\Omega^k)$ is the same as the usual de Rham cohomology defined in (2.6):

Lemma 2.5 *Assume that* $u \in \mathcal{D}'_{E_u^*}(M;\Omega^k)$ *and* $du \in C^\infty(M;\Omega^{k+1})$*. Then there exist* $v \in C^\infty(M;\Omega^k)$*,* $w \in \mathcal{D}'_{E_u^*}(M;\Omega^{k-1})$ *such that* $u = v + dw$*.*

Similarly to [21, §3.3] we introduce the linear map

$$\begin{aligned} \pi_k : \operatorname{Res}_0^k \cap \ker d \to\ H^k(M;\mathbb{C}),\quad \pi_k(u) = [v]_{H^k} \\ \text{where}\quad u = v + dw,\quad v \in C^\infty(M;\Omega^k),\quad w \in \mathcal{D}'_{E_u^*}(M;\Omega^{k-1}). \end{aligned} \tag{2.61}$$

Here v, w exist by Lemma 2.5. To show that the map π_k is well-defined, assume that $u = v + dw = v' + dw'$ where $v, v' \in C^\infty(M;\Omega^k)$ and $w, w' \in \mathcal{D}'_{E_u^*}(M;\Omega^{k-1})$. Then $d(w - w') = v' - v \in C^\infty(M;\Omega^k)$, thus by Lemma 2.5 we may write $w - w' = w_1 + dw_2$ where $w_1 \in C^\infty(M;\Omega^{k-1})$, $w_2 \in \mathcal{D}'_{E_u^*}(M;\Omega^{k-2})$. Then $v' - v = dw_1$ where w_1 is smooth, so $[v]_{H^k} = [v']_{H^k}$.

Similar arguments apply to the spaces $\operatorname{Res}_{0*}^k \cap \ker d$ of closed coresonant k-forms; we denote the corresponding maps by

$$\pi_{k*} : \operatorname{Res}_{0*}^k \cap \ker d \to\ H^k(M;\mathbb{C}).$$

From Lemma 2.4 we see that π_0 is an isomorphism and hence by (2.45) that $\pi_4 = 0$.

We now establish several properties of the spaces $\operatorname{Res}_0^k \cap \ker d$ and the maps π_k; some of these are extensions of the results of [21, §3.3].

Lemma 2.6 *The kernel of* π_k *satisfies*

$$d(\operatorname{Res}_0^{k-1}) \subset \ker \pi_k \subset d(\operatorname{Res}^{k-1,\infty}).$$

Proof The first containment is immediate. For the second one, assume that $u \in \operatorname{Res}_0^k \cap \ker d$ and $\pi_k(u) = 0$. Then $u = v + dw$ where $v \in C^\infty(M;\Omega^k)$ satisfies $[v]_{H^k} = 0$ and $w \in \mathcal{D}'_{E_u^*}(M;\Omega^{k-1})$. We have $v = d\zeta$ for some $\zeta \in C^\infty(M;\Omega^{k-1})$ and by (2.37)

$$u = \Pi_k u = \Pi_k d(\zeta + w) = d\Pi_{k-1}(\zeta + w).$$

Therefore $u \in d(\operatorname{Res}^{k-1,\infty})$. □

We note that the case $k = 0$ of the following lemma holds trivially.

Lemma 2.7 *Assume that for some* k *all the coresonant states in* $\operatorname{Res}_{0*}^{5-k}$ *are exact forms. Then the map* π_k *is onto.*

Proof Take arbitrary $v \in C^\infty(M; \Omega^k)$ such that $dv = 0$. We will construct $u \in \mathrm{Res}_0^k \cap \ker d$ such that $\pi_k(u) = [v]_{H^k}$ by putting

$$u := v + dw \quad \text{for some} \quad w \in \mathcal{D}'_{E_u^*}(M; \Omega_0^{k-1}).$$

Such u is automatically closed, so we only need to choose w so that $\iota_X u = 0$, that is

$$\iota_X dw = \mathcal{L}_X w = -\iota_X v \tag{2.62}$$

where the first equality is immediate because $\iota_X w = 0$.

To solve (2.62), we use Lemma 2.1. It suffices to check that the condition (2.56) holds:

$$\langle\langle \iota_X v, u_* \rangle\rangle = 0 \quad \text{for all} \quad u_* \in \mathrm{Res}_{0*}^{5-k}.$$

We compute

$$\langle\langle \iota_X v, u_* \rangle\rangle = \int_M \alpha \wedge (\iota_X v) \wedge u_* = \int_M v \wedge u_* = 0.$$

Here in the second equality we used that $\iota_X u_* = 0$ (thus ι_X of the 5-forms on both sides are the same) and in the last equality we used that v is closed and, by the assumption of the lemma, u_* is exact. □

Lemma 2.8 *The maps π_1, π_{1*} are isomorphisms, in particular*

$$\dim(\mathrm{Res}_0^1 \cap \ker d) = \dim(\mathrm{Res}_{0*}^1 \cap \ker d) = b_1(M).$$

Proof We only consider the case of π_1, with π_{1*} handled similarly. To show that π_1 is one-to-one, we use Lemma 2.6 and the fact that $\mathrm{Res}^{0,\infty} = \mathrm{Res}_0^0$ consists of constant functions by Lemma 2.4. To show that π_1 is onto, it suffices to use Lemma 2.7: by Lemma 2.4, the space Res_{0*}^4 is spanned by $d\alpha \wedge d\alpha = d(\alpha \wedge d\alpha)$. □

Lemma 2.9 *We have $d(\mathrm{Res}_0^3) = d(\mathrm{Res}_{0*}^3) = 0$.*

Proof We only consider the case of Res_0^3, with Res_{0*}^3 handled similarly. Assume that $u \in \mathrm{Res}_0^3$. Then $du \in \mathrm{Res}_0^4$, so by Lemma 2.4 we have $du = c\, d\alpha \wedge d\alpha$ for some constant c. It remains to use that

$$c \int_M d\,\mathrm{vol}_\alpha = \int_M \alpha \wedge du = \int_M d\alpha \wedge u = 0,$$

where in the second equality we integrated by parts and in the third equality we used that $\iota_X(d\alpha \wedge u) = 0$, thus $d\alpha \wedge u = 0$. □

We also have the following nondegeneracy result for the pairing between closed resonant and coresonant forms when $k = 1$:

Lemma 2.10 *The pairing induced by* $\langle\langle \bullet, \bullet \rangle\rangle$ *on* $(\mathrm{Res}_0^1 \cap \ker d) \times (d\alpha \wedge (\mathrm{Res}_{0*}^1 \cap \ker d))$ *is nondegenerate.*

Proof We show the following stronger statement: for each closed but not exact $v \in C^\infty(M; \Omega^1)$,

$$\mathrm{Re}\langle\langle \pi_1^{-1}([v]_{H^1}), d\alpha \wedge \pi_{1*}^{-1}([\overline{v}]_{H^1}) \rangle\rangle < 0. \tag{2.63}$$

Here we used that the map π_1 is an isomorphism, as shown in Lemma 2.8. We have

$$\pi_1^{-1}([v]_{H^1}) = v + df, \quad \pi_{1*}^{-1}([\overline{v}]_{H^1}) = \overline{v + dg},$$

where $f \in \mathcal{D}'_{E_u^*}(M; \mathbb{C})$, $g \in \mathcal{D}'_{E_s^*}(M; \mathbb{C})$ satisfy

$$Xf = Xg = -\iota_X v. \tag{2.64}$$

We compute

$$\begin{aligned}
\mathrm{Re}\langle\langle \pi_1^{-1}&([v]_{H^1}), d\alpha \wedge \pi_{1*}^{-1}([\overline{v}]_{H^1}) \rangle\rangle \\
&= \mathrm{Re} \int_M \alpha \wedge d\alpha \wedge (v + df) \wedge (\overline{v + dg}) \\
&= \mathrm{Re} \int_M \alpha \wedge d\alpha \wedge (df \wedge \overline{v} + v \wedge d\overline{g} + df \wedge d\overline{g}) \\
&= \mathrm{Re} \int_M d\alpha \wedge d\alpha \wedge (f\overline{v} - \overline{g}v - \overline{g}df) \\
&= \mathrm{Re} \int_M \big(f \iota_X \overline{v} - \overline{g}\iota_X v - (Xf)\overline{g}\big) d\,\mathrm{vol}_\alpha \\
&= -\mathrm{Re}\langle Xf, f \rangle_{L^2(M; d\,\mathrm{vol}_\alpha)}.
\end{aligned}$$

Here in the second line we used that $\mathrm{Re}(v \wedge \overline{v}) = 0$. In the third line we integrated by parts and used that $dv = 0$. In the fourth line we used that $\iota_X d\alpha = 0$ (the 5-forms under the integral are equal as can be seen by taking ι_X of both sides). In the last line we used the identity (2.64).

Thus, if (2.63) fails, we have $\mathrm{Re}\langle Xf, f \rangle_{L^2(M; d\,\mathrm{vol}_\alpha)} \leq 0$ which by Lemma 2.3 implies that $f \in C^\infty(M; \mathbb{C})$ and thus $u := \pi_1^{-1}([v]_{H^1})$ lies in $\mathrm{Res}_0^1 \cap C^\infty(M; \Omega^1)$. Now the fact that u is invariant under the flow φ_t and the stable/unstable decomposition (2.4) imply that $u \in E_0^*$ at each point, and the fact that $\iota_X u = 0$ then gives $u = 0$. This shows that v is exact, giving a contradiction. $\square$

We finally give the following result in the case when all forms in Res_0^1 are closed:

Lemma 2.11 *Assume that* Res_0^1 *consists of closed forms, i.e.* $d(\mathrm{Res}_0^1) = 0$. *Then:*

1. *The semisimplicity condition* (2.41) *holds at* $\lambda_0 = 0$ *for the operators* $P_{1,0}$ *and* $P_{3,0}$.
2. $d(\mathrm{Res}_0^2) = 0$*,* π_2 *is onto, and* $\ker \pi_2$ *is spanned by* $d\alpha$.
3. $m_{1,0}(0) = m_{3,0}(0) = b_1(M)$, $\dim \mathrm{Res}_0^2 = b_2(M) + 1$, *and* $\pi_3 = 0$.

Remark Lemma 2.11 does not provide full information on the resonance at 0 since it does not prove the semisimplicity condition for the operator $P_{2,0}$, and only *assumes* that resonant forms Res_0^1 are closed (in fact we will see that $d(\mathrm{Res}_0^1) \neq 0$ and $P_{2,0}$ is not semisimple in the hyperbolic case when $b_1(M) > 0$, see § 3).

Proof 1. Since $\dim(\mathrm{Res}_0^1 \cap \ker d) = \dim(\mathrm{Res}_{0*}^1 \cap \ker d)$ by Lemma 2.8, and $\dim \mathrm{Res}_0^1 = \dim \mathrm{Res}_{0*}^1$ by (2.54), we have $d(\mathrm{Res}_{0*}^1) = 0$. By (2.49) we have $\mathrm{Res}_{0*}^3 = d\alpha \wedge \mathrm{Res}_{0*}^1$. Now Lemma 2.10 shows that $\langle\langle \bullet, \bullet \rangle\rangle$ defines a non-degenerate pairing on $\mathrm{Res}_0^1 \times \mathrm{Res}_{0*}^3$, which by Lemma 2.2 shows that the semisimplicity condition (2.41) holds at $\lambda_0 = 0$ for the operator $P_{1,0}$. By (2.45) semisimplicity holds for $P_{3,0}$ as well.
2. We first show that Res_0^2 consists of closed forms. Assume that $\zeta \in \mathrm{Res}_0^2$, then $d\zeta \in \mathrm{Res}_0^3$. By (2.45), $d\zeta = d\alpha \wedge u$ for some $u \in \mathrm{Res}_0^1$. Take arbitrary $u_* \in \mathrm{Res}_{0*}^1$. Then

$$\langle\langle u, d\alpha \wedge u_* \rangle\rangle = \int_M \alpha \wedge d\zeta \wedge u_* = \int_M d\alpha \wedge \zeta \wedge u_* = 0 \qquad (2.65)$$

Here in the second equality we integrate by parts and use that $du_* = 0$; in the last equality we use that ι_X applied to the 5-form under the integral is equal to 0. Now by Lemma 2.10 we have $u = 0$, which means that $d\zeta = 0$ as needed.

Next, by Lemma 2.6 we have $\ker \pi_2 \subset d(\mathrm{Res}^{1,\infty})$. By (2.43), Lemma 2.4, and the fact that $\mathrm{Res}_0^{1,\infty} = \mathrm{Res}_0^1$ we have $\mathrm{Res}^{1,\infty} = \mathrm{Res}_0^1 \oplus \mathbb{C}\alpha$. Since $d(\mathrm{Res}_0^1) = 0$ and $d\alpha \in \ker \pi_2$, we see that $\ker \pi_2$ is spanned by $d\alpha$.

Finally, to show that π_2 is onto, it suffices to use Lemma 2.7: since all elements of Res_{0*}^1 are closed, all elements of $\mathrm{Res}_{0*}^3 = d\alpha \wedge \mathrm{Res}_{0*}^1$ are exact.
3. This follows immediately from the above statements and Lemma 2.8. To show that $\pi_3 = 0$ we note that $\mathrm{Res}_0^3 = d\alpha \wedge \mathrm{Res}_0^1$ consists of exact forms. □

2.4.3 Summary

We now briefly summarize the contents of this section. Lemma 2.2 will often be used to interpret the semisimplicity condition (2.41) via the more tractable nondegeneracy of the pairing (2.9). Next, Lemma 2.4 provides us with a definitive understanding of $\mathrm{Res}_0^{0,\infty}$ and $\mathrm{Res}_0^{4,\infty}$, which by the isomorphisms (2.49) reduces the problem to studying $\mathrm{Res}_0^{1,\infty}$ and $\mathrm{Res}_0^{2,\infty}$. As Theorem 1 shows, this is a complicated question, but Lemma 2.8 says that $\mathrm{Res}_0^1 \cap \ker d$ is 'stably topological', that is, it is always mapped isomorphically by π_1 to $H^1(M)$. Moreover, if one can show $d(\mathrm{Res}_0^1) = 0$, Lemma 2.11 shows that semisimplicity for 1-forms is valid, which will be used in the perturbed picture in § 4. Under the same assumption, we also know that Res_0^2 is spanned by the 'topological part' $\pi_2^{-1}(H^2(M))$ and the form $d\alpha$. Thus, to compute (2.59) it suffices to study conditions under which forms in Res_0^1 are closed, and semisimplicity conditions for $P_{2,0}$. This will be done in two steps: in § 3 we will first develop a detailed understanding when φ_t is the geodesic flow of a hyperbolic 3-manifold, and later in § 4 we will study the perturbed picture.

3 Resonant states for hyperbolic 3-manifolds

In this section we study in detail the Pollicott–Ruelle resonant states at 0 for geodesic flows on hyperbolic 3-manifolds. The theorem below summarizes the main results. Here $\mathrm{Res}_0^k = \mathrm{Res}_0^{k,1}$ are the spaces of resonant k-forms, $\mathrm{Res}_0^{k,\ell}$ are the spaces of generalized resonant k-forms (see §2.4), and π_k : $\mathrm{Res}_0^k \cap \ker d \to H^k(M; \mathbb{C})$ are the maps defined in (2.61). The maps $\pi_\Sigma^*, \pi_{\Sigma*}$ are defined in §2.2.2.

Theorem 2 *Let $M = S\Sigma$ where Σ is a hyperbolic 3-manifold and φ_t be the geodesic flow on Σ. Then:*

1. *There exists a 2-form $\psi \in C^\infty(M; \Omega_0^2)$ which is closed but not exact, $\pi_{\Sigma*}(\psi) = -4\pi$, and ψ is invariant under φ_t.*
2. $\mathrm{Res}_0^1 = \mathcal{C} \oplus \mathcal{C}_\psi$ *is $2b_1(\Sigma)$-dimensional where $\mathcal{C} := \mathrm{Res}_0^1 \cap \ker d$ is $b_1(\Sigma)$-dimensional and $\mathcal{C}_\psi$ is another $b_1(\Sigma)$-dimensional space characterized by the identity $d\alpha \wedge \mathcal{C}_\psi = \psi \wedge \mathcal{C}$.*
3. *The semisimplicity condition (2.41) holds at $\lambda_0 = 0$ for the operators $P_{1,0}$ and $P_{3,0}$.*
4. $\mathrm{Res}_0^2 = \mathbb{C}d\alpha \oplus \mathbb{C}\psi \oplus d\mathcal{C}_\psi$ *is $b_1(\Sigma)+2$-dimensional and consists of closed forms. The map π_2 has kernel $\mathbb{C}d\alpha \oplus d\mathcal{C}_\psi$ and range $\mathbb{C}[\psi]_{H^2}$.*
5. $\mathrm{Res}_0^{2,\infty} = \mathrm{Res}_0^{2,2}$ *is $2b_1(\Sigma) + 2$-dimensional. The range of the map $\mathcal{L}_X$: $\mathrm{Res}_0^{2,2} \to \mathrm{Res}_0^2$ is equal to $d\mathcal{C}_\psi$.*
6. $\mathrm{Res}_0^3 = d\alpha \wedge \mathrm{Res}_0^1$ *is $2b_1(\Sigma)$-dimensional and consists of closed forms. The map π_3 has kernel $d\alpha \wedge \mathcal{C}$ and its range is a codimension 1 subspace of $H^3(M; \mathbb{C})$ not containing $[\pi_\Sigma^* d\,\mathrm{vol}_g]_{H^3}$.*
7. *The map $\pi_{\Sigma*}$ annihilates $d\alpha \wedge \mathcal{C}$ and is an isomorphism from $d\alpha \wedge \mathcal{C}_\psi$ onto the space of harmonic 1-forms on Σ.*

Theorem 2 together with Lemma 2.4 and (2.59) give part 1 of Theorem 1:

Corollary 3.1 *Theorem 2, the algebraic multiplicities of 0 as a resonance of the operators $P_{k,0}$ are*

$$\begin{aligned} m_{0,0}(0) = m_{4,0}(0) = 1, \quad m_{1,0}(0) = m_{3,0}(0) = 2b_1(\Sigma), \\ m_{2,0}(0) = 2b_1(\Sigma) + 2 \end{aligned} \tag{3.1}$$

and the order of vanishing of the Ruelle zeta function ζ_{R} at 0 is equal to

$$m_{\mathrm{R}}(0) = 2m_{0,0}(0) - 2m_{1,0}(0) + m_{2,0}(0) = 4 - 2b_1(\Sigma).$$

Previously (3.1) was proved in [16, Proposition 7.7] using different methods. Here we give a more refined description: we construct the resonant forms, prove pairing formulas, and study the existence of Jordan blocks. We emphasize that these properties are of crucial importance for the perturbation arguments in § 4 and were not known prior to this work.

This section is structured as follows: in §3.1 we review the geometric features of hyperbolic 3-manifolds used here. In §3.2 we construct the smooth invariant 2-form ψ and study its properties, proving part 1 of Theorem 2. In §3.3 we study the resonant 1-forms and 3-forms, proving parts 2, 3, and 6 of Theorem 2. In §3.4 we study the resonant 2-forms, proving parts 4 and 5 of Theorem 2. Finally, in §3.5 we show that the pushforward operator $\pi_{\Sigma *}$ maps elements of Res_0^3 to harmonic 1-forms on (Σ, g), proving part 7 of Theorem 2.

3.1 Hyperbolic 3-manifolds

We first review the geometry of hyperbolic 3-manifolds, following [14, §3]. We define a hyperbolic 3-manifold to be a nonempty compact connected oriented 3-dimensional Riemannian manifold Σ with constant sectional curvature -1. Each such manifold can be written as a quotient

$$\Sigma = \Gamma \backslash \mathbb{H}^3,$$

where $\mathbb{H}^3$ is the 3-dimensional hyperbolic space and $\Gamma \subset \mathrm{SO}_+(1,3)$ is a discrete torsion-free co-compact subgroup. We will use the *hyperboloid model*

$$\mathbb{H}^3 = \{x \in \mathbb{R}^{1,3} \mid \langle x, x\rangle_{1,3} = 1,\ x_0 > 0\},$$

where $\mathbb{R}^{1,3} = \mathbb{R}^4$ is the Minkowski space, with points denoted by $x = (x_0, x_1, x_2, x_3)$ and the Lorentzian inner product

$$\langle x, x\rangle_{1,3} := x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

The group $\mathrm{SO}_+(1,3)$ is the group of linear transformations on $\mathbb{R}^{1,3}$ (that is, 4×4 real matrices) which preserve the inner product $\langle \bullet, \bullet\rangle_{1,3}$, have determinant 1, and preserve the sign of x_0 on elements of $\mathbb{H}^3$. The Riemannian metric on $\mathbb{H}^3$ is the restriction of $-\langle \bullet, \bullet\rangle_{1,3}$; the group $\mathrm{SO}_+(1,3)$ acts on $\mathbb{H}^3$ by isometries, so the metric descends to the quotient Σ. Note that we may write $\mathbb{H}^3 \simeq \mathrm{SO}_+(1,3)/\mathrm{SO}(3)$ as a homogeneous space for the $\mathrm{SO}_+(1,3)$-action, since $\mathrm{SO}(3)$ is the stabilizer of the point $(1,0,0,0)\in\mathbb{H}^3$.

3.1.1 Geodesic flow

We now study the geodesic flow on Σ, using the notation of §2.2. The sphere bundle $S\Sigma$ is the quotient

$$S\Sigma = \Gamma\backslash S\mathbb{H}^3, \tag{3.2}$$

where the sphere bundle $S\mathbb{H}^3 \subset \mathbb{R}^{1,3}\times\mathbb{R}^{1,3}$ has the form

$$S\mathbb{H}^3 = \{(x,v)\in\mathbb{R}^{1,3}\times\mathbb{R}^{1,3} \mid \langle x,x\rangle_{1,3}=1,\ \langle v,v\rangle_{1,3}=-1,\ \langle x,v\rangle_{1,3}=0\}.$$

Note that we may write $S\mathbb{H}^3 \simeq \mathrm{SO}_+(1,3)/\mathrm{SO}(2)$ as a homogeneous space for the $\mathrm{SO}_+(1,3)$-action, since $\mathrm{SO}(2)$ is the stabilizer of the point $(1,0,0,0,0,1,0,0)\in S\mathbb{H}^3$. The contact form α, defined in (2.11), and the generator X of the geodesic flow are

$$\alpha = -\langle v, dx\rangle_{1,3}, \quad X = v\cdot\partial_x + x\cdot\partial_v, \tag{3.3}$$

where '$\cdot$' denotes the (positive definite) Euclidean inner product on $\mathbb{R}^{1,3}$. The geodesic flow is then given by

$$\varphi_t(x,v) = (x\cosh t + v\sinh t, x\sinh t + v\cosh t).$$

As a corollary, the distance function on $\mathbb{H}^3$ with respect to the hyperbolic metric is given by

$$\cosh d_{\mathbb{H}^3}(x,y) = \langle x,y\rangle_{1,3} \quad \text{for all} \quad x,y\in\mathbb{H}^3. \tag{3.4}$$

The tangent space $T_{(x,v)}(S\mathbb{H}^3)$ consists of vectors $(\xi_x,\xi_v)\in\mathbb{R}^{1,3}\oplus\mathbb{R}^{1,3}$ such that

$$\langle x,\xi_x\rangle_{1,3} = \langle v,\xi_v\rangle_{1,3} = \langle x,\xi_v\rangle_{1,3} + \langle v,\xi_x\rangle_{1,3} = 0.$$

The connection map (2.14) is given by

$$\mathcal{K}(x,v)(\xi_x,\xi_v) = \xi_v - \langle x,\xi_v\rangle_{1,3}\, x = \xi_v + \langle v,\xi_x\rangle_{1,3} x.$$

Here and throughout we note that the addition of points x and vectors ξ_v (or ξ_x) has to be understood in $\mathbb{R}^{1,3}$. The horizontal and vertical spaces $\mathbf{H}(x, v), \mathbf{V}(x, v) \subset T_{(x,v)}(S\mathbb{H}^3)$ are then

$$\begin{aligned}\mathbf{H}(x, v) &= \{(\xi_x, \xi_v) \mid \langle x, \xi_x\rangle_{1,3} = 0,\ \xi_v = -\langle v, \xi_x\rangle_{1,3}\, x\},\\ \mathbf{V}(x, v) &= \{(0, \xi_v) \mid \langle x, \xi_v\rangle_{1,3} = \langle v, \xi_v\rangle_{1,3} = 0\}\end{aligned}$$

and the horizontal-vertical splitting map (2.15) takes for $\xi = (\xi_x, \xi_v) \in T_{(x,v)}(S\mathbb{H}^3) \subset \mathbb{R}^{1,3} \oplus \mathbb{R}^{1,3}$ the form

$$\xi_H = \xi_x, \quad \xi_V = \xi_v + \langle v, \xi_x\rangle_{1,3}\, x.$$

The Sasaki metric (2.17) is for $\xi, \eta \in T_{(x,v)}(S\mathbb{H}^3)$ given by

$$\langle \xi, \eta\rangle_S = -\langle \xi_x, \eta_x\rangle_{1,3} - \langle \xi_v, \eta_v\rangle_{1,3} + \langle v, \xi_x\rangle_{1,3}\langle v, \eta_x\rangle_{1,3}.$$

The unstable/stable subspaces E_u, E_s from (2.2) on $S\mathbb{H}^3$ are given by

$$\begin{aligned}E_u(x, v) &= \{(w, w) \mid w \in \mathbb{R}^{1,3},\ \langle w, x\rangle_{1,3} = \langle w, v\rangle_{1,3} = 0\},\\ E_s(x, v) &= \{(w, -w) \mid w \in \mathbb{R}^{1,3},\ \langle w, x\rangle_{1,3} = \langle w, v\rangle_{1,3} = 0\}.\end{aligned} \tag{3.5}$$

In terms of the horizontal-vertical splitting (2.15) they can be characterized as follows:

$$E_u = \{\xi_V = \xi_H\}, \quad E_s = \{\xi_V = -\xi_H\}. \tag{3.6}$$

A distinguished feature of hyperbolic manifolds is that the restriction of the differential of the geodesic flow to the unstable/stable spaces is conformal with respect to the Sasaki metric:

$$|d\varphi_t(x, v)\xi|_S = \begin{cases} e^t|\xi|_S, & \xi \in E_u(x, v);\\ e^{-t}|\xi|_S, & \xi \in E_s(x, v).\end{cases} \tag{3.7}$$

The objects discussed above are invariant under the action of $\mathrm{SO}_+(1, 3)$ and thus descend naturally to the quotients $\Sigma, S\Sigma$.

3.1.2 The frame bundle and canonical vector fields

A convenient tool for computations on $M = S\Sigma$ is the *frame bundle* $\mathcal{F}\Sigma$, consisting of quadruples (x, v_1, v_2, v_3) where $x \in \Sigma$ and $v_1, v_2, v_3 \in T_x\Sigma$ form a positively oriented orthonormal basis. We have

$$\mathcal{F}\Sigma = \Gamma\backslash\mathcal{F}\mathbb{H}^3, \quad \mathcal{F}\mathbb{H}^3 \simeq \mathrm{SO}_+(1, 3),$$

where the frame bundle $\mathcal{F}\mathbb{H}^3$ is identified with the group $\mathrm{SO}_+(1,3)$ by the following map (where $e_0 = (1,0,0,0)$, $e_1 = (0,1,0,0), \dots$)

$$\gamma \in \mathrm{SO}_+(1,3) \mapsto (\gamma e_0, \gamma e_1, \gamma e_2, \gamma e_3). \tag{3.8}$$

Under this identification, the action of $\mathrm{SO}_+(1,3)$ on $\mathcal{F}\mathbb{H}^3$ corresponds to the action of this group on itself by left multiplications. Therefore, $\mathrm{SO}_+(1,3)$-invariant vector fields on $\mathcal{F}\mathbb{H}^3$ correspond to left-invariant vector fields on the group $\mathrm{SO}_+(1,3)$, that is to elements of its Lie algebra $\mathfrak{so}(1,3)$. We define the basis of left-invariant vector fields on $\mathrm{SO}_+(1,3)$ corresponding to the following matrices in $\mathfrak{so}(1,3)$:

$$X = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&-1&0 \end{pmatrix}, \qquad U_1^+ = \begin{pmatrix} 0&0&-1&0\\ 0&0&-1&0\\ -1&1&0&0\\ 0&0&0&0 \end{pmatrix},$$

$$U_2^+ = \begin{pmatrix} 0&0&0&-1\\ 0&0&0&-1\\ 0&0&0&0\\ -1&1&0&0 \end{pmatrix}, \qquad U_1^- = \begin{pmatrix} 0&0&-1&0\\ 0&0&1&0\\ -1&-1&0&0\\ 0&0&0&0 \end{pmatrix}, \qquad U_2^- = \begin{pmatrix} 0&0&0&-1\\ 0&0&0&1\\ 0&0&0&0\\ -1&-1&0&0 \end{pmatrix}.$$

Under the identification (3.8), and considering $\mathcal{F}\mathbb{H}^3$ as a submanifold of $(\mathbb{R}^{1,3})^4$, we can write using coordinates $(x, v_1, v_2, v_3) \in (\mathbb{R}^{1,3})^4$ and writing '$\cdot$' for the Euclidean inner product

$$\begin{aligned} X &= v_1 \cdot \partial_x + x \cdot \partial_{v_1}, \quad R = v_2 \cdot \partial_{v_3} - v_3 \cdot \partial_{v_2},\\ U_1^\pm &= -v_2 \cdot \partial_x - x \cdot \partial_{v_2} \pm (v_2 \cdot \partial_{v_1} - v_1 \cdot \partial_{v_2}),\\ U_2^\pm &= -v_3 \cdot \partial_x - x \cdot \partial_{v_3} \pm (v_3 \cdot \partial_{v_1} - v_1 \cdot \partial_{v_3}). \end{aligned}$$

Since the vector fields above are invariant under the action of $\mathrm{SO}_+(1,3)$, they descend to the frame bundle of the quotient, $\mathcal{F}\Sigma$.

The commutation relations between these fields are (as can be seen by computing the commutators of the corresponding matrices, or by using the explicit formulas above)

$$\begin{aligned} &[X, U_i^\pm] = \pm U_i^\pm, \quad [U_i^+, U_i^-] = 2X, \quad [U_1^\pm, U_2^\mp] = 2R,\\ &[X, R] = [U_1^\pm, U_2^\pm] = 0, \quad [R, U_1^\pm] = -U_2^\pm, \quad [R, U_2^\pm] = U_1^\pm. \end{aligned} \tag{3.9}$$

The map

$$\pi_{\mathcal{F}} : (x, v_1, v_2, v_3) \in \mathcal{F}\Sigma \mapsto (x, v_1) \in S\Sigma$$

is a submersion, with one-dimensional fibers whose tangent spaces are spanned by the field R. Thus, if a vector field on $\mathcal{F}\Sigma$ commutes with R then this vector field descends to the sphere bundle $S\Sigma$. In particular, the vector field X descends to the generator of the geodesic flow (which we also denote by X).

The vector fields $U_i^{\pm}$ do not commute with R and thus do not descend to $S\Sigma$. However, the vector space $\operatorname{span}(U_1^+, U_2^+)$ is R-invariant and descends to the stable space E_s on $S\Sigma$. Similarly, the space $\operatorname{span}(U_1^-, U_2^-)$ descends to E_u. Because of this we think of U_1^+, U_2^+ as *stable vector fields* and U_1^-, U_2^- as *unstable vector fields*.

3.1.3 Canonical differential forms

We next introduce the frame of *canonical differential 1-forms* on $\mathcal{F}\Sigma$

$$\alpha,\ R^*,\ U_1^{\pm *},\ U_2^{\pm *}$$

which is defined as a dual frame for the vector fields $X, R, U_1^{\mp}, U_2^{\mp}$, in the sense compatible with the definition of the dual stable/unstable bundles (2.4), as follows:

$$\langle \alpha, X\rangle = \langle R^*, R\rangle = \langle U_1^{\pm *}, U_1^{\mp}\rangle = \langle U_2^{\pm *}, U_2^{\mp}\rangle = 1 \tag{3.10}$$

and all the other pairings between the 1-forms and the vector fields in question are equal to 0. In particular, $\langle U_i^{\pm *}, U_i^{\pm}\rangle = 0$.

Using the following identity valid for any 1-form β and any two vector fields Y, Z

$$d\beta(Y, Z) = Y\beta(Z) - Z\beta(Y) - \beta([Y, Z]), \tag{3.11}$$

the commutation relations (3.9), and the duality relations (3.10), we compute the differentials of the canonical forms:

$$\begin{aligned} &d\alpha = 2(U_1^{+*} \wedge U_1^{-*} + U_2^{+*} \wedge U_2^{-*}), \quad dR^* = 2(U_2^{-*} \wedge U_1^{+*} + U_2^{+*} \wedge U_1^{-*}),\\ &dU_1^{\pm *} = \pm\alpha \wedge U_1^{\pm *} - R^* \wedge U_2^{\pm *}, \quad dU_2^{\pm *} = \pm\alpha \wedge U_2^{\pm *} + R^* \wedge U_1^{\pm *}. \end{aligned} \tag{3.12}$$

It follows that

$$\mathcal{L}_X U_j^{\pm *} = \pm U_j^{\pm *}, \quad \mathcal{L}_R U_1^{\pm *} = -U_2^{\pm *}, \quad \mathcal{L}_R U_2^{\pm *} = U_1^{\pm *}. \tag{3.13}$$

If ω is a differential form on $\mathcal{F}\Sigma$, then ω descends to $S\Sigma$ (i.e. it is a pullback by $\pi_{\mathcal{F}}$ of a form on $S\Sigma$) if and only if $\iota_R\omega = 0$, $\mathcal{L}_R\omega = 0$. In particular the form α on $\mathcal{F}\Sigma$ descends to the contact form on $S\Sigma$, which we also denote by α.

3.1.4 Conformal infinity

Following [14, §3.4] we consider the maps

$$\Phi_\pm : S\mathbb{H}^3 \to (0,\infty), \quad B_\pm : S\mathbb{H}^3 \to \mathbb{S}^2, \tag{3.14}$$

where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$, defined by the identities

$$x \pm v = \Phi_\pm(x,v)(1, B_\pm(x,v)) \quad \text{for all} \quad (x,v) \in S\mathbb{H}^3. \tag{3.15}$$

Note that $B_\pm(x,v)$ is the limit as $t \to \pm\infty$ of the projection to $\mathbb{H}^3$ of the geodesic $\varphi_t(x,v)$ in the compactification of the Poincaré ball model of $\mathbb{H}^3$. Let

$$(\mathbb{S}^2 \times \mathbb{S}^2)_- := \{(\nu_-, \nu_+) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \nu_- \neq \nu_+\}.$$

In fact, the maps $B_\pm$ yield the following diffeomorphism of $S\mathbb{H}^3$ (see [14, (3.24)]):

$$\begin{gathered}\Xi : S\mathbb{H}^3 \ni (y,v) \mapsto (\nu_-, \nu_+, t) \in (\mathbb{S}^2 \times \mathbb{S}^2)_- \times \mathbb{R} \\ \text{with} \quad \nu_\pm = B_\pm(y,v), \quad t = \frac{1}{2}\log\left(\frac{\Phi_+(y,v)}{\Phi_-(y,v)}\right).\end{gathered} \tag{3.16}$$

The geometric interpretation of Ξ is as follows: $\nu_\pm$ are the limits on the conformal boundary $\mathbb{S}^2$ of the geodesic $\varphi_s(y,v)$ as $s \to \pm\infty$ and t is chosen so that $\varphi_{-t}(y,v)$ is the closest point to e_0 on that geodesic (as can be seen from (5.30) below and noting that $Xt = 1$ by (3.22)).

We have the identity [14, (3.23)]

$$\Phi_-(x,v)\Phi_+(x,v)\big|B_-(x,v) - B_+(x,v)\big|^2 = 4, \tag{3.17}$$

where $|\bullet|$ denotes the Euclidean distance on $\mathbb{R}^3 \supset \mathbb{S}^2$.

We also introduce the Poisson kernel

$$P(x,\nu) = \big(\langle x, (1,\nu)\rangle_{1,3}\big)^{-1} > 0, \quad x \in \mathbb{H}^3, \quad \nu \in \mathbb{S}^2 \subset \mathbb{R}^3. \tag{3.18}$$

The following relations hold [14, (3.21)]:

$$\Phi_\pm(x,v) = P(x, B_\pm(x,v)). \tag{3.19}$$

If we fix $x \in \mathbb{H}^3$, then the maps $v \mapsto B_\pm(x,v)$ are diffeomorphisms from the fiber $S_x\mathbb{H}^3$ onto $\mathbb{S}^2$. The inverse maps are given by $\nu \mapsto v_\pm(x,\nu)$ where [14, (3.20)]

$$v_\pm(x,\nu) = \mp x \pm P(x,\nu)(1,\nu) \in S_x\mathbb{H}^3, \quad B_\pm(x, v_\pm(x,\nu)) = \nu. \tag{3.20}$$

The diffeomorphisms $v \mapsto B_\pm(x, v)$ are conformal with respect to the induced metric on $S_x\mathbb{H}^3$ and the canonical metric $| \bullet |_{\mathbb{S}^2}$: by [14, (3.22)]) we have

$$|\partial_v B_\pm(x, v)\eta|_{\mathbb{S}^2} = \frac{|\eta|_g}{\Phi_\pm(x, v)} \quad \text{for all} \quad \eta \in T_v(S_x\mathbb{H}^3). \tag{3.21}$$

Next, we have by (3.3) and (3.5)

$$X\Phi_\pm = \pm\Phi_\pm, \quad d\Phi_-|_{E_u} = d\Phi_+|_{E_s} = 0. \tag{3.22}$$

The maps $B_\pm$ are submersions with connected fibers, the tangent spaces to which are described in terms of the stable/unstable decomposition (2.2) as follows: for each $\nu \in \mathbb{S}^2$

$$T(B_+^{-1}(\nu)) = (E_0 \oplus E_s)|_{B_+^{-1}(\nu)}, \quad T(B_-^{-1}(\nu)) = (E_0 \oplus E_u)|_{B_-^{-1}(\nu)}. \tag{3.23}$$

This can be checked using (3.5), see [14, (3.25)]. The action of the differential dB_+ on E_u, and of dB_- on E_s, can be described as follows: for any $(x, v) \in S\mathbb{H}^3$ and $w \in \mathbb{R}^{1,3}$ such that $\langle x, w\rangle_{1,3} = \langle v, w\rangle_{1,3} = 0$,

$$dB_\pm(x, v)(w, \pm w) = \frac{2(w' - w_0 B_\pm(x, v))}{\Phi_\pm(x, v)} \quad \text{where} \quad w = (w_0, w'). \tag{3.24}$$

We next briefly discuss the action of the group $\mathrm{SO}_+(1, 3)$ on the conformal infinity $\mathbb{S}^2$, referring to [14, §3.5] for details. For any $\gamma \in \mathrm{SO}_+(1, 3)$, define

$$N_\gamma : \mathbb{S}^2 \to (0, \infty), \quad L_\gamma : \mathbb{S}^2 \to \mathbb{S}^2$$

by the identity (where on the left is the linear action of γ on $(1, \nu) \in \mathbb{R}^{1,3}$)

$$\gamma \cdot (1, \nu) = N_\gamma(\nu)(1, L_\gamma(\nu)) \quad \text{for all} \quad \nu \in \mathbb{S}^2.$$

The maps L_γ define an action of $\mathrm{SO}_+(1, 3)$ on $\mathbb{S}^2$. This action is transitive and the stabilizer of $e_1 \in \mathbb{S}^2$ is the group of matrices $A \in \mathrm{SO}_+(1, 3)$ such that $A(1, 1, 0, 0)^T = \tau(1, 1, 0, 0)^T$ for some $\tau > 0$, which may be shown to be isomorphic to the group of similarities of the plane $\mathrm{Sim}(2)$, giving $\mathbb{S}^2 \simeq \mathrm{SO}_+(1, 3)/\mathrm{Sim}(2)$ the structure of a homogeneous space.

This action is by orientation preserving conformal transformations, more precisely

$$|dL_\gamma(\nu)\zeta|_{\mathbb{S}^2} = \frac{|\zeta|_{\mathbb{S}^2}}{N_\gamma(\nu)} \quad \text{for all} \quad (\nu, \zeta) \in T\mathbb{S}^2. \tag{3.25}$$

Moreover, the maps $B_\pm$ have the equivariance property

$$B_\pm(\gamma \cdot (x, v)) = L_\gamma(B_\pm(x, v)) \quad \text{for all} \quad (x, v) \in S\mathbb{H}^3. \tag{3.26}$$

We finally use the maps $B_\pm$ to describe a special class of differential forms on $S\Sigma$ defined as follows (c.f. [14,44]):

Definition 3.2 We call a k-form $u \in \mathcal{D}'(S\Sigma; \Omega_0^k)$ **stable** if it is a section of $\wedge^k E_s^* \subset \Omega_0^k$ where $E_s^* \subset T^*(S\Sigma)$ is the annihilator of $E_0 \oplus E_s$ (see (2.4)). We call u **unstable** if it is a section of $\wedge^k E_u^*$ where E_u^* is the annihilator of $E_0 \oplus E_u$.

We call a form u **totally (un)stable** if both u and du are (un)stable.

The lemma below (see also [44, §§2.3–2.4]) shows that totally (un)stable k-forms on $S\Sigma$, $\Sigma = \Gamma\backslash\mathbb{H}^3$, correspond to Γ-invariant k-forms on $\mathbb{S}^2$. Denote by $\pi_\Gamma : S\mathbb{H}^3 \to S\Sigma$ the covering map.

Lemma 3.3 *Let $u \in \mathcal{D}'(S\Sigma; \Omega_0^k)$ be totally stable. Then the lift $\pi_\Gamma^* u$ has the form*

$$\pi_\Gamma^* u = B_+^* w \quad \textit{where} \quad w \in \mathcal{D}'(\mathbb{S}^2; \Omega^k), \;\; L_\gamma^* w = w \quad \textit{for all} \quad \gamma \in \Gamma \tag{3.27}$$

Conversely, each form $B_+^* w$, where w satisfies (3.27), is the lift of a totally stable k-form on $S\Sigma$. A similar statement holds for totally unstable forms, with B_+ replaced by B_-.

Proof We only consider the case of totally stable forms, with totally unstable forms handled similarly. First of all, note that lifts of totally stable k-forms on $S\Sigma$ are exactly the Γ-invariant totally stable k-forms on $S\mathbb{H}^3$. Next, by (3.23), a k-form $\zeta \in \mathcal{D}'(S\mathbb{H}^3; \Omega^k)$ is totally stable if and only if $\iota_Y \zeta = 0$, $\mathcal{L}_Y \zeta = 0$ for any vector field Y tangent to the fibers of the map B_+, which is equivalent to saying that $\zeta = B_+^* w$ for some $w \in \mathcal{D}'(\mathbb{S}^2; \Omega^k)$. Finally, by (3.26), Γ-invariance of ζ is equivalent to Γ-invariance of w. □

Lemma 3.3 implies that

$$\begin{aligned} \text{every totally stable } u \in \mathcal{D}'(S\Sigma; \Omega_0^k) \quad &\text{lies in } \mathcal{D}'_{E_s^*}(S\Sigma; \Omega_0^k), \\ \text{every totally unstable } u \in \mathcal{D}'(S\Sigma; \Omega_0^k) \quad &\text{lies in } \mathcal{D}'_{E_u^*}(S\Sigma; \Omega_0^k). \end{aligned} \tag{3.28}$$

Indeed, assume that u is totally stable. Write $\pi_\Gamma^* u = B_+^* w$ for some $w \in \mathcal{D}'(\mathbb{S}^2; \Omega^k)$, then we have $\mathrm{WF}(\pi_\Gamma^* u) = \pi_\Gamma^* \mathrm{WF}(u)$ (as π_Γ is a local diffeomorphism). From the behavior of wavefront sets under pullbacks [38, Theorem 8.2.4], we know that $\mathrm{WF}(\pi_\Gamma^* u)$ is contained in the conormal bundle of the fibers of the submersion B_+. From (3.23) and (2.4) we then have $\mathrm{WF}(u) \subset E_s^*$. A similar argument works for the totally unstable case.

3.2 Additional invariant 2-form

The space of smooth flow invariant 2-forms on $S\Sigma$ is known to be 2-dimensional, see Lemma 3.7 below, [40, Claim 3.3] or [36], thus there exists a smooth invariant 2-form which is not a multiple of $d\alpha$. In this section we introduce such a 2-form ψ and study its properties; these are crucial for the study of Pollicott–Ruelle resonances at zero in §§3.3–3.4 below.

3.2.1 A rotation on $E_u \oplus E_s$

Let $x \in \Sigma$. For any two $v, w \in T_x\Sigma$, we may define their *cross product* $v \times w \in T_x\Sigma$, which is uniquely determined by the following properties: $v \times w$ is g-orthogonal to v and w; the length of $v \times w$ is the area of the parallelogram spanned by v, w in $T_x\Sigma$; and $v, w, v \times w$ is a positively oriented basis of $T_x\Sigma$ whenever $v \times w \neq 0$.

For future use we record here an identity true for any $v, w_1, w_2, w_3, w_4 \in T_x\Sigma$ such that $|v|_g = 1$ and w_1, w_2, w_3, w_4 are g-orthogonal to v:

$$\langle v \times w_1, w_2\rangle_g \langle v \times w_3, w_4\rangle_g = \langle w_1, w_3\rangle_g \langle w_2, w_4\rangle_g - \langle w_2, w_3\rangle_g \langle w_1, w_4\rangle_g. \tag{3.29}$$

Using the horizontal/vertical decomposition (2.15), we define the bundle homomorphism

$$\mathcal{I} : TS\Sigma \to TS\Sigma, \quad \mathcal{I}(x, v)(\xi_H, \xi_V) = (v \times \xi_V, v \times \xi_H). \tag{3.30}$$

From (2.18) and (3.6) we see that $\mathcal{I}$ preserves the flow/stable/unstable decomposition (2.2). Moreover, it annihilates $E_0 = \mathbb{R}X$ and it is a rotation by $\pi/2$ on E_u and on E_s (with respect to the Sasaki metric), so in particular it satisfies $\mathcal{I}^2 = -\mathrm{Id}$ on $\ker\alpha = E_u \oplus E_s$; however, the direction of the rotation is opposite on E_u and on E_s if we identify them by (3.5).

The map $\mathcal{I}$ is invariant under the geodesic flow $\varphi_t = e^{tX}$:

$$\mathcal{L}_X \mathcal{I} = 0. \tag{3.31}$$

This follows from the conformal property of the geodesic flow (3.7) and the description of the action of $\mathcal{I}$ on E_0, E_u, E_s in the previous paragraph.

For any point (x, v_1, v_2, v_3) in the frame bundle $\mathcal{F}\Sigma$, we have (using the horizontal/vertical decomposition)

$$\mathcal{I}(x, v_1)(v_2, \pm v_2) = \pm(v_3, \pm v_3), \quad \mathcal{I}(x, v_1)(v_3, \pm v_3) = \mp(v_2, \pm v_2). \tag{3.32}$$

It follows that (see §3.1.2 for the definition of the vector fields $U_i^\pm$ on $\mathcal{F}\Sigma$)

$$\begin{aligned}\mathcal{I}(x, v_1)(d\pi_{\mathcal{F}} U_1^\pm(x, v_1, v_2, v_3)) &= \mp d\pi_{\mathcal{F}} U_2^\pm(x, v_1, v_2, v_3),\\ \mathcal{I}(x, v_1)(d\pi_{\mathcal{F}} U_2^\pm(x, v_1, v_2, v_3)) &= \pm d\pi_{\mathcal{F}} U_1^\pm(x, v_1, v_2, v_3).\end{aligned} \tag{3.33}$$

3.2.2 Relation to conformal infinity

The homomorphism $\mathcal{I}$ lifts to $TS\mathbb{H}^3$. If $B_\pm : S\mathbb{H}^3 \to \mathbb{S}^2$ are the maps defined in (3.14) and '$\times$' denotes the cross product on $\mathbb{R}^3$, then for all $(x, v) \in S\mathbb{H}^3$ and $\xi \in T_{(x,v)}S\mathbb{H}^3$ we have

$$dB_\pm(x, v)(\mathcal{I}(x, v)\xi) = B_\pm(x, v) \times dB_\pm(x, v)(\xi). \tag{3.34}$$

To see this, we use (3.23), and the fact that $\mathcal{I}$ preserves the flow/stable/unstable decomposition, to reduce to the case $\xi = (w, \pm w)$, where x, v, w is an orthonormal set in $\mathbb{R}^{1,3}$. By the equivariance (3.26) of $B_\pm$ under $\mathrm{SO}_+(1, 3)$, the fact that the action L_γ of any $\gamma \in \mathrm{SO}_+(1, 3)$ on $\mathbb{S}^2$ is by orientation preserving conformal maps, and the equivariance of $\mathcal{I}$ under $\mathrm{SO}_+(1, 3)$ we can reduce to the case $x = e_0$, $v = e_1$, $w = e_2$, where e_0, e_1, e_2, e_3 is the canonical basis of $\mathbb{R}^{1,3}$. In the latter case (3.34) is verified directly using (3.24) and (3.32).

Let $\star$ be the Hodge star operator on 1-forms on the round sphere $\mathbb{S}^2$. It may be expressed as follows: for any $w \in C^\infty(\mathbb{S}^2; \Omega^1)$ and $(\nu, \zeta) \in T\mathbb{S}^2$ we have

$$\langle(\star w)(\nu), \zeta\rangle = -\langle w(\nu), \nu \times \zeta\rangle.$$

From (3.34) we get the following relation of $\mathcal{I}$ to $\star$: for any 1-form w on $\mathbb{S}^2$ we have

$$(B_\pm^* w) \circ \mathcal{I} = -B_\pm^*(\star w), \tag{3.35}$$

where for any 1-form β on $S\mathbb{H}^3$ the 1-form $\beta \circ \mathcal{I}$ on $S\mathbb{H}^3$ is defined by

$$\langle(\beta \circ \mathcal{I})(x, v), \xi\rangle = \langle\beta(x, v), \mathcal{I}(x, v)\xi\rangle. \tag{3.36}$$

3.2.3 The new invariant 2-form

We next define the 2-form $\psi \in C^\infty(S\Sigma; \Omega^2)$ as follows: for all $(x, v) \in S\Sigma$ and $\xi, \eta \in T_{(x,v)}S\Sigma$,

$$\psi(x, v)(\xi, \eta) = d\alpha(x, v)(\mathcal{I}(x, v)\xi, \eta). \tag{3.37}$$

To see that ψ is indeed an antisymmetric form, we may use (2.16) and (3.30) to write it in terms of the horizontal/vertical decomposition of ξ, η:

$$\psi(x, v)(\xi, \eta) = \langle v \times \xi_H, \eta_H \rangle_g - \langle v \times \xi_V, \eta_V \rangle_g. \tag{3.38}$$

Using (3.12), (3.33) we may also compute the lift of ψ to the frame bundle $\mathcal{F}\Sigma$, which we still denote by ψ:

$$\psi = 2(U_1^{+*} \wedge U_2^{-*} + U_1^{-*} \wedge U_2^{+*}). \tag{3.39}$$

We have

$$\iota_X \psi = 0, \quad \mathcal{L}_X \psi = 0. \tag{3.40}$$

The first of these statements is checked directly using (2.18). The second statement can be verified using (3.13) and (3.39), or using that $\mathcal{L}_X \mathcal{I} = 0$ and $\mathcal{L}_X d\alpha = 0$.

We now establish several properties of the form ψ. We will use the following corollaries of (2.16), (3.38):

$$d\alpha|_{\mathbf{H}\times\mathbf{H}} = 0, \quad d\alpha|_{\mathbf{V}\times\mathbf{V}} = 0, \quad \psi|_{\mathbf{H}\times\mathbf{V}} = 0 \tag{3.41}$$

where the horizontal/vertical spaces $\mathbf{H}$, $\mathbf{V}$ are defined in §2.2.1.

Lemma 3.4 *We have*

$$d\psi = 0, \tag{3.42}$$

$$\psi \wedge \psi = d\alpha \wedge d\alpha, \tag{3.43}$$

$$d(\alpha \wedge \psi) = 0. \tag{3.44}$$

Proof By (3.40) we have $\iota_X d\psi = 0$. Therefore, $d\psi(x, v)(\xi_1, \xi_2, \xi_3) = 0$ for $\xi_1, \xi_2, \xi_3 \in T_{(x,v)}S\Sigma$ such that one of these vectors lies in E_0. Next, $\mathcal{L}_X d\psi = 0$, that is $d\psi$ is invariant under the geodesic flow. Using this invariance for time $t \to \pm\infty$ together with (3.7) and the fact that 3 is an odd number, we see that $d\psi(x, v)(\xi_1, \xi_2, \xi_3) = 0$ also when each of the vectors ξ_1, ξ_2, ξ_3 lies in either $E_u(x, v)$ or $E_s(x, v)$. It follows that (3.42) holds.

To check (3.43), we first note that ι_X of both sides is zero. Thus it suffices to check that

$$\psi \wedge \psi(x, v)(\xi_1, \xi_2, \xi_3, \xi_4) = d\alpha \wedge d\alpha(x, v)(\xi_1, \xi_2, \xi_3, \xi_4) \tag{3.45}$$

for some choice of basis $\xi_1, \xi_2, \xi_3, \xi_4 \in T_{(x,v)}S\Sigma$ of the kernel of α. We take

$$\xi_1 = (w_1, 0), \quad \xi_2 = (w_2, 0), \quad \xi_3 = (0, w_3), \quad \xi_4 = (0, w_4)$$

under the horizontal/vertical decomposition (2.15), where each $w_j \in T_x\Sigma$ is orthogonal to v. By (2.16), (3.38)

$$\begin{aligned} \psi \wedge \psi(x, v)(\xi_1, \xi_2, \xi_3, \xi_4) &= -2\langle v \times w_1, w_2\rangle_g \langle v \times w_3, w_4\rangle_g, \\ d\alpha \wedge d\alpha(x, v)(\xi_1, \xi_2, \xi_3, \xi_4) &= 2(\langle w_2, w_3\rangle_g \langle w_1, w_4\rangle_g \\ &\quad - \langle w_1, w_3\rangle_g \langle w_2, w_4\rangle_g) \end{aligned}$$

and (3.45) follows from (3.29).

Finally, to show (3.44) it suffices to prove that $d\alpha \wedge \psi = 0$. To show this we may argue similarly to the proof of (3.43) above, using (3.41).

Alternatively, (3.42)–(3.44) can be checked by lifting to the frame bundle $\mathcal{F}\Sigma$ and using (3.12) and (3.39). □

The next lemma studies the relation of ψ to the de Rham cohomology of $M = S\Sigma$; in particular, its first item and (3.40) give the first item of Theorem 2. Recall the pullback and pushforward operators $\pi_\Sigma^*, \pi_{\Sigma*}$ defined in §2.2.2 and denote by $d\,\mathrm{vol}_g$ the volume 3-form on Σ induced by g and the choice of orientation.

Lemma 3.5 *We have:*

1. $\pi_{\Sigma*}(\psi) = -4\pi$. *In particular,* $[\psi]_{H^2} \neq 0$.
2. $\pi_{\Sigma*}(\alpha \wedge \psi) = 0$.
3. $\pi_{\Sigma*}(\alpha \wedge d\alpha) = 0$.
4. $\alpha \wedge d\alpha \wedge d\alpha = 2\psi \wedge \pi_\Sigma^*(d\,\mathrm{vol}_g)$.
5. $[\alpha \wedge \psi]_{H^3} = 2[\pi_\Sigma^*(d\,\mathrm{vol}_g)]_{H^3}$.

Proof 1. Let $(x, v) \in S\Sigma$ and v_2, v_3 be a positively oriented g-orthonormal basis of the tangent space to the fiber $T_v(S_x\Sigma)$. We consider v_2, v_3 as vertical vectors in $T_{(x,v)}S\Sigma$, as well as vectors in $T_x\Sigma$. The triple v, v_2, v_3 is a positively oriented g-orthonormal basis of $T_x\Sigma$, so by (3.38)

$$\psi(x, v)(v_2, v_3) = -\langle v \times v_2, v_3\rangle_g = -1.$$

Thus the restriction of ψ to each fiber $S_x\Sigma$ is -1 times the standard volume form on $S_x\Sigma \simeq \mathbb{S}^2$, which implies that $\pi_{\Sigma*}(\psi) = -4\pi$. It now follows from (2.22) that $[\psi]_{H^2} \neq 0$.

2. Fix $x \in \Sigma$, $v_1 \in T_x\Sigma$. Let $v \in S_x\Sigma$ and v_2, v_3 be a positively oriented g-orthonormal basis of the tangent space $T_v(S_x\Sigma)$ as in part 1 of this proof. Let $\xi_1 = (v_1, 0)$ be the horizontal lift of v_1 to $T_{(x,v)}(S\Sigma)$. By (2.16) and (3.38) we compute

$$\alpha \wedge \psi(x, v)(\xi_1, v_2, v_3) = -\langle v_1, v\rangle_g \langle v \times v_2, v_3\rangle_g = -\langle v_1, v\rangle_g.$$

Since $v \mapsto \langle v_1, v\rangle_g$ is an odd function on $S_x\Sigma$, we have

$$(\pi_{\Sigma*}(\alpha \wedge \psi))(x)(v_1) = \int_{S_x\Sigma} -\langle v_1, v\rangle_g \, d\,\mathrm{vol}(v) = 0.$$

3. If $\xi_1, \xi_2, \xi_3 \in T_{(x,v)}(S\Sigma)$ and ξ_2, ξ_3 are vertical, then by (2.16) we have

$$\alpha \wedge d\alpha(x, v)(\xi_1, \xi_2, \xi_3) = 0$$

which implies that $\pi_{\Sigma*}(\alpha \wedge d\alpha) = 0$.

4. Let $x \in \Sigma$ and v, v_2, v_3 be a positively oriented g-orthonormal basis of $T_x\Sigma$. Let $\xi = X(x, v), \xi_2, \xi_3$ be the horizontal lifts of v, v_2, v_3 to $T_{(x,v)}S\Sigma$; we treat v_2, v_3 as vertical vectors in $T_{(x,v)}S\Sigma$. Using (2.16) and (3.38), we compute

$$\begin{aligned}\alpha \wedge d\alpha \wedge d\alpha(x, v)(\xi, \xi_2, \xi_3, v_2, v_3)\\ = -2 = 2\psi \wedge \pi_\Sigma^*(d\,\mathrm{vol}_g)(x, v)(\xi, \xi_2, \xi_3, v_2, v_3).\end{aligned}$$

5. Using the exact sequence (2.27) and the fact that $\pi_{\Sigma*}(\alpha \wedge \psi) = 0$, we see that

$$[\alpha \wedge \psi]_{H^3} = c[\pi_\Sigma^*(d\,\mathrm{vol}_g)]_{H^3}$$

for some constant c. To determine c, note that $\alpha \wedge \psi \wedge \psi$ has the same integral over $S\Sigma$ as $c\psi \wedge \pi_\Sigma^*(d\,\mathrm{vol}_g)$. Since $\alpha \wedge \psi \wedge \psi = \alpha \wedge d\alpha \wedge d\alpha = 2\psi \wedge \pi_\Sigma^*(d\,\mathrm{vol}_g)$, we get $c = 2$. □

We also have the following identity relating the operators $d\alpha\wedge$ and $\psi\wedge$ on 1-forms in Ω_0^1:

Lemma 3.6 *For any 1-form β on $S\Sigma$ such that $\iota_X\beta = 0$, we have*

$$d\alpha \wedge \beta = \psi \wedge (\beta \circ \mathcal{I}), \tag{3.46}$$

where the 1-form $\beta \circ \mathcal{I}$ is defined by (3.36).

Proof It is easy to see that ι_X of both sides of (3.46) is equal to 0. It is thus enough to check that

$$d\alpha \wedge \beta(x, v)(\xi_1, \xi_2, \xi_3) = \psi \wedge (\beta \circ \mathcal{I})(x, v)(\xi_1, \xi_2, \xi_3) \tag{3.47}$$

for any three vectors ξ_1, ξ_2, ξ_3, each of which is either horizontal or vertical under the decomposition (2.15). Moreover, we may assume that the horizontal components of these vectors lie in the orthogonal complement $\{v\}^\perp$ to v in $T_x\Sigma$. It suffices to consider the following two cases:

Case 1: $\beta(x,v)(\xi) = \langle \xi_H, w_4\rangle_g$ for some $w_4 \in \{v\}^\perp$. By (3.30) and (3.41), both sides of (3.47) are equal to 0 unless two of ξ_1, ξ_2, ξ_3 are horizontal and one is vertical; we write

$$\xi_1 = (w_1, 0), \quad \xi_2 = (w_2, 0), \quad \xi_3 = (0, w_3),$$

where $w_j \in \{v\}^\perp$. We compute using (2.16), (3.30), and (3.38)

$$d\alpha \wedge \beta(x,v)(\xi_1,\xi_2,\xi_3) = \langle w_1, w_3\rangle_g \langle w_2, w_4\rangle_g - \langle w_2, w_3\rangle_g \langle w_1, w_4\rangle_g,$$
$$\psi \wedge (\beta \circ \mathcal{I})(x,v)(\xi_1,\xi_2,\xi_3) = \langle v \times w_1, w_2\rangle_g \langle v \times w_3, w_4\rangle_g$$

and (3.47) follows from (3.29).

Case 2: $\beta(x,v)(\xi) = \langle \xi_V, w_4\rangle_g$ for some $w_4 \in \{v\}^\perp$. By (3.30) and (3.41), both sides of (3.47) are equal to 0 unless two of ξ_1, ξ_2, ξ_3 are vertical and one is horizontal; we write

$$\xi_1 = (0, w_1), \quad \xi_2 = (0, w_2), \quad \xi_3 = (w_3, 0),$$

where $w_j \in \{v\}^\perp$. We compute using (2.16), (3.30), and (3.38)

$$d\alpha \wedge \beta(x,v)(\xi_1,\xi_2,\xi_3) = \langle w_2, w_3\rangle_g \langle w_1, w_4\rangle_g - \langle w_1, w_3\rangle_g \langle w_2, w_4\rangle_g,$$
$$\psi \wedge (\beta \circ \mathcal{I})(x,v)(\xi_1,\xi_2,\xi_3) = -\langle v \times w_1, w_2\rangle_g \langle v \times w_3, w_4\rangle_g$$

and (3.47) again follows from (3.29).

Alternatively, we may lift both sides of (3.46) to the frame bundle $\mathcal{F}\Sigma$: it suffices to consider the cases when β is replaced by one of the forms $U_i^{\pm *}$, in which case (3.46) is checked by a direct calculation using (3.12), (3.33), and (3.39). □

3.2.4 Characterization of all smooth flow-invariant 2-forms

We finally give

Lemma 3.7 *Assume that $u \in C^\infty(S\Sigma; \Omega^2)$ satisfies $\mathcal{L}_X u = 0$. Then u is a linear combination of $d\alpha$ and ψ.*

Proof Without loss of generality we assume that u is real valued. Since $d\alpha \wedge \psi = 0$ and $\psi \wedge \psi = d\alpha \wedge d\alpha$ by (3.43)–(3.44), we may subtract from u a linear combination of $d\alpha$ and ψ to make

$$\int_M \alpha \wedge d\alpha \wedge u = \int_M \alpha \wedge \psi \wedge u = 0. \tag{3.48}$$

We will show that under the condition (3.48) we have $u = 0$.

Since $\alpha \wedge d\alpha \wedge u$, $\alpha \wedge \psi \wedge u$, $\alpha \wedge u \wedge u$ are smooth 5-forms on $S\Sigma$ invariant under the geodesic flow, by Lemma 2.4 (we identify Ω^0 and Ω^5 via the volume form $d\,\mathrm{vol}_\alpha$) we have

$$\alpha \wedge d\alpha \wedge u = \alpha \wedge \psi \wedge u = 0, \quad \alpha \wedge u \wedge u = c\, d\,\mathrm{vol}_\alpha \tag{3.49}$$

for some constant $c \in \mathbb{R}$.

Next, $\iota_X u \in C^\infty(S\Sigma; \Omega^1_0)$ and $\mathcal{L}_X \iota_X u = 0$, so by (2.3) (similarly to the last step of the proof of Lemma 2.10) we get $\iota_X u = 0$. Also by (2.3) we obtain $u|_{E_u \times E_u} = 0$ and $u|_{E_s \times E_s} = 0$. Therefore, it is enough to show that $u|_{E_s \times E_u} = 0$.

Since $d\alpha$ is nondegenerate on $E_s \times E_u$ (as follows for instance from (2.16) and (3.6)), there exists unique smooth bundle homomorphism $A : E_s \to E_s$ such that

$$u(x,v)(\xi,\eta) = d\alpha(A(x,v)\xi,\eta) \quad \text{for all} \quad (x,v) \in S\Sigma,\ \xi \in E_s(x,v),\ \eta \in E_u(x,v).$$

It remains to show that $A = 0$.

Take any $(x,v) \in S\Sigma$, assume that v, w_1, w_2 is a positively oriented orthonormal basis of $T_x\Sigma$, and define using the horizontal/vertical decomposition and (3.6)

$$\xi_j = (w_j, -w_j) \in E_s(x,v), \quad \eta_j = (w_j, w_j) \in E_u(x,v), \quad j = 1,2.$$

Applying (3.49) to the vectors $X(x,v), \xi_1, \xi_2, \eta_1, \eta_2$ and using (2.16), (3.32), and (3.37), we get

$$\operatorname{tr} A(x,v) = 0, \quad A(x,v)^T = A(x,v), \quad \det A(x,v) = c, \tag{3.50}$$

where the transpose is with respect to the restriction of the Sasaki metric to $E_s(x,v)$.

If $c = 0$, then (3.50) implies that $A = 0$. Assume that $c \neq 0$, then by (3.50) we have $c < 0$ and A has eigenvalues $\pm\sqrt{-c}$. The eigenspace of $A(x,v)$ corresponding to the eigenvalue $\sqrt{-c}$ is a one-dimensional subspace of $E_s(x,v)$ depending continuously on (x,v). This is impossible since by restricting to a single fiber $S_x\Sigma \subset S\Sigma$ and projecting E_s onto the vertical space $\mathbf{V}$ we would obtain a continuous one-dimensional subbundle of the tangent space to the 2-sphere. $\square$

3.3 Resonant 1-forms and 3-forms

In this section we apply the properties of the 2-form ψ defined in (3.37) to determine the precise structure of resonant 1-forms on $M = S\Sigma$. Let us introduce some notation for (co-)resonant 1-forms (see (3.36) for the definition of $u \circ \mathcal{I}$)

$$\mathcal{C}_{(*)} := \mathrm{Res}^1_{0(*)} \cap \ker d, \quad \mathcal{C}_{\psi(*)} := \{u \circ \mathcal{I} \mid u \in \mathcal{C}_{(*)}\},$$

where the subscript $(*)$ means we either suppress the star or we include it, respectively corresponding to resonances or co-resonances; we apply this convention to other notions appearing in this section. We remark that the use of subscript ψ in $\mathcal{C}_\psi$ is motivated by the property $d\alpha \wedge \mathcal{C}_\psi = \psi \wedge \mathcal{C}$ demonstrated in (3.58) below; in fact we initially used this relation as the definition of $\mathcal{C}_\psi$, before coming to the interpretation via the map $\mathcal{I}$.

Since $\mathcal{I}$ is invariant under the geodesic flow by (3.31) and annihilates X, we have

$$\mathcal{C}_{\psi(*)} \subset \mathrm{Res}^1_{0(*)} .$$

By Lemma 2.8 and (2.28) we have

$$\dim \mathcal{C}_{(*)} = \dim \mathcal{C}_{\psi(*)} = b_1(\Sigma). \tag{3.51}$$

We next show that *all* resonant 1-forms lie in the direct sum $\mathcal{C} \oplus \mathcal{C}_\psi$. This is done in Lemma 3.9 below but first we need

Lemma 3.8 *Assume that $u \in \mathrm{Res}^1_0$. Then u is totally unstable in the sense of Definition 3.2. Similarly, if $u \in \mathrm{Res}^1_{0*}$, then u is totally stable.*

Remark Lemma 3.8 was previously proved by Küster–Weich [44, §2.6].

Proof We consider the case $u \in \mathrm{Res}^1_0$, with the case $u \in \mathrm{Res}^1_{0*}$ handled in the same way.

We first show that u is unstable in the sense of Definition 3.2. For that it is enough to prove that $u(Y) = 0$ for any $Y \in C^\infty(M; E_0 \oplus E_u)$. Since $\iota_X u = 0$, we may assume that $Y \in C^\infty(M; E_u)$. By the integral formula (2.29) for the Pollicott–Ruelle resolvent $R_{k,0}(\lambda)$, we have for $\operatorname{Im} \lambda \gg 1$ and any $w \in C^\infty(M; \Omega^1_0)$, $\rho \in M$

$$\langle R_{1,0}(\lambda) w, Y\rangle(\rho) = i \int_0^\infty e^{i\lambda t} \langle w(\varphi_{-t}(\rho)), d\varphi_{-t}(\rho) Y(\rho)\rangle \, dt.$$

Since Y is a section of the unstable bundle, by (3.7) we have $|\langle w(\varphi_{-t}(\rho)), d\varphi_{-t}(\rho)Y(\rho)\rangle| \le Ce^{-t}$ for some constant C and all $t \ge 0$, $\rho \in M$. Therefore, the integral above converges uniformly in ρ for $\operatorname{Im}\lambda > -1$, which implies that $\lambda \mapsto \langle R_{1,0}(\lambda)w, Y\rangle$ is holomorphic in $\operatorname{Im}\lambda > -1$. If $\Pi_{1,0}$ is the projector appearing in the Laurent expansion of $R_{1,0}(\lambda)$ at $\lambda = 0$, defined in (2.36), then $\iota_Y \Pi_{1,0} = 0$. Since Res_0^1 is contained in the range of $\Pi_{1,0}$, we get $u(Y) = 0$ as needed.

We now analyze du. First of all, $\iota_X du = 0$ since $u \in \mathrm{Res}_0^1$. Next, we have $du|_{E_u\times E_u} = 0$. This can be seen by following the argument above, or using that $u(Y) = 0$ for any $Y \in C^\infty(M; E_0 \oplus E_u)$, the identity (3.11), and the fact that the class $C^\infty(M; E_0 \oplus E_u)$ is closed under Lie brackets (as follows from (3.23)).

It remains to show that $du|_{E_u\times E_s} = 0$. Let ζ be the restriction of du to $E_u \times E_s$, considered as a section in $\mathcal{D}'_{E_u^*}(M; E_s^* \otimes E_u^*)$. (Here E_s^*, E_u^* are dual to E_u, E_s as in (2.4).) We endow $E_s^* \otimes E_u^*$ with the inner product which is the tensor product of the dual Sasaski metrics on E_s^* and E_u^*. The operator

$$P := -i\mathcal{L}_X : C^\infty(M; E_s^* \otimes E_u^*) \to C^\infty(M; E_s^* \otimes E_u^*)$$

is formally self-adjoint as follows from (3.7), and $P\zeta = 0$. Then by [21, Lemma 2.3] the section ζ is in C^∞.

Let us now consider $\zeta = du|_{E_u\times E_s}$ as a smooth 2-form on M (i.e. $\iota_X\zeta = 0$, $\zeta|_{E_u\times E_u} = \zeta|_{E_s\times E_s} = 0$, and $\zeta|_{E_u\times E_s} = du|_{E_u\times E_s}$), then $\mathcal{L}_X\zeta = 0$ and by Lemma 3.7 we see that $\zeta = a\,d\alpha + b\,\psi$ for some constants a, b. We claim that $a = b = 0$. This follows from (3.43)–(3.44) and the identities

$$\int_M \alpha \wedge d\alpha \wedge \zeta = \int_M \alpha \wedge d\alpha \wedge du = 0, \tag{3.52}$$

$$\int_M \alpha \wedge \psi \wedge \zeta = \int_M \alpha \wedge \psi \wedge du = 0. \tag{3.53}$$

Here the first identity in each line follows from the fact that $d\alpha|_{E_u\times E_u} = \psi|_{E_u\times E_u} = 0$ (which can be verified using (2.16), (3.6), and (3.37)). More precisely, it suffices to observe that $\alpha \wedge d\alpha \wedge (du - \zeta)$ and $\alpha \wedge d\psi \wedge (du - \zeta)$ are pointwise zero, as $du - \zeta$ is supported on $E_s \times E_s$ by definition. The second identity in each line follows by integration by parts and the fact that $d\alpha \wedge d\alpha \wedge u = d\alpha \wedge \psi \wedge u = 0$ (as ι_X of both of these 5-forms is equal to 0). Now, $a = b = 0$ implies that $\zeta = 0$, that is $du|_{E_u\times E_s} = 0$ as needed. □

We are now ready to prove

Lemma 3.9 *We have* $\mathcal{C}_{(*)} \cap \mathcal{C}_{\psi(*)} = \{0\}$ *and* $\mathrm{Res}^1_{0(*)} = \mathcal{C}_{(*)} \oplus \mathcal{C}_{\psi(*)}$.

Proof We consider the case of Res_0^1, with Res^1_{0*} handled similarly. We need to prove that each $u \in \mathrm{Res}_0^1$ can be expressed uniquely as a sum of elements in $\mathcal{C}$ and $\mathcal{C}_\psi$. By Lemma 3.8, u is totally unstable. By Lemma 3.3, the lift of u to $S\mathbb{H}^3$ has the form

$$\pi_\Gamma^* u = B_-^* w \quad \text{for some } \Gamma\text{-invariant} \quad w \in \mathcal{D}'(\mathbb{S}^2; \Omega^1),$$

where $\Gamma \subset \mathrm{SO}_+(1,3)$ is the discrete subgroup such that $\Sigma = \Gamma\backslash\mathbb{H}^3$. Take the Hodge decomposition of w:

$$w = w_1 + \star w_2 \quad \text{where} \quad w_1, w_2 \in \mathcal{D}'(\mathbb{S}^2; \Omega^1), \quad dw_1 = dw_2 = 0. \tag{3.54}$$

Since Γ acts on $\mathbb{S}^2$ by orientation preserving conformal transformations L_γ (see (3.25)), its action commutes with the Hodge star $\star$. Since $H^1(\mathbb{S}^2) = 0$, the Hodge decomposition above is unique, which implies that w_1, w_2 are Γ-invariant. Applying Lemma 3.3 again and using (3.28), we see that

$$B_-^* w_j = \pi_\Gamma^* u_j \quad \text{for some} \quad u_1, u_2 \in \mathcal{D}'_{E_u^*}(M; \Omega_0^1).$$

Since $dw_j = 0$, we have $du_j = 0$, which together with the fact that $\iota_X u_j = 0$ shows that $u_1, u_2 \in \mathcal{C}$. Finally, by (3.35) and (3.54) we may express u uniquely as

$$u = u_1 - u_2 \circ \mathcal{I}, \quad u_1 \in \mathcal{C}, \quad u_2 \circ \mathcal{I} \in \mathcal{C}_\psi,$$

finishing the proof. □

The next lemma establishes semisimplicity on resonant 1-forms:

Lemma 3.10 *The semisimplicity condition* (2.41) *holds at* $\lambda_0 = 0$ *for the operators* $P_{1,0}$ *and* $P_{3,0}$.

Proof By (2.45) it suffices to establish semisimplicity for $P_{1,0}$. By Lemma 2.2 it suffices to show that the pairing $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ on $\mathrm{Res}_0^1 \times \mathrm{Res}_{0*}^3$ is nondegenerate. Recall from (2.49) that $\mathrm{Res}_{0*}^3 = d\alpha \wedge \mathrm{Res}_{0*}^1$. By Lemma 2.10 the pairing $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ is nondegenerate on $\mathcal{C} \times (d\alpha \wedge \mathcal{C}_*)$. Therefore, it suffices to show the following diagonal structure of the pairing with respect to the decompositions $\mathrm{Res}_{0(*)}^1 = \mathcal{C}_{(*)} \oplus \mathcal{C}_{\psi(*)}$ established in Lemma 3.9:

$$\langle\!\langle u, d\alpha \wedge u_* \rangle\!\rangle = 0 \quad \text{for all} \quad u \in \mathcal{C}, \ u_* \in \mathcal{C}_{\psi *} \tag{3.55}$$

$$\langle\!\langle u, d\alpha \wedge u_* \rangle\!\rangle = 0 \quad \text{for all} \quad u \in \mathcal{C}_\psi, \ u_* \in \mathcal{C}_* \tag{3.56}$$

$$\langle\!\langle u, d\alpha \wedge u_* \rangle\!\rangle = - \langle\!\langle u \circ \mathcal{I}, d\alpha \wedge (u_* \circ \mathcal{I}) \rangle\!\rangle \quad \text{for all} \quad u \in \mathcal{C}, \ u_* \in \mathcal{C}_*. \tag{3.57}$$

We first show (3.55). By Lemma 2.5 and (2.25) we may write

$$\begin{aligned} u = \pi_\Sigma^* w + df \quad &\text{for some} \quad w \in C^\infty(\Sigma; \Omega^1),\ dw = 0,\\ &f \in \mathcal{D}'_{E_u^*}(M; \mathbb{C}),\\ u_* \circ \mathcal{I} = \pi_\Sigma^* w_* + df_* \quad &\text{for some} \quad w_* \in C^\infty(\Sigma; \Omega^1),\ dw_* = 0,\\ &f_* \in \mathcal{D}'_{E_s^*}(M; \mathbb{C}). \end{aligned}$$

We now compute

$$\begin{aligned} \langle\langle u, d\alpha \wedge u_* \rangle\rangle &= \langle\langle u, \psi \wedge (u_* \circ \mathcal{I}) \rangle\rangle\\ &= \int_M \alpha \wedge \psi \wedge (\pi_\Sigma^* w + df) \wedge (\pi_\Sigma^* w_* + df_*)\\ &= \int_M \alpha \wedge \psi \wedge \pi_\Sigma^* (w \wedge w_*) = -\int_\Sigma \pi_{\Sigma *}(\alpha \wedge \psi) \wedge w \wedge w_*\\ &= 0. \end{aligned}$$

Here the first equality used Lemma 3.6. The third equality used integration by parts and (3.44). The fourth equality used (2.20) and (2.23), with the negative sign explained in the paragraph following (2.20). The fifth equality used part 2 of Lemma 3.5. A similar argument proves (3.56).

Finally, to show (3.57) we compute

$$\begin{aligned} \langle\langle u, d\alpha \wedge u_* \rangle\rangle = \langle\langle u, \psi \wedge (u_* \circ \mathcal{I}) \rangle\rangle &= \langle\langle \psi \wedge u, u_* \circ \mathcal{I} \rangle\rangle\\ &= -\langle\langle d\alpha \wedge (u \circ \mathcal{I}), u_* \circ \mathcal{I} \rangle\rangle \end{aligned}$$

using Lemma 3.6 and the fact that $u \circ \mathcal{I} \circ \mathcal{I} = -u$. □

We finally discuss the properties of the maps $\pi_{3(*)} : \mathrm{Res}^3_{0(*)} \to H^3(M; \mathbb{C})$; as explained at the top of § 3.3, recall that the subscript $(*)$ denotes the corresponding resonance or co-resonance space, so we can include both in the discussion. Recall that all forms in $\mathrm{Res}^3_{0(*)}$ are closed by Lemma 2.9 and $\mathrm{Res}^3_{0(*)} = d\alpha \wedge \mathrm{Res}^1_{0(*)}$ by (2.45), (2.49). Moreover, by Lemma 3.6 and the definition of $\mathcal{C}_{\psi(*)}$

$$d\alpha \wedge \mathcal{C}_{\psi(*)} = \psi \wedge \mathcal{C}_{(*)}. \tag{3.58}$$

We have $\pi_{3(*)}(d\alpha \wedge \mathcal{C}_{(*)}) = 0$. Assume now that $u \in \mathcal{C}_\psi$, then $u \circ \mathcal{I} \in \mathcal{C}$, and by Lemma 2.5 and (2.25) we may write

$$\begin{aligned} u \circ \mathcal{I} = \pi_\Sigma^* w + df \quad &\text{for some} \quad w \in C^\infty(\Sigma; \Omega^1),\ dw = 0,\\ &f \in \mathcal{D}'_{E_u^*}(M; \mathbb{C}). \end{aligned}$$

Wedging with ψ, taking $\pi_{\Sigma *}$, and using (2.22)–(2.23), part 1 of Lemma 3.5, and Lemma 3.6 we get

$$\pi_{\Sigma *}\pi_3(d\alpha \wedge u) = \pi_{\Sigma *}(\psi \wedge \pi_\Sigma^* w) = -4\pi w,$$

which (together with a similar argument for coresonant states) immediately shows that

$$\pi_{\Sigma *}\pi_{3(*)} : d\alpha \wedge \mathcal{C}_{\psi(*)} \to H^1(\Sigma; \mathbb{C}) \text{ is an isomorphism.} \tag{3.59}$$

This implies that

$$\ker \pi_{3(*)} = d\alpha \wedge \mathcal{C}_{(*)} \tag{3.60}$$

and so by (2.27) the range of $\pi_{3(*)}$ is a codimension 1 subspace of $H^3(M; \mathbb{C})$ which does not contain $[\pi_\Sigma^* d\,\mathrm{vol}_g]_{H^3}$.

Summarizing the contents of § 3.3, we note that the second item of Theorem 2 follows from (3.58), Lemma 3.9, Lemma 2.8, and (2.28), the third item by Lemma 3.10, and the sixth item by the discussion in the preceding paragraph.

3.4 Resonant 2-forms

We next study resonant 2-forms. We start with

Lemma 3.11 *We have $d(\mathrm{Res}^2_{0(*)}) = 0$ and $\ker \pi_{2(*)} = \mathbb{C}d\alpha \oplus d\mathcal{C}_{\psi(*)}$ has dimension $b_1(\Sigma) + 1$.*

Proof We consider the case of resonant 2-forms, with the case of coresonant 2-forms handled similarly. We first show that $d(\mathrm{Res}^2_0) = 0$, arguing similarly to the proof of Lemma 2.11. Take $\zeta \in \mathrm{Res}^2_0$, then by the definition (2.61) of π_3 we have $d\zeta \in \ker \pi_3$. Thus by (3.60), $d\zeta = d\alpha \wedge u$ for some $u \in \mathcal{C}$. Take arbitrary $u_* \in \mathcal{C}_*$, then precisely as in (2.65)

$$\langle\!\langle u, d\alpha \wedge u_* \rangle\!\rangle = \int_M \alpha \wedge d\zeta \wedge u_* = \int_M d\alpha \wedge \zeta \wedge u_* = 0.$$

Now Lemma 2.10 implies that $u = 0$ and thus $d\zeta = 0$ as needed.

Next, if $u \in \mathcal{C}_\psi$, then using the same argument of integration by parts as in (3.52) yields

$$\int_M \alpha \wedge d\alpha \wedge du = 0.$$

Therefore, du cannot be a nonzero multiple of $d\alpha$, which means that $\mathbb{C}d\alpha \cap d\mathcal{C}_\psi = \{0\}$. We have $d\alpha \in \ker \pi_2$ and by Lemma 2.6 we have $d\mathcal{C}_\psi \subset \ker \pi_2$ as well.

It remains to show that $\ker \pi_2 \subset \mathbb{C}d\alpha \oplus d\mathcal{C}_\psi$. By Lemma 2.6, $\ker \pi_2$ is contained in $d(\mathrm{Res}^{1,\infty})$. By (2.43) and Lemmas 2.4, 3.9, and 3.10, we have $\mathrm{Res}^{1,\infty} = \mathbb{C}\alpha \oplus \mathcal{C} \oplus \mathcal{C}_\psi$. Then $d(\mathrm{Res}^{1,\infty}) = \mathbb{C}d\alpha \oplus d\mathcal{C}_\psi$, which finishes the proof. □

We next establish the following auxiliary result:

Lemma 3.12 *Assume that $\eta \in C^\infty(\Sigma; \Omega^2)$, $d\eta = 0$, and $w \in \mathcal{D}'_{E_u^*}(M; \Omega^1)$ satisfy*

$$\iota_X(\pi_\Sigma^* \eta + dw) = 0. \tag{3.61}$$

Then η is exact.

Remark The proof of Lemma 3.12 uses the 2-form ψ which is only available in the case of constant curvature. By contrast, Lemma 3.12 is false if Res^1_{0*} consists of closed forms and $b_1(\Sigma) > 0$; in fact the equation (3.61) then has a solution $w \in \mathcal{D}'_{E_u^*}(M; \Omega^1_0)$ for any closed η. Indeed, in this case $\langle\!\langle \iota_X \pi_\Sigma^* \eta, d\alpha \wedge u_* \rangle\!\rangle = \int_M \pi_\Sigma^* \eta \wedge d\alpha \wedge u_* = 0$ for any $u_* \in \mathrm{Res}^1_{0*}$ by integration by parts, and the existence of w now follows from Lemma 2.1.

Proof Put $\zeta := \pi_\Sigma^* \eta + dw$, then $\iota_X \zeta = 0$. Take arbitrary closed $\eta_* \in C^\infty(\Sigma; \Omega^1)$ and put $u_* := \pi_{1*}^{-1}([\pi_\Sigma^* \eta_*]_{H^1}) \in \mathcal{C}_*$. Then $u_* = \pi_\Sigma^* \eta_* + dw_*$ for some $w_* \in \mathcal{D}'_{E_s^*}(M; \mathbb{C})$. We compute

$$\begin{aligned} 0 &= \int_M \psi \wedge \zeta \wedge u_* = \int_M \psi \wedge \pi_\Sigma^* \eta \wedge \pi_\Sigma^* \eta_* \\ &= -\int_\Sigma (\pi_{\Sigma*}\psi) \eta \wedge \eta_* = 4\pi \int_\Sigma \eta \wedge \eta_*. \end{aligned}$$

Here the first equality follows since the 5-form under the integral lies in the kernel of ι_X. The second equality follows by integration by parts, using that ψ, η, η_* are closed. The third equality follows from (2.20) and (2.23). The fourth equality follows from part 1 of Lemma 3.5.

We see that $\eta \wedge \eta_*$ integrates to 0 on Σ for any closed smooth 1-form η_*. This implies that η is exact; indeed, we can reduce to the case when η is harmonic and take η_* to be the Hodge star of η (we note that this final argument is just a form of Poincaré duality). □

We now describe the space of resonant 2-forms (recalling the convention (∗) at the top of § 3.3):

Lemma 3.13 *The range of* $\pi_{2(*)}$ *is equal to* $\mathbb{C}[\psi]_{H^2}$*, and* $\mathrm{Res}^2_{0(*)} = \mathbb{C}d\alpha \oplus \mathbb{C}\psi \oplus d\mathcal{C}_{\psi(*)}$. *In particular,* $\dim \mathrm{Res}^2_{0(*)} = b_1(\Sigma) + 2$.

Proof We consider the case of resonant 2-forms, with the case of coresonant 2-forms handled similarly. First of all, $\psi \in \mathrm{Res}^2_0$, thus $[\psi]_{H^2} = \pi_2(\psi)$ is in the range of π_2. Next, by (2.26) and part 1 of Lemma 3.5 we have

$$H^2(M; \mathbb{C}) = \pi_\Sigma^* H^2(\Sigma; \mathbb{C}) \oplus \mathbb{C}[\psi]_{H^2}.$$

To show that the range of π_2 is equal to $\mathbb{C}[\psi]_{H^2}$, it remains to prove that the intersection of this range with $\pi_\Sigma^* H^2(\Sigma; \mathbb{C})$ is trivial. Take $u \in \mathrm{Res}^2_0$ and assume that $\pi_2(u) = [\pi_\Sigma^* \eta]_{H^2}$ for some $\eta \in C^\infty(\Sigma; \Omega^2)$, $d\eta = 0$. Then $u = \pi_\Sigma^* \eta + dw$ for some $w \in \mathcal{D}'_{E_u^*}(M; \Omega^1)$. Since $\iota_X u = 0$, Lemma 3.12 implies that η is exact, that is $\pi_2(u) = 0$ as needed.

Finally, the statement that $\mathrm{Res}^2_0 = \mathbb{C}d\alpha \oplus \mathbb{C}\psi \oplus d\mathcal{C}_\psi$ follows from the first statement of this lemma together with Lemma 3.11. □

The next lemma describes the space of generalized resonant states $\mathrm{Res}^{2,2}_0$ (see (2.39) and §2.3.3). It implies in particular that the operator $P_{2,0}$ does not satisfy the semisimplicity condition (2.41), assuming that $b_1(\Sigma) > 0$:

Lemma 3.14 *1. The pairing* $\langle\!\langle \bullet, \bullet \rangle\!\rangle$ *on* $\mathrm{Res}^2_0 \times \mathrm{Res}^2_{0*}$ *has the following form in the decomposition of Lemma 3.13:*

$$\langle\!\langle d\alpha, d\alpha \rangle\!\rangle = \langle\!\langle \psi, \psi \rangle\!\rangle = \mathrm{vol}_\alpha(M) > 0, \quad \langle\!\langle d\alpha, \psi \rangle\!\rangle = \langle\!\langle \psi, d\alpha \rangle\!\rangle = 0, \tag{3.62}$$

$$\langle\!\langle \zeta, \zeta_* \rangle\!\rangle = 0 \quad \textit{for all} \quad \zeta \in d\mathcal{C}_\psi, \ \zeta_* \in \mathrm{Res}^2_{0*}, \tag{3.63}$$

$$\langle\!\langle \zeta, \zeta_* \rangle\!\rangle = 0 \quad \textit{for all} \quad \zeta \in \mathrm{Res}^2_0, \ \zeta_* \in d\mathcal{C}_{\psi*}. \tag{3.64}$$

2. The range of the map

$$\mathcal{L}_X : \mathrm{Res}^{2,2}_{0(*)} \to \mathrm{Res}^2_{0(*)} \tag{3.65}$$

is equal to $d\mathcal{C}_{\psi(*)}$. *We have* $\dim \mathrm{Res}^{2,2}_{0(*)} = 2b_1(\Sigma) + 2$.

Proof 1. The identities (3.62) follow immediately from (3.43) and (3.44). We next show (3.63), with (3.64) proved similarly. Let $\zeta = du$ where $u \in \mathcal{C}_\psi$. We compute

$$\langle\!\langle \zeta, \zeta_* \rangle\!\rangle = \int_M d\alpha \wedge u \wedge \zeta_* = 0.$$

Here in the first equality we integrate by parts and use that $d\zeta_* = 0$ by Lemma 3.11. The second equality follows from the fact that $\iota_X(d\alpha \wedge u \wedge \zeta_*) = 0$.

2. We consider generalized resonant states, with generalized coresonant states handled similarly. First, assume that $\zeta \in \mathrm{Res}_0^{2,2}$, then $\mathcal{L}_X\zeta \in \mathrm{Res}_0^2$. Moreover, since the transpose of $\mathcal{L}_X$ is equal to $-\mathcal{L}_X$ (see §2.3.4) we compute

$$\langle\!\langle \mathcal{L}_X\zeta, \zeta_*\rangle\!\rangle = -\langle\!\langle \zeta, \mathcal{L}_X\zeta_*\rangle\!\rangle = 0 \quad \text{for all} \quad \zeta_* \in \mathrm{Res}_{0*}^2. \tag{3.66}$$

Using this for $\zeta_* = d\alpha$ and $\zeta_* = \psi$ together with (3.62)–(3.63), we see that $\mathcal{L}_X\zeta \in d\mathcal{C}_\psi$. That is, the range of the map (3.65) is contained in $d\mathcal{C}_\psi$.

Now, take arbitrary $\eta \in d\mathcal{C}_\psi$. By (3.63), we have $\langle\!\langle \eta, \zeta_*\rangle\!\rangle = 0$ for all $\zeta_* \in \mathrm{Res}_{0*}^2$. Then by Lemma 2.1 there exists $\zeta \in \mathcal{D}'_{E_u^*}(M; \Omega_0^2)$ such that $\mathcal{L}_X\zeta = \eta$. Since $\eta \in \mathrm{Res}_0^2$, we see that $\zeta \in \mathrm{Res}_0^{2,2}$. This shows that the range of the map (3.65) contains $d\mathcal{C}_\psi$.

Finally, the equality $\dim \mathrm{Res}_0^{2,2} = 2b_1(\Sigma) + 2$ follows from Lemma 3.13 and the fact that the kernel of the map (3.65) is given by Res_0^2. □

We finally show that there are no higher degree Jordan blocks, completing the analysis of the generalized resonant states of $P_{2,0}$ at 0:

Lemma 3.15 *We have* $\mathrm{Res}_{0(*)}^{2,\infty} = \mathrm{Res}_{0(*)}^{2,2}$.

Proof We consider the case of generalized resonant states, with generalized coresonant states handled similarly. It suffices to prove that $\mathrm{Res}_0^{2,3} \subset \mathrm{Res}_0^{2,2}$. Take $\eta \in \mathrm{Res}_0^{2,3}$ and put $\zeta := \mathcal{L}_X\eta \in \mathrm{Res}_0^{2,2}$. Exactly as in (3.66), the pairing of ζ with any element of Res_{0*}^2 is equal to 0. In particular

$$\langle\!\langle \zeta, du_*\rangle\!\rangle = 0 \quad \text{for all} \quad u_* \in \mathrm{Res}_{0*}^1.$$

By part 2 of Lemma 3.14, we have $\mathcal{L}_X\zeta = du$ for some $u \in \mathcal{C}_\psi$. Put

$$\omega := d(\zeta + \alpha \wedge u) \in \mathcal{D}'_{E_u^*}(M; \Omega^3).$$

Then $\iota_X\omega = \iota_X d\zeta - du = 0$. Since ω is exact we have $\mathcal{L}_X\omega = 0$ and moreover that $\omega \in \ker \pi_3 \subset \mathrm{Res}_0^3$. By (3.60), we then have $\omega \in d\alpha \wedge \mathcal{C}$.

We now compute

$$0 = \langle\!\langle \zeta, du_*\rangle\!\rangle = -\int_M \alpha \wedge d\zeta \wedge u_* = \langle\!\langle u, d\alpha \wedge u_*\rangle\!\rangle - \langle\!\langle \omega, u_*\rangle\!\rangle.$$

Here in the second equality we integrated by parts and used that the 5-form $d\alpha \wedge \zeta \wedge u_*$ lies in the kernel of ι_X and thus equals 0. Using the identities (3.55)–(3.57) and Lemma 2.10, recalling that $u \in \mathcal{C}_\psi$, $\omega \in d\alpha \wedge \mathcal{C}$, and using that

u_* can be chosen as an arbitrary element of $\mathcal{C}_*$ or $\mathcal{C}_{\psi *}$, we see that $u = 0$ and $\omega = 0$. Just using that $u = 0$ implies $\mathcal{L}_X^2 \eta = \mathcal{L}_X \zeta = 0$, that is $\eta \in \mathrm{Res}_0^{2,2}$ as needed. ☐

3.5 Relation to harmonic forms

In this section we show that pushforwards of elements of $\mathrm{Res}_0^3 = d\alpha \wedge (\mathcal{C} \oplus \mathcal{C}_\psi)$ to the base Σ are harmonic 1-forms. Recall that a 1-form h is called harmonic if $dh = 0$ and $d \star h = 0$, where $\star$ is the Hodge star on (Σ, g). We will denote the set of such forms by $\mathcal{H}^1(\Sigma)$. We start with the following identity:

Lemma 3.16 *Assume that $u \in \mathcal{D}'_{E_u^*}(M; \Omega_0^1)$ is unstable in the sense of Definition 3.2 and $\beta \in C^\infty(\Sigma; \Omega^1)$. Then*

$$\psi \wedge u \wedge \pi_\Sigma^*(\star\beta) = -\alpha \wedge d\alpha \wedge u \wedge \pi_\Sigma^* \beta, \tag{3.67}$$
$$d\alpha \wedge u \wedge \pi_\Sigma^*(\star\beta) = \alpha \wedge \psi \wedge u \wedge \pi_\Sigma^* \beta. \tag{3.68}$$

Proof We first show (3.67). Take arbitrary $(x, v) \in M = S\Sigma$ and assume that v, w_1, w_2 is a positively oriented g-orthonormal basis of $T_x\Sigma$. It suffices to prove that

$$\begin{aligned}(\psi \wedge u \wedge \pi_\Sigma^*(\star\beta))(x, v)(X, \xi_1, \xi_2, \xi_3, \xi_4)\\ = -(\alpha \wedge d\alpha \wedge u \wedge \pi_\Sigma^*\beta)(x, v)(X, \xi_1, \xi_2, \xi_3, \xi_4)\end{aligned} \tag{3.69}$$

where we write in terms of the horizontal/vertical decomposition (2.15)

$$X = (v, 0), \quad \xi_1 = (w_1, 0), \quad \xi_2 = (w_2, 0), \quad \xi_3 = (0, w_1), \quad \xi_4 = (0, w_2).$$

Using (3.38), (3.41), the fact that $d\pi_\Sigma(x, v)(\xi_H, \xi_V) = \xi_H$, the condition $\iota_X u = 0$, and the identities

$$(\star\beta)(x)(v, w_1) = \beta(x)(w_2), \quad (\star\beta)(x)(v, w_2) = -\beta(x)(w_1)$$

we see that the left-hand side of (3.69) is equal to

$$-u(x, v)(\xi_1)\beta(x)(w_1) - u(x, v)(\xi_2)\beta(x)(w_2).$$

Using (2.16), we next see that the right-hand side of (3.69) is equal to

$$u(x, v)(\xi_3)\beta(x)(w_1) + u(x, v)(\xi_4)\beta(x)(w_2).$$

It remains to note that by (3.6) the vectors $\xi_1 + \xi_3$ and $\xi_2 + \xi_4$ lie in $E_u(x, v)$ and thus $u(x, v)(\xi_1 + \xi_3) = u(x, v)(\xi_2 + \xi_4) = 0$ since u is unstable.

The identity (3.68) is verified by a similar calculation, or simply by applying (3.67) to $u \circ \mathcal{I}$ and using Lemma 3.6 and the fact that $u \circ \mathcal{I} \circ \mathcal{I} = -u$. $\square$

We can now prove item 7 of Theorem 2:

Lemma 3.17 *The map $\pi_{\Sigma*}$ annihilates $d\alpha \wedge \mathcal{C}_{(*)}$ and it is an isomorphism from $d\alpha \wedge \mathcal{C}_{\psi(*)}$ onto the space $\mathcal{H}^1(\Sigma)$. In particular, by Lemma 3.9 we have $\pi_{\Sigma*} : d\alpha \wedge \operatorname{Res}^1_{0(*)} \to \mathcal{H}^1(\Sigma)$.*

Proof We consider the case of resonant 3-forms, with coresonant 3-forms handled similarly (using a version of Lemma 3.16 for stable 1-forms). We first show that for any $u \in \mathcal{C}$, the push-forwards to Σ of $d\alpha \wedge u$ and $\psi \wedge u$ are coclosed, that is

$$d \star \pi_{\Sigma*}(d\alpha \wedge u) = 0, \quad d \star \pi_{\Sigma*}(\psi \wedge u) = 0. \tag{3.70}$$

To show the first equality in (3.70), it suffices to prove that

$$\int_\Sigma \pi_{\Sigma*}(d\alpha \wedge u) \wedge \star df = 0 \quad \text{for all} \quad f \in C^\infty(\Sigma; \mathbb{C}).$$

Using (2.20) and (2.23), we compute this integral as

$$\begin{aligned} -\int_M d\alpha \wedge u \wedge \pi_\Sigma^*(\star df) &= -\int_M \alpha \wedge \psi \wedge u \wedge d(\pi_\Sigma^* f) \\ &= \int_M \pi_\Sigma^* f\, d\alpha \wedge \psi \wedge u = 0 \end{aligned}$$

Here in the first equality we used (3.68), where u is unstable by Lemma 3.8. In the second equality we integrated by parts and used that $d\psi = 0$ and $du = 0$. In the third equality we used that ι_X of the 5-form under the integral is equal to 0. The second equality in (3.70) is proved similarly, using (3.67) instead of (3.68).

Next, by (2.22), since all forms in $d\alpha \wedge \mathcal{C}$ are exact, their pushforwards to Σ are exact as well. Since these pushforwards are also coclosed, we get $\pi_{\Sigma*}(d\alpha \wedge \mathcal{C}) = 0$. Similarly, all forms in $d\alpha \wedge \mathcal{C}_\psi = \psi \wedge \mathcal{C}$ are closed, so their pushforwards are closed as well; since these pushforwards are also coclosed, we get $\pi_{\Sigma*}(d\alpha \wedge \mathcal{C}_\psi) \subset \mathcal{H}^1(\Sigma)$.

Finally, by (3.59) we see that $\pi_{\Sigma*}$ is an isomorphism from $d\alpha \wedge \mathcal{C}_\psi$ onto $\mathcal{H}^1(\Sigma)$. $\square$

We finally remark that for any 1-form $u \in \mathcal{D}'(M; \Omega^1)$ we have

$$\pi_{\Sigma*}(\alpha \wedge u) = 0. \tag{3.71}$$

Indeed, by (2.16) we see that α, and thus $\alpha \wedge u$, vanish when restricted to the tangent spaces of the fibers $S_x\Sigma$. From (3.71) and (2.22) we get for any $u \in \mathcal{D}'(M; \Omega^1)$

$$\pi_{\Sigma*}(d\alpha \wedge u) = \pi_{\Sigma*}(\alpha \wedge du). \tag{3.72}$$

4 Contact perturbations of geodesic flows on hyperbolic 3-manifolds

Let $M = S\Sigma$ where (Σ, g) is a hyperbolic 3-manifold and α_0 be the contact form on M corresponding to the geodesic flow on Σ, see §§2.2,3.1. In this section we study Pollicott–Ruelle resonances at $\lambda = 0$ for perturbations of α_0. Ultimately, we will study perturbations of the metric, but via perturbations of the contact form. In particular, we give the proof of Theorem 1 in §4.4 below, relying on Theorem 5 (in §5) and Proposition A.1 proved later.

Let

$$\alpha_\tau \in C^\infty(M; T^*M), \quad \tau \in (-\varepsilon, \varepsilon)$$

be a family of 1-forms depending smoothly on τ. We may shrink $\varepsilon > 0$ so that each α_τ is a contact form on M and the corresponding Reeb vector field

$$X_\tau \in C^\infty(M; TM)$$

is Anosov; the latter follows from stability of the Anosov condition under perturbations (see for instance [23, Corollary 5.1.12] or [41, Corollary 6.4.7] for the related case of Anosov diffeomorphisms).

We will use first variation methods, introducing the 1-form

$$\beta := \partial_\tau \alpha_\tau|_{\tau=0} \in C^\infty(M; \Omega^1).$$

We use the subscript or superscript (τ) to refer to the objects corresponding to the contact manifold (M, α_τ) and the flow $\varphi_t^{(\tau)} := e^{tX_\tau}$. For example, we use the operators (see §2.3)

$$P_k^{(\tau)} = -i\mathcal{L}_{X_\tau}, \quad P_{k,0}^{(\tau)}, \quad R_k^{(\tau)}(\lambda), \quad \Pi_k^{(\tau)} := \Pi_k^{(\tau)}(0),$$

the spaces of (generalized) resonant states at $\lambda = 0$

$$\mathrm{Res}_{(\tau)}^{k,\ell}, \quad \mathrm{Res}_{0(\tau)}^{k,\ell}, \quad \mathrm{Res}_{(\tau)}^{k}, \quad \mathrm{Res}_{0(\tau)}^{k},$$

and the algebraic multiplicities of 0 as a resonance of the operators $P_k^{(\tau)}$, $P_{k,0}^{(\tau)}$

$$m_k^{(\tau)}(0), \quad m_{k,0}^{(\tau)}(0).$$

When we omit τ this means that we are considering the unperturbed hyperbolic case $\tau = 0$, that is

$$\alpha := \alpha_0, \quad P_k := P_k^{(0)}, \quad R_k := R_k^{(0)}, \quad \mathrm{Res}^{k,\ell} := \mathrm{Res}^{k,\ell}_{(0)}, \quad \Pi_k := \Pi_k^{(0)}, \ldots \tag{4.1}$$

The first result of this section, proved in §4.1 below, is the following theorem. (Here the maps $\pi_k^{(\tau)} : \mathrm{Res}^k_{0(\tau)} \cap \ker d \to H^k(M; \mathbb{C})$ are defined in (2.61).)

Theorem 3 *Let the assumptions above in this section hold. Assume moreover the following nondegeneracy condition:*

$$\langle\!\langle \iota_X \beta \bullet, \bullet \rangle\!\rangle \quad \textit{defines a nondegenerate pairing on} \quad d(\mathrm{Res}^1_0) \times d(\mathrm{Res}^1_{0*}). \tag{4.2}$$

Then there exists $\varepsilon_0 > 0$ such that for all τ with $0 < |\tau| < \varepsilon_0$ we have:

1. $d(\mathrm{Res}^1_{0(\tau)}) = 0$ *and thus by Lemma 2.8 and (2.28) we have* $\dim \mathrm{Res}^1_{0(\tau)} = b_1(\Sigma)$.
2. $d(\mathrm{Res}^2_{0(\tau)}) = 0$*,* $\dim \mathrm{Res}^2_{0(\tau)} = b_1(\Sigma) + 2$*, and the map $\pi_2^{(\tau)}$ is onto and has kernel $\mathbb{C} d\alpha_\tau$.*
3. $d(\mathrm{Res}^3_{0(\tau)}) = 0$ *and the map $\pi_3^{(\tau)}$ is equal to 0.*
4. *The semisimplicity condition (2.41) holds at $\lambda_0 = 0$ for the operators $P_{k,0}^{(\tau)}$ for all $k = 0, 1, 2, 3, 4$.*

Theorem 3 together with Lemma 2.4 and (2.59) give the following

Corollary 4.1 *Under the assumptions of Theorem 3 we have for $0 < |\tau| < \varepsilon_0$*

$$m^{(\tau)}_{0,0}(0) = m^{(\tau)}_{4,0}(0) = 1, \; m^{(\tau)}_{1,0}(0) = m^{(\tau)}_{3,0}(0) = b_1(\Sigma), \; m^{(\tau)}_{2,0}(0) = b_1(\Sigma) + 2$$

and the order of vanishing of the Ruelle zeta function ζ_{R} at 0 is

$$m_{\mathrm{R}}(0) = 2m^{(\tau)}_{0,0}(0) - 2m^{(\tau)}_{1,0}(0) + m^{(\tau)}_{2,0}(0) = 4 - b_1(\Sigma).$$

Corollary 4.1 is in contrast with the hyperbolic case $\tau = 0$, where Corollary 3.1 gives the order of vanishing $4 - 2b_1(\Sigma)$.

To give an application of Theorem 3 which is simpler to prove than Theorem 1, we show in §§4.2–4.3 below that the nondegeneracy condition (4.2) holds for a large set of conformal perturbations of the contact form α: [1]

---

[1] By the Gray Stability Theorem (see [31, Theorem 2.2.2]), any perturbation of a contact form is a conformal perturbation up to pullback by a diffeomorphism.

Theorem 4 *Let $M = S\Sigma$ where (Σ, g) is a hyperbolic 3-manifold. Fix a nonempty open set $\mathscr{U} \subset M$, and denote by $C_c^\infty(\mathscr{U};\mathbb{R})$ the space of all smooth real-valued functions on M with support inside $\mathscr{U}$, with the topology inherited from $C^\infty(M;\mathbb{R})$.*

Then there exists an open dense subset of $C_c^\infty(\mathscr{U};\mathbb{R})$ such that for any $\mathbf{a}$ in this set, the 1-form $\beta := \mathbf{a}\alpha$ satisfies the condition (4.2). It follows that for $\tau \neq 0$ small enough depending on $\mathbf{a}$ the contact flow on M corresponding to the contact form $\alpha_\tau := e^{\tau\mathbf{a}}\alpha$ satisfies the conclusions of Theorem 3, in particular the Ruelle zeta function has order of vanishing $4 - b_1(\Sigma)$ at 0.

4.1 Proof of Theorem 3

We first prove an identity relating the action of the vector field

$$Y := \partial_\tau X_\tau|_{\tau=0} \in C^\infty(M; TM) \tag{4.3}$$

on resonant and coresonant 1-forms to the bilinear form featured in (4.2). It reformulates the pairing (4.2) and will subsequently (see Lemma 4.4) be used to show that the non-closed 1-forms may be perturbed away.

Lemma 4.2 *For all $u \in \mathrm{Res}_0^1$ and $u_* \in \mathrm{Res}_{0*}^1$, we have*

$$\langle\!\langle \Pi_1 \mathcal{L}_Y \Pi_1 u, d\alpha \wedge u_* \rangle\!\rangle = \langle\!\langle \mathcal{L}_Y u, d\alpha \wedge u_* \rangle\!\rangle = \langle\!\langle (\iota_X \beta) du, du_* \rangle\!\rangle. \tag{4.4}$$

Proof 1. To show the first equality in (4.4), we note that by the decomposition (2.44) and Lemma 2.4 we have for all $w \in \mathcal{D}'_{E_u^*}(M;\Omega^1)$

$$\Pi_1 w = \Pi_{1,0}(w - (\iota_X w)\alpha) + \frac{1}{\mathrm{vol}_\alpha(M)}\left(\int_M \iota_X w \, d\,\mathrm{vol}_\alpha\right)\alpha.$$

We now compute

$$\begin{aligned}\int_M \alpha \wedge d\alpha \wedge (\Pi_1 \mathcal{L}_Y \Pi_1 u) \wedge u_* &= \langle\!\langle \Pi_{1,0}(\mathcal{L}_Y u - (\iota_X \mathcal{L}_Y u)\alpha), d\alpha \wedge u_* \rangle\!\rangle \\ &= \langle\!\langle \mathcal{L}_Y u - (\iota_X \mathcal{L}_Y u)\alpha, d\alpha \wedge u_* \rangle\!\rangle \\ &= \int_M \alpha \wedge d\alpha \wedge \mathcal{L}_Y u \wedge u_*.\end{aligned}$$

Here in the first equality we used that $u \in \mathrm{Res}_0^1$ and thus $\Pi_1 u = u$. In the second equality we used that $d\alpha \wedge u_* \in \mathrm{Res}_{0*}^3$ and thus $(\Pi_{1,0})^T(d\alpha \wedge u_*) = d\alpha \wedge u_*$ (see §2.3.4). This proves the first equality in (4.4).

2. We now show the second equality in (4.4). Differentiating the relations $\iota_{X_\tau}\alpha_\tau = 1$ and $\iota_{X_\tau}d\alpha_\tau = 0$ (see (2.1)) at $\tau = 0$, we get

$$\iota_Y\alpha = -\iota_X\beta, \quad \iota_Y d\alpha = -\iota_X d\beta. \tag{4.5}$$

Note also that

$$\alpha \wedge d\alpha \wedge du = \alpha \wedge d\alpha \wedge du_* = 0 \tag{4.6}$$

as follows from Lemma 2.4 as the 5-forms above are in $\mathrm{Res}_0^0\, d\,\mathrm{vol}_\alpha$, respectively $\mathrm{Res}_{0*}^0\, d\,\mathrm{vol}_\alpha$, and integrate to 0 on M using integration by parts (since the 5-forms $d\alpha \wedge d\alpha \wedge u$, $d\alpha \wedge d\alpha \wedge u_*$ lie in the kernel of ι_X and thus are equal to 0).

We have

$$\int_M \alpha \wedge d\alpha \wedge \mathcal{L}_Y u \wedge u_* = \int_M \alpha \wedge d\alpha \wedge \iota_Y du \wedge u_* + \int_M \alpha \wedge d\alpha \wedge d\iota_Y u \wedge u_*. \tag{4.7}$$

We first compute

$$\begin{aligned}\int_M \alpha \wedge d\alpha \wedge \iota_Y du \wedge u_* &= -\int_M \alpha \wedge \iota_Y d\alpha \wedge du \wedge u_* - \int_M (\iota_Y u_*)\alpha \wedge d\alpha \wedge du \\ &= \int_M \alpha \wedge \iota_X d\beta \wedge du \wedge u_* = \int_M d\beta \wedge du \wedge u_* \\ &= \int_M \beta \wedge du \wedge du_* = \int_M (\iota_X\beta)\alpha \wedge du \wedge du_*.\end{aligned} \tag{4.8}$$

Here in the first equality we used that the 5-form $d\alpha \wedge du \wedge u_*$ lies in the kernel of ι_X and is thus equal to 0, implying $\iota_Y(d\alpha \wedge du \wedge u_*) = 0$. In the second equality we used the identities (4.5) and (4.6). In the third equality we used that $\alpha \wedge \iota_X d\beta \wedge du \wedge u_* = d\beta \wedge du \wedge u_*$ as the difference of the two forms belongs to $\ker \iota_X$, by $\iota_X du = 0$ and $\iota_X u_* = 0$. In the fourth equality we integrated by parts, and in the fifth equality we used that ι_X of the integrated 5-forms are equal.

We next compute

$$\int_M \alpha \wedge d\alpha \wedge d\iota_Y u \wedge u_* = \int_M \iota_Y u(d\alpha \wedge d\alpha \wedge u_* - \alpha \wedge d\alpha \wedge du_*) = 0. \tag{4.9}$$

Here in the first equality we integrated by parts and in the second one we used (4.6) and the fact that $d\alpha \wedge d\alpha \wedge u_* = 0$ (as ι_X of this 5-form is equal to 0).

Plugging (4.8)–(4.9) into (4.7), we get the second equality in (4.4). $\square$

The pairing in (4.4) controls how the resonance at 0 for the operator $P_{1,0}^{(\tau)}$ moves as we perturb τ from 0, and the nondegeneracy condition (4.2) roughly speaking means that the multiplicity of 0 as a resonance of $P_{1,0}^{(\tau)}$ drops by $\dim d(\mathrm{Res}_0^1) = b_1(\Sigma)$. This observation is made precise in Lemma 4.4 below, but first we need to review perturbation theory of Pollicott–Ruelle resonances. It will be more convenient for us to use the operators $P_k^{(\tau)}$ rather than $P_{k,0}^{(\tau)}$ since the latter act on the τ-dependent space of k-forms annihilated by ι_{X_τ}. In the rest of this section we assume that $\varepsilon_0 > 0$ is chosen small, with the precise value varying from line to line.

We will use the perturbation theory developed in [7]. For an alternative approach, see [16, §6]. Since we are interested in the resonance at 0, we may restrict ourselves to the strip $\{\mathrm{Im}\,\lambda > -1\}$. Following the notation of [12, §6.1], we consider the τ-independent anisotropic Sobolev spaces

$$\mathcal{H}_{rG,s}(M;\Omega^k) := e^{-r\,\mathrm{Op}(G)} H^s(M;\Omega^k), \quad r \geq 0, \quad s \in \mathbb{R}. \tag{4.10}$$

Here Op is a quantization procedure on M, $G(\rho,\xi) = m(\rho,\xi)\log(1+|\xi|)$ is a logarithmically growing symbol on the cotangent bundle T^*M, $|\xi|$ denotes an appropriately chosen norm on the fibers of T^*M, and the function $m(\rho,\xi)$, homogeneous of order 0 in ξ, satisfies certain conditions [7, (4)] with respect to the vector field X_τ for all $\tau \in (-\varepsilon_0, \varepsilon_0)$. The space H^s is the usual Sobolev space of order s. Denote the domain of $P_k^{(\tau)}$ on $\mathcal{H}_{rG,s}$ by

$$\mathcal{D}_{rG,s}^{(\tau)}(M;\Omega^k) := \{u \in \mathcal{H}_{rG,s}(M;\Omega^k) \mid P_k^{(\tau)} u \in \mathcal{H}_{rG,s}(M;\Omega^k)\}.$$

The following lemma summarizes the perturbation theory used here. For details see for example [7, Theorem 1 and Corollary 2] or [12, Lemma 6.1 and §6.2].

Lemma 4.3 *There exists a constant C_0 such that for $r > C_0 + |s|$ and $\tau \in (-\varepsilon_0, \varepsilon_0)$, the operator*

$$P_k^{(\tau)} - \lambda : \mathcal{D}_{rG,s}^{(\tau)}(M;\Omega^k) \to \mathcal{H}_{rG,s}(M;\Omega^k), \quad \mathrm{Im}\,\lambda > -1 \tag{4.11}$$

is Fredholm and its inverse (assuming λ is not a resonance) is given by $R_k^{(\tau)}(\lambda)$. Moreover, the set of pairs (τ,λ) such that λ is a resonance of $P_k^{(\tau)}$ is closed and the resolvent $R_k^{(\tau)}(\lambda) : \mathcal{H}_{rG,s} \to \mathcal{H}_{rG,s}$ is bounded locally uniformly in τ, λ outside of this set.

Since $R_k^{(\tau)}(\lambda)$ is the inverse of $P_k^{(\tau)} - \lambda$ on anisotropic Sobolev spaces, we have the resolvent identity for all $\tau, \tau' \in (-\varepsilon_0, \varepsilon_0)$

$$R_k^{(\tau)}(\lambda) - R_k^{(\tau')}(\lambda) = R_k^{(\tau)}(\lambda)(P_k^{(\tau')} - P_k^{(\tau)})R_k^{(\tau')}(\lambda), \quad \operatorname{Im}\lambda > -1. \tag{4.12}$$

Here the right-hand side is well-defined since for $r > C_0 + |s| + 1$ the operator $R_k^{(\tau')}(\lambda)$ maps $\mathcal{H}_{rG,s}$ to itself, $P_k^{(\tau)}$ and $P_k^{(\tau')}$ map $\mathcal{H}_{rG,s}$ to $\mathcal{H}_{rG,s-1}$, and $R_k^{(\tau)}(\lambda)$ maps $\mathcal{H}_{rG,s-1}$ to itself. Using (4.12) we see that for $r > C_0 + |s| + 1$ the family $R_k^{(\tau)}(\lambda) : \mathcal{H}_{rG,s} \to \mathcal{H}_{rG,s-1}$ is locally Lipschitz continuous in τ. Next, recalling (4.3) and that $P_k^{(\tau)} = -i\mathcal{L}_{X_\tau}$, we have by (4.12)

$$\partial_\tau R_k^{(\tau)}(\lambda)|_{\tau=0} = i R_k(\lambda)\mathcal{L}_Y R_k(\lambda) \tag{4.13}$$

as operators $\mathcal{H}_{rG,s} \to \mathcal{H}_{rG,s-2}$ when $r > C_0 + |s| + 2$.

Fix a contour γ in the complex plane which encloses 0 but no other resonances of the unperturbed operators $P_k = P_k^{(0)}$. For $|\tau| < \varepsilon_0$, no resonances of $P_k^{(\tau)}$ lie on the contour γ, so we may define the operators

$$\widetilde{\Pi}_k^{(\tau)} := -\frac{1}{2\pi i}\oint_\gamma R_k^{(\tau)}(\lambda)\, d\lambda.$$

Unlike the spectral projectors $\Pi_k^{(\tau)}$ corresponding to the resonance at 0, the operators $\widetilde{\Pi}_k^{(\tau)}$ depend continuously on τ, since $R_k^{(\tau)}(\lambda)$ is continuous in τ. Moreover, the rank of $\widetilde{\Pi}_k^{(\tau)}$ is constant in $\tau \in (-\varepsilon_0, \varepsilon_0)$, see [12, Lemma 6.2]. By (2.36) we have

$$\widetilde{\Pi}_k^{(0)} = \Pi_k := \Pi_k(0)$$

so the rank of $\widetilde{\Pi}_k^{(\tau)}$ can be computed using the algebraic multiplicities of 0 as a resonance in the unperturbed case $\tau = 0$ (using (2.43)):

$$\operatorname{rank} \widetilde{\Pi}_k^{(\tau)} = m_k(0) = m_{k,0}(0) + m_{k-1,0}(0). \tag{4.14}$$

By (2.36), we also have

$$\widetilde{\Pi}_k^{(\tau)} = \sum_{\lambda \in \Upsilon_\tau^k} \Pi_k^{(\tau)}(\lambda) \tag{4.15}$$

where Υ_τ^k is the set of resonances of the operator $P_k^{(\tau)}$ which are enclosed by the contour γ. Note that by (4.15) and (2.42)

$$\widetilde{\Pi}_k^{(\tau)}\Pi_k^{(\tau)}(\lambda) = \Pi_k^{(\tau)}(\lambda) \quad \text{for all} \quad \lambda \in \Upsilon_\tau^k \tag{4.16}$$

and the range of $\widetilde{\Pi}_k^{(\tau)}$ is the direct sum of the ranges $\mathrm{Res}_{(\tau)}^{k,\infty}(\lambda)$ of $\Pi_k^{(\tau)}(\lambda)$ over $\lambda \in \Upsilon_\tau^k$. In particular, using (2.43) we get

$$\operatorname{rank} \widetilde{\Pi}_k^{(\tau)} = \sum_{\lambda\in\Upsilon_\tau^k} \big(m_{k,0}^{(\tau)}(\lambda) + m_{k-1,0}^{(\tau)}(\lambda)\big). \tag{4.17}$$

Together with (4.14) and induction on k this implies for $|\tau| < \varepsilon_0$

$$\sum_{\lambda\in\Upsilon_\tau^k} m_{k,0}^{(\tau)}(\lambda) = m_{k,0}(0). \tag{4.18}$$

We are now ready to show that under the condition (4.2) the space $\mathrm{Res}_{0(\tau)}^1$ of resonant 1-forms at 0 for the perturbed operator $P_{1,0}^{(\tau)}$, $\tau \neq 0$, consists of closed forms:

Lemma 4.4 *Under the assumptions of Theorem 3, there exists $\varepsilon_0 > 0$ such that for $0 < |\tau| < \varepsilon_0$ we have $d(\mathrm{Res}_{0(\tau)}^1) = 0$.*

Proof 1. Define the operator

$$Z(\tau) := P_1^{(\tau)}\widetilde{\Pi}_1^{(\tau)}.$$

Roughly speaking this operator contains information about the nonzero resonances of $P_1^{(\tau)}$ enclosed by γ; in particular, each of the corresponding spaces of generalized resonant states is in the range of $Z(\tau)$ as can be seen from (4.16).

In the hyperbolic case $\tau = 0$, the semisimplicity condition (2.41) holds for the operator P_1 at $\lambda = 0$, as follows from Lemmas 2.4 and 3.10 together with (2.43). Therefore, the range of $\widetilde{\Pi}_1^{(0)} = \Pi_1$ is contained in Res^1, implying that

$$Z(0) = 0. \tag{4.19}$$

By (4.14), the rank of $\widetilde{\Pi}_1^{(\tau)}$ can be computed using the algebraic multiplicities of 0 as a resonance in the hyperbolic case $\tau = 0$, which are known by (3.1):

$$\operatorname{rank} \widetilde{\Pi}_1^{(\tau)} = 2b_1(\Sigma) + 1. \tag{4.20}$$

The intersection of the range of $\widetilde{\Pi}_1^{(\tau)}$ with the kernel of $P_1^{(\tau)}$ is equal to $\mathrm{Res}^1_{(\tau)}$. By (2.43) and Lemma 2.4 we have $\mathrm{Res}^1_{(\tau)} = \mathrm{Res}^1_{0(\tau)} \oplus \mathbb{C}\alpha_\tau$. Next, by Lemma 2.8 and (2.28) we have $\dim \mathrm{Res}^1_{0(\tau)} = b_1(\Sigma) + \dim d(\mathrm{Res}^1_{0(\tau)})$. Therefore

$$\dim \mathrm{Res}^1_{(\tau)} = b_1(\Sigma) + 1 + \dim d(\mathrm{Res}^1_{0(\tau)}).$$

By the Rank-Nullity Theorem and (4.20) we then have

$$\operatorname{rank} Z(\tau) = b_1(\Sigma) - \dim d(\mathrm{Res}^1_{0(\tau)}). \tag{4.21}$$

2. Since $(P_1^{(\tau)} - \lambda)R_1^{(\tau)}(\lambda)$ is the identity operator, we have for all τ

$$Z(\tau) = -\frac{1}{2\pi i}\oint_\gamma \lambda R_1^{(\tau)}(\lambda)\, d\lambda.$$

Using (4.13) we now compute the derivative

$$\partial_\tau Z(0) = -\frac{1}{2\pi}\oint_\gamma \lambda R_1(\lambda)\mathcal{L}_Y R_1(\lambda)\, d\lambda = -i\Pi_1\mathcal{L}_Y\Pi_1.$$

Here in the second equality we used the Laurent expansion (2.36) for $R_1(\lambda)$ at $\lambda_0 = 0$ (recalling that $J_1(0) = 1$ by semisimplicity).

By Lemma 4.2, for any $u \in \mathrm{Res}^1_0$, $u_* \in \mathrm{Res}^1_{0*}$ we have

$$\int_M \alpha \wedge d\alpha \wedge \big(\partial_\tau Z(0)u\big) \wedge u_* = -i\langle\!\langle (\iota_X\beta)du, du_*\rangle\!\rangle. \tag{4.22}$$

By the nondegeneracy assumption (4.2) the bilinear form (4.22) is nondegenerate on $u \in \mathcal{C}_\psi$, $u_* \in \mathcal{C}_{\psi*}$. This implies that

$$\operatorname{rank} \partial_\tau Z(0) \geq \dim \mathcal{C}_\psi = b_1(\Sigma). \tag{4.23}$$

Together (4.19) and (4.23) show that for $0 < |\tau| < \varepsilon_0$

$$\operatorname{rank} Z(\tau) \geq b_1(\Sigma).$$

Then by (4.21) we have $\dim d(\mathrm{Res}^1_{0(\tau)}) = 0$ for $0 < |\tau| < \varepsilon_0$ which finishes the proof. □

Remark Lemma 4.4 holds more generally whenever $P_{1,0}$ is semisimple. If for all contact perturbations $(\alpha_\tau)_\tau$ we would have that (4.2) is trivial, this would imply that $du \wedge du_* = 0$ for all $u \in \mathrm{Res}^1_0$ and $u_* \in \mathrm{Res}^1_{0*}$. When (Σ, g) is hyperbolic, we will show in § 4.2 that this is impossible, while for general (Σ, g) proving such a statement seems out of reach for now.

Together with Lemma 2.4, Lemma 2.9, Lemma 2.11, and (2.28) Lemma 4.4 gives all the conclusions of Theorem 3 except semisimplicity on 2-forms. In particular we have for $0 < |\tau| < \varepsilon_0$ (using (2.43))

$$\dim \operatorname{Res}^2_{0(\tau)} = b_1(\Sigma) + 2, \tag{4.24}$$

$$d(\operatorname{Res}^{1,\infty}_{(\tau)}) = \mathbb{C} d\alpha_\tau. \tag{4.25}$$

To finish the proof of Theorem 3 it remains to establish semisimplicity on 2-forms:

Lemma 4.5 *Under the assumptions of Theorem 3, there exists $\varepsilon_0 > 0$ such that for $0 < |\tau| < \varepsilon_0$ the semisimplicity condition (2.41) holds at $\lambda_0 = 0$ for the operator $P^{(\tau)}_{2,0}$.*

Proof We first claim that for $0 < |\tau| < \varepsilon_0$

$$\operatorname{rank}\big(\alpha_\tau \wedge (\widetilde{\Pi}^{(\tau)}_2 - \Pi^{(\tau)}_2)\big) \geq \operatorname{rank}\big(\alpha_\tau \wedge d(\widetilde{\Pi}^{(\tau)}_1 - \Pi^{(\tau)}_1)\big) \geq b_1(\Sigma). \tag{4.26}$$

Indeed, by (2.37) and (4.15) we have $d(\widetilde{\Pi}^{(\tau)}_1 - \Pi^{(\tau)}_1) = (\widetilde{\Pi}^{(\tau)}_2 - \Pi^{(\tau)}_2)d$ which implies the first inequality in (4.26). Next, we have $\operatorname{rank}(\alpha \wedge d\widetilde{\Pi}^{(0)}_1) = b_1(\Sigma) + 1$ as the range of $d\widetilde{\Pi}^{(0)}_1$ is equal to $d\operatorname{Res}^1 = \mathbb{C} d\alpha \oplus d\mathcal{C}_\psi$. Since $\widetilde{\Pi}^{(\tau)}_1$ depends continuously on τ, we see that $\operatorname{rank}(\alpha_\tau \wedge d\widetilde{\Pi}^{(\tau)}_1) \geq b_1(\Sigma) + 1$ for all small enough τ. On the other hand, for τ small but nonzero we have $\operatorname{rank} d\Pi^{(\tau)}_1 = 1$ by (4.25). Together these imply the second inequality in (4.26).

Now, by (4.15) and (2.43) the range of $\alpha_\tau \wedge (\widetilde{\Pi}^{(\tau)}_2 - \Pi^{(\tau)}_2)$ is contained in the sum of the spaces $\alpha_\tau \wedge \operatorname{Res}^{2,\infty}_{0(\tau)}(\lambda)$ over $\lambda \in \Upsilon^2_\tau \setminus \{0\}$. Therefore (4.26) implies that for $0 < |\tau| < \varepsilon_0$

$$\sum_{\lambda \in \Upsilon^2_\tau \setminus \{0\}} m^{(\tau)}_{2,0}(\lambda) \geq b_1(\Sigma). \tag{4.27}$$

From (4.18) and (3.1) we see that

$$\sum_{\lambda \in \Upsilon^2_\tau} m^{(\tau)}_{2,0}(\lambda) = m_{2,0}(0) = 2b_1(\Sigma) + 2$$

therefore by (4.27) we have $m_{2,0}^{(\tau)}(0) \leq b_1(\Sigma) + 2$. Since $\dim \mathrm{Res}^2_{0(\tau)} = b_1(\Sigma)+2$ by (4.24), we showed that the algebraic and geometric multiplicities for 0 as a resonance of $P_{2,0}^{(\tau)}$ coincide, finishing the proof. $\square$

4.2 The full support property

In this section, we prove a full support statement which will be used in the proof of Theorem 4. In fact, we recall that we need to prove the nondegeneracy assumption (4.2), that is, that $\langle\langle \iota_X \beta \bullet, \bullet \rangle\rangle$ is nondegenerate on $d\,\mathrm{Res}^1_0 \times d\,\mathrm{Res}^1_{0*}$, and the support properties of elements of $d\,\mathrm{Res}^1_{0(*)}$ will be useful. In §§4.2–4.4 we assume that $M = S\Sigma$ where (Σ, g) is a hyperbolic 3-manifold and the contact form α and the spaces of (co-)resonant states at zero Res^1_0, Res^1_{0*} are defined using the geodesic flow on (Σ, g).

Proposition 4.6 *For all $u \in \mathrm{Res}^1_0$, $u_* \in \mathrm{Res}^1_{0*}$ with $du \neq 0$, $du_* \neq 0$, the distributional 5-form $\alpha \wedge du \wedge du_*$ fulfills* $\mathrm{supp}(\alpha \wedge du \wedge du_*) = M$.

To show Proposition 4.6, we first study properties of the 2-forms du and du_*. Define the smooth 2-forms

$$\omega_\pm \in C^\infty(M; \Omega^2_0)$$

by requiring that $E_0 \oplus E_u$ be in the kernel of ω_-, $E_0 \oplus E_s$ be in the kernel of ω_+, and, using the horizontal/vertical decomposition (2.15)

$$\omega_\pm(x, v)\big((w_1, \pm w_1), (w_2, \pm w_2)\big) = \langle v \times w_1, w_2 \rangle_g \quad \text{for all} \quad w_1, w_2 \in \{v\}^\perp \subset T_x\Sigma \tag{4.28}$$

where '$\times$' denotes the cross product on $T_x\Sigma$ defined in §3.2.1. In terms of the canonical 1-forms on the frame bundle $\mathcal{F}\Sigma$ defined in §3.1.3 the lifts of $\omega_\pm$ to $\mathcal{F}\Sigma$ are given by

$$\omega_\pm = U_1^{\pm *} \wedge U_2^{\pm *}. \tag{4.29}$$

One can think of $\omega_\pm$ as canonical volume forms on the stable/unstable spaces.

By (4.29) and (3.12) we compute

$$d\omega_\pm = \pm 2\alpha \wedge \omega_\pm. \tag{4.30}$$

Lemma 4.7 *Assume that* $u \in \mathrm{Res}_0^1$, $u_* \in \mathrm{Res}_{0*}^1$. *Then*

$$du = f_-\omega_-, \quad du_* = f_+\omega_+; \tag{4.31}$$

$$\alpha \wedge du \wedge du_* = -\tfrac{1}{8} f_- f_+ d\,\mathrm{vol}_\alpha \tag{4.32}$$

where the distributions $f_- \in \mathcal{D}'_{E_u^*}(M; \mathbb{C})$, $f_+ \in \mathcal{D}'_{E_s^*}(M; \mathbb{C})$ *satisfy for any vector fields* $U_- \in C^\infty(M; E_u)$, $U_+ \in C^\infty(M; E_s)$

$$(X \pm 2) f_\pm = 0, \quad U_\pm f_\pm = 0. \tag{4.33}$$

Proof We consider the case of du, with du_* studied similarly. From Lemma 3.8 we know that u is a totally unstable 1-form, which implies that du is a section of $E_u^* \wedge E_u^*$. The latter is a one-dimensional vector bundle over M and ω_- is a nonvanishing smooth section of it, so $du = f_-\omega_-$ for some $f_- \in \mathcal{D}'_{E_u^*}(M; \mathbb{C})$. Using (4.30) we compute

$$0 = d(f_-\omega_-) = (df_- - 2f_-\alpha) \wedge \omega_-.$$

Taking ι_X and ι_{U_-} of this identity and using that $\iota_X\omega_- = \iota_{U_-}\omega_- = \iota_{U_-}\alpha = 0$ (recalling the definitions of $U_1^{\pm*}$, $U_2^{\pm*}$ in (3.10) and below), we get (4.33).

Finally, (4.32) follows from (4.31) and the following identity which can be verified using either (4.28) and (2.16) or (4.29) and (3.12):

$$\alpha \wedge \omega_- \wedge \omega_+ = -\tfrac{1}{8} d\,\mathrm{vol}_\alpha\,.$$

□

We can now finish the proof of Proposition 4.6. Given (4.32) it suffices to prove that, assuming that $f_- \neq 0$ and $f_+ \neq 0$,

$$\mathrm{supp}(f_- f_+) = M. \tag{4.34}$$

Let $\pi_\Gamma : S\mathbb{H}^3 \to S\Sigma = M$ be the covering map corresponding to (3.2) and $\Phi_\pm$, $B_\pm$ be defined in (3.14). Then by (3.22) and (4.33) we have for any $U_- \in C^\infty(S\mathbb{H}^3; E_u)$, $U_+ \in C^\infty(S\mathbb{H}^3; E_s)$

$$X(\Phi_\pm^2(f_\pm \circ \pi_\Gamma)) = U_\pm(\Phi_\pm^2(f_\pm \circ \pi_\Gamma)) = 0,$$

that is $\Phi_+^2(f_+ \circ \pi_\Gamma)$ is totally stable and $\Phi_-^2(f_- \circ \pi_\Gamma)$ is totally unstable in the sense of Definition 3.2. Similarly to Lemma 3.3 we can then describe the lifts of $f_\pm$ to $S\mathbb{H}^3$ in terms of some distributions $g_\pm$ on the conformal infinity $\mathbb{S}^2$:

$$f_\pm \circ \pi_\Gamma = \Phi_\pm^{-2}(g_\pm \circ B_\pm) \quad \text{for some} \quad g_\pm \in \mathcal{D}'(\mathbb{S}^2; \mathbb{C}). \tag{4.35}$$

Since $f_\pm$ are resonant states of X, a result of Weich [54, Theorem 1] shows that $\operatorname{supp} f_+ = \operatorname{supp} f_- = M$, which from (4.35) and the facts that $\Phi_\pm > 0$, and that $B_\pm$ are submersions which map $S\mathbb{H}^3$ onto $\mathbb{S}^2$, implies that

$$\operatorname{supp} g_+ = \operatorname{supp} g_- = \mathbb{S}^2. \tag{4.36}$$

We will now use the coordinates $(\nu_-, \nu_+, t) \in (\mathbb{S}^2 \times \mathbb{S}^2)_- \times \mathbb{R}$ on $S\mathbb{H}^3$ introduced in (3.16). Then by (4.35) and (3.17) we can write in these coordinates

$$(f_- f_+) \circ \pi_\Gamma = \tfrac{1}{16} |\nu_- - \nu_+|^4 g_-(\nu_-) g_+(\nu_+).$$

By (4.36), we see that the support of the tensor product $g_- \otimes g_+(\nu_-, \nu_+) = g_-(\nu_-) g_+(\nu_+)$ is equal to the entire $\mathbb{S}^2 \times \mathbb{S}^2$, which implies that $\operatorname{supp}(f_- f_+) \circ \pi_\Gamma = S\mathbb{H}^3$ and thus $\operatorname{supp}(f_- f_+) = M$. This shows (4.34) and finishes the proof.

4.3 Proof of Theorem 4

We first remark that in the special case $\dim d(\mathrm{Res}_0^1) = b_1(\Sigma) = 1$, it is straightforward to see that Proposition 4.6 implies the following simplified version of Theorem 4: for each nonempty open set $\mathscr{U} \subset M$ there exists $\mathbf{a} \in C^\infty(M; \mathbb{R})$ with $\operatorname{supp} \mathbf{a} \subset \mathscr{U}$ and such that $\beta := \mathbf{a}\alpha$ satisfies (4.2). Indeed, it suffices to fix any nonzero $du \in d(\mathrm{Res}_0^1)$, $du_* \in d(\mathrm{Res}_{0*}^1)$, and choose $\mathbf{a}$ such that $\int_M \mathbf{a}\alpha \wedge du \wedge du_* \neq 0$. We note that there are examples of hyperbolic 3-manifolds with $b_1(\Sigma) = 1$, see for instance [24, Theorem 13.4].

For the general case, we will use the following basic fact from linear algebra:

Lemma 4.8 *Denote by $\otimes^2 \mathbb{C}^n$ the space of complex $n \times n$ matrices. Assume that $V \subset \otimes^2 \mathbb{C}^n$ is a subspace such that for each $v_1, v_2 \in \mathbb{C}^n \setminus \{0\}$ there exists $B \in V$ such that $\langle B v_1, v_2 \rangle \neq 0$. (Here $\langle \bullet, \bullet \rangle$ denotes the canonical bilinear inner product on $\mathbb{C}^n$.) Then the set of invertible matrices in V is dense.*

Proof Let $\mathscr{O}$ be a nonempty open subset of V. We need to show that $\mathscr{O}$ contains an invertible matrix. Assume that there are no invertible matrices in $\mathscr{O}$. Let A be a matrix of maximal rank in $\mathscr{O}$, then $k := \operatorname{rank} A < n$ since A cannot be invertible. There exist bases $e_1, \dots, e_n$ and $e_1^*, \dots, e_n^*$ of $\mathbb{C}^n$ such that

$$\langle A e_j, e_\ell^* \rangle = \begin{cases} 1 & \text{if } j = \ell \leq k; \\ 0 & \text{otherwise.} \end{cases}$$

By the assumption of the lemma, there exists $B \in V$ such that $\langle B e_{k+1}, e_{k+1}^* \rangle \neq 0$. Consider the matrix $A_t = A + tB$ which lies in $\mathscr{O}$ for sufficiently small t, and let $b(t)$ be the determinant of the matrix $(\langle A_t e_j, e_\ell^* \rangle)_{j,\ell=1}^{k+1}$. Then $b(0) = 0$ and $b'(0) = \langle Be_{k+1}, e_{k+1}^* \rangle \neq 0$. Therefore, for small enough $t \neq 0$ we have $b(t) \neq 0$, which means that rank $A_t \geq k+1$. This contradicts the fact that k was the maximal rank of any matrix in $\mathscr{O}$. □

We are now ready to give the proof of Theorem 4. For $\mathbf{a} \in C^\infty(M; \mathbb{R})$, define the bilinear form

$$S_\mathbf{a} : d(\mathrm{Res}_0^1) \times d(\mathrm{Res}_{0*}^1) \to \mathbb{C}, \quad S_\mathbf{a}(du, du_*) = \int_M \mathbf{a}\alpha \wedge du \wedge du_*.$$

To prove Theorem 4, it then suffices to show that the set of $\mathbf{a} \in C_\mathrm{c}^\infty(\mathscr{U}; \mathbb{R})$ such that $S_\mathbf{a}$ is nondegenerate is open and dense. Since nondegeneracy is an open condition, this set is automatically open. To show that it is dense, consider the finite dimensional vector space

$$V := \{S_\mathbf{a} \mid \mathbf{a} \in C_\mathrm{c}^\infty(\mathscr{U}; \mathbb{R})\}.$$

Choosing bases of the $b_1(\Sigma)$-dimensional spaces $d(\mathrm{Res}_0^1)$ and $d(\mathrm{Res}_{0*}^1)$, we can identify V with a subspace of $\otimes^2 \mathbb{C}^{b_1(\Sigma)}$. Let $du \in d(\mathrm{Res}_0^1)$, $du_* \in d(\mathrm{Res}_{0*}^1)$ be nonzero, then by Proposition 4.6 we have $\mathrm{supp}(\alpha \wedge du \wedge du_*) = M$, so there exists $\mathbf{a} \in C_\mathrm{c}^\infty(\mathscr{U}; \mathbb{R})$ such that $S_\mathbf{a}(du, du_*) \neq 0$. Then by Lemma 4.8 the set of nondegenerate bilinear forms in V is dense.

Let $\mathbf{U}$ be a nonempty open subset of $C_\mathrm{c}^\infty(\mathscr{U}; \mathbb{R})$. Then $\{S_\mathbf{a} \mid \mathbf{a} \in \mathbf{U}\}$ is a nonempty open subset of V. Thus there exists $\mathbf{a} \in \mathbf{U}$ such that $S_\mathbf{a}$ is nondegenerate, which finishes the proof.

4.4 Proof of Theorem 1

We now give the proof of part 2 of Theorem 1, relying on Theorem 5 (in §5) and Proposition A.1 below, combined together in Corollary 5.1. (Part 1 of Theorem 1 was proved in Corollary 3.1 above.)

We start by computing how a general metric perturbation affects the contact form for the geodesic flow. Let (Σ, g) be any compact 3-dimensional Riemannian manifold and the contact form α and the generator X of the geodesic flow on $S\Sigma$ be defined as in §2.2. Let

$$g_\tau, \quad \tau \in (-\varepsilon, \varepsilon)$$

be a family of Riemannian metrics on Σ depending smoothly on τ, such that $g_0 = g$. The associated geodesic flows act on the τ-dependent sphere bundles

$$S^{(\tau)}\Sigma = \{(x, v) \in T\Sigma \colon |v|_{g_\tau} = 1\}.$$

To bring these geodesic flows to $S\Sigma$, we use the diffeomorphisms

$$\Phi_\tau : S\Sigma \to S^{(\tau)}\Sigma, \quad \Phi_\tau(x, v) = \left(x, \frac{v}{|v|_{g_\tau}}\right).$$

Denote by α_τ the contact form on $S^{(\tau)}\Sigma$ corresponding to g_τ. Then

$$\widetilde{\alpha}_\tau := \Phi_\tau^* \alpha_\tau$$

is a contact 1-form on $S\Sigma$ and the corresponding contact flow is the geodesic flow of (Σ, g_τ) pulled back by Φ_τ.

Let $\pi_\Sigma^{(\tau)} : S^{(\tau)}\Sigma \to \Sigma$ be the projection map. Using (2.11) and the fact that $\pi_\Sigma^{(\tau)} \circ \Phi_\tau$ is equal to $\pi_\Sigma := \pi_\Sigma^{(0)}$, we compute for all $(x, v) \in S\Sigma$ and $\xi \in T_{(x,v)}(S\Sigma)$

$$\langle \widetilde{\alpha}_\tau(x, v), \xi \rangle = \frac{\langle v, d\pi_\Sigma(x, v)\xi \rangle_{g_\tau}}{|v|_{g_\tau}}.$$

Recalling $d\pi_\Sigma(x, v)X(x, v) = v$ (see (2.18)) and using $g_0(v, v) = 1$, it follows that

$$\begin{aligned} \iota_X \partial_\tau \widetilde{\alpha}_\tau|_{\tau=0}(x, v) &= \partial_\tau g_\tau(v, v)|_{\tau=0} - \frac{1}{2} g_0(v, v) \cdot \partial_\tau g_\tau(v, v)|_{\tau=0} \\ &= \partial_\tau |v|_{g_\tau}|_{\tau=0}. \end{aligned} \tag{4.37}$$

In particular, if the metric g_τ is given by a conformal perturbation $g_\tau = e^{-2\tau \mathbf{b}} g$, where $\mathbf{b} \in C^\infty(\Sigma; \mathbb{R})$, then

$$\iota_X \partial_\tau \widetilde{\alpha}_\tau|_{\tau=0}(x, v) = -\mathbf{b} \circ \pi_\Sigma. \tag{4.38}$$

We are now ready to prove Theorem 1. Assume that (Σ, g) is a hyperbolic 3-manifold as defined in §3.1 and put $g_\tau := e^{-2\tau \mathbf{b}} g$. By Theorem 3 applied to the family of contact forms $\widetilde{\alpha}_\tau$, with $\beta = \partial_\tau \widetilde{\alpha}_\tau|_{\tau=0}$ satisfying (4.38), it suffices to show that for $\mathbf{b}$ in an open and dense subset of $C^\infty(\Sigma; \mathbb{R})$ the bilinear form

$$(du, du_*) \mapsto \int_M (\mathbf{b} \circ \pi_\Sigma) \alpha \wedge du \wedge du_*$$

is nondegenerate on $d(\mathrm{Res}_0^1) \times d(\mathrm{Res}_{0*}^1)$.

The space Res_0^1 is preserved by complex conjugation as follows from its definition (2.60); here we use that for any u we have $\mathrm{WF}(\bar{u}) = \{(\rho, -\xi) \mid (\rho, \xi) \in \mathrm{WF}(u)\}$. Denote by $\mathrm{Res}_{0\mathbb{R}}^1$ the space of real-valued 1-forms in Res_0^1 and let $\mathcal{J}(x, v) = (x, -v)$ be the map defined in (2.12). By (2.50), the pullback

$\mathcal{J}^*$ is an isomorphism from Res_0^1 onto Res_{0*}^1. Thus it suffices to show that for $\mathbf{b}$ in an open and dense subset of $C^\infty(\Sigma;\mathbb{R})$ the real bilinear form

$$\widetilde{S}_{\mathbf{b}}(du, du') := \int_M (\mathbf{b}\circ\pi_\Sigma)\alpha\wedge du\wedge\mathcal{J}^*(du')$$

is nondegenerate on $d(\mathrm{Res}_{0\mathbb{R}}^1)\times d(\mathrm{Res}_{0\mathbb{R}}^1)$.

Since $\mathbf{b}\circ\pi_\Sigma$ is $\mathcal{J}$-invariant, $\mathcal{J}^*\alpha=-\alpha$, and $\mathcal{J}$ is an orientation reversing diffeomorphism on M, we see that $\widetilde{S}_{\mathbf{b}}$ is a symmetric bilinear form. Unlike in the contact perturbation case in § 4.3, we will not be able to produce for every pair $(du, du')\in d(\mathrm{Res}_{0\mathbb{R}}^1)\times d(\mathrm{Res}_{0\mathbb{R}}^1)$ an element $\mathbf{b}\in C^\infty(\Sigma;\mathbb{R})$ such that $\widetilde{S}_{\mathbf{b}}(du, du')\neq 0$. Instead, we will only produce $\mathbf{b}$ such that $\widetilde{S}_{\mathbf{b}}(du, du)\neq 0$. Hence, we will need the following variant of Lemma 4.8 for symmetric matrices:

Lemma 4.9 *Denote by $\otimes_S^2\mathbb{R}^n$ the space of real symmetric $n\times n$ matrices. Assume that $V\subset\otimes_S^2\mathbb{R}^n$ is a subspace such that for each $w\in\mathbb{R}^n\setminus\{0\}$ there exists $B\in V$ such that $\langle Bw, w\rangle\neq 0$. Then the set of invertible matrices in V is dense.*

Proof Similarly to the proof of Lemma 4.8, assume that $\mathscr{O}$ is a nonempty open subset of V which does not contain any invertible matrices and A is a matrix in $\mathscr{O}$ of maximal rank $k<n$. Since A is symmetric, it can be diagonalized, i.e. there exists an orthonormal basis $e_1,\dots,e_n$ of $\mathbb{R}^n$ such that $Ae_j=\lambda_j e_j$ where λ_j are real and, since rank $A=k$, we may assume that $\lambda_1,\dots,\lambda_k\neq 0$ and $\lambda_{k+1}=\cdots=\lambda_n=0$.

By the assumption of the lemma, there exists $B\in V$ such that $\langle Be_{k+1}, e_{k+1}\rangle\neq 0$. Consider the matrix $A_t=A+tB$ which lies in $\mathscr{O}$ for sufficiently small t, and let $b(t)$ be the determinant of the matrix $(\langle A_t e_i, e_j\rangle)_{i,j=1}^{k+1}$. Then $b(0)=0$ and $b'(0)=\lambda_1\cdots\lambda_k\langle Be_{k+1}, e_{k+1}\rangle\neq 0$. Therefore, for small enough $t\neq 0$ we have $b(t)\neq 0$, which means that rank $A_t\geq k+1$. This contradicts the fact that k was the maximal rank of any matrix in $\mathscr{O}$. □

Now to show Theorem 1 it remains to follow the argument at the end of §4.3, with Lemma 4.8 replaced by Lemma 4.9 and using the following

Proposition 4.10 *Assume that $u\in\mathrm{Res}_{0\mathbb{R}}^1$ and $du\neq 0$. Then there exists $\mathbf{b}\in C^\infty(\Sigma;\mathbb{R})$ such that $\widetilde{S}_{\mathbf{b}}(du, du)\neq 0$.*

Proof Using the pushforward map $\pi_{\Sigma*}$ defined in (2.19) we compute by (2.20) and (2.23)

$$\widetilde{S}_{\mathbf{b}}(du, du) = -\int_\Sigma \mathbf{b}\pi_{\Sigma*}(\alpha\wedge du\wedge\mathcal{J}^*(du)). \tag{4.39}$$

By Corollary 5.1 below we have $\pi_{\Sigma*}(\alpha \wedge du \wedge \mathcal{J}^*(du)) \neq 0$ which finishes the proof. □

5 The pushforward identity

In this section we prove an identity, Theorem 5, used in Proposition 4.10 above which is a key component in the proof of our main Theorem 1.

We assume throughout this section that (Σ, g) is a compact hyperbolic 3-manifold as defined in §3.1 and write $\Sigma = \Gamma\backslash\mathbb{H}^3$ where $\Gamma \subset \mathrm{SO}_+(1,3)$. For $s > 2$, define the operator

$$\begin{aligned} Q_s &: C_c^\infty(\mathbb{H}^3) \to C^\infty(\mathbb{H}^3), \quad Q_s f(x) \\ &:= \int_{\mathbb{H}^3} \big(\cosh d_{\mathbb{H}^3}(x,y)\big)^{-s} f(y)\, d\,\mathrm{vol}_g(y). \end{aligned} \tag{5.1}$$

As shown in §5.1.2 below, the operator Q_s can be extended to Γ-invariant distributions on $\mathbb{H}^3$ and it is smoothing, so it descends to an operator

$$Q_s : \mathcal{D}'(\Sigma; \mathbb{C}) \to C^\infty(\Sigma; \mathbb{C}). \tag{5.2}$$

Let Δ_g be the (nonpositive) Laplace–Beltrami operator on (Σ, g). Recall the pushforward map on forms $\pi_{\Sigma*}$ defined in (2.19) and the spaces of (co-)resonant k-forms Res_0^k, Res_{0*}^k on $M = S\Sigma$ associated to the geodesic flow on (Σ, g), see §§2.2–2.3.

The main result of this section is the following

Theorem 5 *Assume that $u \in \mathrm{Res}_0^1$, $u_* \in \mathrm{Res}_{0*}^1$. Define the pushforwards*

$$\sigma_- := \pi_{\Sigma*}(d\alpha \wedge u), \quad \sigma_+ := \pi_{\Sigma*}(d\alpha \wedge u_*), \tag{5.3}$$

which are harmonic 1-forms on Σ by Lemma 3.17. Define $F \in \mathcal{D}'(\Sigma; \mathbb{C})$ by

$$\pi_{\Sigma*}(\alpha \wedge du \wedge du_*) = F\, d\,\mathrm{vol}_g\,. \tag{5.4}$$

Then we have

$$Q_4 F = -\tfrac{1}{6}\Delta_g(\sigma_- \cdot \sigma_+), \tag{5.5}$$

where the inner product $\sigma_- \cdot \sigma_+$ is the function on Σ defined by $\sigma_- \cdot \sigma_+(x) = \langle \sigma_-(x), \sigma_+(x)\rangle_g$.

Remark By (4.39) and since Q_4 is self-adjoint we can rewrite (5.5) as follows: for each $\mathbf{b} \in \mathcal{D}'(\Sigma)$,

$$\frac{1}{6}\int_{\Sigma} \mathbf{b}\,\Delta_g(\sigma_- \cdot \sigma_+)\,d\,\mathrm{vol}_g = \int_{S\Sigma} (\pi_\Sigma^* Q_4 \mathbf{b})\alpha \wedge du \wedge du_*. \tag{5.6}$$

One can think of the right-hand side of (5.6) as the integral of $\pi_\Sigma^* Q_4\mathbf{b}$ against a *Patterson–Sullivan distribution* $\alpha \wedge du \wedge du_*$ (note that this distribution is invariant under the geodesic flow) and the left-hand side of (5.6) as a *topological* quantity because it features harmonic 1-forms. Then (5.6) bears some similarity to the result of Anantharaman–Zelditch [2, Theorem 1.1] for the symbol $a := \pi_\Sigma^*\mathbf{b}$; the latter is in the setting when Σ is a surface and the left-hand side there has a *spectral* interpretation because it features an eigenfunction of the Laplacian. However, the operator L_r used in [2] is different in nature from the operator Q_4 featured in (5.6): for our application is crucial that the right-hand side of (5.6) depends only on the pushforward of $\alpha \wedge du \wedge du_*$ to Σ and that does not seem to typically be the case for the right-hand side of [2, Theorem 1.1]. See also the work of Hansen–Hilgert–Schröder [37] giving an asymptotic statement for higher dimensional situations.

The formula (5.6) in the special case $\mathbf{b} \equiv 1$ (which is trivial in our situation because both sides are equal to 0) also has some similarity to the pairing formulas of Dyatlov–Faure–Guillarmou [14, Lemma 5.10] and Guillarmou–Hilgert–Weich [32, Theorem 5]. In this vague analogy between Theorem 5 and the results of [2,14,32] our setting would correspond to an exceptional value of the spectral parameter: comparing (5.32) with [2, (1.3)] gives the value $\mathbf{s} = -2$ (in the notation of [2]).

Together with Proposition A.1, Theorem 5 gives the following statement which is used in the proof of Proposition 4.10. Recall the map $\mathcal{J}(x, v) = (x, -v)$ defined in (2.12).

Corollary 5.1 *Assume that $u \in \mathrm{Res}_0^1$ is real-valued and $du \neq 0$. Then $\pi_{\Sigma *}(\alpha \wedge du \wedge \mathcal{J}^*(du)) \neq 0$.*

Proof Put $u_* = \mathcal{J}^*u \in \mathrm{Res}_{0*}^1$. By (2.13) and (2.24) we have $\sigma_+ = \sigma_-$ where the 1-forms $\sigma_\pm$ are defined in (5.3). By Lemma 3.17, $\sigma = \sigma_+ = \sigma_-$ is a real-valued harmonic 1-form on Σ, and $du \neq 0$ implies that $\sigma \neq 0$.

Let F be defined in (5.4), then by Theorem 5 we have

$$Q_4 F = -\tfrac{1}{6}\Delta_g|\sigma|_g^2. \tag{5.7}$$

Now, by Proposition A.1 we see that $|\sigma|_g^2$ is not constant, that is $\Delta_g|\sigma|_g^2 \neq 0$. Therefore, $Q_4 F \neq 0$ which implies that $F \neq 0$. $\square$

5.1 Preliminary steps

We first prove several preliminary statements. We will use the hyperboloid model of §3.1.

5.1.1 Hyperbolic Laplacian

We first write the Laplacian Δ_g of the hyperbolic metric on $\mathbb{H}^3$ using the hyperboloid model. Consider the open cone

$$\mathcal{C}_+ := \{(\tilde{x}_0, \tilde{x}') \in \mathbb{R}^{1,3} \colon \tilde{x}_0 > |\tilde{x}'|\}.$$

Each point $\tilde{x} \in \mathcal{C}_+$ can be written in polar coordinates as

$$\tilde{x} = rx, \quad r > 0, \quad x \in \mathbb{H}^3.$$

Define the d'Alembert operator on $\mathcal{C}_+$ as $\Box = \partial_{\tilde{x}_0}^2 - \partial_{\tilde{x}_1}^2 - \partial_{\tilde{x}_2}^2 - \partial_{\tilde{x}_3}^2$. In polar coordinates it can be written as

$$\Box = r^{-2}\big((r\partial_r)^2 + 2r\partial_r - \Delta_g\big) \tag{5.8}$$

where the hyperbolic Laplacian Δ_g acts in the x variable.

Using (5.8), we derive the following useful identity: for any $\psi \in C^\infty((0,\infty))$ and $y \in \mathbb{H}^3$

$$\begin{aligned} -\Delta_g \psi(\langle x, y\rangle_{1,3}) &= \widetilde{\psi}(\langle x, y\rangle_{1,3}) \quad \text{where} \quad \widetilde{\psi}(\rho) \\ &:= (1-\rho^2)\psi''(\rho) - 3\rho\psi'(\rho) \end{aligned} \tag{5.9}$$

and the operator Δ_g acts in the x variable (note that $\widetilde{\psi}(\rho)$ is given by the radial part of $-\Delta_g$ applied to $\psi(\rho)$ by (3.4)). Indeed, it suffices to apply (5.8) to the function $f(\tilde{x}) := \psi(\langle \tilde{x}, y\rangle_{1,3})$, $\tilde{x} \in \mathcal{C}_+$, and use that $\Box f(\tilde{x}) = \psi''(\langle \tilde{x}, y\rangle_{1,3})$. Taking in particular $\psi(\rho) = \rho^{-s}$ where $s \in \mathbb{C}$, we get

$$\big(-\Delta_g - s(2-s)\big)\langle x, y\rangle_{1,3}^{-s} = s(s+1)\langle x, y\rangle_{1,3}^{-s-2}. \tag{5.10}$$

Similarly, if $\nu_-, \nu_+ \in \mathbb{S}^2 \subset \mathbb{R}^3$, then by applying (5.8) to the function

$$f_{\nu_-,\nu_+}(\tilde{x}) = \big(\langle \tilde{x}, (1,\nu_-)\rangle_{1,3}\, \langle \tilde{x}, (1,\nu_+)\rangle_{1,3}\big)^{-1}, \quad \tilde{x} \in \mathcal{C}_+$$

and using that $\Box f_{\nu_-,\nu_+} = 2(1-\nu_-\cdot\nu_+) f^2_{\nu_-,\nu_+}$, where we recall '$\cdot$' denotes the Euclidean inner product, we get

$$-\Delta_g\big(P(x,\nu_-)P(x,\nu_+)\big) = 2(1-\nu_-\cdot\nu_+)\big(P(x,\nu_-)P(x,\nu_+)\big)^2 \tag{5.11}$$

where the Poisson kernel $P(x,\nu)$ is defined in (3.18) and the Laplacian Δ_g acts in the x variable.

5.1.2 *Properties of the operators* Q_s

Let $Q_s : C_c^\infty(\mathbb{H}^3) \to C^\infty(\mathbb{H}^3)$ be the operator defined in (5.1). Using (3.4) we can rewrite it as

$$Q_s f(x) = \int_{\mathbb{H}^3} \langle x, y\rangle_{1,3}^{-s} f(y)\, d\,\mathrm{vol}_g(y). \tag{5.12}$$

Note that the operator Q_s is equivariant under the action of the group $\mathrm{SO}_+(1,3)$:

$$Q_s(\gamma^* f) = \gamma^*(Q_s f) \quad \text{for all} \quad \gamma \in \mathrm{SO}_+(1,3). \tag{5.13}$$

For $s > 2$, the function $y \mapsto \langle x, y\rangle_{1,3}^{-s}$ lies in $L^1(\mathbb{H}^3; d\,\mathrm{vol}_g)$ and its L^1 norm is independent of x; indeed, using the $\mathrm{SO}_+(1,3)$-invariance we may reduce to the case $x = (1,0,0,0)$, which can be handled by an explicit computation. Therefore, $Q_s : L^\infty(\mathbb{H}^3) \to L^\infty(\mathbb{H}^3)$.

The space $L^\infty(\Sigma)$ is isomorphic to the space of Γ-invariant functions in $L^\infty(\mathbb{H}^3)$. Using (5.13), we see that Q_s descends to the quotient $\Sigma = \Gamma\backslash\mathbb{H}^3$ as an operator

$$Q_s : L^\infty(\Sigma) \to L^\infty(\Sigma), \quad s > 2. \tag{5.14}$$

Next, using (5.10), we get the following identity relating the operators Q_s with the hyperbolic Laplacian Δ_g on Σ:

$$(-\Delta_g - s(2-s))Q_s = Q_s(-\Delta_g - s(2-s)) = s(s+1)Q_{s+2}. \tag{5.15}$$

Putting together (5.14) and (5.15) and using elliptic regularity, we see that for any $s > 2$, Q_s in fact extends to a smoothing operator $\mathcal{D}'(\Sigma) \to C^\infty(\Sigma)$, proving (5.2).

We now show that for $f \in \mathcal{D}'(\Sigma)$ one can obtain $Q_s f$ as a limit of cutoff integrals:

Lemma 5.2 *Fix a cutoff function $\chi(\rho) \in C_c^\infty(\mathbb{R})$ such that $\chi = 1$ near 0. For $\varepsilon > 0$ and $s > 2$, define the operator*

$$Q_{s,\chi,\varepsilon} : \mathcal{D}'(\mathbb{H}^3) \to C^\infty(\mathbb{H}^3),$$
$$Q_{s,\chi,\varepsilon} f(x) = \int_{\mathbb{H}^3} \chi(\varepsilon\langle x, y\rangle_{1,3})\langle x, y\rangle_{1,3}^{-s} f(y)\, d\,\mathrm{vol}_g(y).$$

Note that $Q_{s,\chi,\varepsilon}$ satisfies the equivariance relation (5.13) and thus descends to an operator $\mathcal{D}'(\Sigma) \to C^\infty(\Sigma)$. Then we have for all $f \in \mathcal{D}'(\Sigma)$

$$Q_{s,\chi,\varepsilon} f \to Q_s f \quad \text{in} \quad C^\infty(\Sigma) \quad \text{as} \quad \varepsilon \to +0. \tag{5.16}$$

Proof It suffices to show that for all $n \geq 0$,

$$\|\Delta_g^n(Q_s - Q_{s,\chi,\varepsilon})\Delta_g^n\|_{L^\infty(\Sigma)\to L^\infty(\Sigma)} \to 0 \quad \text{as} \quad \varepsilon \to +0.$$

By (5.9) with $\psi(\rho) := \rho^{-s}(1 - \chi(\varepsilon\rho))$ we have (with each instance of Δ_g in Δ_g^{2n} below acting in either x or y)

$$\Delta_g^{2n}\big(\langle x, y\rangle_{1,3}^{-s}(1 - \chi(\varepsilon\langle x, y\rangle_{1,3}))\big) = \langle x, y\rangle_{1,3}^{-s}\psi_{s,\chi,\varepsilon}^{(n)}(\langle x, y\rangle_{1,3}),$$

where, putting $T_s := \rho^s\big((1 - \rho^2)\partial_\rho^2 - 3\rho\partial_\rho\big)\rho^{-s}$,

$$\psi_{s,\chi,\varepsilon}^{(n)}(\rho) := T_s^{2n}(1 - \chi(\varepsilon\bullet))(\rho). \tag{5.17}$$

For any $f \in L^\infty(\mathbb{H}^3)$ we have (integrating by parts in y and using the fact that Δ_g is formally self-adjoint)

$$\Delta_g^n(Q_s - Q_{s,\chi,\varepsilon})\Delta_g^n f(x) = \int_{\mathbb{H}^3} \langle x, y\rangle_{1,3}^{-s}\psi_{s,\chi,\varepsilon}^{(n)}(\langle x, y\rangle_{1,3}) f(y)\, d\,\mathrm{vol}_g(y).$$

Estimating the $L_x^\infty L_y^1$ norm of the integral kernel of the latter operator we get for any $\delta \in (0, s-2)$ (we will use that $\delta > 0$ at the end of the proof) and for some $C_{s,\delta} > 0$ depending only on s, δ

$$\|\Delta_g^n(Q_s - Q_{s,\chi,\varepsilon})\Delta_g^n\|_{L^\infty(\Sigma)\to L^\infty(\Sigma)} \leq C_{s,\delta} \sup_{\rho\geq 1} |\rho^{-\delta}\psi_{s,\chi,\varepsilon}^{(n)}(\rho)|. \tag{5.18}$$

For $k \in \mathbb{N}_0$ and $\psi \in C^\infty((0, \infty))$, define the seminorm

$$\|\psi\|_{\delta,k} := \max_{0\leq j\leq k} \sup_{\rho\geq 1} |\rho^{-\delta}(\rho\partial_\rho)^j\psi(\rho)|.$$

We have $\|T_s\psi\|_{\delta,k} \le C_{s,\delta,k}\|\psi\|_{\delta,k+2}$. Therefore

$$\sup_{\rho\ge 1} |\rho^{-\delta}\psi^{(n)}_{s,\chi,\varepsilon}(\rho)| \le C_{s,\delta,n}\|1-\chi(\varepsilon\rho)\|_{\delta,4n} = \mathcal{O}(\varepsilon^{\delta}), \tag{5.19}$$

which finishes the proof. $\square$

5.1.3 Spherical convolution operators

Let $\kappa \in C^\infty([0,4])$. Define the smoothing operator

$$A_\kappa : \mathcal{D}'(\mathbb{S}^2) \to C^\infty(\mathbb{S}^2), \quad A_\kappa f(\nu) = \int_{\mathbb{S}^2} \kappa(|\nu-\nu'|^2) f(\nu')\, dS(\nu'). \tag{5.20}$$

Here $|\nu-\nu'|$ denotes the Euclidean distance between the points $\nu, \nu' \in \mathbb{S}^2 \subset \mathbb{R}^3$.

In this section we prove an estimate on the norm of A_κ between Sobolev spaces, Lemma 5.5, which is used in the regularization argument in §5.2.3 below. Before we state this estimate, we establish a few basic properties of A_κ:

Lemma 5.3 *We have*

$$\|A_\kappa\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \le \pi\,\|\kappa\|_{L^1([0,4])}.$$

Proof By Schur's lemma we have

$$\|A_\kappa\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)} \le \sup_{\nu'\in\mathbb{S}^2} \int_{\mathbb{S}^2} \left|\kappa(|\nu-\nu'|^2)\right| dS(\nu).$$

By SO(3)-invariance we see that the integral above is independent of ν'. Choose $\nu' = (0,0,-1)$ and use spherical coordinates $\nu = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ to compute

$$\int_{\mathbb{S}^2} \left|\kappa(|\nu-\nu'|^2)\right| dS(\nu) = 2\pi \int_0^\pi \left|\kappa(2+2\cos\theta)\right| \sin\theta\, d\theta = \pi \int_0^4 |\kappa(r)|\, dr$$

which finishes the proof. $\square$

Lemma 5.4 *Denote by $\Delta_{\mathbb{S}^2}$ the (nonpositive) Laplace–Beltrami operator on $\mathbb{S}^2$. Then*

$$A_\kappa \Delta_{\mathbb{S}^2} = \Delta_{\mathbb{S}^2} A_\kappa = A_{\tilde\kappa}, \quad \tilde\kappa(r) := (4-r)r\kappa''(r) + (4-2r)\kappa'(r). \tag{5.21}$$

Proof It is enough to show that, with $\Delta_{\mathbb{S}^2}$ acting in the ν variable,

$$\Delta_{\mathbb{S}^2}(\kappa(|\nu-\nu'|^2)) = \tilde{\kappa}(|\nu-\nu'|^2).$$

Similarly to the proof of Lemma 5.3, by SO(3)-invariance we may reduce to the case $\nu' = (0,0,-1)$ and take spherical coordinates (θ,φ) for ν, in which the Laplace operator is $\Delta_{\mathbb{S}^2} = (\sin\theta)^{-1}\partial_\theta \sin\theta\,\partial_\theta + (\sin\theta)^{-2}\partial_\varphi^2$ and $|\nu-\nu'|^2 = 2+2\cos\theta$. Then we compute

$$\begin{aligned}\Delta_{\mathbb{S}^2}(\kappa(|\nu-\nu'|^2)) &= \frac{1}{\sin\theta}\partial_\theta \sin\theta\,\partial_\theta\kappa(2+2\cos\theta)\\ &= 4\sin^2\theta\kappa''(2+2\cos\theta) - 4\cos\theta\kappa'(2+2\cos\theta)\\ &= \tilde{\kappa}(2+2\cos\theta),\end{aligned}$$

which finishes the proof. □

We can now give

Lemma 5.5 *Assume that $s_1, s_2 \in \mathbb{R}$ and $s_2 - s_1 = 2\ell$ for some $\ell \in \mathbb{N}_0$. Then there exists a constant C depending only on s_1, s_2 such that for all $\kappa \in C^\infty([0,4])$*

$$\|A_\kappa\|_{H^{s_1}(\mathbb{S}^2)\to H^{s_2}(\mathbb{S}^2)} \le C\sum_{j=0}^{2\ell} \|r^{\max(j-\ell,0)}\partial_r^j\kappa(r)\|_{L^1([0,4])}. \tag{5.22}$$

Proof Define the differential operator arising from (5.21) (corresponding to $1-\Delta_{\mathbb{S}^2}$)

$$W := (r-4)r\partial_r^2 + (2r-4)\partial_r + 1.$$

Denote by C a constant depending only on s_1, s_2, whose precise value may change from line to line. We have

$$\begin{aligned}\|A_\kappa\|_{H^{s_1}(\mathbb{S}^2)\to H^{s_2}(\mathbb{S}^2)} &\le C\|(1-\Delta_{\mathbb{S}^2})^{s_2/2}A_\kappa(1-\Delta_{\mathbb{S}^2})^{-s_1/2}\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)}\\ &= C\|(1-\Delta_{\mathbb{S}^2})^{\ell}A_\kappa\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)}\\ &= C\|A_{W^\ell\kappa}\|_{L^2(\mathbb{S}^2)\to L^2(\mathbb{S}^2)}\\ &\le C\|W^\ell\kappa\|_{L^1([0,4])}.\end{aligned}$$

Here in the second equality we used that A_κ commutes with $\Delta_{\mathbb{S}^2}$ by Lemma 5.4. In the third inequality we used Lemma 5.4 again. In the last inequality we used Lemma 5.3.

By induction in ℓ we see that W^ℓ is a linear combination with constant coefficients of the operators $r^k\partial_r^j$ where $0 \le j \le 2\ell$ and $k \ge \max(j-\ell, 0)$. Therefore, $\|W^\ell\kappa\|_{L^1([0,4])}$ is bounded by the right-hand side of (5.22), which finishes the proof. □

5.2 Proof of Theorem 5

Here we give the proof of Theorem 5, proceeding in several steps. In §5.2.1 we write both sides of (5.5) as integrals featuring some distributions $g_\pm$ on $\mathbb{S}^2$. In §5.2.2 we introduce a change of variables which shows that the two integrals are formally equal. In §5.2.3 we prove that regularized versions of the two integrals are equal and show convergence of the regularization to finish the proof.

Denote by π_Γ the covering maps $\mathbb{H}^3 \to \Sigma$ and $S\mathbb{H}^3 \to M = S\Sigma$ (which one is meant will be clear from the context). Since we can choose the representation of Σ as the quotient $\Gamma\backslash\mathbb{H}^3$ arbitrarily, for any given $x \in \Sigma$ we may arrange that $\pi_\Gamma(e_0) = x$ where

$$e_0 := (1, 0, 0, 0) \in \mathbb{H}^3. \tag{5.23}$$

Therefore, in order to prove Theorem 5 it suffices to consider the case $x = \pi_\Gamma(e_0)$, i.e. to show that

$$\pi_\Gamma^* Q_4 F(e_0) = -\tfrac{1}{6}\pi_\Gamma^*\Delta_g(\sigma_-\cdot\sigma_+)(e_0). \tag{5.24}$$

5.2.1 Reduction to the conformal boundary

We first express both sides of (5.24) in terms of some distributions $g_\pm$ on the conformal boundary $\mathbb{S}^2$.

Let $u \in \mathrm{Res}_0^1$, $u_* \in \mathrm{Res}_{0*}^1$. By Lemma 4.7 we have

$$du = f_-\omega_-, \quad du_* = f_+\omega_+, \quad \alpha\wedge du\wedge du_* = -\tfrac{1}{8} f_- f_+ d\,\mathrm{vol}_\alpha,$$

where by (4.35), the lifts of $f_- \in \mathcal{D}'_{E_u^*}(M;\mathbb{C})$, $f_+ \in \mathcal{D}'_{E_s^*}(M;\mathbb{C})$ to the covering space $S\mathbb{H}^3$ have the form (recalling the definitions (3.14) of $\Phi_\pm$, $B_\pm$)

$$\pi_\Gamma^* f_\pm = \Phi_\pm^{-2}(g_\pm\circ B_\pm) \quad \text{for some} \quad g_\pm \in \mathcal{D}'(\mathbb{S}^2;\mathbb{C}). \tag{5.25}$$

Arguing similarly to (2.21), we see that the distribution $F \in \mathcal{D}'(\Sigma; \mathbb{C})$ defined in (5.4) can be written as the pushforward

$$F(x) = \frac{1}{4} \int_{S_x \Sigma} f_-(x, v) f_+(x, v)\, dS(v), \quad x \in \Sigma$$

where dS is the canonical volume form on the spherical fiber $S_x\Sigma$. Therefore, the lift of F to $\mathbb{H}^3$ has the form

$$\pi_\Gamma^* F(x) = \frac{1}{4} \int_{S_x \mathbb{H}^3} \big(\Phi_-(x, v)\Phi_+(x, v)\big)^{-2} g_-(B_-(x, v)) g_+(B_+(x, v))\, dS(v). \tag{5.26}$$

We next express the harmonic 1-forms $\sigma_\pm$ defined in (5.3) in terms of the distributions $g_\pm$:

Lemma 5.6 *Using the hyperbolic metric, identify the pullbacks $\pi_\Gamma^*\sigma_\pm$ with vector fields on $\mathbb{H}^3$. Then for any $x \in \mathbb{H}^3$*

$$\pi_\Gamma^* \sigma_\pm(x) = \frac{1}{4} \int_{\mathbb{S}^2} g_\pm(\nu) v_\pm(x, \nu)\, dS(\nu),$$

where $v_\pm(x, \nu) \in S_x\mathbb{H}^3 \subset T_x\mathbb{H}^3$ is defined in (3.20).

Proof By (3.72) and since $du = f_-\omega_-$, $du_* = f_+\omega_+$ we have

$$\sigma_\pm = \pi_{\Sigma *}(f_\pm \alpha \wedge \omega_\pm).$$

Recall the horizontal/vertical decomposition (2.15). For any $(x, v) \in M = S\Sigma$, $\xi = (\xi_H, \xi_V) \in T_{(x,v)}M$, and a positively oriented g-orthonormal basis $v, v_1, v_2 \in T_x\Sigma$ we compute by (2.16) and (4.28)

$$(\alpha \wedge \omega_\pm)(x, v)(\xi, (0, v_1), (0, v_2)) = \tfrac{1}{4} \langle \xi_H, v \rangle_g.$$

Using the metric g, we identify $\sigma_\pm$ with a vector field on Σ. Then

$$\sigma_\pm(x) = \frac{1}{4} \int_{S_x \Sigma} f_\pm(x, v) v\, dS(v), \quad x \in \Sigma.$$

It follows that for each $x \in \mathbb{H}^3$

$$\begin{aligned} \pi_\Gamma^* \sigma_\pm(x) &= \frac{1}{4} \int_{S_x \mathbb{H}^3} \Phi_\pm(x, v)^{-2} g_\pm(B_\pm(x, v)) v\, dS(v) \\ &= \frac{1}{4} \int_{\mathbb{S}^2} g_\pm(\nu) v_\pm(x, \nu)\, dS(\nu). \end{aligned}$$

Here in the first equality we used (5.25). In the second equality we made the change of variables $\nu = B_\pm(x, v)$ and used (3.21). ◻

We note that by the preceding lemma $v_\pm(x, \nu)$ define vector-valued Poisson kernels in the sense of [43,51]. From Lemma 5.6 we get the following formula for the right-hand side of (5.24) in terms of the distributions $g_\pm$:

Lemma 5.7 *We have (here e_0 is defined in (5.23))*

$$-\pi_\Gamma^* \Delta_g(\sigma_- \cdot \sigma_+)(e_0) = \frac{1}{8} \int_{\mathbb{S}^2 \times \mathbb{S}^2} (1 - \nu_- \cdot \nu_+)^2 g_-(\nu_-) g_+(\nu_+) \, dS(\nu_-) dS(\nu_+). \tag{5.27}$$

Proof By (3.20) we have for each $\nu_-, \nu_+ \in \mathbb{S}^2$ and $x \in \mathbb{H}^3$

$$\begin{aligned} \langle v_-(x, \nu_-), v_+(x, \nu_+) \rangle_g &= -\langle v_-(x, \nu_-), v_+(x, \nu_+) \rangle_{1,3} \\ &= P(x, \nu_-) P(x, \nu_+)(1 - \nu_- \cdot \nu_+) - 1. \end{aligned}$$

With the hyperbolic Laplacian Δ_g acting in the x variable, we then compute by (5.11)

$$-\Delta_g \langle v_-(x, \nu_-), v_+(x, \nu_+) \rangle_g = 2(1 - \nu_- \cdot \nu_+)^2 \big(P(x, \nu_-) P(x, \nu_+)\big)^2.$$

Now (5.27) follows from Lemma 5.6 by integration and using that $P(e_0, \nu_\pm) = 1$ by (3.18). ◻

5.2.2 Change of variables

By (5.26) and (5.12) we can formally write the left-hand side of (5.24) as follows:

$$\begin{aligned} \pi_\Gamma^* Q_4 F(e_0) = \frac{1}{4} \int_{S\mathbb{H}^3} & y_0^{-4} \big(\Phi_-(y, v) \Phi_+(y, v)\big)^{-2} \\ & \times g_-(B_-(y, v)) g_+(B_+(y, v)) dS(v) d\,\mathrm{vol}_g(y), \end{aligned} \tag{5.28}$$

where we recall $y = (y_0, y_1, y_2, y_3) \in \mathbb{H}^3$. Note that one has to take care when defining the integral above, as $g_\pm$ are distributions and $S\mathbb{H}^3$ is noncompact, see §5.2.3 below.

On the other hand, the right-hand side of (5.24) can be expressed using (5.27) as an integral over $(\nu_-, \nu_+) \in \mathbb{S}^2 \times \mathbb{S}^2$. To prove (5.24) and relate the two integrals we will use the change of variables $\Xi : (y, v) \mapsto (\nu_-, \nu_+, t)$, where $t \in \mathbb{R}$, introduced in (3.16). The basic properties of Ξ are collected below in

Lemma 5.8 *1. Let* $(\nu_-, \nu_+, t) = \Xi(y, v)$. *Then*

$$\Phi_-(y, v)\Phi_+(y, v) = \frac{4}{|\nu_- - \nu_+|^2} = \frac{2}{1 - \nu_- \cdot \nu_+}, \tag{5.29}$$

$$y_0 = \frac{2\cosh t}{|\nu_- - \nu_+|}. \tag{5.30}$$

(As before, we write elements of $\mathbb{H}^3$ *as* $y = (y_0, y_1, y_2, y_3) \in \mathbb{R}^{1,3}$*.)*

2. The Jacobian of Ξ *at* (y, v) *with respect to the densities* $d\,\mathrm{vol}_g(y)dS(v)$ *and* $dS(\nu_-)dS(\nu_+)dt$ *is equal to* $4\big(\Phi_-(y, v)\Phi_+(y, v)\big)^{-2}$.

Remark The identity in part 2 of the above is well-known, see [50, Theorem 8.1.1 on p. 131].

Proof 1. The identity (5.29) follows immediately from (3.17), noting that $|\nu_- - \nu_+|^2 = 2(1 - \nu_- \cdot \nu_+)$. To see (5.30), we compute by (5.29) and (3.16)

$$\Phi_\pm(y, v) = \frac{2e^{\pm t}}{|\nu_- - \nu_+|},$$

which by (3.15) gives

$$y_0 = \frac{\Phi_-(y, v) + \Phi_+(y, v)}{2} = \frac{2\cosh t}{|\nu_- - \nu_+|}.$$

2. Take $(y, v) \in S\mathbb{H}^3$. Let $w \in T_y\mathbb{H}^3$ satisfy $\langle v, w\rangle_{1,3} = 0$. Then

$$|dB_\pm(y, v)(w, \pm w)|_{\mathbb{S}^2} = 2|dB_\pm(y, v)(0, w)|_{\mathbb{S}^2} = \frac{2|w|_g}{\Phi_\pm(y, v)}. \tag{5.31}$$

Here in the first equality we write $(w, \pm w) = (w, \mp w) \pm 2(0, w)$ and use that by (3.23), $dB_\pm(y, v)(w, \mp w) = 0$. In the second equality we use (3.21). Denoting by X the generator of the geodesic flow and defining t by (3.16), we also have by (3.22) and (3.23)

$$dB_\pm(y, v)(X(y, v)) = 0, \quad dt(X(y, v)) = 1.$$

Fix a g-orthonormal basis v, v_1, v_2 of $T_y\mathbb{H}^3$ and consider the following basis of $T_{(y,v)}S\mathbb{H}^3$:

$$\xi_0 = X(y, v), \quad \xi_1^\pm = (v_1, \pm v_1), \quad \xi_2^\pm = (v_2, \pm v_2).$$

Since $\xi_j^- \wedge \xi_j^+ = 2(v_j, 0) \wedge (0, v_j)$, the value of the density $d\,\mathrm{vol}_g(y)dS(v)$ on

$\xi_0, \xi_1^-, \xi_2^-, \xi_1^+, \xi_2^+$ is equal to 4. On the other hand, writing $(\eta_-(\xi), \eta_+(\xi), \tau(\xi)) = d\Xi(y, v)(\xi)$, we have

$$\eta_\pm(\xi_j^\mp) = \eta_\pm(\xi_0) = 0, \quad \tau(\xi_0) = 1$$

and the vectors $\eta_\pm(\xi_1^\pm), \eta_\pm(\xi_2^\pm)$ are orthogonal to each other and have length $2\Phi_\pm(y, v)^{-1}$ each by (5.31). It follows that the value of the density $dS(\nu_-)dS(\nu_+)dt$ on the images of $\xi_0, \xi_1^-, \xi_2^-, \xi_1^+, \xi_2^+$ under $d\Xi(y, v)$ is equal to $16\big(\Phi_-(y, v)\Phi_+(y, v)\big)^{-2}$. Thus the Jacobian of Ξ at (y, v) is equal to $4\big(\Phi_-(y, v)\Phi_+(y, v)\big)^{-2}$. $\square$

Using Lemma 5.8 and (5.28), we can formally write the left-hand side of (5.24) as

$$\pi_\Gamma^* Q_4 F(e_0) = \frac{1}{64} \int_{(\mathbb{S}^2 \times \mathbb{S}^2)_- \times \mathbb{R}} \frac{(1 - \nu_- \cdot \nu_+)^2}{\cosh^4 t} g_-(\nu_-) g_+(\nu_+)\, dS(\nu_-) dS(\nu_+) dt. \tag{5.32}$$

Using the change of variables $s = \tanh t$, we compute

$$\int_{\mathbb{R}} \frac{dt}{\cosh^4 t} = \int_{-1}^{1} (1 - s^2)\, ds = \frac{4}{3}. \tag{5.33}$$

Comparing (5.32) with (5.27), we formally obtain the identity (5.24). However, our argument is incomplete since the integrals in (5.28) and (5.32) are over the noncompact manifolds $S\mathbb{H}^3$, $(\mathbb{S}^2 \times \mathbb{S}^2)_- \times \mathbb{R}$ and $g_\pm$ are distributions. Thus one cannot immediately apply the change of variables formula to get (5.32) from (5.28), or Fubini's Theorem to get (5.24) from (5.32). To deal with these issues, we will employ a regularization procedure.

5.2.3 Regularization and end of the proof

Fix a cutoff function

$$\chi \in C_c^\infty(\mathbb{R}; [0, 1]), \quad \operatorname{supp} \chi \subset [-2, 2], \quad \chi|_{[-1,1]} = 1.$$

For $\varepsilon > 0$, define the integral

$$I_\varepsilon := \int_{\mathbb{H}^3} \chi(\varepsilon y_0) y_0^{-4} \pi_\Gamma^* F(y)\, d\operatorname{vol}_g(y).$$

(As before, we embed $\mathbb{H}^3$ into $\mathbb{R}^{1,3}$ and we have $y_0 = \langle e_0, y\rangle_{1,3}$ where$e_0 =$

$(1, 0, 0, 0)$.) By Lemma 5.2 with $x = e_0$, I_ε converges to the left-hand side of (5.24):

$$I_\varepsilon \to \pi_\Gamma^* Q_4 F(e_0) \quad \text{as} \quad \varepsilon \to +0. \tag{5.34}$$

By (5.34) and (5.27), the proof of (5.24) (and thus of Theorem 5) is finished once we show that

$$I_\varepsilon \to \frac{1}{48} \int\limits_{\mathbb{S}^2 \times \mathbb{S}^2} (1 - \nu_- \cdot \nu_+)^2 g_-(\nu_-) g_+(\nu_+)\, dS(\nu_-) dS(\nu_+) \quad \text{as} \quad \varepsilon \to +0. \tag{5.35}$$

By (5.26) we have the following regularized version of (5.28):

$$\begin{aligned} I_\varepsilon = \frac{1}{4} \int_{S\mathbb{H}^3} &\chi(\varepsilon y_0) y_0^{-4} \big(\Phi_-(y, v)\Phi_+(y, v)\big)^{-2} \\ &\times g_-(B_-(y, v)) g_+(B_+(y, v))\, dS(v) d\,\mathrm{vol}_g(y). \end{aligned}$$

Making the change of variables $(\nu_-, \nu_+, t) = \Xi(y, v)$ and using Lemma 5.8, we then get the following regularized version of (5.32) (we keep in mind that $g_\pm$ are merely distributions so that all of the integrals around these lines are understood in the distributional sense):

$$\begin{aligned} I_\varepsilon = \frac{1}{64} \int\limits_{\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{R}} &\chi\Big(\frac{2\varepsilon \cosh t}{|\nu_- - \nu_+|}\Big) \frac{(1 - \nu_- \cdot \nu_+)^2}{\cosh^4 t} \\ &\times g_-(\nu_-) g_+(\nu_+)\, dS(\nu_-) dS(\nu_+) dt. \end{aligned}$$

For $r \geq 0$, define the function

$$\psi_\varepsilon(r) := \frac{3}{4} \int_{\mathbb{R}} \chi\Big(\frac{2\varepsilon \cosh t}{\sqrt{r}}\Big) \cosh^{-4} t\, dt. \tag{5.36}$$

Note that $\psi_\varepsilon \in C^\infty([0, \infty))$ and $\psi_\varepsilon(r) = 0$ for $r \ll \varepsilon^2$. We now have

$$I_\varepsilon = \frac{1}{48} \int\limits_{\mathbb{S}^2 \times \mathbb{S}^2} \psi_\varepsilon(|\nu_- - \nu_+|^2)(1 - \nu_- \cdot \nu_+)^2 g_-(\nu_-) g_+(\nu_+)\, dS(\nu_-) dS(\nu_+). \tag{5.37}$$

Recalling that $|\nu_- - \nu_+|^2 = 2(1 - \nu_- \cdot \nu_+)$, we see from (5.37) that it suffices to prove the following version of (5.35):

$$\int\limits_{\mathbb{S}^2 \times \mathbb{S}^2} \big(1 - \psi_\varepsilon(|\nu_- - \nu_+|^2)\big)|\nu_- - \nu_+|^4 g_-(\nu_-) \times g_+(\nu_+)\, dS(\nu_-) dS(\nu_+) \to 0 \quad \text{as} \quad \varepsilon \to +0. \tag{5.38}$$

If $g_\pm$ were smooth functions on $\mathbb{S}^2$, then (5.38) would follow from the Dominated Convergence Theorem since by (5.33) we have $\psi_\varepsilon(r) \to 1$ as $\varepsilon \to +0$ for all $r > 0$. However, $g_\pm$ are merely distributions, so one has to be more careful. We start by establishing the Sobolev regularity of $g_\pm$ by following the standard proof of the Fredholm property in anisotropic Sobolev spaces. (We use the proof in [20]; one could alternatively carefully examine the proof in [29].) See the papers of Adam–Baladi [1, §3.3], Guillarmou–Poyferré–Bonthonneau [30, Appendix A], and Dyatlov [19] for a general discussion of Sobolev regularity thresholds for the Pollicott–Ruelle resolvent.

Lemma 5.9 *We have $g_\pm \in H^{-2-\delta}(\mathbb{S}^2)$ for all $\delta > 0$.*

Proof We show the regularity of g_-, with g_+ handled similarly. Recall that g_- is related to the distribution $f_- \in \mathcal{D}'_{E_u^*}(M; \mathbb{C})$ by (5.25). Since Φ_- is smooth and B_- is a submersion, it suffices to show that $f_- \in H^{-2-\delta}(M)$.

By Lemma 4.7, we have $(X - 2) f_- = 0$, that is f_- is a Pollicott–Ruelle resonant state for the operator $P = -iX$ corresponding to the resonance $\lambda_0 = -2i$, see §2.3.2. Given that Pollicott–Ruelle resonant states are eigenfunctions of P on anisotropic Sobolev spaces (see (4.10)), it suffices to show that one can choose the order function m in the definition of the weight $G(\rho, \xi) = m(\rho, \xi) \log(1 + |\xi|)$ such that the Fredholm property (4.11) holds on the anisotropic Sobolev space $\mathcal{H}_{G,0}$ for $\operatorname{Im} \lambda \geq -2$ and $\mathcal{H}_{G,0} \subset H^{-2-\delta}$; the latter is equivalent to requiring that $m \geq -2 - \delta$ everywhere.

In [20, §§3.3–3.4] the Fredholm property (4.11) is shown using propagation of singularities and microlocal radial estimates. Following the proof of [20, Proposition 3.4], we see that one only needs to check that the low regularity radial estimate [20, Proposition 2.7] applies to the operator $P - \lambda$ (where $\operatorname{Im} \lambda \geq -2$) at the radial sink E_u^* (see (2.4)) in the space $H^{-2-\delta}$. (The high regularity radial estimate [20, Proposition 2.6] would apply once m is sufficiently large on E_s^*, which can be arranged.) The threshold regularity for this estimate is computed in [22, Theorem E.54]. In our setting, since the operator P is symmetric on $L^2(M; d\,\mathrm{vol}_\alpha)$ and it has order $k = 1$, it is enough that

$$2 + (-2 - \delta) \frac{H_p |\xi|}{|\xi|} < 0 \quad \text{on} \quad E_u^*$$

where $p(\rho, \xi) = \langle X(\rho), \xi \rangle$ is the principal symbol of P and its Hamiltonian flow is given by $e^{tH_p}(\rho, \xi) = (\varphi_t(\rho), d\varphi_t^{-T}(\rho)\xi)$, see [20, §3.1]. Choosing the norm $|\xi|$ induced by the Sasaki metric and using (3.7), we see that

$$\frac{H_p|\xi|}{|\xi|} = 1 \quad \text{on} \quad E_u^*,$$

which means that the threshold regularity condition for the radial estimate is satisfied and the proof is finished. □

Coming back to the proof of (5.38), we rewrite it as

$$\langle A_{\kappa_\varepsilon} g_-, \overline{g_+} \rangle_{L^2(\mathbb{S}^2)} \to 0 \quad \text{as} \quad \varepsilon \to +0, \tag{5.39}$$

where the operator A_{κ_ε} is given by (5.20):

$$A_{\kappa_\varepsilon} f(\nu_+) = \int_{\mathbb{S}^2} \kappa_\varepsilon(|\nu_- - \nu_+|^2) f(\nu_-)\, dS(\nu_-)$$

and the function $\kappa_\varepsilon \in C([0,4])$ is given by (using (5.33) and (5.36) in the second equality below)

$$\kappa_\varepsilon(r) := \frac{4}{3} r^2 (1 - \psi_\varepsilon(r)) = r^2 \int_{\mathbb{R}} \left(1 - \chi\left(\frac{2\varepsilon \cosh t}{\sqrt{r}}\right)\right) \cosh^{-4} t\, dt.$$

Using Lemma 5.9, we have in particular $g_\pm \in H^{-5/2}(\mathbb{S}^2)$. Thus to finish the proof of (5.39), and thus of Theorem 5, it remains to prove the norm bound

$$\|A_{\kappa_\varepsilon}\|_{H^{-5/2}(\mathbb{S}^2) \to H^{5/2}(\mathbb{S}^2)} \to 0 \quad \text{as} \quad \varepsilon \to +0. \tag{5.40}$$

To show (5.40), we will bound the norms of A_{κ_ε} between Sobolev spaces using Lemma 5.5. To do this we estimate the derivatives of κ_ε:

Lemma 5.10 *Let $j, k \in \mathbb{N}_0$. Then there exists C depending only on j, k such that for all $\varepsilon \in (0, 1]$*

$$\|r^k \partial_r^j \kappa_\varepsilon(r)\|_{L^1([0,4])} \le \begin{cases} C\varepsilon^4, & k \ge j; \\ C\varepsilon^4 \log(1/\varepsilon), & k = j - 1; \\ C\varepsilon^{2(3+k-j)}, & k \le j - 2. \end{cases} \tag{5.41}$$

Proof Throughout the proof we denote by C a constant depending only on j, k whose precise value might change from line to line.

1. For any $G(s) \in C^\infty([0,\infty))$ which is constant near $s=\infty$ define

$$\Phi_G(\tau) := \int_{\mathbb{R}} G\Big(\frac{2\cosh t}{\sqrt{\tau}}\Big)\cosh^{-4} t\,dt, \quad \tau > 0.$$

We have the identity

$$\tau\partial_\tau \Phi_G = -\tfrac{1}{2}\Phi_{s\partial_s G}. \tag{5.42}$$

Moreover, we have the estimate

$$G|_{[-1,1]} = 0 \quad \Longrightarrow \quad |\Phi_G(\tau)| \le \frac{C\|G\|_{L^\infty}}{1+\tau^2}, \tag{5.43}$$

which can be proved by bounding $|\Phi_G(\tau)|$ by $\|G\|_{L^\infty}$ times the integral of $\cosh^{-4} t\,dt$ over the set of t such that $\cosh t \ge \sqrt{\tau}/2$ and using that $\int \cosh^{-4} t\,dt = \tanh t - \frac{1}{3}\tanh^3 t + C$ and $\sqrt{1-\lambda} - \frac{1}{3}(1-\lambda)^{3/2} = \frac{2}{3} + \mathcal{O}(\lambda^2)$ as $\lambda = \frac{4}{\tau} \to 0$.
2. We have

$$\kappa_\varepsilon(r) = r^2 \Phi_{1-\chi}(\varepsilon^{-2} r).$$

By (5.42) for each $j \ge 0$

$$(r\partial_r)^j \kappa_\varepsilon(r) = r^2 (r\partial_r + 2)^j \big(\Phi_{1-\chi}(\varepsilon^{-2} r)\big) = r^2 \Phi_{G_j}(\varepsilon^{-2} r),$$
$$\text{where} \quad G_j(s) := (2 - \tfrac{1}{2} s\partial_s)^j (1-\chi)(s).$$

Since $\chi|_{[-1,1]} = 1$, we have $G_j|_{[-1,1]} = 0$. Thus by (5.43)

$$|(r\partial_r)^j \kappa_\varepsilon(r)| \le \frac{Cr^2}{1+\varepsilon^{-4} r^2}.$$

Writing $r^j \partial_r^j$ as a linear combination of $(r\partial_r)^q$ with $0 \le q \le j$, we get

$$|\partial_r^j \kappa_\varepsilon(r)| \le \frac{Cr^{2-j}}{1+\varepsilon^{-4} r^2} \le C\varepsilon^4 r^{-j}.$$

Since $\operatorname{supp}\chi \subset [-2,2]$, we have by (5.33)

$$\kappa_\varepsilon(r) = \tfrac{4}{3} r^2 \quad \text{for} \quad 0 \le r \le \varepsilon^2.$$

Therefore

$$\|r^k \partial_r^j \kappa_\varepsilon(r)\|_{L^1([0,4])} \le C \int_0^{\varepsilon^2} r^k \partial_r^j (r^2)\, dr + C\varepsilon^4 \int_{\varepsilon^2}^4 r^{k-j}\, dr$$

which gives (5.41). □

Combining Lemma 5.5 and Lemma 5.10, we get

$$\|A_{\kappa_\varepsilon}\|_{H^{-5/2}\to H^{3/2}} \le C\varepsilon^2, \quad \|A_{\kappa_\varepsilon}\|_{H^{-5/2}\to H^{7/2}} \le C.$$

By interpolation in Sobolev spaces (taking $f \in H^{-5/2}(\mathbb{S}^2)$ and using that $\|v\|^2_{H^1(\mathbb{S}^2)}$ is bounded by $\langle(1-\Delta_{\mathbb{S}^2})v, v\rangle_{L^2(\mathbb{S}^2)} \le C\|v\|_{L^2(\mathbb{S}^2)}\|v\|_{H^2(\mathbb{S}^2)}$ for $v := (1-\Delta_{\mathbb{S}^2})^{3/4} A_{\kappa_\varepsilon} f)$ we then have

$$\|A_{\kappa_\varepsilon}\|_{H^{-5/2}\to H^{5/2}} \le C\varepsilon.$$

This gives (5.40) and finishes the proof of Theorem 5.

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Appendix A. Harmonic 1-forms of constant length

The purpose of this appendix is to give an elementary proof of the fact that there are no harmonic 1-forms of constant nonzero length on closed hyperbolic 3-manifolds:

Proposition A.1 *Let (Σ, g) be a compact hyperbolic 3-manifold (see §3.1). Assume that $\omega \in C^\infty(\Sigma; T^*\Sigma)$ is a harmonic 1-form such that its length $|\omega|_g$ is constant. Then $\omega = 0$.*

Remark Proposition A.1 follows directly from the more general work of [55]. The presentation in the appendix borrows from ideas in [39].

To prove Proposition A.1 we argue by contradiction. Assume that $\omega \neq 0$; dividing ω by its length we arrange that, where $\delta = -\star d\star$ is the formal adjoint of d (here $\star$ is the Hodge star)

$$d\omega = 0, \quad \delta\omega = 0, \quad |\omega|_g = 1.$$

Using the metric g, define the dual vector field to ω,

$$W \in C^\infty(\Sigma; T\Sigma), \quad |W|_g = \omega(W) = 1.$$

Lemma A.2 *There exist one-dimensional smooth subbundles $E_\pm \subset T\Sigma$ such that $T\Sigma = \mathbb{R}W \oplus E_+ \oplus E_-$.*

Proof 1. The Levi-Civita covariant derivative ∇W is an endomorphism on the fibers of $T\Sigma$. This endomorphism is symmetric with respect to the metric g; indeed we compute for any two vector fields $Y, Z \in C^\infty(\Sigma; T\Sigma)$

$$\begin{aligned} 0 = d\omega(Y, Z) &= Yg(W, Z) - Zg(W, Y) - g(W, [Y, Z]) \\ &= g(\nabla_Y W, Z) - g(\nabla_Z W, Y). \end{aligned} \tag{A.1}$$

Taking $Z := W$ and using that $g(\nabla_Y W, W) = \frac{1}{2} Yg(W, W) = 0$ we see that the vector field W is geodesible, that is

$$\nabla_W W = 0. \tag{A.2}$$

Since $\delta\omega = 0$, the vector field W is also divergence free; that is,

$$\mathrm{tr}(\nabla W) = 0. \tag{A.3}$$

2. We next claim that

$$\mathrm{tr}((\nabla W)^2) = 2. \tag{A.4}$$

To see this, take locally defined vector fields Y_1, Y_2 such that W, Y_1, Y_2 is a g-orthonormal frame and $\nabla_W Y_j = 0$. These can be obtained using parallel transport along the flow lines of W (which are geodesics since $\nabla_W W = 0$). We compute

$$\begin{aligned}1 &= g(\nabla_W \nabla_{Y_j} W - \nabla_{Y_j} \nabla_W W + \nabla_{\nabla_{Y_j} W} W - \nabla_{\nabla_W Y_j} W, Y_j)\\ &= W g(\nabla_{Y_j} W, Y_j) - g(\nabla_{Y_j} W, \nabla_W Y_j)\\ &\quad + g(\nabla_{\nabla_{Y_j} W} W, Y_j) - g(\nabla_{\nabla_W Y_j} W, Y_j)\\ &= W g(\nabla_{Y_j} W, Y_j) + g((\nabla W)^2 Y_j, Y_j).\end{aligned}$$

Here in the first line we used that Σ has sectional curvature -1, in the second line we used (A.2), and in the last line we used that $\nabla_W Y_j = 0$. Summing over $j = 1, 2$ and using again (A.2) we get

$$2 = W \operatorname{tr}(\nabla W) + \operatorname{tr}((\nabla W)^2)$$

and (A.4) now follows from (A.3).

3. From (A.2), (A.3), and (A.4) we see that ∇W has eigenvalues $0, 1, -1$. It remains to let $E_\pm$ be the eigenspaces of ∇W with eigenvalues ± 1. □

We are now ready to finish the proof of Proposition A.1. We can approximate the 1-form ω by a closed 1-form with rational periods (integrals over closed curves on Σ); indeed, for an appropriate choice of linear isomorphism $H^1(\Sigma; \mathbb{C}) \simeq \mathbb{C}^{b_1(\Sigma)}$ the forms with rational periods correspond to points in $\mathbb{Q}^{b_1(\Sigma)}$. In particular, we can find a number $q \in \mathbb{N}$ and a closed 1-form $\widetilde{\omega}$ with integer periods such that

$$\sup_\Sigma |\omega - q^{-1}\widetilde{\omega}|_g \le \tfrac{1}{2}. \tag{A.5}$$

Since $\widetilde{\omega}$ has integer periods, we can write $\widetilde{\omega} = df$ for some smooth map f from Σ to the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Since $\omega(W) = 1$, (A.5) implies that $Wf = \widetilde{\omega}(W) > 0$ which in turn gives $df \neq 0$ everywhere, that is f is a fibration. Next, for each $x \in \Sigma$ define the one-dimensional spaces

$$\widetilde{E}_\pm(x) := (\mathbb{R}W(x) \oplus E_\pm(x)) \cap \ker df(x),$$

then the tangent bundle of each fiber $f^{-1}(c)$ decomposes into a direct sum $\widetilde{E}_+ \oplus \widetilde{E}_-$. Since Σ is orientable, so is $f^{-1}(c)$, which implies that $f^{-1}(c)$ is topologically a torus. Then Σ is a torus bundle over a circle, which gives a contradiction because such bundles do not admit hyperbolic metrics: by the homotopy long exact sequence of a fibration the fundamental group of Σ

contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, which is impossible for compact negatively curved manifolds by Preissman's Theorem [45, Theorem 12.19].

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