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Generalized Yukawa couplings and Matrix Models

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Abstract

In this note we investigate $U(N)$ gauge theories with matter in the fundamental and adjoint representations of the gauge group, interacting via generalized Yukawa terms of the form $\mathrm{Tr}\,[Q\Phi^n\tilde{Q}]$. We find agreement between the matrix model and the gauge theory descriptions of these theories. The analysis leads to a partial description of the Higgs branch of the gauge theory. We argue that the transition between phases with different unbroken flavor symmetry groups is related to the appearance of cuts in the matrix model computation.

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1 Introduction

Recently, Dijkgraaf and Vafa have proposed [1, 2, 3] a method for computing perturbatively the effective glueball superpotential of $\mathcal{N} = 1$ theories with fields transforming in the adjoint and bifundamental representations of the gauge group. According to this proposal, the planar free energy of the matrix model whose potential is the tree-level superpotential of the $\mathcal{N} = 1$ theory yields the effective superpotential of this theory.

When fields transforming in the fundamental representation of the gauge group (quarks) are present, one only needs to include the planar free energy coming from diagrams with one quark boundary [4, 5]. More explicitly, the gauge theory effective superpotential is proposed to be

$$W_{\text{eff}}(S, \Lambda) = N_c S(1 - \ln \frac{S}{\Lambda^3}) + N_c \frac{\partial \mathcal{F}_{\chi=2}}{\partial S} + N_f \mathcal{F}_{\chi=1} \quad . \tag{1}$$

This prescription was successfully used to compare matrix model predictions with known gauge theory results for theories with massive and massless flavors, with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry [4]-[10].

For theories with fields transforming in the adjoint representation of the gauge group, proofs that planar graphs are the only ones which contribute to the matrix effective superpotential were presented in [11] (based on the analysis of superspace Feynman diagrams) and [12] (based on holomorphy and symmetries). The latter arguments were extended in [13] to the case of theories with fields transforming in the fundamental representation of the gauge group and it was shown that only planar diagrams with one (appropriately generalized) quark boundary contribute to the gauge theory effective superpotential. Other interesting related work has appeared in [14]-[31].

The correspondence between gauge theories and matrix models has been pushed very far for superpotentials depending only on the adjoint fields. However, these checks have only been performed in the simplest cases of theories with fields transforming in the fundamental representation of the gauge group. While it seems reasonable that the arguments of [12] and [13] generalize to generic superpotentials, it would be interesting to perform some explicit checks, along the lines of [4] and [5] .

In this paper we work out the details of the matrix model and the gauge theory for the $\tilde{Q}\Phi^n Q$ coupling. With these results as a starting point, we then outline how a polynomial of generalized Yukawa couplings $\tilde{Q}P(\Phi)Q$ can be analyzed. We find complete agreement between the matrix model with one boundary and the gauge theory. The Coulomb branch of such theories was discussed in detail in [32]. However, the Higgs branch seems largely unexplored. The matrix model computations suggest a simple way for analyzing it.

In the next section we use the matrix model to compute the effective superpotential of this theory. Since the adjoint field does not interact with itself, this superpotential is just the sum of a Veneziano-Yankielowicz piece (coming from the dynamics of the gauge field) and the free energy given by diagrams with one quark boundary.

We find that the sum of these diagrams gives a free energy identical to that of a theory containing an adjoint and n quarks with regular Yukawa couplings $g_i \tilde{Q}_i \Phi Q_i$, where the coupling constants g_i are proportional to the n roots of the unity. We then show that one can go from the second theory to the first by simply integrating out certain combinations of the n quarks until only one quark and the adjoint are left.

We then discuss the gauge theory origin of the matrix model results. The nonperturbative contribution to the effective superpotential of the theory with a $\tilde{Q}\Phi^n Q$ coupling is hard to obtain by symmetry and holomorphy arguments. One might hope that integrating Φ out might make things better, since only quarks will be left and the nonperturbative contribution to the superpotential would be of Affleck-Dine-Seiberg type [33]. Nevertheless, after integrating out Φ, one is left with a "tree level" superpotential which contains the coupling constant to a negative power. Since this term does not have a well defined limit as $g \to 0$, one can no longer argue that this term cannot mix with the Affleck-Dine-Seiberg contribution. Therefore, one expects nonperturbatively generated terms which contain combinations of Λ and g. These nonperturbative terms cannot be easily found using analyticity and charge conservation.

It appears therefore that in order to find the nonperturbatively generated contribution to the superpotential one has to find a theory (related to the theory of interest by integrating in and integrating out) where the nonperturbative contribution has a simple form. Fortunately, as the matrix result hints also, the theory with Yukawa coupling $\mathrm{Tr}\,[Q\Phi^n\tilde{Q}]$ can be obtained by integrating out $n-1$ nontrivial linear combinations of quarks in a theory with an adjoint, n quarks, and Yukawa couplings $\sum_{l=1}^{n} e^{2\pi i l/n}\mathrm{Tr}\,[Q_l\Phi\tilde{Q}_l]$. At this stage one can integrate out the adjoint field and find a theory with n quarks and interactions of the form $\mathrm{Tr}\,[(\sum_l g_l Q_l\tilde{Q}_l)^2]$. For this theory one can use the usual holomorphy and charge conservation arguments [34] to show that the only nonperturbatively generated superpotential is the Affleck-Dine-Seiberg one.

Once we have the full gauge theory effective potential it is not hard to integrate out all the fields and relate the resulting effective superpotential $W_{\mathrm{eff}}(\Lambda, m_i, g_i)$ with the matrix model computation. The Higgs branch of the original theory can also be analyzed.

We emphasize that his effective superpotential is the same, regardless of the order in which one integrates out the fields[4], and thus regardless of the hierarchy of the m_i. This is quite obvious from the matrix model perspective: by summing all (planar) Feynman diagrams one obtains the same function of mass parameters, regardless of their hierarchy.

One can also see this in the gauge theory. The physical mass of a field depends both on its superpotential mass parameter m, and on the Kähler potential. Since by adjusting the latter any hierarchy can be achieved (regardless of the magnitudes of the m's), and since this adjustment does not affect the superpotential, it follows that the final result is independent on the mass hierarchy and on the order one integrates fields out.

Therefore, the effective superpotential $W_{\mathrm{eff}}(\Lambda, m_i, g_i)$ one finds after integrating out all the fields is the same if the theories we start from can be related to each other by integrating in or integrating out. In the case of the theory we discuss, this effective superpotential is most easily obtained by considering the related theory with n quarks and simple Yukawa interactions, finding its nonperturbatively generated superpotential, and integrating out all the quarks. This computation appears in the last section of this note.

Note Added: When this work was near completion we received the preprint [10] which, while having a different focus, overlaps with the technical details of our work.

[4]We are grateful to Eric D'Hoker for pointing this out to us.

2 Matrix model

As we recalled in the Introduction, the part of the matrix model free energy $\mathcal{F}(S)$ which is related to gauge theory via extensions of the DV prescription is computed by summing all the planar Feynman diagrams with one quark boundary. If the quark-adjoint interaction is of the form $\tilde{Q}_i \Phi Q_i$, the necessary combinatorics of the Feynman diagrams was described in [4, 35]: a diagram with $2k$ vertices comes with a factor of $\frac{1}{(2k)!}$ from the exponential; then, the different ways of contracting the quarks produce a factor of $(2k-1)!$; the number of ways of connecting, in a planar manner, the boundary points with adjoint propagators give rise to a factor $\frac{(2k)!}{(k+1)!k!}$. Since each diagram contains $2k$ quark propagators, k adjoint propagators, and $2k$ vertices, it is multiplied by $\left(\frac{g^2}{M_\Phi m_Q^2}\right)^k$. Finally, the external flavor loop gives a factor of N_f, while the $k+1$ color loops give a factor of S^{k+1}, where S is the 't Hooft coupling of the matrix model, and becomes identified under the correspondence with the gauge theory glueball superfield.

Thus, the free energy contribution of these diagrams is [4]

$$\mathcal{F}^{n=1}_{\chi=1} = -N_f \sum_{k=1}^{\infty} \frac{(2k-1)!}{(k+1)!k!} S^{k+1} \left(\frac{g^2}{M_\Phi m_Q^2}\right)^k \tag{2}$$

In the case of diagrams with a $g_n \tilde{Q}_i \Phi^n Q_i$ interaction, it is necessary to distinguish between the case of odd and even n. If n is odd, only diagrams with even numbers of insertions contribute. The combinatorial factors coming from the quarks are unchanged. Nevertheless since now there are nk Φ lines, there will be $\frac{(2nk)!}{(nk+1)!(nk)!}$ different ways of connecting them, and the overall power of S will be $nk+1$. Thus

$$\mathcal{F}^{\mathrm{odd}}_{\chi=1} = -nN_f \sum_{k=1}^{\infty} \frac{(2nk-1)!}{(nk+1)!(nk)!} S^{nk+1} \left(\frac{g_n^2}{M_\Phi^n m_Q^2}\right)^k \tag{3}$$

If $n = 2p$ is even, diagrams with any number of insertions contribute. The free energy is

$$\mathcal{F}^{\mathrm{even}}_{\chi=1} = -2pN_f \sum_{k=1}^{\infty} \frac{(2pk-1)!}{(pk+1)!(pk)!} S^{pk+1} \left(\frac{g_n}{M_\Phi^p m_Q}\right)^k \tag{4}$$

One might have also expected extra factors of $n!$ coming from the different orderings of the Φ's originating from one vertex. However, since the Φ's are matrices, one cannot interchange them at a vertex because this would make the diagrams nonplanar. Thus, the diagrams give the same answers as if the n Φ's originating at one interaction vertex were separated by tiny propagators of some auxiliary quarks. This is depicted in Figure 1, and is a hint toward the equivalence of our theory to a theory with n quarks and simple Yukawa couplings, equivalence which will be discussed in the next section.

If one introduces the new variable p, such that $p\kappa = n$ where

$$\begin{aligned} \kappa &= 2 \qquad && \text{if } n \text{ is even}, \\ \kappa &= 1 \qquad && \text{if } n \text{ is odd}, \end{aligned} \tag{5}$$

it is not hard to see that the free energy for both even or odd n is given by:

$$\mathcal{F}_{\chi=1} = -\kappa p N_f \mathcal{K}_p, \tag{6}$$

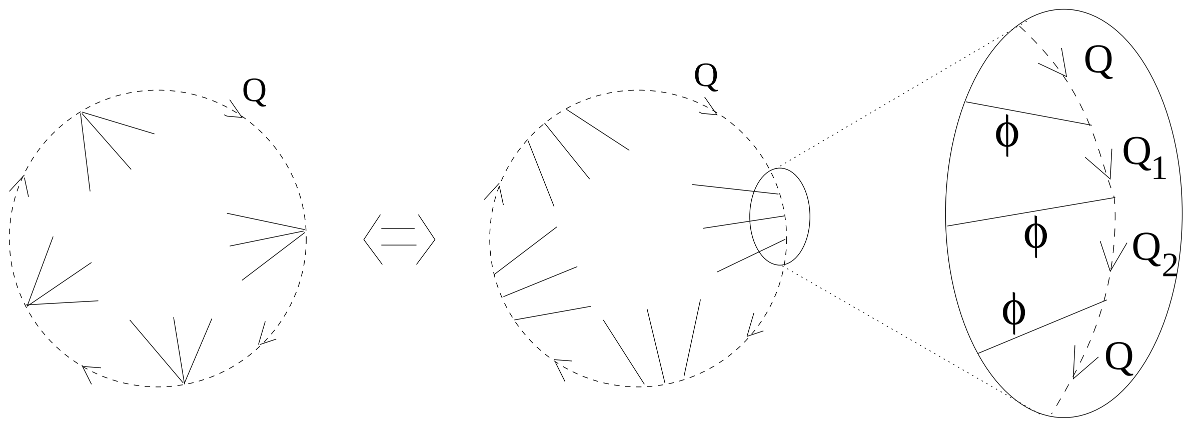

Figure 1: Equivalence between Yukawa and generalized Yukawa couplings.

where $\mathcal{K}_p$ is the value of the sum in (3).

The radius of convergence of the sum $\mathcal{K}_p$ can be easily found to be $S_c = \frac{1}{4\alpha}$. For $S < S_c$ the sum is:

$$\mathcal{K}_p = \frac{1}{p}\sum_{l=0}^{p-1} S\left[\frac{1}{2} + \frac{1}{4(-)^{2\frac{l}{p}}\alpha S}\left[\sqrt{1-4(-)^{2\frac{l}{p}}\alpha S} - 1\right] - \ln\frac{1}{2}(1+\sqrt{1-4(-)^{2\frac{l}{p}}\alpha S})\right] \quad (7)$$

and $\alpha^n = \left(\frac{g_n^2}{M_\Phi^n m_Q^2}\right)$.

For $S \geq S_c$ the sum is divergent, and one must find its value by analytical continuation [10]. Since (7) contains square roots, one expects branch cuts in the complex S plane starting from the points where the square roots become zero and ending at infinity.

Moreover, since we have n square roots, each comes with a choice of branch. Therefore, for S outside the radius of convergence, the sum (7) has 2^n values, depending on the choice of branch for each square root. Furthermore, since the sum (7) applies separately for each of the flavors, there will be a total of 2^{nN_f} branches for the free energy. The result is:

$$\mathcal{K}_p(\epsilon) = \frac{1}{p}\sum_{l=0}^{p-1} S\left[\frac{1}{2} + \frac{1}{4(-)^{2\frac{l}{p}}\alpha S}\left[\epsilon_{l,f}\sqrt{1-4(-)^{2\frac{l}{p}}\alpha S} - 1\right] - \ln\frac{1}{2}\left(1+\epsilon_{l,f}\sqrt{1-4(-)^{2\frac{l}{p}}\alpha S}\right)\right] \quad (8)$$

with $\epsilon_{l,f} = \pm 1$ with $l = 1, \ldots, n$ and $f = 1, \ldots, N_f$.

For the example discussed in [10], the choice of branch in the matrix integral was matched in gauge theory with the choice of roots of a certain the second order equations. However, one can choose the parameters of the theory such that all relevant values of S lie inside the radius of convergence. We limit ourselves to showing agreement in this regime. We will show that the convergence of (7) for $S < S_c$ has a precise meaning in gauge theory. The branch structure and the corresponding phase structure of the theory [10] follows from the careful analytical continuation of our results.

To evaluate the superpotential (1) at its critical point we begin by constructing a single formula which covers both cases $\kappa = 1$ and $\kappa = 2$. Since the set of odd roots of unity goes to itself when squared, it is not hard to see that the free energy can be expressed solely in terms of n:

$$\mathcal{F}_{\chi=1} = -N_f \sum_{l=0}^{n-1} S \left[\frac{1}{2} + \frac{1}{4e^{2\pi i \frac{2l}{n}} \alpha S} \left[\sqrt{1-4e^{2\pi i \frac{2l}{n}} \alpha S} - 1 \right] - \ln \frac{1}{2} (1 + \sqrt{1-4e^{2\pi i \frac{2l}{n}} \alpha S}) \right] \tag{9}$$

Since all roots of order n of unity appear in the above expression, it is clear that all phases in the definition of α below equation (7) are equivalent.

It is quite easy to see that each term in the sum in equation (9) reproduces the 1-boundary free energy of a theory with a single quark and a regular Yukawa coupling. The ratios of the coupling constants of these theories are n-th roots of unity. In the next section we will show how this comes about in the gauge theory.

Using (9), the critical points of the effective superpotential (1) are given by the solution of the equation:

$$\frac{N_c}{N_f} \ln \frac{S}{\Lambda^3} = \sum_{l=0}^{n-1} \ln \frac{1}{2} (1 + \sqrt{1-4e^{2\pi i \frac{2l}{n}} \alpha\, S}) = \ln \prod_{l=0}^{n-1} \frac{1}{2} (1 + \sqrt{1-4e^{2\pi i \frac{2l}{n}} \alpha\, S}) \quad , \tag{10}$$

which can be trivially transformed into:

$$y^{\frac{N_c}{N_f}} = \prod_{l=1}^{n} \frac{1}{2} (1 + \sqrt{1-4e^{2\pi i \frac{2l}{n}} \beta\, y}) \quad , \tag{11}$$

with $y = \frac{S}{\Lambda^3}$ and $\beta = \alpha \Lambda^3$.

Then, the values of the superpotential at its critical points are given by

$$W|_{\rm crit} = \Lambda^3 \left[N_c - \frac{n}{2} N_f + N_f \sum_{l=1}^{n} \frac{1}{1 + \sqrt{1-4e^{2\pi i \frac{2l}{n}} \beta\, y}} \right] y \tag{12}$$

where we have to replace y by a solution of (11). This is the result that we will compare with the gauge theory predictions. We stress that this equation is valid only in the limit of small β, i.e. far from branch points of the series (7).

For generic β the critical values of the superpotential are given by

$$W|_{\rm crit} = \Lambda^3 \left[N_c - \frac{n}{2} N_f + \sum_{f=1}^{N_f} \sum_{l=1}^{n} \frac{1}{1 + \epsilon_{i,f} \sqrt{1-4e^{2\pi i \frac{2l}{n}} \beta\, y}} \right] y \quad , \tag{13}$$

where y is given by an equation similar to (11), except that the square roots are dressed with ϵ coefficients.

3 Gauge theory

Let us begin the gauge theory analysis by showing that the generalized Yukawa coupling $\mathrm{Tr}\,[Q\Phi^n\tilde{Q}]$ is equivalent to a set of n ordinary $\mathrm{Tr}\,[Q\Phi\tilde{Q}]$ terms whose strenghs differ by n-th roots of unity. To this end we notice that, by starting with the superpotential

$$W_{\rm tree} = \frac{1}{2} M\, \mathrm{Tr}\, \Phi^2 + m \sum_{i=1}^{n} \mathrm{Tr}\,[Q_i \tilde{Q}_i] + g_0 \sum_{i=1}^{n} \mathrm{Tr}\,[Q_i \Phi \tilde{Q}_{i+1}] \quad \text{with} \quad \tilde{Q}_{n+1} \equiv \tilde{Q}_1 \tag{14}$$

and integrating out Q_i and $\tilde{Q}_i$ for all $i = 2, \ldots n$ we recover

$$W_{\text{tree}} = \frac{1}{2} M \,\mathrm{Tr}\, \Phi^2 + m \mathrm{Tr}\, [Q\tilde{Q}] + g \mathrm{Tr}\, [Q\Phi^n \tilde{Q}] \tag{15}$$

provided that $m^{n-1} g_0^n = g$.

Equation (14) can be rewritten as

$$W_{\text{tree}} = \frac{1}{2} M \,\mathrm{Tr}\, \Phi^2 + m \sum_i \mathrm{Tr}\, [Q_i \tilde{Q}_i] + m^{(n-1)/n} g_0 \sum_{i,j=1}^{n} \mathrm{Tr}\, [Q_i \Phi \tilde{Q}_j] \mathcal{P}_{ij} \tag{16}$$

with the matrix $\mathcal{P}$ being given by

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \quad . \tag{17}$$

Since we chose the mass matrix of the nN_f quarks to be proportional to the identity matrix, it is clear that the Yukawa and the mass terms can be simultaneously diagonalized. Noticing that eigenvalues of $\mathcal{P}$ are given by the roots of unity, we can immediately rewrite (16) as

$$W_{\text{tree}} = \frac{1}{2} M \,\mathrm{Tr}\, \Phi^2 + m \sum_i \mathrm{Tr}\, [Q_i \tilde{Q}_i] + \sum_{l=1}^{n} \omega_l g_0 \mathrm{Tr}\, [Q_l \Phi \tilde{Q}_l] \quad \text{with} \;\; \omega_l = e^{2\pi i \frac{l}{n}} \quad . \tag{18}$$

We have thus shown that the coupling $\mathrm{Tr}\, [Q\Phi^n \tilde{Q}]$ is equivalent to a set of diagonal Yukawa couplings whose strengths differ by roots of unity.

Unlike equation (14), the superpotential (18) is invariant under global $SU(N_f)^{\otimes n}$ transformations. In constructing the Affleck-Dine-Seiberg superpotential it is useful to think of (18) as a particular case of

$$W_{\text{tree}} = \frac{1}{2} M \mathrm{Tr}\, \Phi^2 + m \,\mathrm{Tr}\, [\mathcal{Q}\tilde{\mathcal{Q}}] + \mathrm{Tr}\, [G \mathcal{Q} \Phi \tilde{\mathcal{Q}}] \quad , \tag{19}$$

where now $\mathcal{Q}$ are in $SU(nN_f)$ and G is a diagonal matrix. The $SU(nN_f)$ invariance if broken either by having different coupling constants or by having different masses for the n sets of N_f quarks.

Since we are interested in comparing this gauge theory to the matrix model of the previous section, we first integrate out the adjoint fields. The effective superpotential is given by:

$$W_{\text{tree}} = m \mathrm{Tr}\, [\mathcal{Q}\tilde{\mathcal{Q}}] - \frac{1}{2M} \mathrm{Tr}\, [G \mathcal{Q} \tilde{\mathcal{Q}} G \mathcal{Q} \tilde{\mathcal{Q}}] \quad . \tag{20}$$

We will later identify the coupling constant matrix with the one given by equation (18), but we will derive the formulae for an arbitrary diagonal $G = \mathrm{diag}(g_1, \ldots, g_n)$.

To this tree level superpotential we have to add the nonperturbative contributions. It is not hard to see that they are given by the ADS superpotential for $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$. Indeed, in the limit of vanishing m and G this is the only possible term. Demanding analyticity as well as preservation of the symmetries leads to the conclusion that no corrections are possible.

Next we want to integrate out all quarks. The easiest way to find the result is to rewrite the above superpotential in terms of mesons and notice that, since the mass matrix as well as G are diagonal, all off-diagonal components of the mesons are constrained to vanish. Writing the remaining components of the meson field as $X_{ii} = Q_i \tilde{Q}_i = x_i \mathbb{1}_{N_f}$ the remaining effective superpotential is:

$$W_{\text{eff}} = N_f\, m \sum_{i=1}^{n} x_i - N_f \sum_{i=1}^{n} a_i x_i^2 + (N_c - nN_f) \left[\frac{\Lambda^{3N_c - nN_f}}{\prod_{i=1}^{n} x_i^{N_f}} \right]^{\frac{1}{N_c - nN_f}} , \tag{21}$$

where $a_i = \frac{g_i^2}{2M}$. Minimizing this superpotential gives

$$m x_i - 2a_i x_i^2 - \Lambda^{\frac{3N_c - nN_f}{N_c - nN_f}} \prod_{i=1}^{n} x_i^{-\frac{N_f}{N_c - nN_f}} = 0 \tag{22}$$

To compare with the matrix model, it is useful to make the following change of variables:

$$y = \prod_{j=1}^{n} y_j^{-\frac{N_f}{N_c - nN_f}} , \tag{23}$$

in terms of which the equations of motion can be rewritten as:

$$y_i - \beta_i y_i^2 - \prod_{j=1}^{n} y_j^{-\frac{N_f}{N_c - nN_f}} = 0 \tag{24}$$

where we defined

$$\beta_i = \frac{2a_i}{m} \left[\frac{\Lambda^{\frac{3N_c - nN_f}{N_c - nN_f}}}{m} \right]^{\frac{N_c - nN_f}{N_c}} , \tag{25}$$

and we recall that $i = 1, \ldots, n$ labels the different types of quarks. This gives a system of n coupled non-linear equations. To proceed it is helpful to introduce a "radial" variable:

$$y = \prod_{j=1}^{n} y_j^{-\frac{N_f}{N_c - nN_f}} . \tag{26}$$

The equation of motion becomes

$$y_i - \beta_i y_i^2 - y = 0 \quad . \tag{27}$$

We can solve for each individual y_i in terms of the couplings and the radial variable:

$$y_i = -\frac{1}{2\beta_i} \left[-1 + \varepsilon_i \sqrt{1 - 4\beta_i y} \right] \quad , \tag{28}$$

where $\varepsilon_i = \pm 1$. Each choice of ε-s one finds an equation for y:

$$y^{\frac{N_c}{N_f} - n} = \prod_{i=1}^{n} \frac{-2\beta_i}{-1 + \varepsilon_i \sqrt{1 - 4\beta_i y}} = \prod_{i=1}^{n} 2\beta_i \frac{1 + \varepsilon_i \sqrt{1 - 4\beta_i y}}{4\beta_i y} \quad . \tag{29}$$

Bringing factors of y together, we get the following equation:

$$y^{\frac{N_c}{N_f}} = \prod_{i=1}^{n} \frac{1}{2}(1 + \varepsilon_i \sqrt{1 - 4\beta_i y}) \tag{30}$$

This algebraic equation can be solved numerically for various values of n and the couplings. Once one has a solution of this equation, one can obtain the y_i's from equation (28) for each of the 2^n choices of ϵ_i.

We next compute the effective superpotential at the minimum:

$$\begin{aligned} W|_{\text{crit}} &= \frac{1}{2} N_f \, m \sum_{i=1}^{n} x_i + \frac{1}{2}(2N_c - nN_f) \left[\frac{\Lambda^{3N_c - nN_f}}{\prod_{i=1}^{n} x_i^{N_f}} \right]^{\frac{1}{N_c - nN_f}} \\ &= m^{\frac{nN_f}{N_c}} \Lambda^{\frac{3N_c - nN_f}{N_c}} \left[\frac{1}{2} N_f \sum_{i=1}^{n} y_i + \frac{1}{2}(2N_c - nN_f) \prod_{i=1}^{n} y_i^{-\frac{N_f}{N_c - nN_f}} \right] . \end{aligned} \tag{31}$$

In fact, the superpotential can be written in terms of the variable y alone by using equation (28)

$$\begin{aligned} W|_{\text{crit}} &= \Lambda_0^3 \left[\frac{1}{2}(2N_c - nN_f)\, y + \frac{1}{2} N_f \sum_{i=1}^{n} \frac{-1}{2\beta_i} \left(-1 + \varepsilon_i \sqrt{1 - 4\beta_i y} \right) \right] \\ &= \Lambda_0^3 \left[N_c - \frac{n}{2}\, N_f + N_f \sum_{i=1}^{n} \frac{1}{1 + \varepsilon_i \sqrt{1 - 4\beta_i y}} \right] y \end{aligned} \tag{32}$$

where the scale Ł$_0$ is defined by

$$\Lambda_0^3 = m^{\frac{nN_f}{N_c}} \Lambda^{\frac{3N_c - nN_f}{N_c}} \tag{33}$$

which is the correct relation between scales when all nN_f quarks are integrated out.

So far the discussion was for general β_l. Now, when all of the couplings β_l, $l = 1, \ldots, n$, are different, the number of solutions to the system (27) is 2^n times the number of solutions of equation (27). If we chose a more general form of the meson, $X_{ii} = \text{diag}(x_i^1, \ldots, x_i^{N_f})$, the number of different vacua would increase to 2^{nN_f} times the number of solutions of the apropriately modified equation (30). This matches the total number of branch cuts in the matrix model computation. Furthermore, it is easy to see that specializing the coupling constants g_l to the ones implied by the equation (18) we find

$$a_l = e^{2\pi i \frac{2l}{n}}\, a = e^{2\pi i \frac{2l}{n}} \, \frac{g_0^2}{m^{2/n}} \frac{m^2}{2M^2} \qquad \Rightarrow \qquad \beta_l = e^{2\pi i \frac{2l}{n}} \beta \ . \tag{34}$$

Therefore, the gauge theory and the matrix model results match in the region where the series (7) develops branch cuts. However, since for $y\beta < \frac{1}{4}$ the series leading to (8) is convergent, all $\epsilon_{i,f}$ coefficients are fixed to unity, while in gauge theory the choice of signs $\epsilon_{i,f}$ seems to persist for all values of y.

To understand the solution of this apparent discrepancy we should note that the appearance of different branch points leads to a spontaneous breaking of the $U(N_f)$ flavor symmetry in a theory with N_f identical quarks. In the regions of parameters where both $\epsilon_{i,f} = +1$ and $\epsilon_{i,f} = -1$ are allowed, there exist vacua with broken flavor symmetry.

However, one expects that, as the coupling constant is reduced, the flavor symmetry will be restored. Indeed, by taking the small β_i limit on equation (28) we find that, for any i, the solution corresponding to $\epsilon_i = -1$ moves off to infinity while the one corresponding to $\epsilon_{i,f} = +1$ remains at finite distance. Thus, in the small coupling limit, only the choice $\epsilon_i = +1$ is allowed.

This has definite meaning in the matrix model computation. In the small coupling limit the radius of convergence of the series (7) becomes very large. Therefore, all branch points move off to infinity and there remains a unique choice for the free energy $\mathcal{F}_{\chi=1}$.

We are therefore led to interpret the appearance of branch cuts in the series (7) as the matrix model version of transitions between domains of the Higgs branch with different flavor symmetry.

4 More generic quark-adjoint interactions

We can generalize the techniques we developed for $\mathrm{Tr}\,[Q\Phi^k\tilde{Q}]$ interaction to study theories with superpotentials of the form $\mathrm{Tr}\,[QP(\Phi)\tilde{Q}]$, where $P(\Phi)$ is an arbitrary polynomial of finite degree.

The matrix prescription is obvious. One must compute quark contribution to the free energy by summing all the one boundary Feynman diagrams with arbitrary types of insertions. The vertices are given by the monomials appearing in $P(\Phi)$, and the quark and adjoint propagators are the inverse of their masses.

While a formal expression for the free energy of the matrix model can be written for a generic $P(\Phi)$, it does not seem of any particular use. Let as only illustrate the procedure by considering a theory with an interaction term of the form

$$\mathrm{Tr}\,[Q(g_1\Phi^{2p_1} + g_2\Phi^{2p_2} + g_3\Phi^{2p_3})\tilde{Q}] \quad , \tag{35}$$

where we have chosen the powers of Φ to be even in order to keep the counting easy. A general diagram containing k_i vertices of type i comes with a factor of $1/(k_1!k_2!k_3!)$ from the expansion of the exponential; the different ways of contracting the quarks give a factor of $(k_1 + k_2 + k_3 - 1)!$; the different ways of connecting the boundary points with non-intersecting adjoint propagators give [35] a factor

$$\frac{(2k_1p_1 + 2k_2p_2 + 2k_3p_3)!}{(k_1p_1 + k_2p_2 + k_3p_3 + 1)!(k_1p_1 + k_2p_2 + k_3p_3)!} \quad . \tag{36}$$

Since each diagram contains $k_1 + k_2 + k_3$ quark propagators, $k_1p_1 + k_2p_2 + k_3p_3$ adjoint propagators, and k_1, k_2, and respectively k_3 vertices, it is multiplied by

$$\frac{g_1^{k_1}g_2^{k_2}g_3^{k_3}}{M_\Phi^{k_1p_1+k_2p_2+k_3p_3}m_Q^{k_1+k_2+k_3}} \quad . \tag{37}$$

Finally, the external flavor loop gives a factor of N_f, while the $(k_1p_1 + k_2p_2 + k_3p_3 + 1)$ color loops give the appropriate power of the glueball superfield. Thus, the 1-boundary contribution to the free energy of this model is simply

$$\mathcal{F}_{\chi=1} = -N_f S \sum_{\substack{k_1,k_2,k_3=0\\ k_1+k_2+k_3\neq 0}}^{\infty} \frac{(k_1 + k_2 + k_3 - 1)!}{(k_1!k_2!k_3!)} \times \tag{38}$$

$$\times \frac{(2k_1p_1 + 2k_2p_2 + 2k_3p_3)!}{(k_1p_1 + k_2p_2 + k_3p_3 + 1)!(k_1p_1 + k_2p_2 + k_3p_3)!} \left(\frac{S}{M_\phi}\right)^{k_1p_1+k_2p_2+k_3p_3} \frac{g_1^{k_1} g_2^{k_2} g_3^{k_3}}{m_Q^{k_1+k_2+k_3}}$$

For odd monomials the above sums have to be only over combinations of terms which give an even number of Φ propagators. It is quite easy to see that in general such sums are hard to compute explicitly.

Nevertheless, it is possible to obtain this free energy by using the vertex splitting procedure we used in section 3. A vertex of the form $g_n \mathrm{Tr}\,[Q\Phi^n\tilde{Q}]$ can be thought of as arising from a theory with n quarks and off diagonal Yukawa interactions $g\mathrm{Tr}\,[\mathcal{Q}_n\Phi\tilde{\mathcal{Q}}_n\mathcal{P}_n]$ of the type (16) by integrating out the auxiliary quarks Q_i.

Therefore, the matrix model of the theory with a polynomial interaction $\mathrm{Tr}\,[QP(\Phi)\tilde{Q}]$ can be related to that of a theory with interactions linear in Φ if one introduces $n-1$ auxiliary quarks for each monomial $g_n\mathrm{Tr}\,[Q\Phi^n\tilde{Q}]$ coming from $P(\Phi)$. One obtains an off-diagonal interaction matrix whose dimension is the sum of the powers of the monomials in $P(\Phi)$ minus the number of monomials plus one. Since the auxiliary quarks have the same mass as Q [5], one can diagonalize the interaction matrix and obtain a theory with diagonal Yukawa interactions $\lambda_j\mathrm{Tr}\,[Q_j\Phi\tilde{Q}_j]$, where the λ_j are the eigenvalues of the interaction matrix.

As in the case discussed in section 2, the partition function will be the sum of 1-quark regular Yukawa partition functions with couplings λ_j. The only difference from the equation (7) will be that the couplings will not be proportional to the roots of unity, but will have a more complicated form.

To treat this theory correctly in the gauge theory, one must perform the same steps, by integrating in the auxiliary quarks and obtaining a theory with only linear couplings of the adjoint field Φ. One can then integrate out Φ and obtains a theory with only quarks. For this theory it is possible to determine that only the regular Affleck-Dine-Seiberg superpotential is generated nonperturbatively; one can then integrate out all the quarks and obtain a superpotential which can be related to the matrix one.

It appears therefore that since integrating in and out work identically on the two sides, the equivalence of the matrix result and the gauge theory result is ensured by the equivalence of these results for theories with quarks with equal mass and different Yukawa couplings. This equivalence is obvious from the computations in sections 2 and 3, and follows also from the results of [10] by making particular choices for the mass parameters and rescaling the quarks.

5 Conclusions

We have investigated an $U(N)$ gauge theory with adjoint and fundamental matter interacting via a coupling of the form $\mathrm{Tr}\,[Q\Phi^n\tilde{Q}]$. We have solved the matrix model and found the exact low energy effective superpotential. This effective superpotential is identical to that of a theory with n quarks minimally coupled to Φ, with coupling constants proportional to the n'th roots of unity. As expected, these two theories are related by integrating in/out $n-1$ quarks.

On the gauge theory side we argued that in order to determine unambiguously the nonperturbatively generated contribution to the superpotential one needs to first integrate

[5] As we explained, this does not affect integrating them out.

in these auxiliary $n-1$ quarks, obtain a theory with minimal couplings between the quarks and the adjoint field Φ, and then integrate out Φ to obtain a theory with n massive quarks and a quartic tree-level superpotential. One can then use standard holomorphy and symmetry arguments to argue that the only nonperturbative superpotential in this theory is the Affleck-Dine-Seiberg one. By integrating out all the quarks we obtained the low energy effective superpotential of our theory, and we found it agrees with the one computed in the matrix model.

We also described a method to investigate theories with more complicated adjoint-quark couplings, of the form $\mathrm{Tr}\,[QP(\Phi)\tilde{Q}]$, where $P(\Phi)$ is a generic polynomial of finite degree. We illustrated this technique by writing down the matrix free energy for a polynomial P built out of three monomials of arbitrary even powers. While the generalization is straightforward, the perturbation theory is difficult to resume. It is also possible to further generalize this discussion by adding an arbitrary superpotential depending only on the the adjoint field.

We then presented a method to solve these theories by relating them to gauge theories with many quarks but only minimal couplings $\lambda_i \mathrm{Tr}\,[Q_i\Phi\tilde{Q}_i]$. When the polynomial contains just one monomial of order n (this is the case discussed in the first two sections of this note), the λ_i are proportional to the n'th roots of unity. For more complicated polynomials, the λ_i are the eigenvalues of the quark interaction matrix.

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