Plectic arithmetic of Hilbert modular varieties

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Abstract

We introduce plectic Galois actions on the set of CM points, on the set of connected components, and on the set of cocharacters of Shimura varieties that differ in the centre from the Hilbert modular variety. By allowing the centre to vary, we extend the plectic framework of Nekovář–Scholl to include such Shimura varieties, thereby also bridging the gap to earlier work of Nekovář. Our main result is that the map that sends a point on the Shimura variety to its connected component is equivariant under the plectic action.

To achieve this, we define a generalisation of the plectic Taniyama element, describe the points of the Shimura varieties in question in terms of abelian varieties with extra structure, and orient ourselves by the main theorem of complex multiplication over the rationals to define the plectic action on CM points. Moreover, we use a description of the set of connected components as a zero-dimensional Shimura variety and then employ class field theoretic techniques to prove the main result.

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Introduction

...schon eine Idee ausdenken, durchdenken, bis in die letzte Weiterung durchrechnen und kombinieren, das ist eine Freude, auf die ich gar nicht mehr gerechnet habe.

– Stefan Zweig, *Rausch der Verwandlung*[1]

1.1 Background and framework

Fix a totally real number field F and denote its degree by $r = [F : \mathbb{Q}]$. The main objects of study in this thesis are variants of the Hilbert modular variety associated to F. These variants are certain Shimura varieties whose points parametrise r-dimensional abelian varieties with real multiplication by F equipped with a polarisation and a level structure. Of special focus will be those points corresponding to abelian varieties with complex multiplication (CM), so let us start with a summary of CM theory.

1.1.1 Galois conjugates of CM abelian varieties

A complex elliptic curve A_0 has CM if its endomorphism algebra is an imaginary quadratic number field K_0. The most powerful result in the theory of CM elliptic curves is the Main Theorem of Complex Multiplication. It describes the Galois conjugates of torsion points of A_0 in terms of the class field theory of K_0. See [Sil94, Ch. II] for a modern account and [Sch98] for a historical perspective. As an application, the Main Theorem implies "Kroneckers Jugendtraum" — an explicit construction of the maximal abelian extension of K_0 using the torsion points of A_0.

The first higher-dimensional analogue of CM theory was established by Shimura and Taniyama [ST61, Shi71]. They study abelian varieties A whose endomorphism

[1] Fischer Taschenbuch Verlag, 21st edition, July 2014, p. 290, l. 13-15.

algebra is a CM field K such that A induces a fixed CM type Φ on K. The Main Theorem in this context gives a very similar description of Galois conjugates of torsion points of A, with two main modifications: First of all, the theorem only gives information about conjugation by Galois elements that fix the reflex field E of the pair (K, Φ). And secondly, the class-field-theoretic behaviour is now governed by the reflex norm

$$N_\Phi \colon \mathbb{A}_{E,f}^\times \longrightarrow \mathbb{A}_{K,f}^\times.$$

This result turned out to be extremely powerful. For example, it allowed Deligne [Del71] to define the notion of canonical models for Shimura varieties. The problem of extending the Main Theorem to conjugation by arbitrary Galois elements, not only those fixing the reflex field, was addressed by Tate [Tat16]. He defined an extension of the reflex norm called the half transfer

$$F_\Phi \colon \Gamma_\mathbb{Q} \longrightarrow \Gamma_K^{\mathrm{ab}}$$

and conjectured (and proved "up to signs") that the formula describing Galois conjugates of CM abelian varieties in the Main Theorem over the reflex field should remain valid for conjugation by arbitrary Galois elements if one replaces the reflex norm by the half transfer. Deligne [Del82] completed Tate's proof and the result is now known as the Main Theorem of CM over $\mathbb{Q}$. It is intricately related to the Taniyama group [Lan79, MS82b], CM motives [Sch94, Far06], and to conjugation of Shimura varieties [MS82a].

As an application, the Main Theorem of CM over $\mathbb{Q}$ yields an explicit formula for the Galois action on the CM points of the PEL Hilbert modular variety in terms of the half transfer.

1.1.2 Plectic conjecture

The plectic Galois group is the semidirect product $S_r \ltimes \Gamma_F^r$ of the symmetric group S_r acting on r-tuples of elements of Γ_F by permuting the coordinates. Choosing representatives s_i for the right Γ_F-cosets in $\Gamma_\mathbb{Q}$, so that $\Gamma_\mathbb{Q} = \bigsqcup_{1 \le i \le r} s_i \Gamma_F$, we get an embedding

$$\rho_s \colon \Gamma_\mathbb{Q} \hookrightarrow S_r \ltimes \Gamma_F^r, \quad \gamma \mapsto (\sigma, (h_i)_{1 \le i \le r})$$

determined by

$$\gamma s_i = s_{\sigma(i)} h_i, \quad 1 \le i \le r.$$

The embedding ρ_s depends on the choice of the s_i; however, there are choice-free versions of the plectic group $S_r \ltimes \Gamma_F^r$ and of the embedding ρ_s, see e. g. (4.1.3).

Nekovář [Nek09] noticed that Tate's half transfer admits a natural extension to a plectic half transfer

$$\widetilde{F}_\Phi\colon S_r \ltimes \Gamma_F^r \longrightarrow \Gamma_K^{\mathrm{ab}}.$$

This allowed him to define an action of a certain large subgroup $(S_r \ltimes \Gamma_F^r)_0$ of $S_r \ltimes \Gamma_F^r$ on the CM points of the PEL Hilbert modular variety. By the formula mentioned at the end of Section 1.1.1, this action of $(S_r \ltimes \Gamma_F^r)_0$ extends the action of $\Gamma_{\mathbb{Q}}$. Moreover, Blake [Bla16] used the plectic half transfer to define a plectic Taniyama group.

Inspired by [Nek09], Nekovář and Scholl [NS16] formulated the plectic conjecture. It predicts that Shimura varieties associated to groups of the form $R_{F/\mathbb{Q}}H$, for H a reductive group over F, carry functorial and canonical extra structures. These so-called plectic structures should conjecturally exist on the level of motives. The prototypical example is the (non-PEL) Hilbert modular variety, which is associated to $R_{F/\mathbb{Q}}\,\mathrm{GL}_2$. In this case, the motives in question are given by abelian varieties with real multiplication and level structure.

For general H, [NS16] also outlines how the plectic structures should manifest themselves in various realisations, defines a plectic analogue of the reflex field called the plectic reflex Galois group, and sketches arithmetic applications e. g. to Stark's conjectures and Beilinson's conjectures. For example, on the étale cohomology groups of Shimura varieties, a plectic structure is simply an action of the plectic group extending the Galois action. In a different direction, [NS17] defines plectic mixed Hodge structures and shows that the singular cohomology of the Hilbert modular variety carries a canonical plectic mixed Hodge structure.

1.2 Motivation and main results

There is a subtle, but fundamental difference between the setups of [Nek09] and [NS16]. Namely [Nek09] considers the PEL Hilbert modular variety $\mathrm{Sh}(G, \mathfrak{h}^r \sqcup (-\mathfrak{h})^r)$ whose underlying group G is not of the form $R_{F/\mathbb{Q}}H$ for any group H. Instead, G is related to the group $R_{F/\mathbb{Q}}\,\mathrm{GL}_2$, which underlies the Hilbert modular variety $\mathrm{Sh}(R_{F/\mathbb{Q}}\,\mathrm{GL}_2, (\mathbb{C}\setminus\mathbb{R})^r)$, by the Cartesian diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d] & R_{F/\mathbb{Q}}\,\mathrm{GL}_2 \arrow[d, "R_{F/\mathbb{Q}}(\det)"] \\ \mathbb{G}_m \arrow[r, hook] & R_{F/\mathbb{Q}}\mathbb{G}_m. \end{tikzcd} \tag{1.2.0.1}$$

This situation motivates the following questions: Is there a way of extending the plectic framework of [NS16] to include Shimura varieties whose underlying groups are not necessarily of the form $R_{F/\mathbb{Q}}H$? Is there a common reason for the phenomena in [Nek09] and [NS16] that explains why it was necessary to restrict to the subgroup

$(S_r \ltimes \Gamma_F^r)_0 \subset S_r \ltimes \Gamma_F^r$ in [Nek09]? As a concrete example, we want to be able to define a plectic reflex Galois group for the PEL Hilbert modular variety in analogy to [NS16], and to define an action of the plectic group $S_r \ltimes \Gamma_F^r$ on the CM points of the non-PEL Hilbert modular variety in analogy to [Nek09].

In this thesis, we address these questions for groups G' that differ only in the centre from a restriction of scalars $R_{F/\mathbb{Q}}H$. Here we say that G' differs only in the centre from $G_1 := R_{F/\mathbb{Q}}H$ if G' embeds into G_1 and the diagram

$$\begin{tikzcd} G' \arrow[r, hook] \arrow[d, "d"] & G_1 \arrow[d, "d_1"] \\ C' \arrow[r, hook] & C_1 \end{tikzcd} \tag{1.2.0.2}$$

is Cartesian, where d and d_1 are the canonical maps $d\colon G' \to C' := G'/(G')^{\mathrm{der}}$ and $d_1\colon G_1 \to C_1 := G_1/G_1^{\mathrm{der}}$. We will see in (6.2.9) that this condition implies that G' and G_1 have the same adjoint and derived groups, but different centres, justifying the terminology.

For example, in the case of the Hilbert modular variety, the groups that differ only in the centre from $R_{F/\mathbb{Q}}\,\mathrm{GL}_2$ are in bijection with algebraic tori R over $\mathbb{Q}$ with $R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$. We denote the group corresponding to R by G^R; it fits into the Cartesian diagram

$$\begin{tikzcd} G^R \arrow[r, hook] \arrow[d] & R_{F/\mathbb{Q}}\,\mathrm{GL}_2 \arrow[d, "R_{F/\mathbb{Q}}(\det)"] \\ R \arrow[r, hook] & R_{F/\mathbb{Q}}\mathbb{G}_m. \end{tikzcd} \tag{1.2.0.3}$$

For technical reasons, we restrict to R with $\mathbb{G}_m \subset R$. For each such R, we define a subset $X^R \subset (\mathbb{C}\backslash\mathbb{R})^r$ such that (G^R, X^R) is a Shimura datum. We think of the family of Shimura varieties $\mathrm{Sh}(G^R, X^R)$, where R ranges over tori with $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$, as interpolating between the Hilbert modular case ($R = R_{F/\mathbb{Q}}\mathbb{G}_m$) and the PEL Hilbert modular case ($R = \mathbb{G}_m$). We therefore call the Shimura varieties $\mathrm{Sh}(G^R, X^R)$ variants of the Hilbert modular variety.

Our main results are as follows: For each R, we define three plectic groups $(S_r \ltimes \Gamma_F^r)^R_{\mathrm{CM}}$, $(S_r \ltimes \Gamma_F^r)^R_{\pi_0}$ and $(S_r \ltimes \Gamma_F^r)^R_{\mathrm{cc}}$ together with actions of

(A) $(S_r \ltimes \Gamma_F^r)^R_{\mathrm{CM}}$ on the set $\mathrm{Sh}(G^R, X^R)_{\mathrm{CM}}$ of CM points of $\mathrm{Sh}(G^R, X^R)$,

(B) $(S_r \ltimes \Gamma_F^r)^R_{\pi_0}$ on the set $\pi_0(\mathrm{Sh}(G^R, X^R))$ of connected components of $\mathrm{Sh}(G^R, X^R)$,

(C) $(S_r \ltimes \Gamma_F^r)^R_{\mathrm{cc}}$ on the set $X_*(G^R)$ of cocharacters of G^R.

More precisely, the groups $(S_r \ltimes \Gamma_F^r)^R_{\mathrm{CM}}$ and $(S_r \ltimes \Gamma_F^r)^R_{\mathrm{cc}}$ are subgroups of $S_r \ltimes \Gamma_F^r$. For many choices of R, the group $(S_r \ltimes \Gamma_F^r)^R_{\pi_0}$ is also a subgroup of $S_r \ltimes \Gamma_F^r$, but we do not know if this is true in general, see (5.2.7).

For (A) we need two ingredients: The first ingredient is a description of the points of $\mathrm{Sh}(G^R, X^R)$ in terms of abelian varieties with real multiplication by F equipped with an $R(\mathbb{Q})$-class of a polarisation and a level structure, see (4.4.5). The second ingredient is a class-field-theoretic preimage (the plectic Taniyama element)

$$\widetilde{f}_\Phi \colon S_r \ltimes \Gamma_F^r \longrightarrow \mathbb{A}_{K,f}^\times / K^\times$$

of the plectic half transfer $\widetilde{F}_\Phi$, see (4.3.6). The group $(S_r \ltimes \Gamma_F^r)_{\mathrm{CM}}^R$ is then precisely defined, in (4.4.11), in such a way that the formula for the Galois action on the CM points of $\mathrm{Sh}(G^R, X^R)$ extends to $(S_r \ltimes \Gamma_F^r)_{\mathrm{CM}}^R$ when Tate's half transfer F_Φ is replaced by the plectic Taniyama element $\widetilde{f}_\Phi$, see (4.4.13).

For (B) we take advantage of the description of $\pi_0(\mathrm{Sh}(G^R, X^R))$ as a zero-dimensional Shimura variety, see (5.1.9) and (5.2.3). We then use the class-field-theoretic description (5.2.4) of the Galois action on $\pi_0(\mathrm{Sh}(G^R, X^R))$ to define the group $(S_r \ltimes \Gamma_F^r)_{\pi_0}^R$ and its action on $\pi_0(\mathrm{Sh}(G^R, X^R))$ in (5.2.6).

Although the groups and actions in (A) and (B) are defined in different ways, we are able to prove the following theorem, which is Theorem (5.2.10) and can be seen as the main result of this thesis.

1.2.1 Theorem. *Let $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ be an intermediate algebraic torus defined over $\mathbb{Q}$. Then $(S_r \ltimes \Gamma_F^r)_{\mathrm{CM}}^R$ canonically embeds into $(S_r \ltimes \Gamma_F^r)_{\pi_0}^R$, and the π_0-map restricted to CM points*

$$\pi_0 \colon \mathrm{Sh}(G^R, X^R)_{\mathrm{CM}} \longrightarrow \pi_0(\mathrm{Sh}(G^R, X^R))$$

is $(S_r \ltimes \Gamma_F^r)_{\mathrm{CM}}^R$-equivariant.

Finally, (C) requires purely group-theoretic considerations: The full plectic group $S_r \ltimes \Gamma_F^r$ acts naturally on the cocharacters of $R_{F/\mathbb{Q}}\,\mathrm{GL}_2$ and by the Cartesian diagram (1.2.0.3) the cocharacters of G^R form a subgroup of the cocharacters of $R_{F/\mathbb{Q}}\,\mathrm{GL}_2$. We thus define $(S_r \ltimes \Gamma_F^r)_{\mathrm{cc}}^R$ as the stabiliser of this subgroup under the action of $S_r \ltimes \Gamma_F^r$, see (6.2.4). It is then straightforward to define the plectic reflex Galois group of (G^R, X^R), see (6.2.6).

To conclude, the group actions in (A), (B) and (C) all extend the natural action of the Galois group $\Gamma_\mathbb{Q}$. Thus they can be viewed as instances of plectic structures in the sense of [NS16] for the groups G^R, which differ only in the centre from $R_{F/\mathbb{Q}}\,\mathrm{GL}_2$. In the case $R = \mathbb{G}_m$ the action (A) is precisely the one discovered in [Nek09], and [NS16] dealt with (C) for $R = R_{F/\mathbb{Q}}\mathbb{G}_m$. In this sense, this thesis both generalises and interpolates between [Nek09] and [NS16].

1.3 Chapter overview

Chapters 2 and 3 are a review of CM theory as outlined in Section 1.1.1 and serve a twofold purpose: In these chapters we introduce the necessary notation, which is quite extensive, and present the results of CM theory in a form most amenable to generalisation. We often only sketch proofs or give references. We invite readers who are familiar with CM theory and the theory of Shimura varieties to skim through these chapters, or to directly start with Chapter 4.

In Chapter 2 we review the theory of complex multiplication over the reflex field. We start by introducing CM fields, CM types, the reflex norm, and CM abelian varieties. We then state the Main Theorem of Complex Multiplication over the reflex field in two versions, Theorem (2.4.7) and Theorem (2.4.10). We finish Chapter 2 with an application to the canonical model of the PEL Shimura variety parametrising polarised CM abelian varieties.

Section 3.1 recalls the definition of Tate's half transfer $F_\Phi\colon \Gamma_{\mathbb{Q}}^{\mathrm{ab}} \to \Gamma_K^{\mathrm{ab}}$ and states the Main Theorem of CM over $\mathbb{Q}$ again in two versions, Theorem (3.1.3) and Theorem (3.1.5). In Section 3.2 we interpret the points of the PEL Hilbert modular variety $\mathrm{Sh}(G, \mathfrak{h}^r \sqcup (-\mathfrak{h})^r)$ in terms of abelian varieties equipped with real multiplication, polarisation and level structure. By applying the Main Theorem over $\mathbb{Q}$, we obtain an explicit formula for the $\Gamma_{\mathbb{Q}}$-action on the CM points of $\mathrm{Sh}(G, \mathfrak{h}^r \sqcup (-\mathfrak{h})^r)$.

Sections 4.1 and 4.2 summarise the results of [Nek09]. Here we introduce the plectic Galois group $S_r \ltimes \Gamma_F^r$ and discuss several of its manifestations. We also recall Nekovář's plectic half transfer and the action of the subgroup $(S_r \ltimes \Gamma_F^r)_0$ on the CM points of the PEL Hilbert modular variety.

The stage is now set for the results announced in Section 1.2. Section 4.3 extends the definition of Nekovář's plectic Taniyama element to the entire plectic group. The main ingredients are a suitable section χ_F of the reciprocity homomorphism $r_F\colon \mathbb{A}_{F,f}^\times / F_{>0}^\times \twoheadrightarrow \Gamma_F^{\mathrm{ab}}$ together with diagram (4.3.1.1) from class field theory. Section 4.4 introduces the variants $\mathrm{Sh}(G^R, X^R)$ of the Hilbert modular variety and proves a moduli description (4.4.5) in terms of abelian varieties. Then it defines the subgroup $(S_r \ltimes \Gamma_F^r)_{\mathrm{CM}}^R$ of $S_r \ltimes \Gamma_F^r$ and an action of this subgroup on the CM points of $\mathrm{Sh}(G^R, X^R)$ in (4.4.13), extending the natural action of $\Gamma_{\mathbb{Q}}$.

Chapter 5 is devoted to the study of $\pi_0(\mathrm{Sh}(G^R, X^R))$, the set of connected components of the Shimura variety $\mathrm{Sh}(G^R, X^R)$. We start by recalling the description of the set of connected components of an arbitrary Shimura variety as a zero-dimensional Shimura variety and then apply it to $\mathrm{Sh}(G^R, X^R)$. We then use a general formula to describe the Galois action of $\Gamma_{\mathbb{Q}}$ on $\pi_0(\mathrm{Sh}(G^R, X^R))$ in terms of a certain reciprocity morphism, see (5.2.4). This allows us to define the plectic group $(S_r \ltimes \Gamma_F^r)_{\pi_0}^R$ and an action of this group on $\pi_0(\mathrm{Sh}(G^R, X^R))$ in (5.2.6), extending the action of $\Gamma_{\mathbb{Q}}$. We then prove Theorem (1.2.1) in (5.2.9) and (5.2.10).

Section 5.3 is an appendix and contains a discussion of cohomological conditions on the torus R that imply that $(S_r \ltimes \Gamma_F^r)^R_{\pi_0}$ is a subgroup of the plectic group $S_r \ltimes \Gamma_F^r$. Among those conditions is the vanishing of the Shafarevich–Tate group Ш(R) of R, see (5.3.4.1). Examples of tori whose Shafarevich–Tate group vanishes include $R = \mathbb{G}_m$ and $R = R_{F/\mathbb{Q}}\mathbb{G}_m$.

Section 6.1 reviews the plectic Galois action of $S_r \ltimes \Gamma_F^r$ on the set of cocharacters $X_*(R_{F/\mathbb{Q}}H)$ of the group $R_{F/\mathbb{Q}}H$. In Section 6.2, we first define the plectic group $(S_r \ltimes \Gamma_F^r)^R_{\mathrm{cc}}$ and an action of this group on $X_*(G^R)$ in (6.2.4) using the Cartesian diagram (1.2.0.3), and then define the plectic reflex Galois group of (G^R, X^R) in (6.2.6). We finish with a discussion of general properties (6.2.9) of groups that differ only in the centre from a group of the form $R_{F/\mathbb{Q}}H$, and generalise the results of this chapter to such groups.

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First and foremost, I would like to thank my supervisor Tony Scholl for introducing me to the beautiful theory of complex multiplication. Thank you very much for your ongoing encouragement and support. From you I learned that complicated mathematics is best understood with easy examples in mind, and I think the influence of this maxim is visible in this work.

Secondly, over the course of four years I have learned a lot of mathematics from attending conferences and exchanging mathematical ideas with many people. It is impossible to name them all, but I would like to highlight a few instances with a particularly large impact on my mathematical career and the contents of this thesis:

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1.4 Notation and conventions

For an algebraic number field k, we write $\mathbb{A}_k$ (resp. $\mathbb{A}_{k,f}$) for the adeles (resp. finite adeles) of k and $\mathbb{A}_k^\times$ (resp. $\mathbb{A}_{k,f}^\times$) for the ideles (resp. finite ideles) of k. When $k = \mathbb{Q}$, we often write $\mathbb{A}$ for $\mathbb{A}_\mathbb{Q}$ etc. Moreover, $k_{>0}^\times$ denotes the subgroup of $k^\times$ consisting of all elements that are positive under every real embedding of k. We write Γ_k for the absolute Galois group of k and Γ_k^{ab} for its abelianisation. For notation regarding class field theory, see (2.4.1). We also write $\Sigma_k = \mathrm{Hom}(k, \overline{\mathbb{Q}}) = \Gamma_\mathbb{Q}/\Gamma_k$ for the set of embeddings of k into $\overline{\mathbb{Q}}$, where $\overline{\mathbb{Q}}$ denotes an algebraic closure of $\mathbb{Q}$.

To avoid notational difficulties, we will usually view all occuring number fields as embedded into $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}$ as embedded into $\mathbb{C}$. We will use the letter c to denote complex conjugation, both as an element of $\mathrm{Aut}(\mathbb{C})$, of $\Gamma_\mathbb{Q}$ or $\Gamma_\mathbb{Q}^{\mathrm{ab}}$, or of Galois groups of CM fields. We will also occasionally use $\overline{z}$ for the complex conjugate of z.

From Section 3.2 onwards, we fix a totally real number field F and write Σ for Σ_F. We denote by S_Σ the symmetric group of the finite set Σ. Since we usually denote elements of S_Σ by σ, we will use x for elements of Σ.

Algebraic groups will usually be defined over $\mathbb{Q}$. For an algebraic group H over a number field k, we write $R_{k/\mathbb{Q}}H$ for the Weil restriction of scalars. It is the algebraic group over $\mathbb{Q}$ whose A-points, for any $\mathbb{Q}$-algebra A, are given by

$$(R_{k/\mathbb{Q}}H)(A) = H(A \otimes_\mathbb{Q} k).$$

To shorten notation, we will occasionally denote the torus $R_{k/\mathbb{Q}}\mathbb{G}_m$ by T_k, where $\mathbb{G}_m$ denotes the multiplicative group. For a torus T, we denote its character group by $X^*(T)$ and its cocharacter group by $X_*(T)$. We often write R for an algebraic torus over $\mathbb{Q}$ with $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$. For more notation on algebraic groups, see e.g. (6.2.8).

If k is a number field, V is a k-vector space and A is a $\mathbb{Q}$-algebra, we sometimes write $V(A)$ for $V \otimes_\mathbb{Q} A$.

Labels to paragraphs, theorems, examples, etc. consist of three numbers, the first two of them indicating the chapter and section in which the paragraph can be found. Labels for equations consist of four numbers, and labels for items in lists of three numbers and a lower case letter. In both cases, the first three numbers indicate the paragraph in which the equation or list can be found. When cross-referencing, except for chapters and sections we will always put the label in parentheses, and we will often simply write "(1.2.1)" when refering to Theorem (1.2.1).

2

Main Theorem of Complex Multiplication over the reflex field

In this chapter, we summarise the theory of complex multiplication over the reflex field. The main goals are to introduce notation for CM fields, CM types, and polarised abelian varieties, and to provide the background to state the Main Theorem of CM over the reflex field. The Main Theorem is the starting point of both the theory of Shimura varieties and the study of Galois conjugates of CM abelian varieties, and both will play a major role in this thesis.

In Section 2.1 we start by defining CM fields and CM types. We then associate a reflex field E and a reflex type to a CM type Φ of a CM field K, and use them to define the reflex norm $N_\Phi \colon E^\times \to K^\times$. Finally, using the Serre group we find that N_Φ extends to a morphism of algebraic tori $R_{E/\mathbb{Q}}\mathbb{G}_m \to R_{K/\mathbb{Q}}\mathbb{G}_m$.

In Section 2.2 we introduce complex abelian varieties as polarisable complex tori and explain how to think of polarisations in terms of Riemann forms. We then define rational Hodge structures and state the equivalence of categories (2.2.10) between the category of abelian varieties up to isogeny and the category of polarisable rational Hodge structures.

In Section 2.3 we take a closer look at abelian varieties A with complex multiplication by a CM field K. By definition, this means A is equipped with an embedding

$$i \colon K \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We define the CM type Φ of K associated to (A, i), and then describe polarised CM abelian varieties (A, i, ψ) of type Φ by associating to (A, i, ψ) a lattice $\mathfrak{a} \subset K$ and a totally imaginary element $t \in K^\times$, see (2.3.5). We call $(K, \Phi; \mathfrak{a}, t)$ the type of (A, i, ψ).

Section 2.4 is devoted to the Main Theorem over the reflex field. We start with a quick summary of results from class field theory, which we will use throughout this thesis. We then state the Main Theorem in two versions: For a polarised CM abelian variety (A, i, ψ) of type $(K, \Phi; \mathfrak{a}, t)$, Theorem (2.4.7) describes the type of the conjugate of (A, i, ψ) under an automorphism of $\mathbb{C}$ fixing the reflex field E of (K, Φ) in terms of the finite adelic points of the reflex norm

$$N_\Phi \colon \mathbb{A}_{E,f}^\times \longrightarrow \mathbb{A}_{K,f}^\times.$$

Moreover, Theorem (2.4.7) also describes the conjugates of the torsion points of A in terms of the reflex norm, and Theorem (2.4.10) rephrases this result in terms of Tate modules.

In Section 2.5 we introduce the PEL Shimura variety $\mathrm{Sh}(T, \{h_\Phi\})$ associated to a pair (K, Φ). The points of the zero-dimensional variety $\mathrm{Sh}(T, \{h_\Phi\})$ are in bijection with isomorphism classes of abelian varieties with CM of type (K, Φ) equipped with polarisation and level structure. We then explain how the Main Theorem over the reflex field can be used to get a canonical model of $\mathrm{Sh}(T, \{h_\Phi\})$.

With small exceptions, the contents of this chapter can be found in [Shi71], which is based on [ST61], and in [Mil07], see also [Lan83]. For this reason, proofs are kept short and often sketchy. For details about the Serre group, we also use [Bla15], [Sch94] and [MS82b]. In Section 2.2, we follow [SD74] and [BL04], and for the background on Shimura varieties we use [Del71] and [Mil17].

2.1 CM fields and the reflex norm

2.1.1 (CM fields)**.** A *CM field* is a totally imaginary quadratic extension K of a totally real field. We will usually denote this totally real field by F and its degree by $r = [F : \mathbb{Q}]$, so that $[K : \mathbb{Q}] = 2r$.

It is useful to also state the equivalent definition [Shi71, Prop. 5.11]: A number field K is a CM field if and only if it has a non-trivial automorphism c that acts as complex conjugation under every embedding $\varphi \in \Sigma_K := \mathrm{Hom}(K, \mathbb{C})$, i. e. $\varphi \circ c = \overline{\varphi}$ where $\overline{(\cdot)}$ denotes complex conjugation on $\mathbb{C}$. Namely, one chooses c to be the non-trivial element of $\mathrm{Gal}(K/F)$. We will often simply denote c by $\overline{(\cdot)}$.

Note that the Galois closure over $\mathbb{Q}$ of a CM field is again a CM field, and similarly the composite of two CM fields is a CM field too.

2.1.2 (CM types)**.** Let K be a CM field. A *CM type* Φ of K is a subset $\Phi \subset \Sigma_K = \mathrm{Hom}(K, \mathbb{C})$ containing precisely one embedding from each complex conjugate pair, i. e.

$$\Sigma_K = \Phi \sqcup \overline{\Phi}.$$

The pair (K, Φ) is called a *CM pair*.

If $K \subset L$ are CM fields and Φ is a CM type of K, then $\Phi_L := \{\varphi \in \Sigma_L \colon \varphi|_K \in \Phi\}$ is a CM type of L. We say that Φ_L is *induced* from (K, Φ). If a CM type Φ of a CM field K is not induced from any proper CM subfield of K, we call Φ (or the pair (K, Φ)) *primitive*. Note that in this way every CM type of K is induced by a unique primitive CM pair (K_0, Φ_0) with $K_0 \subset K$, see [Mil06, Prop. 1.9].

2.1.3 (Reflex field and reflex type)**.** Let (K, Φ) be a CM pair. To (K, Φ), we can associate the *reflex field* $E = E(K, \Phi)$, which will play a crucial role in the following considerations. It is the subfield E of $\mathbb{C}$ given by

$$\mathrm{Aut}(\mathbb{C}/E) := \{\tau \in \mathrm{Aut}(\mathbb{C}) \mid \tau\Phi = \Phi\}.$$

More explicitly, it is given by

$$E = \mathbb{Q}(\mathrm{Tr}_\Phi(x) \mid x \in K),$$

where $\mathrm{Tr}_\Phi(x) := \sum_{\varphi \in \Phi} \varphi(x)$.

Note that the reflex field E is by definition a subfield of $\mathbb{C}$, and is itself a CM field because complex conjugation on $\mathbb{C}$ induces the required automorphism on E. Moreover, the reflex field E comes equipped with a CM type Φ^*, called the *reflex type*. It is defined as follows:

Let L be the Galois closure of K over $\mathbb{Q}$ and let $G = \mathrm{Gal}(L/\mathbb{Q})$. Fix an embedding $L \subset \mathbb{C}$, inducing an identification $G \cong \Sigma_L$. We view E as a subfield of L, so we have a diagram of fields

$$\begin{tikzcd}
& L \arrow[d, dash, "H"] \arrow[ddl, dash, "H^*"'] \arrow[ddd, dash, bend left=60, "G"] \\
& K \arrow[dd, dash] \\
E \arrow[dr, dash] & \\
& \mathbb{Q}.
\end{tikzcd}$$

It is then not hard to see [Shi71, p. 125–126] that

$$\Phi_L^{-1} = \{\varphi \in G \colon \varphi^{-1}|_K \in \Phi\}$$

is a CM type of L and is induced from a CM type Φ^* of E. The pair (E, Φ^*) is called the *reflex pair* of (K, Φ).

Sketch of proof. Let us write $H := \mathrm{Gal}(L/K)$, $H^* := \mathrm{Gal}(L/E)$, and identify $\Sigma_K = G/H$ and $\Sigma_E = G/H^*$. Observe that $H^* = \{g \in G \mid g\Phi_L = \Phi_L\}$.

Let us write $\Phi = \{\varphi_1 H, \dots, \varphi_r H\} \subset \Sigma_K = G/H$, so that $\Phi_L = \sqcup_{i=1}^r \varphi_i H$. Using that complex conjugation gives an element c in the centre of G, one sees that Φ_L^{-1} is a CM type of L and consists of right cosets for H^*, i. e.

$$\Phi_L^{-1} = \bigsqcup_{j=1}^{m} \psi_j H^* \tag{2.1.3.1}$$

for some $\psi_j \in G$. This means that Φ_L^{-1} is induced from $\Phi^* := \{\psi_1 H^*, \dots, \psi_m H^*\} \subset \Sigma_E$, and one can show that Φ^* is indeed a CM type of E. □

However, it is worth pointing out that Φ^* depends on the embedding $K \subset \mathbb{C}$ induced from the chosen embedding $L \subset \mathbb{C}$.

2.1.4 Example. Let us give a few concrete examples of CM types. Let K be a cyclic CM field of degree 6 and write $G = \mathrm{Gal}(K/\mathbb{Q}) = \langle g \rangle$. For a concrete example, one may take $K = \mathbb{Q}(\zeta_7)$. To avoid confusion, let us embed K into $\mathbb{C}$ via $\mathrm{id} = g^0$ and identify Σ_K with G. The maximal totally real subfield F of K is the fixed field of $g^3 = c$. Let us denote by K_0 the subfield of K fixed by the subgroup $H = \langle g^2 \rangle$. Then K_0 is an imaginary quadratic field.

Since K has degree 6, it has precisely $2^3 = 8$ different CM types: two of them are induced from the subfield K_0 and the remaining six are primitive. For example the CM type $\Phi_0 = \{g^0|_{K_0}\} \subset \Sigma_{K_0} = G/H$ of K_0 induces the CM type $\Phi = \{g^0, g^4, g^2\}$ of K. On the other hand, the CM type $\Psi = \{g^0, g^1, g^2\}$ of K is primitive. To calculate their reflex types, we see that $\Phi^{-1} = \{g^0, g^4, g^2\} = \mathrm{id}\, H$, so $\Phi^* = \Phi_0$. Moreover, $\Psi^{-1} = \{g^0, g^4, g^5\} = \Psi^*$.

We observe that the reflex pair is always primitive, and repeating this calculation we see that the "double reflex" of a CM type is equal to its primitive subpair. These two facts actually hold in complete generality, i. e. for any CM pair (K, Φ), see [ST61, p. 71]. For more examples, see [ST61, 8.4]

2.1.5 (Reflex norm)**.** Let (K, Φ) be a CM pair and E its reflex field. Fix an embedding $K \subset \mathbb{C}$ and let Φ^* be the associated reflex type of E. The *reflex norm* is the group homomorphism

$$\begin{aligned} N_\Phi \colon E^\times &\longrightarrow K^\times \\ x &\longmapsto \prod_{\psi \in \Phi^*} \psi(x). \end{aligned}$$

This is well-defined: Using the notation of (2.1.3), note that if $\sigma \in H$ and $\psi \in \Phi_L^{-1} = \sqcup_{i=1}^r H\varphi_i^{-1}$, then $\sigma\psi \in \Phi_L^{-1}$. So σ induces a permutation of Φ_L^{-1}. By (2.1.3.1) σ induces a permutation of $\{\psi_1, \dots, \psi_m\}$, say $\psi_j \mapsto \psi_{k(\sigma,j)}$, determined by $\sigma\psi_j \in$

$\psi_{k(\sigma,j)}H^*$. For $x \in E^\times$ we then conclude that

$$\sigma N_\Phi(x) = \prod_{j=1}^{m} \sigma\psi_j(x) = \prod_{j=1}^{m} \psi_{k(\sigma,j)}(x) = N_\Phi(x).$$

Hence $N_\Phi(x)$ is fixed by all $\sigma \in H$, i. e. $N_\Phi(x) \in K^\times$.

Moreover, one can show that the reflex norm is independent of the choice of embedding $K \subset \mathbb{C}$: Calling this embedding $\iota\colon K \xrightarrow{\sim} \iota(K)$, with $\iota(K) \subset \mathbb{C}$, we see that Φ^* depends on ι, hence the reflex norm $N_\Phi\colon E^\times \to \iota(K^\times)$ depends on ι (in the calculation above, we suppressed ι from notation). In order to get a map $E^\times \to K^\times$ we need to post-compose the reflex norm N_Φ with ι^{-1}, and one can check that this composition is independent of the choice of ι.

We will need the reflex norm also as a map between the (finite) ideles of E and K, so we remark here that the group homomorphism N_Φ extends to a morphism

$$N_\Phi\colon R_{E/\mathbb{Q}}\mathbb{G}_m \longrightarrow R_{K/\mathbb{Q}}\mathbb{G}_m$$

of algebraic tori over $\mathbb{Q}$. The details can be found in (2.1.8) below, but first we need some background about algebraic tori and the Serre group.

2.1.6 (Restriction of scalars of $\mathbb{G}_m$)**.** For any number field M, we write T_M for the algebraic torus $R_{M/\mathbb{Q}}\mathbb{G}_m$. It is defined over $\mathbb{Q}$ and its character and cocharacter groups can be identified with $\mathbb{Z}[\Sigma_M]$ as $\Gamma_\mathbb{Q}$-modules. We write (co-)characters as $\sum_{\sigma\in\Sigma_M} n_\sigma[\sigma]$ with $n_\sigma \in \mathbb{Z}$. Here $[\sigma']$ denotes either the character

$$\begin{aligned} [\sigma']\colon (M \otimes_\mathbb{Q} \overline{\mathbb{Q}})^\times = \prod_{\sigma\in\Sigma_M} \overline{\mathbb{Q}}^\times &\longrightarrow \overline{\mathbb{Q}}^\times, \\ (t_\sigma)_{\sigma\in\Sigma_M} &\longmapsto t_{\sigma'}, \end{aligned}$$

or the cocharacter

$$\begin{aligned} [\sigma']\colon \overline{\mathbb{Q}}^\times &\longrightarrow \prod_{\sigma\in\Sigma_M} \overline{\mathbb{Q}}^\times = (M \otimes_\mathbb{Q} \overline{\mathbb{Q}})^\times \\ t &\longmapsto (1, \dots, 1, t, 1, \dots, 1), \end{aligned}$$

where the t is in the entry corresponding to σ'. It will always be clear from context whether $[\sigma']$ denotes a character or a cocharacter.

The action of $\gamma \in \Gamma_\mathbb{Q}$ on $[\sigma] \in X^*(T_M)$ or $[\sigma] \in X_*(T_M)$ is given by

$$\gamma\colon [\sigma] \mapsto [\gamma \circ \sigma].$$

The canonical perfect pairing

$$\langle\cdot,\cdot\rangle\colon X^*(T_M)\times X_*(T_M)\to\mathbb{Z},\quad \langle f,g\rangle := \deg(f\circ g),$$

where the degree of the morphism $\mathbb{G}_m\to\mathbb{G}_m$, $t\mapsto t^n$, is defined to be n, translates to

$$\mathbb{Z}[\Sigma_M]\times\mathbb{Z}[\Sigma_M]\to\mathbb{Z},\quad \left\langle \sum_{\sigma\in\Sigma_M} m_\sigma[\sigma], \sum_{\sigma\in\Sigma_M} n_\sigma[\sigma]\right\rangle = \sum_{\sigma\in\Sigma_M} n_\sigma m_\sigma.$$

2.1.7 (Serre group)**.** Let $E\subset\mathbb{C}$ be a CM field. The *Serre group $\mathcal{S}^E$ at level E* is the algebraic torus over $\mathbb{Q}$ whose character group is given by

$$X^*(\mathcal{S}^E) := \left\{ \sum_{\sigma\in\Sigma_E} n_\sigma[\sigma] \;\middle|\; n_\sigma+n_{\overline{\sigma}} = n_{\sigma'}+n_{\overline{\sigma}'} \text{ for all } \sigma,\sigma'\in\Sigma_E \right\}\subset X^*(T_E).$$

The equivalence of categories [PR94, Thm 2.1] between algebraic tori and their character groups implies that $\mathcal{S}^E$ is a quotient of T_E, so there is a canonical map $T_E\to\mathcal{S}^E$. Moreover, its cocharacter group is given by

$$X_*(\mathcal{S}^E) = X_*(T_E)/X^*(\mathcal{S}^E)^\perp,$$

where $X^*(\mathcal{S}^E)^\perp := \{\mu\in X_*(T_E)\mid \langle\chi,\mu\rangle = 0 \text{ for all } \chi\in X^*(\mathcal{S}^E)\}$.

The fixed embedding $E\subset\mathbb{C}$, which we will temporarily denote by ι, determines a cocharacter $\mu^E := [\iota]+X^*(\mathcal{S}^E)^\perp$ of $\mathcal{S}^E$. On characters, this corresponds to

$$(\mu^E)^*\colon X^*(\mathcal{S}^E)\to X^*(\mathbb{G}_m)=\mathbb{Z},\quad \sum_{\sigma\in\Sigma_E} n_\sigma[\sigma]\mapsto n_\iota.$$

From this it is not hard to see that μ^E is defined over E and that its *weight* $-(c+1)\mu^E$ is defined over $\mathbb{Q}$, where $c\in\Gamma_\mathbb{Q}$ denotes complex conjugation.

In fact, the pair $(\mathcal{S}^E,\mu^E)$ satisfies the following universal property [Mil81, start of §4]: For every pair (T,μ) consisting of a $\mathbb{Q}$-algebraic torus T and a cocharacter μ of T defined over E such that $-(c+1)\mu$ is defined over $\mathbb{Q}$, there exists a unique morphism $\rho_\mu\colon\mathcal{S}^E\to T$ defined over $\mathbb{Q}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "\mu^E"] \arrow[dr, "\mu"'] & \mathcal{S}^E \arrow[d, "\rho_\mu"] \\ & T. \end{tikzcd}$$

2.1.8 (Reflex norm as a morphism of algebraic groups)**.** Let (K,Φ) be a CM pair with reflex pair (E,Φ^*) as in (2.1.5). There we constructed the reflex norm as a group homomorphism $N_\Phi\colon E^\times\to K^\times$, which we will now upgrade to a morphims of algebraic tori over $\mathbb{Q}$.

Note that the CM type Φ determines a cocharacter $\mu_\Phi := \sum_{\varphi\in\Phi}[\varphi] \in X_*(T_K)$. By definition of the reflex field E, the cocharacter μ_Φ is defined over E, and its weight $-(c+1)\mu_\Phi = \sum_{\varphi\in\Sigma_K}[\varphi]$ is defined over $\mathbb{Q}$. By the universal property of the Serre group, there exists a unique morphism $\rho_\Phi\colon \mathcal{S}^E \to T_K$ defined over $\mathbb{Q}$ such that

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, "\mu^E"] \arrow[dr, "\mu_\Phi"'] & \mathcal{S}^E \arrow[d, "\rho_\Phi"] \\ & T_K \end{tikzcd} \tag{2.1.8.1}$$

commutes.

Composing ρ_Φ with the canonical map $T_E \to \mathcal{S}^E$ gives the desired morphism

$$N_\Phi\colon T_E \longrightarrow \mathcal{S}^E \xrightarrow{\rho_\Phi} T_K,$$

which we will temporarily denote by N'_Φ to avoid confusion.

We claim that N'_Φ agrees, on $\mathbb{Q}$-points, with the reflex norm N_Φ defined in (2.1.5).

Proof of claim. Looking at the morphisms in the commutative diagram (2.1.8.1) on the level of characters, one sees that

$$\rho_\Phi^*\colon X^*(T_K) \longrightarrow X^*(\mathcal{S}^E), \qquad \sum_{\sigma\in\Sigma_K} n_\sigma[\sigma] \longmapsto \left(\sum_{\sigma\in\Phi} n_\sigma\right)[\iota] + (\text{other terms}),$$

where ι denotes the fixed embedding $E \subset \mathbb{C}$.

Using that ρ_Φ is defined over $\mathbb{Q}$, one can determine ρ_Φ^* completely as

$$\rho_\Phi^*\colon X^*(T_K) \longrightarrow X^*(\mathcal{S}^E), \quad [\sigma] \longmapsto \sum_{\tau\in\Sigma_E} m_\tau(\sigma)[\tau]$$

with

$$m_\tau(\sigma) = \begin{cases} 1, & \text{if } \tau = \gamma^{-1}\circ\iota \text{ for some } \gamma\in\Gamma_{\mathbb{Q}} \text{ such that } \gamma\circ\sigma\in\Phi, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $(N'_\Phi)^*$ is given by the same formula, using $X^*(\mathcal{S}^E) \subset X^*(T_E)$. This enables us to write down N'_Φ explicitly on $\overline{\mathbb{Q}}$-points (in terms of the $m_\tau(\sigma)$).

We want to prove that the following diagram commutes

$$\begin{tikzcd}
(E \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times \arrow[r, "N'_\Phi"] & (K \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times \\
E^\times \arrow[u, hook] \arrow[r, "N_\Phi"'] & K^\times. \arrow[u, hook]
\end{tikzcd}$$

Using the above formula for N'_Φ (in terms of the $m_\tau(\sigma)$) and combining it with the notation of (2.1.3), this reduces to proving that for each $\sigma \in \Sigma_K$,

$$\{\tau \in \Sigma_E \mid \sigma \in \tau\Phi\} = \tilde{\sigma}\Phi^*,$$

where $\tilde{\sigma} \in G$ denotes a lift of $\sigma \in \Sigma_K = G/H$. To show this, also denote a lift of $\tau \in \Sigma_E$ to G by $\tilde{\tau}$. Then

$$\begin{aligned}
\{\tau \in \Sigma_E \mid \sigma \in \tau\Phi\} &= \{\tilde{\tau}H^* \in \Sigma_E \mid \sigma \in \tilde{\tau}\Phi\} \\
&= \{\tilde{\tau}H^* \in \Sigma_E \mid \tilde{\sigma} \in \sqcup_{i=1}^r \tilde{\tau}\varphi_i H\} \\
&= \{\tilde{\tau}H^* \in \Sigma_E \mid \tilde{\tau} \in \tilde{\sigma}(\sqcup_{i=1}^r H\varphi_i^{-1}) = \tilde{\sigma}(\sqcup_{j=1}^m \psi_j H^*)\} \\
&= \tilde{\sigma}\Phi^*,
\end{aligned}$$

so we are done. In the penultimate step, we used (2.1.3.1). $\square$

2.2 Abelian varieties

2.2.1 (Complex abelian varieties)**.** Let A be an abelian variety defined over $\mathbb{C}$, of dimension $g = \dim A$. By Riemann's classification of abelian varieties over $\mathbb{C}$, see e. g. [Mil08, Prop. I.2.1], there exists a lattice Λ in a g-dimensional $\mathbb{C}$-vector space V and an isomorphism $A(\mathbb{C}) \cong V/\Lambda$ of complex Lie groups, which we will call a *uniformization* of A.

More intrinsically, one can define V to be the tangent space of A at the origin. In this setting, Λ can be defined to be $\Lambda := H_1(A, \mathbb{Z})$.

Conversely, let V be a finite dimensional complex vector space and Λ a lattice in V. Then [Mum70, §3, Cor. on p. 35] the complex torus V/Λ admits the structure of an abelian variety if and only if it is *polarisable.*

Here, a *polarisation* on V/Λ is defined to be an isogeny between V/Λ and its dual complex torus that is associated to an ample line bundle on V/Λ. We call V/Λ *polarisable* if it admits a polarisation.

2.2.2 (Polarisations and Riemann forms)**.** Let V be a finite dimensional complex vector space and Λ a lattice in V. Because we are working over the complex numbers, the Theorem of Appell–Humbert [Mum70, §2, p. 20] implies that polarisations on

V/Λ are in bijection with *positive definite Hermitian Riemann forms on V with respect to Λ*. Such forms are maps $H\colon V \times V \to \mathbb{C}$ satisfying

- H is Hermitian, i. e. H is $\mathbb{C}$-linear in the first variable and satisfies $H(v,w) = \overline{H(w,v)}$ for all $v, w \in V$.
- H is positive definite, i. e. $H(v,v) > 0$ for all $v \in V \setminus \{0\}$.
- $\operatorname{Im} H$ is $\mathbb{Z}$-valued on $\Lambda \times \Lambda$.

It is not difficult (compare [Mum70, §2, Lemma on p. 19]) to express these properties only in terms of the imaginary part E of H, namely

- $E\colon V \times V \to \mathbb{R}$ is an $\mathbb{R}$-bilinear alternating form satisfying $E(iv, iw) = E(v, w)$ for all $v, w \in V$.
- The form $(x, y) \mapsto E(ix, y)$ is positive definite.
- E is $\mathbb{Z}$-valued on $\Lambda \times \Lambda$.

Such forms E are called *Riemann forms on V with respect to* Λ. Because $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$, it is enough to know E on Λ (or on $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$). In this thesis, we will often simply call E (or $E|_{\Lambda\times\Lambda}$) a polarisation of V/Λ.

2.2.3 (Endomorphisms of abelian varieties)**.** Let A be an abelian variety over $\mathbb{C}$ of dimension g, and let $A = V/\Lambda$ be a uniformization of A. Then the endomorphisms of A are given by

$$\operatorname{End}(A) = \{\Psi \in \operatorname{End}_{\mathbb{C}}(V) \colon \Psi(\Lambda) \subset \Lambda\}.$$

This shows that the endomorphism ring $\operatorname{End}(A)$ has two natural representations:

(a) The *complex representation* is given by the action of $\operatorname{End}(A)$ on the complex vector space V of dimension g.

(b) The *rational representation* is given by the action of $\operatorname{End}(A)$ on the free $\mathbb{Z}$-module $\Lambda = H_1(A, \mathbb{Z})$ of rank $2g$, or (which is essentially the same) as the action of $\operatorname{End}_{\mathbb{Q}}(A) := \operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ on the $2g$-dimensional $\mathbb{Q}$-vector space $H_1(A, \mathbb{Q})$. This action is faithful.

These representations are related by the *Hodge decomposition*

$$H_1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong V \oplus \overline{V}, \tag{2.2.3.1}$$

where $\overline{V}$ denotes a copy of the vector space V on which $\operatorname{End}(A)$ acts by the complex conjugate of the action on V, i. e. if P is the matrix of $\Psi \in \operatorname{End}(A)$ acting on V with respect to some basis, then Ψ acts on $\overline{V}$ by $\overline{P}$ (with respect to the same basis). The proof of this relation is an explicit calculation using 'nice' bases, see [SD74, Lem. 39].

Before moving on to CM abelian varieties, it will be useful to connect the theory of abelian varieties over $\mathbb{C}$ to the theory of rational Hodge structures. We follow [Mil17, §2], but restrict ourselves to the necessary minimum.

2.2.4 ((Rational) Hodge structures)**.** A *(real) Hodge structure* is a finite dimensional real vector space V together with a *Hodge decomposition*

$$V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q},$$

where $V^{p,q}$ are complex subspaces of $V \otimes_{\mathbb{R}} \mathbb{C}$ such that $V^{p,q}$ is the complex conjugate of $V^{q,p}$. The set of $(p,q) \in \mathbb{Z}^2$ with $V^{p,q} \neq 0$ is called the *type* of the Hodge structure.

The condition $V^{p,q} = \overline{V^{q,p}}$ implies that, for each $n \in \mathbb{Z}$, the (complex) subspace $\bigoplus_{p+q=n} V^{p,q}$ is stable under complex conjugation. Hence there is a real subspace V_n of V such that $V_n \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$. We call V_n the *weight space of weight n*.

A *$\mathbb{Z}$-Hodge structure* is a free $\mathbb{Z}$-module Λ of finite rank together with a Hodge decomposition of $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ such that the weight spaces are defined over $\mathbb{Q}$. Similarly, a *$\mathbb{Q}$-Hodge structure* is a finite dimensional $\mathbb{Q}$-vector space M together with a Hodge decomposition of $M \otimes_{\mathbb{Q}} \mathbb{C}$ such that the weight spaces are defined over $\mathbb{Q}$.

2.2.5 (Hodge structures and representations of the Deligne torus)**.** Giving a Hodge structure on a finite dimensional real vector space V is equivalent to giving a representation $h\colon \mathbb{S} \to \mathrm{GL}(V)$ of the *Deligne torus* $\mathbb{S} := R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$, which is an algebraic torus over $\mathbb{R}$. The dictionary between these two points of view is to let $(z_1, z_2) \in \mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$ act on $V^{p,q}$ by multiplication by $z_1^{-p} z_2^{-q}$.

Associated to a Hodge structure $h\colon \mathbb{S} \to \mathrm{GL}(V)$ we also have a cocharacter μ_h of $\mathrm{GL}(V)_{\mathbb{C}}$ given by

$$\mu_h\colon \mathbb{G}_{m,\mathbb{C}} \to \mathrm{GL}(V \otimes_{\mathbb{R}} \mathbb{C}), \quad \mu_h(z) := h_{\mathbb{C}}(z, 1),$$

and this cocharacter determines h.

For example, a Hodge structure of type $\{(-1,0),(0,-1)\}$ on a real vector space V is the same as a complex structure J on V, i. e. an endomorphims $J \in \mathrm{End}(V)$ such that $J^2 = -1$. Namely, if h denotes the associated representation of $\mathbb{S}$, then $J = h(i)$.

2.2.6 (Polarisations of Hodge structures)**.** Let (M,h) be a $\mathbb{Q}$-Hodge structure of weight n. A *polarisation* of (M,h) is a $\mathbb{Q}$-bilinear form $\psi\colon M \times M \to \mathbb{Q}$ satisfying

- $\psi(h(z)v, h(z)w) = (z\overline{z})^{-n}\psi(v, w)$ for all $v, w \in M \otimes_{\mathbb{Q}} \mathbb{R}$, $z \in \mathbb{C}^{\times}$, and
- the pairing $(v, w) \mapsto \psi(v, h(i)w)$ is positive definite (on $M \otimes_{\mathbb{Q}} \mathbb{R} \times M \otimes_{\mathbb{Q}} \mathbb{R}$).

We call (M, h) *polarisable* if it admits a polarisation. Similarly, one defines polarisations for $\mathbb{Z}$-Hodge structures and real Hodge structures.

For example, if (M, h) has type $(-1, 0), (0, -1)$ (i. e. weight -1), then a polarisation of (M, h) is a bilinear form $\psi\colon M \times M \to \mathbb{Q}$ satisfying

- ψ is alternating and $\psi(h(i)v, h(i)w) = \psi(v, w)$ for all $v, w \in M \otimes_{\mathbb{Q}} \mathbb{R}$, and
- $(v, w) \mapsto \psi(v, h(i)w)$ is positive definite.

The connection to abelian varieties is given by the following theorem, see [Mil17, Thm 6.8]:

2.2.7 Theorem (Abelian varieties and Hodge structures)**.** *The category of abelian varieties over $\mathbb{C}$ is equivalent to the category of polarisable $\mathbb{Z}$-Hodge structures of type $\{(-1, 0), (0, -1)\}$ via the functor*

$$A \longmapsto H_1(A, \mathbb{Z}).$$

2.2.8 Remark. If the abelian variety A is given as V/Λ, then $H_1(V/\Lambda, \mathbb{Z}) = \Lambda$. The Hodge decomposition (2.2.3.1) equips Λ with a $\mathbb{Z}$-Hodge structure of type $\{(-1, 0), (0, -1)\}$, namely, $V = V^{-1,0}$ and $\overline{V} = V^{0,-1}$. We mentioned in (2.2.1) that A admits a polarisation, which we described in terms of Riemann forms in (2.2.2). The Riemann form E (restricted to $\Lambda \times \Lambda$) is then precisely a polarisation of the $\mathbb{Z}$-Hodge structure Λ.

2.2.9 (Isogeny category)**.** For large parts of this thesis, we will work in the category of abelian varieties *up to isogeny*. Its objects are abelian varieties, but the morphisms between A_1 and A_2 in this isogeny category are given by

$$\operatorname{Hom}_{\mathbb{Q}}(A_1, A_2) := \operatorname{Hom}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Sometimes, elements of $\operatorname{Hom}(A_1, A_2) \otimes_{\mathbb{Z}} \mathbb{Q}$ are called *quasi-isogenies*, because some integer multiple of a quasi-isogeny is an actual isogeny.

Working in this category has the effect that isogenies are viewed as isomorphisms.

The equivalence of categories (2.2.7) implies:

2.2.10 Theorem. *The category of abelian varieties over $\mathbb{C}$ up to isogeny is equivalent to the category of polarisable $\mathbb{Q}$-Hodge structures of type $\{(-1, 0), (0, -1)\}$ via the functor*

$$A \longmapsto H_1(A, \mathbb{Q}).$$

We will denote the $\mathbb{Q}$-Hodge structure on $H_1(A, \mathbb{Q})$ by $h_A\colon \mathbb{S} \to \mathrm{GL}(H_1(A, \mathbb{R}))$.

2.3 Abelian varieties with complex multiplication

Going back to endomorphisms of abelian varieties, using the ideas of (2.2.3) one can classify the possible division algebras $\mathrm{End}_{\mathbb{Q}}(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ for simple A, see [SD74, Thm. 43]. In this thesis, we are interested in abelian varieties with complex multiplication, which in some sense are those abelian varieties whose endomorphism algebra has largest possible centre. Namely, by [ST61, Prop. 1, p. 39], any commutative semi-simple subalgebra B of $\mathrm{End}_{\mathbb{Q}}(A)$ satisfies $\dim_{\mathbb{Q}} B \leq 2 \dim A$.

2.3.1 (Abelian varieties with complex multiplication)**.** We say an abelian variety A over $\mathbb{C}$ of dimension g has *complex multiplication (CM) by a CM field* K if $[K : \mathbb{Q}] = 2g$ and we are given an embedding

$$i \colon K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A).$$

A few remarks:

(2.3.1.a) An abelian variety A can have CM by several CM fields. Therefore, we include the homomorphism $i \colon K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A)$ as part of the data and say that (A, i) has CM by K, or that A has CM by K via i.

(2.3.1.b) If A is simple, then $\mathrm{End}_{\mathbb{Q}}(A)$ is a division algebra. One can then show that if (A, i) has CM by K with A simple, then i is an isomorphism. In this case, K is determined up to isomorphism.

(2.3.1.c) Some authors allow K to be a *CM algebra* instead of a CM field. A CM algebra is a finite product of CM fields. Most of the theory can be worked out for CM algebras instead of CM fields, but for the purpose of this thesis we will always restrict to the case where K is a field. This is not too restrictive because if A has CM by a CM algebra, then it is isogeneous to a product of abelian varieties, each of which has CM by a CM field.

2.3.2 (CM type determined by A)**.** Let us assume that $(A, i \colon K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A))$ has CM by K. Then the $2g$-dimensional $\mathbb{Q}$-vector space $H_1(A, \mathbb{Q})$ carries a faithful representation of $\mathrm{End}_{\mathbb{Q}}(A)$ and hence by K. So we can view $H_1(A, \mathbb{Q})$ as a one-dimensional vector space over K, and simply write $H_1(A, \mathbb{Q}) = K$.

On the other hand, K also acts, via i and the complex representation of $\mathrm{End}_{\mathbb{Q}}(A)$, on the tangent space V of A at the origin. The vector space V is g-dimensional over $\mathbb{C}$, and diagonalising the K-action we get an isomorphism of complex vector spaces with K-actions

$$V \cong \bigoplus_{\varphi \in \Phi} \mathbb{C}_{\varphi} =: \mathbb{C}^{\Phi},$$

for a subset $\Phi \subset \Sigma_K = \mathrm{Hom}(K, \mathbb{C})$ with g elements (repetitions allowed), where $\mathbb{C}_\varphi$ denotes a one-dimensional complex vector space on which K acts via φ.

It remains to show that Φ is a CM type of K. This follows from the Hodge decomposition (2.2.3.1): We have isomorphisms of complex vector spaces with K-actions

$$V \oplus \overline{V} \cong H_1(A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong K \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\varphi \in \Sigma_K} \mathbb{C}_\varphi.$$

But the left hand side is isomorphic to $\bigoplus_{\varphi\in\Phi} \mathbb{C}_\varphi \oplus \bigoplus_{\varphi\in\overline{\Phi}} \mathbb{C}_\varphi$, thus Φ is a CM type of K.

We say that (A, i) is of *type* (K, Φ).

2.3.3 Remark. Let (A, i) be a CM abelian variety of type (K, Φ). As in (2.3.2), we can view $H_1(A, \mathbb{Q})$ as a one-dimensional K-vector space, and the tangent space V of A at the origin is isomorphic to $\mathbb{C}^\Phi$, which is equipped with CM by K by letting $x \in K$ act as multiplication by $\Phi(x) = (\varphi(x))_{\varphi\in\Phi} \in \mathbb{C}^\Phi$.

Under the equivalence of categories (2.2.10), the abelian variety A hence corresponds to the $\mathbb{Q}$-Hodge structure (K, h_Φ), where h_Φ encodes the isomorphism

$$K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^\Phi.$$

In other words, the endomorphism $h_\Phi(i)$ of $K \otimes_{\mathbb{Q}} \mathbb{R}$ is equal to the pullback of multiplication by $(i, \ldots, i) \in \mathbb{C}^\Phi$ under this isomorphism.

The CM pair (K, Φ) determines the $\mathbb{Q}$-Hodge structure (K, h_Φ), so we conclude that the isogeny class of a CM abelian variety (A, i) is determined by its type (K, Φ). Put differently, there is a bijection between CM types of K and isogeny classes of abelian varieties with CM by K.

We have seen that (A, i) determines the pair (K, Φ), and that in turn (K, Φ) determines the isogeny class of (A, i). In order to get a finer classification, we also need to take polarisations into account. We follow the exposition in [Mil07, p. 5, before Prop. 1.3].

2.3.4 (Compatible polarisations)**.** Let $(A, i\colon K \hookrightarrow \mathrm{End}_{\mathbb{Q}}(A))$ be a CM abelian variety of type (K, Φ). We call (the Riemann form of) a polarisation $\psi\colon H_1(A, \mathbb{Q}) \times H_1(A, \mathbb{Q}) \to \mathbb{Q}$ *(K-)compatible (with i)* if its Rosati involution stabilises $i(K)$ and induces complex conjugation on it. This means that $\psi(i(a)v, w) = \psi(v, i(c(a))w)$ for all $v, w \in H_1(A, \mathbb{Q})$ and $a \in K$.

We can describe compatible polarisations more explicitly as follows: As in (2.3.2) we can view $H_1(A, \mathbb{Q})$ as a 1-dimensional vector space over K. Call a basis element e. Then by [Shi71, (5.5.13)] there exists a unique element $t \in K^\times$ such that

$$\psi(x \cdot e, y \cdot e) = E_t(x, y) := \mathrm{Tr}_{K/\mathbb{Q}}(txc(y)), \quad \text{for all } x, y \in K.$$

The conditions of (2.2.2) (or (2.2.6)) on the polarisation ψ translate to

(2.3.4.a) t being totally imaginary (because ψ is alternating), and

(2.3.4.b) $\operatorname{Im}\varphi(t) > 0$ for all $\varphi \in \Phi$ (because $(x, y) \mapsto \psi(ix, y)$ is positive definite).

2.3.5 (Polarised CM abelian varieties)**.** Let us classify triples (A, i, ψ) consisting of a CM abelian variety (A, i) of type (K, Φ) as in (2.3.2) and a compatible polarisation ψ.

To analyse this situation, let $\Theta\colon \mathbb{C}^\Phi/\Lambda \xrightarrow{\sim} A$ be a uniformization. Choose a K-basis element e of $H_1(A, \mathbb{Q})$. This determines a lattice $\mathfrak{a} \subset K$ by $\mathfrak{a} := \{x \in K \mid x{\cdot}e \in \Lambda\}$, and allows us to identify $\Lambda \subset \mathbb{C}^\Phi$ with $\Phi(\mathfrak{a})$, where we write $\Phi\colon K \hookrightarrow \mathbb{C}^\Phi$ for the map $x \mapsto (\varphi(x))_{\varphi\in\Phi}$.

Together with the discussion in (2.3.4), this shows that the choice of e determines a quadruple $(K, \Phi; \mathfrak{a}, t)$, which we will also call the *type* of (A, i, ψ) relative to the uniformization Θ.

A different choice of basis element for $H_1(A, \mathbb{Q})$ has the form $e' = a^{-1}e$ for some $a \in K^\times$, resulting in changing $\mathfrak{a}$ to $\mathfrak{a}' = a\mathfrak{a}$ and t to $t' = \frac{t}{a\bar{a}}$ (and Θ to $\Theta' = \Theta \circ a^{-1}$, where $a^{-1}\colon \mathbb{C}^\Phi/\Phi(a\mathfrak{a}) \to \mathbb{C}^\Phi/\Phi(\mathfrak{a})$ denotes the map induced from multiplication by $\Phi(a^{-1})$ on $\mathbb{C}^\Phi$).

To summarise: The type $(K, \Phi; \mathfrak{a}, t)$ as above determines the triple (A, i, ψ) up to isomorphism. Here an isomorphism of triples (A, i, ψ) is an isomorphism of abelian varieties respecting the complex multiplication i and the polarisation ψ. Conversely, a triple (A, i, ψ) determines the type $(K, \Phi; \mathfrak{a}, t)$ up to simultaneously changing $\mathfrak{a}$ to $a\mathfrak{a}$ and t to $\frac{t}{a\bar{a}}$, for some $a \in K^\times$.

Finally, a quadruple $(K, \Phi; \mathfrak{a}, t)$, with K a CM field, Φ a CM type of K, $\mathfrak{a}$ a lattice in K and $t \in K^\times$ a totally imaginary element, occurs as the type of some polarized CM abelian variety (A, i, ψ) if and only if $\operatorname{Im}\varphi(t) > 0$ for all $\varphi \in \Phi$.

The following lemma will be very useful.

2.3.6 Lemma. *Let $(A, i\colon K \hookrightarrow \operatorname{End}_{\mathbb{Q}}(A))$ be a CM abelian variety. Let F denote the maximal totally real subfield of K, i. e. $F = K^{\langle c\rangle}$. Assume that $\psi\colon H_1(A, \mathbb{Q}) \times H_1(A, \mathbb{Q}) \to \mathbb{Q}$ is a polarisation of A that is F-compatible, i. e. $\psi(i(a)v, w) = \psi(v, i(a)w)$ for all $v, w \in H_1(A, \mathbb{Q})$ and $a \in F$.*

Then ψ is also K-compatible.

Proof. To shorten notation, let $V := H_1(A, \mathbb{Q})$. As before, we identify V with K and drop i from notation. Note that $\dim_K V = 1$, hence $\dim_F V = 2$. The proof proceeds in three steps.

- The map

$$\operatorname{Hom}_F(V, F) \longrightarrow \operatorname{Hom}_{\mathbb{Q}}(V, \mathbb{Q}), \quad v^* \longmapsto \operatorname{Tr}_{F/\mathbb{Q}} \circ v^*$$

is an F-linear isomorphism, where $\operatorname{Hom}_{\mathbb{Q}}(V,\mathbb{Q})$ is an F-vector space via $(f\cdot\alpha)(v) := \alpha(fv)$ for $f\in F$, $\alpha\in\operatorname{Hom}_{\mathbb{Q}}(V,\mathbb{Q})$, and $v\in V$.

The map is clearly F-linear and both its domain and target have dimension $2[F:\mathbb{Q}]$ over $\mathbb{Q}$, so it is enough to check that the map is injective. So assume that $\operatorname{Tr}_{F/\mathbb{Q}}\circ v^* = 0$ for some $v^*\in\operatorname{Hom}_F(V,F)$. Then the F-vector space $\operatorname{im}(v^*)$ is contained in $\ker(\operatorname{Tr}_{F/\mathbb{Q}})$, which has dimension $[F:\mathbb{Q}]-1$ over $\mathbb{Q}$, hence $v^*=0$.

- Let W denote the $\mathbb{Q}$-vector space of all $\mathbb{Q}$-bilinear, alternating, F-compatible forms $B\colon V\times V\to\mathbb{Q}$. We make W into an F-vector space by defining $(f\cdot B)(v,w) := B(fv,w)$ for $f\in F$ and $B\in W$. Then W has dimension 1 as an F-vector space.

Let us first look at the F-vector space U of all $\mathbb{Q}$-bilinear, F-compatible (not necessarily alternating) forms $B\colon V\times V\to\mathbb{Q}$. Writing U' for the F-vector space of F-bilinear forms $B'\colon V\times V\to F$, identifying a bilinear form B on V with the map $v\mapsto B(v,\cdot)$, and using the previous step we get an F-linear isomorphism

$$\begin{aligned} U' = \operatorname{Hom}_F(V,\operatorname{Hom}_F(V,F)) &\xrightarrow{\sim} \operatorname{Hom}_F(V,\operatorname{Hom}_{\mathbb{Q}}(V,\mathbb{Q})) = U \\ B' &\longmapsto \operatorname{Tr}_{F/\mathbb{Q}}\circ B'. \end{aligned}$$

The F-subspace W of U consists precisely of those $B\in U$ that are alternating. Now $B'\in U'$ is alternating if and only if the linear map $\beta_{B'}\colon V\to\operatorname{Hom}_F(V,F)$ defined by $\beta_{B'}(v) := B'(v,\cdot)+B'(\cdot,v)$ is trivial, which by the previous step happens if and only if $\operatorname{Tr}_{F/\mathbb{Q}}\circ\beta_{B'}$ is trivial. But clearly $\operatorname{Tr}_{F/\mathbb{Q}}\circ\beta_{B'} = \beta_{\operatorname{Tr}_{F/\mathbb{Q}}\circ B'}$, which by the same argument is trivial if and only if $\operatorname{Tr}_{F/\mathbb{Q}}\circ B'$ is alternating. Thus the isomorphism $U\cong U'$ restricts to an isomorphism between W and the F-vector space of F-bilinear alternating forms $V\times V\to F$, i. e. with $\operatorname{Hom}_F(\bigwedge_F^2 V, F)$. Since V has dimension 2 over F, this latter space has dimension 1 over F.

- Note that $\psi\in W$. So finally we claim that any $B\in W$ is K-compatible.

Let W' denote the set of all K-compatible, $\mathbb{Q}$-bilinear, alternating forms $B\colon V\times V\to\mathbb{Q}$. Then W' is an F-subspace of W, and $W'\neq 0$ because the pairing E_t, for any totally imaginary $t\in K^\times$, is non-trivial. But since $\dim_F W = 1$, we must have $W'=W$. $\square$

2.3.7 Remark (Torsion points and uniformizations)**.** Let (A,i) be a CM abelian variety of type (K,Φ), and let $\Theta\colon\mathbb{C}^\Phi/\Phi(\mathfrak{a})\xrightarrow{\sim}A$ be a uniformization of A, with $\mathfrak{a}$ a lattice in K. Since $\mathfrak{a}\otimes_{\mathbb{Z}}\mathbb{Q}=K$, the torsion points of A correspond under Θ to $\Phi(K)/\Phi(\mathfrak{a})$. We will simply write $\Theta\colon K/\mathfrak{a}\xrightarrow{\sim}A_{\text{tors}}$, although this should really be written as $\Theta\circ\Phi$, for $\Phi\colon K\hookrightarrow\mathbb{C}^\Phi$ given by $\Phi(x) := (\varphi(x))_{\varphi\in\Phi}$.

The importance of introducing the reflex field is illustrated by the following observation:

2.3.8 (Change of CM type)**.** Let (A,i) be a CM abelian variety of type (K,Φ). Let $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then the conjugate abelian variety ${}^\sigma A$ also has CM by K via

$$ {}^\sigma i \colon K \xrightarrow{i} \mathrm{End}_{\mathbb{Q}}(A) \xrightarrow{\sigma} \mathrm{End}_{\mathbb{Q}}({}^\sigma A). $$

The same calculation as in (2.3.2) shows that $({}^\sigma A, {}^\sigma i)$ is of type $(K, \sigma\Phi)$, where $\sigma\Phi := \{\sigma \circ \varphi \mid \varphi \in \Phi\}$. In particular, $({}^\sigma A, {}^\sigma i)$ has the same type as (A,i) if and only if $\sigma\Phi = \Phi$, i. e. if and only if σ fixes the reflex field $E(K,\Phi)$.

2.4 Main Theorem over the reflex field

We are now almost ready to state the Main Theorem of Complex Multiplication over the reflex field. But before doing so, we need to introduce more notation, and talk about class field theory.

2.4.1 (Class field theory)**.** Let k be a number field. We write $\mathbb{A}_k^\times := \prod_v' k_v^\times$ for the group of ideles of k and $\mathbb{A}_{k,f}^\times := \prod_{v\nmid\infty}' k_v^\times$ for the group of finite ideles. We embed $k^\times$ diagonally into $\mathbb{A}_k^\times$ and $\mathbb{A}_{k,f}^\times$.

Class field theory [NSW15, Ch. 8] says that there exists a surjective, continuous group homomorphism

$$ \mathrm{art}_k \colon \mathbb{A}_k^\times / k^\times \longrightarrow \mathrm{Gal}(k^{\mathrm{ab}}/k) = \Gamma_k^{\mathrm{ab}} $$

called the *Artin map*. We normalise it by sending uniformizers to *geometric* Frobenius elements. Recall that the kernel of art_k is the connected component of the identity element, which is equal to the closure of $k_{\infty,>0}^\times \cdot k^\times$ inside $\mathbb{A}_k^\times/k^\times$. Here $k_{\infty,>0}^\times := \prod_{v \text{ real}} \mathbb{R}_{>0}^\times \times \prod_{v \text{ complex}} \mathbb{C}^\times$ is the identity component in $k_\infty^\times := \prod_{v|\infty} k_v^\times$. In particular, art_k induces a surjective, continuous group homomorphism

$$ r_k \colon \mathbb{A}_{k,f}^\times / k_{>0}^\times \longrightarrow \Gamma_k^{\mathrm{ab}}, $$

where $k_{>0}^\times := k^\times \cap k_{\infty,>0}^\times$ denotes the elements of $k^\times$ that are positive at all real places of k. For example, if k is totally imaginary, e. g. when k is a CM field, then $k_{>0}^\times = k^\times$. The following commutative diagram summarises the situation

$$ \begin{tikzcd} \mathbb{A}_k^\times/k^\times \arrow[r, two heads] & \mathbb{A}_k^\times/k_{\infty,>0}^\times k^\times \arrow[r, two heads] & \pi_0(\mathbb{A}_k/k^\times) \arrow[r, "\mathrm{art}_k", "\sim"'] & \Gamma_k^{\mathrm{ab}} \\ & \mathbb{A}_{k,f}^\times/k_{>0}^\times . \arrow[u, "\sim"] \arrow[urr, "r_k"'] & & \end{tikzcd} \tag{2.4.1.1} $$

2.4.2 Example (Class field theory of $\mathbb{Q}$)**.** For the field $\mathbb{Q}$, one can describe its class field theory explicitly using roots of unity and the cyclotomic character: The Theorem of Kronecker–Weber states that $\mathbb{Q}^{\mathrm{ab}}$ is obtained by adjoining all roots of unity to $\mathbb{Q}$. Hence the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ is canonically isomorphic to $\hat{\mathbb{Z}}^\times$, the isomorphism being given by the cyclotomic character $\chi_{\mathrm{cyc}}\colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \xrightarrow{\sim} \hat{\mathbb{Z}}^\times$: An element $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ acts on a root of unity ζ by

$$\sigma(\zeta) = \zeta^{\chi_{\mathrm{cyc}}(\sigma)}.$$

On the other hand, the identity component in $\mathbb{Q}_\infty^\times = \mathbb{R}^\times$ is $\mathbb{R}_{>0}^\times$, and in this case $\mathbb{R}_{>0}^\times{\cdot}\mathbb{Q}^\times/\mathbb{Q}^\times$ is a closed subgroup of $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$. Moreover, the quotient $\mathbb{A}_\mathbb{Q}^\times/(\mathbb{R}_{>0}^\times{\cdot}\mathbb{Q}^\times)$ is isomorphic to $\hat{\mathbb{Z}}^\times$, the isomorphism being induced from the inclusion $\hat{\mathbb{Z}}^\times \subset \mathbb{A}_{\mathbb{Q},f}^\times \hookrightarrow \mathbb{A}_\mathbb{Q}^\times$. The Artin map is then given explicitly by the commutative diagram

$$\begin{tikzcd}
\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times \arrow[r, two heads] & \mathbb{A}_\mathbb{Q}^\times/\mathbb{R}_{>0}^\times\mathbb{Q}^\times \arrow[r, equal] & \pi_0(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times) \arrow[r, "\mathrm{art}_\mathbb{Q}", "\sim"'] & \Gamma_\mathbb{Q}^{\mathrm{ab}} \arrow[ddll, "\chi_{\mathrm{cyc}}"', "\sim"] \\
& \mathbb{A}_{\mathbb{Q},f}^\times/\mathbb{Q}_{>0}^\times . \arrow[u, "\sim"] \arrow[urr, "r_\mathbb{Q}"', "\sim"] & & \\
& \hat{\mathbb{Z}}^\times . \arrow[u, "\sim"] & &
\end{tikzcd}$$

2.4.3. (Class field theory and transfer) Let k'/k be a finite extension of number fields. Then we have a commutative diagram [Tat67, (13), p. 197]

$$\begin{tikzcd}
\mathbb{A}_{k'}^\times/k'^\times \arrow[r, "\mathrm{art}_{k'}"] & \Gamma_{k'}^{\mathrm{ab}} \\
\mathbb{A}_k^\times/k^\times \arrow[u] \arrow[r, "\mathrm{art}_k"'] & \Gamma_k^{\mathrm{ab}}, \arrow[u, "V_{k'/k}"']
\end{tikzcd}$$

where the left vertical arrow is induced from the inclusion $k \subset k'$ and the right vertical arrow is the transfer map $V_{k'/k}\colon \Gamma_k^{\mathrm{ab}} \to \Gamma_{k'}^{\mathrm{ab}}$, see (2.4.4). The same diagram with "r" instead of "art" and finite ideles and totally positive elements on the left hand side also commutes. We will very often use the particular case when $k = \mathbb{Q}$.

2.4.4 (Transfer)**.** Let Γ be a (profinite) group and Δ be a (closed) finite index subgroup. Denote (the closure of) the commutator subgroup of Γ by $[\Gamma, \Gamma]$, and the abelianisation of Γ by $\Gamma^{\mathrm{ab}} := \Gamma/[\Gamma, \Gamma]$. The *transfer map*

$$V\colon \Gamma^{\mathrm{ab}} \longrightarrow \Delta^{\mathrm{ab}}$$

is defined as follows: For $x \in \Gamma/\Delta$, let $s_x \in \Gamma$ be a representative of the right Δ-coset x, so that $\Gamma = \bigsqcup_x s_x\Delta$. For any $\gamma \in \Gamma$, we have $\gamma s_x = s_{\gamma x}\delta$ for some $\delta \in \Delta$, i. e.

$s_{\gamma x}^{-1}\gamma s_x \in \Delta$. So we can define

$$V(\gamma) := \prod_{x\in\Gamma/\Delta} (s_{\gamma x}^{-1}\gamma s_x \cdot [\Delta,\Delta]) \in \Delta^{\mathrm{ab}}.$$

One checks that V is a (continuous) group homomorphism, hence factors through Γ^{ab}, and is independent of the chosen system of representatives (s_x).

If k'/k is a finite extension of number fields, then $\Gamma_{k'}$ is a finite index subgroup of Γ_k and we denote the resulting transfer map by $V_{k'/k}$.

2.4.5 ("Idele times lattice")**.** Let K be a number field and $\mathfrak{a} \subset K$ a lattice. For a prime number p, let $K_p := K \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathfrak{a}_p := \mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Z}_p$, a $\mathbb{Z}_p$-lattice in K_p. For an idele $x = (x_v)_v \in \mathbb{A}_K^\times$ (or a finite idele, since the ∞-component will not play a role here), define its p-component to be $x_p := (x_v)_{v|p} \in \prod_{v|p} K_v = K_p$. Then $x_p \cdot \mathfrak{a}_p$ is also a $\mathbb{Z}_p$-lattice in K_p.

Now [Lan83, Ch. 3.6, p. 77-78] one can find a lattice $\mathfrak{b}$ in K such that $\mathfrak{b}_p = x_p\mathfrak{a}_p$ for all prime numbers p. So we define $x \cdot \mathfrak{a} := \mathfrak{b}$. For example, if $\mathfrak{a}$ is a fractional ideal, then we can define $\mathfrak{b} := (x) \cdot \mathfrak{a}$, where (x) is the fractional ideal of K defined by the idele x.

Furthermore, using the isomorphism $K/\mathfrak{a} \cong \bigoplus_p K_p/\mathfrak{a}_p$, we can also define a "multiplication by x" map $K/\mathfrak{a} \to K/x\mathfrak{a}$ in the following way: Let $u \in K$ and choose $v \in K$ such that $v \equiv x_p u \bmod x_p\mathfrak{a}_p$ for all p. Define $x \cdot (u \bmod \mathfrak{a}) := v \bmod x\mathfrak{a}$. The following commutative diagram illustrates this process:

$$\begin{tikzcd}
K/\mathfrak{a} \arrow[r, dashed] \arrow[d, "\sim"'] & K/x\mathfrak{a} \arrow[d, "\sim"] \\
\bigoplus_p K_p/\mathfrak{a}_p \arrow[r, "\cdot x"] & \bigoplus_p K_p/x_p\mathfrak{a}_p \\
K_p/\mathfrak{a}_p \arrow[r, "\cdot x_p"] \arrow[u, hook, "\subset"] & K_p/x_p\mathfrak{a}_p. \arrow[u, hook, "\subset"']
\end{tikzcd}$$

We denote this map by $K/\mathfrak{a} \xrightarrow{x} K/x\mathfrak{a}$ although it is not really a "multiplication by x" map.

If $\mathfrak{a}$ is a fractional ideal, one can avoid working with prime numbers p and instead work with the prime ideals $\mathfrak{p}$ of K. The procedure remains the same.

2.4.6 (Projective vs. inductive limit)**.** Here is another way to think about the construction in (2.4.5): For any lattice $\mathfrak{a}$ in K, we have $\mathfrak{a} \otimes_{\mathbb{Z}} \mathbb{Q} = K$. So on the one hand, we can view $K/\mathfrak{a}$ as the union (inductive limit) of all $\frac{1}{n}\mathfrak{a}/\mathfrak{a}$, and we have constructed a map

$$K/\mathfrak{a} = \varinjlim_n \frac{1}{n}\mathfrak{a} \Big/ \mathfrak{a} \xrightarrow{x} \varinjlim_n \frac{1}{n}x\mathfrak{a} \Big/ x\mathfrak{a} = K/x\mathfrak{a}.$$

On the other hand, we have

$$\mathbb{A}_{K,f} = \left(\varprojlim_n \frac{1}{n}\mathfrak{a} \Big/ \mathfrak{a} \right) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

For every n, the "multiplication by x" map above induces a map $\frac{1}{n}\mathfrak{a}/\mathfrak{a} \to \frac{1}{n}x\mathfrak{a}/x\mathfrak{a}$. For $n|m$, these maps are compatible in the right way, so we get an induced map between the projective limits (also denoted by x)

$$\begin{tikzcd}
\left(\varprojlim_n \frac{1}{n}\mathfrak{a}/\mathfrak{a}\right) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[r, "x"] \arrow[d, equal] & \left(\varprojlim_n \frac{1}{n}x\mathfrak{a}/x\mathfrak{a}\right) \otimes_{\mathbb{Z}} \mathbb{Q} \arrow[d, equal] \\
\mathbb{A}_{K,f} \arrow[r, dotted] & \mathbb{A}_{K,f}.
\end{tikzcd}$$

Then the dotted arrow is precisely given by multiplication by $x \in \mathbb{A}_{K,f}^\times$.

We are now ready to state the Main Theorem of Complex Multiplication over the reflex field. Up to notational differences[1], this is [Shi71, Thm 5.15] and follows from the results in [ST61]. For a full proof, see [Lan83, Ch. 3, Thm 6.1] or [Mil07, Thm 3.13].

2.4.7 Theorem (Main Theorem of CM over the reflex field I). *Let (A, i) be a CM abelian variety of type (K, Φ) equipped with a compatible polarisation ψ. Let (A, i, ψ) be of type $(K, \Phi; \mathfrak{a}, t)$ relative to a uniformization $\Theta\colon \mathbb{C}^\Phi/\Phi(\mathfrak{a}) \xrightarrow{\sim} A$. Furthermore, let E be the reflex field of (K, Φ), let $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ and choose $s \in \mathbb{A}_{E,f}^\times$ such that $r_E(s) = \sigma|_{E^{\mathrm{ab}}}$.*

Then:

(2.4.7.a) The triple $\sigma(A, i, \psi) := ({}^\sigma A, {}^\sigma i, {}^\sigma \psi)$ is of type $(K, \Phi; f\mathfrak{a}, t\frac{\chi_{\mathrm{cyc}}(\sigma)}{f\bar{f}})$ relative to some uniformization Θ', where $f = N_\Phi(s) \in \mathbb{A}_{K,f}^\times$.

(2.4.7.b) There exists a unique uniformization $\Theta'\colon \mathbb{C}^\Phi/\Phi(f\mathfrak{a}) \xrightarrow{\sim} ({}^\sigma A)(\mathbb{C})$ such that the following diagram commutes:

$$\begin{tikzcd}
K/\mathfrak{a} \arrow[r, "\Theta"] \arrow[d, "f"'] & A_{\mathrm{tors}} \arrow[d, "\sigma"] \\
K/f\mathfrak{a} \arrow[r, "\Theta'"'] & ({}^\sigma A)_{\mathrm{tors}}.
\end{tikzcd}$$

[1] Most notably, [Shi71] and [Lan83] use a different normalisation for class field theory by sending uniformizers to *arithmetic* Frobenius elements. We follow the convention in [Mil07], see (2.4.1).

Here the left vertical arrow is the "multiplication by f" map defined in (2.4.5).

2.4.8 Remark. In (2.4.7.a) we write ${}^{\sigma}A$ for the conjugate abelian variety. It has CM by K via ${}^{\sigma}i$, see (2.3.8). The polarisation ${}^{\sigma}\psi$ of ${}^{\sigma}A$ is defined to be associated to the σ-conjugate of the line bundle determining ψ. For more details in the slightly different language of divisors, see [Lan83, Ch. 3.4].

Moreover, by [Mil07, Lemma 3.7] we have $\frac{\chi_{\mathrm{cyc}}(\sigma)}{f\bar{f}} \in \mathbb{Q}_{>0}^{\times}$, so the quadruple $(K, \Phi; f\mathfrak{a}, t\frac{\chi_{\mathrm{cyc}}(\sigma)}{f\bar{f}})$ is a type in the sense of (2.3.5).

The Main Theorem of Complex Multiplication is a statement of the action of $\sigma \in \mathrm{Aut}(\mathbb{C})$ fixing the reflex field on the torsion points of an abelian variety A with CM by K. It will be useful to restate this theorem in terms of the Tate module of A.

2.4.9 (Tate module)**.** The *(full, rational) Tate module* of an abelian variety A is defined to be

$$\widehat{V}(A) := \left(\varprojlim_{n} A[n]\right) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

It is a free $\mathbb{A}_{\mathbb{Q},f}$-module of rank $2\dim A$. If A has complex multiplication by K, then $\widehat{V}(A)$ becomes a free $\mathbb{A}_{K,f}$-module of rank 1.

The Main Theorem now becomes [Mil07, Thm 3.10]:

2.4.10 Theorem (Main Theorem of CM over the reflex field II)**.** *Let (A, i) be an abelian variety with CM of type (K, Φ), and let E be the reflex field of (K, Φ). Let $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ and let $s \in \mathbb{A}_{E,f}^{\times}$ such that $r_E(s) = \sigma|_{E^{\mathrm{ab}}}$. Finally, let $f := N_{\Phi}(s) \in \mathbb{A}_{K,f}^{\times}$.*

Then there exists a unique K-linear quasi-isogeny $\lambda\colon A \to {}^{\sigma}A$ such that the following diagram is commutative:

$$\begin{tikzcd} \widehat{V}(A) \arrow[r, "f"] \arrow[dr, "\sigma"'] & \widehat{V}(A) \arrow[d, "\lambda"] \\ & \widehat{V}({}^{\sigma}A). \end{tikzcd} \tag{2.4.10.1}$$

Moreover, if ψ is a compatible polarisation on (A, i), then for all $v, w \in H_1(A, \mathbb{Q})$ we have

$$({}^{\sigma}\psi)(\lambda(v), \lambda(w)) = \frac{\chi_{\mathrm{cyc}}(\sigma)}{f\bar{f}}\psi(v, w).$$

Sketch of proof that Theorems (2.4.7) *and* (2.4.10) *are equivalent.* Theorem (2.4.7) follows from Theorem (2.4.10) by [Mil07, 3.13], but the argument can be reversed. Namely, the isogeny $\lambda\colon A \to {}^{\sigma}A$ and the uniformizations Θ and Θ' are related by

$$\lambda = \Theta' \circ \Theta^{-1},$$

where we view Θ as a K-linear isomorphism of $\mathbb{Q}$-Hodge structures $(K, h_\Phi) \xrightarrow{\sim} (H_1(A, \mathbb{Q}), h_A)$ and similarly for Θ'. Moreover, the commutative diagrams (2.4.7.b) and (2.4.10.1) are related by (2.4.6). □

2.5 PEL Shimura variety associated to (K, Φ)

We will now explain how the Main Theorem over the reflex field can be used to get a canonical model of a certain (PEL) Shimura variety associated to a CM pair (K, Φ). This is an easy case of a general phenomenon, which will come up again in (3.2.8), and is included here as an illustrative example for the general case.

2.5.1 (Two Shimura data associated to (K, Φ))**.** Let K be a CM field and Φ a CM type of K. Let $h_\Phi\colon \mathbb{S} \to \mathrm{GL}_\mathbb{Q}(K)_\mathbb{R}$ be the associated $\mathbb{Q}$-Hodge structure, see (2.3.3). We view $T_K = R_{K/\mathbb{Q}}\mathbb{G}_m$ as an algebraic subgroup of $\mathrm{GL}_\mathbb{Q}(K)$: On $\mathbb{Q}$-points, we embed $T_K(\mathbb{Q}) = K^\times$ as the K-linear endomorphisms in $\mathrm{GL}_\mathbb{Q}(K)$. By construction, for every $z \in \mathbb{C}^\times$ the map $h_\Phi(z)$ is K-linear on $K \otimes_\mathbb{Q} \mathbb{R}$, hence we may view h_Φ as a morphism $h_\Phi\colon \mathbb{S} \to (T_K)_\mathbb{R}$ of algebraic tori over $\mathbb{R}$.

The pair $(T_K, \{h_\Phi\})$ satisfies the axioms [Del79, (2.1.1.1-3)] because T_K is a torus, hence $(T_K, \{h_\Phi\})$ is a Shimura datum. However, except when $K/\mathbb{Q}$ is imaginary quadratic, it is not of PEL type. We modify T_K slightly in order to get a PEL Shimura datum:

Let F be the maximal totally real subfield of K. Define the $\mathbb{Q}$-algebraic torus T by the Cartesian diagram

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d] & T_K \arrow[d, "N"] \\ \mathbb{G}_m \arrow[r, hook] & T_F, \end{tikzcd} \tag{2.5.1.1}$$

where the map N is the *norm map*, which on the level of $\mathbb{Q}$-points is given by the norm $N_{K/F}\colon K^\times \to F^\times$, and the embedding $\mathbb{G}_m \hookrightarrow T_F$ on the level of $\mathbb{Q}$-points is given by the inclusion $\mathbb{Q}^\times \hookrightarrow F^\times$. In particular, this means that

$$T(\mathbb{Q}) = \{x \in K^\times \mid N_{K/F}(x) \in \mathbb{Q}^\times\}.$$

We describe all these maps on the level of cocharacters in (6.2.2).

It is then not hard to see that the image of h_Φ is contained in the $\mathbb{R}$-torus $T_\mathbb{R}$. Then $(T, \{h_\Phi\})$ is a Shimura datum of PEL type: Fix a totally imaginary element $t \in K^\times$ such that $\operatorname{Im}\varphi(t) > 0$ for all $\varphi \in \Phi$. Then $(T, \{h_\Phi\})$ is the PEL Shimura datum (in the sense of [Del71, 4.9]) associated to the simple $\mathbb{Q}$-algebra K, with involution given by complex conjugation, acting on the $\mathbb{Q}$-vector space K, which we equip with the alternating, $\mathbb{Q}$-bilinear, K-compatible form E_t from (2.3.4).

2.5.2 (Shimura variety associated to $(T, \{h_\Phi\})$)**.** For a compact open subgroup $U \subset T(\mathbb{A}_f)$, we define the *Shimura variety of level* U to be the set

$$\mathrm{Sh}_U(T, \{h_\Phi\}) := T(\mathbb{Q})\backslash \left[\{h_\Phi\} \times T(\mathbb{A}_f)/U\right].$$

It is often more convenient to work with the projective limit over all such U, namely we call the limit

$$\mathrm{Sh}(T, \{h_\Phi\}) := \varprojlim_U \mathrm{Sh}_U(T, \{h_\Phi\})$$

the *Shimura variety* associated to $(T, \{h_\Phi\})$. It carries an action by $T(\mathbb{A}_f)$ by multiplication on the right in the second factor. We mention the formula [Orr18, (1), p. 4] (which is a variant of [Del79, Prop. 2.1.10]):

$$\mathrm{Sh}(T, \{h_\Phi\}) = T(\mathbb{Q})\backslash \left[\{h_\Phi\} \times T(\mathbb{A}_f)/\overline{T(\mathbb{Q})}\right] = \{h_\Phi\} \times T(\mathbb{A}_f)/\overline{T(\mathbb{Q})},$$

where $\overline{T(\mathbb{Q})}$ denotes the closure of $T(\mathbb{Q})$ inside $T(\mathbb{A}_f)$ (in the idelic topology).

The Shimura variety $\mathrm{Sh}(T, \{h_\Phi\})$ is a pro-algebraic variety over $\mathbb{C}$, i. e. the projective limit of the algebraic varieties $\mathrm{Sh}_U(T, \{h_\Phi\})$ over $\mathbb{C}$. However, every Shimura variety has a unique so-called *canonical model*, defined over its *reflex field*, a certain number field. The aim of the following discussion is to describe the canonical model for $\mathrm{Sh}(T, \{h_\Phi\})$ in terms of abelian varieties with complex multiplication.

2.5.3 (Reflex field of $\mathrm{Sh}(T, \{h_\Phi\})$)**.** Associated to the Hodge structure h_Φ we have the cocharacter μ_{h_Φ} (see (2.2.5)). By definition, for $z \in \mathbb{C}^\times$ it is given by

$$\mu_{h_\Phi}(z) = \text{multiplication by } (z, \dots, z, 1, \dots, 1) \text{ on } \mathbb{C}^\Phi \oplus \mathbb{C}^{\overline{\Phi}} = K \otimes_\mathbb{Q} \mathbb{C},$$

i. e. the cocharacter μ_{h_Φ} agrees with the cocharacter μ_Φ of T_K defined in (2.1.8). The description of the cocharacters of T in (6.2.2.4) shows that μ_Φ can be viewed as a cocharacter of T, too. This also means that the reflex norm constructed in (2.1.8) can be viewed as a morphism $N_\Phi \colon T_E \to T$.

By Definition (6.1.11), the reflex field of the Shimura datum $(T, \{h_\Phi\})$ (or of $(T_K, \{h_\Phi\})$) is the field of definition of the cocharacter μ_Φ, and so is equal to the reflex field E of the CM pair (K, Φ).

2.5.4 (Canonical model of $\mathrm{Sh}(T, \{h_\Phi\})$)**.** The *canonical model* of a Shimura variety is defined to be a model over the reflex field, equipped with a (right) action of the adelic points of the underlying group, satisfying a certain *reciprocity law* on its *special points*. For a precise definition, see [Del71, 3.13] or [Mil17, 12.8].

In our case, the underlying algebraic group is the torus T, so all points are special points and the associated Shimura variety is zero-dimensional. Moreover, the reciprocity law is given by a morphism $T_E \to T$ (see [Mil17, (60)]) that agrees, by its definition, with the reflex norm $N_\Phi \colon T_E \to T$.

Spelling out the definition: The canonical model of $\mathrm{Sh}(T, \{h_\Phi\})$ is a pro-variety M defined over E with a (right) action of $T(\mathbb{A}_f)$, together with a $T(\mathbb{A}_f)$-equivariant isomorphism $m \colon M \times_E \mathbb{C} \xrightarrow{\sim} \mathrm{Sh}(T, \{h_\Phi\})$ such that for every point $[h_\Phi, g] \in \mathrm{Sh}(T, \{h_\Phi\})$, where $g \in T(\mathbb{A}_f)$, the point $m^{-1}([h_\Phi, g]) \in M(\mathbb{C})$ is defined over E^{ab}, and for an arbitrary $\sigma \in \mathrm{Aut}(\mathbb{C}/E)$ and an element $s \in \mathbb{A}_{E,f}^\times$ such that $r_E(s) = \sigma|_{E^{\mathrm{ab}}}$, we have

$$\sigma[h_\Phi, g] = [h_\Phi, g N_\Phi(s)]. \tag{2.5.4.1}$$

Now, a zero-dimensional pro-variety defined over any field E is the same as a profinite set equipped with a continuous action of the absolute Galois group Γ_E of E. So here we can simply define the canonical model M by declaring it to be the profinite set $\mathrm{Sh}(T, \{h_\Phi\})$, which we equip with a Γ_E-action by *defining* $\sigma \in \Gamma_E$ to act by the formula (2.5.4.1).

2.5.5 Remark. As explained in (2.5.4), the construction of canonical models for tori is basically tautological. (2.5.4.1) simply defines the correct Γ_E-action. What is less tautological is its relationship with the Main Theorem over the reflex field, which we will explain in the remaining paragraphs of this section and which is the reason for defining canonical models in the way they are defined.

Canonical models are defined by the property that on special points the Galois action is equal to a certain class-field theoretic recipe (called a reciprocity map). This reciprocity map is modelled on the reflex norm. This means that, for PEL Shimura varieties, special points correspond to abelian varieties with complex multiplication, and so the Main Theorem over the reflex field implies, under a few technical assumptions, that the moduli variety representing the associated PEL moduli problem defines a canonical model of the Shimura variety.

We illustrate this procedure by looking at the example of the Shimura variety $\mathrm{Sh}(T, \{h_\Phi\})$. It is related to abelian varieties with complex multiplication of type (K, Φ) by Theorem (2.5.8) below, which can be found in [Del71, 4.11], specialised to the particular choice of PEL data in (2.5.1). For a proof and an explicit description of the bijection, one can use the same strategy as in the proof of (4.4.5).

For the remainder of this section, we fix, as in (2.5.1), a totally imaginary element $t \in K^\times$ such that $\mathrm{Im}\,\varphi(t) > 0$ for all $\varphi \in \Phi$. Temporarily, we write $V = K$ and $\psi \colon V \times V \to \mathbb{Q}$ for the bilinear form E_t defined in (2.3.4), and we write $V(\mathbb{A}_f)$ for $V \otimes_{\mathbb{Q}} \mathbb{A}_f = \mathbb{A}_{K,f}$.

2.5.6 Theorem ($(T, \{h_\Phi\})$ and CM abelian varieties)**.** *The coset space*

$$T(\mathbb{Q}) \backslash [\{h_\Phi\} \times T(\mathbb{A}_f)]$$

is in bijection with the set of isomorphism classes of quadruples $(A, i, \mathbb{Q}^\times s, \eta)$, consisting of

- *(A, i) a CM abelian variety of type (K, Φ),*
- *s a K-compatible polarisation of A, and*
- *$\eta \colon V(\mathbb{A}_f) \xrightarrow{\sim} \widehat{V}(A)$ an isomorphism of $\mathbb{A}_{K,f}$-modules sending $\mathbb{A}_f^\times \psi$ to $\mathbb{A}_f^\times s$,*

satisfying the condition

(2.5.6.a) There exists a K-linear isomorphism $a \colon (H_1(A, \mathbb{Q}), h_A) \xrightarrow{\sim} (V, h_\Phi)$ of $\mathbb{Q}$-Hodge structures that sends $\mathbb{Q}^\times s$ to $\mathbb{Q}^\times \psi$.

Moreover, if we let $g \in T(\mathbb{A}_f)$ act on such a quadruple $(A, i, \mathbb{Q}^\times s, \eta)$ by sending it to $(A, i, \mathbb{Q}^\times s, \eta \circ g)$, then this bijection is equivariant for the $T(\mathbb{A}_f)$-actions.

2.5.7 Remarks. A few comments about the notation in the above theorem are necessary.

(2.5.7.a) We view $\mathbb{Q}^\times s$ as the set of all bilinear forms $\{qs \mid q \in \mathbb{Q}\}$ on $H_1(A, \mathbb{Q})$ (and similarly for $\mathbb{Q}^\times \psi$). Note that not every element of the set $\mathbb{Q}^\times s$ is a polarisation, since for negative q the pairing $(v, w) \mapsto qs(v, h_A(i)w)$ is negative definite, so the last condition of (2.2.6) fails.

The condition on η sending $\mathbb{A}_f^\times \psi$ to $\mathbb{A}_f^\times s$ then translates to: There exists an element $q \in \mathbb{A}_f^\times$ such that $s(\eta(v), \eta(w)) = q\psi(v, w)$ for all $v, w \in V(\mathbb{A}_f)$. Similarly for the last condition on a in (2.5.6.a).

(2.5.7.b) An isomorphism of quadruples $(A, i, \mathbb{Q}^\times s, \eta) \xrightarrow{\sim} (A', i', \mathbb{Q}^\times s', \eta')$ is a quasi-isogeny $f \colon A \to A'$ such that

- f sends $\mathbb{Q}^\times s$ to $\mathbb{Q}^\times s'$,
- f is K-compatible (with respect to i and i'), and
- f sends η to η', i. e. $f \circ \eta = \eta'$.

Note that the $T(\mathbb{A}_f)$-action on quadruples induces a well-defined action on the set of isomorphism classes of such quadruples.

Theorem (2.5.6) immediately implies the following interpretation of the points of the Shimura variety $\mathrm{Sh}_U(T, \{h_\Phi\})$, see [Del71, 4.11]:

2.5.8 Corollary (Complex points of $\mathrm{Sh}(T, \{h_\Phi\})$)**.** *For any compact open subgroup $U \subset T(\mathbb{A}_f)$, the points of the Shimura variety*

$$\mathrm{Sh}_U(T, \{h_\Phi\})(\mathbb{C}) = T(\mathbb{Q})\backslash [\{h_\Phi\} \times T(\mathbb{A}_f)/U]$$

are in bijection with isomorphism classes of quadruples $(A, i, \mathbb{Q}^\times s, \eta U)$ satisfying the conditions of Theorem (2.5.6)*, where ηU denotes a (right) U-coset of isomorphisms $V(\mathbb{A}_f) \xrightarrow{\sim} \widehat{V}(A)$ and isomorphisms between such quadruples are defined as in* (2.5.7.b)*, with the last condition replaced by the analogous equality of (right) U-cosets $(f \circ \eta)U = \eta' U$.*

Similarly, in the projective limit we get a bijection between

$$\mathrm{Sh}(T, \{h_\Phi\})(\mathbb{C}) = T(\mathbb{Q})\backslash \left[\{h_\Phi\} \times T(\mathbb{A}_f)/\overline{T(\mathbb{Q})}\right]$$

and isomorphism classes of quadruples $(A, i, \mathbb{Q}^\times s, \eta\overline{T(\mathbb{Q})})$.

We use this interpretation of the complex points of $\mathrm{Sh}(T, \{h_\Phi\})$ in terms of CM abelian varieties to define another profinite set and equip it with the structure of a pro-variety over E:

2.5.9 (Moduli variety)**.** Let M^+ be the set of isomorphism classes of quadruples $(A, i, \mathbb{Q}^\times s, \eta\overline{T(\mathbb{Q})})$, consisting of

- (A, i) a CM abelian variety of type (K, Φ) defined over $\overline{\mathbb{Q}}$,
- s a K-compatible polarisation of A, also defined over[2] $\overline{\mathbb{Q}}$, and
- $\eta\colon V(\mathbb{A}_f) \xrightarrow{\sim} \widehat{V}(A)$ an $\mathbb{A}_{K,f}$-module-isomorphism sending $\mathbb{A}_f^\times \psi$ to $\mathbb{A}_f^\times s$,

satisfying the condition

(2.5.9.a) There exists a K-linear isomorphism $a\colon (H_1(A, \mathbb{Q}), h_A) \xrightarrow{\sim} (V, h_\Phi)$ of $\mathbb{Q}$-Hodge structures that sends $\mathbb{Q}^\times s$ to $\mathbb{Q}^\times \psi$.

Note that the only difference to the set in Corollary (2.5.8) is that (A, i, s) is defined over $\overline{\mathbb{Q}}$, which enables us to conjugate (A, i, s) by an element of Γ_E. In fact, any (polarised) complex abelian variety with CM is isomorphic to a (polarised) CM abelian variety defined over some number field, see [ST61, Prop. 26, p. 109], so indeed M^+ is equal to the set in (2.5.8).

We endow M^+ with the profinite topology as M^+ is equal to the projective limit, over all compact open subgroups $U \subset T(\mathbb{A}_f)$, of sets M_U^+ defined in the same way

[2]This means that the line bundle associated to s is defined over $\overline{\mathbb{Q}}$, see (2.2.2).

with (right) U-cosets of η instead of $\overline{T(\mathbb{Q})}$-cosets. We let $\sigma \in \Gamma_E$ act on an element $[A, i, \mathbb{Q}^\times s, \eta\overline{T(\mathbb{Q})}]$ of M^+ by

$$\sigma[A, i, \mathbb{Q}^\times s, \eta\overline{T(\mathbb{Q})}] := [{}^\sigma A, {}^\sigma i, \mathbb{Q}^\times({}^\sigma s), (\sigma \circ \eta)\overline{T(\mathbb{Q})}],$$

where ${}^\sigma A$ denotes the conjugate abelian variety, ${}^\sigma i$ and ${}^\sigma s$ are defined as in (2.4.8), and the action on the level structure η is given by composition with the induced map on Tate modules $\sigma\colon \widehat{V}(A) \to \widehat{V}({}^\sigma A)$.

It is not hard to check that this gives a well-defined continuous action of Γ_E on the set M^+, hence gives M^+ the structure of a zero-dimensional pro-variety over E. We call M^+ the *moduli variety* (associated to T, h_Φ and ψ).

Combining this with the Main Theorem of Complex Multiplication over the reflex field, in the version (2.4.10), yields the following theorem, see [Del71, 4.19 & 4.20]:

2.5.10 Theorem. *The moduli variety M^+ defines a canonical model of the Shimura variety* $\mathrm{Sh}(T, \{h_\Phi\})$.

Proof. First of all, Corollary (2.5.8) shows that M^+ indeed defines a model of $\mathrm{Sh}(T, \{h_\Phi\})$ over the reflex field E.

Secondly, let $[A, i, \mathbb{Q}^\times s, \eta\overline{T(\mathbb{Q})}]$ be a point of M^+, and let $\gamma \in \mathrm{Aut}(\mathbb{C}/E)$ and $u \in \mathbb{A}_{E,f}^\times$ such that $r_E(u) = \gamma|_{E^{\mathrm{ab}}}$. Let $\lambda\colon A \to {}^\gamma A$ be the quasi-isogeny in the statement of the Main Theorem over the reflex field (2.4.10). Then

$$\gamma \circ \eta = \lambda \circ N_\Phi(u) \circ \eta = \lambda \circ \eta \circ N_\Phi(u),$$

where the first equality is the commutative diagram (2.4.10.1), and the second equality holds because η is $\mathbb{A}_{K,f}$-linear. Together with the statement about the polarisation in (2.4.10), we see that λ defines an isomorphism of quadruples

$$(A, i, \mathbb{Q}^\times s, \eta \circ N_\Phi(u)) \xrightarrow{\sim} ({}^\gamma A, {}^\gamma i, \mathbb{Q}^\times({}^\gamma s), \gamma \circ \eta).$$

We conclude that

$$\gamma[A, i, \mathbb{Q}^\times s, \eta\overline{T(\mathbb{Q})}] = [A, i, \mathbb{Q}^\times s, (\eta \circ N_\Phi(u))\overline{T(\mathbb{Q})}],$$

which by (2.5.4.1) precisely says that M^+ is a canonical model for $\mathrm{Sh}(T, \{h_\Phi\})$. □

3

Galois conjugates of CM abelian varieties

Let (K, Φ) be a CM pair. In this chapter we generalise the Main Theorem of CM over the reflex field to the Main Theorem of CM over $\mathbb{Q}$. The former gives only information about conjugates of CM abelian varieties by Galois elements fixing the reflex field E of (K, Φ), whereas the latter allows arbitrary Galois elements. The key idea is to replace the reflex norm $N_\Phi \colon \mathbb{A}_{E,f}^\times \to \mathbb{A}_{K,f}^\times$ by the Taniyama element $f_\Phi \colon \Gamma_\mathbb{Q} \to \mathbb{A}_{K,f}^\times / K^\times$.

Section 3.1 starts by defining Tate's half transfer

$$F_\Phi \colon \Gamma_\mathbb{Q} \to \Gamma_K^{\mathrm{ab}}$$

associated to a CM pair (K, Φ). For $\gamma \in \Gamma_\mathbb{Q}$, the Taniyama element $f_\Phi(\gamma) \in \mathbb{A}_{K,f}^\times / K^\times$ is then a suitably normalised class-field-theoretic preimage of $F_\Phi(\gamma)$. Using class field theory, we can think of the Taniyama element as extending the reflex norm. We then state the Main Theorem of CM over $\mathbb{Q}$ in two version: Theorem (3.1.3) describes Galois conjugates of torsion points of CM abelian varieties, and Theorem (3.1.5) describes the same effect on the level of Tate modules.

Fix a totally real number field F. In Section 3.2 we introduce the PEL Hilbert modular variety $\mathrm{Sh}(G, X)$ associated to F. In (3.2.6) we interpret the points of $\mathrm{Sh}(G, X)$ as isomorphism classes of abelian varieties with real multiplication by F equipped with polarisation and level structure, and describe the canonical model of $\mathrm{Sh}(G, X)$ over $\mathbb{Q}$. Finally, in (3.2.9) we apply the Main Theorem of CM over $\mathbb{Q}$ to get an explicit formula describing the Galois action of $\Gamma_\mathbb{Q}$ on the CM points of $\mathrm{Sh}(G, X)$ in terms of the Taniyama element. This formula is extremely useful when generalising this action to an action of the plectic Galois group in Chapter 4.

The results of Section 3.1 can be found in [Mil07, §4], which is based on the observations in [Tat16], see also [Lan83, Ch. 7]. In Section 3.2 we apply the general theory of Shimura varieties, see [Mil17] and [Del71], to the specific example of the PEL Hilbert modular variety.

3.1 Main Theorem over $\mathbb{Q}$

3.1.1 (Tate's half transfer)**.** Let $K \subset \overline{\mathbb{Q}}$ be a CM field and Φ be a CM type of K. Identify $\Sigma_K = \mathrm{Hom}(K, \overline{\mathbb{Q}})$ with $\Gamma_{\mathbb{Q}}/\Gamma_K$. For each $\rho \in \Sigma_K$, choose a representative $w_\rho \in \Gamma_{\mathbb{Q}}$ of the corresponding coset in $\Gamma_{\mathbb{Q}}/\Gamma_K$ in such a way that for all $\rho \in \Sigma_K$ we have

$$w_{c\rho} = c w_\rho,$$

where $c \in \Gamma_{\mathbb{Q}}$ denotes complex conjugation.

The choice of w_ρ gives a partition $\Gamma_{\mathbb{Q}} = \sqcup_{\rho \in \Sigma_K} w_\rho \Gamma_K$, so for any $\gamma \in \Gamma_{\mathbb{Q}}$, we have $\gamma w_\rho = w_{\gamma\rho} h$ for some $h \in \Gamma_K$. In other words, $w_{\gamma\rho}^{-1} \gamma w_\rho \in \Gamma_K$, so we may define *Tate's half transfer* by

$$\begin{aligned} F_\Phi \colon \Gamma_{\mathbb{Q}} &\longrightarrow \Gamma_K^{\mathrm{ab}} \\ \gamma &\longmapsto \prod_{\varphi \in \Phi} \left(w_{\gamma\varphi}^{-1} \gamma w_\varphi \right)\big|_{K^{\mathrm{ab}}}. \end{aligned}$$

These maps have the following properties, which can be checked easily:

(3.1.1.a) [Mil07, Lem. 4.4] $F_\Phi(\gamma)$ is independent of the choice of the set-theoretic section $w \colon \Sigma_K \to \Gamma_{\mathbb{Q}}$, $\rho \mapsto w_\rho$ to $\Gamma_{\mathbb{Q}} \twoheadrightarrow \Gamma_{\mathbb{Q}}/\Gamma_K = \Sigma_K$.

(3.1.1.b) [Mil07, Lem. 4.5] $F_\Phi(\gamma)$ is independent of the choice of embedding of K into $\overline{\mathbb{Q}}$.

(3.1.1.c) From the definition of the transfer map $V_{K/\mathbb{Q}} \colon \Gamma_{\mathbb{Q}}^{\mathrm{ab}} \to \Gamma_K^{\mathrm{ab}}$ in (2.4.4) one immediately sees that

$$F_\Phi(\gamma) \cdot F_{c\Phi}(\gamma) = \prod_{\varphi \in \Sigma_K} \left(w_{\gamma\varphi}^{-1} \gamma w_\varphi \right)\big|_{K^{\mathrm{ab}}} = V_{K/\mathbb{Q}}(\gamma).$$

That is why F_Φ is called a "half-transfer".

(3.1.1.d) [Nek09, (1.4.1.3)] For $\gamma, \gamma' \in \Gamma_{\mathbb{Q}}$ we have the "cocycle relation"

$$F_\Phi(\gamma\gamma') = F_{\gamma'\Phi}(\gamma) F_\Phi(\gamma').$$

We are aiming for a description of the effect of conjugation by $\gamma \in \Gamma_{\mathbb{Q}}$ on the torsion points of a CM abelian variety. Under a uniformization, the torsion points will be given by $K/\mathfrak{a}$ as in (2.3.7), and the effect of γ on torsion points will again be expressed by multiplication by a certain idele $f \in \mathbb{A}_{K,f}^{\times}$. This idele is (a lift of) the Taniyama element — a particular preimage, under the Artin map, of Tate's half transfer. Here we follow the ideas of [Tat16, Prop.-Def.].

3.1.2 (Taniyama element)**.** Let (K, Φ) be as in (3.1.1), and let $\gamma \in \Gamma_{\mathbb{Q}}$. Look at the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \ker(r_K) \arrow[r] \arrow[d, "1+c"] & \mathbb{A}_{K,f}^{\times}/K^{\times} \arrow[r, "r_K"] \arrow[d, "1+c"] & \Gamma_K^{\mathrm{ab}} \arrow[r] \arrow[d, "1+c"] & 0 \\ 0 \arrow[r] & \ker(r_K) \arrow[r] & \mathbb{A}_{K,f}^{\times}/K^{\times} \arrow[r, "r_K"] & \Gamma_K^{\mathrm{ab}} \arrow[r] & 0. \end{tikzcd} \tag{3.1.2.1}$$

Here, $1 + c$ denotes the map that multiplies an element with its conjugate under c. Moreover, c acts by conjugation on Γ_K^{ab}. By [Tat16, Lemma 1], $\ker(r_K)$ is uniquely divisible and complex conjugation c acts trivially on it, so the left vertical arrow is an isomorphism. By an easy diagram chase, this means the right hand square is a pullback square.

Using the action of c by conjugation on Γ_K^{ab}, choosing the coset representatives $w'_{\varphi} := w_{c\varphi}c$ to calculate $F_{c\Phi}(\gamma)$ (see 3.1.1.a), then using (3.1.1.c) and finally class field theory (2.4.3) and (2.4.2), we see that

$$^{1+c}F_{\Phi}(\gamma) = F_{\Phi}(\gamma)(cF_{\Phi}(\gamma)c^{-1}) = F_{\Phi}(\gamma)F_{c\Phi}(\gamma) = V_{K/\mathbb{Q}}(\gamma) = r_K(\chi_{\mathrm{cyc}}(\gamma)).$$

Since the right square in (3.1.2.1) is Cartesian, there exists a unique $f_{\Phi}(\gamma) \in \mathbb{A}_{K,f}^{\times}/K^{\times}$ such that

(3.1.2.a) $r_K(f_{\Phi}(\gamma)) = F_{\Phi}(\gamma)$, and

(3.1.2.b) $^{1+c}f_{\Phi}(\gamma) = \chi_{\mathrm{cyc}}(\gamma)K^{\times}$.

The map $f_{\Phi}\colon \Gamma_{\mathbb{Q}} \to \mathbb{A}_{K,f}^{\times}/K^{\times}$ is called the *Taniyama element* attached to (K, Φ). The cocycle relation for Tate's half transfer translates to

$$f_{\Phi}(\gamma\gamma') = f_{\gamma'\Phi}(\gamma)f_{\Phi}(\gamma'), \quad \gamma, \gamma' \in \Gamma_{\mathbb{Q}}.$$

The relation between the Taniyama element and the reflex norm is given by [Mil07, Prop. 4.9]: If $\gamma \in \Gamma_E$, where E is the reflex field of (K, Φ), then for any $u \in \mathbb{A}_{E,f}^{\times}/E^{\times}$ such that $r_E(u) = \gamma|_{E^{\mathrm{ab}}}$ we have

$$f_{\Phi}(\gamma) = N_{\Phi}(u)K^{\times}.$$

We are now ready to state the Main Theorem. Again, we will present two versions of it, one in terms of torsion points and one in terms of Tate modules. For a proof:

[Tat16] proves the theorem up to a sequence of signs, and the proof was completed as a consequence of [Del82], see [Del82, Rem. 4, p. 263]. The details can be found in [Mil07, §4].

3.1.3 Theorem (Main Theorem of CM over $\mathbb{Q}$ — I)**.** *Let (A, i, s) be a triple consisting of a CM abelian variety (A, i) of type (K, Φ) and a compatible polarisation s. Assume (A, i, s) has type $(K, \Phi; \mathfrak{a}, t)$ relative to a uniformization Θ. Let $\gamma \in \Gamma_{\mathbb{Q}}$ and take $f \in \mathbb{A}_{K,f}^{\times}$ such that $f_{\Phi}(\gamma) = fK^{\times}$. Then:*

(3.1.3.a) The triple $({}^{\gamma}A, {}^{\gamma}i, {}^{\gamma}s)$ has type

$$\left(K, \gamma\Phi; f\mathfrak{a}, t\frac{\chi_{\mathrm{cyc}}(\gamma)}{f \cdot \overline{f}}\right)$$

relative to some uniformization Θ'.

(3.1.3.b) Moreover, one can choose Θ' uniquely such that the following diagram is commutative:

$$\begin{array}{ccc} K/\mathfrak{a} & \xrightarrow{\Theta} & A_{\mathrm{tors}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \gamma} \\ K/f\mathfrak{a} & \xrightarrow[\Theta']{} & ({}^{\gamma}A)_{\mathrm{tors}}. \end{array}$$

3.1.4 Remark. By condition (3.1.2.b) we have $\frac{\chi_{\mathrm{cyc}}(\gamma)}{f \cdot \overline{f}} \in K^{\times}$, and this element is fixed by complex conjugation, hence even lies in $F^{\times}$. Hence the quadruple $\left(K, \gamma\Phi; f\mathfrak{a}, t\frac{\chi_{\mathrm{cyc}}(\gamma)}{f \cdot \overline{f}}\right)$ in (3.1.3.a) makes sense.

Now, part of the assertion of (3.1.3.a) is that

$$\operatorname{Im} \varphi\left(t\frac{\chi_{\mathrm{cyc}}(\gamma)}{f \cdot \overline{f}}\right) > 0, \quad \text{for all } \varphi \in \gamma\Phi, \tag{3.1.4.1}$$

so that the quadruple satisfies the last condition in (2.3.5). We will prove (3.1.4.1) directly and in more generality in (4.2.9).

In terms of Tate modules, this translates to:

3.1.5 Theorem (Main Theorem of CM over $\mathbb{Q}$ — II)**.** *Let (A, i, s) be a triple consisting of a CM abelian variety (A, i) of type (K, Φ) and a compatible polarisation s. Let $\gamma \in \Gamma_{\mathbb{Q}}$ and take $f \in \mathbb{A}_{K,f}^{\times}$ such that $f_{\Phi}(\gamma) = fK^{\times}$. Then:*

(3.1.5.a) $({}^{\gamma}A, {}^{\gamma}i)$ is of type $(K, \gamma\Phi)$.

(3.1.5.b) There exists a unique K-linear isomorphism[1] $\delta\colon H_1(A, \mathbb{Q}) \xrightarrow{\sim} H_1({}^{\gamma}A, \mathbb{Q})$ such that

$$s\left(\frac{\chi_{\mathrm{cyc}}(\gamma)}{f \cdot \overline{f}}x, y\right) = ({}^{\gamma}s)\,(\delta x, \delta y), \quad x, y \in H_1(A, \mathbb{Q}),$$

[1]Note that if $\gamma\Phi \neq \Phi$, then δ cannot be an isomorphism of $\mathbb{Q}$-Hodge structures. So in general δ is only an isomorphism of K-vector spaces.

and the following diagram commutes:

$$\begin{tikzcd} \widehat{V}(A) \arrow[r, "f"] \arrow[dr, "\gamma"'] & \widehat{V}(A) \arrow[d, "\delta"] \\ & \widehat{V}({}^{\gamma}A). \end{tikzcd}$$

Sketch of proof that Theorems (3.1.3) *and* (3.1.5) *are equivalent.* This is similar to the equivalence of (2.4.7) and (2.4.10). Namely, we again view a uniformization Θ as a K-linear isomorphism of $\mathbb{Q}$-Hodge structures $(K, h_\Phi) \xrightarrow{\sim} (H_1(A, \mathbb{Q}), h_A)$. Then the map δ of (3.1.5.b) is related to the uniformizations in (3.1.3) by

$$\delta = \Theta' \circ \Theta^{-1}.$$

Moreover, the commutative diagrams in (3.1.3.b) and (3.1.5.b) are related by (2.4.6). $\square$

3.2 PEL Hilbert modular variety and Galois action on its CM points

In (2.5.8), we saw that the Shimura variety $\mathrm{Sh}(T, \{h_\Phi\})$ parametrises CM abelian varieties of a fixed type (K, Φ). Moreover, in (2.5.10) we then used the Main Theorem over the reflex field E of (K, Φ) to describe the Γ_E-action on $\mathrm{Sh}(T, \{h_\Phi\})$ in terms of the reflex norm. It was essential to only look at Galois elements fixing E, because otherwise, by (2.3.8), the conjugate abelian variety induces a CM type on K that is different from Φ, and therefore does not define a point of $\mathrm{Sh}(T, \{h_\Phi\})$.

From now on, fix a totally real number field F. In this section, we introduce the Hilbert modular variety and the PEL Hilbert modular variety. The latter parametrises abelian varieties with real multiplication by a totally real field F equipped with polarisation and level structure. For any totally imaginary quadratic extension K of F, the PEL Hilbert modular variety contains the points corresponding to abelian varieties with CM by K — of any CM type — as special points. It is the goal of this section to understand the action of $\Gamma_{\mathbb{Q}}$ on these special points using the Main Theorem over $\mathbb{Q}$.

We denote the two-dimensional F-vector space F^2 by V.

3.2.1 (Hilbert modular variety)**.** Let G_1 be the $\mathbb{Q}$-algebraic group given by $G_1 :=$

$R_{F/\mathbb{Q}} \mathrm{GL}_2 = R_{F/\mathbb{Q}} \mathrm{GL}_F(V)$. Define $h_0\colon \mathbb{S} \to (G_1)_{\mathbb{R}}$ by

$$h_0(i) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G_1(\mathbb{R}) = \mathrm{GL}_{F\otimes_{\mathbb{Q}}\mathbb{R}}((F \otimes_{\mathbb{Q}} \mathbb{R})^2),$$

and let X_1 be the $G_1(\mathbb{R})$-conjugacy class of h_0. To describe X_1 more explicitly, we identify $V(\mathbb{R}) = (F^2) \otimes_{\mathbb{Q}} \mathbb{R}$ with $\prod_{x\in\Sigma_F} \mathbb{R}^2$, using $F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{x\in\Sigma_F} \mathbb{R}$, where $\Sigma_F = \mathrm{Hom}(F, \mathbb{C})$. Since the totally real field F will always be fixed, we will very often drop the F from notation and simply write Σ for Σ_F.

Under this identification, we get $G_1(\mathbb{R}) = \prod_{x\in\Sigma_F} \mathrm{GL}_2(\mathbb{R})$, so we write elements $g \in G_1(\mathbb{R})$ as $g = (g_x)_{x\in\Sigma_F}$ with $g_x \in \mathrm{GL}_2(\mathbb{R})$. For example, the element $h_0(i)$ corresponds to

$$h_0(i) = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)_{x\in\Sigma_F} \in G_1(\mathbb{R}) = \prod_{x\in\Sigma_F} \mathrm{GL}_2(\mathbb{R}). \tag{3.2.1.1}$$

If we let $\prod_{x\in\Sigma_F} \mathrm{GL}_2(\mathbb{R})$ act on $(\mathbb{C} \setminus \mathbb{R})^{\Sigma_F}$ by componentwise Möbius transformations, then we may identify X_1 with $(\mathbb{C} \setminus \mathbb{R})^{\Sigma_F}$ by

$$\begin{aligned} X_1 &\xrightarrow{\sim} (\mathbb{C} \setminus \mathbb{R})^{\Sigma_F} \\ gh_0g^{-1} &\longmapsto (g_x \cdot i)_{x\in\Sigma_F}, \quad g = (g_x)_{x\in\Sigma_F} \in G_1(\mathbb{R}), \end{aligned} \tag{3.2.1.2}$$

compare [vdG88, Ch. I.7]. The pair (G_1, X_1) is a Shimura datum, i. e. it satisfies the axioms [Del79, (2.1.1.1-3)]. The associated Shimura variety $\mathrm{Sh}(G_1, X_1)$ is called the *Hilbert modular variety* (associated to F).

In order to get a PEL Shimura datum, we equip V with the $\mathbb{Q}$-bilinear alternating form $\psi\colon V \times V \to \mathbb{Q}$ given by

$$\psi\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \mathrm{Tr}_{F/\mathbb{Q}} \circ \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}.$$

The form ψ is F-compatible, i. e. $\psi(fv, w) = \psi(v, fw)$ for all $v, w \in V$ and $f \in F$.

3.2.2 (PEL Hilbert modular variety)**.** Associated to the PEL datum (of type (C)) consisting of the simple $\mathbb{Q}$-algebra F, with trivial involution, acting on the $\mathbb{Q}$-vector space V equipped with the alternating, $\mathbb{Q}$-bilinear, F-compatible form ψ, we get a PEL Shimura datum (G, X) as in [Mil17, Def. 8.15], [Del71, 4.9]. We call the associated Shimura variety $\mathrm{Sh}(G, X)$ the *PEL Hilbert modular variety.*

We can describe the group $G \subset G_1$ explicitly as having $\mathbb{Q}$-points

$$G(\mathbb{Q}) = \{g \in \mathrm{GL}_2(F) \mid \det(g) \in \mathbb{Q}^\times\}.$$

The proof of this is an easy calculation and is done, in slightly more generality, in (4.4.4). In other words, the diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d] & R_{F/\mathbb{Q}}\,\mathrm{GL}_2 \arrow[d, "R_{F/\mathbb{Q}}(\det)"] \\ \mathbb{G}_m \arrow[r, hook] & R_{F/\mathbb{Q}}\mathbb{G}_m \end{tikzcd} \tag{3.2.2.1}$$

is Cartesian. This is precisely diagram (1.2.0.1). On $\mathbb{Q}$-points, the bottom arrow is given by the inclusion $\mathbb{Q}^\times \hookrightarrow F^\times$ and the right hand arrow is given by the usual determinant $\det\colon \mathrm{GL}_2(F) \to F^\times$. Note the similarity to (2.5.1.1).

On real points, this means

$$G(\mathbb{R}) = \left\{ g = (g_x)_{x\in\Sigma_F} \in G_1(\mathbb{R}) = \prod_{x\in\Sigma_F} \mathrm{GL}_2(\mathbb{R}) \;\middle|\; \det(g_x) = \det(g_{x'})\ \forall x, x' \in \Sigma \right\}.$$

We can also describe X more explicitly: Let $h_0\colon \mathbb{S} \to (G_1)_{\mathbb{R}}$ be as in (3.2.1). First of all, by (3.2.1.1) h_0 lands in $G_{\mathbb{R}} \subset (G_1)_{\mathbb{R}}$. Moreover, the form $(v,w) \mapsto \psi(v, h_0(i)w)$ is positive definite on $V(\mathbb{R})$, hence h_0 lies in X. This means that X is the $G(\mathbb{R})$-conjugacy class of h_0, and it is easy to see that the bijection (3.2.1.2) restricts to

$$\begin{aligned} X &\xrightarrow{\sim} \mathfrak{h}^{\Sigma_F} \sqcup (-\mathfrak{h})^{\Sigma_F} \\ g h_0 g^{-1} &\longmapsto (g_x \cdot i)_{x\in\Sigma_F}, \quad g = (g_x)_{x\in\Sigma_F} \in G(\mathbb{R}), \end{aligned} \tag{3.2.2.2}$$

where $\mathfrak{h}$ denotes the upper half plane in $\mathbb{C}$. For a proof in a slightly more general setting, we refer to (5.1.7).

3.2.3 (Reflex field of $\mathrm{Sh}(G,X)$)**.** In (6.1.11) we define the reflex field $E(G,X)$ of a Shimura datum (G,X). By [Mil17, 12.4(c)], in the case of a PEL Shimura datum the reflex field $E(G,X)$ is given by $\mathbb{Q}(\mathrm{Tr}_X(b) \mid b \in F^\times)$, where $\mathrm{Tr}_X(b)$ denotes the trace of $b \in F^\times$ acting on $V^{-1,0}$, where $V = V^{-1,0} \oplus V^{0,-1}$ is the Hodge decomposition associated to h (compare (2.2.4)).

In the case of the Shimura datum (G,X) from (3.2.2), we can take $h = h_0$, and then a direct calculation shows that $\mathrm{Tr}_X(b) = \mathrm{Tr}_{F/\mathbb{Q}}(b)$ for all $b \in F^\times$. Hence $E(G,X) = \mathbb{Q}$. Moreover, Definition (6.1.11) only depends on the adjoint group of G. By (6.2.9), the adjoint group of G and G_1 are the same, so $E(G_1,X_1) = \mathbb{Q}$ too.

Similar to (2.5.8), the complex points of $\mathrm{Sh}(G,X)$ can be interpreted as isomorphism classes of abelian varieties. The following theorem is a special case of [Mil17, Thm 8.17]. We prove a slightly different, but more general version in (4.4.5).

3.2.4 Theorem ((G,X) and abelian varieties with real multiplication)**.** *The coset* *space*

$$G(\mathbb{Q})\backslash[X \times G(\mathbb{A}_f)]$$

is in bijection with the set of isomorphism classes of quadruples $(A, i, \mathbb{Q}^\times s, \eta)$*, where*

- A *is a complex abelian variety,*
- $i\colon F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is a ring homomorphism,*
- s *is an* F*-compatible polarisation of* A*, and*
- $\eta\colon V(\mathbb{A}_f) \xrightarrow{\sim} \widehat{V}(A)$ *is an* $\mathbb{A}_{F,f}$*-module-isomorphism sending* $\mathbb{A}_f^\times \psi$ *to* $\mathbb{A}_f^\times s$*,*

satisfying the condition

(3.2.4.a) There exists an F*-linear isomorphism* $a\colon H_1(A, \mathbb{Q}) \xrightarrow{\sim} V$ *that sends* $\mathbb{Q}^\times s$ *to* $\mathbb{Q}^\times \psi$ *and satisfies* $a \circ h_A \circ a^{-1} \in X$*.*

Moreover, this bijection is equivariant for the right $G(\mathbb{A}_f)$*-actions.*

3.2.5 Remarks. To clarify the notation, let us make a few comments.

(3.2.5.a) If $i\colon F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ for an abelian variety A of dimension $[F : \mathbb{Q}]$ as in the theorem, then we say that (A, i) has *real multiplication* by F.

(3.2.5.b) A polarisation s of an abelian variety (A, i) with real multiplication by F is called *F-compatible* if $s(i(f)v, w) = s(v, i(f)w)$ for all $f \in F$ and $v, w \in H_1(A, \mathbb{Q})$.

(3.2.5.c) In (3.2.4.a) $h_A\colon \mathbb{S} \to \mathrm{End}(H_1(A, \mathbb{R}))$ denotes the Hodge structure on $H_1(A, \mathbb{Q})$ and $a \circ h_A \circ a^{-1}\colon \mathbb{S} \to \mathrm{GL}(V)_{\mathbb{R}}$ is given (on $\mathbb{R}$-points) by $z \mapsto a \circ h_A(z) \circ a^{-1}$.

(3.2.5.d) We call $(A, i, \mathbb{Q}^\times s, \eta)$ and $(A', i', \mathbb{Q}^\times s', \eta')$ isomorphic if there exists a quasi-isogeny $f\colon A \to A'$ that is F-linear (with respect to i and i'), sends $\mathbb{Q}^\times s$ to $\mathbb{Q}^\times s'$ and satisfies $\eta' = f \circ \eta$.

(3.2.5.e) By [Mil17, 8.19], condition (3.2.4.a) is equivalent to s being F-compatible and i satisfying the trace condition

- For any $b \in F^\times$, the trace of $i(b)$, acting on the tangent space of A at the origin, is equal to $\mathrm{Tr}_X(b) = \mathrm{Tr}_{F/\mathbb{Q}}(b)$.

By a calculation very similar to the one in (2.3.2), we see that this trace condition is automatically satisfied. It is nonetheless useful to mention condition (3.2.4.a) because the existence of an isomorphism a (although automatic) is the key for the proof of Theorem (3.2.4).

As in (2.5.8), Theorem (3.2.4) implies the following interpretation of the points of the Shimura variety $\mathrm{Sh}(G, X)$, see [Del71, 4.11]:

3.2.6 Corollary (Complex points of $\mathrm{Sh}(G, X)$)**.** *The complex points*

$$\mathrm{Sh}(G, X)(\mathbb{C}) := \varprojlim_{U} G(\mathbb{Q})\backslash [X \times G(\mathbb{A}_f)/U] = G(\mathbb{Q})\backslash \left[X \times G(\mathbb{A}_f)/\overline{Z(\mathbb{Q})}\right] \quad (3.2.6.1)$$

of the Shimura variety $\mathrm{Sh}(G, X)$ *are in bijection with isomorphism classes of quadruples* $(A, i, \mathbb{Q}^\times s, \eta\overline{Z(\mathbb{Q})})$, *where*

- A *is a complex abelian variety,*
- $i\colon F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is a ring homomorphism,*
- s *is an* F*-compatible polarization of* A, *and*
- $\eta\colon V(\mathbb{A}_f) \xrightarrow{\sim} \widehat{V}(A)$ *is an* $\mathbb{A}_{F,f}$*-module-isomorphism sending* $\mathbb{A}_f^\times \psi$ *to* $\mathbb{A}_f^\times s$,

satisfying the condition

(3.2.6.a) There exists an F*-linear isomorphism* $a\colon H_1(A, \mathbb{Q}) \xrightarrow{\sim} V$ *that sends* $\mathbb{Q}^\times s$ *to* $\mathbb{Q}^\times \psi$ *and satisfies* $a \circ h_A \circ a^{-1} \in X$.

Here $Z \subset G$ *denotes the* centre *of* G, *and* $\overline{Z(\mathbb{Q})}$ *denotes the closure of* $Z(\mathbb{Q})$ *inside* $G(\mathbb{A}_f)$ *(in the adelic topology). Moreover, the second equality in* (3.2.6.1) *is [Orr18, (1), p. 4].*

3.2.7 (Special points of $\mathrm{Sh}(G, X)$)**.** By definition, a *special point* of the Shimura variety $\mathrm{Sh}(G, X)$ is a point $[h, g] \in \mathrm{Sh}(G, X)$ such that the Mumford–Tate group of h is a torus, where we view $h\colon \mathbb{S} \to G_{\mathbb{R}}$ as a $\mathbb{Q}$-Hodge structure on V via $G \subset \mathrm{GL}_F(V) \subset \mathrm{GL}_{\mathbb{Q}}(V)$.

By [Mil17, 14.11][2], if the point $[h, g]$ corresponds to a quadruple $[A, i, \mathbb{Q}^\times s, \eta\overline{Z(\mathbb{Q})}]$ under the bijection in (3.2.6), then $[h, g]$ is special if and only if A has CM by a CM algebra (compare (2.3.1.c)). In this case, since h takes values in $G_{\mathbb{R}}$, it is in particular F-linear, so the CM structure on A must extend the real multiplication i by F. Hence A has CM by a CM field K that is a totally imaginary quadratic extension of F. We will often abuse notation and denote the associated embedding $K \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ also by i.

3.2.8 (Canonical model of $\mathrm{Sh}(G, X)$)**.** In order to get a canonical model for the Shimura variety $\mathrm{Sh}(G, X)$, one follows the strategy outlined in [Mil17, p. 126], mimicking the Siegel case presented in [Mil17, §14]. Namely, the canonical model is the

[2][Mil17, 14.11] is actually the analogous statement for points of the Siegel variety. But the condition on h being special (resp. A having CM) only depends on the Mumford-Tate group of h (resp. A), hence the same argument works.

(pro-)variety M^+ over $\mathbb{Q}$ representing the moduli functor, modelled on Corollary (3.2.6), of abelian varieties with real multiplication by F equipped with polarisation and level structure. See also [Del71, 4.15, 5.8] or [Kot92, §8].

Corollary (3.2.6) tells us that M^+ is a model for the Shimura variety $\mathrm{Sh}(G,X)$. By (3.2.7), its special points correspond to CM abelian varieties. To show that this defines a canonical model is then similar to the calculation in (2.5.10) and relies only on the Main Theorem over the reflex field.

3.2.9 (Galois action on special points of $\mathrm{Sh}(G,X)$)**.** Let $\mathcal{P} = [A, i, \mathbb{Q}^\times s, \eta\overline{Z(\mathbb{Q})}]$ be a special point of $\mathrm{Sh}(G,X)$ as in (3.2.7). Then A has CM, so in particular can be defined over $\overline{\mathbb{Q}}$. Let $\gamma \in \Gamma_\mathbb{Q}$.

By (3.2.8), the γ-conjugate of the given point corresponds to the quadruple

$$\gamma[A, i, \mathbb{Q}^\times s, \eta\overline{Z(\mathbb{Q})}] = [{}^\gamma\! A, {}^\gamma i, \mathbb{Q}^\times({}^\gamma s), (\gamma\circ\eta)\overline{Z(\mathbb{Q})}].$$

The abelian variety (A,i) has CM, say of type (K,Φ) for some totally imaginary quadratic extension K of F. Note that by Lemma (2.3.6) the polarisation s is automatically K-compatible. We observe that the point $\gamma\mathcal{P}$ for arbitrary $\gamma \in \Gamma_\mathbb{Q}$ is a special point too, because $({}^\gamma\! A, {}^\gamma i)$ is a CM abelian variety (of type $(K, \gamma\Phi)$). Moreover, we can describe it using the Main Theorem over $\mathbb{Q}$: Take $f \in \mathbb{A}_{K,f}^\times$ such that $f_\Phi(\gamma) = fK^\times$. Using the notation in (3.1.5), we can write

$$[{}^\gamma\! A, {}^\gamma i, \mathbb{Q}^\times({}^\gamma s), \gamma\circ\eta] = \left[{}^\gamma\! A, {}^\gamma i, \mathbb{Q}^\times s\left(\frac{\chi_{\mathrm{cyc}}(\gamma)}{f\cdot\overline{f}}\delta^{-1}(\cdot), \delta^{-1}(\cdot)\right), \delta\circ f\circ\eta\right].$$

We can give a similar description in terms of uniformizations: Using the notation of Theorem (3.1.3), we have

$$[A, i, \mathbb{Q}^\times s, \eta] = [\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, \mathbb{Q}^\times E_t, \Theta^{-1}\circ\eta], \tag{3.2.9.1}$$

where $i_\Phi\colon K \to \mathrm{End}(\mathbb{C}^\Phi/\Phi(\mathfrak{a}))$ is given by sending $k \in K$ to the map on $\mathbb{C}^\Phi/\Phi(\mathfrak{a})$ given by multiplication by $(\varphi(k))_{\varphi\in\Phi}$. For the conjugate quadruple, we get

$$\begin{aligned}[{}^\gamma\! A, {}^\gamma i, \mathbb{Q}^\times({}^\gamma s), \gamma\circ\eta] &= [\mathbb{C}^{\gamma\Phi}/\gamma\Phi(f\mathfrak{a}), i_{\gamma\Phi}|_F, \mathbb{Q}^\times E_{\chi t}, (\Theta')^{-1}\circ\gamma\circ\eta] \\ &= [\mathbb{C}^{\gamma\Phi}/\gamma\Phi(f\mathfrak{a}), i_{\gamma\Phi}|_F, \mathbb{Q}^\times E_{\chi t}, f\circ\Theta^{-1}\circ\eta], \end{aligned} \tag{3.2.9.2}$$

where $\chi = \frac{\chi_{\mathrm{cyc}}(\gamma)}{f\cdot\overline{f}} \in F^\times$. Here the first equality follows from (3.1.3.a) and the second equality follows from (3.1.3.b).

3.2.10 Remark (Galois conjugates of special points)**.** It is actually true in full generality that conjugates of special points of an arbitrary Shimura variety by Galois elements that fix its reflex field are again special points, see [Orr18, Thm 5.1].

But here we additionally need the Main Theorem over $\mathbb{Q}$ to get the explicit description in (3.2.9) of the conjugate special point on the PEL Hilbert modular variety in terms of CM abelian varieties.

4

Plectic Galois action on CM points

Fix a totally real number field F. In this chapter, we introduce the plectic Galois group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ associated to F. For every algebraic torus R defined over $\mathbb{Q}$ with $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$, we then show that a certain subgroup $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$ of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ acts naturally on the CM points of the Shimura variety $\mathrm{Sh}(G^R, X^R)$. Here the Shimura variety $\mathrm{Sh}(G^R, X^R)$ is a variant of the Hilbert modular variety. It is associated to the group G^R that fits into the Cartesian diagram (1.2.0.3)

$$\begin{tikzcd} G^R \arrow[r, hook] \arrow[d] & R_{F/\mathbb{Q}}\,\mathrm{GL}_2 \arrow[d, "R_{F/\mathbb{Q}}(\det)"] \\ R \arrow[r, hook] & R_{F/\mathbb{Q}}\mathbb{G}_m. \end{tikzcd}$$

The key idea to define this action of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$ on the CM points is to extend Tate's half transfer $F_\Phi \colon \Gamma_{\mathbb{Q}} \to \Gamma_K^{\mathrm{ab}}$ to a plectic half transfer

$$\widetilde{F}_\Phi \colon \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \longrightarrow \Gamma_K^{\mathrm{ab}}.$$

The goal of Section 4.1 is to familiarise the reader with the properties of the plectic group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$. The absolute Galois group of $\mathbb{Q}$ embeds into the plectic group via

$$\Gamma_{\mathbb{Q}} \hookrightarrow \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}), \quad \gamma \mapsto \mathrm{id} \otimes \gamma.$$

This definition does not involve any choices. However, it is useful to identify $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ with the semi-direct product $S_\Sigma \ltimes \Gamma_F^\Sigma$ mentioned in Section 1.1.2, where $\Sigma := \mathrm{Hom}(F, \overline{\mathbb{Q}}) = \Gamma_{\mathbb{Q}}/\Gamma_F$. Namely, a choice of coset representatives $s = (s_x)_{x \in \Sigma}$ for the right Γ_F-cosets in $\Gamma_{\mathbb{Q}}$ defines an isomorphism $\beta_s \colon \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \xrightarrow{\sim} S_\Sigma \ltimes \Gamma_F^\Sigma$. We finish Section 4.1 by recalling in (4.1.11) the definition of the plectic half transfer $\widetilde{F}_\Phi$, where (K, Φ) denotes a CM pair.

Section 4.2 starts with some remarks about class field theory before defining, for an element γ of a certain subgroup $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$ of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$, the plectic Taniyama element $\widetilde{f}_\Phi(\gamma) \in \mathbb{A}_{K,f}^\times/K^\times$ as a suitably normalised class-field-theoretic preimage of $\widetilde{F}_\Phi(\gamma)$. Inspired by formula (3.2.9) for the $\Gamma_{\mathbb{Q}}$-action, we then show in (4.2.10) how the subgroup $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$ of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$ naturally acts on the set of CM points of the PEL Hilbert modular variety.

Sections 4.1 and 4.2 are a review of the results in [Nek09], most of which are presented without proof. We present them in enough detail to generalise the results to Shimura varieties whose groups differ only in the centre from $R_{F/\mathbb{Q}}\,\mathrm{GL}_2$, i. e. are of the form G^R as above.

In Section 4.3 we extend the plectic Taniyama element $\widetilde{f}_\Phi$ to the entire plectic group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$. To do so, we choose a splitting $\chi_F \colon \Gamma_F^{\mathrm{ab}} \to \mathbb{A}_{F,f}^\times/F_{>0}^\times$ to the reciprocity map $r_F \colon \mathbb{A}_{F,f}^\times/F_{>0}^\times \to \Gamma_F^{\mathrm{ab}}$, and then proceed in analogy to Tate's strategy (3.1.2), see (4.3.6).

In Section 4.4 we study the Shimura varieties $\mathrm{Sh}(G^R, X^R)$. In (4.4.5) we prove a moduli interpretation of the points of $\mathrm{Sh}(G^R, X^R)$ in terms of isomorphism classes of abelian varieties with real multiplication by F equipped with an $R(\mathbb{Q})$-class of a polarisation and a level structure. We define the subgroup $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$ of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ in (4.4.11). Finally, we show in (4.4.13) that the group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$ acts naturally on the CM points of $\mathrm{Sh}(G^R, X^R)$, extending the action of $\Gamma_{\mathbb{Q}}$.

4.1 Plectic Galois group and plectic half transfer

4.1.1. (Plectic Galois group) The *(F-)plectic Galois group* is the group

$$\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$$

of F-algebra automorphisms of $F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. The absolute Galois group of $\mathbb{Q}$ embeds into the plectic Galois group via

$$\Gamma_{\mathbb{Q}} \hookrightarrow \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}), \quad \gamma \mapsto \mathrm{id}_F \otimes \gamma.$$

It is useful to view $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ more concretely as a certain semi-direct product:

4.1.2. (Semi-direct product) Fix an embedding $F \subset \overline{\mathbb{Q}}$ and identify $\Sigma := \Sigma_F = \mathrm{Hom}(F, \overline{\mathbb{Q}})$ with the quotient $\Gamma_{\mathbb{Q}}/\Gamma_F$. Let S_Σ be the symmetric group on the finite set Σ. Let Γ_F^Σ denote the group of Σ-tuples $h = (h_x)_{x\in\Sigma}$ of elements of Γ_F, with the group structure given by pointwise composition.

We introduce the semi-direct product

$$S_\Sigma \ltimes \Gamma_F^\Sigma,$$

where the group operation is given by

$$(\sigma, h)(\sigma', h') := \left(\sigma\sigma', (h_{\sigma'(x)}h'_x)_{x\in\Sigma}\right).$$

Let $s\colon x \mapsto s_x$ be a section to the map $\Gamma_{\mathbb{Q}} \twoheadrightarrow \Gamma_{\mathbb{Q}}/\Gamma_F = \Sigma$. Then we can define an injective group homomorphism

$$\rho_s\colon \Gamma_{\mathbb{Q}} \hookrightarrow S_\Sigma \ltimes \Gamma_F^\Sigma, \quad \rho_s(\gamma) = (\sigma, h),$$

where σ is given by $\sigma\colon x \mapsto \gamma x$ and $h = (h_x)_{x\in\Sigma}$ by $h_x = s_{\gamma x}^{-1}\gamma s_x$. For more details, e. g. on how ρ_s depends on the chosen section s, see [Nek09, (1.1.2)].

4.1.3. (Isomorphism between $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ and $S_\Sigma \ltimes \Gamma_F^\Sigma$) We continue (4.1.2). Let $\overline{\mathbb{Q}}^\Sigma$ denote the ring of Σ-tuples of elements of $\overline{\mathbb{Q}}$. We view $\overline{\mathbb{Q}}^\Sigma$ as an F-algebra via the fixed embedding $F \subset \overline{\mathbb{Q}}$ in each component. By [Nek09, (1.1.3)] the canonical map

$$\begin{aligned} S_\Sigma \ltimes \Gamma_F^\Sigma &\xrightarrow{\sim} \mathrm{Aut}_F\left(\overline{\mathbb{Q}}^\Sigma\right), \\ (\sigma, h) &\longmapsto \left[(q_x)_{x\in\Sigma} \mapsto \left(h_{\sigma^{-1}(x)}(q_{\sigma^{-1}(x)})\right)_{x\in\Sigma}\right], \end{aligned}$$

is a group isomorphism. We will simply identify these two groups.

The section $s\colon \Sigma \to \Gamma_{\mathbb{Q}}$ induces an isomorphism of F-algebras

$$\begin{aligned} \overline{\mathbb{Q}}^\Sigma &\xrightarrow{\sim} \prod_{x\in\Sigma} \overline{\mathbb{Q}}_x = F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \\ (q_x)_{x\in\Sigma} &\longmapsto (s_x(q_x))_{x\in\Sigma}, \end{aligned}$$

where we view $\overline{\mathbb{Q}}_x$ as an F-algebra via the embedding $x \in \Sigma = \mathrm{Hom}(F, \overline{\mathbb{Q}})$. Combining these two observations (see [Nek09, (1.1.4)]), we get an isomorphism

$$\beta_s\colon\ \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{Aut}_F(\overline{\mathbb{Q}}^\Sigma) = S_\Sigma \ltimes \Gamma_F^\Sigma$$

between the plectic Galois group (4.1.1) and the semi-direct product (4.1.2). Moreover, the isomorphism β_s is compatible with the embeddings of $\Gamma_{\mathbb{Q}}$, i. e. we have the commutative diagram

$$\begin{tikzcd} \Gamma_{\mathbb{Q}} \arrow[r, hook] \arrow[dr, "\rho_s"'] & \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \arrow[d, "\beta_s"] \\ & S_\Sigma \ltimes \Gamma_F^\Sigma. \end{tikzcd}$$

4.1.4 Remark. (Plectic groups and induced representations) The semi-direct product $S_\Sigma \ltimes \Gamma_F^\Sigma$ is a purely group-theoretical construction. Replacing $\Gamma_{\mathbb{Q}}$ by an arbitrary group Γ and the subgroup Γ_F of $\Gamma_{\mathbb{Q}}$ by a finite index subgroup Δ of Γ, and denoting $\Sigma := \Gamma/\Delta$, we can define the group $S_\Sigma \ltimes \Delta^\Sigma$ in the same way.

Similarly, a section $s\colon \Sigma \to \Gamma$ will induce an embedding $\rho_s\colon \Gamma \hookrightarrow S_\Sigma \ltimes \Delta^\Sigma$. Now (compare [NS16, §3] and [Nek09, (1.1.1.1)]), if V is a representation of Δ, then the induced representation is the vector space

$$\mathrm{Ind}_\Delta^\Gamma(V) = \bigoplus_{x\in\Sigma} s_x.V, \tag{4.1.4.1}$$

where the right hand side consists of formal sums (or Σ-tuples) of elements of V, and the s_x are only introduced for notational purposes to denote the summand corresponding to x. It comes equipped with a Γ-action given as follows: If $\gamma \in \Gamma$ and $\sum_{x\in\Sigma} s_x.v_x \in \mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(V)$, and $\rho_s(\gamma) = (\sigma, h)$, then

$$\gamma\left(\sum_{x\in\Sigma} s_x.v_x\right) := \sum_{x\in\Sigma} s_{\sigma(x)}.h_x(v_x). \tag{4.1.4.2}$$

This formula clearly makes sense for arbitrary $(\sigma, h) \in S_\Sigma \ltimes \Delta^\Sigma$, and it is straightforward to check that this defines a representation of $S_\Sigma \ltimes \Delta^\Sigma$. Similar remarks apply for the tensor induction of V and other related constructions, see e. g. (4.1.6) below.

Induced representations are one instance where plectic groups arise naturally. Of course, the action of Γ on the induced module is canonical, i. e. given a different section s' there is a canonical isomorphism of Γ-representations between $\bigoplus_{x\in\Sigma} s_x.V$ and $\bigoplus_{x\in\Sigma} s'_x.V$. In other words, there is a description of $\mathrm{Ind}_\Delta^\Gamma(V)$ that is independent of s, so it is desirable to also find a description of $\rho_s\colon \Gamma \hookrightarrow S_\Sigma \ltimes \Delta^\Sigma$ that is independent of s.

4.1.5 (Another version of the plectic group)**.** For general groups Γ and Δ, look at the group (see [NS16, §3] and [Bla15, (3.2.1)])

$$\Gamma\#\Delta := \mathrm{Aut}_{\mathrm{Set}-\Delta}(\Gamma)$$

of right Δ-equivariant bijections of the set Γ with itself.

The group Γ embeds into $\Gamma\#\Delta$ as the subgroup of left translations, i. e. $\gamma \in \Gamma$ is sent to the bijection $g \mapsto \gamma g$, which clearly is right Δ-equivariant. Now [Bla15, (3.2.2)], a section $s\colon \Sigma \to \Gamma$ induces a group isomorphism

$$\pi_s\colon \Gamma\#\Delta \xrightarrow{\sim} S_\Sigma \ltimes \Delta^\Sigma \tag{4.1.5.1}$$

that is compatible with the respective embeddings of Γ, i. e. we have the commutative diagram

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[dr, hook, "\rho_s"'] & \Gamma\#\Delta \arrow[d, "\pi_s"] \\ & S_\Sigma \ltimes \Delta^\Sigma. \end{tikzcd}$$

Moreover, $\Gamma\#\Delta$ acts canonically on the intrinsically defined induction $\mathrm{Ind}_\Delta^\Gamma(V)$, and the isomorphim (4.1.4.1) is compatible with the actions of $\Gamma\#\Delta$ and $S_\Sigma \ltimes \Delta^\Sigma$ under π_s.

4.1.6 Example. (Plectic action on CM types — I) As a concrete example of (4.1.4), let K be a totally imaginary quadratic extension of F, and choose an embedding $K \subset \overline{\mathbb{Q}}$. We continue to use the notation of (4.1.2). The section $s\colon \Sigma = \Sigma_F \to \Gamma_\mathbb{Q}$ determines a CM type $\{\varphi_x \mid x \in \Sigma\}$ of K by $\varphi_x := s_x|_K$. It allows us to write

$$\Sigma_K = \{c^b\varphi_x \mid x \in \Sigma_F, b \in \mathbb{Z}/2\mathbb{Z}\}. \tag{4.1.6.1}$$

We claim that the set Σ_K with its usual $\Gamma_\mathbb{Q}$-action is isomorphic to the induced module (in the category of sets with $\Gamma_\mathbb{Q}$-action rather than the category of $\Gamma_\mathbb{Q}$-representations over some field)

$$\mathrm{Ind}_{\Gamma_F}^{\Gamma_\mathbb{Q}}(\{1, c\}) = \bigsqcup_{x\in\Sigma} s_x.\{1, c\}$$

by sending $c^b\varphi_x = c^b s_x|_K = (s_x c^b)|_K \in \Sigma_K$ to $s_x.c^b \in \mathrm{Ind}_{\Gamma_F}^{\Gamma_\mathbb{Q}}(\{1, c\})$. Here $\{1, c\}$ is a two-element set equipped with the (left) Γ_F-action given by the map $\Gamma_F \twoheadrightarrow \mathrm{Gal}(K/F) = \langle c\rangle = \{1, c\}$, which we will denote by $h \mapsto c^{\overline{h}}$ with $\overline{h} \in \mathbb{Z}/2\mathbb{Z}$.

Proof of claim. The map $c^b\varphi_x \mapsto s_x.c^b$ is clearly a bijection $\Sigma_K \to \mathrm{Ind}_{\Gamma_F}^{\Gamma_\mathbb{Q}}(\{1, c\})$, and for $\gamma \in \Gamma_\mathbb{Q}$ and $\rho_s(\gamma) = (\sigma, h)$ we have

$$\gamma(c^b\varphi_x) = \big[\gamma s_x c^b\big]\big|_K = s_{\gamma x}(s_{\gamma x}^{-1}\gamma s_x)c^b\big|_K = s_{\sigma(x)}h_x c^b\big|_K = s_{\sigma(x)}c^{b+\overline{h}_x}\Big|_K,$$

so

$$\gamma(c^b\varphi_x) \mapsto s_{\sigma(x)}.c^{b+\overline{h}_x} = \gamma(s_x.c^b),$$

where the last equality holds by Definition (4.1.4.2) (without the formal sum). □

We conclude that we can define an action of $S_\Sigma \ltimes \Gamma_F^\Sigma$ on Σ_K by

$$(\sigma, h)(c^b\varphi_x) := c^{b+\overline{h}_x}\varphi_{\sigma(x)}. \tag{4.1.6.2}$$

Clearly this action factors through the finite group $S_\Sigma \ltimes \mathrm{Gal}(K/F)^\Sigma$. Since σ is a permutation of Σ, this action induces an action of $S_\Sigma \ltimes \Gamma_F^\Sigma$ on the set of CM types of K, which depends on the choice of s because the identification (4.1.6.1) does — but see (4.1.8.a) for a result that is independent of s.

4.1.7 Example. Let us give a concrete example of the plectic action on CM types by having another look at the CM types $\Phi = \{g^0, g^4, g^2\}$ and $\Psi = \{g^0, g^1, g^2\}$ of (2.1.4), where $G = \mathrm{Gal}(K/\mathbb{Q}) = \langle g \rangle$ is cyclic of order 6. It is not hard to see that the set of CM types decomposes into precisely two G-orbits, namely the induced types and the primitive types. The former is the G-orbit of Φ, the latter is the G-orbit of Ψ.

We identify $\Sigma = G/\langle c \rangle$ with $\{0, 1, 2\}$ by $g^i \langle c \rangle \mapsto i$ and also S_Σ with $S_{\{0,1,2\}} = S_3$. We thus write elements of $\mathrm{Gal}(K/F)^\Sigma = \langle c \rangle^3$ as $(c^{a_0}, c^{a_1}, c^{a_2})$ with $a_i \in \mathbb{Z}/2\mathbb{Z}$. Moreover, we choose the section $s\colon \Sigma \to G$ given by $s_i := g^i$. Then we have the embedding

$$\rho_s\colon G \hookrightarrow S_3 \ltimes \langle c \rangle^3, \quad g \mapsto \big((0 \quad 1 \quad 2), (c^0, c^0, c^1)\big)\,.$$

Using (4.1.6.2) we directly calculate that

$$\big((0 \quad 1), (c^1, c^0, c^0)\big)\,\Psi = \Phi.$$

In particular the plectic group $S_\Sigma \ltimes \mathrm{Gal}(K/F)^\Sigma$ acts transitively on the set of CM types of K. In fact, this holds for any CM field K.

4.1.8 Remarks. (Independence of the section s)

(4.1.8.a) The actions of $S_\Sigma \ltimes \Gamma_F^\Sigma$ on induced representations in (4.1.4.2) and on the set of CM types in (4.1.6) induce, via β_s, actions of $\mathrm{Aut}_F(F \otimes_\mathbb{Q} \overline{\mathbb{Q}})$, which turn out to be *independent* of the chosen section s. See (4.1.9) for the strategy of the proof.

(4.1.8.b) Combining the isomorphisms β_s of (4.1.3) and π_s of (4.1.5.1), we get the group isomorphism [Bla15, (3.2.12)]

$$\mathrm{Aut}_F(F \otimes_\mathbb{Q} \overline{\mathbb{Q}}) \xrightarrow{\sim} \Gamma_\mathbb{Q} \# \Gamma_F,$$

which turns out to be independent of the chosen section $s\colon \Sigma \to \Gamma_\mathbb{Q}$.

4.1.9 Example. (Quotients of the plectic group)

(4.1.9.a) The composition[1]

$$\mathrm{Aut}_F(F \otimes_\mathbb{Q} \overline{\mathbb{Q}}) \xrightarrow{\beta_s} S_\Sigma \ltimes \Gamma_F^\Sigma \twoheadrightarrow S_\Sigma,$$
$$(\sigma, h) \mapsto \sigma,$$

[1] This is actually a special case of (4.1.8.a), because similar to (4.1.6) we have $\Sigma \cong \mathrm{Ind}_{\Gamma_F}^{\Gamma_\mathbb{Q}}(\{1\}) = \bigsqcup_{x\in\Sigma} s_x.\{1\}$, via the map $x \mapsto s_x.1$, as sets with $\Gamma_\mathbb{Q}$-action, where $\{1\}$ denotes a set with one element and trivial Γ_F-action.

is independent of the choice of section $s\colon \Sigma \to \Gamma_{\mathbb{Q}}$.

(4.1.9.b) The composition

$$\begin{aligned} \operatorname{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \xrightarrow{\beta_s} S_\Sigma \ltimes \Gamma_F^\Sigma &\xrightarrow{(1,\mathrm{prod})} \Gamma_F^{\mathrm{ab}}, \\ (\sigma, h) &\longmapsto \prod_{x\in\Sigma} h_x|_{F^{\mathrm{ab}}}, \end{aligned}$$

is independent of the choice of section $s\colon \Sigma \to \Gamma_{\mathbb{Q}}$.

Moreover, when restricted to $\Gamma_{\mathbb{Q}}$, we get the commutative diagram

$$\begin{tikzcd} \Gamma_{\mathbb{Q}} \arrow[r, hook, "\rho_s"] \arrow[d, two heads] & S_\Sigma \ltimes \Gamma_F^\Sigma \arrow[d, two heads, "(1,\mathrm{prod})"] \\ \Gamma_{\mathbb{Q}}^{\mathrm{ab}} \arrow[r, "V_{F/\mathbb{Q}}"'] & \Gamma_F^{\mathrm{ab}}. \end{tikzcd}$$

Proof. Let $s'\colon \Sigma \to \Gamma_{\mathbb{Q}}$ be another section, so $s'_x = s_x t_x$ for some $t = (t_x)_{x\in\Sigma} \in \Gamma_F^\Sigma$. By [Nek09, (1.1.4)(iv)], we have for all $\gamma \in \operatorname{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$:

$$\beta_{s'}(\gamma) = (1,t)^{-1}\beta_s(\gamma)(1,t).$$

So if $\beta_s(\gamma) = (\sigma, h) \in S_\Sigma \ltimes \Gamma_F^\Sigma$, then

$$\beta_{s'}(\gamma) = \left(\sigma, \left(t_{\sigma(x)}^{-1} h_x t_x\right)_{x\in\Sigma}\right).$$

From this, (4.1.9.a) follows immediately. For (4.1.9.b), note

$$\begin{aligned} (1,\mathrm{prod}) \circ \beta_{s'}(\gamma) &= \prod_{x\in\Sigma} t_{\sigma(x)}^{-1} h_x t_x\Big|_{F^{\mathrm{ab}}} \\ &= \left(\prod_{x\in\Sigma} t_{\sigma(x)}^{-1}\Big|_{F^{\mathrm{ab}}}\right)\left(\prod_{x\in\Sigma} t_x|_{F^{\mathrm{ab}}}\right)\left(\prod_{x\in\Sigma} h_x|_{F^{\mathrm{ab}}}\right) \\ &= \prod_{x\in\Sigma} h_x|_{F^{\mathrm{ab}}}, \end{aligned}$$

so as claimed $(1,\mathrm{prod}) \circ \beta_s$ is independent of s.

The last assertion is [Nek09, (1.1.2.3)] and can be proved as follows. For $\gamma \in \Gamma_{\mathbb{Q}}$, we have $\rho_s(\gamma) = \left([x \mapsto \gamma x], (s_{\gamma x}^{-1}\gamma s_x)_{x\in\Sigma}\right)$, so

$$(1,\mathrm{prod}) \circ \rho_s(\gamma) = \prod_{x\in\Sigma} \left(s_{\gamma x}^{-1}\gamma s_x\right)\big|_{F^{\mathrm{ab}}},$$

which by definition of the transfer map is equal to $V_{F/\mathbb{Q}}(\gamma)$, see (2.4.4). □

So far, we have found two good indications that the plectic theory can be combined with CM theory. Namely, by (4.1.6) the plectic group acts on CM types, extending the action of $\Gamma_{\mathbb{Q}}$, which is a good sign in view of (2.3.8). On the other hand, by the commutative diagram in (4.1.9.b) there is a connection between the plectic group and the transfer map, so we can hope to find an analogue of Tate's half transfer in the plectic world.

4.1.10. (Plectic action on CM types — II) We continue (4.1.6). The identification (4.1.6.1) induces a bijection

$$\begin{aligned}(\mathbb{Z}/2\mathbb{Z})^{\Sigma} &\xrightarrow{\sim} \{\text{CM types of } K\},\\ (a_x)_{x\in\Sigma} &\longmapsto \{c^{a_x} s_x|_K : x \in \Sigma\}.\end{aligned}$$

The plectic action on CM types induced by (4.1.6.2) then translates to an action of $(\sigma, h) \in S_\Sigma \ltimes \Gamma_F^\Sigma$ on $a = (a_x)_{x\in\Sigma} \in (\mathbb{Z}/2\mathbb{Z})^\Sigma$ given by

$$(\sigma, h)a = \big(a_{\sigma^{-1}(x)} + \overline{h}_{\sigma^{-1}(x)}\big)_{x\in\Sigma}. \tag{4.1.10.1}$$

4.1.11. (Nekovář's plectic half transfer) With the notation as in (4.1.10), let Φ be a CM type of K, corresponding to $a = (a_x)_{x\in\Sigma} \in (\mathbb{Z}/2\mathbb{Z})^\Sigma$. As in [Nek09, (2.1.3)], define the map

$$\begin{aligned}{}_s\widetilde{F}_a \colon S_\Sigma \ltimes \Gamma_F^\Sigma &\longrightarrow \Gamma_K^{\mathrm{ab}}\\ (\sigma, h) &\longmapsto \prod_{x\in\Sigma} s_{\sigma(x)}^{-1} c^{a_x+\overline{h_x}} s_{\sigma(x)} h_x s_x^{-1} c^{a_x} s_x\Big|_{K^{\mathrm{ab}}}.\end{aligned}$$

This map depends on s and a.

By the calculations in [Nek09, (2.1.2)], this extends the domain of Tate's half transfer (3.1.1) from $\Gamma_{\mathbb{Q}}$ to $S_\Sigma \ltimes \Gamma_F^\Sigma$ via the embedding $\rho_s \colon \Gamma_{\mathbb{Q}} \hookrightarrow S_\Sigma \ltimes \Gamma_F^\Sigma$, namely

$${}_s\widetilde{F}_a \circ \rho_s = F_\Phi \colon \Gamma_{\mathbb{Q}} \longrightarrow \Gamma_K^{\mathrm{ab}}.$$

Moreover (see [Nek09, (2.1.7)]), the composition

$$\widetilde{F}_\Phi := {}_s\widetilde{F}_a \circ \beta_s \colon \operatorname{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \longrightarrow \Gamma_K^{\mathrm{ab}}$$

is independent of the section s and therefore only depends on the CM type Φ of K. It is called the *(F-)plectic half transfer* and satisfies:

(4.1.11.a) Restricted to $\Gamma_{\mathbb{Q}}$ via the embedding $\Gamma_{\mathbb{Q}} \hookrightarrow \operatorname{Aut}_F(F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}})$ of (4.1.1), the plectic half transfer $\widetilde{F}_\Phi$ agrees with Tate's half transfer F_Φ.

(4.1.11.b) For $\gamma, \gamma' \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$, we have the cocycle relation

$$\widetilde{F}_\Phi(\gamma\gamma') = \widetilde{F}_{\gamma'\Phi}(\gamma)\widetilde{F}_\Phi(\gamma').$$

(4.1.11.c) For $(\sigma, h) \in S_\Sigma \ltimes \Gamma_F^\Sigma$, we have [Nek09, (2.1.4)(ii), first formula]

$$ {}_s\widetilde{F}_a(\sigma,h)\Big|_{F^{\mathrm{ab}}} = \prod_{x\in|(\sigma,h)a|} c_x \prod_{x\in|a|} c_x \prod_{x\in\Sigma} h_x|_{F^{\mathrm{ab}}},$$

where $|a| := \{x \in \Sigma \colon a_x \neq 0\}$ denotes the support of $a = (a_x)_{x\in\Sigma} \in (\mathbb{Z}/2\mathbb{Z})^\Sigma$, and $(\sigma, h)a$ is given by (4.1.10.1) and corresponds to the CM type $(\sigma, h)\Phi$.

(4.1.11.d) For $(\sigma, h) \in S_\Sigma \ltimes \Gamma_F^\Sigma$, we have [Nek09, (2.1.4)(ii), second formula]

$$ {}^{1+c}\left({}_s\widetilde{F}_a(\sigma,h)\right) = V_{K/F} \circ (1, \mathrm{prod})(\sigma, h) \in \Gamma_K^{\mathrm{ab}}.$$

4.2 Plectic Taniyama element and plectic action on CM points of the PEL Hilbert modular variety

As in (3.1.2), in this section we define a suitably normalised preimage $\widetilde{f}_\Phi(\gamma) \in \mathbb{A}_{K,f}/K^\times$ of the plectic half transfer $\widetilde{F}_\Phi(\gamma) \in \Gamma_K^{\mathrm{ab}}$ under the Artin map. To achieve this, we will start with a few remarks about class field theory, see [Nek09, §1.3].

4.2.1. (Complex conjugations) Fix an embedding $F \subset \overline{\mathbb{Q}}$. For each $x \in \Sigma$, we define the complex conjugation corresponding to x to be the element $c_x \in \Gamma_F^{\mathrm{ab}}$ defined as follows: If $s \colon \Sigma \to \Gamma_{\mathbb{Q}}$ is a section as before, then $s_x^{-1} c s_x$ is an element of Γ_F and its image c_x in Γ_F^{ab} is independent of s.

Define the subgroup $\mathfrak{c} := \langle c_x \colon x \in \Sigma\rangle \subset \Gamma_F^{\mathrm{ab}}$. By [Nek09, (1.3.1)], the Artin map $r_F \colon \mathbb{A}_{F,f}^\times/F_{>0}^\times \to \Gamma_F^{\mathrm{ab}}$ induces a bijection

$$\begin{aligned} r_F \colon F^\times/F_{>0}^\times &\xrightarrow{\sim} \mathfrak{c} \\ \alpha F_{>0}^\times &\longmapsto \prod_{x\in\Sigma} c_x^{\alpha_x}, \end{aligned}$$

where the $\alpha_x \in \mathbb{Z}/2\mathbb{Z}$ are determined by $(-1)^{\alpha_x} = \mathrm{sgn}(x(\alpha))$ for each $x \in \Sigma$. Here $\mathrm{sgn} \colon \mathbb{R}^\times \to \{\pm 1\}$ denotes the sign of a real number.

Note that the group $F^\times/F_{>0}^\times$ is isomorphic to $\{\pm1\}^\Sigma$ via $\alpha F_{>0}^\times \mapsto (\mathrm{sgn}(x(\alpha)))_{x\in\Sigma}$.

4.2.2. (Kernel of transfer) By [Nek09, (1.2.5)], for any number field k the kernel of the transfer map $V_{k/\mathbb{Q}} \colon \Gamma_{\mathbb{Q}}^{\mathrm{ab}} \to \Gamma_k^{\mathrm{ab}}$ is either $\langle c\rangle$, if k is totally complex, or trivial otherwise. In particular, one can show [Nek09, (1.3.2.4-6)] that this implies that the preimage of $\mathfrak{c} \subset \Gamma_F^{\mathrm{ab}}$ under $V_{F/\mathbb{Q}}$ is equal to $\langle c \rangle \subset \Gamma_{\mathbb{Q}}^{\mathrm{ab}}$, so that

$$\overline{V}_{F/\mathbb{Q}} \colon \Gamma_{\mathbb{Q}}^{\mathrm{ab}}/\langle c \rangle \longrightarrow \Gamma_F^{\mathrm{ab}}/\mathfrak{c} \tag{4.2.2.1}$$

is injective. Moreover, we have $V_{F/\mathbb{Q}}(c) = \prod_{x \in \Sigma} c_x$.

We state [Nek09, (1.3.4)], which will be extremely useful:

4.2.3 Proposition. *Let K be a CM field and F its maximal totally real subfield.*

(4.2.3.a) The Artin map r_K restricts to a bijection

$$\{y \in \mathbb{A}_{K,f}^{\times} \colon {}^{1+c}y \in \widehat{\mathbb{Z}}^{\times} K^{\times}\}/K^{\times} \xrightarrow{\sim} \{\gamma \in \Gamma_K^{\mathrm{ab}} \colon \gamma|_{F^{\mathrm{ab}}} \in \mathfrak{c} \cdot V_{F/\mathbb{Q}}(\Gamma_{\mathbb{Q}}^{\mathrm{ab}})\}.$$

Denoting its inverse by ℓ_K, we get

$${}^{1+c}\ell_K(\gamma) = \chi_{\mathrm{cyc}}(u(\gamma))K^{\times} \in \mathbb{A}_{K,f}^{\times}/K^{\times},$$

where $u(\gamma) \in \Gamma_{\mathbb{Q}}^{\mathrm{ab}}/\langle c \rangle$ is the unique element satisfying $\overline{V}_{F/\mathbb{Q}}(u(\gamma)) = \gamma|_{F^{\mathrm{ab}}} \cdot \mathfrak{c}$.

(4.2.3.b) One can be more precise: If γ is an element of the domain of ℓ_K, write (non-uniquely)

$$\gamma|_{F^{\mathrm{ab}}} = V_{F/\mathbb{Q}}(u) \prod_{x \in \Sigma} c_x^{\alpha_x},$$

with $u \in \Gamma_{\mathbb{Q}}^{\mathrm{ab}}$ and $\alpha_x \in \mathbb{Z}/2\mathbb{Z}$, $x \in \Sigma$. Then

$$N_{K/F}(\ell_K(\gamma)) = \chi_{\mathrm{cyc}}(u)\alpha F_{>0}^{\times} \in \mathbb{A}_{F,f}^{\times}/F_{>0}^{\times},$$

where $\alpha \in F^{\times}$ has signs

$$\mathrm{sgn}(x(\alpha)) = (-1)^{\alpha_x}, \quad x \in \Sigma.$$

4.2.4 Remarks. A few comments to avoid confusion:

(4.2.4.a) For a finite idele $y \in \mathbb{A}_{K,f}^{\times}$, note that ${}^{1+c}y = i_{K/F} \circ N_{K/F}(y) \in \mathbb{A}_{K,f}^{\times}$, where $i_{K/F}$ denotes the embedding $\mathbb{A}_{F,f}^{\times} \hookrightarrow \mathbb{A}_{K,f}^{\times}$. Since $N_{K/F}(K^{\times}) \subset F_{>0}^{\times}$, we get an induced map

$$N_{K/F} \colon \mathbb{A}_{K,f}^{\times}/K^{\times} \longrightarrow \mathbb{A}_{F,f}^{\times}/F_{>0}^{\times},$$

fitting into the commutative diagram

$$\begin{tikzcd} \mathbb{A}_{K,f}^\times/K^\times \arrow[r, "N_{K/F}"] \arrow[dr, "1+c"'] & \mathbb{A}_{F,f}^\times/F_{>0}^\times \arrow[d, "i_{K/F}"] \\ & \mathbb{A}_{K,f}^\times/K^\times. \end{tikzcd}$$

From this point of view, (4.2.3.a) describes the image of $\ell_K(\gamma)$ under the diagonal arrow, whereas (4.2.3.b) describes the image under the top horizontal arrow.

(4.2.4.b) Moreover, note how the description in (4.2.3.b) depends (very slightly) on the choice of u: By the injectivity of (4.2.2.1), the only other possible choice is $u' = cu$, and this forces α_x to change to $\alpha'_x = 1-\alpha_x$, $x \in \Sigma$, and α to $\alpha' = -\alpha$.

4.2.5 Remark. Recall the definition of the Taniyama element $f_\Phi(\gamma)$ in (3.1.2), for $\gamma \in \Gamma_{\mathbb{Q}}$, from Tate's half transfer $F_\Phi(\gamma)$. From the point of view of (4.2.3), we can rephrase (3.1.2.b) as stating that $F_\Phi(\gamma)$ lies in the domain of ℓ_K, and (3.1.2.a) as

$$f_\Phi(\gamma) = \ell_K(F_\Phi(\gamma)).$$

So we want to use the isomorphism ℓ_K of Proposition (4.2.3) to get a preimage of the plectic half transfer $\widetilde{F}_\Phi(\gamma)$ under the Artin map, for $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$. For this we need $\widetilde{F}_\Phi(\gamma)$ to lie in the domain of ℓ_K, so we need to look at $\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}}$.

4.2.6. (1st subgroup of the plectic group) Using the notation of (4.1.11), recall (4.1.11.c): for $(\sigma, h) \in S_\Sigma \ltimes \Gamma_F^\Sigma$, we have

$$\left.{}_s\widetilde{F}_a(\sigma, h)\right|_{F^{\mathrm{ab}}} = \prod_{x \in |(\sigma,h)a|} c_x \prod_{x \in |a|} c_x \prod_{x \in \Sigma} h_x|_{F^{\mathrm{ab}}}.$$

We want ${}_s\widetilde{F}_a(\sigma, h)|_{F^{\mathrm{ab}}}$ to lie inside $\mathfrak{c} \cdot V_{F/\mathbb{Q}}(\Gamma_{\mathbb{Q}}^{\mathrm{ab}})$, as then ${}_s\widetilde{F}_a(\sigma, h)$ lies in the domain of ℓ_K. However, as the h_x are arbitrary elements of Γ_F, this is not automatically the case. To remedy this, we define the subgroup $(S_\Sigma \ltimes \Gamma_F^\Sigma)_1 \subset S_\Sigma \ltimes \Gamma_F^\Sigma$ by the Cartesian diagram

$$\begin{tikzcd} (S_\Sigma \ltimes \Gamma_F^\Sigma)_1 \arrow[r, hook] \arrow[d] & S_\Sigma \ltimes \Gamma_F^\Sigma \arrow[d, "(1,\mathrm{prod})"] \\ \Gamma_{\mathbb{Q}}^{\mathrm{ab}}/\langle c \rangle \arrow[r, hook, "\overline{V}_{F/\mathbb{Q}}"'] & \Gamma_F^{\mathrm{ab}}/\mathfrak{c}. \end{tikzcd}$$

This precisely means that $(\sigma, h) \in S_\Sigma \ltimes \Gamma_F^\Sigma$ lies in $(S_\Sigma \ltimes \Gamma_F^\Sigma)_1$ if and only if $\prod_{x \in \Sigma} h_x|_{F^{\mathrm{ab}}}$ lies in $\mathfrak{c} \cdot V_{F/\mathbb{Q}}(\Gamma_{\mathbb{Q}}^{\mathrm{ab}})$, i. e. if and only if ${}_s\widetilde{F}_a(\sigma, h)$ lies in the domain of ℓ_K.

We also define

$$\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1 := \beta_s^{-1}\left((S_\Sigma \ltimes \Gamma_F^\Sigma)_1\right) \subset \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}).$$

By (4.1.9.b), this definition is independent of the choice of s.

Finally, observe that the diagram in (4.1.9.b) implies that the embedding ρ_s of $\Gamma_{\mathbb{Q}}$ into $S_\Sigma \ltimes \Gamma_F^\Sigma$ actually lands inside $(S_\Sigma \ltimes \Gamma_F^\Sigma)_1$. In other words,

$$\Gamma_{\mathbb{Q}} \subset \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1.$$

4.2.7. (Plectic Taniyama element) Continuing with the same notation, the *plectic Taniyama element* is the map

$$\begin{aligned} \widetilde{f}_\Phi \colon \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1 &\longrightarrow \mathbb{A}_{K,f}^\times / K^\times \\ \gamma &\longmapsto \ell_K\left(\widetilde{F}_\Phi(\gamma)\right). \end{aligned}$$

It satisfies the following properties [Nek09, (2.2.3)]:

(4.2.7.a) Because ℓ_K is inverse to r_K, we have $r_K \circ \widetilde{f}_\Phi = \widetilde{F}_\Phi\Big|_{\mathrm{Aut}_F(F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}})_1}$.

(4.2.7.b) By (4.1.11.a) and (4.2.5), we have $\widetilde{f}_\Phi(\mathrm{id}_F \otimes \gamma) = f_\Phi(\gamma)$ for $\gamma \in \Gamma_{\mathbb{Q}}$.

(4.2.7.c) By (4.1.11.b), we have the cocycle relation $\widetilde{f}_\Phi(\gamma\gamma') = \widetilde{f}_{\gamma'\Phi}(\gamma)\widetilde{f}_\Phi(\gamma')$ for $\gamma, \gamma' \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$.

(4.2.7.d) By applying (4.2.3.a) to $\widetilde{F}_\Phi(\gamma)$, we see that for $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$ we have

$$^{1+c}\widetilde{f}_\Phi(\gamma) = \chi_{\mathrm{cyc}}(\widetilde{u}(\gamma))K^\times \in \mathbb{A}_{K,f}^\times / K^\times,$$

where $\widetilde{u}(\gamma)$ is the unique element of $\Gamma_{\mathbb{Q}}^{\mathrm{ab}}/\langle c \rangle$ with $\overline{V}_{F/\mathbb{Q}}(\widetilde{u}(\gamma)) = \widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}\mathfrak{c}}$.

It was necessary to restrict to $(S_\Sigma \ltimes \Gamma_F^\Sigma)_1$, because then a combination of (4.1.11.c) and (4.2.3.a) allowed us to define $\widetilde{f}_\Phi$. Moreover, we can achieve a more precise statement in (4.2.7.d) if we use (4.2.3.b) instead of (4.2.3.a). For this to work, it is necessary to restrict to a smaller subgroup:

4.2.8. (0^{th} subgroup of the plectic group) Define the subgroup $(S_\Sigma \ltimes \Gamma_F^\Sigma)_0$ of $S_\Sigma \ltimes \Gamma_F^\Sigma$ by the Cartesian diagram

$$\begin{tikzcd} (S_\Sigma \ltimes \Gamma_F^\Sigma)_0 \arrow[r, hook] \arrow[d] & S_\Sigma \ltimes \Gamma_F^\Sigma \arrow[d, "(1,\mathrm{prod})"] \\ \Gamma_{\mathbb{Q}}^{\mathrm{ab}} \arrow[r, hook, "V_{F/\mathbb{Q}}"'] & \Gamma_F^{\mathrm{ab}}, \end{tikzcd}$$

and let

$$\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0 := \beta_s^{-1}\left((S_\Sigma \ltimes \Gamma_F^\Sigma)_0\right) \subset \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}),$$

which, again by (4.1.9.b), is independent of the chosen section $s\colon \Sigma \to \Gamma_{\mathbb{Q}}$.

Clearly, we have $(S_\Sigma \ltimes \Gamma_F^\Sigma)_0 \subset (S_\Sigma \ltimes \Gamma_F^\Sigma)_1$ and so $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0 \subset \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$. Moreover, by the diagram in (4.1.9.b) we see that

$$\Gamma_{\mathbb{Q}} \subset \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0.$$

Again using the notation of (4.1.11), let $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$ and $(\sigma, h) = \beta_s(\gamma) \in (S_\Sigma \ltimes \Gamma_F^\Sigma)_0$, and let $\widetilde{u} \in \Gamma_{\mathbb{Q}}^{\mathrm{ab}}$ be the unique element such that $V_{F/\mathbb{Q}}(\widetilde{u}) = \prod_{x \in \Sigma} h_x|_{F^{\mathrm{ab}}}$. We calculate

$$\begin{aligned}
\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}} &= {}_s\widetilde{F}_a(\sigma, h)|_{F^{\mathrm{ab}}} \\
&= \prod_{x \in |(\sigma,h)a|} c_x \prod_{x \in |a|} c_x \prod_{x \in \Sigma} h_x|_{F^{\mathrm{ab}}} \\
&= \prod_{x \in \Sigma} c_x^{\alpha_x}\, V_{F/\mathbb{Q}}(\widetilde{u}),
\end{aligned}$$

where $(\alpha_x)_{x \in \Sigma}$ measures the difference between the CM types Φ (which corresponds to a) and $\gamma\Phi$ (which corresponds to $(\sigma, h)a$), namely

$$\alpha_x = \begin{cases} 0, & \text{if } [(\sigma, h)a]_x = a_x, \\ 1, & \text{otherwise.} \end{cases} \tag{4.2.8.1}$$

In the calculation above, the first equality holds by Definition (4.1.11) of $\widetilde{F}_\Phi$, the second equality is (4.1.11.c), and the last equality follows from the definition of $\widetilde{u}$.

Applying (4.2.3.b), we conclude that (see [Nek09, (2.2.3)(vii)], which contains a typo in the definition of $u(g)$ that we fixed here):

(4.2.8.a) For $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$ and $\widetilde{u} \in \Gamma_{\mathbb{Q}}^{\mathrm{ab}}$ defined above, we have

$$N_{K/F}\widetilde{f}_\Phi(\gamma) = \chi_{\mathrm{cyc}}(\widetilde{u})\alpha F_{>0}^\times \in \mathbb{A}_{F,f}^\times / F_{>0}^\times,$$

where $\alpha \in F^\times$ has signs

$$\mathrm{sgn}(x(\alpha)) = (-1)^{\alpha_x} = \begin{cases} 1, & \text{if } \gamma\Phi \text{ and } \Phi \text{ agree at } x, \\ -1, & \text{otherwise,} \end{cases}$$

for $x \in \Sigma$.

Here we say that two CM types Φ, Φ' *agree at* $x \in \Sigma$ if the unique elements $\varphi_x \in \Phi$ and $\varphi'_x \in \Phi'$ extending x are the same.

4.2.9 Remark. The assertion about the signs in (4.2.8.a) implies the following: Let $t \in K^\times$ be totally imaginary such that $\operatorname{Im}\varphi(t) > 0$ for all $\varphi \in \Phi$. Let $\gamma \in \operatorname{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$ and $\widetilde{f} \in \mathbb{A}_{K,f}^\times$ such that $\widetilde{f}_\Phi(\gamma) = \widetilde{f}K^\times$. Then

$$\operatorname{Im}\varphi\left(t\frac{\chi_{\mathrm{cyc}}(\widetilde{u})}{1+c\widetilde{f}}\right) > 0, \quad \text{for all } \varphi \in \gamma\Phi.$$

In particular, for $\gamma \in \Gamma_{\mathbb{Q}}$, this proves (3.1.4.1).

We have seen in (3.2.9) how the action of $\Gamma_{\mathbb{Q}}$ on the special points of the PEL Hilbert modular variety $\operatorname{Sh}(G, X)$ can be described using Tate's half transfer. Namely, recall (3.2.9.2): If K is a CM field whose maximal totally real subfield is F, and $(K, \Phi; \mathfrak{a}, t)$ describes a polarised CM abelian variety as in (2.3.5), then for $\gamma \in \Gamma_{\mathbb{Q}}$ and $f \in \mathbb{A}_{K,f}^\times$ such that $f_\Phi(\gamma) = fK^\times \in \mathbb{A}_{K,f}^\times/K^\times$ we have[2]

$$\gamma\left[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, \mathbb{Q}^\times E_t, \eta\overline{Z(\mathbb{Q})}\right] = \left[\mathbb{C}^{\gamma\Phi}/\gamma\Phi(f\mathfrak{a}), i_{\gamma\Phi}|_F, \mathbb{Q}^\times E_{\chi t}, f \circ \eta\overline{Z(\mathbb{Q})}\right],$$

where $\chi := \frac{\chi_{\mathrm{cyc}}(\gamma)}{f \cdot \overline{f}} \in F^\times$.

Now, we reverse this process and *define* an action of $\operatorname{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$ on the special points of $\operatorname{Sh}(G, X)$ using the plectic half transfer (4.1.11) together with the observations in (4.2.8), compare [Nek09, (2.2.5)]:

4.2.10 (Plectic action on CM points of PEL Hilbert modular variety)**.** Let K be a totally imaginary quadratic extension of F, and let $(K, \Phi; \mathfrak{a}, t)$ be a type as in (2.3.5). We look at the CM point $[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, \mathbb{Q}^\times E_t, \eta\overline{Z(\mathbb{Q})}]$ of the PEL Hilbert modular variety $\operatorname{Sh}(G, X)$.

Let $\gamma \in \operatorname{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$ and $\widetilde{f} \in \mathbb{A}_{K,f}^\times$ such that $\widetilde{f}_\Phi(\gamma) = \widetilde{f}K^\times \in \mathbb{A}_{K,f}^\times/K^\times$. Let $\widetilde{u} \in \Gamma_{\mathbb{Q}}^{\mathrm{ab}}$ be the unique element such that $V_{F/\mathbb{Q}}(\widetilde{u}) = (1, \mathrm{prod}) \circ \beta_s(\gamma)$ as in (4.2.8), and denote $\chi := \frac{\chi_{\mathrm{cyc}}(\widetilde{u})}{1+c\widetilde{f}} \in F^\times$. Define

$$\gamma\left[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, \mathbb{Q}^\times E_t, \eta\overline{Z(\mathbb{Q})}\right] := \left[\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a}), i_{\gamma\Phi}|_F, \mathbb{Q}^\times E_{\chi t}, \widetilde{f} \circ \eta\overline{Z(\mathbb{Q})}\right]. \tag{4.2.10.1}$$

This defines a group action of $\operatorname{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$ on the set of CM points of $\operatorname{Sh}(G, X)$, extending the action of $\Gamma_{\mathbb{Q}}$.

We will prove this in more generality in (4.4.13). Here, we only give a sketch:

Sketch of proof. First of all, one needs to show that (4.2.10.1) is independent of the choice of $\widetilde{f}$ and of choosing a different type $(K, \Phi, \mathfrak{a}', t')$ (of the same CM point), but both are easy to check.

[2] We now ignore the uniformizations Θ and Θ' and directly work with the abelian variety $A = \mathbb{C}^\Phi/\Phi(\mathfrak{a})$.

Secondly, one needs to check that the right hand side of (4.2.10.1) defines a valid CM point of $\mathrm{Sh}(G,X)$. For this, note that $E_{\chi t}$ is a K-compatible polarisation of $\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a})$ by (4.2.9), which by (3.2.5.e) implies that the right hand side of (4.2.10.1) is a point of $\mathrm{Sh}(G,X)$, which is clearly a CM point. Let us also remark that here we identify $\widehat{V}(\mathbb{C}^{\Phi}/\Phi(\mathfrak{a})) = H_1(\mathbb{C}^{\Phi}/\Phi(\mathfrak{a}),\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q},f}$ with $\mathbb{A}_{K,f}$, so we may view $\widetilde{f} \in \mathbb{A}_{K,f}^{\times}$ as a map

$$\widehat{V}(\mathbb{C}^{\Phi}/\Phi(\mathfrak{a})) = \mathbb{A}_{K,f} \xrightarrow{\widetilde{f}} \mathbb{A}_{K,f} = \widehat{V}(\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a})).$$

This shows that (4.2.10.1) is well-defined. Finally, the cocycle relation (4.2.7.c) implies that (4.2.10.1) does indeed define a group action of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$. $\square$

4.2.11 Remark. In Definition (4.2.10.1), it was absolutely essential to restrict to $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$: First of all, by (4.2.6) we need to restrict to $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$ because otherwise the plectic Taniyama element $\widetilde{f}_{\Phi}(\gamma)$ is not defined. However, in (4.3.6) we will find a way to define the plectic Taniyama element on the entire plectic group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$.

Nonetheless, we need to restrict further to $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$ because otherwise we would have no control over the signs of the imaginary parts of χt under the embeddings $\varphi \in \gamma\Phi$. So in view of (2.3.4.b) and (4.2.9), we can only guarantee that $E_{\chi t}$ is a polarisation of $\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a})$ if $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$.

4.3 Plectic Taniyama element on the entire plectic group

In the previous section we recalled Nekovář's definition of the plectic Taniyama element $\widetilde{f}_{\Phi}(\gamma) := \ell_K(\widetilde{F}_{\Phi}(\gamma)) \in \mathbb{A}_{K,f}^{\times}/K^{\times}$ from the plectic half transfer $\widetilde{F}_{\Phi}(\gamma) \in \Gamma_K^{\mathrm{ab}}$ via the isomorphism ℓ_K of (4.2.3). We had to restrict to $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$ so that $\widetilde{F}_{\Phi}(\gamma)$ lies in the domain of ℓ_K.

In this section we will extend the definition of $\widetilde{f}_{\Phi}$ to the entire plectic group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$. We will do this in a way closer to the definition of the (non-plectic) Taniyama element in (3.1.2), using an extension of diagram (3.1.2.1).

From now on, unless stated otherwise, the results and proofs are our own.

4.3.1 (A useful diagram)**.** We look at the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \ker(r_K) \arrow[r] \arrow[d, "N_{K/F}"] & \mathbb{A}_{K,f}^\times/K^\times \arrow[r, "r_K"] \arrow[d, "N_{K/F}"] & \Gamma_K^{\mathrm{ab}} \arrow[r] \arrow[d, "\mathrm{res}"] & 0 \\ 0 \arrow[r] & \ker(r_F) \arrow[r] \arrow[d, "i_{K/F}"] & \mathbb{A}_{F,f}^\times/F_{>0}^\times \arrow[r, "r_F"] \arrow[d, "i_{K/F}"] & \Gamma_F^{\mathrm{ab}} \arrow[r] \arrow[d, "V_{K/F}"] & 0 \\ 0 \arrow[r] & \ker(r_K) \arrow[r] & \mathbb{A}_{K,f}^\times/K^\times \arrow[r, "r_K"] & \Gamma_K^{\mathrm{ab}} \arrow[r] & 0. \end{tikzcd} \tag{4.3.1.1}$$

Here $N_{K/F}$ denotes the norm map, $i_{K/F}$ is induced from the inclusion $F \subset K$, $\mathrm{res}(\gamma) = \gamma|_{F^{\mathrm{ab}}}$ is the restriction and $V_{K/F}$ the transfer map. The vertical composites are $1+c$, so that the outer diagram is precisely (3.1.2.1).

As mentioned in (3.1.2), the map $1+c$ is an isomorphism on $\ker(r_K)$, so that the right hand composite rectangle is Cartesian. By [Nek09, (1.2.2)], we have $\ker(r_K) \cong \mathcal{O}_K^\times \otimes_{\mathbb{Z}} (\mathbb{A}_{\mathbb{Q},f}/\mathbb{Q})$ and $\ker(r_F) \cong \mathcal{O}_{F,>0}^\times \otimes_{\mathbb{Z}} (\mathbb{A}_{\mathbb{Q},f}/\mathbb{Q})$. By Dirichlet's Unit Theorem [Neu92, Thm I.7.4] the groups $\mathcal{O}_K^\times$ and $\mathcal{O}_{F,>0}^\times$ have the same $\mathbb{Z}$-rank. Hence the maps $N_{K/F}\colon \mathcal{O}_K^\times \to \mathcal{O}_{F,>0}^\times$ and $i_{K/F}\colon \mathcal{O}_{F,>0}^\times \to \mathcal{O}_K^\times$ have finite kernel and cokernel, and since $\mathbb{A}_{\mathbb{Q},f}/\mathbb{Q}$ is a $\mathbb{Q}$-vector space, we conclude that both left vertical arrows $N_{K/F}\colon \ker(r_K) \to \ker(r_F)$ and $i_{K/F}\colon \ker(r_F) \to \ker(r_K)$ are in fact isomorphisms. By the same diagram chase as in (3.1.2.1) this means that both right hand squares in (4.3.1.1) are Cartesian. Let us highlight the top right Cartesian square

$$\begin{tikzcd} \mathbb{A}_{K,f}^\times/K^\times \arrow[r, "r_K"] \arrow[d, "N_{K/F}"] & \Gamma_K^{\mathrm{ab}} \arrow[d, "\mathrm{res}"] \\ \mathbb{A}_{F,f}^\times/F_{>0}^\times \arrow[r, "r_F"] & \Gamma_F^{\mathrm{ab}} \end{tikzcd} \tag{4.3.1.2}$$

because it will be extremely useful for the rest of this section.

4.3.2 (Definition of ℓ_K revisited)**.** With the help of diagrams (4.3.1.1) and (4.3.1.2) let us give a different perspective on the definition of ℓ_K: Let $\gamma \in \Gamma_K^{\mathrm{ab}}$ lie in the domain of ℓ_K, i. e. $\gamma|_{F^{\mathrm{ab}}} = V_{F/\mathbb{Q}}(u)\prod_{x\in\Sigma} c_x^{\alpha_x}$ for some (almost unique, see (4.2.4.b)) $u \in \Gamma_{\mathbb{Q}}^{\mathrm{ab}}$ and $\alpha_x \in \mathbb{Z}/2\mathbb{Z}$. By (4.2.1) we have $r_F(\alpha F_{>0}^\times) = \prod_{x\in\Sigma} c_x^{\alpha_x}$ for $\alpha \in F^\times$ with $\mathrm{sgn}(x(\alpha)) = (-1)^{\alpha_x}$ for all $x \in \Sigma$. Moreover, by (2.4.3) we also have $r_F(\chi_{\mathrm{cyc}}(u)) = V_{F/\mathbb{Q}}(u)$. Now the Cartesian square (4.3.1.2) looks like

$$\begin{tikzcd} \ell \arrow[r, mapsto, dashed, "r_K"] \arrow[d, mapsto, dashed, "N_{K/F}"] & \gamma \arrow[d, mapsto, "\mathrm{res}"] \\ \chi_{\mathrm{cyc}}(u)\alpha F_{>0}^\times \arrow[r, mapsto, "r_F"] & V_{F/\mathbb{Q}}(u)\prod_{x\in\Sigma} c_x^{\alpha_x}, \end{tikzcd}$$

hence asserts the existence of a unique $\ell \in \mathbb{A}_{K,f}^\times/K^\times$ such that $r_K(\ell) = \gamma$ and

$N_{K/F}(\ell) = \chi_{\mathrm{cyc}}(u)\alpha F_{>0}^{\times}$. The element ℓ is precisely $\ell_K(\gamma)$ from (4.2.3).

4.3.3 (Definition of $\widetilde{f}_\Phi$ revisited)**.** Let $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$. By definition of the group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$, we have $(1, \mathrm{prod})(\beta_s(\gamma)) = V_{F/\mathbb{Q}}(u) \prod_{x\in\Sigma} c_x^{b_x} \in \Gamma_F^{\mathrm{ab}}$ for some (again almost unique) $u \in \Gamma_{\mathbb{Q}}^{\mathrm{ab}}$ and $b_x \in \mathbb{Z}/2\mathbb{Z}$. Combining this with (4.1.11.c) yields (with α_x given by (4.2.8.1))

$$\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}} = V_{F/\mathbb{Q}}(u) \prod_{x\in\Sigma} c_x^{\alpha_x + b_x}.$$

Using (4.2.1), let $\alpha' \in F^\times$ with $\mathrm{sgn}(x(\alpha')) = (-1)^{\alpha_x + b_x}$ for all $x \in \Sigma$, so that $r_F(\alpha' F_{>0}^\times) = \prod_{x\in\Sigma} c_x^{\alpha_x + b_x}$. Now the Cartesian square (4.3.1.2) looks like

$$\begin{tikzcd} \widetilde{f} \arrow[r, mapsto, dashed, "r_K"] \arrow[d, mapsto, dashed, "N_{K/F}"] & \widetilde{F}_\Phi(\gamma) \arrow[d, mapsto, "\mathrm{res}"] \\ \chi_{\mathrm{cyc}}(u)\alpha' F_{>0}^\times \arrow[r, mapsto, "r_F"] & V_{F/\mathbb{Q}}(u) \prod_{x\in\Sigma} c_x^{\alpha_x+b_x} \end{tikzcd}$$

and hence asserts the existence and uniqueness of an element $\widetilde{f} \in \mathbb{A}_{K,f}^\times / K^\times$. The element $\widetilde{f}$ is precisely the plectic Taniyama element $\widetilde{f}_\Phi(\gamma)$ from (4.2.7).

In other words, the reason for restricting to $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$ is to ensure we have a canonical preimage of $\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}} = V_{F/\mathbb{Q}}(u) \prod_{x\in\Sigma} c_x^{\alpha_x+b_x} \in \Gamma_F^{\mathrm{ab}}$ under r_F, namely $\chi_{\mathrm{cyc}}(u)\alpha' F_{>0}^\times \in \mathbb{A}_{F,f}^\times / F_{>0}^\times$, so that the Cartesian square (4.3.1.2) implies the existence of $\widetilde{f}_\Phi(\gamma)$. Note here that the ambiguity (4.2.4.b) in choosing α_x and u does not affect the product $\chi_{\mathrm{cyc}}(u)\alpha' F_{>0}^\times$.

For general $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ however we only have (4.1.11.c)

$$\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}} = (1, \mathrm{prod})(\beta_s(\gamma)) \prod_{x\in\Sigma} c_x^{\alpha_x} \in \Gamma_F^{\mathrm{ab}}.$$

Since $(1, \mathrm{prod})\colon S_\Sigma \ltimes \Gamma_F^\Sigma \to \Gamma_F^{\mathrm{ab}}$ is surjective, $\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}}$ can be any element of Γ_F^{ab}. Since we are aiming to construct a plectic Taniyama element $\widetilde{f}_\Phi(\gamma)$ for any $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ using the Cartesian diagram (4.3.1.2), this means we need to choose a splitting[3] $\chi_F\colon \Gamma_F^{\mathrm{ab}} \to \mathbb{A}_{F,f}^\times / F_{>0}^\times$ to $r_F\colon \mathbb{A}_{F,f}^\times / F_{>0}^\times \to \Gamma_F^{\mathrm{ab}}$.

4.3.4 (Splitting of r_F)**.** Let us look at the middle row in (4.3.1.1), i. e. the short exact sequence of topological abelian groups

$$0 \longrightarrow \ker(r_F) \xrightarrow{\kappa_F} \mathbb{A}_{F,f}^\times / F_{>0}^\times \xrightarrow{r_F} \Gamma_F^{\mathrm{ab}} \longrightarrow 0,$$

[3] We choose the notation χ_F for this splitting as it is going to play a similar role as the cyclotomic character χ_{cyc}.

where κ_F temporarily denotes the inclusion $\ker(r_F) \hookrightarrow \mathbb{A}_{F,f}^\times/F_{>0}^\times$. The Splitting Lemma [Hat02, p. 147] tells us that (in the category of abelian groups) it is equivalent to find a map χ_F such that $r_F \circ \chi_F = \mathrm{id}_{\Gamma_F^{\mathrm{ab}}}$ or a map ω_F such that $\omega_F \circ \kappa_F = \mathrm{id}_{\ker(r_F)}$. We call either of χ_F and ω_F a splitting, and if they exist they are related by the identity $\mathrm{im}(\chi_F) = \ker(\omega_F)$. This situation is illustrated by the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker(r_F) \arrow[r, "\kappa_F"] & \mathbb{A}_{F,f}^\times/F_{>0}^\times \arrow[r, "r_F"] \arrow[l, bend right, dashed, "\omega_F"'] & \Gamma_F^{\mathrm{ab}} \arrow[r] \arrow[l, bend left, dashed, "\chi_F"] & 0. \end{tikzcd}$$

As mentioned in (4.3.1), $\ker(r_F)$ is (uniquely) divisible, hence an injective object in the category of abelian groups. In turn, the injectivity of $\ker(r_F)$ implies the existence of the dashed arrow in the commutative diagram

$$\begin{tikzcd} \ker(r_F) \arrow[r, hook, "\kappa_F"] \arrow[d, "\mathrm{id}"] & \mathbb{A}_{F,f}^\times/F_{>0}^\times \arrow[dl, dashed, "\omega_F"] \\ \ker(r_F). & \end{tikzcd}$$

This diagram precisely means that ω_F is a splitting of κ_F, and we denote the associated splitting of r_F by χ_F.

We want the splitting χ_F of r_F to be compatible with the lift $\chi_{\mathrm{cyc}}(u)\alpha F_{>0}^\times$ of $V_{F/\mathbb{Q}}(u)\prod_{x\in\Sigma} c_x^{\alpha_x}$ under r_F as in (4.3.2). This is guaranteed by the following lemma.

4.3.5 Lemma (Extra properties of χ_F)**.** *We use the notation of* (4.3.4)*. Then*

(4.3.5.a) Restricted to the subgroup $\mathfrak{c} \subset \Gamma_F^{\mathrm{ab}}$, the splitting χ_F is an inverse to the isomorphism of (4.2.1)

$$r_F \colon F^\times/F_{>0}^\times \xrightarrow{\sim} \mathfrak{c}.$$

(4.3.5.b) We can choose the splitting χ_F so that the following diagram commutes

$$\begin{tikzcd} \hat{\mathbb{Z}}^\times \arrow[d, "i_{F/\mathbb{Q}}"'] & \Gamma_{\mathbb{Q}}^{\mathrm{ab}} \arrow[l, "\chi_{\mathrm{cyc}}"', "\sim"] \arrow[d, "V_{F/\mathbb{Q}}"] \\ \mathbb{A}_{F,f}^\times/F_{>0}^\times & \Gamma_F^{\mathrm{ab}}. \arrow[l, "\chi_F"'] \end{tikzcd}$$

Proof. To prove (4.3.5.a), notice that ω_F restricted to $F^\times/F_{>0}^\times$ is a group homomorphism with domain a finite group and target a uniquely divisible group, hence ω_F must be trivial. This easily translates into the desired property of χ_F: Let

$c' \in \mathfrak{c}$. We need to show $\chi_F(c') = \xi(c')$, where $\xi\colon \mathfrak{c} \to F^\times/F^\times_{>0}$ denotes the inverse of $r_F\colon F^\times/F^\times_{>0} \xrightarrow{\sim} \mathfrak{c}$.

We see that $\xi(c') \in F^\times/F^\times_{>0} \subset \ker(\omega_F) = \operatorname{im}(\chi_F)$, so there exists a $\delta \in \Gamma_F^{\mathrm{ab}}$ such that $\chi_F(\delta) = \xi(c')$. We calculate

$$\chi_F(c') = \chi_F r_F \xi(c') = \chi_F r_F \chi_F(\delta) = \chi_F(\delta),$$

where the first equality uses that ξ is inverse to r_F, the second equality follows from the choice of δ, and the third equality holds since $r_F\chi_F = \mathrm{id}$.

To prove (4.3.5.b), let us first look at the commutative diagram

$$\begin{tikzcd} & & \hat{\mathbb{Z}}^\times \arrow[r, "\sim", "r_{\mathbb{Q}}"'] \arrow[d, hook, "i_{F/\mathbb{Q}}"] & \Gamma_{\mathbb{Q}}^{\mathrm{ab}} \arrow[d, hook, "V_{F/\mathbb{Q}}"] & \\ 0 \arrow[r] & \ker(r_F) \arrow[r, "\kappa_F"] & \mathbb{A}_{F,f}^\times/F^\times_{>0} \arrow[r, "r_F"] & \Gamma_F^{\mathrm{ab}} \arrow[r] & 0. \end{tikzcd} \tag{4.3.5.1}$$

It shows $\operatorname{im}(i_{F/\mathbb{Q}}) \cap \ker(r_F) = 0$, hence the composition

$$\ker(r_F) \xrightarrow{\kappa_F} \mathbb{A}_{F,f}^\times/F^\times_{>0} \longrightarrow \mathbb{A}_{F,f}^\times/\hat{\mathbb{Z}}^\times F^\times_{>0}$$

is injective, where the second arrow is the quotient map. Again since $\ker(r_F)$ is divisible, the dashed arrow in the following diagram

$$\begin{tikzcd} \ker(r_F) \arrow[r, hook, "\kappa_F"] \arrow[d, "\mathrm{id}"] & \mathbb{A}_{F,f}^\times/\hat{\mathbb{Z}}^\times F^\times_{>0} \arrow[dl, dashed, "\omega'_F"] \\ \ker(r_F) & \end{tikzcd}$$

exists. Then the composite

$$\omega_F\colon \mathbb{A}_{F,f}^\times/F^\times_{>0} \longrightarrow \mathbb{A}_{F,f}^\times/\hat{\mathbb{Z}}^\times F^\times_{>0} \xrightarrow{\omega'_F} \ker(r_F)$$

is a splitting of κ_F satisfying $\omega_F(\hat{\mathbb{Z}}^\times) = 0$. We claim that the splitting χ_F associated to such an ω_F makes the diagram (4.3.5.b) commutative, where we proceed in a similar way as at the end of the proof of (4.3.5.a):

Let $\gamma \in \Gamma_{\mathbb{Q}}^{\mathrm{ab}}$ and let $z := \chi_{\mathrm{cyc}}(\gamma) \in \hat{\mathbb{Z}}^\times$, so that $r_{\mathbb{Q}}(z) = \gamma$. Since $i_{F/\mathbb{Q}}(z) \in \ker(\omega_F) = \operatorname{im}(\chi_F)$, there exists an element $\delta \in \Gamma_F^{\mathrm{ab}}$ such that $\chi_F(\delta) = i_{F/\mathbb{Q}}(z)$. We calculate

$$\chi_F V_{F/\mathbb{Q}}(\gamma) = \chi_F V_{F/\mathbb{Q}} r_{\mathbb{Q}}(z) = \chi_F r_F i_{F/\mathbb{Q}}(z) = \chi_F r_F \chi_F(\delta) = \chi_F(\delta) = i_{F/\mathbb{Q}}(z),$$

where the first equality uses the definition of z, the second equality follows from

(4.3.5.1), the third and fifth equality use the definition of δ, and the fourth equality uses $r_F \chi_F = \mathrm{id}$. Finally, using the definition of z one more time, we conclude

$$\chi_F V_{F/\mathbb{Q}}(\gamma) = i_{F/\mathbb{Q}}(z) = i_{F/\mathbb{Q}} \chi_{\mathrm{cyc}}(\gamma),$$

i. e. (4.3.5.b) commutes. $\square$

From now on, we fix a splitting χ_F as in (4.3.4) satisfying the additional property (4.3.5.b). We are ready to define the plectic Taniyama element $\widetilde{f}_\Phi(\gamma)$ for arbitrary $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$.

4.3.6 (Plectic Taniyama element)**.** Let $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$. By the Cartesian diagram (4.3.1.2), there exists a unique element $\widetilde{f}_\Phi(\gamma) \in \mathbb{A}_{K,f}^\times / K^\times$ such that

- $r_K(\widetilde{f}_\Phi(\gamma)) = \widetilde{F}_\Phi(\gamma)$ and
- $N_{K/F}(\widetilde{f}_\Phi(\gamma)) = \chi_F(\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}})$.

We call the map $\widetilde{f}_\Phi\colon \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \to \mathbb{A}_{K,f}^\times / K^\times$ the *plectic Taniyama element.* In diagrammatic form, $\widetilde{f}_\Phi(\gamma)$ is the unique element such that

$$\begin{tikzcd}
\widetilde{f}_\Phi(\gamma) \arrow[r, mapsto, dashed, "r_K"] \arrow[d, mapsto, dashed, "N_{K/F}"] & \widetilde{F}_\Phi(\gamma) \arrow[d, mapsto, "\mathrm{res}"] \\
\chi_F(\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}}) \arrow[r, mapsto, "r_F"] & \widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}}
\end{tikzcd}$$

commutes.

4.3.7 Remark. Clearly this definition of $\widetilde{f}_\Phi$ depends on the choice of χ_F. However let us show that Definition (4.3.6) of $\widetilde{f}_\Phi$ extends Nekovář's definition (4.2.7), and hence by (4.2.7.b) in particular extends the Taniyama element f_Φ of (3.1.2):

Let $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_1$. We use the notation of (4.3.3). There we saw that $\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}} = V_{F/\mathbb{Q}}(u) \prod_{x \in \Sigma} c_x^{\alpha_x + b_x}$ and that (4.2.7) defines $\widetilde{f}_\Phi(\gamma)$ as the unique element such that

$$\begin{tikzcd}
\widetilde{f}_\Phi(\gamma) \arrow[r, mapsto, dashed, "r_K"] \arrow[d, mapsto, dashed, "N_{K/F}"] & \widetilde{F}_\Phi(\gamma) \arrow[d, mapsto, "\mathrm{res}"] \\
\chi_{\mathrm{cyc}}(u)\alpha' F_{>0}^\times \arrow[r, mapsto, "r_F"] & V_{F/\mathbb{Q}}(u) \prod_{x \in \Sigma} c_x^{\alpha_x + b_x}
\end{tikzcd} \tag{4.3.7.1}$$

commutes.

On the other hand, (4.3.5) implies

$$\chi_F\left(V_{F/\mathbb{Q}}(u) \prod_{x \in \Sigma} c_x^{\alpha_x + b_x}\right) = \chi_{\mathrm{cyc}}(u)\alpha' F_{>0}^\times,$$

hence the defining diagram in (4.3.6) is precisely the same as (4.3.7.1), thus the two definitions of $\widetilde{f}_\Phi(\gamma)$ agree.

4.3.8 Remark. Since (4.3.1.2) is Cartesian, the splitting χ_F of the bottom horizontal map induces a splitting, let us denote it by χ_K, of the top horizontal map: For $\gamma \in \Gamma_K^{\mathrm{ab}}$, the element $\chi_K(\gamma) \in \mathbb{A}_{K,f}^\times/K^\times$ is the unique element such that

$$\begin{tikzcd}
\chi_K(\gamma) \arrow[r, mapsto, dashed, "r_K"] \arrow[d, mapsto, dashed, "N_{K/F}"] & \gamma \arrow[d, mapsto, "\mathrm{res}"] \\
\chi_F(\gamma|_{F^{\mathrm{ab}}}) \arrow[r, mapsto, "r_F"] & \gamma|_{F^{\mathrm{ab}}}
\end{tikzcd}$$

commutes. Then

$$\widetilde{f}_\Phi = \chi_K \circ \widetilde{F}_\Phi.$$

4.4 Variants of the Hilbert modular variety, and plectic action on their CM points

Let G^R be a group that differs only in the centre from $R_{F/\mathbb{Q}}\,\mathrm{GL}_2$, i. e. G^R fits into the Cartesian diagram (1.2.0.3) and R denotes an algebraic torus over $\mathbb{Q}$ with $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$. For example, for $R = \mathbb{G}_m$ the group $G^{\mathbb{G}_m}$ is the group G considered in (3.2.2). In (4.2.10) we saw that the subgroup $\mathrm{Aut}_F(F \otimes_\mathbb{Q} \overline{\mathbb{Q}})_0$ acts on the CM points of $\mathrm{Sh}(G, X)$. The goal of this section is to find an analogue of this plectic action on CM points for the Shimura variety $\mathrm{Sh}(G^R, X^R)$, for any R.

Let us start by recalling the notation of Section 3.2 and defining the variants $\mathrm{Sh}(G^R, X^R)$ of the Hilbert modular variety.

4.4.1. (Notation) As in Section 3.2 let

- $V := F^2$ be a 2-dimensional F-vector space.
- $G_1 := R_{F/\mathbb{Q}}\,\mathrm{GL}_2$ be the restriction of scalars of GL_2.
- $\psi\colon V \times V \to \mathbb{Q}$ be the F-compatible, $\mathbb{Q}$-bilinear alternating form given by

$$\psi\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \mathrm{Tr}_{F/\mathbb{Q}} \circ \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}.$$

- $h_0\colon \mathbb{S} \to (G_1)_\mathbb{R}$ be the morphism of algebraic groups that is (on real points) given by

$$h_0(i) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G_1(\mathbb{R}) = \mathrm{GL}_2(F \otimes \mathbb{R}).$$

As noted in (3.2.2), the form $(v, w) \mapsto \psi(v, h_0(i)w)$ is positive definite on $V(\mathbb{R})$.

- X_1 be the $G_1(\mathbb{R})$-conjugacy class of h_0.

4.4.2. (Variants of the Hilbert modular variety) Let R be an algebraic torus over $\mathbb{Q}$ with embeddings $\mathbb{G}_m \hookrightarrow R \hookrightarrow R_{F/\mathbb{Q}}\mathbb{G}_m$. We will always assume that the composition of these two embeddings is given by the 'usual' embedding $\mathbb{G}_m \hookrightarrow R_{F/\mathbb{Q}}\mathbb{G}_m$, which on $\mathbb{Q}$-points is given by the inclusion $\mathbb{Q}^\times \hookrightarrow F^\times$. Define the algebraic group G^R over $\mathbb{Q}$ by the Cartesian diagram

$$\begin{tikzcd} G^R \arrow[r, hook] \arrow[d] & R_{F/\mathbb{Q}}\,\mathrm{GL}_2 \arrow[d, "R_{F/\mathbb{Q}}(\det)"] \\ R \arrow[r, hook] & R_{F/\mathbb{Q}}\mathbb{G}_m. \end{tikzcd} \tag{4.4.2.1}$$

This is diagram (1.2.0.3).

Note that $R = R_{F/\mathbb{Q}}\mathbb{G}_m$ results in $G^{R_{F/\mathbb{Q}}\mathbb{G}_m} = G_1$. For $R = \mathbb{G}_m$, the above diagram is the same as (3.2.2.1), so $G^{\mathbb{G}_m}$ is equal to the group G considered there. Of course, $\mathbb{G}_m \hookrightarrow R \hookrightarrow R_{F/\mathbb{Q}}\mathbb{G}_m$ yields inclusions

$$G^{\mathbb{G}_m} \hookrightarrow G^R \hookrightarrow G_1.$$

In (3.2.2), we also saw that h_0 actually takes values in $(G^{\mathbb{G}_m})_\mathbb{R}$, so in particular we can view h_0 as a morphism $\mathbb{S} \to (G^R)_\mathbb{R}$ and define X^R to be the $G^R(\mathbb{R})$-conjugacy class of h_0. Clearly, we thus have

$$X^{\mathbb{G}_m} \subset X^R \subset X^{R_{F/\mathbb{Q}}\mathbb{G}_m} = X_1.$$

In (6.2.9) we will calculate that G^R has the same adjoint group as G_1. Axioms [Del79, (2.1.1.2-3)] obviously only depend on the adjoint group, and the same is true for [Del79, (2.1.1.1)], see [Mil17, after Def. 5.5, p. 55]. Thus (G^R, X^R) is a Shimura datum and the associated Shimura variety $\mathrm{Sh}(G^R, X^R)$ is the desired *variant of the Hilbert modular variety.*

Moreover, similar to (3.2.3) we see that (G^R, X^R) has the same reflex field as (G_1, X_1), so $E(G^R, X^R) = \mathbb{Q}$.

For the remainder of this section, fix an algebraic torus R as above. We will often drop the R from notation and simply write (G, X) for (G^R, X^R) if there is no danger of confusion.

4.4.3 Remark. The Shimura varieties $\mathrm{Sh}(G^R, X^R)$ interpolate between the two extreme cases, the Hilbert modular variety (for $R = R_{F/\mathbb{Q}}\mathbb{G}_m$) and the PEL Hilbert modular variety (for $R = \mathbb{G}_m$). The groups G^R are precisely the groups that differ only in the centre from $G_1 = R_{F/\mathbb{Q}} \mathrm{GL}_2$ in the sense of (1.2.0.2). We will see in (6.2.9) that this implies that $(G^R)^{\mathrm{der}} = G_1^{\mathrm{der}}$ and $(G^R)^{\mathrm{ad}} = G_1^{\mathrm{ad}}$.

Moreover, similar to (3.2.1.2) and (3.2.2.2), there is a more explicit description of the set X^R in terms of upper and lower half planes, see (5.1.6).

4.4.4. (Action of $F^\times$ on F-compatible bilinear forms) We let $F^\times$ act on the set of all F-compatible, $\mathbb{Q}$-bilinear alternating forms ψ' as follows:

$$(f \cdot \psi')(v, w) := \psi'(fv, w), \quad f \in F^\times,\ v, w \in V.$$

Let us calculate the effect of an element $g \in G^R(\mathbb{Q})$ on the form ψ:

$$\psi(gv, gw) = \mathrm{Tr}_{F/\mathbb{Q}}(\det g \det(v, w)) = \mathrm{Tr}_{F/\mathbb{Q}}(\det(\det(g)v, w)) = \psi(\det(g)v, w).$$

Now by definition of G^R this means that this form is an $R(\mathbb{Q})$-multiple of ψ. The same calculation also shows the converse, i. e. that an element $g \in \mathrm{GL}_2(F)$ preserves $R(\mathbb{Q})\psi$ if and only if $g \in G^R(\mathbb{Q})$. Hence

$$G^R(\mathbb{Q}) = \{g \in \mathrm{GL}_2(F) \mid g^*(\psi) \in R(\mathbb{Q})\psi\}. \tag{4.4.4.1}$$

In (4.2.10), the key to writing down a plectic Galois action on the CM points of the PEL Hilbert modular variety $\mathrm{Sh}(G^{\mathbb{G}_m}, X^{\mathbb{G}_m})$ was to understand the points of $\mathrm{Sh}(G^{\mathbb{G}_m}, X^{\mathbb{G}_m})$ in terms of abelian varieties with real multiplication and extra structure as in (3.2.6). We will now prove (3.2.6) in this more general set-up:

4.4.5 Theorem (Moduli interpretation for $\mathrm{Sh}(G^R, X^R)$)**.** *Let $(G, X) = (G^R, X^R)$ be the Shimura datum constructed in* (4.4.2). *Then the complex points of the Shimura variety*

$$\mathrm{Sh}(G, X)(\mathbb{C}) := \varprojlim_U G(\mathbb{Q})\backslash\left[X \times G(\mathbb{A}_f)/U\right] = G(\mathbb{Q})\backslash\left[X \times G(\mathbb{A}_f)/\overline{Z(\mathbb{Q})}\right]$$

are in bijection with isomorphism classes of quadruples $\left(A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})}\right)$*, where*

- *A is a complex abelian variety of dimension $[F : \mathbb{Q}]$,*
- *$i\colon F \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a ring homomorphism,*
- *s is an F-compatible polarisation of A, and*
- *$\eta\colon V(\mathbb{A}_f) \xrightarrow{\sim} \widehat{V}(A)$ is an $\mathbb{A}_{F,f}$-module-isomorphism sending the class $R(\mathbb{A}_f)\psi$ to $R(\mathbb{A}_f)s$,*

satisfying the condition

(4.4.5.a) *There exists an F-linear isomorphism $a\colon H_1(A,\mathbb{Q}) \xrightarrow{\sim} V$ that sends $R(\mathbb{Q})s$ to $R(\mathbb{Q})\psi$ and satisfies $a \circ h_A \circ a^{-1} \in X$, where $h_A\colon \mathbb{S} \to \mathrm{End}(H_1(A,\mathbb{R}))$ denotes the Hodge structure on $H_1(A,\mathbb{Q})$.*

Here, the projective limit is taken over all compact open subgroups $U \subset G(\mathbb{A}_f)$. Moreover, $Z = Z^R$ denotes the centre of $G = G^R$, and $\overline{Z(\mathbb{Q})}$ denotes the closure of $Z(\mathbb{Q})$ inside $G(\mathbb{A}_f)$.

4.4.6 Remark. In analogy to (3.2.5.d), we call quadruples $(A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})})$ and $(A', i', R(\mathbb{Q})s', \eta'\overline{Z(\mathbb{Q})})$ isomorphic if there exists a quasi-isogeny $f\colon A \to A'$ that is F-linear (with respect to i and i'), sends $R(\mathbb{Q})s$ to $R(\mathbb{Q})s'$ and satisfies $\eta'\overline{Z(\mathbb{Q})} = f \circ \eta\overline{Z(\mathbb{Q})}$.

Proof of Theorem (4.4.5). Let $\mathcal{A} = \mathcal{A}^R$ denote the set of isomorphism classes of quadruples $(A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})})$ as described above. The strategy of the proof is modelled on the (PEL) Siegel case outlined in [Mil17, §6, especially prop. 6.3].

- Definition of α.

Let

$$\begin{aligned} \alpha\colon \mathrm{Sh}(G,X)(\mathbb{C}) &\longrightarrow \mathcal{A} \\ [h,g] &\longmapsto \left[A_h, i_0, R(\mathbb{Q})\psi, g\overline{Z(\mathbb{Q})}\right], \end{aligned}$$

where

- A_h is the abelian variety associated to the $\mathbb{Q}$-rational Hodge structure (V,h). In particular, we identify $H_1(A_h,\mathbb{Q})$ with V.
- $i_0\colon F \to \mathrm{End}(V)$ is the usual F-structure on V, which does respect the Hodge structure h and hence induces $i_0\colon F \to \mathrm{End}(V,h) = \mathrm{End}(A_h) \otimes_{\mathbb{Z}} \mathbb{Q}$,
- ψ is viewed as an F-compatible polarisation of A_h (using the identification $H_1(A_h,\mathbb{Q}) = V$), and
- g is viewed as the map $g\colon V(\mathbb{A}_f) \to V(\mathbb{A}_f) = \widehat{V}(A_h)$, again using $V(\mathbb{A}_f) = H_1(A_h,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}_f = \widehat{V}(A_h)$.

- The map α is well-defined.

First of all, $(A_h, i_0, R(\mathbb{Q})\psi, g\overline{Z(\mathbb{Q})})$ satisfies the necessary conditions: By (the adelic version of) (4.4.4.1) we see that g preserves $R(\mathbb{A}_f)\psi$, and condition (4.4.5.a) holds with $a = \mathrm{id}$ because we identified $H_1(A_h,\mathbb{Q})$ with V. Thus $[A_h, i_0, R(\mathbb{Q})\psi, g\overline{Z(\mathbb{Q})}]$ is an element of $\mathcal{A}$.

Secondly, choosing (h', g') in the same class as (h, g), we see that $h' = q \circ h \circ q^{-1}$ and $g' = qgz$ for some $q \in G(\mathbb{Q})$ and $z \in \overline{Z(\mathbb{Q})}$. Then q defines an F-compatible isomorphism of Hodge structures $H_1(A_h, \mathbb{Q}) \xrightarrow{\sim} H_1(A_{h'}, \mathbb{Q})$, i. e. an F-compatible isogeny $A_h \to A_{h'}$. It moreover preserves $R(\mathbb{Q})\psi$, and together with $g' = qgz$ this shows that q defines an isomorphism of quadruples $(A_h, i_0, R(\mathbb{Q})\psi, g\overline{Z(\mathbb{Q})}) \xrightarrow{\sim} (A_{h'}, i_0, R(\mathbb{Q})\psi, g'\overline{Z(\mathbb{Q})})$. Thus α is well-defined.

- Definition of β.

In the other direction, we define the map

$$\begin{aligned} \beta \colon \mathcal{A} &\longrightarrow \mathrm{Sh}(G, X)(\mathbb{C}) \\ \left[A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})}\right] &\longmapsto [a \circ h_A \circ a^{-1}, a \circ \eta], \end{aligned}$$

where h_A denotes the Hodge structure of A and a is an isomorphism as in (4.4.5.a).

- The map β is well-defined.

First of all, part of condition (4.4.5.a) says that $a \circ h_A \circ a^{-1} \in X$. Also, $[a \circ h_A \circ a^{-1}, a \circ \eta] \in \mathrm{Sh}(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash[X \times G(\mathbb{A}_f)/\overline{Z(\mathbb{Q})}]$ depends only on the class $\eta\overline{Z(\mathbb{Q})}$, i. e. is unchanged if η is replaced by $\eta \circ z$ for any $z \in \overline{Z(\mathbb{Q})}$. Moreover, $a \circ \eta$ is clearly an $\mathbb{A}_{F,f}$-linear endomorphism of $V(\mathbb{A}_f)$, hence an element of $(R_{F/\mathbb{Q}}\,\mathrm{GL}_2)(\mathbb{A}_f)$, and by (4.4.5.a) and the condition on η we see that $a \circ \eta$ preserves $R(\mathbb{A}_f)\psi$. Thus by (the adelic version of) (4.4.4.1) we conclude that $a \circ \eta \in G(\mathbb{A}_f)$ as desired, so $[a \circ h_A \circ a^{-1}, a \circ \eta]$ is an element of $\mathrm{Sh}(G, X)(\mathbb{C})$.

Secondly, if $a' \colon H_1(A, \mathbb{Q}) \xrightarrow{\sim} V$ is another isomorphism satisfying (4.4.5.a), then $q := a' \circ a^{-1} \colon V \to V$ preserves $R(\mathbb{Q})\psi$ (by (4.4.5.a) for both a and a'), hence by (4.4.4.1) is an element of $G(\mathbb{Q})$. This shows

$$[a' \circ h_A \circ (a')^{-1}, a' \circ \eta] = [q \circ a \circ h_A \circ a^{-1} \circ q^{-1}, q \circ a \circ \eta] = [a \circ h_A \circ a^{-1}, a \circ \eta],$$

so β is independent of the choice of a.

Thirdly, choose $(A', i', R(\mathbb{Q})s', \eta'\overline{Z(\mathbb{Q})})$ in the same class as $(A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})})$. This means there is an isogeny $f \colon A \to A'$ with the properties in (4.4.6). Look at $a' := a \circ f^{-1} \colon H_1(A', \mathbb{Q}) \xrightarrow{\sim} V$. Then a' satisfies (4.4.5.a) for the quadruple $(A', i', R(\mathbb{Q})s', \eta'\overline{Z(\mathbb{Q})})$: It sends $R(\mathbb{Q})s'$ to $R(\mathbb{Q})\psi$ by combining (4.4.5.a) for a with (4.4.6); since f is an isogeny, this means $h_{A'} = f \circ h_A \circ f^{-1}$, hence

$$a' \circ h_{A'} \circ (a')^{-1} = a \circ f^{-1} \circ h_{A'} \circ f \circ a^{-1} = a \circ h_A \circ a^{-1} \in X.$$

Thus we may take a' to compute β using the representative $(A', i', R(\mathbb{Q})s', \eta'\overline{Z(\mathbb{Q})})$

of $[A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})}]$, resulting in

$$(a' \circ h_{A'} \circ a', a' \circ \eta') = (a \circ f^{-1} \circ h_{A'} \circ f \circ a^{-1}, a \circ f^{-1} \circ f \circ \eta) = (a \circ h_A \circ a^{-1}, a \circ \eta).$$

Thus β is well-defined.

- α and β are inverses of each other.

Let us first show that $\alpha \circ \beta = \mathrm{id}$: The map β sends $[A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})}]$ to $[a \circ h_A \circ a^{-1}, a \circ \eta]$; applying α to this results in $[A_{a\circ h_A \circ a^{-1}}, i_0, R(\mathbb{Q})\psi, a \circ \eta\overline{Z(\mathbb{Q})}]$. The Hodge structure of $A_{a\circ h_A\circ a^{-1}}$ is given by $h_{A_{a\circ h_A\circ a^{-1}}} = a \circ h_A \circ a^{-1}$, hence a defines an isogeny $A \to A_{a\circ h_A\circ a^{-1}}$ sending $R(\mathbb{Q})s$ to $R(\mathbb{Q})\psi$ by (4.4.5.a). Thus the isogeny a satisfies the conditions of (4.4.6), and we conclude

$$\left[A_{a\circ h_A\circ a^{-1}}, i_0, R(\mathbb{Q})\psi, a \circ \eta\overline{Z(\mathbb{Q})}\right] = \left[A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})}\right],$$

i. e. $\alpha \circ \beta = \mathrm{id}$.

Now let us show that $\beta \circ \alpha = \mathrm{id}$: The map α sends $[h, g]$ to $[A_h, i_0, R(\mathbb{Q})\psi, g\overline{Z(\mathbb{Q})}]$; applying β using $a = \mathrm{id}$, corresponding to the canonical identification $H_1(A_h, \mathbb{Q}) = V$, gives back $[h, g]$. Thus

$$\beta \circ \alpha([h, g]) = [h, g].$$

□

4.4.7 Remark. As mentioned before, the case $R = \mathbb{G}_m$ yields the PEL Hilbert modular variety, in which case Theorem (4.4.5) was already mentioned in (3.2.6), and is proved in [Del71, 4.11].

For most authors, "the" Hilbert modular variety is the Shimura variety associated to $G_1 = R_{F/\mathbb{Q}}\,\mathrm{GL}_2$, which corresponds to the case $R = R_{F/\mathbb{Q}}\mathbb{G}_m$, see e. g. [vdG88, Ch. I.7]. In this case, Theorem (4.4.5) is a special case of [Del71, 4.14] and simplifies significantly:

The Hodge structure $H_1(A, \mathbb{Q})$ of any abelian variety A of dimension $[F : \mathbb{Q}]$ with real multiplication by F is a two-dimensional F-vector space, hence by the proof of Lemma (2.3.6) the $F^\times$-class of any non-zero, alternating, $\mathbb{Q}$-bilinear, F-compatible form s consists of all such forms and hence always contains a polarisation. Moreover, $\bigwedge_F^2 H_1(A, \mathbb{Q}) \cong F$ also implies that any F-linear isogeny will preserve the $F^\times$-class of such a form. In conclusion, we can simply forget about the polarisation as part of the data. Moreover, condition (4.4.5.a) is automatic because the $G_1(\mathbb{R})$-conjugacy class X_1 consists of *all* F-compatible (polarisable) $\mathbb{Q}$-Hodge structures on $V = F^2$.

4.4.8 (Canonical model of $\mathrm{Sh}(G^R, X^R)$)**.** In the PEL case $R = \mathbb{G}_m$, the canonical model of the Shimura variety $\mathrm{Sh}(G^R, X^R)$ is a fine moduli space for a certain moduli problem involving abelian varieties with real multiplication, polarisation, and level structure, see (3.2.8).

For general R, the canonical model of the Shimura variety $\mathrm{Sh}(G^R, X^R)$ is a coarse moduli space for a similar functor modelled on the set $\mathcal{A}^R$ in the proof of Theorem (4.4.5), see [DT04, Remarque 2.8.1)]. The case $R = R_{F/\mathbb{Q}}\mathbb{G}_m$ of the Hilbert modular variety $\mathrm{Sh}(G_1, X_1)$ is explained in more detail in [TX16, §2.3, in particular after Prop. 2.4]. See also [Hid04, §4.2.1], [Liu16, §2.4.1] and [DS17, §2.5 and §2.7].

4.4.9. (Aut($\mathbb{C}$)-equivariance) In particular, (4.4.8) implies that for all R the bijection of Theorem (4.4.5) is Aut($\mathbb{C}$)-equivariant, i. e. conjugate points correspond to conjugate abelian varieties. Since [DT04, 2.8.1)] does not give many details and we will crucially use this equivariance, let us briefly explain an alternative proof of this fact, relying only on the coarse moduli space interpretation in the case $R = R_{F/\mathbb{Q}}\mathbb{G}_m$.

Namely, look at the following commutative diagram

$$\begin{tikzcd} \mathcal{A}^R(\mathbb{C}) \arrow[r, hook] \arrow[d, "\sim"] & \mathcal{A}_1(\mathbb{C}) \arrow[d, "\sim"] \\ \mathrm{Sh}(G^R, X^R)(\mathbb{C}) \arrow[r, hook] & \mathrm{Sh}(G_1, X_1)(\mathbb{C}), \end{tikzcd}$$

where $\mathcal{A}_1(\mathbb{C}) := \mathcal{A}^{R_{F/\mathbb{Q}}\mathbb{G}_m}(\mathbb{C})$. In this diagram, the vertical maps are the bijections of Theorem (4.4.5). The bottom horizontal arrow is induced from the inclusion $G^R \subset G_1$ and is a closed immersion by [Del71, 1.15.1], and the top horizontal arrow is chosen to make the diagram commute; explicitly, it maps $[A, i, R(\mathbb{Q})s, \eta\overline{Z^R(\mathbb{Q})}]$ to $[A, i, F^\times s, \eta\overline{Z_1(\mathbb{Q})}]$.

If we let Aut($\mathbb{C}$) act on $\mathcal{A}^R(\mathbb{C})$ and $\mathcal{A}_1(\mathbb{C})$ by conjugating the abelian variety (and its extra structure), then the top horizontal arrow is clearly Aut($\mathbb{C}$)-equivariant. By [Del71, 5.4], the bottom horizontal arrow is the base change of a map between the canonical models defined over $\mathbb{Q}$, thus this arrow is also Aut($\mathbb{C}$)-equivariant. Finally, by [TX16] the canonical model for $\mathrm{Sh}(G_1, X_1)$ is a coarse moduli space for such a functor, thus the right vertical map is Aut($\mathbb{C}$)-equivariant, too. Hence the left vertical map is Aut($\mathbb{C}$)-equivariant, which is what we wanted to show.

4.4.10 (Special points of $\mathrm{Sh}(G^R, X^R)$)**.** In analogy to (3.2.7), special points correspond to CM abelian varieties. If $[h, g]$ corresponds to $[A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})}]$ under the bijection in (4.4.5), then $[h, g]$ is special if and only if A has CM by a CM algebra, and it again turns out that this CM algebra is automatically a CM field that is a totally imaginary quadratic extension of F.

Now let $[A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})}]$ be a special point of $\mathrm{Sh}(G, X)$, where K is a CM field and $i\colon K \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an extension of $i\colon F \to \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. By Lemma (2.3.6) the polarisation s is automatically K-compatible. Thus (A, i, s) is a triple as in (2.3.4), so we can find $\Phi, \Theta, \mathfrak{a}, t$ such that

$$[A, i, R(\mathbb{Q})s, \eta\overline{Z(\mathbb{Q})}] = [\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, R(\mathbb{Q})E_t, \Theta^{-1} \circ \eta\overline{Z(\mathbb{Q})}].$$

We claim that conversely, as long as t satisfies (2.3.4.b) and η sends $R(\mathbb{A}_f)\psi_0$ to $R(\mathbb{A}_f)E_t$, a quadruple of the form $[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, R(\mathbb{Q})E_t, \eta\overline{Z(\mathbb{Q})}]$ always satisfies (4.4.5.a) and hence defines a point of $\mathrm{Sh}(G, X)$.

Proof of claim. We identify the tangent space at 0 of $\mathbb{C}^\Phi/\Phi(\mathfrak{a})$ with $\mathbb{C}^\Phi$. Then, for $f \in F$, we have

$$\mathrm{Tr}\left(i_\Phi(f)|\,\mathrm{Tgt}_0(\mathbb{C}^\Phi/\Phi(\mathfrak{a}))\right) = \mathrm{Tr}\left((\varphi(f))_{\varphi\in\Phi}|\,\mathbb{C}^\Phi\right) = \sum_{x\in\Sigma_F} x(f),$$

hence the quadruple fulfills condition (3.2.5.e) (E_t is clearly F-compatible because it is even K-compatible). Since this, in turn, is equivalent to (3.2.4.a) (i. e. condition (4.4.5.a) in the PEL case), this means that there exists an F-linear isomorphism $a\colon H_1(A, \mathbb{Q}) \xrightarrow{\sim} V$ that sends $\mathbb{Q}^\times E_t$ to $\mathbb{Q}^\times\psi_0$ and satisfies $a \circ h_A \circ a^{-1} \in X^{\mathbb{G}_m}$. In particular, a sends $R(\mathbb{Q})E_t$ to $R(\mathbb{Q})\psi_0$, and since $X^{\mathbb{G}_m} \subset X^R$, we conclude that $[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, R(\mathbb{Q})E_t, \eta\overline{Z(\mathbb{Q})}]$ is a point of $\mathcal{A}^R$. □

We continue to suppress uniformizations from notation and will henceforth write CM points of $\mathrm{Sh}(G, X)$ as $[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, R(\mathbb{Q})E_t, \eta\overline{Z(\mathbb{Q})}]$. Here it is enough if some element $t' \in R(\mathbb{Q})t$ satisfies (2.3.4.b), but we usually choose t to satisfy (2.3.4.b) itself.

As promised, we will now define a certain subgroup of the plectic group and an action of this group on the set of CM points of $\mathrm{Sh}(G^R, X^R)$, generalising (4.2.10).

4.4.11. (Subgroup of the plectic group associated to R) As before, let R be a $\mathbb{Q}$-algebraic torus with $\mathbb{G}_m \hookrightarrow R \hookrightarrow R_{F/\mathbb{Q}}\mathbb{G}_m$. We have subgroups $R(\mathbb{Q}) \subset F^\times$ and $R(\mathbb{A}_f) \subset \mathbb{A}_{F,f}^\times$. Let $R(\mathbb{Q})_{>0} := R(\mathbb{Q}) \cap F_{>0}^\times$. We then have an injective map

$$R(\mathbb{A}_f)/R(\mathbb{Q})_{>0} \hookrightarrow \mathbb{A}_{F,f}^\times/F_{>0}^\times.$$

We define the subgroup $(S_\Sigma \ltimes \Gamma_F^\Sigma)_{\mathrm{CM}}^R$ of the plectic group $S_\Sigma \ltimes \Gamma_F^\Sigma$ by the Cartesian diagram

$$\begin{tikzcd}
(S_\Sigma \ltimes \Gamma_F^\Sigma)_{\mathrm{CM}}^R \arrow[r, hook] \arrow[d] & S_\Sigma \ltimes \Gamma_F^\Sigma \arrow[d, "\chi_F\circ(1,\mathrm{prod})"] \\
R(\mathbb{A}_f)/R(\mathbb{Q})_{>0} \arrow[r, hook] & \mathbb{A}_{F,f}^\times/F_{>0}^\times.
\end{tikzcd}$$

We also define

$$\mathrm{Aut}_F(F \otimes_\mathbb{Q} \overline{\mathbb{Q}})_{\mathrm{CM}}^R := \beta_s^{-1}\left((S_\Sigma \ltimes \Gamma_F^\Sigma)_{\mathrm{CM}}^R\right) \subset \mathrm{Aut}_F(F \otimes_\mathbb{Q} \overline{\mathbb{Q}}),$$

which is independent of s by (4.1.9.b).

4.4.12 Remark. For $R = R_{F/\mathbb{Q}}\mathbb{G}_m$ we clearly have

$$(S_\Sigma \ltimes \Gamma_F^\Sigma)_{\mathrm{CM}}^{R_{F/\mathbb{Q}}\mathbb{G}_m} = S_\Sigma \ltimes \Gamma_F^\Sigma, \quad \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_{\mathrm{CM}}^{R_{F/\mathbb{Q}}\mathbb{G}_m} = \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}).$$

For $R = \mathbb{G}_m$, we have $\mathbb{A}_{\mathbb{Q},f}^\times/\mathbb{Q}_{>0}^\times = \hat{\mathbb{Z}}^\times$. The bottom left entries of the defining Cartesian diagrams for the groups $(S_\Sigma \ltimes \Gamma_F^\Sigma)_{\mathrm{CM}}^{\mathbb{G}_m}$ and $(S_\Sigma \ltimes \Gamma_F^\Sigma)_0$ in (4.4.11) and (4.2.8), respectively, are related by the isomorphism $r_{\mathbb{Q}} \colon \hat{\mathbb{Z}}^\times \to \Gamma_{\mathbb{Q}}^{\mathrm{ab}}$ or its inverse $\chi_{\mathrm{cyc}} \colon \Gamma_{\mathbb{Q}}^{\mathrm{ab}} \to \hat{\mathbb{Z}}^\times$. Using the commutative diagrams (2.4.3) and (4.3.5.b), we conclude that

$$(S_\Sigma \ltimes \Gamma_F^\Sigma)_{\mathrm{CM}}^{\mathbb{G}_m} = (S_\Sigma \ltimes \Gamma_F^\Sigma)_0, \quad \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_{\mathrm{CM}}^{\mathbb{G}_m} = \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0.$$

For general R, note that $\mathbb{G}_m \hookrightarrow R$ induces

$$(S_\Sigma \ltimes \Gamma_F^\Sigma)_0 \subset (S_\Sigma \ltimes \Gamma_F^\Sigma)_{\mathrm{CM}}^R, \quad \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0 \subset \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_{\mathrm{CM}}^R.$$

4.4.13. (Plectic action on the CM points of $\mathrm{Sh}(G^R, X^R)$) Let $(G, X) = (G^R, X^R)$ and let $(K, \Phi; \mathfrak{a}, t)$ be a type as in (2.3.5), with K a totally imaginary quadratic extension of F. We look at the CM point

$$\mathcal{P} = \left[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, R(\mathbb{Q})E_t, \eta\overline{Z(\mathbb{Q})}\right]$$

of the Shimura variety $\mathrm{Sh}(G, X)$.

Let $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_{\mathrm{CM}}^R$ and $\widetilde{f} \in \mathbb{A}_{K,f}^\times$ such that $\widetilde{f}_\Phi(\gamma) = \widetilde{f}K^\times \in \mathbb{A}_{K,f}^\times/K^\times$. Let $u \in R(\mathbb{A}_f)$ be such that $\chi_F \circ (1, \mathrm{prod}) \circ \beta_s(\gamma) = uF_{>0}^\times$. The existence of u is guaranteed by the definition of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_{\mathrm{CM}}^R$. Finally denote $\chi := \frac{u}{1+c\widetilde{f}}$, which lies in $F^\times$, see (4.4.13.2) below.

Define

$$\gamma\left[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, R(\mathbb{Q})E_t, \eta\overline{Z(\mathbb{Q})}\right] := \left[\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a}), i_{\gamma\Phi}|_F, R(\mathbb{Q})E_{\chi t}, \widetilde{f} \circ \eta\overline{Z(\mathbb{Q})}\right]. \tag{4.4.13.1}$$

This defines a group action of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_{\mathrm{CM}}^R$ on the set of CM points of $\mathrm{Sh}(G, X)$, extending the action of $\Gamma_{\mathbb{Q}}$.

Proof. There are several things to check in order to show that this is well-defined and in fact does give an action of $(\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}))_{\mathrm{CM}}^R$ on the CM points of $\mathrm{Sh}(G, X)$, namely that (4.4.13.1) defines a point of $\mathrm{Sh}(G, X)$, does not depend on the choice of u, $\widetilde{f}$, or the representative $(\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, R(\mathbb{Q})E_t, \eta\overline{Z(\mathbb{Q})})$ of $\mathcal{P}$ (i. e. of the uniformization), that the resulting point is again a CM point with K-compatible ($R(\mathbb{Q})$-

class of) polarisation, and that this defines a group action:

- (4.4.13.1) does not depend on the choice of u:

A different choice for u must be of the form $u' = au$ with $a \in R(\mathbb{Q})_{>0}$. This changes χ to $\chi' = a\chi$, hence the polarisations $E_{\chi t}$ and $E_{\chi' t}$ lie in the same $R(\mathbb{Q})$-class.

- (4.4.13.1) does not depend on the choice of $\widetilde{f}$:

Let $\widetilde{f}'K^\times = \widetilde{f}K^\times = \widetilde{f}_\Phi(\gamma) \in \mathbb{A}_{K,f}/K^\times$. Then $\widetilde{f}' = k\widetilde{f}$ for some $k \in K^\times$, and k defines an isogeny $\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a}) \to \mathbb{C}^{\gamma\Phi}/\gamma\Phi(k\widetilde{f}\mathfrak{a})$ that sends the polarisation $E_{\chi t}$ to $E_{\frac{\chi}{1+c_k}t}$ and the level structure $\widetilde{f} \circ \eta$ to $k \circ \widetilde{f} \circ \eta$, i. e. k induces an equality between $\left[\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a}), i_{\gamma\Phi}|_F, R(\mathbb{Q})E_{\chi t}, \widetilde{f} \circ \eta\overline{Z(\mathbb{Q})}\right]$ and

$$\left[\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}'\mathfrak{a}), i_{\gamma\Phi}|_F, R(\mathbb{Q})E_{\chi' t}, \widetilde{f}' \circ \eta\overline{Z(\mathbb{Q})}\right],$$

where $\chi' = \frac{u}{1+c_{\widetilde{f}'}} = \frac{\chi}{1+c_k}$.

- (4.4.13.1) does not depend on the choice of uniformization:

By (2.3.5), a different choice of uniformization yields a different representative $(\mathbb{C}^\Phi/\Phi(k\mathfrak{a}), i_\Phi|_F, R(\mathbb{Q})E_{t_2}, k \circ \eta\overline{Z(\mathbb{Q})})$ of $\mathcal{P}$, for some $k \in K^\times$ and $t_2 := \frac{t}{1+c_k}$. The right hand side of (4.4.13.1) then becomes

$$\left[\mathbb{C}^{\gamma\Phi}/\gamma\Phi(k\widetilde{f}\mathfrak{a}), i_{\gamma\Phi}|_F, R(\mathbb{Q})E_{\chi\frac{t}{k\cdot\bar{k}}}, \widetilde{f} \circ k \circ \eta\overline{Z(\mathbb{Q})}\right].$$

But again we can view k as an isogeny $k\colon \mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a}) \to \mathbb{C}^{\gamma\Phi}/\gamma\Phi(k\widetilde{f}\mathfrak{a})$, which induces an equality between $\left[\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a}), i_{\gamma\Phi}|_F, R(\mathbb{Q})E_{\chi t}, \widetilde{f} \circ \eta\overline{Z(\mathbb{Q})}\right]$ and

$$\left[\mathbb{C}^{\gamma\Phi}/\gamma\Phi(k\widetilde{f}\mathfrak{a}), i_{\gamma\Phi}|_F, R(\mathbb{Q})E_{\chi\frac{t}{k\cdot\bar{k}}}, k \circ \widetilde{f} \circ \eta\overline{Z(\mathbb{Q})}\right],$$

as desired.

- (4.4.13.1) defines a CM point of $\mathrm{Sh}(G, X)$:

We need to check that the right hand side of (4.4.13.1) lies in $\mathcal{A}^R$ (see Theorem (4.4.5)). But if we show that $\widetilde{f} \circ \eta$ is compatible with the $R(\mathbb{A}_f)$-classes of the polarisations, and that $R(\mathbb{Q})E_{\chi t}$ contains a polarisation of $\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a})$, then this is guaranteed by (4.4.10).

- $\widetilde{f} \circ \eta$ sends $R(\mathbb{A}_f)\psi_0$ to $R(\mathbb{A}_f)E_{\chi t}$:

For $v, w \in V(\mathbb{A}_f)$, we have

$$\begin{aligned} E_{\chi t}\left(\widetilde{f}\circ\eta(v), \widetilde{f}\circ\eta(w)\right) &= \mathrm{Tr}_{K/\mathbb{Q}}\left(ut\eta(v)\overline{\eta(w)}\right) \\ &= E_t\left(\eta(uv), \eta(w)\right) \\ &= \psi_0\left(urv, w\right), \quad \text{for some } r \in R(\mathbb{A}_f). \end{aligned}$$

Here the first equality uses the definition of $E_{\chi t}$ and of χ, the second equality follows from the $\mathbb{A}_{F,f} \subset \mathbb{A}_{K,f}$-linearity of η and the definition of E_t, and the third equality follows because η sends $R(\mathbb{A}_f)\psi_0$ to $R(\mathbb{A}_f)E_t$.

As u by definition lies in $R(\mathbb{A}_f)$, this calculation precisely shows that $\widetilde{f}\circ\eta$ sends $R(\mathbb{A}_f)\psi_0$ to $R(\mathbb{A}_f)E_{\chi t}$. This explains why we had to restrict to $\gamma \in \mathrm{Aut}_F(F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}})^R_{\mathrm{CM}}$ — for arbitrary $\gamma \in \mathrm{Aut}_F(F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}})$ we only have $u \in \mathbb{A}^\times_{F,f}$, but we need $u \in R(\mathbb{A}_f)$.

- $R(\mathbb{Q})E_{\chi t}$ contains a polarisation of $\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a})$:

We need to calculate the signs of the imaginary part of χt under embeddings $\varphi \in \Sigma_K$. Recall that $\mathrm{Im}\,\varphi(t) > 0$ for all $\varphi \in \Phi$. To calculate the signs of χ, recall (4.1.11.c) that

$$\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}} = (1, \mathrm{prod})(\beta_s(\gamma))\prod_{x\in\Sigma} c_x^{\alpha_x},$$

where $(\alpha_x)_{x\in\Sigma}$ are given by (4.2.8.1). By (4.3.6) and (4.3.5.a) we thus have

$$N_{K/F}\left(\widetilde{f}_\Phi(\gamma)\right) = \chi_F\left(\widetilde{F}_\Phi(\gamma)|_{F^{\mathrm{ab}}}\right) = (\chi_F\circ(1,\mathrm{prod})\circ\beta_s(\gamma))(\alpha F^\times_{>0}),$$

where $\alpha \in F^\times$ satisfies $\mathrm{sgn}(x(\alpha)) = (-1)^{\alpha_x}$ for $x \in \Sigma$. By choice of $\widetilde{f}$ and u, the last equality implies that

$$\chi \in \alpha^{-1}F^\times_{>0}. \tag{4.4.13.2}$$

In other words, χ has the same signs as α. These signs are given by (4.2.8.1) and measure the difference between the CM types Φ and $\gamma\Phi$. We conclude that, for $\varphi \in \Sigma_K$, we have

$$\mathrm{Im}\,\varphi(\chi t) > 0 \text{ if and only if } \varphi \in \gamma\Phi. \tag{4.4.13.3}$$

Since we know that $\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a})$ has type $\gamma\Phi$, by (2.3.4.b) we conclude that $[\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a}), i_{\gamma\Phi}|_F, R(\mathbb{Q})E_{\chi t}, \widetilde{f}\circ\eta\overline{Z(\mathbb{Q})}]$ is a point of $\mathrm{Sh}(G, X)$. It is then automatically a CM point.

- (4.4.13.1) defines a group action of $\mathrm{Aut}_F(F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}})^R_{\mathrm{CM}}$ on the CM points of $\mathrm{Sh}(G, X)$:

Let $\mathcal{P} = [\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, R(\mathbb{Q})E_t, \eta\overline{Z(\mathbb{Q})}] \in \mathcal{A}^R$ be a CM point with K-compatible polarisation E_t. Let $\gamma_1, \gamma_2 \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$. We show

$$\gamma_2(\gamma_1\mathcal{P}) = (\gamma_2\gamma_1)\mathcal{P}. \tag{4.4.13.4}$$

To calculate the left hand side, we first let $\widetilde{f}_1 \in \widetilde{f}_\Phi(\gamma_1)$, $u_1 \in \chi_F \circ (1, \mathrm{prod}) \circ \beta_s(\gamma_1)$ and $\chi_1 = \frac{u_1}{{}^{1+c}\widetilde{f}_1} \in K^\times$. Then by (4.4.13.1)

$$\gamma_1\mathcal{P} = \left[\mathbb{C}^{\gamma_1\Phi}/\gamma_1\Phi(\widetilde{f}_1\mathfrak{a}), i_{\gamma_1\Phi}|_F, R(\mathbb{Q})E_{\chi_1 t}, \widetilde{f}_1 \circ \eta\overline{Z(\mathbb{Q})}\right].$$

Now take $\widetilde{f}_2 \in \widetilde{f}_{\gamma_1\Phi}(\gamma_2)$, $u_2 \in \chi_F \circ (1, \mathrm{prod}) \circ \beta_s(\gamma_2)$ and $\chi_2 = \frac{u_2}{{}^{1+c}\widetilde{f}_2}$, so we get

$$\gamma_2(\gamma_1\mathcal{P}) = \left[\mathbb{C}^{\gamma_2\gamma_1\Phi}/\gamma_2\gamma_1\Phi(\widetilde{f}_2\widetilde{f}_1\mathfrak{a}), i_{\gamma_2\gamma_1\Phi}|_F, R(\mathbb{Q})E_{\chi_2\chi_1 t}, \widetilde{f}_2 \circ \widetilde{f}_1 \circ \eta\overline{Z(\mathbb{Q})}\right]. \tag{4.4.13.5}$$

To calculate the right hand side of (4.4.13.4), we use the cocycle relation (4.2.7.c), which allows us to take $\widetilde{f} = \widetilde{f}_2\widetilde{f}_1 \in \widetilde{f}_\Phi(\gamma_2\gamma_1)$ in order to calculate

$$(\gamma_2\gamma_1)\mathcal{P} = \left[\mathbb{C}^{\gamma_2\gamma_1\Phi}/\gamma_2\gamma_1\Phi(\widetilde{f}\mathfrak{a}), i_{\gamma_2\gamma_1\Phi}|_F, R(\mathbb{Q})E_{\chi t}, \widetilde{f} \circ \eta\overline{Z(\mathbb{Q})}\right], \tag{4.4.13.6}$$

where $\chi = \frac{u}{{}^{1+c}\widetilde{f}} \in K^\times$ with $u \in \chi_F \circ (1, \mathrm{prod}) \circ \beta_s(\gamma_1\gamma_2)$. Since $\chi_F \circ (1, \mathrm{prod}) \circ \beta_s$ is a group homomorphism, we may take $u = u_2u_1$, resulting in

$$\chi = \frac{u_2u_1}{{}^{1+c}(\widetilde{f}_2\widetilde{f}_1)} = \chi_2\chi_1.$$

Therefore the right hand sides of (4.4.13.5) and (4.4.13.6) are equal, hence we have shown that (4.4.13.4) holds.

- (4.4.13.1) extends the action of $\Gamma_{\mathbb{Q}}$ to $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$:

The reflex field of (G^R, X^R) is $\mathbb{Q}$, and by (4.4.8) the $\Gamma_{\mathbb{Q}}$-action on the $\overline{\mathbb{Q}}$-points of the Shimura variety $\mathrm{Sh}(G^R, X^R)$ is given by conjugating the corresponding abelian variety and its extra structure. In particular, this applies to CM points as they are defined over $\overline{\mathbb{Q}}$, where the Galois conjugate can again be described using Tate's half transfer. By (4.3.7) this agrees with (4.4.13.1). $\square$

For each $\mathbb{Q}$-algebraic torus R with $\mathbb{G}_m \hookrightarrow R \hookrightarrow R_{F/\mathbb{Q}}\mathbb{G}_m$ as above, we have defined a plectic group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$ and an action of this group on the CM points of $\mathrm{Sh}(G^R, X^R)$. These Shimura varieties are variants of the Hilbert modular variety, and their moduli interpretation (4.4.5) only differ in the degree of precision with which they include polarisations and level structures.

The calculations in the above proof show that it was crucial to restrict to $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$, precisely because then and only then the level structure $\widetilde{f} \circ \eta$ sends $R(\mathbb{A}_f)\psi_0$ to $R(\mathbb{A}_f)E_{\chi t}$.

4.4.14 Remark. Let us make a few comments about the two extreme cases:

(4.4.14.a) If $R = \mathbb{G}_m$, then $\mathrm{Sh}(G^{\mathbb{G}_m}, X^{\mathbb{G}_m})$ is equal to the PEL Hilbert modular variety $\mathrm{Sh}(G, X)$ of (3.2.2). We have defined two plectic actions on its CM points, namely in (4.2.10) and in (4.4.13), which we will compare now:

We already remarked in (4.4.12) that the involved plectic groups $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_0$ and $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^{\mathbb{G}_m}_{\mathrm{CM}}$ are the same. Moreover, by (4.3.5.b) we can take u in (4.4.13.1) to be $\chi_{\mathrm{cyc}}(\widetilde{u})$ in (4.2.10.1), and then the two formulae giving the plectic action agree.

(4.4.14.b) If $R = R_{F/\mathbb{Q}}\mathbb{G}_m$, then $\mathrm{Sh}(G^{R_{F/\mathbb{Q}}\mathbb{G}_m}, X^{R_{F/\mathbb{Q}}\mathbb{G}_m})$ is equal to the Hilbert modular variety $\mathrm{Sh}(G_1, X_1)$ of (3.2.1). In this case, (4.4.13) gives an action of the entire plectic group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ on the CM points of the Hilbert modular variety. This proves [NS16, Prop. 6.8].

4.4.15 Remark. It is an interesting question in what way the group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$ and its action on the CM points of $\mathrm{Sh}(G^R, X^R)$ depend on the choice of splitting χ_F. (4.4.14.a) shows that for $R = \mathbb{G}_m$ the property (4.3.5.b) prescribes the values of the splitting χ_F on the image of $V_{F/\mathbb{Q}} \colon \Gamma^{\mathrm{ab}}_{\mathbb{Q}} \to \Gamma^{\mathrm{ab}}_F$, which is all that is needed. However, for all other choices of R, including $R = R_{F/\mathbb{Q}}\mathbb{G}_m$, this question remains open.

5

Plectic action on the set of connected components of variants of Hilbert modular varieties

Fix a totally real number field F and an algebraic torus R over $\mathbb{Q}$ with $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$. The goal of this chapter is to define an action of a certain plectic group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\pi_0}$ on the set $\pi_0(\mathrm{Sh}(G^R, X^R))$ of connected components of the Shimura variety $\mathrm{Sh}(G^R, X^R)$. Here $\mathrm{Sh}(G^R, X^R)$ is the variant of the Hilbert modular variety defined in (4.4.2). This action of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\pi_0}$ can be viewed as a plectic structure, in the vague sense of Section 1.1.2, on $\pi_0(\mathrm{Sh}(G^R, X^R))$.

In Section 5.1 we start by recalling the description of the set of connected components of an arbitrary Shimura variety as a zero-dimensional Shimura variety. Specialising to $\mathrm{Sh}(G^R, X^R)$, we see in (5.1.9) that

$$\pi_0(\mathrm{Sh}(G^R, X^R)) \xrightarrow{\sim} \mathrm{Sh}(R, \mathrm{VZ}^R),$$

where VZ^R is a certain subset of $\{\pm 1\}^\Sigma$ defined in (5.1.6).

In Section 5.2 we recall that the Galois action on the set of connected components of a Shimura variety is given by a reciprocity homomorphism. Again specialising to $\mathrm{Sh}(G^R, X^R)$, we see in (5.2.4) that $\pi_0(\mathrm{Sh}(G^R, X^R)) \cong \pi_0(R(\mathbb{A})/R(\mathbb{Q}))$ and explicitly describe the action of $\gamma \in \Gamma_{\mathbb{Q}}$ on $\pi_0(R(\mathbb{A})/R(\mathbb{Q}))$ in terms of class field theory. In (5.2.6) we define the group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\pi_0}$ and an action of this group on $\pi_0(R(\mathbb{A})/R(\mathbb{Q}))$, extending the action of $\Gamma_{\mathbb{Q}}$. We then show in (5.2.9) that $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$ embeds canonically into $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\pi_0}$ and prove Theorem (1.2.1) in (5.2.10), namely that the π_0-map is $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\mathrm{CM}}$-equivariant when restricted to CM points.

In the Appendix 5.3 we formulate a cohomological condition on the algebraic torus R that implies that the group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\pi_0}$ is a subgroup of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$.

Namely, we show that if the Shafarevich–Tate group Ш(R) of R vanishes (condition (5.3.4.1)), then $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\pi_0}$ is a subgroup of $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$. Examples of tori R satisfying (5.3.4.1) include $R = \mathbb{G}_m$ and $R = R_{F/\mathbb{Q}}\mathbb{G}_m$. Moreover, if for instance the extension $F/\mathbb{Q}$ is cyclic, then every R satisfies (5.3.4.1).

For the description of the set of connected components of a general Shimura variety, we use [Mil17, §5, §12-13]. For background on the Galois cohomology of tori, see [Tat66, PR94].

5.1 Connected components of Shimura varieties

In this section we recall facts about connected components of Shimura varieties from [Mil17, §5]:

5.1.1. (Notation) Fix a Shimura datum (G, X). We will write

- G^{der} for the *derived group* of G, i. e. the group spanned by all commutators of elements of G.
- $T = G/G^{\mathrm{der}}$ for the *cocentre* of G, and $\nu\colon G \to T$ for the quotient map.
- Z for the *centre* and $G^{\mathrm{ad}} := G/Z$ for the *adjoint group* of G.
- $T(\mathbb{R})^\dagger := \nu(Z(\mathbb{R}))$ and $T(\mathbb{Q})^\dagger := T(\mathbb{Q}) \cap T(\mathbb{R})^\dagger$.

We state [Mil17, Thm 5.17]:

5.1.2 Theorem. *Let (G, X) be a Shimura datum and U be a sufficiently small compact open subgroup of $G(\mathbb{A}_f)$. If moreover G^{der} is simply connected, then the set of connected components of the Shimura variety $\mathrm{Sh}_U(G, X)$ is given by*

$$\pi_0(\mathrm{Sh}_U(G, X)) \xrightarrow{\sim} T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(U).$$

The proof of Theorem (5.1.2) can be found in [Mil17, Thm 5.17]. However, it will be useful to describe the bijection explicitly.

5.1.3. (π_0-map) We use the notation of (5.1.2). As always, we will mostly work in the limit over all compact open subgroups, but for the moment let us fix a sufficiently small compact open subgroup $U \subset G(\mathbb{A}_f)$. Let us describe the map

$$\pi_0\colon\ \mathrm{Sh}_U(G, X) \twoheadrightarrow T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(U).$$

It is given as the composition

$$G(\mathbb{Q})\backslash\left[X \times G(\mathbb{A}_f)/U\right] \xrightarrow{\sim} G(\mathbb{Q})_+\backslash\left[X^+ \times G(\mathbb{A}_f)/U\right] \longrightarrow T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(U) \tag{5.1.3.1}$$

where

- $G(\mathbb{R})_+$ is the preimage in $G(\mathbb{R})$ of the identity component of $G^{\mathrm{ad}}(\mathbb{R})$, and $G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+$.
- X^+ denotes a connected component of X.

The first arrow is induced from the inclusion of X^+ into X, and is a bijection by [Mil17, Lem 5.11]. The second map is induced by $\nu\colon G(\mathbb{A}_f) \to T(\mathbb{A}_f)$, forgetting the X^+-coordinate.

5.1.4 Remark. (Zero-dimensional Shimura varieties) The double quotient

$$T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(U)$$

occuring in (5.1.2) looks similar to a double quotient defined by a Shimura variety. Let us make this resemblance even more striking:

Using the notation of (5.1.2), we also write $Y := T(\mathbb{R})/T(\mathbb{R})^\dagger$. By real approximation[1], we have $Y = T(\mathbb{Q})/T(\mathbb{Q})^\dagger$. Moreover, Y is finite by [Mil17, p. 59, after (34)]. For any compact open subgroup $U \subset T(\mathbb{A}_f)$, we have the bijection

$$\begin{aligned} T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/U &\xrightarrow{\sim} T(\mathbb{Q}) \backslash \left[Y \times T(\mathbb{A}_f)/U\right] \\ [g] &\longmapsto [1, g]. \end{aligned} \tag{5.1.4.1}$$

Proof. It is easy to check that this map is well-defined and surjective. For the injectivity, assume $[1, g] = [1, g'] \in T(\mathbb{Q}) \backslash \left[Y \times T(\mathbb{A}_f)/U\right]$ for $g, g' \in T(\mathbb{A}_f)$. Then there exist $q \in T(\mathbb{Q})$ and $u \in U$ such that $(q, qgu) = (1, g')$. This means in particular that $q = 1 \in T(\mathbb{Q})/T(\mathbb{Q})^\dagger$, i. e. $q \in T(\mathbb{Q})^\dagger$ and therefore $[g] = [g']$ in $T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/U$. □

We denote the double quotient $T(\mathbb{Q}) \backslash \left[Y \times T(\mathbb{A}_f)/U\right]$ by $\mathrm{Sh}_U(T, Y)$ although (T, Y) is not a Shimura datum because T is abelian and hence a $T(\mathbb{R})$-conjugacy class of Hodge structures has to consist of a single element, whereas Y is a finite set with usually more than one element.

We still call such $\mathrm{Sh}_U(T, Y)$ a *zero-dimensional Shimura variety*, using the convention of [Mil17, p. 62-63], namely that in place of the $T(\mathbb{R})$-conjugacy class of Hodge structures we allow a finite set with a transitive $T(\mathbb{R})/T(\mathbb{R})^\dagger$-action. Here, this finite set is precisely $Y = T(\mathbb{R})/T(\mathbb{R})^\dagger$. And, of course, we denote the projective limit over all compact open subgroups $U \subset T(\mathbb{A}_f)$ by $\mathrm{Sh}(T, Y)$.

In this way, Theorem (5.1.2) tells us that the set of connected components of the Shimura variety $\mathrm{Sh}_U(G, X)$ is the zero-dimensional Shimura variety $\mathrm{Sh}_{\nu(U)}(T, Y)$,

[1] This is [Mil17, p. 63, l. 4]. For a slightly more elaborate explanation in the case of interest, see (5.1.8).

and the map

$$\pi_0\colon \operatorname{Sh}_U(G,X) \longrightarrow \operatorname{Sh}_{\nu(U)}(T,Y) \tag{5.1.4.2}$$

is given as the composition of (5.1.3.1) and (5.1.4.1). In the projective limit over all compact open subgroups $U \subset G(\mathbb{A}_f)$ we get

$$\pi_0\colon \operatorname{Sh}(G,X) \longrightarrow \operatorname{Sh}(T,Y). \tag{5.1.4.3}$$

5.1.5 Example. (Hilbert modular variety and PEL Hilbert modular variety) We will explicitly describe the map π_0 for the variants of the Hilbert modular variety introduced in (4.4.2). But let us first look at two familiar examples in detail:

(5.1.5.a) Let $G_1 = R_{F/\mathbb{Q}}\operatorname{GL}_2$ and $X_1 = (\mathbb{C}\setminus\mathbb{R})^\Sigma$ be the Shimura datum associated to the Hilbert modular variety (3.2.1), where we denote $\Sigma = \Sigma_F = \operatorname{Hom}(F,\mathbb{C})$. Then:

- $G_1^{\mathrm{der}} = R_{F/\mathbb{Q}}\operatorname{SL}_2$, so $\nu = R_{F/\mathbb{Q}}(\det)\colon G_1 \to R_{F/\mathbb{Q}}\mathbb{G}_m$. In particular, $T = T_1 := R_{F/\mathbb{Q}}\mathbb{G}_m$. We will also denote ν by $d_F\colon G_1 \to T_1$.
- $Z = Z_1 := R_{F/\mathbb{Q}}(\mathbb{G}_m)$.
- $T(\mathbb{R})^\dagger = \{g = (g_x)_{x\in\Sigma} \in (F\otimes\mathbb{R})^\times = \prod_{x\in\Sigma}\mathbb{R}_x^\times \mid g_x > 0\ \forall x\in\Sigma\} = (F\otimes_\mathbb{Q}\mathbb{R})^\times_{>0} \cong (\mathbb{R}^\times_{>0})^\Sigma$ and hence $T(\mathbb{Q})^\dagger = F^\times_{>0}$ and $Y = F^\times/F^\times_{>0} = \{\pm1\}^\Sigma$.
- $G_1(\mathbb{R})_+ = \{g = (g_x)_{x\in\Sigma} \in \operatorname{GL}_2(F\otimes\mathbb{R}) \mid \det(g_x) > 0\ \forall x\in\Sigma\}$.
- We choose the component $X_1^+ = \mathfrak{h}^\Sigma$ of X_1.

Chasing through the description of the map π_0 in (5.1.3) yields the following: Take $[h,g] \in \operatorname{Sh}_U(G_1,X_1)$, with h corresponding to $(z_x)_{x\in\Sigma} \in (\mathbb{C}\setminus\mathbb{R})^\Sigma$ under (3.2.1.2). The first arrow in (5.1.4.2) is given by

$$[h,g] \longmapsto [aha^{-1}, ag] \in G(\mathbb{Q})_+\backslash\left[X^+ \times G(\mathbb{A}_f)/U\right],$$

where $a = \begin{pmatrix} f & 0 \\ 0 & 1\end{pmatrix} \in G_1(\mathbb{Q}) = \operatorname{GL}_2(F)$ with $f \in F^\times$ an arbitrary element satisfying $\operatorname{sgn}(x(f)) = \operatorname{sgn}(\operatorname{Im} z_x)$ for each $x\in\Sigma$, because then the Hodge structure aha^{-1} corresponds to $(fz_x)_{x\in\Sigma} \in \mathfrak{h}^\Sigma$ under (3.2.1.2). Hence (5.1.3.1) is given by

$$[h,g] \longmapsto [d_F(a)d_F(g)] \in T(\mathbb{Q})^\dagger\backslash T(\mathbb{A}_f)/\nu(U).$$

Finally, composing this with (5.1.4.1) and using $d_F(a) = f \in F^\times = T(\mathbb{Q})$, we get (in the inverse limit over U)

$$\begin{aligned}\pi_0\colon \operatorname{Sh}(R_{F/\mathbb{Q}}\operatorname{GL}_2, (\mathbb{C}\setminus\mathbb{R})^\Sigma) &\longrightarrow \operatorname{Sh}(R_{F/\mathbb{Q}}\mathbb{G}_m, \{\pm1\}^\Sigma),\\ [(z_x)_{x\in\Sigma}, g] &\longmapsto [(\operatorname{sgn}\operatorname{Im} z_x)_{x\in\Sigma}, d_F(g)].\end{aligned}$$

(5.1.5.b) Let (G, X) be the Shimura datum associated to the PEL Hilbert modular variety, see (3.2.2); in particular, we have $X = \mathfrak{h}^\Sigma \sqcup (-\mathfrak{h})^\Sigma$. Then:

- $G^{\mathrm{der}} = G_1^{\mathrm{der}} = R_{F/\mathbb{Q}}\operatorname{SL}_2$, so $T = \mathbb{G}_m$ and $\nu = d_F|_G\colon G \to \mathbb{G}_m \hookrightarrow R_{F/\mathbb{Q}}\mathbb{G}_m$, where the second arrow is the embedding $\mathbb{G}_m \hookrightarrow R_{F/\mathbb{Q}}\mathbb{G}_m$ of (3.2.2).
- $Z = Z_1 \cap G$ has $\mathbb{Q}$-points $Z(\mathbb{Q}) = \{t \in F^\times \mid t^2 \in \mathbb{Q}^\times\}$.
- $T(\mathbb{R})^\dagger = \{g = (g)_{x\in\Sigma} \in \mathbb{R}^\times \subset \prod_{x\in\Sigma}\mathbb{R}_x^\times \mid g > 0\} = \mathbb{R}_{>0}^\times$ and hence $T(\mathbb{Q})^\dagger = \mathbb{Q}_{>0}^\times$.
- $Y = \mathbb{Q}^\times/\mathbb{Q}_{>0}^\times = \{\pm1\}$.
- $G^{\mathrm{ad}} = G_1^{\mathrm{ad}}$, hence $G(\mathbb{R})_+ = \{g = (g_x)_{x\in\Sigma} \in G(\mathbb{R}) \mid \det(g_x) > 0\ \forall x \in \Sigma\}$.
- We choose the component $X^+ = \mathfrak{h}^\Sigma$ of X.

Chasing through (5.1.3) in the same way as in (5.1.5.a) yields

$$\begin{aligned}\pi_0\colon \operatorname{Sh}(G, X) &\longrightarrow \operatorname{Sh}(\mathbb{G}_m, \{\pm1\}),\\ [(z_x)_{x\in\Sigma}, g] &\longmapsto [\operatorname{sgn}\operatorname{Im} z_{x_0}, d_F(g)],\end{aligned}$$

for any choice of $x_0 \in \Sigma$.

Before repeating this procedure for the Shimura varieties $\operatorname{Sh}(G^R, X^R)$ of (4.4.2), let us first decribe X^R in more concrete terms.

5.1.6 Lemma. *The $G^R(\mathbb{R})$-conjugacy class X^R is in bijection with the set*

$$\{(z_x)_{x\in\Sigma} \in (\mathbb{C}\setminus\mathbb{R})^\Sigma \mid (\operatorname{sgn}\operatorname{Im} z_x)_{x\in\Sigma} \in \mathrm{VZ}^R\}, \tag{5.1.6.1}$$

where VZ^R is the subgroup[2] of $\{\pm1\}^\Sigma$ defined by

$$\mathrm{VZ}^R := \left(R(\mathbb{R})\cdot(\mathbb{R}_{>0}^\times)^\Sigma\right)/(\mathbb{R}_{>0}^\times)^\Sigma \subset (\mathbb{R}^\times)^\Sigma/(\mathbb{R}_{>0}^\times)^\Sigma = \{\pm1\}^\Sigma.$$

In other words,

$$\pi_0(X^R) = \mathrm{VZ}^R.$$

[2] "VZ" is short for "Vorzeichen", the German word for signs.

5.1.7 Remark. Before starting the proof, let us remark that in the cases $R = R_{F/\mathbb{Q}}\mathbb{G}_m$ and $R = \mathbb{G}_m$ considered in (5.1.5), we get $\mathrm{VZ}^{R_{F/\mathbb{Q}}\mathbb{G}_m} = \{\pm 1\}^\Sigma$ and $\mathrm{VZ}^{\mathbb{G}_m} = \{(1,\dots,1),(-1,\dots,-1)\} \cong \{\pm 1\}$. Hence $X^{R_{F/\mathbb{Q}}\mathbb{G}_m} = (\mathbb{C}\setminus\mathbb{R})^\Sigma$ and $X^{\mathbb{G}_m} = \mathfrak{h}^\Sigma \sqcup (-\mathfrak{h})^\Sigma$. So (5.1.6) includes (3.2.1.2) and proves (3.2.2.2).

Proof of Lemma (5.1.6). Let us temporarily denote G^R by G and the set in (5.1.6.1) by W. First of all, recall that $(\mathbb{C}\setminus\mathbb{R})^\Sigma$ is identified with the $G_1(\mathbb{R})$-conjugacy class X_1 of $h_0\colon \mathbb{S} \to G_{1,\mathbb{R}}$, see (3.2.1.2). We saw in (3.2.2) that h_0 factors through $G^{\mathbb{G}_m}$, and $G^{\mathbb{G}_m} \subset G^R = G$, so we may view h_0 as a morphism $h_0\colon \mathbb{S} \to G_\mathbb{R}$. Under the identification in (3.2.1.2), it remains to calculate the orbit $G(\mathbb{R})\cdot(i,\dots,i)$. We claim that this orbit is equal to W:

- Step 1: $G(\mathbb{R})\cdot(i,\dots,i) \subset W$.

If $(z_x)_x = (g_x \cdot i)_x$ for some $(g_x)_x \in G(\mathbb{R})$, then $\mathrm{sgn}(\mathrm{Im}\, z_x) = \mathrm{sgn}(\det g_x)$ for all $x \in \Sigma$ by the usual formula for the imaginary part under Möbius transformations. As $(\det g_x)_x \in R(\mathbb{R})$ by definition of G, we get $(\mathrm{sgn}\det g_x)_x \in \mathrm{VZ}^R$. This means $(z_x)_x \in W$.

- Step 2: If $(z_x)_x \in G(\mathbb{R})\cdot(i,\dots,i)$, and $(z'_x)_x$ satisfies $\mathrm{sgn}\,\mathrm{Im}\, z_x = \mathrm{sgn}\,\mathrm{Im}\, z'_x$ for all $x \in \Sigma$, then $(z'_x)_x \in G(\mathbb{R})\cdot(i,\dots,i)$.

As $\mathrm{GL}_2(\mathbb{R})$ acts transitively on $\mathbb{C}\setminus\mathbb{R}$, there exists a $(g_x)_x \in \mathrm{GL}_2(\mathbb{R})^\Sigma$ such that $g_x \cdot z_x = z'_x$ for all x. By assumption on the signs of the imaginary parts, we know that $\det g_x > 0$ for all x. Now rescale every g_x by a scalar in order to get $\det g_x = 1$ and still $g_x \cdot z_x = z'_x$ for all x, which is possible since scalars do not affect the action by Möbius transformations. Thus we get $(g_x)_x \in \mathrm{SL}_2(\mathbb{R})^\Sigma \subset G(\mathbb{R})$, which implies $(z'_x)_x \in G(\mathbb{R})\cdot(i,\dots,i)$.

- Step 3: For any $(\varepsilon_x)_x \in \mathrm{VZ}^R$, there exists $(z_x)_x \in G(\mathbb{R})\cdot(i,\dots,i)$ such that $\mathrm{sgn}\,\mathrm{Im}\, z_x = \varepsilon_x$ for all x.

By definition, there exists $(\lambda_x)_x \in R(\mathbb{R})$ such that $\mathrm{sgn}\,\lambda_x = \varepsilon_x$ for all $x \in \Sigma$. Then $(g_x)_x := \left(\begin{pmatrix}\lambda_x & 0\\ 0 & 1\end{pmatrix}\right)_x$ lies in $G(\mathbb{R})$ by definition of G and satisfies $\det g_x = \lambda_x$ for all x. Hence $(z_x)_x := (g_x)_x \cdot (i,\dots,i) \in G(\mathbb{R})\cdot(i,\dots,i)$ satisfies $\mathrm{sgn}\,\mathrm{Im}\, z_x = \varepsilon_x$.

Together, steps 2 and 3 imply $W \subset G(\mathbb{R})\cdot(i,\dots,i)$. □

We can also describe the set VZ^R in terms of rational points of the torus R:

5.1.8 Lemma. *Let R be a $\mathbb{Q}$-algebraic torus as before. Then the inclusion $R(\mathbb{Q}) \hookrightarrow R(\mathbb{R})$ induces a bijection*

$$(R(\mathbb{Q})\cdot F^\times_{>0})/F^\times_{>0} \xrightarrow{\sim} \left(R(\mathbb{R})\cdot(\mathbb{R}^\times_{>0})^\Sigma\right)/(\mathbb{R}^\times_{>0})^\Sigma = \mathrm{VZ}^R.$$

Proof. The map is well-defined and injective since an element of $R(\mathbb{Q}) \subset F^\times$ that lies in $(\mathbb{R}^\times_{>0})^{\Sigma_F}$ clearly lies in $F^\times_{>0}$. For the surjectivity: Using the usual canonical isomorphisms, we can write the above map as

$$R(\mathbb{Q})/R(\mathbb{Q}) \cap F^\times_{>0} \longrightarrow R(\mathbb{R})/R(\mathbb{R}) \cap (\mathbb{R}^\times_{>0})^\Sigma.$$

The surjectivity now follows because $R(\mathbb{Q})$ is dense in $R(\mathbb{R})$ by real approximation [Mil17, Thm 5.4] for the torus R, and $R(\mathbb{R}) \cap (\mathbb{R}^\times_{>0})^\Sigma$ is open in $R(\mathbb{R})$. $\square$

Now let us describe the π_0-map for $\mathrm{Sh}(G^R, X^R)$:

5.1.9 Example (General Hilbert modular variety)**.** Let $(G, X) = (G^R, X^R)$ be the Shimura datum of (4.4.2), with R a $\mathbb{Q}$-algebraic torus as before, and let us identify X^R with the set in (5.1.6.1). We again calculate the quantities of (5.1.3), see (6.2.9) for the calculation of Z and G^{ad}.

- $G^{\mathrm{der}} = G_1^{\mathrm{der}} = R_{F/\mathbb{Q}}\,\mathrm{SL}_2$, so $T = R$ and $\nu = d_F|_G \colon G \to T$.
- $Z = Z_1 \cap G$.
- $T(\mathbb{R})^\dagger = \{g = (g_x)_{x\in\Sigma} \in R(\mathbb{R}) \subset \prod_{x\in\Sigma} \mathbb{R}^\times_x \mid g_x > 0\ \forall x \in \Sigma\} = R(\mathbb{R}) \cap (\mathbb{R}^\times_{>0})^\Sigma =: R(\mathbb{R})_{>0} \subset (\mathbb{R}^\times)^\Sigma$ and hence $T(\mathbb{Q})^\dagger = R(\mathbb{Q}) \cap F^\times_{>0}$.
- $Y = R(\mathbb{R})/(R(\mathbb{R}) \cap (\mathbb{R}^\times_{>0})^\Sigma) = (R(\mathbb{R}) \cdot (\mathbb{R}^\times_{>0})^\Sigma)/(\mathbb{R}^\times_{>0})^\Sigma = \mathrm{VZ}^R$.
- $G^{\mathrm{ad}} = G_1^{\mathrm{ad}}$, hence $G(\mathbb{R})_+ = \{g = (g_x)_{x\in\Sigma} \in G(\mathbb{R}) \mid \det(g_x) > 0\ \forall x \in \Sigma\}$.
- We choose the component $X^+ = \mathfrak{h}^\Sigma$ of X.

Going through the same steps as in (5.1.5.a), we find that

$$\begin{aligned} \pi_0 \colon \mathrm{Sh}(G, X) &\longrightarrow \mathrm{Sh}(R, \mathrm{VZ}^R), \\ [(z_x)_{x\in\Sigma}, g] &\longmapsto [(\mathrm{sgn}\,\mathrm{Im}\, z_x)_{x\in\Sigma}, d_F(g)]. \end{aligned}$$

5.2 Plectic Galois action on connected components

Under the hypotheses of (5.1.2), we have given in (5.1.4.3) a rather explicit description of the map $\pi_0 \colon \mathrm{Sh}(G, X) \to \mathrm{Sh}(T, Y)$. Moreover, the Shimura variety $\mathrm{Sh}(G, X)$ has a canonical model over the reflex field $E(G, X)$, so it is natural to expect that the π_0-map is equivariant with respect to a certain action of $\mathrm{Aut}(\mathbb{C}/E(G, X))$ on $\mathrm{Sh}(T, Y)$.

We quote the following results from [Mil17, (64), p. 119]. Namely, the action on $\mathrm{Sh}(T,Y)$ is given by a class field theoretic recipe similar to the one giving the action on Shimura varieties associated to tori (compare [Mil17, (60)-(62), p. 114]). The only difference is a component at infinity of the reciprocity map acting on Y.

5.2.1. (Galois action on connected components) Let (G,X) be an arbitrary Shimura datum such that G^{der} is simply connected. We continue to use the notation of (5.1.2). Let $h_0 \in X$ and $h := \nu \circ h_0 \colon \mathbb{S} \to T_{\mathbb{R}}$. Look at the associated cocharacter μ_h of T, see (2.2.5). By Definition (6.1.11), the reflex field $E(G,X)$ is precisely the field of definition of the unique $G(\overline{\mathbb{Q}})$-conjugacy class contained in the $G(\mathbb{C})$-conjugacy class of μ_{h_0}, thus μ_h is certainly defined over $E(G,X)$. [Mil17, (60)] defines a morphism of $\mathbb{Q}$-algebraic groups

$$r = r(T,\mu_h) \colon R_{E(G,X)/\mathbb{Q}}\mathbb{G}_m \longrightarrow T$$

called the *reciprocity morphism*.

The action of $\mathrm{Aut}(\mathbb{C}/E(G,X))$ on $\mathrm{Sh}(T,Y)$ is then defined via the adelic points of r and class field theory: Let $\sigma \in \mathrm{Aut}(\mathbb{C}/E(G,X))$ and $s \in \mathbb{A}^{\times}_{E(G,X)}$ such that $\mathrm{art}_{E(G,X)}(s) = \sigma|_{E(G,X)^{\mathrm{ab}}}$. Write $r(s) \in T(\mathbb{A}_{\mathbb{Q}})$ as $r(s) = (r(s)_\infty, r(s)_f) \in T(\mathbb{R}) \times T(\mathbb{A}_f)$. Then define

$$\sigma[y,g]_U := [r(s)_\infty \cdot y, r(s)_f \cdot g], \quad [y,g]_U \in \mathrm{Sh}_U(T,Y),$$

and similarly in the limit over all compact open subgroups $U \subset T(\mathbb{A}_f)$. Then the map

$$\pi_0 \colon \mathrm{Sh}(G,X) \longrightarrow \mathrm{Sh}(T,Y)$$

is $\mathrm{Aut}(\mathbb{C}/E(G,X))$-equivariant.[3]

Let us see how (5.2.1) applies to the Hilbert modular varieties $\mathrm{Sh}(G^R, X^R)$:

5.2.2 Example (Galois action on $\pi_0(\mathrm{Sh}(G^R,X^R))$)**.** We use the general set-up of Example 5.1.9, so we look at the group G^R associated to an algebraic torus R with $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$.

As mentioned before, the reflex field of (G^R, X^R) is $\mathbb{Q}$, so (5.2.1) gives an action of $\mathrm{Aut}(\mathbb{C})$ described by class field theory and the adelic points of the reciprocity morphism

$$r \colon \mathbb{G}_m \longrightarrow T = R.$$

To calculate r, take $h_0 \in X^{\mathbb{G}_m} \subset X^R$ as defined in (4.4.2). Then the cocharacter

[3] [Mil17, (64), p. 119] states this and sketches a proof. In the case of interest of CM points of variants of the Hilbert modular variety, we will prove a more general result in (5.2.10).

$\mu_h\colon \mathbb{G}_{m,\mathbb{C}} \to R_{\mathbb{C}}$ associated to $h = d_F \circ h_0\colon \mathbb{S} \to R_{\mathbb{R}}$ is given (on $\mathbb{C}$-points) by

$$\begin{aligned} \mu_h\colon \mathbb{G}_m \to R &\hookrightarrow R_{F/\mathbb{Q}}\mathbb{G}_m, \\ z \mapsto \qquad &(z,\dots,z) \in \prod_{x\in\Sigma} \mathbb{C}_x^\times = (F \otimes_{\mathbb{Q}} \mathbb{C})^\times. \end{aligned}$$

In other words, μ_h is equal to the fixed embedding $\mathbb{G}_m \hookrightarrow R$. Here we used the assumption of (4.4.2) that

$$\begin{tikzcd} \mathbb{G}_m \arrow[r, hook] \arrow[dr] & R \arrow[d, hook] \\ & R_{F/\mathbb{Q}}\mathbb{G}_m \end{tikzcd}$$

commutes, where the diagonal arrow is given by $z \mapsto (z,\dots,z) \in \prod_{x\in\Sigma}\mathbb{C}_x^\times$ on complex points.

The definition of r in [Mil17, (60)] then immediately says that $r\colon \mathbb{G}_m \to R$ is equal to μ_h, i. e. r is equal to the fixed embedding $\mathbb{G}_m \hookrightarrow R$.

Before continuing this example, let us prove the following lemma:

5.2.3 Lemma. *Let $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ as before. Then*

$$\pi_0(\mathrm{Sh}(G^R, X^R)) = \pi_0(R(\mathbb{A})/R(\mathbb{Q})).$$

Proof. By (5.1.9) and the definition of $\mathrm{Sh}(R, \mathrm{VZ}^R)$ and of VZ^R we have

$$\begin{aligned} \pi_0(\mathrm{Sh}(G^R, X^R)) &= \mathrm{Sh}(R, \mathrm{VZ}^R) \\ &= \varprojlim_U \; R(\mathbb{Q})\backslash \left[\mathrm{VZ}^R \times R(\mathbb{A}_f)/U\right] \\ &= \varprojlim_U \; R(\mathbb{A}) \left/ \big(R(\mathbb{Q}) \cdot (R(\mathbb{R})_{>0} \times U)\big) \right. , \end{aligned}$$

where $R(\mathbb{R})_{>0} = R(\mathbb{R}) \cap (F \otimes_{\mathbb{Q}} \mathbb{R})^\times_{>0}$, and the projective limit is taken over all compact open subgroups $U \subset R(\mathbb{A}_f)$. Note that $R(\mathbb{R})_{>0} \times U$ is a subgroup of $R(\mathbb{R}) \times R(\mathbb{A}_f) = R(\mathbb{A})$, and that we embed $R(\mathbb{Q})$ diagonally in $R(\mathbb{R}) \times R(\mathbb{A}_f) = R(\mathbb{A})$.

Let us write $C_R := R(\mathbb{A})/R(\mathbb{Q})$, which is a subgroup of $C_F = \mathbb{A}_F^\times/F^\times$ endowed with the subspace topology. Since C_F is a locally compact, complete topological group, so is C_R. Moreover,

$$R(\mathbb{A}) \left/ \big(R(\mathbb{Q}) \cdot (R(\mathbb{R})_{>0} \times U)\big)\right. = \pi_0\left(\frac{R(\mathbb{A})}{R(\mathbb{Q}).[1\times U]}\right) = \pi_0(C_R/U'),$$

where we write U' for the subgroup $(R(\mathbb{Q}).[1\times U])/R(\mathbb{Q})$ of C_R. Taking the projective limit commutes with π_0, so

$$\varprojlim_U \pi_0(C_R/U') = \pi_0\left(\varprojlim_U C_R/U'\right).$$

It remains to show that the canonical map

$$p\colon C_R \longrightarrow \varprojlim_U C_R/U'$$

is an isomorphism.

- p is injective:

The kernel of p is the intersection $\bigcap_U U'$. To show it is trivial is the same as showing that

$$\bigcap_U R(\mathbb{Q}).[1\times U] = R(\mathbb{Q}).$$

Now $R(\mathbb{Q})$ is discrete in $R(\mathbb{A})$, so if U is small enough, then

$$R(\mathbb{Q}).[1\times U] = \coprod_{r\in R(\mathbb{Q})} r.[1\times U].$$

We know that the intersection of all U is 1 (because the intersection of all compact open subgroups of $\mathbb{A}_F^\times$ is 1), hence

$$\bigcap_U\left(\coprod_{r\in R(\mathbb{Q})} r.[1\times U]\right) = R(\mathbb{Q}),$$

as desired.

- p is surjective:

To ease notation, use the fact that the topology on $R(\mathbb{A}_f)$ is second countable and choose a neighbourhood basis of the identity of the form

$$U_1\supset U_2\supset\dots$$

with $U_n\subset R(\mathbb{A}_f)$ compact open subgroups, and $\bigcap_n U_n = 1$. Note that this system of neighbourhoods is cofinal for the system of all compact open subgroups $U\subset R(\mathbb{A}_f)$, so we have

$$\varprojlim_U C_R/U' = \varprojlim_n C_R/U_n'.$$

Similarly, choose auxiliary open subsets

$$V_1 \supset V_2 \supset \dots$$

of $R(\mathbb{R})$ with $\bigcap_n V_n = \{1\}$. Then $W_n := V_n \times U_n$ is a cofinal system of open neighbourhoods of 1 in $R(\mathbb{A})$ (because $\bigcap_n W_n = 1$), and similarly the images W'_n of W_n inside C_R form a cofinal system of open neighbourhoods of 1.

Now assume that $(y_n U'_n)_n \in \varprojlim_n C_R/U'_n$, with $y_n \in C_R$. Fix $N \in \mathbb{N}$. By definition of the projective limit, we have for all $m \geq N$

$$y_m \equiv y_N \mod U'_N,$$

i. e.

$$y_m^{-1} y_N \in U'_N. \tag{5.2.3.1}$$

In particular, for all $m, n \geq N$ we have

$$y_m^{-1} y_n = (y_m^{-1} y_N)(y_n^{-1} y_N)^{-1} \in U'_N \subset W'_N.$$

By the cofinality of the system $(W'_N)_N$, this means that $(y_n)_n$ is a Cauchy sequence inside C_R. But C_R is complete, so the sequence $(y_n)_n$ converges to a limit $y \in C_R$. We claim that

$$p(y) = (y_n U'_n)_n,$$

which then implies that p is surjective. □

Proof of claim. Since $(y_n)_n$ converges to y, we in particular have, for all $N \in \mathbb{N}$,

$$y \equiv y_N \mod W'_N. \tag{5.2.3.2}$$

If we can show this congruence modulo U'_N instead of W'_N, we are done since this then says

$$p(y) = (y_N U'_N)_N.$$

Let $(a_n, b_n) \in R(\mathbb{R}) \times R(\mathbb{A}_f)$ (resp. (a, b)) denote a lift of $y_n \in C_R$ (resp. y). Since $\lim_{n\to\infty} y_n = y$ in C_R, there exist $r_n \in R(\mathbb{Q})$ such that $\lim_{n\to\infty} r_n(a_n, b_n) = (a, b)$ in $R(\mathbb{A})$. Changing (a_n, b_n) to $r_n(a_n, b_n)$, we may assume that

$$(a_n, b_n) \to (a, b), \quad n \to \infty, \tag{5.2.3.3}$$

inside $R(\mathbb{A})$.

(5.2.3.1) implies that for all $m \geq N$ (and all N) we have

$$(a_m^{-1}a_N, b_m^{-1}b_N) \in U'_N = R(\mathbb{Q}).[1 \times U_N].$$

In particular, $a_m = s_m a_1$ for $s_m \in R(\mathbb{Q})$. But $a_m \to a$ as $m \to \infty$, so $s_m \to a/a_1$ as $m \to \infty$. But $R(\mathbb{Q})$ is discrete in $R(\mathbb{R})$, hence $(s_m)_m$ must become constant, so a/a_1 lies in $R(\mathbb{Q})$ and w.l.o.g. we may assume that $s_m = a/a_1$ for all m, i. e. $a_m = a$ for all m.

Now (5.2.3.3) becomes

$$(a_n, b_n) = (a, b_n) \to (a, b), \quad n \to \infty,$$

which when combined with (5.2.3.2) becomes

$$y_N = (a_N, b_N)R(\mathbb{Q}) \equiv (a, b)R(\mathbb{Q}) = y \mod U'_N,$$

so we are done. □

5.2.4 Example. (Galois action on $\pi_0(\mathrm{Sh}(G^R, X^R)) = \pi_0(R(\mathbb{A})/R(\mathbb{Q}))$) (5.2.2) describes an action of $\mathrm{Aut}(\mathbb{C})$ on $\pi_0(\mathrm{Sh}(G^R, X^R))$. Under the identification of (5.2.3), we get an action of $\mathrm{Aut}(\mathbb{C})$ on $\pi_0(R(\mathbb{A})/R(\mathbb{Q}))$, which we will now describe:

The inclusions $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ induce inclusions on the level of adelic points $\mathbb{A}_\mathbb{Q}^\times \hookrightarrow R(\mathbb{A}_\mathbb{Q}) \hookrightarrow \mathbb{A}_F^\times$ and on the level of $\mathbb{Q}$-points $\mathbb{Q}^\times \hookrightarrow R(\mathbb{Q}) \hookrightarrow F^\times$. Hence we also get an induced map on quotients $C_\mathbb{Q} \hookrightarrow C_R \hookrightarrow C_F$, and can pass to the connected components to get

$$\pi_0(\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times) \longrightarrow \pi_0(R(\mathbb{A})/R(\mathbb{Q})) \longrightarrow \pi_0(\mathbb{A}_F^\times/F^\times). \tag{5.2.4.1}$$

We denote the left hand arrow by i and the right hand arrow by j. Then $\gamma \in \mathrm{Aut}(\mathbb{C})$ acts on $\mathrm{Sh}(R, \mathrm{VZ}^R) = \pi_0(R(\mathbb{A}_\mathbb{Q})/R(\mathbb{Q}))$ by multiplication by

$$i\left(\mathrm{art}_\mathbb{Q}^{-1}(\gamma|_{\mathbb{Q}^{\mathrm{ab}}})\right) \in \pi_0(R(\mathbb{A}_\mathbb{Q})/R(\mathbb{Q})).$$

5.2.5 Example. Let us look in more detail at the special cases of Example (5.1.5):

(5.2.5.a) As in (5.1.5.a), let $G_1 = R_{F/\mathbb{Q}}\,\mathrm{GL}_2$ and $X_1 = (\mathbb{C} \setminus \mathbb{R})^\Sigma$, so that $\mathrm{Sh}(G_1, X_1)$ is the Hilbert modular variety. Then the connected components of $\mathrm{Sh}(G_1, X_1)$ are given by

$$\pi_0(\mathrm{Sh}(G_1, X_1)) = \mathrm{Sh}(R_{F/\mathbb{Q}}\mathbb{G}_m, \{\pm 1\}^\Sigma) = \pi_0(\mathbb{A}_F^\times/F^\times),$$

the group of connected components of the idele class group of F, which is isomorphic to Γ_F^{ab} via the Artin map art_F, see (2.4.1).

Let $\gamma \in \mathrm{Aut}(\mathbb{C})$ and $s \in \mathbb{A}_{\mathbb{Q}}^\times$ such that $\mathrm{art}_{\mathbb{Q}}(s) = \gamma|_{\mathbb{Q}^{\mathrm{ab}}}$. Then γ acts on $\mathrm{Sh}(R_{F/\mathbb{Q}}\mathbb{G}_m, \{\pm 1\}^\Sigma) = \pi_0(\mathbb{A}_F^\times/F^\times)$ as multiplication by the element

$$r(s) = i(\mathrm{art}_{\mathbb{Q}}^{-1}(\gamma|_{\mathbb{Q}^{\mathrm{ab}}})) = \mathrm{art}_F^{-1}(V_{F/\mathbb{Q}}(\gamma|_{\mathbb{Q}^{\mathrm{ab}}})) \in \pi_0(\mathbb{A}_F^\times/F^\times),$$

where the second equality uses the commutative diagram (2.4.3) from class field theory.

(5.2.5.b) As in (5.1.5.b), look at the PEL Hilbert modular variety $\mathrm{Sh}(G, X)$. The connected components are given by

$$\pi_0(\mathrm{Sh}(G, X)) = \mathrm{Sh}(\mathbb{G}_m, \{\pm 1\}) = \pi_0(\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times).$$

In this case, $r\colon \mathbb{G}_m \to R = \mathbb{G}_m$ is the identity, so $\gamma \in \mathrm{Aut}(\mathbb{C})$ acts on $\mathrm{Sh}(\mathbb{G}_m, \{\pm 1\}) = \pi_0(\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times)$ as multiplication by the element

$$r(s) = \mathrm{art}_{\mathbb{Q}}^{-1}(\gamma|_{\mathbb{Q}^{\mathrm{ab}}}) \in \pi_0(\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times).$$

By (5.2.4) the action of $\Gamma_{\mathbb{Q}}$ on $\pi_0(\mathrm{Sh}(G^R, X^R)) = \pi_0(R(\mathbb{A})/R(\mathbb{Q}))$ is determined by the map

$$\Gamma_{\mathbb{Q}} \xrightarrow{\mathrm{art}_{\mathbb{Q}}^{-1}} \pi_0(\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times) \xrightarrow{i} \pi_0(R(\mathbb{A})/R(\mathbb{Q})).$$

We would like to extend this action to the plectic group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$. It is natural to do this by extending this map, and it will again not be the full plectic group that acts but rather a group fitting into a Cartesian diagram.

5.2.6. (Plectic action on $\pi_0(\mathrm{Sh}(G^R, X^R))$) Let R be an algebraic torus defined over $\mathbb{Q}$ with $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ as in (4.4.2). We define the group $(S_\Sigma \ltimes \Gamma_F^\Sigma)_{\pi_0}^R$ by the Cartesian diagram

$$\begin{tikzcd}
(S_\Sigma \ltimes \Gamma_F^\Sigma)_{\pi_0}^R \arrow[r] \arrow[d, "(1,\mathrm{prod})_{\pi_0}"'] & S_\Sigma \ltimes \Gamma_F^\Sigma \arrow[d, "(1,\mathrm{prod})"] \\
\pi_0(R(\mathbb{A})/R(\mathbb{Q})) \arrow[r, "\mathrm{art}_F \circ j"'] & \Gamma_F^{\mathrm{ab}},
\end{tikzcd} \tag{5.2.6.1}$$

where $j\colon \pi_0(R(\mathbb{A})/R(\mathbb{Q})) \to \pi_0(\mathbb{A}_F^\times/F^\times)$ denotes the right hand arrow in (5.2.4.1). From now on we denote the left vertical arrow defined by this diagram by $(1, \mathrm{prod})_{\pi_0}$.

We also define $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_{\pi_0}^R$ by the Cartesian diagram

$$\begin{tikzcd}
\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})_{\pi_0}^R \arrow[r] \arrow[d, "(1,\mathrm{prod})_{\pi_0} \circ \beta_s"'] & \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \arrow[d, "(1,\mathrm{prod}) \circ \beta_s"] \\
\pi_0(R(\mathbb{A})/R(\mathbb{Q})) \arrow[r, "\mathrm{art}_F \circ j"'] & \Gamma_F^{\mathrm{ab}},
\end{tikzcd}$$

which is independent of s by (4.1.9.b). Note that the isomorphism $\beta_s\colon \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \xrightarrow{\sim} S_\Sigma \ltimes \Gamma_F^\Sigma$ induces an isomorphism $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\pi_0} \xrightarrow{\sim} (S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0}$, which we also denote by β_s.

Now let $\gamma \in (S_{\Sigma_F} \ltimes \Gamma_F^{\Sigma_F})^R_{\pi_0}$ act on $\pi_0(R(\mathbb{A}_{\mathbb{Q}})/R(\mathbb{Q}))$ by multiplication by the element $(1, \mathrm{prod})_{\pi_0}(\gamma)$. We claim that this defines a group action of $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0}$ on $\pi_0(R(\mathbb{A}_{\mathbb{Q}})/R(\mathbb{Q}))$ that extends the action of $\Gamma_{\mathbb{Q}}$ described in (5.2.4) via the embedding $\rho_s\colon \Gamma_{\mathbb{Q}} \hookrightarrow S_\Sigma \ltimes \Gamma_F^\Sigma$.

Of course, we also let $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\pi_0}$ act on $\pi_0(R(\mathbb{A})/R(\mathbb{Q}))$ by multiplication by $(1, \mathrm{prod})_{\pi_0}(\beta_s(\gamma))$, which is independent of s by (4.1.9.b).

Proof of claim. It is a group action because $(1, \mathrm{prod})_{\pi_0}$ is a group homomorphism. To show that it extends the $\Gamma_{\mathbb{Q}}$-action, look at the diagram

$$\begin{tikzcd}
\Gamma_{\mathbb{Q}} \arrow[dd] \arrow[dr, hook, dotted] \arrow[drr, hook, bend left, "\rho_s"] & & \\
& (S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0} \arrow[r] \arrow[d, "(1,\mathrm{prod})_{\pi_0}"'] & S_\Sigma \ltimes \Gamma_F^\Sigma \arrow[d, "(1,\mathrm{prod})"] \\
\Gamma_{\mathbb{Q}}^{\mathrm{ab}} \arrow[r, "i \circ \mathrm{art}_{\mathbb{Q}}^{-1}"'] & \pi_0(R(\mathbb{A})/R(\mathbb{Q})) \arrow[r, "\mathrm{art}_F \circ j"'] & \Gamma_F^{\mathrm{ab}},
\end{tikzcd}$$

where $i\colon \pi_0(\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times) \to \pi_0(R(\mathbb{A})/R(\mathbb{Q}))$ is the left hand arrow in (5.2.4.1). In this diagram, the composition of the two bottom arrows is equal to $V_{F/\mathbb{Q}}\colon \Gamma_{\mathbb{Q}}^{\mathrm{ab}} \to \Gamma_F^{\mathrm{ab}}$ by (2.4.3), so the commutative diagram in (4.1.9.b) precisely says that the outer trapezium in this diagram commutes. As the bottom right square is Cartesian, the the embedding $\rho_s\colon \Gamma_{\mathbb{Q}} \hookrightarrow S_\Sigma \ltimes \Gamma_F^\Sigma$ factors through $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0}$. The commutativity of the left hand trapezium then precisely shows that the action of $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0}$ restricted to $\Gamma_{\mathbb{Q}}$ coincides with the action in (5.2.4). $\square$

5.2.7 Remark. Note that the group $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0}$ is not necessarily a subgroup of the plectic group $S_\Sigma \ltimes \Gamma_F^\Sigma$. It will be a subgroup if the bottom arrow $\mathrm{art}_F \circ j$ in (5.2.6.1) is injective. Since $\mathrm{art}_F\colon \pi_0(\mathbb{A}_F^\times/F^\times) \xrightarrow{\sim} \Gamma_F^{\mathrm{ab}}$ is an isomorphism, the bottom arrow is injective if and only if the following condition holds:

(5.2.7.a) The map

$$j\colon \pi_0(R(\mathbb{A})/R(\mathbb{Q})) \longrightarrow \pi_0(\mathbb{A}_F^\times/F^\times),$$

induced from the inclusion $R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$, is injective.

In Section 5.3, we will discuss condition (5.2.7.a) and exhibit many examples of tori satisfying it, see (5.3.1) and (5.3.7).

Let us continue by looking at special cases of (5.2.6).

5.2.8 Example. ((PEL) Hilbert modular variety) As always, let us have a closer look at the special cases $R = R_{F/\mathbb{Q}}\mathbb{G}_m$ and $R = \mathbb{G}_m$:

(5.2.8.a) In the case (5.1.5.a) of the Hilbert modular variety, we have $R = R_{F/\mathbb{Q}}\mathbb{G}_m$ and $j = \mathrm{id}\colon \pi_0(\mathbb{A}_F^\times/F^\times) \to \pi_0(\mathbb{A}_F^\times/F^\times)$, so R satisfies (5.2.7.a). Moreover $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0} = S_\Sigma \ltimes \Gamma_F^\Sigma$, so the entire plectic group acts on $\pi_0(\mathrm{Sh}(G_1, X_1))$.

(5.2.8.b) In the case (5.1.5.b) of the PEL Hilbert modular variety, we have $R = \mathbb{G}_m$ and $j\colon \pi_0(\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times) \to \pi_0(\mathbb{A}_F^\times/F^\times)$. By (2.4.3), the map j corresponds to the transfer map $V_{F/\mathbb{Q}}\colon \Gamma_\mathbb{Q}^{\mathrm{ab}} \to \Gamma_F^{\mathrm{ab}}$ under class field theory. Hence j is injective by (4.2.2), thus R satisfies (5.2.7.a).

Moreover, $(S_\Sigma \ltimes \Gamma_F^\Sigma)^{\mathbb{G}_m}_{\pi_0} = (S_\Sigma \ltimes \Gamma_F^\Sigma)_0$ by comparing the defining diagrams in (5.2.6) and (4.2.8): The bottom left entries are isomorphic via the Artin map $\mathrm{art}_\mathbb{Q}\colon \pi_0(\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times) \xrightarrow{\sim} \Gamma_\mathbb{Q}^{\mathrm{ab}}$ and the bottom arrows are related by diagram (2.4.3).

Next, we show that the π_0-map

$$\pi_0\colon \mathrm{Sh}(G^R, X^R)_{\mathrm{CM}} \longrightarrow \mathrm{Sh}(R, \mathrm{VZ}^R)$$

is equivariant with respect to the plectic actions defined in (4.4.13) and (5.2.6). Here $\mathrm{Sh}(G^R, X^R)_{\mathrm{CM}}$ denotes the set of CM points of $\mathrm{Sh}(G^R, X^R)$, see (4.4.10).

First of all, we need to compare the group $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{CM}}$, defined in (4.4.11), which acts on $\mathrm{Sh}(G^R, X^R)_{\mathrm{CM}}$, with the group $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0}$, defined in (5.2.6), which acts on $\mathrm{Sh}(R, \mathrm{VZ}^R)$. In (5.2.8) we saw that for $R = \mathbb{G}_m$ or $R = R_{F/\mathbb{Q}}\mathbb{G}_m$ we have $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{CM}} = (S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0}$. In general, we have

5.2.9 Lemma. *Let $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ be an intermediate algebraic torus over $\mathbb{Q}$. Then*

$$(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{CM}} \hookrightarrow (S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0}.$$

Proof. Look at the diagram

$$\begin{tikzcd}
(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{CM}} \arrow[r, hook] \arrow[d] & S_\Sigma \ltimes \Gamma_F^\Sigma \arrow[d, "\chi_F \circ (1,\mathrm{prod})"] \arrow[ddr, bend left, "(1,\mathrm{prod})"] & \\
R(\mathbb{A}_f)/R(\mathbb{Q})_{>0} \arrow[r, hook] \arrow[d] & \mathbb{A}_{F,f}^\times/F_{>0}^\times \arrow[d, two heads] \arrow[dr, two heads, "r_F"] & \\
\pi_0(R(\mathbb{A})/R(\mathbb{Q})) \arrow[r, "j"] & \pi_0(\mathbb{A}_F^\times/F^\times) \arrow[r, "\mathrm{art}_F", "\sim"'] & \Gamma_F^{\mathrm{ab}}.
\end{tikzcd} \tag{5.2.9.1}$$

Here the top left hand square is the Cartesian square (4.4.11), which defines $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{CM}}$, the top right hand triangle commutes because $r_F \circ \chi_F = \mathrm{id}$, and the bottom right hand triangle commutes by (2.4.1.1). Moreover, the bottom left hand vertical arrow is well-defined because clearly the identity component of $R(\mathbb{A})/R(\mathbb{Q})$ contains $(R(\mathbb{R})_{>0} \cdot R(\mathbb{Q}))/R(\mathbb{Q})$, and the kernel of the map $R(\mathbb{A}_f) \to R(\mathbb{A})/(R(\mathbb{R})_{>0} \cdot R(\mathbb{Q}))$ is $R(\mathbb{Q})_{>0}$. Also j is induced by $R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$, and so the bottom left square commutes by functoriality of taking quotients and applying π_0.

Looking only at the outer edges of (5.2.9.1), we conclude that the embedding $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{CM}} \subset S_\Sigma \ltimes \Gamma_F^\Sigma$ factors through $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\pi_0}$ by the Cartesian diagram (5.2.6.1). $\square$

We are ready to prove our main result Theorem (1.2.1).

5.2.10 Theorem. *Let $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ be an intermediate algebraic torus defined over $\mathbb{Q}$. Then the π_0-map restricted to CM points*

$$\pi_0 \colon \operatorname{Sh}(G^R, X^R)_{\mathrm{CM}} \longrightarrow \operatorname{Sh}(R, \mathrm{VZ}^R)$$

is $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{CM}}$-equivariant.

Proof. The main ingredients of the proof are the description (5.1.9) of the π_0-map, the interpretation (4.4.5) of (special) points of $\operatorname{Sh}(G^R, X^R)$ as (CM) abelian varieties, and of course the definition of the plectic actions on CM points (4.4.13) and on the set of connected components (5.2.6).

So let

$$\mathcal{P} = \left[\mathbb{C}^\Phi/\Phi(\mathfrak{a}), i_\Phi|_F, R(\mathbb{Q})E_t, \eta\overline{Z^R(\mathbb{Q})}\right]$$

be a CM point of $\operatorname{Sh}(G^R, X^R)$, with K a totally imaginary quadratic extension of F and $(K, \Phi; \mathfrak{a}, t)$ as in (2.3.5). Write the CM type Φ as $\Phi = \{\varphi_x \mid x \in \Sigma_F\}$, where $\varphi_x|_F = x$ for all $x \in \Sigma_F$. Let us assume that $\operatorname{Im}\varphi_x(t) > 0$ for all $x \in \Sigma_F$.

(a) Calculation of $\pi_0(\mathcal{P})$:

We start by calculating the point $[h, g] \in \operatorname{Sh}(G^R, X^R)$ corresponding to $\mathcal{P}$. The Hodge structure $h_\Phi \colon \mathbb{S} \to \mathrm{GL}(H_1(\mathbb{C}^\Phi/\Phi(\mathfrak{a}), \mathbb{R}))$ of the abelian variety $\mathbb{C}^\Phi/\Phi(\mathfrak{a})$ is given in (2.3.3): We have $H_1(\mathbb{C}^\Phi/\Phi(\mathfrak{a}), \mathbb{Q}) = K$ and h_Φ on real points is given by

$$\begin{aligned} h_\Phi \colon \mathbb{C}^\times &\longrightarrow \operatorname{Aut}_{K\otimes_\mathbb{Q}\mathbb{R}}(H_1(\mathbb{C}^\Phi/\Phi(\mathfrak{a}), \mathbb{R})) = (K \otimes_\mathbb{Q} \mathbb{R})^\times \subset \mathrm{GL}_\mathbb{R}(\mathbb{C}^\Phi) \\ i &\longmapsto \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix} \end{aligned}$$

with respect to the $\mathbb{R}$-basis

$$\left\{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix},\begin{pmatrix}i\\0\\\vdots\\0\end{pmatrix},\dots,\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix},\begin{pmatrix}0\\\vdots\\0\\i\end{pmatrix}\right\}$$

of $\mathbb{C}^\Phi = K\otimes_{\mathbb{Q}}\mathbb{R}$.

Using the totally imaginary element $\alpha := \frac{1}{t}\in K^\times$, which under $\varphi_x\in\Phi$ has positive imaginary parts $\beta_x := \operatorname{Im}\varphi_x(\alpha) > 0$ for all $x\in\Sigma_F$, we define the isomorphism of F-vector spaces

$$a\colon K \xrightarrow{\sim} F^2, \quad 1\mapsto \begin{pmatrix}1\\0\end{pmatrix}, \alpha\mapsto\begin{pmatrix}0\\1\end{pmatrix}.$$

We conclude that by (4.4.5) we have

$$h = a\circ h_\Phi\circ a^{-1}\colon \mathbb{S}\longrightarrow (G^R)_{\mathbb{R}} \quad\text{and}\quad g = a\circ\eta\in G^R(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{A}_{F,f}),$$

but first we need to check that a satisfies condition (4.4.5.a): Indeed, a direct calculation shows that

$$E_t(u,v) = -2\operatorname{Tr}_{F/\mathbb{Q}}(u_1v_2 - u_2v_1) = -2\psi(a(u),a(v)),$$

where $u = u_1+u_2\alpha, v = v_1+v_2\alpha\in K^\times$ with $u_1,u_2,v_1,v_2\in F^\times$. As $-2\in\mathbb{Q}^\times\subset R(\mathbb{Q})$, this means that a sends E_t to an $R(\mathbb{Q})$-multiple of ψ as desired.

Moreover, $h = a\circ h_\Phi\circ a^{-1}\in X^R$ holds by the following calculation of the image of h under the isomorphism $X_1\cong(\mathbb{C}\backslash\mathbb{R})^\Sigma$ of (3.2.1.2): We see that $h(i) = (h(i)_x)_{x\in\Sigma}\in G_1(\mathbb{R})$ with

$$h(i)_x = \begin{pmatrix}0 & i\varphi_x(\alpha)\\ \frac{-1}{i\varphi_x(\alpha)} & 0\end{pmatrix} = \begin{pmatrix}0 & -\beta_x\\ \frac{1}{\beta_x} & 0\end{pmatrix} = \begin{pmatrix}\beta_x & \\ & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\beta_x & \\ & 1\end{pmatrix}^{-1}.$$

This means that $h = Bh_0B^{-1}$ with $B = \left(\begin{pmatrix}\beta_x & \\ & 1\end{pmatrix}\right)_{x\in\Sigma_F}\in G_1(\mathbb{R})$. Hence h corresponds to $B\cdot(i,\dots,i) = (\beta_x i)_{x\in\Sigma_F}$, which lies in $\mathfrak{h}^\Sigma = X_1^+ = (X^R)^+$. In particular, h lies in X^R and so a satisfies (4.4.5.a).

Using (5.1.9), we conclude that

$$\pi_0(\mathcal{P}) = [(\operatorname{sgn}(\beta_x))_{x\in\Sigma},\det(a\circ\eta)] = [(1)_{x\in\Sigma},\det(a\circ\eta)]\in\operatorname{Sh}(R,\mathrm{VZ}^R).$$

(b) Calculation of $\gamma(\pi_0(\mathcal{P}))$, for $\gamma \in (S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{CM}}$.

By (5.2.6), the image of $[(1)_{x\in\Sigma}, \det(a\circ\eta)]$ under γ is

$$\gamma(\pi_0(\mathcal{P})) = (1, \mathrm{prod})_{\pi_0} \circ \beta_s(\gamma) \cdot [(1)_{x\in\Sigma}, \det(a\circ\eta)]. \tag{5.2.10.1}$$

(c) Calculation of $\pi_0(\gamma\mathcal{P})$:

On the other hand, let us first conjugate $\mathcal{P}$ by γ and then calculate the image under the π_0-map. After choosing $\widetilde{f} \in \mathbb{A}_{K,f}^\times$ such that $\widetilde{f_\Phi}(\gamma) = \widetilde{f}K^\times$ and $u \in R(\mathbb{A}_f)$ such that $\chi_F \circ (1, \mathrm{prod}) \circ \beta_s(\gamma) = uF_{>0}^\times$, (4.4.13.1) gives

$$\gamma\mathcal{P} = \left[\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a}), i_{\gamma\Phi}|_F, F^\times E_{\chi t}, \widetilde{f}\circ\eta\overline{Z^R(\mathbb{Q})}\right],$$

where $\chi = \frac{u}{1+c\widetilde{f}} \in F^\times$.

Doing the exact same steps as before, let us first calculate the point $[h', g'] \in \mathrm{Sh}(G^R, X^R)$ corresponding to this conjugate quadruple: Write $\gamma\Phi = \{\varphi'_x \mid x \in \Sigma_F\}$, where $\varphi'_x|_F = x$ for all $x \in \Sigma_F$. Let $\alpha' := \frac{1}{\chi t} \in K^\times$. The calculation of the signs of χt in (4.4.13.3) shows that we have

$$\beta'_x := \operatorname{Im}\varphi'_x(\alpha') > 0, \quad \text{for all } x \in \Sigma_F.$$

The Hodge structure of $\mathbb{C}^{\gamma\Phi}/\gamma\Phi(\widetilde{f}\mathfrak{a})$ is given by $h_{\gamma\Phi}$, and using the isomorphism of F-vector spaces

$$a'\colon K \xrightarrow{\sim} F^2, \quad 1 \mapsto \begin{pmatrix}1\\0\end{pmatrix}, \alpha' \mapsto \begin{pmatrix}0\\1\end{pmatrix}$$

we see that

$$h' = a' \circ h_{\gamma\Phi} \circ (a')^{-1} \quad \text{and} \quad g' = a' \circ \widetilde{f} \circ \eta.$$

Furthermore, we see that $h'(i) = (h'(i)_x)_{x\in\Sigma} \in G^R(\mathbb{R})$ with

$$h'(i)_x = \begin{pmatrix} 0 & -\beta'_x \\ \frac{1}{\beta'_x} & 0 \end{pmatrix} = \begin{pmatrix} \beta'_x & \\ & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \beta'_x & \\ & 1 \end{pmatrix}^{-1},$$

hence $h' = B'h_0(B')^{-1}$ with $B' = \left(\begin{pmatrix} \beta'_x & \\ & 1 \end{pmatrix}\right)_{x\in\Sigma_F} \in G_1(\mathbb{R})$ and so h' corresponds to $B' \cdot (i, \ldots, i) = (\beta'_x i)_{x\in\Sigma_F} \in \mathfrak{h}^\Sigma \subset X^R$.

Finally, we get

$$\pi_0(\gamma\mathcal{P}) = [(\operatorname{sgn}\beta'_x)_{x\in\Sigma}, \det(a'\circ\widetilde{f}\circ\eta)] = [(1)_{x\in\Sigma}, \det(a'\circ\widetilde{f}\circ\eta)].$$

As $a' = \begin{pmatrix} 1 & \\ & \chi \end{pmatrix} \circ a$, we see that

$$\det(a' \circ \widetilde{f} \circ \eta) = \chi({}^{1+c}\widetilde{f}) \det(a \circ \eta) = u \det(a \circ \eta),$$

so we conclude that

$$\pi_0(\gamma \mathcal{P}) = [(1)_{x \in \Sigma}, u \det(a \circ \eta)]. \tag{5.2.10.2}$$

(d) Comparison of (5.2.10.1) and (5.2.10.2):

We do this by spelling out the action of γ in (5.2.10.1). By the proof of Lemma (5.2.9), in particular by diagram (5.2.9.1), the element $(1, \mathrm{prod})_{\pi_0} \circ \beta_s(\gamma)$ is equal to the image of $uR(\mathbb{Q})_{>0} \in R(\mathbb{A}_f)/R(\mathbb{Q})_{>0}$ inside $\pi_0(R(\mathbb{A})/R(\mathbb{Q}))$. Chasing through the identification $\mathrm{Sh}(R, \mathrm{VZ}^R) \cong \pi_0(R(\mathbb{A})/R(\mathbb{Q}))$ of (5.2.3), we see that (5.2.10.1) equals

$$(1, \mathrm{prod})_{\pi_0} \circ \beta_s(\gamma) \cdot [(1)_{x \in \Sigma}, \det(a \circ \eta)] = [(1)_{x \in \Sigma}, u \det(a \circ \eta)],$$

i. e. is equal to (5.2.10.2). $\square$

5.3 Appendix: When is $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^R_{\pi_0}$ a subgroup of the plectic group?

We conclude this chapter by discussing scenarios in which the torus R satisfies assumption (5.2.7.a). As a motivating example, we look at tori $R = R_{F'/\mathbb{Q}}\mathbb{G}_m$ coming from subfields $F' \subset F$:

5.3.1 Lemma. *Let $F' \subset F$ be a subfield, and let $R = R_{F'/\mathbb{Q}}\mathbb{G}_m$. Let $\mathbb{G}_m \hookrightarrow R \hookrightarrow R_{F/\mathbb{Q}}\mathbb{G}_m$ be the morphisms of $\mathbb{Q}$-algebraic tori that on $\mathbb{Q}$-points are given by the inclusions $\mathbb{Q}^\times \hookrightarrow F'^\times \hookrightarrow F^\times$.*

Then R satisfies (5.2.7.a).

Proof. First of all, note that $R(\mathbb{A}) = \mathbb{A}_{F'}^\times$ and $R(\mathbb{Q}) = F'^\times$, so that $R(\mathbb{A})/R(\mathbb{Q})$ is equal to the idele class group $C_{F'} = \mathbb{A}_{F'}^\times / F'^\times$ of F'.

We want to prove that

$$\pi_0(C_{F'}) \longrightarrow \pi_0(C_F)$$

is injective. We reduce to the case when F/F' is Galois: Indeed, in the general case let $F^\diamond$ be the Galois closure of F over F'. Then the result for Galois extensions implies that $\pi_0(C_{F'})$ and $\pi_0(C_F)$ both embed into $\pi_0(C_{F^\diamond})$. The map $C_{F'} \to C_{F^\diamond}$ factors through C_F, so the embedding $\pi_0(C_{F'}) \hookrightarrow \pi_0(C_{F^\diamond})$ factors through $\pi_0(C_F)$, implying that the map $\pi_0(C_{F'}) \to \pi_0(C_F)$ is injective as desired.

So now assume that F/F' is Galois, say with Galois group H. Look at the short exact sequence

$$0 \longrightarrow C_F^0 \longrightarrow C_F \longrightarrow \pi_0(C_F) \longrightarrow 0,$$

where C_F^0 denotes the identity component of C_F.

Taking H-invariants of this short exact sequence yields the long exact sequence

$$0 \longrightarrow (C_F^0)^H \longrightarrow C_F^H \longrightarrow \pi_0(C_F)^H \longrightarrow H^1(H, C_F^0) \longrightarrow \dots$$

By [Neu13, III.2.7], we have $H^0(H, C_F) = C_{F'}$, and by [NSW15, 8.2.6] we have $H^0(H, C_F^0) = C_{F'}^0$ and $H^1(H, C_F^0) = 0$. Hence the long exact sequence yields

$$(\pi_0(C_F))^H = C_{F'}/C_{F'}^0 = \pi_0(C_{F'}),$$

so in particular the map $\pi_0(C_{F'}) \to \pi_0(C_F)$ is injective. □

Inspired by the proof of (5.3.1), we can formulate a condition in terms of Galois cohomology of the torus R that also implies (5.2.7.a).

5.3.2 Proposition. *Let $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ be as in* (4.4.2). *Denote a Galois closure of F over $\mathbb{Q}$ by $F^\diamond$ and its Galois group by $G := \mathrm{Gal}(F^\diamond/\mathbb{Q})$. Write $\Lambda := X^*(R)$ for the character group of R. The torus R is split over $F^\diamond$, so Λ is a G-module, and we assume that*

$$\ker\left(\mathrm{Ext}^1_G(\Lambda, (F^\diamond)^\times) \longrightarrow \mathrm{Ext}^1_G(\Lambda, \mathbb{A}^\times_{F^\diamond})\right) = 0. \tag{5.3.2.1}$$

Then R satisfies (5.2.7.a).

Before starting with the proof, let us simplify notation and assume that $F/\mathbb{Q}$ is Galois:

5.3.3 Remark. (Reduction to the case when $F/\mathbb{Q}$ is Galois) Assuming the statement of the proposition for all Galois extensions $F/\mathbb{Q}$, we can deduce the general statement in the same way as in the proof of (5.3.1): We then have embeddings $\pi_0(R(\mathbb{A})/R(\mathbb{Q})) \hookrightarrow \pi_0(C_{F^\diamond})$ and $\pi_0(C_F) \hookrightarrow \pi_0(C_{F^\diamond})$, fitting into the commutative diagram

$$\begin{tikzcd} \pi_0(R(\mathbb{A})/R(\mathbb{Q})) \arrow[r, hook] \arrow[dr] & \pi_0(C_{F^\diamond}) \\ & \pi_0(C_F). \arrow[u, hook] \end{tikzcd}$$

This implies that the diagonal arrow is also injective, which is what we wanted to show.

From now on, we assume that $F/\mathbb{Q}$ is Galois with $G = \mathrm{Gal}(F/\mathbb{Q})$. Before completing the proof of (5.3.2), let us make some remarks about the Ext-groups $\mathrm{Ext}^1_G(\Lambda, F^\times)$ and $\mathrm{Ext}^1_G(\Lambda, \mathbb{A}_F^\times)$:

5.3.4 Remark ($\mathrm{Ext}^1_G(\Lambda, -)$)**.** The functors $\mathrm{Ext}^p(\Lambda, -)$ are the right derived functors of the functor $\mathrm{Hom}_G(\Lambda, -)$. As functors from the category of G-modules to the category of abelian groups, we have

$$\mathrm{Hom}_G(\Lambda, -) = (\cdot)^G \circ \mathrm{Hom}(\Lambda, -).$$

The associated Grothendieck spectral sequence

$$E_2^{pq} = H^p(G, \mathrm{Ext}^q(\Lambda, -)) \Longrightarrow \mathrm{Ext}^{p+q}_G(\Lambda, -).$$

degenerates, because Λ is a free abelian group and hence $\mathrm{Hom}(\Lambda, -)$ is exact. Thus

$$H^p(G, \mathrm{Hom}(\Lambda, -)) = \mathrm{Ext}^p_G(\Lambda, -), \quad p \geq 0.$$

In particular, we get

$$\begin{aligned} \mathrm{Ext}^1_G(\Lambda, F^\times) &= H^1(G, \mathrm{Hom}(\Lambda, F^\times)) \\ &= H^1(G, X_*(R) \otimes_{\mathbb{Z}} F^\times) \\ &= H^1(G, R(F)), \end{aligned}$$

where we used the G-isomorphisms $\mathrm{Hom}(\Lambda, F^\times) \cong X_*(R) \otimes_{\mathbb{Z}} F^\times \cong R(F)$, the former being induced from the perfect pairing $\Lambda \times X_*(R) \to \mathbb{Z}$, the latter being [PR94, Lemma 6.7, p. 302].

Similarly, by [Tat66, p. 719] we have $\mathrm{Hom}(\Lambda, \mathbb{A}_F^\times) = R(\mathbb{A}_F)$, thus

$$\mathrm{Ext}^1_G(\Lambda, \mathbb{A}_F^\times) = H^1(G, R(\mathbb{A}_F)).$$

Hence (5.3.2.1) is equivalent to

$$Ш(R) := \ker\big(H^1(G, R(F)) \longrightarrow H^1(G, R(\mathbb{A}_F))\big) = 0, \tag{5.3.4.1}$$

where $Ш(R)$ denotes the *Shafarevich–Tate group* of R, see [PR94, p. 307].

Proof of Proposition (5.3.2). By remark (5.3.3) we assume that $F/\mathbb{Q}$ is Galois with $G := \mathrm{Gal}(F/\mathbb{Q})$. We follow the same strategy as in the proof of (5.3.1) and start by looking at the short exact sequence

$$0 \longrightarrow C_F^0 \longrightarrow C_F \longrightarrow \pi_0(C_F) \longrightarrow 0.$$

We will always denote the identity component of a topological group A by A^0.

Applying the functor $\mathrm{Hom}_G(\Lambda, -)$ yields

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_G(\Lambda, C_F^0) \arrow[r] & \mathrm{Hom}_G(\Lambda, C_F) \arrow[r] & \mathrm{Hom}_G(\Lambda, \pi_0(C_F)) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^1_G(\Lambda, C_F^0) \arrow[r] & \dots \end{tikzcd}$$

We will see below that

(a) $\mathrm{Hom}_G(\Lambda, C_F) = R(\mathbb{A})/R(\mathbb{Q})$,

(b) $\mathrm{Hom}_G(\Lambda, C_F^0) = \mathrm{Hom}_G(\Lambda, C_F)^0$, and

(c) $\mathrm{Ext}^1_G(\Lambda, C_F^0) = 0$,

so the above long exact sequence yields

$$\mathrm{Hom}_G(\Lambda, \pi_0(C_F)) = \frac{R(\mathbb{A})/R(\mathbb{Q})}{\big(R(\mathbb{A})/R(\mathbb{Q})\big)^0} = \pi_0(R(\mathbb{A})/R(\mathbb{Q})).$$

Now note that $R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$, so $\Lambda = X^*(R)$ is a quotient of $X^*(R_{F/\mathbb{Q}}\mathbb{G}_m) = \mathbb{Z}[G]$, say $\Lambda = \mathbb{Z}[G]/I$ for a G-stable $I \subset \mathbb{Z}[G]$. This means Λ is a cyclic $\mathbb{Z}[G]$-module, thus

$$\mathrm{Hom}_G(\Lambda, \pi_0(C_F)) \longrightarrow \pi_0(C_F), \quad \alpha \mapsto \alpha([\mathrm{id}] + I)$$

is injective, i. e. $\pi_0(R(\mathbb{A})/R(\mathbb{Q}))$ embeds into $\pi_0(C_F)$.

- Proof of (a):

Let us look at the short exact sequence

$$0 \longrightarrow F^\times \longrightarrow \mathbb{A}_F^\times \longrightarrow C_F \longrightarrow 0.$$

We apply the functor $\mathrm{Hom}_G(\Lambda, -)$ and, using the assumption (5.3.2.1), get the short exact sequence

$$0 \longrightarrow \mathrm{Hom}_G(\Lambda, F^\times) \longrightarrow \mathrm{Hom}_G(\Lambda, \mathbb{A}_F^\times) \longrightarrow \mathrm{Hom}_G(\Lambda, C_F) \longrightarrow 0.$$

Now we use [Vos01, §2.4, p. 3111] that, because R is split over F, for every $\mathbb{Q}$-algebra A we have

$$R(A) = \mathrm{Hom}_G(\Lambda, (A \otimes_\mathbb{Q} F)^\times).$$

Applying this fact for $A = \mathbb{Q}$ and $A = \mathbb{A} = \mathbb{A}_\mathbb{Q}$ gives

$$R(\mathbb{A})/R(\mathbb{Q}) = \mathrm{Hom}_G(\Lambda, C_F).$$

- Proof of (b) and (c):

Since Λ is $\mathbb{Z}$-free, we have

$$\mathrm{Hom}(\Lambda, C_F^0) = \mathrm{Hom}(\Lambda, C_F)^0. \tag{5.3.4.2}$$

To shorten notation, let us temporarily denote $\mathrm{Hom}(\Lambda, C_F)$ by A. Since C_F^0 is G-stable, so is $\mathrm{Hom}(\Lambda, C_F^0) = A^0$. So we take G-invariants and get

$$\mathrm{Hom}_G(\Lambda, C_F^0) = \left[A^0\right]^G.$$

For (b), we are left to show

$$\left[A^0\right]^G = \left[A^G\right]^0.$$

First of all, $[A^G]^0$ is connected, so $[A^G]^0 \subset A^0$; but also every element of $[A^G]^0$ is G-invariant, hence

$$\left[A^0\right]^G \supset \left[A^G\right]^0.$$

To show equality, we need to show that $[A^0]^G$ is connected, because then the inclusion $[A^0]^G \subset A^G$ implies that

$$[A^0]^G \subset [A^G]^0.$$

To show that $[A^0]^G$ is connected, let us first use the definition of A and (5.3.4.2):

$$\left[A^0\right]^G = \left[\mathrm{Hom}(\Lambda, C_F)^0\right]^G = H^0(G, \mathrm{Hom}(\Lambda, C_F^0)).$$

Using that Λ is a free $\mathbb{Z}$-module, say of rank m, we see that as abelian groups we have

$$\mathrm{Hom}(\Lambda, C_F^0) \cong (C_F^0)^m.$$

By [NSW15, 8.2.5], $C_F^0 = ([\mathbb{R} \times \widehat{\mathbb{Z}}]/\mathbb{Z})^{r-1} \times \mathbb{R}$, where $r = [F : \mathbb{Q}]$, which by [NSW15, 8.2.4] is uniquely divisible. Thus the same is true for $(C_F^0)^m \cong \mathrm{Hom}(\Lambda, C_F^0)$, so by [NSW15, 1.6.2] we see that $\mathrm{Hom}(\Lambda, C_F^0)$ is cohomologically trivial, i. e.

$$\hat{H}^0(G, \mathrm{Hom}(\Lambda, C_F^0)) = 0 \quad \text{and} \quad H^1(G, \mathrm{Hom}(\Lambda, C_F^0)) = 0. \tag{5.3.4.3}$$

The first equality means that

$$H^0(G, \mathrm{Hom}(\Lambda, C_F^0)) = N_G(\mathrm{Hom}(\Lambda, C_F^0)),$$

so the continuity of the norm map $N_G = \sum_{\sigma \in G} \sigma \colon C_F \to C_F$ implies that $[A^0]^G = H^0(G, \mathrm{Hom}(\Lambda, C_F^0))$ is connected because $\mathrm{Hom}(\Lambda, C_F^0) \cong (C_F^0)^m$ is. This finishes the proof of (b).

Finally, using (5.3.4) and (5.3.4.3) we can also deduce (c):

$$\operatorname{Ext}^1_G(\Lambda, C^0_F) = H^1(G, \operatorname{Hom}(\Lambda, C^0_F)) = 0.$$

This finishes the proof of the proposition. □

5.3.5 Remark. For completeness, note that the torus $R = R_{F'/\mathbb{Q}}\mathbb{G}_m$ for a subfield $F' \subset F$ satisfies (5.3.2.1), so that (5.3.2) includes (5.3.1) as a special case:

We still assume that $F/\mathbb{Q}$ is Galois with $G = \operatorname{Gal}(F/\mathbb{Q})$, and let $H := \operatorname{Gal}(F/F')$. Then $\Lambda = X^*(R) = \mathbb{Z}[G/H] = \operatorname{Coind}^G_H \mathbb{Z} = \operatorname{Ind}^G_H \mathbb{Z}$ as G-modules[4], hence by [NSW15, footnote p. 63] we have

$$\operatorname{Hom}_G(\Lambda, -) = \operatorname{Hom}_G(\operatorname{Ind}^G_H \mathbb{Z}, -) = \operatorname{Hom}_H(\mathbb{Z}, \operatorname{Res}^G_H -) = (\cdot)^H \circ \operatorname{Res}^G_H,$$

where Res^G_H is the functor from G-modules to H-modules that simply restricts the G-action to the subgroup H, and $\mathbb{Z}$ is equipped with the trivial H-action.

The functor Res^G_H is clearly exact, hence in the same way as in (5.3.4) we deduce that

$$H^p(H, \operatorname{Res}^G_H -) = \operatorname{Ext}^p_G(\Lambda, -), \quad p \geq 0.$$

Thus by Hilbert 90:

$$\operatorname{Ext}^1_G(\Lambda, F^\times) = H^1(H, F^\times) = 0,$$

hence R satisfies (5.3.2.1).

We finish this section by discussing condition (5.3.4.1) and providing more examples of tori R satisfying it.

5.3.6 Remark. Let $F/\mathbb{Q}$ be Galois with Galois group G and $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ as before. Let $S = S_F$ denote the set of all places of F. We view $\mathbb{Z}[S]$ as a G-module via the usual action of G on places of F, and let A be the kernel of the G-equivariant surjective map

$$\begin{aligned} \mathbb{Z}[S] &\longrightarrow \mathbb{Z}, \\ \sum_{w \in S} a_w[w] &\longmapsto \sum_{w \in S} a_w. \end{aligned}$$

We apply [Tat66, Thm, p. 717] with $M := X_*(R)$ and $r + 2 = 1$ to get the

[4]See [NSW15, footnote p. 61] for the definition of the functor Coind^G_H from H-modules to G-modules. Here it is equal to Ind^G_H (see (4.1.4.1)) because G is finite.

following commutative diagram with exact rows and vertical isomorphisms:

$$\begin{tikzcd}
\dots \arrow[r] & \hat{H}^{-1}(G, A \otimes_{\mathbb{Z}} M) \arrow[r] \arrow[d, "\sim"] & \hat{H}^{-1}(G, \mathbb{Z}[S] \otimes_{\mathbb{Z}} M) \arrow[r] \arrow[d, "\sim"] & \hat{H}^{-1}(G, M) \arrow[d, "\sim"] \\
\dots \arrow[r] & \hat{H}^{1}(G, R(F)) \arrow[r] & \hat{H}^{1}(G, R(\mathbb{A}_F)) \arrow[r] & \hat{H}^{1}(G, C_F \otimes_{\mathbb{Z}} M)
\end{tikzcd}$$

Thus

$$\begin{aligned}
Ш(R) &= \ker\left(\hat{H}^{-1}(G, A \otimes_{\mathbb{Z}} M) \to \hat{H}^{-1}(G, \mathbb{Z}[S] \otimes_{\mathbb{Z}} M)\right) \\
&= \operatorname{coker}\left(\hat{H}^{-2}(G, \mathbb{Z}[S] \otimes_{\mathbb{Z}} M) \to \hat{H}^{-2}(G, M)\right),
\end{aligned}$$

so (5.3.4.1) is equivalent to

$$\operatorname{coker}\left(\hat{H}^{-2}(G, \mathbb{Z}[S] \otimes_{\mathbb{Z}} M) \to \hat{H}^{-2}(G, M)\right) = 0. \tag{5.3.6.1}$$

For certain fields F, all tori R with $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ satisfy (5.3.6.1):

5.3.7 Example. Let $F/\mathbb{Q}$ be Galois and $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ as before. Denote $M := X_*(R)$. Assume that there exists a prime number p with precisely one prime ideal $\mathfrak{p}$ above p in F. For example, if $F/\mathbb{Q}$ is cyclic, there exist inert primes p by Chebotarev's Density Theorem [Neu92, VII.13.4], which have this property.

The G-module $\mathbb{Z}[S]$ decomposes as $\bigoplus_v \mathbb{Z}[S_v]$, where v runs over the places $S_{\mathbb{Q}}$ of $\mathbb{Q}$ and S_v denotes the places of F above v. Let us show that R satisfies (5.3.6.1), i. e. that the map

$$\hat{H}^{-2}(G, \mathbb{Z}[S_F] \otimes_{\mathbb{Z}} M) = \bigoplus_{v \in S_{\mathbb{Q}}} \hat{H}^{-2}(G, \mathbb{Z}[S_v] \otimes_{\mathbb{Z}} M) \longrightarrow \hat{H}^{-2}(G, M)$$

is surjective: By assumption $S_p = \{\mathfrak{p}\}$, so

$$\mathbb{Z}[S_p] \longrightarrow \mathbb{Z}, \quad a[\mathfrak{p}] \longmapsto a,$$

is an isomorphism of G-modules, inducing a G-equivariant isomorphism $\mathbb{Z}[S_p] \otimes_{\mathbb{Z}} M \cong M$. Thus $\hat{H}^{-2}(G, \mathbb{Z}[S_p] \otimes_{\mathbb{Z}} M) \cong \hat{H}^{-2}(G, M)$ is also an isomorphism, hence R satisfies (5.3.6.1).

5.3.8. (Summary) We have seen that the equivalent conditions (5.3.2.1), (5.3.4.1) (vanishing of $Ш(R)$) and (5.3.6.1) are satisfied for many tori R, see (5.3.5) and (5.3.7). We also saw in (5.3.2) that these conditions imply (5.2.7.a).

However, we do not know whether (5.2.7.a) is equivalent to (5.3.2.1), or whether either of these conditions holds for all tori R. There are examples of "norm one tori" T which do not satisfy $Ш(T) = 0$, see e. g. [PR94, Ex. 1, p. 308]. The Shafarevitch–

Tate group of a norm one torus is well-studied because it vanishes if and only if the Hasse norm principle holds for the corresponding field extension. See [MN19] for more details.

Unfortunately, the norm one torus for the extension $F/\mathbb{Q}$ does not contain $\mathbb{G}_m$, so it does not fall into the setup of (4.4.2). Nonetheless the fact that for certain fields F there are tori $T \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ that do not satisfy $Ш(T) = 0$ seems to us to be an indication that (5.3.4.1) probably does not hold for all R with $\mathbb{G}_m \subset R$ either.

6

Plectic action on cocharacters

Fix a totally real number field F. The reflex field of a Shimura datum (G, X) is of fundamental importance in the theory of Shimura varieties. The goal of this chapter is to define the plectic analogue of the absolute Galois group of the reflex field, the so-called plectic reflex Galois group, for algebraic groups G that differ only in the centre from a group of the form $R_{F/\mathbb{Q}}H$.

In Section 6.1, we recall from [NS16, §5] how the plectic group $S_\Sigma \ltimes \Gamma_F^\Sigma$ canonically acts on the cocharacters of the group $R_{F/\mathbb{Q}}H$, and how this action is used to define the plectic reflex Galois group of a Shimura datum $(R_{F/\mathbb{Q}}H, X)$.

In Section 6.2 we look at cocharacters of the groups G^R that differ only in the centre from $R_{F/\mathbb{Q}} \mathrm{GL}_2$, where R is an algebraic torus over $\mathbb{Q}$ with $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$. The Cartesian diagram (1.2.0.3) allows us to view the cocharacter group $X_*(G^R)$ of G^R as a subgroup of $X_*(R_{F/\mathbb{Q}} \mathrm{GL}_2)$. The latter carries an action of $S_\Sigma \ltimes \Gamma_F^\Sigma$ by Section 6.1, so we can define the subgroup $(S_\Sigma \ltimes \Gamma_F^\Sigma)_{\mathrm{cc}}^R$ of $S_\Sigma \ltimes \Gamma_F^\Sigma$ as the stabiliser of the subset $X_*(G^R) \subset X_*(R_{F/\mathbb{Q}} \mathrm{GL}_2)$. In (6.2.6), it is then straightforward to define the plectic reflex Galois group of (G^R, X^R) as the stabiliser of a certain conjugacy class μ_{X^R} of cocharacters of G^R.

We finish Section 6.2 with discussing properties of algebraic groups G that differ only in the centre from an arbitrary group G_1. By definition, this means that diagram (1.2.0.2) is Cartesian. In (6.2.9), we relate the derived group, the adjoint group, the centre, and the cocentre of G to their respective counterparts of G_1 using this Cartesian diagram. Moreover, if G_1 is of the form $R_{F/\mathbb{Q}}H$ and admits a Shimura variety, the same methods as for the groups G^R can be used to define the plectic reflex Galois group associated to G.

6.1 Plectic action on cocharacters of restrictions of scalars

6.1.1. (Cocharacters) Let G be an algebraic group defined over $\mathbb{Q}$. Then a *cocharacter* of G is a morphism of algebraic groups from the multiplicative group $\mathbb{G}_m$ to G, defined over $\overline{\mathbb{Q}}$. We denote the set of cocharacters of G by

$$X_*(G) := \mathrm{Hom}(\mathbb{G}_{m,\overline{\mathbb{Q}}}, G),$$

where "Hom" denotes morphisms of group schemes over $\mathbb{Q}$.

The set $X_*(G)$ comes equipped with an action of the Galois group $\Gamma_{\mathbb{Q}}$. To define this action, let us recall:

6.1.2. (Galois action on schemes) Let k be a field, X_0 a scheme over k and X the base change of X_0 to the separable closure $\overline{k}$ of k. This means we have a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "a"'] & \operatorname{Spec} \overline{k} \arrow[d, "\iota"] \\ X_0 \arrow[r, "\pi_0"'] & \operatorname{Spec} k. \end{tikzcd}$$

Here $\iota = \mathrm{Spec}(i)$ for the fixed inclusion $i\colon k \hookrightarrow \overline{k}$, and π_0 is the structure morphism.

Now for $\gamma \in \Gamma_k$, we have $\gamma \circ i = i$, which translates to $\iota \circ \mathrm{Spec}(\gamma) = \iota$. This yields the unique dotted arrow in the following diagram, defining a (right) action of Γ_k on the scheme X.

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, dotted, "\gamma_X^*"] \arrow[dd, bend right, "a"'] & \operatorname{Spec} \overline{k} \arrow[d, "\mathrm{Spec}(\gamma)"] \\ X \arrow[r, "\pi"] \arrow[d, "a"'] & \operatorname{Spec} \overline{k} \arrow[d, "\iota"] \\ X_0 \arrow[r, "\pi_0"'] & \operatorname{Spec} k. \end{tikzcd}$$

As a concrete example, assume $X_0 = \operatorname{Spec} k[T_1, \dots, T_n]/(f_1, \dots, f_m)$ is an affine variety. The $\overline{k}$-points of X are the simultaneous solutions $(x_i)_i \in \overline{k}^n$ to the polynomial equations $f_j(x_1, \dots, x_n) = 0$, $j = 1, \dots, m$. Then γ_X^* acts on the $\overline{k}$-points of X by $(x_i)_i \mapsto (\gamma(x_i))_i$.

From now on, we will simply write γ (or sometimes γ^*) for γ_X^* or $\mathrm{Spec}(\gamma)$. We hope this does not cause confusion.

6.1.3. (Galois action on cocharacters) We apply (6.1.2) for the scheme $\mathbb{G}_m$ over $\mathbb{Q}$, and the action of $\Gamma_{\mathbb{Q}}$ on $\mathbb{G}_{m,\overline{\mathbb{Q}}}$ induces an action of $\Gamma_{\mathbb{Q}}$ on $X_*(G)$ by pre-composition

$$\gamma\colon \chi \mapsto \chi \circ \gamma, \quad \gamma \in \Gamma_{\mathbb{Q}},\ \chi \in X_*(G). \tag{6.1.3.1}$$

Note that, despite its appearance, this is a *left* action because in reality we are precomposing by $\gamma^*_{\mathbb{G}_{m,\overline{\mathbb{Q}}}}$ and $(\gamma_1\gamma_2)^*_{\mathbb{G}_{m,\overline{\mathbb{Q}}}} = (\gamma_2)^*_{\mathbb{G}_{m,\overline{\mathbb{Q}}}}(\gamma_1)^*_{\mathbb{G}_{m,\overline{\mathbb{Q}}}}$. Also note that $\chi \circ \gamma$ is again a morphism of group schemes, i. e. a cocharacter of G.

There is another way of defining this action, which will be useful for us. First of all, note that by base-change

$$X_*(G) = \mathrm{Hom}_{\mathbb{Q}-\text{Gr-Sch}}(\mathbb{G}_{m,\overline{\mathbb{Q}}}, G) = \mathrm{Hom}_{\overline{\mathbb{Q}}-\text{Gr-Sch}}(\mathbb{G}_{m,\overline{\mathbb{Q}}}, G_{\overline{\mathbb{Q}}}). \tag{6.1.3.2}$$

The absolute Galois group $\Gamma_{\mathbb{Q}}$ acts on both $\mathbb{G}_{m,\overline{\mathbb{Q}}}$ and $G_{\overline{\mathbb{Q}}}$ by (6.1.2). The previous Definition (6.1.3.1) then translates (via base-change) to

$$\gamma\colon \chi \mapsto \gamma^{-1} \circ \chi \circ \gamma, \quad \gamma \in \Gamma_{\mathbb{Q}},\ \chi \in \mathrm{Hom}_{\overline{\mathbb{Q}}-\text{Gr-Sch}}(\mathbb{G}_{m,\overline{\mathbb{Q}}}, G_{\overline{\mathbb{Q}}}). \tag{6.1.3.3}$$

To be precise, the right hand side is equal to $(\gamma^{-1})^*_{G_{\overline{\mathbb{Q}}}} \circ \chi \circ \gamma^*_{\mathbb{G}_{m,\overline{\mathbb{Q}}}}$.

6.1.4 (Plectic action on cocharacters for restrictions of scalars I)**.** Let H be an algebraic group defined over a number field F and let $G := R_{F/\mathbb{Q}}H$. Then G is an algebraic group over $\mathbb{Q}$, hence $X_*(G)$ carries a $\Gamma_{\mathbb{Q}}$-action as explained above. However, because G is a restriction of scalars, it carries an action by the plectic group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ which can be described as follows:

First of all, we can exploit that $G(S) = H(S_F)$ for every $\mathbb{Q}$-scheme S, i. e.

$$\mathrm{Hom}_{\mathbb{Q}-\mathrm{Sch}}(S, G) = \mathrm{Hom}_{F-\mathrm{Sch}}(S_F, H),$$

where S_F denotes the base change $\mathrm{Spec}(F) \times_{\mathrm{Spec}(\mathbb{Q})} S$ of S.

Note that $X_*(G)$ is by definition the subset of $G(\mathbb{G}_{m,\overline{\mathbb{Q}}})$ consisting of those $\mathbb{Q}$-morphisms from $\mathbb{G}_{m,\overline{\mathbb{Q}}}$ to G which are morphisms of group schemes. This property means that a certain diagram commutes, and translates into the analogous property in $H(\mathbb{G}_{m,F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}})$. Thus

$$X_*(G) = \mathrm{Hom}_{\mathbb{Q}-\text{Gr-Sch}}(\mathbb{G}_{m,\overline{\mathbb{Q}}}, G) = \mathrm{Hom}_{F-\text{Gr-Sch}}(\mathbb{G}_{m,F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}}, H). \tag{6.1.4.1}$$

In analogy to diagram (6.1.2), we get for $\gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$

$$\begin{tikzcd}
\mathbb{G}_{m,F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}} \arrow[r] \arrow[d, dotted, "\gamma^*"] \arrow[dd, bend right] & \mathrm{Spec}(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \arrow[d, "\mathrm{Spec}(\gamma)"] \\
\mathbb{G}_{m,F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}} \arrow[r] \arrow[d] & \mathrm{Spec}(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \arrow[d] \\
\mathbb{G}_{m,F} \arrow[r] & \mathrm{Spec}\, F
\end{tikzcd}$$

We conclude that $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ acts (on the left) on $X_*(G)$ by precomposition

$$\gamma\colon \chi \mapsto \chi \circ \gamma^*, \quad \gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}),\ \chi \in X_*(G). \tag{6.1.4.2}$$

It clearly extends the $\Gamma_{\mathbb{Q}}$-action (6.1.3.1).

6.1.5 (Plectic action on cocharacters for restrictions of scalars II)**.** We can also describe the plectic action more explicitly in the following way, using the plectic group $S_\Sigma \ltimes \Gamma_F^\Sigma$: Because G is a restriction of scalars of H, the action of $\Gamma_{\mathbb{Q}}$ on $X_*(G)$ is the induced action of Γ_F on $X_*(H)$, i. e. we have an isomorphism of $\Gamma_{\mathbb{Q}}$-sets

$$X_*(G) \cong \mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}} X_*(H). \tag{6.1.5.1}$$

The action of $S_\Sigma \ltimes \Gamma_F^\Sigma$ is then given by (4.1.4). This definition of the plectic action agrees with the one in (6.1.4) under the isomorphsim $\beta_s\colon \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \xrightarrow{\sim} S_\Sigma \ltimes \Gamma_F^\Sigma$ of (4.1.3).

6.1.6 Remark (Plectic equivariant maps on cocharacters)**.** Let $G_i = R_{F/\mathbb{Q}} H_i$, $i = 1, 2$, be two algebraic groups over $\mathbb{Q}$ that are restriction of scalars, and let $f\colon H_1 \to H_2$ be defined over F. Then f induces $R_{F/\mathbb{Q}}(f)\colon G_1 \to G_2$, which is defined over $\mathbb{Q}$, and thus induces the map

$$R_{F/\mathbb{Q}}(f)_*\colon X_*(G_1) \longrightarrow X_*(G_2), \quad \chi \longmapsto R_{F/\mathbb{Q}}(f) \circ \chi.$$

This map is not only $\Gamma_{\mathbb{Q}}$-equivariant, but by functoriality in the adjunction (6.1.4.1) and by Definition (6.1.4.2) it is also equivariant with respect to the plectic action.

6.1.7 Example. Consider the algebraic torus $R_{F/\mathbb{Q}} \mathbb{G}_m$ over $\mathbb{Q}$. Here we consider $\mathbb{G}_m$ as an algebraic group over F, and $X_*(\mathbb{G}_m)$ is isomorphic to $\mathbb{Z}$ with the trivial Γ_F-action by

$$\mathbb{Z} \xrightarrow{\sim} X_*(\mathbb{G}_m) = \mathrm{Hom}_{F\text{−Gr-Sch}}(\mathbb{G}_{m,\overline{F}}, \mathbb{G}_{m,F}), \quad n \mapsto [t \mapsto t^n].$$

By (6.1.5.1) and (4.1.4.2), the $\Gamma_{\mathbb{Q}}$-module $X_*(R_{F/\mathbb{Q}} \mathbb{G}_m)$ is isomorphic to $\mathbb{Z}[\Sigma]$, see (2.1.6). By (4.1.4.2), the plectic action of $S_\Sigma \ltimes \Gamma_F^\Sigma$ is given as the linear extension of

$$(\sigma, h)[x] = [\sigma(x)], \quad x \in \Sigma,\ (\sigma, h) \in S_\Sigma \ltimes \Gamma_F^\Sigma$$

This means the plectic action on $X_*(R_{F/\mathbb{Q}} \mathbb{G}_m)$ factors through the canonical quotient S_Σ, see (4.1.9.a).

6.1.8 Example. Let K be a totally imaginary quadratic extension of the totally real field F. Let $\mathrm{Gal}(K/F) = \langle c \rangle$, i. e. denote complex conjugation by c. Consider the algebraic torus $R_{K/\mathbb{Q}}\mathbb{G}_m = R_{F/\mathbb{Q}}R_{K/F}\mathbb{G}_m$. The $\Gamma_{\mathbb{Q}}$-module $X_*(R_{K/\mathbb{Q}}\mathbb{G}_m)$ is isomorphic to $\mathbb{Z}[\Sigma_K]$, where $\Sigma_K := \operatorname{Hom}(K, \overline{\mathbb{Q}})$, with $\Gamma_{\mathbb{Q}}$-action given by

$$\gamma[\varphi] = [\gamma\varphi], \quad \varphi \in \Sigma_K.$$

In (4.1.6) we have seen how $S_\Sigma \ltimes \Gamma_F^\Sigma$ acts on Σ_K, and the action on $\mathbb{Z}[\Sigma_K]$ is thus given by

$$(\sigma, h)[c^b\varphi_x] = [c^{b+\overline{h}_x}\varphi_{\sigma(x)}].$$

This shows that the plectic action on $X_*(R_{K/\mathbb{Q}}\mathbb{G}_m)$ factors through the group $S_\Sigma \ltimes \operatorname{Gal}(K/F)^\Sigma$. If K is Galois over $\mathbb{Q}$, then this group is isomorphic to $\operatorname{Aut}_F(F \otimes_{\mathbb{Q}} K)$ (resp. $\operatorname{Gal}(K/\mathbb{Q}) \# \operatorname{Gal}(K/F)$) under a modification of β_s (resp. π_s) from (4.1.3) (resp. (4.1.5.1)), see [Bla15, (3.2.12)].

6.1.9 Remark. Similar to (6.1.3.3), we can also define the plectic action on cocharacters as follows: Let H again be an algebraic group defined over F and $G := R_{F/\mathbb{Q}}H$. Starting from (6.1.4.1), base change yields

$$X_*(G) = \operatorname{Hom}_{F-\text{Gr-Sch}}(\mathbb{G}_{m,F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}}, H) = \operatorname{Hom}_{F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}-\text{Gr-Sch}}(\mathbb{G}_{m,F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}}, H_{F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}}),$$

where $H_{F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}} := H \times_{\operatorname{Spec}(F)} \operatorname{Spec}(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$.

In complete analogy to (6.1.3.3), base-change of (6.1.4.2) yields

$$\gamma\colon \chi \mapsto \gamma^{-1} \circ \chi \circ \gamma, \quad \gamma \in \operatorname{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}),\ \chi \in \operatorname{Hom}_{F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}-\text{Gr-Sch}}(\mathbb{G}_{m,F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}}, H_{F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}}). \tag{6.1.9.1}$$

To be precise, the right hand side is equal to $(\gamma^{-1})^*_{H_{F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}}} \circ \chi \circ \gamma^*_{\mathbb{G}_{m,F\otimes_{\mathbb{Q}}\overline{\mathbb{Q}}}}$.

6.1.10 (Galois action on conjugacy classes of cocharacters)**.** Let G be an algebraic group over $\mathbb{Q}$. In (6.1.3.3) we defined a Galois action on $X_*(G)$. This action induces an action on the $G(\overline{\mathbb{Q}})$-conjugacy classes of cocharacters which we will now explain:

By (6.1.3.2), the cocharacters of G are given by $X_*(G) = \operatorname{Hom}_{\overline{\mathbb{Q}}-\text{Gr-Sch}}(\mathbb{G}_{m,\overline{\mathbb{Q}}}, G_{\overline{\mathbb{Q}}})$. Recall that an element $g \in G(\overline{\mathbb{Q}})$ acts on the scheme $G_{\overline{\mathbb{Q}}}$ by conjugation as follows:

$$\begin{aligned} \operatorname{ad}(g)\colon G_{\overline{\mathbb{Q}}} &\to G_{\overline{\mathbb{Q}}} \times G_{\overline{\mathbb{Q}}} &&\to G_{\overline{\mathbb{Q}}} \\ x &\mapsto (g, x) &&\mapsto gxg^{-1} \end{aligned}$$

To be precise, the map denoted $x \mapsto g$ is shorthand for

$$G_{\overline{\mathbb{Q}}} \longrightarrow \operatorname{Spec}(\overline{\mathbb{Q}}) \xrightarrow{g} G_{\overline{\mathbb{Q}}}$$

and the second map is the conjugation map (built from the group law $G \times G \to G$).

We can therefore use (6.1.3.2) to define the cocharacter conjugate to $\chi \in X_*(G)$ by $g \in G(\overline{\mathbb{Q}})$ as ${}^g\chi := \mathrm{ad}(g) \circ \chi$, i. e.

$$ {}^g\chi \colon \mathbb{G}_{m,\overline{\mathbb{Q}}} \xrightarrow{\chi} G_{\overline{\mathbb{Q}}} \xrightarrow{\mathrm{ad}(g)} G_{\overline{\mathbb{Q}}}. \tag{6.1.10.1} $$

In order to deduce an action on the conjugacy classes of cocharacters, we need to show that the $\Gamma_{\mathbb{Q}}$-action on $X_*(G)$ is compatible with conjugation. We show

$$ \gamma({}^g\chi) = {}^{\gamma g}(\gamma\chi), \quad \gamma \in \Gamma_{\mathbb{Q}}, g \in G(\overline{\mathbb{Q}}), \chi \in X_*(G). \tag{6.1.10.2} $$

The calculation is straight-forward:

$$ \begin{aligned} \gamma({}^g\chi) &= (\gamma^{-1})^*_G \circ ({}^g\chi) \circ \gamma^*_{\mathbb{G}_m} \\ &= (\gamma^{-1})^*_G \circ (\mathrm{ad}(g) \circ \chi) \circ \gamma^*_{\mathbb{G}_m} \\ &= (\gamma^{-1})^*_G \circ \mathrm{ad}(g) \circ \gamma^*_G \circ (\gamma^{-1})^*_G \circ \chi \circ \gamma^*_{\mathbb{G}_m} \\ &= \mathrm{ad}(\gamma g) \circ (\gamma\chi) \\ &= {}^{\gamma g}(\gamma\chi). \end{aligned} $$

In the penultimate step, we used $\mathrm{ad}(\gamma g) = (\gamma^{-1})^*_G \circ \mathrm{ad}(g) \circ \gamma^*_G$ and Definition (6.1.3.3) of $\gamma\chi$. The former is a scheme-theoretic exercise which comes down to the fact that the group law, and hence conjugation, on G is defined over $\mathbb{Q}$.

6.1.11. (Reflex field of a Shimura datum) Let (G, X) be a Shimura datum. We recall the definition [Mil17, Def. 12.2] of the *reflex field* $E(G, X)$ of (G, X): Take any $h \in X$ and look at the associated cocharacter μ_h of G, see (2.2.5). Then the $G(\mathbb{C})$-conjugacy class $\mathcal{C}$ of μ_h is independent of the choice of h, hence only depends on X. Moreover, $\mathcal{C}$ contains a unique $G(\overline{\mathbb{Q}})$-conjugacy class μ_X of cocharacters of G. The reflex field is defined to be the field of definition of μ_X.

6.1.12 (Plectic Galois action on conjugacy classes of cocharacters)**.** With the notation of (6.1.10), assume that $G = R_{F/\mathbb{Q}}H$ for some algebraic group H over F. We use the description (6.1.9.1) of the plectic Galois action on $X_*(G)$. In exactly the same way as in (6.1.10.2), one can show that

$$ \gamma({}^g\chi) = {}^{\gamma g}(\gamma\chi), \quad \gamma \in \mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}), g \in G(\overline{\mathbb{Q}}), \chi \in X_*(G). \tag{6.1.12.1} $$

In particular, the plectic action on $X_*(G)$ induces an action on $G(\overline{\mathbb{Q}})$-conjugacy classes of cocharacters.

6.1.13 (Plectic reflex Galois group)**.** Let (G, X) be a Shimura datum with $G = R_{F/\mathbb{Q}}H$ for some algebraic group H over F. In analogy to (6.1.11), using (6.1.12) we can define the *plectic reflex Galois group* of (G, X) to be the stabiliser of μ_X inside the plectic group $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$, see [NS16, Def. 5.1].

6.1.14 Example. (Hilbert modular variety) The Hilbert modular variety is associated to the group $G_1 = R_{F/\mathbb{Q}} \mathrm{GL}_2$, see (3.2.1). So the plectic group $S_\Sigma \ltimes \Gamma_F^\Sigma$ acts on the set $X_*(G_1)$, and by (6.1.12) also on the set of $G_1(\overline{\mathbb{Q}})$-conjugacy classes of $X_*(G_1)$. We can describe this more explicitly as follows:

We use the $\mathbb{Q}$-torus $S_1 := R_{F/\mathbb{Q}}(\mathbb{G}_m^2)$, embedded as a maximal torus inside $G_1 = R_{F/\mathbb{Q}} \mathrm{GL}_2$ as the diagonal matrices. Then we can identify the set of $G_1(\overline{\mathbb{Q}})$-conjugacy classes of cocharacters of G_1 with

$$X_*(G_1)/G_1(\overline{\mathbb{Q}})\text{-conj.} = X_*(S_1)/W,$$

where W denotes the Weyl group of S_1.

We have $S_1(\overline{\mathbb{Q}}) = \prod_{x \in \Sigma} \begin{pmatrix} * & \\ & * \end{pmatrix} \subset G_1(\overline{\mathbb{Q}})$, hence the cocharacters of S_1 are given by

$$\begin{aligned} \chi = (\chi_x)_{x \in \Sigma} \colon \mathbb{G}_m &\longrightarrow S_1 \\ \chi_x \colon t &\longmapsto \begin{pmatrix} t^{n_x} & \\ & t^{m_x} \end{pmatrix}, \end{aligned}$$

for integers $n_x, m_x \in \mathbb{Z}$. The Weyl group W is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\Sigma$, with the element $w_x = (0, \dots, 0, 1, 0, \dots, 0)$ (with the 1 in the x-th coordinate) acting on the x-th factor of $S_1(\overline{\mathbb{Q}}) = \prod_{x \in \Sigma} \begin{pmatrix} * & \\ & * \end{pmatrix}$ by swapping the two entries and acting trivially on all other factors. Thus in the above notation

$$X_*(S_1)/W \cong \{\chi = (\chi_x)_{x \in \Sigma} \mid \chi_x \colon t \mapsto \begin{pmatrix} t^{n_x} & \\ & t^{m_x} \end{pmatrix} \text{ with } n_x \geq m_x \text{ for all } x\}.$$

We can also describe the plectic Galois action on $X_*(S_1)$ explicitly, which will then also describe the plectic action on the $G_1(\overline{\mathbb{Q}})$-conjugacy classes of $X_*(G_1)$ by (6.1.6): Note that $X_*(\mathbb{G}_{m,F}^2)$ carries the trivial Γ_F-action, hence by (4.1.4.2) the action of $S_\Sigma \ltimes \Gamma_F^\Sigma$ on $X_*(S_1)$ factors through S_Σ and is given by

$$(\sigma, h) \colon \chi = (\chi_x)_{x \in \Sigma} \longmapsto (\chi_{\sigma^{-1}(x)})_{x \in \Sigma}.$$

6.2 Plectic action on cocharacters of more general groups

In this section we define a plectic action on cocharacters of algebraic groups that differ only in the centre from a group of the form $R_{F/\mathbb{Q}}H$. The following purely group-theoretical observations will be very useful:

Assume Γ is a group acting on a set A_1, and let $A \subset A_1$. We denote the largest subgroup of Γ stabilising A by $\mathrm{Stab}_\Gamma(A)$.

6.2.1 Lemma. *Let Γ be a group acting on the sets A_1 and B_1, and let $d\colon A_1 \to B_1$ be Γ-equivariant and surjective. Assume that $B \subset B_1$ is a subset, and that $A \subset A_1$ is given by the following Cartesian diagram of sets*

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & A_1 \arrow[d, "d"] \\ B \arrow[r, hook] & B_1, \end{tikzcd}$$

i. e.

$$A = \{a \in A_1 \mid d(a) \in B\} = d^{-1}(B).$$

Then

$$\mathrm{Stab}_\Gamma(A) = \mathrm{Stab}_\Gamma(B).$$

Proof. An element $\gamma \in \Gamma$ preserves A if and only if for all $a \in A$ we have $\gamma a \in A$, i. e. $d(\gamma a) \in B$. But $d(\gamma a) = \gamma d(a)$ and $d|_A\colon A \to B$ is surjective, so this is equivalent to γ preserving B. $\square$

As a motivating example, we generalise (6.1.8) to certain subtori of $R_{K/\mathbb{Q}}\mathbb{G}_m$, for a totally imaginary quadratic extension K of F:

6.2.2. (Subtori of $R_{K/\mathbb{Q}}\mathbb{G}_m$) We use the notation of (6.1.8), and abbreviate $R_1 := R_{F/\mathbb{Q}}\mathbb{G}_m$ and $T_1 := R_{K/\mathbb{Q}}\mathbb{G}_m = R_{F/\mathbb{Q}}R_{K/F}\mathbb{G}_m$ for the algebraic tori over $\mathbb{Q}$ of (6.1.7) and (6.1.8), respectively. Let

$$N := R_{F/\mathbb{Q}}(N_{K/F})\colon T_1 \to R_1$$

be the *norm map*, which on $\overline{\mathbb{Q}}$-points is given by

$$N\colon (K \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times = \prod_{x\in\Sigma,\ b\in\mathbb{Z}/2\mathbb{Z}} \overline{\mathbb{Q}}^\times \longrightarrow \prod_{x\in\Sigma} \overline{\mathbb{Q}}^\times = (F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times,$$
$$(z_{x,b})_{x,b} \longmapsto (z_{x,0}z_{x,1})_x.$$

It will be useful to calculate the induced map $N_*\colon X_*(T_1) \to X_*(R_1)$ on cocharacters explicitly: Recall that $X_*(R_1) = \mathbb{Z}[\Sigma]$ and $X_*(T_1) = \mathbb{Z}[\Sigma_K]$. By using the description of the cocharacter $[x]$ in (2.1.6), we see that

$$N_*\colon \sum_{x\in\Sigma,\ b\in\mathbb{Z}/2\mathbb{Z}} m_{x,b}[c^b\varphi_x] \longmapsto \sum_{x\in\Sigma}(m_{x,0}+m_{x,1})[x]. \tag{6.2.2.1}$$

Let $R \hookrightarrow R_1$ be an algebraic subtorus over $\mathbb{Q}$ and define T to be the algebraic torus over $\mathbb{Q}$ that makes the following diagram Cartesian

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d] & T_1 \arrow[d, "N"] \\ R \arrow[r, hook] & R_1. \end{tikzcd}$$

Note that in general T will not be of the form $R_{F/\mathbb{Q}}H$ for some algebraic torus H over F, hence (6.1.4) does not give a plectic action on $X_*(T)$.

However, it does give an action on $X_*(T_1)$, see (6.1.8), and hence $X_*(T) \subset X_*(T_1)$ inherits an action by the largest subgroup of the plectic group $S_\Sigma \ltimes \Gamma_F^\Sigma$ which stabilises the subspace $X_*(T)$. This subgroup is $\mathrm{Stab}_{S_\Sigma\ltimes\Gamma_F^\Sigma}(X_*(T))$ in the notation of (6.2.1). It contains, of course, the absolute Galois group of $\mathbb{Q}$ because T is an algebraic torus over $\mathbb{Q}$, but it is in fact much bigger. We can calculate it explicitly as follows:

The above diagram of tori induces a Cartesian diagram of $\Gamma_\mathbb{Q}$-modules

$$\begin{tikzcd} X_*(T) \arrow[r, hook] \arrow[d] & X_*(T_1) \arrow[d, "N_*"] \\ X_*(R) \arrow[r, hook] & X_*(R_1), \end{tikzcd}$$

i. e.

$$X_*(T) = \{\chi \in X_*(T_1) \mid N_*(\chi) \in X_*(R) \subset X_*(R_1)\}. \tag{6.2.2.2}$$

For example, if $R = \mathbb{G}_m$ and $\mathbb{G}_m \hookrightarrow R_1$ is the usual embedding which on $\mathbb{Q}$-points is simply the inclusion $\mathbb{Q}^\times \subset F^\times$, then the corresponding map on cocharacters is given by

$$\begin{aligned} X_*(\mathbb{G}_m) = \mathbb{Z} &\longrightarrow X_*(R_1) = \mathbb{Z}[\Sigma], \\ n &\longmapsto n\sum_{x\in\Sigma}[x] \end{aligned} \tag{6.2.2.3}$$

and thus

$$X_*(T) = \left\{ \sum_{\varphi\in\Sigma_K} m_\varphi[\varphi] \;\middle|\; m_\varphi + m_{\overline{\varphi}} = m_{\varphi'} + m_{\overline{\varphi}'} \text{ for all } \varphi, \varphi' \in \Sigma_K \right\}. \qquad (6.2.2.4)$$

Using (6.1.6) and the fact that N_* is surjective (obvious from (6.2.2.1)), (6.2.1) implies that

$$\mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(T)) = \mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(R)).$$

We can be more explicit as follows: The action of the plectic group on $X_*(R_1)$ factors through its canonical quotient S_Σ, see (6.1.7). Let $\mathcal{S}(R) := \mathrm{Stab}_{S_\Sigma}(X_*(R))$ be the largest subgroup of S_Σ stabilising $X_*(R) \subset X_*(R_1)$, then

$$\mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(R)) = \text{preimage of } \mathcal{S}(R) = \mathcal{S}(R) \ltimes \Gamma_F^\Sigma. \qquad (6.2.2.5)$$

We denote this group by $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{cc}}$.

6.2.3 Example. Let us look at subtori $R = R_{F'/\mathbb{Q}}\mathbb{G}_m$ coming from intermediate fields $\mathbb{Q} \subset F' \subset F$:

(6.2.3.a) Clearly, if $F' = F$, then $R = R_1$ and $T = T_1$, so $\mathcal{S}(R) = S_\Sigma$ and the full plectic group acts on the cocharacters of T.

(6.2.3.b) Let $F = \mathbb{Q}$, so $R = \mathbb{G}_m$, embedded in R_1 diagonally (on $\overline{\mathbb{Q}}$-points). Then the associated T is the algebraic torus introduced in (2.5.1) (the “PEL torus”).

Then $X_*(R) \subset X_*(R_1) = \mathbb{Z}[\Sigma]$ corresponds to $\mathbb{Z}(\sum_{x\in\Sigma}[x])$. Clearly this is stable under the action of S_Σ (in fact, S_Σ acts trivially on it), hence again the full plectic group acts on $X_*(T)$.

This example also shows that $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{cc}} = S_\Sigma \ltimes \Gamma_F^\Sigma$ is possible even though the torus T is not the restriction of scalars of some torus defined over F.

(6.2.3.c) General case: This example will show that it is possible that $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{cc}} \subsetneq S_\Sigma \ltimes \Gamma_F^\Sigma$. We have

$$X_*(R) \hookrightarrow X_*(R_1), \quad [x'] \mapsto \sum_{x|x'} [x],$$

where $x|x'$ means that $x\colon F \hookrightarrow \overline{\mathbb{Q}}$ extends $x'\colon F' \hookrightarrow \overline{\mathbb{Q}}$.

Therefore $\mathcal{S}(R)$ is equal to all permutations of Σ preserving the partition $\Sigma = \bigsqcup_{x'\in\Sigma'}\{x\colon x|x'\}$, where $\Sigma' := \mathrm{Hom}(F', \overline{\mathbb{Q}})$, i. e. $\sigma \in S_\Sigma$ belongs to $\mathcal{S}(R)$ if and only if for all $x' \in \Sigma'$ there exists $y' \in \Sigma'$ such that $\{\sigma(x)\colon x|x'\} = \{x \in \Sigma\colon x|y'\}$.

The cases $F' = F$ and $F' = \mathbb{Q}$ give back the previous examples. But in all other cases, this is a proper subgroup of S_Σ, hence also $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{cc}}$ is a proper subgroup of $S_\Sigma \ltimes \Gamma_F^\Sigma$.

After this introductory example, we will define a plectic action on the cocharacters of a group that differs only in the centre (see (6.2.9)) from a restriction of scalars. As the second and most important example, we look at groups that differ only in the centre from $G_1 := R_{F/\mathbb{Q}} \mathrm{GL}_2$.

6.2.4 Example (Cocharacters of variants of $R_{F/\mathbb{Q}} \mathrm{GL}_2$)**.** Similar to (6.2.2) and as in (4.4.2), let $R \hookrightarrow R_1 = R_{F/\mathbb{Q}}\mathbb{G}_m$ be an algebraic subtorus over $\mathbb{Q}$ and let $G = G^R$. Recall the Cartesian diagram (4.4.2.1)

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d] & G_1 \arrow[d, "d_F"] \\ R \arrow[r, hook] & R_1, \end{tikzcd}$$

where $d_F := R_{F/\mathbb{Q}}(\det)\colon G_1 = R_{F/\mathbb{Q}} \mathrm{GL}_2 \to R_{F/\mathbb{Q}}\mathbb{G}_m = R_1$. Then $X_*(G)$ is a subset of $X_*(G_1)$, fitting into the Cartesian diagram of sets with $\Gamma_\mathbb{Q}$-action

$$\begin{tikzcd} X_*(G) \arrow[r, hook] \arrow[d] & X_*(G_1) \arrow[d, "(d_F)_*"] \\ X_*(R) \arrow[r, hook] & X_*(R_1), \end{tikzcd}$$

i. e.

$$X_*(G) = \{\chi \in X_*(G_1) \mid (d_F)_*(\chi) \in X_*(R)\}. \tag{6.2.4.1}$$

The cocharacters $X_*(G_1)$ of G_1 carry an action by the plectic group by (6.1.4), and so once again $X_*(G)$ inherits an action by $\mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(G))$. In analogy to (6.2.2.2), $(d_F)_*$ is plectic equivariant by (6.1.6), hence (6.2.1) implies that

$$\mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(G)) = \mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(R)) = (S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{cc}}.$$

For Definition (6.1.11) of the reflex field in the theory of Shimura varieties, one does not use the action of the Galois group $\Gamma_\mathbb{Q}$ on cocharacters of G, but the induced action on $G(\overline{\mathbb{Q}})$-*conjugacy classes* of cocharacters of G.

6.2.5 Example (Conjugacy classes of cocharacters of G^R)**.** We continue the previous example. Having defined a plectic action on the cocharacters of G, we now need to make sure this action is compatible with conjugation by $G(\overline{\mathbb{Q}})$. Note that by (6.1.12) the full plectic group acts on the $G_1(\overline{\mathbb{Q}})$-conjugacy classes of cocharacters of G_1.

If two cocharacters $\chi_1, \chi_2 \in X_*(G)$ of G are $G(\overline{\mathbb{Q}})$-conjugate, then they are clearly also $G_1(\overline{\mathbb{Q}})$-conjugate, and hence by what we know about G_1 their plectic images ${}^\gamma\chi_1, {}^\gamma\chi_2$ are $G_1(\overline{\mathbb{Q}})$-conjugate. To conclude that they are $G(\overline{\mathbb{Q}})$-conjugate, note that conjugation factors through the adjoint groups G^{ad} and G_1^{ad}. But these two groups are equal by (6.2.8) below.

Thus we may conclude that $(S_\Sigma \ltimes \Gamma_F^\Sigma)_{\mathrm{cc}}^R$ acts on conjugacy classes of cocharacters of G.

In (6.1.13) we defined the *plectic reflex Galois group* for Shimura data (G, X), with G a group of the form $R_{F/\mathbb{Q}}H$, as the stabiliser inside $\mathrm{Aut}_F(F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ of a certain conjugacy class μ_X of cocharacters. In analogy, we have:

6.2.6. (Plectic reflex Galois group of (G^R, X^R)) We continue (6.2.4). Let $(G, X) := (G^R, X^R)$. Let μ_X be the $G(\overline{\mathbb{Q}})$-conjugacy class of cocharacters of G determined by X, see (6.1.11). Then the *plectic reflex Galois group* of (G, X) is defined to be the stabiliser of μ_X inside $(S_\Sigma \ltimes \Gamma_F^\Sigma)_{\mathrm{cc}}^R$.

6.2.7 Example. We continue (6.2.4). By (6.2.5), we can identify the set of $G(\overline{\mathbb{Q}})$-conjugacy classes in $X_*(G)$ with a subset of $X_*(S_1)/W$, namely those cocharacters χ with $(d_F)_*(\chi) \in X_*(R)$. Here W again denotes the Weyl group of S_1, see (6.1.14). We can explicitly calculate that for $\chi = (\chi_x)$ we have

$$(d_F)_*(\chi) = \sum_{x \in \Sigma} (n_x + m_x)[x] \in \mathbb{Z}[\Sigma],$$

where we identify $X_*(R_1)$ with $\mathbb{Z}[\Sigma]$ as in (6.1.7). Hence

$$X_*(G)/G(\overline{\mathbb{Q}})\text{-conj.} \cong \{\chi = (\chi_x) \mid n_x \geq m_x \text{ and } \sum_{x \in \Sigma} (n_x + m_x)[x] \in X_*(R)\}.$$

Note the resemblance with (6.2.2.1) and (6.2.2.2).

To compare the groups G and G_1 of (6.2.4), let us first recall some facts about algebraic groups:

6.2.8 (Adjoint and derived group)**.** Recall that for any connected reductive group G over $\mathbb{Q}$ one has two short exact sequences fitting into the following diagram of algebraic groups over $\mathbb{Q}$:

$$\begin{tikzcd} & & 1 \arrow[d] & & \\ & & Z \arrow[d] \arrow[dr, dashed] & & \\ 1 \arrow[r] & G^{\mathrm{der}} \arrow[r] \arrow[dr, dashed] & G \arrow[r] \arrow[d] & C \arrow[r] & 1 \\ & & G^{\mathrm{ad}} \arrow[d] & & \\ & & 1 & & \end{tikzcd}$$

Here

- Z denotes the centre of G,
- $G^{\mathrm{ad}} = G/Z$ denotes the adjoint group of G,
- $G^{\mathrm{der}} = [G, G]$ denotes the derived subgroup of G,
- $C := G/G^{\mathrm{der}}$ denotes the cocentre of G,

and the induced dashed arrows are *isogenies*, i. e. surjective morphisms with finite kernel. For example, for $G_1 = R_{F/\mathbb{Q}} \operatorname{GL}_2$ we get:

- $Z_1 = R_{F/\mathbb{Q}}\mathbb{G}_m$, embedded diagonally in G_1,
- $G_1^{\mathrm{ad}} = R_{F/\mathbb{Q}} \operatorname{PGL}_2$,
- $G_1^{\mathrm{der}} = R_{F/\mathbb{Q}} \operatorname{SL}_2$,
- $C_1 = R_{F/\mathbb{Q}}\mathbb{G}_m = R_1$ and the map $G_1 \to C_1$ is equal to $d_F \colon G_1 \to R_1$.

The calculation of C_1 indicates that defining G by the Cartesian diagram (4.4.2.1) means a change in the cocentre, and we can indeed proceed in greater generality:

6.2.9. (Change in the cocentre) Let G_1 be a reductive groups over $\mathbb{Q}$. Fix some subtorus R of C_1. Define the reductive group G over $\mathbb{Q}$ by the Cartesian diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d] & G_1 \arrow[d, "d"] \\ R \arrow[r, hook] & C_1, \end{tikzcd} \tag{6.2.9.1}$$

where $d \colon G_1 \to C_1$ denotes the canonical map. In this case, we say that G *differs only in the centre* from G_1, compare (1.2.0.2).

To justify this terminology, we now successively calculate the groups Z, C, G^{ad}, G^{der} associated to G from the corresponding groups Z_1, C_1, G_1^{ad}, G_1^{der} of G_1 by a diagram chase:

Step 1: The groups G and G_1 have the same derived groups:

By definition, we have $G_1^{\mathrm{der}} = \ker(d)$. Now $d|_G$ factors through R by the diagram (6.2.9.1) defining G, and R is abelian (because C_1 is), hence $\ker(d)$ contains G^{der}. On the other hand, the Cartesian diagram (6.2.9.1) also implies that G contains $\ker(d) = G_1^{\mathrm{der}}$, and hence G^{der} contains $(G_1^{\mathrm{der}})^{\mathrm{der}} = G_1^{\mathrm{der}}$, where the last equality holds because G_1^{der} is semisimple. We conclude:

$$G^{\mathrm{der}} = G_1^{\mathrm{der}}.$$

Step 2: The cocentre of G is R:

Because R is abelian, the map $G \to R$ factors through the cocentre C. This shows that $G^{\mathrm{der}} \subset K := \ker(G \to R)$.

Moreover, since $G^{\mathrm{der}} = G_1^{\mathrm{der}}$ there is a map $C \hookrightarrow C_1$ which factors through R, i. e. we get a commutative diagram

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, hook] & C_1 \\ R \arrow[ur, hook] & \end{tikzcd}$$

This means we get the commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] & G \arrow[r] & R \arrow[r] & 1 \\ 1 \arrow[r] & G^{\mathrm{der}} \arrow[r] \arrow[u, hook] & G \arrow[r] \arrow[u, equal] & C \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

A direct diagram chase shows that $G^{\mathrm{der}} = K$, hence

$$C = R.$$

Step 3: The groups G and G_1 have the same adjoint group:

The adjoint group of a reductive group is the same as the adjoint group of its derived group, hence step 1 implies that G and G_1 have the same adjoint group.

Step 4: The centre Z of G is equal to $Z_1 \cap G$:

By step 3 we have

$$Z = \ker(G \to G^{\mathrm{ad}}) = \ker(G \hookrightarrow G_1 \to G_1^{\mathrm{ad}} = G^{\mathrm{ad}}) = G \cap Z_1.$$

6.2.10 (Plectic reflex Galois group – general case)**.** Assume that G and G_1 are related by the Cartesian diagram (6.2.9.1). We saw that this implies that G and G_1 have the same derived and adjoint groups. They only differ in the cocentre $C = R \hookrightarrow C_1$ and the centre $Z = Z_1 \cap G$.

Now if $G_1 = R_{F/\mathbb{Q}}H_1$ is a restriction of scalars, then C_1 is of the form $R_{F/\mathbb{Q}}D_1$ for some algebraic torus D_1 over F. This means that the cocharacters of both G_1 and C_1 carry plectic actions by the full plectic group, and that the induced map $X_*(G_1) \to X_*(C_1)$ is equivariant with respect to these actions by (6.1.6).

We can now mimic (6.2.4) to define a plectic group acting on $X_*(G)$: Recall that $\mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(R))$ is the largest subgroup of the plectic group stabilising the subset $X_*(R)$ of $X_*(C_1)$. Then (6.2.1) shows that $X_*(G)$ inherits an action by

$$\mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(G)) = \mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(R)) =: (S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{cc}}.$$

Note that this group is not necessarily of the form $\mathcal{S} \ltimes \Gamma_F^\Sigma$ for some subgroup $\mathcal{S} \subset S_\Sigma$ because the action of the plectic group on $X_*(C_1)$ does not necessarily factor through S_Σ like in (6.2.4).[1]

Also note that because $G^{\mathrm{ad}} = G_1^{\mathrm{ad}}$, the same argument as in (6.2.5) shows that the action of $\mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(R))$ on cocharacters induces an action on $G(\overline{\mathbb{Q}})$-conjugacy classes of cocharacters of G. This allows us to define the *plectic reflex Galois group* of a Shimura datum (G, X) as the subgroup of $\mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(R))$ fixing the $G(\overline{\mathbb{Q}})$-conjugacy class μ_X of cocharacters of G.

Let us conclude this section by relating the plectic action on cocharacters of tori defined in (6.2.2) to the action on cocharacters of groups related to $R_{F/\mathbb{Q}}\,\mathrm{GL}_2$:

6.2.11 Remark. (Functoriality of the plectic reflex Galois group) Let $\mathbb{G}_m \subset R \subset R_{F/\mathbb{Q}}\mathbb{G}_m$ be an intermediate algebraic torus over $\mathbb{Q}$, and define $G = G^R$ as in (6.2.4). Moreover, let K/F be a totally imaginary quadratic extension, and let T be the torus defined in (6.2.2).

A F-vector space isomorphism $a\colon K \xrightarrow{\sim} F^2$ as in the proof of (5.2.10) induces an embedding $(T_1, h_\Phi) \hookrightarrow (G_1, X_1)$ of Shimura data. We choose the isomorphism a is so that $a \circ h_\Phi \circ a^{-1}$ lies inside $X = X^R$, hence we also get an embedding $(T, h_\Phi) \hookrightarrow (G, X)$. From now on, we will suppress a from notation, so we view h_Φ as an element of X.

The plectic reflex Galois group of (G, X) is the stabiliser, inside $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{cc}}$, of the $G(\overline{\mathbb{Q}})$-conjugacy class μ_X determined by the cocharacter μ associated to a fixed $h \in X$. For example, we can take the conjugacy class of the cocharacter μ_Φ associated to $h_\Phi \in X$.

[1]However, the action of the plectic group on $X_*(C_1)$ factors through some finite quotient of the plectic group, and so one does at least get a more concrete description of $\mathrm{Stab}_{S_\Sigma \ltimes \Gamma_F^\Sigma}(X_*(R))$.

Now $T \hookrightarrow G$ induces a map $X_*(T) \hookrightarrow X_*(G)$ that is $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{cc}}$-equivariant: Indeed, $X_*(T_1) \hookrightarrow X_*(G_1)$ is $S_\Sigma \ltimes \Gamma_F^\Sigma$-equivariant by (6.1.6), and so by the definition of the plectic action on $X_*(T)$ in (6.2.2) and on $X_*(G)$ in (6.2.4) the assertion follows from the commutative diagram

$$\begin{tikzcd} X_*(T) \arrow[r, hook] \arrow[d, hook] & X_*(G) \arrow[d, hook] \\ X_*(T_1) \arrow[r, hook] & X_*(G_1). \end{tikzcd}$$

So if $\gamma \in (S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{cc}}$ lies in the plectic reflex Galois group of (T, h_Φ), then γ fixes μ_Φ (viewed as a cocharacter of T), so in particular it fixes the conjugacy class of μ_Φ inside $X_*(G)$, hence γ is an element of the plectic reflex Galois group of (G, X). This means that the plectic reflex Galois group of (T, h_Φ) is contained in the plectic reflex Galois group of (G, X).

One can easily prove the plectic analogue of [Mil17, 12.3(c)]:

6.2.12 Lemma. *For $i = 1, 2$, let $G_i = R_{F/\mathbb{Q}} H_i$ for some algebraic group H_i over F. Denote the cocentre of G_i by C_i, let $R_i \subset C_i$ be a subtorus and look at the group $G_i^{R_i} \subset G_i$ defined by the Cartesian diagram*

$$\begin{tikzcd} G_i^{R_i} \arrow[r, hook] \arrow[d] & G_i \arrow[d] \\ R_i \arrow[r, hook] & C_i. \end{tikzcd}$$

Moreover, let $(G_i^{R_i}, X_i)$ be a Shimura datum. Let $f\colon H_1 \to H_2$ be a morphism defined over F such that $R_{F/\mathbb{Q}}(f)\colon G_1 \to G_2$ restricts to a morphism of Shimura data $(G_1^{R_1}, X_1) \to (G_2^{R_2}, X_2)$.

Then the plectic reflex Galois group of $(G_1^{R_1}, X_1)$ is contained in the plectic reflex Galois group of $(G_2^{R_2}, X_2)$.

6.2.13 Remark. [Del71, Thm 5.1] says that the reflex field of a Shimura datum is determined by the reflex fields of its special points. Is there a plectic analogue of this, i. e. is the plectic reflex Galois group determined by the plectic reflex Galois group of its special points? We do not know the answer to this question, not even in the case where $G = R_{F/\mathbb{Q}} H$.

6.2.14 Remark. We are also curious to see how the plectic groups $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{CM}}$ and $(S_\Sigma \ltimes \Gamma_F^\Sigma)^R_{\mathrm{cc}}$ are related.

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