



Derjaguin and the DMT Theory: A Farewell to DMT?

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Abstract

There is a widespread belief that in the 1970s, two conflicting theories of the adhesion of a spherical particle to a substrate were published: the JKR and DMT theories. And that the dispute was resolved when Tabor introduced a parameter μ , such that for small μ , DMT was correct, while for large μ , JKR was correct. We point out that there never were two theories of contact: the dispute was about the magnitude of the pull-off force (with an implication that since the DMT value was obtained by a *thermodynamic* method, it must be correct). And what Tabor actually said was simply that for large μ , the neglect of surface forces in the JKR theory was acceptable, and that he distrusted the neglect of deformation by the large surface forces immediately outside the Hertzian contact in the DMT theory. We point out the errors in both the DMT/MDT thermodynamic method and the MDT force method (preferred by MDT) but also argue that once Derjaguin and his collaborators (MYD) established that Hertzian geometry does not occur, no theory based on that geometry should be taken seriously. Derjaguin's well-merited fame rests on more important contributions.

Keywords Derjaguin · DMT · JKR · Force method · Thermodynamic method

1 Introduction

It sometimes appears that there must be a law requiring any paper-linking adhesion with traditional contact mechanics to begin by explaining that there are two theories describing the “adhesive Hertz problem”; the JKR theory [1] and the DMT theory [2]; and that Tabor [3] reconciled the two conflicting theories by introducing a parameter $(R\gamma^2/E^2\varepsilon^3)^{1/3}$ (Now usually modified to $\mu \equiv (R\Delta\gamma^2/E^*2\varepsilon^3)^{1/3}$) and explaining that for μ small, the DMT theory is correct, while for μ large, the JKR theory is correct.

Very little of this explanation is true!

There is indeed a complete JKR contact theory, which leads to equations relating the load to the approach and the contact radius: a convenient form¹ being the parametric equations $\hat{P} = (4/3)\hat{a}^3 - \sqrt{8\pi}\hat{a}^3$; $\hat{\delta} = \hat{a}^2 - \sqrt{2\pi}\hat{a}$ where $\hat{P} \equiv P/R\Delta\gamma$, $\hat{\delta} = \beta^2(\delta/R)$, $\hat{a} \equiv \beta(a/R)$ with $\beta^3 = E^*R/\Delta\gamma$. The pull-off force ($\hat{P} = -3\pi/2$; $T \equiv (3/2)\pi R\Delta\gamma$) then follows immediately by differentiation. In contrast, but there is no contrast! There never was a DMT *contact* theory: all the

DMT paper provided was a proof that the maximum tensile load occurred when the elastic indentation disappeared, so that contact reverted to the rigid body geometry, and that therefore, the pull-off force regained the value $2\pi R\Delta\gamma$ found previously by Derjaguin [4] (and Bradley [5]) for a contact between rigid spheres. No equations relating load to approach² or to contact area, were given by DMT. They do indeed provide a table labelled “Total Force”, but this is misleading: the values given represent only the relation between the “adhesion force” and the approach: “total” referring merely to the addition of the contributions within and outside the contact area.

After the MYD [6] analysis appeared (1980), in 1983, the DMT “thermodynamic” method was specifically rejected as incorrect by its three authors (reordered as MDT [7]), so is now history.

But did Tabor give his blessing to the DMT theory as the correct description of contact behaviour for small values of his parameter? He did no such thing! Indeed he says rather little about the DMT model except commenting:

¹ The variables used in the self-consistent plots, a^* and $\alpha^* \equiv \delta/\varepsilon$ are such that $a^* = \hat{a}$ but $\alpha^* = \mu\hat{\delta}$. I would prefer the symbol α for approach (or displacement), but too many printers use the same symbol for a and α .

² An asymptotic force-approach relation could be constructed from information given, but it would be valid only for $\delta/\varepsilon \ll 1$.

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This analysis is interesting for its attempt to link up surface forces with the use of the Hertzian elastic equations. Unfortunately, the shape of the deformed zone ignores the deformation due to the attractive forces close to the edge of, but outside, the Hertzian circle.

Tabor adds that these neglected forces are comparable with the forces inside the Hertzian circle, (in fact near pull-off they are far larger). And Maugis, of all people, excuses the neglect as acceptable if the elastic modulus is high, forgetting that the Hertzian deformation uses the same modulus.

Tabor certainly accepted the need to consider non-contact surface forces (Recall that Tabor & Winterton [8] made one of the first measurements of van der Waals forces³!). He argues that (at zero applied load) the neck height in the JKR theory will be of order⁴ $(R\gamma^2/E^2)^{1/3}$, so the JKR theory will be inapplicable unless this is substantially larger than the range of action of the surface forces. Thus, his actual assessment is, in effect, “[if μ falls to $O(1)$], the forces outside the contact zone can no longer be neglected”.

[Interestingly, this was precisely where Derjaguin himself went wrong in his early contribution⁵ [4]! DMT explain the need for their modification: “...because we disregarded the energy of the noncontact adhesion forces acting within the ring-shaped zone surrounding the contact area”].

If the rule which has gone into the literature: “DMT for μ small, JKR for μ large”, was not introduced by Tabor, where did it come from? From Derjaguin himself! To be precise, remarks equivalent to this are made by Muller, Yuschenko and Derjaguin in the MYD paper [6]. And as we argue below, this is wrong: what the Tabor parameter governs is the transition from *rigid body* behaviour to JKR behaviour.

But does that case need arguing? What can be clearer than DMT’s own abstract⁶: “In fact, it remains equal to the attractive force value that is determined when considering the point contact of a nondeformed ball with a plane”.

2 The “Thermodynamic” Method

There is no difference in principle between the DMT paper and the thermodynamic part of the later MDT paper [7]. Neither paper actually calculates the energy; the force is found from the derivative of the energy, and the differentiation of the energy integral is done analytically and the

subsequent numerical integration uses only the derivative of the energy density: a surface force law. The difference is that the DMT paper uses the simple van der Waals law $\varphi(h) = (\Delta\gamma)(\epsilon/h)^3$, so the resulting surface force law is that the force/unit area between two half-spaces a distance h apart is $f = (2\Delta\gamma/\epsilon)(\epsilon/h)^3$ (for $h > \epsilon$). MDT uses an improved law of adhesion energy in which a repulsive energy term is added to the attractive van der Waals energy. This gives the 3–9 surface force law introduced by Derjaguin et al. in their self-consistent numerical solution (MYD [6]. Based on the Lennard–Jones 6–12 potential between individual molecules, the new law is $f = (8\Delta\gamma/3\epsilon)[(\epsilon/h)^3 - (\epsilon/h)^9]$: it will be referred to here as the DerjaguinLJ (or DLJ) law.

The change would of course lead to a different force-approach law for the contact problem, but neither paper offers such a law: the significant result that the maximum tension occurs when the approach is zero, is unchanged.

But could the “thermodynamic” method really have given a force-approach law? It took Pashley [9], a year after the MDT paper appeared, to point out the limitation of the method: the answer is only correct if the configuration is correct.

The method used in both papers will be referred to as the DMT thermodynamic method.

But what is it?

The beginner, wishing to understand the famous theory, might be tempted to try reading the DMT paper [2]. Sadly, all he would find is a helpful reference to Derjaguin’s 1934 paper: “At that time an approach also was suggested to solving this problem by using the virtual displacement (thermodynamic) method taking into account the adhesion energy in combination with the Hertz theory”. And in that paper, he would find the only mathematical statement of the virtual displacement method: there we have

$$P_1 \delta\alpha = U + f_0 \delta S \quad (1)$$

[α is the approach, **our** δ : U is the elastic strain energy, S the contact area and $f_0 = -\Delta\gamma$ {Derjaguin’s δ seems to be just a prefix, meaning small}].

The last term is never referred to again but (verbally) is replaced by the total adhesion energy W_s , found as the integral of the surface energy. Neither the original U nor W_s appears to be small quantities, as required by the virtual displacement equation. But it soon becomes clear that the reference should have been ignored, and that the relevant principle, never stated, is that

$$P_1 = \frac{dU}{d\alpha} + \frac{dW_s}{d\alpha} \quad (2)$$

(the adhesion energy is tensile, so negative).

If our beginner, seeking clarification, consults the later MDT paper, he will then find

³ Derjaguin himself was earlier (1954).

⁴ Tabor uses γ to denote surface tension, so we have $\Delta\gamma = 2\gamma$ (or $\Delta\gamma = \gamma_1 + \gamma_2 - \gamma_{12}$).

⁵ But how could he, in 1934, have calculated the external contribution for his elliptical geometry?

⁶ Note the “it”: the DMT paper is only interested in the pull-off force.

“This force is obviously equal to the sum of the external load, P_1 and the molecular interaction of the sphere with the substrate, F_s , over the whole surface. In accordance with [1] [ie Derjaguin 1934], the latter is defined as the derivative of the total energy of molecular interaction, W , with respect to the approach.” Yes, that is indeed the method of Eq. (2) but is certainly not in accordance with [1].

2.1 Evaluation of the derivatives

Since the strain energy is found as the integral giving the work done in producing the Hertzian indentation, $U = \int P d\delta$, the derivative $dU/d\delta = P$, the Hertz load, and we may forget its origin and simply offset the adhesion force by the Hertz load for the relevant indentation.

Calculating the adhesion force is harder: $F_s = \frac{dW_s}{d\delta} = \frac{d}{d\delta} [2\pi \int_0^\infty \varphi(h) r dr]$, where $h = h(r, a) + \varepsilon$ with $h(r, a) = (1/\pi R)(a\sqrt{r^2 - a^2} + (r^2 - 2a^2) \cdot \arctan \sqrt{(r^2/a^2 - 1)})$.

For the Hertzian geometry postulated, we know⁷ that $a^2 = R\delta$: so that $\frac{d}{d\delta} \equiv \frac{R}{2a} \frac{\partial}{\partial a}$ and $F_s = \frac{2\pi R}{2a} \int_0^\infty \frac{d\varphi(h)}{dh} \frac{\partial h}{\partial a} r dr$. $\frac{d\varphi}{dh}$ is just the surface force $f(h)$ and after some algebra $\frac{\partial h}{\partial a} = \frac{4a}{\pi R} \arctan \sqrt{r^2/a^2 - 1}$ (and is zero for $r < a$). Thus, $F_s^i(\delta) = 2\pi \frac{R}{2a} \frac{4a}{\pi R} \int_a^\infty f(h) \cdot \arctan \sqrt{r^2/a^2 - 1} r dr = 4a^2 \int_0^\infty f(h) \cdot \arctan(t) t dt$.

After setting $t^2 = (r^2 - a^2)/a^2$.

This, however, was not the route DMT took, for now Derjaguin's past caught up with him: he already knew how to calculate the contribution from the contact area! So DMTs partition the energy into $W_s^i + W_s^e = 2\pi \int_0^a \varphi(h) r dr + 2\pi \int_a^\infty \varphi(h) r dr$ and differentiate the two terms separately. But over the contact area, $h = \varepsilon$ and $\varphi(\varepsilon) = \Delta\gamma$, so $W_s^i = \pi a^2 \Delta\gamma$. And for the Hertzian geometry postulated, we have $a^2 = R\delta$: so that $W_s^i = (\pi R \Delta\gamma) \delta$, and differentiating gives the contribution to the adhesion force as $F_s^i = \pi R \Delta\gamma$, avoiding the need for numerical integration.

But this had to be paid for! For the integral for the contribution outside the contact area no longer has fixed limits, and DMT (not knowing how to differentiate an integral wrt. its limits !!) made a change of variable $x^2 = (r^2 - a^2)$ to bring the lower limit to zero before differentiating the integral. They, therefore, needed not $\frac{\partial h(r,a)}{\partial a}$ but $\frac{\partial h(x,a)}{\partial a}$. Using this led them (correctly) to the external contribution to the pull-off force $\frac{dW_s^e}{d\delta} = 2\pi \int_0^\infty f(h) \left[\frac{xa}{x^2+a^2} - \arctan(x/a) \right] x dx$.

Adding to this, the internal contribution $F_s^i = \pi R \Delta\gamma$ gives exactly the same answer as before, with the maximum $2\pi R \Delta\gamma$ as $\delta \rightarrow 0$ and we regain the rigid indenter configuration. But for a rigid indenter, the external contribution, found directly, is the whole $2\pi R \Delta\gamma$, not the $\pi R \Delta\gamma$ we have just found for the same configuration! This seems worth repeating:

The contribution to the adhesive force from the region outside the contact area

(a) tends to $\pi R \Delta\gamma$ as the indentation $\delta \rightarrow 0$

(a) tends to $2\pi R \Delta\gamma$ as the gap $h_0 \rightarrow 0$

Perhaps this is the correct place to emphasise the other half of perhaps the same problem: the contribution from the contact area is $\pi R \Delta\gamma$, whatever, the size of the contact. The mean (tensile) stress over the contact is, therefore, $\pi R \Delta\gamma / \pi a^2 = \Delta\gamma / \delta$: embarrassing when $\delta \rightarrow 0$.

3 The MDT Paper Part II: The “Force” Method

The MDT paper [7] carefully describes the thermodynamic method, the very essence of the DMT paper and of its own first half, and then rejects it in favour of a direct force method (and so relegates the DMT paper to join phlogiston and the æther as interesting history). Not because the thermodynamic method gives only an approximate answer and not for the glaring discontinuity described above, but because where the force–approach curve for elastic contact meets Derjaguin's equation for the adhesion force before contact takes place, it meets it in a cusp, instead of joining smoothly as found in the full numerical solutions of MYD [6] (and surely expected by common sense?). This fundamental decision is illustrated by MYD in Fig. 3, in which the important information is confined to a small corner: Fig. 1 shows the problem more clearly.

The force method simply integrates the surface forces for the assumed geometry, just as is done for the out-of-contact rigid body (using Derjaguin's approximation that the force between two curved, inclined surface elements is the same as if they were elements of two parallel half-spaces). The new answer is now only an estimate and obviously only correct if the assumed geometry is correct.

The calculation of the total adhesive force, now from entirely outside the contact area (which now contributes *none*, instead of $\pi R \Delta\gamma$!), is a straightforward numerical integration: it produces a curve which after a slow start very close to $2\pi R \Delta\gamma$, then *rises* rapidly when the indentation increases. $\delta = 0$ gives the *minimum* value of the adhesion force, not the *maximum* as found in the thermodynamic method. Figure 2 shows the result of subtracting the Hertz load $\hat{P} = (4/3\pi)(\delta/\mu)^{3/2}$. For $\mu < 0.27$ the tensile load is again a maximum at zero approach, so as MDT happily reports (ignoring the limitation on μ), we again get the rigid body configuration at pull-off. But for larger values of μ , as Pashley [9] first pointed out (overlooked(?) by MDT), for $\mu > 0.27$ the pull-off force exceeds the rigid solid value of $2\pi R \Delta\gamma$, and increases monotonically with μ , the value $2.06\pi R \Delta\gamma$ shown for $\mu = 1$ being relatively minor. For $\mu = 5$

⁷ WE know, but one *fifth* of the DMT paper is spent on proving this Hertzian relation. Does that make it a DMT equation?

Fig. 1 Goodbye to DMT? The rigid body curve (Derjaguin 1934) is of course independent of Tabor’s parameter. The elastic curves are typical curves for $\mu < 0.3$

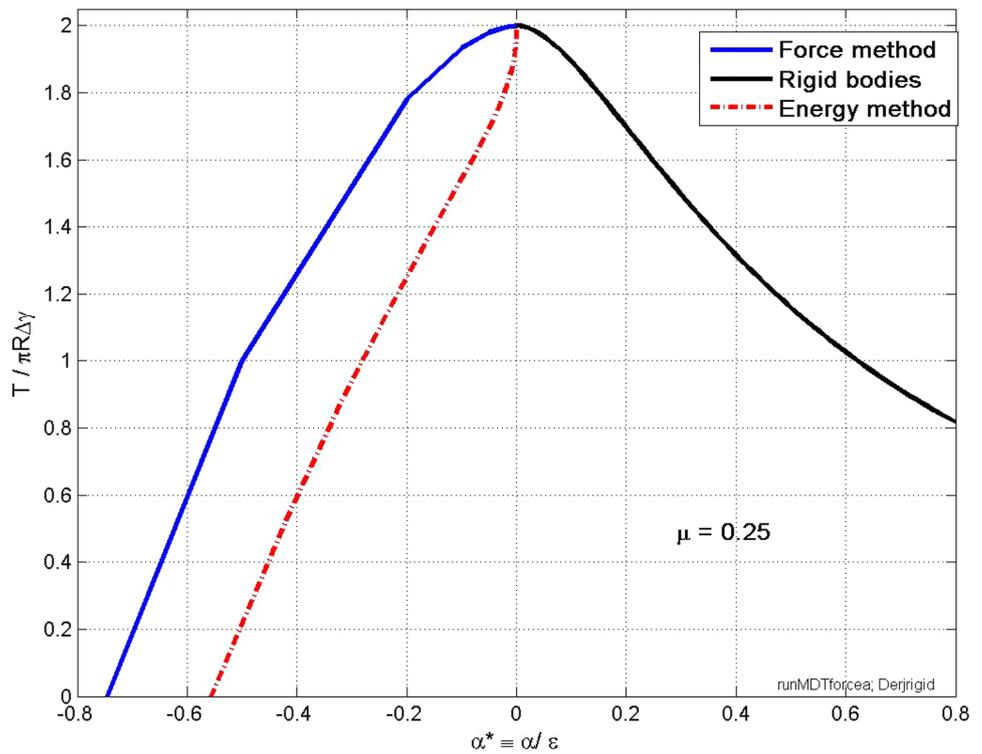
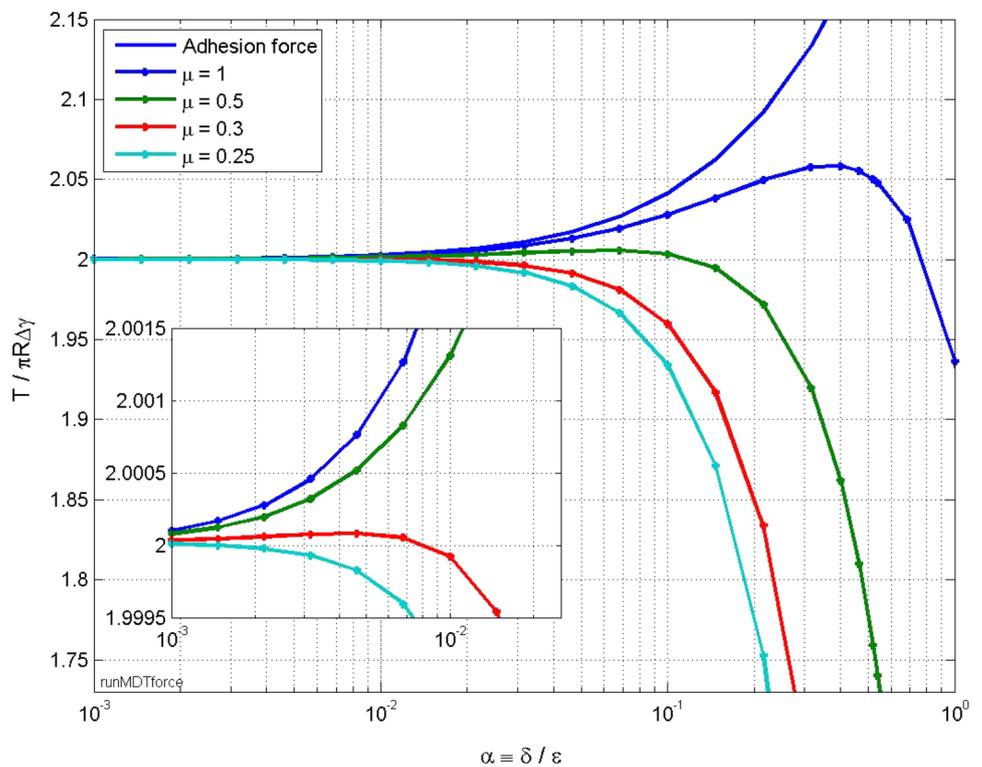


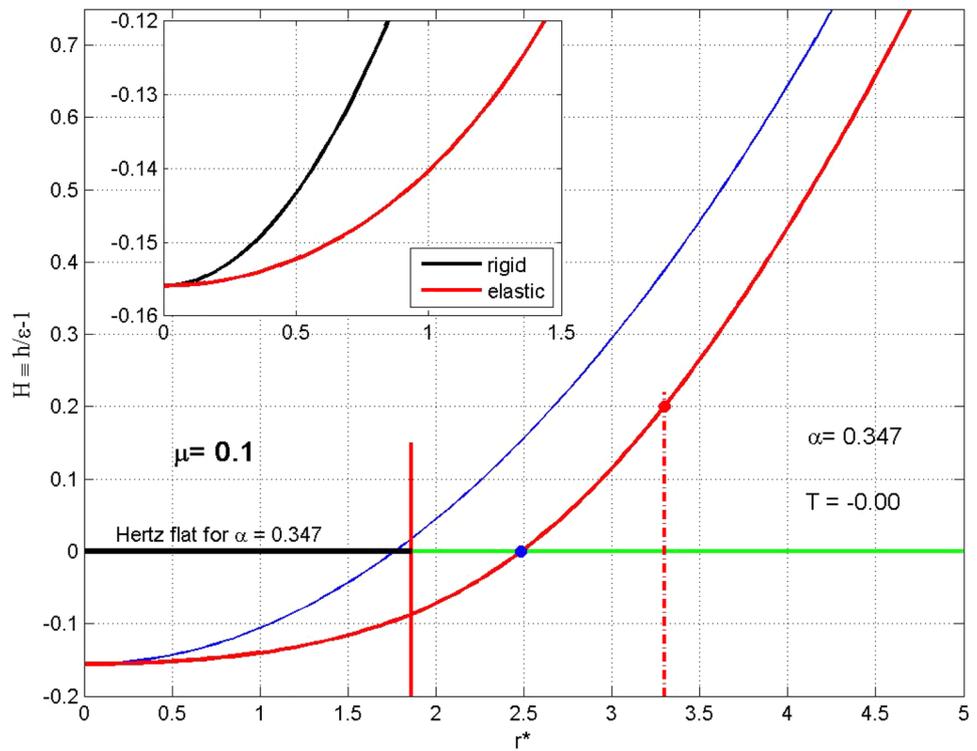
Fig. 2 The MDT (force) solution



the pull-off force is $2.68\pi R\Delta\gamma$, and it increases indefinitely as μ rises. None of the “self-consistent” solutions pioneered by Derjaguin and his colleagues (MYD [6] has ever

exceeded $2\pi R\Delta\gamma$ (or indeed, even quite reached it). (See, for example, Greenwood [10] or Feng [11], or the Maugis “Dugdale” analytical solution [12]).

Fig. 3 Qualitative agreement. The indenter shape at zero load shows no signs of a Hertzian flat



3.1 The Maugis (M-DMT) Theory

Buried in the beautiful Maugis (Dugdale) paper [12] is a quite separate theory describing the adhesive contact between a sphere and an elastic half-space. Rejecting the attempts to calculate the adhesive force either as the derivative of the adhesive energy or by direct integration of the surface forces, Maugis postulates that it has a constant value of $2\pi R\Delta\gamma$, and that we then have Hertzian contact under a load P as if the load was $\{P + 2\pi R\Delta\gamma\}$, so that $a^3 = (3/4)(P + 2\pi R\Delta\gamma)R/E^*$. With the notation used in the introduction, $\hat{P} \equiv P/R\Delta\gamma$, $\hat{\delta} = \beta^2(\delta/R)$, $\hat{a} \equiv \beta(a/R)$ with $\beta^3 = E^*R/\Delta\gamma$, we get the M-DMT equations:

$$\begin{aligned} \hat{a}^3 &= (3/4)(\hat{P} + 2\pi) \\ \hat{a}^2 &= \hat{\delta} \end{aligned} \tag{3}$$

widely, but incorrectly, quoted as due to DMT.

It is interesting to relate this to the MDT force method. MDT (as DMT) starts by choosing the indentation δ , which for a Hertzian contact is equivalent to choosing $a \equiv \sqrt{\delta/R}$. The Hertz solution (rederived by DMT) gives the gap shape outside the contact, and using a surface force law, the adhesion force $F_s(\delta/\epsilon)$ is calculated. The Hertz force is $F_h = (4/3)a^3 E^*/R$, and the (tensile) load is the difference of these: $T = F_s(\delta/\epsilon) - F_h(a)$, or $T/R\Delta\gamma = F_s/R\Delta\gamma - (4/3)a^3 E^*/R^2\Delta\gamma$, which in the notation above is $\hat{T} = \hat{F}_s - (4/3)\hat{a}^3$. Changing from a tensile

force \hat{T} to a compressive load $\hat{P} = -\hat{T}$, this is approximately Eq. (3a). Since in the force method F_s remains very close to $2\pi R\Delta\gamma$ for $\delta/\epsilon < 0.27$, Maugis' hypothesis just corrects the suspicious increase in \hat{F}_s (perhaps due to the omission of a negative contribution from the contact area?), and so, for the whole range where "DMT" might be considered, satisfactorily represents the MDT force answers.⁸

The equations are the same, but the point of view differs: MDT starts from the approach δ and finds the geometry and hence the load: M-DMT takes the load T as fundamental and finds the approach.

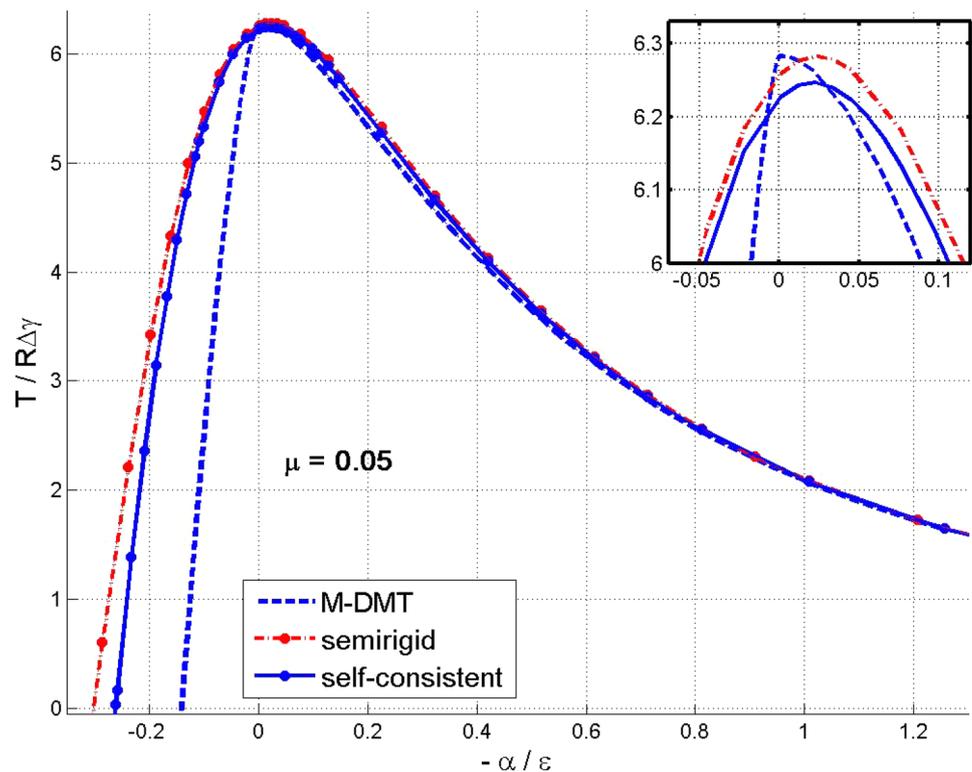
But do either describe the adhesive contact of a sphere and a half-space?

3.2 The Defects of the MDT(Force) Method

Has the introduction of Hertzian geometry into the analysis been a useful step? It was indeed a very natural one in 1934 when Derjaguin first raised the question of how adhesive forces might affect a Hertzian contact. But to

⁸ A pause to think is needed here. I have no difficulty with the signs in the equation $\hat{T} = \hat{F}_s - (4/3)\hat{a}^3$. But when this is reversed and becomes $\hat{a}^3 = (3/4)(\hat{F}_s - \hat{T})$, I suddenly realise that the *tensile* surface stresses have applied a *compressive* load to the sphere. Well, thanks be to Newton! $\hat{T} = \hat{F}_s - (4/3)\hat{a}^3$

Fig. 4 Comparison of the models (i) [M-DMT + rigid] and (ii) semi-rigid, with “exact” solution



continue after the MYD numerical solutions showed no signs of a Hertzian flat?

Feng [13] accepts the MDT claim that the DMT(f) approach gives *qualitative* agreement for the gradual shrinking of the contact area as the tensile load is increased. Figure 3, produced using the MYD self-consistent numerical method, suggests otherwise. Even for zero load, the shape of the sphere is qualitatively still as for a rigid sphere: the indenter curvature is reduced, but there is no sign of a Hertzian flat. For this value of the approach, using the M-DMT equation $\hat{a}^2 = \hat{\delta}$, the approach $\alpha^* = 0.347$ found by the self-consistent solution implies a Hertzian flat of radius 1.86 as shown. For $T = 0$, the M-DMT theory predicts a contact radius $a = 1.68$ and an approach $\alpha^* \equiv \delta/\varepsilon = 0.281$. In fact, the contact pressure falls to zero much further out, at the intersection of the elastic shape curve with the green line. [The maximum tensile stress (the preferred definition of a contact area by MYD (and Greenwood)) occurs at $a = 3.301$ [second vertical line ($H = 0.201$)].

The inset shows more clearly the relation of the deformed shape to the initial rigid shape. (The rigid curve is placed purely to show this: its vertical position should be ignored).

If we leave the DMT region, by moving to $\mu = 1$, incipient flattening can be seen [13] but only by zooming the y-axis.

Recognisable flats require $\mu \geq 2$, and flats with definite edges need $\mu \geq 3$. Indeed, Tabor’s concern at neglecting the deformation due to attractive forces immediately outside the Hertzian flat is seen to be a minor one: there is no Hertzian flat! [We might note that this is not an inadequacy in the numerical solutions: good approximations to a flat are found, but for $\mu \geq 3$.

It might also be noted (see [14]) that Derjaguin’s tension curve for $\alpha \leq 0$, (which by virtue of the assumption of Hertzian geometry, and so, for $\alpha \leq 0$ of zero Hertz force) is just the rigid body adhesion force curve, is wrong even for $\mu = 0.1$, predicting contact areas before contact! Locally the sphere is attracted towards the half-space, resulting (for $\mu = 0.1$) in a “ σ_{\max} ” contact area when $\alpha^* > -0.28$, and a “ $p > 0$ ” contact area when $\alpha^* > -0.07$.

Qualitative is a very loose term!].

4 The “Semi-Rigid” Theory [14, 15]

If even for the substantial deformation of the zero-load contact, there is no sign of a Hertzian flat appearing, the obvious response is to forget Hertz. But what can we do instead? Fig. 4 provides a possible answer: use the original rigid shape! Certainly there has been elastic deformation but is the rigid shape perhaps better than the Hertzian shape? The semi-rigid model ignores the geometry changes,

so uses Derjaguin's equation for the load (Derjaguin 1934, MDT1983). The central displacement due to the same stresses is calculated to find the approach. Thus, starting from the minimum gap $h_0^* \equiv h_0/\epsilon$ the equations⁹ are

$$\begin{aligned} T/\pi R\Delta\gamma &= (8/3)[H_1^{-2} - (1/4)H_1^{-8}] \text{ where } (H_1 = (h_0/\epsilon + 1)) \\ w^*(0) &= \pi \mu^{3/2} \sqrt{2H_1} [H_1^{-3} - 0.5237H_1^{-9}] \\ \delta^* &= h_0^* + w^*(0) \end{aligned} \quad (4)$$

Figure 4 shows a comparison of the force/approach curves for the three theories (M-DMT, semi-rigid, and self-consistent) for $\mu = 0.05$.

Both models predict that the pull-off force is the traditional $2\pi R\Delta\gamma$ for all values of μ , which is not serious here where the true answer is only 4% lower (but would matter for $\mu = 1$ where the answer should be 17% lower). The semi-rigid model finds the approach at pull-off rather well and is good over the whole range. Indeed, not surprisingly, it, not either of the MDT models, is the correct limit as $\mu \rightarrow 0$. But in fact, rather soon the M-DMT model becomes far the better of the two. It gives an excellent answer for $\mu = 0.2$. Greenwood's conclusion [14] that the semi-rigid model is generally the better seems wrong (and conflicts with his own Fig. (10) see [14]): perhaps, it was based on the correct prediction of the approach at pull-off.

5 Discussion

The most important consequence of the appallingly badly named M-DMT theory seems never to have been noticed. Perhaps it is necessary to re-state the basic hypothesis in words. The pull-off force for a rigid sphere, *independently of the surface force law*, is $2\pi R\Delta\gamma$ (Bradley [16], Derjaguin [4]). Assume that this same value of the adhesion force operates; whenever, there is contact. Then if the geometry is Hertzian, we get Eqs. (3): $\hat{a}^3 = (3/4)(\hat{P} + 2\pi)$; $\hat{a}^2 = \delta$. There is no need to choose a surface force law, and, vitally, there is no need for any numerical integrations. The need for MDT to choose between the thermodynamic method and the force method is now simple: do neither! And forget both?

The objections to this theory are those made above in the discussion of Fig. 4. It consistently predicts a pull-off force of $2\pi R\Delta\gamma$ instead of the decreasing values found as μ increases, and, except for μ near 0.2, it either underestimates the zero-load approach (μ small) or overestimates it ($\mu > 0.2$). And worst of all, it predicts a non-existent

Hertzian flat and confirms the idea that Hertz theory plays a part in the study of adhesive point contacts.

The semi-rigid theory turns out to be the proper limiting theory as μ approaches zero, but for $\mu > 0.1$, the M-DMT equations are better. For $\mu > 0.3$ the choice is between the Maugis (Dugdale) theory and the full self-consistent numerical analysis.

Has the DMT method (or more accurately, the MDT (force) method) a future? Is there a DMT (force) method for other geometries? One solution, and perhaps the obvious one, is to start by solving the dry (non-adhesive) contact problem. Then the contact areas are ignored, and the adhesive stresses over the out-of-contact area are calculated (see, for example [17]). Applied to a single spherical indenter, this would exactly reproduce the DMT(force) method. (Applying it to a fractal rough surface would bring in a new difficulty: does Derjaguin's approximation that the force/unit area between two tiny elements is the same as between two infinite, plane, surfaces hold for fractal elements?).

Alternatively, "DMT", or specifically the M-DMT solution, can be used to study rough surface in a different way. Just as Fuller and Tabor [18] used the JKR solution in combination with an asperity model, Maugis [19] used the M-DMT equations. Many elaborations of the asperity model are possible. But are such approaches "using the DMT method"? Is there a difference between making a brick and using bricks to build a wall?

6 Derjaguin's Contribution

Since Derjaguin in the MYD paper specifically rejects the thermodynamic method, the heart of the DMT paper, (added to the difficulty of following the DMT paper!), the obvious course is to ignore it. The present author believes the MDT paper should equally be ignored. (We might add that no one seems ever to have considered generating and using load-approach equations for either model, but that may be because of the temptation to use the simple M-DMT equations $\hat{P} = (4/3)\hat{a}^3 - 2\pi$; $\hat{\delta} = \hat{a}^2$ due to Maugis, not Derjaguin. But it would be absurd to forget Derjaguin's major contributions to solid contact (or indeed that this was a sideline for a major contributor to the electrochemistry of colloids); or that he made one of the earliest (1954) direct measurements of van der Waals forces.¹⁰

[A] In his 1934 paper, Derjaguin introduced.

- i. *The Derjaguin approximation* that the force between a small, curved, inclined element is the same as the

⁹ $w(0)/\epsilon = (16/3)\mu^{3/2} \sqrt{2H_1} [C_4 H_1^{-3} - C_{16} H_1^{-9}]$ with $C_{2n} = \sqrt{\pi}/2 \cdot (n-1/2)!/n!$

¹⁰ cited by Tabor and Winterton [6], but inaccessible, See also the reference in [2].

force across a parallel gap (Rediscovered 44 years later by the physicists, and labelled the “Proximity Force Theorem”. [20]).

- ii. Using this approximation, and assuming that the gap between an ellipsoid and a plane is of the form $h = h_0 + x^2/2R_1 + y^2/2R_2$, there will be an adhesive force between them $T(h_0) = 2\pi\sqrt{R_1R_2} \int_{h_0}^{\infty} f(h) dh$ where $f(h)$ is the force/unit area between two half-spaces a distance h apart. Hence, when h_0 is the equilibrium separation ε between the bodies, so that the integral equals the work function (or Dupré surface energy $\Delta\gamma$, then the pull-off force will be $T_{\max} = 2\pi\sqrt{R_1R_2} \cdot \Delta\gamma$. This is a generalisation of the result found by Bradley [5], that for a sphere $T_{\max} = 2\pi R\Delta\gamma$, but Derjaguin’s proof is for any ellipsoid, for any law of force $f(h)$, and requires enormously less effort.
- iii. Introducing, somewhat unsuccessfully due to his attachment as an experimenter to crossed cylinders and therefore to elliptical contacts (and apparently a distrust of the Hertz theory!) the idea that elastic deformation must be taken into account. (An idea which seems not yet to have reached the physicists (eg [21])
- i. **[B]** (i) In the 1980 (MYD) paper, Derjaguin (and his collaborators) introduced a surface force law based on the Lennard–Jones 6–12 law for the potential between two molecules, the now well-known law $F(h) = (8\Delta\gamma/3\varepsilon)[(\varepsilon/h)^3 - (\varepsilon/h)^9]$ (MYD, but perhaps with help from Rayleigh [22])
- ii. (ii) They then used this surface force law as the surface pressures on an elastic sphere (assuming without comment that the sphere deformation may be found using the half-space equations, as did Hertz), and by an iterative numerical analysis solved the problem of the approach and separation of an elastic sphere to a rigid half-space. They showed that by scaling the governing equations, the solution depends on a single dimensionless parameter. [Which, except for a scale factor, is the parameter μ previously introduced by Tabor by physical reasoning.]
- iii. (iii) They showed that as μ rises, the pull-off force falls from the rigid solid value $2\pi R\Delta\gamma$ towards the JKR value $(3/2)\pi R\Delta\gamma$. [But their plot has no actual points. and the decrease from the rigid solid value is steeper than later workers have found].

7 Conclusion

Derjaguin introduced the study of how Hertzian contact is modified by molecular forces, so naturally as a perturbation of a Hertzian contact. But when he and his collaborators

[MYD] introduced the self-consistent solution of the problem by numerical analysis, they found the configuration to be very far from Hertzian: as a result, no useful answers can be obtained by analyses based on it. The DMT paper should be written off as a gallant failure. And the MDT (force) method, and the idea of finding and integrating surface forces, should equally be written off, and Maugis’ assumption that the rigid body adhesion force applies *during* contact as well as *at* contact accepted, not as an approximation, but as a complete and admirable theory!

DMT and MDT never attempted to provide complete analyses of how contact takes place, so in no way were they competitors to the JKR analysis, which, like M-DMT, is a complete solution, giving relations between load, approach, and contact area (for a specific range of the Tabor parameter). Their aim seems to have been purely to determine the value of the pull-off force and to find the configuration at which separation occurs: it is in no way the equivalent of the JKR theory.

End-note: Derjaguin’s rejection of Dahneke’s theory (from DMT 1975).

It is an obvious truth that passing over from the picture of elastic stresses and pressures to their molecular interpretation, we should make F_{rep} equal to the resultant of all the molecular interactions, and if we want to take into account separately the close-range repelling (repulsive) forces and far-range attraction forces, F_{rep} should have been made equal to the difference between the former and the latter. Limiting ourselves only to the former when interpreting the contact interaction [as was done in (4)], implies making a gross error which is of the same magnitude as the main component of term F_{att} that was taken into account by Dahneke in calculating the total sticking force.

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Declarations

Conflict of interest The author has no relevant financial or non-financial interests to disclose.

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