SOLITONS AND DUALITIES IN $2 + 1$ DIMENSIONS

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DECLARATION

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SOLITONS AND DUALITIES IN 2 + 1 DIMENSIONS · EDWARD NICHOLAS WALTON

We examine three topics in the physics of three-dimensional systems, paying particular attention to the roles of topological field theory, topological solitons, and duality.

First, we consider the dynamics of nonrelativistic Chern–Simons-matter theories in $2+1$ dimensions. The theories we consider support vortex solutions and, at critical coupling, they admit an effective low-temperature description as a topological quantum mechanics on the moduli space of these vortices. Using localisation techniques, we compute the 'expected' dimension of the Hilbert space of this quantum mechanics (that is, the Euler characteristic of the relevant quantum line bundle on the moduli space) in the case of $U(N_c)$ gauge theory with N_f fundamental flavours at arbitrary Chern–Simons level, λ, compactified on an arbitrary closed Riemann surface.

We apply this result to the analysis of the duality between the vortices of these theories and the (composite) fermions that arise in descriptions of strongly correlated electron systems and, in particular, of (nonAbelian, fractional) quantum Hall fluids. In simple cases, where the degeneracies of the fermion fluids are well understood, the results give quantitative evidence for these dualities. For example, when $N_c = N_f = \lambda$ we find strong evidence that the vortices are dual to fermions in the lowest nonAbelian Landau level associated to a background $U(N_f)$ flux. In more complicated examples, the results lead us to quantitative insight into the conjectural dual theory. We find that quantum vortices generally admit a description as composite objects, bound states of (dual) anyons. We comment on potential links with three-dimensional mirror symmetry. We also compute the equivariant expected degeneracy of local Abelian vortices on the Ω-deformed sphere, finding it to be a q-analog of the undeformed version.

By taking the semiclassical limit, we use our result to compute the volumes of vortex moduli spaces (which are closely related to the statistical mechanical partition functions of vortex gases). The volume of the k-vortex moduli space is a polynomial in the volume of physical space and, in the local case of $N_c = N_f$, we find a reduction in the degree of this polynomial, from $N_f k$ to k, resolving a point of confusion in the literature.

We also find new integrability results relating solutions of the exotic vortex equations, which generalise the Jackiw–Pi and Ambjørn–Olesen vortex equations, in theories with $N_f = 1$ to pairs consisting of a flat $PSU(N_c+1)$ connection and a holomorphic section of the associated holomorphic $\mathbb{C}P^{N_c}$-bundle. As well as providing a way to generate exact exotic vortex solutions in nonAbelian gauge theories, this leads to conjectural descriptions of the moduli spaces of exotic vortices and to topology-dependent selection rules for exotic vortex solutions (generalising the fact that Jackiw–Pi vortex solutions on the infinite sphere always have even flux).

Second, we turn our attention to three-dimensional sigma models with $\mathcal{N}=4$ supersymmetry. When its target space is a hyperKähler manifold carrying a (so-called) permuting $Sp(1)$ action, such a sigma model admits two topological twists: the A-twist and the B-twist. The B-twist, also known as Rozansky–Witten theory, is well-understood, while the A-twist is a little more mysterious. The two twists are expected to be related by three-dimensional mirror symmetry.

We construct the A-twist on general three-manifolds and analyse its local and extended operators. We show that compactifying the theory on a circle gives the two-dimensional A-model

in the presence of a certain 'defect operator'. We then outline the construction of the 2-category of boundary conditions in the three-dimensional A-twist. In particular, we find that the A-twist induces a monoidal structure on the Fukaya categories of a certain, restricted, class of Kähler manifolds.

Third, we consider magnetic Skyrmions in ferromagnetic materials. We produce a continuum toy model of fractionalised electrons in three spatial dimensions describing magnetic Skyrmions and their creation and destruction via emergent magnetic monopoles. When an external magnetic field is applied, the model has a critical point where confined monopoles are dynamically stabilised and monopole-antimonopole pairs may condense. We find novel 'BPS-like' equations for these configurations. By tuning the model to critical coupling and then deforming it, we find qualitative agreement with the observed phase structure of chiral ferromagnets.

We then consider the critically-coupled model for magnetic Skyrmions on thin films, generalising it to thin films with curved geometry. We find exact Skyrmion solutions on some curved films with symmetry, namely spherical, conical, and cylindrical films. We prove the existence of Skyrmion solutions in the model on general compact films and investigate the geometry of the (resolved) moduli space of solutions.

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PART I

INTRODUCTION

1.1 STRUCTURE AND DUALITY

Any object is no more and no less than its interactions with every other object. One can understand an object only by probing it with other objects, as one understands a particle by throwing other particles at it in an accelerator, or as one understands a painting by looking at it. This holds in mathematics in the form of the Yoneda lemma, which tells us that any category of 'objects' admits a full and faithful embedding into the category of 'probes' of these objects[1].

Just as anything else, quantum field theories are subject to this. Let us illustrate this in the setting of topological field theories, describing, for example, the dynamics of gapped theories at low energy, where one can hope to make precise statements. One 'probes' a topological quantum field theory Z with another Z' by considering the domain walls between Z and Z', along with the algebraic data defining their fusion, which generally turns the collection of domain walls into a higher category (see [Kap10]). It is a result of Lurie [Lur09] that a topological field theory (in the sense that Lurie defines) is determined entirely by its higher category of domain walls with the trivial theory, which is to say by its higher category of boundary conditions.

A *duality* between two field theories is just a domain wall between them that is 'trivial', in an appropriate sense. This is a rather soulless description. Dualities can be rich, beautiful, and computationally powerful. It is said that poetry is the art of giving different names to the same thing, which might explain the enduring allure of duality in physics: dualities capture the fact that the same physical system can be described in very different ways.

Perhaps the best-known duality is the electric-magnetic duality of pure electromagnetism in four spacetime dimensions. Maxwell's equations for an electromagnetic gauge potential a read

$$\mathrm{d}F(a) = 0 \quad \text{and} \quad \mathrm{d} * F(a) = 0,$$

where $F(a)$ is the field strength of a, containing the data of the electric and magnetic fields, and $*$ is the Hodge star, taking 2-forms to 2-forms in four dimensions. These equations are invariant under the 'electric-magnetic duality transformation'

$$F(a) \leftrightarrow *F(a), \tag{1.1}$$

which exchanges electric and magnetic fields.

This duality is 'broken' by the mathematical formulation of gauge theory. The first of Maxwell's equations is a mathematical triviality, while the second has dynamical content, suggesting that the duality switches the roles of the kinematic and the dynamic.

This can lead to global issues, which are clearly seen by considering Maxwell theory on a closed Riemannian manifold. On such a manifold, the field strength $F(a)$ always defines a class in the integral second cohomology. Even on-shell, where the dual field strength $*F(a)$ defines a closed 2-form, there is no reason for the dual field strength to determine a class in *integral*

[1] Here, a 'probe' is a presheaf on the category of 'objects'.

cohomology. It turns out that the duality is restored when one passes to the quantum theory [Wit95], as one must then sum over all possible line bundles that could carry the gauge potential a. A classical quantisation condition on one side of the duality is related fundamentally to quantum mechanics on the other.

This mathematical asymmetry is also made manifest by the introduction of charged matter into the theory. It is straightforward to introduce electric charges into the theory, but the insertion of a magnetic monopole into Maxwell's theory requires the presence of a topological defect in space [Dir31]. The asymmetry is observed physically: we see electrically charged particles everywhere, but no one has ever seen a magnetic monopole.

electric	magnetic
kinematic	dynamic
local	nonlocal
classical	quantum
weak coupling	strong coupling
elementary particles	solitons

Table 1.1: Some dual concepts.

The essential essence of (1.1), as summarised in Table 1.1, is at the root of a vast number of dualities. Let us highlight briefly two further aspects of electric-magnetic duality which impact our story. First, there is particle-vortex duality [Pes78] and its numerous cousins. These describe the action of electric-magnetic duality on matter. Elementary particles are mapped under these dualities to magnetically charged *solitons*, localised collective excitations of the elementary fields (see [MS04] for a textbook treatment of these objects).

Second, electric-magnetic duality admits a generalisation to nonAbelian gauge theory, known as Montonen–Olive duality, after [MO77]. The most precise form of this duality is the S-duality of four-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry, the mathematical essence of which was captured in [KW07].

This thesis is concerned with magnetically charged objects and electric-magnetic duality in a broad sense. Kontsevich wrote in [DLP08] that

> *'for me the motivation is mostly the desire to understand the hidden machinery in a striking concrete example, around which one can build formalisms...*
>
> *In a deep sense we are all geometers'.*

This encapsulates the ambitions, if not always the results, of this thesis. We aim to explore the 'hidden machinery' of duality for field theories through the analysis of some very concrete examples.

We focus in particular on physical systems in three spacetime dimensions. In rough terms, there are three reasons for this: first, there are interesting physical phenomena that are exclusive to or otherwise prominent in three spacetime dimensions, such as anyonic particles, quantum

Hall phases, magnetic Skyrmions, three-dimensional mirror symmetry; second, three is between two and four, and three-dimensional physics represents a potential bridge between the very well-understood area of two-dimensional physics and the wilder world of four-dimensional physics[2]; third, three is less than four, and the properties of three-dimensional systems can be directly observed in laboratory settings.

In section 1.2 we summarise elements of the topics of our study and set forth some questions which this thesis addresses. Then, in section 1.3, we summarise some of the answers we find.

1.2 PHYSICS IN THREE DIMENSIONS

1.2.1 The quantum Hall effect, anyons, and vortices

When one applies a transverse magnetic field $\boldsymbol{B}$ to a two-dimensional electron gas (that is, a thin slab of material) in which a current $\boldsymbol{j}$ is applied, as in Figure 1.1, a current is generated perpendicular to both the magnetic field and the applied current. This is a consequence of the Lorentz law.

If the sample has finite width, this leads to the formation of a voltage drop across the sample, as charged particles accumulate on one edge. The voltage drop in the steady state that forms is called the *Hall voltage*, V_H, and the phenomenon is called the *Hall effect*.

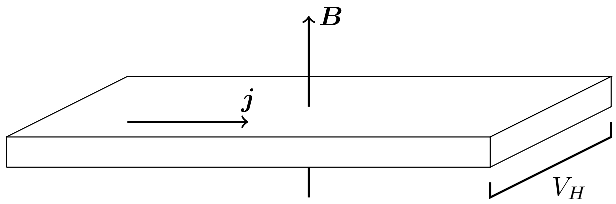


Figure 1.1: The Hall effect.

The *Hall conductivity* is defined to be $\sigma_H := |\boldsymbol{j}|/V_H$. A classical computation in a Drude-type model predicts that σ_H should be inversely proportional to the magnitude of the applied magnetic field. For small magnetic fields, this is indeed observed.

When the applied magnetic field has large magnitude and the temperature is low, something strange happens. Plateaux appear where the Hall conductivity stays constant for some range of magnetic fields. Moreover, on these plateaux, the Hall conductivity is (to great accuracy) quantised to take the form

$$\sigma_{H,\nu} \overset{!}{=} \nu \frac{e^2}{h}$$

where h is Planck's constant, e is the elementary charge, and ν is an integer. Different plateaux have different values for ν. This is the *integer quantum Hall effect*, first observed in [KDP80].

The integer quantum Hall effect is a consequence of quantum mechanics. Charged fermions in a magnetic field have energies quantised into *Landau levels*. As the magnetic field gets larger,

[2]This point is particularly valid in the context of extended field theory, where increasing the dimension of a field theory increases the categorical 'level' of the theory.

the degeneracy of each Landau level increases, as does the energy gap between each level. The plateau corresponding to a given value of ν corresponds to the system with ν filled Landau levels.

For very clean samples, with large magnetic fields, something even stranger happens. Plateaux appear for values of ν which are not integers but which are *rational*. This is the *fractional quantum Hall effect*, discovered in [TSG82].

While the integer quantum Hall effect has a relatively simple explanation, the fractional quantum Hall effect fundamentally depends on interactions between electrons in the material and so is hard to understand. Indeed, the fractional quantum Hall effect is a consequence of electron interactions splitting the degeneracy of a Landau level at partial filling to create a gapped state. The microscopic origin of the effect is not well-understood. However, a great deal of theoretical progress has been made on effective descriptions of fractional quantum Hall fluids. A very nice review of the quantum Hall effect and the various ways to understand it is given in [Ton16]. A more compact review is given in [Rea94]. We will summarise the ideas briefly.

The first step was made by Laughlin, who theorised in [Lau83] that the fractional quantum Hall effect for $\nu = \frac{1}{q}$ with q odd could be explained by introducing 'quasiparticles' and 'quasiholes', emergent particle-like objects of fractional electric charge $\pm e/q$ which appear as a kind of collective excitation of the electrons in the material. It was later shown in [ASW84] that these quasiparticles should have fractional statistics: they should be *anyons* [LM77; Wil82]. Interchanging two of these quasiholes should shift the wavefunction by a phase $\exp(i\pi/q)$. To describe more general rational values of ν, one approach is to use the so-called *hierarchy states* in which one considers quantum Hall states of the quasiholes themselves.

Richer still is the composite fermion approach of Jain [Jai89] (see also [LF91; Jai92; HLR93; Jai07]). This takes the perspective that the correct low energy degrees of freedom in a quantum Hall fluid are *composite fermions*, electrons (or holes) bound to an even number of 'vortices'. These 'vortices' may be thought of as winding modes in the wavefunction of the fluid and should each carry a quantum of 'emergent' magnetic flux. The point is that the composite fermions experience a smaller 'effective' magnetic field than that experienced by the fundamental electrons, given by

$$B^* = B - 2p\rho\Phi_0$$

where B is the magnitude of the true applied field, $2p$ is the number of 'vortices' attached to each electron, ρ is the particle density, and Φ_0 is the flux quantum. The fractional quantum Hall effect can then be understood as an integer quantum Hall effect for composite fermions.

A particular success of the composite fermion model is its prediction of the Fermi sea of composite objects for a half-filled Landau level, as observed in many materials [Wil$^+$93; HLR93]. If each electron has two vortices bound to it, then the effective magnetic field on the composite fermions at $\nu = 1/2$ (meaning that $\rho = \frac{B}{2\Phi_0}$) is zero.

Some of the most subtle aspects of quantum Hall physics require even more subtle theoretical ideas. In particular, in [Wil$^+$87] a quantum Hall state at filling fraction $\nu = \frac{5}{2}$ was observed. Initially, it was theorised that incompressible states at this filling fraction were not fully spin-polarised and so not present in the strong magnetic field limit, but evidence suggests that this is not the case [Tie$^+$12]. Further even-demoninator quantum Hall states have also been observed (see [Li$^+$17]). This poses a problem because the theoretical story so far works well for odd denominators, but not even ones. A theoretical explanation was given in [MR91], where a trial wavefunction describing fermions at even-denominator filling fraction was given using the (so-called) Pfaffian. Physically, the state at $\nu = \frac{5}{2}$ can be understood through the condensation of pairs of composite fermions.

Perhaps the most interesting aspect of the story of [MR91] is that quasihole-like excitations of the quantum Hall state at $\nu = \frac{5}{2}$ can be *nonAbelian anyons*. These have the property that they transform in an irreducible representation of the braid group of dimension higher than one, so that taking two particles around each other can do something more complicated than simply multiplying the wavefunction by a phase: it can act by some more general unitary transformation (hence 'nonAbelian'). Quantum Hall fluids supporting nonAbelian excitations of this kind are called *nonAbelian quantum Hall fluids*. There is hope that they may be useful in quantum computation (see [Nay$^+$08]).

Running parallel to all of these ideas is the idea of *duality*, the idea that the physics of electrons or, indeed, composite fermions in the material may be alternatively described as the physics of magnetically charged objects: *vortices* [Ste94; Ton04a; TT15a; RTT16; Tur17; ER20]. These vortices are not the same as those that appear in the composite fermion wavefunction. Instead they are classical, solitonic configurations in a dual gauge theory.

The relevant dual theories are nonrelativistic Chern–Simons-matter theories, and it is these theories that we study in this thesis. We will consider theories at *critical coupling*, where the parameters are fine-tuned so that the vortices are BPS (Bogomolny–Prasad–Sommerfield) objects. This means that they exert no forces on each other and so there is a degenerate *moduli space* of vortex solutions (see [MS04], for example). Physically, critical coupling is justified by the fact that the critically-coupled theory is an infrared fixed point of the renormalisation group flow [BL94]. Critical coupling can also be justified with supersymmetry [TT15a]. Mathematically, critical coupling facilitates exact computations via exploration of the geometry of the moduli space.

At low temperature, the theory admits an effective description as a one-dimensional topological sigma model with target given by this vortex moduli space. This one-dimensional low-temperature theory is topological, so it is completely determined by the dimension of its Hilbert space. We are led to consider the following fundamental questions, which should be equivalent by duality.

1. What is the dimension of the Hilbert space of critically-coupled nonrelativistic Chern–Simons-matter theory with gauge group $U(N_c)$ and N_f fundamental flavours at level λ compactified on a Riemann surface of genus g at low temperature?

2. What are the degeneracies of quasihole excitations of a (nonAbelian) quantum Hall fluid?

We (approximately) answer these questions (or at least the first one) by first realising the 'expected' dimension of the Hilbert space as an integral of a particular differential form over the vortex moduli space and then using *Coulomb branch localisation* [Wit92; JK95] to transform this integral into a particular residue computation. The validity of this idea, which is well-known in the literature concerning supersymmetric field theories, demonstrates the power of critical coupling.

These questions are interesting not only from the perspective of systems of correlated electrons, but also from the perspective of vortices: they reduce to an *a priori* difficult question about intersection theory on the moduli space of nonAbelian vortices. These moduli spaces are not well understood in general and exact results are not always easy to find.

Moreover, the answers to these questions provide physical insight into the nature of quantum vortices. Indeed, preempting the results which we summarise below, our computation reveals that quantum vortices are generally *composite objects*, built out of (dual) anyons. The vortices are dual to the composite fermions of the electron theory.

There is yet another twist in the story. For certain values of their parameters, the nonrelativistic Chern–Simons-matter theories that we consider admit an unusual class of vortex-like configurations that we call *exotic vortices*. These generalise the well-known Jackiw–Pi [JP90; Dun$^+$91] and Ambjørn–Olesen [AO88; AO89; AO90] vortex configurations. Abelian exotic vortices were studied systematically in [Man17] (see also [CD17]). In [DTT16b; Tur20], the importance of Jackiw–Pi vortices in theories of anyons (and so quantum Hall physics) was pointed out. In [DTT16b], a series of questions were posed, which can be summarised (after widening them to incorporate general exotic vortices) as follows.

3. What is the precise relationship between exotic vortices and bound states of anyons?
4. What is the low-temperature theory of quantum exotic vortices?
5. What is the moduli space of solutions to the exotic vortex equations?

We argue that the answer to the first of these questions is electric-magnetic duality and (as mentioned previously) that this is not exclusive to exotic vortices. The second question can also be answered: the low-temperature theory is (at least formally) the same topological quantum mechanics as for 'non-exotic' vortices, except that it is now on the moduli space of exotic vortex solutions. The formal methods we use in the non-exotic case can be implemented directly in the exotic case.

The third of these questions, question 5, is tricky, because many standard mathematical techniques can not be applied to exotic vortices. We address it by generalising methods of [Wit77; Man13; Man17]. As we summarise in more detail below, we show that exotic vortices in theories with one fundamental flavour (but arbitrarily many colours) can be built from objects that we call *twisted holomorphic maps* into spaces of constant holomorphic sectional curvature. We conjecture, with some evidence, that all solutions are built this way. This provides us with a qualitatively easier to understand description of the moduli space.

A theme of this thesis is the importance of *global effects*. The true understanding of a theory, or a duality, can only really come when one understands what it looks like on arbitrary (admissible) backgrounds. A question that we will be particularly interested in is as follows.

6. What can be said about global effects in fractional quantum Hall fluids?

The importance of global structure in the fractional quantum Hall effect is well-documented (see [Hal11; CLW14], for instance).

We address this question from the perspective of the dual vortex theory, finding the presence of an effective topological contribution to total effective magnetic flux experienced by composite vortices or fermions, as previously realised (in a local form) in [Hal11]. This topological contribution also leads to an interesting breaking of particle-hole symmetry, which can be understood in terms of 'spin-attachment' (induced by BF-type couplings between emergent gauge fields and the spin connection).

We also recover the well-known fact that the quasiobjects that comprise the composite vortex can generally not exist on their own on compact surfaces: the number of anyonic particles must obey a certain quantisation condition, so that the overall configuration is a geometrically viable object (for us, a vortex).

There are some particularly interesting global effects at play in the exotic vortex theory. Our 'twisted holomorphic map' construction suggests that exotic vortices on the sphere obey rather strange flux quantisation rules, which are not present on surfaces of higher genus (generalising the known behaviour of Jackiw–Pi vortices [Dun$^+$91; Ake$^+$11]). This is explained by an understanding of the topology of $PSU(n)$-bundles on surfaces.

1.2.2 Three-dimensional mirror symmetry and the categorification of two-dimensional mirror symmetry

The particle-vortex duality above can, in certain cases, be viewed as a deformed version of a duality called three-dimensional mirror symmetry.

Three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry are rich but highly structured objects. Associated to any such theory are two (generally singular) hyperKähler manifolds, the *Higgs branch* and the *Coulomb branch*. The infrared behaviour of the theory is tied to the geometry of these spaces: at low energies the theory can be effectively described in terms of a sigma model into the moduli space of vacua of the theory, which is the union of the Higgs and Coulomb branches along with additional 'mixed' branches.

The geometry of the Higgs branch is given exactly as the hyperKähler quotient of a linear space (a procedure introduced in [Hit90a]). The geometry of the Coulomb branch, on the other hand, does not have a straightforward description. Finding a satisfactory definition of the Coulomb branch has attracted significant attention [Nak16; BFN18; BDG17].

The three-dimensional mirror symmetry of [IS96] is an infrared duality between pairs of three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry. Mirror symmetry interchanges the Higgs branch of one theory with the Coulomb branch of the other.

When possible, one of the best ways to understand dualities is to associate a pair (or a more general family) of topological field theories to the theory of interest in such a way that they are interchanged under the duality. As we discuss shortly, this provides one of the best ways to understand two-dimensional mirror symmetry [Wit98]. It has also been used effectively in relating the S-duality of four-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry to the geometric Langlands correspondence [KW07].

It is desirable to use this idea to probe three-dimensional mirror symmetry. We will work at the level of three-dimensional $\mathcal{N}=4$ sigma models. Such theories are specified by a hyperKähler target space X. When X admits a (so-called) *permuting* $Sp(1)$ *action*, the corresponding supersymmetric sigma model admits two topological twists. One of these is very well-understood: it is the Rozansky–Witten theory of [RW97]. What about the other one? We consider the following question.

7. What is the mirror to Rozansky–Witten theory?

Many aspects of this question are well-understood. Its spaces of states [Nak16; BFN18; BFK19; Gai16; CDZ20], line operators [CCG19; Dim$^+$20], and boundary conditions [Bul$^+$16] have all attracted study in special cases. However, many global aspects of the theory have not been considered, partly because the theory is rarely considered on arbitrary backgrounds.

Before considering this question in any detail, let us now turn to an ostensibly different topic. One of the best understood dualities is the two-dimensional mirror symmetry between two-dimensional sigma models with $\mathcal{N}=(2,2)$ supersymmetry.

Given a two-dimensional sigma model into a Calabi–Yau manifold Y, one can associate two topological field theories, the A-model and the B-model. These theories arise as topological twists of the original supersymmetric sigma model with target Y [Wit98].

The B-model is sensitive to complex ('shape') moduli[3] of the target space Y. Its category of boundary conditions, or 'B-branes', is (roughly speaking) the derived category of coherent sheaves on Y, $D\mathrm{Coh}(Y)$. An object in this category may be thought of heuristically as a complex submanifold of Y carrying a holomorphic vector bundle (or a complex of vector bundles).

The A-model, on the other hand, is sensitive to symplectic ('size') moduli of Y. Its category of boundary conditions, the 'A-branes', is the Fukaya category $\mathrm{Fuk}(Y)$ of Y (see [Hor$^+$03; Aur13]). This is defined to be the category with objects Lagrangian submanifolds of Y and spaces of morphisms the Floer cohomology groups. Really, for mirror symmetry to work, one should pass to the derived Fukaya category $D\mathrm{Fuk}(Y)$ (see [Sei08]).

The basic statement of two-dimensional mirror symmetry is that the A-model in Y is equivalent to the B-model in a *mirror manifold* $Y^\vee$ and vice versa. This can be formulated mathematically as the homological mirror conjecture of Kontsevich [Kon95] which asserts that, roughly,

$$D\mathrm{Fuk}(Y) \simeq D\mathrm{Coh}(Y^\vee),$$

and similarly with Y and $Y^\vee$ interchanged.

The details of this are complicated. We will not be overly concerned with these details, instead focussing on the following question.

8. The category of B-branes has a natural tensor product. Does this have a mirror and, if it does, what is it?

This question is more tangible, and physical, than it may appear. The existence of a tensor product of boundary conditions indicates that they can be fused as if they were extended operators

[3] This is a slight simplification, as it also depends on a choice of basepoint in the space of Kähler moduli (see [Wit93]).

in a theory one dimension higher. Indeed, the tensor product of B-branes can be (partially) understood as resulting from the existence of Rozansky–Witten theory, which has the property that compactifying it on the circle gives the B-model (with a local operator inserted). The details of this were understood in [KRS09; KR10]. As we see below, the answer to this question seems to generally be negative, albeit in an interesting way.

A more physical form of question 8 is the following.

9. Is there a three-dimensional A-model and, if so, what is it?

By 'three-dimensional A-model' we mean a three-dimensional topological theory which gives the A-model when compactified on a circle. It turns out that this is rarely possible 'on the nose'. Instead, one should ask for a theory which gives the two-dimensional A-model in the presence of a certain defect operator when compactified on the circle.

This discussion suggests that we should consider the hypothetical commutative diagram

$$\begin{array}{ccc} ? & \leftrightarrow & \text{Rozansky–Witten theory} \\ \downarrow & & \downarrow \\ \text{A-model} + \text{defect operator} & \leftrightarrow & \text{B-model} + \text{local operator} \end{array} \tag{1.2}$$

where the horizontal arrows represent mirror symmetry, the vertical arrows represent compactification on a circle, and the question mark represents the 'three-dimensional A-model'.

We get to grips with the above questions and (1.2) by explicitly carrying out the topological twist of three-dimensional $\mathcal{N} = 4$ sigma models into hyperKähler manifolds with permuting $Sp(1)$ actions that is mirror to Rozansky–Witten theory. By studying the boundary conditions in this theory, we find conditions under which this theory induces the existence of a tensor product on the Fukaya category of a Kähler manifold.

We note that a 'three-dimensional A-model' has been constructed in [KV10]. This is distinct from our model, except in special cases. We comment on the difference between the models in section 5.4.

1.2.3 Magnetic Skyrmions, monopoles, and spintronics

Three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry are rather abstract things. There is a physical system where topological solitons, and even some of the mathematical ideas of sophisticated abstract theories, play a very visible role, even in the laboratory: magnets.

Ferromagnetic materials below their critical temperature are well-described at long distances by *micromagnetism*, which characterises the configuration of a sample, modelled by a smooth three-manifold M, by a smooth vector field $\boldsymbol{m}$ of unit length, the *magnetisation*. Because $\boldsymbol{m}$ is of unit length, it defines a map from the sample into the 2-sphere. This allows for the formation of topological solitons, with topological charge characterised by the second cohomotopy group of M. The relevant solitons are called *magnetic Skyrmions*, which are characterised by an integral topological charge. Magnetic Skyrmions were predicted in [PB75; BH94] and first observed in [Müh$^+$09].

Magnetic Skyrmions have attracted significant attention within the theoretical, experimental, and technological communities (see [FRC17] for a review). There have been suggestions that they may a play role in 'next-generation' storage devices and other so-called 'spintronic' applications by virtue of their small size, ease of manipulation, and stability.

At short distances, the micromagnetic approximation is not valid and the topological symmetry protecting Skyrmions is broken (the 'ultraviolet' theory of the magnet generically has no such topological symmetry). This presents the possibility that Skyrmions may be created or destroyed by finite energy processes. This creation or destruction is mediated by so-called *emergent magnetic monopoles*, also known as *Bloch points*. These emergent monopoles were observed (indirectly) in [Mil$^+$13] and more directly in [Bir$^+$20].

We consider the following fundamental questions.

10. What do emergent monopoles look like?
11. Can we explain the phase structure of chiral ferromagnets in a unified way?
12. What is the theory of the phase transition between the Skyrmionic and the (conical, helical, or polarised) ferromagnetic phases?

We address these questions from the perspective of an emergent gauge theory of fractionalised electrons (see [Sac19] for a nice review). The microscopic electron field is decomposed into two parts, one carrying the spin (sometimes called the *spinon*) and the other carrying the charge (known as the *chargon*). There is a redundancy in this description, which is gauged. In three spatial dimensions, this gives an $SU(2)$ gauge theory (this $SU(2)$ is sometimes called *pseudospin*). For the particular setup we construct, micromagnetism emerges in the Higgs phase of this gauge theory.

By breaking the gauge symmetry in the pattern

$$SU(2) \dashrightarrow U(1) \dashrightarrow 0 \tag{1.3}$$

we produce a theory supporting emergent magnetic Skyrmions at low energy (classified by $\pi_1(U(1))$) which are created and destroyed by (unstable) monopoles (classified by $\pi_2(SU(2)/U(1))$ at high energy. This type of symmetry breaking pattern has been explored in the context of (super) quantum chromodynamics in [Auz$^+$04; Auz$^+$03] (see also [Ton05; SY04]).

In vacuum, both the Skyrmions and the monopoles of a theory with the symmetry breaking pattern (1.3) are not stable. However, in condensed matter systems we can vary external parameters. If we apply a magnetic field to the system, the stabilisation of the Skyrmion phase is observed, and we explain how this works in our model.

To analyse this phenomenon more deeply, one can use ideas of criticality. The notion of critical coupling is well-exploited in high energy physics (sometimes for reasons of supersymmetry). A critically-coupled micromagnetic theory was given in [BSRS20], details of which were explored further in [Sch19; Wal20] (see also [Sch20]). For the critically-coupled theory, magnetic Skyrmions are BPS objects, minimising the energy within their topological class and solving a first-order equation.

We extend this critical coupling into our theory of fractionalised electrons, finding novel BPS-type equations for confined monopoles in a magnetic field at the critical point. This gives us some insight into the 'shape' of monopoles and, by deforming the theory away from criticality, into the phase structure of the chiral magnet.

The continuum toy model we produce provides, at best, a coarse-grained picture of a true ferromagnet, which will not have continuous translational symmetry. However, it seems reasonable that the ideas we discuss could be reproduced in a lattice gauge theory and, if not, it is still interesting to see an example of a theory infrared dual to the 'true' theory displaying much of the same phenomenology.

We also consider the following question about magnetic Skyrmions.

13. What happens when magnetic Skyrmions are put on curved thin films?

This is an interesting question for technological reasons, as well as mathematical ones. It has been suggested that one can trap magnetic Skyrmions in position using curvilinear defects on thin films [Kra+18], which may have implications for spintronic applications.

We consider this question using the ideas of [BSRS20; Sch19; Sch20], generalising the critically coupled model of micromagnetism given in [BSRS20] to curved thin films.

1.3 SUMMARY

1.3.1 *Part II: Vortices and the quantum Hall effect*

In chapter 2 (which is adapted from and builds on the author's sole-author paper [Wal21b]), we compute the Hilbert polynomials of quantum line bundles on moduli spaces of vortices in $U(N_c)$ gauge theories with N_f fundamental flavours on arbitrary compact Riemann surfaces using localisation techniques. This is equivalently the computation of the index (that is, the 'expected dimension' of the Hilbert space) of the low-temperature effective topological quantum mechanics of nonrelativistic $(2+1)$-dimensional Chern–Simons-matter theories tuned to critical coupling. The Abelian result, computed directly, is given in theorem 2.3.6. The nonAbelian result, computed using Coulomb branch localisation, is given as (2.24). This represents an approximate answer to question 1. We argue, but do not prove, that it becomes an exact answer when the area of the spatial surface is sufficiently large.

The output of this computation is generally complicated, but it simplifies dramatically in special cases. When $N_c = N_f =: N$ and the Chern–Simons level λ is also N, we find strong quantitative evidence for the idea that vortices at low temperature have a dual description as charged fermions in the lowest (nonAbelian) Landau level in a background $U(N)$ flux. When $N_f > N_c = 1$ and $\lambda = N_f$, we find that vortices behave as (minimal) bound states of anyons in a magnetic field. We give a description of these dualities, which seem to be valid at all two-dimensional length scales (provided one sits at criticality), representing progress towards answering question 2.

Our results also allow us to probe the action of the Seiberg duality $N_c \mapsto N_f - N_c$ in these theories (and on vortex moduli spaces), at least in special cases. In particular, in section 2.6, we show that the Hilbert polynomial enjoys a refactorisation property, allowing one to express it in 'Seiberg-dual form'.

In chapter 3 (which is adapted from the author's sole-author paper [Wal21a]), we address question 3, question 4, and question 5. We recall that nonrelativistic Chern–Simons-matter theories with negative quartic coupling may support exotic vortex-like configurations. We give the basic general theory of these objects and study their moduli spaces using direct and indirect methods. Many of the standard techniques do not work in the exotic case.

We show that, when $N_f = 1$, (exotic) vortex configurations can be constructed from (twisted) holomorphic maps into spaces of constant holomorphic sectional curvature. We conjecture that all solutions are of this form, giving coarse evidence for this. This leads to interesting topology-dependent constraints on the possible vortex number of solutions. It also allows us to give (conjectural) descriptions of the moduli spaces of exotic vortices in special cases.

We then (formally) study quantum mechanics on the exotic vortex moduli space, adapting the localisation computations of chapter 2 to (formally) compute the Euler characteristic of a quantum line bundle on the moduli space.

In chapter 4, we use the results of chapter 2 to compute the volumes of vortex moduli spaces. We show that when $N_f = N_c =: N$ the volume of the k-vortex moduli space is a polynomial in the volume of physical space of degree k rather than the naïve degree Nk, resolving a point of confusion in the literature.

The main original results of this part, in order of appearance, are as follows.

- Theorem 2.3.6 represents the rigorous computation of the index for Abelian theories with an arbitrary number of fundamental flavours and arbitrary Chern–Simons level. The proof of this result is based on projective bundle methods, which allow us to give a description of the cohomology ring of the moduli space of Abelian vortices with arbitrary numbers of flavours. The result provides evidence for the duality between vortices and composite fermions in electron fluids. An important application of this result is given by the conjectural duality (**??**), which provides tantalising links with three-dimensional mirror symmetry.

- The result (2.18) gives the index for local Abelian vortices on the Ω-deformed sphere. The result is the q-analog of the undeformed result.

- The result (2.24) gives the general form of the index for nonAbelian vortices. This represents a 'K-theoretic' generalisation of the work of [MOS11] on the volumes of vortex moduli spaces.

- In the special case of $N_c = N_f = \lambda$, we show that the vortex index simplifies dramatically. The result is given in (2.25), which provides strong evidence that vortices in these theories are dual to fermions in nonAbelian Landau levels. In subsection 2.5.3, we give a heuristic argument for this result.

- Through Theorem 3.3.1, we show constructively that (exotic) vortices in nonAbelian gauge theories with one flavour can be built from objects that we call twisted holomorphic maps.

This generalises work of Manton [Man17], Witten [Wit77], and Olesen [Ole91]. It is an integrability result, providing (in principle) a method by which solutions to vortex equations in nonAbelian gauge theories can be given. It also gives conjectural insight into the nature of the exotic vortex moduli space.

- With (4.2), we give volumes of vortex moduli spaces, previously computed in [MOS11], in a novel form. In subsection 4.2.3, we use this to show that the statistical mechanical partition function of a gas of nonAbelian vortices has degree in surface area that is independent of the genus of the underlying surface. This resolves some confusion in the literature.

1.3.2 Part III: Mirror symmetry in two and three dimensions

In chapter 5, we address question 7, the question of writing down the theory mirror to Rozansky–Witten theory. We show that one can associate such a mirror theory to any hyperKähler manifold X carrying an isometric $Sp(1)$ action permuting its complex structures. Indeed, given this data, we construct the mirror theory, which we call the *A-twist*, as a cohomological field theory. The theory can be defined on arbitrary framed three-manifolds.

We go on to study the observables of the A-twist. Its path integral localises to an integral over the moduli space of solutions to a certain nonlinear Dirac equation. As already well-known (see [Nak16; BFK19]), its spaces of quantum states are described by the de Rham cohomology spaces of moduli spaces of holomorphic sections of certain X-bundles (sometimes known as *twisted quasimaps*). We show that its category of line operators may be described as the category of boundary conditions for a two-dimensional A-model deformed by a geometrical defect operator.

We then study boundary conditions in the model, finding that they are (in a particular 'gauge') given by complex Lagrangian submanifolds of the target space (in a fixed complex structure) obeying a certain equivariance property (refer to subsection 5.4.5 for details). By studying the theory that is obtained on compactification on an interval with given boundary conditions, we address question 8. We find that the A-twist induces a tensor product on the Fukaya category of a Kähler manifold Y only if Y can be embedded into a hyperKähler manifold carrying a suitable $Sp(1)$ action in a particular 'invariant' way (again, refer to subsection 5.4.5 for the details). We relate this observation to previously known descriptions of tensor product structures on Fukaya categories in special cases (see [Pas18]).

To summarise, the main original results of this part are as follows.

- With (5.9) we give a general formulation of the 'three-dimensional A-model', for the first time. This builds on [KV10], which gives the theory in a special case.

- In subsection 5.4.5, we describe general boundary conditions in our three-dimensional A-model for the first time. We use this to characterise he conditions under which a Fukaya category may admit a natural monoidal structure.

1.3.3 Part IV: Magnetic Skyrmions

In chapter 6, we produce a model of fractionalised electrons which supports Skyrmion and monopole-like configurations. We use this to explain qualitatively the phase structure of chiral magnets. In doing this, we provide a possible answer to question 11.

In the course of this, we find a critical point where confined monopoles are dynamically stabilised by the applied magnetic field. At this critical point, confined monopoles obey a novel BPS-like equation and experience no local forces. This leads us to answers to the question of the shape of emergent monopoles, question 10 and, more interestingly, the question of the theory of the phase transition between Skyrmionic and ferromagnetic phases, question 12. At the phase transition point, we suggest that monopole-antimonopole pairs may condense.

In chapter 7 (which is adapted from the author's sole-author paper [Wal20]), we consider question 13, producing a model of critically-coupled micromagnetism on curved thin films. This generalises a model introduced in [BSRS20]. We give families of exact Skyrmion solutions in the model on certain curved surfaces with symmetry, namely spheres, cones, and cylinders. The axially symmetric solutions on the cylinder resemble kinks between vacua.

We further give an existence result for Skyrmion solutions in the critically-coupled micromagnetic model on compact surfaces and investigate the (resolved) moduli space by appealing to a theory of semilocal vortices in a background gauge field (which are also investigated in chapters 2 and 6).

Specifically, the main original contributions of this part are as follows.

- A novel general continuum model for chiral ferromagnetic materials is given in (6.3), based on an effective gauge theory description. We show that the model supports magnetic Skyrmion strings and confined monopoles on which such strings end. We show that there is a critical point, given by (6.7), at which these confined monopoles are stabilised. Such configurations are described by the novel BPS-like equations (6.10). This BPS-like model is qualitatively different to known models which support confined monopoles. This is for the technical reason that the gauge action permutes the complex structures on the target space, which is unusual. We show that the general model reproduces the qualitative phase structure of chiral ferromagnets.

- We derive a critically-coupled model for Skyrmions on curved films, given in (7.14). This represents a special case of the general model of [Sch19], using a particular background connection that captures the extrinsic geometry of the thin film in three-dimensional space. We solve the corresponding Skyrmion equation exactly on the sphere, with solution given by (7.23). We also find general axially symmetric solutions the cone, given in (7.27), and the cylinder, given in (7.28).

- With proposition 7.2.1 we show that Skyrmion solutions in the general model of [Sch19] always exist on compact surfaces.

PART II

VORTICES AND THE QUANTUM HALL EFFECT

2.1 INTRODUCTION AND SUMMARY

In this chapter we investigate the low-temperature behaviour of nonrelativistic gauge theory in $2+1$ dimensions. We focus particularly on $U(N_c)$ Chern–Simons gauge theories with N_f fundamental scalar fields compactified on Riemann surfaces. In the Higgs phase, these theories are dynamically rich, supporting vortex solutions. Indeed, the particular theories we study have an effective low-temperature description in terms of Hamiltonian dynamics on moduli spaces of (nonAbelian) vortices.

A priori, studying the dynamics of these theories, even at low temperature, looks like a difficult task. Vortices are complicated objects, solving systems of coupled, nonlinear differential equations to which analytic solutions are known only in extremely special cases. As soon as $N_c > 1$, the geometry of the moduli space is - in the best cases - only partially understood, making it hard to study quantum mechanics on it directly. With all that said, there are two remarkable strands of knowledge which allow us to get to grips with these theories, as follows.

- *Coulomb branch localisation.* Also known as Jeffrey–Kirwan–Witten localisation and originally due to Witten [Wit92] and Jeffrey–Kirwan [JK95], this technique allows one to express integrals over symplectic quotient spaces in terms of certain residues. It has led to enormous progress in the analysis of supersymmetric field theories (see [MNS00; Pes$^+$17], for instance). The vortex moduli space is a symplectic quotient (of an infinite-dimensional space by an infinite-dimensional group), so this is available to us (at least formally). This technique has been used in [MOS11; OS19] to compute the volumes of vortex moduli spaces.

- *Electric-magnetic duality.* It is expected that vortices, these magnetically charged particles in the background of an electrically charged condensate, admit a dual description as electrically charged particles in the background of a magnetic field. This idea has found itself to be particularly important in understanding the physics of the quantum Hall effect: it is expected that fluids of vortices in bosonic Chern–Simons-matter theories with gauge group $U(N_c)$ at Chern–Simons level λ describe (nonAbelian) quantum Hall fluids at filling fraction N_c/λ [Ste94; Ton04a; RTT16; DTT16a; Kim10].

We will weave these strands together, using the first to provide quantitative evidence for the second and using the second to guide our interpretation of the results of the first.

Our main result is the computation of the Hilbert polynomials $\chi(\mathcal{L}^\lambda)$ of quantum line bundles $\mathcal{L} \to \mathcal{M}$ on moduli spaces of nonAbelian vortices for arbitrary N_c and N_f on an arbitrary compact Riemannian surface Σ. This corresponds to computing the index (that is, the 'expected' dimension of the Hilbert space) of the effective topological quantum mechanics of the $2+1$ dimensional theories on $\Sigma \times S^1$ or $\Sigma \times \mathbb{R}$ at arbitrary Chern–Simons level, which is the power λ. We expect, but do not prove, that this index exactly computes the Hilbert space dimension when the area of Σ is sufficiently large.

In the Abelian case, where the moduli spaces are relatively easy to describe, we compute the index directly, using the Hirzebruch–Riemann–Roch theorem to express $\chi(\mathcal{L}^\lambda)$ as an integral over the moduli space and doing this integral directly (this is sometimes called 'Higgs branch localisation'). In the nonAbelian case, where the moduli spaces are not sufficiently well-understood to do this, we use Coulomb branch localisation techniques to carry out the computation. A key step in the calculation is the identification of the correct residue prescription when negative powers of Vandermonde-type determinants appear (which is whenever the genus of Σ is one or higher). The form that the result takes is generally complicated, but in special cases it simplifies dramatically.

The simplest, and perhaps most striking, upshot of our study is strong evidence for the notion that nonrelativistic vortices in $U(N)$ gauge theories with N fundamental Higgs fields (so-called *local* vortices) at level $\lambda = N$ 'are' fermions in a background flux quantum mechanically, at least at low temperature. This can be expressed as a low-temperature duality

$$Z_{\mathrm{SCS},N} \;\leftrightarrow\; Z_{\mathrm{Fermi},N} \tag{2.1}$$

where $Z_{\mathrm{SCS},N}$ is a critically-coupled nonrelativistic Chern–Simons-matter theory (a Schrödinger–Chern–Simons theory, hence SCS) with gauge group $U(N)$, Chern–Simons level $\lambda = N$ and N fundamental flavours, and $Z_{\mathrm{Fermi},N}$ is a theory of N fermion flavours critically coupled to a particular background gauge field for the $U(N)$ flavour symmetry. Vortices in the theory on the left are mapped to elementary fermionic excitations on the right. The matching of various objects across the duality is summarised in Table 2.1.

Z_{SCS}	Z_{Fermi}
vortices	fermionic particles
vortex charges	fermion flavours
monopole operators	particle creation operators
vortex-hole symmetry	particle-hole symmetry
$(U(1)_{\mathrm{top}} \times SU(N)_{\mathrm{flavour}})/\mathbb{Z}_N$ global symmetry	$U(N)$ global symmetry
Fayet–Iliopoulos parameter	background $U(N)$ gauge field
Weil prequantisation condition	flux quantisation condition
saturated Bradlow bound	saturated Landau level

Table 2.1: A summary of the local Fermi-vortex duality.

The duality (2.1) was more-or-less proved in the Abelian case of $N = 1$ in [ER20]. In particular, it was shown in [ER20] that the low-temperature topological quantum mechanics on both sides of the duality were equivalent: an isomorphism of the vector spaces of states was given (although the question of whether this introduced an isomorphism of 'natural' Hilbert space structures was not resolved, see [ER20, Remark 7.3]). At the level of indices, the same result was found in [DG17].

Both [ER20] and [DG17] use knowledge of the geometry of the moduli space of Abelian vortices in their calculations. As we resort to the use of rather 'soft' techniques, we demonstrate a weaker statement for general N, finding an equality of the 'expected' dimension of the Hilbert space (that is, the relevant index) on both sides of the duality[1]. In particular, on both sides of the duality we find the index (at fixed particle number)

$$\binom{N\mathcal{A}}{k} \tag{2.2}$$

where $N\mathcal{A} = N\frac{e^2\tau\mathrm{vol}(\Sigma)}{4\pi}$ is the area of the two-dimensional surface Σ in units of vortex size (it is related linearly to the Fayet–Iliopoulos parameter τ of the vortex theory, and to the gauge coupling constant e^2) and k is the number of particles or vortices. We see that the inverse of the vortex size, which is roughly the charge density of the bosonic condensate, sets the strength of the effective magnetic field in the Fermi theory. The parameter $N\mathcal{A}$ must be an integer: in the vortex theory this is necessary for the quantum theory to exist, in the Fermi theory this is a flux quantisation condition.

On the Fermi side, the computation of the index is a simple calculation: these are fermions in the lowest (nonAbelian) Landau level at filling fraction $\nu = k/N\mathcal{A}$. On the vortex side it is tricky: even after one has carried out the localisation calculation, it involves apparently miraculous cancellations in a polynomial in $N\mathcal{A}$ of generic degree Nk (there is also a simple, but highly heuristic, argument for the result just from symmetry arguments, which we give in subsection 2.5.3).

The result (2.2) does not depend on the topology or local geometry of the surface Σ, which may be thought of as reflecting the local nature of vortices when $N_f = N_c$. The dependence of the result on the choice of two-dimensional scale $\mathcal{A}$ is simple. Note also that (2.2) captures the so-called Bradlow bound [Bra90], which is the selection rule $0 \leq k \leq N\mathcal{A}$ for vortices on compact surfaces.

The fact that our evidence takes the form of an equality of indices may lead one to suspect that the duality only holds as a result of some hidden supersymmetry. However, the evidence of [ER20] in the Abelian case suggests that this is not the case. In that case there is a genuine isomorphism of non-supersymmetric Hilbert spaces. Our methods are not powerful enough to show this in the nonAbelian case, but it seems reasonable to conjecture that it is true; this conjecture then leads to conjectures on the geometry of the moduli space of nonAbelian vortices.

Our general result, given as (2.24), contains much more than (2.2). We can independently vary λ, which is conjectured to change the filling fraction of the dual quantum Hall fluid, N_f, and N_c. In general, these changes lead to significant complications in the form of the index, involving genus-dependent contributions. This reflects just how special local vortices at the right level are.

So as to not bombard the reader with unnecessarily complicated formulae at this early stage, we will illustrate the ideas in the Abelian case. Setting $N_c = 1$ but allowing for a general number

[1] We expect that the index exactly computes the dimension of the Hilbert space when the area of Σ is sufficiently large. This expectation is based on the general lore of positivity for vector bundles and is proved in the Abelian case in [ER20].

N_f of charged scalars and general level λ, we find the index

$$\sum_{j=0}^{g} \binom{g}{j} \lambda^j N_f^{g-j} \binom{\lambda(\mathcal{A}-k) + N_f k + (N_f - 1)(1-g) - g}{N_f k + (N_f - 1)(1-g) - j} \tag{2.3}$$

at vortex number k, where g is the genus of Σ. We derive this result rigorously: it is given as Theorem 2.3.6.

We extract some lessons from this, as follows.

- *Composite vortices and fractional statistics.* The way that the vortex number k enters (2.3) indicates that vortices can be given a 'composite' interpretation. If we squint and ignore topology-dependent effects for a moment, comparison with counting formulae for particles with fractional statistics (see [Wu94], for example) suggests that we should think of a single vortex in this Abelian theory as a bound state of particles with exchange phase $\exp(\mathrm{i}\pi N_f/\lambda)$ (and, dually, vortex holes are bound states of particles with exchange phase $\exp(\mathrm{i}\pi\lambda/N_f)$). There are some subtle topological issues in play, but: *Quantum vortices are made of anyons!* This idea is not new: in [TT15a] it was pointed out that (local) Abelian vortices can admit quasihole-like excitations with fractional spin and charge, which were evaluated using a Berry phase argument.

  The composite vortices can be thought of as experiencing the 'effective magnetic flux'

  $$\lambda\mathcal{A} - (\lambda - N_f)k + \cdots = N_f\mathcal{A} - (N_f - \lambda)(\mathcal{A} - k) + \cdots$$

  where $\cdots$ denotes topological contributions. This is highly reminiscent of the effective magnetic flux experienced by the composite fermions of Jain [Jai89; Jai07], with $\mathcal{A}$ identified with the total applied magnetic flux (as in the duality above). Indeed, modulo topology, at $N_f = 1$ and λ odd, vortices should be dual to composite fermions, electrons bound to $\lambda - 1$ flux quanta.

  We note further that the topological contributions that we have suppressed above are related in the dual theory to the geometrical phases of [WZ92] that arise when coupling the quantum Hall fluid to the curvature of space (see also [Hal11] for an approach to understanding quantum Hall fluids where similar phases arise, albeit for slightly different reasons).

- *Vortex-hole (a)symmetry and bosonisation.* An interesting feature of the local result (2.2) is that it demonstrates an exact symmetry between the theory of the vortices in the Higgs phase, which have number k, and the theory of vortex holes in a vortex fluid (which is a Coulomb phase), which have number $N_c\mathcal{A} - k$, via the symmetry $\binom{N\mathcal{A}}{k} = \binom{N\mathcal{A}}{N\mathcal{A}-k}$.

  The index (2.3) illustrates the fact that when $N_c \neq N_f$ this symmetry is generally broken by topology-dependent effects. In the general case, this breaking is characterised by the quantity[2] $N_c(N_f - N_c)(1-g)$. The symmetry is partially restored at $g = 1$: in this case,

[2] This quantity is, not by coincidence, reminiscent of that characterising the ghost anomaly in the theory of two-dimensional sigma models into $\mathrm{Gr}(N_c, N_f)$ with $\mathcal{N} = (2, 2)$ supersymmetry.

(2.3) is unchanged if one interchanges

$$k \leftrightarrow \mathcal{A} - k \text{ and } \lambda \leftrightarrow N_f. \tag{2.4}$$

That one must interchange λ with N_f is perhaps surprising and a little mysterious. A potential explanation, which we do not develop in any detail here, comes from the consideration of more general three-dimensional theories. The Chern–Simons level may be viewed as being induced by integrating out massive fermionic fields in an auxiliary ultraviolet theory. Then the genus one vortex-hole duality of (2.4) might be viewed as a shadow of (a nonrelativistic version of) the 'master' *bosonisation* duality of [Jen18] (which builds, in particular, on [Aha16]; see also [RTT16] for comments on bosonisation in nonrelativistic theories). This duality interchanges fermions and bosons, so by applying the duality and then integrating out the fermions on both sides, one recovers the right kind of picture.

- *Semilocal duality and three-dimensional mirror symmetry.* In the special case of $\lambda = N_f$, the duality between vortices in Abelian theories and composite particles is reminiscent of the three-dimensional mirror symmetry of [IS96].

  At this special point, the pseudoparticles which comprise a vortex do not have fractional statistics and so a direct description of the dual theory is more accessible. We find in section 2.8 that these vortices may be regarded as bound states of bosons or bound states of fermions attached to flux and spin. The particles are forced into bound states by Chern–Simons terms for an Abelian gauge field.

  The fermionic version of the dual theory has N_f fermion flavours and a gauge group $U(1)^{N_f}$, with each $U(1)$ factor attaching flux to one of the flavours. The overall diagonal $U(1)$ factor is 'gauged again' to constrain the particles into bound states. This has the effect of inducing a constraint on the form of the overall diagonal gauge field and so the theory is, morally speaking, a theory with gauge group $U(1)^{N_f}/U(1)$. (That said, there are certain subtleties, particularly on surfaces of genus greater than zero.)

  The duality in this case therefore has schematic form

  $$U(1)_{N_f} + N_f \text{ scalars} \leftrightarrow U(1)^{N_f}/U(1) + N_f \text{ fermions} + \text{CS and BF couplings}$$

  (where the subscript on the gauge group denotes the Chern–Simons level), under which vortices are mapped to bound states of fermions, and so bears a resemblance to the mirror duality

  $$U(1) + N_f \text{ hypermultiplets} \leftrightarrow U(1)^{N_f}/U(1) + N_f \text{ hypermultiplets}$$

  of (relativistic) three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry [IS96]. It seems natural to expect that our nonrelativistic story can, in this special case, be viewed as a certain deformation of three-dimensional mirror symmetry, although we do not go so far as to construct this deformation here.

  The dualities we consider have the flavour of mirror symmetry beyond the superficial similarities. Mirror symmetry is an electric-magnetic duality, exchanging magnetically charged vortices with electrically charged particles, just as our does. Our version of the duality interchanges the parameter $\mathcal{A} = \frac{e^2\tau\mathrm{vol}(\Sigma)}{4\pi}$ of the vortex theory with the flux of a background magnetic field (just as in the local duality discussed above). At the level of global symmetries, this corresponds to the fact that the duality interchanges the topological $U(1)$ symmetry of the vortex theory with the phase rotation symmetry of the fermion theory.

- *Topological degeneracy.* Consider the vortex fluid at maximal density, corresponding to $k = \mathcal{A}$. Then the index (2.3) tells us that the vortex fluid has expected quantum degeneracy λ^g.

  This is not surprising. When $k = \mathcal{A}$, the vortex centres spread over the whole surface, the Higgs fields vanish everywhere, and the theory is in a Coulomb phase, with a massless photon. The theory is then simply a $U(1)$ Chern–Simons theory at level λ. This theory has classical degeneracy characterised by the moduli space of flat $U(1)$ connections. It is well-known, and not too hard to show, that the corresponding quantum theory has degeneracy λ^g.

The above ideas extend to the nonAbelian case, but now there are some wrinkles. The outcome of the computation becomes significantly more complicated away from the special cases of $N_c = N_f = \lambda$ or $N_c = 1$. We expect that the general result leads to understanding of nonAbelian fractional quantum Hall fluids.

An interesting use of our general result is the analysis of Seiberg-like dualities for these theories. In the limit of strong gauge coupling, vortices in these theories become holomorphic curves in the complex Grassmannian $\mathrm{Gr}(N_c, N_f)$, which is the Higgs branch of vacua in the theory. The symmetry $\mathrm{Gr}(N_c, N_f) \cong \mathrm{Gr}(N_f - N_c, N_f)$ implies that there must be a kind of duality for these theories under $N_c \leftrightarrow N_f - N_c$ at strong coupling. In general, the vortex moduli space resolves the (generally singular) moduli space of holomorphic maps into the Higgs branch and the resolutions on each side of the duality may differ.

Our approach runs parallel to matrix model approaches to understanding quantum mechanics on the vortex moduli space [Pol01; HR01; HT03; TT15a; DTT16a]. The advantage of our approach is that it is rather general (in particular, we can work on surfaces of arbitrary genus) and, in a sense, geometrically clearer. On the other hand, we are restricted to the computation of rather 'soft' observables (like the Euler characteristic) and may have to work hard for concrete insight.

Our story is also closely related to that of vortex partition functions in two- and three-dimensional supersymmetric theories (see, for example, [Sha07; DGH11; Yos11; BC14; FHY14; Fuj+12; Fuj+15; Bul+18]). The quantity we compute is slightly different from these partition functions, essentially because the vortex moduli space is the *phase space* of our nonrelativistic theory at low temperature, rather than the *configuration space*. In particular, the quantity we compute is sensitive to the Kähler structure on the moduli space. We also benefit from the fact that we do not require the presence of a supersymmetric theory. This means that we can consider the theory on arbitrary backgrounds (with arbitrary Riemannian structure) without having to worry about supersymmetry. It is certainly conceivable that certain nonrelativistic deformations of low-energy partition functions in three-dimensional supersymmetric theories could reproduce our results, at least in special cases. Indeed, this could lead to a clearer understanding of the relationships between our results and both bosonisation and mirror symmetry that we alluded to above.

The rest of this chapter is structured as follows. In section 2.2 we outline the mathematical background behind nonrelativistic Chern–Simons-matter theories and describe how they can be effectively described in terms of quantum mechanics on the vortex moduli space. We also describe the process of geometric quantisation and some of the subtleties that arise. In sections 2.3 to 2.5 we carry out the computation of the Hilbert polynomial of the quantum line bundle on the vortex moduli space. This computation has three parts:

- Rigorously quantise the moduli space of Abelian vortices. We do this by direct integration over the moduli space, which we describe as a projective bundle over the Jacobian variety of Σ for sufficiently large k. The result is given in Theorem 2.3.6. When $N_c = N_f = 1$ and $\Sigma \cong S^2$, we go further, quantising the theory in the presence of a 'harmonic trap'.
- Use Coulomb branch localisation to reduce the computation in the nonAbelian case to an integral over a Cartan subalgebra of $\mathfrak{u}(N_c)$ and, correspondingly, to a sum of 'Abelian' contributions, which we now understand. The result of this is the general formula (2.24) for the Hilbert polynomials of quantum line bundles on moduli spaces of vortices in the theories we study.
- Simplify the output in special cases. This is 'elementary' but tricky, requiring us to prove some combinatorial identities. Our main weapon in these proofs is simply the comparison of generating functions. The result in the case $N_c = N_f = \lambda$ is given in (2.25).

We go on to interpret the results in the context of duality in section 2.7 and section 2.8. In section 2.6, we describe a refactorisation of the index for nonAbelian semilocal vortices, which we argue reflects an underlying Seiberg-like duality.

2.2 Chern–Simons-matter theories and vortices

2.2.1 *Kähler vortices*

Let (Y, ω_Y) be a Kähler manifold carrying the isometric, Hamiltonian action of a compact Lie group G. Write

$$\mu : Y \to \mathfrak{g}^\vee$$

for the Kähler moment map. Here $\mathfrak{g}$ is the Lie algebra associated to G. We will identify $\mathfrak{g}$ with its dual using the Killing form.

2.2.1 Example. The example to which we specialise shortly is the case where Y consists of N_f copies of the fundamental representation of $G = U(N_c)$. The moment map is then

$$\mu((z_1, \cdots, z_{N_f})) = \frac{\mathrm{i}}{2} z_i z_i^\dagger$$

where $z_i \in \mathbb{C}^{N_c}$ for $i = 1, \cdots, N_f$ and summation is implied.

Let Σ be a Riemann surface, representing physical space, with a volume form ω_Σ. Let $P \to \Sigma$ be a principal G-bundle. We can then form the associated Y-bundle

$$p : \mathcal{Y} := P \times_G Y \to \Sigma.$$

We consider a theory of a connection A on P and a section ϕ of $\mathcal{Y}$. The connection A on P induces a connection on $\mathcal{Y}$, which is a splitting of the Atiyah exact sequence

$$T_V\mathcal{Y} \to T\mathcal{Y} \xrightarrow{\mathrm{d}p} p^*T\Sigma, \tag{2.5}$$

where $T_V\mathcal{Y}$ is the bundle of vertical vector fields (defined as those in the kernel of dp). Indeed, the connection A induces a map

$$v_A : T\mathcal{Y} \to T_V\mathcal{Y}$$

which splits the exact sequence (2.5). The *covariant derivative* of a section ϕ of $\mathcal{Y}$ with respect to the connection A is then defined by

$$\mathrm{d}_A\phi = v_A(\mathrm{d}\phi) \in \Omega^1(\Sigma, T_V\mathcal{Y}).$$

The symplectic form ω_Y on Y induces a vertical 2-form on the fibres of $\mathcal{Y}$, but not a true 2-form. For that, we need to use the connection to tell us how to feed non-vertical vectors into the vertical 2-form (this is essentially minimal coupling to a gauge field). This gives a true 2-form, which we call $\omega_{\mathcal{Y}}$.

The natural energy density functional on the space of pairs (A, ϕ) is the Yang–Mills–Higgs energy density

$$\mathcal{E}_{\mathrm{YMH}} = \frac{1}{e^2}|F(A)|^2 + |\mathrm{d}_A\phi|^2 + e^2(\xi - \mu(\phi))^2,$$

where ξ is an element of the (dual of the) centre of $\mathfrak{g}$ (known in the context of supersymmetric field theory as the *Fayet-Iliopoulos parameter*), and e^2 is the gauge coupling constant, a scale factor on the dual Killing form on $\mathfrak{g}^\vee$ (we call the coupling constant e^2 even in the nonAbelian case rather than the usual g^2 to avoid confusion with the genus g of Σ). As shown in [CGS00], this functional admits the following *Bogomolny rearrangement*

$$\mathcal{E}_{\mathrm{YMH}} = \frac{1}{e^2}| * F(A) - e^2\,(\xi - \mu(\phi))\,|^2 + |\bar{\partial}_A\phi|^2 + *(\phi, A)^*[\omega_Y^G](\xi), \tag{2.6}$$

where the Dolbeault operator $\bar{\partial}_A$ is defined by

$$\bar{\partial}_A\phi = \frac{1}{2}\left(\mathrm{d}_A\phi + J_Y \circ \mathrm{d}_A\phi \circ j_\Sigma\right),$$

and where ω_Y^G is the G-equivariant symplectic form on Y (see [AB84]).

The *vortex equations* ask that

$$F(A) = e^2\,(\xi - \mu(\phi))\,\omega_\Sigma \tag{2.7}$$

$$\bar{\partial}_A\phi = 0. \tag{2.8}$$

Solutions to these equations are absolute minimisers of $\mathcal{E}_{\mathrm{YMH}}$ within their topological class.

2.2.2 Vortex moduli

Write $\mathcal{C}$ for the infinite-dimensional space of (reasonable) pairs (A, ϕ). This is an infinite-dimensional Kähler manifold, with Kähler form[3]

$$\omega_{\mathcal{C}}((\dot{A}_1, \dot{\phi}_1), (\dot{A}_2, \dot{\phi}_2)) = e^2 \int_\Sigma \left(\frac{1}{e^2} \mathrm{tr}(\dot{A}_1 \wedge \dot{A}_2) + \omega_{\mathcal{Y}}(\dot{\phi}_1, \dot{\phi}_2) \omega_\Sigma \right)$$

where $(\dot{A}_i, \dot{\phi}_i) \in \Omega^1(\Sigma, \mathrm{ad}_P) \times \Gamma(\phi^* T_V \mathcal{Y})$, $i = 1, 2$ are tangent vectors to $\mathcal{C}$ at a point (A, ϕ). The space $\mathcal{C}$ has a natural complex structure and the corresponding Riemannian metric is the natural kinetic energy functional for the relativistic $(2+1)$-dimensional gauge theory associated to this data.

Let $\mathcal{C}_0$ be the space of pairs $(A, \phi) \in \mathcal{C}$ solving the equation

$$\bar{\partial}_A \phi = 0.$$

This is (formally) a symplectic space, with symplectic form $\omega_{\mathcal{C}_0}$. The group $\mathcal{G} \coloneqq \Gamma(\mathrm{Ad}_P)$ of gauge transformations acts on $\mathcal{C}$ and on $\mathcal{C}_0$ in the usual way, preserving the symplectic forms $\omega_{\mathcal{C}}$ and $\omega_{\mathcal{C}_0}$. One can show [CGS00] (see also [AB83]) that the moment map for this action is

$$\nu(A, \phi) = *F(A) + e^2 \mu(\phi)$$

which takes values in the dual of the Lie algebra of $\mathcal{G}$. This tells us that the vortex moduli space is the Kähler quotient

$$\mathcal{M} = \mathcal{C}_0 //_\xi \mathcal{G}.$$

We have been very sloppy: the functional analysis of this was worked out in the Abelian case in [GP94], and it works out as advertised. One upshot of this result is that the vortex moduli space is a Kähler manifold (at least for generic ξ).

What does the moduli space look like? In general it is hard to say, but a great deal is known in special cases. In particular, if $G = U(1)$ and Y is the fundamental representation of $U(1)$, then the moduli space of charge k vortex solutions on Σ is well-known [JT80; Nog87; Bra90; GP94] to be the symmetric product

$$\mathrm{Sym}^k(\Sigma) \coloneqq \Sigma^{\times k} / S_k$$

where the symmetric group S_k acts by permuting the various factors in $\Sigma^{\times k}$. This tells us that a charge k vortex solution in this case is uniquely specified by an unordered list of k points in Σ, which we interpret as the positions of the k vortices.

More generally, when $G = U(N_c)$ and Y is N_f copies of the fundamental representation, it is well known that the moduli space of charge k vortex solutions on Σ modulo gauge transformations is a Kähler manifold of complex dimension

$$N_f k + N_c(N_f - N_c)(1 - g),$$

[3] We have inserted an extra factor of e^2 into the definition of the Kähler form so as to simplify notation later, in line with our convention that Chern–Simons levels are pure integers.

as follows from an index calculation.

The case of $N_c = N_f =: N$ is particularly pleasant. In this case the Higgs field ϕ is an $N \times N$ matrix, so that one can take its determinant. Vortices then have a precise position - their centres are located at the zeroes of the determinant. They are then exponentially localised around their centre which justifies the fact that they are known as *local vortices*.

In the local case, Baptista produced an attractive description of the vortex moduli space [Bap09, Theorem 1.3]. He showed, using the Hitchin–Kobayashi correspondence, that a configuration of $k < \mathcal{A}$ local vortices on a compact surface Σ is uniquely determined by k pairs of the form (z, V), where $z \in \Sigma$ and V is a *vortex internal structure*, which captures the internal moduli of the vortex. Baptista's result does not give information in the regime $\mathcal{A} \leq k \leq N\mathcal{A}$. The theorem might be viewed as a formalisation of some of the arguments of [HT03], which used symmetry arguments to understand the internal moduli of vortices.

There are many more results on vortex moduli spaces. Two particularly successful approaches from the physics literature are the moduli matrix approach [Eto+06a; Eto+06b; Eto+08a; Eto+07], which is closely related to the Hitchin–Kobayashi correspondence, and the matrix model approach [HT03]. The matrix model can be derived from D-brane constructions and gives an approximation to the worldvolume theory of the vortices. Both the moduli matrix and the matrix model describe the moduli space of local vortices on flat space as a finite-dimensional Kähler quotient.

2.2.3 Schrödinger–Chern–Simons theories

The Chern–Simons matter theories which play a central role in our story are the 'natural' nonrelativistic $(2+1)$-dimensional extensions of the theory of two-dimensional Yang–Mills–Higgs theory we considered above. As we will show, these theories admit an (exact) description as gauged Hamiltonian mechanics on the infinite-dimensional configuration space $\mathcal{C}$ and an effective low-temperature description in terms of trivial Hamiltonian mechanics on the vortex moduli space $\mathcal{M}$. The particular theories we consider are generalisations of the Abelian theory considered by Manton in [Man97] (see also [MN99a; Rom01; RS04; TT15a] for further considerations of Manton's model relevant to our study). We will give the general formulation to illustrate the geometry of the theories, but will later specialise to familiar linear examples, where the description simplifies.

The theory is specified by the data of a Lie group G, a Hamiltonian G-space Y, as above, and a Chern–Simons level in $H^4(BG)$, which we represent as λ. The action of the theory on $\Sigma \times S^1$ takes the form

$$S = \frac{\lambda}{2\pi}\int_{\Sigma\times D^2}\left(\mathrm{tr}\left(F(\tilde{A})\wedge F(\tilde{A})\right) + e^2\tilde{\phi}^*\omega_{\mathcal{Y}}\wedge\omega_\Sigma + e^2(\mu(\tilde{\phi})-\xi, F(\tilde{A}))\wedge\omega_\Sigma\right) \\ -\frac{\beta}{2\pi}\int_{\Sigma\times S^1}\mathcal{E}_{\mathrm{YMH}}\mathrm{d}t\wedge\omega_\Sigma \tag{2.9}$$

There are a few things to explain here. First, $\tilde{A}$ and $\tilde{\phi}$ are extensions of A and ϕ respectively from $\Sigma \times S^1$ to $\Sigma \times D^2$. In general, such extensions may not exist, so in defining the theory we should restrict to loops $(A|_\Sigma, \phi) : S^1 \to \mathcal{C}$ which are simply connected. Provided an extension exists, the

Lagrangian depends only on the homotopy class of the extension provided the parameter λ is an integer (or more accurately, that the class in $H^4(BG)$ induced by λ is integral).[4]

The coordinate $t \in [0, 2\pi)$ is a periodic coordinate on the circle factor. The constant β is a scale factor on the length of the circle. Notice that it appears only in front of the second integral, as this is the only place where a 'bare' $\mathrm{d}t$ sits.

For reasons that we'll describe shortly, one can (up to a ground state energy) replace the critically-coupled two-dimensional energy density $\mathcal{E}_{\text{YMH}}$, which is the Hamiltonian of the theory of (2.9), with $|\bar{\partial}_A \phi|^2$, which looks a little less 'fine-tuned'.

To connect (2.9) with more familiar things, we can write it purely in $(2+1)$-dimensional notation. To do this, it is useful to specialise to the case of $G = U(N_c)$ and $Y = \mathbb{C}^{N_c N_f}$. Then

$$\begin{aligned}\tilde{\phi}^* \omega_{\mathcal{Y}} - (\mu(\tilde{\phi}), F(A)) &= \mathrm{i}\, \mathrm{d}_{\tilde{A}} \tilde{\phi}_i^\dagger \wedge \mathrm{d}_{\tilde{A}} \tilde{\phi}_i - \mathrm{i} \phi_i^\dagger F(A) \phi_i \\ &= \frac{\mathrm{i}}{2} \mathrm{d} \left(\tilde{\phi}_i^\dagger \mathrm{d}_{\tilde{A}} \tilde{\phi}_i - (\mathrm{d}_{\tilde{A}} \tilde{\phi}_i^\dagger) \tilde{\phi}_i \right)\end{aligned}$$

where $i = 1, \cdots, N_f$ is the flavour index and summation is implied. This allows us to use Stokes' theorem, leaving us with an integral over the boundary of $\Sigma \times D^2$, which is $\Sigma \times S^1$. This is

$$S = \frac{\lambda}{2\pi} \int_{\Sigma \times S^1} \left(*\mathrm{CS}(A) - e^2 (\xi, A_t) + \frac{\mathrm{i} e^2}{2} \left(\phi_i^\dagger D_t \phi_i - (D_t \phi_i)^\dagger \phi_i \right) + \frac{\beta}{\lambda} \mathcal{E}_{\text{YMH}} \right) \mathrm{d}t \wedge \omega_\Sigma \tag{2.10}$$

where D_t is the time component of the covariant derivative and $\mathrm{CS}(A)$ is the Chern–Simons 3-form,

$$\mathrm{CS}(A) \coloneqq \mathrm{tr} \left(A \wedge \mathrm{d}A + \frac{1}{3} A \wedge [A, A] \right),$$

which has the defining property that it is a primitive of $\mathrm{tr}(F(\tilde{A}) \wedge F(\tilde{A}))$.

In the form (2.10), the action looks quite familiar. It is the obvious generalisation of the model considered by Manton in [Man97] in the Abelian case, at least for certain choices of coupling. The nonAbelian version of this theory has previously been considered in [RTT16; Tur17], for example. The reason that we initially wrote (2.9) rather than (2.10) is that the latter is not generally well-defined (or rather, it is well-defined only when one defines it as (2.9)).

2.2.4 Classical vortex mechanics

The action (2.9) can be reinterpreted as an action for gauged Hamiltonian dynamics on the infinite-dimensional configuration space $\mathcal{C}$. The gauge group is the group $\mathcal{G} = \Gamma(\Sigma, \mathrm{Ad}_P)$ of G-valued gauge transformations on Σ.

In $(0+1)$-dimensional language, the field content of the theory consists of a path in the space of configurations on Σ, which we write as

$$(A_\Sigma, \phi) : S^1 \to \mathcal{C}$$

[4] There is a quirk of conventions here. A Chern–Simons level is sometimes thought of as a dimensionful quantity taking values in $\hbar e^{-2}\mathbb{Z}$, rather than a dimensionless quantity taking values in $\mathbb{Z}$. We will prefer to take our Chern–Simons levels to be pure integers. We will usually work in units with $\hbar = 1$, but we have to be careful with factors of e^2.

where A_Σ is the two-dimensional gauge field and ϕ is the scalar field, as before, and a gauge potential a for the infinite-dimensional gauge group $\mathcal{G}$. The gauge potential a is the t-component of the original gauge potential A of the $(2+1)$-dimensional theory. Indeed, $A = a + A_\Sigma$.

More precisely, we take a $\mathcal{G}$-bundle $\mathcal{P} \to S^1$ with connection a. There is an associated $\mathcal{C}$-bundle given by

$$\underline{\mathcal{C}} \coloneqq \mathcal{P} \times_{\mathcal{G}} \mathcal{C} \to S^1,$$

and we take a section (A_Σ, ϕ) of this bundle. The space $\mathcal{C}$ has a symplectic form $\omega_{\mathcal{C}}$, which defines a vertical 2-form along the fibres of $\underline{\mathcal{C}}$. By minimally coupling this to the gauge field a, this can be extended to a true 2-form $\omega_{\underline{\mathcal{C}}}$ on $\underline{\mathcal{C}}$.

We suppose that this data can be extended over the disk and we write $(\tilde{A}_\Sigma, \tilde{\phi})$ and $\tilde{a}$ respectively for the corresponding extensions of (A_Σ, ϕ) and a.

The action (2.9) becomes

$$S = \frac{\lambda}{2\pi} \int_{D^2} \Big((\tilde{A}, \tilde{\phi})^* \omega_{\underline{\mathcal{C}}} + (\nu(\tilde{A}, \tilde{\phi}) - e^2 \xi, F(\tilde{a})) \Big) - \beta \int_{S^1} E_{\mathrm{YMH}}(A, \phi) \mathrm{d}t, \tag{2.11}$$

where we recall that the moment map ν is

$$\nu(\tilde{A}, \tilde{\phi}) = *_\Sigma F(\tilde{A}) + e^2 \mu(\tilde{\phi}),$$

and have used bracket notation to denote the inner product on $\mathrm{Lie}(\mathcal{G}) = \Gamma(\mathrm{ad}_P)$, which can be thought of as being induced by the inner product on the original Lie algebra $\mathfrak{g}$ and integration over Σ.

Applying Stokes' theorem to (2.11) gives

$$S = \frac{\lambda}{2\pi} \int_{D^2} (\tilde{A}, \tilde{\phi})^* \omega_{\underline{\mathcal{C}}} + \frac{1}{2\pi} \int_{S^1} \left(\lambda(\nu(A, \phi) - \xi, a_t) - \beta E_{\mathrm{YMH}} \right) \mathrm{d}t, \tag{2.12}$$

where $a = a_t \mathrm{d}t$.

The particular Lagrangian (2.11) takes the standard form of gauged Hamiltonian mechanics (as in [Xu16]) in $\mathcal{C}$. The gauge group is $\mathcal{G}$ and the $\mathcal{G}$-invariant Hamiltonian is E_{YMH}. The gauge field a of this mechanics enters only as a Lagrange multiplier, and integrating it out imposes the 'Gauss law'

$$*F(A) = e^2(\xi - \mu(\phi)),$$

which we identify as the first vortex equation. With this done, we may quotient by the gauge action, after which the action (2.11) becomes (in the temporal gauge, $a = 0$)

$$\frac{\lambda}{2\pi} \int_{D^2} (\tilde{A}, \tilde{\phi})^* \omega_{\mathcal{C}//\mathcal{G}} - \frac{\beta}{2\pi} \int_{S^1} \left(\int_\Sigma (|\bar{\partial}_A \phi|^2 \omega_\Sigma) + E_{\mathrm{top}} \right) \mathrm{d}t \tag{2.13}$$

which is a Hamiltonian mechanics on the symplectic quotient $\mathcal{C}//\mathcal{G}$, with Hamiltonian (up to the topological term, E_{top}, which is just a ground state energy)

$$\int_\Sigma |\bar{\partial}_A \phi|^2 \omega_\Sigma,$$

which we view as a function of gauge equivalence classes of pairs (A, ϕ). In the limit of low energy[5], or low temperature (that is, rescaling $t \mapsto \beta t$ and taking $\beta \to \infty$), we are led to consider the configurations which minimise this energy, which is the locus of

$$\bar{\partial}_A \phi = 0.$$

Imposing this equation puts on the vortex moduli space, with restricted action

$$S_{\text{vortex}} = \frac{\lambda}{2\pi} \int_{D^2} \tilde{z}^* \omega_{\mathcal{M}} \tag{2.14}$$

where z is a simply-connected loop $S^1 \to \mathcal{M}$, $\tilde{z}$ is an extension of z to the disk, the 2-form $\omega_{\mathcal{M}}$ is the Kähler form on the vortex moduli space, and we have subtracted off the dynamically unimportant topological term. The reason that this all works out is the fact that the vortex moduli space is a symplectic quotient.

The action (2.14) describes Hamiltonian mechanics on the vortex moduli space with zero Hamiltonian. Notice that the vortex moduli space is the phase space of the theory, not the configuration space (as it would be in a theory with second-order dynamics). The theory is classically 'trivial' (modulo the difficult problem of finding a vortex solution), with no dynamics. The vortices sit still, reflecting their BPS nature. This theory is interesting quantum mechanically, however: it is a (generally nontrivial) topological quantum mechanics on the vortex moduli space.

Note that we could have taken the low-temperature limit before integrating out the one-dimensional gauge field. In this case, we get the low-temperature gauged quantum mechanics

$$S_{\beta\to\infty} = \frac{\lambda}{2\pi} \int_{D^2} \Big((\tilde{A}, \tilde{\phi})^* \omega_{\underline{\mathcal{C}_0}} + (\nu(\tilde{A}, \tilde{\phi}) - e^2\xi, F(\tilde{a})) \Big).$$

Integrating out a and taking the quotient by the group $\mathcal{G}$ leaves one with the theory (2.14).

2.2.5 Quantum vortex mechanics

The most natural way to quantise the classical theory of (2.14) is by *geometric quantisation* (see [Woo92]). This proceeds by finding a Hermitian, holomorphic line bundle

$$\mathcal{L}^\lambda \to \mathcal{M}$$

over the moduli space with first Chern class

$$c_1(\mathcal{L}^\lambda) = \frac{\lambda}{2\pi} [\omega_{\mathcal{M}}] \in H^2(\mathcal{M}, \mathbb{Z}).$$

This imposes a Bohr–Sommerfeld quantisation condition (also known as a Weil prequantisation condition) on the Kähler form $\omega_{\mathcal{M}}$.

The Hilbert space of states is then identified with the space of holomorphic sections of $\mathcal{L}^\lambda$:

$$\mathcal{H}_0(\lambda) := H^0(\mathcal{L}^\lambda).$$

[5] In condensed matter applications, one could imagine dissapative effects driving the system towards low energy. In pure Hamiltonian mechanics, the energy is conserved.

In general, even the dimension of this space is hard to access. It generally depends on the precise choice of line bundle $\mathcal{L}$. A more accessible quantity is the dimension of the graded space

$$\mathcal{H}(\lambda) := \sum_i (-1)^i H^i(\mathcal{L}^\lambda),$$

The graded dimension of this is the *Euler characteristic* $\chi(\mathcal{L}^\lambda)$ of $\mathcal{L}^\lambda$. If there is a well-defined bundle $\mathcal{L}$ at $\lambda = 1$, we may also call $\chi(\mathcal{L}^\lambda)$ the *Hilbert polynomial* of $\mathcal{L}$ when viewed as a function of λ.

The Euler characteristic is rendered accessible by virtue of the Hirzebruch–Riemann–Roch theorem

$$\chi(\mathcal{L}^\lambda) \stackrel{!}{=} \int_{\mathcal{M}} \mathrm{ch}(\mathcal{L}^\lambda)\mathrm{td}(\mathcal{M}) \tag{2.15}$$

where $\mathrm{ch}(\mathcal{L}^\lambda) = \exp(\frac{\lambda}{4\pi}[\omega_{\mathcal{M}}])$ is the Chern character of $\mathcal{L}^\lambda$ and $\mathrm{td}(\mathcal{M})$ is the Todd class of the tangent bundle of $\mathcal{M}$. This is an entirely topological decription of the *a priori* analytic quantity $\chi(\mathcal{L}^\lambda)$. The main aim of this note is to compute $\chi(\mathcal{L}^\lambda)$ in the case of $G = U(N_c)$ and $Y = \mathbb{C}^{N_c N_f}$.

There are two subtleties which should be addressed, as follows.

- *Polarisation dependence.* Generally, geometric quantisation requires a choice of polarisation. This is, roughly speaking, a choice between 'position' and 'momentum' representations for the quantum states. In our context, the fact that the moduli space $\mathcal{M}$ is Kähler allows us to gloss over this issue: the polarisation here is the choice of holomorphic structure on the bundle $\mathcal{L}$. Polarisations of this type are called *Kähler polarisations.*

  It is an interesting question to ask how the Hilbert space changes as one varies the choice of polarisation. For pure Chern–Simons theory quantised on the product of a smooth, closed two-dimensional surface Σ and the real line, the space of Kähler polarisations is the moduli space of complex structures on Σ and the bundle of Hilbert spaces over this space carries a projectively flat connection, the *Hitchin connection* [Hit90b] (see also [Wit89; ADPW91; And12]). This demonstrates that, in this instance, the choice of Kähler polarisation is unimportant.

  On the other hand, it was argued in [ER20] that for quantisations of moduli spaces of local Abelian vortices it is generally not possible to construct a similar projectively flat connection on the moduli space of Kähler polarisations.

  For us, the details of different polarisations are relatively unimportant, as we simply compute a topological invariant.

- *The metaplectic correction.* The space of quantum states should have a natural inner product. One way to ensure that this is so is to 'upgrade' the quantum states to $\mathcal{L}$-valued half-densities on $\mathcal{M}$, so that they pair to give a density that can be integrated over $\mathcal{M}$. Mathematically, this means taking the quantum states to be sections of $\mathcal{L} \otimes K_{\mathcal{M}}^{\frac{1}{2}}$ rather than sections of $\mathcal{L}$, where $K_{\mathcal{M}}^{\frac{1}{2}}$ is a square root of the canonical bundle of n-forms on $\mathcal{M}$, if such a square root exists.

The metaplectic correction is often necessary in the case of non-Kähler polarisations, as otherwise the Hilbert space that one obtains is often empty. In general though, the metaplectic correction may not even exist, as is generally the case for Chern–Simons theory [ADPW91] and for quantisations of Abelian vortex moduli spaces [ER20]. The fact that metaplectic corrections do not generally exist for the theories which we are interested in suggests that we should not consider them and so we will not attempt to include a metaplectic correction in our quantisation.

2.2.6 The path integral

There is another approach to quantising the theory: the path integral approach. The partition function for the low-temperature theory on $\Sigma \times S^1$ is written schematically as

$$Z = \int_{\mathcal{L}_0\mathcal{M}} \mathcal{D}z \exp\left(\mathrm{i}\frac{\lambda}{2\pi}\int_{S^1} z^*\theta_{\mathcal{M}}\right)$$

where $\theta_{\mathcal{M}}$ is a local symplectic potential on $\mathcal{M}$ and $\mathcal{L}_0\mathcal{M}$ is the space of contractible loops on $\mathcal{M}$.

On general grounds, the partition function is the dimension of the Hilbert space of the theory. One can compute it directly using localisation techniques (see, for example, [Sza96]). Localisation allows one to reduce the path integral to an integral over the space of constant loops, which is simply $\mathcal{M}$. Up to some potential topological subtleties, this integral over $\mathcal{M}$ is exactly that on the right-hand-side of (2.15).

2.3 QUANTISATION OF ABELIAN VORTEX MODULI

2.3.1 Characteristic classes

In the Abelian case of $N_c = 1$, we will compute the Euler characteristic $\chi(\mathcal{L}^\lambda)$ rigorously by a direct integration over the moduli space using the Hirzebruch–Riemann–Roch theorem (2.15). To do this, we need to get to grips with the integrand. Here we recall some relevant information about characteristic classes.

Let X be a complex manifold and let $E \to X$ be a complex vector bundle or rank n. If

$$c_t(E) = 1 + c_1(E)t + c_2(E)t^2 + \cdots + c_n(E)t^n$$

is the Chern polynomial of E, the *Chern roots* $\{\alpha_i(E)\}_{i=1,\cdots,n}$ of E are defined by

$$c_t(E) = \prod_{i=1}^{n}(1+\alpha_i(E)t).$$

Each of the α_i has cohomological degree 2. The fact that the Chern polynomial can be written in this way is a result of the splitting principle.

The *Todd class* of E is then defined by

$$\mathrm{td}(E) \coloneqq \prod_{i=1}^{n} Q(\alpha_i(E))$$

where
$$Q(x) = \frac{x}{1 - e^{-x}},$$
which we interpret in terms of a formal power series. It is immediate from this definition that the Todd class is multiplicative over exact sequences of vector bundles. We write $\mathrm{td}(X) \coloneqq \mathrm{td}(T_X)$.

2.3.1 Example (Projective space)**.** The tangent bundle of $\mathbb{P}^n$ fits into the Euler exact sequence
$$0 \to \mathcal{O} \to \mathcal{O}^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0.$$
Thus
$$\mathrm{td}(\mathbb{P}^n) = \left(\frac{\xi}{1 - e^{-\xi}}\right)^{n+1}$$
where $\xi = c_1(\mathcal{O}(1))$ is the generator of $H^2(\mathbb{P}^n)$.

2.3.2 Example (Projective bundles)**.** Let $V \to Y$ be a vector bundle of rank r, and let $X \coloneqq \mathbb{P}(V) \xrightarrow{p} Y$ be the corresponding projective bundle, given by projectivising the fibres of V. The space X carries the natural tautological line bundle $\mathcal{O}_X(1)$.

There are two useful exact sequences, the relative Euler sequence
$$0 \to \mathcal{O}_X \to p^*V \otimes \mathcal{O}_X(1) \to T_{X/Y} \to 0$$
and the (holomorphic) Atiyah sequence
$$0 \to T_{X/Y} \to T_X \to p^*T_Y \to 0.$$
These reveal that
$$\mathrm{td}(X) = \mathrm{td}(p^*V \otimes \mathcal{O}_X(1))p^*\mathrm{td}(Y).$$

2.3.2 Local Abelian vortices

Let us begin in the local Abelian case of $N_c = N_f = 1$. In this case, our discussion retreads some of the ideas of [Mac62] on the cohomology rings of symmetric products of Riemann surfaces. The results of [Mac62] were used in [DG17] to compute the expected dimension of the Hilbert space of the vortex quantum mechanics in the special case $\lambda = 1$ and in [MN99b] to compute the volume of the moduli space of local Abelian vortices.

The moduli space of local Abelian vortices on a Riemann surface Σ of genus g is well-known to be the space of effective divisors on Σ [Tau90; Nog87; Bra90; GP94]. This can be thought of as the space of holomorphic line bundles with normalised holomorphic section, which facilitates the following construction of the vortex moduli space as a complex manifold[6].

Write $P^k(\Sigma)$ for the moduli space of holomorphic line bundles on Σ of degree k. This is a torus of complex dimension g. Let
$$\mathcal{U} \to P^k(\Sigma) \times \Sigma$$

[6] Another, direct, way to think of vortices in these terms follows from looking at the dissolving vortex limit [Weh08; BM03], a kind of (shifted) weak-coupling limit, where the vortex equations explicitly become the equations for a line bundle with a flat connection with normalised holomorphic section.

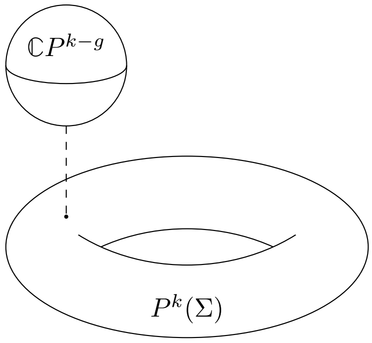


Figure 2.1: A sketch of the moduli space of charge k local Abelian vortices for $k > 2g-2$. The moduli space fibres over the torus $P^k(\Sigma)$, which is the moduli space of charge k holomorphic line bundles, with fibre $\mathbb{C}P^{k-g}$, which is the space of normalised sections. When $k \leq 2g-2$, there are degenerate fibres (see [MR11]).

be the universal degree k line bundle. This is well-defined only after choosing a point $x \in \Sigma$ and asking that $\mathcal{U}|_{P^k(\Sigma)\times\{x\}}$ be trivial.

Writing $q : P^k(\Sigma) \times \Sigma \to P^k(\Sigma)$ for the projection, we consider the direct image sheaf $q_!\mathcal{U}$ over $P^k(\Sigma)$. If $k > 2g-2$, this is locally free of constant rank and defines a vector bundle V over $P^k(\Sigma)$ with fibre over a holomorphic line bundle $\mathcal{L}$ the space of holomorphic sections of $\mathcal{L}$. The moduli space of vortices is then

$$\mathcal{M} \cong \mathbb{P}(V),$$

which admits a projection map $p : \mathcal{M} \to P^k(\Sigma)$ and supports the tautological line bundle $\mathcal{O}_{\mathcal{M}}(1)$.

Combining this with example 2.3.2 tells us that the Todd class of $\mathcal{M}$ is

$$\mathrm{td}(\mathcal{M}) = \mathrm{td}(p^*V \otimes \mathcal{O}_{\mathcal{M}}(1))p^*\mathrm{td}(P^k(\Sigma)).$$

The space $P^k(\Sigma)$ is a torus and so has trivial tangent bundle, so the Todd class becomes

$$\mathrm{td}(\mathcal{M}) = \mathrm{td}(p^*V \otimes \mathcal{O}_{\mathcal{M}}(1)).$$

To compute the Chern roots of $V = q_!\mathcal{U}$, we use the Grothendieck–Riemann–Roch theorem. We have

$$\begin{aligned}\mathrm{ch}(V) &= \mathrm{ch}(q_!\mathcal{U}) \\ &\overset{!}{=} q_*\left(\mathrm{ch}(\mathcal{U})\mathrm{td}(\Sigma)\right),\end{aligned} \tag{2.16}$$

where the first equality follows from the definition of V and the second follows from the Grothendieck–Riemann–Roch theorem. Write $\hat{\omega}$ for the generator of $H^2(\Sigma, \mathbb{Z})$. Then

$$\mathrm{td}(\Sigma) = 1 + (1-g)\hat{\omega}$$

and

$$c_1(\mathcal{U}) = k\hat{\omega} + t$$

where $t \in H^1(\Sigma) \otimes H^1(P^k(\Sigma))$ is the class dual to the natural evaluation map.

Using the fact that $\mathrm{ch}(\mathcal{U}) = \exp(c_1(\mathcal{U}))$ and the Grothendieck–Riemann–Roch formula (2.16), we see that

$$\mathrm{ch}(V) = k + 1 - g + \frac{1}{2} q_* t^2.$$

The class $\sigma = -\frac{1}{2} q^* t^2 \in H^2(P^k(\Sigma))$ is that of the theta divisor.

This reveals that $\mathrm{rank}(V) = k + 1 - g$, that $c_1(V) = -\sigma$, and that all of the higher Chern classes conspire to cancel the higher contributions to the Chern character. This means that

$$c_i(V) = \frac{1}{i!}(-\sigma)^i,$$

and the Chern polynomial is $c_t(V) = e^{-\sigma t}$ (note that this truncates because $\sigma^{g+1} = 0$ for dimensional reasons).

The projective bundle formula reveals that the cohomology ring of $\mathcal{M}$ takes the form

$$H^\bullet(\mathcal{M}) = H^\bullet(P^k(\Sigma))[\xi]/\mathcal{R}$$

where the relation $\mathcal{R}$ is

$$\sum_{i=0}^{g} \frac{1}{i!}(-\sigma)^i \xi^{k-g+1-i} = 0.$$

Write σ_i, $i = 1, \cdots, g$ for the Chern roots of V. These obey $\sigma_i^2 = 0$ for all i. Then the Todd class of $\mathcal{M}$ is

$$\begin{aligned} \mathrm{td}(\mathcal{M}) &= \mathrm{td}(p^* V \otimes \mathcal{O}_{\mathcal{M}}(1)) \\ &= \left(\frac{\xi}{1 - e^{-\xi}}\right)^{k-2g+1} \prod_{i=1}^{g} \frac{\xi - \sigma_i}{1 - e^{-(\xi - \sigma_i)}} \\ &= \left(\frac{\xi}{1 - e^{-\xi}}\right)^{k-g+1} e^{-\sigma X} \end{aligned}$$

where, formally, $X = \frac{1}{\xi} - \frac{e^{-\xi}}{1-e^{-\xi}}$ and we interpret $e^{-\sigma X}$ as a formal power series which truncates.

The Kähler class of the moduli space is

$$\frac{1}{2\pi}[\omega_{\mathcal{M}}] = d\xi + \sigma$$

where $d = \mathcal{A} - k$. To see this, one can write out representatives for ξ and σ directly (see [Per06; Bap11], for example).

Let $\mathcal{L}$ be a holomorphic line bundle with $c_1(\mathcal{L}) = d\xi + \sigma$. There is a natural choice for this quantum line bundle $\mathcal{L}$, namely

$$\mathcal{L} = \mathcal{O}_{\mathcal{M}}(d) \otimes p^* \det(V),$$

although we do not insist on this choice (indeed, V was defined with respect to an arbitrary choice of normalisation for the Poincaré line bundle, so the bundle above is not canonically determined).

Ideally, we would like to compute

$$h^0(\mathcal{L}^\lambda) = \dim H^0(\mathcal{L}^\lambda)$$

where λ is a parameter (physically, it is the Chern–Simons level). This is generally not accessible, and may depend on the particular choice of $\mathcal{L}$. Instead, we would like to compute the Euler characteristic

$$\chi(\mathcal{L}^\lambda) = \sum_i (-1)^i h^i(\mathcal{L}^\lambda).$$

The Riemann–Roch theorem tells us that

$$\chi(\mathcal{L}^\lambda) = \int_{\mathcal{M}} \mathrm{ch}(\mathcal{L}^\lambda)\mathrm{td}(\mathcal{M})$$

which is

$$\int_{\mathcal{M}} e^{\lambda(d\xi+\sigma)} \left(\frac{\xi}{1-e^{-\xi}}\right)^{k-g+1} e^{-\sigma X}.$$

This becomes

$$\begin{aligned}
\chi(\mathcal{L}^\lambda) &= \sum_{j=0} \frac{\lambda^j}{j!} \int_{\mathcal{M}} \sigma^j e^{\lambda d\xi} \left(\frac{\xi}{1-e^{-\xi}}\right)^{k-g+1} e^{-\sigma\left(\frac{1}{\xi}-\frac{e^{-\xi}}{1-e^{-\xi}}\right)} \\
&= \sum_{j=0} \frac{\lambda^j}{j!} \int_{\mathcal{M}} \sigma^j e^{\lambda d\xi} \left(\frac{\xi}{1-e^{-\xi}}\right)^{k-g+1} e^{\sigma\left(\frac{e^{-\xi}}{1-e^{-\xi}}\right)}
\end{aligned}$$

using the relation $\mathcal{R}$. Expanding the exponential, collecting the powers of σ, and removing terms that do not contribute to the integral, we find

$$\begin{aligned}
\chi(\mathcal{L}^\lambda) &= \sum_{j=0}^{g} \frac{\lambda^j}{j!(g-j)!} \int_{\mathcal{M}} \sigma^g \xi^{k-g+1} \frac{e^{(\lambda d-g+j)\xi}}{(1-e^{-\xi})^{k-j+1}} \\
&= \sum_{j=0}^{g} \lambda^j \binom{g}{j} \mathrm{Res}_{x=0} \frac{e^{(\lambda d-g+j)x}}{(1-e^{-x})^{k-j+1}} \mathrm{d}x,
\end{aligned}$$

where we have used that the integral of σ^g over the Jacobian is $g!$, which is a classical result.

To compute a residue of the form

$$\mathrm{Res}_{x=0} \frac{e^{px}}{(1-e^{-x})^{q+1}} \mathrm{d}x$$

one may introduce a variable y by $1-e^{-x} = y$. Then $e^x = (1-y)^{-1}$ and $\mathrm{d}x = \frac{1}{1-y}\mathrm{d}y$. Then the residue is

$$\mathrm{Res}_{y=0} \frac{(1-y)^{-p-1}}{y^{q+1}} \mathrm{d}y = \binom{p+q}{q}.$$

Putting the pieces together gives us the following result.

2.3.3 Theorem. *Let $\mathcal{L} \to \mathcal{M}$ be the quantum line bundle over the moduli space of local Abelian vortices of charge k on a closed Riemannian surface Σ of genus g and dimensionless 'area' $\mathcal{A} = \frac{e^2\tau vol(\Sigma)}{4\pi} \in \mathbb{Z}$. Suppose that $k > 2g-2$. Then the Hilbert polynomial of $\mathcal{L}$ is*

$$\chi(\mathcal{L}^\lambda) = \sum_{j=0}^{g} \lambda^j \binom{g}{j} \binom{\lambda(\mathcal{A}-k)+k-g}{k-j}.$$

2.3.4 Remark. Our proof of this result only holds literally in the case that $k \geq 2g-1$. However, the method still goes through in a virtual sense for all k, and the result holds in general. Note that if $k < g$, the terms with $j = k+1, \cdots, g$ all vanish.

2.3.5 Remark. One can express the Euler characteristic as a single residue as

$$\chi(\mathcal{L}^{\otimes\lambda}) = \mathrm{Res}_{x=0} \left(e^{-x} + \lambda(1-e^{-x})\right)^g \frac{e^{\lambda dx}}{(1-e^{-x})^{k+1}} \mathrm{d}x.$$

This is the form that one might find the answer in if one exploited Coloumb branch localisation. This makes particularly clear the simplification that occurs if $\lambda = 1$.

This theorem mediates between two known results. Setting $\lambda = 1$ computes the Euler characteristic of the quantum line bundle $\mathcal{L}$, which is the expected dimension of the Hilbert space of the vortex quantum mechanics at Chern–Simons level 1. This is

$$\begin{aligned}\chi(\mathcal{L}) &= \sum_{j=0}^{g} \binom{g}{j} \binom{\mathcal{A}}{k-j} \\ &= \binom{\mathcal{A}}{k},\end{aligned}$$

as previously found in [ER20] and in [DG17]. In the full quantum theory, one should sum over topological sectors with a fugacity t for the topological symmetry. The resulting *grand canonical partition function* is

$$\begin{aligned}Z(t) &= \sum_{k=0}^{\infty} \binom{\mathcal{A}}{k} t^k \\ &= (1+t)^{\mathcal{A}}.\end{aligned}$$

This result exhibits a kind of 'vortex-hole duality'. By virtue of the Bradlow bound, a configuration of k vortices in a normalised area $\mathcal{A}$ might also be viewed as a configuration of $\mathcal{A} - k$ vortex holes in a sea of $\mathcal{A}$ vortices. This is reflected in the vortex quantum mechanics by the equality

$$\binom{\mathcal{A}}{k} = \binom{\mathcal{A}}{\mathcal{A}-k}.$$

On the other hand, the volume of the vortex moduli space is (up to constant factors)

$$\mathrm{vol}(\mathcal{M}) = \int_{\mathcal{M}} e^{[\omega_{\mathcal{M}}]}.$$

Because the leading term in the Todd class is unity, this is the coefficient of $\lambda^{\dim(\mathcal{M})}$ in $\chi(\mathcal{L}^\lambda)$ (this is the highest power of λ that appears). We can read this off to be

$$\sum_{j=0}^{\min(k,g)} \binom{g}{j} \frac{(\mathcal{A}-k)^{k-j}}{(k-j)!}$$

in agreement with the computations of [MN99b], [MOS11] and [OS19].

2.3.3 Semilocal Abelian vortices

Introducing more flavours does not significantly alter the nature of the above calculation. As revealed by the dissolving vortex limit, the moduli space is again a projective bundle. Indeed, if the number of flavours is N_f, the moduli space of charge k vortices is, for k sufficiently large,

$$\mathcal{M}_{N_f} = \mathbb{P}(V^{\oplus N_f})$$

where $V \to P^k(\Sigma)$ is the vector bundle introduced previously. The dimension of the moduli space is $N_f(k-g+1)+g-1$, in agreement with an index calculation.

Multiplicativity of the Chern class means that

$$\begin{aligned} c(V^{\oplus N_f}) &= c(V)^{N_f} \\ &= e^{-N_f\sigma}. \end{aligned}$$

Again, the formal power series implied by $e^{-N_f\sigma}$ truncates because $\sigma^{g+1}=0$.

The cohomology ring of the moduli space can be described using the projective bundle formula to be

$$H^\bullet(\mathcal{M}_{N_f}) = H^\bullet(P^k(\Sigma))[\xi]/\mathcal{R}$$

where the relation $\mathcal{R}$ is

$$\sum_{i=0}^{g} \frac{1}{i!}(-N_f\sigma)^i \xi^{N_f(k-g+1)-i} = 0.$$

The Todd class of $\mathcal{M}_{N_f}$ is

$$\mathrm{td}(\mathcal{M}_{N_f}) = \left(\frac{\xi}{1-e^{-\xi}}\right)^{N_f(k-g+1)} e^{-N_f\sigma X},$$

where $X = \frac{1}{\xi} - \frac{e^{-\xi}}{1-e^{-\xi}}$ as before.

The cohomology class of the L^2 Kähler form on the vortex moduli space is (just as before)

$$\frac{1}{2\pi}[\omega_{\mathcal{M}_{N_f}}] = d\xi + \sigma,$$

where $d = \mathcal{A} - k$. (Note that the second cohomology group of the moduli space has two generators, so all that one needs to do to verify this is to check the coefficients.)

We again compute the Euler characteristic of a quantum line bundle $\mathcal{L} \to \mathcal{M}_{N_f}$ with $2\pi c_1(\mathcal{L}) = [\omega_{\mathcal{M}_{N_f}}]$. There is again a natural choice, given by

$$\mathcal{L} = \mathcal{O}_{\mathcal{M}_{N_f}}(d) \otimes p^*\det(V),$$

although, again, we do not insist on this, noting that it is not canonically determined.

Going through the motions as we did for the case of $N_f = 1$ leads to the result

$$\begin{aligned} \chi(\mathcal{L}^\lambda) &= \sum_{j=0}^{g} \lambda^j N_f^{g-j} \binom{g}{j} \mathrm{Res}_{x=0} \frac{e^{(\lambda d-(g-j))x}}{(1-e^{-x})^{N_f(k-g+1)+g-j}} \mathrm{d}x \\ &= \sum_{j=0}^{g} \lambda^j N_f^{g-j} \binom{g}{j} \binom{\lambda d + N_f(k-g+1)-1}{N_f(k-g+1)+g-j-1}. \end{aligned}$$

Substituting in $d = \mathcal{A} - k$ and introducing the notation $\delta = \dim_{\mathbb{C}}(\mathcal{M}_{N_f})$ leads to the following theorem.

2.3.6 Theorem. *Let $\mathcal{M}_{N_f}$ be the moduli space of charge k vortex configurations in a $U(1)$ gauge theory with N_f fundamental flavours on a closed Riemannian surface Σ of genus g and dimensionless 'area' $\mathcal{A} = \frac{e^2 \tau vol(\Sigma)}{4\pi} \in \mathbb{Z}$, supposing that $k > 2g-2$. Let $\mathcal{L} \to \mathcal{M}_{N_f}$ be a quantum line bundle for the L^2 vortex Kähler form on $\mathcal{M}_{N_f}$. Then*

$$\chi(\mathcal{L}^\lambda) = \sum_{j=0}^{g} \lambda^j N_f^{g-j} \binom{g}{j} \binom{\lambda(\mathcal{A}-k) + \delta - g}{\delta - j},$$

where $\delta = N_f(k-g+1) + g - 1$ is the dimension of the moduli space.

2.3.7 Remark. While our proof only literally holds for k sufficiently large, the argument should go through in a virtual sense in the general case.

2.3.8 Remark. Again, one can express the Euler characteristic as a single residue. Indeed, one has

$$\chi(\mathcal{L}^\lambda) = \mathrm{Res}_{x=0}(N_f e^{-x} + \lambda(1-e^{-x}))^g \frac{e^{\lambda d x}}{(1-e^{-x})^{\delta+1}} \mathrm{d}x,$$

which simplifies in the case that $\lambda = N_f$.

As in the local case, there are two particularly interesting corollaries of this result. Setting $\lambda = N_f$, one finds the visually simple result

$$\begin{aligned}\chi(\mathcal{L}^{N_f}) &= N_f^g \sum_{j=0}^{g} \binom{g}{j} \binom{N_f(\mathcal{A}-k) + \delta - g}{\delta - j} \\ &= N_f^g \binom{N_f(\mathcal{A}-k) + \delta}{\delta}.\end{aligned}$$

We interpret this result in section 2.8.

On the other hand, identifying the coefficient of the highest power of λ in $\chi(\mathcal{L}^\lambda)$ gives us the volume of the vortex moduli space, up to factors of 2π. We read this off to be

$$\sum_{j=0}^{\min(\delta,g)} N_f^{g-j} \binom{g}{j} \frac{(\mathcal{A}-k)^{\delta-j}}{(\delta-j)!},$$

in agreement with the result of [MOS11; OS19].

2.3.4 Abelian vortices in a harmonic trap

There is a special class of deformations of the theory that 'preserve critical coupling'. These are deformations induced from Hamiltonian actions of groups on the configuration space of the theory preserving the vortex equations. Such deformations induce a potential on the moduli space for the Hamiltonian mechanics controlling the low-temperature behaviour of the theory. The potential is such that the Hamiltonian flow generates the symmetry.

Here we consider an example in the local Abelian case, that of a harmonic trap. The symmetry generating this is a rotational symmetry of space, so it requires that Σ has a rotational symmetry. We will take Σ to be the round sphere. This has previously been considered in [Rom05]. This is a version of the Ω-deformation, originally considered in four dimensional flat space in [Nek03]. In the case that Σ is the two-dimensional plane, the analogous story was considered in [TT15b] and plays an important role in the dual approach to quantum Hall physics on the plane [Ton04a] (as, in that case, there is no Bradlow bound and a different mechanism is necessary to induce the creation of a region of Coulomb phase).

Suppose that $\Sigma = S^2$ with a round metric and consider $U(1)$ gauge theory with one fundamental flavour. The moduli space of k-vortex solutions is then the complex projective space $\mathcal{M} = \mathbb{C}P^k$. A useful coordinate system for the moduli spaces can be given as follows. Let z be a complex coordinate on $\Sigma \cong \mathbb{C}P^1$. A k-vortex configuration is uniquely specified by k points $(z_1, \cdots, z_k)$ on Σ and so we may parameterise the moduli space (away from configurations with vortices at $z = \infty$) with the polynomials

$$\prod_{i=1}^{k}(z - z_i).$$

Expanding this polynomial gives an alternative parameterisation in terms of (an affine patch of) homogeneous coordinates $[w_0 : w_1 : \cdots : w_k]$ as

$$\prod_{i=1}^{k}(z - z_i) = w_0 z^k + w_1 z^{k-1} + \cdots + w_k.$$

In this patch, $w_0 = 1$, $w_1 = \sum_i z_i$, and so on.

The theory on the round sphere S^2 has a global $U(1)$ symmetry given by rotating the sphere around a fixed axis. This induces an action of $U(1)$ on the space $\mathcal{C}$ of vortex configurations in the obvious way, rotating the configuration. This action preserves the vortex equations and so descends to the moduli space of vortices.

We may align our complex coordinate z on Σ in such a way that $e^{\mathrm{i}\theta} \in U(1)$ acts via

$$z \mapsto e^{\mathrm{i}\theta} z,$$

so that $z = 0$ and $z = \infty$ are the fixed points of the action. This action induces the action

$$z_j \mapsto e^{\mathrm{i}\theta} z_j$$

for $j = 1, \cdots, k$ on the moduli space coordinates, which in turn implies that

$$w_j \mapsto e^{\mathrm{i}\theta j} w_j \tag{2.17}$$

for $j = 1, \cdots, k$.

We can now carry out an equivariant geometric quantisation of the moduli space. Again, we will consider the simpler question of computing the equivariant index. See [Pes+17, Section 2] for an introduction to these ideas, which we summarise here.

The quantum line bundle is $\mathcal{O}(d)^\lambda \to \mathbb{C}P^k$, which lifts to an equivariant line bundle. We are interested in computing the equivariant Euler characteristic

$$\chi_{U(1)}(\mathcal{O}(d)^\lambda; q) = \sum_i (-1)^i \mathrm{tr}_{H^i(\mathbb{C}P^k, \mathcal{O}(d)^\lambda)}(q)$$

of this equivariant line bundle. Here, for V a representation of $U(1)$, $\mathrm{tr}_V(q)$ is the trace of $q \in U(1)$ in the representation V. If $U(1)$ acts trivially, then any element of $U(1)$ is represented as the identity and the trace recovers the dimension of the vector space, so that the equivariant Euler characteristic is the usual Euler characteristic in that case.

One way to compute this is using the equivariant Hirzebruch–Riemann–Roch theorem, which realises the equivariant Euler characteristic as an integral of an element of the equivariant cohomology of the moduli space. One has

$$\chi_{U(1)}(\mathcal{O}(\lambda d); q) = \int_{\mathbb{C}P^k} \left(\mathrm{ch}_{U(1)}(\mathcal{O}(\lambda d))\mathrm{td}_{\mathrm{U(1)}}(\mathbb{C}P^k)\right)(q),$$

where $\mathrm{ch}_{U(1)}(\mathcal{O}(\lambda d))$ is the equivariant Chern character of the bundle $\mathcal{O}(\lambda d)$ and $\mathrm{td}_{U(1)}(\mathbb{C}P^k)$ is the equivariant Todd class of $T_{\mathbb{C}P^k}$.

As recalled in [Pes+17; Sza96], this integral can be reduced to a sum of contributions from the fixed points of the group action. The outcome of this (when the fixed point set is discrete, which it is for us) is the Lefshetz formula

$$\chi_{U(1)}(\mathcal{O}(\lambda d); q) = \sum_{p \in F} \frac{\mathrm{tr}_{\mathcal{O}(\lambda d)_p}(q)}{\det_{T_p^{1,0}\mathbb{C}P^k}(1 - q^{-1})}.$$

We can apply this to the group action (2.17). This action has $k+1$ fixed points $(p_0, \cdots, p_k)$. Physically, the fixed point p_i corresponds to i vortices sitting at $z = 0$ and $k - i$ vortices sitting at $z = \infty$. The character of the $U(1)$ representation $\mathcal{O}(\lambda d)_{p_i}$ is given by

$$\mathrm{tr}_{\mathcal{O}(\lambda d)_{p_i}}(q) = q^{\lambda d i}$$

and the determinant is

$$\det_{(T_{\mathbb{C}P^k})_{p_i}}(1 - q^{-1}) = \prod_{j \neq i}(1 - q^{j-i}).$$

The index is then

$$\chi_{U(1)}(\mathcal{O}(\lambda d); q) = \sum_{i=0}^{k} \frac{q^{i\lambda d}}{\prod_{j\neq i}(1 - q^{j-i})}$$

(see also [Pes+17, section 2.9] for a very similar computation). Rearranging this sum gives

$$\begin{aligned}\chi_{U(1)}(\mathcal{O}(\lambda d); q) &= \prod_{i=0}^{k} \frac{(1 - q^{i+\lambda d})}{(1 - q^i)} \\ &=: \binom{\lambda d + k}{k}_q. \end{aligned} \tag{2.18}$$

This is the *q-binomial coefficient*, sometimes known as the *Gaussian binomial coefficient.* It has the property that, for any n and k,

$$\lim_{q\to 1} \binom{n}{k}_q = \binom{n}{k}$$

so that we recover our previous result Theorem 2.3.3 in the genus zero case.

Substituting in $d = \mathcal{A} - k$, we have at $\lambda = 1$,

$$\chi_{U(1)}(\mathcal{O}(d); q) = \binom{\mathcal{A}}{k}_q .$$

The Ω-deformation on flat space regularises infrared divergences. In the limit that the area of the sphere diverges, corresponding to $\mathcal{A} \to \infty$, we have (for $\lambda > 0$)

$$\lim_{\mathcal{A}\to\infty} \binom{\lambda(\mathcal{A}-k)+k}{k}_q = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}.$$

In this limit, the only contribution to the index comes from the configuration with all vortices sitting at the minimum of the potential. This result may therefore be thought of as a flat space result, with all vortices sitting at the origin. In this special case, we recover a special case of [Yos11, Eq. 5.16], which gives the 'K-theoretic' (that is, $(2+1)$-dimensional) vortex partition function on the Ω-deformed plane.

2.4 QUANTISATION OF NONABELIAN VORTEX MODULI

2.4.1 Coulomb branch localisation of the Euler characteristic

After localising with respect to time translations and integrating out the time component of the gauge field, the (super) partition function of our theory is

$$\chi(\mathcal{L}^\lambda) = \int_{\mathcal{M}} \mathrm{td}(\mathcal{M})\mathrm{ch}(\mathcal{L}^\lambda), \tag{2.19}$$

where $\mathcal{L} \to \mathcal{M}$ is a quantum line bundle on the moduli space of vortices. When the gauge group is nonAbelian, we do not have a generators-and-relations description of the cohomology ring of the moduli space $\mathcal{M}$, so it is hard to compute this integral directly. Instead, we will exploit the Jeffrey–Kirwan–Witten method of equivariant localisation (or 'Coulomb branch localisation') to compute this integral. We do not give a complete account of this technique here, instead giving a fairly high-level exposition of the computation in the context relevant to us. We refer the reader to the original works [JK95; JK97; Wit92] and to [MOS11; OS19], which give a clear exposition of the technique in the context of computing the volumes of vortex moduli spaces. Our computation of $\chi(\mathcal{L}^\lambda)$ generalises the results of these latter works.

The moduli space $\mathcal{M}$ is the formal symplectic quotient of the space $\mathcal{C}_0 = \{(A_\Sigma, \phi) \mid \bar\partial_A \phi = 0\} \subset \mathcal{C} = \{(A_\Sigma, \phi)\}$ by the group $\mathcal{G}$ of gauge transformations. The lift of the tangent bundle of $\mathcal{M}$ to $\mathcal{C}_0$ is the equivariant virtual bundle

$$\mathrm{ad}_{\mathcal{G}} \to T\mathcal{C}_0 \to \mathrm{ad}_{\mathcal{G}}^\vee$$

where $\mathrm{ad}_{\mathcal{G}}$ is the equivariant bundle $\mathcal{C}_0 \times \mathrm{Lie}(\mathcal{G}) \to \mathcal{C}_0$, carrying the adjoint action on the fibres. Here the first arrow is the derivative of the group action and the second arrow is the derivative of the moment map. The equivariant Todd class of this virtual bundle is

$$\mathrm{td}_{\mathcal{G}}(\mathcal{C}_0)\mathrm{td}_{\mathcal{G}}^{-1}(\mathrm{ad}_{\mathcal{G}} \oplus \mathrm{ad}_{\mathcal{G}}^{\vee}).$$

In general, we must regard $\mathcal{C}_0$ itself as a derived space, as the space of solutions to $\bar{\partial}_A \phi = 0$ does not have constant dimension as A varies, so that $T\mathcal{C}_0$ is itself a graded bundle.

The lift of the quantum bundle $\mathcal{L}$ to $\mathcal{C}_0$ is an equivariant line bundle $\hat{\mathcal{L}}^{\lambda}$ with equivariant first Chern class equal to

$$c_1^{\mathcal{G}}(\hat{\mathcal{L}}^{\lambda}) = \frac{\lambda}{2\pi}[\omega_{\mathcal{C}_0}^{\mathcal{G}}] \in H_{\mathcal{G}}^2(\mathcal{C}_0),$$

where $\omega_{\mathcal{C}_0}^{\mathcal{G}} = \omega_{\mathcal{C}_0} - \nu$ is the equivariant symplectic form (see [AB84]).

The super partition function is then the integral

$$\int_{\mathcal{C}_0 \times \mathrm{Lie}(\mathcal{G})} \left(\mathrm{td}_{\mathcal{G}}(\mathcal{C}_0)\mathrm{td}_{\mathcal{G}}^{-1}(\mathrm{ad}_{\mathcal{G}} \oplus \mathrm{ad}_{\mathcal{G}}^{\vee})\mathrm{ch}_{\mathcal{G}}(\hat{\mathcal{L}}^{\lambda})\right)(\Phi)\, \mathcal{D}\Phi.$$

Here Φ is valued in the Lie algebra $\mathrm{Lie}(\mathcal{G})$ and $\mathcal{D}\Phi$ is a measure on this space. Physically, this could be derived by computing the (super) partition function of the three-dimensional theory after localising with respect to time translations. The field Φ is then the constant mode of the time-component of the gauge field.

We will use the notion of equivariant integration to localise this to an integral over the Lie algebra of the maximal torus of the group G. This integral can then be expressed as a residue.

To do this, we note that the space $\mathcal{C}_0 \times \mathrm{Lie}(\mathcal{G})$ carries an infinitesimal $U(1)$ action, generated by the variations

$$\delta((A_\Sigma, \phi), \Phi) = ((\mathrm{d}_{A_\Sigma}\Phi, \Phi(\phi)), 0).$$

Our integrand is invariant under this action and we may localise with respect to it. The space of fixed points decomposes into a number of branches, including the Higgs branch, where $\Phi = 0$, and the Coulomb branch, where $\phi = 0$. The space of fixed points carries an action of the torus $T \cong U(1)^{N_c}$ of global Abelian gauge transformations. The fixed point locus of this T action is the Coulomb branch, where $\phi = 0$ and $\mathrm{d}_{A_\Sigma}\Phi = 0$.

A generic point on the Coulomb branch defines a holomorphic vector bundle of the form

$$E = \bigoplus_a L_a \to \Sigma$$

where L_a has degree k_a and $\sum_a k_a = k$, the total vortex number. Deformations of these T-invariant solutions within the space of configurations are given by pairs consisting of a holomorphic section ψ of E^{N_f} and a deformation of the holomorphic structure of E, which is an element of

$$H^1(E \otimes E^{\vee}) = H^1\left(\mathcal{O}^{N_c} \oplus \bigoplus_{a \neq b} L_a \otimes L_b^{-1}\right).$$

Such a deformation remains T-invariant if $\psi = 0$ and if the variation of holomorphic structure lies in $H^1(\mathcal{O}^{N_c})$.

Recall from above that the equivariant virtual bundle we need to consider is

$$\mathrm{ad}_{\mathcal{G}} \to T\mathcal{C}_0 \to \mathrm{ad}_{\mathcal{G}}^{\vee}.$$

Over a connected component of the T-invariant locus, the moment map is constant (as $\phi = 0$ and by the quantisation of the Abelian flux). The image of the second map is therefore zero. We are left with a T-equivariant bundle whose fibre at the bundle $E \in F$ is

$$V_F = \left(H^0(E \otimes E^{\vee}) \to T_E F \oplus (\nu_F)_E \to 0\right)$$

where the bundle on the left is the bundle of infinitesimal automorphisms of a point E on F (that is, a residual gauge transformation) and ν_F is the equivariant normal bundle to F in $\mathcal{C}_0$, which takes the form

$$\nu_F = \left(0 \to H^1(\bigoplus_{a \neq b} L_a \otimes L_b^{-1}) \oplus H^0(E^{N_f}) \to H^1(E^{N_f})\right)$$

where we recall that we are working with the derived locus of the space of solutions to $\bar{\partial}_A \phi = 0$.

The space F is torus of complex dimension $N_c g$, with trivial tangent bundle. Putting this together, the class of V_F in equivariant K-theory is

$$[V_F] = [TF] + \sum_a [H^\bullet(L_a^{N_f})] - \sum_{a \neq b} [H^\bullet(L_a \otimes L_b^{-1})].$$

where by $H^\bullet$ we mean the graded space $H^0 - H^1$.

We now compute the Todd class of each of these summands divided by the Euler class, include the contribution of the Chern character of the quantum line bundle restricted to the component, and sum over the components of the fixed point locus. Let $x_a, a = 1, \cdots, N_c$ be standard coordinates on the Lie algebra of T. From section 2.3, we know that the first and second summands integrated over F give a contribution of

$$\prod_a \left(N_f \frac{e^{-x_a}}{1 - e^{-x_a}} + \lambda\right)^g \frac{1}{(1 - e^{-x_a})^{N_f(k_a - g + 1)}}$$

to the residue. The second summand gives a contribution of

$$\prod_{a \neq b} \left(1 - e^{-(x_a - x_b)}\right)^{k_a - k_b - g + 1}.$$

The differences $x_a - x_b$ are the roots of $\mathfrak{u}(N_c)$.

The restriction of the equivariant Chern character of the quantum bundle to the fixed point locus is

$$\prod_a e^{\lambda d_a x_a}$$

where $d_a = \mathcal{A} - k_a$ is the action $-\int_\Sigma \left(F(A)_a - e^2(\tau - \mu(\phi))_a \omega_\Sigma\right)$ when $\phi = 0$.

Putting these pieces together and summing over the partitions of k, which correspond to the fixed point loci of the T-action, we come to one of our main results:

$$\begin{aligned}\frac{(-1)^\sigma}{N_c!}\chi(\mathcal{L}^\lambda) = \sum_{\sum k_a = k} \mathrm{Res}_{\boldsymbol{x}=0}\Bigg[\Bigg(\prod_{a<b} e^{x_a}e^{x_b}(e^{-x_a}-e^{-x_b})^2\Bigg)^{1-g} \\ \times \prod_a (N_f e^{-x_a} + \lambda(1-e^{-x_a}))^g \frac{e^{\lambda d_a x_a}}{(1-e^{-x_a})^{\delta_a+1}}\mathrm{d}x_a\Bigg]\end{aligned} \tag{2.20}$$

where $d_a = \mathcal{A} - k_a$ and $\delta_a = N_f(k_a - g + 1) + g - 1$, the sum is over partitions of k of length N_c, and where $\sigma = (N+1)(k+1)$. We comment on precisely how we mean to take this residue below.

Introducing $y_a = 1 - e^{-x_a}$, the residue (2.20) can alternatively be written as

$$\begin{aligned}\frac{(-1)^\sigma}{N_c!}\sum_{\sum k_a = k} \mathrm{Res}_{\boldsymbol{y}=0}\Bigg(\prod_{a<b}(y_a-y_b)^{2-2g} \\ \times \prod_a (N_f(1-y_a)+\lambda y_a)^g \frac{(1-y_a)^{-\lambda d_a -(1-g)(N_c-1)-1}}{y_a^{\delta_a+1}}\mathrm{d}y_a\Bigg).\end{aligned} \tag{2.21}$$

This is often a more convenient form.

The general structure of the quantity in the residue in (2.20) is

$$\underbrace{\prod_{a>b} V(x_a,x_b)^{\chi(\Sigma)}}_{\substack{\text{integration over off-diagonal}\\ \text{components of gauge group}}} \underbrace{\sum_{\sum k_a = k}\prod_a \underbrace{H(x_a)^g}_{\substack{\text{contribution from}\\ \text{cycles in Jacobian}}} Z(x_a,k_a)\mathrm{d}x_a}_{\text{Abelian contribution}},$$

where $V(x_a,x_b) = (e^{-x_a} - e^{-x_b})$ comes from the off-diagonal part of the gauge group, $H(x_a) = (N_f e^{-x_a} + \lambda(1-e^{-x_a})$ comes from integrating over the cycles in the Jacobian (as we saw in section 2.3), and $Z(x_a,k_a) = e^{(\lambda d_a + (1-g)(N_c-1)+1)x_a}(1-e^{-x_a})^{-(\delta_a+1)}$. This is of the same structure as [MOS11, Eq. 4.21], which gives the analogous residue formula for the volume of the vortex moduli space (in our language, the volume is $\int_{\mathcal{M}} \mathrm{ch}(\mathcal{L})$), although the precise details differ.

2.4.2 The Vandermonde determinant

The evaluation of the residues (2.21) requires us to understand the meaning of the Vandermonde-type contribution

$$\prod_{a<b}(y_a-y_b)^{2-2g}. \tag{2.22}$$

The meaning is clear for positive powers (that is, for $g = 0$), but needs thought otherwise, as the residue requires us to understand how it behaves when all the $y_a \to 0$.

We argue that the correct series expansion of

$$(y_a - y_b)^{-2n}$$

for $n \in \mathbb{Z}_{\geq 0}$ for our purposes is the symmetric one:

$$\begin{aligned}(y_a - y_b)^{-2n} &= (-1)^{-n}(y_a - y_b)^{-n}(y_b - y_a)^{-n} \\ &= (-y_a y_b)^{-n}\left(1 - \frac{y_a}{y_b}\right)^{-n}\left(1 - \frac{y_b}{y_a}\right)^{-n} \\ &= (-y_a y_b)^{-n}\left(2 - \left(\frac{y_a}{y_b} + \frac{y_b}{y_a}\right)\right)^{-n} \\ &= (-2y_a y_b)^{-n}\sum_{i=0}^{\infty}(-1)^i\binom{-n}{i}\left(\frac{1}{2}\left(\frac{y_a}{y_b} + \frac{y_b}{y_a}\right)\right)^i.\end{aligned}$$

If one views this as a power series in the real numbers, it never converges, which is a cost of insisting on symmetry.

In general, we say that

$$\begin{aligned}(y_a - y_b)^{2-2g} &= (y_a - y_b)^2(y_a - y_b)^{-2g} \\ &= (y_a - y_b)^2(-2y_a y_b)^{-g}\sum_{i=0}^{\infty}(-1)^i\binom{-g}{i}\left(\frac{1}{2}\left(\frac{y_a}{y_b} + \frac{y_b}{y_a}\right)\right)^i.\end{aligned} \tag{2.23}$$

2.4.3 The general result

We can evaluate the general residue (2.21) using the expansion of the Vandermonde determinant given above:

$$\prod_{a<b}(y_a - y_b)^{2-2g} = 2^{(N_c-1)g}\sum_{\sum l_a = N_c(N_c-1)}\ \sum_{c_1,\cdots,c_{N_c}=-\infty}^{\infty} a_{l_1,\cdots,l_{N_c}}A_{c_1,\cdots,c_{N_c}}\prod_a y_a^{l_a+c_a-g(N_c-1)}$$

where $a_{l_1,\cdots,l_{N_c}}$ is the coefficient of $\prod_a y_a^{l_a}$ in $\prod_{a<b} y_a^{l_a}$ and $A_{c_1,\cdots,c_{N_c}}$ is the coefficient of $\prod_a y_a^{c_a}$ in $\prod_{a<b}\left(1 - 1/2(y_a y_b^{-1} + y_b y_a^{-1})\right)^{-g}$.

Evaluating the residue, we find that

$$\begin{aligned}\chi(\mathcal{L}^\lambda) = \frac{(-1)^\sigma 2^{(N_c-1)g}}{N_c!}&\sum_{\sum k_a = k}\ \sum_{\sum l_a = N_c(N_c-1)}\ \sum_{c_1,\cdots,c_{N_c}} a_{l_1,\cdots,l_{N_c}}A_{c_1,\cdots,c_{N_c}} \\ \times\prod_a&\left(\sum_{j=0}^{g}\binom{g}{j}\lambda^j N_f^{g-j}\binom{\lambda(\mathcal{A}-k_a) + (N_c-1) + N_f k_a + (N_f-1)(1-g) - l_a - c_a - g}{N_f k_a + (N_f-1)(1-g) - l_a - c_a + g(N_c-1) - j}\right),\end{aligned} \tag{2.24}$$

which is the general result.

As written, this is not too meaningful (or, at least, not too useful). In what follows, we will attempt to simplify (2.24) in special cases.

2.5 SIMPLIFICATIONS OF THE INDEX FOR LOCAL VORTICES

2.5.1 *NonAbelian local vortices on the sphere*

Warm-up: The case of $N = 2$

In this section, we will simplify the form of the index (2.24) in the case that $N_c = N_f =: N$. We will find ourselves able to simplify the index dramatically when we take $\lambda = N$.

To warm up, let us consider the case of $N = 2$ and $g = 0$. Results in the general case will follow from similar methods, but it is instructive to begin in this restricted situation as everything is more-or-less directly computationally tractable.

The relevant residue is

$$\chi(\mathcal{L}^{\lambda}_{k;2,2}) = \frac{(-1)^{k+1}}{2} \sum_{k_1+k_2=k} \mathrm{Res}_{x_1=x_2=0}(e^{-x_1} - e^{-x_2})^2 \frac{e^{(\lambda d_1+1)x_1} e^{(\lambda d_2+1)x_2}}{(1-e^{-x_1})^{\delta_1+1}(1-e^{-x_2})^{\delta_2+1}} \mathrm{d}x_1 \mathrm{d}x_2,$$

where $d_a = \mathcal{A} - k_a$ and $\delta_a = 2k_a + 1$.

This can be evaluated: it is

$$\frac{(-1)^{k+1}}{2} \sum_{k_1+k_2=k} \left[\binom{\lambda d_1 + \delta_1 - 1}{\delta_1 - 2}\binom{\lambda d_2 + \delta_2 + 1}{\delta_2} \right.$$
$$\left. - 2\binom{\lambda d_1 + \delta_1}{\delta_1 - 1}\binom{\lambda d_2 + \delta_2}{\delta_2 - 1} + \binom{\lambda d_1 + \delta_1 + 1}{\delta_1}\binom{\lambda d_2 + \delta_2 - 1}{\delta_2 - 2} \right].$$

In general, this appears to be rather complicated.

There is a drastic simplification if $\lambda = 2$, so that $\lambda = N$, as the k_a dependence of the binomial coefficients simplifies. In this case, we have

$$\chi(\mathcal{L}^{2}_{k;2,2}) = (-1)^{k+1} \sum_{i=0}^{k} \left[\binom{2\mathcal{A}}{2i-1}\binom{2\mathcal{A}+2}{2(k-i)+1} - \binom{2\mathcal{A}+1}{2i}\binom{2\mathcal{A}+1}{2(k-i)} \right].$$

This can be (rather strikingly) simplified further to

$$\chi(\mathcal{L}^{2}_{k;2,2}) \stackrel{!}{=} \binom{2\mathcal{A}}{k}$$

as we show with the following lemmas.

2.5.1 Lemma. *The identity*

$$\sum_{i=0}^{k} \left[\binom{n}{2i-1}\binom{n+2}{2(k-i)+1} - \binom{n+1}{2i}\binom{n+1}{2(k-i)} \right] = \sum_{j=0}^{2k} (-1)^{j+1} \binom{n}{j}\binom{n}{2k-j}$$

holds.

Proof. It is possible to derive this result by a liberal application of Pascal's identity for binomial coefficients and a certain rearrangement of the sum, but a much neater proof comes from comparing generating functions on both sides. The sum on the left hand side is the coefficient of x^{2k} in

$$\frac{1}{4}\left[-\left((1+x)^{n+1} + (1-x)^{n+1}\right)^2 + \left(x^{-1}(1-x)^n - x^{-1}(1+x)^n\right)\left(x(1-x)^{n+2} - x(1-x)^{n+2}\right)\right],$$

where we have used that $\frac{1}{2}\left((1+x)^p+(1-x)^p\right)$ isolates the even-power terms of $(1+x)^p$, as well as the analogous result for the odd-power terms.

Expanding this out gives

$$-\frac{1}{4}((1-x)+(1+x))^2(1-x)^n(1+x)^n = -(1-x)^n(1+x)^n.$$

The coefficient of x^{2k} in this final polynomial is

$$-\sum_{j=0}^{2k}(-1)^j\binom{n}{j}\binom{n}{2k-j}$$

which gives the claimed result. □

2.5.2 Lemma. *The identity*

$$\sum_{j=0}^{2k}(-1)^j\binom{n}{j}\binom{n}{2k-j} = (-1)^k\binom{n}{k}$$

holds.

Proof. Comparison of the coefficients of x^{2k} on both sides of the identity

$$(1-x)^n(1+x)^n = (1-x^2)^n$$

gives the result. □

These lemmas lead us to the result that

$$\chi(\mathcal{L}^2_{k;2,2}) = \binom{2\mathcal{A}}{k}.$$

General N on the sphere

When $g=0$ but N is allowed to be general, recall that the relevant residue can be evaluated to give

$$\chi(\mathcal{L}^\lambda_{k;N,N}) = \frac{(-1)^\sigma}{N!}\sum_{\sum k_a=k}\sum_{\sum l_a=N(N-1)} a_{l_1\cdots l_N}\prod_a\binom{\lambda d_a+\delta_a+N-1-l_a}{\delta_a-l_a}$$

where $a_{l_1\cdots l_N}$ is the coefficient of $y_1^{l_1}\cdots y_N^{l_N}$ in the polynomial $\prod_{a<b}(y_a-y_b)^2$.

This is, again, a complicated sum. Once more, it simplifies if we take $\lambda=N$. In this case, it becomes

$$\chi(\mathcal{L}^N_{k;N,N}) = \frac{(-1)^\sigma}{N!}\sum_{\sum k_a=k}\sum_{\sum l_a=N(N-1)} a_{l_1\cdots l_N}\prod_a\binom{N\mathcal{A}+2(N-1)-l_a}{Nk_a+N-1-l_a}.$$

To understand this sum, we will consider its generating function. The basic result that underlies our computation is the following elementary lemma, which tells us how to pick out the relevant binomial coefficients in our expressions.

2.5.3 Lemma. *For any j, one has*

$$\sum_{i=0}^{\infty}\binom{n}{Ni+j}x^{Ni}=\frac{x^{-j}}{N}\sum_{a=1}^{N}r^{-aj}(1+r^a x)^n,$$

where r generates the group of N^{th} roots of unity.

Proof. On the right hand side, we have

$$\begin{aligned}\frac{x^j}{N}\sum_{a=1}^{N}r^{-aj}(1+r^a x)^n &= \frac{1}{N}\sum_{a=1}^{N}\sum_{l=0}^{\infty}\binom{n}{l}r^{a(l-j)}x^{l-j}\\ &= \frac{1}{N}\sum_{l=0}^{\infty}\binom{n}{l}x^{l-j}\sum_{a=1}^{N}r^{a(l-j)}\end{aligned}$$

Now, by virtue of the properties of roots of unity, the sum

$$\sum_{a=1}^{N}r^{a(l-j)}=r^{l-j}\frac{1-r^{N(l-j)}}{1-r^{l-j}}$$

vanishes unless $l=Ni+j$ for some i, in which case it is N. Our sum thus becomes

$$\sum_{i=0}^{N}\binom{n}{Ni+j}x^{Ni}$$

as claimed. □

Using this lemma and writing $n=N\mathcal{A}$, the Euler characteristic $\chi(\mathcal{L}^N_{k;N,N})$ is, up to the overall factor $(-1)^\sigma N!^{-1}N^{-N}$, the coefficient of x^{Nk} of the polynomial

$$P(x)=\sum_{\sum_a l_a=N(N-1)}a_{l_1\cdots l_N}\prod_{a=1}^{n}x^{-(N-1)+l_a}\left(\sum_{b=1}^{N}r^{-(1-l_a)b}(1+r^b x)^{n+2(N-1)-l_a}\right)$$

where r generates the group of N^{th} roots of unity. Because $\sum l_a=N(N-1)$, we see that $\prod_a x^{-(N-1)+l_a}=1$. We then have

$$P(x)=\sum_{\sum_a l_a=N(N-1)}a_{l_1\cdots l_N}\prod_a\left(\sum_b r^{-(1-l_a)b}(1+r^b x)^{n+2(N-1)-l_a}\right).$$

Expanding the product gives

$$\begin{aligned}P(x)&=\sum_{\sum l_a=N(N-1)}a_{l_1\cdots l_N}\sum_{b_1,\cdots,b_N=1}^{N}\prod_{a=1}^{N}r^{-(1-l_a)b_a}(1+r^{b_a}x)^{n+2(N-1)-l_a}\\ &=\sum_{b_1,\cdots,b_N=1}^{N}\sum_{\sum l_a=N(N-1)}a_{l_1\cdots l_N}\prod_{a=1}^{N}r^{-(1-l_a)b_a}(1+r^{b_a}x)^{n+2(N-1)-l_a}.\end{aligned}$$

The only non-zero contributions to the sum come from those terms where all the b_a are different. To see this, sum over the terms where $b_j = \cdots = b_N =: b$ for some $N > j \geq 1$. Write $a' = 1, \cdots, j-1$ and $\tilde{a} = j, \cdots, N$. The sum over the restricted configurations with $b_{\tilde{a}} = b$ is

$$\sum_{\sum l_a = N(N-1)} a_{l_1 \cdots l_N} \sum_{b=1}^{N} \sum_{b_{a'}=1}^{N} \prod_{\tilde{a}=j}^{N} r^{-(1-l_{\tilde{a}})b}(1+r^b x)^{n+2(N-1)-l_{\tilde{a}}} \prod_{a'=1}^{j-1} r^{-(1-l_{a'})b_{a'}}(1+r^{b_{a'}}x)^{n+2(N-1)-l_{a'}}.$$

We can now do the product over $\tilde{a}$, giving

$$\sum_{\sum l_a = N(N-1)} a_{l_1 \cdots l_N} \sum_b \left(r^{\sum_{\tilde{a}} l_{\tilde{a}} b}(1 + r^b x)^{n+2(N-1)-\sum_{\tilde{a}} l_{\tilde{a}}} \right)$$
$$\times \sum_{b_{a'}} \prod_{a'=1}^{j-1} r^{-(1-l_{a'})b_{a'}}(1 + r^{b_{a'}} x)^{n+2(N-1)-l_{a'}}.$$

Write $\tilde{L} = \sum_{\tilde{a}} l_{\tilde{a}}$. Then this becomes

$$\sum_{\tilde{L}=0}^{N(N-1)} \sum_{\sum l_{\tilde{a}} = \tilde{L}} \sum_{\sum l_{a'} = N(N-1)-\tilde{L}} a_{l_1, \cdots, l_N} \sum_b r^{\tilde{L} b}(1 + r^b x)^{n+2(N-1)-\tilde{L}}$$
$$\times \sum_{b_{a'}} \prod_{a'=1}^{j-1} r^{-(1-l_{a'})b_{a'}}(1 + r^{b_{a'}} x)^{n+2(N-1)-l_{a'}}$$

which vanishes because, for each value of $\tilde{L}$, the sum

$$\sum_{\sum_j^N l_{\tilde{a}} = \tilde{L}} a_{l_1 \cdots l_{j-1} l_j \cdots l_N} = 0,$$

as long as $N - j > 1$. This follows from the definition of the coefficient $a_{l_1 \cdots l_N}$ as the coefficient of $\prod_a y_a^{l_a}$ in $\prod_{a<b}(y_a - y_b)^2$.

Returning now to our generating function $P(x)$, restricting to all of the b_a different allows us to write

$$\begin{aligned} \frac{P(x)}{N!} &= \sum_{\sum l_a = N(N-1)} a_{l_1 \cdots l_N} \prod_a r^{-(1-l_a)a}(1 + r^a x)^{n+2(N-1)-l_a} \\ &= \pm(1 \pm x^N)^{n+2(N-1)} \prod_a (r^{-a}(1 + r^a x))^{-l_a}, \end{aligned}$$

where we have used that $\prod_a (1 + r^a x) = (1 \pm x^N)$ where the sign is negative for N even and positive for N odd. We can now use the definition of the $a_{l_1 \cdots l_N}$ to repackage the sum as follows

$$\begin{aligned} \frac{P(x)}{N!} &= \pm(1 \pm x^N)^{n+2(N-1)} \prod_{a<b} \left(\frac{r^a}{(1 + r^a x)} - \frac{r^b}{(1 + r^b x)} \right)^2 \\ &= \pm(1 \pm x^N)^n \prod_{a<b} (r_a - r_b)^2 \\ &= \pm N^N (1 \pm x^N)^n. \end{aligned}$$

By taking the coefficient of x^{Nk} and reinserting the relevant factors, we see that the Euler characteristic is

$$\chi(\mathcal{L}^N_{k;N,N}) = \binom{N\mathcal{A}}{k}$$

when $g = 0$.

2.5.2 NonAbelian local vortices on general compact surfaces

Warming up on general surfaces: The case of $N = 2$

To see through the eventual forest of sums and products that arises in general, when g is general it is again useful to begin in the case of $N = 2$, which is the first nontrivial case.

The residue that we need to compute is

$$\frac{(-1)^\sigma 2^{2g}}{2!} \sum_{k_1+k_2=k} \mathrm{Res}_{y_1=y_2=0}\left[(y_1-y_2)^{2-2g}\frac{(1-y_1)^{-2d_1-(1-g)-1}(1-y_2)^{-2d_2-(1-g)-1}}{y_1^{\delta_1+1}y_2^{\delta_2+1}}\mathrm{d}y_1\mathrm{d}y_2\right].$$

Our prescription for evaluating this residue is based on the series expansion (2.23) for the Vandermonde-type contribution. Plugging this back into the residue and interchanging the sum and the integral gives

$$\begin{aligned}(-1)^\sigma 2^{g-1} &\sum_{k_1+k_2=k}\sum_{i=0}^{\infty}\binom{-g}{i}\left(-\frac{1}{2}\right)^i\sum_{j=0}^{i}\binom{i}{j}\\ &\times \mathrm{Res}\left[\left(\frac{y_1}{y_2}-2+\frac{y_2}{y_1}\right)\frac{(1-y_1)^{-2d_1-(1-g)-1}(1-y_2)^{-2d_2-(1-g)-1}}{y_1^{\delta_1+(g-1)-2j+i+1}y_2^{\delta_2+(g-1)-i+2j+1}}\mathrm{d}y_1\mathrm{d}y_2\right],\end{aligned}$$

where $d_a = \mathcal{A} - k_a$ and $\delta_a = N_f k_a + (N_f - 1)(1 - g)$.

The residues can be evaluated and the terms grouped, giving

$$\begin{aligned}(-1)^\sigma 2^{g} &\sum_{k_1+k_2=k}\sum_{i=0}^{\infty}\binom{-g}{i}\left(-\frac{1}{2}\right)^i\sum_{j=0}^{i}\binom{i}{j}\\ &\times\left[\binom{2\mathcal{A}+(1-g)-1-2j+i}{2k_1-1-2j+i}\binom{2\mathcal{A}+(1-g)+1+2j-i}{2k_2+1+2j-i}-\right.\\ &\left.\binom{2\mathcal{A}+(1-g)-2j+i}{2k_1-2j+i}\binom{2\mathcal{A}+(1-g)+2j-i}{2k_2+2j-i}\right].\end{aligned}$$

Writing $S(g, k, 2\mathcal{A})$ for this quantity, we will show that the identity

$$S(g,k,n) = \binom{n}{k}$$

holds.

Introducing the space-saving notation $n = 2\mathcal{A}$ and $l = 2j - i$, the quantity $S(g,k,n)$ is the coefficient of x^{2k} of the polynomial

$$
\begin{aligned}
(-1)^{\sigma} 2^{g-2} \sum_{i=0}^{\infty} \binom{-g}{i} \left(-\frac{1}{2}\right)^{i} \sum_{j=0}^{i} \binom{i}{j} \\
\Big[\Big((1+x)^{n+(1-g)-1-l} + (-1)^{-g-l}(1-x)^{n+(1-g)-1-l} \Big) \\
\times \Big((1+x)^{n+(1-g)+1+l} + (-1)^{-g-l}(1-x)^{n+(1-g)+1+l} \Big) \\
- \Big((1+x)^{n+(1-g)-l} + (-1)^{1-g-l}(1-x)^{n+(1-g)-l} \Big) \\
\times \Big((1+x)^{n+(1-g)+l} + (-1)^{1-g-l}(1-x)^{n+(1-g)+l} \Big) \Big],
\end{aligned}
$$

which simplifies to give

$$
\begin{aligned}
(-1)^{\sigma-g} 2^{g-1} \sum_{i=0}^{\infty} \binom{-g}{i} \left(\frac{1}{2}\right)^{i} \sum_{j=0}^{i} \binom{i}{j} (1-x^2)^{n+(1-g)-1-l} \left((1+x)^{2l+1} + (1-x)^{2l+1} \right) \\
&= (-1)^{\sigma-g} 2^{g} (1-x^2)^{n-g} \sum_{i=0}^{\infty} \binom{-g}{i} (-1)^i \left(\frac{x^2+1}{x^2-1}\right)^i \\
&= (-1)^{\sigma-g} 2^{g} (1-x^2)^{n-g} \left(\frac{1}{1-\frac{x^2+1}{x^2-1}} \right)^g \\
&= (-1)^k (1-x^2)^n .
\end{aligned}
$$

The coefficient of x^{2k} in $(1-x^2)^n$ is $(-1)^k \binom{n}{k}$, so we have shown that

$$
S(g,k,n) = \binom{n}{k}
$$

as claimed.

Modulo technical questions regarding the mathematical validity of the residue prescription we give, we have demonstrated that

$$
\chi(\mathcal{L}^2_{g;k;2,2}) = \binom{2\mathcal{A}}{k}
$$

for all g, k and $\mathcal{A}$.

General N on general surfaces

We now turn to the most general case of local vortices in $U(N)$ gauge theory with N flavours on arbitrary compact surfaces.

Our results for general N_c and $g = 0$ and for $N_c = 2$ and general g lead us to suspect that

$$
\chi(\mathcal{L}^N_{g;k;N,N}) \stackrel{?}{=} \binom{N\mathcal{A}}{k}
$$

for all N and g.

We again simplify by setting $\lambda = N_f$. The residue we need to compute is then

$$\frac{(-1)^\sigma N_f^{N_c g}}{N_c!} \sum_{\sum k_a = k} \mathrm{Res}_{\boldsymbol{y}=0} \left[\prod_{a<b} (y_a - y_b)^{2-2g} \prod_a \frac{(1-y_a)^{-N_f d_a - (N_c-1)(1-g)-1}}{y_a^{\delta_a+1}} \mathrm{d}y_a \right],$$

where, as always, $d_a = \mathcal{A} - k_a$, $\delta_a = N_f k_a + (N_f - 1)(1-g)$, and $a, b = 1, \cdots N_c$.

Our prescription for computing this residue is based on the expansion of (2.23), which gives

$$\begin{aligned}
\prod_{a<b}(y_a - y_b)^{2-2g} &= \prod_{a<b}(y_a - y_b)^2 \prod_{a<b}(y_a - y_b)^{-2g} \\
&= 2^{(N_c-1)g} \prod_{a<b}(y_a - y_b)^2 \prod_a y_a^{-g(N_c-1)} \prod_{a<b} \left(1 - \frac{1}{2}\left(\frac{y_a}{y_b} + \frac{y_b}{y_a}\right)\right)^{-g} \\
&= 2^{(N_c-1)g} \prod_{a<b}(y_a - y_b)^2 \prod_a y_a^{-g(N_c-1)} \prod_{a<b} \sum_{i_{a,b}=0}^{\infty} \binom{-g}{i_{a,b}} \left(-\frac{1}{2}\right)^{i_{a,b}} \left(\frac{y_a}{y_b} + \frac{y_b}{y_a}\right)^{i_{a,b}} \\
&= 2^{(N_c-1)g} \prod_{a<b}(y_a - y_b)^2 \prod_a y_a^{-g(N_c-1)} \prod_{a<b} \sum_{i_{a,b}=0}^{\infty} \binom{-g}{i_{a,b}} \left(-\frac{1}{2}\right)^{i_{a,b}} \\
&\quad \times \left(\sum_{j_{a,b}=0}^{i_{a,b}} \binom{i_{a,b}}{j_{a,b}} y_a^{2j_{a,b} - i_{a,b}} y_b^{-2j_{a,b}+i_{a,b}} \right) \\
&= 2^{(N_c-1)g} \sum_{\sum l_a = N_c(N_c-1)} a_{l_1 \cdots l_{N_c}} \prod_a y_a^{l_a - g(N_c-1)} \prod_{a<b} \sum_{i_{a,b}=0}^{\infty} \binom{-g}{i_{a,b}} \left(-\frac{1}{2}\right)^{i_{a,b}} \\
&\quad \times \left(\sum_{j_{a,b}=0}^{i_{a,b}} \binom{i_{a,b}}{j_{a,b}} y_a^{2j_{a,b} - i_{a,b}} y_b^{-2j_{a,b}+i_{a,b}} \right) \\
&= 2^{(N_c-1)g} \sum_{\sum l_a = N_c(N_c-1)} a_{l_1, \cdots, l_{N_c}} \sum_{c_1, \cdots, c_{N_c} = -\infty}^{\infty} A_{c_1, \cdots, c_{N_c}} \prod_a y_a^{l_a + c_a - g(N_c-1)}
\end{aligned}$$

where the $a_{l_1 \cdots l_N}$ is the coefficient of $\prod_a y_a^{l_a}$ in $\prod_{a<b}(y_a - y_b)^2$ as before and $A_{c_1, \cdots, c_{N_c}}$ is the coefficient of $\prod_a y_a^{c_a}$ in $\prod_{a<b} \sum_{i_{a,b}} \binom{-g}{i_{i,b}} (-1/2(y_a y_b^{-1} + y_b y_a^{-1}))^{1/2}$.

The idea is to unwind this expansion, compute the residue, pass to the generating function, and then wind the expansion back up to obtain a neat form.

Substituting this into the residue gives, up to the overall prefactor $(-1)^\sigma 2^{(N_c-1)g} N_f^{N_c g} N_c!^{-1}$,

$$\sum_{\sum k_a = k} \sum_{\sum l_a = N_c(N_c-1)} \sum_{c_1, \cdots, c_{N_c}} a_{l_1, \cdots, l_{N_c}} A_{c_1, \cdots, c_{N_c}} \prod_a \binom{N_f d_a + (N_c - 1) + \delta_a - l_a - c_a}{\delta_a - l_a - c_a + g(N_c - 1)}$$

where, as always, $d_a = \mathcal{A} - k_a$ and $\delta_a = N_f k_a + (N_f - 1)(1-g)$. This constitutes the general result for the Euler characteristic $\chi(\mathcal{L}^{N_f}_{k;N_c,N_f})$.

We now set $N_c = N_f =: N$. In this case, the residue becomes

$$\sum_{\sum k_a = k} \sum_{\sum l_a = N(N-1)} \sum_{c_1, \cdots, c_N} a_{l_1, \cdots, l_N} A_{c_1, \cdots, c_N} \prod_a \binom{N\mathcal{A} + (N-1)(2-g) - l_a - c_a}{N k_a + (N-1) - l_a - c_a}.$$

Using the lemma 2.5.3 and writing $n = N\mathcal{A}$, we find that these numbers are the coefficients of x^{Nk} of the polynomial

$$P_g(x) = \sum_{\sum l_a = N(N-1)} \sum_{c_1,\cdots,c_N} a_{l_1,\cdots,l_N} A_{c_1,\cdots,c_N} \\ \times \prod_a x^{-(N-1)+l_a+c_a} \left(\sum_{b=1}^{N} r^{-(1-l_a-c_a)b} (1+r^b x)^{n+(N-1)(2-g)-l_a-c_a} \right)$$

where r generates the N^{th} roots of unity, as before, and we have multiplied through by N^N.

The method now is similar to the genus zero case. That $\sum l_a = N(N-1)$ gives $\prod_a x^{-(N-1)+l_a} = 1$. We then expand the product to give

$$P_g(x) = \sum_{\sum l_a = N(N-1)} \sum_{c_1,\cdots,c_N} a_{l_1,\cdots,l_N} A_{c_1,\cdots,c_N} \\ \times \sum_{b_1,\cdots,b_N=1}^{N} \prod_a x^{c_a} r^{-(1-l_a-c_a)b_a} (1+r^{b_a} x)^{n+(N-1)(2-g)-l_a-c_a}.$$

For exactly the same reasons as before the only nonzero contributions come when all the b_a are distinct, at least after regularisation. Thus

$$\begin{aligned} \frac{P_g(x)}{N!} &= \sum_{\sum l_a = N(N-1)} \sum_{c_1,\cdots,c_N} a_{l_1,\cdots,l_N} A_{c_1,\cdots,c_N} \prod_a x^{c_a} r^{-(1-l_a-c_a)a} (1+r^a x)^{n+(N-1)(2-g)-l_a-c_a} \\ &= (-1)^{N+1} (1 \pm x^N)^{n+(N-1)(2-g)} \sum_{l_a} \sum_{c_a} a_{l_1,\cdots l_N} A_{c_1,\cdots,c_N} \\ &\qquad \times \prod_a (x^{-1} r^{-a} (1+r^a x))^{-c_a} (r^{-a}(1+r^a x))^{-l_a}. \end{aligned}$$

We now wind up the series, using the definition of the coefficients $a_{l_1,\cdots,l_N}$ and $A_{c_1,\cdots,c_N}$, which leads to the polynomial

$$\frac{P_g(x)}{N!} = (-1)^{N+1} (1 \pm x^N)^{n+(N-1)(2-g)} \prod_{a<b} \left(\frac{r^a}{(1+r^a x)} - \frac{r^b}{(1+r^b x)} \right)^2 \\ \times \prod_{a<b} \left(1 - \frac{1}{2} \left(\frac{r^a(1+r^b x)}{r^b(1+r^a x)} + \frac{r^b(1+r^a x)}{r^a(1+r^b x)} \right) \right)^{-g}.$$

This is

$$\begin{aligned} \frac{2^{(N-1)g}}{N!} P_g(x) &= \pm (1 \pm x^N)^{n+(N-1)(2-2g)} \prod_{a<b} \left(\frac{r^a}{(1+r^a x)} - \frac{r^b}{(1+r^b x)} \right)^{2-2g} \\ &= \pm N^{N(1-g)} (1 \pm x^N)^n. \end{aligned}$$

Now, $P_g(x)$ was defined to so that the coefficient of x^{Nk} in

$$(-1)^\sigma N^{N(g-1)} N!^{-1} 2^{(N-1)g} P_g(x)$$

is the quantity we are after. Collecting all of the various prefactors that we have discarded and comparing coefficients, this gives the final result

$$\chi(\mathcal{L}^N_{g;k;N,N}) = \binom{N\mathcal{A}}{k}, \tag{2.25}$$

as expected.

2.5.3 A heuristic argument for the local vortex count

The computation that we have done is interesting in that it is rather involved but has, in the local case (at least), a very simple output. One simple, but heuristic, way to see that the local vortex count might be

$$\binom{N\mathcal{A}}{k}$$

follows by considering symmetry arguments based on colour-flavour locking, as used in [HT03; Ton05] to understand the internal moduli of vortices. Let us summarise the construction. Let (A^*, ϕ^*) be a solution to the local *Abelian* vortex equations. Then one can embed this Abelian solution into a solution (A, ϕ) to the local nonAbelian vortex equations as

$$A = \begin{pmatrix} A^* & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi^* & & & \\ & \sqrt{\tau} & & \\ & & \ddots & \\ & & & \sqrt{\tau} \end{pmatrix}.$$

Of course, there is nothing special about the top left entry: a local nonAbelian vortex can be given by putting a local Abelian vortex in any colour-flavour 'slot'.

One could then build a charge k nonAbelian vortex by distributing k Abelian vortices into the N 'slots'. Given that the number of charge k_i local Abelian vortices is

$$\binom{\mathcal{A}}{k_i},$$

the number of ways to distribute k Abelian vortices among the N slots is

$$\sum_{k_1+\dots+k_N=k} \binom{\mathcal{A}}{k_1} \cdots \binom{\mathcal{A}}{k_N} = \binom{N\mathcal{A}}{k}.$$

This is in agreement with the result (2.25).

Note that, while the result of the localisation computation also came as a sum over partitions of k, the form that the summands take is very different to this and there are cancellations between the contributions from the various partitions.

It is quite remarkable that objects as complicated as vortices might be amenable to such simple symmetry arguments (at least in special cases). It would be interesting to see how to make the above argument rigorous. The results of Baptista in [Bap09] show that aspects of these symmetry-based arguments can be made rigorous in some generality when applied to understanding the geometry of the vortex moduli space.

2.6 INDEX REFACTORISATION FOR SEMILOCAL VORTICES

2.6.1 Seiberg duality

When $N_f > N_c$, the infrared limit (corresponding to $e^2 \to \infty$) of the critically-coupled Yang–Mills–Higgs theory in two dimensions can be effectively described by a sigma model into the Higgs branch of the theory, which is the complex Grassmannian

$$\mathrm{Gr}(N_c, N_f).$$

This is the same as the Higgs branch for the theory with $N_f - N_c$ colours:

$$\mathrm{Gr}(N_c, N_f) \cong \mathrm{Gr}(N_f - N_c, N_f),$$

which leads one to the idea of *Seiberg duality* (originally considered in [Sei95] in the context of four-dimensional theories with $\mathcal{N} = 1$ supersymmetry), the notion that the theory with N_c colours and the theory with $N_f - N_c$ colours are equivalent in the infrared.

The BPS objects of the infrared theory are holomorphic curves in the Grassmannian. As one moves away from the infrared, these become vortices. The moduli space of vortices resolves the moduli space of holomorphic maps into the Grassmannian, which is generally a singular space. The Seiberg dual theories generally lead to different resolutions of the moduli space of holomorphic maps.

Our results allow us, in principle, to probe the action of Seiberg duality on the vortex moduli space. We make only surface level inroads here. To do this, we should get to grips with the meaning of the index (2.24) when $N_c \neq N_f$.

Our main result is the reformulation of the index in 'Seiberg dual' form, with certain sums over N_c indices, each taking values from 1 to N_f, replaced with sums over $N_f - N_c$ indices. We carry this out in detail for $N_f - N_c = 1$.

2.6.2 The index for semilocal vortices

There are two natural generalisations to the simplifying computations we have done, changing λ or changing $N_f - N_c$. Changing λ leads to rather fundamental calculational difficulties, relating to the fact that it becomes hard to write down generating functions. We make some progress in the case of $\lambda \to \infty$ in chapter 4.

On the other hand, moving away from the local case does not lead to such significant problems. The general result for the Euler characteristic $\chi(\mathcal{L}^{N_f}_{g;k;N_c,N_f})$ is, up to a prefactor,

$$\sum_{\sum k_a = k} \sum_{\sum l_a = N_c(N_c-1)} \sum_{c_a=-\infty}^{\infty} a_{l_1,\cdots,l_{N_c}} A_{c_1,\cdots,c_{N_c}} \prod_a \binom{N_f \mathcal{A} + (N_c-1) + (N_f-1)(1-g) - l_a - c_a}{N_f k_a + (N_f-1)(1-g) + g(N_c-1) - l_a - c_a},$$

where $a = 1, \cdots, N_c$.

To understand this, we proceed as before, by looking for a generating function. Indeed, this quantity is the coefficient of $x^{N_f k}$ in the polynomial

$$P_{g;N_c,N_f}(x) = \sum_{\sum l_a = N_c(N_c-1)} \sum_{c_a} a_{l_1 \cdots l_{N_c}} A_{c_1 \cdots c_{N_c}} \prod_a x^{-(N_f-1)(1-g)-g(N_c-1)+l_a+c_a}$$
$$\times \left(\sum_{b=1}^{N_f} r^{-b((N_f-1)(1-g)+g(N_c-1)-l_a-c_a)} (1+r^b x)^{n+(N_c-1)+(N_f-1)(1-g)-l_a-c_a} \right)$$

up to a factor of $N_f^{N_c}$, where $n = N_f \mathcal{A}$ and r generates the group of the ${N_f}^{\text{th}}$ roots of unity. Expanding the product gives

$$P_{g;N_c,N_f}(x) = \sum_{\sum l_a = N_c(N_c-1)} \sum_{c_a} a_{l_1 \cdots l_{N_c}} A_{c_1 \cdots c_{N_c}} x^{-N_c(N_f-N_c)(1-g)} \sum_{b_1, \cdots, b_{N_c}=1}^{N_f} \prod_a x^{c_a}$$
$$\times r^{-b_a((N_f-1)(1-g)+g(N_c-1)-l_a-c_a)} (1+r^{b_a} x)^{n+(N_c-1)+(N_f-1)(1-g)-l_a-c_a}.$$

We now do the l_a and c_a sums. This leaves us with

$$\begin{aligned} P_{g;N_c,N_f}(x) = x^{-N_c(N_f-N_c)(1-g)} & \sum_{b_1, \cdots, b_{N_c}=1}^{N_f} \prod_{a_1<a_2} \left(\frac{r^{b_{a_1}}}{1+r^{b_{a_1}} x} - \frac{r^{b_{a_2}}}{1+r^{b_{a_2}} x} \right)^2 \\ & \times \left(1 - \frac{1}{2} \left(\frac{r^{b_{a_1}}(1+r^{b_{a_2}} x)}{r^{b_{a_2}}(1+r^{b_{a_1}} x)} + \frac{r^{b_{a_2}}(1+r^{b_{a_1}} x)}{r^{b_{a_1}}(1+r^{b_{a_2}} x)} \right) \right)^{-g} \\ & \times \prod_a r^{-b_a((N_f-1)(1-g)+g(N_c-1))} (1+r^{b_a} x)^{n+(N_c-1)+(N_f-1)(1-g)} \\ = x^{-N_c(N_f-N_c)(1-g)} & 2^{-g(N_c-1)} \\ & \times \sum_{b_1, \cdots, b_{N_c}=1}^{N_f} \prod_{a_1<a_2} \left(r^{b_{a_1}} - r^{b_{a_2}} \right)^{2-2g} \\ & \times \prod_a r^{-b_a(N_f-1)(1-g)} (1+r^{b_a} x)^{n+(N_f-N_c)(1-g)}. \end{aligned}$$

As before, we should regularise by asking that all the b_a are distinct.

Simplifying this in general remains difficult. It does have an interesting duality property, which can be exploited. We illustrate this in the case of $N_f - N_c = 1$.

The case of $N_f - N_c = 1$

The b_a should all be distinct so, at the cost of a factor of $N_c!$, we may assume that

$$1 \leq b_1 < \cdots < b_{N_c} \leq N_f.$$

When $N_f = N_c + 1$, we can replace the sum

$$S := \sum_{b_1 < \cdots < b_{N_c}} \prod_{a_1<a_2} (r^{b_{a_1}} - r^{b_{a_2}})^{2-2g} \prod_a r^{-b_a(1-g)} (1+r^{b_a} x)^{n+(1-g)}$$

that appears in $P_{g;N_c,N_f}$ with a sum over the number $\hat{b}$ in $\{1,\cdots,N_f\}$ which none of the b_a take. That is, we have

$$\begin{aligned}
S &= \pm N_f^{N_f(1-g)}(1\pm x^{N_f})^{n+(1-g)}\sum_{\hat{b}=1}^{N_f}(1+r^{\hat{b}}x)^{-n-(1-g)}\prod_{b\neq\hat{b}}\left(r^b-r^{\hat{b}}\right)^{2g-2}\\
&= \pm N_f^{(N_f-1)(1-g)}(1\pm x^{N_f})^{n+(1-g)}\sum_{\hat{b}=1}^{N_f}r^{\hat{b}(1-g)}(1+r^{\hat{b}}x)^{-n-(1-g)}\\
&= \pm N_f^{(N_f-1)(1-g)+1}(1\pm x^{N_f})^{n+(1-g)}\sum_{i=0}^{\infty}\binom{-n-(1-g)}{N_f i-(1-g)}x^{N_f i-(1-g)},
\end{aligned}$$

where we have used lemma 2.5.3.

In this case, the full generating function for the Euler characteristics, including all the right factors, is

$$\pm N_f^{N_c g}x^{-(N_f-1)(1-g)}S = \pm N_f^{N_f}\sum_{i,j=0}^{\infty}(-1)^{j(N_f+1)}\binom{n+(1-g)}{j}\binom{-n-(1-g)}{N_f i-(1-g)}x^{N_f(i+j+g-1)}$$

Given a vortex number k, we're interested in the coefficient of $x^{N_f k}$, which is

$$\chi(\mathcal{L}^{N_f}_{g;k;N_c=N_f+1}) = \pm N_f^{N_f}\sum_{j=0}^{\infty}(-1)^{j(N_f+1)}\binom{n+1-g}{j}\binom{-n-(1-g)}{N_f(k-j)+(N_f-1)(1-g)}.$$

This is substantially simpler than the original form of the index.

The general case

More generally, the sum over the $\{b_a\}_{a=1,\cdots,N_c}$ can be replaced with a sum over those values in $\{1,\cdots,N_f\}$ that none of the b_a take. Just as above, this allows us to 'refactorise' the index in 'Seiberg-dual form'.

2.7 Local duality

2.7.1 *What are local vortices?*

In our discussion of duality we will restrict ourselves to two special cases where our results take a particularly simple form: the local case of $N_f = N_c = \lambda$ and the Abelian semilocal case of $N_c = 1$, $N_f = \lambda$. (Recall that working at Chern–Simons level $\lambda = N_f$ simplifies our formulae significantly, especially on surfaces of genus greater than one.)

We begin here in the local case, where we conjecture that quantum vortices 'are' fermions in a background flux, generalising the results of [ER20].

More precisely, we will find that our vortex quantum mechanics seems to agree with the theory of $N := N_f = N_c = \lambda$ flavours of (spin half) fermion coupled to a background gauge field for the flavour symmetry. Matching of these theories requires the matching of choices made in defining the theory. We will see that both theories depend on the choice of a holomorphic vector bundle.

2.7.2 The Fermi theory

The basic Fermi theory dual to the theory of local $U(N)$ vortices at level N takes the following simple form, based on the Abelian example of [ER20].

Let $V \to \Sigma$ be a (flavour) complex vector bundle of rank N and degree $c_1(V)[\Sigma] = N\mathcal{A}$, which should be an integer. The field of the static theory is a section

$$\psi = (\psi_1, \cdots, \psi_N) \in \Gamma(V \otimes K_\Sigma^{1/2})$$

where $K_\Sigma^{1/2}$ is a spin bundle on Σ. Each of the ψ_i is interpreted as an individual flavour. Note, however, that for general $N\mathcal{A}$ there is no splitting

$$V \stackrel{?}{=} L^{\oplus N}$$

for a line bundle L, so the individual flavours do not really have an independent existence. Of course, one can make sense of them locally. If $\mathcal{A}$ is an integer, so that $N\mathcal{A} \in N\mathbb{Z}$, then L does exist, and one can make sense of the individual flavours.

We give the flavour bundle $V \otimes K_\Sigma^{1/2}$ a backgound connection B. The total flux of B through Σ is fixed by the topology of the bundle to be $N(\mathcal{A} - 1 + g)$. The connection B induces a holomorphic structure on $V \otimes K_\Sigma^{1/2}$ via the Dolbeault operator $\bar{\partial}_B$.

We extend the bundle $V \otimes K_\Sigma^{1/2}$ to $\Sigma \times S^1$ in the trivial way, pulling it back along the projection to Σ. We then think of ψ as being a function of a periodic time component t. We should also introduce a time component B_t to the background gauge field B so as to preserve gauge invariance. However, because the topology of the bundle is trivial along the time direction, we can make the gauge choice $B_t = 0$. This is sometimes called the temporal gauge. As B is a background gauge field, this is completely harmless.

In this gauge, the Lagrangian is the Pauli Lagrangian

$$\psi^\dagger(\mathrm{i}\partial_t + \Delta_B - *F(B))\psi + \text{c.c.}$$

where Δ_B is the usual B-coupled Laplacian and the combination $\Delta_B - *F(B)$ gives the Dolbeault Laplacian $\Delta_{\bar{\partial}_B} = \bar{\partial}_B^\dagger \bar{\partial}_B$.

Note that if $\mathcal{A} \in \mathbb{Z}$ and B diagonalises, this is the Lagrangian for N fermion flavours in the background of a (genuine) magnetic field of total flux $\mathcal{A}$.

At low temperature, the theory becomes fermionic Hamiltonian mechanics on the space of zero modes

$$\bar{\partial}_B \psi = 0,$$

which is the zeroth vector bundle cohomology $H^0(V \otimes K_\Sigma^{1/2})$. The Riemann–Roch theorem tells us that the 'expected' dimension of this space is

$$\begin{aligned} h^0(V \otimes K_\Sigma^{1/2}) - h^1(V \otimes K_\Sigma^{1/2}) &= N(\mathcal{A} - 1 + g) + N(1 - g) \\ &= N\mathcal{A}. \end{aligned}$$

When is $h^1(V \otimes K_\Sigma^{1/2}) = 0$? Suppose that $\mathcal{A} \in \mathbb{Z}$ and that V splits as $L^{\oplus N}$ as a holomorphic bundle where L is a holomorphic line bundle of degree $\mathcal{A}$. In that case, the first cohomology of $V \otimes K_\Sigma^{1/2} = (L \otimes K_\Sigma^{1/2})^{\oplus N}$ vanishes whenever that of $L \otimes K_\Sigma^{1/2}$ vanishes. By positivity, this happens for $\mathcal{A} > g - 1$. It seems likely that in the general case, there will be similar vanishing results for $N\mathcal{A} \gg Ng$.

Suppose we are in this regime. Then we have a nonrelativistic quantum mechanics on the odd vector space

$$\Pi\mathbb{C}^{N\mathcal{A}},$$

where Π denotes parity reversal. The geometric quantisation prescription tells us that quantum states should be holomorphic functions on this space. Such functions are analytic functions of the holomorphic coordinates $(z_1, \cdots, z_{N\mathcal{A}})$. Because the z_i are anticommuting, such a function is the linear combination of monomials of the form

$$z_{i_1} \cdots z_{i_k}$$

where $0 \le k \le N\mathcal{A}$. The total Hilbert space is therefore the exterior algebra

$$\mathcal{H}^\bullet_{\text{Fermi}} = \Lambda^\bullet \mathbb{C}^{N\mathcal{A}}$$

with total dimension $2^{N\mathcal{A}}$.

There is an overall $U(1)$ action on the space of zero modes, acting by $z_i \mapsto e^{\mathrm{i}\theta} z_i$ for each i. This grades the Hilbert space by particle number. The k-particle Hilbert space is

$$\mathcal{H}^k_{\text{Fermi}} = \Lambda^k \mathbb{C}^{N\mathcal{A}},$$

which has dimension $\binom{N\mathcal{A}}{k}$. This is the expected dimension of the k-vortex Hilbert space.

This is the basic result underpinning the duality (2.1) which we gave at the start of this chapter:

$$Z_{\text{SCS},N} \;\leftrightarrow\; Z_{\text{Fermi},N}.$$

We now understand the theories on both sides a little better. When $\mathcal{A} \in \mathbb{Z}$, so that the Fermi theory becomes the theory of N fermions in a magnetic field, this exactly realises the low energy vortex theory as the theory of the lowest Landau level for a system of N Fermi flavours in a magnetic field of flux $\mathcal{A}$. Really though, for general $\mathcal{A}$, this should be interpreted as a Landau level of charge $1/N$ objects (or, more accurately, a nonAbelian Landau level).

There is a subtlety in the matching of the global symmetries in the Fermi-vortex duality above. In the Fermi theory, the global symmetry is

$$U(N) = \frac{U(1) \times SU(N)}{\mathbb{Z}_N}.$$

The $U(1)$ factor allows us to count the number of particles, and the $SU(N)$ factor rotates the flavours among themselves. The discrete quotient allows us to couple to background gauge fields for the flavour symmetry which would not be viable as $U(1) \times SU(N)$ gauge fields. This is the reason that the 'effective Abelian flux' $\mathcal{A}$ need not be an integer, as long as $N\mathcal{A}$ is.

Naïvely, the vortex theory has a topological global symmetry

$$U(1)_{\mathrm{top}}$$

which allows us to count the number of vortices, and a flavour symmetry

$$SU(N)_{\mathrm{flavour}}.$$

In the Higgs phase, the flavour symmetry is locked to the colour group as

$$SU(N)_{\mathrm{diag}} \subset U(N)_{\mathrm{colour}} \times SU(N)_{\mathrm{flavour}}.$$

which rotates the possible vortex charges among themselves at the same time as the scalar field flavours.

How do these global symmetries fit together? One way to understand this is to start in the vortex theory and couple the scalar fields to a background gauge field for a $U(N)$ flavour symmetry. This may look illegal, because the overall $U(1)$ is gauged, but there is nothing stopping one from thinking of the scalar field as taking values in

$$E \otimes V \to \Sigma$$

where E is the colour bundle on which there is a dynamical gauge field and V is a flavour bundle which could have nonzero degree, d. This nonzero degree shifts the overall Chern class of the bundle $E \otimes V$ by Nd and so is equivalent to increasing the degree of the colour bundle by d. This means shifting the vortex number $k \mapsto k + d$. This slightly roundabout argument implies that the global symmetries fit together as

$$\frac{U(1)_{\mathrm{top}} \times SU(N)_{\mathrm{flavour}}}{\mathbb{Z}_N} = U(N),$$

so that the global symmetries match across the duality.

The equality of indices that we have demonstrated (as well as the matching of global symmetries and various parameters) is good evidence for the duality, but does not constitute a proof of the duality. To prove the duality one needs to show more.

In the Abelian case $N = 1$, Eriksson–Romão showed in [ER20] that the k-vortex Hilbert space was

$$\mathcal{H}^k_{N=1} \cong \Lambda^k H^0(\Sigma, L \otimes K_\Sigma^{1/2})$$

where L is a holomorphic line bundle of degree $\mathcal{A}$ as above. This result holds for all $\mathcal{A} \in \mathbb{Z}$ and does not rely on vanishing theorems (in particular, the dimension of the Hilbert space may be larger than the expected dimension for small $\mathcal{A}$). Note that the vortex Hilbert space is defined with respect to a line bundle on Σ which must be taken to be $L \otimes K_\Sigma^{1/2}$ for the above isomorphism to be natural and even to exist in general.

This leads us to conjecture in the nonAbelian case that the k-vortex Hilbert space is

$$\mathcal{H}^k_N \overset{?}{\cong} \Lambda^k H^0(\Sigma, V \otimes K_\Sigma^{1/2}) \tag{2.26}$$

where V is a (semi)stable holomorphic bundle of rank N and degree $N\mathcal{A}$ as above, and the vortex Hilbert space is defined with respect to the bundle $V \otimes K_\Sigma^{1/2}$. Proving this conjecture would essentially constitute a proof of the Fermi-vortex duality as a genuine duality of non-supersymmetric field theories.

The conjecture would also imply that the true dimension of the Hilbert space obeys the inequality

$$\dim(\mathcal{H}_N^k) \geq \binom{N\mathcal{A}}{k},$$

which follows because

$$\begin{aligned} h^0(\Sigma, V \otimes K_\Sigma^{1/2}) &= N\mathcal{A} + h^1(\Sigma, V \otimes K_\Sigma^{1/2}) \\ &\geq N\mathcal{A}. \end{aligned}$$

2.8 Semilocal duality

2.8.1 *What are semilocal vortices?*

The outcome of the vortex count at $N_c = 1$ is (see Theorem 2.3.6)

$$\chi(\mathcal{L}^\lambda_{k;1,N_f}) = \sum_{j=0}^{g} \lambda^j N_f^{g-j} \binom{g}{j} \binom{\lambda(\mathcal{A}-k) + \delta - g}{\delta - j}$$

where $\delta = N_f k + (N_f - 1)(1-g)$.

The simplest case is $\lambda = N_f$, in which case the count is

$$N_f^g \binom{N_f\mathcal{A} + (N_f-1)(1-g)}{N_f k + (N_f-1)(1-g)} = N_f^g \binom{N_f\mathcal{A} + (N_f-1)(1-g)}{N_f(\mathcal{A}-k)}. \tag{2.27}$$

The first thing to note here is the asymmetry between vortices and vortex-holes for $g \neq 1$: it appears that vortex-holes are more natural from the perspective of state counting, reflecting the strange-looking selection rule $N_f k \geq (N_f - 1)(g-1)$ for semilocal Abelian vortices (this is the same as the condition that the expected dimension of the vortex moduli space be nonnegative). The second thing to note is that the count of vortex-holes is reminiscent of the counting of states consisting of N_f anticommuting objects (that is, fermions). The third thing to note is the appearance of the factor N_f^g, which is familiar from state counting in Abelian Chern–Simons and in theories of anyons.

In this case, we conjecture that the vortex theory is dual to a $U(1)^{N_f}/U(1)$ theory coupled to N_f fermion flavours, each attached to flux and spin. We give a description of this theory.

More generally, away from $\lambda = N_f$, topological effects become more important, exemplified by the fact that the index generally comes as a sum over $0, \cdots, g$. We do not give a general field-theoretic description in this case, but conjecture that the vortex theory is dual to a theory of anyons with fractional statistics $\exp(\mathrm{i}\pi\lambda/N_f)$.

2.8.2 Global symmetries

To understand what a potential dual theory looks like, we consider the symmetries of the semilocal theory. There is the flavour symmetry

$$SU(N_f)$$

which is broken to $S(U(N_c)_{\text{diag}} \times U(N_f - N_c))$ in the Higgs phase, and the vortex-counting topological symmetry

$$U(1)_{\text{top}}.$$

To see how the $U(1)$ and $SU(N)$ symmetries fit together, we again think of the scalar fields as taking values in a bundle

$$E \otimes V \to \Sigma$$

where E is the colour bundle and V is the flavour bundle. Shifting the degree of the flavour bundle by d is formally equivalent to a shift of E by $\frac{N_f}{N_c}d$.

In general, the fractional nature of the shift means that there is something to think about here[7], but in the Abelian case this means that the topological symmetry binds to the original flavour symmetry as

$$U(1)_{\text{top}} \times SU(N_f), \tag{2.28}$$

which is the global symmetry we should aim for.

2.8.3 Flux attachment, spin attachment, and Chern–Simons theory

Consider the theory of a spin 1/2 fermion ψ coupled to a $U(1)$ gauge potential a at Chern–Simons level n, with Lagrangian

$$\frac{n}{2\pi} * a \wedge \mathrm{d}a - \frac{1}{2}\left(\psi^\dagger(\mathrm{i}D_t + \Delta_{\bar{\partial}_{a\otimes B}})\psi + \text{c.c.}\right),$$

where B is a background Abelian connection, including the spin connection (often, one incorporates B into the dyanamical connection a by including a BF coupling of the form $a \wedge \mathrm{d}B$).

The equation of motion for a_t is the Gauss law constraint

$$\frac{n}{2\pi}F(a)_\Sigma = |\psi|^2\omega_\Sigma. \tag{2.29}$$

The object on the right is the ψ particle density on Σ. This equation requires the particle number, which is the integral of this particle density, to be quantised in units of n. It also tells us that for a given configuration of ψ particles, the moduli space of classical gauge fields on Σ is the Jacobian of Σ with symplectic form scaled by n. It is well-known that quantising this gives n^g states associated to the gauge field a.

On the other hand, the low-temperature equation of motion for ψ on a compact surface is the lowest Landau level equation

$$\bar{\partial}_{a\otimes B}\psi = 0.$$

[7] In a sense, the fact that the shift simplifies for $N_f = N_c$ or for $N_c = 1$ is the reason why these two cases are simple enough for us to study in detail.

The Riemann–Roch theorem tells us that, if the background magnetic flux is sufficiently strong, the number of possible states for the ψ particle is

$$\mathcal{B} + m$$

where $\mathcal{B}$ is the integrated magnetic flux, and m is the number of ψ particles divided by n. The genus-dependent terms cancel because ψ has spin $1/2$.

Because ψ is a fermion, the occupation number of each state is one or zero. This means that, for a given compatible configuration of the gauge field, the number of ψ states is

$$\binom{\mathcal{B}+m}{nm}.$$

This does not take into account the number of states associated to the gauge field, which to a first approximation, gives an additional factor of n^g. In general, a more detailed calculation is necessary to account for potential effects associated to nontrivial topology of the moduli space.

Proceeding for now, if we set $n = 1$ then $n^g = 1$ and the (approximate) number of quantum states in the full theory is

$$\binom{\mathcal{B}+m}{m}. \tag{2.30}$$

This is the count of states in a Landau level of *bosons*.

This carries the essence of *flux attachment* [Wil82]. The equation (2.29) 'attaches' $1/n$ units of flux to each ψ particle. Each ψ particle is also electrically charged and the flux attachment turns it into a flux-charge composite. By considering the Aharanov–Bohm phase associated to a rotation of this composite, one can see that the statistical exchange phase of ψ is shifted by $\exp(\mathrm{i}\pi n^{-1})$. As we started with a fermion, the new statistical phase is

$$\exp(\mathrm{i}\pi(1/n - 1)).$$

When $n = 1$, the composite is a boson, which agrees with what we saw with the state count in the lowest Landau level.

There is a subtlety here. The state count (2.30) is *not* the count of states for spin 0 bosons in the lowest Landau level in general, but rather the count of states for spin $1/2$ bosons. Indeed, the number of states for a system of q spin 0 bosons in the lowest Landau level instead takes the form

$$\binom{\mathcal{B}+q+(1-g)}{q}.$$

Of course, one can absorb $1 - g$ into $\mathcal{B}$, shifting the total effective magnetic flux. This is a little unsatisfactory from a geometrical perspective. It is a little more natural to think of flux attachment as changing the statistics, but not the geometrical spin.

If we want to really change the spin of the fermion, we should introduce a BF coupling between the dynamical connection a and the background spin connection Γ on $K_\Sigma^{1/2}$ so as to 'eat' the spin degrees of freedom. This term takes the general form

$$\frac{p}{2\pi} * a \wedge \mathrm{d}\Gamma,$$

for some level p. Note that Γ is not dynamical: it is a background connection. The new flux attachment equation is

$$nF(a)_\Sigma + pF(\Gamma)_\Sigma = 2\pi|\psi|^2\omega_\Sigma.$$

Integrating this, we see that the allowed number of ψ particles now takes the form

$$q = nm - p(1-g)$$

for $m \in \mathbb{Z}$. We have used that Γ is a connection on $K_\Sigma^{\text{-}1/2}$ and so has degree $g-1$.

Setting $n = p = 1$, the count of states with $q = m - 1 + g$ of the fermionic ψ particles is then

$$\binom{\mathcal{B} + q + (1-g)}{q}$$

which is the count of states for q spinless bosons, as desired.

2.8.4 The dual of a semilocal vortex

We now turn these ideas towards the Abelian semilocal vortex theory. For simplicity, consider the case of $\lambda = N_f$ so that the vortex count is

$$N_f^g\binom{N_f\mathcal{A} + (N_f-1)(1-g)}{N_f(\mathcal{A}-k)}.$$

When $N_f = 1$, this is simply a theory of fermions in a magnetic field. When $N_f > 1$, we must do something a bit more clever.

Although, for consistency, we will eventually start with a theory of fermions, it is actually easier to write down a bosonic dual in the general case. Take a theory of N_f spinless bosons $\Phi = (\Phi_1, \cdots, \Phi_{N_f})$ coupled to a background $U(1) \times SU(N_f)$ connection b (this differs from our usual B as it does not contain the spin connection, because our bosons are spinless) and a dynamical $U(1)$ gauge potential a. The purpose of this gauge potential is simply to attach flux and spin to the bosons. In flat space, it does not enter the dynamics in any interesting way - its equations of motion are constraints. However, its dynamics do become important on topologically non-trivial backgrounds.

We introduce the Chern–Simons coupling

$$\frac{N_f}{2\pi} * a \wedge \mathrm{d}a - \frac{1}{2\pi} * a \wedge \mathrm{d}\Gamma$$

where Γ is the non-dynamical spin connection on $K_\Sigma^{\text{-}1/2}$, and the usual matter terms

$$\frac{1}{2}\left(\Phi^\dagger(\mathrm{i}D_t + \Delta_{\bar\partial_{a\otimes b}})\Phi + \text{c.c.}\right)$$

into the Lagrangian.

The Gauss law is

$$-N_f F(a)_\Sigma + F(\Gamma)_\Sigma = 2\pi|\Phi|^2\omega_\Sigma,$$

which restricts the number of Φ particles to take the form

$$q = -N_f m + (1 - g),$$

for $m \in \mathbb{Z}$.

At low temperatures, the equation of motion for Φ is

$$\bar{\partial}_{a \otimes b} \Phi = 0$$

which has a space of solutions of expected dimension

$$N_f \mathcal{B} + N_f m + N_f (1 - g)$$

where $\mathcal{B}$ is the total flux associated to the background magnetic field b. In terms of the particle number q, this is

$$N_f \mathcal{B} - q + (N_f - 1)(1 - g).$$

Then, because Φ is bosonic, the count of q-particle states is, ignoring the gauge field for now,

$$\binom{\text{number of states } + \text{ number of particles}}{\text{number of particles}} = \binom{N_f \mathcal{B} + (N_f - 1)(1 - g)}{-N_f m + (1 - g)}.$$

Assuming that the gauge field states enter in the simplest way, the expected quantum degeneracy of the low-temperature theory is then

$$N_f^g \binom{N_f \mathcal{B} + (N_f - 1)(1 - g)}{-N_f m + (1 - g)}.$$

If we identify the vortex number k with $-m + 1 - g$ and $\mathcal{A}$ with $\mathcal{B}$, this is the index for semilocal vortices at $\lambda = N_f$. This is a version of bosonic particle-vortex duality.

It would be nice to give this a fermionic description, which we can do by regarding the bosonic fields in the above description as fermions attached to flux and spin. To do this, we start with N_f spin-half fermion flavours $\psi = (\psi_1, \cdots, \psi_{N_f})$. The fermions are charged under the background field B, which consists of a fixed magnetic field and a fixed spin connection. We gauge the $U(1)$ phase of each fermion flavour individually, writing α_i for the gauge field associated to rotations of ψ_i. We introduce an extra gauge field a, which is associated to simultaneous rotations of all of the ψ_i. This might look a bit strange, as we have already gauged each $U(1)$, but one can think of it as a shift of each of the α_i. As before, the purpose of the gauge field a is simply to implement a constraint. In this case, the constraint can be thought of as a constraint on the sum of the α_i. Thus, on flat space, we could regard the dynamical gauge field as $U(1)^{N_f}/U(1)$. On general backgrounds, the situation is slightly more complicated because of the presence of nontrivial flat connections.

To attach the correct fluxes and spins, we consider the Chern–Simons terms

$$\frac{N_f}{2\pi} a \wedge \mathrm{d}a - \frac{1}{2\pi} a \wedge \mathrm{d}\Gamma - \sum_{i=1}^{N_f} \left(\frac{1}{2\pi} \alpha_i \wedge \mathrm{d}\alpha_i + \frac{1}{2\pi} \alpha_i \wedge \mathrm{d}\Gamma \right),$$

in addition to the usual matter terms for ψ.

The Gauss laws for the various gauge fields impose the equations

$$\begin{aligned} F(\alpha_i) + F(\Gamma) &= 2\pi\psi_i^\dagger\psi_i \text{ (no summation)} \\ -N_f F(a) + F(\Gamma) &= 2\pi\sum_i \psi_i^\dagger\psi_i. \end{aligned}$$

These equations imply that

$$\sum_i F(\alpha_i) = -N_f F(a) + (1-N_f)F(\Gamma).$$

This can be thought of as the constraint that including the field a buys us.

The Riemann-Roch theorem then tells us that expected number of solutions to the lowest Landau level equation $\bar{\partial}_{\alpha\otimes a\otimes B}\psi = 0$ is

$$N_f\mathcal{B} + (N_f-1)(1-g),$$

The total number of particles is constrained to take the form

$$q = -N_f m + (1-g)$$

for an integer m. Thus the count of q-particle fermion states is

$$\binom{\text{number of states}}{\text{number of particles}} = \binom{N_f\mathcal{B} + (N_f-1)(1-g)}{-N_f m + (1-g)}$$

for an integer m.

Incorporating the N_f^g states associated to the gauge field, which we again assumes enters in the simplest way, the total number of states for a given q is then

$$N_f^g\binom{N_f\mathcal{B} + (N_f-1)(1-g)}{-N_f m + (1-g)}.$$

This again agrees with the vortex count.

This leads us to a conjectural electric-magnetic duality, between the nonrelativistic Chern–Simons-matter theory

$$U(1)_{N_f} + N_f \text{ fundamental scalars}$$

and the fermionic theory we have just described. The Fermi theory has gauge group $U(1)^{N_f}$, but the overall $U(1)$ is gauged twice and subject to a constraint. The gauge group is 'morally' $U(1)^{N_f}/U(1)$. Indeed, the overall $U(1)$ should survive as a global symmetry for the duality to hold. However, the need to bind the fermions together and the presence of the factor N_f^g in the index make it difficult to give a direct description of the dual theory as a $U(1)^{N_f-1}$ gauge theory. In any case, we schematically denote the dual theory as

$$U(1)^{N_f}/U(1) + N_f \text{ fermions} + \text{CS and BF couplings}.$$

The duality maps vortices to bound states of fermions.

The background gauge flux for the $U(1)_{\text{top}}$ global symmetry in the vortex theory is $\mathcal{A}$, which is mapped to the background gauge flux $\mathcal{B}$ for the overall $U(1)$ global symmetry in the Fermi theory.

Stated in these terms, the duality is highly reminiscent of the mirror symmetry duality

$$U(1) + N_f \text{ hypermultiplets } \leftrightarrow U(1)^{N_f}/U(1) + N_f \text{ hypermultpliets}$$

of three-dimensional gauge theories with $\mathcal{N} = 4$ supersymmetry [IS96]. Mirror symmetry is also an electric-magnetic duality, mapping vortices to electrically charged particles.

The appearance of a Chern–Simons term in the vortex theory can be thought of from this perspective as arising from integrating out the fermionic fields of the supersymmetric theory. It would be interesting to understand more precisely the nature of three-dimensional mirror symmetry in this 'quantum Hall regime' (a regime considered for three-dimensional Chern–Simons theories with $\mathcal{N} = 2$ supersymmetry in [LLM92]).

In assuming that the index factorises into gauge and matter contributions we have been rather quick. It would be good to clarify the form of the index of the dual theory to verify the duality. It would also be nice to clarify precisely the role of the overall $U(1)$ symmetry in the construction of the dual gauge theory: is there a direct construction of the theory as a $U(1)^{N_f}/U(1)$ theory?

3.1 INTRODUCTION AND SUMMARY

Exotic vortices are vortex-like configurations in physical systems which exhibit a kind of 'anti-Meissner–Higgs effect'. Rather than acting to screen magnetic fields, the Higgs fields in these theories conspire to accumulate in regions of magnetic flux. This means that 'inside' a vortex, where the Higgs fields become small, the magnetic flux is at its weakest.

The equations for BPS-like exotic vortices are exactly as the usual vortex equations, constraining the configuration of a gauge potential A and a Higgs field ϕ living on a Riemann surface Σ, with schematic form

$$\begin{aligned} *F(A) &= \frac{\mathrm{i}}{2}e^2(\tau - \phi\phi^\dagger) \\ \bar{\partial}_A\phi &= 0, \end{aligned} \tag{3.1}$$

except that the constant e^2, which we usually think of as a gauge coupling constant, is allowed to be nonpositive (we give a more complete description below).

Integrating the trace of the first vortex equation over a compact surface tells us that vortices 'take up' some area proportional to e^2. For conventional vortices, with $e^2 > 0$, this leads to the Bradlow bound, constraining the number of vortices that can fit on a compact surface of finite area. When $e^2 < 0$, the vortices 'produce' area: there is no upper bound on the number of vortices.

As we summarise below, exotic vortices arise most naturally in nonrelativistic Chern–Simons-matter theories. Abelian examples include the Jackiw–Pi vortex equation [JP90] and the equations studied by Popov [Pop12] and Manton [Man13; Man17]. Among other equations, including that of Jackiw–Pi, Manton studied the so-called Ambjørn–Olesen equation, which found use in describing the electroweak phase transition [AO88; AO89; AO90].

While Abelian examples are most prominent in the literature, exotic vortices can be considered for the data of a general gauge theory [Dun$^+$91; Tur20] and we recall the basic theory here, extending it to allow for matter living in general nonlinear symplectic representations of the gauge group.

We will address the problems of finding solutions to these equations, understanding the moduli spaces of solutions, and understanding the (quantum) dynamics of exotic vortices. In doing this we address some of the questions raised in [DTT16b] in the context of Jackiw–Pi vortices.

Our main technical result, which is based on results of Manton [Man17], Witten [Wit77], and Olesen [Ole91], relates solutions to the (exotic) vortex equations to objects that we call *twisted holomorphic maps* into complex space forms (complete Kähler manifolds of constant holomorphic sectional curvature). For us, a twisted holomorphic map from a Riemann surface Σ into a Kähler manifold X is a flat $\mathrm{Isom}(X)$ bundle equipped with a holomorphic section of the associated holomorphic X-bundle. Our main result can then be summarised as follows.

Our main technical result, which is based on results of Manton [Man17], Witten [Wit77], and Olesen [Ole91], relates solutions to the (exotic) vortex equations to objects that we call *twisted*

holomorphic maps into complex space forms (complete Kähler manifolds of constant holomorphic sectional curvature). For us, a twisted holomorphic map from a Riemann surface Σ into a Kähler manifold X is a flat $\mathrm{Isom}(X)$-bundle equipped with a holomorphic section of the associated holomorphic X-bundle. We explicitly construct a map from the space of equivalence classes of twisted holomorphic maps from Σ to X, where X has constant holomorphic sectional curvature κ, to the space of equivalence classes of solutions to the exotic vortex equations (3.1) on Σ with gauge group $U(\dim_{\mathbb{C}}(X))$, one fundamental flavour, $e^2 = -\kappa$, and $\tau = \frac{2-2g}{2\pi \mathrm{vol}(\Sigma)\kappa}\mathbb{1}$.

We further conjecture, with both local and global evidence, that the converse holds: every solution to the relevant vortex equation arises in this way and the moduli space of the vortex solutions is in one-to-one correspondence with the moduli space of relevant twisted holomorphic maps. As the moduli space of twisted holomorphic maps is rather more accessible that that of exotic vortices, this conjecture gives us conjectural information about the moduli space, and therefore the quantum mechanics, of exotic vortices. In particular, our results give additional context to results of Turner on nonnormalisable modes in the quantum mechanics of Jackiw–Pi vortices [Tur20]. The conjecture also implies certain selection rules for exotic vortices on the sphere, generalising the known fact that Jackiw–Pi vortices on the sphere have even vortex number [JP90]. In our context, such results are related to the topology of flat $\mathrm{Isom}(X)$-bundles.

Our results have something to say about conventional (nonexotic) vortices in $U(N_c)$ gauge theories with a single fundamental flavour, generalising Witten's work [Wit77] for $N_c = 1$. We focus our attention primarily on the exotic case of $e^2 < 0$, as, in this case, the twisted holomorphic map moduli problem is more accessible.

The chapter continues as follows. In the rest of the introduction, we summarise the story in more detail, outlining our main results and pointing out some related work. In section 3.2, we describe the exotic vortex equations in detail and give some basic facts about the structure of their moduli spaces of solutions. In section 3.3, we prove our main result and use it to give conjectural descriptions of exotic vortex moduli spaces. We show how previous results, such as Olesen's single-vortex solution to the Jackiw–Pi equation on the torus [Ole91], fit into our work. In section 3.4, we describe the (quantum) dynamics of gauge theories supporting exotic vortices. The natural classical dynamics is first order. At low temperature, it is described by Hamiltonian mechanics on the moduli space of vortices. Using results of chapter 2, we (formally) compute the 'expected' (or 'supersymmetric') partition function of the dynamical exotic Abelian theory on Riemann surfaces of arbitrary genus.

3.1.1 Exotic vortices from twisted holomorphic curves

Mathematically, the equations (3.1) with $e^2 < 0$ are harder to analyse than the usual vortex equations, which partially reflects the fact that they do not appear to be stable in relativistic physical theories. In fact, the natural relativistic theories that one would write down which exhibit the kind of anti-Meissner effect that is necessary are completely sick, with Euclidean energy functionals unbounded from below. The unboundedness of the relevant functionals from below makes the usual functional analysis techniques hard, or impossible, to implement.

While this sounds unpleasant, there is beauty in the equations. It was noticed by Manton [Man17] that, on constant curvature backgrounds, special cases of the exotic vortex equations in theories with gauge group $U(1)$ and one fundamental scalar flavour could be solved by meromorphic functions. This is the exotic analogue of Witten's solution of the hyperbolic vortex equations by Blaschke functions [Wit77] and it generalises the fact that the Jackiw–Pi equations on the plane are equivalent to the Liouville equation, which is solved by meromorphic functions [JP90; HY98]. It was shown in [CD17] that these results could be understood in terms of dimensional reduction of the self-duality equations in four dimensions, in analogy to Witten's argument in [Wit77].

We generalise Manton's result in several ways. First, we consider the same vortex systems as Manton, showing that one can produce additional solutions by considering meromorphic functions 'twisted' by a flat gauge field for the $PSU(2)$ symmetry that rotates the target sphere (we explain what this means in detail below). In particular, this includes Olesen's single vortex solution to the Jackiw–Pi equation on the torus [Ole91]. We conjecture that all of the relevant exotic vortex solutions can be found in this way, giving some coarse evidence for this in the form of an equality of indices. This would be useful, because the moduli problem for a flat $PSU(2)$ gauge field and twisted meromorphic function is qualitatively easier to understand than that of the exotic vortex system.

We then generalise this story of Abelian exotic vortices to nonAbelian gauge theories. We show that, on backgrounds of constant curvature, solutions to exotic vortex equations in $U(N_c)$ theories with $N_f = 1$ fundamental flavour can be produced from twisted holomorphic maps into $\mathbb{C}P^{N_c}$, 'twisted' by a flat $PSU(N_c+1)$ connection. More generally, (not necessarily exotic) vortex solutions can be produced from 'twisted' holomorphic maps into Kähler manifolds of constant holomorphic section curvature. Usually one asks that $N_f \geq N_c$ in order to see (nonexotic) vortices, as this is necessary to completely Higgs the gauge group. In the exotic case things are rather different, and we expect to find families of solutions even for $N_f < N_c$.

Indeed, we show that the moduli space of pairs consisting of a flat $PSU(N_c+1)$ connection and a twisted holomorphic curve in the associated $\mathbb{C}P^{N_c}$-bundle has the same expected dimension as that of exotic vortices in the relevant $U(N_c)$ gauge theory with 1 fundamental flavour. This leads us to conjecture that all such exotic vortex solutions on constant curvature backgrounds arise in this way.

We also show how to alleviate the constraint of constant curvature, using 'uniformising functions'. While this is formally useful, it takes us a little way from the original simple solution generating techniques of [Man17].

This construction is novel, but it does not represent the first time that nonAbelian connections have played a role in the story of Abelian exotic vortices. Indeed, the dimensional reduction picture of [Wit77; CD17] mentioned above involves nonAbelian gauge theory in four dimensions (with gauge group a real form of $SL(2,\mathbb{C})$ depending on the form of the vortex equation).

In [RS18], solutions to the Abelian exotic vortex equations on the two-dimensional sphere (known as Popov vortices, after [Pop12]) were related to flat $SU(2)$ gauge potentials on the three-sphere.

A unifying picture might be provided by Hitchin's work [Hit90a] relating solutions to a kind of 'exotic Hitchin system' of equations for an $SU(2)$ gauge theory to (twisted) harmonic maps into the three-sphere. Our result in the Abelian case can be thought of as arising from this when the $SU(2)$ connection is reducible to $U(1)$.

3.1.2 Exotic selection rules

One of the strange things about Jackiw–Pi vortices on the infinite sphere is that there are no solutions with odd flux: the total vortex charge is always even [HY98]. This has no straightforward topological explanation and if one studies the same equation on the flat torus, one can find solutions with odd flux. Such a solution was found by Olesen in [Ole91].

While this might look mysterious from the perspective of the vortex equations, it can be understood clearly in the context of twisted meromorphic functions. For us, a twisted meromorphic function is a section ψ of a $\mathbb{C}P^1$-bundle S over our base Riemann surface Σ. The vortex number associated to the section ψ is

$$k = (2g - 2) + \deg(\psi^* T_S).$$

This is a kind of generalised Riemann–Hurwitz relation.

If the bundle S is topologically trivial, then the second term on the right hand side is always even, because $\deg(T_{\mathbb{C}P^1}) = 2$. Then k is always even. When S is nontrivial, the second term on the right hand side may be odd.

In general, topological S^2-bundles on a surface Σ are classified by a Stiefel–Whitney-type class in

$$H^1(\Sigma, \mathbb{Z}_2).$$

The point here is that the group of oriented diffeomorphisms of the sphere is homotopic to $PSU(2)$. A $PSU(2)$-bundle is locally an $SU(2)$-bundle, but one has an additional choice of sign around each noncontractible cycle, giving rise to an element of $\mathbb{Z}_2$ for each such cycle.

When $\Sigma = \mathbb{C}P^1$, there are no topologically nontrivial bundles, so k must be even. When Σ has higher genus, topologically nontrivial bundles are available and so solutions with odd vortex number may exist.

In subsection 3.3.2, we show that this generalises to the nonAbelian case of a $U(N_c)$ theory with one flavour. Now the selection rules are determined by the topology of $PSU(N_c+1)$-bundles. When $\Sigma = \mathbb{C}P^1$, solutions arising from the relevant analogue of the twisted meromorphic functions above are forced to have

$$k = -2N_c + (N_c + 1)p$$

for $p \in \mathbb{Z}$. Nontrivial $PSU(N_c + 1)$-bundles, which exist on surfaces with $g > 0$, allow for $k - N_c(2g - 2)$ to take nonzero values modulo $N_c + 1$.

3.1.3 Quantum exotic vortices or: Probing the moduli space with quantum mechanics

While lots of things that we are used to when thinking about vortices break when one takes $e^2 < 0$, the construction of the moduli space as a symplectic quotient does not (although it is no longer a Kähler quotient). This means that the moduli space should generically be a symplectic manifold, which makes it a good home for Hamiltonian mechanics.

From the point of view of $(2+1)$-dimensional physics, Hamiltonian mechanics on the vortex moduli space is realised as a low-temperature limit of certain nonrelativistic Chern–Simons-matter theories. We have already studied this in detail in the non-exotic case in chapter 2. Formally, the calculations go through in the same way, with certain parameters changing sign.

Because we know very little about the existence of solutions in the general case, these calculations are fundamentally formal and so are highly speculative. However, they are interesting. We compute, for example, the expected degeneracy of quantum Jackiw–Pi vortices at low temperature.

3.1.4 Vortices as degenerate metrics

In [Bap14], Baptista observed that the usual Abelian vortex equation could be viewed as an equation for a degenerate metric obeying a certain curvature condition. This idea was carried forward by Manton in [Man17], who noted that solutions to Abelian exotic vortex equations produce metrics with conical singularities with cone angle a multiple of 2π.

We will not develop this idea in any detail here, only noting that our results can be used to understand the moduli spaces of constant curvature metrics on Riemann surfaces with conical singularities of the given type (see also [Che$^+$15; EGT14; MZ21]).

3.2 Exotic vortices and their moduli

3.2.1 The exotic vortex equations

The exotic vortex equations take the same data as the usual, nonexotic, vortex equations. We fix a Kähler manifold Y carrying the Hamiltonian action of a compact Lie group G. We write

$$\mu : Y \to \mathfrak{g}^\vee$$

for the corresponding moment map, where $\mathfrak{g}$ is the Lie algebra of G.

We now take a smooth two-dimensional surface Σ with volume form ω_Σ. We fix a principal G-bundle

$$P \to \Sigma$$

and form the associated Y-bundle

$$\mathcal{Y} \coloneqq Y \times_G P \to \Sigma.$$

We take a section ϕ of $\mathcal{Y}$ and a connection A on P. For a given element ξ in the dual of the centre of $\mathfrak{g}$, the vortex equations for (A, ϕ) are

$$\begin{aligned} *F(A) &= e^2(\xi - \mu(\phi)) \\ \bar{\partial}_A \phi &= 0 \end{aligned}$$

where e^2 is a scale factor on the Killing form on $\mathfrak{g}$.

The *exotic vortex equations* are these equations with $e^2 \leq 0$. This subtle change from the usual case has dramatic consequences, as we will see.

The case of $e^2 = 0$ is rather trivial, with the equations becoming the equations for a flat connection and a section of an associated holomorphic bundle. Vortices in these theories are sometimes called *Bradlow vortices*, because they resemble vortices 'at the Bradlow bound'. They have been considered in [GN17]. Because they are rather simple from our perspective, we will focus our attention primarily on the case of $e^2 < 0$.

3.2.2 The moduli space as a symplectic quotient

An important fact about the moduli space of non-exotic vortices is that it is a symplectic, and in fact Kähler, quotient. It is this fact that captured the inherent structure in the Chern–Simons-matter dynamics that we studied in chapter 2.

Some mathematical insight into the nature of exotic vortices comes by considering how this construction of the moduli space differs (or rather, how it does not) when $e^2 \leq 0$.

In a sense, setting $e^2 \leq 0$ is all there is to it. Write $\mathcal{C}$ for the space of (reasonable) pairs (A, ϕ). Recall that this space has a symplectic form

$$\omega_{\mathcal{C}}((\dot{A}_1, \dot{\phi}_1), (\dot{A}_2, \dot{\phi}_2)) = \int_\Sigma \left(\operatorname{tr}\left(\dot{A}_1 \wedge \dot{A}_2\right) + e^2 \omega_{\mathcal{Y}}(\dot{\phi}_1, \dot{\phi}_2)\omega_\Sigma \right).$$

There is nothing stopping us from taking $e^2 \leq 0$, and this is what we do.

This does have consequences, though. When $e^2 > 0$, the symplectic form $\omega_{\mathcal{C}}$ forms part of a Kähler structure. The corresponding Riemannian metric can then be interpreted as a relativistic kinetic energy functional. When $e^2 \leq 0$, this 'metric' is, at best, semiRiemannian (see [Man17; CD17]). It may be rather more degenerate than that. A metric with negative norm states can never make for a good kinetic energy functional, because the system will run away to negative energy.

We do not need to worry about this though: we have a symplectic structure on $\mathcal{C}$ and the story of gauged Hamiltonian mechanics goes through as in the non-exotic case. The group $\mathcal{G}$ of gauge transformations acts symplectically on $\mathcal{C}$. Restricting to the space $\mathcal{C}_0$ of solutions to $\bar{\partial}_A \phi = 0$, we take the symplectic quotient to get the vortex moduli space

$$\mathcal{M} = \mathcal{C}_0 // \mathcal{G}.$$

In particular, this implies that the exotic vortex moduli space is 'generically' symplectic. However, as we will see explicitly, the exotic vortex equations are much less well behaved than the usual vortex equations, and simple examples may fail to be generic in this sense.

3.3 Exotic vortices from twisted holomorphic curves

3.3.1 The Abelian case

In [Man13; Man17], Manton found beautiful integrability results for Abelian exotic vortex equations in the case that the Riemannian surface Σ has constant scalar curvature κ_0, building on [Wit77; Pop12]. In that case, he produced solutions to the equations

$$\begin{aligned} *F(A) &= -\kappa\left(\frac{\kappa_0}{\kappa} - |\phi|^2\right) \\ \bar{\partial}_A\phi &= 0, \end{aligned}$$

where A is a $U(1)$ connection, ϕ is a single scalar field of unit charge, and κ is a positive real number. He did this, roughly speaking, by setting $|\phi|^2 = *\psi^*\omega_{\mathbb{C}P^1}$, where $\psi : \Sigma \to \mathbb{C}P^1$ is holomorphic and $\omega_{\mathbb{C}P^1}$ is the Fubini–Study form, and solving for ϕ in some gauge, showing that the result solved the vortex equations in a second-order form (which is obtained by eliminating the gauge field locally).

We start by generalising the ideas of [Man17] to find more general solutions. To do this, we consider sections of possibly nontrivial $\mathbb{C}P^1$-bundles over Σ. Indeed, suppose $S \to \Sigma$ is a $\mathbb{C}P^1$-bundle carrying a flat $PSU(2)$ connection a. Let ψ be a section of S obeying the equation

$$\bar{\partial}_a\psi = 0.$$

Our claim is that one can build an Abelian exotic vortex solution from this data. We recover Manton's result when the connection a is trivial.

We will take a slightly different approach to that of [Man17]. We will work in the 'holomorphic gauge', as in [Bra90], where one views the Abelian vortex equation as an equation for a holomorphic line bundle $(L, \bar{\partial}_L)$ with section ϕ and Hermitian metric h obeying

$$\begin{aligned} *F(h) &= \frac{\mathrm{i}e^2}{2}(\tau - |\phi|_h^2) \\ \bar{\partial}_L\phi &= 0, \end{aligned} \tag{3.2}$$

where $F(h)$ is the curvature of the Chern connection uniquely associated to $(\bar{\partial}_L, h)$. One can get back and forth between this and the usual vortex equation by the complex gauge transformation g that sets $g(h) = 1$.

By the holomorphicity of ψ and the flatness of a, we have that

$$\mathrm{d}_a\psi \in \Omega^0(\Sigma, K_\Sigma \otimes \psi^*T_S)$$

is holomorphic with respect to the natural holomorphic structure, induced by a and the holomorphic structure of Σ. We set $L \coloneqq K_\Sigma \otimes \psi^*T_S$.

The round Kähler structure (g_{S^2}, ω_{S^2}) defines a vertical Kähler structure on the fibres of S. The connection a allows us to extend the domain of g_{S^2} and ω_{S^2} to include horzontal vector fields, and we write g_S and ω_S for the resulting tensor fields. The flatness of a means that it can be trivialised locally, so the distinction between (g_S, ω_S) and (g_{S^2}, ω_{S^2}) is locally insignificant. Note that, despite the notation, g_S is not a metric on the total space of S and ω_S is not a symplectic form on the total space of S.

The combination of the round Hermitian structure on the fibres of S and the metric on Σ defines a Hermitian structure h on L. With this structure, one has

$$|\mathrm{d}_a\psi|_h^2 = (g_S)_{ij}(g_\Sigma^{-1})^{\mu\nu} D_\mu\psi^i D_\nu\psi^j$$

where we have written $D \coloneqq \mathrm{d}_a$ to accommodate the indices. This admits a Bogomolny rearrangement

$$|\mathrm{d}_a\psi|_h^2 = 2|\bar{\partial}_a\psi|^2 + *\psi^*\omega_S$$

where we have used that a is flat (if a is not flat, an extra curvature dependent term appears). Here $*$ is the Hodge star on Σ. Now, the fact that $\bar{\partial}_a\psi = 0$ tells us that

$$|\mathrm{d}_a\psi|_h^2 = *\psi^*\omega_S. \tag{3.3}$$

On the other hand, associated to the Hermitian structure h and the holomorphic structure on L is the uniquely defined Chern connection ∇_h. By definition, this has the properties that $\nabla_h^{0,1} = \bar{\partial}_L$, and ∇_h preserves h. The curvature of ∇_h is

$$F(h) = -F(g_\Sigma) + \psi^* F(g_S),$$

where by $F(g_S)$ we mean the fibrewise curvature of g_S extended to horizontal vectors using the flat connection a.

Now, g_S is fibrewise the round metric, so in particular it has constant curvature κ, so that $F(g_S) = \mathrm{i}\kappa\omega_S$. Combining this with (3.3) tells us that

$$F(h) = -F(g_\Sigma) + \mathrm{i}\kappa * |\mathrm{d}_a\psi|_h^2.$$

Defining ϕ to be $\mathrm{d}_a\psi$, which is a holomorphic section of L, this is starting to look like a vortex equation. The only thing left to do is to ask g_Σ to have constant curvature, so that

$$F(g_\Sigma) = \mathrm{i}\kappa_0\omega_\Sigma.$$

As $F(g_\Sigma)$ is the curvature of a connection on $T\Sigma$, the Gauss–Bonnet theorem means that we must have

$$\kappa_0 = \frac{2-2g}{2\pi \mathrm{vol}(\Sigma)}$$

where g is the genus of Σ.

In this case, we have

$$\begin{aligned} F(h) &= -\mathrm{i}\kappa\left(\frac{\kappa_0}{\kappa} - |\phi|_h^2\right)\omega_\Sigma \\ \bar{\partial}_L\phi &= 0, \end{aligned}$$

where κ, being the curvature of a round sphere, is necessarily positive. These are exotic vortex equations in the holomorphic gauge (3.2). Thus $(\bar{\partial}_L, h, \phi)$ defined this way give solutions to the exotic vortex equations. We can put them into the unitary form by acting with a complex gauge transformation $g : \Sigma \to \mathbb{C}^*$ with $gg^* = \frac{1}{h}$.

The vortex number k is the degree of the bundle $L = K_\Sigma \otimes \psi^* T_S$. This is given by a generalised Riemann–Hurwitz formula, which we can derive by considering the exact sequences

$$\begin{aligned} 0 \to \mathcal{O}_S \to p^*E \otimes \mathcal{O}_S(1) \to T_{S/\Sigma} \to 0, \\ 0 \to T_{S/\Sigma} \to T_S \to p^*T_\Sigma \to 0 \end{aligned}$$

of holomorphic vector bundles, where $p : S \to \Sigma$ is the projection, E is a choice of rank 2 vector bundle such that S is the projectivisation of E (which always exists for dimensional reasons), and $\mathcal{O}_S(1)$ is the corresponding tautological bundle. The bundle E is not uniquely specified: it can be shifted by any line bundle. This is accompanied by a corresponding shift of the tautological bundle, so that $p^*E \otimes \mathcal{O}_S(1)$ is uniquely specified by S, as it must be. The first sequence is the relative Euler sequence, and the second is the Atiyah sequence.

Using these sequences and the multiplicativity of the Chern character, we have that

$$\mathrm{ch}(\psi^* T_S) = \mathrm{ch}(T_\Sigma)\mathrm{ch}(E \otimes \psi^* \mathcal{O}_S(1))$$

where we have pulled the sequences back along ψ. From this we can read off the degree of $\psi^* T_S$. Without loss of generality, we can assume that $\deg(E)$ is either 0 or 1 (twisting with a line bundle allows us to bring it into one of these forms). If E has degree 0, this gives

$$\deg(\psi^* T_S) = 2\deg(\psi) + 4 - 4g$$

so that the vortex number is

$$\begin{aligned} k &= \deg(K_\Sigma \otimes \psi^* T_S) \\ &= 2\deg(\psi) + 2 - 2g. \end{aligned}$$

This is the usual Riemann–Hurwitz formula.

We have shown to any pair consisting of a flat $PSU(2)$ connection (equivalently, a projectively flat $U(2)$ connection) and a holomorphic section of the corresponding $\mathbb{C}P^1$ bundle gives rise to an Abelian exotic vortex equation on a constant curvature surface.

Note that we could replace the target sphere with the flat plane or the hyperbolic plane. Correspondingly, one should change $PSU(2)$ to $E(2)$ or $PSU(1, 1)$ respectively. This allows one to find solutions with $\kappa = 0$ or κ negative. Indeed, this was the original insight of Witten [Wit77] for (nonexotic) hyperbolic vortices.

Moduli

It is conceivable that every solution to the relevant exotic vortex equation on a constant curvature surface arises from this construction. Showing this directly seems to be tricky as it involves integrating the equation $\phi = \mathrm{d}_a\psi$. Standard arguments as in [Man17], for example, show that it is true locally. By studying the moduli problem for the pair (a, ψ) we give some coarse evidence for the idea in more generality.

The first question is: what is the dimension of the moduli space of pairs of a flat $PSU(2)$ connection a and a holomorphic section ψ of the associated $\mathbb{C}P^1$-bundle? The equations are

$$\begin{aligned} F(a) &= 0 \\ \bar{\partial}_a \psi &= 0. \end{aligned} \tag{3.4}$$

The linearisation of the first equation lives in the elliptic complex

$$\Omega^{\bullet-1}(\mathfrak{su}(2))$$

and so has real index

$$6g - 6.$$

On the other hand, the linearisation of the second equation lives in the complex

$$\Omega^0(\psi^* T_{S/\Sigma}) \xrightarrow{\bar{\partial}_A} \Omega^1(\psi^* T_{S/\Sigma})$$

and so has real index

$$2\deg(\psi^* T_{S/\Sigma}) + (2 - 2g).$$

The vortex number is $k = \deg(\psi^* T_{S/\Sigma}) + 2g - 2$, so this index is

$$2k + 6 - 6g$$

The combined system of equations (3.4) therefore has total index $2k$, which is the same as the index of the vortex problem. This means that the 'expected dimensions' of the two moduli problems agree.

The sphere, bad behaviour, and symmetry breaking

The integrable Abelian vortex equation on the sphere is known as the Popov equation, after [Pop12]. In the limit that the radius of the domain sphere becomes large, the constant $\kappa_0 \to 0$, and the equation becomes the Jackiw–Pi equation of [JP90].

The moduli space of pairs (a, ψ) is directly accessible in this case. When $\Sigma = S^2$, the only flat connection is the trivial one. The moduli space is then the space of rational functions $\psi : \mathbb{C}P^1 \to \mathbb{C}P^1$, modulo the action of $PSU(2)$ rotations of the target sphere.

This tells us that the moduli space of vortex number k solutions that can be constructed in this way is, as a manifold,

$$\left(\mathbb{C}P^{k+3} - \Delta\right)/PSU(2)$$

if k is even and empty if k is odd, where Δ is the resultant hypersurface of rational functions that degenerate to one of lower degree.

The real dimension of this space is

$$2k + 3,$$

not the expected $2k$. This isn't even even! If this is the full vortex moduli space, then it can't be symplectic, even though we expect it to be. What's going on?

The answer is that the moduli problem for flat $PSU(2)$ connections on the sphere is not well-behaved. It has dimension 0, or dimension -3 as a stack, rather than the expected dimension -6 (which is its dimension as a derived stack).

One way to think about this is well illustrated in the zero vortex sector, which is now nontrivial. When $k = 0$ there are still solutions coming from rational functions ψ of degree 1. These form a group, the group $PSL(2)$ of Möbius transformations. Taking the quotient by $PSU(2)$ gives

$$PSL(2)/PSU(2) \cong H^3,$$

hyperbolic 3-space. This is a kind of vacuum moduli space, which can be thought of as coming from the symmetry breaking $PSL(2) \dashrightarrow PSU(2)$ implemented by the choice of round metric on the target sphere.

This issue does not arise for nonexotic hyperbolic vortices, as in [Wit77], because all the holomorphic automorphisms of the hyperbolic plane (which is the target space) are isometries. The holomorphic symmetry is therefore unbroken by the metric and this strange $k = 0$ moduli space is not present.

A related and interesting point is raised in [Tur20]. There, Turner argues that the size modulus of a single Jackiw–Pi vortex should not be considered as a fluctuating mode in the quantum mechanical theory, because an infrared divergence renders it non-normalisable. Instead, Turner argues that the size modulus should be regarded as a fixed dimensionful parameter in the quantum theory. It seems plausible that this argument applies also to the other residual $PSL(2)$ moduli and that, at least for the purposes of quantum mechanics, one should regard the moduli space as

$$\left(\mathbb{C}P^{k+3} - \Delta\right) / PSL(2).$$

Pleasantly, this has complex dimension k, which is the expected dimension of the moduli problem. The details surrounding the physical and mathematical implementation of this more severe quotient deserve further study.

There is another issue that we have not yet addressed: when $\Sigma = S^2$, no vortex solutions with odd vortex number can be found with this construction. As discussed above, this is because there are no topologically nontrivial $PSU(2)$-bundles on the sphere and so the relevant moduli space of flat connections is empty.

The Jackiw–Pi equation on the torus and Olesen's solution

When Σ is a flat torus, the integrable exotic vortex equation is the Jackiw–Pi equation [JP90]

$$F(A) = \mathrm{i}\kappa|\phi|^2.$$

(Notice that this equation also arises as the infinite radius limit of the Popov equation considered previously. The difference here is in the boundary conditions.)

On the torus there exists a topologically nontrivial $PSU(2)$-bundle. This allows for the construction of solutions with odd vortex number, as in [Ole91]. As illustrated in Figure 3.1, the idea of [Ole91] is to find a particular four-vortex solution on the torus (which can be built from an elliptic function) with the property that it defines a doubly-periodic single vortex solution on the quarter cell.

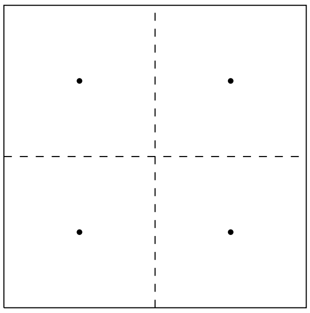


Figure 3.1: A sketch of Olesen's construction of a single vortex solution to the Jackiw–Pi equation on the torus [Ole91]. One starts with a symmetric four-vortex solution, which can be produced from the elliptic function (3.5), and then cuts the domain into four, giving single vortex solutions on the quarter cells.

It is illustrative to consider the details of this. Let Σ be the torus, thought of as a rectangle with side lengths a and b in the upper half plane with opposite sides identified. Letting z be a complex coordinate on the upper-half plane, the symmetric elliptic function found by Olesen can be written

$$\psi_4(z) = \frac{\wp(z) - e_3}{\sqrt{(e_3 - e_1)(e_2 - e_3)}}, \tag{3.5}$$

where $\wp$ is the Weierstrass elliptic function with periods a and b and

$$e_1 = \wp(a/2),\ e_2 = \wp(b/2),\ e_3 = \wp(-(a+b)/2). \tag{3.6}$$

Then the square of the Higgs field is

$$\frac{|\psi_4'|^2}{(1+|\psi_4|^2)^2},$$

which has four zeroes and is periodic with periods $(a/2, b/2)$, not just (a, b), as illustrated in Figure 3.1.

What about ψ_4? As observed in [Ole91], this is not periodic in $a/2$ or $b/2$. Instead, it varies by a Möbius transformation as one moves by a half-period. This means that it is a section of a $\mathbb{C}P^1$-bundle with nontrivial transition functions, which is the basic construction of this chapter.

The general solution to the problem of solving the Jackiw–Pi equation on the torus was given in [Ake$^+$11], generalising the ideas of Olesen. Indeed, the solutions all arise from 'twisted' elliptic functions, referred to as Ω-quasi-elliptic functions in [Ake$^+$11]. The observations that we have made about the topology of $PSU(2)$ bundles resolves a question raised in [Ake$^+$11] about the topology-dependent nature of selection rules for Jackiw–Pi vortices.

3.3.2 The nonAbelian case

Let X be a Kähler manifold with complex structure J and metric g, and let ∇ be the corresponding Levi-Civita connection. Writing R for the curvature of ∇, we can then define a symmetric curvature tensor G by

$$G(v,w) = R(v, Jw) \in \Omega^0(\mathrm{End}(TX))$$

for any pair of vectors v, w. We will proceed as in the Abelian case, where $X = \mathbb{C}P^1$, now using the tensor G to contract indices.

For now, let us not worry about the additional flat connection, and so work with a standard nonlinear sigma model. We consider holomorphic maps

$$\psi : \Sigma \to X.$$

Holomorphicity means that

$$\mathrm{d}\psi + J \circ \mathrm{d}\psi \circ j_\Sigma = 0,$$

where j_Σ is the complex structure on Σ.

Then $\mathrm{d}\psi$ is a holomorphic section of the Hermitian, holomorphic vector bundle $V = K_\Sigma \otimes \psi^* T_X$. The Chern connection A on V is built from those on K_Σ and T_X, giving

$$F(A) = -F(\Sigma) \otimes \mathbb{1} + \psi^* F(\nabla)$$

where $\mathbb{1}$ is the identity endomorphism of $\psi^* T_X$.

We now consider

$$(g_\Sigma^{-1})^{\mu\nu} G(\partial_\mu \psi, \partial_\nu \psi). \tag{3.7}$$

Using the holomorphicity of ψ and the definition of R in terms of $F(\nabla)$, a Bogomolny-type rearrangement tells us that this is

$$*\psi^* F(\nabla).$$

In analogy with the Abelian case, we would like to define

$$\phi \overset{?}{:=} \mathrm{d}\psi$$

as a section of the holomorphic bundle V. As the bundle V has rank n, we expect this to be a Higgs field in a theory with $N_c = n$ and $N_f = 1$.

Writing h for the Hermitian structure on V, we have

$$\begin{aligned}(\phi\phi^{\dagger_h})^i_j &= \phi^i h_{jk} \bar{\phi}^k \\ &= (h_\Sigma^{-1})^{z\bar{z}} \partial_z \psi^i (h_X)_{jk} \partial_{\bar{z}} \bar{\psi}^k\end{aligned}$$

where h_Σ is the Hermitian structure on T_Σ and $h_X = (h_X)_{ij} \mathrm{d}\psi^i \mathrm{d}\bar{\psi}^j$ is the Hermitian structure on $\psi^* T_X$. By contracting the above expression with arbitrary vectors and expanding the Hermitian structure in terms of g and J, we see that this is the same object as (3.7) up to a factor κ if

$$\begin{aligned}&g(R(u,v)w,x) = \\ &\quad \frac{\kappa}{4}\left(g(u,w)g(v,x) - g(v,w)g(u,x) + g(u,Jw)g(v,Jx) - g(u,Jx)g(v,Jw) + 2g(u,Jv)g(w,Jx)\right)\end{aligned}$$

for vectors u, v, w, x, which is equivalent to X having constant *holomorphic sectional curvature* κ [KN96]. The holomorphic sectional curvature is defined by

$$H(\xi) := g(R(\xi,\xi)\xi,\xi)/|\xi|^4$$

for ξ a holomorphic tangent vector.

Thus we find exotic vortex solutions from holomorphic curves in Kähler manifolds of constant holomorphic sectional curvature. In particular, the solutions that one finds are exotic vortex solutions in theories with $N_c = \dim_{\mathbb{C}}(X)$ colours and $N_f = 1$ flavour.

Manifolds with a complete Kähler metric of constant holomorphic sectional curvature are called *complex space forms*, in analogy with the space forms of Riemannian geometry (which are complete Riemannian manifolds of constant sectional curvature). Every simply connected complex space form is either a complex projective space $\mathbb{C}P^n$ (with $H > 0$), a flat space $\mathbb{C}^n$ (with $H = 0$), or a hyperbolic space $\mathbb{H}^n$ (with $H < 0$) [KN96]. This generalises the well-known uniformatisation theorem for $n = 1$.

In the context of exotic vortices, we are interested in the case of positive curvature, so we take $X = \mathbb{C}P^n$. Just as before, we can generalise this by introducing an additional flat connection for the isometry group of the target space X. In the case that $X = \mathbb{C}P^n$, the isometry group of X is $PSU(n+1)$.

Notice that this method also provides a way to generate solutions to the *non-exotic* nonAbelian vortex equations, using holomorphic maps into the hyperbolic space $\mathbb{H}^{N_c+1}$ or quotients thereof. This generalises the result of [Wit77]. Such maps do exist and it would be interesting to investigate the nature of the vortex solutions that can be generated in this way when $N_c > 1$.

On moduli

Fix a Riemann surface Σ and consider the theory of a flat $PSU(n+1)$ connection on Σ and a section ψ of the associated $\mathbb{C}P^n$-bundle.

The real index of the corresponding moduli problem is

$$((n+1)^2 - 1)(2g-2) + 2\psi^* c_1(T_{\mathbb{C}P^n})[\Sigma] + n(2-2g).$$

The vortex number is the degree of the bundle V which is

$$k = n(2g-2) + \psi^* c_1(T_X)[\Sigma]. \tag{3.8}$$

In terms of k, the real index of the moduli problem is

$$2k + n(1-n)(2-2g). \tag{3.9}$$

On the other hand, the real expected dimension of the moduli space of vortices in a $U(N_c)$ gauge theory with N_f flavours is

$$2k + N_c(N_f - N_c)(2-2g). \tag{3.10}$$

This is formally independent of the sign of the coupling constant, so it applies in the exotic case.

The solutions we find from twisted holomorphic curves are meant to be solutions in a theory with $N_c = n$ and $N_f = 1$. Comparing (3.9) and (3.10), we see that the expected dimensions of the moduli spaces of the relevant vortices and the relevant twisted holomorphic curves agree. This leads us to conjecture that all nonAbelian exotic vortex solutions with $\kappa > 0$ on constant curvature backgrounds arise in this way.

Selection rules

In the Abelian case, we saw that integrable exotic vortices on the sphere must have even vortex number. This followed because of the non-existence of topologically nontrivial $SO(3)$ bundles on the sphere and because of the Riemann–Hurwitz relation.

Here we have a similar story. Bundles with structure group $PSU(n+1)$ are topologically classified by a $\mathbb{Z}_{n+1}$-valued invariant. More precisely, $PSU(n+1)$ bundles on a Riemann surface Σ are classified topologically by $H^1(\Sigma, \mathbb{Z}_{n+1})$. On the sphere, any $PSU(n+1)$ bundle is necessarily trivial.

For the trivial bundle, the Riemann–Hurwitz-type formula (3.8) tells us that the vortex number k must take the form

$$k = n(2g-2) + (n+1)p$$

for $p \in \mathbb{Z}$. When nontrivial bundles exist, they allow for $k - n(2g-2)$ to take nontrivial values modulo $n+1$.

3.3.3 Introducing curvature through uniformisation

So far, our results require us to consider the vortex equations on constant curvature backgrounds. We can lift this requirement by introducing additional 'uniformising data' in the solution.

Let Σ be a Riemann surface with Hermitian metric h_Σ. Any other such metric can be given as

$$h_f = e^f h_\Sigma$$

for f some real function. The curvature of this metric is

$$F(h_f) = F(h_\Sigma) + \bar{\partial}\partial f.$$

We would like to set $F(h_f) = \mathrm{i}\kappa_0\omega_\Sigma$, where κ_0 is a constant. This leads to a Poisson-type equation

$$\bar{\partial}\partial f = \mathrm{i}\kappa_0\omega_\Sigma - F(h_\Sigma)$$

for f, which can be solved provided that $[\mathrm{i}\kappa_0\omega_\Sigma] = [F(h_\Sigma)]$ in $H^2(\Sigma)$, which fixes κ_0. The sign of κ_0 is determined by topology through the Gauss–Bonnet theorem. To solve the equation we note that, if the condition above is satisfied, then $\mathrm{i}\kappa_0\omega_\Sigma - F(h_\Sigma)$ is an exact form on Σ and we can use the $\bar{\partial}\partial$-lemma. Write f_0 for a solution to this equation and write $h_0 := h_{f_0}$.

We now turn to the vortex equations. Previously, we found a solution $(V, \bar{\partial}_V, h, \phi)$ in the holomorphic gauge by taking

$$V = K_\Sigma \otimes \psi^* T_{\mathcal{X}}$$

for ψ a section of a certain X-bundle $\mathcal{X}$, with X a Kähler manifold of constant holomorphic sectional curvature, and letting h be the natural tensor product metric $h_\Sigma^{-1} \otimes h_{\psi^* T_\mathcal{X}}$.

When h_Σ does not have constant curvature, we simply let the Hermitian structure h be $h_0^{-1} \otimes h_{\psi^* T_\mathcal{X}}$.

While this saves us from constant curvature, it does not save us from global issues: the Fayet–Iliopoulos parameter in the vortex equation that this construction solves is still fixed by virtue of the Gauss–Bonnet theorem (although it can be varied by scaling the volume of Σ).

Putting everything together, we can now state our main technical result.

3.3.1 Theorem. *Let X be a complete Kähler manifold with constant holomorphic sectional curvature κ and isometry group H. Then every pair consisting of a flat H-connection and a holomorphic section of the associated holomorphic X-bundle on a Riemannian surface Σ of genus g gives rise to a solution to the exotic vortex equations* (3.1) *in a $U(dim_\mathbb{C}(X))$ gauge theory with one fundamental flavour with $e^2 = -\kappa$ and $\tau = \frac{2-2g}{2\pi vol(\Sigma)\kappa}$. Moreover, gauge equivalent pairs of this form give rise to gauge equivalent vortex solutions.*

Perhaps more important than this result itself is that our proof is constructive. It therefore represents an integrability-type result for vortices in nonAbelian gauge theories.

3.4 THE DYNAMICS OF EXOTIC VORTICES

3.4.1 Nonrelativistic Chern–Simons-matter theories

The fact that the configuration space $\mathcal{C}$ for the theory with $e^2 < 0$ is naturally symplectic but not naturally Kähler implies that the dynamical $(2+1)$-dimensional theory of exotic vortices should come from Hamiltonian dynamics on the configuration space. Thus the space $\mathcal{C}$ is the phase space of the $(2+1)$-dimensional theory.

This is exactly the setup of chapter 2, where we studied nonrelativistic Chern–Simons-matter theories, realising them as theories of Hamiltonian mechanics on $\mathcal{C}$. The story goes through *mutatis mutandi* with $e^2 < 0$, so we do not repeat it here. The only subtle point concerns the sign of the Chern–Simons level. For our conventions, it turns out to be natural to accompany a flip in the sign of e^2 with a flip in the sign of the Chern–Simons level.

The low-temperature theory is given by Hamiltonian mechanics on the moduli space of vortices with zero Hamiltonian, capturing the fact that the classical theory is dynamically trivial at low energy. As before, though, there is an interesting topological quantum mechanics at low temperature, obtained by geometric quantisation of the vortex moduli space $\mathcal{M}$.

Even though our understanding of the exotic vortex moduli space is limited, we can formally apply the ideas of chapter 2 to compute the Euler characteristic of a (possibly virtual) quantum line bundle $\mathcal{L} \to \mathcal{M}$ on the moduli space. Recall that such a bundle is one with first Chern class equal to the class of the symplectic form on $\mathcal{M}$ divided by 2π.

To properly carry out the geometric quantisation so as to obtain a space of states, one must choose a polarisation for the line bundle $\mathcal{L}$ (corresponding, roughly, to a choice of ‘position’ or 'momentum' representation of the quantum states). In the non-exotic case, there is a natural way to do this, by giving $\mathcal{L}$ the structure of a holomorphic line bundle. In the exotic case, $\mathcal{M}$ is not necessarily a complex manifold in any natural way (or at all) so this is not a natural procedure. The quantity we compute is sufficiently 'soft' that we need not worry about this. We will simply assume that a reasonable procedure exists and that the required index theorem still holds (to some extent, this follows formally from loop space supersymmetry methods). A more detailed study would have to deal with this potentially thorny issue.

This is not the only thorny issue in play. We saw that the moduli space of Popov-type vortices on the sphere was not even symplectic, even though it is 'expected' to be. We must be aware of this kind of phenomenon. As discussed above, the arguments of [Tur20] suggest that one should omit the extra modes associated with non-rigid Möbius transformations. If one does this, the moduli space does become even-dimensional and so could be symplectic in a natural way. The details of implementing this idea mathematically deserves study. Treating these badly behaved moduli spaces as symplectic manifolds in a formal sense presumably requires a kind of derived symplectic geometry. Physically, this could be understood by including 'ghost' and 'antighost' zero-modes in the moduli space (or, more-or-less equivalently, by working with the supersymmetric completion). We will not worry too much about this, proceeding in a rather formal fashion (the Euler characteristic formally includes the ghost-like fields, but does not allow us to disentangle the bosonic vortex moduli from the ghost moduli).

Such blindness to technical issues makes things rather easy: the general answers we found in chapter 2 did not (formally) require us to take $e^2 > 0$.

3.4.2 Exotic vortex counting

For simplicity, we will illustrate the idea only for the Abelian case. Recall from theorem 2.3.6 that the formal Euler characteristic for k Abelian vortices with N_f flavours at Chern–Simons level λ on a Riemann surface of genus g is

$$\sum_{j=0}^{g} \lambda^j N_f^{g-j} \binom{g}{j} \binom{\lambda(\mathcal{A}-k) + N_f(k-g+1) - 1}{N_f(k-g+1) + (g-j) - 1}$$

where $\mathcal{A} = \frac{e^2 \tau \mathrm{vol}(\Sigma)}{4\pi}$.

It is natural to accompany a flip in the sign of e^2 with a flip in the sign of λ. We will see this explicitly in a moment. The fundamental physical reason for this is discussed in [Tur20]: it is related to the (in)stability of the theory under renormalisation group flow at the relevant fixed point. Thus, while in the non-exotic case we found that the theory was best-behaved when $\lambda = N_f$, here we will consider the case of $\lambda = -N_f$.

Indeed, the case of $\lambda = -1$ provides a good window into nature the theory. When $g = 0$, one finds the index

$$\binom{-\mathcal{A} + 2k}{k}.$$

This captures the 'Bradlow bound'

$$-k < -\mathcal{A}$$

for this theory correctly.

In particular, we see that Jackiw–Pi vortices on the sphere at level -1 have expected degeneracy

$$\binom{2k}{k}. \tag{3.11}$$

Of course, this should be supplemented with the knowledge that there are no vortex solutions when k is odd. To some extent, this result gets to the heart of the question of why Jackiw–Pi vortices are interesting. When $\tau = 0$, the theory is (at least naïvely) gapless. However, this result indicates the potential existence of an interesting topological quantum mechanics at low temperature, which is usually a feature of gapped theories.

The result (3.11) is altered on surfaces of higher genus. For example, when $g = 1$, it is

$$\binom{2k-1}{k} - \binom{2k-1}{k-1}.$$

This is always zero! In fact, whenever g is odd, the terms in the index cancel pairwise, and the result is zero. This is probably because we are computing an index and not the true dimension of the Hilbert space, but it is intriguing nonetheless.

4.1 INTRODUCTION AND SUMMARY

In chapter 2, we were concerned with quantities of the form

$$\chi(\mathcal{L}^\lambda)$$

where $\mathcal{L}^\lambda \to \mathcal{M}$ is a line bundle on the vortex moduli space $\mathcal{M}$ with the property that

$$c_1(\mathcal{L}^\lambda) = \frac{\lambda}{2\pi}[\omega_{\mathcal{M}}],$$

where $\omega_{\mathcal{M}}$ is the natural Kähler form on the vortex moduli space.

We consider the semiclassical limit $\lambda \to \infty$. Recall that, in previous examples with λ finite, we had to impose some kind of quantisation condition on the dimensionless area scale $\mathcal{A}$. In general, though, the quantity that really must be integral for the quantum theory to exist is $\lambda\mathcal{A}$, so that in the limit $\lambda \to \infty$ the area quantisation condition becomes vacuous.

The Hirzebruch–Riemann–Roch theorem tells us what the semiclassical limit of the vortex count is. One has

$$\chi(\mathcal{L}^\lambda) = \int_{\mathcal{M}} \exp\left(\frac{\lambda}{2\pi}[\omega_{\mathcal{M}}]\right) \mathrm{td}(\mathcal{M})$$

so that as $\lambda \to \infty$, the leading term is

$$\left(\frac{\lambda}{2\pi}\right)^{\dim\mathcal{M}} \int_{\mathcal{M}} \frac{\omega_{\mathcal{M}}^{\dim(\mathcal{M})}}{\dim(\mathcal{M})!} = \left(\frac{\lambda}{2\pi}\right)^{\dim\mathcal{M}} \mathrm{vol}(\mathcal{M})$$

where the volume is defined with respect to the Kähler structure and we have used that the zeroeth order part of the Todd class is 1.

Thus, the number of states in the semiclassical theory is measured by the volume of the vortex moduli space. For related reasons, the volume of the vortex moduli space computes the statistical mechanical partition function of a gas of vortices in a *relativistic* theory [Man93; MN99b]. The volumes of vortex moduli spaces are therefore interesting quantities that have attracted attention. In [MN99b], Manton and Nasir computed the volume of the moduli space of local Abelian vortices by direct integration of the Kähler form (or a representative of its cohomology class). This built on previous work [Man93] of Manton and aspects of the calculation were later clarified by Perutz [Per06] and Baptista [Bap11]. In [Eto$^+$08b], the statistical mechanics of a gas of local nonAbelian vortices on the torus was studied by giving a D-brane realisation of vortices and using T-duality to relate them to domain walls. In [MOS11] and [OS19], Miyake–Ohta–Sakai and Ohta–Sakai computed the volumes of moduli spaces of vortices in $U(N_c)$ gauge theories with N_f flavours on arbitrary Riemann surfaces using localisation (although the authors followed the calculation through to the end only for the genus zero and genus one cases). The work of this part was greatly inspired by this calculation.

In general, the volume of the vortex moduli space is a polynomial in $\mathcal{A}$ of degree $\dim(\mathcal{M})$. One of the most striking results of [MOS11; OS19] was that in the local case $N_c = N_f =: N$ the volume of the moduli space of k vortices on a genus zero surface is a polynomial in $\mathcal{A}$ of degree k, which represents a striking reduction from the expected degree of Nk. An analogous result was seen in [Eto+08b] for the vortex gas on the torus.

In this part, we have seen a similarly striking reduction in the Euler characteristics of quantum line bundles on moduli spaces of local vortices. We saw such a reduction for every genus, but [MOS11] saw such a reduction of the volume only for genus zero, and not for genus one (the authors of [MOS11] did not go into detail in the $g > 1$ case, as the calculation becomes very involved).

This is rather strange, because it implies that the thermodynamics of a vortex gas depends strongly on the topology of the underlying surface, which conflicts with general lore that thermodynamics depend weakly on such global effects (see [MN99b], for example). It also disagrees with the result of [Eto+08b]. It seems unlikely that adding a handle to a surface should create a potentially large thermodynamic degeneracy.

We argue that this is due to the particular residue prescription used by [MOS11]. More precisely, we argue that the interpretation of negative powers of the Vandermonde determinant in [MOS11] is incorrect (or at least leads to incorrect answers). Here we give an alternative calculation.

4.2 THE VOLUMES OF VORTEX MODULI SPACES

4.2.1 Abelian vortices

Let us first consider the volume of the moduli space of k-vortex solutions in an Abelian theory with gauge group $U(1)$ and N_f fundamental flavours. This case is well-studied: we will aim to recover known results in this case.

The Euler characteristic $\chi(\mathcal{L}^\lambda)$ is, in the Abelian case,

$$\sum_{j=0}^{g} \binom{g}{j} N_f^{g-j} \lambda^j \binom{\lambda(\mathcal{A}-k) + N_f k + (N_f-1)(1-g) - g}{N_f k + (N_f-1)(1-g) - j} \tag{4.1}$$

as we found in theorem 2.3.6.

Using that

$$\binom{\lambda n + p}{q} = \frac{(\lambda n)^q}{q!} + \cdots,$$

we see that (4.1) is

$$\lambda^{N_f k + (N_f-1)(1-g)} \sum_{j=0}^{g} N_f^{g-j} \frac{(\mathcal{A}-k)^{N_f k + (N_f-1)(1-g) - j}}{(N_f k + (N_f-1)(1-g) - j)!} + \cdots$$

where $\cdots$ denotes terms of lower order in λ. We read off the volume of the moduli space to be

$$\sum_{j=0}^{g} N_f^{g-j} \frac{(\mathcal{A}-k)^{N_f k + (N_f-1)(1-g) - j}}{(N_f k + (N_f-1)(1-g) - j)!}$$

at least up to a constant scale factor. This recovers the results of [MOS11; OS19].

Taking $N_f = 1$ gives the relatively simple formula

$$\sum_{j=0}^{g} \frac{(\mathcal{A}-k)^{k-j}}{(k-j)!}$$

for the volume of the moduli space of local Abelian vortices (up to constant scale factors). This recovers the original result of [Man93; MN99b].

4.2.2 The general result

The Euler characteristic for general N_c and N_f is

$$\begin{aligned}\chi(\mathcal{L}^\lambda) =& \frac{(-1)^\sigma 2^{(N_c-1)g}}{N_c!} \sum_{\sum k_a=k} \sum_{\sum l_a=N_c(N_c-1)} \sum_{c_1,\cdots,c_{N_c}} a_{l_1,\cdots,l_{N_c}} A_{c_1,\cdots,c_{N_c}} \\ &\times \prod_a \left(\sum_{j=0}^{g} \binom{g}{j} \lambda^j N_f^{g-j} \binom{\lambda(\mathcal{A}-k_a) + (N_c-1) + N_f k_a + (N_f-1)(1-g) - l_a - c_a - g}{N_f k_a + (N_f-1)(1-g) - l_a - c_a + g(N_c-1) - j} \right),\end{aligned}$$

as given in (2.24).

The volume of the vortex moduli space is the coefficient of the leading term in λ. Taking the leading term in λ of each of the binomial coefficients, we find that

$$\begin{aligned}\chi(\mathcal{L}^\lambda) = \lambda^{N_f k + N_c(N_f-N_c)(1-g)} & \frac{(-1)^\sigma 2^{(N_c-1)g}}{N_c!} \sum_{\sum k_a=k} \sum_{\sum l_a=N_c(N_c-1)} \sum_{c_1,\cdots,c_{N_c}} a_{l_1,\cdots,l_{N_c}} A_{c_1,\cdots,c_{N_c}} \\ \times \prod_a & \left(\sum_{j=0}^{g} \binom{g}{j} \lambda^{-c_a} N_f^{g-j} \frac{(\mathcal{A}-k_a)^{N_f k_a + (N_f-1)(1-g) - l_a - c_a + g(N_c-1) - j}}{(N_f k_a + (N_f-1)(1-g) - l_a - c_a + g(N_c-1) - j)!} \right) + \cdots .\end{aligned}$$

where the dots again denote lower order terms. The definition of the $A_{c_1,\cdots,c_{N_c}}$ reveals that they are nonzero only when the sum of the c_a vanishes. The factors of λ^{-c_a} therefore cancel and we see that this is the leading term in λ, proportional to $\lambda^{\dim(\mathcal{M})}$. Following our earlier discussion, we identify the coefficient as the volume of the moduli space:

$$\begin{aligned}\mathrm{Vol}(\mathcal{M}) \sim & \frac{(-1)^\sigma 2^{(N_c-1)g}}{N_c!} \sum_{\sum k_a=k} \sum_{\sum l_a=N_c(N_c-1)} \sum_{c_1,\cdots,c_{N_c}} a_{l_1,\cdots,l_{N_c}} A_{c_1,\cdots,c_{N_c}} \\ & \times \prod_a \left(\sum_{j=0}^{g} \binom{g}{j} N_f^{g-j} \frac{(\mathcal{A}-k_a)^{N_f k_a + (N_f-1)(1-g) - l_a - c_a + g(N_c-1) - j}}{(N_f k_a + (N_f-1)(1-g) - l_a - c_a + g(N_c-1) - j)!} \right),\end{aligned} \tag{4.2}$$

at least up to constant factors. This result is the general one, but it is not particularly amenable to further analysis. In the general case, there is no reason to expect it to be simple at all.

4.2.3 Local vortices

Our claim is that, when $N_c = N_f =: N$ the volume of the k-vortex moduli space is a polynomial in $\mathcal{A}$ of degree k. The form of (4.2) reveals that it is a polynomial of maximal degree Nk, so this would represent a significant reduction.

For $N_c = N_f =: N$, the volume of the k-vortex moduli space (4.2) is

$$\begin{aligned} \frac{(-1)^\sigma 2^{(N-1)g}}{N!} &\sum_{\sum k_a = k} \sum_{\sum l_a = N(N-1)} \sum_{c_1,\cdots,c_N} a_{l_1,\cdots,l_N} A_{c_1,\cdots,c_N} \\ &\times \prod_a \left(\sum_{j=0}^{g} \binom{g}{j} N^{g-j} \frac{(\mathcal{A} - k_a)^{Nk_a + (N-1) - l_a - c_a - j}}{(Nk_a + (N-1) - l_a - c_a - j)!} \right) \end{aligned} \tag{4.3}$$

A clean way to deal with expressions such as this is to consider the generating function (as in [OS19] and in chapter 2). In general though, it is not easy to obtain simple expressions for the relevant generating functions.

For this reason, we will give a softer argument, reducing the question of the degree of the polynomial to the results of chapter 2. Our argument can be extended to show that the Euler characteristic $\chi(\mathcal{L}^\lambda)$ is a polynomial in $\mathcal{A}$ of degree k for all λ when $N_c = N_f$.

We note first that the degree of (4.3) as a polynomial in $\mathcal{A}$ is bounded above by the degree of

$$\begin{aligned} \frac{(-1)^\sigma 2^{(N-1)g}}{N!} &\sum_{\sum k_a = k} \sum_{\sum l_a = N(N-1)} \sum_{c_1,\cdots,c_N} a_{l_1,\cdots,l_N} A_{c_1,\cdots,c_N} \\ &\times \prod_a \left(\sum_{j=0}^{g} \binom{g}{j} N^{g-j} \binom{(\mathcal{A} - k_a) + k_a + (N-1)(2-g) - l_a - c_a - g}{Nk_a + (N-1) - l_a - c_a - j} \right) \end{aligned}$$

as a polynomial in $\mathcal{A}$. To see this, one can expand this latter polynomial and realising that it contains (4.3) without cancellations.

Expanding out the product gives

$$\begin{aligned} \frac{(-1)^\sigma 2^{(N-1)g}}{N!} &\sum_{\sum k_a = k} \sum_{\sum l_a = N(N-1)} \sum_{c_1,\cdots,c_N} \sum_{j_1,\cdots,j_N=0}^{g} a_{l_1,\cdots,l_N} A_{c_1,\cdots,c_N} \alpha_{j_1,\cdots,j_N} \\ &\times \prod_a \binom{\mathcal{A} + (N-1)(2-g) - l_a - c_a - g}{Nk_a + (N-1) - l_a - c_a - j_a} \end{aligned}$$

where

$$\alpha_{j_1,\cdots,j_N} = \prod_{a=1}^{N} \binom{g}{j_a} N_f^{g-j_a}.$$

We analyse this by passing to the generating function. Using lemma 2.5.3, this quantity is the coefficient of x^{Nk} in the polynomial

$$\begin{aligned} \frac{(-1)^\sigma 2^{(N-1)g}}{N!N^N} &\sum_{\sum k_a = k} \sum_{\sum l_a = N(N-1)} \sum_{c_1,\cdots,c_N} \sum_{j_1,\cdots,j_N=0}^{g} a_{l_1,\cdots,l_N} A_{c_1,\cdots,c_N} \alpha_{j_1,\cdots,j_N} \\ &\times \prod_a x^{(N-1)-l_a-c_a-j_a} \left(\sum_{b=0}^{N} r^{(1-l_a-c_a-j_a)b} (1 + r^b x)^{\mathcal{A}+(N-1)(2-g)-l_a-c_a} \right), \end{aligned}$$

where r generates the N^{th} roots of unity. We have $\prod_a x^{N-1-l_a} = 1$, which simplifies the product. Expanding the product gives

$$\frac{(-1)^\sigma 2^{(N-1)g}}{N!N^N} \sum_{\sum l_a = N(N-1)} \sum_{c_1,\cdots c_N} \sum_{j_1,\cdots,j_N=0}^{g} a_{l_1,\cdots,l_N} A_{c_1,\cdots,c_N} \alpha_{j_1,\cdots,j_N}$$
$$\times \sum_{b_1,\cdots,b_N=1}^{N} \prod_a x^{-c_a-j_a} r^{(1-l_a-c_a-j_a)b_a} (1+r^{b_a}x)^{\mathcal{A}+(N-1)(2-g)-l_a-c_a}.$$

As we have seen multiple times now, the coefficients $a_{l_1,\cdots,l_N}$ conspire to make it so that the only nonzero contributions come when all of the b_a are different. This gives

$$\frac{(-1)^\sigma 2^{(N-1)g}}{N^N} \sum_{\sum l_a = N(N-1)} \sum_{c_1,\cdots c_N} \sum_{j_1,\cdots,j_N=0}^{g} a_{l_1,\cdots,l_N} A_{c_1,\cdots,c_N} \alpha_{j_1,\cdots,j_N}$$
$$\times \prod_a x^{-c_a-j_a} r^{(1-l_a-c_a-j_a)a} (1+r^a x)^{\mathcal{A}+(N-1)(2-g)-l_a-c_a-g}.$$

We now wind the expansions back up using the definition of the various coefficients. We find

$$\begin{aligned} 2^{(N-1)g} N^{-N} N! (1-x^N)^{\mathcal{A}+(N-1)(2-2g)-g} \prod_{a<b} \left(\frac{1}{r^a(1+r^a x)} - \frac{1}{r^b(1+r^b x)} \right)^{2-2g} \prod_a (r^a x + N)^g \\ = \pm 2^{(N-1)g} N^{-gN} N! (1-x^N)^{\mathcal{A}-g} \prod_a (r^a x + N)^g \end{aligned}$$

which is

$$\begin{aligned} F(x) &:= (-1)^k (1-x^N)^{\mathcal{A}-g} \prod_a \left(1 + \frac{r^a x}{N}\right)^g \\ &= (-1)^k (1-x^N)^{\mathcal{A}-g} \left(1 - \left(\frac{x}{N}\right)^N\right)^g. \end{aligned}$$

The coefficient of x^{Nk} in $F(x)$ is a coarse approximation to the volume of the k-vortex moduli space. This is the quantity

$$\sum_{i+j=k} \binom{\mathcal{A}-g}{i} \binom{g}{j} \left(\frac{1}{N}\right)^{Nj}. \tag{4.4}$$

If $k > g$, the terms with $j > g$ do not contribute.

This is a polynomial in $\mathcal{A}$ of degree k for all g and all N. As the degree of the polynomial defining the volume is bounded above by the degree of (4.4), we have shown that the volume of the k-vortex moduli space is a polynomial in $\mathcal{A}$ of degree at most k.

PART III

MIRROR SYMMETRY IN THREE DIMENSIONS

5 THE THREE-DIMENSIONAL A-MODEL

5.1 INTRODUCTION AND SUMMARY

Given a quaternionic representation V of a compact Lie group G, one can form a three-dimensional gauge theory with $\mathcal{N} = 4$ supersymmetry. The infrared behaviour of such a theory is determined in part by its moduli space of vacua, which decomposes into a number of branches, each of which is a (possibly singular) hyperKähler space. Of particular interest are the *Higgs branch*

$$\mathcal{M}_H = V///G,$$

parameterising vacua in which the gauge group is completely broken, and the *Coulomb branch*

$$\mathcal{M}_C \simeq (\mathbb{R}^3 \times S^1)^{\mathrm{rank}(G)}/\mathrm{Weyl}(G),$$

parameterising vacua in which the gauge group is broken to a maximal torus. Other 'mixed' branches mediate between these two extremes. The geometry of the Higgs branch is determined exactly by the classical hyperKähler quotient construction of [Hit90a], but the geometry of the Coulomb branch receives perturbative and non-perturbative quantum corrections [SW96].

In this chapter, we consider some aspects of a remarkable story which has emerged, that of a deep link between the geometry of the Higgs and Coulomb branches. From a classical perspective, there is little reason to think that there should be any relationship at all between the two spaces. Indeed, the classical Coulomb branch does not even see the representation V. However, there is significant evidence for such a relationship (see [Tau90; RW97; MM99; BT02]) culminating with the program of Braverman–Finkelberg–Nakajima to construct the Coulomb branch in terms of Higgs branch data [Nak16; BFN18] and in the program of symplectic duality [Bra$^+$14; Bul$^+$16].

Let us offer a sketch of the story. By topologically twisting a three-dimensional $\mathcal{N} = 4$ sigma model into a hyperKähler manifold (or stack) X carrying an isometric $Sp(1)$ action permuting its complex structures, one can attempt to produce a pair of distinct three-dimensional topological field theories, which we call Z_A^X and Z_B^X. The theory Z_A^X is known as the *A-twist*[1] and Z_B^X is *Rozansky–Witten theory* [RW97], also known as the *B-twist*[2]. The relationship to which we allude between the Higgs branch, $\mathcal{M}_H$, and the Coulomb branch, $\mathcal{M}_C$, should then be captured (at least schematically) by the duality

$$Z_A^{\mathcal{M}_H} \overset{?!}{\simeq} Z_B^{\mathcal{M}_C} \tag{5.1}$$

and similarly with $\mathcal{M}_H$ and $\mathcal{M}_C$ switched. As we argue shortly, this duality can be understood as stating the equivalence of two different descriptions of a single topological twist of the ultraviolet gauge theory. The richest realisation of this duality would be an equivalence between the 2-categories of boundary conditions of the two theories (see [Kap10]).

[1]Sometimes this A-twisted theory is called 'twisted' Rozansky–Witten theory, but we will refrain from doing this to avoid overuse of the word 'twisted'.

[2]There is significant potential for confusion here, because what is sometimes called the Rozansky–Witten twist of a gauge theory can be described in terms of the A-twisted sigma model into the Coulomb branch (or in terms of the B-twist in the Higgs branch). We will refer to the twists of the ultraviolet gauge theory as the H- or C-twists, for 'Higgs' and 'Coulomb' respectively. The situation is illustrated in Figure 5.1.

One half of (5.1), that of the Rozansky–Witten theory, is rather well-understood. In [KRS09; KR10], a description of the 2-category of boundary conditions in Rozansky–Witten theory was given. There, it was shown that boundary conditions in Z_B^X are given by complex Lagrangian submanifolds of X, thought of as a complex symplectic manifold in a fixed complex structure, generally carrying additional degrees of freedom (given in the general case by complexes of sheaves of categories; see also [Tel14]).

Many elements of the A-twist are also well-understood, with significant recent attention focussing on its spaces of quantum states [Nak16; BFN18; Gai16; BFK19; Bul$^+$18; CDZ20], on its category of line operators [CCG19; Dim$^+$20], and on its local boundary conditions [Bul$^+$16; CO17; CG19]. In [Set13; KSV10], the A-twist of the circle compactification of four-dimensional $\mathcal{N}=4$ gauge theory was constructed. However, some aspects of the theory Z_A^X, including its general definition, remain rather mysterious. This is partly because it is rarely considered on general backgrounds, meaning that some of the more subtle geometrical issues related to the twist do not raise their heads.

Our aim in this chapter is to describe the structure of Z_A^X as an extended three-dimensional (framed) topological quantum field theory. In section 5.4, we formulate the theory on general three-manifolds, describing its local and extended operators. Computations of correlation functions reduce to integrals over moduli spaces of solutions to a certain nonlinear Dirac equation (which becomes an equation for certain quasimaps when reduced to two dimensions, as familiar from [Nak16; BFK19; Kim16]). Compactifying the theory on the circle gives a two-dimensional topological theory that may be thought of as the two-dimensional A-model with a certain defect operator inserted. This is mirror to the fact that Rozansky–Witten theory gives the two-dimensional B-model with a certain local operator inserted when compactified on the circle [RW97; Tho98; KRS09].

Our interest is particularly in the boundary conditions of the theory. We show that boundary conditions in the A-twist are determined by complex Lagrangian submanifolds of the target space with a certain invariance property.

We then analyse the effective theory obtained on compactification on an interval in special cases, which represents an early step towards understanding the collection of boundary conditions as a 2-category. In special cases, the compactified theory is the two-dimensional A-model in a Kähler manifold. We use our construction to understand the conditions under which Fukaya categories of Kähler manifolds may admit monoidal structures (see subsection 5.1.3 below).

We remark that a three-dimensional A-model has been constructed previously by Kapustin–Vyas in [KV10] as a generalisation of the matter sector of a three-dimensional topological Abelian gauge theory considered in [KSV10]. Our theory and that of Kapustin–Vyas are distinct, but agree in certain special cases. We compare the two theories in subsection 5.4.6.

We also note that the A-twist can be viewed as a reduction of a four-dimensional topological sigma model considered by Anselmi and Fré [AF94; AF95] (modulo the 'topological' term in that theory).

In the rest of this introduction we summarise the story of the duality (5.1) in more detail. In section 5.2, we review some fundamental elements of three-dimensional gauge theories with $\mathcal{N}=4$

supersymmetry and topological twists thereof. We offer a (possibly novel) description of the space of twists as a holomorphic symplectic quotient. In section 5.3, we consider the nonlinear Dirac equations that arise in the description of the A-twist. We consider the geometry of solutions on certain backgrounds, including nontrivial circle bundles over surfaces. In section 5.4, we construct and analyse the theory Z_A^X as a cohomological field theory. Finally in section 5.5, we give a brief sketch of a gravity-coupled version of the theory. This has the benefit of allowing for the excision of singularities in the target space (at the cost of dealing with strongly-coupled gravity).

5.1.1 Mirror symmetry and flows

Certain pairs of three-dimensional $\mathcal{N} = 4$ gauge theories are related by an infrared duality called three-dimensional mirror symmetry [IS96]. The group of global symmetries of an undeformed three-dimensional $\mathcal{N} = 4$ theories has a subgroup given by the *R-symmetry group*

$$SO(4) = (Sp(1)_H \times Sp(1)_C)/\mathbb{Z}_2,$$

which rotates the supercharges among themselves. In a gauge theory, the group $Sp(1)_H$ acts isometrically on the Higgs branch, permuting its complex structures, while the group $Sp(1)_C$ acts isometrically on the Coulomb branch, permuting its complex structures. The R-symmetry group has an outer automorphism of order two given by switching $Sp(1)_H$ and $Sp(1)_C$. In nice cases, this automorphism induces mirror symmetry of pairs of linear gauge theories. It interchanges the Higgs and Coulomb branches.

If T is a theory with Higgs branch $\mathcal{M}_H$ and Coulomb branch $\mathcal{M}_C$ and $T^\vee$ is a theory mirror to T, with Higgs branch $\mathcal{M}_H^\vee \cong \mathcal{M}_C$ and Coulomb branch $\mathcal{M}_C^\vee \cong \mathcal{M}_H$, the duality (5.1) implies that

$$Z_A^{\mathcal{M}_H} \overset{(5.1)}{\simeq} Z_B^{\mathcal{M}_C} \overset{\text{mirror}}{\simeq} Z_B^{\mathcal{M}_H^\vee},$$

and simliarly with $\mathcal{M}_C$. This tells us that, just as the Higgs and Coulomb branches are interchanged by mirror symmetry, so are the A-twist and the B-twist.

This is exactly what one expects from the topological twisting procedure (of [Wit88a; Wit88b]) used to produce these topological theories. We are interested primarily in sigma models, so suppose that we have an effective description of a gauge theory as a sigma model into its Higgs branch (say). Then, identifying the (Euclidean) spacetime spin group $Sp(1)_E$ with $Sp(1)_H$ leads to the A-twist with target space the Higgs branch. If one carries out the mirror twist, identifying $Sp(1)_E$ with $Sp(1)_C$, one obtains the B-twist with the same target space.

This can be thought of in the following way. The ultraviolet gauge theory can be described in the infrared by a sigma model into the Higgs branch or the Coulomb branch (or a mixed branch, or a union of branches). Then, roughly speaking, the $Sp(1)_H$ twist, say, of the ultraviolet gauge theory can be described by $Z_A^{\mathcal{M}_H}$ in the Higgs phase, and by $Z_B^{\mathcal{M}_C}$ in the Coulomb phase. Similarly, the $Sp(1)_C$ twist can be described by either $Z_A^{\mathcal{M}_C}$ or $Z_B^{\mathcal{M}_H}$. We illustrate the situation in Figure 5.1.

This is a kind of heuristic 'derivation' of the duality (5.1). By considering only a single topological twist of the ultraviolet gauge theory, one can obtain both of the theories $Z_A^{\mathcal{M}_H}$ and

$$\begin{array}{ccccc} Z_A^{\mathcal{M}_H} & \longleftarrow & Sp(1)_H \text{ twist of } T & \longrightarrow & Z_B^{\mathcal{M}_C} \\ & & \updownarrow \text{ mirror symmetry} & & \\ Z_A^{\mathcal{M}_C^\vee} & \longleftarrow & Sp(1)_C \text{ twist of } T^\vee & \longrightarrow & Z_B^{\mathcal{M}_H^\vee} \end{array}$$

Figure 5.1: A sketch of the relationship between (5.1) and mirror symmetry. In principle, a single topological twist of an ultraviolet theory T has two descriptions, one in terms of $Z_A^{\mathcal{M}_H}$ and one in terms of $Z_B^{\mathcal{M}_C}$. One can equivalently take the mirror twist of the mirror theory $T^\vee$. We note that the story surrounding the horizontal arrows is complicated by the presence of singularities in the moduli spaces of vacua.

$Z_B^{\mathcal{M}_C}$ (choosing the other twist of the ultraviolet theory leads to the same story with the Higgs and Coulomb branches switched), which are then expected to be equivalent (modulo certain technical issues, related to singularities in the moduli space and residual gauge sectors, see [GW09]). This is essentially the argument of [BT02], also alluded to in [RW97], although in that case it was applied to a theory with a trivial Higgs branch. In general, this argument (and the story as a whole) can run into substantial problems in the form of singularities in the moduli space, a point on which we comment in subsection 5.1.4.

5.1.2 The Coulomb branch

While the geometry of the Higgs branch is given exactly by its classical description in the quantum theory, the geometry of the Coulomb branch receives quantum corrections. This makes it hard to understand.

The conjectural duality of (5.1) captures the idea behind Nakajima's approach to understanding the geometry of Coulomb branches [Nak16; BFN18]. The key point is that Rozansky–Witten theory has the property that[3]

$$Z_B^X(S^2) \cong \mathbb{C}[X],$$

which is to say that its algebra of local operators is the chiral ring of X (in a particular complex structure - all the complex structures are essentially equivalent in the undeformed theory) [RW97]. This means that the problem of understanding the chiral ring of the Coulomb branch (and therefore its structure as an affine variety) is the problem of understanding local operators in Rozansky–Witten theory. This is little more than a rephrasing of the problem, but following the conjectural duality (5.1) allows us to suppose that

$$Z_A^{\mathcal{M}_H}(S^2) \simeq \mathbb{C}[\mathcal{M}_C].$$

Thus the structure of the Coulomb branch as an affine variety can be expressed in terms of 'Higgs branch data'. This is the insight of the papers [Nak16; BFN18], which also describe how to

[3]We use the notation of topological field theory: an n-dimensional field theory acts on a k-manifold to produce an $(n-k)$-dimensional field theory (via compactification), which is determined by its linear $(n-k-1)$-category of boundary conditions (see [Lur09; Kap10]). When $n = 3$ and $k = 2$, one gets a linear 0-category, which is a vector space.

construct the space of local operators of the theory $Z_A^{\mathcal{M}_H}$ in terms of moduli spaces of twisted quasimaps.

In principle, there is much more in (5.1) than simply the matching of spaces of local operators, which means that one can reconstruct more details about the geometry of the Coulomb branch by studying operators in Z_A. For example, compactifying the Rozansky–Witten theory Z_B^X on a circle gives a monoidal category which is a deformation of the category of coherent sheaves on X, with the deformation involving a complex symplectic structure on X [KRS09; KR10]. It is plausible therefore that by applying this in the case $X = \mathcal{M}_C$, one could recover details about the complex symplectic structures on $\mathcal{M}_C$. To get to this, one could use (5.1) to realise this monoidal category as $Z_A^{\mathcal{M}_H}(S^1)$, expressing it in terms of Higgs branch data.

Aspects of the monoidal categories $Z_A^X(S^1), Z_B^X(S^1)$ have been studied for gauge theories in [Dim+20], where they are viewed as categories of topological line operators. We show that the category $Z_A^X(S^1)$ can the thought of as the category of boundary conditions in a defect-deformed two-dimensional A-model.

We note for completeness that this is not the only approach to understanding the Coulomb branch. In [BDG17], the ring $\mathbb{C}[\mathcal{M}_C]$ was constructed by a direct parameterisation in terms of expectation values of 'dressed monopole operators' on an 'Abelian patch', $\mathcal{M}_C^{\text{abel}} \hookrightarrow \mathcal{M}_C$, which is the complement of the regions where nonAbelian gauge symmetry is restored.

5.1.3 On categorifying the A-model

A striking fact about Rozansky–Witten theory is that it 'explains' why one can take the tensor product of coherent sheaves on a complex manifolds. Given a complex manifold Y, one can consider Rozansky–Witten theory in the cotangent bundle $T^\vee Y$ (this may not be hyperKähler, but it is complex symplectic, which is enough to define Rozansky–Witten theory [Kap99]). The zero section $Y \subset T^\vee Y$ defines a complex Lagrangian submanifold and is therefore a good boundary condition for the Rozansky–Witten theory [KRS09].

Compactifying the theory on an interval and asking that the boundary condition at both ends of the interval be given by Y leads to an effective two-dimensional sigma model in Y. It was shown in [KRS09] that this effective theory is the two-dimensional B-model in Y (see [Wit98; Hor+03]), which has as *its* boundary conditions coherent sheaves on Y.

From the perspective of the three-dimensional theory, the boundary conditions in the B-model are interpreted as defect line operators separating the boundary conditions at either end of the interval (see [Kap10]). Fusion of these line operators in the three-dimensional theory defines a tensor product operation betwen them. In the effective two-dimensional theory, this defines a tensor product of boundary conditions and hence a tensor product on the category of coherent sheaves on Y. In fact, this is not quite the usual tensor product of coherent sheaves: it is instead a quantum deformation thereof [KRS09].

In constructing the A-twist, we put ourselves in a position to ask what the mirror of this story is. In subsection 5.4.5, we find that boundary conditions in the theory can be described as $U(1)$-equivariant complex Lagrangian submanifolds of X, considered as a complex symplectic manifold in a fixed complex structure. The $U(1)$ action factors through the permuting $Sp(1)$ action, preserving the fixed complex structure.

In general, when one compactifies the theory on an interval and asks the boundary condition at both ends to be specified by a fixed complex Lagrangian submanifold Y, the effective two-dimensional theory that one finds is (semiclassically) a deformed version of the two-dimensional A-model, with the deformation being of a global nature. Only when Y can be embedded into X as a $U(1)$-*invariant* complex Lagrangian submanifold (meaning that $U(1)$ acts trivially on it) does this deformation disappear.

Thus, the A-twist induces a tensor product on the Fukaya category only in special cases, one of which was considered in [KSV10]. This seems to square with the results of the mathematical literature, where natural tensor product structures have been found only in special cases (see [Pas18; NZ09]).

5.1.4 Life at the tip of the cone

One of the most substantial technical difficulties associated to understanding the infrared behaviour of the A-twist in the Higgs branch, or in hyperKähler cones more generally, is the presence of singularities. This is a rather pervasive issue: there has been substantial work [DS18; BGD18; BG18] dedicated to defining indices of superconformal quantum mechanical theories on hyperKähler cones, for example.

If one is willing to restrict oneself to studying the A-twist on spacetime manifolds with spin group broken to $U(1)$ (for example, manifolds which are the product of a surface and a circle), then one can resolve the singularities by introducing an Fayet–Iliopoulos parameter. This is not compatible with the full topological twist, as it breaks the symmetry that is used to define the twisted theory.

We offer a sketch of a possible solution to this issue. The idea is that one can excise the tip of the cone by critically coupling the theory to gravity and taking a strong-coupling limit.

The point is that, when the $Sp(1)$ action on the target space X has the technical property of being what we call a *Swann action* (after [Swa90]), one can form the *Swann quotient*

$$X/_{\mathrm{Swann}}Sp(1)$$

which is quaternion-Kähler manifold of quaternionic dimension one lower than X, which is typically nonsingular. By coupling the theory to gravity and taking a strong-coupling limit, one finds a sigma model into this quotient space. We describe how this works in section 5.5, although we remark that the story we give is by no means complete. In particular, the strongly-coupled gravitational sector remains, leading to a classically underdetermined theory.

This idea provides a partial realisation of an suggestion of Singleton, who argued that one might work with the equivariant cohomology of the homothetic action on a hyperKähler cone to define suitable invariants in [Sin16, Chapter 8].

5.2 Twists of three-dimensional theories with $\mathcal{N} = 4$ supersymmetry

5.2.1 Three-dimensional gauge theories at critical coupling

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $(V, \boldsymbol{\omega}_V)$ be a hyperKähler manifold carrying an isometric, hyperHamiltonian G action, with hyperKähler moment map

$$\boldsymbol{\mu} : V \to \mathfrak{g}^\vee \otimes \mathfrak{sp}(1)_H.$$

(We refer the reader to the paper [Hit+87] for details about the meaning of this.) The subscript H here refers to *Higgs*. The corresponding group $Sp(1)_H$ acts on the endormorphism bundle of the tangent bundle of V (but not necessarily on V itself), rotating the complex structures as a 3-vector. Later we will specialise to the case that the group $Sp(1)_H$ does act on V, as this is the case relevant in examples. We have written V for the hyperKähler manifold to indicate that it is typically taken to be a linear representation.

To this data we can associate a three-dimensional gauge theory. This theory will have an additional global symmetry $Sp(1)_C$, where the C here refers to *Coulomb*. We will use bold symbols to denote $Sp(1)_H$ vectors, and over-arrows to denote $Sp(1)_C$ vectors.

Take a three-manifold M and a principal G-bundle

$$P \to M.$$

Form the associated bundle

$$\mathcal{V} := P \times_G V.$$

The bosonic field content of the theory consists of a connection A on P, a section ψ of $\mathcal{V}$ (the *hypermultiplet scalars*), and an $\mathfrak{sp}(1)_C$-valued adjoint scalar $\vec{\phi} \in \Omega^0(\mathrm{ad}_P \otimes \mathfrak{sp}(1)_C)$ (the *vectormultiplet scalars*).

The connection A induces covariant derivatives on the adjoint bundle ad_P and the bundle $\mathcal{V}$. Sections of the adjoint bundle can act on sections of $\mathcal{V}$ via vertical vector fields induced by the derivative of the G action. (When V is a linear representation of G, this is the usual story.)

Writing g^2 for the coupling constant of the theory, we consider the bosonic energy density

$$\mathcal{E} = \frac{1}{g^2}\left(|F(A)|^2 + |\mathrm{d}_A\vec{\phi}|^2 + |[\vec{\phi}, \vec{\phi}]|^2\right) + |\mathrm{d}_A\psi|^2 + |(\vec{m} - \vec{\phi})\psi|^2 + g^2|\boldsymbol{\xi} - \boldsymbol{\mu}(\psi)|^2,$$

which depends on the parameters $\boldsymbol{\xi}$, an $\mathfrak{sp}(1)_H$-valued element of the dual of the centre of $\mathfrak{g}$, and $\vec{m}$, an $\mathfrak{sp}(1)_C$-valued element of the centre of $\mathfrak{g}$. The parameter $\boldsymbol{\xi}$, known as the *Fayet–Iliopoulos parameter*, breaks $Sp(1)_H$ to $U(1)$ if it is a symmetry of the theory (in general, $Sp(1)_H$ may not act on V). On the other hand, the mass parameter $\vec{m}$ breaks $Sp(1)_C$ to a $U(1)$ subgroup.

The form of $\mathcal{E}$ is fixed by the $\mathcal{N} = 4$ supersymmetry that we will eventually impose. The supersymmetric theory can generally be defined only on $M = \mathbb{R}^3$, although by twisting the theory we will eventually be able to place it on general manifolds.

Vacua

In general, the energy density $\mathcal{E}$ vanishes for solutions to the *vacuum equations*

$$F(A) = 0, \;\; \mathrm{d}_A\vec{\phi} = 0, \;\; \mathrm{d}_A\psi = 0,$$
$$\boldsymbol{\mu}(\psi) = \boldsymbol{\xi}, \;\; \vec{m}(\psi) = \vec{\phi}(\psi), \;\; [\vec{\phi}, \vec{\phi}] = 0.$$

The nature of the space of solutions depends strongly on the parameters $\boldsymbol{\xi}$ and $\vec{m}$. There are two particularly interesting branches of vacua, describing qualitatively different physical behaviour.

- *The Coulomb branch.* Suppose that ψ sits at a constant fixed point of the G action. Then $\vec{\phi}$ is free to take any value for which its three components mutually commute. Moreover, A need not be trivial. The condensation of $\vec{\phi}$ generically breaks the gauge group G to a maximal torus $T \cong U(1)^{\mathrm{rank}(\mathfrak{g})}$. Provided space is simply connected, the only moduli associated with the equation $F(A) = 0$ in this phase are the global gauge transformations for the unbroken gauge group T, which are often interpreted in terms of so-called dual photons.

  These vacua describe a Coulomb phase, because the gauge bosons associated with the residual T gauge group are massless.

  There is then a *Coulomb branch* of these vacua, parameterised by the expectation values of $\vec{\phi}$ and the dual photons,
  $$\mathcal{M}_C \simeq (\mathbb{R}^3 \times S^1)^{\mathrm{rank}(\mathfrak{g})}/W$$
  where W is the Weyl group. This description is classically valid, at least whenever $\vec{\phi} \neq 0$, but generally receives quantum corrections.

  The classical Coulomb branch has a hyperKähler structure induced from $S^1 \times \mathbb{R}^3 \cong \mathbb{H}/\mathbb{Z}$. Supersymmetry constrains the possible quantum corrections to ensure that the quantum Coulomb branch is hyperKähler.

  The undeformed Coulomb branch carries an isometric action of $Sp(1)_C$, permuting its complex structures. Classically, this factors through the vector action of $SO(3)$ on the vectormultiplet scalars $\vec{\phi}$. Mass parameters resolve the Coulomb branch, breaking $Sp(1)_C$ to a $U(1)$ subgroup.

- *The Higgs branch.* Suppose that A is trivial and $\vec{m} = 0$. Then we can set $\vec{\phi} = 0$. There is then a *Higgs branch* of vacua
  $$\mathcal{M}_H = \boldsymbol{\mu}^{-1}(\boldsymbol{\xi})/G.$$
  This phase is called the Higgs phase because the scalar ψ generically breaks the gauge group completely (it 'Higgses' the gauge group).

  The geometry of the Higgs branch as determined by the hyperKähler quotient construction is exact in the quantum theory. When V carries a permuting $Sp(1)_H$ action (for example, when V is a linear representation of the gauge group) the Higgs branch also carries such an action when $\boldsymbol{\xi} = 0$ (we assume that the moment map is normalised so as to be $Sp(1)_H$ equivariant), but a nonzero value of $\boldsymbol{\xi}$ breaks $Sp(1)_H$ to $U(1)$.

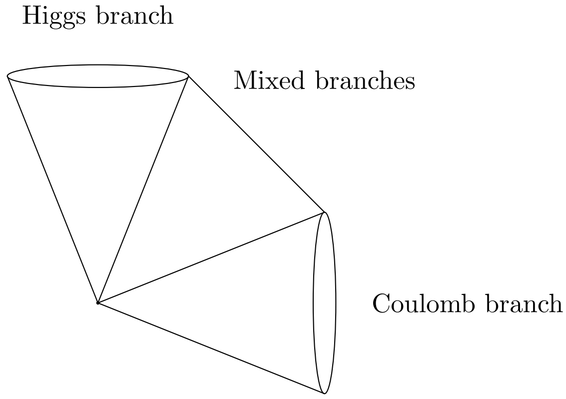


Figure 5.2: A sketch of the vacuum moduli space of a three-dimensional theory with $\mathcal{N} = 4$ supersymmetry.

In general, there are also mixed phases, where the gauge group is partially Higgsed. A schematic sketch of the vacuum moduli space is given in Figure 5.2.

Solitons

Three-dimensional $\mathcal{N} = 4$ gauge theories admit a rich spectrum of BPS excitations. These can be classified by the symmetries they preserve. On $\mathbb{R}^3$, the global symmetry group of an $\mathcal{N} = 4$ theory includes

$$Sp(1)_E \times (Sp(1)_H \cdot Sp(1)_C)$$

where $\cdot$ denotes the product of groups as subgroups of $SO(4)$. Then we can classify BPS configurations in terms of how this symmetry group is (explicitly) broken. We summarise some of the story in Table 5.1. We have included local, 'electric' objects in Table 5.1 to make manifest the nature of the mirror duality.

This also tells us which BPS objects are relevant in the various possible twists of the supersymmetric theory: as we recall below in subsection 5.2.3, the twists of the theory are related to possible identifications of $Sp(1)_E$ (or subgroups thereof) with subgroups of $Sp(1)_H \cdot Sp(1)_C$.

Our interest in the A-twist leads us to consider the (generalised) Seiberg–Witten equations in more detail. Their definition requires one to identify $Sp(1)_E$ with $Sp(1)_H$, as one might expect if they are to play a role in the A-twist in the Higgs branch.

We also briefly recall the notion of complexified flat connections in the Coulomb phase. They require one to identify $Sp(1)_E$ with $Sp(1)_C$, which leads one to expect that they play a role in the A-twist in the Coulomb branch (which should be related to the B-twist in the Higgs branch).

We then propose equations which mediate between the Seiberg–Witten equations and the complex flat connection equations, the *complexified Seiberg–Witten equations*, which require one to identify all three of $Sp(1)_E$, $Sp(1)_H$, and $Sp(1)_C$.

Solitons	**Unbroken symmetry**
(Undeformed) Seiberg–Witten monopoles	$Sp(1)_{E,H} \cdot Sp(1)_C$
(Undeformed) Complexified flat connections	$Sp(1)_H \cdot Sp(1)_{E,C}$
(Undeformed) Complexified Seiberg–Witten monopoles	$Sp(1)_{E,H,C}$
BPS critically-coupledopoles	$Sp(1)_E \times (Sp(1)_H \cdot U(1)_C)$
Particle creation operators	$Sp(1)_E \times (U(1)_H \cdot Sp(1)_C)$
Vortex lines	$U(1)_E \times (U(1)_H \cdot Sp(1)_C)$
Wilson (particle) lines	$U(1)_E \times (Sp(1)_H \cdot U(1)_C)$
BPS monopoles in the Higgs phase [Ton04b]	$U(1)_E \times (U(1)_H \cdot U(1)_C)$
Particle operators in the Coulomb phase	$U(1)_E \times (U(1)_H \cdot U(1)_C)$

Table 5.1: A partial classification of BPS objects in three-dimensional $\mathcal{N} = 4$ theories in terms of the symmetries they preserve (note that solutions may spontaneously break this 'preserved' symmetry). The objects are grouped into pairs related by mirror symmetry. Here $Sp(1)_{E,H}$ is the (anti)diagonal subgroup of $Sp(1)_E \times Sp(1)_H$ and similarly for $Sp(1)_{E,C}$ and $Sp(1)_{E,H,C}$.

- *Complexified flat connections.* The effective theory in the Coulomb phase has energy density

  $$\mathcal{E}_{\text{Coulomb}} = \frac{1}{g^2}\left(|F(A)|^2 + |\mathrm{d}_A\vec{\phi}|^2 + |[\vec{\phi},\vec{\phi}]|^2\right).$$

  Fixing an identification of $Sp(1)_E$ with $Sp(1)_C$, which we write as $\delta^{\dot{a}}_{\mu}$ for $\dot{a} = 1,2,3$ an $\mathfrak{sp}(1)_C$ index and $\mu = 1,2,3$ a spatial index, we can introduce a 1-form field $\Phi = \delta^{\dot{a}}_{\mu}\phi_{\dot{a}}\mathrm{d}x^{\mu}$. We consider solutions to

  $$F(A) = [\Phi \wedge \Phi],\ \ \mathrm{d}_A\Phi = 0,\ \ \mathrm{d}^{\dagger}_A\Phi = 0.$$

  The first two equations assert that the complex connection $A + \mathrm{i}\Phi$ is flat, while the third is a kind of complex gauge fixing equation for this complex connection, necessary to ensure that the system is elliptic modulo real gauge transformation.

  There are generally no nontrivial solutions to these equations on flat space, so a solution defines a constant point in the Coulomb branch.

- *Generalised Seiberg–Witten monopoles.* The effective theory in the Higgs phase has energy density

  $$\mathcal{E}_{\text{Higgs}} = \frac{1}{g^2}|F(A)|^2 + |\mathrm{d}_A\psi|^2 + g^2(\boldsymbol{\xi} - \boldsymbol{\mu}(\psi))^2.$$

  Suppose now that V is a linear representation (the story works in the general case, but is simplified slightly in the linear case). In flat space, $\mathcal{E}_{\text{Higgs}}$ admits the rearrangement

  $$\mathcal{E}_{\text{Higgs}} = \frac{1}{g^2}(\boldsymbol{F}(A) - g^2(\boldsymbol{\xi} - \boldsymbol{\mu}(\psi))^2 + |\not{\mathrm{d}}_A\psi|^2 + 2\boldsymbol{F}(A)\cdot\boldsymbol{\xi} + \mathrm{d}^{\dagger}j$$

where

$$\not{d}_A \psi = \delta^{\mu a} I_a (\mathrm{d}_A)_\mu \psi$$

is a Dirac operator, we define $\boldsymbol{F}^a = \delta^a_\mu (*F)^\mu$, and $j^\mu = \delta^a_\nu \epsilon^{\mu\nu\rho}(\psi, I_a(\mathrm{d}_A\psi)_\rho)$ for the I_a the complex structures on V.

The three-dimensional Seiberg–Witten equations are

$$\boldsymbol{F}(A) = g^2(\boldsymbol{\xi} - \boldsymbol{\mu}), \ \ \not{d}_A \psi = 0.$$

For solutions to the second of these equations, the form j becomes

$$j = (\psi, \mathrm{d}_A \psi),$$

which is a local current, while $\boldsymbol{\xi} \cdot \boldsymbol{F}$ can be thought of as the tension of a vortex line in the direction $\delta^a_\mu \xi_a$. When $\boldsymbol{\xi} = 0$ rotational symmetry is restored.

In the infrared limit, given by taking $g^2 \to \infty$, Seiberg–Witten monopoles become maps into the Higgs branch $V///G$ obeying a nonlinear Dirac equation. The story becomes more complicated on general backgrounds. We consider this in section 5.3.

- *Complexified Seiberg–Witten monopoles.* The previous two classes of BPS configurations can be mashed together. To do this, we identify all of $Sp(1)_E$, $Sp(1)_H$, and $Sp(1)_C$. We do this locally with δ^a_μ and $\delta^{\dot a}_\mu$ as above. We consider the equations

$$\boldsymbol{F}(A + \mathrm{i}\Phi) = g^2 \boldsymbol{\mu}(\psi), \ \ \not{d}_{A+\mathrm{i}\Phi}\psi = 0, \ \ \mathrm{d}^\dagger_A \Phi = 0. \tag{5.2}$$

These equations have solutions given by complexified flat connections (with $\psi = 0$) and from Seiberg–Witten monopoles (with $\Phi = 0$).

5.2.2 Supersymmetry in three dimensions

The (complexified) $\mathcal{N}$-extended superEuclidean algebra in three dimensions takes the form

$$\mathfrak{A} = (\mathfrak{iso}_\mathbb{C}(n) \oplus \mathfrak{so}_\mathbb{C}(\mathcal{N})) \ltimes \Pi S^\mathcal{N}$$

where $S \cong \mathbb{C}^2$ is the spin representation. The factor $\mathfrak{so}(\mathcal{N})$ is the Lie algebra of the R-symmetry group, which acts by rotating the factors of S in $S^\mathcal{N}$.

To define the algebra structure $[\cdot,\cdot]$ on $\mathfrak{A}$, we introduce an $\mathfrak{sp}_\mathbb{C}(1)$-equivariant, non-degenerate map

$$\Gamma : \mathrm{Sym}^2(S^\mathcal{N}) \to \mathbb{C}^3$$

where $\mathbb{C}^3$ is the complexified vector representation. In the absence of central charges, the algebra structure is then given by

$$[\alpha, \beta] = \Gamma(\alpha, \beta)$$

for $\alpha, \beta \in S^\mathcal{N}$.

The map Γ defines an $\mathfrak{sp}_{\mathbb{C}}(1)$-equivariant quadratic form

$$\nu : S^{\mathcal{N}} \to \mathbb{C}^3$$

via $\nu(\alpha) \coloneqq \Gamma(\alpha, \alpha)$.

The standard choice is

$$\Gamma(\alpha, \beta)_\mu = \sigma_\mu^{ab} \alpha_a^i \beta_b^j \delta^{ij}, \tag{5.3}$$

where $\mu = 1, 2, 3$ is an index on the vector representation, a, b are indices on the spin representation $S \cong \mathbb{C}^2$ and $i, j = 1, 2, \cdots, \mathcal{N}$ are indices for the R-symmetry representation $\mathbb{R}^{\mathcal{N}}$. Here the matrices $(\sigma_\mu)_b^a$ are the Pauli matrices, where indices are raised and lowered with the two-dimensional epsilon tensor.

The space $S^{\mathcal{N}}$ is naturally a hyperKähler manifold, with hyperKähler structure induced from that on S. In writing down the complexified supersymmetry algebra, we chose a complex structure on S, regarding it as $\mathbb{C}^2 = \{(z_1, z_2)\}$. This distinguishes the complex symplectic structure

$$\Omega = \sum_{i=1}^{\mathcal{N}} \mathrm{d}z_1^i \wedge \mathrm{d}z_2^i$$

on $S^{\mathcal{N}}$. The spin group $Sp(1)_E$ acts holomorphically on $S^{\mathcal{N}}$, preserving Ω. The holomorphic moment map is then just the quadratic form ν, with respect to the choice (5.3).

5.2.3 Twists of three-dimensional theories

On can unleash the power of homological algebra when one has in one's possession an operator that squares to zero. We define a *twisting supercharge* to be an element $Q \in S^{\mathcal{N}}$ such that

$$Q^2 \coloneqq \frac{1}{2}[Q, Q] = 0.$$

When one has a twisting supercharge Q and supersymmetric field theory representing the supersymmetry algebra, to *twist* the field theory is to work in the cohomology of Q. Twisting of supersymmetric theories was introduced by Witten [Wit88a; Wit88b]. See [ES19] for a comprehensive discussion of twists of supersymmetric theories. In the context of three-dimensional $\mathcal{N} = 4$ theories, see also [CG19].

We would like to classify the twisting supercharges. We have just seen that they are given by solutions to the holomorphic moment map equation

$$\nu(Q) = 0.$$

The complexified spin group acts on the space of solutions to this equation, and the quotient is the holomorphic symplectic quotient

$$\nu^{-1}(0)/Sp(1)_E^{\mathbb{C}} \eqqcolon S^{\mathcal{N}} /\!/ Sp(1)_E^{\mathbb{C}}.$$

This is a singular complex symplectic space of expected complex dimension $2\mathcal{N} - 6$.

In addition to the spin action of $Sp(1)_E$, the space $S^{\mathcal{N}}$ carries an $SO(\mathcal{N})$ action which preserves the hyperKähler structure, and another action of $Sp(1)$, which permutes the hyperKähler structures of S. This second $Sp(1)$ action does not preserve the holomorphic moment map equations and so does not act on the quotient, but there is a subgroup $U(1) \subset Sp(1)$ which fixes our choice of complex structure and so does act on the quotient.

Consider the case that $\mathcal{N} = 4$. Then the quotient above has three strata: one of complex dimension 2, given by generic solutions; one of complex dimension 1, given by the so-called holomorphic twists; and one of dimension 0, consisting of the trivial solution $Q = 0$. When removes the singularities and takes the quotient by the $\mathbb{C}^*$-action which complexifies the $U(1)$ action considered above (that is, takes the geometric invariant theory quotient by $\mathbb{C}^*$), one gets the space $S^2 \sqcup S^2$. In gauge theory representations of the supersymmetry algebra, this is sometimes thought of as the union of the twistor spheres of the Higgs branch and the Coulomb branch, so that a twist is thought of as being specified by a complex structure on either the Higgs branch or the Coulomb branch of the theory.

That the space of twisting supercharges can be realised as a holomorphic symplectic quotient suggests a quaternionic approach to the story. One can replace the holomorphic symplectic quotient

$$S^{\mathcal{N}}//Sp(1)_E^{\mathbb{C}}$$

with the hyperKähler quotient

$$S^{\mathcal{N}}///Sp(1)_E.$$

If the holomorphic quotient were non-singular, one might expect these two spaces to be isomorphic. In fact, the hyperKähler quotient partially resolves the singularities of the holomorphic quotient by removing the so-called holomorphic twists. The hyperKähler quotient carries the permuting action of $Sp(1)$, not just a subgroup. This is related to the fact that this approach does not distinguish a complex structure on the spinor representation.

If $\mathcal{N} = 4$, one can show that the hyperKähler quotient is

$$\mathbb{H}/\mathbb{Z}_2 \cup_{\{0\}} \mathbb{H}/\mathbb{Z}_2,$$

a pair of hyperKähler cones joined at their tips. A (nonzero) point on this space defines a *frame* of complex structures on either the Higgs or Coulomb branches of the theory. The group $SO(4) = Sp(1)_H \cdot Sp(1)_C$ acts on this space, with $Sp(1)_H$ acting in the standard way on one of the $\mathbb{H}/\mathbb{Z}_2$ summands and $Sp(1)_C$ acting on the other.

Topological twists

A twisting supercharge Q is called *topological* if the map $[Q, \cdot\,] : S^{\mathcal{N}} \to \mathbb{R}^3$ is surjective. This means that the generators of translations are exact in the Q-cohomology and so 'trivial' in the twisted theory. The twists that are preserved in the hyperKähler quotient are all nondegenerate in this sense and so are topological.

A twisted theory is a genuine topological field theory, in the sense that it is independent of a choice of metric on spacetime, if its stress-energy tensor is trivial in Q-cohomology. Often, Q

being topological is enough to guarantee this (see [ES19, Section 3.5]). In examples of Lagrangian field theories, it is usually possible to check whether this is the case explicitly, as one can check whether the action of the theory is Q-exact modulo metric-independent terms.

Twisting homomorphisms

One can interpret (topological) twists in terms of a deformation of the action of the spacetime spin group $Sp(1)_E$. Indeed, this was the original approach of [Wit88a; Wit88b]. A generic twisting supercharge Q breaks the usual spacetime spin group, because the supercharges of a supersymmetric field theory are spinor-valued. It is often the case, however, that one can compensate for the action of spin rotations with the action of elements of the R-symmetry group in a consistent way so that Q is invariant under the combined group action.

In our context, this means asking for a homomorphism

$$\varphi : Sp(1)_E \to Sp(1)_H \cdot Sp(1)_C$$

such that Q is invariant under the action of $\{(g, \varphi(g)) \in Sp(1)_E \times (Sp(1)_H \cdot Sp(1)_C)\}$. If such a homomorphism can be found, it is called a *twisting homomorphism*. The existence of such a twisting homomorphism for a twisting supercharge Q implies that Q is topological in the sense above [ES19, Theorem 3.18] (the analogue of this statement holds in all spacetime dimensions greater than two).

All of the topological twists of the three-dimensional $\mathcal{N} = 4$ spupersymmetry algebra considered above admit twisting homomorphisms, given by the $\mathbb{Z}_2$ quotient of the identification of $Sp(1)_E$ with $Sp(1)_H$ or with $Sp(1)_C$. When $Sp(1)_E$ is identified with $Sp(1)_C$, one gets the so-called C-twist, which can be thought of in terms of Rozansky–Witten theory on the Higgs branch. When $Sp(1)_E$ is identified with $Sp(1)_H$, one finds the H-twist, which can be thought of in terms of the A-twist on the Higgs branch.

As we discuss below, the twisting homomorphism for the A-twisted theory exists for nonlinear sigma models only when the target space carries an isometric $Sp(1)$ action permuting its complex structures. If this is not the case, the R-symmetry group is broken to $Sp(1)$ and the twisting homomorphism for the A-twist can not be defined.

Ghost symmetries and $\mathbb{Z}$-gradings

Generally, the operators in a topological theory obtained by twisting a supersymmetric theory come with a $\mathbb{Z}_2$-grading, corresponding to their overall parity (with the fermionic fields, and therefore Q, being odd). Sometimes it is possible to lift this $\mathbb{Z}_2$-grading to a $\mathbb{Z}$-grading, often called a *ghost number* grading, obtained by finding a $U(1)$ symmetry of the theory with respect to which Q has weight one. This symmetry is *compatible* with a twisting homomorphism φ if the $U(1)$ symmetry commutes with the image of φ.

For an $\mathcal{N} = 4$ theory with unbroken R-symmetry group $Sp(1)_H \cdot Sp(1)_C$, all of the twists considered above have such a $U(1)$ symmetry, given by a choice of $U(1)$ subgroup of the $Sp(1)$

factor of the R-symmetry group which is not in the image of the twisting homomorphism. Clearly this still holds if that $Sp(1)$ factor is broken to $U(1)$ (as happens in deformed gauge theories).

Three-dimensional sigma models with $\mathcal{N} = 4$ supersymmetry can be defined with target space a generic hyperKähler manifold (without an isometric $Sp(1)$ action permuting its complex structures). In this case, however, the R-symmetry group is not $Sp(1)_H \cdot Sp(1)_C$, but is broken to a single factor of $Sp(1)$. (We will often think of the hyperKähler manifold as if it were the Higgs branch of a gauge theory, in which case one can think of this group as $Sp(1)_C$.) For such a theory, the Rozansky–Witten theory has a compatible twisting homomorphism, given by a choice of isomorphism

$$Sp(1)_E \xrightarrow{\sim} Sp(1),$$

but can not be given a $\mathbb{Z}$-grading, while the A-twisted theory is not compatible with any twisting homomorphism (see [ES19, Example 4.13]).

One can restore the full global symmetry group by insisting that the target space carries an isometric $Sp(1)$ action permuting its complex structures. This is precisely the situation that arises in the infrared limit of undeformed gauge theories and the one that we consider in this chapter.

In particular, this allows one to give Rozansky–Witten theory a $\mathbb{Z}$-grading. (In fact, if one is happy to consider Rozansky–Witten theory in a single complex structure, one need only ask there to be a permuting $U(1)$ action preserving that complex structure to define the $\mathbb{Z}$-grading.) Importantly, it also allows us to define a twisting homomorphism for the A-twist. If one has only a permuting $U(1)$ action, then the twisting homomorphism for the A-twist can be defined only on manifolds on which the spin group is broken to $U(1)$. This case is often considered in the literature, as Fayet–Iliopoulos parameters break $Sp(1)_H \dashrightarrow U(1)$.

Incidentally, this is why symplectic duality is only a relationship between 1-categories and not 2-categories. It is concerned with complex symplectic manifolds carrying permuting $\mathbb{C}^*$ actions and so does not necessarily require or see the full $Sp(1)$ action that is necessary to turn the A-twist into a genuine three-dimensional topological field theory.

5.3 DIRAC EQUATIONS

5.3.1 HyperKähler manifolds with permuting $Sp(1)$ actions

As discussed above, we can only find two distinct topological twists of a three-dimensional $\mathcal{N} = 4$ sigma model if the target space carries an isometric $Sp(1)$ action permuting its complex structures. We refer to [Cal16] for details about such actions, recalling only the most vital aspects here.

Let (X, g_X) be a hyperKähler manifold with complex structures $\boldsymbol{I} = (I_1, I_2, I_3)$ and Kähler structures $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$. An isometric $Sp(1)$ action on X is called a *permuting $Sp(1)$ action* if the induced action on the 2-sphere of complex structures factors through the standard action of $SO(3)$ on S^2.

There are two basic classes of permuting $Sp(1)$ actions, which might physically be thought of as being of 'hypermultiplet'-type or 'vectormultiplet'-type.

5.3.1 Example ('Hypermultiplet')**.** Let $X = \mathbb{H}$, thought of as a right $\mathbb{H}$-module. Then the action of unit quaternions on the right is a permuting $Sp(1)$ action.

5.3.2 Example ('Vectormultiplet')**.** Consider $X = \mathbb{H}$ and consider the conjugation action of unit quaternions. This defines a permuting $Sp(1)$ action, which splits X into a sum of the scalar and vector representations: $X \cong \mathbb{R} \times \mathbb{R}^3$. Note that the same permuting action descends to $\mathbb{H}/\mathbb{Z} \cong S^1 \times \mathbb{R}^3$.

Hyperexactness

A vital fact is that hyperKähler manifolds with permuting $Sp(1)$ actions are *hyperexact*, meaning that there exists $\boldsymbol{\theta} \in \Omega^1(X, \mathfrak{sp}(1))$ such that

$$\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\theta}.$$

In particular, this implies that all such manifolds are noncompact. The existence of a suitable $\boldsymbol{\theta}$ was shown constructively in [Pid04, Section 2.2] by an argument we now summarise.

The $Sp(1)$ action induces the action of $\mathfrak{sp}(1)$ on X via vector fields. Write $\boldsymbol{i} : \Omega^2(X) \to \Omega^1(X, \mathfrak{sp}(1))$ for the contraction with these vector fields and $\boldsymbol{\mathcal{L}} : \Omega^2(X) \to \Omega^2(X, \mathfrak{sp}(1))$ for the Lie derivative. One has $\boldsymbol{\mathcal{L}} = \mathrm{d}\boldsymbol{i} + \boldsymbol{i}\mathrm{d}$. By definition of the permuting action, one has

$$\mathcal{L}_a \omega_b = \epsilon_{abc} \omega_c,$$

where we have introduced indices $a, b, c, \dots = 1, 2, 3$ on $\mathfrak{sp}(1)$.

One then has

$$\theta_a = \frac{1}{2} \epsilon_{abc} i_b \omega_c,$$

as follows from the fact that

$$\begin{aligned} \mathrm{d}\epsilon_{abc} i_b \omega_c &= \epsilon_{abc} \mathcal{L}_b \omega_c \\ &= \epsilon_{abc} \epsilon_{bcd} \omega_d \\ &= 2\omega_a. \end{aligned}$$

Moment maps and the Higgs branch

Suppose that, in addition to a permuting $Sp(1)$ action, X carries an action of a compact Lie group G which commutes with the $Sp(1)$ factor and preserves its hyperKähler structure. Write $\mathfrak{g}$ for the Lie algebra, which acts on X through the action of vector fields. We write $i_{\mathfrak{g}} : \Omega^\bullet(X) \to \Omega^{\bullet-1}(X, \mathfrak{g}^\vee)$ for the corresponding contraction operation.

Then, as shown in [Pid04, Section 2.2.1] and recalled in [Cal16], the action is hyperHamiltonian, with hyperKähler moment map

$$\boldsymbol{\mu} : X \to \mathfrak{g}^\vee \otimes \mathfrak{sp}(1)$$

given by

$$\boldsymbol{\mu} = i_{\mathfrak{g}} \boldsymbol{\theta}. \tag{5.4}$$

Moreover, because the actions of $Sp(1)$ and G commute, the permuting $Sp(1)$ action descends to the hyperKähler quotient

$$X///G = \boldsymbol{\mu}^{-1}(0)/G.$$

Note that this is only true if the moment map is $Sp(1)$-equivariant (as is the case for (5.4)).

Let us apply this to the case of the Higgs branch. Let $V = \mathbb{H}^n$ with the permuting action of unit quaternions acting on the left (this is an action of 'hypermultiplet' type). Suppose that a compact group G acts through a representation $G \hookrightarrow Sp(n)$ acting on the left. This commutes with the permuting $Sp(1)$ action and preserves the natural hyperKähler structure on V. If one chooses an $Sp(1)$-equivariant moment map, the permuting action descends to the hyperKähler quotient $V///G$. Equivariance of the moment map is broken by Fayet–Iliopoulos parameters.

5.3.2 Dirac operators

In the infrared, the three-dimensional Seiberg–Witten equations reduce to a certain nonlinear Dirac equation for a field valued in the hyperKähler quotient $V///G$. We review here the nature of this nonlinear Dirac equation. In this we follow [Tau99; Pid04; Hay06; Cal16; Doa19] (some of these references deal primarily with the four-dimensional case, but the three-dimensional case is similar). The only possible novelty here is the presentation of a 1-form-valued version of the Dirac operator, which makes clear the sense in which the operator is a three-dimensional version of the Cauchy–Riemann operator.

Let $(X, g_X, \boldsymbol{I}, \boldsymbol{\omega})$ be a hyperKähler manifold carrying a permuting action of $Sp(1)$. Let M be a three-manifold with a spin structure. For us, a spin structure on a three-manifold is a principal $Sp(1)$-bundle $P_{\text{spin}} \to M$ equipped with an isomorphism

$$P_{\text{spin}} \times_{Sp(1)} \mathfrak{sp}(1) \xrightarrow{\sim} TM.$$

We can now form the associated bundle

$$\mathcal{X} \coloneqq P_{\text{spin}} \times_{Sp(1)} X \to M.$$

Why consider this bundle? The point is that, in forming this bundle, we have identified the $Sp(1)_{\text{E}}$ spin group with the $Sp(1)_{\text{H}}$ group of R-symmetries rotating the complex structures on the target space. This is the essence of the topological twist.

5.3.3 Example (Twist of a free hypermultiplet)**.** Let $X = \mathbb{H}$ with the permuting $Sp(1)$ action of unit quaternions on the right. This is the three-dimensional spinor representation and a section of the corresponding bundle $\mathcal{X} \to M$ is a spinor field on M.

5.3.4 Example (Twist of a free vectormultiplet)**.** Take $X = S^1 \times \mathbb{R}^3$ with hyperKähler structure induced from $S^1 \times \mathbb{R}^3 \cong \mathbb{H}/\mathbb{Z}$ and with $Sp(1)$ acting in the vector representation on the $\mathbb{R}^3$ factor. Then, using the isomorphism defining the spin structure, $\mathcal{X} \cong S^1 \times TM$ and a section of $\mathcal{X}$ may be thought of as a pair consisting of a map $M \to S^1$ and a vector field on M. We will return to this example. As we will review later, it is relevant in the Abelian gauged A-model of [KSV10].

Note that, because the spin group $Sp(1)$ is 2-connected (it is diffeomorphic to the three-sphere), any $Sp(1)$-bundle on an oriented three-manifold M is trivial. This is because all maps $M \to BSp(1)$ are homotopic to the identity. Thus the bundle P_{spin} topologically trivial. A trivialisation of P_{spin} is a choice of (spin) framing of M. Note that topological triviality is not geometrical triviality, which corresponds to flatness of the metric induced by the framing.

Now, the endomorphism bundle of TX has a natural subbundle generated by the three complex structures comprising the hypercomplex structure of X. Write $H \to X$ for this bundle, the fibres of which can be naturally identified with $\mathrm{Im}(\mathbb{H})$ and so with $\mathfrak{sp}(1)$, so we can find an isomorphism

$$\mathfrak{sp}(1) \times X \cong \psi^* H.$$

Pulling this isomorphism back along a section ψ of $\mathcal{X}$ induces an isomorphism

$$c : TM \xrightarrow{\sim} \psi^* H$$

using the isomorphism $P_{\text{spin}} \times_{Sp(1)} \mathfrak{sp}(1) \cong TM$ defining the spin structure. We call the isomorphism c a *Dirac structure*. It assigns a complex structure on X to each direction in space. Concretely, we can introduce a frame of 1-forms $\boldsymbol{e} = (e^1, e^2, e^3)$, trivialising the bundle of frames of M, and define

$$c = \psi^* \boldsymbol{I} \cdot \boldsymbol{e}. \tag{5.5}$$

The group $SO(3)$ acts to rotate both $\boldsymbol{I}$ and $\boldsymbol{e}$ separately (that is, through the $Sp(1)_H$ and $Sp(1)_E$ actions). The Dirac structure c is invariant under the diagonal of these actions, which is consistent with the nature of the twist.

The Dirac structure defines a decomposition of the bundle $T^\vee M \otimes \psi^* T_V \mathcal{X}$, where $T_V \mathcal{X}$ is the bundle of vertical vector fields on $\mathcal{X}$, via the operator

$$\mathcal{P} = \frac{1}{3}\left(1 + {*c}\wedge\right),$$

where the Hodge star $*$ is defined with respect to the metric induced by the framing $\boldsymbol{e}$. This operator is a projection operator, obeying

$$\mathcal{P}^2 = \mathcal{P},$$

which follows from the fact that

$$*c \wedge *(c \wedge \alpha) = *c \wedge \alpha + 2\alpha$$

for any $\alpha \in \Omega^1(\psi^* T_V \mathcal{X})$. This fact is a consequence of the quaternionic relations among the complex structures on X. The projection $\mathcal{P}$ is orthogonal with respect to the natural metric on $T^\vee M \otimes \psi^* T_V \mathcal{X}$ induced by the metric on X and the metric on M induced by the frame $\boldsymbol{e}$. The complementary projector to $\mathcal{P}$ is given by

$$\mathcal{P}^\perp = \frac{1}{3}(2 - {*c}\wedge).$$

As a projection operator, the only eigenvalues of P are one and zero. We can therefore decompose $E \coloneqq T^\vee M \otimes \psi^* T_V \mathcal{X}$ as

$$E = E_0 \oplus E_1$$

where E_0 contains the elements in the kernel of $\mathcal{P}$ (that is, the range of $\mathcal{P}^\perp$) and E_1 contains the elements in the range of $\mathcal{P}$ (that is, the kernel of $\mathcal{P}^\perp$).

We can characterise the kernel of $\mathcal{P}$ in the following way. Let α be in $\Omega^1(M, T_V\mathcal{X})$ and consider

$$c \cdot \alpha,$$

where by $\cdot$ we mean the inner product of 1-forms on M induced by the inverse of the metric $\boldsymbol{e}^2$ and the standard action of endomorphisms of TX on TX. More concretely, using the decomposition (5.5), we can write

$$c \cdot \alpha = (I^A)^a_b (\alpha_A)^b$$

where $\alpha = \alpha_A e^A$ and a, b are indices for $T_V\mathcal{X}$. The kernel of $\mathcal{P}$ is the kernel of the operator $c \cdot (\cdot)$.

Indeed, suppose that $\mathcal{P}\alpha = 0$. This implies that

$$\begin{aligned} 0 &= \frac{1}{3}\left(c \cdot \alpha + c \cdot *(c \wedge \alpha)\right) \\ &= \frac{1}{3}\left(c \cdot \alpha + \epsilon_{ABC} I_A I_B \alpha_C\right) \\ &= c \cdot \alpha, \end{aligned}$$

where we have used the quaternionic relations among the I^A. Conversely, suppose that $c \cdot \alpha = 0$. Then

$$\begin{aligned} 0 &= c(c \cdot \alpha) \\ &= I_A I_B \alpha^B \\ &= -\alpha_A - \epsilon^{ABC} I_B \alpha_C \\ &= -3\mathcal{P}\alpha. \end{aligned}$$

Thus the kernel of $\mathcal{P}$ is the kernel of $c \cdot (\cdot)$.

This construction should be compared to the analogous one in complex geometry. Let Σ be a Riemann surface with complex structure j_Σ and let (Y, j_Y) be a complex manifold. Given a map $u : \Sigma \to Y$, the bundle $T^\vee\Sigma \otimes u^*TY$ can be decomposed into holomorphic and antiholomorphic parts, which are respectively the elements in the kernel and range of the projection operator defined by

$$\beta \mapsto \frac{1}{2}\left(1 + j_Y \circ \beta \circ j_\Sigma\right)$$

for $\beta \in T^\vee\Sigma \otimes u^*TY$.

A trivialisation of P_{spin} defines a spin connection γ by declaring the trivialisation to be horizontal with respect to γ. This connection (indeed, any connection) induces a covariant derivative d_γ on sections of $\mathcal{X}$. We now consider the operator

$$\mathcal{D}\psi \coloneqq \mathcal{P}(\mathrm{d}_\gamma \psi).$$

This is the analogue of the Cauchy–Riemann operator in three dimensions, twisted by the spin connection γ.

We can also define the *Dirac–Fueter operator* $\not{\mathrm{d}}_\gamma$ by

$$\not{\mathrm{d}}_\gamma \psi \coloneqq c \cdot \mathrm{d}_\gamma \psi,$$

which has the same kernel as $\mathcal{D}$ by the result of the discussion above.

5.3.3 *The energy of solutions*

Let us consider the energy of solutions to the nonlinear Dirac equation $\not{\mathrm{d}}_\gamma \psi = 0$. Using the concepts of subsection 5.3.1, we will find a relationship between the energy of solutions and the curvature of the underlying manifold M.

By using the quaternionic relations among the complex structures on X, one can show that

$$|\not{\mathrm{d}}_\gamma \psi|^2 = |\mathrm{d}_\gamma \psi|^2 - *e_a \wedge \psi^* \omega^a_{\mathcal{X}}$$

where $\boldsymbol{\omega}_{\mathcal{X}}$ is the $\mathfrak{sp}(1)$-valued 2-form on $\mathcal{X}$ given by minimally coupling the 2-form $\boldsymbol{\omega}$ on the fibres of X to the connection γ, which is necessary for it to be a genuine 2-form on the total space of the bundle. In components, we have

$$\psi^* \boldsymbol{\omega}_{\mathcal{X}} = \boldsymbol{\omega}_{ij} \mathrm{d}_\gamma \psi^i \wedge \mathrm{d}_\gamma \psi^j.$$

The term $*e_a \wedge \psi^* \omega^a_{\mathcal{X}}$ is interesting. If γ were flat, one could use the fact that $(X, \boldsymbol{\omega})$ is hyperexact and that $\mathrm{d}_\gamma \boldsymbol{e} = 0$ to write the term $e^a \wedge \psi^*(\omega_{\mathcal{X}})_a$ as a total derivative. In general though, γ is not flat. Indeed, while we saw in subsection 5.3.1 that one could write down $\boldsymbol{\theta}$ such that $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\theta}$, in general one has

$$\boldsymbol{\omega}_{\mathcal{X}} = \mathrm{d}_\gamma(\boldsymbol{\theta}_{\mathcal{X}}) - (\boldsymbol{F}(\gamma) \cdot \boldsymbol{i})(\boldsymbol{\theta}_{\mathcal{X}}). \tag{5.6}$$

Here, we have used the notation of subsection 5.3.1, writing $\boldsymbol{i}$ for contraction with the vector fields generating the permuting $Sp(1)$ action. We have written $\boldsymbol{\theta}_{\mathcal{X}}$ for the minimal coupling of $\boldsymbol{\theta}$ to the gauge field.

It is perhaps easiest to see why (5.6) holds by working locally. One can write

$$\boldsymbol{\theta}_{\mathcal{X}} = \boldsymbol{\theta}_i \mathrm{d}_\gamma \psi^i$$

where the components $\boldsymbol{\theta}_i$ obey $\partial_i \boldsymbol{\theta}_j - \partial_j \boldsymbol{\theta}_i = \boldsymbol{\omega}_{ij}$. Acting with d_γ on $\boldsymbol{\theta}_{\mathcal{X}}$ gives

$$\begin{aligned} \mathrm{d}_\gamma \boldsymbol{\theta}_{\mathcal{X}} &= \mathrm{d}_\gamma(\boldsymbol{\theta}_i \mathrm{d}_\gamma \psi^i) \\ &= \boldsymbol{\omega}_{ij} \mathrm{d}_\gamma \psi^i \wedge \mathrm{d}_\gamma \psi^j + \boldsymbol{\theta}_i F(\gamma)(\psi)^i \\ &= \boldsymbol{\omega}_{\mathcal{X}} + F(\gamma)^a i_a \boldsymbol{\theta}_{\mathcal{X}}. \end{aligned}$$

Using (5.6), we can write

$$\begin{aligned} e_a \wedge (\omega_{\mathcal{X}})^a &= e_a \wedge \left(\mathrm{d}_\gamma \theta^a_{\mathcal{X}} - F(\gamma)^b i_b(\theta^a_{\mathcal{X}})\right) \\ &= \mathrm{d}(e_a \wedge \theta^a_{\mathcal{X}}) - e_a \wedge F(\gamma)^b i_b(\theta^a_{\mathcal{X}}), \end{aligned}$$

where we have used that $\mathrm{d}_\gamma \boldsymbol{e} = 0$. Now, the quantity

$$S_a^b \coloneqq *e_a \wedge F(\gamma)^b$$

is a tensor on the manifold $(M, \boldsymbol{e})$ that completely determines the Riemann curvature.

Thus, on closed manifolds, one has

$$|\mathrm{d}_\gamma \psi|^2 = |\not{\mathrm{d}}_\gamma \psi|^2 - S_a^b i_b(\theta_{\mathcal{X}}^a) \tag{5.7}$$

so that the energy of a solution to $\not{\mathrm{d}}_\gamma \psi = 0$ is controlled in part by the curvature of M.

Generally, the quantity $i_a \boldsymbol{\theta}_b$ is symmetric [Pid04] and so can be decomposed as

$$\nu \delta_{ab} + A_{ab}$$

where A is a symmetric tensor obeying $\mathrm{tr}(A) = 0$. When $A = 0$, the quantity ν is a *hyperKähler potential* for $(X, \boldsymbol{\omega})$, meaning that $\mathrm{d}\boldsymbol{I}\mathrm{d}\nu = \boldsymbol{\omega}$. This is what happens, for example, for the 'hypermultiplet' type action of $Sp(1)$ on $\mathbb{H}$. When this happens the energy of solutions to $\not{\mathrm{d}}_\gamma \psi = 0$ becomes

$$-R\nu$$

where $R = S_a^a$ is the Ricci scalar of M. This kind of fact is familiar from the original vanishing theorems of [Wit94] for Seiberg–Witten monopoles in four dimensions.

5.3.4 The Dirac equation on circle bundles

As we will soon see, the BPS configurations of the A-twist are solutions to the nonlinear Dirac equation

$$\not{\mathrm{d}}_A \psi = 0.$$

To get a feel for these configurations, we consider solutions in some special cases.

Ultimately, the quantum theory will not depend on the precise choice of frame that one uses to define the Dirac equation. In general, it will depend only on its topological class. For this reason, we can make convenient choices of frames (and corresponding Riemannian metrics) when considering solutions.

Let M be the total space of a circle bundle $p : M \to \Sigma$ over a compact two-dimensional surface Σ. Let Σ have a Riemannian metric g_Σ. One can then give M a metric of the form

$$g_M = \eta^2 + p^* g_\Sigma$$

where $\eta \in \Omega^1(M)$ is a connection form on the bundle (note that a connection can be thought of as a globally defined 1-form on the total space of a bundle, generally valued in the space of vertical vector fields on the bundle). This is a 'Kaluza–Klein' approach to Riemannian geometry on circle bundles.

Writing φ for a local periodic coordinate on the circle fibres of p, one can write

$$g_M \simeq (\mathrm{d}\varphi + p^* A)^2 + p^* g_\Sigma$$

locally, where A is a local connection 1-form on Σ given by pulling η back along a local section of p. The local section may be given by taking ψ to be constant. We write $A = A_x \mathrm{d}x + A_y \mathrm{d}y$.

In a local patch $U \subset \Sigma$, we can write

$$g_\Sigma \simeq e^{2u}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right)$$

for some local coordinates x, y and function $u : U \to \mathbb{R}$. We may then choose a local orthonormal coframe for $T\Sigma|_U$ given by

$$f^1 = e^u \mathrm{d}x,\ f^2 = e^u \mathrm{d}y.$$

Over this patch, we can give the coframe

$$(e^1, e^2, e^3) = (p^* f^1, p^* f^2, \mathrm{d}\varphi + p^* A)$$

for $TM|_{U \times S^1}$.

The inverse metric is

$$g_M^{-1} = (v_1)^2 + (v_2)^2 + (v_3)^2$$

where, locally,

$$v_1 = e^{-u}\partial_x - e^{-u} A_x \partial_\varphi, \quad v_2 = e^{-u}\partial_y - e^{-u} A_y \partial_\varphi, \quad v_3 = \partial_\varphi,$$

which obey $e^a(v_b) = \delta^a_b$. Here, and elsewhere, the indices $a, b, c, \dots = 1, 2, 3$ are indices for the bundle $M \times \mathfrak{so}(3)$, which is identified with TM by the frame v.

The spin connection γ is determined by the structure equation $\mathrm{d}e^a = -\gamma^a_b \wedge e^b$ and by antisymmetry. Solving this gives

$$\gamma_{abc} = \epsilon_{3bc}(\delta_{a1} v_2(u) - \delta_{a2} v_1(u)) + \frac{1}{2}\epsilon_{abc} F_{12} \tag{5.8}$$

where F_{12} is defined by $F(A) = \frac{1}{2} F_{12} e^1 \wedge e^2$.

Now let us consider the Dirac equation on this background. We begin with a simple example, one which serves as a local model for more general cases.

5.3.5 Example (Free hypermultiplet)**.** Let $X = \mathbb{H}$, thought of as a right $\mathbb{H}$-module, with permuting $Sp(1)$ action given by the action of unit quaternions on the right. The Dirac equation is then the usual Dirac equation for a spinor field.

We may regard X as $\mathbb{C}^2$ with complex structures $(\mathrm{i}\sigma_1, \mathrm{i}\sigma_2, \mathrm{i}\sigma_3)$. The permuting action is generated by the action of the complex structures themselves.

Let ψ be a section of the bundle $\mathcal{X}$. The covariant derivative is

$$\mathrm{d}_\gamma \psi = \mathrm{d}\psi + \frac{\mathrm{i}}{2}\epsilon_{abc}\gamma^{ab}\sigma^c \psi.$$

Substituting in the spin connection (5.8) gives

$$\begin{aligned}\mathrm{d}_\gamma \psi &= \mathrm{d}\psi + \frac{\mathrm{i}}{2}\epsilon_{abc}\left(\epsilon_{3bc}(\delta_{d1} v_2(u) - \delta_{d2} v_1(u)) + \frac{1}{2}\epsilon_{dbc} F_{12}\right) e^d \sigma^a \psi \\ &= \mathrm{d}\psi + \mathrm{i}(v_2(u) e^1 - v_1(u) e^2)\sigma^3 \psi + \frac{\mathrm{i}}{2} F_{12} e^a \sigma_a \psi\end{aligned}$$

The Dirac equation associated to the Dirac structure $\mathrm{i}e^a\sigma_a$ is then

$$\begin{aligned} 0 &= \not{\mathrm{d}}_\gamma \psi \\ &= \mathrm{i}\sigma^a v_a(\mathrm{d}_\gamma \psi) \\ &= \mathrm{i}\sigma^a \left(\partial_a \psi + \mathrm{i}(v_2(u)\delta_{a1} - v_1(u)\delta_{a2})\sigma^3 \psi + \frac{\mathrm{i}}{2} F_{12}\sigma_a \psi \right) \\ &= \mathrm{i}\sigma^a \partial_a \psi + \mathrm{i}(v_1(u)\sigma_1 + v_2(u)\sigma_2)\psi - \frac{3}{2} F_{12}\psi, \end{aligned}$$

where $\partial_a(\cdot) = v_a(\mathrm{d}(\cdot))$. We can express this in terms of the coordinates (x, y, φ) and components $\psi = (\psi^1, \psi^2)$ as

$$\begin{pmatrix} \mathrm{i}\partial_\varphi \psi^1 - \frac{3}{2}F_{12}\psi^1 + e^{-u}\left(\partial_x \psi^2 - A_x \partial_\varphi \psi^2 + \partial_x(u)\psi^2 + \mathrm{i}(\partial_y \psi^2 - A_y \partial_\varphi \psi^2 + \partial_y(u)\psi^2)\right) \\ -\mathrm{i}\partial_\varphi \psi^2 - \frac{3}{2}F_{12}\psi^2 + e^{-u}\left(\partial_x \psi^1 - A_x \partial_\varphi \psi^1 + \partial_x(u)\psi^1 - \mathrm{i}(\partial_y \psi^1 - A_y \partial_\varphi \psi^1 + \partial_y(u)\psi^1)\right) \end{pmatrix} = 0.$$

If the bundle $M \to \Sigma$ is trivial and one takes $F = 0$, there are solutions to this equation with $\partial_\varphi \psi = 0$ coming from holomorphic sections of the bundle

$$S_\Sigma \times_{U(1)} X \to \Sigma,$$

where S_Σ is the principal $U(1)$ spin bundle on Σ and $X \simeq \mathbb{C}^2$ is thought of as a complex symplectic manifold with complex structure $-\mathrm{i}\sigma_3$. One sees this by realising that the derivatives of u give the spin connection on Σ.

This idea extends to the general case. Whenever the bundle $M \to \Sigma$ is trivial, there are solutions to the Dirac equation given by pulling back holomorphic sections of a bundle of the form $S_\Sigma \times_{U(1)} X \to \Sigma$. This is well-known [Nak16; BFK19] and such sections are often called quasimaps [Kim16].

The reason for the appearance of the spin bundle on Σ is that the choice of gauge for the frame that we have used trivialises the normal bundle to Σ in M. This implements the decomposition $TM \cong L \oplus p^*T\Sigma$ where L is trivial. Thus, the topology of $T\Sigma$ becomes important.

5.4 THE A-TWIST

5.4.1 *The theory*

The field content of the A-twisted theory consists of a section ψ of the bundle $\mathcal{X}$, a scalar fermionic field $\eta \in \Omega^0(\Pi\psi^* T_V \mathcal{X})$, a 1-form fermionic field $\chi \in \Omega^1(\Pi\psi^* T_V \mathcal{X})$, and an auxiliary bosonic 1-form field $H \in \Omega^1(\psi^* T_V \mathcal{X})$. Here Π denotes parity reversal of a vector bundle. As we will see, the number of fermionic degrees of freedom can be consistently halved via a certain reality condition.

The bundle $T_V \mathcal{X}$ has a natural connection $\tilde{\Gamma}$ given as the sum of γ and the Levi-Civita connection Γ on the tangent bundle of X.

The twisting supercharge Q acts as

$$\begin{aligned} Q\psi &= \eta \\ Q\eta &= 0 \\ Q\chi^i_\mu &= \mathcal{P}(H)^i_\mu + \tilde{\Gamma}^i_{jk}\mathcal{P}(\chi)^j_\mu \eta^k \\ QH^i_\mu &= \frac{1}{2}F(\tilde{\Gamma})^i_{jkl}\mathcal{P}(\chi)^j_\mu \eta^k\eta^l + \tilde{\Gamma}^i_{jk}\mathcal{P}(H)^j_\mu \eta^k, \end{aligned} \tag{5.9}$$

which squares to zero as required. Here and throughout the Latin indices $i, j, k, \cdots$ are indices on $T_V\mathcal{X}$ and Greek indices $\mu, \nu, \cdots = 1, 2, 3$ are indices on TM.

Notice that $Q\mathcal{P}^\perp\chi = 0$ and so that $\mathcal{P}^\perp\chi$ decouples from the algebra completely. One can consistently set it to be zero, if one wishes, which is a kind of reality condition for the fermionic degrees of freedom. One may then replace χ by the scalar field $\beta \coloneqq c \cdot \chi$. One can make a similar statement for the auxiliary field H and a scalar field $h = c \cdot H$. Then, the interesting part of the algebra is determined by

$$\begin{aligned} Q\psi &= \eta \\ Q\eta &= 0 \\ Q\beta^i &= h^i + \tilde{\Gamma}^i_{jk}\beta^j\eta^k \\ Qh^i &= -\frac{1}{2}F(\tilde{\Gamma})^i_{jkl}\beta^j\eta^k\eta^l + \tilde{\Gamma}^i_{jk}h^j\eta^k. \end{aligned}$$

There is a classical ghost symmetry inducing a $\mathbb{Z}$-grading on the fields. The supercharge Q has ghost number 1 and we give ψ ghost number 0, η ghost number 1, χ (and so β) ghost number -1, and H (and so h) ghost number 0.

Naïvely, one would write the action density of the twisted supersymmetric sigma model as

$$|\mathrm{d}_\gamma\psi|^2 + \cdots$$

where $\cdots$ denotes fermionic terms. This is not the right form. The results of subsection 5.3.3 tell us that this should be deformed by a term involving the curvature of the background, taking the form

$$S_{ab}(i^a\theta^b_{\mathcal{X}}) = S(\boldsymbol{i}, \boldsymbol{\theta}_{\mathcal{X}})$$

where $S_{ab} = *e_a \wedge F(\gamma)_b$ and the symmetric tensor $i^a\theta^b_{\mathcal{X}} = \eta\delta^{ab} + A^{ab}$ was defined previously (we have suppressed the pullbacks by ψ). As the original supersymmetric theory was defined only on flat space, there is nothing stopping us from introducing curvature-dependent deformations: they vanish on flat space.

The twisted action is then, up to possible boundary terms,

$$S = \int_M * \left(|\mathrm{d}_\gamma\psi|^2 + S(\boldsymbol{i}, \boldsymbol{\theta}_{\mathcal{X}}) + 2(\beta, \not{\mathrm{d}}_{\tilde{\Gamma}}\eta) + \frac{1}{2}F(\tilde{\Gamma})(\beta, \beta, \eta, \eta) \right).$$

Importantly, this action is Q-exact modulo boundary terms and the equations of motion of the auxiliary field h. We have

$$-Q\int_M *g_X(\beta, h - 2\not{\mathrm{d}}_\gamma\psi) = \int_M * \left(-|h|^2 + 2(h, \not{\mathrm{d}}_\gamma\psi) + 2(\beta, \not{\mathrm{d}}_{\tilde{\gamma}}\eta) + \frac{1}{2}F(\tilde{\Gamma})(\beta, \beta, \eta, \eta) \right)$$

where the various covariant derivative terms are induced by the derivatives of the metric on X. The field h enters algebraically and can be integrated out by imposing its equation of motion, which is

$$h = \not{d}_\gamma \psi.$$

Substituting this in gives

$$\int_M * \left(|\not{d}_\gamma \psi|^2 + 2(\beta, \not{d}_{\tilde{\Gamma}} \eta) + \frac{1}{2} F(\tilde{\Gamma})(\beta, \beta, \eta, \eta) \right).$$

The energy identity (5.7) then reveals that this is the same as S up to a boundary term (we could include this boundary term in the original action - it is not metric-independent).

5.4.2 The partition function

Compactifying the theory on a closed three-manifold M gives the partition function $Z_A^X(M)$ which, in the absence of anomalies, is a complex number.

The partition function of the theory can be written as

$$Z_A^X(M;\lambda) = \int_{\mathcal{C}} \mathcal{D}\mu \exp(-\lambda[Q,V])$$

where $V = -(\beta, h - 2\not{d}_\gamma \psi)$, the space $\mathcal{C}$ is the configuration space of the theory, $\mathcal{D}\mu$ is a path integral measure on $\mathcal{C}$, and λ is a coupling constant.

By the usual yoga of localisation, the partition function is independent of the coupling constant λ and can be computed exactly: it is the integral of the Euler class of the moduli space of Q-invariant configurations. These are those configurations on which $\eta = \beta = h = 0$ and

$$\not{d}_\gamma \psi = 0.$$

Writing $\mathcal{M}(M)$ for the moduli space of solutions to this equation on the compact manifold M, one can show that

$$Z_A^X(M;\lambda) = \int_{\mathcal{M}(M)} e(\mathcal{M}(M)),$$

where $e(\mathcal{M}(M)$ is the Euler class of $T\mathcal{M}(M)$. If $\mathcal{M}(M)$ is compact, this integral is the Euler characteristic of $\mathcal{M}(M)$.

Notice that homotopically trivial variations of the frame defining the nonlinear Dirac equation induce Q-exact variations of the action and therefore do not change the result of the computation. This argument does not apply to 'large' variations of the frame, those which change the topological class of the frame. We therefore expect this theory to have a framing anomaly.

Generically, one expects the moduli space $\mathcal{M}(M)$ to be zero-dimensional, so that the Euler characteristic is a signed count of points in the moduli space. This requires an understanding of the orientation of the points in the moduli space. Without this, one can only define the partition function modulo 2, as an unsigned count of points in the moduli space.

Another question is that of compactness. It is not necessarily the case that the moduli space of solutions is compact. The failure of the moduli space to be compact leads to difficulties in the counting problem. It is plausible that suitable perturbations of the equations lead to good behaviour.

We do not address these analytic issues here, noting that they have been considered in detail for the closely related Seiberg–Witten invariants. Instead, we move on to study local and extended operators in the theory.

5.4.3 Quantum states

The quantum states of the A-twist have been considered in detail in [BFK19; Bul+18; Nak16; BFN18; CDZ20] and elsewhere. We summarise the story briefly.

Compactifying the theory Z_A^X on a closed two-dimensional surface Σ gives an effective one-dimensional topological theory $Z_A^X(\Sigma)$. On general grounds, we expect the theory to be a quantum mechanics on a certain moduli space associated to Σ. In section 5.3, we saw that this moduli space is the space of holomorphic sections of a bundle of the form $S_\Sigma \times_{U(1)} X$, where S_Σ is a principal spin bundle on Σ. These sections are sometimes called *quasimaps*.

Writing $\mathcal{M}(\Sigma)$ for the moduli space of these quasimaps, the space of quantum states is then given by the de Rham cohomology

$$H^\bullet(\mathcal{M}(\Sigma), \mathbb{C}),$$

as follows from standard arguments of Q-invariance.

In principle, this space could receive non-perturbative quantum corrections due to instanton tunnelling between states, the instantons being solutions to the nonlinear Dirac equation $\not{d}_\gamma \psi = 0$ on $\Sigma \times \mathbb{R}$ which tunnel between a pair of distinct quasimaps 'at' $\Sigma \times \{-\infty\}$ and $\Sigma \times \{\infty\}$.

BPS monopoles in the ambient gauge theory can act to create or destroy topological charge associated to a quasimap solution. Algebraic aspects of this story from the perspective of the space of quantum states considered here were studied in [Bul+18].

5.4.4 Line operators

We have considered the theory compactified on a three-manifold and on a two-manifold. We continue here by considering the theory compactified on the circle. Compactifying a three-dimensional topological theory on the circle gives an effective two-dimensional topological theory.

In general, compactification on the circle (or indeed any manifold) gives a theory with infinitely many degrees of freedom. These can be parameterised in terms of Fourier modes of fields around the circle. When one takes the circle to be small, the nonconstant modes typically decouple leaving one with a finite number of degrees of freedom. As we are dealing with a topological theory, the small circle limit is exact and we may neglect nonconstant modes, at least semiclassically. In principle, this semiclassical theory may receive quantum corrections due to BPS solutions winding the circle.

Locally and semiclassically, the compactified theory is the two dimensional A-model (see [Wit98]). Indeed, it is not hard to check that the field content is locally that of the A-model and that the reduced action of Q agrees with that of the A-model twisting supercharge. At the level of the BPS equation, this corresponds to the fact that in coordinates $((x,y),\varphi)$ on $\mathbb{R}^2 \times S^1$,

$$I_1\partial_\varphi + I_2\partial_x + I_3\partial_y \overset{\partial_1=0}{\rightsquigarrow} I_2\partial_x + I_3\partial_y = I_2(\partial_x - I_1\partial_y),$$

so that the reduced Dirac operator is locally the Cauchy–Riemann operator for the complex structure $-I_1$.

However, as revealed by the discussion of section 5.3, the theory is not the A-model globally. Instead, it is the A-model in the presence of a defect (in the sense of [CKW18]), which twists the fields by the spin bundle on Σ. More precisely, the main bosonic field of the reduced theory is a section of

$$S_\Sigma \times_{U(1)} X$$

where S_Σ is a principal spin bundle on Σ and X is thought of as a complex manifold with holomorphic $U(1)$ symmetry in a fixed complex structure. The BPS equation then asks that this section be holomorphic, a story which is familiar from [BFK19].

This concept was realised in section 5.3 using the 'holomorphic gauge', in which the frame field used to define the theory is given locally in terms of a framing of Σ and a 1-form in the circle direction.

A slightly different way to think about this defect is as follows. Globally, it is impossible to frame $\Sigma \times S^1$ in such a way that one of the frame fields is everywhere tangent to the S^1 factor unless the genus of Σ is one. Therefore, as one moves around on the surface of Σ, the complex structure assigned to the S^1-direction by the Dirac structure varies. This defines a map

$$\mathcal{J} : \Sigma \to S^2_X$$

where S^2_X is the 2-sphere of complex structures comprising the hypercomplex structure on X (the *twistor sphere* of X). The topology is such that this map must have topological degree

$$\deg \mathcal{J} = 1 - g,$$

which is minus the degree of the spin bundle on Σ. Composing this with the antipodal map on S^2_X (to account for the minus sign in front of the complex structure in the reduced Dirac equation) gives a map of degree $g-1$, the degree of the spin bundle on Σ.

5.4.5 Boundary conditions

Good boundary conditions for field theories

Boundary conditions and their fusion play a vital role in the understanding of field theories. Here, we recall what makes a good boundary condition for a field theory in the spirit of [Wit86; HW19].

Consider a field theory on an n-manifold M. Part of the data of a field theory is the configuration space $\mathrm{Conf}(M)$ of physical configurations on M. We may also form the universal space

$$\mathcal{U} = M \times \mathrm{Conf}(M).$$

The de Rham complex of $\mathcal{U}$ factors as

$$\Omega^\bullet(\mathcal{U}) \cong \Omega^\bullet(M) \otimes \Omega^\bullet(\mathrm{Conf}(M)),$$

where the tensor product is taken over the ring of smooth functions on $\mathcal{U}$. We call elements of $\Omega^p(M) \otimes \Omega^q(\mathrm{Conf}(M))$ forms of type (p,q) on $\mathcal{U}$. In line with standard notation, we write d for the exterior derivative on $\Omega^\bullet(M)$ and δ for the exterior derivative on $\Omega^\bullet(\mathrm{Conf}(M))$, so that the overall exterior derivative on $\mathcal{U}$ may be written as $\mathrm{d} + \delta$.

The *Lagrangian density* of our field theory is a differential form L on $\mathcal{U}$ of type $(n, 0)$. The *action* is then

$$S = \int_M L,$$

where $\int_M$ represents the pushforward along the projection $\mathcal{U} \to \mathrm{Conf}(M)$. The action S is then a function on $\mathrm{Conf}(M)$.

The equations of motion are determined by

$$\delta S = 0.$$

At the level of L, one can write

$$\delta L = E + \mathrm{d}\Theta$$

where E is an $(n, 1)$-form representing the equations of motion and Θ is an $(n-1, 1)$-form. The form Θ determines the symplectic form on the phase space of the theory.

When M has a boundary, one has

$$\delta S = \int_M E + \int_{\partial M} \Theta.$$

To have boundary conditions compatible with the variational principle and with the equations of motion, one asks that the second term on the right hand side vanishes.

5.4.1 Example (Sigma model)**.** Consider a theory of maps $\psi : M \to N$ for M, N Riemannian manifolds with sigma model Lagrangian density

$$L(\psi) = |\mathrm{d}\psi|^2 \mathrm{vol}_M.$$

Acting on this with δ, one sees that the equation of motion is $\Delta\psi = 0$, where Δ is the Hodge Laplacian coupled to the Levi-Civita connection on the target space, and

$$\Theta = *(\mathrm{d}\psi, \delta\psi).$$

In the presence of a boundary ∂M, we ask that $\Theta|_{\partial M} = 0$ to be sure that the boundary contribution to the variation of the action vanishes. There are many ways to do this in general, two of them are

$$\partial_n \psi|_{\partial M} = 0 \text{ and } \delta\psi|_{\partial M} = 0,$$

where ∂_n is the derivative normal to the boundary, corresponding to Neumann and Dirichlet boundary conditions respectively. Other types of boundary conditions may be mixed Dirichlet–Neumann conditions, with some components of ψ obeying Neumann conditions and the others obeying Dirichlet conditions.

5.4.2 Example (Gauged sigma model)**.** Consider the theory of a sigma model coupled to a background gauge field, with Lagrangian density

$$L = |\mathrm{d}_\gamma \psi|^2 \mathrm{vol}_M.$$

Then varying ψ leads to the boundary term

$$\Theta = *(\delta\psi, \mathrm{d}_\gamma \psi).$$

In general, one can further include boundary (or even more general corner) terms in the action. These can alter the allowed boundary conditions. Moreover, when a field theory has symmetries, one can ask for boundary conditions which are compatible with the symmetry. We have a symmetry to consider, namely that generated by Q, so we will look for boundary conditions compatible with this symmetry.

Boundary conditions in the A-twist

Now let us apply this to the situation at hand. We would like to consider boundary conditions for our twisted sigma model compatible with the symmetry generated by Q.

Let us first work on flat space with boundary, taking $M = \mathbb{R}_{\geq 0} \times \mathbb{R}^2$ with coordinates (x^1, x^2, x^3) where $x^1 \geq 0$ and flat frame $\boldsymbol{e} = (\mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3)$. We take the Dirac structure

$$c = I_a \mathrm{d}x^a,$$

for $a = 1, 2, 3$. In the Cartesian gauge, the spin connection γ is trivial and so one may regard ψ as a map $M \to X$.

The boundary conditions compatible with the twisting supercharge Q are those boundary conditions which are compatible with the BPS equation

$$\mathrm{d}_\gamma \psi + *c \wedge \mathrm{d}_\gamma \psi = 0.$$

Note that the spin connection γ associated to the frame $\boldsymbol{e}$ is trivial. Substituting this equation into the form Θ of example 5.4.2 gives

$$\Theta_{\mathrm{twist}} = - * (*c \wedge \mathrm{d}_\gamma \psi, \delta\psi).$$

This is, for our choice of Dirac structure,

$$\Theta_{\mathrm{twist}} = -\mathrm{d}x^a \wedge (\omega_a)_{ij} \mathrm{d}_\gamma \psi^i \delta\psi^j.$$

We have neglected to consider the fermions, whose behaviour is controlled by Q-invariance.

5.4.3 Remark. Another way to see that this is the correct boundary variation term is to vary the energy identity (5.7).

Good twisted boundary conditions are those for which $(\Theta_{\mathrm{twist}})|_{\{x^1=0\}} = 0$. We therefore require

$$\left(\mathrm{d}x^2(\omega_2)_{ij} + \mathrm{d}x^3(\omega_3)_{ij}\right) \mathrm{d}_\gamma \psi^i \delta\psi^j = 0 \tag{5.10}$$

along $x^1 = 0$. We may choose the Cartesian gauge, where γ is trivial and ψ is just a map $M \to X$. Then suppose that $\psi|_{\partial M} : \partial M \to Y$ for $Y \subseteq X$ a submanifold. Using the fact that $\delta\psi$ and $\mathrm{d}\psi|_{\partial M}$ lie normal to Y, the equation (5.10) then tells us that generic boundary conditions require that Y be isotropic with respect to both ω_2 and ω_3. In fact, Q-invariance also implies that Y be coisotropic with respect to these structures. Thus Y is a *complex Lagrangian submanifold* of X with respect to the complex symplectic structure $\omega_2 + \mathrm{i}\omega_3$ (or a phase rotation thereof). These are mixed Dirichlet–Neumann boundary conditions: half of the degrees of freedom obey Dirichlet conditions and the other half obey Neumann conditions.

5.4.4 Example (Free vectormultiplet)**.** Take $X = \mathbb{H}/\mathbb{Z} \cong S^1 \times \mathbb{R}^3$ with $Sp(1)$ acting in the vector representation on the $\mathbb{R}^3$ factor. In a fixed complex structure we may write $X = \mathbb{C}^* \times \mathbb{C}$ with complex coordinates (z, w) and complex symplectic structure $\mathrm{d}z \wedge \mathrm{d}w$. The complex Lagrangian manifolds either take the form

$$Y_{w_0} = \mathbb{C}^* \times \{w_0\}$$

for some $w_0 \in \mathbb{C}$ or take the form

$$Y_{z_0} = \{z_0\} \times \mathbb{C}$$

for some $z_0 \in \mathbb{C}^*$. See also [KSV10], where these boundary conditions are considered as boundary conditions for the matter sector of a topological Abelian gauge theory. In fact, the authors of that paper insist on setting $w_0 = 0$ when dealing with the first class of complex Lagrangian submanifolds above, with the point being that $w_0 = 0$ is the fixed point of the residual $U(1) \subset Sp(1)$ permuting action preserving the boundary. (As we will see, this is forced in the general case.)

This provides a good local model for general boundary conditions. It is an oversimplified characterisation for two reasons. First, we have fixed the gauge in a way that is generally not possible. Second, in general, the particular complex symplectic structure on the target space which is relevant depends on the normal direction to the boundary in physical space. Directions in physical space and complex structures are inextricably linked by the twisting procedure.

Perhaps the easiest way to understand the effect of this is to work locally in the 'transverse holomorphic gauge'. As before, this works by trivialising the normal bundle of ∂M in M at the cost of working with the spin bundle on the boundary. (We will not consider global issues associated with completing the relevant frame to the whole of M.)

In general, the boundary of M is a two-dimensional surface Σ. Close to the boundary, M looks like $\Sigma \times [0, \epsilon)$. Here, we can choose the metric

$$g_{\Sigma \times [0,\epsilon)} = \mathrm{d}t^2 + g_\Sigma$$

where $t \in [0, \epsilon)$ is a coordinate along the interval and g_Σ is a metric on Σ. Locally, we can choose the orthonormal frame of 1-forms

$$(e^1, e^2, e^3) \simeq (\mathrm{d}t, e^u \mathrm{d}x, e^u \mathrm{d}y)$$

where (x, y) are local conformal coordinates on Σ such that $g_\Sigma = e^{2u}(\mathrm{d}x^2 + \mathrm{d}y^2)$. Write (v_1, v_2, v_3) for the dual frame of orthonormal vector fields. We work with the local Dirac structure $c = I_a e^a$.

Good boundary conditions are then found by solving

$$((\omega_2)_{ij} D_2 \psi^i - (\omega_3)_{ij} D_3 \psi^i) \delta\psi^j|_{\partial M} = 0 \tag{5.11}$$

locally on the boundary, where $D_i = v_i(\mathrm{d}_\gamma)$.

To interpret this, let us regard ψ as an $Sp(1)$-equivariant map $P_{\mathrm{spin}} \to X$. In the gauge we have chosen, which trivialises the normal bundle to the boundary, the boundary breaks $Sp(1)$ to $U(1)$ and we may think of the boundary map as a $U(1)$-equivariant map

$$S_\Sigma \to X \tag{5.12}$$

where S_Σ is the principal $U(1)$ spin bundle on Σ and we regard X as a complex manifold in the complex structure I_1 with holomorphic $U(1)$ action induced from the subgroup of $Sp(1)$ preserving I_1. The complex symplectic structure $\Omega_1 = \omega_2 + \mathrm{i}\omega_3$ on X transforms with weight 2 under the permuting $U(1)$ action on X preserving I_1.

The form of (5.11) tells us that complex Lagrangian submanifolds, on which Ω_1 vanishes, are again the correct boundary conditions to consider. They should also be $U(1)$ equivariant, so that the boundary field (5.12) is well-defined.

5.4.5 Example (Free hypermultiplet)**.** The $U(1)$-equivariant complex Lagrangian submanifolds of $\mathbb{C}^2$ are $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$. Both of these submanifolds transform nontrivially under the $U(1)$ action.

5.4.6 Example (Free vectormultiplet)**.** The $U(1)$-equivariant submanifold of $\mathbb{C}^* \times \mathbb{C}$ are $\{z_0\} \times \mathbb{C}$ for any $z_0 \in \mathbb{C}^*$ and $\mathbb{C}^* \times \{0\}$. Notice that the first of these submanifolds transforms trivially under $U(1)$, while the second transforms nontrivially (see also [KSV10]).

Another description: hyperLagrangians

In the above construction, we have chosen a convenient gauge for the problem. If we do not do this, we should extend our notion of boundary condition to allow for the relevant complex structure to vary as one moves along the boundary. This can be achieved by recalling the notion of *hyperLagrangian submanifolds*, introduced in [LW07].

5.4.7 Definition ([LW07])**.** Let $(X, \omega_1, \omega_2, \omega_3)$ be a hyperKähler manifold and write S^2_X for the sphere of complex structures comprising the hypercomplex structure on X. A *hyperLagrangian submanifold* $(Y, \mathcal{J})$ of X is a submanifold $Y \subset X$ and a map $\mathcal{J} : Y \to S^2_X$ with the property that

$$(\epsilon_{abc} \mathcal{J}_b \omega_c)|_Y = 0.$$

Said another way, a hyperLagrangian submanifold is pointwise complex symplectic, but the relevant complex symplectic structure is allowed to vary. As made clear in [Leu02], hyperLagrangian submanifolds are to hyperKähler manifolds as Lagrangian submanifolds are to Calabi–Yau manifolds.

As a hyperLagrangian submanifold is locally complex Lagrangian, its dimension is half that of the ambient hyperKähler manifold.

There is a topological invariant associated to a hyperLagrangian submanifold $(Y, \mathcal{J})$, given by the topological degree of the map $\mathcal{J} : Y \to S^2$. We will not worry about torsion-type effects and so can think of this invariant as living in the de Rham cohomology $H^2(Y, \mathbb{R})$. It is then represented by the pullback of the unit volume form on S^2. This invariant plays an important role in the story of boundary conditions. The unit normal vector field to the boundary ∂M of M defines a map $\partial M \to S^2$. The degree of this map must agree with the pullback of the degree of $\mathcal{J}$ if the boundary condition is to make sense.

A remarkable fact (albeit one that is physically expected) is that hyperLagrangian submanifolds of hyperKähler manifolds are Kähler [LW07]: $\mathcal{J}$ defines an integrable complex structure on Y and the 2-form $\mathcal{J}^a \omega_a|_Y$ defines a Kähler form on Y.

As before, it is necessary to impose a kind of local equivariance condition on the hyperLagrangians. In general, the property of being hyperLagrangian feels softer than that of being complex Lagrangian, but the equivariance condition seems strong enough to allow us to deal with complex Lagrangians. This deserves further study. The results of [KT20], showing that there is an 'energy gap' between hyperLagrangians which are complex Lagrangian and more general hyperLagrangians, may be relevant.

Compactification on an interval

Compactifying the theory on an interval with a pair of boundary conditions (Y, Y') at either end allows one to associate a two-dimensional topological field theory (and so a category) to (Y, Y'). By gluing pairs of intervals together, one can give a gluing map for these two-dimensional theories. Taken together, this gives the collection of boundary conditions in the three-dimensional theory the structure of a 2-category. We refer the reader to [KV10; KSV10; KRS09; Lur09] for more on this notion.

Thus, getting to grips with the algebraic structure of the A-twist Z_A^X requires us to understand the compactified theory $Z_A^X([0, \epsilon]_{(Y,Y')})$ for arbitrary pairs (Y, Y') of boundary conditions. This was carried out for the B-twist in [KRS09; KR10] (see also [Tel14]).

Let $M = \Sigma \times [0, \epsilon]$ with Σ a closed surface. Trivialising the normal bundle to Σ, we regard the theory as the theory of a $U(1)$-equivariant map

$$\tilde{\psi} : p^* S_\Sigma \to X$$

where $p : M \to \Sigma$ is the projection, S_Σ is the principal $U(1)$ spin bundle on Σ, and X is thought of as a complex manifold in a fixed complex structure.

We impose the boundary conditions

$$\tilde{\psi}[S_\Sigma \times \{0\}] \subseteq Y$$

and

$$\tilde{\psi}[S_\Sigma \times \{\epsilon\}] \subseteq Y'$$

for Y and Y' $U(1)$-equivariant complex Lagrangian submanifolds of X.

Using the fact that theory is topological, we may take the limit $\epsilon \to 0$. Then, semiclassically, we are left with the theory of an equivariant map

$$S_\Sigma \to Y \cap Y'.$$

This theory is a deformed version of the A-model in $Y \cap Y'$.

Generically, the intersection $Y \cap Y'$ is a disjoint union of points and the reduced theory is more-or-less trivial, at least semiclassically. In principle, there remains the possibility of non-peturbative corrections, from solutions to the nonlinear Dirac equation on M with the given boundary conditions. These are instantons, tunneling between the semiclassical vacua.

At the opposite extreme to genericity, there is the case that $Y = Y'$. Then the theory is then a version of the A-model in Y. The details depend on the nature of the $U(1)$ action on Y.

First suppose that the $U(1)$ action on Y is trivial, so that the equivariance condition becomes vacuous. Then the reduced theory is (up to a topological term in the action) exactly the two-dimensional A-model. Then the category defined by the theory $Z_A^X([0,\epsilon]_{(Y,Y)})$ is the Fukaya category of the Kähler manifold Y (this the category of branes in the two-dimensional A-model; see [Sei08; Aur13] for details about this category). This was observed for the case of $Y = \mathbb{C}^* \times \{0\} \subset \mathbb{C}^* \times \mathbb{C}$ in [KSV10].

In the general case, where $U(1)$ acts non-trivially on Y, the reduced theory is the spin-deformed version of the A-model that we saw previously when compactifying on the circle (although in that case the target space was X). In this case, it is the theory of quasimaps, $U(1)$-equivariant maps,

$$S_\Sigma \to Y$$

with the BPS equation asserting that the corresponding section of the bundle $S_\Sigma \times_{U(1)} Y$ be holomorphic with respect to the holomorphic structure induced by the spin connection. This was observed for the case of $Y = \{z_0\} \times \mathbb{C} \subset \mathbb{C}^* \times \mathbb{C}$ in [KSV10].

We have not discussed in any detail the nature of possible non-perturbative deformations to the compactified theory. Such nonperturbative corrections play a vital role in understanding the category of A-branes in the two-dimensional A-model and may well play an important role here.

On monoidal structures on the Fukaya category

The gluing of intervals induces a monoidal structure (that is, a tensor product) on the category defined by the theory $Z_A^X([0,\epsilon]_{(Y,Y)})$. Physically, one can think of this as arising from the fusion of defect line operators separating boundary conditions. In the dual case of the B-twist, this 'explains' the existence of a monoidal structure on the category of branes in the two-dimensional B-model (the branes are coherent sheaves).

It would be nice to make an analogous statement about the category of branes in the two-dimensional A-model (the Fukaya category), but we see that this is not so simple because of the spin deformation. Instead, this construction only gives a monoidal structure on the Fukaya category of a Kähler manifold Y if Y can be realised as a $U(1)$-invariant submanifold of a hyperKähler manifold with permuting $Sp(1)$-action on which the $U(1)$ action is trivial. This seems to be quite a serious restriction.

In some ways, this is not surprising. There are few examples of symplectic manifolds with Fukaya categories with a known monoidal structure and they generally take special forms [Pas18; Tel14]. We suggest that the reason for this (at least in the Kähler case) is precisely the fact that it is not always possible to embed a Kähler manifold into a hyperKähler manifold in the way mentioned above.

There are instances where it can be done. For example, it seems plausible that if L is a real manifold, a formal neighbourhood in the total space of the bundle $(T^\vee L)^{\oplus 3} \simeq T^\vee(T^\vee L)$ would carry a suitable hyperKähler structure (along the lines of [Kal99; Fei03]) with the permuting action permuting the factors of $T^\vee L$. Then this construction would induce a monoidal structure on the Fukaya category of $T^\vee L$.

It is a result of [NZ09] that if one interprets the Fukaya category of $T^\vee L$ in a suitable way, it is equivalent to an object called the category of constructible sheaves on L. The category of contructible sheaves has a monoidal structure, inducing a monoidal structure on the Fukaya category of $T^\vee L$ (see also [Pas18]). It would be interesting to see if this is the same as the monoidal structure induced by the three-dimensional A-twist.

5.4.6 Comparison with the Kapustin–Vyas theory

In [KV10], Kapustin and Vyas associated to any real manifold L a three-dimensional topological theory. The bosonic field content of the Kapustin–Vyas theory consists of a map

$$\phi : M \to L$$

and a $T^\vee L$-valued vector field $\tilde{\tau}$ (in [KV10], these degrees of freedom are instead packaged into a TL-valued 1-form τ - we have raised and lowered some indices for later convenience). The model was inspired by the gauged A-type model considered in [KSV10; Set13].

The point is that, when $L = S^1$, the fields $(\phi, \tilde{\tau})$ can naturally be organised into a 'vectormultiplet'-type field, taking values in $S^1 \times \mathbb{R}^3 \cong \mathbb{H}/\mathbb{Z}$. The fact that τ is a 1-form and not just an $\mathbb{R}^3$-valued field corresponds to the fact that the theory has been twisted, with $Sp(1)_E$ being identified with the $Sp(1)$ that acts on $S^1 \times \mathbb{R}^3$ via the vector representation.

The BPS configurations of the Kapustin–Vyas model are solutions to

$$\nabla\phi + \nabla \times \tilde{\tau} = 0\,,\ \nabla \cdot \tilde{\tau} = 0$$

where ∇ is the covariant derivative on M. When $L = S^1$, these are exactly the nonlinear Dirac equations for map $\psi = (\phi, \tilde{\tau}) : M \to S^1 \times \mathbb{R}^3$ with the vector permuting $Sp(1)$ action.

A success of the Kapustin–Vyas model is that it recovers the two-dimensional A-model in $T^\vee L$ when compactified on an interval with suitable boundary conditions. When $L = S^1$, this corresponds to the fact that $T^\vee L \cong \mathbb{C}^*$ is a complex Lagrangian submanifold of $\mathbb{C}^* \times \mathbb{C} \simeq S^1 \times \mathbb{R}^3$.

For general L though, the total space of the bundle $(T^\vee L)^{\oplus 3} \to L$ is not hyperKähler (as we commented above, there is a sense in which it is locally hyperKähler in nice cases [Kal99; Fei03]).

Moreover, the Kapustin–Vyas model is not flexible enough to incorporate general hyperKähler targets or permuting $Sp(1)$ actions of 'hypermultiplet' type. This means that the Kapustin–Vyas model can not quite be the theory underlying the A-twist of an $\mathcal{N} = 4$ sigma model.

5.5 A GRAVITATING THEORY

5.5.1 *The Swann quotient*

In 'real life' examples, such as those induced from gauge theories, the target space of our A-twisted theory is likely to be a singular space. The singularities generally induce significant complications. For example, when one hits the root of the Higgs branch of an undeformed gauge theory, one could conceivably 'spill out' onto the Coulomb branch.

We have not dealt seriously with the myriad technical issues that arise when considering the theory of the nonlinear Dirac equation into a singular space.

Here we offer a brief, speculative, and incomplete sketch of how the issue might be resolved when the theory is coupled to background gravity. The A-twisted theory already sees background gravity, so it is not too significant a step to pass to dynamical gravity.

We restrict ourselves to hyperKähler manifolds X with permuting $Sp(1)$ actions for which the quantity $i_a\theta_b$ considered in subsection 5.3.1 takes the form

$$i_a\theta_b = \nu\delta_{ab}$$

for some ν. We call such actions *Swann actions*, after [Swa90]. In this case, ν is a hyperKähler potential for X [Pid04; Cal16].

We define the *Swann quotient* of X at level $\alpha \in \mathbb{R}$ to be

$$X/_{\mathrm{Swann}}Sp(1) = \nu^{-1}(\alpha)/Sp(1).$$

This can be viewed as a kind of desingularisation of the quotient of a hyperKähler cone X by $\mathbb{H}$. It is a result of [Swa90] that the Swann quotient is a quaternion-Kähler manifold.

5.5.2 *The gravity-coupled theory*

We will consider only a 'critically-coupled' bosonic theory here. We take a theory of a dynamical frame $\boldsymbol{e}$, along with the field ψ as before. We will view the spin connection γ as a function of the frame e, as determined by the structure equation $\mathrm{d}_\gamma e = 0$.

The idea of the critically-coupled theory comes from considering the term $R\nu$ in the action of the A-twisted theory as a cross term. We consider the action density

$$\mathcal{E} = -\alpha R + \frac{1}{l^2}|F(\gamma)|^2 + |\mathrm{d}_\gamma\psi|^2 + l^2(\alpha - \nu(\psi))^2.$$

The first term is a standard Einstein–Hilbert term, while the second is a kind of R^2 term. We have introduced two gravitational coupling constants, l^2 and α. The point of the particular choice of coupling that we have made is that one can write

$$\mathcal{E} = \frac{1}{l^2}|F(\gamma) - l^2(\alpha - \nu(\psi))e|^2 + |\not{\mathrm{d}}_\gamma\psi|^2,$$

leading one to the BPS equations

$$F(\gamma) = l^2(\alpha - \nu(\psi))e, \;\; \not{d}_\gamma \psi = 0.$$

In the limit $l^2 \to 0$, one finds the A-twisted sigma model in the background of semiclassical gravity, with the first of the BPS equations above becoming the Einstein equation (one can introduce a cosmological constant if one wishes). On the other hand, in the limit $l^2 \to \infty$, one ends up with a classically underdetermined theory for a map

$$M \to \nu^{-1}(\alpha)/Sp(1) = X/_{\mathrm{Swann}} Sp(1).$$

By passing to the Swann quotient, we have 'excised the tip of the cone'. However, this strongly-coupled theory is classically underdetermined because there is no equation of motion for the gravitational field. In the absence of global anomalies, one might be able to 'divide by the gravitational path integral' to give a well-defined sigma model theory.

PART IV

MAGNETIC SKYRMIONS

6 MONOPOLES AND MAGNETIC SKYRMIONS

6.1 INTRODUCTION AND SUMMARY

Finding effective methods of storing data is one of the great challenges of the modern era. The aim of the game is to store ones and zeroes in a way that is stable against unwanted fluctuations and yet easily writable and readable. Importantly, one wants to store these ones and zeroes as densely as possible.

Magnetic Skyrmions have emerged as a potential player in high-density, 'next-generation' storage devices. Predicted theoretically in [PB75; BH94] and first observed in [Müh+09], they are 'almost-topological' solitons in chiral ferromagnets. The 'almost' arises because they are associated to a $U(1)$ topological symmetry[1] which emerges at long distances, but is not present at short distances. Physically, this is manifested by the fact that magnetic Skyrmions can be created and destroyed by finite energy microscopic processes, mediated by sources or sinks for the emergent magnetic field that magnetic Skyrmions carry. These sources (or sinks) are known as *emergent (anti)monopoles* or *Bloch points*.

Originally considered in [Fel65; Dör68], emergent monopoles were observed indirectly in [Mil+13] and have subsequently attracted significant theoretical and experimental attention [Kan+16; SR14; Im+19; Kan+20; Bir+20; Fuj+19; Zha+16; Mül+20], as have related objects called *bobbers* [Ryb+15; Zhe+18].

Perhaps the most obvious approach to understanding the microscopic breaking of the infrared topological symmetry is to regard the microscopic theory as a Heisenberg model, with the spins comprising the magnetisation field tightly bound to atoms in a lattice. Emergent monopoles can then be regarded as sitting in between the lattice points.

In this chapter we take a rather different approach, motivated both by the fact that the lattice theory does not give a deep understanding of the nature of the electron field close to a monopole and by ideas of criticality. We introduce a continuum toy model of chiral ferromagnets in three spatial dimensions to understand their phase structure close to the critical point at which monopoles and antimonopoles may appear. The model is based on the ideas of electron fractionalisation and emergent $SU(2)$ gauge theory (see [Sac19] for a summary of these ideas). This gives us a natural continuum mechanism with which to break the topological symmetry of the infrared description of the ferromagnet at intermediate scales. The model we produce supports Skyrmion strings and confined monopoles, although neither are generally stable. By tuning the applied magnetic field and other parameters in the model we can (approximately) stabilise the Skyrmion phase. The usual micromagnetic theory emerges in the infrared. The model supports arbitrary Dzyaloshinskii–Moriya interaction.

Our main technical contribution is the identification of a critically-coupled point in our model, where confined monopoles are dynamically stabilised. At this point the monopoles obey first-order equations of Bogomolny–Prasad–Sommerfield (BPS) type. The equations are given in (6.10) and the discussion of the dynamic stabilisation of their solutions follows in sections 6.3.5 and 6.3.6.

[1] The 'topological' symmetry is the symmetry for which magnetic Skyrmion number is the conserved charge.

The equations that we find are novel and represent an interesting mix of the BPS Skyrmion equations [BSRS20] and the BPS monopole equations [Bog76; PS75]. They are fundamentally distinct from similar mixes that have appeared in the high-energy physics literature [Ton04b; Auz$^+$03; SY04] for the technical reason that the complex structure(s) used to define the relevant Dolbeault-type operator does not commute with the gauge group. The incorporation of chiral interactions into a confined monopole equation is also novel.

BPS-type descriptions facilitate exact analysis and there has been recent interest in one- and two-dimensional models admitting BPS equations [BSRS20; Sch19; Wal20; RSN20; RSN21; Hon$^+$20]. The three-dimensional theory that we introduce is rather more complicated: while exact BPS solutions can be found in lower dimensions, even the existence of solutions to our system appears to be a difficult question to address. A more complete understanding of the equations and their solutions represents an potential avenue for future work.

In the rest of this introduction we summarise the fundamental ideas in greater detail. We introduce the general gauge theoretic description of chiral magnets in section 6.2, showing that it supports Skyrmions strings and monopoles on which such strings may end. We explore the phase structure of the model. In section 6.3 we find a point in the parameter space where the Hamiltonian of the theory admits a BPS-type rearrangement. At this point, Skyrmion strings have vanishing tension and metastable monopole-antimonopole pairs may appear. We use this to qualitatively explore the phase structure of chiral magnets close to this critical point. We show that the model reproduces micromagnetism at long distances.

6.1.1 Micromagnetism, chiral interactions, and magnetic Skyrmions

Micromagnetism is an effective theory of magnetic materials, valid at scales much larger than the lattice separation, so that a continuum approximation is valid. Below the critical temperature, the configuration of a magnetic material in the micromagnetic regime is characterised by a three-dimensional vector field of constant magntitude $\boldsymbol{m}$, the *magnetisation field*.

Because the magnetisation field $\boldsymbol{m}$ has constant magnitude, it defines a map into a 2-sphere (or a section of a bundle of 2-spheres). The nontrivial topology of the 2-sphere allows for the formation of topological solitons: Skyrmions, classified by $\pi_2(S^2) \cong \mathbb{Z}$, and Hopfions, classified by $\pi_3(S^2) \cong \mathbb{Z}$. Hopfions are essentially (knotted) Skyrmion lines, and have been considered in [Sut18], for example.

In general, however, these solitonic structures may be unstable against bubbling phenomena (for example, due to Derrick-type scaling instabilities [Der64]). It is therefore important to consider the energetics of the system.

If one ignores the so-called demagnetising energy, which is non-local but often unimportant, the effective theory in this regime in the presence of an applied magnetic field $\boldsymbol{b}$ has an energy density of the general form

$$\mathcal{E}_{\text{micro}} = J(\nabla \boldsymbol{m})^2 + D_{ijk} m_i \nabla_j m_k - \mu_0 \boldsymbol{m} \cdot \boldsymbol{b} + \mathcal{V}(\boldsymbol{m}) \tag{6.1}$$

where $\mathcal{V}(\boldsymbol{m})$ is a local potential energy density, zeroth order in derivatives. Here J is the *exchange constant*, which we assume to be positive, and D_{ijk} is a *Lifshitz tensor*, antisymmetric in its last two indices. The term involving D is a chiral term, often called the *Dzyaloshinskii–Moriya* (DM) term, after [Dzy58; Mor60], arising from an antisymmetric interaction between neighbouring spins.

Without the DM term, the Derrick argument renders Skyrmions unstable against scalings. The DM term provides respite, allowing dynamically stable Skyrmions to exist. On such configurations, the DM term is necessarily negative.

The energy density (6.1) has the general form of that of a nonlinear sigma model into the target 2-sphere. How should one interpret the DM term in this language? As exploited in [BSRS20], it naturally arises when one couples the theory to a background orthogonal connection K via an exchange term involving the coupled covariant derivative

$$\nabla_K \boldsymbol{m} = \nabla \boldsymbol{m} + \boldsymbol{K} \times \boldsymbol{m}.$$

The symmetries of the Lifshitz tensor D are the natural symmetries of an orthogonal connection K (or, better, the difference between K and the Levi-Civita connection Γ). Among other things, this gives an easy lower bound on the DM term (which can be negative) in terms of the exchange term.

An important fact about magnetic Skyrmions is that they are charged under an *emergent magnetic field* (note that this is not a true magnetic field). In the micromagnetic theory, this emergent magnetic field is given generally by the pullback of the $SO(3)$-equivariant symplectic form on the target 2-sphere (as determined by the chiral connection K; see Appendix A) which is just the Skyrmion density.

6.1.2 Emergent magnetic monopoles

In [Mil$^+$13], the creation and destruction of Skyrmions was indirectly observed and argued to be mediated by emergent magnetic monopoles, also known as Bloch points.

Magnetic Skyrmions are protected by a topological $U(1)$ symmetry in the micromagnetic theory. The corresponding charge is the Skyrmion number. The existence of emergent monopoles shows that the 'true' theory of the magnetic material generically does not have this symmetry: it emerges in the infrared.

One way to model this is by taking the theory of the magnet to be given by a tight-binding Heisenberg model, in which the spins comprising the magnetisation are held at lattice points. This clearly breaks the topological symmetry: a 'Dirac-type' monopole can sit in between the lattice points.

Another theoretical approach is to consider magnetic Skyrmions as the infrared limit of Abelian vortices and then to embed the $U(1)$ gauge group (supporting the emergent magnetic field) into a gauge group $SU(2)$, invisible except at high energy. This allows us to consider 't Hooft–Polyakov-type monopoles [Hoo74; Pol74] supported by a high energy symmetry breaking $SU(2) \dashrightarrow U(1)$. This is the approach we pursue.

6.2 A MODEL OF (CHIRAL) FERROMAGNETS

6.2.1 Fractionalisation, pseudospin, and charge

Let M be a three-dimensional manifold representing a sample of material. We are interested in the behaviour of electrons in this sample. We introduce an electromagnetically charged spinor q on M representing the (static) electron field. This lives in the spin representation of $\mathrm{Spin}^c(3)$. We may write

$$q = \begin{pmatrix} q_1 & -\bar{q}_2 \\ q_2 & \bar{q}_1 \end{pmatrix},$$

where q_1, q_2 are complex. The spin group acts as $SU(2)_\mathrm{s}$ on the left, with the subscript s standing for 'spin', and the electromagnetic $U(1)$ acts via multiplication of matrices of the form $\exp(\mathrm{i}\theta\sigma_3)$ on the right. This makes the quaternionic nature of the spin representation in three dimensions partially manifest. The magnetisation order parameter is then given by

$$\boldsymbol{m} = \mathrm{tr}\left(\boldsymbol{\sigma} q \sigma_3 q^\dagger\right)$$

(up to factors which we have absorbed into the fields).

We may introduce a 'rotated spin reference frame' for q, by writing

$$q = \psi f \tag{6.2}$$

where ψ and f are both 2×2 matrices, which are clearly not uniquely defined by this equation. The matrix ψ is taken to be a bosonic $SU(2)$ matrix, while

$$f = \begin{pmatrix} f_1 & -\bar{f}_2 \\ f_2 & \bar{f}_1 \end{pmatrix}$$

is fermionic. The field ψ carries the left action of the spin group $SU(2)_\mathrm{s}$ and is known as the *spinon*. The fermionic field f carries the right action of the electromagnetic $U(1)$ gauge group and is known as the *chargon*.

The physical object q is unchanged under

$$\psi \mapsto \psi U, \; f \mapsto U^\dagger f$$

for U an $SU(2)$ valued function. This $SU(2)$ redundancy is known as *pseudospin* and so we refer to this group as $SU(2)_\mathrm{ps}$. This redundancy is, of course, not physical and so it is gauged.

For our purposes, it is convenient to modify the field content slightly. First, we need not deal with all of the information captured by the field f, instead choosing to work with the adjoint field ϕ defined by

$$\phi \cdot \sigma = f \sigma_3 f^\dagger.$$

This has the advantage of leaving us with a theory of bosons.

While passing from f to ϕ simplifies things, it also means that no fields in the theory carry electromagnetic charge. To understand how background electromagnetic fields couple to the theory, it is useful to pass back to a description in terms of the charged field f.

Also, for our purposes, it is useful to allow ψ to have arbitrary determinant, so that it becomes a linear field charged under $SU(2)_{\mathrm{s}} \times SU(2)_{\mathrm{ps}}$. Practically, this allows us to implement the symmetry breaking procedure in a straightforward way. The additional redundancy that this introduces is dealt with dynamically.

6.2.2 A general model

The model we consider is built from the following data[2].

- A three-dimensional manifold M carrying a fixed, background spin connection K, characterising spin-orbit interactions in the material.
- A dynamical $SU(2)_{\mathrm{ps}}$ gauge field A.
- A $SU(2)_{\mathrm{ps}}$-charged fundamental spinor ψ. We will represent ψ as a 2×2 matrix

$$\psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 \\ \psi_2 & \bar{\psi}_1 \end{pmatrix},$$

  where ψ_1, ψ_2 are complex. Hermitian conjugation corresponds to quaternionic conjugation. The action of the spin and pseudospin groups are represented by the fundamental action of $SU(2)$ on the left and right respectively.
- An $SU(2)_{\mathrm{ps}}$ adjoint scalar $\phi = \mathrm{i}\vec{\phi} \cdot \vec{\sigma}$. This is the simplest extra field that one can include so as to form the gauge-invariant magnetisation field. It can also be motivated through electron fractionalisation, as above.

In ferromagnetic materials, the pseudoparticles represented by the fields ψ, ϕ, and A are not macroscopically visible. Hence there must be some mechanism by which they are confined into a macroscopically observable $SU(2)_{\mathrm{ps}}$ singlet. There is essentially one such observable that one can write: the magnetic order parameter,

$$\boldsymbol{m} = \phi_i \mathrm{tr}(\sigma_i \psi^\dagger \boldsymbol{\sigma} \psi).$$

In the presence of an applied magnetic field $\boldsymbol{b}$, with gauge potential α, we consider a Hamiltonian density of the general form

$$\mathcal{E} = \frac{1}{g^2}\left(F(A)^2 + f_1(\psi)\mathrm{d}_A\phi^2\right) + f_2(\phi)|\mathrm{d}_{K\otimes A}\psi|^2 + \mathcal{V}(\phi, \psi, \boldsymbol{b}) + \cdots, \tag{6.3}$$

where f_1, f_2 are positive, gauge-invariant functions, $\mathcal{V}(\phi, \psi, \mathbf{b})$ is a gauge-invariant potential energy density, g^2 is a positive coupling constant, and $\cdots$ denotes additional terms involving derivatives that one can write.

It is useful to note that $\frac{1}{g^2}$ has dimensions of length in a three-dimensional gauge theory. We can therefore use the coupling as a proxy for 'magnification': the theory is strongly coupled at long distances, and weakly coupled at short distances.

[2]In principle, it is natural to phrase the data of the theory in terms of $\mathrm{Spin}^{SU(2)}$-structures. However, in three dimensions this does not lead to increased generality.

6.2.3 Symmetry breaking and the phases of materials

The energy density (6.3) is rather flexible, and can describe several states of matter at low energy depending on the form of the potential $\mathcal{V}$. We can classify these low energy phases in terms of symmetry breaking. In this discussion, we assume that the spin-orbit interaction, represented by K, is relatively small, and that the applied magnetic field $\boldsymbol{b}$ is turned off.

- If neither ϕ or ψ take on a vacuum expectation value, and they are given large masses, the low energy theory is pure $SU(2)_{\mathrm{ps}}$ Yang–Mills theory. The magnetic order parameter does not condense, so this describes a disordered state, perhaps an $SU(2)$ spin liquid (see [Wen02]).
- If ϕ takes on a positive vacuum expectation value, it breaks the gauge group from $SU(2)$ to the circle of rotations preserving ϕ, leaving an emergent $U(1)$ gauge theory. If ψ does not condense, this $U(1)$ is unbroken and we are in a Coulomb phase.
- If ϕ and ψ both condense, the gauge group is completely broken. The condensation of ψ means that the magnetisation $\boldsymbol{m}$ takes non-zero magnitude so the theory has magnetic order. This is a Higgs phase.

  To describe ferromagnetic materials, the theory should sit in the Higgs phase and be very strongly coupled at the length scales that we observe. Then the gauge flucatuations, which are not observed, are screened.

6.2.4 The long and short of things: Monopoles, confinement, and Skyrmion strings

Rather generally, the spontaneous breaking of a gauge group G to a subgroup H can allow for the presence of topological solitons, classified by the homotopy groups of G/H. The point is that, when studing the theory on Euclidean space $\mathbb{R}^n$ the condensed field must take values in G/H at the 'sphere at infinity', S^{n-1}_∞ for a configuration to have a chance at having finite energy. Thus any finite energy configuration should define a class in $\pi_{n-1}(G/H)$.

To describe a ferromagnet, we know that we should completely break the gauge group $SU(2) \dashrightarrow 0$. To understand the topological structures that are observed, we suppose that this breaking is carried out in stages in the pattern

$$SU(2) \dashrightarrow U(1) \dashrightarrow 0.$$

The first stage is associated with the condensation of ϕ and takes place at very high energy. The second stage is associated with the condensation of ψ and takes place at lower energy (although still at higher energies than those we probe macrosopically). Similar symmetry breaking patterns (and the topological solitons they do or do not support) have been considered in the context of (super) quantum chromodynamics and nonAbelian superconductors, see [Auz+04].

A high-energy observer will not see the condensation of ψ and so sees only the breaking

$$SU(2) \dashrightarrow U(1).$$

Such an observer may expect to see emergent monopoles, of 't Hooft–Polyakov type, classified by $\pi_2(SU(2)/U(1)) \cong \mathbb{Z}$.

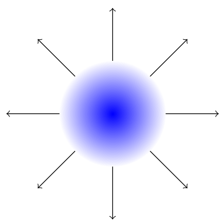
On the other hand, a low-energy observer would see a theory where ϕ takes on its vacuum expectation value and the tranverse gauge bosons are too heavy to be excited. They would see a theory of the field ψ coupled to the emergent $U(1)$ gauge field and the corresponding symmetry breaking

$$U(1) \dashrightarrow 0.$$

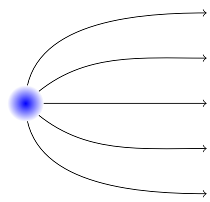
This low-energy effective theory supports semilocal vortex strings, classified by $\pi_1(U(1))$. In the macroscopic (strongly coupled) limit, these vortices appear as magnetic Skyrmions. Importantly, the vortex number becomes the total emergent magnetic flux.

Despite what some high- and low-energy observers may believe, the theory has *no stable topological solitons*, as follows from the fact that $\pi_1(SU(2)) \cong \pi_2(SU(2)) \cong 0$. What happened to the monopoles and the vortices?

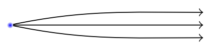
Close to the centre of a monopole, one has $\psi \approx 0$ and the monopole has an approximately Coulombic flux pattern. As one zooms out from the monopole, the Higgs potential encourages the condensation of ψ. This gives mass to the emergent photon, highly discouraging the radial, Coulombic flux pattern. However, the flux has to go somewhere, so it gets confined into a tube. These tubes are the vortex strings of the low-energy theory. We illustrate what one sees as one zooms out from a monopole in Figure 6.1.



(a) The flux pattern is Coulombic at short distances.



(b) Higgs condensation at intermediate length scales discourages a radial flux pattern and confines flux.



(c) At long distances the flux is confined into vortex (or Skyrmion) string.

Figure 6.1: Zooming out from a monopole in the Higgs phase. The shading denotes regions where the gauge group is unbroken. The arrows denote lines of emergent $U(1)$ flux.

The vortex string has some energy per unit length, which means that a monopole is attached to a string under tension. It is favourable to shorten the string, so the monopole gets dragged along in the direction of the string. This leads to monopole confinement - monopoles and antimonopoles in this theory are joined by a string under tension which pulls them together.

On the other hand, a vortex string is unstable against monopole-antimonopole pair production. Once the monopole and antimonopole comprising such a pair are produced, they are pulled off to infinity in opposite directions by the vortex lines.

6.2.5 The stabilisation of the Skyrmion phase

In the context of magnetic materials, we have control over the external parameters of the system. In particular, we can apply a magnetic field to our sample.

The model we have produced provides a theoretical explanation for the existence of emergent monopoles and also explains the mechanism which renders Skyrmions unstable at zero field. However, we know that there are situations in which the Skyrmion phase is stabilised once one applies a magnetic field, $\boldsymbol{b}$, to the system.

As we address quantitatively below, the fractionalisation scheme (6.2) tells us how to couple the applied field $\boldsymbol{b}$ to our effective theory. One natural contribution is the well-known Zeeman term, of the form

$$-\mu_0 \boldsymbol{m} \cdot \boldsymbol{b}.$$

This encourages the magnetisation $\boldsymbol{m}$ to align with the applied magnetic field (assuming that the sign of the constant μ_0 is positive). As realised in [BSRS20] and exploited below, this term can be 'compensated for' by chiral interactions in the theory.

There is another natural term which should be included in the effective theory, of the form

$$-\mu_1 \boldsymbol{b} \cdot \boldsymbol{b}_{\text{emerge}} \tag{6.4}$$

where $\boldsymbol{b}_{\text{emerge}}$ is the 'emergent magnetic field' carried by magnetic Skyrmions. Here μ_1 is a constant.

The term (6.4) looks a little unusual, but it is natural in the fractionalisation scheme we have employed and, as we see below, including it leads to the correct phenomenology. It has the effect of encouraging magnetic Skyrmion strings to align with the applied field, as observed, and it helps us to understand the phase transition into the Skyrmion phase as $|\boldsymbol{b}|$ is increased: it provides a compensating force to the tension of Skyrmion strings.

6.3 CRITICALLY-COUPLED MAGNETISM

6.3.1 Critically-coupled micromagnetism

In [BSRS20], a 'critically-coupled' model for micromagnetism in two dimensions was found. It is characterised by the fact that it admits a moduli space of energy-minimising configurations within each topological charge. These configurations obey a first order differential equation. Strikingly, the authors of [BSRS20] were able to find a general solution to the equation. In chapter 7, the material of which appears in [Wal20], we extend the theory to curved surfaces. For certain particularly symmetric surfaces, we find exact Skyrmion solutions.

The original model of [BSRS20] is a model of a two-dimensional sample, but in summarising it here we will extend it to three dimensions in a trivial way.

The theory can be defined on three-manifolds of the form[3] $\Sigma \times I$, for Σ a two-dimensional Riemannian surface and I a one-manifold, and has the simple energy density

$$\mathcal{E}_{\mathrm{BSRS}} = \frac{1}{2}|\mathrm{d}_K \boldsymbol{m}|^2 - \boldsymbol{F}_{12}(K) \cdot \boldsymbol{m} \tag{6.5}$$

where (as before) $\boldsymbol{m}$ is the micromagnetic order parameter, a unit vector field in three dimensions, and K is the spin connection (or rather the connection on the tangent bundle induced by the spin connection) capturing spin-orbit interactions. Here $\boldsymbol{F}_{12}(K)$ is the Σ-component of the curvature of K.

Expanding (6.5) gives

$$\mathcal{E}_{BSRS} = \frac{1}{2}|\nabla \boldsymbol{m}|^2 + K^{\nu}{}_{\mu\rho} m^{\mu} \nabla_{\nu} m^{\rho} + \frac{1}{2} K_{\mu}{}^{\nu}{}_{\rho} K^{\mu}{}_{\nu\sigma} m^{\rho} m^{\sigma} - F(K)_{12\mu} m^{\mu}.$$

The first of the terms on the right hand side is a standard exchange interaction, the second is a kind of DM interaction, while the third and final terms form an effective potential energy term.

The energy density (6.5) is called ‘critically-coupled’ because it admits the Bogomolny-style rearrangement (see [BSRS20; Sch20] and Appendix A)

$$|\mathrm{d}_K \boldsymbol{m}|^2 = |\bar{\partial}_K^{\Sigma} \boldsymbol{m}|^2 + |\mathrm{d}_K^I \boldsymbol{m}|^2 + *_{\Sigma}(\boldsymbol{m}, K)^* \omega_{S^2}^{SO(3)}.$$

Here $\bar{\partial}_K^{\Sigma}$ is the K-coupled Dolbeault derivative on Σ, while d_K^I is the K-coupled derivative along the one-manifold I, and $*_{\Sigma}$ is the Hodge star on Σ. The object $\omega_{S^2}^{SO(3)}$ is the $SO(3)$-equivariant symplectic form on the target sphere S^2. The meaning of this is explained in Appendix A.

The final term in this rearrangement is a kind of topological term, providing finite energy per unit distance along I. This encourages us to consider the equations

$$\bar{\partial}_K \boldsymbol{m} = 0, \; \mathrm{d}_K^I \boldsymbol{m} = 0. \tag{6.6}$$

The first of these equations always has solutions (see chapter 7), but the second may not. As we discuss later, this is why Skyrmions can be harder to form in ‘thick’ samples, where the length of I is comparable to, or larger than, the characteristic length scale set by the connection K.

Something special happens for special choices of DM interaction. Suppose that our sample is modelled by $\mathbb{R}^3$ with Euclidean metric and Cartesian coordinates (x^1, x^2, x^3) and let K take the form

$$K_{\mu\nu} = \kappa \left(\epsilon_{\mu\nu 1} \mathrm{d}x^1 + \epsilon_{\mu\nu 2} \mathrm{d}x^2 \right)$$

in the Cartesian gauge, for a real parameter κ. Then

$$*\boldsymbol{F}_{12}(K) \cdot \boldsymbol{m} = \kappa^2 m^3$$

where m^3 is the component of $\boldsymbol{m}$ in the x^3 direction. This then resembles a Zeeman term associated to a magnetic field aligned along the x^3 axis. Because the curvature of K has no 23 or 31 components, the equations (6.6) do admit solutions.

We would like to extend this notion of critical coupling to our emergent gauge theory. To do this, we will first consider the high- and low-energy regimes. We will then glue these regimes together.

[3]Slightly more general situations can be considered, but this setup suffices for us.

6.3.2 The low-energy Abelian theory

We know that, at low energy, we have a theory of an Abelian gauge field $a \approx \mathrm{tr}(A\phi)$ and the field ψ. The Abelian gauge potential a is the gauge potential of the emergent magnetic field. We can choose $\phi \propto \sigma_3$, so that a acts on ψ as $\psi a \sigma_3$.

The critically-coupled low-energy gauge theory was studied in [Wal20], where it was viewed as a means to resolve the magnetic Skyrmion moduli space. We now view it as a physical model.

The theory has energy

$$\mathcal{E}_{\text{low}} = \frac{1}{e^2}F(a)^2 + |\mathrm{d}_{K\otimes a}\psi|^2 + e^2(\tau^2 - |\psi|^2)^2 - \mathrm{tr}(\psi F_{12}(K)\psi^\dagger).$$

We now think of K as a true spin connection. We may consider the case of $K = \mathrm{i}\kappa\left(\sigma_1 \mathrm{d}x^1 + \sigma_2 \mathrm{d}x^2\right)$ so that the final term is the Zeeman term for a critical applied magnetic field in the x^3 direction.

As promised, this theory supports vortex strings aligned in the x^3 direction obeying the Bogomolny equations

$$\begin{aligned} *F(a) &= e^2(\tau^2 - |\psi|^2)\mathrm{d}x^3 \\ \bar{\partial}_{K\otimes a}\psi &= 0 \\ D_3\psi &= 0. \end{aligned}$$

Solutions to these equations (with sufficiently fast fall off at infinity) have energy density

$$\tau^2 F_{12}(a)$$

and so the vortex strings have energy $4\pi\tau^2 N$ per unit length, where N is the vortex number. This means the energy of the solution scales linearly with the width of the sample of the x^3 direction. In the full theory, it becomes favourable to destroy the vortex string via the production of a monopole-antimonopole pair when the width becomes sufficiently large (this is one reason why Skyrmions are generally found in relatively thin samples). In this low-energy theory, however, there is no mechanism by which this can occur, so the vortices appear as stable objects.

Micromagnetism emerges in the strong coupling limit, where $e^2 \to \infty$. Then the gauge field decouples (it becomes infinitely strongly screened by the Higgs effect) and we are forced to set $|\psi|^2 = \tau^2$. This constraint defines a three-sphere. Taking the quotient by the $U(1)$ gauge action realises the Hopf fibration $S^3 \to S^2$, which acts as

$$\psi \mapsto \mathrm{tr}(\phi^A \sigma_A \psi \boldsymbol{\sigma} \psi^\dagger) = \boldsymbol{m}$$

(we have reinserted ϕ here to give a gauge-invariant statement, although one can happily fix $\phi^A\sigma_A \equiv \phi_0\sigma_3$ by a gauge transformation). The theory we are left with is the theory of a vector field $\mathbf{m}$ of constant magnitude $|\mathbf{m}| = |\phi_0|\tau^2$. It is not hard to see that this theory is the critically-coupled micromagnetism considered above. The vortex equations become the Skyrmion equations.

The vortex equations can not be solved analytically, except in the limit of strong coupling. They are known to have solutions for any coupling. See Part II for more details about vortices.

6.3.3 The high-energy theory

At short distances, close to a monopole, we can take $g^2 \to 0$ and set $\psi \approx 0$, leaving us with an effective theory of the form

$$\mathcal{E}_{\text{high}} = \frac{1}{g^2}\left(F(A)^2 + \mathrm{d}_A\phi^2\right).$$

As written, this theory has the trivial vacuum $\phi = 0$. One often excludes this possibility with the addition of a Higgs potential of the form

$$\lambda(\phi_0^2 - |\phi|^2)^2.$$

We may then take the limit $\lambda \to 0$, remembering its presence by imposing boundary conditions at infinity.

The theory supports monopoles. The lowest energy monopole configurations solve the BPS equations [Bog76]

$$*F(A) = \mathrm{d}_A\phi.$$

We refer to [MS04] for a textbook treatment of these monopole equations. In general, finite energy solutions to these equations describe a localised region where the gauge group is unbroken, outside of which the gauge group is broken to $U(1)$. Solutions have a topological charge N, which is the total emergent $U(1)$ flux that passes through the sphere at infinity. The moduli space of BPS monopole solutions of charge N is a hyperähler manifold of real dimension $4N$.

6.3.4 A critical model

We know what the theory should look like at high and low energies. We just have to glue these regimes together.

There is a technical issue that renders this a little harder than one might expect. The spin representation, in which ψ takes values, is naturally a hyperKähler space, with hypercomplex structures acting on the right in our parameterisation. The action of the pseudospin group does not commute with the action of the complex structures, instead it is a *permuting action* (see chapter 5). The condensation of ϕ breaks $SU(2)_{\text{ps}}$ to $U(1)$, which fixes a complex structure on the spin representation.

Because the action of $SU(2)_{\text{ps}}$ does not preserve the hyperKähler structure on the target space, there is no hyperKähler moment map associated to it. However, in the language of chapter 5, the $SU(2)_{\text{ps}}$ action is a *Swann action* (see [Swa90; Pid04; Cal16]). What does this mean? Every subgroup $U(1) \subset SU(2)_{\text{ps}}$ preserves a Kähler structure on the target space and defines a Kähler moment map. That the action is Swann means that the Kähler moment map you get does not depend on the choice of subgroup $U(1) \subset SU(2)_{\text{ps}}$. We call this moment map the *Swann moment map*. In our case, it is

$$\nu(\psi) = \frac{\mathrm{i}}{2}|\psi|^2,$$

which might be viewed as 'explaining' the form of potential term in the low-energy theory above.

Similarly, when ϕ is condensed to a vacuum expectation value (meaning that it is covariantly constant), it defines a Dolbeault operator on the field ψ restricted to two-dimensional surfaces, which was necessary to define the vortex equations in the low-energy Abelian theory.

The technical difficulty that we must address is how these ideas are implemented in the regions where ϕ is not condensed. This problem is absent in other similar systems, such as those in [Ton04b; Auz$^+$03; Auz$^+$04; SY04], because in those cases the gauge action commutes with all of the relevant complex structures.

We work on a Riemannian three-manifold (M, g_M) of the form $\Sigma \times I$, for Σ a two-dimensional surface and I a one-manifold. We will write v for the unit 1-form tangent to I which we assume to be closed (this restricts the form of the metric on I).

The theory we come to has energy density

$$\mathcal{E}_{\mathrm{crit}} = \frac{1}{g^2} F(A)^2 + \mathcal{K}(\phi, \psi) + (\phi_0^2 - |\phi|^2)^2 |\psi|^2 + g^2(\tau^2 - |\psi|^2)|\phi|^2, \tag{6.7}$$

where

$$\mathcal{K}(\phi, \psi) = \frac{1}{g^2} |\mathrm{d}_A \phi|^2 + |\mathrm{d}_A \psi|^2 + \frac{1}{\phi_0^2} |\mathrm{d}_A^\Sigma(\psi\phi)|^2$$

is the kinetic term for the matter fields. We have introduced the notation d_A^Σ for the covariant derivative restricted to Σ. The presence of the third term above, breaking the three-dimensional rotational symmetry of the theory, is rather unattractive. However, it seems to be necessary in order to deal with the technical issues raised above. In applications, it is typical to consider systems with rotational symmetry broken (often the samples are taken to be thin, for instance), so it may be that this term is physically justifiable.

The form of $\mathcal{E}_{\mathrm{crit}}$ is such that it admits a Bogomolny-type rearrangement. To describe this, we introduce the operator

$$\mathcal{D}^{(\Sigma,\phi)}\psi := \mathrm{d}_A\psi + \frac{1}{\phi_0} j_\Sigma \circ \mathrm{d}_A(\psi\phi).$$

where j_Σ is the complex structure on Σ. Notice that, if ϕ is condensed to its vacuum expectation value, with $|\phi| = \phi_0$ and $\mathrm{d}_A\phi = 0$, this operator is the usual Dolbeault operator with respect to the complex structure on the spin bundle defined by ϕ, in line with our earlier discussion. We also define the operator $D_v = g_M^{-1}(v, \mathrm{d}_A)$, the covariant derivative in the direction of v.

A short calculation reveals that one can write

$$\begin{aligned}\mathcal{E}_{\mathrm{crit}} = |*F(A) - \mathrm{d}_A\phi - g^2(\tau^2 - |\psi|^2)\phi v|^2 + |D_v\psi - (\phi_0^2 - |\phi|^2)\psi|^2 \\ + |\mathcal{D}^{(\Sigma,\phi)}\psi|^2 + 2\tau^2 * v \wedge \mathrm{tr}(F(A)\phi) + \mathcal{E}_{\mathrm{bdry}}\end{aligned} \tag{6.8}$$

where

$$\begin{aligned}\mathcal{E}_{\mathrm{bdry}} = *\mathrm{d}\frac{2}{g^2}\mathrm{tr}(F(A)\phi) + v^\mu\partial_\mu\left(\tau^2|\phi|^2 + (\phi_0^2 - |\phi|^2)|\psi|^2\right) \\ - *\frac{1}{\phi_0}\mathrm{d}\left(v \wedge \mathrm{tr}(\psi^\dagger \mathrm{d}_A(\psi\phi) + (\mathrm{d}_A(\psi\phi))^\dagger\psi\right)\end{aligned} \tag{6.9}$$

is a boundary term. The first term on the right hand side of (6.9) gives the total monopole flux when integrated, while we expect the second to be small by virtue of the fact that we expect $|\phi|$ to approach ϕ_0 rapidly outside of a monopole. The integral of the third term measures a kind of effective Abelian current carried by the fields through infinity on Σ (note that, in our conventions, the field ϕ is antiHermitian, which explains the signs that arise). If Σ is closed, this term vanishes by Stokes' theorem.

We are encouraged by (6.8) to consider the equations

$$* F(A) = \mathrm{d}_A\phi + g^2(\tau^2 - |\psi|^2)\phi v, \;\; \mathcal{D}^{(\Sigma,\phi)}\psi = 0, \;\; D_v\psi = (\phi_0^2 - |\phi|^2)\psi. \tag{6.10}$$

These equations mix the vortex equations, which arise in the low-energy theory, and the monopole equations, which arise in the high-energy theory. Solutions to these equations do *not* define stable solutions in the theory. In addition to the boundary energy, solutions have energy density

$$\tau^2 * v \wedge \mathrm{tr}(F(A)\psi).$$

In regions where ϕ is condensed, this contribution is caused by the tension of vortex lines (it is necessarily positive for the usual reasons). Indeed, we can identify

$$\boldsymbol{b}_{\text{emerge}} = g_M^{-1}(*\mathrm{tr}(F(A)\phi))$$

as the emergent magnetic vector field (of course, it is only really an emergent magnetic field in regions where ϕ is condensed to its vacuum expectation value). Then, we can write the energy due to vortex line tension as

$$\tau^2 \boldsymbol{v} \cdot \boldsymbol{b}_{\text{emerge}},$$

where $\boldsymbol{v} = g_M^{-1}(v)$.

6.3.5 Turning on a magnetic field

Suppose for now that M is $\mathbb{R}^3$ (or a domain thereof) and let $\boldsymbol{b}$ be a uniform applied magnetic field, thought of as a constant vector field. We do not have a natural way to couple to electromagnetism in our particular fractionalisation scheme. However, by following our nose from the general scheme (6.2), we know to consider the natural couplings of the form

$$-\mu_1 \boldsymbol{b} \cdot \boldsymbol{b}_{\text{emerge}} \quad \text{and} \quad -\mu_0 \boldsymbol{b} \cdot \boldsymbol{m},$$

where μ_0 and μ_1 are coupling constants and we recall that $\boldsymbol{m} = \mathrm{tr}\left(\phi\psi^\dagger \boldsymbol{\sigma}\psi\right)$. Roughly speaking, the magnetic field should couple to ϕ-dependent vector fields and these are the minimal such couplings. While this is a little ad-hoc, we will see that adding these terms allows us to explain observed phase structures of magnetic materials.

Consider the effect of including the first of these terms. We can still consider solutions to the equations (6.10), which now have energy

$$(\tau^2 \boldsymbol{v} - \mu_1 \boldsymbol{b}) \cdot \boldsymbol{b}_{\text{emerge}} + \mathcal{E}_{\text{bdry}}.$$

We are now encouraged to align $\boldsymbol{v}$ in such a way as to minimise the first of these terms, which is to say that it should be parallel to $\boldsymbol{b}$. This means that vortex strings will naturally align along the magnetic field, as observed.

Moreover, this leads to an explanation of the phase structure of magnetic materials (although until we include the chiral interaction, this is something of a caricature). If the magnitude of $\tau^2 \boldsymbol{v}$ is larger than that of $\mu_1 \boldsymbol{v}$, the tension of the vortex string dominates and the string will want to shorten. Thus the stable phase is a polarised phase, with no vortices. On the other hand, if the magnitude of $\mu_1 \boldsymbol{b}$ is larger than that of $\tau^2 \boldsymbol{v}$, the vortex strings have negative energy density and want to lengthen. This leads to the formation of a vortex (or Skyrmion) phase.

There is then a critical point at $|\mu_1 \boldsymbol{b}| = \tau^2$, where a vortex string is massless (modulo boundary terms). At this point, monopoles may be stabilised.

To understand the Zeeman-type term $\boldsymbol{b} \cdot \boldsymbol{m}$, we turn on chiral interactions in the theory.

6.3.6 Turning on chiral interactions

As before, we capture spin-orbit interactions by introducing a spin connection K, acting on ψ on the left. We couple the theory minimally to this connection, taking $\mathrm{d}_A\psi \rightsquigarrow \mathrm{d}_{K\otimes A}\psi$.

We can again consider the equations (6.10), although the operator $\mathcal{D}^{(\Sigma,\phi)}$ is now defined with respect to the connection $K \otimes A$, as is D_v. This leads to an extra contribution to the energy of solutions to the equations, taking the form

$$* \, v \wedge \mathrm{tr}(\phi \psi^\dagger F(K) \psi), \tag{6.11}$$

and also alters $\mathcal{E}_{\mathrm{bdry}}$ in an obvious way.

Suppose we are on flat space, $M = \mathbb{R}^3$, with coordinates (x_1, x_2, x_3) and consider the case of

$$K_{\mu\nu\rho} = \kappa \epsilon_{\mu\nu\rho} \tag{6.12}$$

where we have identified spin indices with spatial indices using an implicit spin frame. This particular choice of spin connection leads to the standard bulk DM interaction in the low energy micromagnetic theory.

For this choice, the term (6.11) becomes

$$\kappa^2 \boldsymbol{v} \cdot \boldsymbol{m}.$$

We see that this takes the same form as the Zeeman term. Indeed, if we include the Zeeman term, there is a term in the energy of solutions to (6.10) of the form

$$(-\mu_0 \boldsymbol{b} + \kappa^2 \boldsymbol{v}) \cdot \boldsymbol{m}.$$

Minimising this again encourages $\boldsymbol{v}$ to be aligned with $\boldsymbol{b}$. Then, if $\mu_0 |\boldsymbol{b}| = \kappa^2$, one could hope to have critical phenomena. This is the mechanism used in [BSRS20; Sch19] to achieve a critical theory in the two-dimensional case.

The connection (6.12) is not the only reasonable choice. Another particularly useful choice is

$$K_{ab} = \kappa(\epsilon_{1ab}\mathrm{d}x^1 + \epsilon_{2ab}\mathrm{d}x^2). \tag{6.13}$$

This breaks three-dimensional rotational symmetry. If $v = \mathrm{d}x^3$, the details of the Bogomolny rearrangement are not significantly changed. However, the Bogomolny equations are changed:

as we saw in the discussion of the low-energy theory, the Bogomolony equations with the chiral interaction specified by the connection of (6.12) have no Skyrmion solutions (except in the thin film limit), while those with chiral interaction specified by (6.13) do.

6.3.7 Phases of bulk chiral magnets

We see that there are two interesting ratios

$$r_1 := \frac{\mu_1 |\boldsymbol{b}|}{\tau^2} \quad \text{and} \quad r_0 := \frac{\mu_0 |\boldsymbol{b}|}{\kappa^2},$$

which are dimensionless (in our units, where $|v| = 1$), characterising the phase of the system.

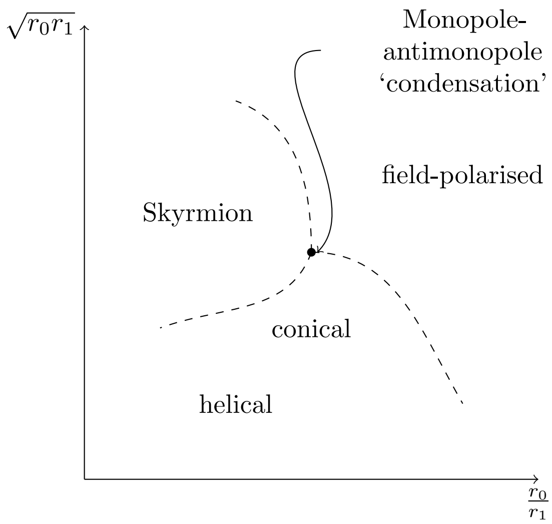


Figure 6.2: A sketch of the phases of a chiral magnet with DM interaction determined by (6.13) as $\sqrt{r_0 r_1} \propto |\boldsymbol{b}|$ and $\frac{r_0}{r_1} = \frac{\mu_0 \tau^2}{\mu_1 \kappa^2}$ are varied. For a standard bulk DM interaction, as determined by (6.12), the Skyrmion phase is further destabilised by thickness effects.

The ratio r_1 characterises the tension of vortex strings, while r_0 characterises the residual Zeeman effect on a solution to (6.10). When $r_0 \gg 1$, the ground state is field polarised.

There is a critical point at $r_1 = r_0 = 1$, where all bulk contributions to the energy of solutions to (6.10) vanish. In principle, the mass of vortex strings vanishes and monopoles may be dynamically stable. Some potentially subtle boundary issues remain, however. We will not address them here, happy for now with the fact that 'local' forces on monopoles cancel.

At this point, there is no local energy cost to the formation of monopole-antimonopole pairs: the length of the vortex string joining them has no effect on the energy of the configuration. The dynamics of the critical point is therefore likely to be complicated, with monopole-antimonopole pairs condensing and displaying intricate behaviours.

In Figure 6.2, we sketch the phase structure of the model as a function of $\sqrt{r_0 r_1} \propto |\boldsymbol{b}|$ and the ratio $\frac{r_0}{r_1} = \frac{\mu_0 \tau^2}{\mu_1 \kappa^2}$. Skyrmions are encouraged to form when r_1 is large, as the Skyrmion lines then contribute negative energy density, while the field-polarised configuration wins out for r_0 large due to the Zeeman term. When $|\boldsymbol{b}|$ is small, the ground state will not support Skyrmions, due to the tension of the Skyrmion lines and instead a helical or conical state will form. In general, the Zeeman term encourages the conical phase to form at small, but non-zero, $\boldsymbol{b}$. This picture agrees qualitatively with the observed phase structure of chiral magnets close to the critical point (see, for example, [Müh+09]).

7 SOME EXACT SKYRMION SOLUTIONS ON CURVED THIN FILMS

7.1 INTRODUCTION AND SUMMARY

In this chapter, we consider magnetic Skyrmions on curved thin films from the perspective of the 'BPS-type' methods that we have used throughout this thesis.

There is mathematical and physical interest in magnetic Skyrmions on curved thin films (see [GKS14; Str$^+$16], for example). By manipulating the geometry of the film, one may be able to manipulate the Skyrmions. Indeed, it has been demonstrated that by creating a curvilinear defect, one can pin Skyrmions in place, at least within a certain family of micromagnetic models [Kra$^+$18]. Ideas like this hint at interesting mathematical links between the extrinsic geometry of thin films and magnetic Skyrmion solutions.

In [BSRS20], a BPS model of magnetic Skyrmions on the plane was constructed. In section 7.2, we generalise this model to curved thin films, possibly with interesting topology. The model has a Dzyaloshinskii–Moriya (DM) antisymmetric exchange interaction[1]. As in the models studied in [GKS14], the geometry induces an apparent magnetic field normal to the film in our model. We write down the first order BPS equation for energy minimising solutions in the theory and prove the existence of solutions on general compact films (it was shown directly in [Sch19] that local solutions can always be constructed; our global proof follows a simple and well-known exercise in complex algebraic geometry). We interpret solutions to this equation as describing Skyrmion states, with number of Skyrmions N given by the topological degree of the solution (as usual). As is typical for BPS models, the energy of solutions to the BPS equation is linear in N. In general, Skyrmion and anti-Skyrmion density in the model is trapped in regions where the film has (extrinsic) curvature.

We go on in section 7.3 to write down explicit solutions on certain symmetric films. In particular, we solve generally the BPS equation for Skyrmions on a round spherical film for every Skyrmion number. Among the solutions, we find the 'hedgehog' solution. We also study the case of axially symmetric Skyrmions on certain axially symmetric films and find exact solutions on cones, which we view as a model for solutions on films with bumps. We further solve the theory on the cylinder. In this case, there are two axially symmetric vacuum solutions, and non-trivial axially symmetric Skyrmion solutions are kinks mediating between these vacua.

In section 7.4, we consider the moduli space of solutions on general compact films. This is a singular complex manifold of complex dimension $2N + 1 - g$, where N is the number of Skyrmions, and g is the genus of the film. It admits a natural resolution given by a certain moduli space of semi-local vortices in a background gauge field, which we describe. We give a direct construction of the moduli space of these vortices (at least for sufficiently high vortex number) using a 'dissolving vortex' limit.

Later, in A.2, we collect some relevant basic results (possibly of general interest) regarding two-dimensional nonlinear sigma models with target a symplectic manifold in the presence of

[1] This is in contrast to some other models of Skyrmions on curved thin films, which generate the DM interaction as a kind of fictional force using the geometry (see [GKS14], for example).

a background gauge field for a Hamiltonian group action on the target space. In particular, we observe that the BPS energy bound for models of this type is given by the pull back of the equivariant symplectic form on the target space evaluated on the funadamental class of the two-dimensional domain. This clarifies the geometrical meaning of the terms that appear in the context of magnetic Skyrmion models.

7.2 MAGNETIC SKYRMIONS ON CURVED THIN FILMS AT CRITICAL COUPLING

7.2.1 *An effective description of chiral magnetism from torsional geometry*

In [BSRS20], it was realised that the theory of planar micromagnetic thin films with DM interaction could be captured by the theory of a two-dimensional sigma model with target space the sphere in the background of a particular $SO(3)$ gauge field.

To understand why this is a worthwhile approach, we should recall first that, in the micromagnetic regime, the magnetisation field (divided by the saturation magnetisation) is a true unit vector field, m, in three dimensions. Hence, it is acted on naturally by the covariant derivative associated to an orthogonal connection (the connection should be orthogonal so that it preserves the unit length property) on the tangent bundle of a Riemannian three-manifold (usually $\mathbb{R}^3$ or a domain thereof). It becomes clear that chiral interactions can be effectively modelled by allowing the connection to have torsion.

As a very simple illustration of this, one can approximate the magnetisation field in the helical phase of a chiral magnet in $\mathbb{R}^3$ to be a solution to the first order equations

$$\begin{aligned}\partial_1 m^a &= -\epsilon_{1ab} m^b \\ \partial_2 m^a &= 0 \\ \partial_3 m^a &= 0,\end{aligned}$$

where $a, b = 1, 2, 3$. This can be written

$$\mathrm{d}_{\mathcal{A}} m = 0$$

where $\mathrm{d}_{\mathcal{A}}$ is the covariant derivative associated to the affine connection $\mathcal{A}$ with components

$$\mathcal{A}_{ab} \equiv \epsilon_{1ab} \mathrm{d}x^1. \tag{7.1}$$

in Cartesian coordinates. (Here and throughout we will use $\equiv$ to denote equality modulo gauge transformations – that is, local coordinate transformations.)

In general, an orthogonal connection $\mathcal{A}$ on the tangent bundle of a Riemannian manifold M can be characterised by its *contorsion tensor*

$$K := \mathcal{A} - \Gamma \in \Omega^1(M, \mathfrak{so}(3))$$

where Γ is the Levi-Civita connection (the unique torsion-free orthogonal connection). In Cartesian coordinates on flat $\mathbb{R}^3$, the components of the Levi-Civita connection vanish. In the case of the connection defined in (7.1), the contorsion tensor in the given Cartesian coordinates is simply $K_{ab} \equiv \epsilon_{1ab}\mathrm{d}x^1$. In what follows, this contorsion tensor will determine the antisymmetric exchange interaction of the theory.

Another natural gauge covariant object that one can associate to a connection $\mathcal{A}$ is its curvature. In general, the curvature of $\mathcal{A}$ can be given in terms of K as

$$\begin{aligned} F(\mathcal{A}) &= F(\Gamma + K) \\ &= F(\Gamma) + \mathrm{d}_\Gamma K + \frac{1}{2}[K, K]. \end{aligned}$$

To see how the DM interaction is related to the choice of K we consider the following. In three dimensions, the micromagnetic energy density functional should take the form

$$\mathcal{E}_{\mathcal{A}}[m] = |\mathrm{d}_{\mathcal{A}} m|^2 + V(m)$$

where $\mathrm{d}_{\mathcal{A}}$ is the covariant derivative and V is a choice of potential. Now, writing Greek indices for vector indices and Latin indices for matrix indices for the adjoint representation of $\mathfrak{so}(3)$ [2], we have

$$|\mathrm{d}_{\mathcal{A}} m|^2 = |\mathrm{d}m|^2 + 2\mathcal{A}_{\mu}{}^{a}_{b} m^b \partial^\mu m_a + \mathcal{A}_{\mu}{}^{a}_{b} m^b \mathcal{A}^{\mu}_{ac} m^c. \tag{7.2}$$

The story makes sense for arbitrary three-manifolds, but from now on we restrict ourselves to the (physically relevant) case that the three-manifold is $\mathbb{R}^3$ with its standard flat metric. In this case, we may choose a gauge (given by Cartesian coordinates) in which $\mathcal{A} \equiv K$. Thus, in these coordinates, the contorsion tensor is precisely the tensor, sometimes called the *DM vector*, defining the DM term, which is the second term on the right-hand side of (7.2). (The third term on the right-hand side of (7.2) contributes to the effective potential energy of the theory.) It is natural therefore to choose K to be translationally invariant, meaning that $\mathrm{d}_\Gamma K = 0$. In Cartesian coordinates $\{x^\mu\}_{\mu=1,2,3}$, we choose

$$K \equiv K_\mu \mathrm{d}x^\mu$$

where K_μ are constant elements of $\mathfrak{so}(3)$. For such choices, the fact that $F(\Gamma) = 0$ and $\mathrm{d}_\Gamma K = 0$ implies that

$$F(\mathcal{A}) = \frac{1}{2}[K, K],$$

which is also translationally invariant.

We might further restrict to connections of the form

$$\mathcal{A} \equiv \kappa O_\mu^A T_A \mathrm{d}x^\mu \tag{7.3}$$

where κ is a real number, O is an orthogonal matrix of determinant 1, and the T_A form the fixed basis for the adjoint representation of $\mathfrak{so}(3)$ given by

$$(T_A)_{ab} = \epsilon_{Aab}$$

[2]In three dimensions, these indices should be identified, but it is useful to keep them separate in the name of maintaining sanity when discussing the two-dimensional theory.

(of course, different choices can be made by a redefinition of O).

These connections are characterised more invariantly by their parallel transport. Given any straight line $l \subset \mathbb{R}^3$, the parallel transport along l is given by fixed rate rotations (with frequency given by κ) in a plane perpendicular to the line defined by acting on l with O. The rate of rotation κ introduces a length scale into the theory.

This shows us that, in general, these connections are symmetric with respect to the subgroup $U(1)$ of the group $SO(3)$ of global spatial rotations consisting of those rotations in the plane preserved by O.

If one takes the special case $O = 1$, one obtains the connection known as the *Cartan spiral staircase*. This leads to the usual bulk DM term, of the form $\kappa\mathbf{m} \cdot \nabla \times \mathbf{m}$, and will be the case we study primarily. It is the most symmetric choice, enjoying full rotational invariance, as $O = 1$ preserves every plane. It is for this reason that we focus on it: it treats every tangent plane of an arbitrary embedded two-dimensional surface in the same way. It will favour Bloch-type Skyrmions in two-dimensions.

An interesting different choice is

$$O = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.4}$$

which would give an 'interfacial' DM term on reduction to the two-dimensional (x_1, x_2) plane. This choice would favour Néel-type Skyrmions on that plane.

Now, if $K_\mu = \kappa O_\mu^A T_A$, then

$$\begin{aligned} F(\mathcal{A})_{\mu\nu} &= \frac{1}{2}[K_\mu, K_\nu] \\ &= \frac{\kappa^2}{2}\epsilon_{ABC} O_\mu^A O_\nu^B T_C \\ &= \frac{\kappa^2}{2}\epsilon_{\mu\nu\lambda} O^{\lambda C} T_C. \end{aligned}$$

Here we raise and lower Greek indices using the metric on $\mathbb{R}^3$, which is just $\delta_{\mu\nu}$ in these coordinates. We then see that

$$\begin{aligned} *F(\mathcal{A})_\mu &= \kappa^2 O_\mu^A T_A \\ &= \kappa K_\mu. \end{aligned} \tag{7.5}$$

This implies, after a similar computation, that

$$\mathrm{d}_A^\dagger F(\mathcal{A}) = \kappa^2 K, \tag{7.6}$$

so that $\mathcal{A}$ is not a solution to the Yang–Mills equation, instead obeying a kind of non-Abelian Proca equation.

We also observe that, for $\kappa \neq 0$, there is no adjoint Higgs field Φ such that $F(\mathcal{A}) = *\mathrm{d}_\mathcal{A}\Phi$, which is to say that $\mathcal{A}$ does not form part of a BPS monopole solution. This follows because the BPS equation implies that

$$\mathrm{d}_\mathcal{A}^\dagger F(\mathcal{A}) = *[F(\mathcal{A}), \Phi]$$

which gives, using (7.5) and (7.6),

$$\kappa^2 K = \kappa[K, \Phi]$$

which can't be solved componentwise in $\mathfrak{so}(3)$ for $\kappa \neq 0$.

Let us note briefly that while it is usually natural to impose translation invariance, rotationally covariant choices of connection are not the only interesting ones. Indeed, magnetic materials are often not isotropic in this way. An interesting example of this was studied recently in [Hon+20].

7.2.2 The critical micromagnetic energy functional on curved thin films

Let us consider a thin film in $\mathbb{R}^3$, which we idealise as a smoothly embedded two-dimensional surface

$$i : \Sigma \hookrightarrow \mathbb{R}^3.$$

The Euclidean metric g on $\mathbb{R}^3$ induces a metric $g_\Sigma := i^*g$ on Σ. In two dimensions, a metric induces a complex structure, so we may regard Σ as a Riemann surface with complex structure j_Σ and compatible metric g_Σ. We write $\mathbf{N} = N^\mu \partial_\mu$ for the unit normal vector field to $\Sigma \subset \mathbb{R}^3$, and

$$n = \frac{N^1 + \mathrm{i}N^2}{1 + N^3} : \Sigma \to \mathbb{C}P^1 \tag{7.7}$$

for the corresponding Gauss map.

The metric connection $\mathcal{A}$ on $T\mathbb{R}^3$ induces an orthogonal connection

$$A := i^*\mathcal{A}$$

on the pullback bundle $i^*T\mathbb{R}^3 \to \Sigma$, the restriction of the tangent bundle of $\mathbb{R}^3$ to Σ. This is a topologically trivial vector bundle of rank 3 on Σ. We refer to [Sch19] and to A.2 for comments on the model for a general connection. We will specialise from now on to the case that $\mathcal{A}$ is the spiral staircase connection, so that

$$A \equiv \kappa T_\mu \mathrm{d}i^\mu$$

where $i^\mu = i^* x^\mu$ are the components of the map i.

We consider the natural energy functional for the sigma model

$$E_A[m] = \frac{1}{2}\int_\Sigma * \left(|\mathrm{d}_A m|^2 + V(m)\right) \tag{7.8}$$

where V is a local potential energy function, $*$ is the Hodge star on (Σ, g_Σ) and $|\cdot|^2$ denotes the square norm given by the combination of g_Σ^{-1} and the dot product of vectors. To relate (7.8) to a two-dimensional micromagnetic energy functional, so as to understand the role of the connection A, we expand

$$\begin{aligned} |\mathrm{d}_A m|^2 &= |\mathrm{d}m + A(m)|^2 \\ &= |\mathrm{d}m|^2 + 2(A(m), \mathrm{d}m) + |A(m)|^2 \end{aligned} \tag{7.9}$$

where the brackets $(\cdot,\cdot)$ denote the metric induced by the combination of g_Σ^{-1} and the dot product of vectors. The meaning of the first term on the right-hand side of (7.9) is clear: it is the usual symmetric exchange interaction. What about the other terms?

Being of zeroth order, the third term on the right-hand side of (7.9) contributes to the effective potential energy of the theory. To see what it is, it suffices to work locally on Σ. Let (x,y) be coordinates on a local patch $U \subset \Sigma$ such that the embedding i takes the local form

$$i(x,y) = (x,y,f(x,y)) \tag{7.10}$$

for $f : U \to \mathbb{R}$ a smooth real function (for sufficiently small U, such local coordinates always exist, possibly after rotating the target space). The metric g_Σ takes the local form, writing $\partial_x f = f_x$ and $\partial_y f = f_y$,

$$g_\Sigma|_U = (1+f_x^2)\mathrm{d}x^2 + 2f_x f_y \mathrm{d}x\mathrm{d}y + (1+f_y^2)\mathrm{d}y^2$$

(as usual, the juxtaposition of basis 1-forms denotes the symmetric product) so that the inverse is

$$g_\Sigma^{-1}|_U = \frac{1}{1+f_x^2+f_y^2}\left((1+f_y^2)\partial_x^2 - 2f_x f_y \partial_x\partial_y + (1+f_x^2)\partial_y^2\right).$$

The unit normal vector to $i[U]$ in $\mathbb{R}^3$ has the form

$$\mathbf{N} = N^\mu \partial_\mu = \frac{\partial_3 - f_x\partial_1 - f_y\partial_2}{\sqrt{1+f_x^2+f_y^2}}. \tag{7.11}$$

We also have

$$\begin{aligned} A|_U &= \kappa T_\mu \mathrm{d}i^\mu \\ &= (T_1 + f_x T_3)\mathrm{d}x + (T_2 + f_y T_3)\mathrm{d}y. \end{aligned}$$

Direct computation then reveals that, on U,

$$\begin{aligned} |A(m)|^2 &= \kappa^2\left(1 + \frac{1}{1+f_x^2+f_y^2}\left(f_x^2 m_1^2 + f_y^2 m_2^2 + m_3^2 - 2f_x m_1 m_3 - 2f_y m_2 m_3 + 2f_x f_y m_1 m_2\right)\right) \\ &= \kappa^2\left(1 + \left(\frac{1}{\sqrt{1+f_x^2+f_y^2}}(m_3 - f_x m_1 - f_y m_2)\right)^2\right) \\ &= \kappa^2\left(1+m_N^2\right) \end{aligned} \tag{7.12}$$

where $m_N := \mathbf{m}\cdot\mathbf{N}$ is the normal component of the magnetisation field. Thus, the connection A induces a term in the effective potential which favours the tangential alignment of the magnetisation (it is an *easy plane* contribution).

A similar computation reveals that the second term on the right-hand side of (7.9) is the standard DM term, $2\kappa\mathbf{m}\cdot\nabla\times\mathbf{m}$, where $\nabla\times$ here denotes the tangential curl.

As shown in [BSRS20; Sch19] and recalled in A.2, there is a very special choice of potential for which the theory admits a so-called 'Bogomolny rearrangement'. The choice is

$$\begin{aligned} V_{\mathrm{crit}}(m) &= -2(*F(A))^B m_B \\ &= -2\kappa^2 m_N, \end{aligned} \tag{7.13}$$

where we have identified the tangent spaces to M with $\mathfrak{so}(3)$ to view m as an $\mathfrak{so}(3)$-valued object $m^B T_B$. To verify the second equality it again suffices to work locally. Again, suppose that the embedding i takes the local form of (7.10), determined by a function f. Then we have, using (7.5),

$$\begin{aligned} F(A) &= i^* F(\mathcal{A}) \\ &= \frac{\kappa^2}{2} i^* (\epsilon_{A\mu\nu} T^A \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) \\ &= \kappa^2 (T_3 - f_x T_1 - f_y T_2) \mathrm{d}x \wedge \mathrm{d}y \\ &= \kappa^2 N^\mu T_\mu \sqrt{1 + f_x^2 + f_y^2}\, \mathrm{d}x \wedge \mathrm{d}y \\ &= \kappa^2 N^\mu T_\mu \omega_\Sigma \end{aligned}$$

where

$$\omega_\Sigma = \sqrt{1 + f_x^2 + f_y^2}\, \mathrm{d}x \wedge \mathrm{d}y$$

is the Riemannian volume form on Σ (the quantity under the square root is the determinant of the metric g_Σ). Thus, $*F(A)^B = \kappa^2 N^B$, so

$$*(F(A))^B m_B = \kappa^2 m_N$$

as claimed. This contribution to the potential resembles a Zeeman term for an applied magnetic field normal to the thin film.

The overall effective potential, combining V_{crit} and the zeroth order contribution (7.12) from $|\mathrm{d}_A m|^2$, is

$$\kappa^2(1 + m_N^2) - 2\kappa^2 m_N = \kappa^2(1 - m_N)^2.$$

This favours the alignment of the magnetisation with the normal vector to the surface. It is sensitive to the orientation of the surface.

The critically-coupled energy functional can therefore be written as

$$\begin{aligned} E_\kappa(m) &= \frac{1}{2}\int_\Sigma * \left(|\mathrm{d}_A m|^2 + V_{\text{crit}}(m)\right) \\ &= \frac{1}{2}\int_\Sigma * \left(|\mathrm{d}\mathbf{m}|^2 + 2\kappa \mathbf{m} \cdot \nabla \times \mathbf{m} + \kappa^2 (1 - m_N)^2\right) \end{aligned} \tag{7.14}$$

where, as previously mentioned, the curl is the tangential curl.

Some comments on (7.14) can be made.

- The energy functional E_κ is the natural generalisation of the functional of [BSRS20] to curved films. In fact, in [BSRS20], a family of energy functionals on planar films was given, parameterised by $U(1)$. Different members of this family can be produced by starting with different three-dimensional contorsion tensors which are symmetric under rotations in the plane parallel to the embedded planar film. The spiral staircase connection is one such choice, another is the example given in (7.4). A general curved film has no global symmetry and so we do not naturally obtain a continuous family of models in our more general setting.

- The theory has a gauge symmetry, given by $SO(3)$ gauge transformations of A and m. From a three-dimensional perspective, this is related to invariance under local coordinate transformations, but in two dimensions we may view it as an abstract gauge symmetry. Both $|\mathrm{d}_A m|^2$ and $V_{\text{crit}}(m)$ are separately invariant under these gauge transformations, although the decomposition of (7.14) into a gradient energy, DM energy and effective potential energy is not invariant. Notice that, because A has curvature, it is not possible to set $A \equiv 0$ and so remove the DM term using a smooth gauge transformation.

- It is not necessarily obvious that E_κ is bounded below, because the DM interaction term (and V_{crit}) can be negative. However, we show in subsection 7.2.3 that it is bounded below (on compact films) by a topological energy contribution.

- The energy functional E_κ depends on the single real parameter κ, which introduces a length scale $\frac{1}{\kappa}$ into the theory. If $\kappa = 0$, then the model becomes the basic sigma model, which is conformally invariant and does not see the extrinsic geometry of the thin film. If one takes $\kappa \to \infty$, then the potential energy dominates and one expects solutions to be normal to the film almost everywhere. In general, $\frac{1}{\kappa}$ is the Skyrmion size, measuring the length scale over which a configuration deviates from the normal field.

- In the case that $\kappa \neq 0$, the theory is highly sensitive to the extrinsic geometry of the thin film $\Sigma \hookrightarrow \mathbb{R}^3$. This is unusual for soliton models, which often depend on intrinsic, but not extrinsic, geometry.

  This idea is clearly illustrated by consideration of the configuration $\mathbf{m} = \mathbf{N}$. One might expect this to be a good 'ground state' for the theory. After all, it has $m_N = 1$ and so minimises the potential energy $\frac{\kappa^2}{2}\int_\Sigma *(1 - m_N)^2$. However, its gradient energy density is generally non-zero, being given by

$$\frac{1}{2}|\mathrm{d}\mathbf{N}|^2 = \frac{1}{2}(\rho_1^2 + \rho_2^2) = 2H^2 - G$$

  where ρ_1, ρ_2 are the principal curvatures of the embedded thin film, $H = \frac{1}{2}(\rho_1 + \rho_2)$ is the (extrinsic) mean curvature of the film, and $G = \rho_1\rho_2$ is its (intrinsic) Gaussian curvature. Here we have used that $\mathrm{d}\mathbf{N}$ is the *shape operator* of the embedded surface Σ and has eigenvalues ρ_1, ρ_2.

  The DM energy density for the normal vector field is a total derivative and so the DM energy vanishes on compact surfaces. Then the overall energy of the configuration $\mathbf{m} = \mathbf{N}$ on a compact Riemann surface Σ of genus g is

$$\begin{aligned} E_\kappa(\mathbf{N}) &= \int_\Sigma * \left(2H^2 - G\right) \\ &= 2\int_\Sigma *(H^2) + 4\pi(g-1) \\ &= 2\int_\Sigma *(H^2 - G) + 4\pi(1-g) \end{aligned} \tag{7.15}$$

where we have used the Gauss–Bonnet theorem to find the second equality, and we have added and subtracted $2G$ and used the Gauss–Bonnet theorem to find the third equality. Notice that this does not depend on the parameter κ. This result pre-empts part of the discussion of subsection 7.2.3 – the Bogomolny argument given there might be viewed as a natural generalisation of these ideas about the curvature of embedded surfaces. In particular, we see that if $H^2 = G$, which is true for the round 2-sphere, then the Gauss map minimises the energy within its topological class (the degree of the Gauss map is $1 - g$) and so it will give us a Skyrmion solution in the sense of subsection 7.2.3.

Let us remark that the value of the energy functional (7.15) for $\mathbf{N}$ gives rise to a functional on the space of smooth embeddings $\Sigma \hookrightarrow \mathbb{R}^3$ (which one may subject to some further constraints). Interpreted in this way, it is a natural energy functional which is well-studied in the context of elastic membranes (see [HKS92], for example). Allowing for noncompact surfaces, solutions to the corresponding Euler–Lagrange equations include minimal surfaces, which have $H = 0$.

7.2.3 The Bogomolny equation and magnetic Skyrmions

The purpose of choosing the critical potential V_{crit} is, as shown in [Sch19] and in A.2, that one can rearrange the energy functional as follows (note that this rearrangement can be made for any choice of connection A). One has

$$\frac{1}{2}\int_\Sigma * \left(|\mathrm{d}_A m|^2 + V_{\text{crit}}(m)\right) = \int_\Sigma *|\bar{\partial}_A m|^2 + \int_\Sigma \left(m^*\omega_{S^2} + \mathrm{d}(m\cdot A)\right), \tag{7.16}$$

where ω_{S^2} is the standard symplectic form on the target 2-sphere with area 4π, and by $m \cdot A$ we mean $m_B A^B_\mu \mathrm{d}x^\mu$. Here,

$$\bar{\partial}_A m := \frac{1}{2}\left(\mathrm{d}_A m + J_{S^2} \circ \mathrm{d}_A m \circ j_\Sigma\right)$$

where j_Σ is the complex structure on Σ induced by its metric, and J_{S^2} is the standard complex structure on the target sphere. Locally, thinking of m as a unit three-vector $\mathbf{m}$, this has a component

$$(\bar{\partial}_A \mathbf{m})_1 = \frac{1}{2}\left(D_1\mathbf{m} + \mathbf{m}\times D_2\mathbf{m}\right) \tag{7.17}$$

where D_k is the k^{th} component of d_A in local conformal coordinates on Σ, and the second local component of $\bar{\partial}_A m$ is not independent.

Note that the second integral on the right-hand side of (7.16) is topological (on a compact surface or with reasonable boundary conditions): it is the sum of 4π times the Skyrmion number (the degree N of the map m) and an integrated 'vorticity', which vanishes on closed surfaces. We give a more precise understanding of this topological energy, as a pairing in equivariant (co)homology, in A.2. It has been argued in [BSRS20] and elsewhere that the integrated vorticity contribution should be removed.

The rearrangement (7.16) implies that the energy within a topological class is minimised for solutions of the first order *Bogomolny equation*

$$\bar{\partial}_A m = 0. \tag{7.18}$$

The main aim of this note is to understand the solutions to this equation, which we abuse language to call *(magnetic) Skyrmions*, and their moduli. The energy of solutions to this equation is $4\pi N$, linear in the Skyrmion number N, at least when the boundary term vanishes. This linear dependence of the energy on the topological degree is typical for solutions to Bogomolny equations in BPS models.

The cross product in (7.17) is often inconvenient to deal with, and so it is useful to change into coordinates in which J_{S^2} is simply i. We do this by stereographic projection of the sphere to the extended complex plane, which carries a natural complex coordinate

$$v \coloneqq \frac{m_1 + \mathrm{i}m_2}{1 + m_3}.$$

In this coordinate, the Bogomolny equation becomes

$$\bar{\partial} v = -A^{0,1}(v) \tag{7.19}$$

where $A^{0,1}$ is the $(0,1)$ part (that is, the part proportional to $\mathrm{d}\bar{z}$ for z a complex conformal coordinate on Σ) of the connection A. Of course one can not truly escape the nonlinearity inherent to equation (7.18). Here it is tied into the action of $A^{0,1}$ on v, which is not a linear action: $\mathfrak{so}(3)$ acts on the Riemann sphere by linearised (real) Möbius transformations, or equivalently by (real) holomorphic vector fields. A particular advantage of this coordinate choice is that it allows us to use the tools of complex geometry, which we exploit in subsection 7.2.4.

For now, let us compute the form of the Bogomolny equation in terms of the data of the embedding $i : \Sigma \to \mathbb{R}^3$. In terms of our choice of connection $A \equiv \kappa\,(\mathrm{d}i^\mu)\,T_\mu$, the Bogomolny equation in the form of equation (7.19) is

$$\partial_{\bar{z}} v = -\kappa(\partial_{\bar{z}} i^\mu) T_\mu(v)$$

where z is a (local) conformal complex coordinate on Σ. Recall that a local complex coordinate z is *conformal* if, locally,

$$g_\Sigma = \Omega^2(z, \bar{z})\mathrm{d}z\mathrm{d}\bar{z}$$

for a real positive function Ω^2, the *conformal factor*.

Once more, it is convenient to work in a local patch $U \subset \Sigma$ with coordinates (x, y) such that the embedding i takes the form of (7.10), determined by the single function f. Then the Bogomolny equation becomes

$$\partial_{\bar{z}} v = -\frac{1}{2}\kappa\left(\partial_{\bar{z}}(x + \mathrm{i}y)(T_1 - \mathrm{i}T_2) + \partial_{\bar{z}}(x - \mathrm{i}y)(T_1 + \mathrm{i}T_2) + 2(\partial_{\bar{z}} f)T_3\right)(v). \tag{7.20}$$

The obvious complex coordinate $u \coloneqq x + \mathrm{i}y$ on U is generally not conformal.

The question of how to produce a conformal coordinate z from a general coordinate u is answered by Gauss's theory of isothermal coordinates. It can be shown that a coordinate $z(u, \bar{u})$ is conformal if it obeys the *Beltrami equation*

$$\frac{\partial z}{\partial u} = \mu \frac{\partial z}{\partial \bar{u}} \tag{7.21}$$

where (for our choice of embedding)

$$\mu := \frac{f_x^2 - f_y^2 + 2\mathrm{i} f_x f_y}{2 + f_x^2 + f_y^2 + 2\sqrt{1 + f_x^2 + f_y^2}}$$

is the *Beltrami coefficient* of the metric g_Σ. One may note that

$$\frac{f_x^2 - f_y^2 + 2\mathrm{i} f_x f_y}{2 + f_x^2 + f_y^2 + 2\sqrt{1 + f_x^2 + f_y^2}} = \left(\frac{f_x + \mathrm{i} f_y}{1 + \sqrt{1 + f_x^2 + f_y^2}}\right)^2$$

which reveals that $\mu = n^2$, the square of the Gauss map (7.7) (recall from (7.11) the form of the unit normal vector field).

Now let us get to grips with the pieces of (7.20). Equation (7.21) implies that

$$\partial_{\bar{z}} u = -\mu \partial_{\bar{z}} \bar{u}.$$

Also,

$$\partial_{\bar{z}} f = \bar{u}_{\bar{z}} \left(-\mu f_u + f_{\bar{u}}\right).$$

A computation then uncovers the fact that

$$-\mu f_u + f_{\bar{u}} = -n.$$

The only piece of the puzzle left to find is the action of the T_A on the Riemann sphere coordinate v. The action of $\mathfrak{so}(3)$ is by (real) linearised Möbius transformations, or equivalently, by the action of (real) holomorphic vector fields. One can show that

$$\begin{aligned} T_3(v) &= -2\mathrm{i}v \\ (T_1 - \mathrm{i}T_2)(v) &= 2\mathrm{i} \\ (T_1 + \mathrm{i}T_2)(v) &= -2\mathrm{i}v^2. \end{aligned}$$

We can now write down the Bogomolny equation in the simple form

$$\begin{aligned} \partial_{\bar{z}} v &= -\mathrm{i}\kappa \bar{u}_{\bar{z}} \left(-\mu - v^2 - 2(-\mu f_u + f_{\bar{u}})v\right) \\ &= -\mathrm{i}\kappa \bar{u}_{\bar{z}} \left(-n^2 - v^2 + 2nv\right) \\ &= \mathrm{i}\kappa \bar{u}_{\bar{z}} \left(v - n\right)^2. \end{aligned} \tag{7.22}$$

In what follows, we will compute the geometrical prefactor $\bar{u}_{\bar{z}}$ explicitly in examples. Note that, while we fixed the local form of u, we did not fix z, instead just asking that it solve the Beltrami equation (7.21). One needs to pick a particular solution to compute $\bar{u}_{\bar{z}}$ (this must be true, as the left-hand side of (7.22) depends on the choice). However, we can say something general: direct computation reveals that

$$|\bar{u}_{\bar{z}}| = \frac{\Omega(z, \bar{z})}{1 + |n|^2},$$

where Ω is the square root of the conformal factor Ω^2.

As one would expect, this equation depends sharply on the extrinsic geometry of the thin film. It captures the interplay in the energy functional (7.14) between the symmetric exchange, or gradient, energy $|\mathrm{d}m|^2$ and the potential energy $(1-m_N)^2$. If the Gauss map n is holomorphic, then there is a natural 'ground state' $v = n$. This clearly minimises the potential energy. In general, however, the Gauss map is not holomorphic, and the potential energy minimising configuration $v = n$ has too much gradient energy to be an overall energy minimiser. On the other hand, a constant configuration (which minimises the gradient energy) generally has too much potential energy to be a solution.

On a flat film the Gauss map is constant, and so trivially holomorphic. If one introduces a smooth bump into the film, then the constant solution is no longer an energy minimiser and so one expects the formation of some Skyrmion–anti-Skyrmion density in the bump. The idea that Skyrmion density might be pinned in place by bumps in a film is familiar from [Kra$^+$18].

Writing $\tilde{v} = v - n$, the equation (7.22) becomes

$$\partial_{\bar{z}}\tilde{v} = \mathrm{i}\kappa\bar{u}_{\bar{z}}\tilde{v}^2 + \partial_{\bar{z}}n$$

The quantity $\partial_{\bar{z}}n$ is related linearly to the difference between the principal curvatures of the film.

7.2.4 Projective bundles, their sections, and the existence of Skyrmions

We have seen that solutions to the Bogomolny equation are sections of a particular bundle over Σ with fibre S^2 satisfying the Dolbeault-type equation (7.18). We will address this structure from a complex analytic point of view. Namely, we regard these solutions as sections of a projective line bundle (that is, a holomorphic fibre bundle with fibre $\mathbb{C}P^1$), where the holomorphic structure is determined by the operator $\bar{\partial}_A$ (that $\bar{\partial}_A$ defines an integrable holomorphic structure follows simply for dimensional reasons: there are no $(0,2)$-forms on a Riemann surface). We can make some progress simply by recalling some basic, and well-known, facts about these objects.

A useful way to build projective line bundles is to take a rank 2 complex vector bundle and projectivise its fibres (that is, we replace the linear fibre by the space of its one-dimensional subspaces). If $\mathcal{E}$ is a holomorphic vector bundle of rank 2, we write $P(\mathcal{E})$ for the projective line bundle obtained by projectivising the fibres, which we call the *projectivisation* of $\mathcal{E}$.

At the level of transition functions, projectivisation corresponds to passing along the quotient map

$$GL(2,\mathbb{C}) \to PGL(2,\mathbb{C}),$$

which has kernel $\mathbb{C}^*$. More precisely, the transition functions of $\mathcal{E}$ live in the first (Čech) cohomology of the sheaf of holomorphic $GL(2,\mathbb{C})$-valued functions, while those of the projective bundle live in the first cohomology of the sheaf of holomorphic $PGL(2,\mathbb{C})$-valued functions. Taking the cohomology of the short exact sequence induced by the above map gives the long exact sequence

$$\cdots \to H^1(\mathcal{O}^*) \to H^1(\mathcal{O}_{GL(2,\mathbb{C})}) \to H^1(\mathcal{O}_{PGL(2,\mathbb{C})}) \to H^2(\mathcal{O}^*) \to \cdots$$

where by $\mathcal{O}_G$ we mean the sheaf of holomorphic G-valued functions for a group G, and $\mathcal{O}^*$ is the sheaf of holomorphic $\mathbb{C}^*$-valued functions.

We first notice that $P(\mathcal{E} \otimes L) \cong P(\mathcal{E})$ for any line bundle L (this is simply the fact that multiplying the transition functions of $\mathcal{E}$ by $\mathbb{C}^*$-valued transition functions does not affect the projectivisation). We further see that any obstruction to lifting a projective bundle P to a vector bundle V such that $P = P(V)$ lives in $H^2(\mathcal{O}^*)$.

In our case, Σ is compact and of real dimension two and therefore $H^2(\mathcal{O}^*) = 0$ (one can use the exponential exact sequence to see this). Hence, every projective line bundle on a Riemann surfaces arises as the projectivisation of some rank 2 vector bundle (in fact, a similar argument shows that every projective bundle on a compact Riemann surface arises as the projectivisation of a vector bundle).

We are interested in sections of projective line bundles. Now, holomorphic sections of the projectivisation $P(\mathcal{E})$ of a vector bundle $\mathcal{E}$ are equivalent to holomorphic line sub-bundles of $\mathcal{E}$ – a line in a fibre of $\mathcal{E}$ is a point in the corresponding fibre of the projectivised bundle. In one complex dimension, every rank 2 bundle has a holomorphic line sub-bundle (because Serre vanishing tells us that $H^0(\mathcal{E} \otimes N) \neq 0$ for a line bundle N of sufficiently high degree) and so, independently of any details of the model at hand, we have the following basic existence result.

7.2.1 Proposition. *Solutions to the Bogomolny equation* (7.18) *on a compact surface always exist. More precisely, there exists an integer N_0 such that there exist Skyrmion solutions of Skyrmion number $N \geq N_0$.*

Note that this implies that solutions may exist with arbitrarily high degree - one can have arbitrarily large Skyrmion density. This is in contrast to vortices, which have some non-zero size and so have finite maximum density. We will see in section 7.4 that Skyrmions can be interpreted as particular two-flavour Abelian vortices in the limit that the vortex size goes to zero. Of course, from a physical perspective, the continuum approximation will break down at very high Skyrmion densities.

Notice that the proof of proposition 7.2.1 makes no reference to the choice of connection A, which determines the holomorphic structure $\bar{\partial}_A$. Thus, the result holds for any choice of DM vector (that is, any choice of three-dimensional contorsion tensor).

7.2.5 Emergent electromagnetism

From (7.16), the topological Skyrmion energy density is

$$\mathcal{E}_{\text{top}}(m) = m^* \omega_{S^2} + \mathrm{d}\,(m \cdot A)\,.$$

This may be interpreted as an emergent magnetic field, generated by the magnetically charged Skyrmions. The corresponding emergent Abelian gauge potential is

$$a = m^* \lambda_{S^2} + m \cdot A$$

where $\mathrm{d}\lambda_{S^2} = \omega_{S^2}$, so that λ_{S^2} is the usual Dirac monopole connection on the target sphere. Under this interpretation, the Skyrmion density becomes the magnetic flux density. In this picture, Skyrmions are the two-dimensional shadow of 'emergent' Dirac monopoles.

One can create or destroy Skyrmions on a thin film by passing an emergent monopole through the film. This follows simply from flux conservation. This provides a 'low energy' realisation of the Skyrmion (un)winding via monopoles, as described in [Mil+13].

A 'UV-completion' of this theory (to allow for films with non-zero thickness) would presumably involve replacing these Dirac monopoles with BPS-type monopoles. It would be interesting to understand possible theories of this type, and in particular to study the expected pair creation of confined monopole-anti-monopole pairs mediating the phase transition between the Skyrmion and helical phases [Mil+13; Kan+16].

7.3 SKYRMION SOLUTIONS

7.3.1 *Skyrmions on spherical films*

Solutions to the Bogomolny equation (7.18) on the flat plane were constructed in [BSRS20]. The next simplest case to study is the round sphere.

The sphere is a good testing ground for these ideas due to its high degree of symmetry. Moreover, its Gauss map is holomorphic so, as we know from (7.22), the Bogomolny equation will separate and there is a good chance that we can find a general solution.

Solutions on the sphere are also interesting in that they may provide insight into Bloch points in three dimensions: If the Skyrmion number on the sphere is non-zero, any attempt to continue the solution into the interior of the sphere inevitably forces one to allow the magnetisation to go to zero at one or more points. We might therefore view these solutions on the sphere as the two-dimensional shadow of a smooth description of Bloch points viewed from large distances.

Let

$$i : S^2 \to \mathbb{R}^3$$

be the standard round embedding of a sphere of radius R. In terms of Cartesian coordinates (x_1, x_2, x_3) on $\mathbb{R}^3$, we introduce a complex coordinate on the sphere:

$$z = \frac{x_1 + \mathrm{i}x_2}{R + x_3}$$

which is 0 at the north pole and ∞ at the south pole. The round sphere has the property that the Gauss map is the identity map, so we may write $n = z$.

To write down the Bogomolny equation (7.22) locally, we need to compute $\bar{u}_{\bar{z}}$ where $u = x_1 + \mathrm{i}x_2$ (recall that, in the derivation of (7.22), the form of u was imposed from the start - there is no freedom in this definition). Computing the derivative reveals that

$$\bar{u}_{\bar{z}} = \frac{2R^2}{(R^2 + |z|^2)^2}.$$

This is the conformal factor of the induced metric on the sphere. That this is the case is special to the sphere. Then, the Bogomolny equation is

$$\partial_{\bar{z}} v = \frac{2\mathrm{i}\kappa R^2}{(R^2 + |z|^2)^2} (v - z)^2 .$$

(a) The 'hedgehog' solution. (b) The degree zero solution (7.24).

Figure 7.1: Some Skyrmion solutions on the sphere (for $\kappa = 1$), visualised in the Runge colour scheme. In this scheme, the argument of the complex-valued function v is represented by the hue of the colour, while the modulus is represented by the brightness (so that black corresponds to zero and white to ∞).

This equation is separable and can be solved explicitly. The solution is

$$v(z, \bar{z}) = z\left(1 + \frac{R^2 + |z|^2}{2\mathrm{i}\kappa R^2 - zf(z)(R^2 + |z|^2)}\right), \tag{7.23}$$

for any meromorphic function f. Notice that if $\kappa = 0$, one is left simply with a meromorphic function, as one should be.

If we choose $1/f = 0$, then we obtain the natural 'ground state', $v = z$. This is the 'hedgehog' solution, where the magnetisation points radially. It is the Gauss map of the embedded sphere. While we call this the 'ground state', it actually has degree 1, and so has positive energy 4π. If the radius is large, then, in a neighbourhood of the north pole, the Gauss map z appears constant and zero which is the natural ground state on the plane.

If $f = 1/z$, we obtain

$$\begin{aligned} v &= z\left(1 + \frac{R^2 + |z|^2}{2\mathrm{i}\kappa R^2 - (R^2 + |z|^2)}\right) \\ &= z\frac{2\mathrm{i}\kappa R^2}{2\mathrm{i}\kappa R^2 + (R^2 + |z|^2)}. \end{aligned} \tag{7.24}$$

This is a degree zero solution with zeroes at the north and south poles. As a degree zero solution, this configuration has zero energy and is therefore a vacuum of the theory.

It would be interesting to explore more deeply the shapes of the solutions one obtains as one varies the free meromorphic function f and also as one varies the parameters of the model.

7.3.2 *Skyrmions on axially symmetric films*

The next step is to consider more general symmetric surfaces. Consider an axially symmetric embedding

$$i(r, \theta) = (r, \theta, f(r))$$

where (r, θ) are plane polar coordinates. This becomes more complicated than the case of the sphere because explicitly finding a good complex coordinate, and computing the function $\bar{u}_{\bar{z}}$ in (7.22), is more difficult. We compute directly, using the symmetry of the problem to simplify matters.

Using primes to indicate derivatives with respect to r, the induced metric on the surface is

$$i^*g = \left(1 + f'^2\right) \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2.$$

The obvious complex coordinate on the surface is

$$u = re^{\mathrm{i}\theta},$$

which is generally not conformal. We introduce another complex coordinate

$$z = \tilde{r}e^{\mathrm{i}\theta}$$

where $\tilde{r}$ is a function of r only. For this to be compatible with the complex structure on the surface induced by the embedding, we must have

$$i^*g = \Omega^2(z, \bar{z})\mathrm{d}z\mathrm{d}\bar{z}$$

for some conformal factor Ω^2. We see that

$$\mathrm{d}z\mathrm{d}\bar{z} = \tilde{r}'^2 \mathrm{d}r^2 + \tilde{r}^2 \mathrm{d}\theta^2,$$

which is conformal to i^*g if

$$\tilde{r}' = \frac{\sqrt{1 + (f')^2}}{r}\tilde{r}$$

with conformal factor

$$\frac{r^2}{\tilde{r}^2}.$$

With this knowledge, we can write down the Bogomolny equation (7.22). On the left-hand side we have

$$\frac{\partial}{\partial \bar{z}}v = \frac{1}{2}\frac{r}{\tilde{r}e^{-\mathrm{i}\theta}}\left(\frac{1}{\sqrt{1 + (f')^2}}\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial \theta}\right)v.$$

On the right, we compute

$$\begin{aligned}\bar{u}_{\bar{z}} &= \frac{\partial}{\partial \bar{z}}re^{\mathrm{i}\theta} \\ &= \frac{1}{2}\frac{r}{\tilde{r}}\frac{2}{1 + |n|^2}\end{aligned}$$

leaving us with

$$\frac{1}{2}\frac{r}{\tilde{r}e^{-\mathrm{i}\theta}}\left(\frac{1}{\sqrt{1 + (f')^2}}\frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial \theta}\right)v = \mathrm{i}\kappa\frac{r}{\tilde{r}}\frac{1}{1 + |n|^2}\left(v - n\right)^2.$$

We can cancel the unknown $\frac{r}{\bar{r}}$ from both sides, giving

$$\left(\frac{1-|n|^2}{1+|n|^2}\partial_r + \frac{\mathrm{i}}{r}\partial_\theta\right) v = \mathrm{i}e^{-\mathrm{i}\theta}\frac{2}{1+|n|^2}\left(v-n\right)^2 . \tag{7.25}$$

In general, the Gauss map n is not holomorphic and so $v = n$ is not a solution in general. However, if the film is asymptotically conical, so that $\partial_r n \to 0$ as $r \to \infty$, then one expects there to be solutions which slowly approach the Gauss map $v(r) \to n$ as $r \to \infty$ by virtue of the $\frac{1}{r}$ suppression of the angular derivative.

The equation (7.25) is not easy to solve in general. The complication, compared to the case of the sphere, is that the Gauss map is not holomorphic and so the equation does not separate. However, we can make explicit progress by looking for axially symmetric solutions for particularly simple choices of f.

First, let us make the symmetric ansatz

$$v = h(r)e^{\mathrm{i}p\theta}$$

for p an integer and h a complex-valued function (it is too much to ask that h be real, as we do not expect solutions to point radially everywhere). Substituting this into (7.25), we have

$$\left(\frac{1-|n|^2}{1+|n|^2}h'(r) - \frac{p}{r}h(r)\right) e^{\mathrm{i}p\theta} = \mathrm{i}e^{\mathrm{i}(2p-1)\theta}\frac{2}{1+|n|^2}\left(h(r)-|n|\right)^2 ,$$

where we have used the axial symmetry of the Gauss map. We see immediately that this can only be solved for $p = 1$. In this case, we have

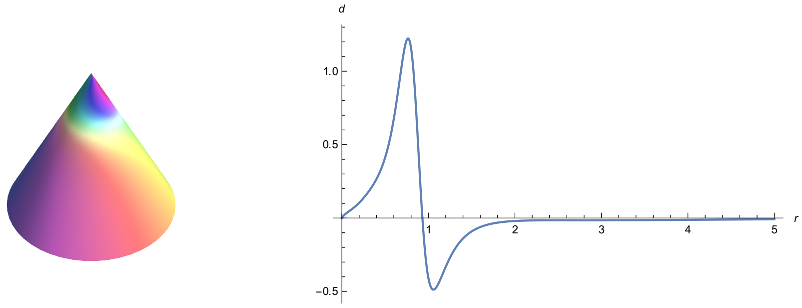
$$\frac{1-|n|^2}{1+|n|^2}h'(r) - \frac{1}{r}h(r) = \mathrm{i}\frac{2}{1+|n|^2}\left(h(r)-|n|\right)^2 . \tag{7.26}$$

The simplest example outside of the plane and the sphere is to take $f = ar$, for a real non-zero constant a. This describes a cone. Then $|n| = \frac{a}{1+\sqrt{1+a^2}}$ does not depend on r. In this case the equation (7.26) can be solved exactly, for any choice of a. In general the solution involves (modified) Bessel functions of the first and second kind of orders determined by a.

For simplicity, we reproduce the solution here only for the value $a = \sqrt{3}$, corresponding to an apex angle of $\pi/3$. The solution is

$$\begin{aligned} h_c(r) = {} & \frac{1}{2\sqrt{2}3^{3/4}\sqrt{-ir}\left(cI_3\left(2\sqrt{2}\sqrt[4]{3}\sqrt{-ir}\right) - K_3\left(2\sqrt{2}\sqrt[4]{3}\sqrt{-ir}\right)\right)} \\ & \times \Bigg(2cI_2\left(2\sqrt{2}\sqrt[4]{3}\sqrt{-ir}\right) + \frac{\sqrt[4]{3}\sqrt{2}c\left(\sqrt{3}-2ir\right)I_3\left(2\sqrt{2}\sqrt[4]{3}\sqrt{-ir}\right)}{\sqrt{-ir}} + 2cI_4\left(2\sqrt{2}\sqrt[4]{3}\sqrt{-ir}\right) \\ & + 2K_2\left(2\sqrt{2}\sqrt[4]{3}\sqrt{-ir}\right) - \frac{\sqrt[4]{3}\sqrt{2}\left(\sqrt{3}-2ir\right)K_3\left(2\sqrt{2}\sqrt[4]{3}\sqrt{-ir}\right)}{\sqrt{-ir}} + 2K_4\left(2\sqrt{2}\sqrt[4]{3}\sqrt{-ir}\right)\Bigg), \end{aligned} \tag{7.27}$$

where c is the constant of integration, which we allow to be complex, the I_α are the modified Bessel functions of the first kind and the K_α are the modified Bessel functions of the second



(a) The solution at $c = 0.1$, visualised in the Runge colour scheme.

(b) The radial Skrymion density d of the solution at $c = 0.1$ plotted against r.

Figure 7.2: Visualisations of the solution (7.27) at $c = 0.1$. The solution describes a ring of Skyrmion–anti-Skyrmion density around the tip of the cone.

kind (note that the arguments of these functions in the solution are complex, so it would be equally reasonable to write the solution in terms of the usual Bessel functions). Introducing $\xi = 2\sqrt{2}\sqrt[4]{3}\sqrt{-ir}$, the solution takes the slightly more compact form:

$$\frac{1}{\sqrt{3}\xi\left(cI_3(\xi) - K_3(\xi)\right)}\left(2cI_2(\xi) + c\left(\frac{12}{\xi} + \xi\right)I_3(\xi) + 2cI_4(\xi)\right.$$
$$\left. + 2K_2(\xi) - \left(\frac{12}{\xi} + \xi\right)K_3(\xi) + 2K_4(\xi)\right).$$

One can compute the limit of these solutions as $r \to \infty$, finding that

$$\lim_{r\to\infty} h_c(r) = \frac{1}{\sqrt{3}} = |n|,$$

so that the magnetisation tends towards the Gauss map as $r \to \infty$, as expected. Despite this, one should note that these solutions do not have finite energy - the solution does not tend to the normal fast enough and there is an infinite contribution to the energy from the integrated vorticity. In spite of this, we provide a short study of these solutions as they may provide a model for solutions on bumps in thin films and they exhibit an interesting confinement phenomenon. Moreover, it has been suggested in [BSRS20] that it is correct to remove the integrated vorticity contribution. If one does this, the solutions we find on the cone have finite energy.

We define the radial Skyrmion density

$$d(r) \coloneqq \frac{h'\bar{h} + h\bar{h}'}{(1 + |h|^2)^2}.$$

This is the quantity such that the 'Skyrmion number' N (which may not be well-defined as an integer in this non-compact case) is the integral $\int d(r)\,\mathrm{d}r$. In Figure 7.2b, we plot d against r (in general, r is not the right coordinate to use, but for the cone it differs from the radial distance along the cone just by a scaling), and see that the solution describes a band of Skyrmion–anti-Skyrmion density around the tip of the cone.

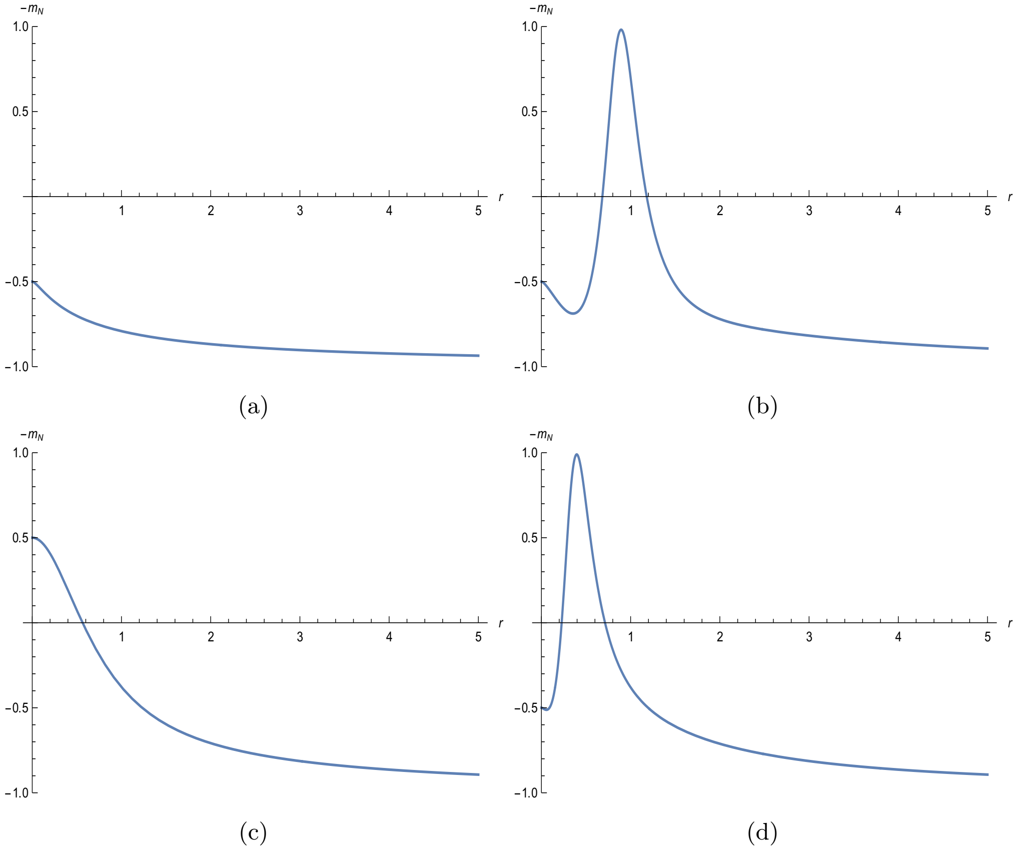


Figure 7.3: The negative normal component of the magnetisation field, $-m_N$, against the radial distance r for axially symmetric solutions of the form (7.27) on the cone of slope $a = \sqrt{3}$ at various values of the parameter c: (a) $c = 0$, (b) $c = 0.1$, (c) $c = \infty$, and (d) $c = -\mathrm{i}$.

To understand how the solution varies as one varies the modulus c, we plot in Figure 7.3 the negative of the normal component of the magnetisation vector field, $-m_N$, against r for several values of c.

Among the values we have plotted are the two special values $c = 0$ and $c = \infty$. The case of $c = 0$ describes the natural 'ground state', where the system remains as close as possible to the normal vector field (note that at $r = 0$, the 'normal' would point straight along the axis of the cone, which corresponds – for the cone angle we have chosen – to $m_N = 0.5$). We have also plotted the case of $c = 0.1$, which can be compared to Figure 7.2.

On the other hand, the solution for $c = \infty$ describes a pointlike Skyrmion at the tip of the cone (again, a value of $m_N = -0.5$ at $r = 0$ implies that the field points along the axis of the cone in the 'anti-normal' direction there). This is the special case in which the radius of the ring of Skyrmion density goes to zero.

It was demonstrated in [Kra+18] that Skyrmions can be pinned in place by bumps in a thin film. It would be desirable to take our analysis further, by studying the theory on a smooth, asymptotically Euclidean 'bump'. Unfortunately, reasonable choices of bump lead to Bogomolny equations that seem to be very difficult to solve. One could instead try to make progress numerically, although we do not do so here.

7.3.3 Skyrmions on cylindrical films

Perhaps more interesting than the case of conical films is that of cylindrical films. The parameterisation that we used above does not naturally allow for cylindrical films, so we treat them separately here. Skyrmions on cylinders may be of interest in the study of nanowires.

A symmetric embedding of a cylinder in $\mathbb{R}^3$ around the x_3-axis takes the form

$$i(t,\theta) = (R\cos\theta, R\sin\theta, t)$$

where $t \in \mathbb{R}$, $\theta \in [0, 2\pi)$ and R is a positive constant. The induced metric on the film is

$$g_\Sigma = \mathrm{d}t^2 + R^2\mathrm{d}\theta^2.$$

The local complex coordinate $z = t + \mathrm{i}R\theta$ is then conformal.

A short computation, along the lines of those we have already carried out, reveals that the Bogomolny equation (7.19) is

$$\frac{\partial v}{\partial t} + \frac{\mathrm{i}}{R}\frac{\partial v}{\partial \theta} = \mathrm{i}e^{-\mathrm{i}\theta}\left(v - e^{\mathrm{i}\theta}\right)^2.$$

Note that the Gauss map of the cylinder is $n = e^{\mathrm{i}\theta}$, so this equation takes a similar form to those we have seen previously.

Again making the axial ansatz $v = h(t)e^{\mathrm{i}\theta}$, we find the separable equation

$$h'(t) = \frac{1}{R}h(t) + \mathrm{i}(h(t)-1)^2$$

which can be solved to give

$$h_c(t) = \frac{i\sqrt{1-4iR}}{2R}\tanh\left(\frac{1}{2}\left(\sqrt{1-4iR}\,(c + t/R)\right)\right) + \frac{\mathrm{i}}{2R} + 1 \tag{7.28}$$

which depends on the complex parameter c. This describes a 'kink', mediating between the asymptotic constant solutions

$$h_\pm = \frac{1}{2R}\left(\mathrm{i} + 2R \pm i\sqrt{1-4iR}\right) \tag{7.29}$$

which solve the t-independent Bogomolny equation

$$\frac{1}{R}h + \mathrm{i}(h-1)^2 = 0.$$

Notice that these asymptotic solutions do not describe the Gauss map for finite R. Because the 'asymptotic boundary' of the cylinder at $t \to \pm\infty$ has finite length $2\pi R$, this does not necessarily cause issues when evaluating the energy contribution of the integrated vorticity. The

t-independent solutions (7.29) are true vacua: they have vanishing Skyrmion density and the integrated vorticity vanishes, as the contribution from each end of the cylinder cancels, and so have zero energy. This occurs in spite of the fact that they have positive potential energy and gradient energy, because the DM energy contributes negatively.

The complex modulus c controls the position and shape of the kink. In Figure 7.4, we plot some solutions and their Skyrmion density profiles.

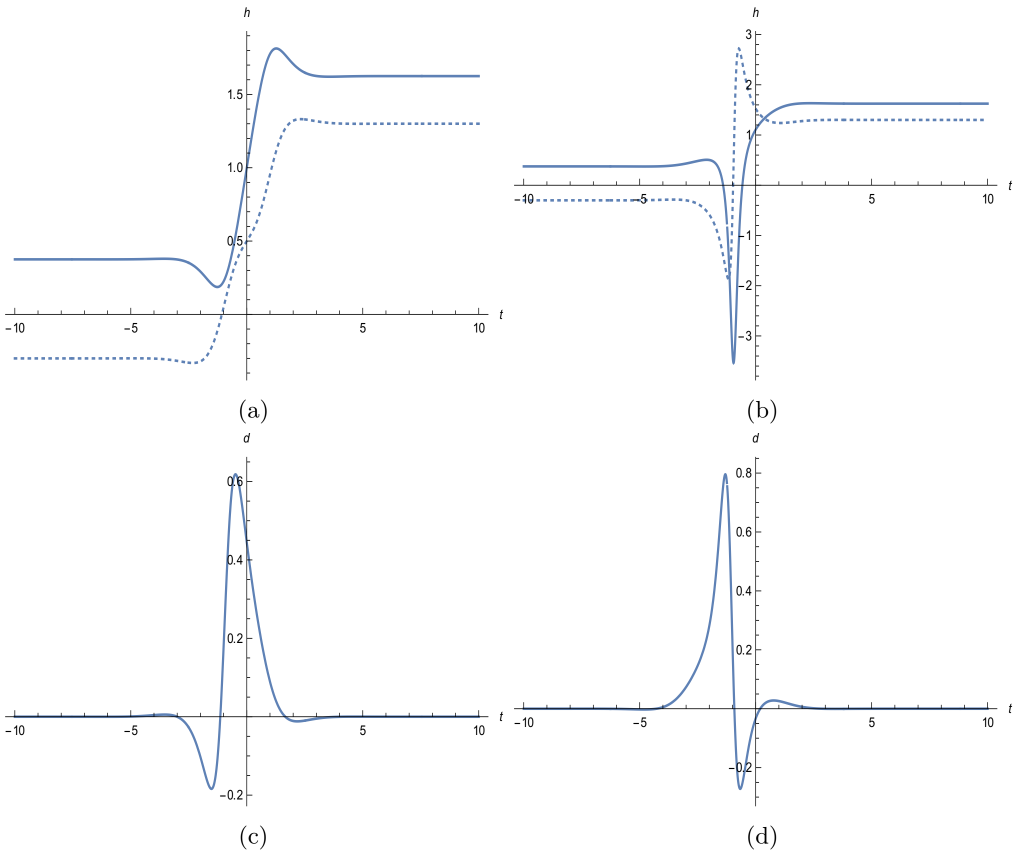


Figure 7.4: The figures (a) and (b) are plots of the real (solid line) and imaginary parts (dashed line) of $h_c(t)$ at $c = 0$ and $c = i$ respectively for $R = 1$. Figures (c) and (d) are the corresponding plots of Skyrmion density d against t for the solutions h_0 and h_i respectively. Notice that changing c moves the centre of the kink and changes its shape.

By evaluating the integrated Skyrmion density and integrated vorticity, one can compute the energy of a Skyrmion kink solution. One finds that it is finite and does not depend on c. However, it depends on the radius R of the cylinder.

Indeed, by examining the solution (7.28), one can see that for large R, the integrated Skyrmion density goes like $1/\sqrt{R}$ (note that the asymptotic solution as $R \to \infty$ is the Gauss map which has no Skyrmion density), while the integrated vorticity goes like $\sqrt{R}$ (it behaves like $R(h_+ - h_-)$). One can further compute the constants, and finds that the integrated Skyrmion density has leading term

$$4\pi \frac{2}{\sqrt{R}}$$

as $R \to \infty$, while the integrated vorticity has leading term

$$4\pi \frac{\sqrt{R}}{2}$$

as $R \to \infty$. (Recall that we are working in units where the Skyrmion size $\frac{1}{\kappa}$ has been set to one.)

7.4 THE MODULI SPACE OF SKYRMIONS AND ITS RESOLUTION

7.4.1 The moduli space of Skyrmions

One of the crucial facts about the critically-coupled model is that it admits a *moduli space* of degenerate energy minimising solutions within each topological class. These moduli include the positions of the basic Skyrmions (which exert no net force on each other at critical coupling) as well as certain internal 'orientational' moduli.

The moduli space of solutions to the Bogomolny equation (7.18) is the space of sections of a (topologically trivial) projective bundle with holomorphic structure determined by $A^{0,1}$. Let us restrict to the case that Σ is compact without boundary. An index theory argument shows that the component of the moduli space corresponding to sections of degree N has expected complex dimension

$$2N + 1 - g$$

where g is the genus of Σ. This is the same as the expected dimension of the space of meromorphic functions on Σ.

The moduli space of sections of a projective bundle contains singular points, at which the section degenerates to one of lower degree. From the point of view of Skyrmions, these singularities are points at which a Skyrmion shrinks to zero size and disappears. The aim of this section is to resolve these singularities.

7.4.2 Semilocal vortices in a background

To resolve the singularities of the moduli space we may use the idea, well-known in the literature of high energy physics, of realising a nonlinear sigma model as a certain strong coupling limit of a gauge theory with linear Higgs fields. The Skyrmions of the nonlinear theory are then the limits of vortices in the gauge theory. The vortices of the gauge theory have a well-defined size and a non-singular moduli space. We may, at least formally, do computations on the vortex moduli space and then take the strong coupling limit to find information about the Skyrmion theory.

From a two-dimensional perspective, the theory we consider is an Abelian gauge theory with a pair of complex scalar Higgs fields coupled to a background gauge potential for the $SU(2)$ flavour symmetry, which rotates the scalar fields among themselves. However, just as before, our theory has a three-dimensional origin - really the two scalar fields arise as the restriction of a spinor field in three dimensions and the background gauge field comes from a spin connection with torsion. The coupling to this connection and the relation to the theory of magnetic Skyrmions are the only potential novelties in the following discussion.

Let M be a three-dimensional Riemannian manifold. To define the vortex theory we will replace the magnetisation vector field m with a spinor on M. To do this, we assume that M is oriented and let $S \to M$ be a spinor bundle. This is a Hermitian complex vector bundle of rank 2, with vanishing Chern class.

We again let $i : \Sigma \hookrightarrow M$ be an embedded thin film. We introduce a Hermitian complex line bundle $L \to \Sigma$ and define $E := i^* S \otimes L$. The fields of our gauge theory are then a section $\Psi \in \Gamma(E)$ and a $U(1)$ connection a on L. The background $SO(3)$ connection A can be lifted to an $SU(2)$ connection on i^*S which we also call A. We may then form the tensor product $U(2)$ connection

$$A \otimes 1 + \mathbb{1} \otimes a$$

on E, which we write as $A + a$.

We introduce the energy functional

$$E_{A,e^2} = \frac{1}{2} \int_\Sigma * \left(|\mathrm{d}_{A+a}\Psi|^2 + \frac{1}{e^2}|F(a)|^2 + e^2(v^2 - |\Psi|^2)^2 - *2\Psi^\dagger F(A)\Psi \right).$$

This functional depends on two positive real parameters: the 'saturation magnetisation' $|v|$, which may as well be set to 1, and the gauge coupling constant e^2 (which may be thought of as a choice of inner product on the Lie algebra of $U(1)$).

A well-known rearrangement (see A.2) shows that, up to boundary terms,

$$E_{A,e^2} = \int_\Sigma \left(|\bar{\partial}_{A+a}\Psi|^2 + \frac{1}{e^2}|F(a) - \mathrm{i}e^2(v^2 - |\Psi|^2)|^2 \right) + \int_\Sigma \mathrm{i}v^2 F(a).$$

Thus, the energy is minimised within a topological class by solutions to the *vortex equations*

$$\bar{\partial}_{A+a}\Psi = 0 \tag{7.30}$$

$$*F(a) = \mathrm{i}e^2(v^2 - |\Psi|^2). \tag{7.31}$$

Exact solutions to equations of this type are usually very hard to come by. However, the moduli space of solutions to the analogue of these equations with $A = 0$, in the case that E is replaced with an arbitrary sum of line bundles (the case of *Abelian semi-local vortices*), is well-studied, as is the case in which A is dynamical and E is general (the case of *non-Abelian vortices*).

The moduli space is a Kähler manifold. A simple index calculation, using the Riemann–Roch theorem, tells us that the expected complex dimension of the moduli space is

$$c_1(E)[\Sigma] + 2 - 2g + (g-1) = 2N_L + 1 - g$$

where $c_1(E)$ is the first Chern class of E, N_L is the degree of L, and g is the genus of Σ.

If one takes the strong coupling limit $e^2 \to \infty$, any finite energy configuration must obey

$$v^2 = |\Psi|^2,$$

which requires that Ψ takes values in a three-sphere. In this limit, the Abelian gauge field a decouples. When we take the quotient by the $U(1)$ gauge group we are left with a nonlinear sigma model into

$$S^2 = S^3/U(1),$$

the *Higgs branch* of vacua of the gauge theory. The theory that results is precisely the critical magnetic Skyrmion theory (indeed, this point of view on the Skyrmion theory is exactly that of subsection 7.2.4).

The vortex theory at finite e^2 is such that the moduli space resolves that of magnetic Skyrmions. In particular, the vortex moduli space is non-singular and, if Σ is closed, closed. Vortices have a size, they take up an area proportional to

$$\frac{1}{v^2 e^2},$$

as can be seen by integrating (7.31) over the surface Σ. In the limit $e^2 \to \infty$, this goes to zero.

We can construct the vortex moduli space, as a manifold, by using instead the 'dissolving vortex' limit (see chapter 2, [Weh08, §7], and [BM03]). One may think of this as the weak coupling limit, where $e^2 \to 0$, although one must be careful to enforce the topological constraints of the Bogomolny equations. To do this, we first fix the vortex number N and introduce $\tilde{v}^2 = v^2 - \frac{2\pi N}{e^2 \mathrm{vol}(\Sigma)}$ so as to rewrite the equations (7.30), (7.31) as

$$\begin{aligned} \bar{\partial}_{A+a}\Psi &= 0 \\ *F(a) - \frac{2\pi \mathrm{i} N}{\mathrm{vol}(\Sigma)} &= \mathrm{i}e^2(\tilde{v}^2 - |\Psi|^2). \end{aligned}$$

Integrating the second of these equations tells us that, for any $e^2 \neq 0$,

$$\int_\Sigma *|\Psi|^2 = \tilde{v}^2 \mathrm{vol}(\Sigma).$$

It turns out that the system remains well-behaved as $e^2 \to 0$ [Weh08]. In this limit, the vortex equations become the *dissolving vortex equations*:

$$\bar{\partial}_{A+a}\Psi = 0 \tag{7.32}$$

$$*\frac{1}{2\pi \mathrm{i}} F(a) = \frac{N}{\mathrm{vol}(\Sigma)} \tag{7.33}$$

$$\frac{1}{\mathrm{vol}(\Sigma)} \int_\Sigma *|\Psi|^2 = \tilde{v}^2. \tag{7.34}$$

So solutions consist of projectively flat line bundles (L, a) equipped with a normalised holomorphic section Ψ of the holomorphic bundle $(S \otimes L, (A \otimes a)^{0,1})$. We expect that, as a complex manifold, the moduli space of dissolving vortices is isomorphic to the moduli space of vortices at positive e^2. Of course, these spaces come with a natural Riemannian structure, which does vary with e^2.

Let us construct the moduli space of dissolving vortices. First, there is a one-to-one correspondence between projectively flat line bundles on a Riemann surface and holomorphic line bundles on the surface, modulo equivalence. Letting $\mathrm{Pic}^N(\Sigma)$ be the Picard group, the moduli space of holomorphic line bundles on Σ of degree N, which is a torus of complex dimension g, we form the universal holomorphic bundle

$$\mathcal{V}_N \to \Sigma \times \mathrm{Pic}^N(\Sigma)$$

which has as fibre over $\Sigma \times \{\mathcal{L}\}$ the bundle $\mathcal{S} \otimes \mathcal{L}$, where $\mathcal{S}$ is the pulled back spinor bundle i^*S with holomorphic structure defined by $A^{0,1}$. We fix the bundle $\mathcal{V}_N$ by choosing a point $x \in \Sigma$ and asking that $\mathcal{V}_N|_{\{x\}\times \mathrm{Pic}^N(\Sigma)}$ be trivial.

Regarding $\mathcal{V}_N$ as a locally free sheaf, we may push it forward along the projection map $p : \Sigma \times \mathrm{Pic}^N(\Sigma) \to \mathrm{Pic}^N(\Sigma)$. If N is sufficiently large, Serre vanishing tells us that the resulting sheaf is locally free of constant rank

$$2N + 2 - 2g$$

(explicit bounds on N may be accessible if one computes the holomorphic type of $\mathcal{S}$). In this case, the pushforward defines a vector bundle over $\mathrm{Pic}^N(\Sigma)$ with fibre over $\mathcal{L}$ the space of holomorphic sections of $\mathcal{S} \otimes \mathcal{L}$.

To impose the normalisation condition (7.34) and to take the quotient of the action of gauge transformations on Ψ, we should take the projectivisation of this vector bundle. The moduli space is then the total space of the corresponding projective bundle. Provided N is sufficiently large, this is a compact complex manifold of dimension

$$2N + 1 - g$$

as we expected.

In the case that $g = 0$, this construction is particularly simple. In that case, the Picard group is a point and the moduli space is simply

$$\mathbb{C}P^{2N+1}.$$

Notice that this resolves the moduli space of rational maps, which is

$$\mathbb{C}P^{2N+1} - \Delta$$

where Δ is the resultant hypersurface, consisting of rational maps which degenerate to a lower degree map.

Note that, at least for N sufficiently large relative to g, the background connection A does not affect the resolved moduli space, at least as a manifold.

PART V

CONCLUSIONS

8.1 SUMMARY

In this thesis we have studied some apparently disparate topics in the pursuit of an understanding of the 'hidden machinery' of magnetically charged objects and electric-magnetic duality. While the topics are distinct, they are thematically and technically tied together in our study, as outlined in Figure 8.1.

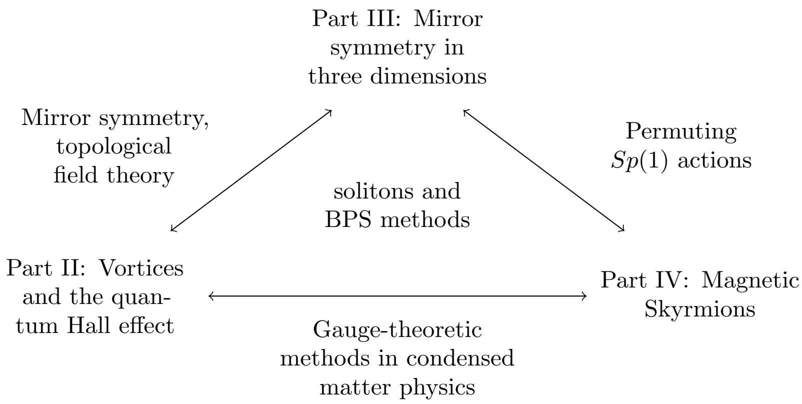


Figure 8.1: The topics of this thesis and some of the themes linking them.

The story of parts II and III was, fundamentally, a story of electric-magnetic duality. In chapter 2, we gave a very explicit realisation of electric-magnetic duality, giving strong evidence for the idea that vortices in nonrelativistic Chern–Simons-matter theories admit a dual description as electrically charged particles, elementary or composite, in a background flux. We did this by (approximately) solving the low-temperature topological quantum mechanics for both sides of the putative duality.

We went on to explore more aspects of this story, considering so-called 'exotic vortex' solutions in these theories in chapter 3. There, we saw that such solutions could be produced from what we called twisted holomorphic maps into spaces of constant holomorphic sectional curvature. We conjectured that all exotic vortex solutions in $U(N_c)$ theories with one fundamental flavour could be produced in this way, giving some evidence for this. This leads to interesting constraints on the possible vortex number of exotic vortex solutions. In chapter 4, we considered the semiclassical limit of the story, using our results to compute the volumes of vortex moduli spaces.

We suggested that, in special cases, this duality between vortices and electrically charged particles could be understood as a deformation of three-dimensional mirror symmetry, a topic that we explored from a loftier perspective in chapter 5. By topologically twisting a three-dimensional $\mathcal{N} = 4$ sigma model with target space a hyperKähler manifold with permuting $Sp(1)$ action, one can produce two topological field theories: the A-twist and the B-twist. The two theories are mirror, in the sense that the A-twist in the Higgs branch of a gauge theory is, in principle, the same as the B-twist in the Coulomb branch of the gauge theory. The B-twist is well-understood, but the A-twist has evaded a complete description. We built the A-twist as a cohomological field theory and sketched it as an extended topological field theory. By outlining a construction of its 2-category of boundary conditions, we offered an understanding of when the Fukaya category of a Kähler manifold admits a monoidal structure (and when it might not).

Vitally important in the construction of the A-twist is the existence of a permuting $Sp(1)$ action on the target space. This idea also played a crucial role in chapter 6, where we built a gauge-theoretic model of magnetic materials supporting magnetic Skyrmions and emergent magnetic monopoles. We studied the thin film limit in chapter 7, finding exact Skyrmion solutions on some surfaces with symmetry at 'critical coupling'. Pleasantly, this use of gauge-theoretic and 'BPS-type' methods (which traditionally find their home in the domain of high energy physics) to understand a condensed matter system brings us back to the ideas of Part II, where we applied these notions to understand another condensed matter system.

8.2 OUTLOOK

Many of the results and ideas of this thesis would benefit from further investigation. Obvious avenues for further work include understanding more deeply the Seiberg-like duality of section 2.6, generalising the twisted holomorphic map construction of chapter 3 to allow for more flavours, studying the quantum corrections to the 2-category of boundary conditions in the A-twist of chapter 5, and sharpening the analysis of the phase structure of the gauge-theoretic model of magnetic materials in chapter 6. Rather than considering the details of these ideas, we content ourselves by making some more general comments about the themes of this thesis.

The idea of using 'BPS-type' methods in condensed matter physics is not new. It has been used successfully in understanding quantum Hall physics in the program of Tong, Turner, and others [Ton04a; Tur17] and in the understanding of chiral magnets in [BSRS20]. In this thesis, we hope that we have demonstrated further the power of this idea. Sophisticated methods like Coulomb branch localisation can be applied to understand the behaviour of systems of electrons in a magnetic field. It seems likely that both particle physics and condensed matter physics can benefit considerably from continuing to find applications for these ideas.

Despite being of mathematical interest, permuting $Sp(1)$ actions on hyperKähler manifolds have played a rather fringe role in physics. In this thesis, we hope that we have demonstrated their importance. Indeed, we have seen that they are crucial in the understanding of the three-dimensional mirror symmetry of sigma models. They also play an intriguing part in chapter 6, which is suggestive of their potential importance more generally in gauge theories. For example, the unusual dual confinement mechanism of chapter 6 could be applied to understand confinement in more general gauge theories.

Finally, electric-magnetic duality remains a fascinating and deep subject. By continuing to understand concrete examples of it, as we have attempted to do in this thesis, we can continue to lift the curtain obscuring its 'hidden machinery'.

A BACKGROUND GAUGE FIELDS

A.1 INTRODUCTION AND SUMMARY

At several points in this thesis, we have considered theories coupled to background gauge fields. This can be done whenever a theory has a global symmetry. Here, we summarise the critical coupling of a two-dimensional nonlinear sigma model with symplectic target to a background flavour symmetry. This is particularly relevant to the material of chapter 7, but also plays a role in chapters 5 and 6.

A.2 NONLINEAR SIGMA MODELS IN A BACKGROUND GAUGE FIELD

We recall some aspects of the theory of sigma models in a background gauge field. The general set up is as follows. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ with a given Killing form. Let (X, ω_X) be a symplectic manifold with a compatible almost complex structure J_X. Let X carry a Hamiltonian G-action. Physically, X will be the target space and G will be the gauge group.

Associated to the Hamiltonian action of G on X is an equivariant moment map

$$\mu : X \to \mathfrak{g}^\vee$$

with the (defining) property that the equivariant 2-form

$$\omega_X - \mu \in \Omega^2_G(X)$$

is equivariantly closed [AB84], which is to say that

$$\mathrm{d}\mu(\xi) = \omega(v_\xi)$$

for all $\xi \in \mathfrak{g}$, where v_ξ is the fundamental vector field on X associated to ξ. Hence, $\omega_X - \mu$ determines a class $[\omega_X - \mu]$ in the equivariant cohomology $H^2_G(X)$.

Now, let Σ be a compact Riemannian 2-manifold, playing the role of physical space. We would like to set up a background gauge theory, and so we take a principal G-bundle $P \to \Sigma$. Our nonlinear field is a gauged map into the target space X (i.e. an X-valued field, charged under G), which is to say that it is a section of the associated bundle

$$\underline{X} \coloneqq X \times_G P \to \Sigma.$$

A section of this bundle is equivalently a G-equivariant map $P \to X$.

Now, we should introduce a background gauge field, which is a connection A on P. This allows us to define the covariant derivative $\mathrm{d}_A\phi \in \Omega^2_V(\underline{X})$ of a section $\phi \in \Gamma(\underline{X})$. Here, by $\Omega^2_V(\underline{X})$ we mean the space of vertical 2-forms. We also define the antiholomorphic derivative

$$\bar{\partial}_A\phi = \frac{1}{2}\left(\mathrm{d}_A\phi + J_X \circ \mathrm{d}_A\phi \circ j_\Sigma\right).$$

The energy functional we consider is the usual one:

$$E[A][\phi] = \frac{1}{2}\int_\Sigma \left(|\mathrm{d}_A\phi|^2 + V(\phi, A)\right) \mathrm{vol}_\Sigma.$$

The critically-coupled model arises for a special potential, namely,

$$V_{\mathrm{crit}}(\phi, A) = -2\mu(\phi)(*F(A)) \tag{A.1}$$

where $*$ is the Hodge star on Σ. The equivariance of the moment map implies that this potential is gauge invariant.

The reason that this is called 'critically-coupled' is that, for this choice of potential, the energy admits a Bogomolny rearrangement. Indeed, it was shown in [CGS00] that

$$\frac{1}{2}\int_\Sigma * \left(|F(A)|^2 + |\mathrm{d}_A\phi|^2 + |\mu(\phi)|^2\right) = \int_\Sigma * \left(|\bar{\partial}_A\phi|^2 + \frac{1}{2}| * F(A) + \mu^\sharp(\phi)|^2\right) + [\omega - \mu]([\phi]) \tag{A.2}$$

where the second term on the right-hand side is the natural pairing of $[\omega_X - \mu] \in H^2_G(X)$ with $[\phi] \in H_2^G(X)$ and $\mu^\sharp(\phi)$ is the moment map with its index raised using the Killing form, so that it takes values in $\mathfrak{g}$. This rearrangement is of interest in the study of (symplectic) vortices.

Rearranging (A.2), we see that

$$\begin{aligned}\int_\Sigma *|\bar{\partial}_A\phi|^2 + [\omega - \mu]([\phi]) &= \frac{1}{2}\int_\Sigma * \left(|\mathrm{d}_A\phi|^2 + |F(A)|^2 + |\mu(\phi)|^2 - | * F(A) + \mu^\sharp(\phi)|^2\right) \\ &= \frac{1}{2}\int_\Sigma * \left(|\mathrm{d}_A\phi|^2 - 2\mu(\phi)(*F(A))\right).\end{aligned}$$

The right-hand side is our energy functional. The energy of a configuration is therefore bounded below within a topological sector by the topological energy

$$[\omega - \mu]([\phi])$$

and the bound is saturated by configurations ϕ obeying

$$\bar{\partial}_A\phi = 0.$$

For the model studied in chapter 7, we take X to be the sphere S^2 with its standard symplectic form ω_{S^2}. We take $G = SO(3)$, which acts on S^2 by rotations in the usual way preserving ω_{S^2}. We now write m for the field ϕ, as we now interpret it as the magnetisation field. It can be shown that the moment map for this action

$$\mu : S^2 \to \mathfrak{so}(3)$$

is the inclusion of the unit sphere. Hence, thinking of $m = (m^1, m^2, m^3)$ as a unit vector in $\mathbb{R}^3 \cong \mathfrak{so}(3)$, we have

$$\mu(m)(\xi) = m^B \xi_B$$

for $\xi \in \mathfrak{so}(3)$ and where $B = 1, 2, 3$ labels coordinates on $\mathfrak{so}(3)$. Substituting this into the critical potential (A.1) leads to the potential term (7.13).

[AB83] M. F. Atiyah and R. Bott. The Yang–Mills equations over Riemann surfaces. *Philos. Trans. R. Soc. A* **308**.1505 (1983), 523–615.

[AB84] M. F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology* **23**.1 (1984), 1–28.

[ADPW91] S. Axelrod, S. Della Pietra, and E. Witten. Geometric quantization of Chern–Simons gauge theory. *J. Differential Geom.* **33**.3 (1991), 787–902.

[AF94] D. Anselmi and P. Frè. Topological sigma-models in four dimensions and triholomorphic maps. *Nucl. Phys. B* **416**.1 (1994), 255–300.

[AF95] D. Anselmi and P. Fré. Gauged hyperinstantons and monopole equations. *Phys. Lett. B* **347**.3 (1995), 247–254.

[Aha16] O. Aharony. Baryons, monopoles and dualities in Chern–Simons-matter theories. *J. High Energy Phys.* **2016**.02 (2016), 093.

[Ake+11] N. Akerblom, G. Cornelissen, G. Stavenga, and J. W. van Holten. Nonrelativistic Chern–Simons vortices on the torus. *J. Math. Phys.* **52**.7 (2011), 072901.

[And12] J. E. Andersen. Hitchin's connection, Toeplitz operators, and symmetry invariant deformation quantization. *Quantum Topol.* **3**.3-4 (2012), 293–325.

[AO88] J. Ambjørn and P. Olesen. Anti-screening of large magnetic fields by vector bosons. *Phys. Lett. B* **214**.4 (1988), 565–569.

[AO89] J. Ambjørn and P. Olesen. A magnetic condensate solution of the classical electroweak theory. *Phys. Lett. B* **218**.1 (1989), 67–71.

[AO90] J. Ambjørn and P. Olesen. A condensate solution of the electroweak theory which interpolates between the broken and the symmetric phase. *Nucl. Phys. B* **330**.1 (1990), 193–204.

[ASW84] D. Arovas, J. R. Schrieffer, and F. Wilczek. Fractional statistics and the quantum Hall effect. *Phys. Rev. Lett.* **53**.7 (1984), 722–723.

[Aur13] D. Auroux. A beginner's introduction to Fukaya categories. 2013. arXiv: `1301.7056 [math.SG]`.

[Auz+03] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung. NonAbelian superconductors: Vortices and confinement in $\mathcal{N} = 2$ SQCD. *Nucl. Phys. B* **673**.1 (2003), 187–216.

[Auz+04] R. Auzzi, S. Bolognesi, J. Evslin, and K. Konishi. Non-Abelian monopoles and the vortices that confine them. *Nucl. Phys. B* **686**.1 (2004), 119–134.

[Bap09] J. Baptista. Non-Abelian vortices on compact Riemann surfaces. *Commun. Math. Phys.* **291** (2009), 799–812.

[Bap11] J. Baptista. On the L^2-metric of vortex moduli spaces. *Nucl. Phys. B* **844**.2 (2011), 308–333.

[Bap14] J. M. Baptista. Vortices as degenerate metrics. *Lett. Math. Phys.* **104** (2014), 731–747.

[BC14] F. Benini and S. Cremonesi. Partition functions of $\mathcal{N} = (2,2)$ gauge theories on S^2 and vortices. *Commun. Math. Phys.* **334**.3 (2014), 1483–1527.

[BDG17] M. Bullimore, T. Dimofte, and D. Gaiotto. The Coulomb branch of 3d $\mathcal{N} = 4$ theories. *Commun. Math. Phys.* **354**.2 (2017), 671–751.

[BFK19] M. Bullimore, A. Ferrari, and H. Kim. Twisted indices of 3d $\mathcal{N} = 4$ gauge theories and enumerative geometry of quasi-maps. *J. High Energy Phys.* **2019**.07 (2019), 014.

[BFN18] A. Braverman, M. Finkelberg, and H. Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. *Adv. Theor. Math. Phys.* **22** (2018), 1071–1147.

[BG18] A. E. Barns-Graham. *Much ado about nothing: The superconformal index and Hilbert series of three-dimensional $\mathcal{N} = 4$ vacua.* PhD thesis. 2018.

[BGD18] A. E. Barns-Graham and N. Dorey. A superconformal index for hyperKähler cones. 2018. arXiv: `1812.04565 [hep-th]`.

[BH94] A. Bogdanov and A. Hubert. Thermodynamically stable magnetic vortex states in magnetic crystals. *J. Magn. Magn. Mater.* **138**.3 (1994), 255–269.

[Bir+20] M. T. Birch, D. Cortés-Ortuño, L. A. Turnbull, M. N. Wilson, F. Groß, N. Träger, A. Laurenson, N. Bukin, S. H. Moody, M. Weigand, G. Schütz, H. Popescu, R. Fan, P. Steadman, J. A. T. Verezhak, G. Balakrishnan, J. C. Loudon, A. C. Twitchett-Harrison, O. Hovorka, H. Fangohr, F. Y. Ogrin, J. Gräfe, and P. D. Hatton. Real-space imaging of confined magnetic Skyrmion tubes. *Nat. Commun.* **11**.1 (2020).

[BL94] O. Bergman and G. Lozano. Aharonov–Bohm scattering, contact interactions, and scale invariance. *Ann. Phys.* **229**.2 (1994), 416–427.

[BM03] J. M. Baptista and N. S. Manton. The dynamics of vortices on S^2 near the Bradlow limit. *J. Math. Phys.* **44**.8 (2003), 3495–3508.

[Bog76] E. B. Bogomolny. The stability of classical solutions. *Sov. J. Nucl. Phys.* **24**.4 (1976), 449.

[Bra+14] T. Braden, A. Licata, N. Proudfoot, and B. Webster. Quantizations of conical symplectic resolutions II: Category $\mathcal{O}$ and symplectic duality. *Astérisque* **384** (2014), 75–179.

[Bra90] S. B. Bradlow. Vortices in holomorphic line bundles over closed Kähler manifolds. *Commun. Math. Phys.* **135**.1 (1990), 1–17.

[BSRS20] B. Barton-Singer, C. Ross, and B. J. Schroers. Magnetic Skyrmions at critical coupling. *Commun. Math. Phys.* **375**.3 (2020), 2259–2280.

[BT02] M. Blau and G. Thompson. On the relationship between the Rozansky–Witten and the three-dimensional Seiberg–Witten invariants. *Adv. Theor. Math. Phys.* **5** (2002), 483–498.

[Bul+16] M. Bullimore, T. Dimofte, D. Gaiotto, and J. Hilburn. Boundaries, mirror symmetry, and symplectic duality in 3d $\mathcal{N} = 4$ gauge theory. *J. High Energy Phys.* **2016**.10 (2016), 108.

[Bul+18] M. Bullimore, T. Dimofte, D. Gaiotto, J. Hilburn, and H.-C. Kim. Vortices and Vermas. *Adv. Theor. Math. Phys.* **22** (2018), 803–917.

[Cal16] M. Callies. *Permuting actions, moment maps and the generalized Seiberg–Witten equations*. PhD thesis. 2016.

[CCG19] K. Costello, T. Creutzig, and D. Gaiotto. Higgs and Coulomb branches from vertex operator algebras. *J. High Energy Phys.* **2019**.3 (2019), 066.

[CD17] F. Contatto and M. Dunajski. Manton's five vortex equations from self-duality. *J. Phys. A* **50**.37 (2017), 375201.

[CDZ20] S. Crew, N. Dorey, and D. Zhang. Factorisation of 3d $\mathcal{N} = 4$ twisted indices and the geometry of vortex moduli space. *J. High Energy Phys.* **2020**.08 (2020), 015.

[CG19] K. Costello and D. Gaiotto. Vertex operator algebras and 3d $\mathcal{N} = 4$ gauge theories. *J. High Energy Phys.* **2019**.5 (2019), 018.

[CGS00] K. Cieliebak, A. R. Gaio, and D. A. Salamon. J-holomorphic curves, moment maps, and invariants of Hamiltonian group actions. *Internat. Math. Res. Notices* **10**.16 (2000), 831–882.

[Che+15] Q. Chen, W. Wang, Y. Wu, and B. Xu. Conformal metrics with constant curvature one and finitely many conical singularities on compact Riemann surfaces. *Pac. J. Math.* **273**.1 (2015), 75–100.

[CKW18] C. Closset, H. Kim, and B. Willett. Seifert fibering operators in 3d $\mathcal{N} = 2$ theories. *J. High Energy Phys.* **2018**.11 (2018), 004.

[CLW14] T. Can, M. Laskin, and P. Wiegmann. Fractional quantum Hall effect in a curved space: Gravitational anomaly and electromagnetic response. *Phys. Rev. Lett.* **113**.4 (2014), 046803.

[CO17] H.-J. Chung and T. Okazaki. $(2, 2)$ And $(0, 4)$ supersymmetric boundary conditions in 3d $\mathcal{N} = 4$ theories and type IIB branes. *Phys. Rev. D* **96**.8 (2017), 086005.

[Der64] G. H. Derrick. Comments on nonlinear wave equations as models for elementary particles. *J. Math. Phys.* **5**.9 (1964), 1252–1254.

[DG17] R. Dey and S. Ganguli. The dimension of the Hilbert space of geometric quantization of vortices on a Riemann surface. *Int. J. Geom. Methods Mod. Phys.* **14**.10 (2017), 1750144.

[DGH11] T. Dimofte, S. Gukov, and L. Hollands. Vortex counting and Lagrangian 3-manifolds. *Lett. Math. Phys.* **98**.3 (2011), 225–287.

[Dim+20] T. Dimofte, N. Garner, M. Geracie, and J. Hilburn. Mirror symmetry and line operators. *J. High Energy Phys.* **2020**.02 (2020), 075.

[Dir31] P. A. M. Dirac. Quantised singularities in the electromagnetic field. *Proc. Roy. Soc. Lond. A* **A133**.821 (1931), 60–72.

[DLP08] J.-F. Dars, A. Lesne, and A. Papillault, eds. *The Unravelers: Mathematical Snapshots*. A K Peters, Ltd., Wellesley, MA, 2008.

[Doa19] A. Doan. *Monopoles and Fueter sections on three-manifolds*. PhD thesis. 2019.

[Dör68] W Döring. Point singularities in micromagnetism. *J. Appl. Phys.* **39**.2 (1968), 1006–1007.

[DS18] N. Dorey and A. Singleton. An index for superconformal quantum mechanics. 2018. arXiv: `1812.11816 [hep-th]`.

[DTT16a] N. Dorey, D. Tong, and C. Turner. Matrix model for non-Abelian quantum Hall states. *Phys. Rev. B* **94**.8 (2016), 085114.

[DTT16b] N. Doroud, D. Tong, and C. Turner. On superconformal anyons. *J. High Energy Phys.* **2016**.01 (2016), 138.

[Dun+91] G. V. Dunne, R. Jackiw, S.-Y. Pi, and C. A. Trugenberger. Selfdual Chern–Simons solitons and two-dimensional nonlinear equations. *Phys. Rev. D* **43** (1991), 1332. Erratum: *Phys. Rev. D* **45** (1992), 3012.

[Dzy58] I. Dzyaloshinskii. A thermodynamic theory of “weak” ferromagnetism of antiferromagnetics. *J. Phys. Chem. Solids* **4**.4 (1958), 241–255.

[EGT14] A. Eremenko, A. Gabrielov, and V. Tarasov. Metrics with conic singularities and spherical polygons. *Illinois J. Math.* **58**.3 (2014), 739–755.

[ER20] D. Eriksson and N. Romão. Kähler quantization of vortex moduli. *Lett. Math. Phys.* **110** (2020), 659–693.

[ES19] C. Elliott and P. Safronov. Topological twists of supersymmetric algebras of observables. *Commun. Math. Phys.* **371**.2 (2019), 727–786.

[Eto+06a] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai. Moduli space of non-Abelian vortices. *Phys. Rev. Lett.* **96** (2006), 161601.

[Eto+06b] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, and N. Sakai. Solitons in the Higgs phase: The moduli matrix approach. *J. Phys. A* **39**.26 (2006), R315–R392.

[Eto+07] M. Eto, J. Evslin, K. Konishi, G. Marmorini, M. Nitta, K. Ohashi, W. Vinci, and N. Yokoi. On the moduli space of semilocal strings and lumps. *Phys. Rev. D* **76**.10 (2007), 105002.

[Eto+08a] M. Eto, T. Fujimori, S. B. Gudnason, K. Konishi, M. Nitta, K. Ohashi, and W. Vinci. Constructing non-Abelian vortices with arbitrary gauge groups. *Phys. Lett. B* **669**.1 (2008), 98–101.

[Eto+08b] M. Eto, T. Fujimori, M. Nitta, K. Ohashi, K. Ohta, and N. Sakai. Statistical mechanics of vortices from D-branes and T-duality. *Nucl. Phys. B* **788**.3 (2008), 120–136.

[Fei03] B. Feix. Twistor spaces of hyperKähler manifolds with S^1-actions. *Differ. Geom. Appl.* **19**.1 (2003), 15–28.

[Fel65] E. Feldtkeller. Mikromagnetisch stetige und unstetige magnetisierungskonfigurationen. *Z. Angew. Phys.* **19**.6 (1965), 530–536.

[FHY14] M. Fujitsuka, M. Honda, and Y. Yoshida. Higgs branch localization of 3d $\mathcal{N} = 2$ theories. *Prog. Theor. Exp. Phys* **2014**.12 (2014), 123B02–123B02.

[FRC17] A. Fert, N. Reyren, and V. Cros. Magnetic Skyrmions: Advances in physics and potential applications. *Nat. Rev. Mater.* **2**.7 (2017), 1–15.

[Fuj+12] T. Fujimori, T. Kimura, M. Nitta, and K. Ohashi. Vortex counting from field theory. *J. High Energy Phys.* **2012**.6 (2012), 028.

[Fuj+15] T. Fujimori, T. Kimura, M. Nitta, and K. Ohashi. 2d partition function in Ω-background and vortex/instanton correspondence. *J.High Energy Phys.* **2015**.12 (2015), 110.

[Fuj+19] Y Fujishiro, N Kanazawa, T Nakajima, X. Yu, K Ohishi, Y Kawamura, K Kakurai, T Arima, H Mitamura, A Miyake, et al. Topological transitions among Skyrmion- and hedgehog-lattice states in cubic chiral magnets. *Nat. Commun.* **10**.1 (2019), 1–8.

[Gai16] D. Gaiotto. Twisted compactifications of 3d $\mathcal{N} = 4$ theories and conformal blocks. *J. High Energy Phys.* **2019**.2 (2016), 061.

[GKS14] Y. Gaididei, V. P. Kravchuk, and D. D. Sheka. Curvature effects in thin magnetic shells. *Phys. Rev. Lett.* **112**.25 (2014), 257203.

[GN17] S. B. Gudnason and M. Nitta. Some exact Bradlow vortex solutions. *J. High Energy. Phys.* **2017**.5 (2017), 039.

[GP94] O. García-Prada. A direct existence proof for the vortex equations over a compact Riemann surface. *Bull. London Math. Soc.* **26**.1 (1994), 88–96.

[GW09] D. Gaiotto and E. Witten. S-duality of boundary conditions in $\mathcal{N} = 4$ super Yang–Mills theory. *Adv. Theor. Math. Phys.* **13**.3 (2009), 721–896.

[Hal11] F. D. M. Haldane. Geometrical description of the fractional quantum Hall effect. *Phys. Rev. Lett.* **107**.11 (2011), 116801.

[Hay06] A. Haydys. *Generalized Seiberg–Witten equations and hyperKähler geometry.* PhD thesis. 2006.

[Hit+87] N. J. Hitchin, A. Karlhede, U. Lindstrom, and M. Rocek. Hyperkahler metrics and supersymmetry. *Commun. Math. Phys.* **108** (1987), 535.

[Hit90a] N. J. Hitchin. Harmonic maps from a 2-torus to the 3-sphere. *J. Differential Geom.* **31**.3 (1990), 627–710.

[Hit90b] N. J. Hitchin. Flat connections and geometric quantization. *Commun. Math. Phys.* **131** (1990), 347–380.

[HKS92] L. Hsu, R. Kusner, and J. Sullivan. Minimizing the squared mean curvature integral for surfaces in space forms. *Exp. Math.* **1**.3 (1992), 191–207.

[HLR93] B. I. Halperin, P. A. Lee, and N. Read. Theory of the half-filled Landau level. *Phys. Rev. B* **47**.12 (1993), 7312–7343.

[Hon+20] M. Hongo, T. Fujimori, T. Misumi, M. Nitta, and N. Sakai. Instantons in chiral magnets. *Phys. Rev. B* **101**.10 (2020), 104417.

[Hoo74] G. 't Hooft. Magnetic monopoles in unified gauge theories. *Nucl. Phys. B* **79** (1974), 276–284.

[Hor+03] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow. *Mirror symmetry.* Vol. 1. Clay Mathematics Monographs. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2003.

[HR01] S. Hellerman and M. V. Raamsdonk. Quantum Hall physics = noncommutative field theory. *J. High Energy Phys.* **2001**.10 (2001), 039.

[HT03] A. Hanany and D. Tong. Vortices, instantons and branes. *J. High Energy Phys.* **2003**.07 (2003), 037.

[HW19] D. Harlow and J. Wu. Covariant phase space with boundaries. 2019. arXiv: `1906.08616 [hep-th]`.

[HY98] P. Horvathy and J. Yera. Vortex solutions of the Liouville equation. *Lett. Math. Phys.* **46** (1998), 111–120.

[Im+19] M.-Y. Im, H.-S. Han, M.-S. Jung, Y.-S. Yu, S. Lee, S. Yoon, W. Chao, P. Fischer, J.-I. Hong, and K.-S. Lee. Dynamics of the Bloch point in an asymmetric permalloy disk. *Nat. Commun.* **10** (2019), 593.

[IS96] K. Intriligator and N. Seiberg. Mirror symmetry in three-dimensional gauge theories. *Phys. Lett. B* **387**.3 (1996), 513–519.

[Jai07] J. K. Jain. *Composite fermions.* Cambridge University Press, 2007.

[Jai89] J. K. Jain. Composite-fermion approach for the fractional quantum Hall effect. *Phys. Rev. Lett.* **63**.2 (1989), 199–202.

[Jai92] J. K. Jain. Microscopic theory of the fractional quantum Hall effect. *Adv. Phys.* **41**.2 (1992), 105–146.

[Jen18] K. Jensen. A master bosonization duality. *J. High Energy Phys.* **2018**.01 (2018), 031.

[JK95] L. C. Jeffrey and F. C. Kirwan. Localization for nonAbelian group actions. *Topology* **34**.2 (1995), 291–327.

[JK97] L. C. Jeffrey and F. C. Kirwan. Localization and the quantization conjecture. *Topology* **36**.3 (1997), 647–693.

[JP90] R. Jackiw and S.-Y. Pi. Soliton solutions to the gauged nonlinear Schrödinger equation on the plane. *Phys. Rev. Lett.* **64**.25 (1990), 2969–2972.

[JT80] A. Jaffe and C. Taubes. *Vortices and monopoles: Structure of static gauge theories.* Progress in physics. Birkhäuser, 1980.

[Kal99] D. Kaledin. A canonical hyperKähler metric on the total space of a cotangent bundle. *Quaternionic structures in mathematics and physics (Rome, 1999).* Univ. Studi Roma "La Sapienza", Rome, 1999, 195–230.

[Kan+16] N Kanazawa, Y Nii, X.-X. Zhang, A. Mishchenko, G De Filippis, F Kagawa, Y Iwasa, N Nagaosa, and Y Tokura. Critical phenomena of emergent magnetic monopoles in a chiral magnet. *Nat. Commun.* **7**.1 (2016), 1–7.

[Kan+20] N. Kanazawa, A. Kitaori, J. S. White, V. Ukleev, H. M. Rønnow, A. Tsukazaki, M. Ichikawa, M. Kawasaki, and Y. Tokura. Direct observation of the statics and dynamics of emergent magnetic monopoles in a chiral magnet. *Phys. Rev. Lett.* **125** (13 2020), 137202.

[Kap10] A. Kapustin. Topological field theory, higher categories, and their applications. *Proceedings of the International Congress of Mathematicians. Volume III.* Hindustan Book Agency, New Delhi, 2010, 2021–2043.

[Kap99] M. Kapranov. Rozansky–Witten invariants via Atiyah classes. *Compositio Math.* **115**.1 (1999), 71–113.

[KDP80] K. v. Klitzing, G. Dorda, and M. Pepper. New method for high-accuracy determination of the fine-structure constant based on quantized Hall resistance. *Phys. Rev. Lett.* **45**.6 (1980), 494–497.

[Kim10] T. Kimura. Vortex description of quantum Hall ferromagnets. *Int. J. Mod. Phys. A* **25**.05 (2010), 993–1008.

[Kim16] B. Kim. Stable quasimaps to holomorphic symplectic quotients. *Schubert Calculus - Osaka 2012.* Tokyo, Japan: Mathematical Society of Japan, 2016, 139–160.

[KN96] S. Kobayashi and K. Nomizu. *Foundations of differential geometry. Volume II.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996.

[Kon95] M. Kontsevich. Homological algebra of mirror symmetry. *Proceedings of the International Congress of Mathematicians.* Ed. by S. D. Chatterji. Basel: Birkhäuser Basel, 1995, 120–139.

[KR10] A. Kapustin and L. Rozansky. Three-dimensional topological field theory and symplectic algebraic geometry II. *Commun. Num. Theor. Phys.* **4** (2010), 463–550.

[Kra+18] V. P. Kravchuk, D. D. Sheka, A. Kákay, O. M. Volkov, U. K. Rößler, J. van den Brink, D. Makarov, and Y. Gaididei. Multiplet of Skyrmion states on a curvilinear defect: Reconfigurable Skyrmion lattices. *Phys. Rev. Lett.* **120**.6 (2018), 067201.

[KRS09] A. Kapustin, L. Rozansky, and N. Saulina. Three-dimensional topological field theory and symplectic algebraic geometry I. *Nucl. Phys. B* **816** (2009), 295–355.

[KSV10] A. Kapustin, K. Setter, and K. Vyas. Surface operators in four-dimensional topological gauge theory and Langlands duality. 2010. arXiv: `1002.0385 [hep-th]`.

[KT20] K. Kunikawa and R. Takahashi. Convergence of mean curvature flow in hyper-Kähler manifolds. *Pac. J. Math.* **305**.2 (2020), 667–691.

[KV10] A. Kapustin and K. Vyas. A-models in three and four dimensions. 2010. arXiv: `1002.4241 [hep-th]`.

[KW07] A. Kapustin and E. Witten. Electric-magnetic duality and the geometric Langlands program. *Commun. Num. Theor. Phys.* **1** (2007), 1–236.

[Lau83] R. B. Laughlin. Anomalous quantum Hall effect: An incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.* **50**.18 (1983), 1395–1398.

[Leu02] N. C. Leung. Riemannian geometry over different normed division algebras. *J. Differential Geom.* **61**.2 (2002), 289–333.

[LF91] A. Lopez and E. Fradkin. Fractional quantum Hall effect and Chern–Simons gauge theories. *Phys. Rev. B* **44**.10 (1991), 5246.

[Li+17] J. I. A. Li, C. Tan, S. Chen, Y. Zeng, T. Taniguchi, K. Watanabe, J. Hone, and C. R. Dean. Even-denominator fractional quantum Hall states in bilayer graphene. *Science* **358**.6363 (2017), 648–652.

[LLM92] M Leblanc, G Lozano, and H Min. Extended superconformal Galilean symmetry in Chern–Simons matter systems. *Ann. Phys.* **219**.2 (1992), 328–348.

[LM77] J. Leinaas and J. Myrheim. On the theory of identical particles. *Nuovo Cim. B* **37** (1977), 1–23.

[Lur09] J. Lurie. On the classification of topological field theories. *Current Developments in Mathematics, 2008*. International Press of Boston, 2009, 129–280.

[LW07] N. C. Leung and T. Wan. Hyper-Lagrangian submanifolds of hyperKähler manifolds and mean curvature flow. *J. Geom. Anal.* **17** (2007), 343–364.

[Mac62] I. MacDonald. Symmetric products of an algebraic curve. *Topology* **1**.4 (1962), 319–343.

[Man13] N. S. Manton. Vortex solutions of the Popov equations. *J. Phys. A* **46**.14 (2013), 145402.

[Man17] N. S. Manton. Five vortex equations. *J. Phys. A* **50**.12 (2017), 125403.

[Man93] N. S. Manton. Statistical mechanics of vortices. *Nucl. Phys. B* **400**.1 (1993), 624–632.

[Man97] N. S. Manton. First order vortex dynamics. *Ann. Phys.* **256**.1 (1997), 114–131.

[Mil+13] P. Milde, D. Köhler, J. Seidel, L. M. Eng, A. Bauer, A. Chacon, J. Kindervater, S. Mühlbauer, C. Pfleiderer, S. Buhrandt, C. Schütte, and A. Rosch. Unwinding of a Skyrmion lattice by magnetic monopoles. *Science* **340**.6136 (2013), 1076–1080.

[MM99] M. Mariño and G. Moore. Three-manifold topology and the Donaldson–Witten partition function. *Nucl. Phys. B* **547**.3 (1999), 569–598.

[MN99a] N. S. Manton and S. M. Nasir. Conservation laws in a first-order dynamical system of vortices. *Nonlinearity* **12**.4 (1999), 851–865.

[MN99b] N. S. Manton and S. M. Nasir. Volume of vortex moduli spaces. *Commun. Math. Phys.* **199** (1999), 591–604.

[MNS00] G. Moore, N. Nekrasov, and S. Shatashvili. Integrating over Higgs branches. *Commun. Math. Phys.* **209**.1 (2000), 97–121.

[MO77] C. Montonen and D. Olive. Magnetic monopoles as gauge particles? *Phys. Lett. B* **72**.1 (1977), 117–120.

[Mor60] T. Moriya. Anisotropic superexchange interaction and weak ferromagnetism. *Phys. Rev.* **120**.1 (1960), 91–98.

[MOS11] A. Miyake, K. Ohta, and N. Sakai. Volume of moduli space of vortex equations and localization. *Prog. Theor. Phys.* **126**.4 (2011), 637–680.

[MR11] N. S. Manton and N. M. Romão. Vortices and Jacobian varieties. *J. Geom. Phys.* **61**.6 (2011), 1135–1155.

[MR91] G. Moore and N. Read. Nonabelions in the fractional quantum Hall effect. *Nucl. Phys. B* **360**.2 (1991), 362–396.

[MS04] N. S. Manton and P. Sutcliffe. *Topological solitons.* Cambridge University Press, 2004.

[Müh+09] S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, and P. Böni. Skyrmion lattice in a chiral magnet. *Science* **323**.5916 (2009), 915–919.

[Mül+20] G. P. Müller, F. N. Rybakov, H. Jónsson, S. Blügel, and N. S. Kiselev. Coupled quasimonopoles in chiral magnets. *Phys. Rev. B* **101**.18 (2020), 184405.

[MZ21] R. Mazzeo and X. Zhu. Conical metrics on Riemann surfaces, II: Spherical metrics. *Int. Math. Res. Not.* (2021).

[Nak16] H. Nakajima. Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, I. *Adv. Theor. Math. Phys.* **20** (2016), 595–669.

[Nay+08] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma. Non-Abelian anyons and topological quantum computation. *Rev. Mod. Phys.* **80**.3 (2008), 1083–1159.

[Nek03] N. A. Nekrasov. Seiberg-Witten prepotential from instanton counting. *Adv. Theor. Math. Phys.* **7**.5 (2003), 831–864.

[Nog87] M. Noguchi. Yang–Mills–Higgs theory on a compact Riemann surface. *J. Math. Phys.* **28**.10 (1987), 2343–2346.

[NZ09] D. Nadler and E. Zaslow. Constructible sheaves and the Fukaya category. *J. Amer. Math. Soc.* **22**.1 (2009), 233–286.

[Ole91] P. Olesen. Soliton condensation in some self-dual Chern–Simons theories. *Phys. Lett. B* **265**.3 (1991), 361–365.

[OS19] K. Ohta and N. Sakai. Higgs and Coulomb branch descriptions of the volume of the vortex moduli space. *Prog. Theor. Exp. Phys.* **2019**.4 (2019), 043B01.

[Pas18] J. Pascaleff. Poisson geometry, monoidal Fukaya categories, and commutative Floer cohomology rings. 2018. arXiv: `1803.07676 [math.SG]`.

[PB75] A. M. Polyakov and A. A. Belavin. Metastable states of two-dimensional isotropic ferromagnets. *JETP Lett.* **22** (1975), 245–248.

[Per06] T. Perutz. Symplectic fibrations and the Abelian vortex equations. *Commun. Math. Phys.* **278** (2006), 289–306.

[Pes+17] V. Pestun, M. Zabzine, F. Benini, T. Dimofte, T. T. Dumitrescu, K. Hosomichi, S. Kim, K. Lee, B. L. Floch, M. Mariño, J. A. Minahan, D. R. Morrison, S. Pasquetti, J. Qiu, L. Rastelli, S. S. Razamat, S. S. Pufu, Y. Tachikawa, B. Willett, and K. Zarembo. Localization techniques in quantum field theories. *J. Phys. A* **50**.44 (2017), 440301.

[Pes78] M. E. Peskin. Mandelstam–'t Hooft duality in Abelian lattice models. *Ann. Phys.* **113** (1978), 122.

[Pid04] V. Y. Pidstrigach. Hyper-Kähler manifolds and the Seiberg–Witten equations. *Tr. Mat. Inst. Steklova* **246** (2004), 263–276.

[Pol01] A. P. Polychronakos. Quantum Hall states as matrix Chern–Simons theory. *J. High Energy Phys.* **2001**.04 (2001), 011.

[Pol74] A. M. Polyakov. Particle spectrum in quantum field theory. *JETP Lett.* **20** (1974), 194–195.

[Pop12] A. D. Popov. Integrable vortex-type equations on the two-sphere. *Phys. Rev. D* **86**.10 (2012), 105044.

[PS75] M. K. Prasad and C. M. Sommerfield. Exact classical solution for the 't Hooft monopole and the Julia–Zee dyon. *Phys. Rev. Lett.* **35** (12 1975), 760–762.

[Rea94] N. Read. Theory of the half-filled Landau level. *Semicond. Sci. Technol.* **9**.11S (1994), 1859–1864.

[Rom01] N. M. Romão. Quantum Chern–Simons vortices on a sphere. *J. Math. Phys.* **42**.8 (2001), 3445–3469.

[Rom05] N. M. Romão. Gauged vortices in a background. *J. Phys. A* **38**.41 (2005), 9127–9144.

[RS04] N. M. Romão and J. M. Speight. Slow Schrödinger dynamics of gauged vortices. *Nonlinearity* **17**.4 (2004), 1337–1355.

[RS18] C. Ross and B. J. Schroers. Magnetic zero-modes, vortices and Cartan geometry. *Lett. Math. Phys.* **108** (2018), 949–983.

[RSN20] C. Ross, N. Sakai, and M. Nitta. Exact phase structure of one dimensional chiral magnets. 2020. arXiv: `2012.08800 [cond-mat.mes-hall]`.

[RSN21] C. Ross, N. Sakai, and M. Nitta. Skyrmion interactions and lattices in chiral magnets: Analytical results. *J. High Energy Phys.* **2021**.2 (2021), 095.

[RTT16] D. Radičević, D. Tong, and C. Turner. Non-Abelian 3d bosonization and quantum Hall states. *J. High Energy Phys.* **2016**.12 (2016), 067.

[RW97] L. Rozansky and E. Witten. HyperKähler geometry and invariants of three manifolds. *Selecta Math.* **3** (1997), 401–458.

[Ryb+15] F. N. Rybakov, A. B. Borisov, S. Blügel, and N. S. Kiselev. New type of stable particlelike states in chiral magnets. *Phys. Rev. Lett.* **115**.11 (2015), 117201.

[Sac19] S. Sachdev. Topological order, emergent gauge fields, and Fermi surface reconstruction. *Rept. Prog. Phys.* **82**.1 (2019), 014001.

[Sch19] B. J. Schroers. Gauged sigma models and magnetic Skyrmions. *SciPost Phys.* **7**.3 (2019), 30.

[Sch20] B. Schroers. Solvable Models for Magnetic Skyrmions. *Quantum theory and symmetries: Proceedings of the 11th international symposium, Montreal, Canada*. 2020.

[Sei08] P. Seidel. *Fukaya categories and Picard-Lefschetz theory*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2008.

[Sei95] N. Seiberg. Electric-magnetic duality in supersymmetric non-Abelian gauge theories. *Nucl. Phys. B* **435**.1-2 (1995), 129–146.

[Set13] K. Setter. *Topological quantum field theory and the geometric Langlands correspondence*. PhD thesis. 2013.

[Sha07] S. Shadchin. On F-term contribution to effective action. *J. High Energy Phys.* **2007**.08 (2007), 052.

[Sin16] A. Singleton. *The geometry and representation theory of superconformal quantum mechanics*. PhD thesis. 2016.

[SR14] C. Schütte and A. Rosch. Dynamics and energetics of emergent magnetic monopoles in chiral magnets. *Phys. Rev. B* **90**.17 (2014), 174432.

[Ste94] A. Stern. Quantum Hall fluid of vortices in a two-dimensional array of Josephson junctions. *Phys. Rev. B* **50**.14 (1994), 10092–10106.

[Str+16] R. Streubel, P. Fischer, F. Kronast, V. P. Kravchuk, D. D. Sheka, Y. Gaididei, O. G. Schmidt, and D. Makarov. Magnetism in curved geometries. *J. Phys. D* **49**.36 (2016), 363001.

[Sut18] P. Sutcliffe. Hopfions in chiral magnets. *J. Phys. A* **51**.37 (2018), 375401.

[SW96] N. Seiberg and E. Witten. Gauge dynamics and compactification to three dimensions. *Conference on the Mathematical Beauty of Physics (In Memory of C. Itzykson)*. 1996, 333–366.

[Swa90] A. F. Swann. *HyperKähler and quaternionic geometry*. PhD thesis. 1990.

[SY04] M. Shifman and A. Yung. Non-Abelian string junctions as confined monopoles. *Phys. Rev. D* **70**.4 (2004), 045004.

[Sza96] R. J. Szabo. Equivariant localization of path integrals. 1996. arXiv: `hep-th/9608068 [hep-th]`.

[Tau90] C. H. Taubes. Casson's invariant and gauge theory. *J. Differential Geom.* **31**.2 (1990), 547–599.

[Tau99] C. H. Taubes. Nonlinear generalizations of a 3-manifold's Dirac operator. *Trends in mathematical physics*. AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI, 1999, 475–486.

[Tel14] C. Teleman. Gauge theory and mirror symmetry. 2014. arXiv: `1404.6305 [math-ph]`.

[Tho98] G. Thompson. On the generalized Casson invariant. *Adv. Theor. Math. Phys.* **3** (1998), 249–280.

[Tie+12] L. Tiemann, G. Gamez, N. Kumada, and K. Muraki. Unraveling the spin polarization of the $\nu = 5/2$ fractional quantum Hall state. *Science* **335**.6070 (2012), 828–831.

[Ton04a] D. Tong. A quantum Hall fluid of vortices. *J. High Energy Phys.* **2004**.02 (2004), 046.

[Ton04b] D. Tong. Monopoles in the Higgs phase. *Phys. Rev. D* **69**.6 (2004), 065003.

[Ton05] D. Tong. TASI lectures on solitons. 2005. arXiv: `hep-th/0509216 [hep-th]`.

[Ton16] D. Tong. Lectures on the quantum Hall effect. 2016. arXiv: `1606.06687 [hep-th]`.

[TSG82] D. C. Tsui, H. L. Stormer, and A. C. Gossard. Two-dimensional magnetotransport in the extreme quantum limit. *Phys. Rev. Lett.* **48**.22 (1982), 1559–1562.

[TT15a] D. Tong and C. Turner. Quantum Hall effect in supersymmetric Chern–Simons theories. *Phys. Rev. B* **92**.23 (2015), 235125.

[TT15b] D. Tong and C. Turner. Vortices and monopoles in a harmonic trap. *J. High Energy Phys.* **2015**.12 (2015), 098.

[Tur17] C. Turner. *BPS approaches to anyons, quantum Hall states and quantum gravity*. PhD thesis. 2017.

[Tur20] C. Turner. The quantum origins of non-topological vortices. *J. High Energy Phys.* **2020**.3 (2020), 189.

[Wal20] E. Walton. On the geometry of magnetic Skyrmions on thin films. *J. Geom. Phys.* **156** (2020), 103802.

[Wal21a] E. Walton. Exotic vortices and twisted holomorphic maps. 2021. arXiv: `2108.00315 [math-ph]`.

[Wal21b] E. Walton. Vortex counting and the quantum Hall effect. 2021. arXiv: `2107.14190 [hep-th]`.

[Weh08] J. Wehrheim. *Vortex invariants and toric manifolds.* PhD thesis. 2008.

[Wen02] X.-G. Wen. Quantum orders and symmetric spin liquids. *Phys. Rev. B* **65**.16 (2002), 165113.

[Wil82] F. Wilczek. Quantum mechanics of fractional-spin particles. *Phys. Rev. Lett.* **49**.14 (1982), 957–959.

[Wil+87] R. L. Willett, J. P. Eisenstein, H. L. Störmer, D. C. Tsui, A. C. Gossard, and J. H. English. Observation of an even-denominator quantum number in the fractional quantum Hall effect. *Phys. Rev. Lett.* **59**.15 (1987), 1776–1779.

[Wil+93] R. L. Willett, R. R. Ruel, K. W. West, and L. N. Pfeiffer. Experimental demonstration of a Fermi surface at one-half filling of the lowest Landau level. *Phys. Rev. Lett.* **71**.23 (1993), 3846–3849.

[Wit77] E. Witten. Some exact multipseudoparticle solutions of classical Yang–Mills theory. *Phys. Rev. Lett.* **38**.3 (1977), 121–124.

[Wit86] E. Witten. Interacting field theory of open superstrings. *Nucl. Phys. B* **276**.2 (1986), 291–324.

[Wit88a] E. Witten. Topological quantum field theory. *Commun. Math. Phys.* **117** (1988), 353.

[Wit88b] E. Witten. Topological sigma models. *Commun. Math. Phys.* **118**.3 (1988), 411–449.

[Wit89] E. Witten. Quantum field theory and the Jones polynomial. *Commun. Math. Phys.* **121** (1989), 351–399.

[Wit92] E. Witten. Two-dimensional gauge theories revisited. *J. Geom. Phys.* **9**.4 (1992), 303–368.

[Wit93] E. Witten. Quantum background independence in string theory. *Conference on Highlights of Particle and Condensed Matter Physics (SALAMFEST)*. 1993, 0257–275.

[Wit94] E. Witten. Monopoles and four-manifolds. *Math. Res. Lett.* **1** (1994), 769–796.

[Wit95] E. Witten. On S-duality in Abelian gauge theory. *Selecta Math.* **1** (1995), 383.

[Wit98] E. Witten. Mirror manifolds and topological field theory. *Mirror symmetry I*. Ed. by S.-T. Yau. AMS/IP Studies in Advanced Mathematics. American Mathematical Society, Providence, RI, 1998, 121–160.

[Woo92] N. M. J. Woodhouse. *Geometric quantization.* Second edition. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.

[Wu94] Y.-S. Wu. Statistical distribution for generalized ideal gas of fractional-statistics particles. *Phys. Rev. Lett.* **73**.7 (1994), 922–925.

[WZ92] X. G. Wen and A. Zee. Shift and spin vector: New topological quantum numbers for the hall fluids. *Phys. Rev. Lett.* **69** (6 1992), 953–956.

[Xu16] G. Xu. Gauged Hamiltonian Floer homology I: Definition of the Floer homology groups. *Trans. Amer. Math. Soc.* **368**.4 (2016), 2967–3015.

[Yos11] Y. Yoshida. Localization of vortex partition functions in $\mathcal{N} = (2,2)$ super Yang–Mills theory. 2011. arXiv: `1101.0872 [hep-th]`.

[Zha+16] X.-X. Zhang, A. S. Mishchenko, G. De Filippis, and N. Nagaosa. Electric transport in three-dimensional Skyrmion/monopole crystal. *Phys. Rev. B* **94**.17 (2016), 174428.

[Zhe+18] F. Zheng, F. N. Rybakov, A. B. Borisov, D. Song, S. Wang, Z.-A. Li, H. Du, N. S. Kiselev, J. Caron, A. Kovács, M. Tian, Y. Zhang, S. Blügel, and R. E. Dunin-Borkowski. Experimental observation of chiral magnetic bobbers in B20-type FeGe. *Nat. Nanotechnol.* **13**.6 (2018), 451–455.