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Geometric general solution to the U(1) anomaly equations

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Abstract: Costa et al. [*Phys. Rev. Lett.* **123** (2019) 151601] recently gave a general solution to the anomaly equations for n charges in a U(1) gauge theory. 'Primitive' solutions of chiral fermion charges were parameterised and it was shown how operations performed upon them (concatenation with other primitive solutions and with vector-like solutions) yield the general solution. We show that the ingenious methods used there have a simple geometric interpretation, corresponding to elementary constructions in number theory. Viewing them in this context allows the fully general solution to be written down directly, without the need for further operations. Our geometric method also allows us to show that the only operation Costa et al. require is permutation. It also gives a variety of other, qualitatively similar, parameterisations of the general solution, as well as a qualitatively different (and arguably simpler) form of the general solution for n even.

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Contents

1 Introduction **1**

2 Geometric method **2**
2.1 The method of chords 3
2.2 Application for $n = 4$ 4
2.3 Arbitrary $n \geq 4$ 5
2.4 Comparison with CDF 6

3 Discussion **7**

A Any solution via permutations **8**
A.1 Even $n \geq 4$ 8
A.2 Odd $n \geq 4$ 9

B Alternative solution for n-even **9**

1 Introduction

The local anomaly cancellation equations for a U(1) gauge theory with n left-handed chiral fermions of charge z_i, which may be taken to be integers, are

$$\sum_{i=1}^{n} z_i = 0, \tag{1.1}$$

$$\sum_{i=1}^{n} z_i^3 = 0. \tag{1.2}$$

The first of these, (1.1), comes from a one-loop triangle diagram with two external gravitons and one external U(1) gauge boson [1], whilst (1.2) comes from the similar diagram with three external U(1) gauge bosons [2–6]. Although written for left-handed chiral fermions, these equations are general for a theory with both left-handed and right-handed chiral fermions since we can charge conjugate any right-handed representation, reversing the sign of its charge and giving a left-handed representation. Eq. (1.2) is a cubic diophantine equation in n variables; since it is not yet known how to solve a generic such equation even in 2 variables (corresponding to an elliptic curve [7]), one might expect that finding the general solution to (1.1)–(1.2) is a difficult problem. However, a recent paper by Costa, Dobrescu and Fox (CDF) [8] managed to do so, in the following way.

JHEP05(2020)065

CDF observed that given two integer solutions $\underline{x} := (x_1, \ldots, x_n)$ and $\underline{y} := (y_1, \ldots, y_n)$, of (1.1), and (1.2), a third could be constructed from a 'merger' operation, which they denoted '$\oplus$'

$$\underline{x} \oplus \underline{y} := \left(\sum_{i=1}^{n} x_i y_i^2\right) \underline{x} - \left(\sum_{i=1}^{n} x_i^2 y_i\right) \underline{y}. \tag{1.3}$$

Some solutions to (1.1) and (1.2) are easy to find, having for each charge z_i another charge $z_j = -z_i$. Using solutions of this form, which we call vector-like solutions, and the merger CDF showed that one can construct chiral sets of charges, namely those where $z_i + z_j \neq 0$ for all i and j. They then showed (via rather lengthy algebra) that any solution can be constructed from these chiral sets of charges by permutation of charges or concatenation with each other or with vector-like solutions. For n even the specific mergers they considered were

$$(l_1, k_1, \ldots, k_m, -l_1, -k_1, \ldots, -k_m) \oplus (0, 0, l_1, \ldots, l_m, -l_1, \ldots, -l_m), \tag{1.4}$$

where $m = n/2 - 1 \geq 2$ and $k_i, l_i \in \mathbb{Z},\ i \in \{1, \ldots, m\}$. Whilst for n odd they were

$$(0, k_1, \ldots, k_{m+1}, -k_1, \ldots, -k_{m+1}) \oplus (l_1, \ldots, l_m, k_1, 0, -l_1, \ldots, -l_m, -k_1), \tag{1.5}$$

where $m = (n-3)/2 \geq 1$. CDF showed that if one wants to avoid zero charges or vector-like copies of charges then conditions have to be applied to k_i's and l_i's.

Here, we show that the ingenious methods of CDF have a simple geometric interpretation, corresponding to elementary constructions long known to number theorists [9]. Viewing them in this context allows a fully general solution to be written down in one fell swoop. The geometric interpretation allows us to give a variety of other, qualitatively similar, parameterisations of the general solution, as well as a qualitatively different form of the general solution for even n. It also allows us to show that to generate all solutions from CDF's parameterisation only requires permutations and not the other operations.

The paper proceeds as follows: in § 2, we review the geometric method that we employ to solve (1.1) and (1.2), generalising a number-theoretic result of Mordell to dimensions higher than 3 in the process. We detail two solutions that our method yields directly, but which require permutations of CDF's solutions, and show that for CDF's parameterisation permutations is the only operation required. We conclude in § 3. There is one potential inconvenience in our parameterisation, in that there are special solutions generated differently from others, which we circumvent in appendix A. We present the different form of the general solution for even n in appendix B.

2 Geometric method

By way of motivation, consider the $n = 6$ solution $(0, -9, 7, -1, 8, -5)$ to (1.1), and (1.2). The only way to get this solution using the method outlined in CDF is by permutation. Our geometric solution will, on the other hand, be able to generate such solutions without

JHEP05(2020)065

resorting to permutations.[1] The reasoning behind this, as we shall see later, lies in our use of a geometrical approach, namely that of projective geometry over the field $\mathbb{Q}$ of rational numbers. Before seeing how geometry makes an appearance in the problem at hand, let us recall the basic definitions.

For a field k, the projective space $\mathrm{P}k^{n-1}$ is the space of all lines through the origin in the affine space k^n. In other words, it is $(k^n - \{0\})/\sim$ where $\sim$ is the equivalence relation $m_1 \sim m_2$ with $m_1, m_2 \in k^n$ if and only if there exists a $\lambda \in k$ such that $m_1 = \lambda m_2$. We denote a point in $\mathrm{P}k^{n-1}$ by the equivalence classes $[a_1, \cdots, a_n]$ for $a_i \in k$.

Within the projective space $\mathrm{P}k^{n-1}$ we can define d-planes. By a d-plane (for $d < n-1$) we mean a d-dimensional projective subspace of $\mathrm{P}k^{n-1}$, which can be written as

$$\Gamma = \sum_{i=1}^{d+1} \alpha_i p_i, \tag{2.1}$$

where $[\alpha_1 : \cdots : \alpha_{d+1}] \in \mathrm{P}\mathbb{Q}^d$ parameterise the d-plane and $p_i \in \mathrm{P}\mathbb{Q}^{n-1}$ are fixed. A 1-plane, for example, is just a (projective) line, homeomorphic to a circle.

To motivate the use of projective space on physical grounds, we note that the Lie algebra of the U(1) gauge group is isomorphic to $\mathbb{R}$. Given that U(1) is compact, this implies that our charges z_i are not only real-valued, but also commensurate, meaning that if $z_j \neq 0$, then z_i/z_j is rational for all i. We can scale every z_i by a single real parameter without changing the physics, as long as the coupling constant is also appropriately scaled. This, along with the fact that the z_i's are commensurate, allows us to undertake a scaling such that all charges are rational, *viz.* $z_i \in \mathbb{Q}$.[2] It also tells us that we should think of the set of all charges as living in projective space, specifically $\mathrm{P}\mathbb{Q}^{n-1}$ and indeed, (1.1), and (1.2), being homogeneous, define loci therein.

It is convenient for us to eliminate z_n in our equations from the cubic equation in (1.2) to get

$$\sum_{i=1}^{n-1} z_i^3 - \left(\sum_{i=1}^{n-1} z_i\right)^3 = 0. \tag{2.2}$$

This equation is homogenous, meaning it is well defined on our equivalence classes in $\mathrm{P}\mathbb{Q}^{n-2}$, and as such it defines a cubic hypersurface (given it is co-dimension 1) of $\mathrm{P}\mathbb{Q}^{n-2}$. In order to make progress in solving this equation, we review some geometric methods used in diophantine analysis.

2.1 The method of chords

Consider a homogenous cubic in n-variables, with rational coefficients, defining a locus in $\mathbb{Q}^n$. Let a and b be two points in $\mathbb{Q}^n$ on the locus. A result from antiquity[3] tells us that a

[1] Though, as we indicate, utilising permutations can be useful.

[2] In the end, we can scale them all so they are integer, as we previously claimed. But working with the field $\mathbb{Q}$, rather than the ring $\mathbb{Z}$, allows us to do geometry.

[3] The result certainly goes back at least to Fermat and Newton in the 17th century and may go back even further to Diophantus in the 3rd century. A historical account is given in [10].

JHEP05(2020)065

chord between a and b will intersect the surface at a third point in $\mathbb{Q}^n$. One can understand this result as follows, let $L(t) = a + t(b-a)$ be the chord joining a and b. Points both lying on this chord and in the cubic surface must satisfy the equation $kt(t-1)(t-t_0) = 0$ where $k, t_0 \in \mathbb{Q}$. This result comes from considering the cubic along the chord and noting that a cubic has one or three (possibly degenerate) real roots. Hence within $\mathbb{Q}^n$, there is a third point of intersection, corresponding to $t = t_0$ and given by $L(t_0)$. We note that this result is equally valid in projective space, $\mathrm{P}\mathbb{Q}^n$. We will call this construction the 'method of chords'.

Further, a rather more recent (though equally elementary) result of Mordell [9] states that *all* rational points in a cubic surface in $\mathrm{P}\mathbb{Q}^2$ can be constructed from chords in this way, starting from a projective line, L, and a point, $p \notin L$ that both lie in the surface. It follows from the realisation that in fact *any* point in $\mathrm{P}\mathbb{Q}^2$ (*ergo* any point on the cubic) is on a chord from p_1 to a point in L. As we will see, this result generalises in a straightforward way to $\mathrm{P}\mathbb{Q}^n$, but there is no analogous result in affine space. In $\mathbb{Q}^3$ for example, the analogous result would have to involve two skew lines, L_1 and L_2. However, points forming a plane with L_2 which is parallel to L_1 will be missed. In $\mathrm{P}\mathbb{Q}^2$, there is no concept of parallel lines — pairs of lines are either disjoint or intersecting — and indeed the aforementioned points all lie on a chord connecting a point on L to p.

This simple observation, when generalised to higher n, underlies the fact that the point $(0, -9, 7, -1, 8, -5)$ is missing from CDFs $n = 6$ parameterisation, but is included when we work in projective space, as we will discuss in detail in § 2.4.

Before actually using any of these results, we note that our general method will not work in the cases for $n = 1$, and $n = 2$. This is because for $n = 1$ and $n = 2$ it would require a notion of a (-1)-plane! Part of the discussion, namely that in appendix A, is also valid only for $n \geq 4$. Happily, the solutions to the $n = 1, 2, 3$ cases can be found directly, allowing us to restrict our general discussion to $n \geq 4$. Namely for $n = 1$ the solution is $z_1 = 0$. For $n = 2$, (2.2) results in no effective constraint (one obtains that the left-hand side is identically zero for any z_1) and so the solution of (1.1), (1.2) is the point $[z_1 : z_2] = [1 : -1] \in \mathrm{P}\mathbb{Q}$. We have three solutions for $n = 3$: $[1 : 0 : -1]$, $[0 : 1 : -1]$ and $[1 : -1 : 0]$. Eqs. (1.1) and (1.2) are invariant under permutations of the z_i and so these three solutions are all in one equivalence class under such permutations.

We now consider higher n where the results above are more useful. For illustrative purposes, we will start with a rather explicit discussion of the case $n = 4$.

2.2 Application for $n = 4$

Let us consider the cubic anomaly-free surface in $\mathrm{P}\mathbb{Q}^2$,

$$z_1^3 + z_2^3 + z_3^3 - (z_1 + z_2 + z_3)^3 = 0, \tag{2.3}$$

corresponding to the $n = 4$ case of our problem, where we remember that $z_4 = -(z_1+z_2+z_3)$ from the gravitational mixed anomaly constraint. Using Mordell's result within this surface we take the line $\Gamma_1 = [k_1 : k_2 : -k_1]$ and the point $\Gamma_2 = [0 : l_1 : -l_1]$ in $\mathrm{P}\mathbb{Q}^2$, which are easily seen to lie on the cubic. Using the overall scaling of projective space, we could rescale such that $l_1 = 1$. At this stage, however, we refrain from doing so, preferring a slightly redundant parameterisation in order to stay closer to our analysis of the higher n cases below. We then construct a line passing through a generic point on each of Γ_1 and Γ_2 as $L_1 = \alpha_1[k_1 : k_2, : -k_1] + \alpha_2[0 : l_1 : -l_1]$, where $k_{1,2}, l_1 \in \mathbb{Q}$. The homogeneous parameter $[\alpha_1 : \alpha_2] \in \mathrm{P}\mathbb{Q}^1$ parameterises L_1, which must intersect the cubic surface at a third point, assuming that L_1 is not wholly in the cubic surface. On substituting the chord into (2.3) we obtain the constraint on α_1 and α_2 at intersections of the line and the cubic surface:

$$-3(k_1 - k_2)l_1\alpha_1\alpha_2\left[(k_1 + k_2)\alpha_1 + l_1\alpha_2\right] = 0. \tag{2.4}$$

If L_1 were entirely in the cubic surface, the left-hand side would have evaluated to zero independently of the values of k_1, k_2 or l_1. The third point of intersection is specified by setting the square bracket in (2.4) to zero, i.e.

$$[\alpha_1 : \alpha_2] = [l_1 : -(k_1 + k_2)], \tag{2.5}$$

a rational point.

Now consider an arbitrary point $[a_1 : a_2 : a_3] \in \mathrm{P}\mathbb{Q}^2$ *not* in Γ_2. We can define a line between this point and one on Γ_2: $L_2 = \beta_1[0 : l_1 : -l_1] + \beta_2[a_1 : a_2 : a_3]$. It can be seen that this line intersects Γ_1 at $[\beta_1 : \beta_2] = [a_3 - a_1 : l_1]$. This, combined with (2.5), tells us that every such rational solution to the cubic equation can be found by considering lines between points on Γ_1 and Γ_2. What we have done here is apply Mordell's result to solve the $n = 4$ case of our problem.

2.3 Arbitrary $n \geq 4$

To consider arbitrary values of $n \geq 4$ we must generalise Mordell's result to an arbitrary cubic *hyper*surface X in $\mathrm{P}\mathbb{Q}^{n-2}$. The generalisation is immediate and gives the following

Theorem. *Let $\Gamma_1, \Gamma_2 \subset X$ be disjoint planes of dimensions $d_1, d_2 = m_o := (n-3)/2$, if n is odd and of dimensions $d_1 = m_e := (n-2)/2$ and $d_2 = m_e - 1$ if n is even. Every rational point $p \in \mathrm{P}\mathbb{Q}^{n-2}$ (*ergo *every $p \in X$) lies on a chord joining a point in Γ_1 to a point in Γ_2.*

Proof. The result is obvious if $p \in \Gamma_2$. If $p \notin \Gamma_2$, then p and Γ_2 define a $(d_2 + 1)$-plane, which intersects Γ_1 in a point p^1. The line through p and p^1 intersects Γ_2 in a point p^2, yielding a chord. □

In the case of interest, the (projective) line $L = \alpha_1 p^1 + \alpha_2 p^2$ through $p^{1,2}$ with homogeneous parameter $[\alpha_1 : \alpha_2] \in \mathrm{P}\mathbb{Q}^1$ intersects the cubic hypersurface X defined by (2.2) when

$$3\alpha_1\alpha_2 \sum_{i=1}^{n-1} \left(\alpha_1 p_i^2 P_i^1 + \alpha_2 p_i^1 P_i^2\right) = 0, \; P_i^a := (p_i^a)^2 - \left(\sum_{j=1}^{n-1} p_j^a\right)^2. \tag{2.6}$$

Thus, along with the points $p^{1,2}$ (corresponding to $\alpha_{2,1} = 0$) we get either a third rational point on X at

$$[\alpha_1 : \alpha_2] = \left[\sum_{i=1}^{n-1} p_i^1 P_i^2 : -\sum_{i=1}^{n-1} p_i^2 P_i^1\right], \tag{2.7}$$

or, if the terms on the right-hand side both vanish, we have that every rational point on L is on X. Lines which lie in X may be regarded as slightly awkward to deal with. Happily, it is possible, as we show in appendix A, to find every solution on such a line by a permutation of the coordinates of a solution arising as the unique third point of intersection on a line not lying in X. A comparison of (2.7) with (1.3) shows that the 'merger' operation is really nothing but the finding of the third rational point starting from two others.

To get a fully general solution, we just need to find suitable Γ_1, Γ_2. To wit,

$$\begin{aligned}
\Gamma_1^e &= [k_1 : \cdots : k_{m_e} : k_{m_e+1} : -k_1 : \cdots : -k_{m_e}] \\
\Gamma_2^e &= [0 : l_1 : \cdots : l_{m_e} : -l_1 : \cdots : -l_{m_e}] \\
\Gamma_1^o &= [k_1 : \cdots : k_{m_o+1} : -k_1 : \cdots : -k_{m_o+1}] \\
\Gamma_2^o &= [l_2 : \cdots : l_{m_o} : l_{m_o+1} : 0 : -l_1 : \cdots : -l_{m_o} : -l_{m_o+1}].
\end{aligned} \tag{2.8}$$

These planes are disjoint (only meeting at the origin, which is not in $\mathrm{P}\mathbb{Q}^{n-2}$), so by the Theorem they yield all rational solutions of (1.1).

2.4 Comparison with CDF

The parameterisations of CDF, in contrast to ours, have $k_{m_e+1} = -l_1$ and $l_{m_o+1} = k_1$. We have already discussed above that CDF's solution misses the point $(0, -9, 7, -1, 8, -5)$, for $n = 6$ and that for them this has to be found by permuting another solution, for example that generated with $k_1 = 14$, $k_2 = 2$, $l_1 = -18$, $l_2 = -9$ after scaling. In our parameterisation $(0, -9, 7, -1, 8, -5)$ can be obtained directly with, for example, $k_3 = 0$, $k_1 = 3$, $k_2 = -2$, $l_1 = 1$, and $l_2 = -1$ in (2.8), giving $p^1 = [3, -2, 0, -3, 2]$ and $p^2 = [0 : 1 : -1 : -1 : 1]$ and the correct third point of intersection.

It is easy to see why CDF's parameterisation misses this point; they cannot set both k_3 and l_3 to zero. Viewing things in the affine space $\mathbb{Q}^5$, the geometric nature of such missed points becomes manifest. The planes for $n = 6$ in (2.8) can be seen as corresponding to

$$\begin{aligned}
\tilde{\Gamma}_1^e &= (k_1, k_2, 1, -k_1, -k_2) \\
\tilde{\Gamma}_2^e &= (0, l_1, l_2, -l_1, l_2).
\end{aligned} \tag{2.9}$$

in $\mathbb{Q}^5$. The $3-d$ plane defined by $\tilde{\Gamma}_2^e$ and the point $(-9, 7, -1, 8, -5)$ does not intercept the $2-d$ plane $\tilde{\Gamma}_1^e$, which is the same reason why Mordell's result fails to catch all the points in $\mathbb{Q}^3$. CDF go halfway to allowing such points, but by fixing $k_3 = l_1$ they don't quite catch them all.

We can be more specific and ask: given the planes in (2.8) where we force $k_{m_e+1} = -l_1$ and $l_{m_0+1} = k_1$ to retrieve CDF's solution, what points don't lie on lines between them? It is easy to see that for even n this would require either k_{m_e+1} or l_1 to be zero and for odd n either l_{m_0+1} or k_1, but not both. Thus, for the point $[a_1 : \cdots : a_n]$ to not lie on such a line, we need, for even n,

$$a_1 + \cdots + a_{n-1} = 0 \text{ or } a_1 + a_{n-2} = 0, \tag{2.10}$$

or, for odd n,

$$a_1 + \cdots + a_{n-1} = 0 \text{ or } a_{m_0+2} = 0. \tag{2.11}$$

For a non-zero solution we can always rearrange the charges so that none of these conditions are satisfied.

The only other points CDF miss are those where the line between the two planes in (2.8) lies within X. For example for $n = 4$ setting $k_2 = k_1$ gives a line L which lies in X. As an explicit example, consider $k_{1,2}, l_1 = 1$. This line is given by

$$L = \alpha_1[1 : 1 : -1] + \alpha_2[0 : 1 : -1]. \tag{2.12}$$

For CDF, points on this line correspond to solutions of the form $(A, -A, B, -B)$ for $A, B \in \mathbb{Z}$. However, CDF's $n = 4$ parameterisation

$$(-l_1^3(k_1 + l_1), -k_1 l_1^2(k_1 + l_1), k_1 l_1^2(k_1 + l_1), l_1^3(k_1 + l_1)) \tag{2.13}$$

can never land on such solutions. Nevertheless, CDF's parameterisation can get these by permutations, for the same reason that the parameterisation given here can, as we discuss in appendix A.

The above two points not only show when CDF's parameterisation fails to reach a specific point but also proves that their parameterisation produces every point up to permutations.

3 Discussion

The pioneering work of CDF finds solutions to the local U(1) anomaly cancellation constraints. This allows the construction of the general solution, provided one allows permutations. Our geometric method provides the general solution directly without having to perform additional steps. The geometric method also explains how some of the otherwise obscure features of CDF's construction (particularly the 'merging' procedure of two solutions) come about. Due to an immediate generalisation of a theorem by Mordell, the geometric method is guaranteed to find *all* rational solutions for a fixed number of charges n. Therefore (after clearing all denominators), it finds all integer solutions.

Two further remarks are in order. Firstly, as we have seen, our parameterisation of the general solution is somewhat distasteful, in that occasionally the chord L joining points on $\Gamma_{1,2}$ lies in X, and so yields not one, but infinitely many solutions. Another way to find these solutions is to permute the coordinates z_i of solutions arising as the unique third intersection of a line L which is not in X, as shown in appendix A. Secondly, in the case where n is even, a completely different, and arguably even simpler, construction of a general solution is possible. Indeed, in such cases, the cubic hypersurface has double points, where both the left-hand side of (2.2) and its partial derivatives vanish (e.g. the rational point $[+1 : -1 : +1 : -1 : \ldots : +1 : -1 : +1]$). A line through such a double point intersects the cubic in one other rational point (or the line lies entirely in X) and thus all solutions can be obtained by constructing all lines through just a single double point, as it were. This is worked through explicitly in appendix B.

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A Any solution via permutations

Here, we give a proof of the statement that any solution sitting on a line in X between the d-planes defined in (2.8) can be found by the permutation of the coordinates of a solution which is on a line not in X. The proof of this statement follows similar reasoning to the proof regarding permutations of solutions in [8]. We must distinguish between n odd and even, so we do them each in turn.

A.1 Even $n \geq 4$

We redefine variables such that

$$\begin{aligned} x_i &= z_1, && \text{for } i = 1, \\ x_i &= z_i + z_{m_e+i}, && \text{for } i = 2, \cdots, m_e + 1, \\ y_i &= z_i + z_{m_e+1+i}, && \text{for } i = 1, \cdots, m_e. \end{aligned} \tag{A.1}$$

The d-planes in (2.8) are defined in our new variables by $y_i = 0$ for Γ_1^e and $x_i = 0$ for Γ_2^e. Consider a point $p = [x_i : y_i] \notin \Gamma_1^e \cup \Gamma_2^e$. There is a unique line

$$L_p = \alpha p^1 + \beta p^2, \tag{A.2}$$

through p, $p^1 \in \Gamma_1^e$ and $p^2 \in \Gamma_2^e$. Under the permutation $\phi^e : z_{m_e+1} \leftrightarrow z_{2m_e+1}$, only y_{m_e} changes and

$$L_{\phi^e(p)} = \alpha p^1 + \beta \phi^e(p^2). \tag{A.3}$$

A necessary condition for L_p to be in X is that

$$\begin{aligned} -3y_{m_e}x_{m_e+1}\left(2\sum_{i=1}^{m_e} x_i + x_{m_e+1}\right) + \cdots = 0 \Leftrightarrow \\ -3(z_{m_e} + z_{2m_e+1})(z_{m_e+1} + z_{2m_e+1})\left(2\sum_{i=1}^{2m_e+1} z_i - z_{m_e+1} - z_{2m_e+1}\right) + \cdots = 0, \end{aligned} \tag{A.4}$$

where the dots indicate terms which are independent of y_{m_e}.

Thus if L_p is in X, for a solution p with coordinates permuted such that

$$|z_{m_e+1}| \neq |z_{2m_e+1}| \quad \text{and} \quad z_{m_e+1} + z_{2m_e+1} - \sum_{i=1}^{2m_e+1} z_i \neq 0, \tag{A.5}$$

then $L_{\phi^e(p)}$ will not be in X. The only case where this cannot be done is where all $|z_i|$ are equal, but such solutions already occur in Γ_1^e after permutations of the z_i.

JHEP05(2020)065

A.2 Odd $n \geq 4$

Here,

$$\begin{aligned} x_i &= z_{m_0+1}, && \text{for } i = 1, \\ x_i &= z_{i-1} + z_{m_0+1+i}, && \text{for } i = 2, \cdots, m_o + 1, \\ y_i &= z_i + z_{m_o+1+i}, && \text{for } i = 1, \cdots, m_o + 1. \end{aligned} \tag{A.6}$$

Again, Γ_1^o is simply defined by $y_i = 0$ and Γ_2^o is defined by $x_i = 0$. Similar to the even n case, we take a point $p = [x_i : y_i] \notin \Gamma_1^o \cup \Gamma_2^o$. There is again a unique line

$$L_p = \alpha p^1 + \beta p^2, \tag{A.7}$$

through p, where $p^1 \in \Gamma_1^o$ and $p^2 \in \Gamma_2^o$. Taking $\phi^o : z_1 \leftrightarrow z_{m_o+2}$, only x_2 changes, where

$$L_{\phi^o(p)} = \alpha \phi^o(p^1) + \beta p^2. \tag{A.8}$$

A necessary condition for L_p to be in X is then

$$\begin{aligned} -3x_2 y_1 \left(2 \sum_{i=2}^{m_o+1} z_i + y_1 \right) + \cdots = 0 \Leftrightarrow \\ -3(z_1 + z_{m_o+3})(z_1 + z_{m_o+2}) \left(2 \sum_{i=1}^{2m_0+2} z_i - z_1 - z_{m_0+2} \right) + \cdots = 0, \end{aligned} \tag{A.9}$$

where now the dots indicate terms which are independent x_2.

If L_p is in X for a solution p with coordinates permuted such that

$$|z_1| \neq |z_{n_o+2}| \quad \text{and} \quad z_1 + z_{m_o+2} - 2 \sum_{i=1}^{2m_o+2} z_i \neq 0, \tag{A.10}$$

then $L_{\phi^o(p)}$ will not be in X. We may use this construction for all solutions and n odd.

B Alternative solution for n-even

For even n, the cubic equation in (2.2) has double points; that is points where all of the partial derivatives of the left-hand side vanish, as well as the left-hand side itself. An example of such a double point is

$$d = [+1 : -1 : +1 : -1 : \ldots : +1 : -1 : +1] \in \mathrm{P}\mathbb{Q}^{n-2}. \tag{B.1}$$

So for e.g. $n = 6$, we have $[+1 : -1 : +1 : -1 : +1]$.

Consider a line through our double point d, $L = \gamma_1 d + \gamma_2 r$, for $r \in \mathrm{P}\mathbb{Q}^{n-2}$ a fixed point and $[\gamma_1 : \gamma_2]$ specifying the position along the line. Any point in $\mathrm{P}\mathbb{Q}^{n-2}$ lies on such a line, and further every such line is either in the hypersurface X (defined by (2.2)) or passes through that hypersurface at exactly one other point.

This other point of intersection can be found by substituting L into (2.2):

$$\gamma_2^2\left(3\gamma_1\sum_{i=1}^{n-1}d_iR_i+\gamma_2\sum_{i=1}^{n-1}r_iR_i\right)=0,\quad R_i:=r_i^2-\left(\sum_{j=1}^{n-1}r_j\right)^2. \tag{B.2}$$

Either $\gamma_2=0$ (the original point d), the l.h.s. is zero independently of γ_1 and γ_2 (corresponding to L being in X) or

$$[\gamma_1:\gamma_2]=\left[\sum_{i=1}^{n-1}r_iR_i:-3\sum_{i=1}^{n-1}d_iR_i\right], \tag{B.3}$$

giving the second point of intersection. As such we can see that the lines L can be used to find all solutions to (2.2) parameterised by r_i, and if L is in X by $[\gamma_1:\gamma_2]$.

Continuing our example, for $n=6$, we have that (B.2) becomes

$$\begin{aligned}&3\gamma_1(r_1^2-r_2^2+r_3^2-r_4^2+r_5^2-(r_1+r_2+r_3+r_4+r_5)^2)\\&\qquad+\gamma_2(r_1^3+r_2^3+r_3^3+r_4^3+r_5^3-(r_1+r_2+r_3+r_4+r_5)^3)=0\end{aligned} \tag{B.4}$$

implying the second point of intersection is at

$$\begin{aligned}[\gamma_1:\gamma_2]=&[(r_1^3+r_2^3+r_3^3+r_4^3+r_5^3-(r_1+r_2+r_3+r_4+r_5)^3)\\&:-3(r_1^2-r_2^2+r_3^2-r_4^2+r_5^2-(r_1+r_2+r_3+r_4+r_5)^2)].\end{aligned} \tag{B.5}$$

References

[1] T. Eguchi, P.B. Gilkey and A.J. Hanson, *Gravitation, gauge theories and differential geometry*, *Phys. Rept.* **66** (1980) 213 [INSPIRE].

[2] S.L. Adler, *Axial vector vertex in spinor electrodynamics*, *Phys. Rev.* **177** (1969) 2426 [INSPIRE].

[3] W.A. Bardeen, *Anomalous Ward identities in spinor field theories*, *Phys. Rev.* **184** (1969) 1848 [INSPIRE].

[4] C. Bouchiat, J. Iliopoulos and P. Meyer, *An anomaly free version of Weinberg's model*, *Phys. Lett.* **B 38** (1972) 519.

[5] D.J. Gross and R. Jackiw, *Effect of anomalies on quasirenormalizable theories*, *Phys. Rev.* **D 6** (1972) 477 [INSPIRE].

[6] H. Georgi and S.L. Glashow, *Gauge theories without anomalies*, *Phys. Rev.* **D 6** (1972) 429 [INSPIRE].

[7] G. Hardy and E. Wright, *An introduction to the theory of numbers*, Oxford University Press, Oxford U.K. (2008).

[8] D.B. Costa, B.A. Dobrescu and P.J. Fox, *General solution to the* U(1) *anomaly equations*, *Phys. Rev. Lett.* **123** (2019) 151601.

[9] L. Mordell, *Diophantine equations*, Academic Press, New York U.S.A. (1969).

[10] J. Stillwell, *Mathematics and its history*, Springer, Germany (2010).

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