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# Random Scattering by Rough Surfaces with Spatially Varying Impedance

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**Abstract**—A method is given for evaluating electromagnetic scattering by an irregular surface with spatially-varying impedance. This uses an operator expansion with respect to impedance variation and allows examination of its effects and the resulting modification of the field scattered by the rough surface. For a fixed rough surface and randomly varying impedance, expressions are derived for the scattered field itself, and for the coherent field with respect to impedance variation for both flat and rough surfaces in the form of effective impedance conditions.

## 1. INTRODUCTION

Many applications of wave scattering from rough surfaces are complicated by the involvement of further scattering mechanisms [1–6]. Radar propagating over a sea surface, for example, may encounter spatially varying impedance due to surface inhomogeneities [7–9], or refractive index variations in the evaporation duct [10, 11]. This is an even greater problem in remote sensing over forest or urban terrain [3, 12]. Roughness is often the dominant feature, but impedance variation may produce further multiple scattering. The great majority of theoretical and numerical studies nevertheless treat such effects in isolation [1, 2, 13–15]. Of particular note are the elegant studies of admittance variation by [1], which obtain analytical solutions by applying Bourret approximation to a Dyson equation, and of impedance variation by [2] which derive intensity fluctuation statistics. Experimental validation of scattering models in complex environments remains a major difficulty, exacerbated by the lack of detailed environmental information, and it is therefore crucial to distinguish and identify sources of scattering. In addition, while numerical computation in these cases may be feasible for the perfectly reflecting surface, it can become prohibitive for more complex environments, particularly in seeking statistics from multiple realisations.

These considerations are the motivation for this paper. The main purpose is to provide an efficient means to evaluate the effect of impedance variation and its interaction with surface roughness; in addition, we derive descriptions of the resulting coherent or mean field (averaged with respect to impedance variation) for an irregular surface. (For random surfaces the field may be averaged further with respect to the rough surface in special cases, although this will be tackled more fully in a later paper and is only sketched here.) In order to do this, an operator expansion is used: Surface currents from which scattered fields are determined are expressed as the solution of an integral equation, in which the effect of impedance variation is separated from the mean impedance. The solution is written in terms of the inverse of the governing integral operator, and provided impedance variation about its mean is moderate. This inversion can be expanded about the leading term. This is carried out here for 2-d problems, for a TE incident field. For the coherent field, this also leads to expressions for equivalent effective impedance conditions.

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The paper is organised as follows: Governing equations are set out in Section 2, and the operator expansion is given in Section 3. Section 4 gives mean field with respect to impedance variation for a fixed rough surface. The procedure for extending averages over randomly rough surfaces is briefly outlined. Some remarks are given in 5 regarding the generalisation to TM and to the fully 3-dimensional case. The work here is based in part on results originally presented in [16].

## 2. GOVERNING EQUATIONS

Consider the wavefield above a rough surface with varying impedance in a 2-dimensional medium, with coordinates $(x, z)$ where $x$ is the horizontal and $z$ the vertical, directed upwards. The incident electric field $E$ is assumed to be time-harmonic, with time-dependence $\exp(-i\omega t)$, say, and can be taken to be either horizontally (TE) or vertically (TM) plane polarized. We can suppress the time-dependence and consider the time-reduced component, and for the moment will restrict attention to an incident TE field. Denote the surface profile by $\zeta(x)$, with impedance $Z = Z_0 + Z_r$ where $Z_0$ is a constant reference value, and $Z_r$ is spatially-varying.

The variation of $Z_r$ is due to varying (known) material properties in the adjacent medium or along the boundary. When ensemble averages are taken, it will be assumed that $Z_r$ is continuous and statistically stationary in $x$, with mean zero and scaled variance $\langle (Z_r/Z_0)^2\rangle = \sigma_I^2$. It will also be assumed that $Z_r$ is not large compared with $Z_0$, in the sense that the root mean square of its modulus is less than $|Z_0|$. Consequently $\sigma_I < 1$. This corresponds to a relatively high-contrast interface. For convenience we assume that $Z_r$ is integrable and essentially bounded and therefore has an $L^2$ norm, and $\zeta(x)$ has continuous first derivative.

We treat the surface $\zeta(x)$ as being random and will assume that it has mean zero and is statistically stationary, and we denote its variance by $\sigma_S^2$ and its autocorrelation function by $\rho(\xi)$, where $\xi$ is the spatial separation. Thus the mean surface plane lies in $z = 0$. We will also assume that the surface and impedance functions are independent.

Here and below, single angled brackets $\langle\cdot\rangle$, or for compactness an overbar, denote ensemble averages with respect to impedance variation. Ensemble averages with respect to both impedance variation and randomly rough surface may be denoted by double angled-brackets $\langle\langle\cdot\rangle\rangle$.

The field $E$ in the upper medium obeys the Helmholtz wave equation $(\nabla^2 + k^2)E = 0$ where $k$ is the wavenumber. Denote by $G$ the free space Green's function, so that (in the 2-dimensional case) $G$ is the zero order Hankel function of the first kind,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4i} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}'|). \tag{1}$$

The total field $E$ along the surface is then given by the solution of a Helmholtz integral equation (see also [1, 17]) as follows:

$$E_{inc}(\mathbf{r}_s) = \frac{1}{2}E(\mathbf{r}_s) - \int_{z=\zeta(x)} \left[\frac{\partial G(\mathbf{r}_s, \mathbf{r}')}{\partial n} + \frac{ik_0 G(\mathbf{r}_s, \mathbf{r}')}{Z_0 + Z_r(x')}\right] E(\mathbf{r}')dS'. \tag{2}$$

where $\mathbf{r}_s$ is an arbitrary surface point $(x, \zeta(x))$, and $\mathbf{r}' = (x', \zeta(x'))$. Elsewhere in the upper half space the field can be written as a boundary integral:

$$E(\mathbf{r}) = \int_{z=\zeta(x)} \left[\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} + \frac{ik_0 G(\mathbf{r}, \mathbf{r}')}{Z_0 + Z_r(x')}\right] E(\mathbf{r}')dS'. \tag{3}$$

where now $\mathbf{r} = (x, z)$ represents a general point in the upper medium. (The right-hand-side of Equation (2) is an operator from functions on the real line to itself, and the same holds for Eq. (3) if $\mathbf{r}$ is, for example, restricted to a line at fixed $z$ parallel to $x$.).

## 3. ROUGH SURFACE WITH VARYING IMPEDANCE

### 3.1. General Case

We first derive the operator expansion for the general case of an irregular variable impedance boundary and will later deal with the special case of a flat variable-impedance boundary. This has been studied by many authors in various parameter regimes. Analytical treatment for the statistical averages will be discussed in the subsequent section.

We first write

$$\frac{1}{Z_0 + Z_r} \equiv \frac{1}{Z_0} - \frac{Z_r}{Z_0(Z_0 + Z_r)}. \tag{4}$$

For a rough surface $z = \zeta(x)$ with impedance $Z = Z_0 + Z_r(x)$ integral Equation (2) then becomes

$$E_{inc}(\mathbf{r}) = (\mathcal{C}_0 + \mathcal{C}_1)E(\mathbf{r}), \tag{5}$$

where

$$\mathcal{C}_0(\cdot) = \frac{1}{2}(\cdot) - \int_{z=\zeta(x)} \left[ \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} + \frac{ik_0 G(\mathbf{r}, \mathbf{r}')}{Z_0} \right] (\cdot) dS'. \tag{6}$$

and $\mathcal{C}_1$ contains the dependence on impedance variation $Z_r$,

$$\mathcal{C}_1(\cdot) = \frac{ik_0}{Z_0} \int_{z=\zeta(x)} \frac{Z_r(x')G(\mathbf{r}, \mathbf{r}')}{Z_0 + Z_r(x')} (\cdot) dS'. \tag{7}$$

Even when the impedance is constant, so that $\mathcal{C}_1 = 0$, there is no closed-form analytical solution, and in general for individual realisation $\mathcal{C}_0^{-1} E_{inc}(\mathbf{r})$ must be evaluated numerically.

The solution of Eq. (5) can be written

$$E(\mathbf{r}) = (\mathcal{C}_0 + \mathcal{C}_1)^{-1} E_{inc}(\mathbf{r}). \tag{8}$$

The inverse can be formally expanded to give

$$(\mathcal{C}_0 + \mathcal{C}_1)^{-1} \equiv \mathcal{C}_0{}^{-1} - (\mathcal{C}_0{}^{-1}\mathcal{C}_1)\mathcal{C}_0{}^{-1} + (\mathcal{C}_0{}^{-1}\mathcal{C}_1)^2 \mathcal{C}_0{}^{-1} - \ldots \tag{9}$$

The expansion applies to arbitrary operators $\mathcal{C}_0, \mathcal{C}_1$, provided that $\mathcal{C}_0^{-1}$ exists, which holds in our case provided that the nonrandom problem has a solution. Since, by assumption, the effect of the term $\mathcal{C}_1$ is not large, by a suitable choice of the variance of $Z_r$, we can enforce that $||\mathcal{C}_1||/||\mathcal{C}_0|| < 1$. This ensures that the series converges uniformly, and the resulting equation may be truncated to obtain an approximation to the field $E(\mathbf{r})$ along the surface:

$$E(\mathbf{r}) \cong \mathcal{C}_0^{-1} E_{inc}(\mathbf{r}) - \mathcal{C}_0^{-1} \left[ \mathcal{C}_1 \mathcal{C}_0^{-1} E_{inc}(\mathbf{r}) \right]. \tag{10}$$

The first term $\mathcal{C}_0^{-1} E_{inc}$ in this expression corresponds to constant impedance $Z_0$, but in general it is non-specular due to the irregular surface. The second term accounts for the diffraction arising from interaction between impedance variation $Z_r$ and surface profile $\zeta$. Once the first term has been obtained, the remaining term is evaluated by applying $\mathcal{C}_1$ and solving again for $\mathcal{C}_0^{-1}$, with the term in square brackets acting as a new driving field. From this, the field away from the surface is obtained from boundary integral Equation (3).

Note that the requirement for uniform (i.e., norm) convergence of Eq. (9) is unnecessarily strong. In practice, we have found that even for large $\mathcal{C}_1$ the truncated series may closely approximate the exact solution. In series of this type semi-convergence may be observed, in which the first two terms of Equation (9) closely reconstruct the exact solution, but later terms diverge.

This formulation conveniently captures the balance between scattering mechanisms, and in important cases is efficient for numerical calculation of the field statistics with respect to impedance variation, as well as allowing theoretical estimates of the field statistics to be obtained. For a single realisation of $\zeta(x)$ and $Z(x)$, numerical evaluation is generally needed. Inversion of the integral Equation (5) is highly costly computationally. However, in several important regimes including low grazing angles highly efficient methods are available (e.g., [18–21]) which cannot be applied directly to the full integral Equation (5). In addition, Equation (10) allows analytical treatment in special cases for the mean field due to a random impedance and either a fixed surface or a randomly rough surface. In such cases, this approach may provide a considerable computational advantage over brute-force calculations, especially since it allows ensemble averaging. Such advantages are potentially much greater in 3-dimensional settings to be addressed in subsequent work.

Figure 1 compares the surface field term $-\mathcal{C}_0^{-1}\mathcal{C}_1\mathcal{C}_0^{-1}E_{inc}(\mathbf{r})$ with the corresponding component of the 'exact' numerical solution of Eq. (5). Here the angle of incidence (with respect to the normal)

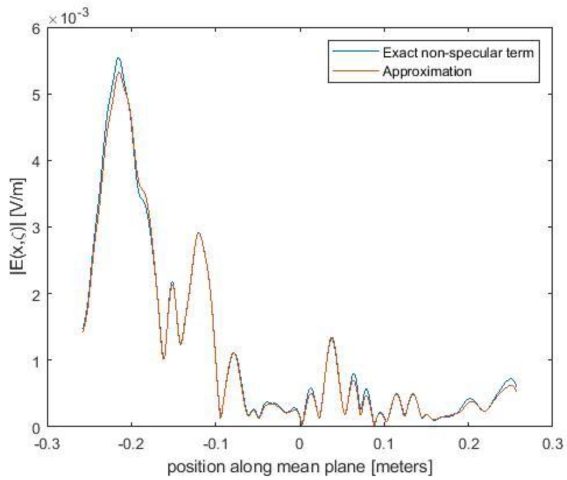


**Figure 1.** Comparison of exact numerical solution and approximation for scattered field on rough profile $z = \zeta$ with varying impedance $Z = Z_0 + Z_r$ for TE polarisation.

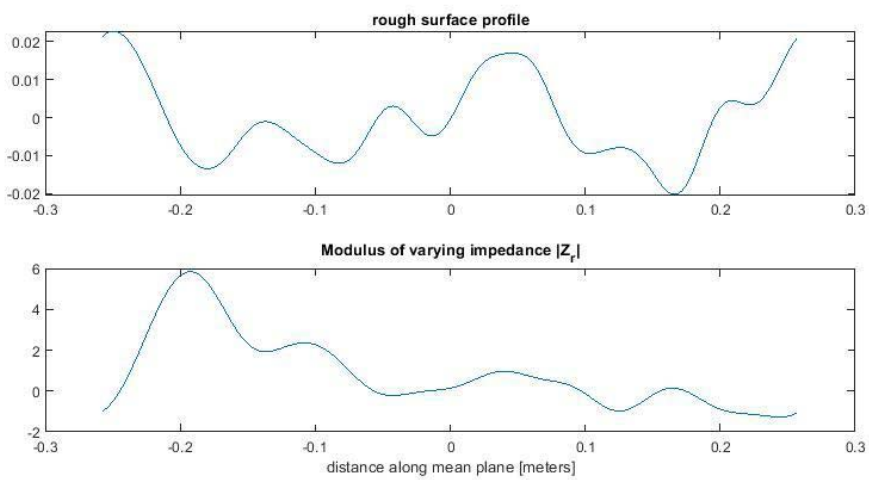


**Figure 2.** The rough surface $z = \zeta$ (upper curve) and the impedance perturbation $|Z_r|$ (lower curve) applied in the example Figure 1.

is around 5°; the ratio of r.m.s. surface height to wavelength $\langle\zeta^2\rangle^{1/2}/\lambda = 2/3$, and the ratio of r.m.s. impedance variation to reference value $Z_0$ is around 1/6. Agreement is seen to be very close. The surface profile and the modukus $|Zr|$ of the varying component of impedance for this example are shown in Figure 2.

### 3.2. Plane Boundary with Varying Impedance

We now consider the special case of a planar surface $\zeta(x) \equiv 0$ with variable impedance, using the above operator expansion. We should mention here the elegant method of [1] and that of [2] which also considers ensemble averages and could be alternatively employed. To simplify notation we will denote the operators in this case by $\mathcal{A}_0$ and $\mathcal{A}_1$ so that Equation (2) becomes

$$E_{inc}(\mathbf{r}) = (\mathcal{A}_0 + \mathcal{A}_1)E(\mathbf{r}), \tag{11}$$

where $\mathbf{r}$ lies on the surface, and $\mathcal{A}_0$ and $\mathcal{A}_1$ are now given by

$$\mathcal{A}_0(\cdot) = \frac{1}{2}(\cdot) - \int_{z=0} \left[\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial z} + \frac{ik_0 G(\mathbf{r}, \mathbf{r}')}{Z_0}\right](\cdot)dS' \tag{12}$$

and

$$\mathcal{A}_1(\cdot) = \frac{ik_0}{Z_0} \int_{z=0} \frac{Z_r(x')G(\mathbf{r}, \mathbf{r}')}{Z_0 + Z_r(x')} (\cdot) dS'. \tag{13}$$

Equation (10) then becomes

$$E(\mathbf{r}) \cong \mathcal{A}_0^{-1} E_{inc}(\mathbf{r}) - \mathcal{A}_0^{-1} \left[ \mathcal{A}_1 \mathcal{A}_0^{-1} E_{inc}(\mathbf{r}) \right]. \tag{14}$$

The solution to Eq. (11) represents the total field at $z = 0$; from this the field elsewhere can be obtained by writing $E(\mathbf{r})$ as a superposition of plane waves without recourse to the integral Equation (3).

Suppose for the moment that the impedance is constant, $Z = Z_0$, so that $\mathcal{A}_1$ vanishes. For an incident plane wave, say $E_\theta(x, z) = \exp(ik[\sin\theta x - \cos\theta z])$ at an angle $\theta$ with respect to the normal, the solution is explicitly

$$\mathcal{A}_0^{-1} E_\theta(x, 0) = [1 + R(\alpha)] \exp(i\alpha x), \tag{15}$$

where $\alpha = k \sin\theta$, $\beta = \sqrt{k^2 - \alpha^2}$, and $R$ is the reflection coefficient

$$R(\alpha) = \frac{\beta Z_0 - k_0}{\beta Z_0 + k_0}. \tag{16}$$

Thus $\mathcal{A}_0^{-1} f$ can be found for *arbitrary* $f(x)$ by expressing $f$ as a superposition of plane waves and applying Eq. (15). If impedance variation $Z(x) = Z_0 + Z_r(x)$ is now reintroduced, then Eq. (11) has formal solution

$$E(\mathbf{r}) = (\mathcal{A}_0 + \mathcal{A}_1)^{-1} E_{inc}(\mathbf{r}). \tag{17}$$

The first term on the right of Equation (10) is the known specular reflection from a constant impedance surface at $z = 0$; the second models its diffuse modification due to $Z_r$, i.e., diffraction effects due to impedance variation.

Specifically, from Eq.s (15) and (13) we obtain

$$\mathcal{A}_1(\mathcal{A}_0^{-1} E_{inc}(\mathbf{r})) = \frac{ik_0(1 + R(\alpha))}{Z_0} \int_{z=0} \frac{Z_r(x')G(\mathbf{r}, \mathbf{r}')}{Z_0 + Z_r(x')} e^{i\alpha x'} dS'. \tag{18}$$

As $\mathcal{A}_0^{-1}$ represents reflection by constant impedance, Eq. (18) can be thought of as a secondary 'driving field' for the diffuse term in Eq. (10). This field consists of a set of plane waves determined by the Fourier transform of the integral in Eq. (18). An example is shown in Figure 3 comparing this term in Eq. (10) with the diffuse part of the exact numerical solution. $Z_r$ here has an rms value of around $Z_0/4$. (Note that in these simulations the incident field has been tapered to zero at the edges to minimise spurious edge-effects.).

As the solution of $\mathcal{A}_0^{-1}$ is known Equation (10) can be evaluated directly, for one or many realisations, and avoids a potentially expensive numerical inversion.

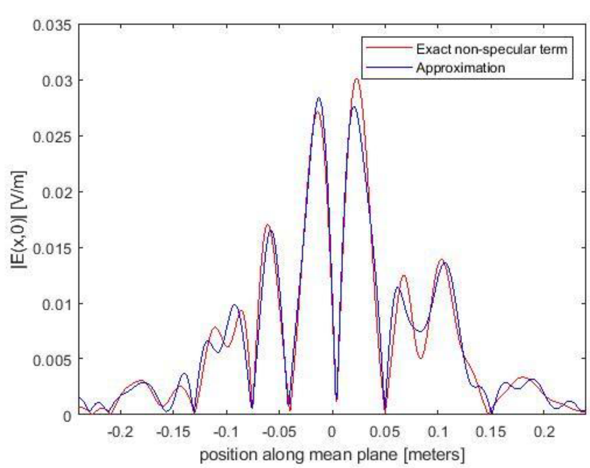


**Figure 3.** Comparison of exact and approximate solutions for non-specular component of scattered field for a flat surface with varying impedance.

## 4. COHERENT FIELD

In this section, we consider statistics of the scattered field (a) when the average is taken over the ensemble of varying impedance functions, and the profile is deterministic but arbitrary, and (b) the more general case of a randomly rough surface, taking the average over both impedance and surface profiles.

As the impedance is often known statistically rather than individually, evaluation of the mean field is important. For *flat* surfaces, the mean field with respect to an ensemble of impedance realisations obeys an effective impedance condition, for which an approximation is derived in Section 4.1. Thus for an incident plane wave, the mean scattered field is specular, but with an 'effective reflection coefficient' depending on incident angle. For a given *rough* surface, the coherent field is no longer specular, and its description is therefore more complex.

### 4.1. Mean Field for Plane Boundary

We first consider the coherent field for a flat varying-impedance surface, §3.2. A low-order approximation for the mean field due to scattering by the randomly varying impedance is easily derived from the expansion (10) in this case. As $Z_r(x)$ is statistically stationary, the coherent field is specular and takes the form of a constant effective impedance whose value we seek. Averaging Equation (14) gives the mean field at the surface

$$\langle E(\mathbf{r})\rangle \cong \mathcal{A}_0^{-1}E_{inc}(\mathbf{r}) - \mathcal{A}_0^{-1}\langle\mathcal{A}_1\rangle\mathcal{A}_0^{-1}E_{inc}(\mathbf{r}) \tag{19}$$

as the term $\mathcal{A}_0^{-1}E_{inc}(\mathbf{r})$ on which $\mathcal{A}_1$ acts is independent of $Z_r$. From Equation (13), $\langle\mathcal{A}_1\rangle$ is given by

$$\langle\mathcal{A}_1(\cdot)\rangle = \frac{ik_0}{Z_0}\int_{z=0}\Lambda G(\mathbf{r},\mathbf{r}')(\cdot)dS' = \Lambda\frac{ik_0}{Z_0}\int_{z=0}G(\mathbf{r},\mathbf{r}')(\cdot)dS' \tag{20}$$

where

$$\Lambda = \left\langle\frac{Z_r}{Z_0+Z_r}\right\rangle = 1 - Z_0\left\langle\frac{1}{Z_0+Z_r}\right\rangle \tag{21}$$

Note that this quantity is a one-point average, which does not depend on the impedance autocorrelation $\rho(\xi) = \langle Z_r(x)Z_r(x+\xi)\rangle$. This scalar can be found analytically for a wide range of distributions, and in any case numerical averaging is straightforward and rapid for arbitrary statistics. For analytical evaluation, $Z_r$ is most commonly assumed to obey a modified form of complex Gaussian distribution. If the distribution is *exactly* Gaussian, then the probability integral has a pole at $Z_r = -Z_0$, but also a well-defined Cauchy principal value, and can be obtained analytically. However, the singularity at $-Z_0$ corresponds to vanishing impedance which may be excluded on physical grounds, and the distribution can be replaced by a Gaussian with cut-off.

Alternatively, to fourth order in the ratio $Z_r/Z_0$, $\Lambda$ can be written

$$\Lambda \cong \left\langle Z_r/Z_0 - Z_r^2/Z_0^2 + Z_r^3/Z_0^3 - Z_r^4/Z_0^4\right\rangle \tag{22}$$

giving a simple high-contrast second order approximation:

$$\Lambda \cong \left\langle\frac{Z_r^2}{Z_0^2}\right\rangle = -\sigma_I^2 \tag{23}$$

valid for any distribution, or a fourth order approximation for the Gaussian case:

$$\Lambda \cong -\left\langle\frac{Z_r^2}{Z_0^2}\right\rangle = -\sigma_I^2 + 3\sigma^4 \tag{24}$$

In any case, when Equations (20) and (21) are substituted back into Equation (19), it is easily seen that the mean field is equivalent to a solution of the original problem with a modified or 'effective' constant impedance $Z_e = Z_0 + Z_{mod}$, with $Z_{mod}$ given by $Z_{mod} = Z_0\Lambda/(1-\Lambda)$. This immediately gives an effective reflection coefficient

$$R_e(\alpha) = \frac{\beta Z_e - k_0}{\beta Z_e + k_0} \tag{25}$$

where

$$Z_e = \frac{Z_0}{1-\Lambda}. \tag{26}$$

### 4.2. Mean Field for a Rough Surface

At a point $\mathbf{r}$ along a given horizontal line above the surface, the field is related to the surface values via the integral Equation (3), which is written

$$E(\mathbf{r}) = \mathcal{C}'E \tag{27}$$

where $\mathcal{C}'$ is the integral operator in Eq. (3), and the prime is simply to distinguish the operator $\mathcal{C}$ evaluated away from the surface from its value along the surface as occurring in Eq. (2). If $\mathcal{C}'$ is split as before into its constant and varying impedance parts $\mathcal{C}'_0$ and $\mathcal{C}'_1$, then, using Equation (10), Equation (27) can be written as

$$\begin{aligned} E &= \mathcal{C}'\mathcal{C}^{-1}E_{inc} \\ &= (\mathcal{C}'_0 + \mathcal{C}'_1)(\mathcal{C}_0 + \mathcal{C}_1)^{-1}E_{inc} \\ &\cong (\mathcal{C}'_0 + \mathcal{C}'_1)(\mathcal{C}_0^{-1} - \mathcal{C}_0^{-1}\mathcal{C}_1\mathcal{C}_0^{-1})E_{inc} \\ &\cong \left[\mathcal{C}'_0\mathcal{C}_0^{-1} - \mathcal{C}'_0\mathcal{C}_0^{-1}\mathcal{C}_1\mathcal{C}_0^{-1} + \mathcal{C}'_1\mathcal{C}_0^{-1})\right] E_{inc} \end{aligned} \tag{28}$$

where we have neglected a term of higher order in $\mathcal{C}_1$. We can now take an ensemble average of Eq. (28) with respect to impedance variation, to get the mean modification by impedance variation of the scattered fields.

$$\langle E(\mathbf{r})\rangle \cong \left[\mathcal{C}'_0\mathcal{C}_0^{-1} + \overline{\mathcal{C}'_1}\mathcal{C}_0^{-1} - \mathcal{C}'_0\mathcal{C}_0^{-1}\overline{\mathcal{C}_1}\mathcal{C}_0^{-1})\right] E_{inc} \tag{29}$$

where $\mathbf{r}$ is in the medium, and for compactness $\overline{\mathcal{C}_1}$ denotes the mean $\langle\mathcal{C}_1\rangle$ and thus

$$\overline{\mathcal{C}_1}(\cdot) = \frac{ik_0}{Z_0}\int_{z=\zeta(x)} \Lambda G(\mathbf{r},\mathbf{r}')(\cdot)dS' = \Lambda\mathcal{C}_1, \tag{30}$$

$\overline{\mathcal{C}'_1}$ is defined similarly, and $\Lambda$ is given by Equation (21).

Expression (29) gives the mean field for an arbitrary irregular surface with randomly varying impedance, but as $\overline{\mathcal{C}_1}$, $\overline{\mathcal{C}'_1}$ depend on the surface profile $\zeta(x)$, numerical evaluation cannot be avoided in general. In particular, this gives rise to a coherent field spectrum with effective coefficients depending on the surface profile. The approximation in Eq. (29) is equivalent to the solution $E_e$, say, for scattering by the surface $\zeta(x)$ but with constant effective impedance $Z_e$. This is easily seen by formulating this equivalent problem in terms of integral operators where it becomes

$$E_e(\mathbf{r}) = (\mathcal{C}'_0 + \Lambda\mathcal{C}')(\mathcal{C}_0 + \Lambda\mathcal{C}_1)^{-1}E_{inc}, \tag{31}$$

and then solving as before and comparing terms with Eq. (29). In terms of the effective field evaluated along the surface which we can denote $E_{se}$, Eq. (31) becomes

$$E_e(\mathbf{r}) = (\mathcal{C}'_0 + \Lambda\mathcal{C}')E_{se} \tag{32}$$

In other words, the effective field $E_e$, i.e., average over impedance realisations, is the solution to the boundary problem given by Equations (2) and (3) with varying impedance replaced by effective impedance $Z_e$:

$$E(\mathbf{r}) = \int_{z=\zeta(x)} \left[\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n} + \frac{ik_0 G(\mathbf{r},\mathbf{r}')}{Z_e}\right] E(\mathbf{r}')dS'. \tag{33}$$

### 4.3. TM Case

The results above apply to a TE incident field. It is straightforward to derive equivalent results for TM incidence as follows. Integral Equations (2) and (3) for $E$ are replaced by the following equations for the field $H$:

$$H_{inc}(\mathbf{r}_s) = \frac{1}{2}H(\mathbf{r}_s) - \int_{z=\zeta(x)} \left[\frac{\partial G(\mathbf{r}_s,\mathbf{r}')}{\partial n} + ik_0 G(\mathbf{r}_s,\mathbf{r}')(Z_0 + Z_r(x'))\right] H(\mathbf{r}')dS'. \tag{34}$$

where $\mathbf{r}_s$ is an arbitrary surface point $(x, \zeta(x))$, and $\mathbf{r}' = (x', \zeta(x'))$. Elsewhere the field can be written as a boundary integral:

$$H(\mathbf{r}) = \int_{z=\zeta(x)} \left[\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n} + ik_0 G(\mathbf{r},\mathbf{r}')(Z_0 + Z_r(x'))\right] H(\mathbf{r}')dS'. \tag{35}$$

Following analogous reasoning we obtain

$$H(\mathbf{r}) \cong \mathcal{D}_0^{-1} H_{inc}(\mathbf{r}) - \mathcal{D}_0^{-1} \left[\mathcal{D}_1 \mathcal{D}_0^{-1} H_{inc}(\mathbf{r})\right]. \tag{36}$$

where now

$$\mathcal{D}_0(\cdot) = \frac{1}{2}(\cdot) - \int_{z=\zeta(x)} \left[\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n} + ik_0 Z_0 G(\mathbf{r},\mathbf{r}')\right](\cdot)dS'. \tag{37}$$

$$\mathcal{D}_1(\cdot) = -ik \int_{z=\zeta(x)} Z_r(x') G(\mathbf{r},\mathbf{r}')(\cdot)dS'. \tag{38}$$

It is immediately clear, however, that when taking the mean with respect to impedance variation the term $\langle\mathcal{D}_1\rangle$ vanishes so that to this order the effective impedance coincides with $Z_0$. Thus $\mathcal{D}_1$ has no effect on $\langle H\rangle$. The effect on the autocorrelation of $H$ and therefore on mean intensity will be non-zero. However, in order to calculate this we need to include a further term $\mathcal{D}_0^{-1}[\mathcal{D}_1\mathcal{D}_0^{-1}]^2$ in the operator expansion of Eq. (36); taking the autocorrelation of $H$ yields two stochastic double integrals, evaluation of which is feasible but is beyond the scope of the present study.

### 4.4. Averaging over Rough Surfaces

Provided that the surface profile and impedance are statistically independent, the above results in Eq. (32) or (33) can be used to examine the double average $\langle\langle E\rangle\rangle$ with respect to rough surface and impedance variation. This may be done using results existing in various regimes, which we will not reproduce in detail here. To illustrate this, consider an incident plane wave at the angle of $\theta$ for small surface height $\sigma_S^2 < 1$. Perturbation theory to the first order in surface height can be applied along the lines of [15]. This allows the first order (in $\sigma_S$) component to be written as $\zeta(x)u(\theta, Z_e)$ where $u(\theta, Z_e)$ is a known function of incident angle and effective impedance.

From this the field everywhere can be expressed in the standard way in terms of the spectral components via the Fourier transform of $\zeta$, with the function $u(\theta, Z_e)$ present as a multiplying factor. Using this we can obtain coherent field and field correlation within the small surface height regime. Similarly, the mean field for low grazing angle incident waves may be obtained by extending results such as [19] although these require further development.

## 5. CONCLUSIONS

Wave scattering by a rough surface with random spatially varying impedance has been considered. We have sought an efficient method for calculating the field while allowing convenient estimation of the effects of impedance variation and its interaction with the surface profile.

The expressions obtained also provide estimates of the mean field with respect to impedance variation. For rough surfaces these are semi-analytical in the sense that numerical evaluation of integrals is needed. (In the case of a flat surface, for which the coherent field is specular, this takes the form of an effective impedance; this is also approximately true for a given irregular surface, but the behaviour is more complicated because of the non-specular nature of the scattered field.).

For simplicity we have restricted attention to 2-dimensional problems, but the extension to full 3-dimensional scattering is straightforward. Similarly, equivalent results for a TM field are easily obtained, and the acoustic case follows immediately. A further question which is not addressed here is of the coherent field which results from the ensemble of randomly rough surfaces with varying impedance. A key difficulty is that typically when this occurs in practice the roughness and impedance are not statistically independent. We note that the impedance condition is a local approximation which breaks down for high order multiple scattering. The approach here can in principle be extended to nonlocal impedance by allowing the operator $\mathcal{C}_1$ in Equation (7) to depend implicitly on $E$ and treating the resulting implicit integral equation.

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