

# Einstein–Weyl Spaces and Near-Horizon Geometry

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## Abstract

We show that a class of solutions of minimal supergravity in five dimensions is given by lifts of three-dimensional Einstein–Weyl structures of hyper-CR type. We characterise this class as most general near-horizon limits of supersymmetric solutions to the five-dimensional theory. In particular we deduce that a compact spatial section of a horizon can only be a Berger sphere, a product metric on  $S^1 \times S^2$  or a flat three-torus.

We then consider the problem of reconstructing all supersymmetric solutions from a given near-horizon geometry. By exploiting the ellipticity of the linearised field equations we demonstrate that the moduli space of transverse infinitesimal deformations of a near-horizon geometry is finite-dimensional.

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# 1 Introduction

In dimension four the topology of a black-hole horizon is necessarily spherical [13], but in dimensions five and higher this restriction needs not to hold [8]. The local differential geometric structure of horizons in higher dimensional gravity theories is also rich. The aim of this paper is to demonstrate that in case of supersymmetric solutions of minimal supergravity in dimension five, the horizon geometry is that of a three-dimensional Riemannian Einstein–Weyl structure of hyper-CR type. We shall also show that the moduli space of linearised transverse deformations of near-horizon geometries with compact spatial sections of horizons is finite-dimensional.

In the first part of the paper we shall demonstrate that a class of solutions of minimal supergravity in five dimensions is given by lifts of three-dimensional Einstein–Weyl structures of hyper-CR type. We characterise this class as most general near-horizon limits of supersymmetric solutions to the five-dimensional theory. In particular we deduce that a compact spatial section of a horizon can only be a Berger sphere, a product metric on  $S^1 \times S^2$  or a flat three-torus.

Space times containing Killing horizons with vanishing surface gravity admit a limiting procedure leading to a near horizon geometry [20, 16, 19, 15]. In the second part of the paper we consider the problem of reconstructing all supersymmetric solutions with a given near-horizon geometry.

In the next two sections we shall review the  $D = 5$  and  $N = 2$  supergravity with its near horizon limit, and the hyper-CR Einstein Weyl equations respectively. In section 4 we shall establish (Theorem 4.1) a correspondence between 3D hyper-CR Einstein–Weyl structures and near-horizon limits of supersymmetric solutions to  $D = 5$ ,  $N = 2$  supergravity, and deduce the allowed topologies of horizons in this case. We shall also construct an explicit local fibration of the 5D metric over a hyper-Kähler four-manifold with a homothetic Killing vector field.

In section 5 we shall prepare the ground for the moduli space calculation, and derive the Bianchi identities resulting from supersymmetry. In section 6 (Theorem 6.1) we shall exploit the ellipticity of the linearised field equations to prove that, if the spatial horizon 3-surface  $\Sigma \subset M$  is compact, then the moduli space of infinitesimal transverse deformations of near-horizon geometry is finite-dimensional.

In [12] it was demonstrated that the near-horizon geometries of minimal gauged five-dimensional supergravity preserve at least half of the supersymmetry. These results are of topological nature, and rely on compactness. In the current work we reveal the rich local structure of the near-horizon equations and its connection with conformal geometry of 3D spatial slices.

The long formulae involving the Ricci tensor in the bulk, and the analysis of the gravitino equation have been relegated to Appendices.

## 2 5D Minimal supergravity and its near-horizon limit

Let  $M$  be a five-dimensional manifold with pseudo-Riemannian metric  $g$  of signature  $(4, 1)$  and a Maxwell potential  $A$ . The five-dimensional action for the Einstein–Maxwell theory with a Chern–Simons term is

$$S = \int_M \mathcal{R} \text{vol}_M - \frac{3}{2} H \wedge *_5 H - H \wedge H \wedge A, \quad (2.1)$$

where  $\mathcal{R}$  is the Ricci scalar of  $g$ ,  $H = dA$  is the  $U(1)$  Maxwell field,  $\text{vol}_M$  is the volume element induced by  $g$  and  $*_5 : \Lambda^\alpha(M) \rightarrow \Lambda^{5-\alpha}(M)$  is the associated Hodge endomorphism. The resulting Einstein–Maxwell–Chern–Simons equations coincide with the bosonic sector of minimal five-dimensional supergravity. These equations are

$$\begin{aligned} dH &= 0, & d *_5 H + H \wedge H &= 0, \\ \mathcal{R}_{\alpha\beta} - \frac{3}{2} H_{\alpha\gamma} H_{\beta}{}^\gamma + \frac{1}{4} g_{\alpha\beta} H^2 &= 0. \end{aligned} \quad (2.2)$$

In [10] it was shown that all timelike supersymmetric solutions of this theory admit local fibrations over hyper-Kähler four manifolds, i. e. locally there exists a function  $u : M \rightarrow \mathbb{R}$  such that

$$g = -f^2(du - \Theta)^2 + f^{-1}g^{HK}, \quad (2.3)$$

where  $g^{HK}$  is a  $u$ -independent hyper-Kähler metric on some four-manifold  $X$ , and  $(\Theta, f)$  are respectively a one-form and a function on which do not depend on  $u$  and satisfy some system of equations on the base  $X$ .

### 2.1 Near-horizon limit

Let  $(u, r, y^i)$ , where  $i = 1, 2, 3$ , be Gaussian null coordinates [19] defined in the neighbourhood of a Killing horizon  $g(V, V) = 0$  where  $V = \partial/\partial u$  is a stationary Killing vector. In these coordinates the horizon is given by  $r = 0$ , and  $y^i$  are local coordinates on three-dimensional Riemannian manifold  $\Sigma$  which is the spatial section of the horizon. The metric and the Maxwell potential are given by

$$g = 2du \left( dr + rh - \frac{1}{2} r^2 \Delta du \right) + \gamma, \quad A = r\Phi du + B, \quad (2.4)$$

where

$$\gamma = \gamma_{ij}(r, y^k) dy^i dy^j, \quad h = h_i(r, y^k) dy^i, \quad B = B_i(r, y^k) dy^i, \quad \Delta = \Delta(r, y^k), \quad \Phi = \Phi(r, y^k)$$

are all real-analytic in  $r$ . The near horizon limit [20, 16, 19, 15] arises by replacing

$$u \longrightarrow u/\epsilon, \quad r \longrightarrow r\epsilon,$$

and taking the limit  $\epsilon \rightarrow 0$ . The resulting metric and Maxwell potential on the neighbourhood of the horizon in  $M$  are given by (2.4), where now  $\gamma$  is a Riemannian metric,  $(h, B)$  are one-forms and  $(\Phi, \Delta)$  are functions on  $\Sigma$ . Thus in the limit  $(\gamma, h, B, \Phi, \Delta)$  depend on the local coordinates  $y^k$ , but not on  $(r, u)$ . In the  $\epsilon \rightarrow 0$  limit, the Euler–Lagrange equations of the functional (2.1) yield a set of equations on the three-dimensional data on  $\Sigma$ . The Maxwell–Chern–Simons equations are

$$d *_3 dB + *_3(d\Phi - \Phi h) - h \wedge *_3 dB - 2\Phi dB = 0, \quad (2.5)$$

and the non-trivial components of the Einstein equations are

$$\frac{1}{2}\nabla^i h_i - \frac{1}{2}h^i h_i + \frac{1}{4}dB_{ij}dB^{ij} + \Phi^2 - \Delta = 0, \quad (2.6)$$

and

$$R_{ij} + \nabla_{(i} h_{j)} - \frac{1}{2}h_i h_j - \frac{3}{2}dB_{ik}dB_j{}^k + \gamma_{ij}\left(\frac{1}{4}dB_{kl}dB^{kl} - \frac{1}{2}\Phi^2\right) = 0. \quad (2.7)$$

Here  $\nabla$  is the Levi–Civita connection of the metric  $\gamma = \gamma_{ij}(y)dy^i dy^j$  and  $*_3 : \Lambda^i(\Sigma) \rightarrow \Lambda^{3-i}(\Sigma)$  is the Hodge operator on  $\Sigma$ . The necessary conditions for the near-horizon geometry to be supersymmetric are [12]

$$h + *_3 dB = 0, \quad \Delta = \Phi^2. \quad (2.8)$$

### 3 Hyper–CR Einstein–Weyl geometry

A Riemannian Weyl structure on a three-dimensional manifold  $\Sigma$  consists of a positive-definite conformal structure  $[\gamma] = \{c\gamma, c : \Sigma \rightarrow \mathbb{R}^+\}$ , and a torsion-free connection  $D$  which is compatible with  $[\gamma]$  in the sense that

$$D_i \gamma_{jk} = 2h_i \gamma_{jk},$$

for some one-form  $h$  on  $\Sigma$ . This compatibility condition is invariant under the transformation

$$\gamma \rightarrow e^{2\Omega}\gamma, \quad h \rightarrow h + d\Omega, \quad (3.9)$$

where  $\Omega$  is a function on  $\Sigma$ . A choice of the conformal factor  $\Omega$  such that

$$\nabla^i h_i = 0 \quad (3.10)$$

is called the Gauduchon gauge.

A Weyl structure is said to be Einstein–Weyl [2, 14] if the symmetrised Ricci tensor of  $D$  is proportional to some metric  $\gamma \in [\gamma]$ . This conformally invariant condition can be formulated directly as a set of non-linear PDEs on the pair  $(\gamma, h)$ :

$$R_{ij} + \nabla_{(i}h_{j)} + h_i h_j - \frac{1}{3}(R + \nabla^k h_k + h^k h_k)\gamma_{ij} = 0, \quad (3.11)$$

where  $\nabla$ ,  $R_{ij}$ , and  $R$  are respectively the Levi–Civita connection, the Ricci tensor and the Ricci scalar of  $\gamma$ .

A tensor object  $T$  which transforms like  $T \rightarrow \exp(m\Omega)T$  when  $\gamma \rightarrow \exp(2\Omega)\gamma$  is said to be conformally invariant with weight  $m$ . The Ricci scalar  $W$  and the Ricci tensor  $W_{ij}$  of the Weyl connection have weights  $-2$  and  $0$  respectively. The Ricci scalar is given by

$$W = R + 4\nabla^i h_i - 2h^i h_i. \quad (3.12)$$

An Einstein–Weyl space is called\* hyper-CR if [9] there exists a scalar function  $\Phi$  of weight  $-1$  which, together with the EW one-form  $h$ , satisfies the monopole equation

$$*_3(d\Phi + h\Phi) = dh \quad (3.13)$$

together with an algebraic constraint

$$W = \frac{3}{2}\Phi^2. \quad (3.14)$$

The hyper-CR Einstein–Weyl spaces can be equivalently characterised by the existence of a holomorphic fibration of the associated mini-twistor spaces over  $\mathbb{CP}^1$ , or by existence of two-parameter family of shear-free, divergence-free geodesic congruences [1]. The only compact examples are the Berger sphere,  $S^1 \times S^2$  or  $T^3$  with the flat EW structure.

In the real-analytic category, a hyper-CR EW structure locally depends on two arbitrary functions of two variables. This can be seen by reformulating the hyper-CR condition in terms of a single second order PDE for one function of three variables [3]:

$$\gamma = dzd\bar{z} + \frac{1}{16}(Fdv - i(F_z dz - F_{\bar{z}}d\bar{z}) + dF_v)^2, \quad h = \frac{(F_z + iF_{vz})dz + (F_{\bar{z}} - iF_{v\bar{z}})d\bar{z}}{F + F_{vv}}, \quad (3.15)$$

where  $F = F(z, \bar{z}, v)$  satisfies

$$F_{z\bar{z}}(F + F_{vv}) - (F_z + iF_{vz})(F_{\bar{z}} - iF_{v\bar{z}}) = 4. \quad (3.16)$$

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\*This class of Einstein–Weyl spaces has originally been called ‘special’ in [9], and it has also been referred to as ‘Gauduchon–Tod’. The current terminology (see e. g. [3, 1, 4, 7]) reflects the fact that the hyper-CR EW spaces arise as symmetry reductions of four-dimensional hyper-complex conformal structures by tri-holomorphic isometry. The three-dimensional quotients admit a sphere of Cauchy–Riemann structures.

The Cauchy–Kowalewskaya Theorem implies that the arbitrary data

$$F(z, \bar{z}, v = 0), \quad \text{and} \quad F_v(z, \bar{z}, v = 0)$$

specifies the solution uniquely.

## 4 From 3D Einstein–Weyl to 5D minimal supergravity

We shall now show that hyper–CR Einstein–Weyl spaces give rise to solutions of minimal  $D = 5$  SUGRA, and characterise the solutions which are obtained from this procedure. Roughly speaking, any hyper–CR Einstein–Weyl structure lifts to a solution of the five-dimensional theory, provided that the Gauduchon gauge is chosen. This reflects the fact that (unlike the EW equations) the 5D SUGRA equations are not conformally invariant. To overcome this we will need to introduce and solve a linear PDE (4.17) on the EW background to achieve the right gauge fixing.

**Theorem 4.1** *Let the metric  $\gamma$  and the one–form  $h$  on a three–dimensional manifold  $\Sigma$  solve the hyper–CR Einstein–Weyl equations, and let  $W$  be the Ricci scalar of the Weyl connection of  $(\gamma, h)$  given by (3.12). Let  $\Omega$  be a function on  $\Sigma$  which satisfies the linear PDE*

$$d *_3 (de^\Omega) + d *_3 (e^\Omega h) = 0 . \quad (4.17)$$

Then

$$g = e^{2\Omega} (2du(dr + rh - \frac{1}{3}r^2Wdu) + \gamma + 6rdud\Omega), \quad A = \sqrt{\frac{2}{3}}e^\Omega r\sqrt{W}du + \alpha \quad (4.18)$$

is a solution to the 5D Einstein–Maxwell–Chern–Simons supergravity (2.2). Here  $\alpha$  is a one–form on  $\Sigma$  such that

$$d\alpha = -e^\Omega *_3 (h + d\Omega) . \quad (4.19)$$

All near–horizon geometries for 5D SUSY back holes/rings/strings (2.4) are locally of the form (4.18). Moreover if the three–manifold corresponding to the spatial sections of the horizon is compact, then  $\gamma$  is a metric on the Berger sphere, a product metric on  $S^1 \times S^2$  or a flat metric on  $T^3$ .

**Proof.** Consider the field equations (2.5, 2.6, 2.7) and additionally assume that the SUSY constrains (2.8) hold. Thus  $(dB)_{jk} = -\epsilon_{ijk}h_i$  and

$$dB_{im}dB_j{}^m = (\gamma_{ij}|h|^2 - h_i h_j) .$$

The Maxwell–Chern–Simons condition (2.5) now reduces to the monopole equation (3.13). We regard  $\Phi$  as a weighted scalar with conformal weight  $(-1)$  on the three–manifold  $\Sigma$ . The  $(ur)$  component (2.6) of the Einstein equation becomes (3.10). Thus the five–dimensional field equations force the Gauduchon gauge on the Weyl geometry. Finally the  $(ij)$  components (2.7) of the Einstein equations yield

$$R_{ij} + \nabla_{(i}h_{j)} + h_i h_j = \left(\frac{1}{2}\Phi^2 + h^k h_k\right)\gamma_{ij}. \quad (4.20)$$

Taking a trace of this condition and using the Gauduchon gauge (3.10) gives a constraint on the Ricci scalar of  $\gamma$

$$R = \frac{1}{2}(3\Phi^2 + 4|h|^2).$$

Thus  $R \geq 0$  with the equality iff both  $\gamma$  and  $D$  on  $\Sigma$  are flat. We can now use the expression (3.12) for the Ricci scalar of the Weyl connection to re-express this constraint as (3.14). Now equations (4.20) are equivalent to the Einstein–Weyl equations (3.11) in the Gauduchon gauge, subject to two constraints (3.14) and (3.13). These two constraints are (as we have explained in the previous section) the defining property of the hyper–CR Einstein–Weyl conditions.

To recover the form of the five–dimensional solution from the hyper–CR EW geometry we must make sure that the latter is given in the Gauduchon gauge. This appears to break the conformal invariance of our procedure, but it was to be expected as the five–dimensional theory is not conformally invariant. Assume that a hyper–CR EW structure  $(\hat{\gamma}, \hat{h})$  is given in the Gauduchon gauge. Substituting the Einstein–Weyl data into (2.4) we find that the construction explained so far gives the lift to five dimensions of the form

$$g = 2d\hat{u}(d\hat{r} + \hat{r}\hat{h} - \frac{1}{3}\hat{r}^2\hat{W}d\hat{u}) + \hat{\gamma}, \quad (4.21)$$

where  $(\hat{r}, \hat{u})$  are some local Gaussian coordinates in the neighbourhood of the horizon. Now consider a hyper–CR EW structure  $(\gamma, h)$  in an arbitrary gauge. To put it in a Gauduchon gauge we need to find a function  $\Omega$  on  $\Sigma$  such that (4.17) holds. The solution of this equation always exists locally on  $\Sigma$ , and it follows from the work of Tod [21] that it also exists globally on compact EW manifolds. Thus the EW structure

$$\hat{h} = h + d\Omega, \quad \hat{\gamma} = e^{2\Omega}\gamma \quad (4.22)$$

is in the Gauduchon gauge. The constraints (3.14) and (3.13) are preserved under the conformal rescaling if  $\Phi$  has conformal weight  $(-1)$ . Therefore we substitute (4.22) together with

$$\hat{\Phi} = e^{-\Omega}\Phi$$

into (4.21). We also make coordinate changes

$$(\hat{r}, \hat{u}) = (e^{2\Omega}r, u).$$

This yields (4.18). To compare it with (2.4) if  $\Omega = 1$  use (2.8) (3.14).

We now consider the gauge potential. We substitute the expression (2.8) for  $dB$  into  $H = dA$ , where  $A$  is given by (2.4). Using the equation (3.13) and tracking down the effect of conformal resealing needed to enforce the Gauduchon gauge yields  $A$  in (4.18).

□

### Remarks

- As a spin-off from this analysis we have established the transformation rule for the five-dimensional structures  $(g, A)$  under the conformal rescalings (3.9) of the underlying EW geometry. We see that the metric  $g$  does not transform by a simple scaling but also picks up an inhomogeneous term

$$g \longrightarrow e^{2\Omega}(g + 6rdud\Omega).$$

This additional term of course vanishes on the horizon.

- Three-dimensional Einstein–Weyl equations are integrable by twistor transform [14], and can be regarded as a master dispersionless integrable system in 3 and 2+1 dimensions - all other known dispersionless systems arise as special cases. It is remarkable that in case of super-symmetric solutions the non-integrable equations of 5D minimal supergravity reduce to integrable Einstein–Weyl structures. This phenomenon has been observed in other supergravity theories [11, 5, 6, 18] supporting the evidence that that the super-symmetric sectors of non-integrable classical field theories can be described by integrable models.

## 4.1 Fibration over a hyper-Kähler manifold

Finally we shall show how to put the metric  $g$  (4.18) in the form of the fibration (2.3) over a hyper-Kähler manifold. Comparing the expressions (4.18) and (2.3), and completing the square yields

$$f^2 = \frac{2}{3}e^{2\Omega}r^2W.$$

We now define a new coordinate  $\rho$  by  $r = \exp(\rho - 3\Omega)$  and find

$$g^{HK} = e^\rho(\Phi\gamma + \Phi^{-1}(d\rho + h)^2), \quad f = \Phi e^{\rho-2\Omega}, \quad \Theta = \Phi^{-2}e^{3\Omega-\rho}(d\rho + h), \quad (4.23)$$

where  $\Phi^2 = 2W/3$  and  $W$  is the scalar curvature of the Weyl connection of  $(\gamma, h)$  given by (3.12). Moreover, it follows from the work of [9] that any hyper-Kähler metric with a tri-holomorphic homothety<sup>†</sup> is of the form (4.23) for some Hyper-CR EW structure  $(\gamma, h)$ . In our coordinate system the homothety is generated by  $\partial/\partial\rho$ , and the horizon in five dimensions corresponds to  $\rho = -\infty$ .

## 5 Extension into the Bulk

In this section we briefly summarise some of the conditions imposed on the gauge field strength and the geometry by supersymmetry which we shall use in the moduli calculation in section 6. The gravitino Killing spinor equation is given by:

$$\left[ \nabla_\alpha - \frac{i}{8} \Gamma_\alpha H_{\nu_1\nu_2} \Gamma^{\nu_1\nu_2} + \frac{3i}{4} H_\alpha{}^\nu \Gamma_\nu \right] \epsilon = 0, \quad (5.24)$$

where  $\epsilon$  is a Dirac spinor. Here we work with a metric of mostly plus signature  $(-, +, +, +, +)$ .

We have already introduced the Gaussian null coordinates. In what follows, it is convenient to adopt the basis  $\{\mathbf{e}^+, \mathbf{e}^-, \mathbf{e}^i : i = 1, 2, 3\}$ , adapted to the Gaussian null co-ordinate system in which

$$g = 2\mathbf{e}^+ \mathbf{e}^- + \delta_{ij} \mathbf{e}^i \mathbf{e}^j \quad (5.25)$$

and

$$\begin{aligned} \mathbf{e}^+ &= du \\ \mathbf{e}^- &= dr + rh - \frac{1}{2} r^2 \Delta du \end{aligned}$$

and take a  $u$ -independent basis of  $\Sigma$  to be given by  $\mathbf{e}^i = \mathbf{e}^i_j dy^j$  for  $i, j = 1, 2, 3$ ; where  $\mathbf{e}^i_j$  depends analytically on  $r$ , but not on  $u$  and  $y^j$  are local co-ordinates on  $\Sigma$ . The spin connection associated with the above frame is computed in Appendix C.

Furthermore, in what follows,  $\dot{\phantom{x}}$  denotes the Lie derivative with respect to  $\frac{\partial}{\partial r}$ ;  $\hat{d}$  denotes the restriction of the exterior derivative to surfaces of constant  $r$ , i.e.

$$\hat{d}\Delta = \partial_i \Delta dy^i, \quad \hat{d}h = \frac{1}{2} (\partial_i h_j - \partial_j h_i) dy^i \wedge dy^j, \quad \partial_i = \frac{\partial}{\partial y^i}. \quad (5.26)$$

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<sup>†</sup>Recall that a tri-holomorphic homothety is a conformal Killing vector  $K$  which preserves the sphere of complex structures, i. e.

$$\mathcal{L}_K(g^{HK}) = \eta g^{HK}, \quad \mathcal{L}_K I_i = 0,$$

where  $\eta$  is a non-zero constant, and  $I_i, i = 1, 2, 3$  are the complex structures satisfying the quaternionic algebra.

## 5.1 Conditions obtained from Supersymmetry

Supersymmetry imposes a number of conditions on the gauge field strength, as well as conditions on the geometry. The explicit analysis of the Killing spinor equations is given in Appendix C. Here we shall summarise the results which will be of use in the moduli space analysis to follow. In particular, the gauge field strength is given by

$$H = \eta du \wedge d(r\Delta^{\frac{1}{2}}) + \frac{1}{3} \left( \star_3 Y + (dr + rh) \wedge W \right) \quad (5.27)$$

where  $W$  is a  $r$ -dependent 1-form on  $\Sigma$ , and

$$Y = -3h - 3r\dot{h} + 2\eta r\sqrt{\Delta}W. \quad (5.28)$$

The Bianchi identity associated to this expression for  $H$  implies

$$\hat{d}(rh \wedge W + \star_3 Y) = 0 \quad (5.29)$$

and

$$\hat{d}W - \mathcal{L}_{\frac{\partial}{\partial r}}(rh \wedge W + \star_3 Y) = 0. \quad (5.30)$$

A number of further useful identities obtained from the supersymmetry analysis are

$$\hat{d}h = rh \wedge \dot{h} - \eta \star_3 \left( \Delta^{\frac{1}{2}}h + 2r\Delta^{\frac{1}{2}}\dot{h} - \eta r\Delta W + \frac{1}{2}\Delta^{-\frac{1}{2}}\hat{d}\Delta - \frac{1}{2}\eta\Delta^{-\frac{1}{2}}\dot{\Delta}h \right) \quad (5.31)$$

$$\begin{aligned} \hat{d}Y &= -3\mathcal{L}_{\frac{\partial}{\partial r}} \left( r^2h \wedge \dot{h} - \eta r \star_3 \left( \Delta^{\frac{1}{2}}h + 2r\Delta^{\frac{1}{2}}\dot{h} - \eta r\Delta W + \frac{1}{2}\Delta^{-\frac{1}{2}}\hat{d}\Delta - \frac{1}{2}r\Delta^{-\frac{1}{2}}\dot{\Delta}h \right) \right) \\ &+ 2\eta r\hat{d}(\Delta^{\frac{1}{2}}W). \end{aligned} \quad (5.32)$$

Also, the gauge field equations reduce to the following:

$$\hat{d} \star_3 W - \mathcal{L}_{\frac{\partial}{\partial r}}(rh \wedge \star_3 W) + \frac{2}{3}W \wedge \star_3 Y - 3\eta\mathcal{L}_{\frac{\partial}{\partial r}}(\partial_r(r\Delta^{\frac{1}{2}}) \text{dvol}_\Sigma) = 0. \quad (5.33)$$

## 6 Moduli Space Calculation

We shall now consider the moduli space of infinitesimal supersymmetric transverse deformations of the near-horizon data, and prove that, for compact  $\Sigma$ , this is finite-dimensional by establishing that the moduli are constrained by certain elliptic second order differential operators. This analysis follows that done in [17] for the case of non-supersymmetric vacuum horizons with a

cosmological constant, though for the solutions we consider, there is some modification due to the inclusion of a 2-form field strength, as well as supersymmetry.

In particular, suppose that we consider the metric written in Gaussian null coordinates as (2.4) and Taylor expand the metric data  $(\Delta, h, \gamma)$  as

$$\begin{aligned}\Delta &= \overset{\circ}{\Delta}(y) + r\delta\Delta(y) + \mathcal{O}(r^2), \\ h &= \overset{\circ}{h}(y) + r\delta h(y) + \mathcal{O}(r^2), \\ \gamma &= \overset{\circ}{\gamma}(y) + r\delta\gamma(y) + \mathcal{O}(r^2)\end{aligned}\tag{6.34}$$

where  $\overset{\circ}{\Delta}, \overset{\circ}{h}, \overset{\circ}{\gamma}$  are the near-horizon metric data, and the metric moduli are  $\delta\Delta, \delta h, \delta\gamma$ . There is some gauge ambiguity in this choice of metric moduli, although the near horizon data is unique. As noted in [17], the vector field

$$\xi = \frac{1}{2}f\left(dr + r\overset{\circ}{h} - \frac{1}{2}r^2\overset{\circ}{\Delta}du\right) - \frac{1}{4}r^2\left(\overset{\circ}{\Delta}f + \mathcal{L}_{\overset{\circ}{h}}f\right)du - \frac{1}{2}rdf\tag{6.35}$$

for an arbitrary smooth function  $f$  on  $\Sigma$ , maps the near-horizon data  $(\delta\Delta, \delta h, \delta\gamma)$  to  $(\delta\tilde{\Delta}, \delta\tilde{h}, \delta\tilde{\gamma})$  where

$$\begin{aligned}\delta\tilde{\gamma}_{ij} &= \delta\gamma_{ij} + \overset{\circ}{\nabla}_i\overset{\circ}{\nabla}_j f - \overset{\circ}{h}_{(i}\overset{\circ}{\nabla}_{j)} f \\ \delta\tilde{h}_i &= \delta h_i + \frac{1}{2}\overset{\circ}{\Delta}\overset{\circ}{\nabla}_i f - \frac{1}{4}(\overset{\circ}{\nabla}_i\overset{\circ}{h}_j)\overset{\circ}{\nabla}^j f - \frac{1}{4}\overset{\circ}{h}_i\overset{\circ}{h}_j\overset{\circ}{\nabla}^j f + \frac{1}{2}(\overset{\circ}{\nabla}_j\overset{\circ}{h}_i)\overset{\circ}{\nabla}^j f + \frac{1}{4}\overset{\circ}{h}_j\overset{\circ}{\nabla}_i\overset{\circ}{\nabla}^j f \\ \delta\tilde{\Delta} &= \delta\Delta + \frac{1}{2}\overset{\circ}{\nabla}^i f\left(\overset{\circ}{\nabla}_i\overset{\circ}{\Delta} - \overset{\circ}{h}_i\overset{\circ}{\Delta}\right).\end{aligned}\tag{6.36}$$

In addition to the metric, we also have the Maxwell 2-form, which we have shown can be decomposed as

$$H = \eta du \wedge d(r\Delta^{\frac{1}{2}}) + \frac{1}{3}\left(\star_3\left(-3h - 3r\dot{h} + 2\eta r\sqrt{\Delta}W\right) + (dr + rh) \wedge W\right)\tag{6.37}$$

where  $W$  is a 1-form on  $\Sigma$ . The  $--$  component of the Einstein equations implies that  $W$  is of the same order as  $(\delta\Delta, \delta h, \delta\gamma)$ . We therefore take the transverse moduli to be  $(\delta\Delta, \delta h, \delta\gamma, W)$

**Theorem 6.1** *The moduli space of supersymmetric transverse deformations of supersymmetric near horizon solutions with compact spatial sections of horizons, corresponding to the moduli  $(\delta\Delta, \delta h, \delta\gamma, W)$ , modulo the gauge transformations of the type (6.36), is finite dimensional.*

**Proof.** To begin, we shall consider the trace of the metric moduli  $\delta\gamma_{ij}$ , and prove that by choosing an appropriate gauge transformation (6.36), this modulus satisfies an elliptic PDE which decouples from the remaining transverse moduli.

In particular, the trace transforms as

$$\delta\gamma_k{}^k \rightarrow \delta\gamma_k{}^k + \mathcal{D}f \quad (6.38)$$

where

$$\mathcal{D} \equiv \overset{\circ}{\nabla}{}^2 - \overset{\circ}{h}{}^i{}_{\circ} \overset{\circ}{\nabla}_i, \quad (6.39)$$

and the adjoint is given by

$$\mathcal{D}^\dagger = \overset{\circ}{\nabla}{}^2 + \overset{\circ}{h}{}^i{}_{\circ} \overset{\circ}{\nabla}_i \quad (6.40)$$

because  $\overset{\circ}{\nabla}_i \overset{\circ}{h}{}^i = 0^\ddagger$ . We decompose  $\delta\gamma_k{}^k$  as

$$\delta\gamma_k{}^k = \phi + \phi^\perp \quad (6.41)$$

where  $\phi \in \text{Im}\mathcal{D}$ , and  $\phi^\perp \in (\text{Im}\mathcal{D})^\perp$ . It follows that

$$\phi = \mathcal{D}(\tau) \quad (6.42)$$

for some smooth function  $\tau$ , and  $\phi^\perp \in (\text{Im}\mathcal{D})^\perp$  must satisfy

$$\mathcal{D}^\dagger \phi^\perp = 0, \quad (6.43)$$

which is an elliptic PDE. So, choosing  $f = -\tau$  in the transformation (6.38), with this choice of gauge

$$\delta\gamma_k{}^k = \phi^\perp \quad (6.44)$$

and so in this gauge

$$\mathcal{D}^\dagger \delta\gamma_k{}^k = 0. \quad (6.45)$$

In fact, this implies that  $\delta\gamma_k{}^k$  must be constant. To see this, note that (6.45) implies that

$$\delta\gamma_k{}^k \overset{\circ}{\nabla}{}^2 \delta\gamma_j{}^j + \frac{1}{2} \overset{\circ}{h}{}^i{}_{\circ} \overset{\circ}{\nabla}_i \left( (\delta\gamma_k{}^k)^2 \right) = 0. \quad (6.46)$$

---

<sup>‡</sup>This was established in [20].

On integrating this expression over  $\mathring{\Sigma}$ , and using  $\mathring{\nabla}_i \mathring{h}^{\circ i} = 0$ , implies (under the additional assumption that the near-horizon spatial cross section has no boundary) that  $\delta\gamma_k^k$  is constant.

To analyse the remaining transverse moduli, we linearize the field equations in terms of the moduli  $\delta\Delta$ ,  $\delta h$ ,  $\delta\gamma$  and  $W$ , making use of the conditions imposed by supersymmetry which we have previously obtained. We first use the  $-i$  and  $+-$  components of the Einstein equations to fix the moduli  $\delta h$  and  $\delta\Delta$  in terms of  $W$  and  $\delta\gamma$  as:

$$\delta h_i = -\frac{1}{2}\mathring{\nabla}_j \delta\gamma_i^j - \frac{1}{4}(\delta\gamma_k^k)\mathring{h}_i + \frac{1}{2}\mathring{h}^{\circ j} \delta\gamma_{ij} + \frac{1}{2}\eta\mathring{\Delta}^{\frac{1}{2}} W_i - \frac{1}{2}W_j \epsilon_i^{jk} \mathring{h}_k \quad (6.47)$$

$$\begin{aligned} \delta\Delta &= \mathring{\nabla}_i \delta h^i - \mathring{h}^{\circ i} \delta h_i - (\mathring{\nabla}_i \delta\gamma^{ij})\mathring{h}_j + \frac{1}{4}\delta\gamma_k^k \mathring{h}_i \mathring{h}^{\circ i} \\ &- \frac{1}{2}\mathring{\Delta} \delta\gamma_k^k + \frac{1}{4}\mathring{h}^{\circ i \circ j} \delta\gamma_{ij} - \frac{1}{2}\eta\mathring{\Delta}^{\frac{1}{2}} \mathring{h}^{\circ i} W_i - \frac{1}{6}\eta W^i \mathring{\nabla}_i \mathring{\Delta}^{\frac{1}{2}} - \frac{1}{2}\delta\gamma^{ij} \mathring{\nabla}_i \mathring{h}_j. \end{aligned} \quad (6.48)$$

We remark that as a consequence of the analysis in [20], the last two terms in (6.48) vanish, because

$$\mathring{\Delta} = \text{const.}, \quad \mathring{\nabla}_{(i} \mathring{h}_{j)} = 0. \quad (6.49)$$

Having fixed these moduli, we shall construct elliptic systems of PDEs constraining  $W$  and  $\delta\gamma$ . To begin, consider the  $W$  moduli. We consider the Bianchi identity (5.30), which when linearized implies

$$\mathring{\nabla}_i W_j - \mathring{\nabla}_j W_i = \left( \mathring{h} \wedge W + \mathcal{L}_{\frac{\partial}{\partial r}} \star_3 Y \right)_{ij}. \quad (6.50)$$

On taking the divergence, we then obtain the condition

$$\mathring{\nabla}^2 W_j = -\mathring{R}_{ij} W^i + \mathring{\nabla}_j \left( \mathring{\nabla}_i W^i \right) + \mathring{\nabla}^i \left( \mathring{h} \wedge W + \mathcal{L}_{\frac{\partial}{\partial r}} \star_3 Y \right)_{ij} \quad (6.51)$$

and the term  $\mathring{\nabla}_i W^i$  is given by linearizing (5.33), as

$$\mathring{\nabla}_i W^i = 3\mathring{h}^{\circ i} W_i + \frac{3}{2}\eta\mathring{\Delta}^{\frac{1}{2}} \delta\gamma_i^i + 6\eta\delta(\Delta^{\frac{1}{2}}) \quad (6.52)$$

and hence

$$\begin{aligned} \mathring{\nabla}^2 W_j &= -\mathring{R}_{ij} W^i + \mathring{\nabla}^i \left( \mathring{h} \wedge W \right)_{ij} + \mathring{\nabla}_j \left( 3\mathring{h}^{\circ i} W_i + \frac{3}{2}\eta\mathring{\Delta}^{\frac{1}{2}} \delta\gamma_i^i \right) \\ &+ 6\eta\mathring{\nabla}_j \delta(\Delta^{\frac{1}{2}}) + \mathring{\nabla}^i \left( \mathcal{L}_{\frac{\partial}{\partial r}} \star_3 Y \right)_{ij} \Big|_{r=0}. \end{aligned} \quad (6.53)$$

It is clear that the first three terms on the RHS of this expression give no contribution to the principle symbol of the differential operator acting on  $W$ . However, due to the presence of  $\delta\Delta$  in the remaining terms, it may appear that the RHS contains terms of order  $\overset{\circ}{\nabla}^2 \delta g$  on using (6.47) and (6.48). Such terms arise in the combination

$$6\eta\hat{d}\delta\Delta^{\frac{1}{2}} - \star_3(\hat{d}\delta Y). \quad (6.54)$$

However, on making use of (5.32), it follows that the  $\delta\Delta$  contribution to this expression vanishing, and hence in (6.53) the RHS depends on the moduli linearly in  $W, \overset{\circ}{\nabla}W, \delta\gamma, \overset{\circ}{\nabla}\delta\gamma$ .

Next, we consider the  $ij$  components of the Einstein equations, which imply

$$R_{ij} = \frac{3}{2}(H_{i+}H_{j-} + H_{j+}H_{i-} + H_{i\ell}H_j^\ell) - \frac{1}{4}\delta_{ij}(-2(H_{+-})^2 + 4H_{-\ell}H_+^\ell + H_{\ell_1\ell_2}H^{\ell_1\ell_2}) \quad (6.55)$$

On making use of (5.28), all of the terms quadratic in  $H$  on the RHS of this expression depend linearly in  $W, \overset{\circ}{\nabla}W, \delta\gamma, \overset{\circ}{\nabla}\delta\gamma$ , with the exception of the  $(H_{+-})^2$  term, which gives rise to a  $\delta\Delta$  term. Taking this into account, and making use of (6.47) and (6.48) we find

$$\overset{\circ}{\nabla}^2 \delta\gamma_{ij} - \delta_{ij}\overset{\circ}{\nabla}_k\overset{\circ}{\nabla}_\ell\delta\gamma^{\ell k} - (\overset{\circ}{\nabla}_\ell\overset{\circ}{\nabla}_j - \overset{\circ}{\nabla}_j\overset{\circ}{\nabla}_\ell)\delta\gamma^\ell{}_i - (\overset{\circ}{\nabla}_\ell\overset{\circ}{\nabla}_i - \overset{\circ}{\nabla}_i\overset{\circ}{\nabla}_\ell)\delta\gamma^\ell{}_j = \mathcal{A}_{ij} \quad (6.56)$$

where  $\mathcal{A}_{ij}$  depends linearly on  $W, \overset{\circ}{\nabla}W, \delta\gamma, \overset{\circ}{\nabla}\delta\gamma$ . This expression can be simplified by first noting that the terms on the second line of the LHS can be rewritten in terms of  $\overset{\circ}{R}$  curvature terms, and hence incorporated into the algebraic term on the RHS, i.e.

$$\overset{\circ}{\nabla}^2 \delta\gamma_{ij} - \delta_{ij}\overset{\circ}{\nabla}_k\overset{\circ}{\nabla}_\ell\delta\gamma^{\ell k} = \mathcal{B}_{ij} \quad (6.57)$$

where  $\mathcal{B}_{ij}$  depends linearly on  $W, \overset{\circ}{\nabla}W, \delta\gamma, \overset{\circ}{\nabla}\delta\gamma$ . On taking the trace of (6.57) we also find

$$\overset{\circ}{\nabla}_k\overset{\circ}{\nabla}_\ell\delta\gamma^{\ell k} = -\frac{1}{3}\mathcal{B}_i{}^i \quad (6.58)$$

and so the second term on the LHS of (6.57) can be eliminated in favour of  $\mathcal{B}_i{}^i$ , to give

$$\overset{\circ}{\nabla}^2 \delta\gamma_{ij} = \mathcal{C}_{ij} \quad (6.59)$$

where  $\mathcal{C}_{ij}$  depends linearly on  $W, \overset{\circ}{\nabla}W, \delta\gamma, \overset{\circ}{\nabla}\delta\gamma$ .

The condition (6.59) is an elliptic constraint on the traceless part of  $\delta\gamma_{ij}$ . So, we have proven that there exists a gauge in which the system of PDEs (6.59), (6.45) together with (6.53) constitute an elliptic set of PDEs which constrain the moduli  $W, \delta\gamma_k{}^k$  and the traceless part of  $\delta\gamma_{ij}$ .

□

## Appendix A Spinorial Geometry Conventions

The space of Dirac spinors consists of the space of complexified forms on  $\mathbb{R}^2$ , which has basis  $\{1, e_1, e_2, e_{12} = e_1 \wedge e_2\}$ . We define the action of the Clifford algebra generators on this space via

$$\gamma_i = -e_i \wedge -i_{e_i}, \quad \gamma_{i+2} = i(-e_i \wedge +i_{e_i}) \quad i = 1, 2 \quad (\text{A.1})$$

and set

$$\gamma_0 = i\gamma_{1234} \quad (\text{A.2})$$

which acts as

$$\gamma_0 1 = i1, \quad \gamma_0 e_{12} = i e_{12}, \quad \gamma_0 e_i = -i e_i. \quad (\text{A.3})$$

We then define generators adapted to the frame (5.25) as

$$\Gamma_{\pm} = \frac{1}{\sqrt{2}}(\gamma_3 \pm \gamma_0), \quad \Gamma_1 = \gamma_1, \quad \Gamma_2 = \sqrt{2}e_2 \wedge, \quad \Gamma_{\bar{2}} = \sqrt{2}i_{e_2} \quad (\text{A.4})$$

where we take a basis  $\{\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^{\bar{2}}\}$  for  $\Sigma$  such that  $\mathbf{e}^{\bar{2}} = (\mathbf{e}^2)^*$  and the metric on  $\Sigma$  is

$$\gamma = (\mathbf{e}^1)^2 + 2\mathbf{e}^2 \mathbf{e}^{\bar{2}}. \quad (\text{A.5})$$

With these conventions, the space of positive chirality spinors is spanned by  $\{1 - e_1, e_2 + e_{12}\}$ , and the space of negative chirality spinors is spanned by  $\{1 + e_1, e_2 - e_{12}\}$  and we remark that  $Spin(3)$ , with generators  $i\Gamma_{2\bar{2}}, \Gamma_1(\Gamma_2 + \Gamma_{\bar{2}}), i\Gamma_1(\Gamma_2 - \Gamma_{\bar{2}})$  form a representation of  $SU(2)$  acting on  $\{1 - e_1, e_2 + e_{12}\}$ .

A  $Spin(4, 1)$  invariant inner product  $\beta$  on the space of spinors is then given by

$$\beta(\epsilon_1, \epsilon_2) = \langle \gamma_0 \epsilon_1, \epsilon_2 \rangle = \frac{1}{\sqrt{2}} \langle (\Gamma_+ - \Gamma_-) \epsilon_1, \epsilon_2 \rangle \quad (\text{A.6})$$

where  $\langle, \rangle$  denotes the canonical inner product on  $\mathbb{C}^4$  equipped with basis  $\{1, e_1, e_2, e_{12}\}$ .

The charge conjugation operator  $C$  is defined by

$$C.1 = -e_{12}, \quad C.e_{12} = 1, \quad C.e_i = \epsilon_i^j e_j \quad (\text{A.7})$$

and satisfies

$$C * \Gamma_{\mu} + \Gamma_{\mu} C * = 0. \quad (\text{A.8})$$

With respect to the *real* frame (5.25),

$$\Gamma_i^\dagger = \Gamma_i, \quad \Gamma_+^\dagger = \Gamma_-, \quad \Gamma_-^\dagger = \Gamma_+. \quad (\text{A.9})$$

We also note the following useful identities:

$$\Gamma_{ij}\epsilon_\pm = \mp i\epsilon_{ij}{}^k\Gamma_k\epsilon_\pm, \quad \Gamma_{ijk}\epsilon_\pm = \mp i\epsilon_{ijk}\epsilon_\pm. \quad (\text{A.10})$$

The relationship between the 5-dimensional volume form  $\epsilon_5$  and the volume form  $\epsilon_\Sigma$  of  $\Sigma$  is

$$\epsilon_5 = \mathbf{e}^+ \wedge \mathbf{e}^- \wedge \epsilon_\Sigma. \quad (\text{A.11})$$

## Appendix B Ricci Tensor

The components of the Ricci tensor in this basis are:

$$\begin{aligned} \mathcal{R}_{++} &= r^2 \left( -\frac{3}{2}h^i\hat{\nabla}_i\Delta - \frac{1}{2}\Delta\hat{\nabla}_i h^i + \frac{1}{2}\hat{\nabla}_i\hat{\nabla}^i\Delta + \Delta h_i h^i + \frac{1}{4}(\hat{d}h)_{ij}(\hat{d}h)^{ij} \right) \\ &+ r^3 \left( \frac{1}{2}\dot{h}^i\hat{\nabla}_i\Delta - h^i\hat{\nabla}_i\dot{\Delta} + \frac{1}{2}h^i\hat{\nabla}^j\Delta\dot{g}_{ij} - \frac{1}{4}\dot{g}_k{}^k h^i\hat{\nabla}_i\Delta + 2\dot{\Delta}h_i h^i \right. \\ &- \Delta h_i \dot{h}^i - h^i \dot{h}^j (\hat{d}h)_{ij} - \frac{1}{4}\Delta h^i h^j \dot{g}_{ij} + \frac{1}{4}\Delta h_i h^i \dot{g}_k{}^k + \frac{1}{2}\Delta\hat{\nabla}_i \dot{h}^i - \frac{1}{2}\dot{\Delta}\hat{\nabla}_i h^i \left. \right) \\ &+ r^4 \left( -\frac{1}{2}\Delta h^i \ddot{h}_i + \frac{1}{2}\Delta h^i \dot{h}^j \dot{g}_{ij} - \frac{1}{2}\dot{\Delta} h^i h^j \dot{g}_{ij} - \frac{1}{8}\Delta^2 (\partial_r \dot{g}_k{}^k + \frac{1}{2}\dot{g}_{ij}\dot{g}^{ij}) \right. \\ &+ \left. \frac{1}{2}\ddot{\Delta} h_i h^i + \frac{1}{4}\dot{\Delta} h_i h^i \dot{g}_k{}^k - \frac{1}{4}\Delta h_i \dot{h}^i \dot{g}_k{}^k + \frac{1}{2}h_i h^i \dot{h}_j \dot{h}^j - \frac{1}{2}h^i \dot{h}_i h^j \dot{h}_j \right) \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} \mathcal{R}_{+-} &= \frac{1}{2}\hat{\nabla}_i h^i - \frac{1}{2}h_i h^i - \Delta \\ &+ r \left( \frac{1}{2}\hat{\nabla}_i \dot{h}^i - \frac{1}{4}\dot{g}_k{}^k h_j h^j - 2\dot{\Delta} - \frac{1}{2}\Delta\dot{g}_k{}^k + \frac{1}{2}h^i h^j \dot{g}_{ij} - 2h^i \dot{h}_i \right) \\ &+ r^2 \left( -\frac{1}{2}h^i \ddot{h}_i + \frac{1}{2}h^i \dot{h}^j \dot{g}_{ij} - \frac{1}{2}\dot{h}_i \dot{h}^i - \frac{1}{4}\dot{g}_k{}^k h_j h^j \right. \\ &- \left. \frac{1}{2}\ddot{\Delta} - \frac{1}{4}\dot{\Delta}\dot{g}_k{}^k - \frac{1}{4}\Delta(\partial_r \dot{g}_k{}^k + \frac{1}{2}\dot{g}_{ij}\dot{g}^{ij}) \right) \end{aligned} \quad (\text{B.2})$$

$$\mathcal{R}_{--} = -\frac{1}{2}(\partial_r \dot{g}_k{}^k + \frac{1}{2}\dot{g}_{ij}\dot{g}^{ij}) \quad (\text{B.3})$$

$$\begin{aligned}
\mathcal{R}_{-i} &= \dot{h}_i - \frac{1}{2}\hat{\nabla}_i \dot{g}_k^k + \frac{1}{4}h_i \dot{g}_k^k - \frac{1}{2}h^j \dot{g}_{ij} + \frac{1}{2}\hat{\nabla}_j \dot{g}_i^j \\
&+ r \left( \frac{1}{2}\ddot{h}_i - \frac{1}{2}h^j \ddot{g}_{ij} + h_i \left( \frac{1}{2}\partial_r \dot{g}_k^k + \frac{1}{4}\dot{g}_{jk} \dot{g}^{jk} \right) \right. \\
&\left. + \frac{1}{4}\dot{g}_k^k \dot{h}_i - \frac{1}{2}\dot{h}^j \dot{g}_{ij} - \frac{1}{4}\dot{g}_k^k h^j \dot{g}_{ij} \right) \tag{B.4}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{+i} &= r \left( \frac{1}{2}\hat{\nabla}_j (\hat{d}h)_i^j - h^j (\hat{d}h)_{ij} + \Delta h_i - \hat{\nabla}_i \Delta \right) \\
&+ r^2 \left( -\frac{1}{2}\Delta \dot{h}_i - \frac{1}{2}h_i \hat{\nabla}_j \dot{h}^j + \frac{1}{2}\dot{h}_i \hat{\nabla}_j h^j + h^j \hat{\nabla}_j \dot{h}_i - \frac{1}{2}\hat{\nabla}_i (h_j \dot{h}^j) + \frac{1}{2}\dot{g}_i^j \hat{\nabla}_j \Delta \right. \\
&+ \frac{1}{4}\Delta \hat{\nabla}_j \dot{g}_i^j + \frac{1}{2}\dot{g}_i^k (\hat{d}h)_k^j h_j + \frac{1}{2}(\hat{d}h)_{ik} h^j \dot{g}_j^k + \frac{3}{2}h_j h_i \dot{h}^j - \frac{3}{2}h_j h^j \dot{h}_i - \frac{3}{4}\Delta \dot{g}_i^j h_j \\
&- \frac{1}{2}\hat{\nabla}_i \dot{\Delta} - \frac{1}{4}\hat{\nabla}_i (\Delta \dot{g}_k^k) + 2\dot{\Delta} h_i + \frac{3}{8}\Delta h_i \dot{g}_k^k - \frac{1}{4}h^j (\hat{d}h)_{ij} \dot{g}_k^k \left. \right) \\
&+ r^3 \left( -\frac{1}{4}\Delta \ddot{h}_i + \frac{1}{2}h_i h^j \ddot{h}_j - \frac{1}{2}h_j h^j \ddot{h}_i + \frac{1}{2}h_i \dot{h}_j \dot{h}^j - \frac{1}{2}h_j \dot{h}^j \dot{h}_i + \frac{1}{4}\Delta \dot{g}_i^j \dot{h}_j \right. \\
&+ \frac{1}{2}\dot{g}_i^k (h_j h^j \dot{h}_k - h_j \dot{h}^j h_k) + \frac{1}{2}(\dot{h}_i h^j h^k - h_i h^j \dot{h}^k) \dot{g}_{jk} - \frac{1}{4}\Delta \dot{g}_i^j h_j \\
&+ \frac{1}{2}\ddot{\Delta} h_i + \frac{1}{4}h_i \dot{\Delta} \dot{g}_k^k + \frac{1}{2}h_i \Delta \left( \frac{1}{2}\partial_r \dot{g}_k^k + \frac{1}{4}\dot{g}_{jk} \dot{g}^{jk} \right) + \frac{1}{4}\dot{h}_i \dot{\Delta} - \frac{1}{4}\Delta \dot{g}_i^j h_j - \frac{1}{8}\Delta \dot{h}_i \dot{g}_k^k \\
&\left. + \frac{1}{2}\dot{g}_k^k h_j \left( \frac{1}{2}h_i \dot{h}^j - \frac{1}{2}h^j \dot{h}_i - \frac{1}{4}\Delta \dot{g}_i^j \right) - \frac{1}{4}\Delta \dot{g}_{ij} h^j \right) \tag{B.5}
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}_{ij} &= \hat{\mathcal{R}}_{ij} + \hat{\nabla}_{(i} h_{j)} - \frac{1}{2}h_i h_j \\
&+ r \left( \hat{\nabla}_{(i} \dot{h}_{j)} - 3h_{(i} \dot{h}_{j)} + (-\Delta + \frac{1}{2}\hat{\nabla}_k h^k - h_k h^k) \dot{g}_{ij} - \dot{g}_{(i}^k \hat{\nabla}_{|k|} h_{j)} \right. \\
&- h^k \hat{\nabla}_{(i} \dot{g}_{j)k} + h^k \hat{\nabla}_k \dot{g}_{ij} - h_{(i} \hat{\nabla}^k \dot{g}_{j)k} + 2h_k h_{(i} \dot{g}_{j)}^k + h_{(i} \hat{\nabla}_{j)} \dot{g}_k^k \\
&\left. + \frac{1}{2}\dot{g}_k^k (\hat{\nabla}_{(i} h_{j)} - h_i h_j) \right) \\
&+ r^2 \left( -\frac{1}{2}(\Delta + h_k h^k) (\ddot{g}_{ij} - \dot{g}_i^n \dot{g}_{nj}) + \frac{1}{2}h_i h_n (\ddot{g}_j^n - \dot{g}_j^k \dot{g}_k^n) + \frac{1}{2}h_j h_n (\ddot{g}_i^n - \dot{g}_i^k \dot{g}_k^n) \right. \\
&- \frac{1}{2}\Delta \dot{g}_{ij} - \frac{1}{2}h_i \ddot{h}_j - \frac{1}{2}h_j \ddot{h}_i - \frac{1}{2}h_i h_j (\partial_r \dot{g}_k^k + \frac{1}{2}\dot{g}_{nk} \dot{g}^{nk}) - \frac{1}{2}\dot{h}_i \dot{h}_j - \frac{1}{4}\Delta \dot{g}_k^k \dot{g}_{ij} \\
&+ \frac{1}{2}\dot{g}_i^k (h_j \dot{h}_k + h_k \dot{h}_j) + \frac{1}{2}\dot{g}_j^k (h_i \dot{h}_k + h_k \dot{h}_i) - h_k \dot{h}^k \dot{g}_{ij} - \frac{1}{4}\dot{g}_k^k (h_i \dot{h}_j + h_j \dot{h}_i) \\
&\left. - \frac{1}{4}h_n h^n \dot{g}_k^k \dot{g}_{ij} + \frac{1}{4}\dot{g}_k^k h_i h_n \dot{g}_j^n + \frac{1}{4}\dot{g}_k^k h_j h_n \dot{g}_i^n + \frac{1}{2}h^n h^k \dot{g}_{nk} \dot{g}_{ij} - \frac{1}{2}h^n h^k \dot{g}_{in} \dot{g}_{jk} \right) \tag{B.6}
\end{aligned}$$

Here  $\hat{\mathcal{R}}_{ij}$  denotes the Ricci tensor of  $\Sigma$ , and  $\dot{\phantom{x}}$  denotes the Lie derivative with respect to  $\frac{\partial}{\partial r}$ , so

$$\dot{h} = \mathcal{L}_{\frac{\partial}{\partial r}} h, \quad \dot{\mathbf{e}}^i = \mathcal{L}_{\frac{\partial}{\partial r}} \mathbf{e}^i, \quad \ddot{h} = \mathcal{L}_{\frac{\partial}{\partial r}} \dot{h}, \quad \text{etc.} \quad (\text{B.7})$$

and

$$\partial_r(h_i) = \dot{h}_i - (\dot{\mathbf{e}}^j)_i h_j, \quad \partial_r(\dot{h}_i) = \ddot{h}_i - (\dot{\mathbf{e}}^j)_i \dot{h}_j. \quad (\text{B.8})$$

## Appendix C Analysis of Gravitino Equation

In this Appendix we present the analysis of the gravitino Killing spinor equation. We first list the components of the spin connection, and then investigate the gravitino equation

### C.1 The spin connection

With respect to the frame (5.25) we have

$$\begin{aligned} de^+ &= 0, \\ de^- &= (\mathbf{e}^- - \frac{1}{2}r^2\Delta\mathbf{e}^+) \wedge h + r\dot{h} + r\Delta\mathbf{e}^+ \wedge \mathbf{e}^- - \frac{1}{2}r^2\hat{\Delta}\Delta \wedge \mathbf{e}^+ \\ &\quad + \frac{1}{2}r^2\dot{\Delta}\mathbf{e}^+ \wedge \mathbf{e}^- + \frac{1}{2}r^3\dot{\Delta}h \wedge \mathbf{e}^+ + r(\mathbf{e}^- - rh + \frac{1}{2}r^2\Delta\mathbf{e}^+) \wedge \dot{h}, \\ de^i &= (\mathbf{e}^- - rh + \frac{1}{2}r^2\Delta\mathbf{e}^+) \wedge \dot{\mathbf{e}}^i + \hat{d}\mathbf{e}^i. \end{aligned} \quad (\text{C.1})$$

Also, if  $g$  is any function, then the relationship between frame and co-ordinate indices is:

$$\begin{aligned} \partial_+ g &= \partial_u g + \frac{1}{2}r^2\Delta\dot{g} \\ \partial_- g &= \dot{g} \\ \partial_i g &= \hat{\partial}_i g - r\dot{g}h_i \end{aligned} \quad (\text{C.2})$$

where  $\hat{\partial}_i = e^j{}_i \partial_j$ . It follows that the components of the spin connection are given by

$$\begin{aligned} \omega_{+,+-} &= -r\Delta - \frac{1}{2}r^2\dot{\Delta} \\ \omega_{+,+i} &= \frac{1}{2}r^2\Delta h_i - \frac{1}{2}r^2\hat{\partial}_i\Delta + \frac{1}{2}r^3\dot{\Delta}h_i - \frac{1}{2}r^3\Delta\dot{h}_i \\ \omega_{+,-i} &= -\frac{1}{2}h_i - \frac{1}{2}r\dot{h}_i \end{aligned}$$

$$\begin{aligned}
\omega_{+,ij} &= -\frac{1}{2}r(\hat{d}h)_{ij} + \frac{1}{2}r^2(h_i\dot{h}_j - h_j\dot{h}_i) - \frac{1}{4}r^2\Delta(\dot{\mathbf{e}}^i)_j + \frac{1}{4}r^2\Delta(\dot{\mathbf{e}}^j)_i \\
\omega_{-,+-} &= 0 \\
\omega_{-,+i} &= -\frac{1}{2}h_i - \frac{1}{2}r\dot{h}_i \\
\omega_{-,-i} &= 0 \\
\omega_{-,ij} &= -\frac{1}{2}(\dot{\mathbf{e}}^i)_j + \frac{1}{2}(\dot{\mathbf{e}}^j)_i \\
\omega_{i,+} &= \frac{1}{2}h_i + \frac{1}{2}r\dot{h}_i \\
\omega_{i,+j} &= -\frac{1}{4}r^2\Delta((\dot{\mathbf{e}}^i)_j + (\dot{\mathbf{e}}^j)_i) - \frac{1}{2}r(\hat{d}h)_{ij} + \frac{1}{2}r^2(h_i\dot{h}_j - h_j\dot{h}_i) \\
\omega_{i,-j} &= -\frac{1}{2}((\dot{\mathbf{e}}^i)_j + (\dot{\mathbf{e}}^j)_i) \\
\omega_{i,jk} &= \Omega_{i,jk} + \frac{1}{2}rh_i((\dot{\mathbf{e}}^j)_k - (\dot{\mathbf{e}}^k)_j) + \frac{1}{2}rh_j((\dot{\mathbf{e}}^i)_k + (\dot{\mathbf{e}}^k)_i) - \frac{1}{2}rh_k((\dot{\mathbf{e}}^i)_j + (\dot{\mathbf{e}}^j)_i) \quad (\text{C.3})
\end{aligned}$$

where  $\Omega_{i,jk}$  is the spin connection of  $\Sigma$  with basis  $\mathbf{e}^i$  (restricting to constant  $r$ ).

## C.2 Analysis of the KSE

Next, we consider the gravitino KSE (5.24). We shall first analyse these equations acting on a  $u$ -independent spinor  $\epsilon = \epsilon_+ + \epsilon_-$ , and then apply the conditions (C.20), (C.21) and (C.29) imposed by the bi-linear matching. We begin by analysing the  $\alpha = -$  and  $\alpha = +$  components of (5.24).

From the  $\alpha = -$  component of (5.24) we obtain the conditions

$$\partial_r\epsilon_+ = \left(\frac{1}{4}(\dot{\mathbf{e}}^i)_j\Gamma_i{}^j - \frac{3i}{4}H_{-i}\Gamma^i\right)\epsilon_+ \quad (\text{C.4})$$

and

$$\begin{aligned}
\partial_r\epsilon_- &= \Gamma_- \left(\frac{1}{4}(h_i + r\dot{h}_i)\Gamma^i + \frac{i}{2}H_{+-} + \frac{i}{8}H_{ij}\Gamma^{ij}\right)\epsilon_+ \\
&+ \left(\frac{1}{4}(\dot{\mathbf{e}}^i)_j\Gamma_i{}^j - \frac{i}{4}H_{-i}\Gamma^i\right)\epsilon_-. \quad (\text{C.5})
\end{aligned}$$

Furthermore, on substituting (C.4) and (C.5) into the  $\alpha = +$  component of (5.24), and using the condition  $\partial_u\epsilon = 0$  which we have obtained previously, we find the following algebraic conditions

$$\begin{aligned}
&\left(-\frac{3i}{8}r^2\Delta H_{-i}\Gamma^i + \frac{1}{2}(r\Delta + \frac{1}{2}r^2\dot{\Delta}) + \left(-\frac{1}{8}r(\hat{d}h)_{ij} + \frac{1}{4}r^2h_i\dot{h}_j\right)\Gamma^{ij} + \frac{i}{4}H_{+i}\Gamma^i\right)\epsilon_+ \\
&+ \Gamma_+ \left(-\frac{1}{4}(h_i + r\dot{h}_i)\Gamma^i - \frac{i}{8}H_{ij}\Gamma^{ij} + \frac{i}{2}H_{+-}\right)\epsilon_- = 0 \quad (\text{C.6})
\end{aligned}$$

and

$$\begin{aligned} & \Gamma_- \left( \frac{i}{4} r^2 \Delta H_{+-} + \frac{i}{16} r^2 \Delta H_{ij} \Gamma^{ij} + \left( \frac{3}{8} r^2 \Delta h_i - \frac{1}{4} r^2 \hat{\nabla}_i \Delta + \frac{1}{4} r^3 \dot{\Delta} h_i - \frac{1}{8} r^3 \Delta \dot{h}_i \right) \Gamma^i \right) \epsilon_+ \\ & + \left( -\frac{i}{8} r^2 \Delta H_{-i} \Gamma^i - \frac{1}{2} (r \Delta + \frac{1}{2} r^2 \dot{\Delta}) + \left( -\frac{1}{8} r (\hat{d}h)_{ij} + \frac{1}{4} r^2 h_i \dot{h}_j \right) \Gamma^{ij} + \frac{3i}{4} H_{+i} \Gamma^i \right) \epsilon_- = 0. \end{aligned} \quad (\text{C.7})$$

This exhausts the content of the  $\alpha = +$  and  $\alpha = -$  components of (5.24). The  $\alpha = i$  component of (5.24) is equivalent to

$$\begin{aligned} \hat{\nabla}_i \epsilon_+ + \left( \frac{3i}{4} r h_i H_{-j} \Gamma^j - \frac{1}{4} (h_i + r \dot{h}_i) + \frac{1}{4} r h_j \dot{\gamma}_{ik} \Gamma^{jk} + \frac{i}{4} \Gamma_i H_{+-} - \frac{i}{8} \Gamma_i^{jk} H_{jk} + \frac{i}{2} H_{ij} \Gamma^j \right) \epsilon_+ \\ + \left( \frac{1}{4} \dot{\gamma}_{ij} \Gamma^j + \frac{i}{4} H_{-j} \Gamma_i^j - \frac{i}{2} H_{-i} \right) \Gamma_+ \epsilon_- = 0 \end{aligned} \quad (\text{C.8})$$

$$\begin{aligned} \hat{\nabla}_i \epsilon_- + \left( \frac{i}{4} r h_i H_{-j} \Gamma^j + \frac{1}{4} (h_i + r \dot{h}_i) + \frac{1}{4} r h_j \dot{\gamma}_{ik} \Gamma^{jk} - \frac{i}{4} \Gamma_i H_{+-} - \frac{i}{8} \Gamma_i^{jk} H_{jk} + \frac{i}{2} H_{ij} \Gamma^j \right) \epsilon_- \\ + \left( \frac{1}{4} r h_i h_j \Gamma^j - \frac{i}{2} r h_i H_{+-} - \frac{i}{8} r h_i H_{jk} \Gamma^{jk} + \left( \frac{1}{8} r^2 \Delta \dot{\gamma}_{ij} + \frac{1}{4} r (\hat{d}h)_{ij} + \frac{1}{4} r^2 \dot{h}_i h_j \right) \Gamma^j \right. \\ \left. + \frac{i}{4} H_{+j} \Gamma_i^j - \frac{i}{2} H_{+i} \right) \Gamma_- \epsilon_+ = 0 \end{aligned} \quad (\text{C.9})$$

where

$$\dot{\gamma}_{ij} = \mathbf{e}^m{}_i \mathbf{e}^n{}_j \delta_{kl} \left( \dot{\mathbf{e}}^k{}_m \mathbf{e}^\ell{}_n + \mathbf{e}^k{}_m \dot{\mathbf{e}}^\ell{}_n \right). \quad (\text{C.10})$$

### C.3 $u$ -dependence of the Spinor and Bi-linear Matching

To proceed, note first that if  $\epsilon$  is a Killing spinor then so is  $C * \epsilon$ , where  $C *$  denotes the charge conjugation operator. Also,  $\epsilon$  and  $C * \epsilon$  are linearly independent (over  $\mathbb{C}$ ). We shall assume that the bulk black hole solution is half-supersymmetric.

As all the bosonic fields, and the frame, are  $u$ -independent, it follows that if  $\epsilon$  is a Killing spinor then so is  $\partial_u \epsilon$ . This implies that there exist constants  $k_1, k_2 \in \mathbb{C}$  such that

$$\partial_u \epsilon = k_1 \epsilon + k_2 C * \epsilon. \quad (\text{C.11})$$

Now consider the Killing spinor

$$\tilde{\epsilon} = \alpha \epsilon + \beta C * \epsilon \quad (\text{C.12})$$

for constant  $\alpha, \beta \in \mathbb{C}$ . By choosing  $\alpha, \beta$  appropriately (not both zero), it follows that there exists a Killing spinor  $\tilde{\epsilon}$  such that

$$\partial_u \tilde{\epsilon} = k \tilde{\epsilon} \quad (\text{C.13})$$

for constant  $k \in \mathbb{C}$ , and hence

$$\tilde{\epsilon} = e^{ku} \phi \quad (\text{C.14})$$

where  $\partial_u \phi = 0$ . If the solution is exactly half-supersymmetric, then any Killing spinor  $\epsilon$  can be written as a linear combination of  $\tilde{\epsilon}$  and  $C * \tilde{\epsilon}$ , it follows that

$$\epsilon = \ell_1 e^{ku} \phi + \ell_2 e^{\bar{k}u} C * \phi \quad (\text{C.15})$$

for complex constants  $\ell_1, \ell_2$ . We shall require that the spinor has a well-defined near-horizon limit, which implies that  $k = 0$ , and hence the Killing spinor  $\epsilon$  is  $u$ -independent.

Now, we shall assume that there exists a Killing spinor  $\epsilon$  such that the  $Spin(4, 1)$ -invariant 1-form Killing spinor bi-linear  $Z$ , where

$$Z_\alpha = \langle (\Gamma_+ - \Gamma_-) \epsilon, \Gamma_\alpha \epsilon \rangle \quad (\text{C.16})$$

is proportional to the 1-form dual to the black hole Killing vector

$$V = \frac{\partial}{\partial u}. \quad (\text{C.17})$$

We shall identify  $V$  with  $Z$ , and set, without loss of generality

$$Z = -2V = r^2 \Delta \mathbf{e}^+ - 2\mathbf{e}^-. \quad (\text{C.18})$$

In order to impose the bi-linear matching condition  $Z = -2V$ , we decompose the spinor  $\epsilon$  as

$$\epsilon = \epsilon_+ + \epsilon_-, \quad \Gamma_\pm \epsilon_\pm = 0. \quad (\text{C.19})$$

The condition  $Z_- = -2V_-$  implies

$$\| \epsilon_+ \|^2 = 1 \quad (\text{C.20})$$

and the condition  $Z_+ = -2V_+$  implies

$$\| \epsilon_- \|^2 = \frac{1}{2} r^2 \Delta \quad (\text{C.21})$$

We also require  $Z_i = 0$ , or equivalently

$$\text{Re} \left( \langle \epsilon_+, \Gamma_i \Gamma_+ \epsilon_- \rangle \right) = 0. \quad (\text{C.22})$$

We shall use spinorial geometry methods to analyse this condition. First, note that we can apply a  $Spin(3)$  gauge transformation as described in the Appendix A, to set

$$\epsilon_+ = f(1 - e_1) \quad (\text{C.23})$$

where  $f \in \mathbb{R}$ . Note that (C.20) implies that  $2f^2 = 1$ . In addition, we set

$$\epsilon_- = p(1 + e_1) + q(e_2 - e_{12}) \quad (\text{C.24})$$

for  $p, q \in \mathbb{C}$ . So (C.22) can be rewritten as

$$\text{Im} \left( \langle \Gamma_i(1 + e_1), p(1 + e_1) + q(e_2 - e_{12}) \rangle \right) = 0. \quad (\text{C.25})$$

It is straightforward to note that these conditions imply that

$$\text{Im}(p) = 0, \quad q = 0. \quad (\text{C.26})$$

Hence

$$\epsilon_- = h(1 + e_1) \quad (\text{C.27})$$

for  $h \in \mathbb{R}$ . The condition (C.21) implies that  $2h^2 = \frac{1}{2}r^2\Delta$ . In particular, note that  $\Delta \geq 0$ . These conditions can be rewritten as

$$\epsilon_- = \frac{i}{\sqrt{2}} \frac{h}{f} \Gamma_- \epsilon_+ \quad (\text{C.28})$$

and on using (C.20) and (C.21), this can be further rewritten as

$$\epsilon_- = \frac{i}{2} \eta r \Delta^{\frac{1}{2}} \Gamma_- \epsilon_+ \quad (\text{C.29})$$

where  $\eta^2 = 1$ . We then compute the gauge-invariant scalar bi-linear, to obtain

$$\beta(\epsilon, \epsilon) = -\sqrt{2} i \eta r \sqrt{\Delta}. \quad (\text{C.30})$$

We require that all spinor bilinears are analytic functions of  $r$ , and so  $r\sqrt{\Delta}$  is analytic in  $r$ .

## C.4 Bi-linear Matching: Further Simplification of the KSE

Next we substitute the bi-linear matching conditions (C.20), (C.21) and (C.29) into the KSE conditions (C.5), (C.6), (C.7), (C.8), (C.9). We rewrite these conditions in terms of conditions solely on  $\epsilon_+$ .

First, note that on using (C.4) it follows that (C.5), (C.6), (C.7) are equivalent to

$$H_{+-} = \eta(\Delta^{\frac{1}{2}} + \frac{1}{2}r\Delta^{-\frac{1}{2}}\dot{\Delta}) \quad (\text{C.31})$$

$$\frac{1}{4}(h_i + r\dot{h}_i) + \frac{1}{8}\epsilon_i^{mn}H_{mn} - \frac{1}{2}\eta r\Delta^{\frac{1}{2}}H_{-i} = 0 \quad (\text{C.32})$$

$$-\frac{3}{8}r^2\Delta H_{-i} + \left(\frac{1}{8}r(\hat{d}h)_{mn} - \frac{1}{4}r^2h_m\dot{h}_n\right)\epsilon_i^{mn} + \frac{1}{4}H_{+i} + \eta r\Delta^{\frac{1}{2}}\left(\frac{1}{4}(h_i + r\dot{h}_i) - \frac{1}{8}\epsilon_i^{mn}H_{mn}\right) = 0 \quad (\text{C.33})$$

$$\begin{aligned} & \frac{1}{16}r^2\Delta\epsilon_i^{mn}H_{mn} + \frac{3}{8}r^2\Delta h_i - \frac{1}{4}r^2\hat{\nabla}_i\Delta + \frac{1}{4}r^3\dot{\Delta}h_i - \frac{1}{8}r^3\Delta\dot{h}_i \\ & - \frac{1}{16}\eta r^3\Delta^{\frac{3}{2}}H_{-i} - \frac{1}{2}\eta r\Delta^{\frac{1}{2}}\left(\frac{1}{8}r(\hat{d}h)_{mn} - \frac{1}{4}r^2h_m\dot{h}_n\right)\epsilon_i^{mn} + \frac{3}{8}\eta r\Delta^{\frac{1}{2}}H_{+i} = 0 \end{aligned} \quad (\text{C.34})$$

and it is straightforward to show that the KSE (C.9) is implied by (C.8) together with (C.31), (C.32), (C.33) and (C.34). The conditions (C.31), (C.32), (C.33) and (C.34) determine all of the components of  $H$ , as

$$H_{+-} = \eta(\Delta^{\frac{1}{2}} + \frac{1}{2}r\Delta^{-\frac{1}{2}}\dot{\Delta}) \quad (\text{C.35})$$

$$\begin{aligned} H_{-i} &= \frac{1}{3}\eta r^{-1}\Delta^{-\frac{1}{2}}h_i + \frac{2}{3}\eta\Delta^{-\frac{1}{2}}\dot{h}_i + \frac{1}{6}\eta r^{-1}\Delta^{-\frac{3}{2}}\hat{\nabla}_i\Delta \\ &- \frac{1}{6}\eta\Delta^{-\frac{3}{2}}\dot{\Delta}h_i + \frac{4}{3}r^{-2}\Delta^{-1}\left(\frac{1}{8}r(\hat{d}h)_{mn} - \frac{1}{4}r^2h_m\dot{h}_n\right)\epsilon_i^{mn} \end{aligned} \quad (\text{C.36})$$

$$\begin{aligned} \frac{1}{2}\epsilon_i^{mn}H_{mn} &= -\frac{1}{3}h_i + \frac{1}{3}r\dot{h}_i + \frac{1}{3}\Delta^{-1}\hat{\nabla}_i\Delta - \frac{1}{3}r\Delta^{-1}\dot{\Delta}h_i \\ &+ \frac{8}{3}\eta r^{-1}\Delta^{-\frac{1}{2}}\left(\frac{1}{8}r(\hat{d}h)_{mn} - \frac{1}{4}r^2h_m\dot{h}_n\right)\epsilon_i^{mn} \end{aligned} \quad (\text{C.37})$$

$$\begin{aligned}
H_{+i} &= -\frac{5}{6}\eta r \Delta^{\frac{1}{2}} h_i + \frac{1}{3}\eta r^2 \Delta^{\frac{1}{2}} \dot{h}_i + \frac{7}{12}\eta r \Delta^{-\frac{1}{2}} \hat{\nabla}_i \Delta - \frac{7}{12}\eta r^2 \Delta^{-\frac{1}{2}} \dot{\Delta} h_i \\
&+ \frac{2}{3} \left( \frac{1}{8} r (\hat{d}h)_{mn} - \frac{1}{4} r^2 h_m \dot{h}_n \right) \epsilon_i^{mn}
\end{aligned} \tag{C.38}$$

which implies that

$$\begin{aligned}
H &= \eta du \wedge d(r\Delta^{\frac{1}{2}}) + \frac{2}{3}\eta d(\Delta^{-\frac{1}{2}}h) + \frac{1}{3}\eta \Delta^{-\frac{3}{2}} r^{-1} dr \wedge d\Delta \\
&+ \frac{1}{3} r^{-1} (dr + rh) \wedge \left( \Delta^{-1} \star_3 (\hat{d}h - rh \wedge \dot{h}) + \eta \Delta^{-\frac{1}{2}} h - \frac{1}{2} \eta \Delta^{-\frac{3}{2}} d\Delta \right) \\
&+ \frac{1}{3} \star_3 \left( -h + r\dot{h} + \Delta^{-1} \hat{d}\Delta - r\Delta^{-1} \dot{\Delta} h \right),
\end{aligned} \tag{C.39}$$

or equivalently

$$H = \eta du \wedge d(r\Delta^{\frac{1}{2}}) + \frac{1}{3} \left( \star_3 Y + (dr + rh) \wedge W \right) \tag{C.40}$$

where

$$W_i = 3H_{-i}, \quad Y_i = \frac{3}{2} \epsilon_i^{mn} H_{mn}. \tag{C.41}$$

## C.5 Gauge Field Equations

Here, we briefly summarize some details required for the evaluation of the gauge field equations. In particular, we have

$$\begin{aligned}
\star H &= \frac{1}{3} r du \wedge dh - \frac{1}{3} du \wedge dr \wedge h - \eta \left( \Delta^{\frac{1}{2}} + \frac{1}{2} r \Delta^{-\frac{1}{2}} \dot{\Delta} \right) \text{dvol}_S \\
&+ du \wedge (dr + rh) \wedge \left( \frac{1}{3} \Delta^{-1} d\Delta + \frac{2}{3} \eta \Delta^{-\frac{1}{2}} \star_3 (\hat{d}h - rh \wedge \dot{h}) \right) \\
&- \frac{2}{3} \eta r \Delta^{\frac{1}{2}} du \wedge \star_3 \left( h - r\dot{h} - \Delta^{-1} \hat{d}\Delta + r\Delta^{-1} \dot{\Delta} h \right) \\
&- (dr + rh) \wedge \left( \star_3 \left( \frac{1}{3} \eta r^{-1} \Delta^{-\frac{1}{2}} h + \frac{2}{3} \eta \Delta^{-\frac{1}{2}} \dot{h} + \frac{1}{6} \eta r^{-1} \Delta^{-\frac{3}{2}} \hat{d}\Delta - \frac{1}{6} \eta \Delta^{-\frac{3}{2}} \dot{\Delta} h \right) \right. \\
&\left. + \frac{1}{3} r^{-1} \Delta^{-1} dh \right),
\end{aligned} \tag{C.42}$$

or equivalently

$$\begin{aligned}
\star H &= -du \wedge d(rh) + \frac{2}{3} \eta r \Delta^{\frac{1}{2}} du \wedge \left( \star_3 Y + (dr + rh) \wedge W \right) \\
&- \frac{1}{3} (dr + rh) \wedge \star_3 W - \eta \partial_r (r\Delta^{\frac{1}{2}}) \text{dvol}_\Sigma.
\end{aligned} \tag{C.43}$$

Then the  $(wrij)$  component of the gauge equations is equivalent to (5.29), and the  $(wijk)$  component is equivalent to (5.30). The only remaining  $(rijk)$  component of the gauge equations then reduces to (5.33).

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