New Perspectives on Operator Deformations and T-duality in String Theory

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Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the preface and specified in the text. It is not substantially the same as any work that has already been submitted before for any degree or other qualification except as declared in the preface and specified in the text.

The material in chapter 6 as well as sections 4.3.4 and 4.3.5 is based on [1] and is original work done by the author in collaboration with R. A. Reid-Edwards.

The material in chapters 7, 8 and 9 is based on [2] and is original work done by the author in collaboration with R. A. Reid-Edwards.

The material in chapter 10 is based on unpublished work by the author.

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Abstract

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The moduli space of string theories has been the subject of intense research efforts for many years now. Much of these efforts are focused on string compactifications, and in particular worldsheet conformal field theories (CFTs) embedded in toroidal target spaces. The CFTs defined at each point in this moduli space consist of an algebra of operators which define the content of the theory. In this thesis, we will investigate methods of traversing this moduli space, and we will attempt to elucidate the relations and symmetries that exist between different points in this space.

Specifically, we study the deformation of operators in string compactifications, as well as T-duality from the worldsheet CFT perspective. We review, and bring into a contemporary context, a construction based on universal coordinates that can be used to define operators at one point in moduli space in terms of the operators at some reference point. We also review how this construction can be used to perform T-duality algebraically, thus providing an alternative perspective to the Buscher construction.

Using the language of connections and parallel transport on the space of backgrounds, we discuss how to deform general operators in a given space of theories, including quantum field theories (QFTs) lacking conformal symmetry. We find that, for a general operator, there are two sources of deformation. The first is the usual deformation operator derived from the worldsheet sigma model. The second, less familiar part is a deformation directly induced as a result of the change in the background, which depends on the tensor structure of the operator of interest. In particular, scalar operators are invariant under this deformation. In the literature, since it is usually scalar operators such as the stress tensor that are of interest, this part of the deformation has not previously been addressed to our knowledge.

Throughout, we apply our formalisms to well-known torus bundle examples such as the nilfold, the T^3 with H-flux and the T-fold, and we also employ the doubled

geometry construction. Initially, we utilise an 'adiabatic' approximation, where we neglect the worldsheet interactions arising from the coordinate dependence of the background. We investigate how the gauge algebra of the torus bundle with doubled fibres, pulled back to the worldsheet, compares with the algebra of the zero modes. Surprisingly, we find that the algebra of the worldsheet theory reproduces the doubled twisted torus algebra, i.e. the algebra where the base is also doubled.

We also consider worldsheet sigma models corresponding to these torus bundles in their entirety, away from the adiabatic limit, and derive operator deformations in this context. We discuss T-duality between these backgrounds and we explain how our formalism could be used to construct T-dual backgrounds in more general settings. We also consider how the formalism applies to the $N = 1$ superstring in the NS-NS sector.

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بِسْمِ ٱللَّهِ ٱلرَّحْمَٰنِ ٱلرَّحِيمِ

For my parents . . .

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Chapter 1

Introduction

String theory has been the most promising candidate for a theory of quantum gravity for some decades now. Over the past half-century or so, there have been great strides made in our understanding of the theory, and at every turn it has revealed itself to be richer and deeper than we previously thought. As is well-known, the Standard Model is the best theory we have for describing almost all known interactions in nature, and it has been the gold standard in physics for many years. However, the problems with the Standard Model highlight the need for a theory such as string theory. In particular, its incompatibility with gravity is its most glaring problem. There are also other issues though, such as the apparent arbitrariness of its construction. Why the particular gauge group $SU(3) \times SU(2) \times U(1)$? Why the specific parameters of the theory? Such questions naturally lead us to consider a theory which has a more 'natural' construction and more easily incorporates *all* of the interactions we observe in nature. Over the years, string theory has maintained its status as the foremost theory potentially capable of filling this role. From a simple construction of a worldsheet embedded in spacetime, we get many appealing properties that often appear to arise very naturally. For example, string theory has no free parameters, so it does not have the arbitrariness of the standard model. It also naturally incorporates gravity and has no issues with UV divergences. Furthermore, not only does string theory admit a supersymmetric description, but supersymmetry is *required* for consistency of the theory. Additionally, and perhaps surprisingly, there have been many connections made with rich areas of mathematics which have broadened our understanding of the interplay between mathematics and physics, such as mirror symmetry between Calabi-Yau manifolds [3, 4]. Such unexpected connections are ubiquitous in string theory, and as such it is clear that there are many deep implications of the theory that are yet to be understood.

It is apparent, therefore, that string theory is worth studying. Though it has been the subject of intense focus for many years, there are still numerous open questions and our understanding of the theory is far from complete. It is generally thought that string theories are 2d conformal field theories embedded in a spacetime ('target space') of 26 (bosonic case) or 10 (supersymmetric case) dimensions. Understanding this space of theories better, and in particular constructing tools that allow us to traverse this space, is one of the key goals of string theory. This will largely be the focus of this thesis. More specifically, we will be interested in how operators in the CFT at one string theory can be deformed to another string theory, i.e. another point in the moduli space of theories. We will also be interested in the symmetries on this space and we will pay particular attention to the phenomenon of T-duality [5–7]. In short, the aim of the research undertaken and presented in this thesis is to elucidate the matter of operator deformations and to better understand the symmetries existing between operators and the theories to which they belong.

T-duality is a symmetry on the moduli space which relates two apparently different sigma models by constructing a relation between the background tensors which define the theories. These relations are usually known as the *Buscher rules* [7], and this has been the established understanding of T-duality for some time. The Buscher construction is often the most useful way of proving that two apparently different theories are really just different T-dual descriptions of the same theory, and is the isometry gauging procedure originally introduced in [7] and further refined in [8, 9]. The validity of the Buscher procedure rests on the existence of a compact abelian isometry of the target space which can be used to generate a rigid symmetry of the worldsheet theory. This symmetry can then be gauged in the sigma model and the gauge fields integrated out to give a dual description of the gauged theory. If one can then show that the gauged theory is equivalent to the ungauged theory, then one has a pair of worldsheet theories that describe the same physics. There are many cases where the required symmetry of the target space does not exist (or may exist only locally), but there is still evidence that a dual description of the theory exists and there is a sense that the relationship between these two descriptions should still be thought of as a form of T-duality. Examples include the SYZ description of mirror symmetry [10] that involves torus bundles where the fibres can degenerate, as well as T-folds [11], whose non-geometric nature can be traced to an obstruction to extending a local circle isometry globally in the geometric dual [12].

Progress has been made in casting the Buscher construction in more general (and exotic) settings [13], but we always fall back to the same restrictions when attempting to prove the duality rigorously, at the level of the worldsheet theory. This stands in contrast to the remarkable progress that has taken place in incorporating duality symmetries directly into supergravity-inspired field theory constructions[1]. In this thesis, instead of attempting to generalise the Buscher construction, we revisit the perspective of T-duality as an automorphism of the operator algebra. Using ideas discussed in [15] and with a view to moving beyond the isometric torus bundle paradigm, whilst remaining within the context of a symmetry that is recognisable as T-duality, we reconsider the operator algebra approach in a contemporary context. One of the main questions concerning T-duality is whether there is a valid way to extend its applicability to non-isometric cases. We will not settle the issue of non-isometric T-duality in this thesis, but we hope that the constructions presented and the observations made will help to indicate a possible alternative approach, one which we feel has not yet been fully explored. In particular, we hope to frame T-duality in a more fundamental perspective. For a general CFT, it is the operator algebra which is the defining content of the theory, not the spacetime. Indeed, there may not even be a proper spacetime interpretation of a given theory, but we will always have an operator algebra, and so we are of the opinion that the algebraic approach better gets to the heart of T-duality and is more fundamental than the Buscher procedure. In general, there has been renewed interest in understanding QFT from the operator algebra perspective [16, 17]. This thesis takes a similar view to the primacy of the the worldsheet operator algebra in string theory as a bundle over the space of backgrounds.

As indicated, our approach in this thesis will be very much from an operator perspective, and our primary tools will be the algebra on the worldsheet CFT. As well as T-duality, our main focus will be on how these operators are deformed as we move through moduli space. One of our aims will be to describe how the use of a connection on the space of backgrounds can be used to define a parallel transport that allows one to write the operators of a theory in one background in terms of the operators in another background (at least perturbatively). Ultimately, one of the aims of this endeavour is to gain a more background independent view of string theory, or at least a given family of string backgrounds. In relating operators at different backgrounds to each other, we hope to obtain a broader picture of these backgrounds which perhaps makes it easier to study more general background independent properties.

Much attention has been given to deformations of the holomorphic worldsheet stress tensor $T(z)$ under a change of background. This is of course a spacetime scalar, but there has not been much attention given to the deformation of non-scalar operators.

[1]See [14] for a nice review of some of the more recent advances.

Thus, a key aim of this thesis is to develop a formalism that correctly treats operators, such as $\partial X^\mu(z)$, which are not target space scalars.

A rough outline of the framework is as follows (details will be given in the text). We start with a family of sigma models, characterised by a metric, B-field and any other fields in the target space. We assume that, near a theory of interest, this data defines a space of backgrounds $\mathcal{M}$. Over $\mathcal{M}$ there is a bundle $\mathcal{E} \to \mathcal{M}$ whose fibres are operators of the worldsheet theory defined at the point in $\mathcal{M}$. We will really only be interested in a subset of these operators relevant to the deformation we are considering.

The starting point is a string theory at a point of enhanced symmetry $p_0 \in \mathcal{M}$, where this symmetry acts as an automorphism on the operator algebra

$$\mathcal{A}_0 \to U^{-1}\mathcal{A}_0 U, \tag{1.1}$$

for any operator $\mathcal{A}_0$ in the fibre at p_0, where U is an element of the symmetry group. We then define a connection Γ on $\mathcal{E}$ and use it to parallel transport to a background of interest $p \in \mathcal{M}$. This allows us to describe the theory at p in terms of the operators of the theory at p_0. The advantage of this is that, if we know the action of the symmetry group on operators at p_0, we can use this to deduce the action on operators at p. In general, the enhanced symmetry is spontaneously broken in the new background, but a subgroup may remain. As we will show, T-duality is one of these residual gauge symmetries and the group of all such symmetries forms a discrete subgroup of the space of backgrounds which we quotient by to obtain the physical moduli space.

We will initially consider toroidal backgrounds which are exact CFTs, where we will show that our formalism reproduces the known deformation results obtained in [15, 18] to all orders. However, one of the benefits of this approach is that, even if the background we start at is a CFT, we can deform the theory 'off-shell' to a background which is not a CFT, so it can be used for a large class of backgrounds, including a number of toy models. When we come to the nilfold and T^3 with H-flux in chapter 6, we will see that this is useful for investigating the T-duality between these backgrounds. Initially, we will study such backgrounds in the context of an 'adiabatic approximation'[19], which essentially allows us to regard these backgrounds as CFTs, and later we will move away from this approximation and see how the parallel transport formalism can be applied in more general cases. Although these toy models are not string theories on their own, we still take an interest in them because they are used as components in the construction of legitimate string theories. It is also interesting to see how the case with conformal symmetry differs from the more general case. In particular, we will review an idea from string field theory known as 'universal coordinates' [20], which makes the CFT case particularly simple. This is essentially the statement that the string embedding field $X(\sigma, \tau)$ is independent of the background (at least for fixed τ), and we will see that such a construction is unique to the CFT case.

Although we find the conceptual framework is compelling, there are several drawbacks to this approach: the main issue is that the calculations are hard. This is, in part, reflective of the fact that the worldsheet quantum field theory is at the heart of this formalism and calculations in interacting quantum field theories are hard. The challenges of doing explicit calculations are a reflection of this fact. There is a sense in which this is not the whole story. Even the free theory calculations in this formalism can be involved and there are examples of dualities between toy models that may be done straightforwardly in the Buscher construction [7, 8], but are technically challenging in the formalism presented here. We also avoid issues of topology change and degenerations in the background and we generally lift to a covering space to perform the parallel transport.

What *is* to be gained from this formalism is a way to clearly understand how individual operators in a CFT or QFT may be deformed, as opposed to correlators, which is usually the focus in the literature. The formalism we describe is widely applicable to many theories and is a step forward in being able to traverse the space of string theories and QFTs more generally.

Additionally, we gain a different perspective on T-duality, one that is potentially applicable to a wider class of examples than the Buscher construction. For example, the challenges with non-isometric duality are particularly clear from this perspective and no longer seem insurmountable.

1.1 Outline

The outline of this thesis is as follows. Chapters 2, 3, 4 and 5 will largely be review material of topics of relevance for later chapters. In chapter 2, we briefly review toroidal compactifications in string theory and the conventional understanding of T-duality. We also look more closely at symmetries in the space of string backgrounds and we explain how T-duality can be understood in this context.

In chapter 3, we explore an idea that will be important in many of our discussions, namely doubled geometry [21, 11]. We will start by reviewing the setup of the formalism and then we will show how it can be applied to twisted torus bundles. These backgrounds, the T^3 with constant H-flux, the nilfold and the T-fold, will be introduced here and in particular we will explore the T-dualities between them. We will also show how these dualities fit naturally in the doubled formalism, and we will end by reviewing the sigma model of the doubled torus bundle.

In chapter 4, we introduce the notion of algebraic T-duality as an alternative to the Buscher procedure. First, we review the notion of automorphisms of CFTs in general, from the perspective of the worldsheet operator algebra. We then explain how T-duality can be constructed as such an automorphism, and we explore this transformation in some detail. We introduce the idea of universal coordinates and we apply the algebraic T-duality construction to simple toroidal cases.

In chapter 5, we review the connection story of [22], first in the more general context of [23, 24], and then specifically for the CFT case. We show how connections on the space of backgrounds are constructed and how these can be used to define a notion of parallel transport of operators. We briefly review the QCD example of [23] and then discuss the stress tensor deformation.

In chapter 6, we apply the ideas of the previous two chapters in particular and discuss how universal coordinates can be viewed from the connection perspective. We then apply the universal coordinate construction to the twisted torus bundles discussed in chapter 3, where we employ the adiabatic limit to justify this application to backgrounds which are not strictly CFTs. We look at the T-duality between these backgrounds from the algebraic perspective. A significant amount of attention is paid to how the algebra defining the backgrounds of chapter 3 looks from the worldsheet CFT perspective. In some sense, this is the pullback of the algebra to the worldsheet. What we find is that the algebra on the worldsheet is a central extension of the full doubled twisted torus algebra from [12] which is classically obtained by doubling both the fibres and the base. Given that we only double the base, this result is highly intriguing. We briefly explore how one might tackle non-isometric T-duality in this formalism and we show how, though the problems in the Buscher procedure are still present here, it seems less insurmountable and that progress on non-isometric T-duality could possibly be made in this algebraic perspective. We end the chapter by exploring some generalisations of twisted torus bundles to target spaces which are still constructed from the same structure constants, but may not be torus bundles. We derive the relevant algebras here and discuss associativity. We make some brief comments on the R-flux case.

In chapter 7, we discuss a formalism which allows us to extend our analysis thus far to more general theories and operators. In particular, our construction will be valid for generic connections on theory space and takes into account the tensor structure of the operator of interest. We will apply the formalism to the flat torus case first, where we recover the results derived earlier. Here, for the case when we have a non-zero B-field, we will see that we need to use the doubled formalism to recover the results derived earlier, and we will discuss in detail why this is. We will also explore the relationship between universal coordinates and doubled geometry.

In chapter 8, we continue directly from the previous chapter and apply our formalism to the torus bundle backgrounds considered earlier, but this time without making any simplifying assumptions. We show that this results in the deformations being more complicated and we derive these extra terms to first order. We also briefly explore a covariant construction, based on methods described, for example, in [25].

In chapter 9, we look at the T-duality between the torus bundles once again, but this time with the extra corrections from the previous chapter taken into account. Our approach is essentially a generalisation of [15] where we use the stress tensor to derive the duality, and we show that we recover the expected duality. We also explain how this method would be used more generally to construct T-dual backgrounds.

In chapter 10, we look at how the formalism discussed in the previous few chapters applies to the $N = 1$ supersymmetric case in the NS-NS sector. We derive the deformation for a fermion field both in the flat torus case and in more general cases. We briefly discuss universal coordinates for fermions and why such a construction is not readily available. We once again derive the T-duality between the T^3 with H-flux and the nilfold, but this time with the fermionic deformations taken into account, and we find that the expected duality does indeed hold. We also make some interesting observations regarding picture changing and how this fits in with our formalism.

Finally, in chapter 11, we make some concluding remarks and we discuss some questions that would be very interesting to explore, but unfortunately we did not have time for during the research period.

Chapter 2

String Theory on a Flat Torus

We begin by reviewing string compactifications on the flat torus. In particular, we will be interested in the symmetries of these theories and we will review the T-duality construction of [7, 8], as well as the gauge symmetry discussions of [26]. We also lay out much of the conventions that we will use throughout this thesis.

2.1 Toroidal compactifications in string theory

Much of this thesis will be focused on target spaces containing some compactified toroidal space T^d with a flat metric $g_{\mu\nu}$ and constant antisymmetric B-field $B_{\mu\nu}$. In particular, we will be interested in the conformal field theory on such spaces (we will not pay any attention to the uncompactified dimensions in this thesis and our focus will be entirely on string compactifications). The space of such backgrounds, modulo symmetries, is

$$\mathcal{M} = O(d) \times O(d) \backslash O(d,d)/O(d,d;\mathbb{Z}), \tag{2.1}$$

where d is the dimension of the torus. The discrete group $O(d,d;\mathbb{Z})$ is of particular interest to us and we will discuss its presence in more detail in section 2.4. The worldsheet sigma model embedded in the torus, with worldsheet coordinates (σ,τ), is given by the action

$$S = -\frac{1}{4\pi}\int_0^{2\pi} d\sigma \int d\tau \left(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} + \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}\right), \tag{2.2}$$

where the metric signature is $(+,-)$, $\epsilon^{\tau\sigma} = -1$ and we have chosen conventions such that the α' factors have all been absorbed into the fields. This ensures that all fields appearing here are dimensionless. At some points it will be useful to reintroduce the α' dependence, and we will make it clear whenever we do so.

It is convenient to work in conformal gauge, where the action becomes

$$S = \frac{1}{\pi} \int d^2\sigma E_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu, \tag{2.3}$$

where $E_{\mu\nu} := g_{\mu\nu} + B_{\mu\nu}$ specifies the background[1] and we have defined $\partial := \frac{1}{2}(\partial_\tau - \partial_\sigma), \bar{\partial} := \frac{1}{2}(\partial_\tau + \partial_\sigma)$. The equation of motion is then

$$\Box X = 0, \tag{2.4}$$

where $\Box \equiv \partial\bar{\partial}$. X then has the mode expansion

$$X^\mu(\sigma, \tau) = x^\mu + \omega^\mu \sigma + \tau g^{\mu\nu}(p_\nu - B_{\nu\rho}\omega^\rho) + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-in(\tau-\sigma)} + \bar{\alpha}_n^\mu e^{-in(\tau+\sigma)} \right), \tag{2.5}$$

and the conjugate momentum $P_\mu = \frac{1}{2\pi}(g_{\mu\nu}\dot{X}^\nu + B_{\mu\nu}X'^\nu)$, where $\dot{X} \equiv \partial_\tau X, X' \equiv \partial_\sigma X$, has the mode expansion

$$\Pi_\mu(\sigma, \tau) \equiv 2\pi P_\mu(\sigma, \tau) = p_\mu + \frac{1}{\sqrt{2}} \sum_{n\neq 0} \left(E^T_{\mu\nu} \alpha_n^\nu e^{-in(\tau-\sigma)} + E_{\mu\nu} \bar{\alpha}_n^\nu e^{-in(\tau+\sigma)} \right). \tag{2.6}$$

At fixed τ, these fields obey the equal time commutation relations

$$[X^\mu(\sigma, \tau), \Pi_\nu(\sigma', \tau)] = 2\pi i \delta(\sigma - \sigma') \delta^\mu{}_\nu. \tag{2.7}$$

We can normally take $\tau = 0$ wlog because, if τ takes some fixed non-zero value, we can simple redefine

$$\alpha_n^\mu \to \alpha_n^\mu e^{in\tau}. \tag{2.8}$$

We also have the mode commutation relations

$$[\alpha_n^\mu, \alpha_m^\nu] = n g^{\mu\nu} \delta_{n+m}, \tag{2.9}$$

which can be verified using the mode expansion, or equivalently the inverse relation

$$\alpha_n^\mu(\tau) = \frac{\sqrt{2}}{2\pi} \oint d\sigma\, \partial X^\mu(\sigma, \tau)\, e^{-in\sigma}, \tag{2.10}$$

[1] $E_{\mu\nu}$ can be seen as a parameterisation of the moduli space (2.1).

where $\alpha_n(\tau) = \alpha_n e^{-in\tau}$. It will also be useful to split X into left and right-moving parts as

$$X^\mu_L(\sigma^-) = x^\mu_L + \frac{1}{\sqrt{2}}\alpha^\mu_0\sigma^- + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n e^{-in\sigma^-}, \tag{2.11}$$

$$X^\mu_R(\sigma^+) = x^\mu_R + \frac{1}{\sqrt{2}}\bar{\alpha}^\mu_0\sigma^+ + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}^\mu_n e^{-in\sigma^+}, \tag{2.12}$$

where $\sigma^\pm = \tau \pm \sigma$ and

$$g_{\mu\nu}\alpha^\nu_0 = \frac{1}{\sqrt{2}}\Big(p_\mu - E_{\mu\nu}\omega^\nu\Big), \qquad g_{\mu\nu}\bar{\alpha}^\nu_0 = \frac{1}{\sqrt{2}}\Big(p_\mu + E^T_{\mu\nu}\omega^\nu\Big). \tag{2.13}$$

As we will discuss in chapter 4, for these flat toroidal backgrounds, we will think of X'^μ, Π_μ as being *universal coordinates* [20], in the sense that they are independent of E.

We will very often be interested in the weight $(1,0)$ and $(0,1)$ fields $\partial X_\mu(\sigma^-), \bar{\partial}X(\sigma^+)$ respectively, where $\sigma^\pm = \tau \pm \sigma$. These can be expressed in terms of X', Π as

$$\partial X_\mu(\sigma^-) = \frac{1}{2}\Big(\Pi_\mu(\sigma,\tau) - E_{\mu\nu}X'^\nu(\sigma,\tau)\Big), \qquad \bar{\partial}X_\mu(\sigma^+) = \frac{1}{2}\Big(\Pi_\mu(\sigma,\tau) + E^T_{\mu\nu}X'^\nu(\sigma,\tau)\Big). \tag{2.14}$$

Often, we will prefer to work on the plane as opposed to the cylinder, i.e. on the Euclidean worldsheet Σ. To go to Euclidean signature, we must do the Wick rotation $\tau \to -i\tau$ and define complex worldsheet coordinates $z = e^{\tau - i\sigma}$. If we define the Euclidean action as $S_E = iS$, then, dropping the subscript E, we get

$$S = \int_\Sigma d^2z E_{\mu\nu}\partial X^\mu(z)\bar{\partial}X^\nu(\bar{z}), \tag{2.15}$$

where now

$$\partial_{\text{Plane}} := \partial_z = -\frac{i}{z}\partial_{\text{Cylinder}}, \qquad \bar{\partial}_{\text{Plane}} := \partial_{\bar{z}} = -\frac{i}{\bar{z}}\bar{\partial}_{\text{Cylinder}}, \tag{2.16}$$

and our conventions are such that $d^2z = dz \wedge d\bar{z}/2\pi i$. In this signature, the mode expansions are now

$$X^\mu(z,\bar{z}) = x^\mu - \frac{i}{\sqrt{2}}\Big(\alpha^\mu_0\log(z) + \bar{\alpha}^\mu_0\log(\bar{z})\Big) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha^\mu_n z^{-n} + \bar{\alpha}^\mu_n\bar{z}^{-n}\right), \tag{2.17}$$

$$\Pi_\mu(z,\bar{z}) \equiv 2\pi P_\mu(z,\bar{z}) = p_\mu + \frac{1}{\sqrt{2}}\sum_{n\neq 0}\left(E^T_{\mu\nu}\alpha^\nu_n z^{-n} + E_{\mu\nu}\bar{\alpha}^\nu_n\bar{z}^{-n}\right). \tag{2.18}$$

Instead of commutation relations, we now have the OPEs

$$: X^\mu(z) :: X^\nu(w) :=: X^\mu(z) X^\nu(w) : -\frac{1}{2} g^{\mu\nu} \log |z-w|^2, \tag{2.19}$$

and

$$z\partial X_\mu(z) = -\frac{i}{2}\left(\Pi_\mu(z,\bar{z}) - E_{\mu\nu} X'^\nu(z,\bar{z})\right), \qquad \bar{z}\bar{\partial} X_\mu(\bar{z}) = -\frac{i}{2}\left(\Pi_\mu(z,\bar{z}) + E^T_{\mu\nu} X'^\nu(z,\bar{z})\right), \tag{2.20}$$

with mode expansions

$$\partial X_\mu(z) = -\frac{i}{\sqrt{2}} \sum_{n=-\infty}^{\infty} g_{\mu\nu} \alpha_n^\nu z^{-n-1}, \tag{2.21}$$

$$\bar{\partial} X_\mu(\bar{z}) = -\frac{i}{\sqrt{2}} \sum_{n=-\infty}^{\infty} g_{\mu\nu} \bar{\alpha}_n^\nu \bar{z}^{-n-1}. \tag{2.22}$$

Let us now look more closely at the $d = 1$ case, i.e. where the compact target space is simply a circle. We will illustrate the phenomenon of T-duality and then we will generalise this to the T^d case.

2.2 String theory on a circle & T-duality

Consider string theory on a circle of radius R, where we have the identification

$$X \sim X + 2\pi R. \tag{2.23}$$

In addition to compactifying the momentum to take integer values, we also have *winding sectors* defined by

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi R w. \tag{2.24}$$

The mode expansions are as given in (2.17) with $E = g = R^2$, and in Minkowski signature the action on the circle is

$$S[X] = -\frac{1}{4\pi} \int d^2\sigma R^2 (-\dot{X}^2 + X'^2). \tag{2.25}$$

On the plane, we have action

$$S = \int_\Sigma d^2 z R^2 \partial_z X(z) \partial_{\bar{z}} X(\bar{z}). \tag{2.26}$$

To discuss T-duality for string theory on a circle, an illuminating approach is to look at the physical state (or mass-shell) conditions, derived from the Virasoro modes $L_0, \bar{L}_0$. These can be written as

$$m^2 = \frac{p^2}{R^2} + w^2R^2 + 2(N + \bar{N} - 2), \tag{2.27}$$

$$0 = pw + N - \bar{N}, \tag{2.28}$$

where $N, \bar{N}$ are the level number operators given by

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n, \qquad \bar{N} = \sum_{n=1}^{\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n, \tag{2.29}$$

where the contraction of the modes includes all directions, including any external (non-compact) directions. The mass-shell conditions can be used to compute the physical states, e.g. if $m^2 = 0$, we get the usual metric, B-field and dilaton states corresponding to the symmetric, anti-symmetric and trace parts of a general rank 2 tensor respectively. Now, under the transformation

$$R \to \frac{1}{R}, \qquad p \leftrightarrow w, \tag{2.30}$$

we see that these physical state conditions are invariant, i.e. this transformation is a symmetry of the theory. However, it is not a symmetry of the action and therefore it takes us to a different, but physically equivalent, sigma model. Thus, we think of this symmetry as a kind of 'duality' between different string backgrounds, and this is what we call *T-duality*. Note that, when $R = 1$, this is a special case where the duality is in fact a symmetry of the sigma model. This is called the 'self-dual radius', and will be extremely important throughout this thesis. Let us discuss this point in more detail.

2.2.1 The self-dual radius

At generic R, the theory has a $U(1)^2$ gauge symmetry (i.e. $U(1)_L \times U(1)_R$) which is exhibited by the fact that there is just a single $(1, 0)$ current, the massless gauge boson $J^3 = i\partial X$, and a single $(0, 1)$ current $\bar{J}^3 = i\bar{\partial} X$.

However, at $R = 1$, (2.27) has an enlarged set of solutions since the momentum and winding terms can now combine. In particular, there are additional massless gauge bosons, and it can be shown that the states at each mass-level now form multiplets of

$SU(2)$. Thus, there is a symmetry enhancement

$$U(1)_L \times U(1)_R \to SU(2)_L \times SU(2)_R. \tag{2.31}$$

The (left-moving) $SU(2)$ is generated by the currents

$$\begin{aligned} J^1(z) &= \cos(2X_L(G)(z)), \\ J^2(z) &= \sin(2X_L(G)(z)), \\ J^3(z) &= i\partial X(G)(z), \end{aligned} \tag{2.32}$$

where the background $E = G$ is the self-dual radius point, and we have labelled it explicitly here for clarity. Note that, throughout this thesis, we will use $E = G$ to refer to the background with metric $g = G \equiv \mathbb{1}$ and $B = 0$, and lowercase g will always refer to a general metric. The currents (2.32) can be shown to form a level 1 $SU(2)$ affine lie algebra

$$J^i(z)J^j(w) \sim \frac{\delta^{ij}}{2(z-w)^2} + i\frac{\epsilon^{ijk}}{z-w}J^k(w). \tag{2.33}$$

In fact, it was shown in [27] that, at the self-dual radius, string theory on a circle is equivalent to the level 1 $SU(2)$ WZW model. Furthermore, the $\mathbb{Z}_2$ T-duality symmetry group is a subgroup of this $SU(2)$, i.e. we can think of T-duality as originating from a symmetry group of the $R = 1$ theory that is broken away from this point. We will discuss this idea in more detail in section 2.4.

The fact that we can view string theory at the self-dual radius (SDR) as either a string on a circle or an embedding into an $SU(2)$ group manifold, a 3-dimensional geometry, shows that strings see geometry very differently to point particles, and such equivalences come up frequently in string theory. Let us now discuss how these ideas generalise to higher dimensions.

2.3 T-duality on a flat torus: the Buscher rules

Consider the case where the compactified spacetime (or at least part of it) is a torus of some dimensions $d \geq 1$. Instead of using the above approach of looking at the physical state conditions, we will instead use the *Buscher procedure* [7, 8] to derive the T-duality transformation for a general background. As we briefly mentioned in chapter 1, this is a procedure by which we gauge a global, compact isometry in the target space. As we will see, the gauged action can be reduced to either the original action or a new action, the latter defining the dual theory.

It is useful to see the α' dependence explicitly here, so we will put it back in. Start with the action

$$S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} \left[g_{\mu\nu} dX^\mu \wedge \star dX^\nu - i B_{\mu\nu} dX^\mu \wedge dX^\nu + \alpha' R\phi * 1 \right], \tag{2.34}$$

where ϕ is the dilaton. Note that the dilaton only appears at $O(\alpha'^0)$, whereas the metric and B-field are at $O(\alpha'^{-1})$. We suppose that this is the action on some compact target space manifold Σ embedded in some larger spacetime. The requirement of compactness is crucial to the validity of this procedure. The basis of the Buscher procedure, aside from compactness, is the existence of globally defined continuous isometries, i.e. we suppose that there is some symmetry

$$\delta_\epsilon X^\mu = \epsilon k^\mu(X), \tag{2.35}$$

which is a symmetry of the action as long as

$$\mathcal{L}_k g = 0, \quad \mathcal{L}_k B = dv, \quad \mathcal{L}_k \phi = 0, \tag{2.36}$$

where v is some globally defined one-form on Σ. We now gauge this symmetry, i.e. we allow ϵ to depend on the worldsheet coordinates, by introducing the gauge field A. For the metric term, we simply make the replacement $dX^\mu \to dX^\mu + k^\mu A$. For the B-field, due to the one-form v we must also introduce an extra scalar field χ which will in fact play an important role when we deduce the dual action. The gauged action is then

$$\begin{aligned} \hat{S} = -\frac{1}{2\pi\alpha'} \int_{\Sigma} \Big[& \frac{1}{2} g_{\mu\nu} (dX^\mu + k^\mu A) \wedge \star (dX^\nu + k^\nu A) - \frac{i}{2} B_{\mu\nu} dX^\mu \wedge dX^\nu \\ & - i(v - \iota_k B + d\chi) \wedge A \Big], \end{aligned} \tag{2.37}$$

where we have dropped the dilaton term since it plays no role in this gauging procedure (we will come back to it at the end). This gauged action is indeed invariant under the local symmetry transformations

$$\hat{\delta}_\epsilon X^\mu = \epsilon k^\mu, \qquad \hat{\delta}_\epsilon A = -d\epsilon, \qquad \hat{\delta}_\epsilon = -\epsilon \iota_k v. \tag{2.38}$$

Now, before deriving the duality, we make some simplifying assumptions. Specifically, we assume we are using coordinates where $k^\mu = (1, 0, ..., 0)^T$, and that we are in a B-field gauge where $v = 0$. Overall, these assumptions reduce our initial conditions on the background fields to

$$\partial_1 g = \partial_1 B = 0, \tag{2.39}$$

i.e. the metric and B-field are independent of the coordinate X^1. Now, to deduce the T-duality, there are two things we must do: firstly, we must be able to return to our original, ungauged action. Secondly, we should be able to arrive at some sort of dual action which has the same form as the original action, but with a different metric and B-field.

Returning to the ungauged action

To return to the ungauged action, we must be able to set $A = 0$ via a gauge transformation. To do this, A must be pure gauge. We will show that this is the case under the equations of motion of χ, i.e. χ ensures that no additional degrees of freedom are introduced during the gauging procedure. To show this, we first introduce a basis of harmonic one-forms on the worldsheet,

$$\omega^m \in \mathcal{H}^1(\Sigma, \mathbb{R}), \qquad m = 1, ..., 2g, \tag{2.40}$$

where g denotes the genus of Σ. This group of harmonic one-forms is isomorphic to the first de Rham cohomology group $H^1(\Sigma, \mathbb{R})$, and if we let the basis of the corresponding first homology group be $\gamma_m \in H_1(\Sigma, \mathbb{Z})$, we have

$$\int_{\gamma_m} \omega^n = \delta_m{}^n. \tag{2.41}$$

Now, using the Hodge decomposition theorem, we can express the closed one-form $d\chi$ as

$$d\chi = d\chi_{(0)} + \chi_{(m)}\omega^m, \tag{2.42}$$

and substituting this into the gauged action and varying it wrt $\chi_{(0)}$ gives the equation of motion

$$F = dA = 0. \tag{2.43}$$

Thus, in general, we have

$$A = da_{(0)} + a_{(m)}\omega^m. \tag{2.44}$$

In order to be able to set $A = 0$, we need A to be pure gauge, i.e. we need the $a_{(m)}$ to vanish. The $a_{(m)}$ correspond to possible Wilson loops $W_\gamma = \exp(2\pi i \oint_\gamma A)$ of A around one-cycles $\gamma \int H_1(\Sigma, \mathbb{Z})$. If we now vary the gauged action wrt $\chi_{(m)}$, we indeed find that $a_{(m)} = 0$ and that A is pure gauge. Thus, we can set $A = 0$ and we recover the ungauged action, as required.

The dual action

To obtain the dual action, we now integrate out A by varying wrt χ. Since χ only appears as a lagrange multiplier, we simply obtain an algebraic expression for A which we can substitute back into the gauged action. Doing so, we obtain

$$\begin{aligned}\tilde{S} = -\frac{1}{2\pi\alpha'}\int_{\Sigma}\Bigg[&\frac{1}{2}\left(g_{\mu\nu} - \frac{g_{\mu 1}g_{\nu 1} - B_{\mu 1}B_{\nu 1}}{g_{11}}\right)dX^\mu \wedge \star dX^\nu + \frac{1}{2}\frac{\alpha'^2}{g_{11}}d\tilde{X}^1 \wedge d\tilde{X}^1 \\ &+ \alpha'\frac{B_{\mu 1}}{g_{11}}d\tilde{X}^1 \wedge \star dX^\nu - \frac{i}{2}\left(B_{\mu\nu} - \frac{B_{\mu 1}g_{\nu 1} - g_{\mu 1}B_{\nu 1}}{g_{11}}\right)dX^\mu \wedge dX^\nu \\ &-i\alpha'\frac{g_{\mu 1}}{g_{11}}dX^\mu \wedge d\tilde{X}^1 - i\alpha' dX^1 \wedge d\tilde{X}^1\Bigg], \end{aligned} \tag{2.45}$$

where we have defined

$$d\tilde{X}^1 = \frac{1}{\alpha'}d\chi. \tag{2.46}$$

From this, we can read off the dual background as

$$\begin{aligned} \tilde{g}_{11} &= \frac{\alpha'^2}{g_{11}}, \\ \tilde{g}_{\mu 1} &= \alpha'\frac{B_{\mu 1}}{g_{11}}, \qquad \tilde{B}_{\mu 1} = \alpha'\frac{g_{\mu 1}}{g_{11}}, \\ \tilde{g}_{\mu\nu} &= g_{\mu\nu} - \frac{g_{\mu 1}g_{\nu 1} - B_{\mu 1}B_{\nu 1}}{g_{11}}, \qquad \tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu 1}g_{\nu 1} - g_{\mu 1}B_{\nu 1}}{g_{11}}. \end{aligned} \tag{2.47}$$

These are called the *Buscher rules* and we can see that, in the $d = 1$ case, they simply reduce to $\tilde{R} = \frac{\alpha'}{R}$, i.e. we recover the results we derived previously for the circle. Note that we could have done this for the circle case, but it is enlightening to see how T-duality works from the perspective of the state space since the physics is clearer.

Finally, we note that everything we have done here has been at the level of the supergravity, i.e. lowest order in α'. At this order, there is no change to the dilaton. However, if we look at higher loop calculations, we find that there are corrections to the dilaton. For example, at one-loop, we find that [28]

$$\tilde{\phi} = \phi - \frac{1}{4}\frac{\det g}{\det \tilde{g}}, \tag{2.48}$$

and further corrections can be obtained from higher loop order calculations. We only include the dilaton here for completeness, and it will not concern us in what follows.

2.4 Discrete symmetries and gauge transformations of toroidal backgrounds

We now discuss symmetries in toroidal compactifications in more detail. Earlier, we described the symmetry enhancement of the $d = 1$ circle compactification at the so-called self-dual radius $R = 1$ from $U(1)$ to $SU(2)$. Furthermore, we described how, at this point, there are extra gauge bosons and the states at each mass level form multiplets of $SU(2)$. We also mentioned how it was shown [27] that, at this point, the theory is equivalent to a level 1 $SU(2)$ WZW model.

This structure can be generalised and put in a more geometrical setting. Here we review this construction, based largely on [26] and references therein. We will not directly make use of everything presented here, but it is useful to know in order to understand the general picture.

In chapter 2, we described T-duality as a discrete symmetry between points in the moduli space of toroidal compactifications. In fact, note that, in (2.1), we quotient by the discrete symmetry group $O(d, d; \mathbb{Z})$, not just T-duality. T-duality is actually just a subgroup of this larger group, and there are other discrete symmetries that together form $O(d, d; \mathbb{Z})$ (this group is sometimes called the 'generalised T-duality group', but we will usually refrain from such terminology to avoid confusion). In this section, we will use the notation $\mathcal{G} \equiv O(d, d; \mathbb{Z})$. What we will show is that T-duality is part of a special subgroup of $\mathcal{G}$ which can be interpreted as gauge transformations on the moduli space (2.1).

First, let us discuss gauge transformations in toroidal compactifications in more generality [26]. In the moduli space (2.1), there are points at which the gauge symmetry is enhanced to some $g_L \times g_R$, and we say such points have *maximally extended* symmetry if $g_L = g_R = g$, where g is a semi-simple group of rank d. For example, in $d = 1$ we have the enhanced symmetry point where $g = SU(2)$. In what follows we will often use $d = 2$ as an illustrative example that is simple enough to work with explicitly, but still manages to capture many of the complications that may arise in the general case. In this case, in addition to the $g_L = g_R = SU(2)^2$ point with metric $G = 1$ and B-field $B = 0$, there is also a point where we have $g = SU(3)$ (corresponding to A_2 in the

ADE classification), with background

$$E(SU_3) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.49}$$

In fact, at a given point with maximally extended symmetry, the metric (i.e. the symmetric part of E) is (one half of) the Cartan matrix of the lie algebra corresponding to the symmetry group g. Furthermore, this point in moduli space can always be described by a level 1 WZW model with group manifold g, or as a theory of d free bosons with background E. Throughout this thesis, we will often be considering deformations of operators starting from the $SU(2)$ point of enhanced symmetry. Such deformations can be considered as deformations of the action S_E at the point of enhanced symmetry with background E. In general, a deformation of the theory at the point of enhanced symmetry with background E looks like

$$S(E,\varepsilon) = S_E + \Delta S, \qquad \Delta S = \varepsilon_{ij} J^i \bar{J}^j, \tag{2.50}$$

where the J^i are the $(1,0)$ currents, $i = 1, ...\mathrm{dim}(g)$, and $\bar{J}^i$ are the $(0,1)$ currents. However, not all of these deformations are independent, and in fact it turns out that the moduli space in the neighbourhood of E is spanned by the currents which are elements of the Cartan subalgebra of g, so

$$\Delta S = \varepsilon_{ab} H^a \bar{H}^b, \tag{2.51}$$

$a = 1, ..., d$ (since $\mathrm{rank}(g) = d$), where H^a are the Cartan subalgebra elements of g_L, and $\bar{H}^a$ those for g_R. For example, recall from the $d = 1$ case that the moduli space is spanned by deformations of the radius of the circle, which are generated by $J^3 = i\partial X(G)$. If we have two deformations $S(E,\varepsilon), S(E,\varepsilon')$ such that $\varepsilon, \varepsilon'$ are related by a gauge transformation, such a transformation is a residual symmetry of the gauge group at the point E (e.g. T-duality is a residual gauge symmetry of $SU(2)$). Such gauge symmetries are then symmetries of the entire moduli space, and, as we will explain, the product of such symmetries form a subgroup of $\mathcal{G}$.

First, we must review the structure of $\mathcal{G}$. Recall that these are the matrices which leave

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.52}$$

invariant under conjugation. The generators of the group are as follows. We have P_{ab}, which is a permutation of the coordinates a and b. In $d = 2$, this is

$$P_{12} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \qquad p = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.53}$$

We also have B_{ab}, which has the matrix $\Theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the (a, b) position, e.g. for $d = 2$,

$$B_{12} = \begin{pmatrix} 1 & \Theta \\ 0 & 1 \end{pmatrix}. \tag{2.54}$$

We also have

$$T_{12} = \begin{pmatrix} t & 0 \\ 0 & t^{-T} \end{pmatrix}, \qquad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.55}$$

For general d, we can get any T_{ab} and any B_{ab} by conjugating with the appropriate $P's$. We also have reflections, such as

$$R_1 = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \qquad r = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.56}$$

which is a reflection in the first coordinate. For $d = 2$, $S_{12} = RP_{12}$ and T_{12} are generators of $SL(2, \mathbb{Z})$. The final generators are given by

$$D_a = \begin{pmatrix} 1 - e_a & e_a \\ e_a & 1 - e_a \end{pmatrix}, \tag{2.57}$$

where e_a has 1 in the aa component and 0 everywhere else. These D_a are the T-duality transformations, e.g. in $d = 1$ we just have a single D_a, namely

$$D_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.58}$$

which induces the $R \to 1/R$ duality (this can be viewed as swapping the embedding coordinate X with its dual).

These are the generators of $\mathcal{G}$, and now we discuss how they relate to gauge transformations on the moduli space. Let Λ be such that $T^d = \mathbb{R}^d/\Lambda$ is the torus lattice at a point of enhanced symmetry with background E. Then, [26] proved the following statements:

S1: For every pair of Weyl transformations $w, \bar{w}$ of Λ, there is a corresponding element $g(w, \bar{w}, E) \in \mathcal{G}$.

In [26], $g(w, \bar{w}, E)$ was constructed explicitly. If we have Weyl reflections $w, \bar{w}$, these act on the Cartan elements as

$$H \to w^T H, \qquad \bar{H} \to \bar{w}^T \bar{H}, \tag{2.59}$$

and this is a symmetry of the deformed action $S(E, \varepsilon) = S_E + H^T \varepsilon \bar{H}$ because S_E is invariant under Weyl reflections. Furthermore, this can be written as a transformation of ε,

$$\varepsilon \to w \varepsilon \bar{w}^T \equiv \varepsilon'. \tag{2.60}$$

Thus, $g(w, \bar{w}, E)$ should be such that $g(E + \varepsilon) = E + \varepsilon'$, and indeed it was found that

$$g(w, \bar{w}, E) = A(E) \begin{pmatrix} \bar{w}^{-T} & 0 \\ 0 & w \end{pmatrix} A^{-1}(E), \tag{2.61}$$

where

$$A(E) = \begin{pmatrix} \frac{1}{2}E & -E^T(\mathrm{sym}(E))^{-1} \\ \mathbf{1}/2 & (\mathrm{sym}(E))^{-1} \end{pmatrix}, \qquad A^{-1}(E) = \begin{pmatrix} (\mathrm{sym}(E))^{-1} & (\mathrm{sym}(E))^{-1}E^T \\ -1/2 & \frac{1}{2}E \end{pmatrix}, \tag{2.62}$$

where $\mathrm{sym}(E)$ is the metric at the point E, and we have written it like this to avoid any confusion, since we are using g here for groups and elements of $\mathcal{G}$.

S2: If $g \in \mathcal{G}$, then

$$g = \prod_a g(w_a, \bar{w}_a, E_a), \tag{2.63}$$

for enhanced symmetry points E_a, with $w_a, \bar{w}_a$ Weyl transformations of Λ satisfying

$$w_a(\mathrm{sym}(E))w_a^T = \bar{w}_a(\mathrm{sym}(E))\bar{w}_a^T = \mathrm{sym}(E). \tag{2.64}$$

The proof of this was given by construction for $d = 2$. In general, this can be proven by explicitly writing the generators in the form (2.63), and for $d = 2$ we have

$$\begin{aligned} D_1 = g(r, 1, \mathbf{1}), \qquad R = g(r, r, \mathbf{1}), \qquad P = g(p, p, \mathbf{1}), \\ B = g(p, p, E(SU_3)), \qquad T = RPDB^{-1}g(p, 1, E(SU_3))PR, \end{aligned} \tag{2.65}$$

where $D = D_1 D_2$ is the transformation that does $E \to E^{-1}$.

S3: The residual symmetries of the broken gauge groups (i.e. the symmetries which remain after we deform away from a point of enhanced symmetry, such as T-duality) are precisely the elements

$$\mathcal{T} = \left\{ \prod_a g(w_a, \bar{w}_a, E_a) \right\} \subset \mathcal{G}. \tag{2.66}$$

Furthermore, these are the elements of $\mathcal{G}$ which contain even numbers of the generators B, P and T. This can be proven for $d = 2$ using (2.65).

S4: The E_a appearing in $S3$ can be chosen to correspond to the root lattice of the same group. Furthermore, elements of $\mathcal{T}$ can be written in terms of any semi-simple simply laced group of rank d (recall that simply laced groups have root vectors all of the same length and that the ADE groups are all simply laced). It was argued in [26] that $\mathcal{T}$ is generated solely by $g \in \mathcal{G}$ corresponding to Weyl reflections at the points with $SU(2)^d$ enhanced symmetry.

The transformation P is not itself a gauge transformation as it is an outer automorphism of $SU(2)^2$, and so, overall, we have that $\mathcal{G} = \mathcal{T} \times \mathbb{Z}_2$, where $\mathbb{Z}_2$ is the outer automorphism generated by P. Thus, we can relate elements of $O(d,d;\mathbb{Z})$ to residual gauge transformations from the points of enhanced symmetry.

2.4.1 Example: the space of compact $c = 1$ CFTs

The space of compact $c = 1$ CFTs (i.e. a single compact boson) is simple enough to describe in its entirety, and can also be easily visualised. The $c = 1$ CFTs are essentially of two types: toroidal CFTs and orbifold CFTs. Before describing this in detail, let us briefly review orbifold CFTs.

Orbifold CFTs

We know that string compactifications can be obtained by considering a periodic identification of the string embedding. In a similar way, we can identify the embedding under reflections,

$$X \sim -X. \tag{2.67}$$

If we are considering a non-compact worldsheet, then we could have $X \in \mathbb{R}$, in which case the identification (2.67) would effectively reduce the worldsheet to the half-line $\mathbb{R}^{\geq 0}$. Of particular interest for us are the *compact* $c = 1$ CFTs. Thus, we combine this reflection with the periodic identification $X \sim X + 2\pi Rw$, so that we obtain the worldsheet $S^1/\mathbb{Z}_2$. This space is known as an *orbifold*, and in general dimensions we would have $T^d/\mathbb{Z}_2$, though we could also have other orbifolds in higher dimensions depending on the lattice symmetry of the torus.

As we will see, the effect that this extra identification has on the theory is that it projects more states out and it also introduces a new sector. This can be seen by considering what happens on the cylinder as we move around the periodic direction. Since we have (2.67), we do not need the string to come back to itself up to winding, but rather we can also have

$$X(\sigma + 2\pi) = -X(\sigma), \tag{2.68}$$

i.e. we only need to come back up to a reflection. This sector is known as the *twisted sector*. In this sector, the mode expansion of X changes to reflect the antisymmetry, and we now have

$$X(z, \bar{z}) = \frac{i}{\sqrt{2}} \sum_n \frac{1}{n + \frac{1}{2}} \left(\alpha_{n+1/2} z^{-n-1/2} + \bar{\alpha}_{n+1/2} \bar{z}^{-n-1/2} \right). \tag{2.69}$$

Note that there are no zero modes here because such terms would not be consistent with the antiperiodicity. Note also that, in general, we have such twisted sectors wherever we have fixed points of the twist action. In this case, we have fixed points at $X = 0$ and $X = 2\pi R$, so there will be another twisted sector at $2\pi R$ with a similar mode expansion, but with a constant $2\pi R$ term.

More generally, constructions such as the orbifolding described here are examples of *twisting*. This is where we have an underlying CFT, such as the circle CFT in the case considered above, and we quotient by some symmetry of the CFT, H. The resulting theory will have multiple sectors. First, we have the states of the underlying theory. These will be projected onto the space of states which are invariant under the action of H. Additionally, we have the twisted sectors defined by

$$X(\sigma + 2\pi) = hX(\sigma), \tag{2.70}$$

where $h \in H$. Starting from this, we would find the physical states as usual and then project onto H-invariant states. Not all elements of H give unique twisted sectors, and in fact the independent twisted sectors are determined by the conjugacy classes of H. For $c = 1$ CFTs, the two main types of twisting are periodic identification, giving us toroidal backgrounds, and reflection, giving us orbifolds. As we will see, there are also three other isolated types of compact $c = 1$ CFTs, which utilise the extra symmetry attained at the self-dual radius.

There is much more to be said about orbifolds and twisted sectors in string theory more generally, and there is much discussion of such backgrounds in the literature (see, for example, [26, 29]). Such backgrounds are central to string compactifications in superstring theories, such as Calabi-Yau constructions [30]. In this thesis however, we will not be particularly interested in orbifolds, though there are certainly interesting questions one can ask given the results we will present. For now, let us describe the space of compact CFTs with central charge $c = 1$.

Compact $c = 1$ CFTs

As we saw, we have two families of compact $c = 1$ CFTs, toroidal target spaces and orbifold target spaces. In both cases, due to T-duality, the radius of the circle is in the range $R \in [1, \infty]$. Let us see how the two families are related. Starting from the circle at the self-dual radius, we can scale the radius by twisting by the action

$$r : X \to X + \pi. \tag{2.71}$$

This effectively halves the radius of the circle, which gives radius $R = 1/2$, but this is T-dual to $R = 2$, so we have ultimately doubled the radius. Now, consider the action of r on the $SU(2)$ currents (2.32). r flips the signs of $J^{1,2}$ and leaves J^3 invariant, so, thinking of the SDR theory as a WZW model on the 3-sphere (the group manifold of $SU(2)$), this action is a rotation by π around the 3-axis. Consider also the orbifold action

$$s : X \to -X. \tag{2.72}$$

This flips the signs of $J^{2,3}$ and leaves J^1 invariant, so is a rotation by π around the 1-axis. However, r and s must be equivalent due to the symmetry of the 3-sphere, so taking the quotient of the theory at the SDR by these actions must give the same result. As we saw, the action of r induces the $R = 2$ toroidal theory, and the action of s induces the $R = 1$ orbifold theory. Thus, these two theories are equal, as can also be checked explicitly using the partition functions. This is the only point at which the two families coincide, which can be seen from the fact that, at generic radius, since the toroidal theory has only $U(1)^2$ gauge symmetry and the orbifold does not have any gauge symmetry, there is no way to relate the two families. Note that the orbifolding does not remove the $J^3\bar{J}^3$ modulus, so at any radius of either family we are able to deform the radius of the circle. Thus, we should think of this space as two semi-infinite (semi because of T-duality) lines meeting at the point described above.

There are in fact three other special CFTs with $c = 1$, but these are isolated points. They are obtained by quotienting the theory at the SDR by the tetrahedral, octahedral or icosahedral groups. Note that these symmetries are only available at the SDR, so when we quotient by them we lose the modulus which allows us to deform the radius (in fact, the only scalar which survives is the dilaton). Thus, starting from any of these three theories, we cannot deform to any other theory, so they are isolated points in the moduli space. This completes the description of compact $c = 1$ CFTs.

Much of the above $O(d, d)$ discussion can be better understood when viewed from an explicitly $O(d, d)$ covariant formalism. We now turn to such a formalism: *doubled geometry.*

Chapter 3

Doubled Geometry & Torus Bundles

The doubled geometry was constructed as a way of making the $O(d,d;\mathbb{Z})$ symmetry of toroidal compactifications explicit. The formalism was originally developed in [21, 11] and applied to toroidal flux compactifications in [12, 31–33]. As we will see, the T-duality becomes particularly simple in this formalism because it is manifestly covariant under $O(d,d;\mathbb{Z})$ transformations, so T-dualities amount to simple linear transformations.

3.1 General description

Here, we describe the doubled geometry for toroidal backgrounds fibred over a circle. The basic construction involves doubling the fibres (i.e. the torus), but, as explained below, this can be seen as a special case of the more general 'twisted torus' background where all coordinates are doubled.

First, we will look at the doubled torus without the fibration over the base. This is a torus T^{2d} with $O(d,d)$-covariant coordinates $\mathbb{X}^I = (z^\mu, \tilde{z}_\mu)$, where $\tilde{z}_\mu$ are usually referred to as the dual coordinates.

An $O(d,d)$-invariant metric on the doubled space, L_{IJ}, is given by [12]

$$ds^2 = \frac{1}{2} L_{IJ} d\mathbb{X}^I d\mathbb{X}^J = dz^\mu d\tilde{z}_\mu, \tag{3.1}$$

where $\mu = 1, ..., d$. This is the metric that is used to raise/lower indices.

For a given constant background $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$, we have the doubled metric

$$\mathcal{H}_{IJ} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\rho} g^{\rho\lambda} B_{\lambda\nu} & B_{\mu\rho} g^{\rho\nu} \\ -g^{\mu\rho} B_{\rho\nu} & g^{\mu\nu} \end{pmatrix}. \tag{3.2}$$

This combines the metric and B-field into a single $O(d,d)$-covariant tensor.

Let us now look at the fibration of this doubled torus over a circle. There are two cases here: the first is where we simply take the T^{2d} as above and fibre over a circle, and the second is where we also double the base, so that all coordinates are doubled. We start with the doubled torus bundle with fibre T^{2d} and a base with coordinate $x \sim x + 2\pi$. The doubled torus bundle over a circle

$$T^{2d} \hookrightarrow \mathcal{T} \to S^1, \tag{3.3}$$

with monodromy $e^N \in O(d,d)$, may be thought of as a $2d+1$ dimensional twisted torus [11, 34, 35]. That is, a manifold that is locally a group $\mathcal{G}_{2d+1}$, but is globally of the form $\mathcal{T} = \mathcal{G}_{2d+1}/\Gamma_{2d+1}$, where $\Gamma_{2d+1} \subset \mathcal{G}_{2d+1}$ is a discrete (cocompact) group acting from the left such that $\mathcal{T}$ is compact. $\mathcal{T}$ is parallelisable and as such has globally defined, left-invariant one forms

$$P^x = dx, \qquad \mathcal{P}^I = (e^{Nx})^I{}_J d\mathbb{X}^J. \tag{3.4}$$

The fact that they are left-invariant under the action of Γ_{2d+1} means that they are well-defined on the quotient $\mathcal{T}$ and not just the manifold $\mathcal{G}_{2d+1}$. As we will see when we look at the nilfold, we can also construct right-invariant one-forms and vector fields which are locally, but not necessarily globally, defined on $\mathcal{T}$.

A metric on $\mathcal{T}$ is

$$ds^2 = dx \otimes dx + \mathcal{H}_{IJ}(x) d\mathbb{X}^I \otimes d\mathbb{X}^J, \tag{3.5}$$

where $\mathcal{H}_{IJ}(x) = (e^{Nx})_I{}^K G_{KL} (e^{N^T x})^L{}_J$ and G_{IJ} is the metric on the untwisted torus fibre. The isometries of $\mathcal{T}$ are generated by the vector fields

$$Z_x = \frac{\partial}{\partial x}, \qquad T_I = (e^{-Nx})_I{}^J \frac{\partial}{\partial \mathbb{X}^J}, \tag{3.6}$$

and close to give the algebra

$$[T_I, T_J] = 0, \qquad [Z_x, T_I] = -N_I{}^J T_J. \tag{3.7}$$

If we compactify supergravity on $\mathcal{T}$, the consistent truncation has gauge algebra which includes

$$[T_I, T_J] = N_{IJ}X^x, \qquad [Z_x, T_I] = -N_I{}^J T_J. \tag{3.8}$$

We see that (3.7) is a contraction of (3.8), where the missing generator is $X^x = \partial/\partial\tilde{x}$ and is associated with B-field transformations with gauge parameter lying along the base circle [12].

The algebra (3.8) may be realised as the isometry algebra of the twisted torus $\mathcal{G}_{2d+2}/\Gamma_{2d+2}$, where $\mathcal{G}_{2d+2}$ is a $2d+2$ dimensional group and Γ_{2d+2} is a cocompact subgroup of $\mathcal{G}_{2d+2}$ which acts from the left. The left-invariant one forms on $\mathcal{G}_{2d+2}/\Gamma_{2d+2}$ are

$$P^x = dx, \qquad Q_x = d\tilde{x} + \frac{1}{2}N_{IJ}\mathbb{X}^I d\mathbb{X}^J, \qquad \mathcal{P}^I = (e^{Nx})^I{}_J d\mathbb{X}^J, \tag{3.9}$$

and a natural metric on $\mathcal{G}_{2d+2}/\Gamma_{2d+2}$ is given by

$$ds^2 = dx \otimes dx + \mathcal{H}_{IJ}(x) d\mathbb{X}^I \otimes d\mathbb{X}^J + Q_x \otimes Q_x. \tag{3.10}$$

We see that the metric depends on *all* of the coordinates of $\mathcal{T}$ (the $\mathbb{X}^I$ in addition to the base circle) except $\tilde{x}$. In particular, Q_x may depend on all of the z^μ and $\tilde{z}_\mu$ coordinates. The isometry group of the $2d+2$ dimensional twisted torus is generated by the vector fields dual to these one-forms,

$$Z_x = \frac{\partial}{\partial x}, \qquad X^x = \frac{\partial}{\partial \tilde{x}}, \qquad T_I = (e^{-Nx})_I{}^J \left(\frac{\partial}{\partial \mathbb{X}^J} - \frac{1}{2}N_{JK}\mathbb{X}^K \frac{\partial}{\partial \tilde{x}} \right). \tag{3.11}$$

A gauge transformation[1] brings the vector fields to the convenient form

$$Z_x = \frac{\partial}{\partial x} + N^I{}_J \mathbb{X}^J \frac{\partial}{\partial \mathbb{X}^I}, \qquad X^x = \frac{\partial}{\partial \tilde{x}}, \qquad T_I = \frac{\partial}{\partial \mathbb{X}^I} - \frac{1}{2}N_{IJ}\mathbb{X}^J \frac{\partial}{\partial \tilde{x}}. \tag{3.12}$$

In some sense, the $2d+1$ dimensional twisted torus is less fundamental than the $2d+2$ dimensional construction, as the former arises, from the perspective of the underlying Lie algebra, as a contraction of the latter. However, the route from the physical space to $\mathcal{T}$ is intuitively very clear - the physical space, in any polarisation, is a torus bundle with a given monodromy and the $\mathcal{T}$ encodes that monodromy geometrically. By contrast, $\mathcal{G}_{2d+2}/\Gamma_{2d+2}$ does not follow in an obvious way from the physical construction, making it difficult to generalise to non-parallelisable cases. Specifically, when considering the original construction - a T^2 bundle over S^1 with monodromy e^N - it is far from obvious

[1] $\mathbb{X}^I \to (e^{Nx})^I{}_J \mathbb{X}^J$.

a priori that the metric on $\mathcal{G}_{2d+2}/\Gamma_{2d+2}$ will depend explicitly on the $\mathbb{X}^I$, whereas the metric on $\mathcal{T}$ depends *only* on x and is determined by the monodromy.

In chapter 6, we will show that, when we consider the algebra of the pullback of the Z_x, T_I of (3.6) to the worldsheet, we actually recover the full doubled algebra (3.8) even though (3.6) themselves only generate the contracted algebra of $\mathcal{T}$, (3.7).

3.2 A chain of T-dualities: H-flux, nilfold, T-fold

Here, we briefly describe some twisted torus backgrounds and recover a familiar chain of T-dualities [12]. We will pause our discussion of doubled geometry here briefly, but in section 3.3 we will see how these backgrounds, and the dualities between them, are realised in the explicit doubled construction. We will see that the T-duality in particular is much easier to understand in the doubled formalism.

It is easiest, geometrically, to start with the nilfold. This is a T^2 fibred over a circle with some monodromy twist in the fibration, where the monodromy is an element of $SL(2,\mathbb{Z})$, i.e. the mapping class group of the T^2 fibre. We will describe two ways of constructing this background explicitly. The first way involves considering a general matrix in the lie algebra $\mathfrak{sl}(2)$ which generates the monodromy, $f^\mu{}_\nu$, where

$$f^\mu{}_\nu = \begin{pmatrix} 0 & -m \\ 0 & 0 \end{pmatrix}, \qquad \left(e^f\right)^\mu{}_\nu = \begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix}. \tag{3.13}$$

Note that this monodromy is in the fibre coordinates (y,z). Thus, we have a globally defined basis of one-forms given by

$$P^x = dx, \qquad P^y = dy - mxdz, \qquad P^z = dz. \tag{3.14}$$

From this, we can construct a metric on the bundle, $g = P^\mu P^\nu \delta_{\mu\nu}$, which gives the nilfold metric

$$ds^2 = dx^2 + (dy - mxdz)^2 + dz^2, \tag{3.15}$$

and we have the global coordinate identifications

$$(x,y,z) \sim (x+2\pi, y+mz, z), \qquad (x,y,z) \sim (x, y+2\pi, z), \qquad (x,y,z) \sim (z, y, z+2\pi). \tag{3.16}$$

Now, there is a second way we can construct this background. Namely, via a twisted torus construction similar to that described for the doubled formalism above. In the case of the nilfold, we may construct it using the Heisenberg group G. The generators of the Heisenberg group satisfy

$$[t_x, t_z] = m t_y, \qquad [t_y, t_z] = 0, \qquad [t_x, t_y] = 0, \tag{3.17}$$

i.e. $t_{x,y,z}$ span the Lie algebra, and a general element of the Heisenberg group can be written as $e^{xt_x+yt_y+zt_z}$ for $(x, y, z) \in \mathbb{R}^3$. The nilfold is a compact group though, so we also need to quotient G by a discrete subgroup Γ, where we choose $\Gamma \subset G$ such that

$$\Gamma = \{e^{xt_x+yt_y+zt_z} \in G : x, y, z \in \mathbb{Z}\}. \tag{3.18}$$

A general matrix representation of an element $g \in G$ can be taken to be

$$g = \begin{pmatrix} 1 & mx & y \\ 0 & 0 & z \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.19}$$

From this, we can construct left-invariant one forms $P = gdg^{-1}$. Note that, under the left action $g \to g'g$, for $g' \in G$, P is indeed invariant. The above matrix representation reproduces the one-forms of (3.14). The left-invariant vector fields dual to these one-forms are

$$Z_x = \frac{\partial}{\partial x}, \qquad Z_y = \frac{\partial}{\partial y}, \qquad Z_z = \frac{\partial}{\partial z} + mx\frac{\partial}{\partial y}, \tag{3.20}$$

and these generate the right action G_R of the Heisenberg group on G. Similarly, we can construct the right-invariant one-forms $\tilde{P} = dgg^{-1}$ (which can easily be seen to be invariant under $g \to gg'$), and the right-invariant vector fields

$$\tilde{Z}_x = \frac{\partial}{\partial x} + mz\frac{\partial}{\partial y}, \qquad \tilde{Z}_y = \frac{\partial}{\partial y}, \qquad \tilde{Z}_z = \frac{\partial}{\partial z}. \tag{3.21}$$

Note that the vector fields $Z_y, \tilde{Z}_z$ generate symmetries of the nilfold metric (3.15), which means they are potential candidates for generators of T-duality transformations, as discussed in the Buscher construction. Starting with Z_y, we can apply the Buscher rules in the y-direction to obtain the T^3 *with H-flux*, with metric and B-field given by

$$ds^2 = dx^2 + dy^2 + dz^2, \qquad B = mxdy \wedge dz. \tag{3.22}$$

Note that the general construction of the T^3 with H-flux is

$$ds^2 = dx^2 + dy^2 + dz^2, \qquad H = mdx \wedge dy \wedge dz, \tag{3.23}$$

but we automatically have a specific B-field gauge due to the choice of matrix representation for the nilfold from which we constructed the left-invariant one-forms.

Note also that these background do not constitute string theories on their own, but we usually imagine them to be embedded in larger backgrounds which overall *are* string theories.

Now, going back to the nilfold, we see that we also have an isometry generated by $\tilde{Z}_z$ (i.e. constant shifts in z), but since $\tilde{Z}_z$ is only right-invariant, it is not globally defined on the quotient G/Γ, since this is defined by the *left* action of Γ. In fact, under a shift in the base $x \to x + 2\pi$, we find that $\tilde{Z}_z \to \tilde{Z}_z - m\tilde{Z}_y$. Thus, technically, the conditions under which we may apply the Buscher procedure are not met because we need a *globally defined* compact isometry. However, we usually get around this problem by arguing that we go to a covering space where the isometry is globally defined. In this case, this amounts to decompactifying the x-coordinate so that the space now has topology $T^2 \times \mathbb{R}$. Now $\tilde{Z}_z$ is globally defined and we can apply the Buscher rules, which gives what we refer to as the *T-fold*, with background

$$ds^2 = dx^2 + \frac{1}{1+(mx)^2}(dy^2 + dz^2), \qquad B = -\frac{mx}{1+(mx)^2}(dy \wedge dz). \tag{3.24}$$

Since we are trying to compute the T-dual to the nilfold, we suppose that we should recompactify x after this duality. However, notice that neither the metric nor the B-field for the T-fold are periodic under $x \to x + 2\pi$, so this is not a smooth geometry in the usual sense. However, this *is* a well-defined geometry modulo T-duality transformations, i.e. if we allowed for T-duality transformations in the transition functions, then we can patch together local coordinates without issue (hence the name 'T-fold'). Thus, we describe this as a 'non-geometric background' in that it is not a manifold in the usual geometric sense. From the perspective of a string moving on this background, if we consider T-duality to be a symmetry of string theory, then the string does not 'see' this background any differently from the geometric backgrounds since T-dual backgrounds are equivalent. This is an example of how strings see geometry differently to point particles.

3.2.1 R-flux

The chain of dualities described here (H-flux $\to$ nilfold $\to$ T-fold) is suggestive of a third T-duality; namely, that of a T-duality in the base coordinate x. Of course, there is a problem here in that these backgrounds have explicit x-dependence, and as such we do not have an isometry in the base. However, the structure that emerges here, and which is particularly apparent in doubled geometry, suggests that such a duality still exists, even if it is unattainable via the Buscher procedure. If we try to handle this x-dependence by saying that we simply replace x with the dual coordinate $\tilde{x}$, we can imagine the duality as replacing the base circle with a dual base with coordinate $\tilde{x}$, and the gauge algebra would then become

$$[T_I, T_J] = N_{IJ} Z_x, \qquad [X^x, T_I] = -N_I{}^J T_J, \tag{3.25}$$

i.e. $X^x \leftrightarrow Z_x$, giving the so-called *R-flux* algebra. This may seem promising, but when we discuss polarisations below, we will see that it is a non-trivial matter of going from a doubled gauge algebra to a valid, undoubled background. In any case, since the starting point lacks any theoretical justification, i.e. since we cannot apply the Buscher procedure and we have no other way of justifying the duality, any discussion around the R-flux is necessarily speculative. We will revisit the idea of non-isometric T-duality more generally in sections 6.9 and 9.1.3, though we will not settle the issue in this thesis.

Let us now view these backgrounds from the perspective of doubled geometry. Here, we will see that, when we double the T^2 fibres, the T-fold background can be described in a fully geometric sense.

3.3 Torus bundles in doubled geometry

It is illuminating to consider the above backgrounds from the perspective of doubled geometry. The monondromy matrix N defining the fibration of a T^{2d} over an S^1 can be decomposed as

$$N_I{}^J = \begin{pmatrix} f_\mu{}^\nu & K_{\mu\nu} \\ Q^{\mu\nu} & -f_\nu{}^\mu \end{pmatrix}, \tag{3.26}$$

where $N_{IJ} = -N_{JI}, Q^{\mu\nu} = -Q^{\nu\mu}, K_{\mu\nu} = -K_{\nu\mu}$, and we have fibre indices $\mu = 1, .., d$. We refer to f, K, Q respectively as geometric flux, H-flux and non-geometric flux. f can be thought of as the monodromy of a T^d fibration over a circle, and K should be thought of as the H-flux for such a torus bundle. If we only have these two fluxes, then this background is a geometric background in the sense that we can describe it with a well-defined geometry [12]. If the matrices (3.26) generate $O(d,d)$, then matrices with $Q = 0$ are said to be in the geometric subgroup of $O(d,d)$. Matrices with $Q \neq 0$ are not equivalent to any kind of twisted torus bundle. Instead, they are equivalent to non-geometric backgrounds where the transition functions defining the background include $O(d,d;\mathbb{Z})$ transformations (generalised T-duality transformations). As seen above, such backgrounds are referred to as T-folds. Turning to the T^2 fibrations considered above, when we double the fibres to a T^4, note that all $O(2,2;\mathbb{Z})$ actions have a geometric action on the fibres now since $O(2,2;\mathbb{Z}) \subset GL(4;\mathbb{Z})$. Starting from the nilfold again, we have torus coordinates $\mathbb{X}^I = (y, z, \tilde{y}, \tilde{z})$, where the monodromy now acts as

$$(x, y, z, \tilde{y}, \tilde{z}) \to (x + 2\pi, y + mz, z, \tilde{y}, \tilde{z} - m\tilde{y}). \tag{3.27}$$

Note that the monodromy does not mix the 'original' and 'dual' coordinates, which means that the action can be projected onto a T^2 submanifold, as we already know. This can also be seen from the fact that the doubled metric (3.2) in this case is simply

$$\mathcal{H} = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \tag{3.28}$$

where g is the nilfold metric given by (3.15). So far, this is all a straightforward doubling of the usual nilfold construction. However, we can instead construct the nilfold as a quotient $\mathcal{T} = \mathcal{G}/\Gamma$, for some group manifold $\mathcal{G}$ and discrete subgroup Γ. As we did for the undoubled case, we can construct a matrix representation for $\mathcal{G}$ and define a Γ such that the global identifications are those of (3.27). Once again, we can then construct left and right invariant one-forms and vector fields. The left-invariant one forms are

$$\begin{aligned} P^x &= dx, \qquad P^y = dy - mxdz, \qquad P^z = dz, \\ Q_y &= d\tilde{y}, \qquad Q_z = d\tilde{z} + mxd\tilde{y}. \end{aligned} \tag{3.29}$$

Note that there is no Q_x since we have not doubled the base. The dual vector fields are

$$\begin{aligned} Z_x &= \frac{\partial}{\partial x}, \qquad Z_y = \frac{\partial}{\partial y}, \qquad Z_z = \frac{\partial}{\partial z} + mx\frac{\partial}{\partial y}, \\ X^y &= \frac{\partial}{\partial \tilde{y}} - mx\frac{\partial}{\partial \tilde{z}}, \end{aligned} \tag{3.30}$$

which obey the commutation relations

$$[Z_x, Z_z] = mZ_y, \qquad [Z_x, X^y] = mX^z, \tag{3.31}$$

which is indeed of the form (3.7).

Now, we would like to obtain the H-flux and T-fold backgrounds from this via T-duality. Looking at $\mathcal{T}$, we find that we can recover the standard nilfold background by taking the quotient $\mathcal{T}/\tilde{G}_L$, where $\tilde{G}_L$ is generated by the right-invariant vector fields $\tilde{X}^y, \tilde{X}^z$. Since T-duality in the doubled formalism simply amounts to a geometric $O(d,d;\mathbb{Z})$ transformation, we can in fact obtain the T^3 with H-flux and T-fold backgrounds from $\mathcal{T}$ as well, through different choices of *polarisation*, which we now demonstrate.

We define the polarisation as follows. On the doubled T^4, we have coordinates $\mathbb{X}^I$. To project out a T^2 from this, we must construct two 'physical' coordinates (y,z) which are linear combinations of the $\mathbb{X}^I$. We would then choose $(\tilde{y},\tilde{z})$ to be different linear combinations for the dual torus. Denoting these polarisations by $\Pi^\mu{}_I, \tilde{\Pi}_{\mu I}$, we have full polarisation

$$\Theta^{\hat{I}}{}_I = \left(\Pi^\mu{}_I, \tilde{\Pi}_{\mu I}\right), \tag{3.32}$$

and the coordinates for the T^2 are then given by

$$\mathbb{X}^I = \Theta^{\hat{I}}{}_I \mathbb{X}^{\mathbb{I}} = \begin{pmatrix} y \\ z \\ \tilde{y} \\ \tilde{z} \end{pmatrix}. \tag{3.33}$$

It can be shown that a T-duality transformation $\mathcal{O} \in O(2,2;\mathbb{Z})$ transforms the polarisation as

$$\Theta \to \Theta\mathcal{O}, \tag{3.34}$$

and the generalised metric $\mathcal{H}$ transforms as

$$\mathcal{H}_{IJ} \to (\mathcal{O}^T)_I{}^K \mathcal{H}_{KL} \mathcal{O}^L{}_J, \tag{3.35}$$

and thus we can read off the new background using this transformation. This is essentially the Buscher rules written in an $O(d,d)$ covariant manner.

For the nilfold, from our starting point, the polarisation is simply the identity, $\Theta = 1$, so that we have

$$(y,z,\tilde{y},\tilde{z}) = (\mathbb{X}^1, \mathbb{X}^2, \mathbb{X}^3, \mathbb{X}^4). \tag{3.36}$$

As we saw, we can read off the nilfold metric from the subsequent doubled metric $\mathcal{H}$. For the H-flux, since this is obtained by T-duality in the y-direction, the $O(2,2;\mathbb{Z})$

element that we need is

$$\mathcal{O} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.37}$$

This gives the polarisation

$$\Theta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.38}$$

i.e. the same as $\mathcal{O}$ (since we started with the identity matrix), which gives the coordinates

$$(y, z, \tilde{y}, \tilde{z}) = (\mathbb{X}^3, \mathbb{X}^2, \mathbb{X}^1, \mathbb{X}^4). \tag{3.39}$$

We can also read off the new generalised metric as

$$\mathcal{H} = \begin{pmatrix} 1+(mx)^2 & 0 & 0 & mx \\ 0 & 1+(mx)^2 & -mx & 0 \\ 0 & -mx & 1 & 0 \\ mx & 0 & 0 & 1 \end{pmatrix}. \tag{3.40}$$

From this, we can read off the metric and B-field from the general form of (3.2), and we do indeed recover the H-flux background (3.22). If instead we T-dualise the nilfold in the z-direction, we have

$$\mathcal{O} = \Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{3.41}$$

with coordinates

$$(y, z, \tilde{y}, \tilde{z}) = (\mathbb{X}^1, \mathbb{X}^4, \mathbb{X}^3, \mathbb{X}^2), \tag{3.42}$$

and generalised metric

$$\mathcal{H} = \begin{pmatrix} 1 & 0 & 0 & -mx \\ 0 & 1 & mx & 0 \\ 0 & mx & 1+(mx)^2 & 0 \\ -mx & 0 & 0 & 1+(mx)^2 \end{pmatrix}. \tag{3.43}$$

Once again, we can read off the background and we find that we recover the T-fold (3.24). The monodromy matrix for the T-fold can also be read off as

$$Q^{\mu\nu} = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}. \tag{3.44}$$

Of course, this is not in the geometric subgroup of $O(d,d;\mathbb{Z})$ and, as stated earlier, the monodromy includes a T-duality transformation. This is exemplified by the global structure

$$(x, y, z, \tilde{y}, \tilde{z}) \sim (x + 2\pi, y + m\tilde{z}, z - m\tilde{y}, \tilde{y}, \tilde{z}), \tag{3.45}$$

i.e. the (y, z) and $(\tilde{y}, \tilde{z})$ mix. Thus, even though locally we may identify coordinates (y, z) for the 'physical' T^2, globally these cannot be separated from the dual coordinates $(\tilde{y}, \tilde{z})$. However, notice that, in its doubled form, the T-fold background is periodic in x and is on exactly the same footing as the nilfold and H-flux. This is because, as stated above, the full $O(d,d;\mathbb{Z})$ has a geometric action on the T^4, and so the mixing of the coordinates is no different, from the doubled perspective, from the nilfold, say. When we read off the metric and B-field, we are essentially projecting out the dual coordinates, and the non-geometric nature of the T-fold is equivalent to saying that we cannot consistently define such a projection globally. For the R-flux, if we attempted a similar construction, we would find that such a projection cannot even be made locally [12].

3.4 Sigma model formulation for twisted tori

It is possible to frame all of the above starting from a worldsheet sigma model [12]. We will start with the undoubled case, i.e. we will see how to embed the worldsheet sigma model in torus bundle target spaces. Then, we will see how this generalises to the doubled case and how we can recover the undoubled torus bundles from this.

3.4.1 The undoubled torus bundle sigma model

In general, suppose we have some torus bundle $\mathcal{N}$ which is a T^d fibred over some k-dimensional base M. We introduce local coordinates on M x^u, $u = 1, ..., k$, and, as usual, we have coordinates z^μ on the T^d. We write the metric on $\mathcal{N}$ as

$$\mathcal{K}_{\alpha\beta} = \begin{pmatrix} g_{uv} + g_{\mu\nu} A^\mu{}_u A^\nu{}_v & g_{\mu\rho} A^\rho{}_v \\ g_{\nu\rho} A^\rho{}_u & g_{\mu\nu} \end{pmatrix}, \tag{3.46}$$

where α, β are coordinates on $\mathcal{N}$, with $x^\alpha = (x^u, z^\mu)$, $\alpha = 1, ..., k+d$. Here, the one-forms $A^\mu = A^\mu{}_u dx^u$ can be viewed as $U(1)^d$ connections of the T^d fibration.

Then, the sigma model on $\mathcal{N}$ is given by[2]

$$S_\mathcal{N} = \frac{1}{2}\int_\Sigma \mathcal{K}_{uv} dx^u \wedge *dx^v + \frac{1}{2}\int_\Sigma g_{\mu\nu} dz^\mu \wedge *dz^\nu + \int_\Sigma dz^\mu \wedge *J_\mu, \tag{3.47}$$

where $\mathcal{K}_{uv} = g_{uv} + g_{\mu\nu} A^\mu{}_u A^\nu{}_v$ and $J_\mu = g_{\mu\nu} A^\nu{}_u dx^u$, and all objects are pulled back to the worldsheet as appropriate. In the case when the base is a circle and we have some geometric monodromy e^f (we are only dealing with geometric monondromy for the undoubled case), we have the $\mathcal{K}$-invariant one-forms

$$P^\mu = (e^{fx})^\mu{}_\nu dz^\nu, \qquad P^x = dx, \tag{3.48}$$

where x is the circle coordinate. These one-forms satisfy the Maurer-Cartan equations

$$dP^x = 0, \qquad dP^\mu - f^\mu{}_\nu P^x \wedge P^\nu = 0, \tag{3.49}$$

and we can write the action as

$$S_\mathcal{N} = \frac{1}{2}\int_\Sigma \mathcal{K} P^x \wedge *P^x + \frac{1}{2}\int_\Sigma h_{\mu\nu} P^\mu \wedge *P^\nu + \int_\Sigma P^\mu \wedge *J_\mu, \tag{3.50}$$

where $\mathcal{K} = 1 + h_{\mu\nu} A^\mu{}_x A^\nu{}_x$ and $g_{\mu\nu}(x) = (e^{fx})_\mu{}^\rho h_{\rho\sigma} (e^{fx})^\sigma{}_\nu$, i.e. h is the x-independent metric obtained when we factor out the twists. Let us now see how this works in the doubled formalism.

3.4.2 The doubled torus bundle sigma model

We now have a target space $\mathcal{T}$ with doubled torus fibres T^{2d}, and the setup is similar to the above. We have metric

$$\mathcal{K} = \begin{pmatrix} g_{uv} + \frac{1}{2}\mathcal{H}_{KL}\mathcal{A}^K{}_u \mathcal{A}^L{}_v & \frac{1}{2}\mathcal{H}_{IK}\mathcal{A}^K{}_v \\ \frac{1}{2}\mathcal{H}_{JK}\mathcal{A}^K{}_u & \frac{1}{2}\mathcal{H}_{IJ} \end{pmatrix}, \tag{3.51}$$

where $\mathcal{H}$ is the usual doubled metric on the doubled torus fibres and the doubled connection $\mathcal{A}^I = (A^\mu, B_\mu)$, where B is the B-field, i.e. the B-field is now incorporated into the geometry, as we would expect for the doubled formalism, though this is only

[2] Where $*dz = -dz, *d\bar{z} = \bar{d}z$.

for the B-field in the toroidal directions, since the base remains undoubled. The presence of the B-field means we now have a Wess-Zumino term, and we also have a topological term which is necessary for the quantum theory, so overall the action is $S_{\mathcal{T}} = S_{\mathcal{K}} + S_{wz} + S_{\Omega}$, i.e.

$$S_{\mathcal{T}} = \frac{1}{2}\int_{\Sigma} \mathcal{K}_{uv} dx^u \wedge *dx^v + \frac{1}{4}\int_{\Sigma} \mathcal{H}_{IJ} d\mathbb{X}^I \wedge *d\mathbb{X}^J - \frac{1}{2}\int_{\Sigma} d\mathbb{X}^I \wedge *J_I + \frac{1}{4}\int_{\Sigma} \Omega_{IJ} d\mathbb{X}^I \wedge d\mathbb{X}^J, \tag{3.52}$$

where $J_I = \mathcal{H}_{IJ}\mathcal{A}^J - L_{IJ} * \mathcal{A}^J$. We have the self-duality constraints

$$d\mathbb{X}^I = L^{IJ}\left(\mathcal{H}_{JK} * d\mathbb{X}^K + *J_J\right), \tag{3.53}$$

which are needed to ensure the correct number of degrees on freedom (i.e. we need the same number of degrees of freedom as the undoubled case).

Once again focusing on the circle case, we have one-forms

$$P^x = dx, \qquad \mathcal{P}^I = (e^{Nx})^I{}_J d\mathbb{X}^J, \tag{3.54}$$

where we now have a general twist N, and these once again obey the Maurer-Cartan equations

$$dP^x = 0, \qquad d\mathcal{P}^I - N^I{}_J P^x \wedge \mathcal{P}^J = 0. \tag{3.55}$$

The action in this case becomes

$$S_{\mathcal{T}} = \frac{1}{2}\int_{\Sigma} \mathcal{K} P^x \wedge *P^x + \frac{1}{4}\int_{\Sigma} \mathcal{M}_{IJ} P^I \wedge *P^J + \frac{1}{2}\int_{\Sigma} L_{IJ}\mathcal{P}^I \wedge *J^J + \frac{1}{4}\int_{\Sigma} \Omega_{IJ} d\mathbb{X}^I \wedge d\mathbb{X}^J, \tag{3.56}$$

where $\mathcal{H}_{IJ}(x) = (e^{Nx})_I{}^K \mathcal{M}_{KL}(e^{Nx})^L{}_J$ defines the x-independent doubled metric $\mathcal{M}$.

In order to recover an undoubled background from this, we need to select a polarisation. As discussed earlier, if the background is geometric, we can specify coordinates z^μ for the 'physical' torus and $\tilde{z}_\mu$ for the dual (or 'auxiliary') torus, and such a choice can be globally defined. If we have a non-geometric background, such a choice can only be made locally at best.

Recall that $\mathcal{T} = \mathcal{G}/\Gamma$, and that we have an isometry group $\tilde{G}_L \subset \mathcal{G}$ generated by the $\tilde{X}^a$, which themselves are chosen by a polarisation $\tilde{X}^a = \Pi^{aA}\tilde{T}_A$. The undoubled action corresponding to this polarisation choice is recovered by gauging this subgroup $\tilde{G}_L$. We will not give the details here, but they can be found, for example, in [12].

Finally, we will mention that it is indeed possible to also double the base to have a fully doubled target space, and the generalisation is mostly as one would expect from the doubled fibre case.

Chapter 4

Algebraic T-duality

The Buscher procedure discussed in chapter 2 can be thought of as the ‘traditional’ approach to T-duality and is certainly the most established. Here, we discuss a different approach to T-duality; namely, a worldsheet CFT approach. To do so, we will also discuss ideas such as universal coordinates. First, we will introduce the ideas developed in [26] that explain how we can use automorphisms of the CFT algebra to construct symmetry deformations of the CFT fields such as the stress tensor. Then, we will discuss algebraic T-duality in detail. Our discussions regarding ∂X_μ deformations and T-duality in this chapter will be largely based on [15], though much of the discussion surrounding the T-duality charge and nuances such as gauge equivalence of different charges is original work done in collaboration in [1].

4.1 Symmetries and automorphisms

A given worldsheet CFT is defined by the stress tensor and its commutation relations with the primary fields of the CFT. Thus, given two CFT algebras $\mathcal{A}_1, \mathcal{A}_2$, and an isomorphism between them $\iota : \mathcal{A}_1 \to \mathcal{A}_2$ which maps stress tensor to stress tensor, the two CFTs corresponding to the algebras $\mathcal{A}_1, \mathcal{A}_2$ are isomorphic. Additionally, we require that equal time commutation relations are preserved, i.e.

$$\iota([A, B]) = [\iota(A), \iota(B)], \tag{4.1}$$

for any $A, B \in \mathcal{A}_1$.

We will be interested in constructing new CFTs from some starting point, and in particular we will be interested in deformations which can be described as deformations of the fields in the CFT. Given a CFT with algebra of operators $\mathcal{A}$, if we define the map

$$\iota(A) = A + i[Q, A], \tag{4.2}$$

for *any* fixed, infinitesimal operator Q, then this is a valid CFT isomorphism and so such transformations generate CFT deformations (this can be verified using the Jacobi identity). In particular, given a stress tensor T_Φ, the CFT with stress tensor $T_\Phi + i[Q, T_\Phi]$ is isomorphic to the original theory. Furthermore, if we can find a Q such that

$$T_\Phi + i[Q, T_\Phi] = T_{\Phi+\delta\Phi}, \tag{4.3}$$

then the field deformation

$$\Phi \to \Phi + \delta\Phi \tag{4.4}$$

is a symmetry transformation of the spacetime fields which make up the CFT. We therefore would like to find such operators Q which correspond to spacetime field deformations.

As is well-known, the addition of a $(1,1)$ primary field (i.e. a vertex operator) V to the stress tensor, $T \to T + V$, is induced by turning on a deformation in the action corresponding to the vertex operator V, and this deformation can be described by a deformation in the fields. Such deformations are called *canonical* deformations. Thus, if we have a canonical deformation which also happens to be of the form (4.3), then we have a symmetry of the theory given by field deformations. Such a deformation is found by defining Q such that

$$Q = \oint J, \tag{4.5}$$

where J is some $(1,0)$ primary field. In this case, $[Q, T]$ is a $(1,1)$ primary field and therefore Q induces a canonical deformation that can be interpreted as a deformation of the spacetime fields and is a symmetry of the theory.

Such observations have been used extensively in the literature. For example, [36] used this to demonstrate a Higgs mechanism for string theory. Specifically, they showed how the $SU(2)$ gauge symmetry is dynamically broken by turning on a massless scalar deformation at the self-dual radius, and they showed how massless fields at the SDR gain mass via Goldstone bosons. They used integrated deformations of the form (4.5) to determine how the stress tensor was deformed under such symmetry breaking.

The above discussion regarding Q was for infinitesimal deformations, but this can easily be extended to finite deformations by considering the exponential e^{iQ} acting on operators as

$$\mathcal{A} \to e^{iQ}\mathcal{A}e^{-iQ}, \tag{4.6}$$

where $\mathcal{A}$ is an operator in the worldsheet theory.

Any valid operator Q will generate a symmetry, but the interesting cases are when the automorphism admits an alternative description of the physics. We will specifically be interested in describing T-duality via such an (inner) automorphism. For us, the starting point is the key observation of [27] that T-duality on a circle may be understood as a residual discrete symmetry that endures after the breaking of a larger enhanced gauge symmetry. The larger symmetry is only manifest for special backgrounds and it is in this sense that T-duality should be thought of as a gauge symmetry of the target space. If the operator $\mathcal{A}$ is the stress tensor of a string theory and Q is such that the automorphism maps it to a stress tensor of an apparently different string theory, then the symmetry can be thought of as a string-string duality.

The automorphism (4.6) is a symmetry of the theory and so any infinitesimal deformation of the form

$$\delta\mathcal{A} = i[Q, \mathcal{A}] \tag{4.7}$$

gives a symmetry of the theory. This can be contrasted with general transformations, generated by vertex operators, that correspond to genuine physical deformations of the theory. This description of gauge symmetry is elegantly encoded in the BRST framework of String Field Theory, wherein symmetries of the target space are generated by BRST transformations of the string field[1]

$$\delta|\Psi\rangle = Q_B|\Lambda\rangle + ..., \tag{4.8}$$

where +... denotes non-linear terms. As mentioned, we will be interested in conserved charges of the form

$$Q = \oint dz\, \Lambda\, J(z), \tag{4.9}$$

where $J(z)$ is a weight $(1,0)$ holomorphic worldsheet current, or the obvious antiholomorphic counterpart, or combinations of both. Examples include B-field transformations and diffeomorphisms of the target space, where the appropriate conserved currents are $J = \xi_i \partial X^i(z) + \zeta_i \bar{\partial} X^i(\bar{z})$. Questions of conservation (commutation with the worldsheet Hamiltonian) and the fact that the natural symmetries from the target space perspective can be simply understood in terms of the combinations $X^i_R(\bar{z}) \pm X^i_L(z)$, rather than $X^i_R(\bar{z})$ and $X^i_L(z)$ separately, indicate that the most natural language in which to discuss these symmetries is the canonical one in which the worldsheet has a Lorentzian metric.

[1] See [37] for details.

As discussed above, at special points in the moduli space of string backgrounds we see an enhancement of the target space symmetry as additional states become massless and form multiplets of non-abelian gauge symmetries. In particular, we studied the classic example of the Halpern-Frenkel-Kac-Segal (HFKS) mechanism [38–40] in which the d commuting currents $H^i = i\partial X^i_L(z)$, where $i = 1, ..d$, are joined by the currents $E_{\pm\alpha} =: e^{\pm i\alpha_i X^i_L(z)}$:, which are weight $(1,0)$ at this enhancement point in the moduli space. The α_i are the root vectors of the enhanced group, which has rank d. One perspective on this [20] is that the enhanced symmetry is a gauge symmetry of the background independent theory that is generically broken by a choice of background.

4.2 T-duality as a gauge symmetry

Throughout this thesis, we will often refer to the self-dual point as a kind of reference background with respect to which we consider operators of other backgrounds. As such, it will be useful to distinguish this background, so we define

$$\partial\phi_\mu := \partial X_\mu(E = G), \qquad \bar{\partial}\phi_\mu := \bar{\partial} X_\mu(E = G), \tag{4.10}$$

$$\phi^\mu_L := X^\mu_L(E = G), \qquad \phi^\mu_R := X^\mu_R(E = G), \tag{4.11}$$

where we recall that the background $E = G$ refers to the enhanced symmetry point $G = 1, B = 0$. At this point, it may be confusing to the reader why we do not simply say that $\phi^\mu := X^\mu(E = G)$. As we will see shortly, and as alluded to earlier, we will be working in a regime where we consider the coordinates X^μ to be *universal*, i.e. background independent. It is therefore more appropriate to state the definitions (4.10), (4.11), since the objects that appear here are *not* universal. Note that in the $d = 1$ case, we will simply write $\partial\phi$, etc. rather than $\partial\phi_x$, etc. for ease of notation.

Let us consider the $d = 1$ case where, at the SDR, the $U(1)_L \times U(1)_R$ symmetry is enhanced to $SU(2)_L \times SU(2)_R$, generated by currents $\partial\phi$, $e^{\pm i2\phi_L}$ for $SU(2)_L$ and similarly for $SU(2)_R$. As discussed, away from the SDR, the gauge symmetry breaks to the Cartan $U(1) \times U(1)$ with a residual discrete $\mathbb{Z}_2$ gauge symmetry which could be identified with T-Duality in the circle. [15] showed that the charge responsible for the action of the $\mathbb{Z}_2 \subset SU(2)_L$ is

$$Q = \frac{1}{2}\oint d\sigma \sin\left(2\phi_L(\sigma)\right). \tag{4.12}$$

Furthermore, it was shown that this could still be used to generate the T-duality transformation away from the self-dual radius, even though the current $\sin(2\phi_L(\sigma))$ is generally not conserved[2]. The key was to write the fields of the theory at radius R in terms of the fields defined at the self-dual radius. This then ensured that the action of the charge Q could be computed on fields defined away from the self-dual radius. Thus the effect of an automorphism, by this charge, on the operator algebra of the theory at generic radius could be computed. It was shown in [15] that this procedure correctly reproduces the Buscher rules. In fact, all discrete $O(d,d;\mathbb{Z})$ transformations can be expressed as worldsheet operator algebra automorphisms, and this was demonstrated explicitly for $d=2$ in [41][3]. Importantly, if

$$Q_\Lambda = \Lambda \oint d\sigma \sin(2\phi_L(\sigma)), \tag{4.13}$$

the current is not of weight $(1,0)$ and this charge will not be conserved for general values of Λ. Surprisingly, the automorphism still makes sense away from the self-dual radius if $\Lambda = \frac{1}{2}$. We shall discuss why this is the case in section 4.3.

The main result of [15] stems from the fact that, at the self-dual radius,

$$e^{iQ}\partial\phi(\sigma)e^{-iQ} = -\partial\phi(\sigma), \qquad e^{iQ}\bar{\partial}\phi(\sigma)e^{-iQ} = \bar{\partial}\phi(\sigma). \tag{4.14}$$

Away from the self-dual radius, the transformation is more complicated. If one knows how to write the fields of the theory at a particular background, e.g. at $R=1$, in terms of the Hilbert space of another background, then (4.14) can be used to deduce the duality transformations. We can define a basis for the operators at a given point. Most of our considerations will involve the operators constructed from combinations of $\partial\phi$ and $\bar{\partial}\phi$. The natural way to do this is to define a connection on the space of backgrounds and then to parallel transport, with respect to that connection, the basis of states or operators at a point of enhanced symmetry to the background of interest. This may sound like a tall order, but some progress has been made on this general issue [42, 20, 22, 43, 44] and, as we shall see later, this issue simplifies greatly in a certain class of backgrounds. We will discuss this in more detail in chapter 5, but for now we

[2]Since the coordinates are dimensionless in our conventions, the charge looks different to that given in [15].

[3]Note that, whilst the T-duality automorphism involves the currents $J^{2,3}$, the automorphisms corresponding to other generators of $O(d,d;\mathbb{Z})$ only include the currents $J^1, \bar{J}^1$. Thus, even within the $O(d,d;\mathbb{Z})$ gauge symmetries, T-duality is still special in that it is inherently a stringy symmetry since it originates from the $SU(2)$ symmetry enhancement, whereas the other symmetries are abelian [41].

will recover the results of [15] for the flat torus. Later, we will show how we can apply these results to the backgrounds considered above, i.e. the nilfold, H-flux and T-fold.

4.3 An algebraic approach to T-duality

Most work on T-duality has focused on the Buscher construction, which gives central importance to the existence of compact abelian isometries [7]. The observation of [27], that T-duality may be thought of as a residual discrete gauge symmetry, provides a framework in which to think about T-duality without reference to isometries of the target space.

In this section we review the approach to realising T-duality as a $\mathbb{Z}_2$ automorphism of the operator algebra of the worldsheet theory. For illustrative purposes we focus on the simplest case where the target space is a circle at the self-dual radius $R = 1$. In this special case, the $\mathbb{Z}_2$ is a discrete subgroup of the larger $SU(2)_L \times SU(2)_R$ automorphism that appears at the self-dual radius. We shall discuss more general cases in the following section.

4.3.1 Universal coordinates

Recall from chapter 2 that for the string embedding in a flat torus, we have the equal-time commutation relations

$$[X^\mu(\sigma), \Pi_\nu(\sigma')] = i\delta(\sigma - \sigma')\delta^\mu{}_\nu, \tag{4.15}$$

where we take $\tau = 0$. Given that this relation is independent of the background $E_{\mu\nu}$, we say that, for fixed τ, the fields $X(\sigma), \Pi(\sigma)$ are *universal*, i.e. independent of background [20].[4] This idea is an old one and is discussed for example in [20], and it allows us to compare objects constructed out of these *universal coordinates* at different backgrounds.

The requirement that $X^\mu(\sigma)$ and $\Pi_\mu(\sigma)$ remain the same as we change the background $E_{\mu\nu}$ means that the oscillator modes must be background *dependent*. Identifying $X^\mu(\sigma)$ and $\Pi_\mu(\sigma)$ on different backgrounds, $E_{\mu\nu}$ and $E'_{\mu\nu}$, requires that the modes are

[4] We are really only requiring that these fields are universal in a local region of the space of backgrounds that we are interested in.

related by[5]

$$2g'_{\mu\nu}\alpha_n^\nu(E') = \Big(E^T_{\mu\nu} + E'_{\mu\nu}\Big)\alpha_n^\nu(E) + \Big(E_{\mu\nu} - E'_{\mu\nu}\Big)\bar{\alpha}^\nu_{-n}(E), \tag{4.16}$$

$$2g'_{\mu\nu}\bar{\alpha}_n^\nu(E') = \Big(E^T_{\mu\nu} - E'^T_{\mu\nu}\Big)\alpha_{-n}^\nu(E) + \Big(E_{\mu\nu} + E'^T_{\mu\nu}\Big)\bar{\alpha}^\nu_{n}(E), \tag{4.17}$$

where $\alpha_n^\mu(E), \bar{\alpha}_n^\mu(E)$ are the oscillator modes of the string embedding into the background specified by $E_{\mu\nu}$. The momenta and winding p_μ and ω^μ do not change (they are defined as background independent in our conventions). We can see that the α_0^μ and $\bar{\alpha}_0^\mu$ transform in line with (4.16), (4.17) from their definitions

$$g_{\mu\nu}\alpha_0^\nu = \frac{1}{\sqrt{2}}\Big(p_\mu - E_{\mu\nu}\omega^\nu\Big), \qquad g_{\mu\nu}\bar{\alpha}_0^\nu = \frac{1}{\sqrt{2}}\Big(p_\mu + E^T_{\mu\nu}\omega^\nu\Big). \tag{4.18}$$

This notion of universal coordinates is useful in understanding how the $\mathbb{Z}_2 \subset SU(2) \times SU(2)$ gauge symmetry at the self-dual radius generalises to other backgrounds.

4.3.2 The T-duality charge

We shall look for a charge Q that generates an automorphism $\mathcal{A} \to e^{iQ}\mathcal{A}e^{-iQ}$ which has the required $\mathbb{Z}_2$ effect[6]: $e^{iQ}\partial\phi e^{-iQ} = -\partial\phi$ and $e^{iQ}\bar{\partial}\phi e^{-iQ} = \bar{\partial}\phi$. The fields $\phi_L(\sigma)$ and $\phi_R(\sigma)$ do not have to produce a nice operator algebra independently, but the combinations $\phi_R(\sigma) \pm \phi_L(\sigma)$ do. The $\mathbb{Z}_2$ automorphism simply exchanges these two linear combinations, giving a sigma model description in both cases.

Working in $d = 1$ for now, we define

$$Q_\Lambda = \Lambda \oint d\sigma \cos(2\phi_L(\sigma)), \tag{4.19}$$

[5]This quantisation at $\tau = 0$ is related to a quantisation at generic τ by the usual relations $\alpha_n(\tau) = e^{-i\tau L_0}\alpha_n(0)e^{i\tau L_0} = e^{-in\tau}\alpha_n(0)$, where the second equality follows from standard commutation relations. We notice that $X(\sigma, 0) \to X(\sigma, \tau)$ if we also change $x \to x^\mu + \tau g^{\mu\nu}(p_\nu - B_{\nu\rho}w^\rho) \equiv x^\mu(\tau)$, which has a clear interpretation as a translation arising from a unitary time evolution, i.e. $X^\mu(\sigma, \tau)$ takes the same algebraic form as $X^\mu(\sigma)$, but with the replacements of $x^\mu(\tau)$ and $\alpha_n^\mu(\tau)$ for x^μ and α_n^μ. Thus, universal coordinates defined for generic fixed τ have the same algebraic form as the $\tau = 0$ case, so the notion of a universal coordinate does not depend on the choice of τ (but a choice must be made).

[6]Note that we could equivalently choose the duality to act as $\bar{\partial}\phi \to -\bar{\partial}\phi$ and leave $\partial\phi$ invariant. This is equivalent to swapping X' and Π with an extra minus sign inserted, which also preserves the canonical commutation relations.

which we note is invariant under the periodicity $\phi_L \to \phi_L + \pi$. The action of Q_Λ on $\partial\phi(\sigma)$ may be written as

$$e^{iQ_\Lambda}\partial\phi(\sigma)e^{-iQ_\Lambda} = \partial\phi(\sigma) + i[Q_\Lambda, \partial\phi(\sigma)] + \frac{i^2}{2!}[Q_\Lambda, [Q_\Lambda, \partial\phi(\sigma)]] + ..., \tag{4.20}$$

with the ellipsis denoting nested commutators at higher order in Λ. Since we are in one dimension and at the self-dual radius, we do not need to worry about indices being raised or lowered (we take the relevant component of the metric to be normalised to unity). Computing the leading contributions gives[7]:

$$\begin{aligned}
[Q_\Lambda, \partial\phi(\sigma)] =& \Lambda \oint d\sigma' [\cos(2\phi_L(\sigma')), -{\phi_L}'(\sigma)] \\
=& -\Lambda \oint d\sigma' \left[1 - \frac{2^2}{2!}({\phi_L}')^2(\sigma') + ..., {\phi_L}'(\sigma)\right] \\
=& -2\pi\Lambda i \sin(2\phi_L(\sigma)),
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
[Q_\Lambda, [Q_\Lambda, \partial\phi(\sigma)]] =& -2\pi\Lambda^2 i \oint d\sigma' \, [\cos(2\phi_L(\sigma)), \sin(2\phi_L(\sigma'))] \\
=& (2\pi\Lambda)^2 \oint d\sigma' \, \delta(\sigma - \sigma')\partial\phi(\sigma') \\
=& (2\pi\Lambda)^2 \partial\phi(\sigma).
\end{aligned} \tag{4.23}$$

It is clear that the next term $[Q_\Lambda, [Q_\Lambda, [Q_\Lambda, \partial\phi(\sigma)]]]$ is proportional to $\sin(2\phi_L(\sigma))$. Continuing in this fashion, we see that successive nested commutators will alternately give terms proportional to $\sin(2\phi_L)$ and $\partial\phi$ with coefficients that are straightforward to determine. Putting this all together gives [15]

$$\begin{aligned}
& e^{iQ_\Lambda}\partial\phi(\sigma)e^{-iQ_\Lambda} \\
=& \left(1 + \frac{(2\pi\Lambda i)^2}{2!} + \frac{(2\pi\Lambda i)^4}{4!} + ...\right)\partial\phi(\sigma) - i\left(2\pi\Lambda i + \frac{(2\pi\Lambda i)^3}{3!} + \frac{(2\pi\Lambda i)^5}{5!} + ...\right)\sin(2\phi_L(\sigma)) \\
=& \cos(2\pi\Lambda)\partial\phi(\sigma) + \sin(2\pi\Lambda)\sin(2\phi_L(\sigma)).
\end{aligned} \tag{4.24}$$

[7] Note also that

$$\partial\phi = \frac{1}{2}(\Pi - X') = \frac{1}{2}(-{\phi_L}' + {\phi_R}' - {\phi_L}' - {\phi_R}') = -{\phi_L}'. \tag{4.21}$$

In our conventions, the choice $\Lambda = 1/2$ gives the required transformation and so if we define $Q \equiv Q_{1/2}$, we have

$$e^{iQ}\partial\phi(\sigma)e^{-iQ} = -\partial\phi(\sigma). \tag{4.25}$$

Thus, Q is a suitable T-duality charge. Of course, at the self-dual radius, any choice of Λ generates a symmetry of the theory as Q_Λ is built from a $(1,0)$ current. What is interesting is that, even away from the self-dual radius, the charge Q defined above still gives a symmetry of the theory, as we discuss below. Note that we can equivalently use the charge

$$Q = \frac{1}{2}\oint d\sigma \sin(2\phi_L(\sigma)). \tag{4.26}$$

We get exactly the same transformation either way. We shall discuss the relationships between these charges in section 4.3.5.

For general dimensions, if we wish to compute the T-duality in the direction x^μ, the corresponding charge would be

$$Q = \frac{1}{2}\oint d\sigma \cos(2\phi_L^\mu(\sigma)), \tag{4.27}$$

or with sin instead of cos.

4.3.3 Automorphisms away from the self-dual radius

Given that the $\mathbb{Z}_2$ duality described above is, at the self-dual radius, a subgroup of an exact gauge symmetry group of the target space theory, it is hardly surprising that the duality is a symmetry of the theory. What is more surprising is that the $\mathbb{Z}_2$ continues to hold as an exact symmetry of the theory for any radius. In this section we review the arguments that lead to this conclusion, but from the algebraic perspective rather than the usual Buscher construction.

From the derivatives (2.20), we have

$$\partial\phi_\mu = \frac{1}{2}(\Pi_\mu - G_{\mu\nu}X'^\nu). \tag{4.28}$$

We can relate $\partial X_\mu(E)$ and $\bar{\partial}X_\mu(E)$ at different backgrounds using their expressions in terms of the background independent fields given above. Rearranging to get

$$\Pi_\mu - B_{\mu\nu}X'^j = \frac{1}{2}\Big(\partial X_\mu(E) + \bar{\partial}X_\mu(E)\Big), \qquad g_{\mu\nu}X'^\nu = -\frac{1}{2}\Big(\partial X_\mu(E) - \bar{\partial}X_\mu(E)\Big), \tag{4.29}$$

we find that, for $\tau = 0$,

$$\partial X_\mu(E) = \frac{1}{2} g'^{\nu\rho} \Big((E_{\mu\nu} + E'^{T}_{\mu\nu}) \partial X_\rho(E') + (-E_{\mu\nu} + E'_{\mu\nu}) \bar{\partial} X_\rho(E') \Big), \tag{4.30}$$

$$\bar{\partial} X_\mu(E) = \frac{1}{2} g'^{\nu\rho} \Big((-E^{T}_{\mu\nu} + E'^{T}_{\mu\nu}) \partial X_\rho(E') + (E^{T}_{\mu\nu} + E'_{\mu\nu}) \bar{\partial} X_\rho(E') \Big). \tag{4.31}$$

These expressions allow us to determine how composite operators defined in terms of $\partial X_\mu(E')$ and $\bar{\partial} X_\mu(E')$ transform under the automorphism generated by charge Q if we know how the operators $\partial X_\mu(E)$ and $\bar{\partial} X_\mu(E)$ transform. So, for example, if $d = 1$,

$$\partial X(E) = \frac{1}{2} G^{-1} \Big((g + G) \partial \phi + (G - g) \bar{\partial} \phi \Big), \tag{4.32}$$

where we use ϕ without index in the $d = 1$ case. From this, we deduce

$$e^{iQ} \partial X(E) e^{-iQ} = \frac{1}{2} G^{-1} \Big(-(g + G) \partial \phi + (G - g) \bar{\partial} \phi \Big). \tag{4.33}$$

More generally, if we know how an operator $\mathcal{F}(E)$ defined at a background E transforms under T-duality, then, if we know the relationship between $\mathcal{F}(E')$ and $\mathcal{F}(E)$, we can use the definition of the charge Q in the background E to determine how the symmetry acts on $\mathcal{F}$ defined at the background E'. For example, the chiral stress tensor

$$T(\sigma) = g^{\mu\nu} \partial X_\mu(E) \partial X_\nu(E) \tag{4.34}$$

is clearly invariant at the self-dual point $E = G$, but transforms in a more complicated manner at other points of $\mathscr{M}_d$. It was shown in [15] that the charge of the kind (4.19) maps a stress tensor to a stress tensor, for generic radius, only for the value of the parameter $\Lambda = 1/2$ (in our conventions). Thus, for this value of the parameter, the automorphism is not only a symmetry of the conformal field theory, but relates one stress tensor to an apparently different one, yielding a duality between string theories.

Furthermore, it was also shown in [15] that this mapping of stress tensors under T-duality can be used to reproduce the Buscher rules. Given some generic background E with an isometry in a given direction, we can use the relations (4.30), (4.31) to write the stress tensor in terms of objects at the point of enhanced symmetry. This gives us a general form from which we can read off components of the metric and B-field. If we perform T-duality in the direction in which we have an isometry and then read off the metric and B-field of our new stress tensor, we find that we precisely reproduce the Buscher rules (2.47).

4.3.4 Symplectomorphisms and charge conservation

All automorphisms of the operator algebra are symmetries of the theory, but those that preserve the Hamiltonian play a special role. As such, we would like to know under what conditions the charge $Q(\Lambda) = e^{i\Lambda h}$ is conserved. This is important as, if the notion of universal coordinates and canonical commutation relations[8] is to survive the automorphism, it must make sense at each fixed value of τ. The time evolution of $Q(\Lambda)$ is given by the Hamiltonian

$$Q_\tau(\Lambda) = e^{-iH\tau} Q_0(\Lambda) e^{iH\tau}, \tag{4.35}$$

or $e^{iQ_\tau} = e^{-iH\tau} e^{iQ_0} e^{iH\tau}$. For the charge to be conserved, we require $Q_\tau(\Lambda) = Q_0(\Lambda)$, i.e.

$$e^{i\Lambda h} = e^{-iH\tau} e^{i\Lambda h} e^{iH\tau}. \tag{4.36}$$

We can rewrite this as

$$e^{iH\tau} = e^{-i\Lambda h} e^{iH\tau} e^{i\Lambda h}, \tag{4.37}$$

which says that the automorphism preserves the Hamiltonian. This can happen in two ways. The most obvious way is if $[H, h] = 0$, i.e. the functional h is constant in time. This will be true if we can write h as

$$h = \oint d\sigma J(\sigma), \tag{4.38}$$

where $J(\sigma)$ is a weight $(1,0)$ or $(0,1)$ current. In that case, $[H, J(\sigma)] = 0$ and the current is conserved. This is the case when $J(\sigma)$ generates a continuous symmetry of the theory, such as B-field gauge transformations, spacetime diffeomorphisms, or the $SU(2) \times SU(2)$ gauge symmetry at the self-dual radius. As the symmetry is continuous, Λ can take any value in the parameter space of the corresponding Lie group.

This is not the only way to preserve the Hamiltonian. Consider a theory with action (2.2), taking the worldsheet metric to be $\gamma_{\alpha\beta} = \mathrm{diag}(1, -1)$. The Hamiltonian may be written as

$$H = \oint d\sigma\, \mathcal{S}^T \mathcal{H} \mathcal{S}, \tag{4.39}$$

[8]The requirement that the automorphism preserves the commutation relations is already a sign that it must be a symplectomorphism.

where $\mathcal{S}_I = (\Pi_\mu, X'^\mu)$, where Π_μ is the canonical momentum and

$$\mathcal{H}^{IJ} = \begin{pmatrix} g^{\mu\nu} & g^{\mu\rho}B_{\rho\nu} \\ B_{\nu\rho}g^{\rho\nu} & g_{\mu\nu} + B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} \end{pmatrix}. \tag{4.40}$$

Indices are raised and lowered by the invariant of $O(d,d)$; $\mathcal{S}^I = L^{IJ}\mathcal{S}_J = (X'^\mu, \Pi_\mu)$, where

$$L^{IJ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.41}$$

The Hamiltonian is invariant under the transformations

$$\mathcal{S} \to \mathcal{O}\mathcal{S}, \qquad \mathcal{H} \to \mathcal{O}\mathcal{H}\mathcal{O}^{-1}, \tag{4.42}$$

where $\mathcal{O} \in O(d,d;\mathbb{Z})$ (the discreteness is required to preserve integer-valuedness of the zero modes). Thus, an automorphism that generates an $O(d,d;\mathbb{Z})$ transformation,

$$e^{-iQ}\mathcal{S}_I e^{iQ} = \mathcal{O}_I{}^J\mathcal{S}_J, \tag{4.43}$$

will also preserve the Hamiltonian (provided $\mathcal{H}^{IJ}$ is also transformed). The requirement that $\mathcal{O} \in O(d,d;\mathbb{Z})$ means that Λ may only take certain discrete values. Thus, we see that the condition for charge conservation is that the charge generates a symplectomorphism.

4.3.5 Gauge equivalence of T-duality charges

In this section, it will be more convenient to use OPEs instead of commutation relations. Therefore, working in $d = 1$ for convenience, the charge is now[9,10]

$$Q^c = \pi \oint dz \cos(2\phi_L(z)). \tag{4.44}$$

[9]The superscript in Q^c refers to the cosine function that is used to define the charge. It is included here to distinguish it from a similar construction using the sine function.

[10]We define the contour integral so that the $2\pi i$ factor is absorbed into the integral, so that

$$\oint_{|z|=\text{const.}} \frac{dz}{z} = -\oint_{|z|=\text{const.}} \frac{d\bar{z}}{\bar{z}} = 1.$$

We also use conventions where the XX OPE is given by

$$X(z,\bar{z})X(w,\bar{w}) \sim -\frac{1}{2}\log|z-w|^2 \tag{4.45}$$

at the self-dual radius, which is the only radius we will be interested in for this section. To show gauge equivalence of the charges, we need to show that the transformations they induce are the same up to a $U(1)_L \times U(1)_R$ gauge transformation. It is sufficient to look at general derivatives of X and general exponentials of X[11]. The transformation of higher derivatives of X under an automorphism follows straightforwardly once the transformation of $\partial\phi$ is known,

$$e^{iQ}\partial^n\phi e^{-iQ} = \partial^{n-1}e^{iQ}\partial\phi e^{-iQ} = -\partial^n\phi, \tag{4.47}$$

and this is the same for both charges. However, the transformation of exponentials $e^{in\phi_L}$ is more difficult. In [15] the transformation of such exponentials is found using a point-splitting argument, but we present a slightly different approach, via induction, and the details are presented in appendix A. The result of the automorphsim is

$$e^{in\phi_L(z)} \to \begin{cases} ie^{-in\phi_L(z)}, & \text{n odd,} \\ e^{-in\phi_L(z)}, & \text{n even.} \end{cases} \tag{4.48}$$

We see that, in the n odd case, there is an extra factor of i compared to expectations. This technical detail is discussed in appendix A.

Thus far, we have made use of the $\mathbb{Z}_2$ symmetry generated by the charge (4.19) with $\Lambda = 1/2$; however, as mentioned earlier, we could have equally well used the charge

$$Q^s = \frac{1}{2}\oint dz \sin(2\phi_L(z)). \tag{4.49}$$

Indeed, it was this charge that was used in [15]. Both charges give rise to the same action $\partial\phi \to -\partial\phi$, but they do not act in the same way on exponentials. This may seem strange at first, but it is not hard to see that these charges are related by a $U(1)_L \times U(1)_R \subset SU(2)_L \times SU(2)_R$ gauge transformation generated by the currents $\partial\phi(z)$ and $\bar{\partial}\phi(\bar{z})$. Moving away from the self-dual radius, this symmetry is preserved, so this equivalence of the charges holds throughout moduli space. If we use the sine

[11] This follows from the fact that

$$e^{iQ}ABe^{-iQ} = e^{iQ}Ae^{-iQ}e^{iQ}Be^{-iQ} \tag{4.46}$$

for any operators A, B, where we rely on the associativity of the OPE.

charge, via a similar process to the above, we find that

$$e^{inX_L} \to \begin{cases} ie^{in(-\phi_L+\pi/2)}, & \text{n odd}, \\ e^{in(-\phi_L+\pi/2)}, & \text{n even}, \end{cases} \tag{4.50}$$

where it is instructive to write an n-dependent phase on the right hand side as a shift in ϕ_L. Written in this way, it seems that the effects of the two charges on $e^{in\phi_L}$ are related by a $U(1)_L \times U(1)_R$ transformation that gives the required shift in ϕ_L. To see this, consider the automorphism generated by the charge

$$Q^3_\Lambda \equiv \Lambda \oint dz \partial\phi(z), \tag{4.51}$$

where the 3 superscript indicates that it is the charge corresponding to the current J^3. By a simple OPE calculation, we can show that

$$e^{iQ^3_\Lambda} e^{in\phi_L} e^{-iQ^3_\Lambda} = e^{-in\pi\Lambda} e^{in\phi_L}, \tag{4.52}$$

or simply $e^{iQ^3_\Lambda}\phi_L e^{-iQ^3_\Lambda} = \phi_L - \pi\Lambda$. Now, denote the T-duality cosine and sine charges by Q^c, Q^s respectively.

If we set $\Lambda = -1/2$ in Q^3_Λ, we find that $e^{in\phi_L} \to (-1)^{\frac{n}{2}} e^{in\phi_L}$. Thus, we have

$$e^{in\phi_L} \xrightarrow{Q^s} \begin{cases} ie^{in(-\phi_L+\pi/2)} \\ e^{in(-\phi_L+\pi/2)} \end{cases} \xrightarrow{Q^3_{-\frac{1}{2}}} \begin{cases} ie^{-in\phi_L}, & \text{n odd}, \\ e^{-in\phi_L}, & \text{n even}. \end{cases} \tag{4.53}$$

The right-hand side is the same as the Q^c transformation, and thus we have shown that the effects of the two charges are related by $U(1)_L$ gauge transformations. The difference between the cases for n odd or even can be traced to the way in which the highest weight states in the modules $L_{[1,0]}$ and $L_{[0,1]}$ transform under the automorphisms generated by $Q^{c,s}$ (see the discussion in appendix A). It is not hard to show that the charges Q^c and Q^s themselves are directly related by a similar $U(1)$ gauge transformation, generated by Q^3_Λ with parameter $\Lambda = -1/4$, i.e. $UQ^cU^{-1} = Q^s$, where $U = \exp\left(-\frac{1}{4}\oint dz\, J^3(z)\right)$.

A note on charge conservation on orbifolds

As mentioned, orbifold backgrounds are highly important in string theory for a variety of reasons, and the T-duality of such backgrounds is well-established in many cases, e.g. $\mathbb{Z}_n$ orbifolds of toroidal backgrounds. Indeed, [29] explicitly construct the duality between the state spaces of toroidal orbifolds, including the twisted sectors. There is a question of whether the charges discussed in this chapter are preserved under such dualities and whether this matters with regards to the T-duality. For a simple $\mathbb{Z}_2$ orbifold, for example, the sine charge Q^s is clearly *not* preserved under the action $X \to -X$, but the cosine charge Q^c *is.* Thus, we might say that this orbifolding removes the ambiguity associated with the choice of charge and forces us to choose Q^c. However, if we consider a $\mathbb{Z}_3$ orbifold of the T^2 say, here we clearly see that neither charge is well-defined, but we know that the T-duality is still well-defined. If we were to use the T-duality charges naively regardless of whether or not they were globally defined on the orbifold, we would of course get the correct duality transformations, so it is unclear whether the action of the orbifold on the charges matters.

Indeed, it is possible to reconstruct the nilfold with elliptic monodromy as an orbifold [29], and given that the charge is well-defined on the nilfold it must also be well-defined on the orbifold formulation.

We will not be focusing on orbifolds in this thesis, but this would be interesting to clarify.

Chapter 5

Connections and Deformations on the Space of Backgrounds

In the previous chapter, we illustrated how, using knowledge of how the operator algebra defined at a point of enhanced symmetry transforms under the automorphism generated by Q, one can determine the effect of the automorphism at a general radius.

[15, 45, 46] also consider torus backgrounds with constant B-field. These are exact string backgrounds and have an explicit description in terms of a worldsheet CFT. However, the realisation of the approach to T-duality proposed in [15] is contingent on being able to describe the Hilbert space of the CFT corresponding to a particular background in terms of the Hilbert space of the theory at the self-dual point.

We are interested in finding the appropriate framework to discuss and generalise the construction outlined in section 4.3.3. We shall argue that the identification of a connection on the space of backgrounds with which parallel transport may be performed (from a point of enhanced symmetry to a background of interest) provides a natural generalization of the technique of section 4.3.3. Initially we will focus on the general QFT case, where we will work with correlation functions and describe how we can describe their deformations in theory space - indeed, this was the case that was first dealt with by Sonoda [23]. Then, in section 5.2, we will see how this formalism has been adapted to spaces of CFT backgrounds as well. It is important to do this for a couple of reasons. Firstly, in the CFT case we can make use of surface states, and it is useful to see how the formalism can be constructed using these instead of correlation functions. Secondly, in the initial formulation of correlator deformations on a space of QFTs [23], the construction was discussed in the context of the RG equations, although they are not central to the discussion and will not be particularly relevant for us. Therefore, it is not immediately obvious that it can be extended to the CFT case, but we will show that this indeed possible, based on the construction of [22]. One of the key differences between the QFT and CFT cases is that, for CFTs, we will primarily work with the connection corresponding to universal coordinates, which as we will see is termed the $\hat{\Gamma}$ connection, whereas we will keep the connection more general for QFTs. When we have conformal symmetry, the $\hat{\Gamma}$ connection is very convenient, but for a general QFT it is at the very least unnatural and it is not clear whether it is even a valid connection at all orders. In chapter 6, we will show how these ideas relate to the universal coordinate construction considered earlier, and we will apply these ideas to the twisted torus backgrounds considered in chapter 3. Let us first discuss the general case.

5.1 Connections on the space of QFT backgrounds

We assume the existence of a space of backgrounds $\mathcal{M}$, each point of which corresponds to a nonlinear sigma model. The data that defines the sigma model - its coupling constants, in the form a metric, B-field, etc. - define a point $p \in \mathcal{M}$. We can define a fibre bundle with base $\mathcal{M}$ and fibre given by the Hilbert space of states $\mathcal{H}_p$ of the theory at p [22]. Here, we choose to work with the bundle $\mathcal{E} \to \mathcal{M}$ with fibres given by the operators[1] of the theory.

The space of backgrounds we consider is that of renormalized theories parameterised by $\mathfrak{m}^\alpha(p)$, with α indexing the dimensions of the theory space. These parameters are taken as local coordinates on theory space, with $\mathfrak{m}^\alpha = 0\, \forall\, \alpha$ corresponding to the UV fixed point. The parameters obey the RG equations

$$\frac{d}{dl}\mathfrak{m}^\alpha = \beta^\alpha(\mathfrak{m}), \tag{5.1}$$

where the $\beta^\alpha(\mathfrak{m})$ are the beta functions of the parameters and are vector fields on the theory space, i.e. they span the fibres of the tangent space $T\mathcal{M}$.

The fibres of the tangent space $T\mathcal{M}$ are spanned by the beta-functions of the theory. Similarly, and in a sense made precise in [47, 23], the fibres of the dual space $T^*\mathcal{M}$ are spanned by the deformation operators $\mathcal{O}_\alpha$. These operators are conjugate to the local coordinates $\mathfrak{m}^\alpha(p)$ in the neighbourhood of a point $p \in \mathcal{M}$. We shall be interested in studying connections on $\mathcal{E}$ and the associated deformations of the sigma models given by parallel transport in $\mathcal{E}$ [23, 24, 47, 48].

[1] We have in mind here some generating set of operators $\{\mathcal{O}_\alpha\}$ that are, at the least, rich enough to construct the deformation operators in the cotangent bundle $T^*\mathcal{M}$.

In particular, we will be studying correlation functions of operators at a point in theory space and how they change as we move through theory space. We can define such movement in a covariant manner as follows. At a given point p in the theory space, we have a countably infinite set of linearly independent composite operators $\{\Phi_a\}_p$ which form a basis of all operators of the theory at p. Such a basis is not unique and can be changed by an invertible linear transformation

$$\Phi_a \to \mathcal{S}_a{}^b \Phi_b. \tag{5.2}$$

Thus, a correlation function $\left\langle \Phi_1(z_1)...\Phi_n(z_n) \right\rangle_p$ is a rank-n tensor on theory space, and as such we would like to construct a deformation of this tensor that is covariant under basis transformations of the form (5.2) [47].

To make the above discussion precise, we can think of the correlation function $\left\langle \Phi_1(z_1)...\Phi_n(z_n) \right\rangle_p$ as a formal Taylor expansion about a reference point $p = p_0$,

$$\left\langle \Phi_1(z_1)...\Phi_n(z_n) \right\rangle_p = \left\langle \Phi_1(z_1)...\Phi_n(z_n) \right\rangle_{p_0} + \delta\mathfrak{m}^\alpha \frac{\partial}{\partial \mathfrak{m}^\alpha} \left\langle \Phi_1(z_1)...\Phi_n(z_n) \right\rangle_{p_0} + ... \tag{5.3}$$

Given the above discussion, the naive guess would be to associate the derivative with an insertion of the conjugate deformation operator

$$\mathcal{O}_\alpha = \int d^2 z\, O_\alpha(z, \bar{z}). \tag{5.4}$$

It has been proposed that the derivative should be more properly understood to mean[2]

$$\begin{aligned} -\frac{\partial}{\partial \mathfrak{m}^\alpha} \left\langle \Phi_{a_1}(z_1)...\Phi_{a_n}(z_n) \right\rangle_p = \lim_{\epsilon \to 0} \Bigg[\int\limits_{\Sigma - \bigcup_i \mathcal{D}_i^\epsilon} d^2 z \left\langle O_\alpha(z) \Phi_{a_1}(z_1)...\Phi_{a_n}(z_n) \right\rangle_p \\ + \sum_{i=1}^n \left(\Gamma_{\alpha, a_i}{}^b(p) - \int\limits_{\mathcal{D}_i^1 - \mathcal{D}_i^\epsilon} d^2 z C_{\alpha, a_i}{}^b(z, p) \right) \left\langle \Phi_{a_1}(z_1)...\Phi_b(z_i)...\Phi_{a_n}(z_n) \right\rangle_p \Bigg], \end{aligned} \tag{5.5}$$

where $\mathcal{D}_i^\epsilon$ is a disk of radius ϵ around z_i, and $\mathcal{D}_i^1$ is the unit disk [23, 24, 22]. The insertion of the conjugate operator $\mathcal{O}_\alpha$ is expected. What of the other terms? The divergences caused by the short-distance singularities are removed by extracting the OPE contribution on a finite disc. The covariance of the expression under field

[2]The variational formula (5.5) was proposed in [23], but has not has not been derived from first principles as far as we are aware. A similar formula was suggested in [49].

transformations (5.2) is ensured by introducing the counter-terms $\Gamma_\alpha(p)$. These terms also compensate for the arbitrariness of how we subtract the potential divergences. Indeed, the C_α are not unique. There are different ways in which one may remove divergences in (5.5) and different prescriptions correspond to different choices of connection in $\mathcal{E}$. These are independent of z and transform as connections on the space of backgrounds,

$$\Phi_a \to \mathcal{S}_a{}^b(p)\Phi_b, \qquad \Gamma_{\alpha a}{}^b(p) \to \mathcal{S}_a{}^c(p)\Big(\Gamma_{\alpha c}{}^d(p) + \delta_c^d \partial_\alpha\Big)(\mathcal{S}^{-1})_d{}^b(p). \tag{5.6}$$

Defining the covariant derivative $D_\alpha = \partial_\alpha + \Gamma_\alpha$, this can be written in a manifestly covariant manner as

$$\begin{aligned} -D_\alpha\Big\langle \Phi_{a_1}(z_1)...\Phi_{a_n}(z_n)\Big\rangle_p = \lim_{\epsilon\to 0}\Bigg[&\int\limits_{\Sigma-\bigcup_i \mathcal{D}_i^\epsilon} d^2z \Big\langle O_\alpha(z)\Phi_{a_1}(z_1)...\Phi_{a_n}(z_n)\Big\rangle_p \\ &- \sum_{i=1}^n \int\limits_{\mathcal{D}_i^1-\mathcal{D}_i^\epsilon} d^2z\, C_{\alpha,a_i}{}^b(z,p)\Big\langle \Phi_{a_1}(z_1)...\Phi_b(z_i)...\Phi_{a_n}(z_n)\Big\rangle_p\Bigg]. \end{aligned} \tag{5.7}$$

The choice of connection is, in part, given by a choice of how we deal with the divergences. One may also combine this transformation with an automorphism of the operator algebra (a symmetry of the theory at a given point) and write

$$\begin{aligned} D_\alpha\Big\langle \Phi_{a_1}(z_1)...\Phi_{a_n}(z_n)\Big\rangle_p &= \lim_{\epsilon\to 0}\Bigg[-\int\limits_{\Sigma-\bigcup_i \mathcal{D}_i^\epsilon} d^2z \Big\langle O_\alpha(z)\Phi_{a_1}(z_1)...\Phi_{a_n}(z_n)\Big\rangle_p \\ &+ \sum_{i=1}^n \Omega_{\alpha,a_i}{}^b(p)\Big\langle \Phi_{a_1}(z_1)...\Phi_b(z_i)...\Phi_{a_n}(z_n)\Big\rangle_p\Bigg], \end{aligned} \tag{5.8}$$

where

$$\Omega_{\alpha,a_i}{}^b(p) = \int\limits_{\mathcal{D}_i^1-\mathcal{D}_i^\epsilon} d^2z\, C_{\alpha,a_i}{}^b(z,p) + \omega_{\alpha,a_i}{}^b(p). \tag{5.9}$$

ω generates the symmetry around each puncture. For CFTs, one can relate one value of ϵ to another by a dilation and so there is no need to take the $\epsilon \to 0$ limit. Indeed, different choices of ϵ may be compensated by including an appropriate conformal transformation in Ω.[3]

This construction is related to renormalization of the sigma model and renormalization flow defines a trajectory on $\mathcal{M}$. The relationship between this construction and renormalization group flow was explored in [23, 24][4] (see also [47]). In particular, compatibility of the renormalization flow of the operator $\mathcal{O}$ with the variational formula (5.5) places constraints on the coefficients $C_{\alpha,a_i}{}^b(z,p)$, which are detailed in [23, 24]. This relationship with renormalization is central. The deformation operator $\mathcal{O}$ relates one theory to a theory of a similar kind, with a different value of the background fields - the theory is qualitatively the same, but quantitatively different. This quality of self-similarity, familiar from renormalization, is the feature that defines a particular path between two points in $\mathcal{M}$.

5.1.1 Choices of connection

In [22], three connections were highlighted; denoted by $\hat{\Gamma}, c$ and $\bar{c}$. Here we will focus mainly on the $\hat{\Gamma}$ connection and the c connection (which is the connection originally used in [50] to computing the deformation of the stress tensor for string theory on a circle and is the natural choice for a general QFT). Each of these connections is described by the pair $(\mathcal{D}, \Omega)$, where Ω is as described above, and $\mathcal{D} = \bigcup_i \mathcal{D}_i$ is the region around the punctures that we remove to regulate the divergences. We briefly describe these connections here.

[3]The general form of the correlation function for a theory in D dimensions is

$$D_\alpha \Big\langle \prod_{i=1}^n \Phi_{a_i}(r_i) \Big\rangle_p = \lim_{\epsilon\to 0} \Bigg[- \int_{\Sigma - \bigcup_i \mathcal{D}_i^\epsilon} d^D r \Big\langle \widehat{O}_\alpha(r) \prod_{i=1}^n \Phi_{a_i}(r_i) \Big\rangle_p + \sum_{i=1}^n \int_{\mathcal{D}_i^1 - \mathcal{D}_i^\epsilon} \frac{d^D r}{\mathrm{Vol}(S^{D-1})} C_{\alpha,a_i}{}^b \Big\langle \Phi_{a_1}(r_1) ... \Phi_b(r_i) ... \Phi_{a_n}(r_n) \Big\rangle_p \Bigg]. \quad (5.10)$$

There are some differences in higher dimensions, such as the appearance of $\widehat{O}_\alpha(r) = O_\alpha(r) - \langle O_\alpha(r) \rangle$ in the correlation function and the smearing over angular directions. For details, we encourage the reader to consult [23, 24, 47]. The limit given here reflects the fact that the connection was defined in terms of balls of radius ϵ whose size was taken to zero. The prescription chosen to absorb divergences in the integral as $\epsilon \to 0$ into $C_{\alpha i}{}^k$ is part of what specifies a choice of connection. The prescription of [20] excised *unit* discs $\mathcal{D}^1$ from the worldsheet and hence exhibits no divergences in the correlation function corresponding to the limit $\epsilon \to 0$. As such, Ω_α may be taken to vanish, although one could choose Ω_α to include a finite transformation as discussed above.

[4]In fact, the requirement that the connection was compatible (in a way made precise in [23, 24]) with renormalization group flow was one of the main considerations in constructing it.

The $\hat{\Gamma}$ connection is defined by the pair $(\bigcup_i \mathcal{D}_i^1, 0)$, i.e. we remove a disk of radius 1 around each puncture z_i. The radius 1 is arbitrary and is conformally equivalent to any other radius. In this case, there are no divergences to subtract since we are not shrinking the discs to zero, so $\Omega = 0$. Note that this is only possible for a CFT since, due to the conformal symmetry, any calculation will be independent of the radii of the disks, so these radii may be freely chosen. It was pointed out in [22] that, using uniformising coordinates on Σ, this connection preserves the metric[5] $\mathscr{G}_{\alpha\beta}$ on the space of CFTs. Indeed, as observed in [44], string field theory seems to favour this connection. Note that we have not yet shown that the above construction is valid for CFTs, but we will show in section 5.2 that it is. Although the formulae will look quantitatively the same, the interpretation will of course be different as there is no RG flow in this case.

The connection c is defined to subtract away any divergences as we approach the punctures. Therefore, the disk around each puncture has radius $\epsilon \to 0$, and for a given puncture z_i, ${\Omega_{\alpha,a_i}}^b$ is the coefficient of Φ_b in the OPE between the deformation operator O_α and the operator at z_i which diverges as $z \to z_i$. This divergent part is computed in an annulus $\epsilon < |z - z_i| < 1$ (1 chosen here to match up with $\hat{\Gamma}$), and therefore we can also think of the c connection as follows: we always integrate up to the disk of radius 1 around a given puncture. Then, operators in the OPE which diverge as we take $z \to z_i$ are not integrated any further, and operators which do not diverge are integrated fully within the annulus. Practically, the way we would use this connection is that we would work with disks of radius ϵ removed around each puncture, and at the end of the calculation we try to take the limit $\epsilon \to 0$. If this limit exists, we take it, and if not, we set $\epsilon = 1$. This is done for each operator appearing in the OPE.

The $\bar{c}$ connection is a kind of 'minimal subtraction' variation on the c connection. We get this by noticing that not all terms in the divergent part of the OPE between the deformation operator and an operator insertion need be divergent in the limit $\epsilon \to 0$. We therefore define Ω such that we subtract only the parts of the OPE which diverge in the limit. In [22], it is explained that this is essentially just the diagonal part of the c connection, which is itself the upper triangular part of the $\hat{\Gamma}$ connection. We will not be particularly interested in the $\bar{c}$ connection in this thesis. The focus will be on the deformation of the sigma model and, in general, a specific choice of connection (regularisation of divergences) will not be made, but we will state explicitly if we ever do so.

[5]The metric is given by $\mathscr{G}_{\alpha\beta}(p) = \langle \Phi_\alpha \Phi_\beta \rangle_p$, where $\{\Phi_\alpha\}$ is a basis describing the theory, $p \in \mathcal{M}$ and Σ is a sphere.

5.1.2 Parallel transport

Given a suitable path in $\mathcal{M}$ and a covariant derivative of a correlation function in terms of some deformation operator $\mathcal{O}$, one can parallel transport operators from one theory to another. In other words, we can describe an operator at one point in $\mathcal{M}$ in terms of a basis of operators at another. We saw an example of this in the form of universal coordinates earlier, but the discussion here is more general. We recall the salient points of the discussion in [22] for completeness.

Suppose we have a path in moduli space $\mathfrak{m}^\alpha(s')$, where $s' \in [0, s]$ parameterises the path and the $\mathfrak{m}^\alpha$ can be thought of as local coordinates in $\mathcal{M}$. We are interested in expressing operators $\Phi(s)$ in terms of operators $\Phi(0)$ via parallel transport. This is defined by the vanishing of the covariant derivative $\frac{D}{Ds'}\Phi(s')$, which can be written in terms of the connection as

$$\frac{D}{Ds'}\Phi(s') = \frac{\partial}{\partial s}\Phi(s') + \frac{d\mathfrak{m}^\alpha(s')}{ds'}\Gamma_\alpha(s')\Phi(s') = 0, \tag{5.11}$$

where the connection Γ can be written in terms of $(\mathcal{D}, \Omega)$. [22] showed that the solution to this is given by

$$\begin{aligned}\Phi(s) &= \Phi(0) P \exp\left(-\int_0^s ds' \frac{d\mathfrak{m}^\alpha(s')}{ds'}\Gamma_\alpha(s')\right) \\ &= \Phi(0)\left(1 - s\frac{d\mathfrak{m}^\alpha}{ds}\Gamma_\alpha(0) - \frac{s^2}{2}\left(\frac{d^2\mathfrak{m}^\alpha}{ds^2}\Gamma_\alpha(0) + \frac{d\mathfrak{m}^\alpha}{ds}\frac{d\mathfrak{m}^\beta}{ds}\Big(\partial_\alpha\Gamma_\beta - \Gamma_\alpha\Gamma_\beta\Big)(0)\right) + ...\right),\end{aligned} \tag{5.12}$$

where P denotes path ordering and $\mathfrak{m}^\alpha, \Gamma_\alpha$ and their derivatives are all evaluated at $s = 0$ on the right hand side. For our purposes, $\mathfrak{m}^\alpha$ will essentially be the background tensor $E_{\mu\nu}$ and our connection will be defined by the deformation operator $\mathcal{O}$ and a choice of regularisation prescription. Note that this path ordered exponential is invariant under reparameterisations of s', so that the parallel transport of $\Phi(0)$ is in fact independent of the parameterisation we choose, as we would expect. In chapter 7, we will see this explicitly, in particular for the circle deformation, where we can either choose to work with δg, the full metric variation, or δR, the deformation of the radius of the circle, and both choices give the same results.

Alternatively, suppose we locally have a section of $\mathcal{E}$, given by a choice of some $\Phi(\mathfrak{m})$ in some subset of $\mathcal{M}$. Take some path through moduli space $\mathfrak{m}(s')$, $s' \in [0, s]$, and a connection $\Gamma_\alpha(s')$, as before. Unless $\Phi(\mathfrak{m})$ was generated by parallel transport using this connection, it is not covariantly constant along this path. However, it is possible to define a tensor along this path, $\tilde{\Phi}(s')$, such that $\tilde{\Phi}$ is covariantly constant, given by [22]

$$\tilde{\Phi}(s) = P\exp\left(-\int_0^s ds' \frac{d\mathfrak{m}^\alpha(s')}{ds'} D_\alpha\right)\Phi(s). \tag{5.13}$$

It can be easily checked that $D\tilde{\Phi}/Ds = 0$. It is also easily seen that, in the case where $D\Phi/Ds = 0$, i.e. Φ is already covariantly constant along the path $\mathfrak{m}(s')$, the above equation reduces to $\tilde{\Phi}(s) = \Phi(s)$, as we would expect. (5.13) can be used to relate parallel transport by different connections. Given two distinct connections Γ_1, Γ_2, use Γ_1 to compute the deformation of some operator Φ_0 along some path between points in $\mathcal{M}$ defined by parameters $s' = 0$ and $s' = s$. Call this parallel transport $\Phi_1(s')$, so that $\Phi_1(0) = \Phi_0$. This deformation is defined by parallel transport with respect to Γ_1, but it is not defined by parallel transport with respect to Γ_2. However, using (5.13), we can define a new deformation of Φ_0, $\Phi_2(s')$, which is covariantly constant wrt Γ_2, and in this way we can relate deformations defined by different connections. If we write $D_2 = \partial + \Gamma_2 = D_1 + \Gamma_2 - \Gamma_1$, then (5.13) becomes

$$\Phi_2(s) = P\exp\left(-\int_0^s ds' \frac{d\mathfrak{m}^\alpha(s')}{ds'}\Big(\Gamma_2 - \Gamma_1\Big)_\alpha\right)\Phi_1(s), \tag{5.14}$$

so the difference between the two is given by the difference between connections. In general, the difference between connections will include differences in regularisation and any local automorphisms. We will not take these to be physically significant and so take $\Phi_1(p) \sim \Phi_2(p)$.

A point that should be made clear here is that there is a difference between the choice of connection and the choice of deformation operator $\mathcal{O}$. Given two points $p_1, p_2 \in \mathcal{M}$, the first choice to be made is the path between the points. This essentially defines the deformation operator $\mathcal{O}$. Then, given the path, we then need to define a connection on that path, and, as discussed above, there are many different choices of connection one can make.

Chapters 6 and 8 will discuss the specific case of a flat background[6] with constant H-flux. There, the gauge transformations of the B-field play an important role. It is important to bear in mind that these gauge transformations are different to the redundancies discussed here that arise from different connection choices. This is because, if we have two B-fields that differ by a large gauge transformation, $B_1 \sim B_2$,

[6] We will have in mind a T^3, but we will work with $\mathbb{R}^3$ and impose identifications on the coordinates after the deformation.

these correspond to *different* points $p_1 \neq p_2 \in \mathcal{M}$, unless we impose appropriate identifications in $\mathcal{M}$. Therefore, this choice of B-field gauge is made before any connection considerations. Another example is the physical equivalence of two sigma models under a diffeomorphism that is not isometric, which would be described by different points in $\mathcal{M}$ unless an appropriate identification is imposed.

In general, there are two types of symmetry here. On the one hand, there are symmetries which preserve a given sigma model, i.e. which do not change the point in $\mathcal{M}$. (5.14) is an example of this, since $\Phi_{1,2}$ must be physically equivalent, so the relation between them can be described as a symmetry. Since this symmetry is induced by a change of connection on the same path $\mathfrak{m}(s')$, the point in $\mathcal{M}$, $\mathfrak{m}(s)$, is unchanged. On the other hand, there are symmetries between *distinct* points on $\mathcal{M}$. Physically, sigma models related by such symmetries are the same, but this is not apparent on the worldsheet.

We shall be interested in the case where the starting point for the parallel transport is a free theory, with the destination the theory with non-trivial interaction. In chapter 6, we will initially overlook these interactions by making use of the so-called *adiabatic limit* [19]. Later on, in chapter 8, we will pay particular attention to such interactions and the role they play in the parallel transport of operators of the theory. In either case, the OPE coefficients which appear in (5.9), and play a key role in the connection[7] Γ, are those of the free theory.

5.1.3 Example: QCD

As a concrete example of the above setup, let us look at one of the original examples demonstrating this method of correlator deformations, that of QCD given in [23]. This theory is characterised by parameters g_1, m and g_E. g_1 is an additive constant to the lagrangian density with scaling dimension 4, m is the quark mass parameter with scaling dimension 1, and g_E is the strong fine structure constant with scaling dimension zero. For each of these parameters, we have a conjugate operator, so-called because the variation of a correlation function wrt a given parameter is given by the formula (5.5) with O corresponding to the conjugate operator. For g_1, the conjugate operator is the identity operator $\mathbb{1}$. For the mass parameter m it is the mass density operator $O_m = \bar{\psi}\psi$, where ψ is the quark field, and the operator conjugate to g_E is $O_E = F_{\mu\nu}F_{\mu\nu}$, the energy density operator.

[7] The connection coefficients are determined by the OPE coefficients, the regularisation procedure used and any additional symmetries that are included. Specific examples are derived in [22].

Since g_1 has conjugate operator equal to the identity operator, it does not play a role in correlator deformations, so we will mostly focus on the other two operators. Since these conjugate operators are scalars, we can apply the formula (5.5), which gives

$$-\frac{\partial}{\partial m}\Big\langle\Phi_{a_1}(r_1)...\Phi_{a_n}(r_n)\Big\rangle_p = \lim_{\epsilon\to 0}\Bigg[\int\limits_{\Sigma-\bigcup_i\mathcal{D}_i^\epsilon} d^4r\Big\langle O_m(r)\Phi_{a_1}(r_1)...\Phi_{a_n}(r_n)\Big\rangle_p$$
$$+\sum_{i=1}^n\left(\Gamma_{m\,a_i}{}^b(p)-\int\limits_{\mathcal{D}_i^1-\mathcal{D}_i^\epsilon} d^2z C_{m\,a_i}{}^b(r,p)\right)\Big\langle\Phi_{a_1}(r_1)...\Phi_b(r_i)...\Phi_{a_n}(r_n)\Big\rangle_p\Bigg], \tag{5.15}$$

$$-\frac{\partial}{\partial g_E}\Big\langle\Phi_{a_1}(r_1)...\Phi_{a_n}(r_n)\Big\rangle_p = \lim_{\epsilon\to 0}\Bigg[\int\limits_{\Sigma-\bigcup_i\mathcal{D}_i^\epsilon} d^4r\Big\langle O_E(r)\Phi_{a_1}(r_1)...\Phi_{a_n}(r_n)\Big\rangle_p$$
$$+\sum_{i=1}^n\left(\Gamma_{E\,a_i}{}^b(p)-\int\limits_{\mathcal{D}_i^1-\mathcal{D}_i^\epsilon} d^2z C_{E\,a_i}{}^b(r,p)\right)\Big\langle\Phi_{a_1}(r_1)...\Phi_b(r_i)...\Phi_{a_n}(r_n)\Big\rangle_p\Bigg]. \tag{5.16}$$

The C_m, C_E are the OPE coefficients which are at least as singular as $1/r^4$, given by

$$\begin{aligned}\mathcal{O}_m(r)\Phi(0) &= C_m(r,p)\Phi(0)+o\left(\frac{1}{r^4}\right),\\ \mathcal{O}_E(r)\Phi(0) &= C_E(r,p)\Phi(0)+o\left(\frac{1}{r^4}\right),\end{aligned} \tag{5.17}$$

where the $o(1/r^4)$ terms are less singular that $1/r^4$ and do not contribute to the deformation in the limit $\epsilon\to 0$. Note that we have averaged over the radial direction here so that the operators appearing in the OPE expansions are scalar operators only. The coefficients C_m, C_E can be written as an infinite sum in powers of g_E, which can be shown using the RG equations given below. The details can be found in [23].

As mentioned, although these formulae are not derived from scratch, it is shown that they are consistent with the RG equations. Specifically, these are [23]

$$\begin{aligned}
&\frac{d}{dl}g_1 = 4g_1 + \frac{m^4}{4!}\beta_1(g_E), \\
&\frac{d}{dl}m = (1+\beta_m(g_E))m, \\
&\frac{d}{dl}g_E = \beta_E(g_E),
\end{aligned} \tag{5.18}$$

where the beta functions can be expanded in powers of g_E near $g_E = 0$. The conjugate operators $\mathcal{O}_m$, $\mathcal{O}_E$ have the RG equations

$$\begin{aligned}
&\frac{d}{dl}\mathcal{O}_m = (3-\beta_m(g_E))\,\mathcal{O}_m - \frac{m^3}{3!}\beta_1(g_E)\mathbb{1}, \\
&\frac{d}{dl}\mathcal{O}_E = (4-\beta_E'(g_E))\,\mathcal{O}_E - m\beta_m'(g_E)\mathcal{O}_m - \frac{m^4}{4!}\beta_1'(g_E)\mathbb{1}.
\end{aligned} \tag{5.19}$$

If $\mathcal{O}_i$ is a basis of gauge invariant scalar operators, where $\mathcal{O}_i$ has scaling dimension x_i, we also have the RG equations

$$\frac{d}{dl}\mathcal{O}_i = x_i\mathcal{O}_i + \sum_j \gamma_{i,j}\mathcal{O}_j, \tag{5.20}$$

where $\gamma_{i,j}$ are the anomalous dimensions. Using these equations, if we substitute (5.18), (5.19), (5.20) into the LHS and RHS of (5.15),(5.16), we find that we obtain consistency conditions analogous to the relation between the anomalous dimensions and the $1/\epsilon$ poles in dimensional regularization with minimal subtraction [23]. What is also found in [23] is that (5.15), (5.16) can be iterated to produce consistency conditions at higher orders. For example, at second order we get consistency conditions involving the curvature. Thus, as discussed at the beginning of section 5.1, the deformation formulae can be shown to be consistent with the RG equations, at least to second order.

5.2 Connections on the space of CFT backgrounds

We now consider the case where we are specifically dealing with on-shell deformations of CFT correlators. As mentioned above, the $\hat{\Gamma}$ connection is the natural choice here, though many other connections are valid. The main difference with the above will be that we can now work with surface states instead of correlation functions, which are more subtle objects.

5.2.1 CFT connections and surface states

The choice of which connection to use depends on what we want to preserve under parallel transport. Given that a string background (in the perhaps limited sense of satisfying the string equations) is equivalent to the existence of an associated worldsheet CFT, natural things to preserve under parallel transport are gluing properties of CFTs [22]. In this way, the parallel transport of the state space of a CFT will yield another CFT and in principle could be used to explore the CFT moduli space.

A perturbative string theory defined on a certain background, in the form of a string field theory, may be described by the BRST charge $\mathcal{Q}_B$ and the collection of surface states $\langle\Sigma_n|$. As such, a simple way to understand the effect of a connection on a CFT is to see how the stress tensor and surface states transform under parallel transport. Let $\Sigma_{n,g}$ be an n-pointed Riemann surface of genus g. A surface state $\langle\Sigma_{n,g}|$ may be defined as follows. Given the states $|\Phi_i\rangle$ in the string Hilbert space at the i'th puncture, the correlation functions $\langle\Phi_i(z_1)...\Phi_n(z_n)\rangle$ are given by[8]

$$\langle\Phi_i(z_1)...\Phi_n(z_n)\rangle = \langle\Sigma_{n,g}||\Phi_1\rangle...|\Phi_n\rangle, \tag{5.22}$$

where $|\Phi_i\rangle$ is the state corresponding to the operator Φ_i inserted at the puncture located at $(z_i, \bar{z}_i)$.

If we have a CFT, the general framework for constructing connections on the space of backgrounds has been mapped out [42] (see also [22]) in terms of the surface state $\Big|\Sigma, z_i\Big\rangle$. Such connections give a manifestly conformally-invariant way of moving between backgrounds. As we saw for the QFT case, moving from the point $p(\mathfrak{m}^\alpha)$ to $p'(\mathfrak{m}^\alpha + \delta\mathfrak{m}^\alpha)$ with marginal operator $\mathcal{O}_\alpha$, conjugate to the deformation, can be described in terms of a covariant derivative. However, here the covariant derivative acts on the surface states as

$$\Big|\Sigma_{n,g}\Big\rangle_{p'} = \Big|\Sigma_{n,g}\Big\rangle_p + \delta\mathfrak{m}^\alpha D_\alpha\Big|\Sigma_{n,g}\Big\rangle_p + ..., \tag{5.23}$$

[8] The surface state for the matter sector may be written as

$$\langle\Sigma_{n,g}| = \langle\vec{0}|\exp\left(\frac{1}{2}\sum_{a,b=1}^{N}\sum_{m,p\geq 0}\mathcal{N}_{mp}(z_a,z_b)\alpha_m^{(a)}\cdot\alpha_p^{(b)} + \text{a.h.}\right), \tag{5.21}$$

where $\mathcal{N}_{mp}(z_a, z_b)$ are Neumann coefficients (see, for example [51]) and $\langle\vec{0}| = \langle 0_1|...\langle 0_n|(2\pi)^D\delta^D\left(\sum_a p_a\right)$, and similarly for the right-moving sector. Ghost parts have been neglected but may be straightforwardly incorporated.

where

$$D_\alpha|\Sigma_{n,g}\Big\rangle_p = -\int\limits_{\Sigma - \cup_i \mathcal{D}_i} d^2z \langle O_\alpha|\Sigma_{n+1,g}; z\rangle_p - \sum_{i=1}^{n} \Omega_{\alpha,a_i}|\Sigma_{n,g}\rangle_p, \tag{5.24}$$

where $|\Sigma_{n+1,g}; z\rangle$ is a surface state with an extra puncture at the point z where the marginal operator $\mathcal{O}_\alpha$ is inserted. Covariance here is defined in exactly the same way, i.e. with respect to the basis transformations of the form (5.2) [9]. For the QFT case, the requirement of covariance came from the fact that we wanted the correlation functions to transform in a covariant way with respect to basis redefinitions. Here, the same is true, but we also demand covariance because a deformation defined by the above covariant derivative will always generate a CFT definition, in the sense that the deformed surface states will satisfy the necessary CFT sewing relations, as shown in [22]. When written in terms of correlation functions, the deformation equations look exactly the same for both QFTs and CFTs, with only the space of backgrounds and the interpretation being different. Thus, in practice, there is not much to distinguish between the two, and we will generally not be interested in these differences in this thesis. Instead, we will be interested in how this correlator deformation formalism can be used to extract information about the deformation of specific operators defined in a particular space of theories, whether that be a space of CFTs or renormalized QFTs.

The effect of parallel transport by different connections on the surface states provides a useful language in which to frame the issue of background independence in string field theory [44]. However, it is difficult to find suitably explicit, yet interesting, string solutions which are non-trivial torus fibrations and for which the explicit CFT is known. One can learn a lot by studying explicit toy models which, though not CFT descriptions of complete string backgrounds, may play an interesting role as building blocks for such backgrounds. Well studied examples include T^2 bundles with monodromies in the $SL(2;\mathbb{Z})$ modular group of the fibre and T^3 backgrounds with constant volume-filling NS flux, like the backgrounds considered in chapter 3. Such examples have been recently studied [52] and do play a role in what are thought to be string solutions. We

[9]In this framework, one can see how the stress tensor transforms in such a way as to preserve conformal invariance [42]. For the stress tensor, we can get the Virasoro generators from the two-punctured sphere

$$L_n = \frac{d}{d\epsilon_n}\langle\Sigma; z', 1/z'\big|_{\epsilon_n=0}, \tag{5.25}$$

where $z' = z + \sum_n \epsilon_n z^n$. The change in the modes of the stress tensor under a change in background may be extracted by

$$\epsilon\delta L_n = \int_{\mathcal{D}'-\mathcal{D}} d^2z\mathcal{O}_{z\bar{z}}, \tag{5.26}$$

where the discs $\mathcal{D}'$ and $\mathcal{D}$ are related by a conformal transformation generated by the stress tensor and $\mathcal{O}$ is a marginal operator relating the two backgrounds under consideration.

will study such examples both in the adiabatic limit, where they can be viewed as having approximate conformal symmetry, and with all worldsheet interactions taken into account. When these interactions are taken into account, such luxuries as universal coordinates are no longer available to us. We will study this in detail in chapter 8. These backgrounds also play a role in the well-known NS5-brane and Kaluza-Klein monopole backgrounds [53–56].

5.2.2 A connection from universal coordinates

Which connection we choose depends on what aspects of the model we care about. It was shown in [44] that String Field Theory selects, as a natural connection, that found in [20], i.e. the $\hat{\Gamma}$ connection. In this case, the connection arises not from the principle of gluing being a CFT operation, but from the requirement that the universal coordinates $(\Pi_\mu(\sigma), X^\mu(\sigma))$ are preserved on backgrounds of the form $M_D \times T^d$. Put another way, at each point on the space of backgrounds there exists a canonical set of fields $(\Pi_\mu(\sigma), X^\mu(\sigma))$. This connection transports the $(\Pi_\mu(\sigma), X^\mu(\sigma))$ at one background to the corresponding $(\Pi_\mu(\sigma), X^\mu(\sigma))$ at another background, where they are identified (up to possible symmetry transformations at that point). This is the $\hat{\Gamma}$ connection studied in section 5.1 [22] and is the simplest choice of connection intuitively from the perspective of string field theory. However, there are technical challenges to integrating infinitesimal deformations up to finite changes of background.

It was shown in [43] that this CFT connection is equivalent to a connection proposed in [20] when considering string field theory on toroidal backgrounds. The virtue of the derivation of this connection given in [20] is that it does not rely on CFT concepts and so generalises to sigma models that are not exact CFTs. In this sense, it is a connection that can allow us to study backgrounds that are, from a string field theory perspective, off-shell, opening up the possibility of connecting a very wide class of sigma models to one another by parallel transport of this connection.

Let us briefly see how this connection relates to the universal coordinate results we reviewed in 4 for CFTs. The basic idea is to consider an object, such as the surface state defined at a point p, and then to consider how it changes under a change of background. To first order, this gives the connection

$$|\Sigma_n\rangle_{p'} - |\Sigma_n\rangle_p = \delta\mathfrak{m}^\alpha \Gamma_\alpha |\Sigma_n\rangle + \dots. \tag{5.27}$$

As discussed in section 4.3, $\partial X_\mu(E)$ defined at the background $E_{\mu\nu}$ is related to that defined at the background with metric $G_{\mu\nu}$ and zero B-field by

$$\partial X_\mu(E) = \frac{1}{2}(\partial\phi_\mu + \bar{\partial}\phi_\mu) + \frac{1}{2}E_{\mu\nu}G^{\nu\rho}(\partial\phi_\rho - \bar{\partial}\phi_\rho). \tag{5.28}$$

Given that the metric and B-field are constant, the associated mode transformation is given by (4.16). In terms of the circle at radius $R = 1 + \delta R$, this latter result gives

$$\delta\alpha_n = \delta R\Big(\alpha_n + \bar{\alpha}_{-n}\Big), \tag{5.29}$$

to leading order in δR, as found in [20]. The surface state $|\Sigma_n\rangle$ depends explicitly on these modes and so might be expected to transform under the change of background. The Neumann coefficients do not change, but the α_n do, and with the transformation (5.29) comes a corresponding transformation in the worldsheet vacuum. The vacuum is defined by $\alpha_n|0\rangle = 0 = \bar{\alpha}_n|0\rangle$ for $n \geq 0$, and so a change in background also changes the vacuum,

$$|0\rangle \to e^{\Delta}|0\rangle, \qquad \Delta = \sum_{n\neq 0}\frac{1}{2n}\alpha_n^\mu \delta E_{\mu\nu}\bar{\alpha}_n^\nu. \tag{5.30}$$

Using these results, it follows [20, 44] that the Ω-independent part of this connection can be seen to preserve the surface states (for $n > 2$). This can be understood from the perspective of the CFT connection as that connection which takes $\{\mathcal{D}_i\}$ to be unit discs in terms of projective coordinates on $\Sigma_{n,g}$. The sewing relation $w = 1/z$ then tells us that the domain of integration $\Sigma - \cup_i \mathcal{D}_i$ in (5.24) is empty and the connection is of the form

$$\Gamma_\alpha|\Sigma_{n,g}\Big\rangle_p = \sum_{i=1}^{n}\Omega_{\alpha,a_i}|\Sigma_{n,g}\rangle_p, \tag{5.31}$$

i.e. just a point-wise automorphism on the Hilbert space.

As we have seen, the difference between using correlation functions and deforming surface states is really just a difference in approach, rather than principle.

Thus, a connection such as that defined by universal coordinates gives a way to clearly describe the transport of a Hilbert space from one point on the space of backgrounds to another. Note that the Buscher procedure does not require the worldsheet theory to be a CFT and so this allows us to make contact with the predictions of T-duality applied to more general non-linear sigma models. This observation will form the basis of the application of the procedure of [15] to various toy models based on backgrounds such as the nilfold, T^3 with constant H-flux and their non-geometric relatives, amongst others.

5.3 Example: stress tensor deformation

Let us look at a relatively simple CFT example. We start with a worldsheet embedding into S^1. In [50], the deformation of the stress tensor of the free boson CFT to first order under an infinitesimal deformation of the background was derived[10]. Here, we briefly recap this derivation as it will be important for much of our later discussions. We start by considering the addition of the term to the action

$$\mathcal{O} = \lambda \int_{\Sigma} d^2z \, O(z, \bar{z}), \tag{5.32}$$

where O is some $(1,1)$ primary field and λ is a constant. The correlation function in the deformed background, denoted by a prime, is

$$\begin{aligned}\Big\langle \Phi_1(z_1, \bar{z}_1)...\Phi_N(z_N, \bar{z}_N)\Big\rangle' &= \Big\langle \Phi_1(z_1, \bar{z}_1)...\Phi_N(z_N, \bar{z}_N)\Big\rangle \\ &+ \lambda \int_{\Sigma^\epsilon} d^2z \Big\langle O(z, \bar{z})\Phi_1(z_1, \bar{z}_1)...\Phi_N(z_N, \bar{z}_N)\Big\rangle,\end{aligned} \tag{5.33}$$

where we have defined the shorthand

$$\Sigma^\epsilon = \Sigma - \bigcup_i \mathcal{D}_i^\epsilon. \tag{5.34}$$

Note that we are not taking the limit $\epsilon \to 0$ here since we are dealing with a CFT. Of course, we may still choose to take the limit if we wish, but we have the freedom to leave ϵ as it is for now. However, if we insert our operator of interest at some point $z = w$, we will always choose our connection such that the radius of the disk around $z = w$ goes to zero, and we will assume that this is understood throughout the thesis. Thus, given the correlator[11]

$$\Big\langle T(w) \prod_{i=1}^{N} \Phi_i(z_i, \bar{z}_i)\Big\rangle = \sum_{i=1}^{N} \sum_{n \geq -1} (w - z_i)^{-n-2} \Big\langle \prod_{j \neq i} \Phi_j(z_j, \bar{z}_j) L_n \Phi_i(z_i, \bar{z}_i)\Big\rangle, \tag{5.35}$$

[10] See also [18, 57] for further discussion on the deformation of the stress tensor.

[11] We have used

$$(w - z)^{-2} = \sum_{n \geq 0} n(w - z_i)^{-n-1}(z - z_i)^{n-1}.$$

we insert the marginal operator $\mathcal{O}$ and compute the OPEs of T with $\mathcal{O}$ and with the Φ_i (see [50] for the full details). We find that

$$\begin{aligned}\Big\langle T(w)\prod_{i=1}^{N}\Phi_i(z_i,\bar{z}_i)\Big\rangle' &= \sum_{i=1}^{N}\sum_{n\geq -1}(w-z_i)^{-n-2}\Big\langle\prod_{j\neq i}\Phi_j(z_j,\bar{z}_j)L_n\Phi_i(z_i,\bar{z}_i)\Big\rangle' \\ &\quad -\lambda\oint_{C^\epsilon} d\bar{z}\frac{1}{w-z}\Big\langle O(z,\bar{z})\Phi_1(z_1,\bar{z}_1)...\Phi_N(z_N,\bar{z}_N)\Big\rangle, \end{aligned} \tag{5.36}$$

where C^ϵ denotes the collection of anticlockwise contours of radius ϵ around each of the points (with $\epsilon\to 0$ for the contour around w). Noting then that the RHS contains no singularities at $z=w$ and therefore that the integral around w vanishes in the $\epsilon\to 0$ limit, we are finally left with

$$\begin{aligned}\Big\langle T(w)\prod_{i=1}^{N}\Phi_i(z_i,\bar{z}_i)\Big\rangle' = \sum_{i=1}^{N}\sum_{n\geq -1}(w-z_i)^{-n-2}\Big\langle\prod_{j\neq i}\Phi_j(z_j,\bar{z}_j)\Big(L_n\Phi_i(z_i,\bar{z}_i) \\ -\lambda\oint_{C_i^\epsilon} d\bar{z}(z-z_i)^{n+1}O(z,\bar{z})\Phi_i(z_i,\bar{z}_i)\Big)\Big\rangle', \end{aligned} \tag{5.37}$$

where C_i^ϵ is the contour around the point z_i of radius ϵ, from which we compare coefficients of $(w-z_i)^{-n-2}$ and read off the shift in the Virasoro modes as[12]

$$L_n\to L_n-\lambda\oint_{C_0^\epsilon} d\bar{z}z^{n+1}O(z,\bar{z}), \tag{5.38}$$

where C_0^ϵ is the contour of radius ϵ around $z=0$. Beyond first order, the prescription must be more carefully defined since we have multiple $\mathcal{O}$ insertions. We must ensure that we define these insertions in such a way that we are able to deal with potential singularities when different $\mathcal{O}$ insertions coincide. The details of our prescription for this are given in appendix E, and we show that this does indeed give the expected deformation for ∂X.[13]

[12]Naively, it looks as though this will lead to non-trivial commutation relations between L_n and $\bar{L}_n$ in the deformed CFT. It was shown in [42] that this is not the case.

[13]Note that the formalism discussed in appendix E is that of chapter 7. This is necessary since in the appendix we take a general approach where we are not simply using universal coordinates, as we do in chapter 6, and we are considering a general connection on the space of backgrounds. This is similar to how we computed the stress tensor deformation above, but, since ∂X is not a scalar operator, its deformation is more complicated than that of the stress tensor. We will discuss this in more detail in chapter 7.

We note that the natural objects with which to describe the deformation in $\mathcal{M}$ are, as anticipated, non-local objects - the modes - rather than local fields on Σ.

Chapter 6

Connections and Torus Bundles

Having gained an understanding of how to construct deformations on the space of theories via parallel transport, we now apply this formalism to some concrete examples. In particular, we study the torus bundle examples introduced in chapter 3, but from the perspective of parallel transport on the space of theories containing these backgrounds. We consider the role CFT connections and their generalizations can play in understanding a special class of torus bundle backgrounds. Note that the backgrounds considered in this chapter are in an 'adiabatic limit' where we neglect worldsheet interactions arising from the base of the torus bundle. We will explain this in more detail in section 6.4. This limit essentially allows us to use the universal coordinate formalism discussed in chapter 4 without having to worry about the specifics of the parallel transport construction of the previous chapter. In section 6.1, we will explain our general philosophy for this chapter in more detail.

Towards the end of this chapter we consider how the formalism discussed extends to backgrounds which are not torus bundles, but can still be described as flux compactifications. We also consider a further generalisation where the flux components are non-constant. The content of this chapter is based on original work published in collaboration in [1].

6.1 Twisting as a relationship between backgrounds

Consider string theory on a T^2 background with constant metric $g_{\mu\nu}$ and B-field $B_{\mu\nu}$. The space of such backgrounds is the orbifold

$$\mathscr{M} = O(2,2;\mathbb{Z})\backslash O(2,2)/O(2)\times O(2) \tag{6.1}$$

with local coordinates $\mathfrak{m}^\alpha$. As we saw in chapter 5, we can parallel transport elements of the Hilbert space along curves in (6.1) using a connection, as given by (5.12). We would like to make this more specific to the twisted torus backgrounds we considered earlier. Consider a path γ parameterised by $s \in [s_1, s_2]$ such that any two points on γ are related by an action of a particular generator of $O(2,2)$. For simplicity, we take the parameterisation to be aligned with one of the local coordinates $\mathfrak{m}$ on $\mathscr{M}$ and also $(s_1, s_2) = (0, \mathfrak{m})$. The mode operators $(\alpha_n, \bar{\alpha}_n)$ provide illustrative examples of operators we might transport from one point to another. Alternatively, we could consider the case where the operators in (5.12) consist of the worldsheet currents ∂X^μ and $\bar{\partial} X^\mu$. Suppose we start at $E = G$, the point of enhanced symmetry. In general, the transformation of a basis of operators given by such a parallel transport will mix the fields together. Matters are simpler for the deformations that move us around the space of torus compactifications. The transformation of the modes is given by (4.16), where we note that the zero modes transform amongst themselves and so the transformations of the fields ∂X_μ and $\bar{\partial} X_\mu$ tell us exactly how the target space changes under the transformation and we shall focus on those fields. For different deformations of the theory, different subsets of fields may be of interest[1]. The relationship (5.12) can be written as

$$\widehat{\mathcal{A}}_I = \Big(e^{-\Gamma(\mathfrak{m})}\Big)_I{}^J \mathcal{A}_J, \tag{6.2}$$

where we have written $\mathcal{A}_I = (\partial X_\mu, \bar{\partial} X_\mu)$, and $\hat{\mathcal{A}}_I = (\partial \hat{X}_\mu, \bar{\partial} \hat{X}_\mu)$ are the objects at another background. In particular,

$$\partial X_\mu = \frac{1}{2}\Big(\delta^\nu_\mu + E_{\mu\rho} G^{\rho\nu}\Big)\partial\phi_\nu + \frac{1}{2}\Big(\delta^\nu_\mu - E^T_{\mu\rho} G^{\rho\nu}\Big)\bar{\partial}\phi_\nu, \tag{6.3}$$

where $\partial\phi_\mu, \bar{\partial}\phi_\mu$ are defined according to (4.10). For example, we could take $\partial\phi_\mu = (\partial\phi_y, \partial\phi_z)$ and the deformation to generate a constant B-field

$$B = m d\phi_y \wedge d\phi_z, \tag{6.4}$$

giving

$$\partial Y = \partial\phi_y + \frac{m}{4}(\partial\phi_z - \bar{\partial}\phi_z), \qquad \partial Z = \partial\phi_z - \frac{m}{4}(\partial\phi_y - \bar{\partial}\phi_y). \tag{6.5}$$

Such a deformation can be written as an automorphism

$$\widehat{\mathcal{A}}_I = \exp\Big(-\Gamma_I{}^J\Big)\mathcal{A}_J = U^{-1}\mathcal{A}_I U, \tag{6.6}$$

[1]This would be the case if, for example, some additional fields became massless along the path.

where
$$U = \exp\left(\frac{m}{4}\oint d\sigma\Big(Y(\sigma)Z'(\sigma) - Y'(\sigma)Z(\sigma)\Big)\right). \tag{6.7}$$
Since $\widehat{\mathcal{A}}_I$ can be written in this way, such a deformation is pure gauge, provided m is appropriately quantised[2].

A second example is the deformation
$$U = \exp\left(-\frac{m}{2}\oint d\sigma\Big(Z(\sigma)\Pi_y(\sigma)\Big)\right). \tag{6.8}$$
This too is a gauge transformation and may be understood as follows. In terms of the action of Γ along γ, we might consider the element of the parabolic conjugacy class of $SL(2)$
$$\Pi_\mu \to \left(e^{-\Gamma}\right)_\mu{}^\nu \Pi_\nu, \qquad X^{\mu\prime} \to \left(e^{-\Gamma^T}\right)^\mu{}_\nu X^{\nu\prime}, \qquad \Gamma = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \qquad m \in \mathbb{R}, \tag{6.9}$$
where Γ^T denotes the transpose of Γ and $e^\Gamma \in SL(2)$. Thus, the algebra of operators of the theory at the background at one point on γ is related to that at another point by an $SL(2)$ transformation. In other words, as we move along the curve γ we change the complex structure of the torus. In this way, we can think of γ as a curve generated by a particular element of the Lie algebra of $SL(2)$. In particular, if we chose the parameterisation such that the metric at $m = 0$ is given by $g_{\mu\nu} = G_{\mu\nu}$, then the metric at a point $m \neq 0$ would be given by $g(m) = e^{-\Gamma m} G e^{-\Gamma^T m}$. Alternatively, we can describe this by a change in the complex structure,
$$\tau(m) = i + m, \tag{6.10}$$
on the T^2 fibre, and by a general element of the Mobius group for a γ generated by a general element of $SL(2)$. Any two backgrounds related by $SL(2;\mathbb{Z}) \subset O(2,2;\mathbb{Z})$ are identified in $\mathscr{M}$ and so are equivalent. This is just the action of the modular group of the T^2 in this simple case and we can consider transporting around closed loops, provided the monodromy is in $O(2,2;\mathbb{Z})$.

6.1.1 Twisted backgrounds

Where things become interesting is when the above discussion inspires constructions of non-trivial backgrounds that may be part of exact string solutions. In this case,

[2] Setting $m \in \mathbb{Z}$ sets $e^{\Gamma(m)} \in O(2,2;\mathbb{Z})$.

we take the closed curve γ as a physical direction in spacetime with coordinate x and fibre the T^2 over the line, varying the complex structure in the above way as we do so. Moreover, x can be made periodic as long as the resulting monodromy is an element of $O(2,2;\mathbb{Z})$.

Thus, we have a locally smooth geometry given by a T^2 fibred over a circle with coordinate x. The monodromy of the bundle is an element of $O(2,2;\mathbb{Z})$ acting on the fields $\Pi_I := (\Pi_\mu, X'^\mu)$ of the T^2 fibre, and here the monodromy along the base direction x must be an element of $O(2,2;\mathbb{Z})$.

We can use the CFT connection to transport the theory in the fibres from one point to another on the base and, if the fibres contain circles that are at the self-dual radius at some point, then T-duality may be performed as outlined in section 4.3.3 following the procedure of [15]. Treating the CFT in the fibres separately from the base gives a construction in which the duality is performed fibrewise. Such a construction is useful, but may not always give the full story as the base coordinate may play an important role. We will come to this in chapter 8, where we attempt to consider the base coordinate dependence in its entirety, but for now we shall see how far we can go without taking this into account.

The theory in the fibres is a CFT, but by including the base direction with non-trivial monodromy, the background described by the bundle is not a CFT.[3] However, using the connection construction of chapter 5, we can still make use of the same operator deformations as we had in section 6.1. We shall apply this general formalism to the torus bundles described above. In what follows, we shall focus only on the $(\partial X_\mu, \bar{\partial} X_\mu)$ sector and ignore any possible mixing with other fields. The rationale for this is that, unless modes become massless under parallel transport, such mixing is not expected to play a central role in the description of the target space of the backgrounds we consider to leading order. Neglecting such mixing is equivalent to considering the duality to be performed fibrewise, which we can see more concretely by considering the mode expansion of the base coordinate. Taking the base circle to have radius R and re-introducing the R and α' dependence explicitly, the field X in our expressions is replaced by X/R, which may be written as

$$R^{-1}X(z,\bar{z}) = R^{-1}x - iw\ln(z/\bar{z}) - i\lambda^2 p\ln|z| + \frac{i}{\sqrt{2}}\lambda\sum_{n\neq 0}\frac{1}{n}\Big(\alpha_n z^{-n} + \tilde{\alpha}_n\bar{z}^{-n}\Big), \quad (6.11)$$

where $\lambda = \sqrt{\alpha'/R^2}$ and p and w are momentum and winding numbers respectively. The fibrewise construction, in which the field X is taken as a parameter x, is recovered

[3] Though it may be an important part of an exact string background[52].

in the $w = 0$ sector by taking the $\lambda \to 0$ limit. Thus, we see that the operator mixing that signals a departure from the fibrewise construction enters, at least in the $w = 0$ sector, when the supergravity approximation ($\lambda \ll 1$) can no longer be trusted. This can also be understood from a physical perspective in terms of an 'adiabatic' argument of the type considered in [19], and we shall refer to it as such. We shall assume that we are working in this 'adiabatic limit' for the remainder of this chapter, but from chapter 7 we will move away from this limit and consider the full base coordinate dependence. In section 6.4, before we consider torus bundle examples in detail, we will give a brief overview of the adiabatic limit and how it applies here.

An example of such a construction is the nilfold, with monodromy given by (6.9). In this case, the analogue of (6.3) is given by

$$\begin{gathered}\partial X = \partial\phi_x, \qquad \partial Y = \partial\phi_y - \frac{1}{2}m\phi^x(\partial\phi_z - \bar{\partial}\phi_z), \\ \partial Z = \left(1 + \frac{1}{2}(m\phi^x)^2\right)\partial\phi_z - \frac{1}{2}(m\phi^x)^2\bar{\partial}\phi_z - \frac{1}{2}m\phi^x(\partial\phi_y - \bar{\partial}\phi_y),\end{gathered} \tag{6.12}$$

and the corresponding expressions for $\bar{\partial}X_\mu$ given in (6.40). One can show that it is not possible to find a U such that this transformation can be produced by a similarity transformation of the form (6.6) and so this can be thought of as a physical deformation, rather than a gauge transformation. The key difference between the two cases is that here the presence of $X(\sigma)$ (or $\phi^x(\sigma)$) multiplying m means that the exponent in U would have to depend linearly on $mX(\sigma)$; however, such a dependence is incompatible with $\partial X = \partial\phi_x$. By contrast, in the pure gauge case, m is a parameter - a real number - and so its presence in the exponent of U does not affect the qualitative transformation properties of the fields under automorphisms with U.

The above discussion is a rather tortuous way of thinking about familiar duality twist backgrounds [29, 58]. What is gained by framing the construction in this way is an interpretation of the monodromy as a map between different backgrounds. In particular, we can think of the duality twist (6.9) as a map from a way of describing a given background $E_{\mu\nu}$ in terms of a reference background $G_{\mu\nu}$. If we take the reference metric to be at a point of enhanced symmetry in $\mathscr{M}$ then we can use this relationship to describe that background in terms of a Hilbert space of fields at the point of enhanced symmetry. This seems reminiscent of [15] and indeed we can see that, for geometric backgrounds, this is the same construction as found there. This construction gives a framework in which to describe fibrewise T-duality in torus bundles, which we will come to when we discuss the nilfold in section 6.6.

6.2 Twisting the hamiltonian

The worldsheet Hamiltonian (density) of a theory related by a geometric duality twist is

$$\mathcal{H}(X) = \begin{pmatrix} \Pi, & X' \end{pmatrix} \begin{pmatrix} e^{-1} & 0 \\ -Be^{-1} & e \end{pmatrix} \mathcal{H}_0 \begin{pmatrix} e^{-T} & e^{-T}B \\ 0 & e^{T} \end{pmatrix} \begin{pmatrix} \Pi \\ X' \end{pmatrix}, \tag{6.13}$$

where $\mathcal{H}_0$ is the Hamiltonian density at a point of enhanced symmetry, and the metric g can be written in terms of a vielbein as $g_{\mu\nu} = e_{\mu}{}^{a}\delta_{ab}e^{b}{}_{\nu}$. Here, the twist is composed as a product of an $SL(d;\mathbb{Z})$ transformation and a B-field shift, which may be written as

$$\begin{pmatrix} e^{-1} & 0 \\ -Be^{-1} & e \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix}. \tag{6.14}$$

For example, for the $SU(2)$ enhanced symmetry we set the radii of the circles to be 1 and the B-field to be zero. We may write this as

$$\mathcal{H}_0 = \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix}. \tag{6.15}$$

Then, $\mathcal{H}(X) = \mathcal{Z}^T G^{-1}\mathcal{Z} + \mathcal{X}^T G \mathcal{X}$, where

$$\mathcal{Z}_a = e_a{}^{\mu}\Big(\Pi_\mu - B_{\mu\nu}X'^{\nu}\Big), \qquad \mathcal{X}^a = e^{a}{}_{\mu}X'^{\mu}. \tag{6.16}$$

We can think of $(\mathcal{Z}_a, \mathcal{X}^a)$ as twisted versions of (Π_μ, X'^μ). Defining $\partial X_a(E)$ and $\bar{\partial} X_a(E)$ by

$$\mathcal{Z}_a = \partial X_a(E) + \bar{\partial} X_a(E), \qquad G_{ab}\mathcal{X}^b = -\partial X_a(E) + \bar{\partial} X_a(E), \tag{6.17}$$

one can show that $\partial X_\mu = e_\mu{}^a \partial X_a$ may be written as

$$\partial X_\mu(E) = \frac{1}{2}\left(\partial\phi_\mu + \bar{\partial}\phi_\mu\right) + \frac{1}{2}E_{\mu\nu}G^{\nu\rho}\left(\partial\phi_\rho - \bar{\partial}\phi_\rho\right). \tag{6.18}$$

This is the same relationship between backgrounds as found in chapter 4. For non-geometric backgrounds, where we can still write the Hamiltonian as a duality twist of a reference background $\mathcal{H} = \mathcal{O}\mathcal{H}_0\mathcal{O}^T$, $\mathcal{Z}_a$ and $\mathcal{X}^a$ may not take the form (6.16). This will be relevant when we discuss the T-fold in section 6.8.

6.3 Degenerating fibres

One might consider an example where the curve γ passes through a point in the moduli space where the fibres degenerate. An example of this would be for the T^2 bundle where the Deligne-Mumford compactification of the moduli space allows us to include points on the boundary corresponding to a cycle in the T^2 fibre degenerating. The transformation of the chiral fields is

$$\begin{pmatrix} \partial X \\ \bar{\partial} X \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + EG^{-1} & 1 - EG^{-1} \\ 1 - E^T G^{-1} & 1 + E^T G^{-1} \end{pmatrix} \begin{pmatrix} \partial \phi \\ \bar{\partial} \phi \end{pmatrix}. \tag{6.19}$$

We see that, if the fibres degenerate ($E \to 0$), then the matrix appearing above has a non-trivial kernel and we can no longer transport the chiral fields past this degeneration (the transformation is no longer invertible).

A virtue of this framework is that the emphasis is placed upon the integrability of the connection along a path connecting two backgrounds, rather than the existence of globally defined compact isometries. The connection of [20], defined on $\mathscr{M}$, is flat and so we can connect all torus backgrounds by parallel transport.

An interesting example where such degenerations are important is the $SU(2)_n$ WZW model, which can be formulated as a degenerating torus bundle over an interval with metric [59]

$$ds^2_{S^3} = \frac{1}{4}(d\psi^1)^2 + \sin^2\left(\frac{\psi^1}{2}\right)(d\xi^2)^2 + \cos^2\left(\frac{\psi^1}{2}\right)(d\xi^3)^2, \tag{6.20}$$

where $\psi^1 \in [0, \pi], \xi^{2,3} \in [0, 2\pi]$. The full conventions can be found in [59], but we can see that the torus fibres of this metric degenerate at $\psi^1 = 0, \pi$, where one of the circles shrinks to zero. Although such examples would be interesting to understand better, in this thesis we shall restrict ourselves to cases where this does not happen.

We now consider how the construction we have outlined would be applied to the examples considered in chapter 5, i.e. the T^3 with H-flux, the nilfold and the T-fold. As we know, these backgrounds are not CFTs on their own, and they contain explicit x-dependence either in the metric or B-field (or both for the T-fold). As discussed, we will ignore this coordinate dependence on the worldsheet for now using a so-called adiabatic argument. Let us briefly explain the origins of this argument and how we intend to apply it here.

6.4 The adiabatic limit

Backgrounds such as the nilfold have explicit coordinate dependence in their backgrounds, and, when pulled back to the worldsheet, these coordinates become operators which in general have complicated interactions with the other fields in the theory. When dealt with fully, these interactions are generally expected to play a significant role in operator deformations. As mentioned above, we will deal with this background coordinate dependence explicitly in chapter 7, but for now let us see what we can say without doing so. This is useful because it allows us to maintain contact with the literature where such backgrounds are considered from the perspective of supergravity reductions [12], but, since we will be looking at these backgrounds from the worldsheet CFT perspective, we will still be able to derive new insight.

The adiabatic argument initially came from [19], where it was proposed in the context of finding dualities between Type II and Heterotic string orbifolds. The issue there was that it was not clear that the string-string dualities were compatible with the orbifolding. The resolution to the problem was to consider an observer on the target space moving around an S^1 in the space, e.g. the Type II theory on $\mathbb{R}^5 \times S^1 \times K3$. From the observer's perspective, if the radius of the S^1 is large, then locally the observer cannot distinguish between the target space they are on and $\mathbb{R}^6 \times K3$. On this space, one can perform the duality without issue, and so, repeating this argument locally whilst moving around the S^1, one can argue that the duality can be constructed on the orbifold space.

We use a similar argument here, but in the context of operator deformations on the space of CFTs (or more general spaces of QFTs). Suppose we have a target space like the nilfold, where the metric has explicit dependence on the base coordinate x. From the worldsheet perspective, the pullback of this metric will be an operator involving X, and therefore will have non-trivial interactions with the fields we are deforming. However, suppose we have an observer on the nilfold moving slowly around the base. We can suppose that the effect of the twisting is smooth enough so that, locally, the fibration can be treated as trivial and the X-dependence treated as constant x-dependence, i.e. small deviations in x correspond to small deviations in the metric on the fibre. Thus, we neglect interactions of X with the operators we are considering.

We could also describe this as the supergravity limit, since this is the limit where we take $\alpha' \to 0$, as discussed earlier. An issue with this is that we will discuss backgrounds, such as the T^3 with constant H-flux, which do not actually solve the supergravity equations of motion (i.e. the first order string equations of motion), so this is not an entirely accurate description. Also, the adiabatic description is nice because it presents a more intuitive picture and comes from a physical perspective rather than a mathematical one. Either way, we will assume in what follows that we are justified in neglecting the coordinate dependence of the backgrounds we consider.

6.5 Setup for twisted torus bundles

We start with some definitions. Consider the T^d bundle over S^1 with monodromy $e^N \in O(d,d;\mathbb{Z})$. $X(\sigma)$ is the base coordinate and $\mu = 1, 2...d$ indexes the fibre directions[4]. Recall that we can decompose the twist in the fibres as

$$N_I{}^J = \begin{pmatrix} f_\mu{}^\nu & K_{\mu\nu} \\ Q^{\mu\nu} & -f_\nu{}^\mu \end{pmatrix}. \tag{6.21}$$

Note that, in general, the vielbein decomposition of a metric is written in terms of objects with one frame index and one spacetime index, $e_\mu{}^a$. However, the matrix $(e^{NX})^\mu{}_\nu$ has only spacetime indices. Therefore, technically, to use it as a vielbein we would need to contract it with a trivial vielbein $\delta_\mu{}^a$. We thus define objects such as

$$(e^{NX})_\mu{}^a := (e^{NX})_\mu{}^\nu \delta_\nu{}^a, \tag{6.22}$$

and similarly for e^{-NX}, etc.

The fields $\mathcal{A}_A(\sigma) = (e^{-NX})_A{}^I \Pi_I(\sigma)$ are defined as $\mathcal{A}_A(\sigma) = (\mathcal{Z}_a(\sigma), \mathcal{X}^a(\sigma))$, where

$$\begin{aligned} \mathcal{Z}_a &= (e^{-NX})_a{}^\mu \Pi_\mu + (e^{-NX})_{a\mu} X'^\mu, \\ \mathcal{X}^a &= (e^{-NX})^{a\mu} \Pi_\mu + (e^{-NX})^a{}_\mu X'^\mu, \end{aligned} \tag{6.23}$$

and $\Pi_I = (\Pi_\mu, X'^\mu)$. Note that the $\mathcal{A}_A(\sigma)$ are well defined as $X(\sigma)$ commutes with the Π_I in the fibres. In the fibres we have (taking $B_{\mu\nu} = 0$)

$$\Pi_\mu = \partial\phi_\mu + \bar{\partial}\phi_\mu, \qquad G_{\mu\nu} X'^\nu = -\partial\phi_\mu + \bar{\partial}\phi_\mu. \tag{6.24}$$

It will be useful to define the twisted analogues of $\partial\phi_\mu$ and $\bar{\partial}\phi_\mu$ as

$$\mathcal{J}_a = \frac{1}{2}\left(\mathcal{Z}_a - G_{ab}\mathcal{X}^b\right), \qquad \bar{\mathcal{J}}_a = \frac{1}{2}\left(\mathcal{Z}_a + G_{ab}\mathcal{X}^b\right). \tag{6.25}$$

[4]Such constructions have received much attention as toy models to study duality and to address issues of moduli stabilization in flux compactifications. See [60–63, 29, 33] for further details.

Explicitly,

$$\begin{aligned}\mathcal{J}_a &= \frac{1}{2}\Big((e^{-NX})_a{}^\mu - G_{ab}(e^{-NX})^{b\mu}\Big)\left(\partial\phi_\mu + \bar{\partial}\phi_\mu\right)\\ &+\frac{1}{2}\Big((e^{-NX})_{a\mu} - G_{ab}(e^{-NX})^b{}_\nu\Big)G^{\nu\mu}\left(-\partial\phi_\mu + \bar{\partial}\phi_\mu\right).\end{aligned} \tag{6.26}$$

There are similar expressions for $\bar{\mathcal{J}}_a$, which are the natural twisted versions of the $\bar{\partial}\phi_\mu$. In the given polarization, we write the Hamiltonian density $\mathcal{H} = \mathcal{H}^{IJ}(X)\Pi_I\Pi_J$ in terms of a metric $g_{\mu\nu}$ and B-field $B_{\mu\nu}$ in the fibres,

$$\mathcal{H}^{IJ} = (e^{-N^TX})^I{}_K G^{KL}(e^{-NX})_L{}^J = \begin{pmatrix} g^{\mu\nu} & g^{\mu\rho}B_{\rho\nu} \\ B_{\mu\rho}g^{\rho\nu} & g_{\mu\nu} + B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu}\end{pmatrix}. \tag{6.27}$$

An interesting object to consider is the chiral stress tensor

$$T(\sigma) = G^{ab}\mathcal{J}_a\mathcal{J}_b. \tag{6.28}$$

From the results derived earlier, we also know that

$$\partial\phi_\mu + \bar{\partial}\phi_\mu = E^T_{\mu\nu}\partial X^\nu + E_{\mu\nu}\partial X^\nu, \tag{6.29}$$

$$\partial\phi_\mu - \bar{\partial}\phi_\mu = G_{\mu\nu}\left(\partial X^\mu - \bar{\partial}X^\nu\right). \tag{6.30}$$

Using these, after a lot of tedious but straightforward algebra and using the fact that $e^{-NX} \in O(d,d)$, one can show that $T(\sigma)$ may also be written as

$$T(\sigma) = g^{\mu\nu}\partial X_\mu(E)\partial X_\nu(E), \tag{6.31}$$

where we recall that $\partial X_\mu(E)$ is given by (5.28), $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$ is the background tensor and $g_{\mu\nu}$ and $B_{\mu\nu}$ are *defined* by (6.27). This is true regardless of whether or not the background admits a global geometric description. If the twist matrix e^{-NX} may be written in the same form as the vielbein for $\mathcal{H}^{IJ}$ (as in (6.13)), then the background will admit a geometric interpretation globally, otherwise it will not in general. We see that $T(\sigma)$ is the parallel transport of the untwisted chiral stress tensor to the background with a duality twist. The examples we consider in the following sections are toy models, but they do play a role in building honest string backgrounds, where there is reason to believe a CFT description exists. In such cases the stress tensor plays an important role in defining the CFT. Here we restrict our attention to toy models and consider[5] $T(\sigma)$ and, in particular, how it transforms under T-duality given by the automorphism $T(\sigma) \to e^{iQ}T(\sigma)e^{-iQ}$.

Note that, in general, if we have a background where $g_{\mu\nu} = e_\mu{}^a \delta_{ab} e^b{}_\nu$ and $B = 0$, we can write

$$T = g_{\mu\nu}\partial X^\mu \partial X^\nu = e_\mu{}^a \delta_{ab} e^b{}_\nu \partial X^\mu \partial X^\nu = \mathcal{J}^a \delta_{ab} \mathcal{J}^b, \tag{6.32}$$

where $\mathcal{J}^a = \partial X^\mu e_\mu{}^a$, i.e. the $\mathcal{J}$ are simply the frame objects $\partial X^a \equiv \partial X^\mu e_\mu{}^a$. When there is a non-trivial B-field, we must instead use the doubled formalism with the doubled vielbein given in (6.14).

6.6 The nilfold

We once again study the duality sequence involving the nilfold that we reviewed in chapter 3, but this time from the perspective of the worldsheet in the adiabatic approximation. This has also been studied in [12]. The nilfold metric is a T^2 bundle over S^1 with monodromy e^{-f}, where

$$f_\mu{}^\nu = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \tag{6.33}$$

and the pullback of the metric to the worldsheet is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -mX \\ 0 & -mX & 1 + (mX)^2 \end{pmatrix}. \tag{6.34}$$

Following (6.23), we introduce $\mathcal{Z}_a = (e^{-fX})_a{}^\mu \Pi_\mu$ in the fibres,

$$\mathcal{Z}_x(\sigma) = \Pi_x(\sigma), \qquad \mathcal{Z}_y(\sigma) = \Pi_y(\sigma), \qquad \mathcal{Z}_z(\sigma) = \Pi_z(\sigma) + mX(\sigma)\Pi_y(\sigma), \tag{6.35}$$

where Π_μ are the momenta[6]. Note that $[X(\sigma), \Pi_y(\sigma)] = 0$, so these objects are well-defined. A rationale for introducing the $\mathcal{Z}_a$ is the $O(d,d)$ covariant form of the Hamiltonian,

$$\mathcal{H}(\sigma) = \mathcal{S}_I(\sigma)\mathcal{H}^{IJ}(\sigma)\mathcal{S}_J(\sigma) = \mathcal{A}_A(\sigma) G^{AB} \mathcal{A}_B(\sigma), \tag{6.36}$$

[5] Similar sentiments hold also for $\bar{T}(\sigma) = G^{ab}\bar{\mathcal{J}}_a\bar{\mathcal{J}}_b$.

[6] These are also momenta of the untwisted backgrounds - they are conjugate to the coordinates on the fibres. The universality of $(\Pi_\mu(\sigma), X'^\mu(\sigma))$ means that we identify the two sets of momenta; $\Pi_\mu(\sigma)|_E = \Pi_\mu(\sigma)|_G$.

with $\mathcal{A}_A = (e^{NX})_A{}^I \mathcal{S}_I$, where $\mathcal{S}_I := (\Pi_\mu, X'^\mu)$, and the monodromy has been included explicitly and $\mathcal{A}_A = (\mathcal{Z}_a, \mathcal{X}^a)$. Here, G^{AB} denotes (6.27) evaluated at the self-dual background (with frame indices) and may be taken to be proportional to the identity. The $\mathcal{Z}_a$ obey the Heisenberg-like (loop) algebra

$$[\mathcal{Z}_x(\sigma), \mathcal{Z}_z(\sigma')] = -im\delta(\sigma-\sigma')\mathcal{Z}_y(\sigma'), \qquad [\mathcal{Z}_y(\sigma), \mathcal{Z}_z(\sigma')] = 0, \qquad [\mathcal{Z}_y(\sigma), \mathcal{Z}_x(\sigma')] = 0, \tag{6.37}$$

by virtue of the standard canonical commutation relations on the torus fibres. If we use the relations (6.24), then the $\mathcal{Z}_a$ of (6.23) may be written in terms of the $\partial\phi_\mu, \bar{\partial}\phi_\mu$ as

$$\mathcal{Z}_x(\sigma) = \partial\phi_x + \bar{\partial}\phi_x, \qquad \mathcal{Z}_y(\sigma) = \partial\phi_y + \bar{\partial}\phi_y, \qquad \mathcal{Z}_z(\sigma) = \partial\phi_z + \bar{\partial}\phi_z + m\phi^x(\partial\phi_y + \bar{\partial}\phi_y), \tag{6.38}$$

where we reiterate that the base coordinate ϕ^x is still considered to be universal, so we could just as well write X, but we write ϕ^x for clarity and since it ties in better with the calculations of chapter 8 where we move away from the adiabatic limit and therefore no longer have universal coordinates.

Using the relationship (5.28), the change in the fields in going from the background with identity metric $G_{\mu\nu}$ to the nilfold background is

$$\begin{gathered} \partial X = \partial\phi_x, \qquad \partial Y = \partial\phi_y - \frac{1}{2}m\phi^x(\partial\phi_z - \bar{\partial}\phi_z), \\ \partial Z = \left(1 + \frac{1}{2}(m\phi^x)^2\right)\partial\phi_z - \frac{1}{2}(m\phi^x)^2\bar{\partial}\phi_z - \frac{1}{2}m\phi^x(\partial\phi_y - \bar{\partial}\phi_y). \end{gathered} \tag{6.39}$$

The stress tensor is given by (6.31) with $\partial X_\mu(E)$ given by (6.39). Similarly,

$$\begin{gathered} \bar{\partial} X = \bar{\partial}\phi_x, \qquad \bar{\partial} Y = \bar{\partial}\phi_y + \frac{1}{2}m\phi^x(\partial\phi_z - \bar{\partial}\phi_z), \\ \bar{\partial} Z = \left(1 + \frac{1}{2}(m\phi^x)^2\right)\bar{\partial}\phi_z - \frac{1}{2}(m\phi^x)^2\partial\phi_z + \frac{1}{2}m\phi^x(\partial\phi_y - \bar{\partial}\phi_y). \end{gathered} \tag{6.40}$$

Note that the change in background is a twisting of the torus, i.e. an $SL(2)$ action on the coordinates (Y, Z). Since the $\mathcal{J}_a$ are $SL(2)$-invariant, they take the same functional form when written using the $\partial\phi_\mu$ or the ∂X_μ. This may be checked explicitly.

6.6.1 T-duality

The stress tensor may be written schematically as $T(\sigma) = (\partial X)^T g^{-1}(\partial X)$. The T-dual expression is given by $V^{-1}TV$, where $V = e^{-iQ}$, and so, writing $\partial X = U\partial\phi$,

$$V^{-1}TV = (V^{-1}\partial\phi^T V)(V^{-1}U^T V)(V^{-1}g^{-1}V)(V^{-1}UV)(V^{-1}\partial\phi V). \tag{6.41}$$

Assuming g does not contain any dependence on the coordinate we want to dualise along, $(V^{-1}gV) = g$, and then all we need to understand is $V^{-1}UV$. Equivalently, we need to understand $V^{-1}\Gamma V$, i.e. how $\Gamma_I{}^J(X)$ transforms under T-duality. We shall start by studying T-duality along the Y and Z directions. Since $\Gamma_I{}^J = N_I{}^J X(\sigma)$ depends only on $X(\sigma)$, T-duality along the Y and Z directions has no effect and $V^{-1}UV = U$. Thus, we need only consider the factor $V^{-1}\partial\phi V$ which, as discussed at length in section 4.3, is well understood. We see that complications in understanding T-duality arise when g and/or U have explicit dependence on the direction we are performing the duality in. Taken together, the conditions that $V^{-1}UV = U$ and $(V^{-1}g^{-1}V) = g^{-1}$ are that the background is invariant under shifts along the direction in which we are performing the T-duality. The requirement of such invariance is the key ingredient from the Buscher perspective.

The stress tensor for the nilfold is given by $T(\sigma) = g^{\mu\nu}\partial X_\mu \partial X_\nu$,

$$\begin{aligned} T(\sigma) &= (\partial X)^2 + (\partial Y)^2 + (\partial Z + mX\partial Y)^2 \\ &= (\partial\phi_x)^2 + \left(\partial\phi_y - \frac{1}{2}m\phi^x(\partial\phi_z - \bar{\partial}\phi_z)\right)^2 + \left(\partial\phi_z + \frac{1}{2}m\phi^x(\partial\phi_y + \bar{\partial}\phi_y)\right)^2. \end{aligned} \tag{6.42}$$

The T-dual stress tensor is given by $e^{iQ}T(\sigma)e^{-iQ}$, where we perform a T-duality along the Y direction using the charge

$$Q = \frac{1}{2}\oint d\sigma \cos(2\phi^y_L(\sigma)). \tag{6.43}$$

The effect of this automorphism on $\partial\phi_y(\sigma), \bar{\partial}\phi_y(\sigma)$ is $e^{iQ}\partial\phi_y(\sigma)e^{-iQ} = -\partial\phi_y(\sigma)$ and $e^{iQ}\bar{\partial}\phi_y(\sigma)e^{-iQ} = \bar{\partial}\phi_y(\sigma)$. The stress tensor of the dual theory is then

$$T(\sigma) = (\partial\phi_x)^2 + \left(\partial\phi_y + \frac{1}{2}m\phi^x(\partial\phi_z - \bar{\partial}\phi_z)\right)^2 + \left(\partial\phi_z - \frac{1}{2}m\phi^x(\partial\phi_y - \bar{\partial}\phi_y)\right)^2. \tag{6.44}$$

It is not hard to check that the background is that of the T^3 with H-flux. Using (6.24) and the expression (6.31), the stress tensor for the background with $g_{\mu\nu} = \delta_{\mu\nu}$,

$B = mxdy \wedge dz$ is given by $T(\sigma) = (\partial X)^2 + (\partial Y)^2 + (\partial Z)^2$, where

$$\partial X = \partial\phi_x, \qquad \partial Y = \partial\phi_y + \frac{1}{2}m\phi^x(\partial\phi_z - \bar{\partial}\phi_z), \qquad \partial Z = \partial\phi_z - \frac{1}{2}m\phi^x(\partial\phi_y - \bar{\partial}\phi_y). \tag{6.45}$$

This is precisely the stress tensor found as the dual of the nilfold stress tensor, as expected. It is more straightforward to construct the related currents $\mathcal{J}_a$ for the nilfold, given by (6.26),

$$\mathcal{J}_x = \partial\phi_x, \qquad \bar{\mathcal{J}}_x = \bar{\partial}\phi_x, \tag{6.46}$$

$$\mathcal{J}_y = \partial\phi_y - \frac{1}{2}m\phi^x(\partial\phi_z - \bar{\partial}\phi_z), \qquad \bar{\mathcal{J}}_y = \bar{\partial}\phi_y + \frac{1}{2}m\phi^x(\partial\phi_z - \bar{\partial}\phi_z), \tag{6.47}$$

$$\mathcal{J}_z = \partial\phi_z + \frac{1}{2}m\phi^x(\partial\phi_y + \bar{\partial}\phi_y), \qquad \bar{\mathcal{J}}_z = \bar{\partial}\phi_z + \frac{1}{2}m\phi^x(\partial\phi_y + \bar{\partial}\phi_y). \tag{6.48}$$

The stress tensor of the nilfold may then be written as (6.28).

6.6.2 A doubled algebra

The twisted versions of the X'^{μ} are given by $\mathcal{X}^a = (e^{f^T X})^a{}_{\mu}X'^{\mu}$, where f^T denotes the transpose of (6.33),

$$\mathcal{X}^x(\sigma) = X'(\sigma), \qquad \mathcal{X}^y(\sigma) = Y' - mX(\sigma)Z'(\sigma), \qquad \mathcal{X}^z = Z'(\sigma). \tag{6.49}$$

The $\mathcal{Z}_a(\sigma)$ and $\mathcal{X}^a(\sigma)$ close to form an algebra under commutation. The non-trivial commutators are[7]

$$[\mathcal{Z}_x(\sigma), \mathcal{Z}_z(\sigma')] = -im\delta(\sigma-\sigma')\mathcal{Z}_y(\sigma'), \qquad [\mathcal{Z}_x(\sigma), \mathcal{X}^y(\sigma')] = im\delta(\sigma-\sigma')\mathcal{X}^z(\sigma'),$$

$$[\mathcal{Z}_z(\sigma), \mathcal{X}^y(\sigma')] = -im\delta(\sigma-\sigma')\mathcal{X}^x(\sigma'), \tag{6.50}$$

and the central extensions

$$[\mathcal{Z}_a(\sigma), \mathcal{X}^b(\sigma')] = i\delta_a{}^b\delta'(\sigma-\sigma'). \tag{6.51}$$

This algebra is reminiscent of the Lie algebras (3.7), (3.8) that appear in flux compactification of supergravity on the nilfold [33, 32] and also the description of the nilfold in doubled geometry [12, 31]. We shall comment on this in section 6.10.

[7]Seen by using delta function manipulations and the fact that $(X(\sigma) - X(\sigma'))\delta(\sigma-\sigma') = 0$.

The commutator algebra may be seen to be a centrally extended analogue of the doubled algebra,

$$
\begin{aligned}
&[\mathcal{Z}_a(\sigma), \mathcal{Z}_b(\sigma')] = -if_{ab}{}^c \mathcal{Z}_c(\sigma'),\\
&[\mathcal{Z}_a(\sigma), \mathcal{X}^b(\sigma')] = i\delta_a{}^b \delta'(\sigma-\sigma') - if_{ac}{}^b \mathcal{X}^c(\sigma'),\\
&[\mathcal{X}^a(\sigma), \mathcal{X}^b(\sigma')] = 0,
\end{aligned} \tag{6.52}
$$

where $f_{zx}{}^y = -f_{xz}{}^y = m$. This may be written in an $O(d,d;\mathbb{Z})$-covariant way as

$$
[\mathcal{A}_A(\sigma), \mathcal{A}_B(\sigma')] = iL_{AB}\delta'(\sigma-\sigma') - it_{AB}{}^C \mathcal{A}_C(\sigma')\delta(\sigma-\sigma'), \tag{6.53}
$$

where $\mathcal{A}_A = (\mathcal{Z}_a, \mathcal{X}^a)$, $t_{xz}{}^y = -m$ (and zero otherwise) and L_{AB} is the invariant of $O(d+1,d+1)$.

6.7 T^3 with H-flux

We consider the case where the monodromy matrix is of the form (6.21) with $f^\mu{}_\nu = 0 = Q^{\mu\nu}$ and $K_{yz} = m \in \mathbb{Z}$. Note that the $\mathcal{J}_a = e_a{}^\mu \partial X_\mu$ for this background are given by (5.28) (with $g_{\mu\nu} = \delta_{\mu\nu}$ and $B_{yz} = m\phi_x$) or read off from the stress tensor (6.44) found by dualising the nilfold,

$$
\begin{aligned}
\mathcal{J}_x &= \partial\phi_x, & \bar{\mathcal{J}}_x &= \bar\partial\phi_x,\\
\mathcal{J}_y &= \partial\phi_y + \frac{1}{2}m\phi^x(\partial\phi_z - \bar\partial\phi_z), & \bar{\mathcal{J}}_y &= \bar\partial\phi_y - \frac{1}{2}m\phi^x(\partial\phi_z - \bar\partial\phi_z),\\
\mathcal{J}_z &= \partial\phi_z - \frac{1}{2}m\phi^x(\partial\phi_y - \bar\partial\phi_y), & \bar{\mathcal{J}}_z &= \bar\partial\phi_z - \frac{1}{2}m\phi^x(\partial\phi_y - \bar\partial\phi_y),
\end{aligned} \tag{6.54}
$$

which gives the components of $\mathcal{A}_A(\sigma)$ as

$$
\begin{aligned}
\mathcal{Z}_x &= \Pi_x, & \mathcal{Z}_y &= \Pi_y + mXZ', & \mathcal{Z}_z &= \Pi_z - mXY',\\
\mathcal{X}^x &= X', & \mathcal{X}^y &= Y', & \mathcal{X}^z &= Z'.
\end{aligned} \tag{6.55}
$$

The commutation relations are then[8]

$$[\mathcal{Z}_a(\sigma), \mathcal{Z}_b(\sigma')] = -im\varepsilon_{abc}\mathcal{X}^c(\sigma')\delta(\sigma - \sigma'), \qquad [\mathcal{Z}_a(\sigma), \mathcal{X}^b(\sigma')] = i\delta_a{}^b\delta'(\sigma - \sigma'),$$
$$[\mathcal{X}^a(\sigma), \mathcal{X}^b(\sigma')] = 0, \tag{6.58}$$

which, again, is of the form (6.53).

6.8 T-fold

Alternatively, T-duality of the nilfold along the Z-direction gives the T-fold. It will be instructive to see how the background arises from the currents $\mathcal{J}_a$. An automorphism with the duality charge amounts to the exchange $\partial\phi_z \to -\partial\phi_z$, $\bar{\partial}\phi_z \to \bar{\partial}\phi_z$. There is no explicit Z-dependence, so we need not worry about how Z transforms. The resulting currents are

$$\begin{aligned}
\mathcal{J}_x &= \partial\phi_x, & \bar{\mathcal{J}}_x &= \bar{\partial}\phi_x,\\
\mathcal{J}_y &= \partial\phi_y + \frac{1}{2}m\phi^x(\partial\phi_z + \bar{\partial}\phi_z), & \bar{\mathcal{J}}_y &= \bar{\partial}\phi_y - \frac{1}{2}m\phi^x(\partial\phi_z + \bar{\partial}\phi_z),\\
\mathcal{J}_z &= \partial\phi_z - \frac{1}{2}m\phi^x(\partial\phi_y + \bar{\partial}\phi_y), & \bar{\mathcal{J}}_z &= \bar{\partial}\phi_z + \frac{1}{2}m\phi^x(\partial\phi_y + \bar{\partial}\phi_y),
\end{aligned} \tag{6.59}$$

which gives the fields $\mathcal{Z}_a$ and $\mathcal{X}^a$ as

$$\begin{aligned}
\mathcal{Z}_x &= \Pi_x, & \mathcal{Z}_y &= \Pi_y, & \mathcal{Z}_z &= \Pi_z,\\
\mathcal{X}^x &= X', & \mathcal{X}^y &= Y' - mX\Pi_z, & \mathcal{X}^z &= Z' + mX\Pi_y,
\end{aligned} \tag{6.60}$$

from which we see that $N_I{}^J$ is of the form (6.21) with all entries zero except $Q^{yz} = m$. The stress tensor is given by

$$T = \sum_i \mathcal{J}_a\mathcal{J}_a. \tag{6.61}$$

[8]Using

$$\begin{aligned}
X(\sigma)\frac{d}{d\sigma'}\delta(\sigma-\sigma') - X(\sigma')\frac{d}{d\sigma'}\delta(\sigma-\sigma') &= \frac{d}{d\sigma'}\Big((X(\sigma)-X(\sigma'))\delta(\sigma-\sigma')\Big) - X'(\sigma')\delta(\sigma-\sigma')\\
&= -X'(\sigma')\delta(\sigma-\sigma'),
\end{aligned} \tag{6.56}$$

and also the fact that

$$\frac{d}{d\sigma'}\delta(\sigma-\sigma') = -\frac{d}{d\sigma}\delta(\sigma-\sigma'), \tag{6.57}$$

which can be easily seen from the Fourier series representation of the periodic delta function.

The Hamiltonian may be constructed from $\mathcal{Z}_a$ and $\mathcal{X}^a$ as in (6.36), and the metric and B-field read off for this background,

$$ds^2 = dx^2 + \frac{1}{1+(mx)^2}\Big(dy^2 + dz^2\Big), \qquad B = -\frac{mx}{1+(mx)^2} dy \wedge dz. \tag{6.62}$$

The non-trivial commutation relations for the algebra of the $\mathcal{Z}_a$ and $\mathcal{X}^a$ are

$$[\mathcal{Z}_x(\sigma), \mathcal{X}^z(\sigma')] = -im\mathcal{Z}_y(\sigma')\delta(\sigma-\sigma'), \qquad [\mathcal{Z}_x(\sigma), \mathcal{X}^y(\sigma')] = im\mathcal{Z}_z(\sigma')\delta(\sigma-\sigma'),$$

$$[\mathcal{X}^y(\sigma), \mathcal{X}^z(\sigma')] = im\mathcal{X}^x(\sigma')\delta(\sigma-\sigma'), \tag{6.63}$$

and the central extension term

$$[\mathcal{Z}_a(\sigma), \mathcal{X}^b(\sigma')] = i\delta_a{}^b\delta'(\sigma-\sigma'). \tag{6.64}$$

This is the central extension of an algebra with structure constant $Q_x{}^{yz} = m$, as we might expect for the T-fold. Again, this algebra is of the general form (6.53).

6.9 On non-isometric T-duality

In this section, we briefly consider a simple case where, in the language of section 6.6.1, $(V^{-1}UV) \neq U$. This can occur when U depends explicitly on the direction we are performing the duality in. Duality involving functions of $\partial\phi_\mu$ and $\bar{\partial}\phi_\mu$ are well understood. What if we have a more general function of $X(\sigma)$? The only such functions that appear at the self-dual radius are of the form $e^{in\phi^x_L(\sigma)}$, which we have already studied and transform in a well-defined way. However, in considering the obstacles one might need to overcome to apply the operator formalism to non-isometric torus fibrations, it may be instructive to study how functions of $X(\sigma)$ that are not invariant under isometries transform. As a first step we compute $e^{iQ}\phi^x_L(\sigma)e^{-iQ}$. In the framework presented here, $X(\sigma)$ is a universal coordinate. Thus, if we know how $\phi^x(\sigma)$ transforms under T-duality at the self-dual radius, we can infer how it transforms in backgrounds related to that one by parallel transport. It is not hard to show that

$$[\phi^\mu_L(\sigma), \phi^\nu_L(\sigma')] = i\pi\Theta(\sigma-\sigma')\delta^{\mu\nu}, \tag{6.65}$$

where[9]

$$\Theta(\sigma - \sigma') = \frac{1}{2\pi}(\sigma - \sigma') - i \sum_{n \neq 0} \frac{1}{n} e^{in(\sigma - \sigma')}. \tag{6.67}$$

The fact that Θ is not a periodic function and so is not well defined on the worldsheet will be the source of the difficulty in making sense of applying the T-duality automorphism to $\phi^x_L(\sigma)$. Using the charge

$$Q = \frac{1}{2} \oint d\sigma \cos(2\phi^x_L(\sigma)), \tag{6.68}$$

with a little work one finds

$$[Q, \phi^x_L(\sigma)] = -i\pi \oint d\sigma' \Theta(\sigma' - \sigma) \sin(2\phi^x_L(\sigma')),$$
$$[Q^{(2)}, \phi^x_L(\sigma)] = -\pi^2 \oint d\sigma' \Theta(\sigma' - \sigma) {\phi^x_L}'(\sigma'),$$

where we use the notation

$$[Q^{(n)}, \phi^x_L] \equiv [Q, [Q, ...[Q, \phi^x_L]]...], \tag{6.69}$$

where there are n nested commutators on the RHS. We might think that we can do this integral by parts and get rid of the boundary term in the second expression. Assuming this, you would end up with $\phi^x_L \to -\phi^x_L$ as the transformation. However, since ϕ^x_L and Θ are not periodic, the boundary term does not vanish. The easiest way to compute the integral is in fact to use the mode expansions and do the integrals directly. Doing this, we have

$$[Q^{(2)}, \phi^x_L(\sigma)] = -\pi^2 \left(-\phi^x_L(\sigma) + \frac{1}{2} \left(\phi^x_L(0) + \phi^x_L(2\pi) \right) \right).$$

[9]Note that

$$\frac{d}{d\sigma}\Theta(\sigma - \sigma') = \sum_n e^{in(\sigma - \sigma')}, \tag{6.66}$$

which is the periodic delta-function.

Since the charge acts in the same way regardless of the value of σ, we can now just write down the successive commutators:

$$
\begin{aligned}
[Q^{(3)}, \phi^x_L(\sigma)] =& i\pi^3 \oint d\sigma' \sin(2\phi^x_L(\sigma'))(-\Theta(\sigma'-\sigma) + \frac{1}{2}\Theta(\sigma') + \frac{1}{2}\Theta(\sigma'-2\pi)), \quad (6.70)\\
[Q^{(4)}, \phi^x L(\sigma)] =& \pi^4 \left(\phi^x_L(\sigma) - \frac{1}{2}(\phi^x_L(0) + \phi^x_L(2\pi))\right),\\
&\vdots \qquad (6.71)
\end{aligned}
$$

There is a repeating pattern and so the full transformation may be written down,

$$
\begin{aligned}
e^{iQ}\phi^x_L(\sigma)e^{-iQ} =& \left(-\phi^x_L(\sigma) + \frac{1}{2}\left(\phi^x_L(0) + \phi^x_L(2\pi)\right)\right)\left(\frac{\pi^2}{2!} - \frac{\pi^4}{4!} + ...\right)\\
&+ \oint d\sigma' \sin(2\phi^x_L(\sigma'))\left(-\Theta(\sigma'-\sigma) + \frac{1}{2}\Theta(\sigma') + \frac{1}{2}\Theta(\sigma'-2\pi)\right)\left(\frac{\pi^3}{3!} - \frac{\pi^5}{5!} + ...\right)\\
&+ \phi^x_L(\sigma) + \pi \oint d\sigma' \sin(2\phi^x_L(\sigma'))\Theta(\sigma'-\sigma)\\
=& -\phi^x_L(\sigma) + \phi^x_L(0) + \phi^x_L(2\pi) + \frac{\pi}{2}\oint d\sigma' \sin(2\phi^x_L(\sigma'))\left(\Theta(\sigma') + \Theta(\sigma'-2\pi)\right).
\end{aligned} \tag{6.72}
$$

We can use $\Theta(\sigma'-2\pi) = \Theta(\sigma') - 1$ and $\mathrm{sgn}(\sigma') = 2\Theta(\sigma') - 1$ to slightly simplify the last term, so that

$$
e^{iQ}\phi^x_L(\sigma)e^{-iQ} = -\phi^x_L(\sigma) + \phi^x_L(0) + \phi^x_L(2\pi) + \frac{\pi}{2}\oint d\sigma' \sin(2\phi^x_L(\sigma'))\mathrm{sgn}(\sigma').
$$

Note that, if we use the sine charge instead, we arrive at a different result, namely

$$
e^{iQ}\phi^x_L(\sigma)e^{-iQ} = -\phi^x_L(\sigma) + \phi^x_L(0) + \phi^x_L(2\pi) - \frac{\pi}{2}\oint d\sigma' \cos(2\phi^x_L(\sigma'))\mathrm{sgn}(\sigma').
$$

Taking an optimistic view of this rather messy result, we note that it is of the form

$$
e^{iQ}\phi^x_L(\sigma)e^{-iQ} = -\phi^x_L(\sigma) + \mathcal{C}, \tag{6.73}
$$

where $\mathcal{C}$ is a constant operator. This operator depends on the charge one uses to perform the duality and points to the fact that such a shift can be removed by a $U(1)_L \times U(1)_R$ gauge transformation. Put another way, this result is suggestive of the possibility of considering the correct action of the duality on $\phi^x_L(\sigma)$ as $\phi^x_L(\sigma) \to -\phi^x_L(\sigma)$ (equivalently $X(\sigma) \to \tilde{X}(\sigma)$), with the shift by $\mathcal{C}$ dropping out of all gauge-invariant results. Such a proposal has received support from other quarters (see for example

[53, 55, 29, 12]). Of course, in those cases the emphasis was on performing T-duality in the absence of isometries in the target space. Here, we see that what is required to be able to neglect $\mathcal{C}$ is the unbroken $U(1)_L$ symmetry acting on ϕ^x_L. This is a particular linear combination of isometry and B-field transformation; the diagonal $U(1)_L \subset U(1)_Z \times U(1)_X$, where the isometry is $U(1)_Z$. An obvious argument against this interpretation is that $\mathcal{C}$ is an operator and will not commute with other generators in the $SU(2)$ gauge symmetry. As such, it seems a poor candidate for a parameter of a translation symmetry, though it would be interesting to investigate this further. The transformations above may be written as

$$\delta_{\mathcal{Z}} X = 2\alpha, \qquad \delta_{\mathcal{Z}} \tilde{X} = 0, \qquad \delta_{\mathcal{Z}} \phi^x_L = \alpha = \delta_{\mathcal{Z}} \phi^x_R \tag{6.74}$$

for the T-duality and

$$\delta_{\mathcal{X}} X = 0, \qquad \delta_{\mathcal{X}} \tilde{X} = 2\tilde{\alpha}, \qquad \delta_{\mathcal{X}} \phi^x_L = -\tilde{\alpha} = \delta_{\mathcal{X}} \phi^x_R \tag{6.75}$$

for the B-field transformation. We see that, if we choose $\alpha = 0$ and $\tilde{\alpha} = \mathcal{C}$, i.e. if we couple the T-duality transformation with a B-field transformation, then we may have $\phi^x_L(\sigma)$ transforming in the expected way, even if there is no isometry in $X(\sigma)$, provided there is B-field symmetry present. This is not a concrete prescription, but it suggests that the formalism considered in this chapter may admit more general notions of T-duality if generalised to more interesting backgrounds.

As an application, consider T-duality of the T-fold along the X-direction, neglecting the presence of $\mathcal{C}$. We are therefore, somewhat artificially, elevating the variable $X(\sigma)$ along the base circle from a parameter that characterises the bundle (in the spirit of the construction outlined in section 6.1) to a full quantum field. This pushes us out of the realm of toy models that should in principle be part of a bona fide CFT and into somewhat uncertain territory. Nonetheless, we shall press on. An automorphism with the duality charge amounts to the exchange $\partial\phi^x \to -\partial\phi^x$, $\bar{\partial}\phi^x \to \bar{\partial}\phi^x$. From the above, we shall *assume* that $X \to \tilde{X}$ (i.e. $\phi^x_L \to -\phi^x_L$).

The resulting currents are

$$\begin{aligned}
\mathcal{J}_x &= \partial\phi_x, & \bar{\mathcal{J}}_x &= \bar{\partial}\phi_x, \\
\mathcal{J}_y &= \partial\phi_y + \frac{1}{2} m\widetilde{\phi}^x(\partial\phi_z + \bar{\partial}\phi_z), & \bar{\mathcal{J}}_y &= \bar{\partial}\phi_y - \frac{1}{2} m\widetilde{\phi}^x(\partial\phi_z + \bar{\partial}\phi_z), \\
\mathcal{J}_z &= \partial\phi_z - \frac{1}{2} m\widetilde{\phi}^x(\partial\phi_y + \bar{\partial}\phi_y), & \bar{\mathcal{J}}_z &= \bar{\partial}\phi_z + \frac{1}{2} m\widetilde{\phi}^x(\partial\phi_y + \bar{\partial}\phi_y),
\end{aligned} \tag{6.76}$$

where $\widetilde{\phi}^x = \tilde{X}$, and so the fields $\mathcal{Z}_a$ and $\mathcal{X}^a$ are

$$\begin{aligned} \mathcal{Z}_x &= \Pi_x, & \mathcal{Z}_y &= \Pi_y, & \mathcal{Z}_z &= \Pi_z, \\ \mathcal{X}^x &= X', & \mathcal{X}^y &= Y' - m\tilde{X}\Pi_z, & \mathcal{X}^z &= Z' + m\tilde{X}\Pi_y. \end{aligned} \tag{6.77}$$

These can be obtained by a T^2 bundle over the dual circle with $\mathcal{A}_A(\sigma) = (e^{-N\tilde{X}})_A{}^I\Pi_I(\sigma)$, where $N_I{}^J$ takes the same form as for the T-fold above. The key point is that $X(\sigma)$ is replaced by $\tilde{X}(\sigma)$.

The non-trivial commutation relations are then

$$\begin{aligned} [\mathcal{X}^x(\sigma), \mathcal{X}^z(\sigma')] &= -2\pi i m\delta(\sigma-\sigma')\mathcal{Z}_y(\sigma'), \\ [\mathcal{X}^x(\sigma), \mathcal{X}^y(\sigma')] &= 2\pi i m\delta(\sigma-\sigma')\mathcal{Z}_z(\sigma'), \\ [\mathcal{X}^z(\sigma), \mathcal{X}^y(\sigma')] &= -2\pi i m\delta(\sigma-\sigma')\mathcal{Z}_x(\sigma'), \end{aligned} \tag{6.78}$$

and the central extension is

$$[\mathcal{Z}_a(\sigma), \mathcal{X}^b(\sigma')] = 2\pi i\delta_a{}^b\delta'_{\sigma'}(\sigma-\sigma'). \tag{6.79}$$

This is reminiscent of the expected R-flux algebra (3.25) [61, 12]. To be clear, the above discussion does not in any way provide a proof that the R-flux background is dual to the T-fold. It does, however, demonstrate that this formalism gives rise to similar algebraic structures seen in the supergravity [33, 32, 29, 62] and doubled geometry [12] discussions of such backgrounds, *assuming* $X(\sigma) \to \tilde{X}(\sigma)$. It also makes clearer the assumptions that are required for such backgrounds to map into each other under T-duality, defined as an automorphism of the operator algebra.

When we come to chapter 8, we will see that addressing the X-dependence fully requires more care and makes the above transformations significantly more complicated. We will focus more on the nilfold and H-flux cases there, but even in those cases the situation is very different to the adiabatic limit. When we have a better understanding of how to deal with more complicated worldsheet coordinate dependence, we will briefly return to this non-isometric story in section 9.1.3, where we re-examine an idea of [15] and apply it to a simple case.

6.10 Relationship to the doubled formalism & the doubled algebra

Here we discuss the relationship of the universal coordinate formalism presented above to the doubled formalism [12], and we discover an interesting feature of the algebra of worldsheet operators $\mathcal{Z}_a, \mathcal{X}^a$.

Recall our setup of the doubled formalism from chapter 3. Here, we have doubled embedding coordinates $\mathbb{X}^M = (X^m, \tilde{X}_m)$ in the fibres, with base X, i.e. $X^\mu = (X, X^m)$.

In the previous section we have seen that the natural objects that relate the Hamiltonian of a given background to that of the background at an enhanced point involve only the monodromy encoded in the Hamiltonian density

$$\mathcal{H}_{IJ}(X(\sigma))\Pi^I(\sigma)\Pi^J(\sigma) = (\Pi_x(\sigma))^2 + (X'(\sigma))^2 + \mathcal{M}^{MN}(X(\sigma))\Pi_M(\sigma)\Pi_N(\sigma), \quad (6.80)$$

where we have introduced

$$\Pi_M(\sigma) = \frac{1}{\sqrt{2\pi}}\Big(\Pi_m(\sigma), X'^m(\sigma)\Big) \quad (6.81)$$

in the torus fibres and $\mathcal{M}^{MN}(X)$ is the inverse of the metric in the fibres of $\mathcal{T}$, which is determined by the monodromy and depends only on the base coordinate. The $\Pi_M(\sigma)$ obey the commutation relations

$$[\Pi_M(\sigma), \Pi_N(\sigma')] = iL_{MN}\delta'(\sigma - \sigma'), \quad (6.82)$$

by virtue of the canonical commutation relations. The natural action of $O(d,d)$ on the coordinates and momenta of the T^d fibres suggests the natural $O(d,d)$-covariant objects on the bundle[10]

$$\mathcal{A}_P(\sigma) = (e^{-NX(\sigma)})_P{}^M\Pi_M(\sigma). \quad (6.83)$$

The algebra of these objects is given by the commutator

$$\begin{aligned}[][\mathcal{A}_P(\sigma), \mathcal{A}_Q(\sigma')] &= (e^{-NX(\sigma)})_P{}^M(e^{-NX(\sigma')})_Q{}^N[\Pi_M(\sigma), \Pi_N(\sigma')] \\ &= i(e^{-NX(\sigma)})_P{}^M L_{MN}(e^{-NX(\sigma')})_Q{}^N\delta'(\sigma - \sigma'). \end{aligned} \quad (6.84)$$

[10] All products of operators are assumed normal ordered.

Integrating by parts in σ and using the fact that L_{MN} is invariant under the action of $e^{-NX(\sigma)}$ gives

$$[\mathcal{A}_P(\sigma), \mathcal{A}_Q(\sigma')] = iL_{PQ}\delta'(\sigma - \sigma') - iN_{PQ}X'(\sigma)\delta(\sigma - \sigma'), \tag{6.85}$$

where $N_{PQ} = N_P{}^R L_{RQ} = -N_{QP}$. Following the previous section, it is useful to split the fibre fields as $\mathcal{A}_P(\sigma) = (\mathcal{Z}_p, \mathcal{X}^p)$ and similarly for the base circle. The only other non-trivial commutator is $[\Pi_x(\sigma), \mathcal{A}_P(\sigma')]$, which is easily evaluated to give the algebra

$$\begin{aligned}
[\mathcal{A}_P(\sigma), \mathcal{A}_Q(\sigma')] &= iL_{PQ}\delta'(\sigma - \sigma') - iN_{PQ}\mathcal{X}(\sigma')\delta(\sigma - \sigma'), \\
[\mathcal{Z}_x(\sigma), \mathcal{A}_P(\sigma')] &= iN_P{}^Q\mathcal{A}_Q(\sigma')\delta(\sigma - \sigma'), \qquad [\mathcal{X}^x(\sigma), \mathcal{A}_P(\sigma')] = 0, \\
[\mathcal{X}^x(\sigma), \mathcal{Z}_x(\sigma')] &= i\delta(\sigma - \sigma').
\end{aligned} \tag{6.86}$$

This may be written compactly as

$$[\mathcal{A}_A(\sigma), \mathcal{A}_B(\sigma')] = iL_{AB}\delta'(\sigma - \sigma') - it_{AB}{}^C\mathcal{A}_C(\sigma')\delta(\sigma - \sigma'), \tag{6.87}$$

where $t_{xP}{}^Q = N_P{}^Q$. This is a central extension of the loop algebra based on the Lie algebra (3.8). The interesting fact is that it arises in a very direct way from the intuitive torus bundle geometry. This is an important and unexpected result since we are able to derive (a central extension of) the *full* doubled algebra by just doubling the torus fibres. At the level of the supergravity, when we doubled the base the algebra generated by the vector fields dual to the left-invariant one-forms could only at most generate the contraction (3.7), since the information of the B-field along the base was not part of the geometric structure of the doubled torus bundle. Here, we find that we are able to recover the full algebra (3.8) due to the relationship between the worldsheet fields, encoded in the commutation relations.

We can see this explicitly by looking at the zero modes, which effectively projects us down to the supergravity limit. The zero modes of the fibre fields are

$$X'^{\mu}(\sigma) = \omega^{\mu} + ..., \qquad \Pi_{\mu}(\sigma) = p_{\mu} + ..., \tag{6.88}$$

where ω^{μ} is the (dimensionless) winding and p_{μ} is the (dimensionless) momentum zero mode. The ellipsis denote terms with non-trivial σ-dependence. The coordinates conjugate to these zero modes are $\tilde{z}_\mu$ and z^μ respectively, and we can write

$$X'^\mu(\sigma) = -i\frac{\partial}{\partial \tilde{z}_\mu} + ..., \qquad \Pi_\mu(\sigma) = -i\frac{\partial}{\partial z^\mu} + ..., \tag{6.89}$$

and $\Pi_I(\sigma) = -i\partial_I + ...$ Thus, the zero modes of the fields are given by

$$\mathcal{A}_A(\sigma) = -i(e^{-NX(\sigma)})_A{}^I\frac{\partial}{\partial \mathbb{X}^I} + ..., \qquad \mathcal{Z}_x(\sigma) = -i\frac{\partial}{\partial x} + ..., \qquad \mathcal{X}^x(\sigma) = -i\frac{\partial}{\partial \tilde{x}} + ... \tag{6.90}$$

Truncating to the zero modes (neglecting the +... terms) gives a set of vector fields that generate the isometry algebra (3.7) with an additional $U(1)$ factor corresponding to the isometry around the dual circle with coordinate $\tilde{x}$. This is a contraction of the algebra (3.8).

Thus, we see that it is precisely the non-trivial σ-dependence that gives rise to the full doubled algebra of the twisted doubled torus. Put another way, it is the extended nature of the string that takes us from the algebra (3.7) that we might expect from particle mechanics on the geometry $\mathcal{T} \times \widetilde{S}^1$ to the full doubled geometry corresponding to the algebra (3.8). This suggests that the two doubled geometries are perhaps more closely related than previously thought, and it would be interesting to investigate this further, though we will not have anything more to say on the subject here.

6.11 Beyond torus bundles

Our focus in this section is on seeing how the framework described in previous sections might generalise to backgrounds that are not necessarily torus bundles. The character of this section will be formal and rather speculative and we do not consider explicit examples, although it would be interesting to do so and there is a clear connection with the flux compactifications of [62, 32]. We do not need to worry about normal ordering issues and a more careful treatment may alter the general scheme outlined here. We shall see that the general algebraic structure that mirrors that of doubled geometry seems to emerge in very general cases. The absence of a detailed understanding of worldsheet theories on such backgrounds makes this section necessarily schematic. In particular, we do not consider normal ordering issues that might arise or potential α' corrections which would alter the form of the Hamiltonian. We do find that algebraic structures reminiscent of (parallelisable) flux compactifications of supergravity [32] emerge in this approximation; however, unlike those cases, these constructions do not seem to be limited by the requirement that the doubled geometry be locally a group manifold. They are perhaps more reminiscent of compactifications inspired by generalised complex geometry [64, 65] and its generalisations [66, 67].

The starting point is the Hamiltonian at the point of enhanced symmetry, denoted by $H(p_0)$,

$$H(p_0) = \oint d\sigma\, \mathcal{S}(p_0)\mathcal{H}(p_0)\mathcal{S}^T(p_0), \tag{6.91}$$

where the point p_0 on the space of backgrounds $\mathcal{M}$ is an enhanced symmetry point. The generalised metric may be written in terms of generalised vielbeins $\mathcal{H}(p_0) = \mathcal{E}(p_0)\mathcal{E}^T(p_0)$ as in (6.13) and $\mathcal{S}_I(p_0;\sigma) = (\Pi_\mu(\sigma), X'^\mu(\sigma))_{p_0}$.

The Hamiltonian density at a point $p \in \mathcal{M}$ could be given in terms of the generalised metric

$$\mathcal{H}(p) = \mathcal{U}(p,p_0)\mathcal{H}(p_0)\mathcal{U}^T(p,p_0), \tag{6.92}$$

where $\mathcal{U}(p,p_0) = \mathcal{E}(p)\mathcal{E}^{-1}(p_0)$. The Hamiltonian for the theory at p may then be written as

$$\begin{aligned} H(p) &= \oint d\sigma\, \mathcal{S}\mathcal{U}(p,p_0)\mathcal{H}(p_0)\mathcal{U}^T(p,p_0)\mathcal{S}^T \\ &= \oint d\sigma\, \mathcal{A}(p)\mathcal{H}(p_0)\mathcal{A}^T(p). \end{aligned} \tag{6.93}$$

Since $\mathcal{S}_I$ is taken to be a universal coordinate, $\mathcal{S}_I(p;\sigma) = \mathcal{S}_I(p_0;\sigma)$, and so we can drop the explicit p-dependence, and we have defined $\mathcal{A}(p) = \mathcal{S}(p_0)\mathcal{U}(p,p_0)$.

A polarization is a (maximally isotropic) choice of splitting $\mathcal{S}$ into Π and X'. Similarly, we define a polarization of $\mathcal{A}_A(\sigma)$ as a splitting $\mathcal{A}_A(\sigma) = \Big(\mathcal{Z}_a(\sigma), \mathcal{X}^a(\sigma)\Big)$.

6.11.1 Flux compactification on a twisted torus

To begin, we consider the example, familiar from many supergravity constructions, of a constant H-flux on a parallelisable[11] background. Consider a background that is generated from the reference background by the action of a geometric subgroup of $O(d,d;\mathbb{Z})$, i.e.

$$\begin{pmatrix} e & Be^{-T} \\ 0 & e^{-T} \end{pmatrix}, \tag{6.94}$$

[11]The background need only locally be a group. More generally it could be the quotient of a group by a cocompact subgroup [33].

where $e \in SL(d)$. If the reference background is the identity, then the metric and B-field for this background are

$$g_{\mu\nu} = \delta_{ab} e^a{}_\mu e^b{}_\nu, \qquad B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{6.95}$$

The twisted torus with constant flux is a simple example of such a background. Let us take, at the point p, the metric and H-field to be

$$ds^2 = \delta_{ab} e^a \otimes e^b, \qquad H = \frac{1}{6} K_{abc} e^a \wedge e^b \wedge e^c, \tag{6.96}$$

where $e^a = e^a{}_\mu dx^\mu$ is a left-invariant one-form for the group manifold G with structure constants $f_{ab}{}^c = -f_{ab}{}^c$, i.e.

$$de^a + \frac{1}{2} f_{bc}{}^a e^b \wedge e^c = 0. \tag{6.97}$$

The condition $dH = 0$ then requires $K_{[ab|c} f_{|de]}{}^c = 0$ [62].

The doubled algebra

With this polarization, we have

$$\mathcal{Z}_a(\sigma) = (e^{-1})_a{}^\mu \left(\Pi_\mu(\sigma) - B_{\mu\nu} X'^\nu(\sigma)\right), \qquad \mathcal{X}^a(\sigma) = e^a{}_\mu X'^\mu(\sigma), \tag{6.98}$$

which we may think of as

$$\mathcal{Z}_a(\sigma) = (e^{-1})_a{}^\mu \left(-i \frac{\delta}{\delta X^\mu(\sigma)} - B_{\mu\nu} X'^\nu(\sigma)\right), \qquad \mathcal{X}^a(\sigma) = e^a{}_\mu X'^\mu(\sigma). \tag{6.99}$$

Using the canonical commutation relations, we have the algebra (details of the calculations in appendix C)

$$\begin{aligned}
&[\mathcal{Z}_a(\sigma), \mathcal{Z}_b(\sigma')] = -f_{ab}{}^c \mathcal{Z}_c(\sigma') 2\pi i \delta(\sigma - \sigma') - K_{abc} \mathcal{X}^c(\sigma') 2\pi i \delta(\sigma - \sigma'), \\
&[\mathcal{Z}_a(\sigma), \mathcal{X}^b(\sigma')] = f_{ac}{}^b \mathcal{X}^c(\sigma') 2\pi i \delta(\sigma - \sigma') + \delta_a^b 2\pi i \delta'(\sigma - \sigma'), \\
&[\mathcal{X}^a(\sigma), \mathcal{X}^b(\sigma')] = 0,
\end{aligned} \tag{6.100}$$

which is of the form (6.53).

We next verify here that the algebras that we work with are indeed associative. We also discuss generalisations of the above to (non-constant) structure functions, and we show that, at least for geometric flux compactifications, associativity is preserved.

We compute (details in appendix C)

$$[\mathcal{Z}_a(\sigma), [\mathcal{Z}_b(\sigma'), \mathcal{Z}_c(\sigma'')]] = -4\pi^2\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')\Big(f_{bc}{}^d f_{ad}{}^e \mathcal{Z}_e(\sigma'') + f_{bc}{}^d K_{ade}\mathcal{X}^e(\sigma'') - f_{ae}{}^d K_{bcd}\mathcal{X}^e(\sigma'')\Big) + 4\pi^2\delta'(\sigma-\sigma'')\delta(\sigma'-\sigma'')K_{bcd}\delta^d_a. \tag{6.101}$$

Now we now sum over cyclic permutations. Care must be taken as we are also moving around the $\sigma, \sigma', \sigma''$ dependence. In the first term, this does not matter since the delta functions are only supported on $\sigma = \sigma' = \sigma''$, so we can just antisymmetrise on the indices a, b, c without any issues. However, for the second term, since there is a derivative of a delta function and the contributing terms do not appear on the same footing, we have to be more careful. We get

$$\begin{aligned}
&\frac{1}{3}\Big([\mathcal{Z}_a(\sigma), [\mathcal{Z}_b(\sigma'), \mathcal{Z}_c(\sigma'')]] + \text{cyclic}\Big) \\
&= 4\pi^2\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')\left(f_{d[a}{}^e f_{bc]}{}^d \mathcal{Z}_e(\sigma'') - 2K_{d[ea}f_{bc]}{}^d\mathcal{X}^e(\sigma'')\right) \\
&+ \frac{4}{3}\pi^2 K_{abc}\left(\delta'(\sigma-\sigma'')\delta(\sigma'-\sigma'') + \delta'(\sigma'-\sigma)\delta(\sigma''-\sigma) + \delta'(\sigma''-\sigma')\delta(\sigma-\sigma')\right).
\end{aligned} \tag{6.102}$$

The last line here vanishes by delta function manipulations. In order to establish that the other terms vanish, we compute the constraints that arise from $d^2e^a = 0$ and the fact that $dH = 0$. Taking the exterior derivative of (6.97) gives $f_{b[c}{}^a f_{de]}{}^b = 0$. Similarly, $dH = 0$ gives $K_{a[bc}f_{de]}{}^a = 0$. Taken together, these two constraints tell us that the right hand side of (6.102) vanishes.

Since the $\mathcal{X}^a$ commute with each other, the only other case we have to consider is

$$[\mathcal{X}^a(\sigma), [\mathcal{Z}_b(\sigma'), \mathcal{Z}_c(\sigma'')]] + \text{cyclic}. \tag{6.103}$$

We find that, after a short calculation,

$$[\mathcal{Z}_b(\sigma'), [\mathcal{Z}_c(\sigma''), \mathcal{X}^a(\sigma)]] = -4\pi^2\delta(\sigma'-\sigma)\delta(\sigma''-\sigma)\left(f_{cd}{}^a f_{be}{}^d \mathcal{X}^e(\sigma)\right) - 4\pi^2\delta'(\sigma'-\sigma)\delta(\sigma''-\sigma)f_{cb}{}^a. \tag{6.104}$$

When we add the cyclic permutations, the first term will vanish by the Maurer-Cartan equation constraint (6.97). The second term vanishes by delta function manipulations, as with the previous case. Thus, we find that

$$[\mathcal{X}^a(\sigma), [\mathcal{Z}_b(\sigma'), \mathcal{Z}_c(\sigma'')]] + [\mathcal{Z}_b(\sigma'), [\mathcal{Z}_c(\sigma''), \mathcal{X}^a(\sigma)]] + [\mathcal{Z}_c(\sigma''), [\mathcal{X}^a(\sigma), \mathcal{Z}_b(\sigma')]] = 0, \tag{6.105}$$

and so the algebra is indeed associative.

6.11.2 From structure constants to structure functions

We would like to see how far we can generalise this, and in particular we would like to see if we can relax the condition that ${f_{ab}}^c$ and K_{abc} are constant and allow them to be functions

$${f_{ab}}^c \to {f_{ab}}^c(X), \qquad K_{abc} \to K_{abc}(X). \tag{6.106}$$

In particular, we take $g_{\mu\nu}$ and $B_{\mu\nu}$ to be general (we assume the metric is torsion-free). It is fairly easy to see that the algebra will still go through without any problems. This is essentially because there are no derivatives of f or K in the derivation of the algebra. However, where there might be problems is associativity. Since associators have nested commutators, we *do* have derivatives of f and K appearing. However, we will find that the algebra is in fact still associative. This is in contrast to the doubled geometry construction, where the group structure plays a prominent role. However, we suspect a formal generalisation of doubled geometry along similar lines is possible. Imposing the self-duality constraint there might require gauging an algebroid structure along the lines of [68] and we shall not comment on this further here.

Firstly, we should derive the modified constraints that now arise from the Maurer-Cartan equation (6.97) and the flux condition $dH = 0$. Going through the same process we find that

$$e_{[a}{}^{\mu}\partial_\mu {f_{bc]}}^d - {f_{e[ab}}^d {f_{bc]}}^e = 0, \qquad e_{[a}{}^{\mu}\partial_\mu K_{bcd]} - \frac{3}{2} K_{e[ab} {f_{cd]}}^e = 0. \tag{6.107}$$

How does this alter the calculations checking associativity? Essentially, there are two changes to consider: the explicit derivatives of f and K, and the changes to the δ' terms (which are themselves a result of the σ dependence of f and K). For example, we now have

$$\begin{aligned}
[\mathcal{Z}_a(\sigma), [\mathcal{Z}_b(\sigma'), \mathcal{Z}_c(\sigma'')]] = &-4\pi^2\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')\Big\{ {f_{bc}}^d {f_{ad}}^e \mathcal{Z}_e(\sigma'') + {f_{bc}}^d K_{ade}\mathcal{X}^e(\sigma'') \\
&-{f_{ae}}^d K_{bcd}\mathcal{X}^e(\sigma'') + e_a{}^{\mu}\partial_\mu {f_{bc}}^d \mathcal{Z}_d(\sigma'') + e_a{}^{\mu}\partial_\mu K_{bce}\mathcal{X}^e(\sigma'')\Big\} \\
&+ 4\pi^2\delta'(\sigma-\sigma'')\delta(\sigma'-\sigma'')K_{bcd}(\sigma'')\delta^d_a.
\end{aligned} \tag{6.108}$$

The new terms are the last two terms in the braces. Now, when we antisymmetrise this, the term we have to be extra careful with is the last one, i.e. the term outside of the braces. This is because the K now has σ'' dependence. We can write this term as

$$\begin{aligned}-\delta'(\sigma-\sigma'')\delta(\sigma'-\sigma'')K_{bcd}(\sigma'')\delta^d_a =&\partial_{\sigma''}\left(\delta(\sigma-\sigma'')K_{abc}(\sigma'')\right)\delta(\sigma'-\sigma'')\\ &-\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')K'_{abc}(\sigma'')\\ =&-\delta'(\sigma-\sigma'')\delta(\sigma'-\sigma'')K_{abc}(\sigma)\\ &-\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')X'^{\mu}\partial_\mu K_{abc}(\sigma''). \end{aligned} \quad (6.109)$$

Thus, after antisymmetrising, the total contribution from the δ' terms is

$$\begin{aligned}&\frac{4}{3}\pi^2 K_{abc}\left(\delta'(\sigma-\sigma'')\delta(\sigma'-\sigma'')+\delta'(\sigma'-\sigma)\delta(\sigma''-\sigma)+\delta'(\sigma''-\sigma')\delta(\sigma-\sigma')\right)\\ &+4\pi^2\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')\frac{1}{6}e_e{}^\mu\partial_\mu K_{abc}\mathcal{X}^e(\sigma'')\\ &=4\pi^2\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')\frac{1}{3}e_e{}^\mu\partial_\mu K_{abc}\mathcal{X}^e(\sigma''), \end{aligned} \quad (6.110)$$

the first term vanishing as in the constant case. We will see that the remaining term contributes in such a way as to ensure associativity. The full expression can now be written as

$$\begin{aligned}[\mathcal{Z}_{[a}(\sigma),[\mathcal{Z}_b(\sigma'),\mathcal{Z}_{c]}(\sigma'')]] =&4\pi^2\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')\Big\{f_{e[a}{}^d f_{bc]}{}^e\mathcal{Z}_d(\sigma'')-2K_{d[ea}f_{bc]}{}^d\mathcal{X}^e(\sigma'')\\ &-e_{[a}{}^\mu\partial_\mu f_{bc]}{}^d\mathcal{Z}_d(\sigma'')-e_{[a}{}^\mu\partial_\mu K_{bc]e}\mathcal{X}^e(\sigma'')+\frac{1}{3}e_e{}^\mu\partial_\mu K_{abc}\mathcal{X}^e(\sigma'')\Big\}, \end{aligned} \quad (6.111)$$

and we can write the last two terms in this expression as

$$-e_{[a}{}^\mu\partial_\mu K_{bc]e}\mathcal{X}^e(\sigma'')+\frac{1}{3}e_e{}^\mu\partial_\mu K_{abc}\mathcal{X}^e(\sigma'')=\frac{4}{3}e_{[e}{}^\mu\partial_\mu K_{abc]}\mathcal{X}^e. \quad (6.112)$$

Thus, (6.111) becomes

$$\begin{aligned}[\mathcal{Z}_{[a}(\sigma),[\mathcal{Z}_b(\sigma'),\mathcal{Z}_{c]}(\sigma'')]] =&4\pi^2\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')\Big\{f_{e[a}{}^d f_{bc]}{}^e\mathcal{Z}_d(\sigma'')-2K_{d[ea}f_{bc]}{}^d\mathcal{X}^e(\sigma'')\\ &-e_{[a}{}^\mu\partial_\mu f_{bc]}{}^d\mathcal{Z}_d(\sigma'')+\frac{4}{3}e_{[e}{}^\mu\partial_\mu K_{abc]}\mathcal{X}^e(\sigma'')\Big\}=0, \end{aligned} \quad (6.113)$$

where we notice that the expressions in the braces are precisely the constraints imposed by the equations (6.107) and therefore vanish. The calculation of $[\mathcal{Z},[\mathcal{Z},\mathcal{X}]]$ also works out in a similar, though slightly simpler, way. Thus, we conclude that the algebra is still associative even if $f_{ab}{}^c$ and K_{abc} are not constant.

A comment on associativity of the R-flux background

In this section we have focused only on geometric flux compactifications because this is the simplest case to approach in the general setting which we have laid out. It would be interesting to extended to include other, possibly non-geometric, backgrounds, but we have not considered this here. For the specific case of the R-flux background discussed earlier, it is straightforward to study the associativity of the algebra and we find that the algebra is indeed associative. The details are similar to those given in appendix C. There has been much discussion of the R-flux background giving rise to a non-associative structure[12], but we find no sign of any such structure in our construction. It may be that such a structure emerges when we take the X-dependence into account fully away from the adiabatic limit. Unfortunately, to look into this properly would require an understanding of non-isometric T-duality.

[12] See, for example, [69–73].

Chapter 7

General Deformations and Connections

Having gained a thorough understanding of twisted torus bundles in the universal coordinate construction, we now attempt to generalise the discussion of the previous chapter to more general connections. In particular, we will see how to construct a formalism that allows one to take any operator at some starting point in moduli space and deform it to a nearby point. We will mainly look at deformations of ∂X_μ, but the formalism we present is one that can be applied to a wide class of operators and theories, both with and without conformal invariance. As we discussed, the formalism of [22] entailed the deformation of correlation functions of CFTs, based on the formalism of [23, 24] for more general QFT deformations. In the previous chapter, we showed how universal coordinates was tied to this connection story, and we applied this to the torus bundles introduced in chapter 3, employing an adiabatic approximation to neglect worldsheet interactions involving the coordinate dependence in the background.

Here, we will show how we can proceed without this approximation. We will start by discussing the construction for flat toroidal backgrounds, where we demonstrate how to recover (4.30). As we will see, this will involve taking the tensor structure of the operator of interest into account. In chapter 8, we will then apply these ideas to the H-flux and nilfold cases, where we will derive first order corrections to the deformations derived in the adiabatic limit previously. Of course, the deformation of the stress tensor was already derived in section 5.3, but the method used there would not work for non-scalar operators since it would miss contributions arising from the change in the background, as we explain below. In chapter 9, we will examine the T-duality between these backgrounds once again, this time with the worldsheet interactions taken into account. The content of chapters 7, 8 and 9 is based on original work written in collaboration in [2].

Let us explain our approach. We shall consider a family of sigma models $\{E_{\mu\nu}\}$, where $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$ as usual, parameterised in some convenient way. Let $\mathcal{E}$ be a bundle of operators $\mathcal{A}$ over this space, where an element $A \in \mathcal{A}$ may be written schematically in terms of a local worldsheet operator as $A = \int \Phi$. Then, the integrated correlation functions will change as we move about this space, not simply because of any direct metric dependence of the field, but also due to the change in the measure $e^{S[\Phi]}$ used to define the correlation function. If

$$\Big\langle \Phi_1 ... \Phi_n \Big\rangle = \int \mathcal{D}\Phi e^{-S[\Phi]} \; \Phi_1 ... \Phi_n, \tag{7.1}$$

where Φ_i is a local operator inserted at the point z_i on a worldsheet Σ embedded in the target space[1], then to leading order,

$$\delta \Big\langle \Phi_1 ... \Phi_n \Big\rangle = \sum_{i=1}^{n} \int \mathcal{D}\Phi e^{-S[\Phi]} \; \Phi_1 ... \delta_E \Phi_i ... \Phi_n - \int \mathcal{D}\Phi e^{-S[\Phi]} \delta S[\Phi] \; \Phi_1 ... \Phi_n + ..., \tag{7.2}$$

where $\delta_E \Phi$ is a 'classical' change in Φ - a change in the background field that preserves $S[\Phi]$. Note that this was missing from the analysis of chapter 5 since there we were only dealing with scalar operators. As we will discuss, the δ_E transformation is only present for non-scalar operators.

If we want to understand how an individual operator, say Φ_i, changes we need a more subtle tool. The above expressions suggest that this would be given schematically, to first order, by

$$\delta \Phi_i = -\mathcal{O}_i[X](\Phi_i) + \delta_E \Phi_i + ..., \tag{7.3}$$

where the effect of the change in the action on the contribution Φ_i makes to the correlation function is given by the insertion of a non-local operator $\mathcal{O}_i[X]$ that has the same functional form as $\delta S[X]$,

$$\mathcal{O}_i[X] = \int_{\Sigma_i^\epsilon} : \delta \widehat{\mathcal{L}} :, \tag{7.4}$$

[1] To compute the correlation functions, the locations of the operators would ultimately be integrated over and it is probably more natural to think in terms of integrated operators $\int_\Sigma \Phi$ when talking about objects in the fibres of $\mathcal{E}$. We will also not explicitly include the ghost contributions to such correlation functions.

where $\delta\widehat{\mathcal{L}}$ is the integrand[2] of $\delta S[X]$, lifted to an operator expression and $\Sigma_i^\epsilon \subset \Sigma$ has holes of size $\epsilon > 0$ cut out around the locations of the $\Phi_{j\neq i}$ fields. We recover the variation of the correlation function when all fields Φ are allowed to change. Here, we will mainly be interested in cases where the starting point is a free theory. In this case, we can use Wick's theorem to evaluate the correlation function, and the action of $\mathcal{O}$ that affects the field Φ_i directly is given by the contraction

$$\delta_{\mathcal{O}}\Phi_i(z_i) = \overbracket{\mathcal{O}_i[X]\Phi}_i(z_i). \tag{7.5}$$

Sequential applications of $\mathcal{O}$ will be discussed in detail in chapter 7 and appendix E.

This construction extends simply to cases where the sigma model is specified by other target space fields. We will flesh out what the terms in (7.5) mean in the following sections. If Φ is a target space scalar, then this is the whole story. However, if Φ is a vector or higher tensor, then there may be an additional contribution which preserves the action and therefore is not included in the transformation generated by $\mathcal{O}$. For example, in a flat torus background, $\partial X^\mu = e^\mu{}_a \partial X^a$ has an explicit dependence on the background metric through the vielbein $e^\mu{}_a$. We now imagine changing the radius R of one of the circles. With all of the R dependence in $e^\mu{}_a$, we think of ∂X^a as a universal field for such backgrounds and the change in ∂X^μ may be written as

$$\partial X^\mu \to \partial X^\mu + \delta R (\partial_R e^\mu{}_a) e^a{}_\nu \partial X^\nu + ... \tag{7.6}$$

Such contributions are absent in the previously studied fields such as the worldsheet stress tensor, but play an important role in recovering the correct transformation properties of non-scalar fields. We denote such contributions to the variation by δ_E.[3] It will be helpful to think of δ_E as the part of the deformation that leaves the action invariant and $\mathcal{O}$ as that part which changes the action.

Let us now apply this formalism to the flat torus.

7.1 $d = 1$: circle deformations by parallel transport

We begin by discussing the space of CFTs on toroidal backgrounds (i.e. the space given by (2.1) and parallel transport around this space generated by marginal deformations, and in this section we will focus on the $d = 1$ case. Earlier, we argued that universal

[2] $\mathcal{O}_i$ is not local and the deformation need not be of Lagrangian type, but for simplicity we shall assume it does take the form $\delta S = \int\limits_\Sigma \delta\mathcal{L}$.

[3] Schematically, δ_E may be thought of as being of the form $\delta_E \sim \delta g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}$.

coordinates could be seen as coming from the $\hat{\Gamma}$ connection and we used this to recover the results (4.30), (4.31). Here, we shall keep our procedure more general with a view to applying it to QFTs without conformal symmetry. Looking at CFT cases first provides a helpful toy example since we can compare with known results.

For convenience, we will recall here the universal coordinate results derived earlier. We have, for the $d=1$ case, on the contour $|z|=1$,[4]

$$\partial X(R+\delta R) = \partial X(R) + \frac{\delta R}{R}\left(\partial X(R) - \frac{\bar{z}}{z}\bar{\partial} X(R)\right) + \frac{1}{2}\left(\frac{\delta R}{R}\right)^2 \left(\partial X(R) - \frac{\bar{z}}{z}\bar{\partial} X(R)\right), \tag{7.7}$$

$$\bar{\partial} X(R+\delta R) = \bar{\partial} X(R) + \frac{\delta R}{R}\left(\bar{\partial} X(R) - \frac{z}{\bar{z}}\partial X(R)\right) + \frac{1}{2}\left(\frac{\delta R}{R}\right)^2 \left(\bar{\partial} X(R) - \frac{z}{\bar{z}}\partial X(R)\right), \tag{7.8}$$

and using the mode expansion

$$\partial X(R)(z) = -\frac{iR^2}{\sqrt{2}}\sum_n \alpha_n z^{-n-1}, \tag{7.9}$$

we obtain

$$\alpha_n(R+\delta R) = \alpha_n(R) - \frac{\delta R}{R}(\alpha_n(R) + \bar{\alpha}_{-n}(R)) + \frac{3}{2}\left(\frac{\delta R}{R}\right)^2 (\alpha_n(R) + \bar{\alpha}_{-n}) + ..., \tag{7.10}$$

$$\bar{\alpha}_n(R+\delta R) = \bar{\alpha}_n(R) - \frac{\delta R}{R}(\bar{\alpha}_n(R) + \alpha_{-n}(R)) + \frac{3}{2}\left(\frac{\delta R}{R}\right)^2 (\bar{\alpha}_n(R) + \alpha_{-n}) + ..., \tag{7.11}$$

where the ... indicates higher order terms arising from the expansion of the inverse metric $(R+\delta R)^{-2}$. In this relatively simple $d=1$ case, we can easily obtain the finite transformation as

$$\alpha_n(R+\delta R) = \alpha_n(R) - \frac{\lambda}{2(1+\lambda)}(\alpha_n(R) + \bar{\alpha}_{-n}(R)), \tag{7.12}$$

where $\lambda = g^{-1}\delta g = (2R\delta R + \delta R^2)/R^2$. Note that, in this finite case, δR need not be a small deformation of R.

These results may be recovered using the connection formalism and the $\mathcal{O}$ and δ_E operators written down previously, which we now do.

[4] We have removed the explicit $z, \bar{z}$ dependence to keep these equations uncluttered.

To compute the deformation, we first need to write down the action and in particular the marginal operator.

7.1.1 The action & deformation

Under a shift $E_{\mu\nu} \to (E + \delta E)_{\mu\nu}$, we know from (2.15) that the marginal operator is

$$\delta E_{\mu\nu} \int_\Sigma \partial X^\mu \bar{\partial} X^\nu. \tag{7.13}$$

For the circle of radius $R \to R + \delta R$, the marginal operator required is thus

$$\mathcal{O} = \frac{2\delta R}{R^3} \int\limits_\Sigma d^2 z \partial X(z) \bar{\partial} X(\bar{z}). \tag{7.14}$$

Unlike the stress tensor deformation, this is not the full story and this is where the spacetime tensor structure of the operator of interest is important. The full deformation of the operator is given by the sum of two parts: a part coming from the marginal operator ($\mathcal{O}$) and a part coming from the deformation of the background, which for the circle is given by

$$\delta_E = \delta R \frac{\partial}{\partial R}. \tag{7.15}$$

The full deformation operator is given by the path ordered exponential

$$P \exp\left(\int\limits_0^s ds' \frac{d\mathfrak{m}}{ds'} \frac{1}{R^4} \int\limits_\Sigma d^2 z \partial X(z) \bar{\partial} X(\bar{z}) + \int\limits_0^s ds' \frac{d\mathfrak{m}}{ds'} \frac{1}{2R} \frac{\partial}{\partial R} \right), \tag{7.16}$$

where $\mathfrak{m}$ is our parameterisation of moduli space. As discussed in section 5.1.2, we are free to choose this parameterisation, but the one which we will use here is $\mathfrak{m} = r^2$ for a circle of radius r, i.e. the metric. Thus, if our path is between the radii R and $R' = R + \delta R$, parameterised by $s' \in [0, R' - R]$, so that $s' = r - R$, where r is the radius at s', we have

$$\int\limits_0^s ds_1 ... \int\limits_0^{s_{n-1}} ds_n \frac{d\mathfrak{m}}{ds_1} ... \frac{d\mathfrak{m}}{ds_n} = \frac{1}{n!} (R'^2 - R^2)^n, \tag{7.17}$$

so that our deformation operator is

$$P \exp\left((2R\delta R + \delta R^2) \left(\frac{1}{R^4} \int\limits_\Sigma d^2 z \partial X(z) \bar{\partial} X(\bar{z}) + \frac{1}{2R} \frac{\partial}{\partial R} \right) \right). \tag{7.18}$$

We now compute the transformation of ∂X to first order in δR. The higher order case will be discussed properly in section 7.1.3; there are subtleties about how we define multiple $\mathcal{O}$ insertions due to potential divergences when different $\mathcal{O}$ coincide.

7.1.2 ∂X deformation using parallel transport in $\mathcal{M}$

We now show how (7.7) can be recovered from the parallel transport construction outlined in the previous section. We are interested in the correlator $\Big\langle \partial X(w)\Phi(0)\Big\rangle$, where Φ is a generic operator (inserted at $z=0$ for convenience). We will use this correlation function to deduce the deformation of $\partial X(w)$. From the discussion that led to (5.8), we have, to first order,

$$\Big\langle \partial X(w)\Phi(0)\Big\rangle' = \Big\langle \partial X(w)\Phi(0)\Big\rangle + \Big\langle \mathcal{O}\partial X(w)\Phi(0)\Big\rangle + \Big\langle \delta_E \partial X(w)\Phi(0)\Big\rangle. \tag{7.19}$$

First consider the action of $\mathcal{O}$. We have

$$\begin{aligned}
&\Big\langle \partial X(w)\Phi(0)\Big\rangle' \\
&= \Big\langle \partial X(w)\Phi(0)\Big\rangle + \frac{2\delta R}{R^3}\int_{\Sigma^\epsilon} d^2z \Big\langle \partial X(z)\bar\partial X(\bar z)\partial X(w)\Phi(0)\Big\rangle + \Big\langle \delta_E \partial X(w)\Phi(0)\Big\rangle \\
&= -\frac{iR^2}{\sqrt{2}}\sum_{i=1}^{N}\sum_{n\geq 0}(w-z_i)^{-n-1}\left(\Big\langle \alpha_n\Phi(0)\Big\rangle + \frac{2\delta R}{R^3}\Big\langle \alpha_n\Phi(0)\int_{\Sigma^\epsilon} d^2z\,\partial X(z)\bar\partial X(\bar z)\Big\rangle\right) \\
&- \frac{2\delta R}{R}\int_{\Sigma^\epsilon} d^2z\frac{1}{(w-z)^2}\Big\langle \bar\partial X(\bar z)\Phi(0)\Big\rangle + \Big\langle \delta_E \partial X(w)\Phi(0)\Big\rangle \\
&= -\frac{iR^2}{\sqrt{2}}\sum_{i,n}(w-z_i)^{-n-1}\Big\langle \alpha_n\Phi(0)\Big\rangle' \\
&+ \frac{\delta R}{R}\int_{\Sigma^\epsilon} d^2z\,\partial_z\left(\frac{1}{z-w}\right)\Big\langle \bar\partial X(\bar z)\Phi(0)\Big\rangle + \Big\langle (\delta_E\partial X(w))\Phi(0)\Big\rangle,
\end{aligned} \tag{7.20}$$

where the action of δ_E on Φ has been absorbed into the first term in (7.20), and we recall the definition of Σ^ϵ from (5.34). Comparing this first term with the LHS, we see that it is simply the zeroth order term, so we will focus on the second term. We can write it as

$$\frac{\delta R}{R}\int_{\Sigma^\epsilon} d^2z\,\partial_z\left(\frac{1}{z-w}\right)\Big\langle \bar\partial X(\bar z)\Phi(0)\Big\rangle = -\frac{\delta R}{R}\oint_{C^\epsilon}\frac{d\bar z}{z-w}\Big\langle \bar\partial X(\bar z)\Phi(0)\Big\rangle, \tag{7.21}$$

where the contour C^ϵ simply consists of circles around $z=0,w$ of radius ϵ. Around w there is no contribution since there are no negative powers of $\bar z - \bar w$.

Around $z = 0$, we use

$$(w - z)^{-1} = \sum_{n \geq 0} (w - z_i)^{-n-1} (z - z_i)^n \tag{7.22}$$

with $z_i = 0$ to expand the integrand around zero to get

$$\frac{\delta R}{R} \sum_{i,n} w^{-n-1} \oint_{C_0^\epsilon} d\bar{z} z^n \left\langle \bar{\partial} X(\bar{z}) \Phi(0) \right\rangle . \tag{7.23}$$

Comparing this to what we have on the left hand side, as in [50], gives

$$\begin{aligned}
\left\langle \partial X(w) \Phi(0) \right\rangle' &= \sum_{n \geq 0} w^{-n-1} \left(\frac{-i}{\sqrt{2}} (R^2 + 2R\delta R) \right) \left\langle \alpha_n' \Phi(0) \right\rangle \\
&= \sum_{n \geq 0} w^{-n-1} \left\langle \left(\frac{-iR^2}{\sqrt{2}} \right) \alpha_n \Phi(0) + \frac{\delta R}{R} \oint_{C_0^\epsilon} d\bar{z} z^n \bar{\partial} X(\bar{z}) \Phi(0) \right\rangle + \left\langle (\delta_E \partial X(w)) \Phi(0) \right\rangle \\
&= \sum_{n \geq 0} w^{-n-1} \left\langle \left[\left(\frac{-iR^2}{\sqrt{2}} \right) \alpha_n + \frac{\delta R}{R} \oint_{C_0^\epsilon} d\bar{z} z^n \bar{\partial} X(\bar{z}) + \frac{\delta R}{R} \partial X(w) \right] \Phi(0) \right\rangle ,
\end{aligned} \tag{7.24}$$

where $\delta_E \partial X(R) = R^{-1} \delta R \, \partial X(R)$ has been used. This can be seen by noting that δ_E is a derivative with respect to the einbein which acts trivially on spacetime scalars. In one dimension, the einbein is simply R. We can rearrange (7.24) to read off

$$\delta \alpha_n = -\frac{\delta R}{R} \left(\alpha_n + \bar{\alpha}_{-n} \epsilon^{2n} \right) . \tag{7.25}$$

Note that at this point we would usually make a specific choice of connection, for example by taking the limit $\epsilon \to 0$ and dropping divergent terms, or setting $\epsilon = 1$. In this case, in order to make contact with the results of chapter 4, we set $\epsilon = 1$ (the $\hat{\Gamma}$ connection), which recovers their result. We would like to write this as a first order shift of ∂X itself as opposed to just the modes of ∂X. To do this, we substitute the above transformation into $\partial X(R + \delta R)$, working to first order. Initially we will leave ϵ as it is, but we will see that it is necessary to choose $|z| = \epsilon$ for the deformation of

$\partial X(z)$. We have

$$\begin{aligned}
\partial X(R+\delta R)(z) &= -\frac{i}{\sqrt{2}}\sum_n (R+\delta R)^2\,(\alpha_n+\delta\alpha_n)\,z^{-n-1} \\
&= -\frac{i}{\sqrt{2}}\sum_n \left(R^2\alpha_n + R^2\delta\alpha_n + 2R\delta R\alpha_n + O(R^2)\right) z^{-n-1}, \\
&= -\frac{i}{\sqrt{2}}\sum_n \left((R^2+R\delta R)\alpha_n z^{-n-1} - R\delta R\bar{\alpha}_{-n}\bar{z}^{n-1}\frac{\bar{z}}{z}\left(\frac{\epsilon}{\eta}\right)^{2n}\right), \quad (7.26)
\end{aligned}$$

where $|z|=\eta$. Thus, we see that, in order to be able to write the last term as a $\bar{\partial}X$, we need to set $\epsilon=\eta$. As discussed earlier, the choice of $\epsilon=1$ for the $\hat{\Gamma}$ connection is arbitrary and for a CFT all choices are equivalent under the conformal symmetry. Note also that the metric factor of $(R+\delta R)^2$ gives us an extra α_n contribution, so overall, setting $\epsilon=|z|=1$, we have

$$\partial X(R+\delta R)(z) = \partial X(R)(z) + \frac{\delta R}{R}\left(\partial X(R)(z) - \frac{\bar{z}}{z}\bar{\partial}X(R)(\bar{z})\right). \quad (7.27)$$

Note that it is reassuring that the requirement that $|z|=1$ (or $|z|=$ const. more generally) appears here, since it also appeared in the universal coordinate approach. There, it came from having equal-time commutation relations, which translated to constant $|z|$ on the plane. Here we see the same requirement coming from the transition from the mode transformation to the full local operator transformation. In section 7.1.4 we will come back to the relationship between universal coordinates and the $\hat{\Gamma}$ connection in the context of the formalism presented in this chapter. We will show that the existence of universal coordinates is indeed special to the $\hat{\Gamma}$ connection and that there does not seem to be an analogue for general connections. First, we extend the analysis of [50] and recover the results of [15] to all orders in δR.

7.1.3 Higher orders

For ∂X, we can simply obtain the deformation to all orders in δR by replacing $2R\delta R \to \delta g = 2R\delta R + \delta R^2$. If we then work to first order in δg instead of δR, we will obtain the same results as above, but with $2R\delta R$ replaced by δg:

$$\delta\partial X(z) = \frac{1}{2}\delta g\left(\partial X(z) - \frac{\bar{z}}{z}\bar{\partial}X(\bar{z})\right), \quad (7.28)$$

for a background deformation $g \to g + \delta g$. This is in fact the full deformation, since we already know that the finite transformation is given by (4.30), and indeed we see that this is only first order in the metric deformation.

However, the story for the mode transformation is different because this transformation does *not* truncate at first order in δg (or second order in δR), but has corrections to all orders, as may be seen by expanding the right hand side of (7.12) in powers of λ. This is essentially because the mode deformation involves the inverse metric, the deformation of which involves an infinite expansion in δg. Therefore, we need a way of computing $\delta_{\mathcal{O}}$ and δ_E transformations to all orders.

δ_E has a straightforward action as a local transformation, so there is no difficulty in computing higher order variations $(\delta_E)^n$ simply as repeated applications of δ_E. Higher order $\mathcal{O}$ insertions require more care. In order to generalise the path integral derivation of the first order stress tensor deformation in section 5.3 to higher orders, a prescription is needed to specify how to treat the otherwise ambiguous insertions of higher powers of $\mathcal{O}$. Our prescription for computing higher order deformations is given by

$$\mathcal{O}^n \partial X(w) = \int_{\Sigma_n} d^2 z_n ... \int_{\Sigma_1} d^2 z_1 O(z_n, \bar{z}_n) ... O(z_1, \bar{z}_1) \partial X(w), \tag{7.29}$$

where

$$\Sigma_i = \{z_i \in \mathbb{C} | \quad |z_i| \geq \epsilon, |z_i - w| > 0, |z_i - z_j| \geq \epsilon \quad \forall j > i\}, \tag{7.30}$$

where the order of the integral signs denotes the order in which the integrations are done, i.e. we remove discs around all punctures corresponding to $O(z_j, \bar{z}_j)$ insertions which have not yet been integrated out. For example, in the above integral we would compute the z_1 integral first, and therefore we would need to remove discs around $z_2, ..., z_n$. For the z_2 integral, since z_1 has already been integrated out, we now only need to remove discs around $z_3, ..., z_n$, and so on.

The evaluation of the resulting deformation is potentially complicated. However, we claim[5] that the only term that gives a finite contribution at order δg^n is the one where the contractions are taken by order of integration. Schematically, this is

$$\int_{\Sigma_n} ... \int_{\Sigma_1} [O_n [O_{n-1} [...[O_2 [O_1, \partial X]]...]], \tag{7.31}$$

i.e. we first contract ∂X with O_1, then contract the result with O_2, and so on until O_n has been contracted. Here, we use commutator notation to avoid the confusing

[5] Further details and a justification of this prescription are given in appendix E.

notation of multiple Wick contractions. Since a contraction of O_i with ∂X or $\bar{\partial}X$ leads to a $\bar{\partial}X$ or ∂X respectively, this prescription is unambiguous. For other deformation operators, such as those we shall consider in chapter 8, matters are more complicated. This prescription is verified explicitly to second order in appendix E. In terms of the modes, this gives a justification for being able to apply $\mathcal{O}$ sequentially to α_n, which is how we will compute the higher order corrections to $\delta\alpha_n$. With this justification, we now come to the calculation itself.

Working with δg instead of δR, we have seen that

$$\delta_{\mathcal{O}}(R^2\alpha_n) = \frac{i}{\sqrt{2}}\delta g g^{-1} \oint_{C_0^\epsilon} d\bar{z} z^n \bar{\partial}X(\bar{z}) = -\frac{1}{2}\epsilon^{2n}(2R\delta R + \delta R^2)\bar{\alpha}_{-n} = -\frac{1}{2}\epsilon^{2n}\delta g\bar{\alpha}_{-n}, \tag{7.32}$$

which we can simply divide by $g = R^2$, since the metric is unaffected by $\mathcal{O}$, to get

$$\delta_{\mathcal{O}}\alpha_n = -\frac{1}{2}\epsilon^{2n}\lambda\bar{\alpha}_{-n}, \tag{7.33}$$

where we introduce $\lambda := g^{-1}\delta g$. We note that $\delta_E\, g = \delta g$ and $\delta_E\delta g = 0$, and so $\delta_E\lambda = -\lambda^2$. Therefore, acting with δ_E on the mode gives

$$\delta_E(R^2\alpha_n) = \left(R\delta R + \frac{1}{2}\delta R^2\right)\alpha_n = \frac{1}{2}\delta g\alpha_n, \tag{7.34}$$

which we rearrange to get

$$\delta_E\alpha_n = -\frac{1}{2}\lambda\alpha_n. \tag{7.35}$$

Thus,

$$(\delta_{\mathcal{O}} + \delta_E)\alpha_n = -\frac{1}{2}\lambda\Big(\alpha_n + \epsilon^{2n}\bar{\alpha}_{-n}\Big). \tag{7.36}$$

This first order result, coupled with the prescription described above, allows us to systematise the calculation of higher order terms. We can thus iterate these results to obtain the transformation to all orders. We will work with δg here because it is simpler than working with δR, but it is easy to switch between the two (the two are related by a reparameterisation of the path on $\mathcal{M}$ connecting the initial and final backgrounds). The advantage of working with δg is that we can say $\mathcal{O}, \delta_E$ are first order in δg, and the nth order operator insertion is simply $\frac{1}{n!}(\mathcal{O} + \delta_E)^n$, as we can see from the path ordered exponential (7.18).

It is easy to show that[6]

$$\frac{1}{m!}\Big(\delta_{\mathcal{O}} + \delta_E\Big)^m \alpha_n = (-1)^m \frac{1}{2}\lambda^m(\alpha_n + \epsilon^{2n}\bar{\alpha}_{-n}). \tag{7.39}$$

Summing over m recovers the all orders result (7.12) found using the universal coordinate method. From (7.9), we can also recover the transformation of ∂X, evaluated on the contour $|z| = \epsilon$, to all orders,

$$\partial X(R + \delta R)(z) = \partial X(R)(z) + \delta g g^{-1}\left(\partial X(R)(z) - \frac{\bar{z}}{z}\bar{\partial} X(R)(\bar{z})\right), \tag{7.40}$$

which indeed truncates at second order (since the transformation of $R^2\alpha_n$ truncates at second order) and agrees with chapter 4.

7.1.4 Interlude: connections and universal coordinates

It is interesting to consider the relationship between universal coordinates and choice of connection. As we shall see, X' and Π only remain universal under parallel transport in $\mathcal{M}$ (given by (2.1)) with connection $\hat{\Gamma}$. For a fixed background, we have $X'(z,\bar{z}) = \frac{i}{R^2}(z\partial X(z) - \bar{z}\bar{\partial} X(\bar{z}))$ and $\Pi(z,\bar{z}) = i(z\partial X(z) + \bar{z}\bar{\partial} X(\bar{z}))$. In terms of modes,

$$X'(z,\bar{z}) = \frac{1}{\sqrt{2}}\sum_n(\alpha_n z^{-n} - \bar{\alpha}_n \bar{z}^{-n}), \qquad \Pi(z,\bar{z}) = \frac{R^2}{\sqrt{2}}\sum_n(\alpha_n z^{-n} + \bar{\alpha}_n \bar{z}^{-n}), \tag{7.41}$$

and so using the results of the previous section we can compute how X', Π change to first order.

[6]This may be shown via induction:

$$\frac{1}{(m+1)!}(\delta_{\mathcal{O}} + \delta_E)^{m+1}\alpha_n = \frac{1}{m+1}\frac{(-1)^m}{2}\lambda^m(\delta_{\mathcal{O}} + \delta_E)\left(\alpha_n + \epsilon^{2n}\bar{\alpha}_{-n}\right). \tag{7.37}$$

Then, using $\delta_E(g^{-1}\delta g)^m = m(g^{-1}\delta g)^{m-1}\delta_E(g^{-1}\delta g)$, we get to the desired result, i.e.

$$\frac{1}{(m+1)!}(\delta_{\mathcal{O}} + \delta_E)^{m+1}\alpha_n = (-1)^{(m+1)}\frac{1}{2}(g^{-1}\delta g)^{m+1}(\alpha_n + \epsilon^{2n}\bar{\alpha}_{-n}), \tag{7.38}$$

as required. □

The $\hat{\Gamma}$ connection

For this connection, where $\epsilon = 1$,

$$\delta X'(z,\bar{z}) = \frac{1}{\sqrt{2}}\sum_n (\delta\alpha_n z^{-n} - \delta\bar{\alpha}_n \bar{z}^{-n}) = 0, \tag{7.42}$$

on the contour $z\bar{z} = 1$. The R^2-dependence makes the calculation for Π only slightly more involved,

$$\Pi(z,\bar{z}) + \delta\Pi(z,\bar{z}) = \frac{(R+\delta R)^2}{\sqrt{2}}\sum_n \Big((\alpha_n + \delta\alpha_n)z^{-n} + (\bar{\alpha}_n + \delta\bar{\alpha}_n)\bar{z}^{-n}\Big). \tag{7.43}$$

Substituting the transformations for the modes in, we find that $\delta\Pi = 0$. Thus, we see that X', Π are indeed universal for the $\hat{\Gamma}$ connection. Given our earlier discussion in section 5.2.2, this is as we would have expected. Although we expect this not to hold in generality for the $c, \bar{c}$ connections, it is interesting to see whether these operators in particular retain their universal nature for other connections. If not, it is interesting to see precisely where this fails. Let us look at the $c, \bar{c}$ connections now.

The c connection

For the c connection, we integrate up to $\epsilon = 1$ and then, for those operator coefficients in the OPE for which the integral gives a finite result, we take the limit $\epsilon \to 0$. In the case at hand, the only OPE which may have potential singularities that need subtracting is the OPE between the marginal operator and $\partial X(w)$, which gives

$$\oint_{C_w^1 - C_w^\epsilon} \frac{d\bar{z}}{z-w}\Big(\bar{\partial}X(\bar{w}) + (\bar{z}-\bar{w})\bar{\partial}^2 X(\bar{w}) + ...\Big), \tag{7.44}$$

where C_w^1 and C_w^ϵ are circles around $z = w$ of radius 1 and ϵ respectively. Evaluating (7.44) gives zero since there are no negative powers of $(\bar{z} - \bar{w})$, so there are in fact no divergences to subtract. This is also evident in the fact that (7.25) is finite in the limit $\epsilon \to 0$, since we are only summing over $n \geq 0$. We see in this limit that all of the terms vanish apart from $n = 0$. Extending the result to $n < 0$, the transformation of the modes for the c connection is given by

$$\delta\alpha_n = -\frac{\delta R}{R}\alpha_n, \qquad \delta\bar{\alpha}_n = -\frac{\delta R}{R}\bar{\alpha}_n, \qquad n \neq 0,$$

$$\delta\alpha_0 = \delta\bar{\alpha}_0 = -\frac{\delta R}{R}(\alpha_0 + \bar{\alpha}_0). \tag{7.45}$$

In this case, after some brief calculation, we find that

$$\delta X'(z,\bar{z}) = -\frac{\delta R}{R\sqrt{2}}\sum_{n\neq 0}(\alpha_n z^{-n} - \bar{\alpha}_n \bar{z}^{-n}), \qquad \delta\Pi(z,\bar{z}) = \frac{R\delta R}{\sqrt{2}}\sum_{n\neq 0}(\alpha_n z^{-n} + \bar{\alpha}_n \bar{z}^{-n}), \tag{7.46}$$

i.e. they are not universal with respect to the c connection. Note that the variation of the zero modes still cancels out in both cases. In fact, what we notice is that the transformation of $\alpha_0, \bar{\alpha}_0$ is independent of the ϵ-dependent part of the connection. We should perhaps have expected this since we recall that $\alpha_0, \bar{\alpha}_0$ commute with L_0^+, the generator of dilations. Since changes in the radii of the disks, ϵ, correspond to dilations, we would expect $\alpha_0, \bar{\alpha}_0$ to be independent of ϵ.

The $\bar{c}$ connection

In this case, since there are no divergences to subtract, the c and $\bar{c}$ connections are actually the same, so X', Π are only universal for the $\hat{\Gamma}$ connection, as suggested by our calculations earlier. We shall see in chapter 8 that, for more general non-CFT deformations, the X^μ will not be universal even for the $\hat{\Gamma}$ connection. This is essentially why things are so much more complicated when we move away from the adiabatic limit. The existence of universal coordinates makes all of the calculations much simpler, and when we do not have them we must use the more general parallel transport construction.

We now look at the higher dimensional analogue of the previous section, i.e. when we have a toroidal target space. In some ways things are clearer in this more general case since the vielbein structure is more explicit and thus it is easier to see the contrast between the $\mathcal{O}$ and δ_E transformations and why we need both of them to get the full transformation.

7.2 $d > 1$: torus deformations by parallel transport

For $d > 1$, the above discussion generalises straightforwardly. The only new feature is the possible presence of a constant B-field. Due to a subtlety in the δ_E transformation, it will be helpful to distinguish between the $B = 0$ and $B \neq 0$ cases.

We recall the results from universal coordinates for convenience, which on the plane are

$$\partial X_\mu(E')(z) = \frac{1}{2} g^{\nu\rho}\Big((E'_{\mu\nu} + E^T_{\mu\nu})\partial X_\rho(E)(z) + (-E'_{\mu\nu} + E_{\mu\nu})\frac{\bar z}{z}\bar\partial X_\rho(E)(\bar z)\Big), \quad (7.47)$$

$$\bar\partial X_\mu(E')(\bar z) = \frac{1}{2} g^{\nu\rho}\Big((-E'^T_{\mu\nu} + E^T_{\mu\nu})\frac{z}{\bar z}\partial X_\rho(E)(z) + (E'^T_{\mu\nu} + E_{\mu\nu})\bar\partial X_\rho(E)(\bar z)\Big), \quad (7.48)$$

and the mode transformations are

$$2g'_{\mu\nu}\alpha^\nu_n(E') = \Big(E^T_{\mu\nu} + E'_{\mu\nu}\Big)\alpha^\nu_n(E) + \Big(E_{\mu\nu} - E'_{\mu\nu}\Big)\bar\alpha^\nu_{-n}(E), \quad (7.49)$$

$$2g'_{\mu\nu}\bar\alpha^\nu_n(E') = \Big(E^T_{\mu\nu} - E'^T_{\mu\nu}\Big)\alpha^\nu_{-n}(E) + \Big(E_{\mu\nu} + E'^T_{\mu\nu}\Big)\bar\alpha^\nu_n(E). \quad (7.50)$$

As with the $d = 1$ case, we can choose a parameterisation of our path along which we deform. We have deformation operator

$$P\exp\left(\int_0^s ds' \frac{dg_{\mu\nu}(s')}{ds'}\left(\int_\Sigma d^2z\,\partial X^\mu(z)\bar\partial X^\nu(\bar z) + \frac{\partial}{\partial g_{\mu\nu}}\right)\right), \quad (7.51)$$

where $g(s') = g + \frac{s'}{s}\delta g$, so that

$$\int_0^s ds_1 ... \int_0^{s_{n-1}} ds_n \frac{dg_{\mu\nu}}{ds_1} ... \frac{dg_{\rho\sigma}}{ds_n} = \frac{1}{n!}\delta g_{\mu\nu} ... \delta g_{\rho\sigma}, \quad (7.52)$$

giving deformation operator

$$P\exp\left(\delta g_{\mu\nu}\left(\int_\Sigma d^2z\,\partial X^\mu(z)\bar\partial X^\nu(\bar z) + \frac{\partial}{\partial g_{\mu\nu}}\right)\right). \quad (7.53)$$

7.2.1 Metric deformations

We start with the simpler case where $B_{\mu\nu} = 0$ and the only change is due to the metric. It is useful to introduce vielbeins ${e_\mu}^a$ such that

$$g_{\mu\nu} = {e_\mu}^a e_{a\nu}. \quad (7.54)$$

The generalisation of the δ_E transformation to higher dimensions is given by

$$\delta_E = \delta g_{\mu\nu}\frac{\partial}{\partial g_{\mu\nu}}, \quad (7.55)$$

and we identify $\delta_E e_\mu{}^a = \delta e_\mu{}^a$, where δe is induced by δg (this could also be taken as a definition of δ_E in this case). To compute this, we vary (7.54) and rearrange to get to

$$\delta e_\mu{}^a = \delta g_{\mu\nu} e^{\nu a} - e_\mu{}^b \delta e_{b\nu} e^{\nu a}. \tag{7.56}$$

Now, if we define $U_a{}^b = \delta e_{a\nu} e^{\nu b}$, then we see that the second term in the vielbein variation looks like a frame transformation except for the fact that U is not antisymmetric, which is required to give a Lorentz transformation. Therefore, we will extract the symmetric part from it and take the anti-symmetric part as the Lorentz transformation. Doing so, we find that

$$\begin{aligned} U_{(ab)} =& \delta e_{(a|\mu} e^\mu{}_{|b)} \\ =& e_a{}^\mu \delta g_{\mu\nu} e^\nu{}_b - V_{(ab)}, \end{aligned} \tag{7.57}$$

where $V_{(ab)} = e_{(a|}{}^\mu \delta e_{\mu|b)}$, i.e. we have

$$U_s = \delta e^T e^{-T} + e^{-1}\delta e, \tag{7.58}$$

$$V_s = e^{-1}\delta e + \delta e^T e^{-T}, \tag{7.59}$$

and so $U_s = V_s$. Thus, rearranging the above equation gives

$$U_{ab} = \frac{1}{2} e_a{}^\mu \delta g_{\mu\nu} e^\nu{}_b + \Lambda_{ab}, \tag{7.60}$$

where $\Lambda_{ab} = U_{[ab]}$, and the variation of the vielbein can be written as

$$\delta e_\mu{}^a = \frac{1}{2}\delta g_{\mu\nu} e^{\nu a} - e_\mu{}^b \Lambda_b{}^a, \tag{7.61}$$

where $\Lambda_{ab} = U_{[ab]}$ is a local frame transformation. The transpose can similarly be written as

$$\delta e^a{}_\mu = \frac{1}{2} e^{a\nu} \delta g_{\nu\mu} + \Lambda^a{}_b e^b{}_\mu, \tag{7.62}$$

where the sign change comes from the fact that Λ is antisymmetric. We can also invert these to get

$$\delta e^\mu{}_a = -\frac{1}{2} g^{\mu\nu} \delta g_{\nu\rho} e^\rho{}_a - e^\mu{}_b \Lambda^b{}_a, \qquad \delta e_a{}^\mu = -\frac{1}{2} e_a{}^\rho \delta g_{\rho\nu} g^{\nu\mu} + \Lambda_a{}^b e_b{}^\mu. \tag{7.63}$$

For ∂X_a, since this is a frame one-form, there should be a similar Lorentz transformation for its variation as well, so we take

$$\delta_E \partial X_a = \Lambda_a{}^b \partial X_b, \tag{7.64}$$

and so overall we find that

$$\delta_E \partial X_\mu(w) = \frac{1}{2}\delta g_{\mu\nu}\partial X^\nu(w), \tag{7.65}$$

i.e. the Lorentz transformations cancel, as we would expect for objects without frame indices. In general, we will usually ignore these Lorentz transformations and say, for example, that $\delta_E \partial X_a = 0$, since, by construction, they should always cancel for the spacetime operators we are interested in. A general frame object $\mathcal{A}_{a\ldots b}{}^{c\ldots d}$ would be expected to transform under δ_E as

$$\delta_E \mathcal{A}_{a\ldots b}{}^{c\ldots d} = \Lambda_a{}^e \mathcal{A}_{e\ldots b}{}^{c\ldots d} + \Lambda_b{}^e \mathcal{A}_{a\ldots e}{}^{c\ldots d} - \Lambda_e{}^c \mathcal{A}_{a\ldots b}{}^{e\ldots d} - \Lambda_e{}^d \mathcal{A}_{a\ldots b}{}^{c\ldots e}. \tag{7.66}$$

For the $\mathcal{O}$ insertion, the deformation operator is

$$\mathcal{O}[X] = \delta g_{\mu\nu}\int_\Sigma d^2z \partial X^\mu(z)\bar{\partial}X^\nu(\bar{z}), \tag{7.67}$$

and taking the OPE with $\partial X_\mu(w)$ gives

$$\begin{aligned} O[X]\partial X_\mu(w) &\sim -\frac{1}{2}\delta g_{\mu\nu}\int_{\Sigma'}\frac{d^2z}{(z-w)^2}\bar{\partial}X^\nu(\bar{z}) \\ &= -\frac{1}{2}\delta g_{\mu\nu}\oint_{C'}\frac{d\bar{z}}{z-w}\bar{\partial}X^\nu(\bar{z}). \end{aligned} \tag{7.68}$$

Thus, similarly to the $d = 1$ case, we find that

$$\delta(g_{\mu\nu}\alpha_n^\nu) = -\frac{1}{2}\delta g_{\mu\nu}\Big(\alpha_n^\nu + \epsilon^{2n}\bar{\alpha}_{-n}^\nu\Big). \tag{7.69}$$

If we are on the contour $|z| = \epsilon$ (see the discussion for the circle case for details), we can also deduce the transformation for $\partial X_\mu(z)$ from this, which is

$$\delta\partial X_\mu(z) = \frac{1}{2}\delta g_{\mu\nu}\left(\partial X^\nu(z) - \frac{\bar{z}}{z}\bar{\partial}X^\nu(\bar{z})\right), \tag{7.70}$$

agreeing with (7.47).

7.2.2 Deformations with a constant B-field

The doubled formalism [34, 11] provides an efficient way to generalise (7.61) in the presence of non-vanishing constant B-field. This will shed light on how to understand this in the 'undoubled' case, which we explain in detail in 7.2.3.

The embedding coordinate in the original (X^μ) and dual ($\tilde{X}_\mu$) descriptions are related as [12, 11, 20]

$$\partial \tilde{X}_\mu = -E^T_{\mu\nu} \partial X^\nu, \qquad \bar{\partial} \tilde{X}_\mu = E_{\mu\nu} \bar{\partial} X^\nu. \tag{7.71}$$

In terms of the $O(d,d)$-covariant doubled coordinate $\mathbb{X}^I = (X^\mu, \tilde{X}_\mu)$, the $O(d,d)$-invariant metric on the doubled space L_{IJ} is given by [12]

$$ds^2 = \frac{1}{2} L_{IJ} d\mathbb{X}^I d\mathbb{X}^J = dX^\mu d\tilde{X}_\mu, \tag{7.72}$$

where $\mu = 1, ..., d$. This is the metric that is used to raise/lower indices. Additionally, for a given background $E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}$, recall that we have the doubled metric

$$\mathcal{H}_{IJ} = \begin{pmatrix} g_{\mu\nu} - B_{\mu\rho} g^{\rho\lambda} B_{\lambda\nu} & B_{\mu\rho} g^{\rho\nu} \\ -g^{\mu\rho} B_{\rho\nu} & g^{\mu\nu} \end{pmatrix} \tag{7.73}$$

combining the metric and B-field into a single $O(d,d)$-covariant tensor. Thus, the case with B-field may be found by applying the methodology of section 7.2.1 to the doubled target space sigma-model

$$S = \frac{1}{4} \int_\Sigma \mathcal{H}_{IJ} d\mathbb{X}^I \wedge * d\mathbb{X}^J + \frac{1}{2} \int_\Sigma \Omega_{IJ} d\mathbb{X}^I \wedge d\mathbb{X}^J, \tag{7.74}$$

where Ω_{IJ} is the constant antisymmetric tensor we introduced in (3.52) and will play no further role[7]. The relations (7.71) can be derived from the self-duality constraints (3.53), which here simply become

$$d\mathbb{X}^I = L^{IJ} (\mathcal{H}_{JK} * d\mathbb{X}^K). \tag{7.75}$$

A deformation of the target space $\mathcal{H}_{IJ} \to \mathcal{H}_{IJ} + \delta\mathcal{H}_{IJ}$ is given by the marginal operator

$$\mathcal{O} = \frac{1}{2} \int_\Sigma \delta\mathcal{H}_{IJ} \partial \mathbb{X}^I \bar{\partial} \mathbb{X}^J. \tag{7.76}$$

[7] See [11] and [12] for a more general discussion of such terms

Note that L_{IJ} does not change. Using the undoubled OPEs, it is straightforward to verify that

$$\partial\mathbb{X}^I(z)\partial\mathbb{X}^J(w) \sim -\frac{1}{2}\frac{\mathcal{H}^{IJ} - L^{IJ}}{(z-w)^2}. \tag{7.77}$$

As in the previous section, we can write the doubled metric in terms of doubled vielbeins as

$$\mathcal{H}_{IJ} = \mathcal{V}_I{}^A \delta_{AB} \mathcal{V}^B{}_J, \tag{7.78}$$

where δ_{AB} is the doubled frame metric (which does *not* raise/lower frame indices, just as $\mathcal{H}$ does not raise/lower spacetime indices) and

$$\mathcal{V}_I{}^A = \begin{pmatrix} e^T & Be^{-1} \\ 0 & e^{-1}, \end{pmatrix}, \qquad \mathcal{V}^A{}_I = \begin{pmatrix} e & 0 \\ -e^{-T}B & e^{-T} \end{pmatrix}. \tag{7.79}$$

We want $\delta\partial\mathbb{X}_I = \delta(\mathcal{V}_I{}^A\partial\mathbb{X}_A)$. Given some deformation of the doubled metric $\delta\mathcal{H}_{IJ}$, we have the relations

$$\delta\mathcal{H}^{IJ} = -\mathcal{H}^{IK}\delta\mathcal{H}_{KL}\mathcal{H}^{LJ} = L^{IK}\delta\mathcal{H}_{KL}L^{LJ}, \tag{7.80}$$

and we can combine these two to obtain the often more useful identity $\mathcal{H}_{IJ}\delta\mathcal{H}^{JK} = -\delta\mathcal{H}_{IJ}\mathcal{H}^{JK}$. Note also that we can safely define such objects as $\delta\mathcal{H}_I{}^J = \delta\mathcal{H}_{IK}L^{KJ}$, since $\delta L = 0$. We also have

$$\delta\mathcal{V}_I{}^A = \frac{1}{2}\delta\mathcal{H}_{IJ}\mathcal{V}^J{}_B\delta^{BA} - \mathcal{V}_I{}^B\Lambda_B{}^A, \tag{7.81}$$

analogous to the undoubled case. Thus, the $\partial\mathbb{X}_I$ transformation is largely the same as in the undoubled case, except for the fact that there are two metrics, each playing a different role. We are interested in the OPE of $\mathcal{O}$, given by (7.76), and $\partial\mathbb{X}_I(w)$. Evaluating the OPE using (7.77) and comparing coefficients as we did for the circle deformation gives the doubled mode deformation as

$$\delta_{\mathcal{O}}(\mathcal{H}_{IJ}\mathbb{a}^J_n) = \frac{1}{4}\Big(\delta\mathcal{H}_{IJ} - \mathcal{H}^K{}_I\delta\mathcal{H}_{KJ}\Big)\,\epsilon^{2n}\bar{\mathbb{a}}^J_{-n}, \tag{7.82}$$

where we have introduced the mode expansion

$$\partial\mathbb{X}_I(z) = \frac{i}{\sqrt{2}}\sum_n \mathcal{H}_{IJ}\,\mathbb{a}^J_n\,z^{-n-1}, \tag{7.83}$$

and $\mathbb{a}^I_n = (\alpha^\mu_n, \tilde{\alpha}_{n\mu})$ are the doubled oscillator modes. Using the above identities, we can rewrite (7.82) as

$$\delta_{\mathcal{O}}(\mathcal{H}_{IJ}\mathbb{a}^J_n) = \frac{1}{2}\delta\mathcal{H}_{IJ}\epsilon^{2n}\bar{\mathbb{a}}^J_{-n}. \tag{7.84}$$

For the $\hat{\Gamma}$ connection with $|z| = 1$, we can then deduce the transformation of $\partial\mathbb{X}_I$ as

$$\delta_{\mathcal{O}}\partial\mathbb{X}_I(z) = \frac{1}{2}\frac{\bar{z}}{z}\delta\mathcal{H}_{IJ}\bar{\partial}\mathbb{X}^J(\bar{z}). \tag{7.85}$$

The $\delta_E\partial\mathbb{X}_I$ calculation also has a subtlety,

$$\delta_E\partial\mathbb{X}_I = \delta\mathcal{V}_I{}^A\partial\mathbb{X}_A = \frac{1}{2}\delta\mathcal{H}_{IJ}\mathcal{V}^J{}_B\delta^{BA}\partial\mathbb{X}_A. \tag{7.86}$$

It is not the case that $\mathcal{V}^J{}_B\delta^{BA}\partial\mathbb{X}_A = \partial\mathbb{X}^J$, since δ^{BA} does not raise/lower indices, since it is not the $O(d,d)$-invariant metric L. We have

$$\mathcal{V}^J{}_B\delta^{BA}\partial\mathbb{X}_A = \begin{pmatrix} e^\nu{}_b & 0 \\ B_{\nu\rho}e^\rho{}_b & e_\nu{}^b \end{pmatrix}\begin{pmatrix} \delta^{ba} & 0 \\ 0 & \delta_{ba} \end{pmatrix}\begin{pmatrix} \partial\tilde{X}_a \\ \partial X^a \end{pmatrix} = -\partial\mathbb{X}^J, \tag{7.87}$$

where we have used that $\partial\tilde{X}_\nu = -E^T_{\nu\rho}\partial X^\rho$ and $\partial\tilde{X}_a = -\partial X_a$ (since the background for the frame is just δ_{ab} and this raises/lowers indices in the undoubled case). Thus, overall, we find that

$$\delta\partial\mathbb{X}_I(z) = \frac{1}{2}\delta\mathcal{H}_{IJ}\left(-\partial\mathbb{X}^J(z) + \frac{\bar{z}}{z}\bar{\partial}\mathbb{X}^J(\bar{z})\right), \tag{7.88}$$

the components of which can be checked to recover the results of chapter 4.

7.2.3 The utility of the doubled formalism when $B \neq 0$

Once the effects of a metric deformation are understood, the doubled formalism provides an efficient way to generalise to include a B-field. In particular, the doubled formalism provides a natural way to incorporate the B-field deformation into the δ_E part of the transformation. In the undoubled case, we have

$$\delta_E\partial X^\mu = \delta e^\mu{}_a\partial X^a. \tag{7.89}$$

This clearly only gives the metric contribution and there does not seem to be any way to introduce the B-field. However, we recall that

$$\delta e^{\mu}{}_{a} = -\frac{1}{2} g^{\mu\nu} \delta g_{\nu\rho} e^{\rho}{}_{a} - e^{\mu}{}_{b} \Lambda^{b}{}_{a}, \tag{7.90}$$

where, importantly, Λ_{ab} is antisymmetric. If we now write this as

$$\begin{aligned}\delta e^{\mu}{}_{a} \partial X^{a} =& -\frac{1}{2} g^{\mu\nu} \delta g_{\nu\rho} e^{\rho}{}_{a} \partial X^{a} + \frac{1}{2} g^{\mu\nu} \delta B_{\nu\rho} \partial X^{\rho} - \frac{1}{2} g^{\mu\nu} \delta B_{\nu\rho} \partial X^{\rho} - e^{\mu}{}_{b} \Lambda^{b}{}_{a} \partial X^{a} \\ =& -\frac{1}{2} g^{\mu\nu} \delta g_{\nu\rho} e^{\rho}{}_{a} \partial X^{a} + \frac{1}{2} g^{\mu\nu} \delta B_{\nu\rho} \partial X^{\rho} - e^{\mu}{}_{b} \Lambda'^{b}{}_{a} \partial X^{a},\end{aligned} \tag{7.91}$$

where we have absorbed the $-\frac{1}{2} g^{\mu\nu} \delta B_{\nu\rho} \partial X^{\rho}$ term into the new frame transformation Λ', we see that we obtain the correct transformation. It is not obvious why we should do this. However, in the doubled formalism, this is exactly what happens, except it occurs naturally due to the $O(d,d)$ structure. If we compute the first term of (7.81) in components, we find that

$$\frac{1}{2} \delta \mathcal{H}_{IJ} \mathcal{V}^{J}{}_{B} \delta^{BA} = \frac{1}{2} \begin{pmatrix} \delta g_{\mu\nu} e^{\nu a} - B_{\mu\nu} g^{\nu\rho} \delta B_{\rho\sigma} e^{\sigma a} & \delta B_{\mu\nu} e^{\nu}{}_{a} - B_{\mu\nu} g^{\nu\rho} \delta g_{\rho\sigma} e^{\sigma}{}_{a} \\ -g^{\mu\nu} \delta B_{\nu\rho} e^{\rho a} & -g^{\mu\nu} \delta g_{\nu\rho} e^{\rho}{}_{a} \end{pmatrix}. \tag{7.92}$$

Since we want $\delta \partial X^{\mu}$, it is the second row we are interested in (since $\partial \mathbb{X}_I = (\partial \tilde{X}_{\mu}, \partial X^{\mu})$). We see that the bottom left entry gives us the additional contribution of

$$\frac{1}{2} g^{\mu\nu} \delta B_{\nu\rho} e^{\rho a} \partial X_{a}, \tag{7.93}$$

using $\partial \tilde{X}_a = -\partial X_a$. This is precisely the contribution that we included arbitrarily in (7.91), but now we see that it arises naturally in the doubled formalism. Thus, it seems as though the $O(d,d)$ structure is precisely the 'extra information' we need in order to recover (7.47). This is somewhat unsatisfactory though, since we intuitively expect that we can derive the correct results without having to resort to the doubled formalism. We will see next that this is indeed possible precisely in the case of the $\hat{\Gamma}$ connection, which is the context in which (7.47) is derived in [15], and that there is a close connection with universal coordinates.

7.2.4 Universal coordinates and doubled geometry

Let us look at the universality of X' in the $\hat{\Gamma}$ connection (we could also look at Π, but X' is simpler). We have $X'(z,\bar{z}) = -\frac{1}{2} g^{-1}(z\partial X(z) - z\bar{\partial} X(\bar{z}))$. Now, let us assume that

$$\delta e_\mu{}^a = \frac{1}{2}\delta g_{\mu\nu} g^{\nu\rho} e_\rho{}^a + \gamma \delta B_{\mu\nu} e^{\nu a}, \tag{7.94}$$

for some c-number γ. As we saw earlier, there is the potential to have such a B-field term since the B-field is antisymmetric. From the doubled treatment above, we saw that the extra B-field contribution to the δ_E transformation did not seem to naturally come from the vielbein transformation, but did arise naturally in the doubled formalism as a requirement of the $O(d,d)$-covariance of the doubled formalism. If we require that all (target space) scalar operators are killed by δ_E, this specifies the action of δ_E on the operators. Thus, we have

$$\delta_E \partial X_\mu = \frac{1}{2}\delta g_{\mu\nu}\partial X^\nu + \gamma\delta B_{\mu\nu}\partial X^\nu, \qquad \delta_E \bar\partial X_\mu = \frac{1}{2}\delta g_{\mu\nu}\bar\partial X^\nu + \bar\gamma\delta B_{\mu\nu}\bar\partial X^\nu, \tag{7.95}$$

where $\bar\gamma$ is *not* the complex conjugate of γ. Then, substituting this into X', we have

$$\begin{aligned}\delta X'^\mu(z,\bar z) &= \frac{1}{2}g^{\mu\rho}g^{\sigma\nu}\delta g_{\rho\sigma}(z\partial X_\nu(z) - \bar z\bar\partial X_\nu(\bar z)) - \frac{1}{2}g^{\mu\nu}\left(\frac{1}{2}\delta g_{\nu\rho} z\partial X^\rho(z) + \gamma\delta B_{\nu\rho} z\partial X^\rho(z)\right. \\ &\quad \left. -\frac{1}{2}\delta E_{\nu\rho}\bar z\bar\partial X^\rho(\bar z) - \frac{1}{2}\delta g_{\nu\rho}\bar z\bar\partial X^\rho(\bar z) - \bar\gamma\delta B_{\nu\rho}\bar z\bar\partial X^\rho(\bar z) + \frac{1}{2}\delta E^T_{\nu\rho} z\partial X^\rho(z)\right)\right) \\ &= \frac{1}{4}g^{\mu\nu}\delta B_{\nu\rho}\Big(z\partial X^\rho(z)(1-2\gamma) + \bar z\bar\partial X^\rho(\bar z)(1+2\bar\gamma)\Big),\end{aligned} \tag{7.96}$$

and so we require $\gamma = -\bar\gamma = \frac{1}{2}$ for X' to be universal. Thus, we see that the condition of universality is sufficient to give the correct B-field contributions to recover (7.47), (7.48). These are also precisely the contributions that arose naturally in the doubled geometry. This is as we would expect because universality and doubled geometry both preserve the $O(d,d)$ structure. With universality, this comes from the fact that the canonical commutation relations are preserved under parallel transport, and with doubled geometry the $O(d,d)$ structure is explicit in its construction. Given that the $\hat\Gamma$ connection seems to precisely correspond to the existence of universal coordinates, we thus conclude that the $\hat\Gamma$ connection preserves the natural $O(d,d)$-covariance of the embedding fields under parallel transport and gives the same deformation results as the doubled geometry.

It is instructive to look at the general case where we have not yet chosen a connection, i.e. we do not specify a regime for ϵ (such as keeping ϵ fixed and finite, or taking a limit $\epsilon \to 0$ with some counter-terms). We find that

$$\delta\alpha^\mu_n = -\frac{1}{2}g^{\mu\nu}\Big((\delta g_{\nu\rho} - 2\gamma\delta B_{\nu\rho})\alpha^\rho_n + \epsilon^{2n}\delta E_{\nu\rho}\bar\alpha^\rho_{-n}\Big). \tag{7.97}$$

Now, we would like to compare this to the doubled geometry. In the doubled case, we have doubled oscillator modes $\mathbb{a}^I_n = (\alpha^\mu_n, \tilde{\alpha}_{n\mu})$, and the self-duality constraints give us

$$\tilde{\alpha}_{n\mu} = -E^T_{\mu\nu}\alpha^\nu_n, \qquad \tilde{\bar{\alpha}}_{n\mu} = E_{\mu\nu}\bar{\alpha}^\nu_n. \tag{7.98}$$

From our earlier calculations, we have

$$\delta_{\mathcal{O}}\mathbb{a}^I_n = \frac{1}{2}\epsilon^{2n}\delta\mathcal{H}^I_J\bar{\mathbb{a}}^J_{-n}, \qquad \delta_E\mathbb{a}^I_n = -\frac{1}{2}\delta\mathcal{H}^I_J\mathbb{a}^J_n, \tag{7.99}$$

where we have used $\delta\mathcal{V}^I{}_A \sim -\frac{1}{2}\mathcal{H}^{IJ}\delta\mathcal{H}_{JK}\mathcal{V}^K{}_A$. Thus, by taking the appropriate components and using the self-duality constraints, we get

$$\delta_E\alpha^\mu_n = -\frac{1}{2}g^{\mu\nu}(\delta g_{\nu\rho} - \delta B_{\nu\rho})\alpha^\rho_n. \tag{7.100}$$

Thus, we see that, for agreement with the doubled geometry, we need to set $\gamma = \frac{1}{2}$, and this is independent of connection. Similarly, we would find that $\bar{\gamma} = -\frac{1}{2}$ by looking at $\bar{\alpha}_n$.

Thus, in general we require $\gamma = -\bar{\gamma} = \frac{1}{2}$ to preserve the $O(d,d)$ structure, but it is only for the $\hat{\Gamma}$ connection that these values are required, and in particular it is the existence of universal coordinates which fixes them. As discussed earlier, universal coordinates are special to the $\hat{\Gamma}$ connection and in general we do not have this additional structure. For a general connection, there is some freedom to choose how the B-field enters into the transformation, but if the $O(d,d)$ structure is in place then the symmetry between the metric and B-field removes this freedom.

Chapter 8

Nonlinear Sigma Models and Off-shell Deformations

So far, our discussion of operator deformations using $\mathcal{O}$ and δ_E have been in the context of genuine CFTs only and the calculations have all been fairly tractable. However, we now once again look at the torus bundle examples of chapter 3, where we no longer have conformal invariance, though these backgrounds can be used as building blocks for honest string backgrounds[1]. The difficulty of working with toy models which are not full string theory solutions is that we can no longer rely on worldsheet conformal invariance and we are forced to consider off-shell correlation functions. The difference between what we did in chapter 6 and what we will do now is that here we take the x-dependence of the metrics into account when pulled back to the worldsheet, i.e. we now move away from the adiabatic limit. This is something that has been largely ignored in the literature, in part due to the computational difficulty of the problem.

The approach taken in String Field Theory will be our guide. The worldsheet theory will be taken to be Weyl-invariant, allowing for the local decoupling of worldsheet metric degrees of freedom, but at the cost of a loss in diffeomorphism invariance. The ghost sector will not be changed under parallel transport and will not concern us further. Instead of dealing with a complicated worldsheet metric, we imagine a local coordinate system w_i around each puncture. The coordinates around each puncture are related to a reference coordinate system z, with respect to which any integration may be done, by functions $z = f_i(w_i)$. It is conventional to choose the locations of the punctures as the origins of the local coordinate systems, so that $z_i := f_i(0)$. The details of the metric on the worldsheet Σ are then encoded in the set of functions f_i.

[1] See [52] for a recent example.

On-shell correlation functions are independent of the choice of f_i and may be written as the familiar integral over the moduli space of punctured Riemann surfaces $\mathcal{M}_{g,n}$,

$$\int_{\mathcal{M}_{g,n}} \mu_{g,n} \Big\langle \Phi_1(z_1)...\Phi_n(z_n) \Big\rangle, \tag{8.1}$$

where $\mu_{g,n}$ is the usual measure on $\mathcal{M}_{g,n}$ built from ghosts and Beltrami differentials. More generally, off-shell correlation functions will depend on the choice of f_i. One way to address this [37, 74] is to replace the usual integral over $\mathcal{M}_{g,n}$ with an integral over $\mathcal{S}_{g,n} \subset \mathcal{P}_{g,n}$. Here $\mathcal{P}_{g,n}$ is the infinite-dimensional bundle with finite-dimensional base $\mathcal{M}_{g,n}$ and infinite-dimensional fibres describing the possible choices of local coordinate about each puncture. $\mathcal{S}_{g,n}$ is a section of $\mathcal{P}_{g,n}$ with the same dimension as $\mathcal{M}_{g,n}$. If we employ this construction for an on-shell correlation function, the choice of section $\mathcal{S}_{g,n}$ does not matter and we recover the standard prescription.

In practice, we will only be interested in following the deformation of a small number of insertions (typically one at the point $z = w$ and another, a spectator field, at the origin) and so the details of the maps $z = f_i(w_i)$ and which section of $\mathcal{P}_{g,n}$ we are working with will not be of immediate concern.

We will only be interested in deformations that preserve $L_0^- = L_0 - \bar{L}_0$, the generator of rotations on the worldsheet[2]. This is true for CFT cases [22] and we have shown in appendix F that this is true for the constant H-flux background we consider here[3]. Even though we shall often discuss the transformation of operators that are not in the kernel of L_0^-, such as ∂X^μ, ultimately one would be interested in operators which would be taken to lie in the kernel of L_0^-, as they would have a more direct relevance for physical states.

In these more general cases where we are considering non-CFT deformations, we no longer necessarily have the luxury of universal coordinates. As before, it will prove helpful in this context to be more explicit about the objects at the self-dual point. In particular, we will denote embedding fields of the theory before deformation by ϕ^μ (usually a free theory at some point of enhanced symmetry) and those of the deformed theory by the more traditional X^μ. However, note that the theory with embedding field ϕ^μ may not simply be the T^d with background $E = G$, since, as we will shortly, we will usually work on the cover where ϕ^x is decompactified, so the target space will be $\mathbb{R} \times T^{d-1}$. This is more of a technical point and will not have any effect on

[2] $L_0^- = 0$ need only be preserved modulo gauge.

[3] Neglecting the effect of terms not invariant under rotations was a property built in to the original formalism of [23, 24]. Here it arises naturally from the vanishing of L_0^-.

the calculations themselves, but, as we now explain, it means we can avoid issues of topology change.

We will have in mind a worldsheet Σ of genus zero, but we expect our considerations to generalise to higher genus.

A note on topology change

For much of this section, our starting point will be a T^3 with one or more circles tuned to the self-dual radius $R = \sqrt{\alpha'}$ and a free worldsheet CFT describing the embedding into this background. The backgrounds of interest will include target spaces with constant curvature or constant H-field, described by interacting worldsheet theories. In principle, one would have to contend with a change in the topology of the spacetime (or in the doubled space) when switching on such constant fluxes. We shall sidestep this issue by working in a covering space (such as $T^2 \times \mathbb{R}$), where the deformation can be smoothly turned on, and then an identification on the coordinates may be imposed to recover the desired compact background[4].

8.1 H-flux deformation

We start with the T^3 with constant H-field $H_{\mu\nu\rho}$. This background may be thought of as a T^2 bundle over S^1 in which the B-field in the fibres undergoes a large gauge transformation upon circumnavigating the base. In this case, we choose to work in the cover $T^2 \times \mathbb{R}$ where an identification on the base coordinate is taken to be imposed at a later point. The action is

$$S[X] = \frac{1}{2} \int_\Sigma d^2 z \partial X_\mu(z) \bar{\partial} X^\mu(\bar{z}) + \int_V H + ..., \tag{8.2}$$

where $\partial V = \Sigma$ and the ellipsis denotes ghosts and other terms that will not be relevant to our discussion. The classical equation of motion is

$$\partial \bar{\partial} X^\mu(z, \bar{z}) + H^\mu{}_{\nu\lambda} \partial X^\nu(z, \bar{z}) \bar{\partial} X^\lambda(z, \bar{z}) = 0. \tag{8.3}$$

Integrating gives

$$X^\mu(z, \bar{z}) = \phi^\mu(z, \bar{z}) - H^\mu{}_{\nu\lambda} \int_\Sigma d^2 w\, G(z, w) \partial X^\nu(w) \bar{\partial} X^\lambda(\bar{w}), \tag{8.4}$$

[4]This is similar in spirit to the perspective often taken in discussions of the T-fold [12].

where $\phi^\mu(z,\bar{z}) \in \mathrm{Ker}(\partial\bar{\partial})$, i.e. is a solution on the torus with $H = 0$, and $G(z,w) \sim -\ln|z-w|^2$ is the Green's function for $\partial\bar{\partial}$ at genus 0. In the specific examples below, we will introduce λ such that $H = \lambda * 1_3$. From this classical consideration the deformation of $\partial\phi_\mu(z)$ at first order is

$$\partial\phi_\mu(z) \to \partial X_\mu(z) = \partial\phi_\mu(z) + \lambda\epsilon_{\mu\nu\lambda}\int_\Sigma \frac{d^2w}{z-w}\partial\phi^\nu(w)\bar{\partial}\phi^\lambda(\bar{w}) + ... \tag{8.5}$$

If we put the α' dependence back in, we find that this simply corresponds to $\alpha' \to \alpha'\lambda^2$. It is significant that α' and λ^2 appear together[5] here and we will comment upon this further in section 8.6. At higher order in λ, the full quantum calculation will include higher order contractions not given by classical considerations, but at leading order, where only single contractions contribute, we expect the classical considerations to be reliable. We shall see that this is true.

If one takes the base coordinate $X(z,\bar{z}) = x + ...$, as given by (2.17), and only considers the constant piece x, the deformation operator may be written in the form

$$\mathcal{O} = \frac{1}{3}H_{\mu\nu\rho}x^\rho\int_\Sigma d^2z\partial\phi^\mu(z)\bar{\partial}\phi^\nu(\bar{z}) + ..., \tag{8.6}$$

and we are in the adiabatic (free CFT) regime studied in chapter 6. Such deformations fall into the class of toroidal deformations considered in chapter 7. We now turn to determining the leading corrections to this adiabatic approximation.

8.1.1 Deformation at first order

As usual, we have a T^2 with coordinates (Y,Z) fibred over $\mathbb{R}$ with coordinate X and an identification $X \sim X + 2\pi$ later imposed. For this background, the important information appears as a large gauge transformation monodromy of the B-field in the fibres as $X \to X + 2\pi$.[6] We usually choose the gauge such that

$$g = dx^2 + dy^2 + dz^2, \qquad B = mxdy \wedge dz, \tag{8.7}$$

where $m \in \mathbb{Z}$, and the deformation operator, given by the pullback of the B-field to the worldsheet, is

$$\mathcal{O}[\phi^x] = m\int_\Sigma d^2z\ \phi^x(z,\bar{z})F^-_{yz}(z,\bar{z}), \tag{8.8}$$

[5] To see this one must redefine the fields as $X^\mu \to \lambda^{-1}X^\mu$.

[6] Note that, apart from the circle case, we will always use X^μ to refer to a general coordinate component, so there should not be any confusion in calling the first component X.

where we have introduced

$$F^{\pm}_{\mu\nu}(z,\bar{z}) \equiv \partial\phi_\mu(z)\bar{\partial}\phi_\nu(\bar{z}) \pm \partial\phi_\nu(z)\bar{\partial}\phi_\mu(\bar{z}). \tag{8.9}$$

Calculation of δ_E

For the H-flux, since we have a non-zero B-field, we must evaluate the δ_E transformation using one of the approaches described earlier. Since we have already gone through the doubled geometry derivation, it is easiest to just plug in the H-flux background into the result we derived, which we recall is

$$\delta_E \partial \mathbb{X}_I = -\frac{1}{2}\mathcal{H}_{IJ}\partial\mathbb{X}^J. \tag{8.10}$$

Doing so, we find that

$$\delta_E\partial\phi_x = 0, \qquad \delta_E\partial\phi_y = \frac{1}{2}m\phi^x\partial\phi_z, \qquad \delta_E\partial\phi_z = -\frac{1}{2}m\phi^x\partial\phi_y. \tag{8.11}$$

We have the δ_E transformation, but if we want to compute the mode transformation we would like to write this in integral form. The reason for this is that, in order to read off the mode transformation $\delta\alpha^y_n$, we would like to have an expression of the form $\sum_m w^{-m-1}f_m$, where f_m is some expression in terms of the modes, which can then be read off as the deformation of α^y_n, as we did for the stress tensor deformation (5.38). Such expressions are most easily obtained from integral expressions like the ones we have seen already (from expanding $(z-w)^{-1}$ in powers of z). We can show (details in appendix D) that

$$\delta_E\phi_y(w) = \frac{1}{2}m\int\limits_{\Sigma'}\frac{d^2z}{z-w}\bar{\partial}\phi_x(\bar{z})\partial\phi_z(z) - \frac{1}{2}m\oint\limits_{C_0'}\frac{dz}{z-w}\phi^x(z,\bar{z})\partial\phi_z(z), \tag{8.12}$$

where we define

$$\int\limits_{\Sigma'} = \lim_{\epsilon\to 0}\int\limits_{\Sigma^\epsilon}, \qquad \int\limits_{C_0'} = \lim_{\epsilon\to 0}\int\limits_{C_0^\epsilon} \tag{8.13}$$

A similar result follows for $\delta_E\partial\phi_z$. We can then use this integral representation to compute the mode deformation. The calculation is fairly involved and the details are in appendix D.

Calculation of $\delta_{\mathcal{O}}$

The deformation operator, that part of the transformation which changes the action, is given by (8.8). Before proceeding, we need to think about whether this object makes sense as it stands. If we require ϕ^x to be a compact direction, there will be a branch point on the worldsheet whenever the string wraps the ϕ^x direction. This is manifest in the periodicity condition

$$\phi^x(z,\bar{z}) \sim \phi^x\left(e^{2\pi i}z, e^{-2\pi i}\bar{z}\right) = \phi^x(z,\bar{z}) + 2\pi\omega, \tag{8.14}$$

where $\omega \in \mathbb{Z}$ is the winding number around the ϕ^x direction. This leads to an ambiguity $\mathcal{O} \sim \mathcal{O} + \Delta\mathcal{O}$, where

$$\Delta\mathcal{O} = 2\pi wm \int_{\Sigma} d^2z\ F^-_{yz}(z,\bar{z}). \tag{8.15}$$

There is no problem here though. $\Delta\mathcal{O}$ generates a large gauge transformation of the B-field and is a symmetry of the theory [45]. Thus, the ambiguity simply reflects the fact that the B-field is only locally defined, as the presence of the non-trivial H field strength indicates. We shall see a more general way to deal with such symmetries in section 8.5. For now, we shall assume our definition of the connection is augmented to include an appropriate gauge transformation to account for the branch point. Alternatively (and in practice), we could work in the cover (where ϕ^x is not compact), deform our theory and then impose the relevant identification on ϕ^x in the new background.

Contraction of $\mathcal{O}$ with $\partial\phi_y(w)$ gives

$$\delta_{\mathcal{O}}\partial\phi_y(w) = -\frac{1}{2}m\int_{\Sigma'} d^2z\frac{1}{(z-w)^2}\ \phi^x(z,\bar{z})\bar{\partial}\phi_z(\bar{z}), \tag{8.16}$$

which we can more conveniently write as

$$\delta_{\mathcal{O}}\partial\phi_y(w) = -\frac{1}{2}m\int_{\Sigma'} \frac{d^2z}{z-w}\ \partial\phi_x(z)\bar{\partial}\phi_z(\bar{z}) - \frac{1}{2}m\oint_{C'_0} \frac{d\bar{z}}{z-w}\ \phi^x(z,\bar{z})\bar{\partial}\phi_z(\bar{z}), \tag{8.17}$$

where we have dropped the C'_w integral since this can be shown to vanish. The first order change in the modes is then given by[7]

$$\delta_{\mathcal{O}}\alpha^y_n = \frac{i}{\sqrt{2}}m\int_{\Sigma'} d^2z\ z^n\partial\phi_x(z)\bar{\partial}\phi_z(\bar{z}) + \frac{i}{\sqrt{2}}m\oint_{C'_0} d\bar{z}\ z^n\phi^x(z,\bar{z})\bar{\partial}\phi_z(\bar{z}) + ..., \tag{8.18}$$

[7] Using $(z-w)^{-1} = -\sum_{n\geq 0} w^{-n-1}z^n$ when $|z| < |w|$.

where the ellipsis denotes divergent terms, which are dealt with according to the choice of connection. As with δ_E, the details of this mode calculation are given in appendix D. Similar results also follow for $\delta\partial\phi_x$ and $\delta\partial\phi_z$.

The first order deformation

Putting the $\mathcal{O}$ and δ_E parts together gives the first order changes (evaluated on a contour $|w|$=constant)[8]

$$\delta\partial\phi_x(w) = \frac{1}{2}m\int_{\Sigma'}\frac{d^2z}{z-w}\,F^-_{yz}(z,\bar{z}), \tag{8.19}$$

$$\delta\partial\phi_y(w) = \frac{1}{2}m\int_{\Sigma'}\frac{d^2z}{z-w}\,F^-_{zx}(z,\bar{z}) - \frac{1}{2}m\oint_{C'_0}\frac{\phi^x(z,\bar{z})d\phi_z(z,\bar{z})}{z-w}, \tag{8.20}$$

$$\delta\partial\phi_z(w) = \frac{1}{2}m\int_{\Sigma'}\frac{d^2z}{z-w}\,F^-_{xy}(z,\bar{z}) + \frac{1}{2}m\oint_{C'_0}\frac{\phi^x(z,\bar{z})d\phi_y(z,\bar{z})}{z-w}, \tag{8.21}$$

where we have written $d\phi_z(z,\bar{z}) = dz\partial\phi_z(z) + d\bar{z}\bar{\partial}\phi_z(\bar{z}), d\phi_y(z,\bar{z}) = dz\partial\phi_y(z) + d\bar{z}\bar{\partial}\phi_y(\bar{z})$. In terms of modes, the first order deformation for α^x_n is

$$\delta\alpha^x_n = -\frac{1}{2}m\int_{\Sigma'} d^2z\, z^n\, F^-_{yz}(z,\bar{z}), \tag{8.22}$$

with the mode transformations of α^y_n and α^z_n taking a similar form. With a little work, the integrals can be done and the ϵ dependence made explicit. This is done in detail in appendix D, where it is shown that the deformation in the $\partial\phi_y$ modes may be written as

$$\delta_{\mathcal{O}}\alpha^y_n = x\mathcal{A}_n\bar{\alpha}^z_{-n} + \sum_p \mathcal{B}_{np}\bar{\alpha}^z_p\alpha^x_{n+p} + \sum_p \mathcal{C}_{np}\bar{\alpha}^z_p\bar{\alpha}^x_{-n-p}, \qquad n \geq 0, \tag{8.23}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are ϵ-dependent constants. Only the first term contributes in the adiabatic limit. These expressions give some insight into how turning on the H-field deforms the free field algebra of operators.

[8] Note that we could explicitly extract the corrections of the adiabatic limit if we made the radius of the base explicit, along the lines of the discussion around (6.11), though here we fix the base radius to 1 for simplicity.

8.1.2 Branch points and gauge-invariance

A few comments are in order. We claim that the second, branch-dependent, terms in (8.20) and (8.21) are gauge-dependent. Note that, in this gauge, $\partial\phi_x$ is in the kernel of δ_E. This would not necessarily be the case had we chosen to work in a different gauge and is a reflection of the fact that δ_E preserves the classical action. Had we chosen to work in the gauge where $B = mydz \wedge dx$, then $\partial\phi_y$ would have been in the kernel of δ_E. The difference between these two gauges is the large gauge transformation $B \to B + \Lambda$, where $\Lambda = \frac{1}{2}md(xydz)$. In terms of the worldsheet, this is generated by

$$\Lambda[\phi^x, \phi^y] = \frac{1}{2}m \oint_{\partial\Sigma} \phi^x \phi^y d\phi_z. \tag{8.24}$$

Contracting $\Lambda[\phi^x, \phi^y]$ with $\partial\phi_y(w)$ gives the second term in (8.20), suggesting this term really is a gauge artifact.

There is further evidence for this if we look at the solution obtained from considering the classical equation of motion (8.3), which for Y is

$$\partial\bar{\partial}Y = -\frac{1}{2}m(\partial X\bar{\partial}Z - \partial Z\bar{\partial}X). \tag{8.25}$$

Given a solution ϕ_y to the flat equation of motion $\partial\bar{\partial}\phi_y = 0$, this can be solved iteratively. Doing so to first order gives

$$\delta\partial\phi_y(w) = -\frac{1}{2}m \int \frac{d^2z}{z-w} F^-_{xz}(z,\bar{z}), \tag{8.26}$$

where $\partial Y = \partial\phi_y + \delta\partial\phi_y + O(m^2)$. This can be seen, for example, by noting that $\bar{\partial}\left(\frac{1}{z-w}\right) = \delta^2(z-w)$. Thus, up to the branch-dependent terms, $(\delta_{\mathcal{O}} + \delta_E)\partial\phi_y$ is of the same form as the classical result, which is what we would expect at first order, and further suggests that the branch-dependent term is a gauge artifact. At higher order, we expect our formalism to give the full quantum corrections and so the results will differ from the classical case.

8.1.3 Absence of universal coordinates

Throughout, we have mentioned how the existence of universal coordinates is specific to CFTs and is intimately related to the $\hat{\Gamma}$ connection. The deformations we derived above for the H-flux were in the c connection, so we do not expect there to be any universal coordinate construction there anyway (see section 7.1.4). However, what about when we use the $\hat{\Gamma}$ connection? As mentioned, the $\hat{\Gamma}$ connection is only naturally defined in a CFT context, since we can always dilate the discs around the punctures to arbitrary radius. However, it is still worth seeing explicitly the lack of existence of universal coordinates in off-shell cases. We will take the deformation of $\partial\phi_x$, since this is simplest and the form of the deformation is exactly the same in any connection, the only difference being the integration domain. For the $\hat{\Gamma}$ connection, we have

$$\delta\partial\phi_x(w) = \frac{1}{2}m \int\limits_{\Sigma-(\mathcal{D}_0^1\cup\mathcal{D}_w^1)} \frac{d^2z}{z-w}\, F^-_{yz}(z,\bar{z}). \tag{8.27}$$

Note that, in the CFT case, we took the disc around w to have radius $\epsilon \to 0$, which we had the freedom to do due to conformal symmetry. Here however, we do not have this freedom and so we fix the radius of the disc around w to be 1. Now, recall that we have

$$X'^{\mu}(E) = -ig^{\mu\nu}\left(z\partial X_\nu(E) - \bar{z}\bar{\partial}X_\nu(E)\right), \tag{8.28}$$

and that this is universal for a CFT with the $\hat{\Gamma}$ connection. Given (8.27) and the corresponding transformation for $\bar{\partial}\phi_x$,[9]

$$\delta\bar{\partial}\phi_x(\bar{w}) = \frac{1}{2}m \int\limits_{\Sigma-(\mathcal{D}_0^1\cup\mathcal{D}_w^1)} \frac{d^2z}{\bar{z}-\bar{w}}\, F^-_{yz}(z,\bar{z}), \tag{8.29}$$

we have

$$\begin{aligned}\delta X'(G)(w,\bar{w}) &= -i(w\partial\phi_x(w) - \bar{w}\delta\bar{\partial}\phi_x(\bar{w}))\\ &= -\frac{im}{2}\int\limits_{\Sigma-(\mathcal{D}_0^1\cup\mathcal{D}_w^1)} d^2z F^-_{yz}(w,\bar{w})\left(\frac{w}{z-w} + \frac{\bar{w}}{\bar{z}-\bar{w}}\right).\end{aligned} \tag{8.30}$$

Clearly, this does not vanish, and so X' is not universal. Similar calculations can be done for Y' and Z', as well as Π_μ. Thus, as claimed, we no longer have universal coordinates for these off-shell deformations.

8.1.4 Higher order contributions

In the adiabatic approximation, the deformation of $\partial\phi_\mu$ truncates at first order in m, since the H-flux deformation δE is first order in m. However, if we are taking the X-dependence in the B-field into account, we expect that there will be stringy

[9]We have used that, under complex conjugation, $dz \wedge d\bar{z} \to -dz \wedge d\bar{z}$.

corrections at all orders in m. Therefore, it is of interest to find a way to calculate these corrections. In the CFT case, i.e. when $\mathcal{O}$ has no ϕ^x-dependence, we described a way to systematically generalise the first order procedure to all orders in section 7.1.3 and in more detail in appendix E. However, when there is ϕ^x-dependence in $\mathcal{O}$, this procedure would need significant modification. Nevertheless, this is still a starting point to consider the challenges that arise at higher order. For example, at second order, if we are looking at $\delta\partial\phi_y(w)$, we would be interested in the integral

$$\mathcal{O}_2\mathcal{O}_1\partial\phi_y(w) = \int\limits_{\Sigma_2} d^2z_2 \int\limits_{\Sigma_1} d^2z_1 O(z_2,\bar{z}_2)O(z_1,\bar{z}_1)\partial\phi_y(w), \tag{8.31}$$

where

$$\mathcal{O}_i = m\int\limits_{\Sigma_i} d^2z \phi^x(z,\bar{z})F^-_{yz}(z,\bar{z}), \tag{8.32}$$

with Σ_i given by (7.30). Naively following the prescription used in the free field case, we would first contract $\partial\phi_y(w)$ with $O(z_1,\bar{z}_1)$, and then contract the result with $O(z_2,\bar{z}_2)$, i.e. we would have the sequential contractions

$$\begin{aligned}
&\delta_{\mathcal{O}_2}\Big(\delta_{\mathcal{O}_1}\Big(\partial\phi_y(w)\Big)\Big)\\
&= -\frac{m^2}{2}\int\limits_{\Sigma_2} d^2z_2 \int\limits_{\Sigma_1} \frac{d^2z_1}{(z_1-w)^2}\phi^x(z_2,\bar{z}_2)\Big(\partial\phi_y(z_2)\bar{\partial}\phi_z(\bar{z}_2) - \partial\phi_z(z_2)\bar{\partial}\phi_y(\bar{z}_2)\Big)\phi^x(z_1,\bar{z}_1)\bar{\partial}\phi_z(\bar{z}_1),\\
&= \frac{m^2}{4}\int\limits_{\Sigma_2} d^2z_2 \int\limits_{\Sigma_1} \frac{d^2z_1}{(z_1-w)^2}\quad\Bigg(F^-_{yz}(z_2,\bar{z}_2)\bar{\partial}\phi_z(\bar{z}_1)\log|z_1-z_2|^2\\
&\qquad\qquad + \frac{\phi^x(z_2,\bar{z}_2)\partial\phi_y(z_2)\phi^x(z_1,\bar{z}_1)}{(\bar{z}_1-\bar{z}_2)^2} - \frac{1}{2}\frac{\log|z_1-z_2|^2}{(\bar{z}_1-\bar{z}_2)^2}\partial\phi_y(z_2)\Bigg).
\end{aligned} \tag{8.33}$$

We could then compute this integral to obtain the result. Note that here we have included both the single and double contractions between the z_1 and z_2 fields, though the double contraction term is most likely a divergent term that we would remove via a regularisation procedure. In appendix E, we show that, as expected, the deformation resulting from $(\mathcal{O}+\delta_E)^2$ vanishes for $\partial\phi_\mu$ when we have a constant background E. However, in this case, since there is X-dependence in E, we expect in general that the same cancellations will not occur and that there will be second (and higher) order corrections. As we can see from the above integral, the ϕ^x-dependence is explicit and makes the calculation significantly more complicated, and will undoubtedly introduce new terms.

If we looked at higher order contractions, what we would see is that, in addition to the terms that we would ordinarily get without the ϕ^x-dependence, we also get additional terms coming from the ϕ^x contractions. The contractions which do not involve ϕ^x follow the expected pattern, e.g. at third order, we have the contractions

$$\begin{aligned}
\delta_{\mathcal{O}_3}\left(\delta_{\mathcal{O}_2}\left(\delta_{\mathcal{O}_1}\left(\partial\phi_y(w)\right)\right)\right) = \frac{m^2}{4}\int_{\Sigma_3} d^2z_3 \int_{\Sigma_2} d^2z_2 \int_{\Sigma_1} \frac{d^2z_1}{(z_1-w)^2}\Bigg[\\
&\phi^x(z_3,\bar z_3)\partial\phi_y(z_3)\bar\partial\phi_z(\bar z_3)\partial\phi_y(z_2)\bar\partial\phi_z(\bar z_2)\bar\partial\phi_z(\bar z_1)\log|z_1-z_2|^2 \\
&+\Big(\phi^x(z_3,\bar z_3)\partial\phi_y(z_3)\bar\partial\phi_z(\bar z_3)\partial\phi_y(z_2)\bar\partial\phi_z(\bar z_2)\bar\partial\phi_z(\bar z_1) \\
&+\phi^x(z_3,\bar z_3)\partial\phi_z(z_3)\bar\partial\phi_y(\bar z_3)\partial\phi_z(z_2)\bar\partial\phi_y(\bar z_2)\bar\partial\phi_z(\bar z_1)\Big)\log|z_1-z_2|^2 \\
&+\frac{1}{(\bar z_1-\bar z_2)^2}\Big(\phi^x(z_3,\bar z_3)\partial\phi_y(z_3)\bar\partial\phi_z(\bar z_3)\phi^x(z_2,\bar z_2)\partial\phi_y(z_2)\phi^x(z_1,\bar z_1) \\
&+\phi^x(z_3,\bar z_3)\partial\phi_y(z_3)\bar\partial\phi_z(\bar z_3)\phi^x(z_2,\bar z_2)\partial\phi_y(z_2)\phi^x(z_1,\bar z_1)\Big) \\
&-\frac{\log|z_1-z_2|^2}{2(\bar z_1-\bar z_2)^2}\phi^x(z_3,\bar z_3)\partial\phi_y(z_3)\bar\partial\phi_z(\bar z_3)\partial\phi_y(z_2)\Bigg],
\end{aligned} \tag{8.34}$$

and the term that we would have in the flat torus case would be

$$\phi^x(z_3,\bar z_3)\partial\phi_y(z_3)\bar\partial\phi_z(\bar z_3)\phi^x(z_2,\bar z_2)\partial\phi_y(z_2)\phi^x(z_1,\bar z_1) = -\frac{\bar\partial\phi_z(\bar z_3)\phi^x(z_3,\bar z_3)\phi^x(z_2,\bar z_2)\phi^x(z_1,\bar z_1)}{2(z_3-z_2)^2}, \tag{8.35}$$

whereas all other contractions would involve log terms from the '$\phi^x\phi^x$' contractions. Of course, when it comes to doing the integration, the ϕ^x-dependence will have a significant effect here as well, so even terms which would still be there in the constant background case may give extra contributions.

These calculations make it clear that it is better to make a specific choice of connection, one designed to reduce the complexity of the calculations from the start. As discussed above, the presence of the branch points is symptomatic of winding modes around the base, and either working in the cover or incorporating an appropriate large gauge transformation in the definition of the connection should deal with such terms. It is unlikely that the calculation can be systematised in the way achieved for the CFT deformations, as $\mathcal{O}$ can no longer be thought of as a map that preserves the subspace spanned by $\partial\phi$ and $\bar\partial\phi$. Instead, the $\mathcal{O}$ considered here mixes in other operators. It would be interesting to see if there is a relatively simple subspace that $\mathcal{O}$ preserves. It is also clear that any candidate for a universal coordinate for this deformation would need to be a more general vector within this subspace than that considered in [15].

8.2 Nilfold deformation

As a second example, we consider the T-dual of the H-flux, the nilfold. As above, we shall take the base to be $\mathbb{R}$ and then impose an identification in the coordinates. We shall see agreement at first order with the classical result here as well. We shall investigate the T-duality between the H-flux and the nilfold in chapter 9 from our parallel transport perspective.

The metric is

$$ds^2 = dx^2 + (dy - mx dz)^2 + dz^2. \tag{8.36}$$

The nilfold is usually taken to be compact, with identifications on all coordinates. For clarity, we lift to the cover (the three-dimensional Heisenberg group manifold mentioned in chapter 3), perform the deformation, and then impose the appropriate identifications on the coordinates.

The deformation operator is[10]

$$\mathcal{O}[\phi^x] = -m \int_\Sigma \phi^x F^+_{yz} + m^2 \int_\Sigma (\phi^x)^2 \partial\phi_z \bar{\partial}\phi_z. \tag{8.37}$$

As before, we focus on one of the coordinates. We shall only consider the first order deformation and so neglect the m^2 term. The worldsheet equation of motion for ∂Y for the nilfold is given by

$$\partial\bar{\partial}Y = \frac{1}{2}m(\partial X \bar{\partial}Z + \partial Z \bar{\partial}X) + O(m^2), \tag{8.38}$$

and integrating out the $\bar{\partial}$ gives

$$\delta\partial\phi_y(w) = - m\phi^x(w,\bar{w})\partial\phi_z(w) + \frac{1}{2}m \int \frac{d^2z}{z-w} F^+_{xz}(z,\bar{z}) = \frac{1}{2}m \int \frac{d^2z}{z-w} F^-_{xz}(z,\bar{z}), \tag{8.39}$$

where we have used $\phi^x(w,\bar{w})\partial\phi_z(w) = \int d^2z \bar{\partial}\left(\frac{1}{z-w}\right)\phi^x(z,\bar{z})\partial\phi_z(z)$. We will reproduce this up to contour integrals using our formalism. The calculations are qualitatively the

[10]Unlike the previous example, there is a quadratic part to $\mathcal{O}$ in which normal ordering of $(\phi^x)^2$ is understood.

same as for the H-flux, so we will not give much detail. We have:

$$\delta_{\mathcal{O}}\partial\phi_y(w) = \frac{1}{2}m\int_{\Sigma'}\frac{d^2z}{z-w}\partial\phi_x(z,\bar{z})\bar{\partial}\phi_z(z) + \frac{1}{2}\oint_{C_0'}\frac{d\bar{z}}{z-w}\phi^x(z,\bar{z})\bar{\partial}\phi_z(\bar{z}), \tag{8.40}$$

$$\begin{aligned}\delta_E\partial\phi_y(w) &= -\frac{1}{2}m\phi^x(w,\bar{w})\partial\phi_z(w)\\ &= \frac{1}{2}m\oint_{C_0'}\frac{dz}{z-w}\phi^x(z,\bar{z})\partial\phi_z(z) - \frac{1}{2}m\int_{\Sigma'}\frac{d^2z}{z-w}\partial\phi_z(z)\bar{\partial}\phi_x(\bar{z}),\end{aligned} \tag{8.41}$$

and so overall we have

$$(\delta_{\mathcal{O}}+\delta_E)\partial\phi_y(w) = \frac{1}{2}m\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xz}(z,\bar{z}) + \frac{1}{2}m\oint_{C_0'}\frac{\phi^x(z,\bar{z})d\phi_z(z,\bar{z})}{z-w}, \tag{8.42}$$

which agrees with the classical result up to the branch-dependent terms. The calculation follows similarly for $\partial\phi_x, \partial\phi_z$, and overall we have, to first order in m,

$$\delta\partial\phi_x(w) = -\frac{1}{2}m\int_{\Sigma'}\frac{d^2z}{z-w}F^+_{yz}(z,\bar{z}), \tag{8.43}$$

$$\delta\partial\phi_y(w) = \frac{1}{2}m\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xz}(z,\bar{z}) + \frac{1}{2}m\oint_{C_0'}\frac{\phi^x(z,\bar{z})d\phi_z(z,\bar{z})}{z-w}, \tag{8.44}$$

$$\delta\partial\phi_z(w) = \frac{1}{2}m\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xy}(z,\bar{z}) + \frac{1}{2}m\oint_{C_0'}\frac{\phi^x(z,\bar{z})d\phi_y(z,\bar{z})}{z-w}. \tag{8.45}$$

As in the previous example, we view the first terms in the above expressions - those containing $F^{\pm}_{\mu\nu}(z,\bar{z})$ - as the physical deformations of the fields. The gauge ambiguity, represented by the contour integral terms, correspond to target space diffeomorphisms that are T-dual to the gauge transformations generated by (8.24).

8.3 T-fold deformation

For completeness, let us also look at the T-fold. Recall that this is a globally non-geometric background in the sense that the metric and B-field are not well-defined under the global base coordinate identifications. We will include the background here again for convenience, which is

$$ds^2 = dx^2 + \frac{1}{1+(mx)^2}(dy^2+dz^2), \qquad B = -\frac{mx}{1+(mx)^2}(dy\wedge dz). \tag{8.46}$$

To first order, this is simply

$$ds^2 = dx^2 + dy^2 + dz^2 + O(m^2), \qquad B = -mx dy \wedge dz + O(m^2), \tag{8.47}$$

i.e. the same as the T^3 with H-flux, but with an overall minus sign in the B-field. Thus, the first order deformations are exactly the same as for the H-flux, but with overall minus signs, i.e.

$$\delta\partial\phi_x(w) = -\frac{1}{2}m\int_{\Sigma'} \frac{d^2z}{z-w}\, F^-_{yz}(z,\bar{z}), \tag{8.48}$$

$$\delta\partial\phi_y(w) = -\frac{1}{2}m\int_{\Sigma'} \frac{d^2z}{z-w}\, F^-_{zx}(z,\bar{z}) + \frac{1}{2}m\oint_{C_0'} \frac{\phi^x(z,\bar{z})d\phi_z(z,\bar{z})}{z-w}, \tag{8.49}$$

$$\delta\partial\phi_z(w) = -\frac{1}{2}m\int_{\Sigma'} \frac{d^2z}{z-w}\, F^-_{xy}(z,\bar{z}) - \frac{1}{2}m\oint_{C_0'} \frac{\phi^x(z,\bar{z})d\phi_y(z,\bar{z})}{z-w}. \tag{8.50}$$

When we come to the T-duality of these backgrounds in this formalism in chapter 9, we will show how these deformations are used to demonstrate the dualities.

8.4 Deformations of the algebra of twisted torus bundles

Recall from chapter 3 that the twisted torus reductions had gauge algebra (3.7). These were generated by the left-invariant one-forms (3.6). On the worldsheet, we found that the analogous objects, $\mathcal{Z}_a, \mathcal{X}^a$, generated a central extension of this algebra. Additionally, we found that the algebra of the worldsheet operators $\mathcal{Z}_a, \mathcal{X}^a$ obtained from doubling the torus fibres was in fact (a central extension of) the full doubled algebra (3.8) and not just the contraction (3.7).

Given the deformations we have derived above with the worldsheet interactions taken into account, there is a question of how this affects the algebra of the $\mathcal{Z}_a, \mathcal{X}^a$. Here, we briefly address this question. We will not explicitly derive the algebra in full since it is not particularly enlightening, but we will go far enough to show that there are extra terms, and it will be clear where these extra terms come from. The fact that the $O(d,d)$ covariant algebra no longer holds is as we would expect, since the worldsheet interactions break the $O(d,d)$ covariance of the deformations.

To derive the deformed algebra, we need the $\mathcal{Z}_a, \mathcal{X}^a$. Recall that, in section 6.2, we wrote down the $\mathcal{Z}_a, \mathcal{X}^a$ as twisted versions of Π_μ, X'^μ. We then used universal coordinates to derive their algebra at various backgrounds. Here, since we no longer have universal coordinates, we need to be more careful.

In the most general case, suppose that we have a background E with corresponding doubled metric $\mathcal{H}$ and doubled vielbein $\mathcal{V}$. We would define

$$\mathcal{A}_A = \mathcal{V}_A{}^I \Pi_I \equiv (\mathcal{Z}_a, \mathcal{X}^a), \tag{8.51}$$

where $\Pi_I \equiv (\Pi_\mu, X'^\mu)$. Now, when we have a non-zero B-field, we need this doubled setup in order to get the correct $\mathcal{Z}_a, \mathcal{X}^a$, much like how we needed the doubled geometry to get the correct δ_E transformation in section 7.2. We will look specifically at the nilfold case here, where there is no B-field deformation, to simplify things. In this case, we simply have

$$\mathcal{Z}_a = e_a{}^\mu \Pi_\mu. \tag{8.52}$$

It will be useful to invert the equations (2.20) to get

$$X'^\mu(z,\bar{z}) = -ig^{\mu\nu}\left(z\partial X_\nu(z) - \bar{z}\bar{\partial}X_\nu(\bar{z})\right), \qquad \Pi_\mu(z,\bar{z}) = izE^T_{\mu\nu}\partial X^\nu(z) + i\bar{z}E_{\mu\nu}\bar{\partial}X^\nu(\bar{z}). \tag{8.53}$$

Note that these equations are always true at any given background and do *not* rely on the existence of universal coordinates. For the nilfold, this gives[11]

$$\Pi_x = i(\partial X + \bar{\partial}X), \tag{8.54}$$
$$\Pi_y = i(\partial Y + \bar{\partial}Y) - imX(\partial Z + \bar{\partial}Z), \tag{8.55}$$
$$\Pi_z = i(\partial Z + \bar{\partial}Z) - imX(\partial Y + \bar{\partial}Y). \tag{8.56}$$

If we use vielbeins such that $\delta e_\mu{}^a = \frac{1}{2}\delta g_{\mu\nu}e^{\nu a}$, we have

$$\mathcal{Z}_x = \Pi_x, \tag{8.57}$$
$$\mathcal{Z}_y = \Pi_y + \frac{1}{2}mX\Pi_z, \tag{8.58}$$
$$\mathcal{Z}_z = \Pi_z + \frac{1}{2}mX\Pi_y. \tag{8.59}$$

We now substitute (8.54), (8.55), (8.56) into this and compute their OPEs. Up to now, this is all as we would have in the adiabatic limit. However, the key point here is that the Π_μ are no longer universal, so we must replace the $\partial X_\mu, \bar{\partial}X_\mu$ terms in (8.54), (8.55), (8.56) with the nilfold deformations (8.43), (8.44), (8.45) we derived earlier.

[11] We drop the $z, \bar{z}$ factors here for brevity. They can be reintroduced via $\partial X_\mu(z) \to z\partial X_\mu(z), \bar{\partial}X_\mu(\bar{z}) \to \bar{z}\bar{\partial}X_\mu(\bar{z})$

Let us have a look at one of the relevant OPEs. Recall from (6.37) that we had the commutator

$$[\mathcal{Z}_x(\sigma), \mathcal{Z}_z(\sigma')] = -2\pi i m\delta(\sigma - \sigma')\mathcal{Z}_y(\sigma'). \tag{8.60}$$

Using all of the above, we can write $\mathcal{Z}_x$ and $\mathcal{Z}_z$ as

$$\mathcal{Z}_x(w) = i\left(\partial\phi_x(w) + \bar{\partial}\phi_x(\bar{w}) - \frac{1}{2}m\int_{\Sigma'} d^2z\left(\frac{F^+_{yz}(z,\bar{z})}{z-w} + \frac{\bar{F}^+_{yz}(z,\bar{z})}{\bar{z}-\bar{w}}\right)\right), \tag{8.61}$$

$$\begin{aligned}\mathcal{Z}_z(w) = i\Bigg(&\partial\phi_z(w) + \bar{\partial}\phi_z(\bar{w}) + \frac{1}{2}m\int_{\Sigma'} d^2z\left(\frac{F^-_{xy}(z,\bar{z})}{z-w} + \frac{\bar{F}^-_{xy}(z,\bar{z})}{\bar{z}-\bar{w}}\right)\\ &+\frac{1}{2}m\oint \phi^x(z,\bar{z})d\phi_y(z,\bar{z})\left(\frac{1}{z-w}+\frac{1}{\bar{z}-\bar{w}}\right) - \frac{1}{2}m\phi^x(w,\bar{w})\left(\partial\phi_y(w)+\bar{\partial}\phi_y(\bar{w})\right)\Bigg).\end{aligned} \tag{8.62}$$

Taking a closer look at these expressions, we see that the non-integral terms are essentially what we had in the adiabatic approximation (6.38). To first order in m, the OPE of these terms gives

$$\frac{1}{2}m\left(\partial\phi_x(z)+\bar{\partial}\phi_x(\bar{z})\right)\phi^x(w,\bar{w})\left(\partial\phi_y(w)+\bar{\partial}\phi_y(\bar{w})\right) \sim -\frac{1}{4}m\left(\frac{\partial\phi_y(w)}{z-w}+\frac{\bar{\partial}\phi_y(\bar{w})}{\bar{z}-\bar{w}}\right), \tag{8.63}$$

and, if we are on contours where $|z| = |w| = 1$, this is just $\mathcal{Z}_y$ to first order in m. The other terms in (8.61), (8.62) will therefore give us corrections which vanish in the adiabatic limit, and thus we see that we will recover the algebra (6.37) in this limit. We will not do the calculations explicitly here. The above was simply to demonstrate that the algebra (3.7) is not preserved when worldsheet interactions are taken into account, but it is recovered in the adiabatic limit, as we would hope.

8.5 The covariant construction

Our discussion of the H-flux (and subsequently its T-duals) is always in the context of a particular gauge choice. A gauge-invariant construction of the H-flux deformation (or covariant construction for the nilfold) may be obtained using the *background field* *method*[12], which we describe briefly. The starting point is the Polyakov action[13]

$$S_P[X] = \frac{1}{4\pi\alpha'} \int\limits_\Sigma d^2\sigma \sqrt{\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X). \tag{8.64}$$

The first step is to split the worldsheet embedding X into two parts; $X = X_0 + \pi$, where X_0 obeys the classical equations of motion and π can be thought of as a quantum fluctuation as the path integral over X reduces to a path integral over π. Working with Riemann normal coordinates simplifies the problem and we define new coordinates for the quantum fluctuations, η^μ, where

$$\pi^\mu = \eta^\mu - \frac{1}{2}\Gamma^\mu_{\nu\rho}(X_0)\eta^\nu\eta^\rho + ... \tag{8.65}$$

We consider the background field formulation which preserves the manifest covariance of the theory. Substituting in the new coordinates X_0, η, gives [25]

$$\begin{aligned} S_P[X_0+\eta] = S_P[X_0] &+ \frac{1}{2\pi\alpha'} \int\limits_\Sigma d^2\sigma \sqrt{\gamma}\gamma^{ab} g_{\mu\nu}(X_0)\partial_a X_0^\mu \nabla_b \eta^\nu \\ &+ \frac{1}{4\pi\alpha'} \int\limits_\Sigma d^2\sigma \sqrt{\gamma}\gamma^{ab} \Big\{ g_{\mu\nu}(X_0)\nabla_a\eta^\mu\nabla_b\eta^\nu + R_{\mu\lambda\sigma\nu}(X_0)\Big(\partial_a X_0^\mu \partial_b X_0^\nu \eta^\lambda\eta^\sigma \\ &+ \frac{4}{3}\partial_a X_0^\mu \eta^\lambda\eta^\sigma\nabla_b\eta^\nu + \frac{1}{3}\eta^\lambda\eta^\sigma\nabla_a\eta^\mu\nabla_b\eta^\nu\Big) + ...\Big\}. \end{aligned} \tag{8.66}$$

Note that this is now explicitly gauge-covariant. Note that $S_P[X_0]$ and the terms linear in η can be discarded if we choose X_0 to obey the classical equations of motion[14]. The kinetic term is awkward in that it involves coupling with the background fields and so a potentially complicated Greens function. One way around this is to work with frame fields $\eta^a = e^a{}_\mu(X_0)\eta^\mu$. Instead, we expand the metric $g_{\mu\nu}(X_0)$ around a flat reference background

$$g_{\mu\nu}(X_0) = \eta_{\mu\nu} + h_{\mu\nu}(X_0). \tag{8.67}$$

The covariant deformation operator, with respect to this background, is then

$$\begin{aligned} \mathcal{O}_g[X_0,\eta] = \frac{1}{4\pi\alpha'} \int\limits_\Sigma d^2\sigma \sqrt{\gamma}\gamma^{ab} \Big\{ & h_{\mu\nu}(X_0)\nabla_a\eta^\mu\nabla_b\eta^\nu + R_{\mu\lambda\sigma\nu}(X_0)\Big(\partial_a X_0^\mu \partial_b X_0^\nu \eta^\lambda\eta^\sigma \\ &+ \frac{4}{3}\partial_a X_0^\mu \eta^\lambda\eta^\sigma\nabla_b\eta^\nu + \frac{1}{3}\eta^\lambda\eta^\sigma\nabla_a\eta^\mu\nabla_b\eta^\nu\Big) + ...\Big\}. \end{aligned} \tag{8.68}$$

[12] For example, see [25].
[13] We reinsert the α' factors in this section for clarity.
[14] If X_0 is a classical instanton solution, then $S_P[X_0]$ will still give a finite contribution.

To make contact with the discussion in section 7.2, we allow the possibility of a non-trivial B-field. Suppose we have an anti-symmetric part of the action given by

$$S_{AS}[X] = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X). \tag{8.69}$$

As with $S_P[X]$, we can substitute in the background field expansion and obtain the action in a covariant form. We will not write it down in full here, but the result is given in [25]. The terms that are of relevance for us are

$$\frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \epsilon^{ab} H_{\mu\nu\rho}(X_0) \partial_a X_0^\mu \nabla_b \eta^\nu \eta^\rho + \frac{1}{12\pi\alpha'} \int_\Sigma d^2\sigma \epsilon^{ab} H_{\mu\nu\rho}(X_0) \eta^\mu \nabla_a \eta^\nu \nabla_b \eta^\rho. \tag{8.70}$$

For a flat background with constant H, these are the only contributions to $\mathcal{O}$.

The background fields $g_{\mu\nu}(X_0)$, $H_{\mu\nu\lambda}(X_0)$ and their derivatives play the role of the deformation parameters $\mathfrak{m}^\alpha$. The action is that of an interacting theory with couplings specified by the covariant functions $H_{\mu\nu\rho}(X_0)$, $R_{\mu\nu\lambda\rho}(X_0)$ and their covariant derivatives. A natural construction would be to consider a natural basis of functions $f_I(X_0)$ on the reference spacetime and to then decompose the background metric deformation and B-field in terms of this basis, i.e.

$$h_{\mu\nu}(X_0) = \sum_I c^I_{\mu\nu} f_I(X_0). \tag{8.71}$$

The coefficients c^I might then provide suitable local coordinates on $\mathcal{M}$ with which to parameterise the deformation. In cases where the initial and final backgrounds have different topology, it is natural to pass to the cover as discussed above[15]. Given a path in $\mathcal{M}$ between two backgrounds, the classical solution X_0 varies as the action changes as we move along the path. As such, the basis f_I will also change along the path; however, the expression (5.12) only requires knowledge of the moduli and their derivatives along the path evaluated at the start of the path (where the theory is free in most cases).

A connection on the space of such backgrounds is given by the variational formula (5.8), where the OPEs can in principle be computed in perturbation theory.[16] The construction given in [22] and outlined in chapter 5 then gives the connection associated

[15] There are of course well-known cases where one can smoothly change the topology in string theory [75, 76, 9]. In such cases, a continuous path, without degenerations in the fibres, is expected to exist between the two topologically distinct backgrounds.

[16] Or, by first finding the beta-functions, the OPE coefficients can be computed using the explicit construction given in [23, 24].

with deforming the theory by changing the value of the background metric and B-field. Taking the reference background as the free theory, we only require knowledge of the OPE of the free theory and we recover the previous construction of chapter 5, but now in a manifestly covariant form.

By way of example, in the case of constant H-flux $H_{\mu\nu\rho} = \lambda\epsilon_{\mu\nu\rho}$ on a flat background $h_{\mu\nu}(X_0) = 0$, the deformation operator is

$$\mathcal{O}_H[X_0, \eta] = \frac{\lambda}{4\pi\alpha'} \int\limits_{\Sigma} d^2\sigma \epsilon^{ab}\Big(\epsilon_{\mu\nu\rho}\partial_a X_0^\mu \partial_b \eta^\nu \eta^\rho + \frac{1}{3}\epsilon_{\mu\nu\rho}\eta^\mu \partial_a \eta^\nu \partial_b \eta^\rho\Big), \tag{8.72}$$

where X_0 is the classical solution at $\lambda \neq 0$[17]. This is manifestly gauge invariant. Thus, we see that the choice of branch cut for the X dependence in the B-field does not make any difference to the physics, since these choices of branch correspond to B-field gauge transformations. The interaction terms then, order by order in λ, perturbatively describe the deformation of the theory away from the $\lambda = 0$ point. Incorporating non-perturbative effects, which will be inaccessible via these techniques, is briefly discussed in the following section.

8.6 Non-perturbative effects

What can go wrong? As alluded to in chapter 5, there is a close relationship between parallel transport and conventional interaction picture perturbation theory, with the deformation $\mathcal{O}$ playing a role akin to an interaction Hamiltonian and the parallel transport (5.12) akin to a Dyson series. It is well-known that not all physics is accessible via perturbation theory and non-perturbative effects can play an important role. The issue of convergence of the perturbative expansion has an obvious analogue in attempting to parallel transport from one background to another. We can only hope to access that part of the deformed theory that is analytic in the deformation parameter λ.

In the H-flux case, one can show that, up to field redefinitions, λ and α' always appear together and so one can use α' as a proxy for λ. Thus, non-perturbative effects in α' will also be non-perturbative effects in λ. The parallel transport will be blind to phenomena like worldsheet instantons. This is not always the case and, in chapter 11, we briefly consider an example where non-perturbative effects in α' are in fact perturbative in λ.

[17] To relate this to the background field description of the free theory at $\lambda = 0$, one could write X_0 explicitly in terms of λ and the classical solution at $\lambda = 0$ using (8.4).

Chapter 9

T-duality Revisited

We now have an understanding of how the $\partial\phi_\mu$ operators are deformed in the twisted torus bundle backgrounds we have been considering throughout. In this chapter, we consider how T-duality appears in the construction of the previous chapter and how the requirements evident in the Buscher construction [7, 8] emerge in this framework. As demonstrated earlier, we can use this method of CFT deformation to do the T-duality at any background by deforming to it from the point of enhanced symmetry. We then use a charge to compute the T-duality by acting on the stress tensor via an automorphism. In some sense, this is a rather trivial process since, provided the automorphism has a well-defined action on the operator algebra of the theory, it will obviously produce a new description of the same underlying physics. Where this is interesting is when the new description is also a conventional string theory, but with a different interpretation of the target space.

We will also revisit T-duality of the torus bundles of chapter 3, but this time away from the adiabatic limit. What we will find is that, though the calculations to verify the dualities are more involved, the basic principle carries thorough in the same way. What we uncover is a method for constructing T-dual backgrounds for more general non-linear sigma model (NLSM) deformations, which potentially has greater applicability than the Buscher procedure. We also briefly return to non-isometric T-duality, discussing an idea of [15].

9.1 Conditions for the Buscher rules

Recall that the Buscher construction requires the existence of a globally-defined[1], compact isometry in the target space which preserves all non-trivial field strengths in the background. Do we see such requirements in this formalism and in what capacity? We discuss the existence and compactness of the isometry in turn.

9.1.1 Existence of a continuous isometry

The requirement of an isometry can be seen from the fact that the action of the T-duality charge Q is not well-defined on X itself. In section 6.9, we saw that

$$e^{iQ}\phi_L(\sigma)e^{-iQ} = -\phi_L(\sigma) + \mathcal{C}, \tag{9.1}$$

where $Q = \frac{1}{2}\oint d\sigma \cos(2\phi_L(\sigma))$ and $\mathcal{C}$ was a constant operator that was dependent on the T-duality charge used. Thus, when we have explicit X-dependence, we have to deal with the troubling operator $\mathcal{C}$ and it is not clear how to proceed.[2] In the Buscher procedure, we cannot do anything without an isometry, whereas here the situation is less clear. Subsequently, there is a question of whether we can make sense of the non-isometric case regardless of the aforementioned difficulties. In section 9.1.3, we will discuss how one might be able to make some progress on non-isometric T-duality using a simple concrete example. Though the result is not conclusive, it is an interesting approach to explore and it is clear that there is some mileage in this parallel transport formalism with regards to non-isometric T-duality.

We can appreciate why the non-isometric case is much harder to understand if we consider what is happening from the perspective of theory space. A point on the space $\mathcal{M}$ is given by a choice of metric and B-field. Since an isometry will preserve the sigma model, it is reasonable to identify different points on $\mathcal{M}$ if they correspond to the same sigma model. Hence, an isometry will preserve the sigma model and keep us at the same point. A more general diffeomorphism, which is not an isometry, takes us to a different point on $\mathcal{M}$ representing a different sigma model. One would need some non-local (from the sigma model perspective) deformation $\mathcal{O}$ to relate the original fields with those after the diffeomorphism. This gauge transformation in the target space is thus a non-trivial deformation (or parallel transport) in $\mathcal{M}$. Therefore,

[1]It may be that the requirement that the isometry is globally defined may be dropped [11].

[2]It is interesting to note that the transformation is well-defined on those ‘self-dual’ states for which $p = w$. If we set $p = w$, we actually find that the anomalous term $\mathcal{C}$ vanishes and we get $e^{iQ}\phi_L(\sigma)e^{-iQ} = -\phi_L(\sigma)$.

from the worldsheet perspective, there is a significant difference between isometries and non-isometric diffeomorphisms. Understanding this difference better is key to understanding T-duality in the absence of continuous isometries in this framework. For example, it may be that some kind of construction which is manifestly T-duality invariant, even just for the cases where the Buscher procedure is valid, is what is needed to make progress on the non-isometric case.

9.1.2 Compactness of the isometry

The Buscher prescription also requires the isometry to be compact. The necessity of this requirement can be illustrated in the one-dimensional case, where we can consider a limit in which the circle 'decompactifies'. Recall that we can write $\partial X(R)$ in terms of objects at the self-dual radius as

$$\partial X(R)(z) = \frac{1}{2}(1+R^2)\partial X(z) + \frac{1}{2}(1-R^2)\bar{\partial} X(\bar{z}), \tag{9.2}$$

where $\partial X, \bar{\partial} X$ are at the self-dual radius. We might try to think of the 'decompactified' case as taking the limit $R \to \infty$ or, by T-duality, $R \to 0$. However, we can see from the above that, if we take this limit, we get

$$\partial X(0) = \frac{1}{2}(\partial X + \bar{\partial} X) = \Pi, \tag{9.3}$$

and doing the same for $\bar{\partial} X(R)$ gives $\bar{\partial} X(0) = \Pi = \partial X(0)$, i.e. holomorphic and antiholomorphic derivatives seem to coincide. Another way of saying this is that if we have

$$\begin{pmatrix} \partial X(R) \\ \bar{\partial} X(R) \end{pmatrix} = M(R) \begin{pmatrix} \partial X \\ \bar{\partial} X \end{pmatrix}, \tag{9.4}$$

then the matrix $M(R)$ degenerates in the limit $R \to 0$. This failure of the self-dual basis to extend to this case is unsurprising and suggests a new ingredient would be needed to extend the duality to this unlikely case. One suspects the curvature of the connection would be badly behaved at this point, although we have not checked this.

9.1.3 Non-isometric T-duality via Fourier expansions

Given the above discussion of the requirement of an isometry, it seems like, in the parallel transport formalism we have presented here, there may be some leeway regarding this requirement. In [15], an interesting idea was presented where, given some metric with X-dependence, the T-duality of the corresponding stress tensor could still be computed by considering the Fourier expansion of the metric. The idea here was that exponentials had a well-defined transformation under the T-duality automorphism Q and so, by expressing the metric in terms of such exponentials, we could compute the T-duality.

The main issue with this is that the construction in [15] still assumes the existence of universal coordinates. As we saw, if there is explicit coordinate dependence in the background then we do not have universal coordinates, and so this idea does not seem to be quite right. However, even in the absence of universal coordinates, we will try to see if there is any mileage in it. For simplicity, we will just work in the $d = 1$ case and we will assume that we start from the self-dual radius point $E = G$. Let us suppose we deform to some metric $g = G + \lambda\delta g(X)$, where

$$\delta g(X) = e^{ikX}, \tag{9.5}$$

where we use the shorthand $kX \equiv k_L X_L + k_R X_R$, $k_{L,R} \in \mathbb{Z}$, and λ is some expansion parameter, i.e. we have a single Fourier mode in our deformation. In order to compute the duality, we need to compute the deformation of $\partial\phi$ and then use this to compute the stress tensor T_g and subsequently the T-duality. Given the above metric deformation, we suppose that the deformation operator can be taken to be

$$\mathcal{O} = \lambda \int d^2z \partial\phi(z)\bar{\partial}\phi(\bar{z})e^{ik\phi(z,\bar{z})}. \tag{9.6}$$

Thus, to compute the deformation of $\partial\phi(w)$, we take the OPE

$$\begin{aligned}
&\int d^2z \partial\phi(z)\bar{\partial}\phi(\bar{z})e^{ik\phi(z,\bar{z})}\partial\phi(w) \\
&\sim \int d^2z \left[-\frac{1}{2(z-w)^2}\bar{\partial}\phi(\bar{z})e^{ik\phi(z,\bar{z})} + \frac{ik_L}{2(z-w)}\partial\phi(z)\bar{\partial}\phi(\bar{z})e^{ik\phi(z,\bar{z})} \right] \\
&= \int d^2z \left[-\frac{1}{2}\left(\partial\left(\frac{\bar{\partial}\phi(\bar{z})e^{ik\phi(z,\bar{z})}}{w-z} \right) - \frac{\partial\phi(z)\bar{\partial}\phi(\bar{z})ik_L e^{ik\phi(z,\bar{z})}}{w-z} \right) + \frac{ik_L}{2(z-w)}\partial\phi(z)\bar{\partial}\phi(\bar{z})e^{ik\phi(z,\bar{z})} \right] \\
&= -\frac{1}{2}\oint \frac{d\bar{z}}{z-w}\bar{\partial}\phi(\bar{z})e^{ik\phi(z,\bar{z})},
\end{aligned} \tag{9.7}$$

i.e. we see that the contraction of $\partial\phi(w)$ with the exponential is cancelled by a contribution from the contraction with $\partial\phi(z)$. Thus, the final result looks very similar to what we obtained in chapter 7 for the constant circle deformation and we find that, for $|z| = 1$,

$$\delta_{\mathcal{O}}\partial\phi = -\frac{\lambda}{2}\frac{\bar{z}}{z}\bar{\partial}\phi(\bar{z})e^{ik\phi(z,\bar{z})}. \tag{9.8}$$

We expect δ_E to act in the same way as usual, i.e.

$$\delta_E\partial\phi = \frac{\lambda}{2}\partial\phi e^{ik\phi}, \tag{9.9}$$

so that overall we have

$$\delta\partial\phi(z) = \frac{\lambda}{2}e^{ik\phi(z,\bar{z})}\left(\partial\phi(z) - \frac{\bar{z}}{z}\bar{\partial}\phi(\bar{z})\right). \tag{9.10}$$

Now, the stress tensor is given by $T_g = g^{\mu\nu}(X)\partial X_\mu \partial X_\nu$, so the exponential e^{ikX} will appear explicitly as well as $e^{ik\phi}$. From earlier (and appendix A), we know how exponentials in ϕ transform, but, due to the lack of universal coordinates, we do *not* know how exponentials transform away from the self-dual point. This seems to be a major obstacle in deriving the duality, but we will try to proceed anyway, albeit in a rather speculative way. Suppose the deformation of e^{ikX} can be written as

$$\delta e^{ikX} = \sum_{n_R, n_R} f_n(k)e^{in\phi}, \tag{9.11}$$

where $n\phi \equiv n_L\phi_L + n_R\phi_R$ and the f_n are constants. Recall from (4.48) that, for n_L even, we simply have

$$e^{iQ}e^{in_L\phi_L}e^{-iQ} = e^{-in_L\phi_L}. \tag{9.12}$$

Thus, let us simplify things by supposing that $k_{L,R}, n_{L,R} \in 2\mathbb{Z}$. In this case, under the duality, we have

$$e^{iQ}e^{ikX}e^{-iQ} = \sum_n f_n(k)e^{-in\phi}. \tag{9.13}$$

Ideally, what we would like at this point is to be able to say

$$f_{-n}(k) \propto f_n(\tilde{k}), \tag{9.14}$$

where $\tilde{k}X = -k_L X_L + k_R X_R$. This is because, if we can say this, then we essentially have

$$e^{iQ}e^{ikX}e^{-iQ} = e^{i\tilde{k}X}. \tag{9.15}$$

Thus, the stress tensor would be

$$T_g(z) = \frac{1}{1+\lambda e^{ikX(z,\bar{z})}} \left(\partial\phi + \frac{\lambda}{2} e^{ik\phi(z,\bar{z})} \left(\partial\phi(z) - \frac{\bar{z}}{z}\bar{\partial}\phi(\bar{z})\right)\right)^2 , \tag{9.16}$$

and if the duality does $\partial\phi \to -\partial\phi$ as well as (9.15), overall we simply have the usual Buscher rules, but with $k \to \tilde{k}$, i.e. the dual metric is $\tilde{g}$, where

$$\tilde{g} = \frac{1}{1+\lambda e^{i\tilde{k}X}}. \tag{9.17}$$

Another way one might be able to arrive at this result is to suppose that the exponential at g can be written as a simple deformation of the corresponding exponential at G, i.e.

$$e^{ik_L X_L} = e^{ik_L \phi_L} + \delta e^{ik_L \phi_L}. \tag{9.18}$$

If we assume that there is no δ_E transformation, then we only need to look at the OPE of $\mathrm{e}^{ik_L\phi_L}$ with the deformation operator, which is

$$\begin{aligned}
&\int d^2z : \partial\phi(z)\bar{\partial}\phi(\bar{z}) e^{ik\phi(z,\bar{z})} :: e^{ik_L\phi_L(w)} : \\
\sim &\int d^2z \left[-\frac{i}{2(z-w)} : \bar{\partial}\phi(\bar{z}) e^{ik\phi(z,\bar{z})} e^{ik_L\phi_L(w)} : + (z-w)^{K_L^2/2} : \partial\phi(z)\bar{\partial}\phi(\bar{z}) e^{ik\phi(z,\bar{z})} e^{ik_L\phi_L(w)} : \right],
\end{aligned} \tag{9.19}$$

where we have put the normal ordering in explicitly for clarity. If this integral is what gives us the deformation $\delta e^{ik_L\phi_L}$, then the effect of T-duality on $\delta e^{ik_L X_L}$ can be deduced from the T-duality transformation of the above integral, which would once again be the usual Buscher rules together with $k_L \to -k_L$.

Note that we have made many assumptions for the exponential transformation along the way, so the above does not constitute a proof of the duality, but it at least goes some way to showing how progress could be made on non-isometric T-duality in this formalism. It may be that the deformation of exponentials under generic deformations of the metric is a solvable problem and can be done properly, at least in these simple cases. If so, then the T-duality should be doable and, if this method is valid, this would be a concrete example of non-isometric T-duality. It should be noted though that, if the deformation is simply a target space diffeomorphism, then the background is physically the same as an isometric one, even if it appears to be non-isometric. Since all circle backgrounds in the $d = 1$ case are diffeomorphic, the above example could be argued to be 'secretly isometric'.

Additionally, there are some issues with this approach that need to be straightened out. In particular, there is obviously some tension in going from an object like X, which is not periodic on the worldsheet, to the pullback of the Fourier series of x to the worldsheet, which *is* periodic on the worldsheet. There is a kind of non-commutativity of taking the Fourier series with pulling back to the worldsheet, and this must be understood properly. In particular, this approach would probably not be applicable to the R-flux case. This is because the R-flux background is not globally defined, whereas fourier modes always are. Another issue here is the ambiguity of the T-duality transformation of exponential operators. As we saw in chapter 6, the precise transformation depends on the charge used. In general this may not matter since it is just the overall factor that is affected. However, if we are interpreting exponentials as terms in a Fourier series, then such factors are important, and different charges might give different functions after T-duality. There is then a question of whether these different backgrounds are equivalent, or if there is some gauge symmetry relating them. Thus, there are many issues that need to be understood in more detail before this Fourier expansion approach can be trusted.

This is all we will have to say on non-isometric T-duality. For now, we turn our attention to how T-duality generally fits into our framework and how we can derive T-dual backgrounds using the formalism discussed in this thesis.

9.2 Deriving T-dual backgrounds in the parallel transport formalism

The transformations we have obtained using the parallel transport formalism are different to what we would expect from the universal coordinate methods considered in chapter 6, which give an adiabatic approximation. The application of this formalism to T-duality in trivial torus bundles was also discussed there. We turn now to study T-duality in the torus bundles discussed above.

For a CFT associated with toroidal backgrounds, we know that the left-moving component of the stress tensor is

$$T_E = g^{\mu\nu}\partial X_\mu(E)\bar{\partial}X_\nu(E), \tag{9.20}$$

but for general sigma models there is no clean split into left- and right-moving sectors. Nonetheless, T_E is a useful object to consider as it is the simplest composite operator that is a target space scalar and so invariant under δ_E. This invariance under δ_E streamlines the analysis somewhat.

By writing the stress tensor in terms of fields defined at the self-dual radius, [15] showed how to use the stress tensor to compute the T-dual of a given background: starting with some background E and writing T_E in terms of the fields at the self-dual point, one can use the action of T-duality at the self-dual point to determine the stress tensor for the dual background, written in terms of the self-dual basis. T_E transforms as

$$T_E \to T_{\tilde{E}} = U T_E U^{-1} \tag{9.21}$$

under the T-duality automorphism $U = e^{iQ}$ discussed in chapter 4. Knowledge of how the stress tensor changes under a general enough class of deformations (e.g. marginal deformations) then allows one to read off $\tilde{E}$ from the dual tensor $T_{\tilde{E}}$.

For the toroidal backgrounds considered in [15], this was essentially a novel derivation of the familiar Buscher rules that we wrote down in (2.47). Much of this structure carries over to the more general constructions considered in the last chapter. In this section, we sketch how known dualities are realised in the framework presented in this thesis and to what extent one may use it to generalise beyond the cases where the Buscher construction is valid.

It is important to stress that it is not true that the deformation operators for T-dual backgrounds $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ are related simply as $e^{iQ}\mathcal{O}e^{-iQ}$. That this cannot be true is easily seen if we consider the explicit deformation parameters of the T^3 with constant H-flux, nilfold and T-fold backgrounds with respect to the T^3 CFT:

$$\mathcal{O}_H = m \int_\Sigma \phi^x F^-_{yz}, \tag{9.22}$$

$$\mathcal{O}_N = -m \int_\Sigma \phi^x F^+_{yz} + m^2 \int_\Sigma (\phi^x)^2 \partial\phi_z \bar{\partial}\phi_z, \tag{9.23}$$

$$\mathcal{O}_{\text{T-fold}} = -m \int_\Sigma \phi^x F^-_{yz} + \sum_{n\geq 1} (-1)^n (n+1)(m)^{2n} \int_\Sigma (\phi^x)^{2n} \left(\partial\phi_y \bar{\partial}\phi_y + \partial\phi_z \bar{\partial}\phi_z - m\phi^x F^-_{yz}\right). \tag{9.24}$$

These are clearly not related simply by a change of sign of the chiral field along the direction the duality is being performed[3]. The reason is that the dual descriptions of the theories involved different parameterisations of the backgrounds. In general, if we have two dual backgrounds $E = G + \delta E$ and $\tilde{E} = G + \delta\tilde{E}$, where G is the background

[3] Although they are related in this simple way to first order in the parameter m.

metric at the self-dual point, the deformation operators are given by

$$\mathcal{O}_E = \delta E_{\mu\nu} \int_\Sigma \partial\phi^\mu \bar{\partial}\phi^\nu, \qquad \mathcal{O}_{\tilde{E}} = \delta \tilde{E}_{\mu\nu} \int_\Sigma \partial\phi^\mu \bar{\partial}\phi^\nu. \tag{9.25}$$

If we try to T-dualise $\mathcal{O}_E$ explicitly in the y-direction say, this would simply correspond to $\partial\phi_y \to -\partial\phi_y$. However, this is not sufficient to relate $\mathcal{O}_E$ to $\mathcal{O}_{\tilde{E}}$, since we also need $\delta E \to \delta\tilde{E}$, which is *not* induced by the simple automorphism on the fields alone. From the worldsheet perspective, δE and $\delta\tilde{E}$ are coupling constants for perturbations of two different, but related, theories. In order to relate these deformations, we must understand how the coupling constants are related.

9.2.1 Trivial torus bundles

First, let us briefly verify the constant case, i.e. when E has no coordinate dependence. This is essentially the same as [15], though we include it for completeness and to illustrate the general idea. Specifically, we want to use the stress tensor to show that the $\mathcal{O}$ transformation in the dual background can be deduced from the $\mathcal{O}$ transformation in the original background. As above, suppose we have some background $E = G + \delta E$, where δE is not necessarily small (i.e. this is to arbitrary order), as well as the corresponding deformation operator $\mathcal{O}_E$. From this (together with δ_E), we can find $\partial X_\mu(E)$ in terms of the self-dual point objects, and we know in this constant case that we simply reproduce the results of [15]. Thus, the stress tensor, T_E, can then be T-dualised and we obtain the dual tensor $T_{\tilde{E}}$. We can then read off $\tilde{E}$ and therefore deduce the dual operator $\mathcal{O}_{\tilde{E}}$, where $\tilde{E} = G + \delta\tilde{E}$. Of course, in this constant background case, we already know that the relation between E and $\tilde{E}$ is given by the fractional-linear transformation

$$\tilde{E} = (aE+b)(cE+d)^{-1}, \qquad U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d;\mathbb{Z}), \tag{9.26}$$

which leads to a complicated relationship between the deformation parameters in $\mathcal{O}_E$ and $\mathcal{O}_{\tilde{E}}$ of the form

$$\delta\tilde{E} = (a(G+\delta E)+b)(c+d)^{-1} \sum_{n\geq 0} (-1)^n \left((c+d)^{-1} c\delta E\right)^n - G. \tag{9.27}$$

Note that, in general, there is a non-trivial zeroth order term here, but for the cases we are considering (i.e. T-duality), it is always true that $a+b = c+d = 1$ [12], and so we have

$$\delta\tilde{E} = a\delta E\sum_{n\geq 0}(-1)^n\,(c\delta E)^n\,. \tag{9.28}$$

The complication caused by the transformation of δE reflects the fact that the way in which the data $\mathfrak{m}^\alpha$ parameterise the space of backgrounds depends on the duality frame chosen.

Circle example

It is easiest to see this in a concrete example, so we briefly review how this works for the familiar case of the circle. The metric G at the self-dual radius $\sqrt{G}$ is changed to g, with a corresponding change in the field ϕ to X, so that, using (7.7),

$$\partial X = \frac{1}{2}\Big(\partial\phi + \bar{\partial}\phi\Big) + \frac{1}{2}gG^{-1}\Big(\partial\phi - \bar{\partial}\phi\Big). \tag{9.29}$$

As seen in chapter 7, this can be constructed from the deformation operator

$$\mathcal{O} = \lambda\int_{\Sigma'} d^2z\,\partial\phi(z)\bar{\partial}\phi(\bar{z}), \qquad \lambda := \frac{g-G}{G^2}. \tag{9.30}$$

The deformed stress tensor $T = g^{-1}\partial X\partial X$ may be written as

$$T = \frac{1}{4}g^{-1}\Big((1+gG^{-1})\partial\phi + (1-gG^{-1})\bar{\partial}\phi\Big)^2. \tag{9.31}$$

Under the T-duality automorphism, $U(\partial\phi,\bar{\partial}\phi)U^{-1} = (-\partial\phi,\bar{\partial}\phi)$, so the dual stress tensor is (note the relative sign change between terms)

$$\widetilde{T} = \frac{1}{4}g^{-1}\Big((1+gG^{-1})\partial\phi - (1-gG^{-1})\bar{\partial}\phi\Big)^2. \tag{9.32}$$

$\widetilde{T}$ may be written in the form of (9.31), but with g replaced by $\tilde{g} = G^2/g$, thus recovering the standard Buscher rule for $d=1$. The dual theory can also be expressed as a deformation of the self-dual theory by the operator

$$\widetilde{\mathcal{O}} = \widetilde{\lambda}\int_{\Sigma'} d^2z\,\partial\phi(z)\bar{\partial}\phi(\bar{z}), \qquad \widetilde{\lambda}(\lambda) = -\frac{\lambda}{1+\lambda}. \tag{9.33}$$

The relationship between the deformation parameters λ and $\widetilde{\lambda}$ is an alternative writing of the Buscher rules and is an example of the transformation (9.26).

9.2.2 Stress tensor deformations and T-duality

In this section, we consider a more general case where the background has some coordinate dependence. The reason why the stress tensor was so useful for the constant case was because we could write down a general form for T_E for any constant background E. When there is coordinate dependence, to do the same using the parallel transport method, we need to know explicitly what the coordinate dependence is.

As always, we start with a reference background, which we take to be a torus, tuned to the self-dual radius with coordinates ϕ^i and a single base direction with coordinate ϕ^x, such that $\phi^\mu = (\phi^x, \phi^i)$. We assume that the base is $\mathbb{R}$, but may impose identifications so that it is an S^1. The action and stress tensor are

$$S_0[X] = \int_\Sigma \partial\phi^x\bar{\partial}\phi^x + G_{ij}\partial\phi^i\bar{\partial}\phi^j, \qquad T_G = \partial\phi^x\partial\phi^x + G_{ij}\partial\phi^i\partial\phi^j. \tag{9.34}$$

This background is then deformed to a background of interest. The associated deformation operator is

$$\mathcal{O}[\phi^x] = \int_\Sigma d^2z\, \delta E_{ij}(\phi^x)\partial\phi^i\bar{\partial}\phi^j, \tag{9.35}$$

where we define $\delta E_{ij} := E_{ij} - G_{ij}$ and we allow this to be a function of ϕ^x. It would be natural to express $\delta E_{ij}(\phi^x)$ in terms of a basis of functions on the base $f_I(\phi^x)$, so that there is a well-defined decomposition

$$\delta E_{ij}(\phi^x) = \sum_n \lambda_{ij}^{(I)} f_I(\phi^x), \tag{9.36}$$

where the $\lambda_{ij}^{(I)}$ give a set of coupling constants for the deformation. Our approach will be to work on the cover of the base (in this case $\mathbb{R}$) and then impose identifications $\phi^x \sim \phi^x + 2\pi$ after the deformation. This leads to a natural (although not unique) decomposition of the deformation operators $\mathcal{O} = \sum_I \mathcal{O}_I$, where

$$\mathcal{O}_I[\phi^x] = \int_\Sigma \lambda_{ij}^{(I)} f_I(\phi^x)\partial\phi^i\bar{\partial}\phi^j. \tag{9.37}$$

One then uses this deformation operator to deform the operators of the theory, such as the stress tensor:

$$T_E = T_G + \delta_{\mathcal{O}}(T_G) + \delta_{\mathcal{O}^2}(T_G) + ... \tag{9.38}$$

The T-dual stress tensor is given by applying the automorphism[4] $T_{\tilde{E}} = e^{iQ}T_E e^{-iQ}$. This then gives a perturbative description of the dual stress tensor. To leading order,

[4]We assume the duality is performed along one of the fibre directions.

the deformations are related by

$$T_G + \delta_{\widetilde{\mathcal{O}}}(T_G) + ... = T_G + e^{iQ}\delta_{\mathcal{O}}(T_G)e^{-iQ} + ..., \tag{9.39}$$

where we note that T_G is invariant under the action of the automorphism. This is hard to calculate in practice and tends to yield complicated expressions on both sides, as the stress tensor written in terms of the reference background is likely to be a complicated and unfamiliar object. It is possible, in principle, to extract information on the dual background $\tilde{E}_{ij}$, from which one could construct a dual sigma model. Given a generic enough deformation, one for which both the original and dual theories are particular examples, one can in principle construct the stress tensor of this generic theory in terms of the fields of the reference background. The couplings $\widetilde{\lambda}_{ij}^{(I)}$ of the dual theory can then be read off from $T_{\tilde{E}}$, and the dual deformation $\delta\tilde{E}_{ij}$ constructed as with (9.36). The dual couplings will be functions of the couplings of the original theory $\widetilde{\lambda}_{ij}^{(I)}(\lambda)$ and this relationship is, in essence, the Buscher rules relating the two backgrounds. This generalises the torus case (9.27), a simple example of which is the relationship between the radii of a circle background and its dual.

It is worth stressing that the dual theory may be found in terms of its operators; however, identifying the explicit background for a sigma model construction is more involved. The procedure we have outlined here is the straightforward generalisation of that used in [15] for toroidal target spaces. The simplifying feature there was the existence of universal coordinates with which to calculate.

Leading order deformations and duality

In general, using this approach to deduce T-dual backgrounds is difficult to do computationally, so we will only demonstrate this to first order for a relatively simple case. In particular, we will suppose that we start from a trivial torus bundle as described above and that we deform the background in the fibres to some background E_{ij}, where

$$E_{ij} = G_{ij} + \lambda\phi^x\delta E_{ij} + O(\lambda^2), \tag{9.40}$$

where δE is constant and λ is some small parameter. We will deduce a general form for the stress tensor T_E. Additionally, we will assume that the dual background $\tilde{E}$ is of the same form, i.e.

$$\tilde{E}_{ij} = G_{ij} + \lambda\phi^x\delta\tilde{E}_{ij} + O(\lambda^2). \tag{9.41}$$

This is of course true for the nilfold and H-flux, for which we will verify this method explicitly. To compute T_E, let us first compute the deformation of $\partial\phi_\mu$. We have

$$\mathcal{O}_E = \lambda\delta E_{ij}\int_\Sigma \phi^x\partial\phi^i\bar{\partial}\phi^j. \tag{9.42}$$

The calculation of the first order change in $\partial\phi_\mu$ closely follows the method set out in chapter 8 and so we will be brief here. Taking the OPE with $\partial\phi_i$ and including the δ_E contribution

$$\delta_E\partial\phi_i = \frac{\lambda}{2}\delta E_{ij}\phi^x\partial\phi^j, \tag{9.43}$$

which can be written in integral form as similar to (D.29), gives

$$\delta\partial\phi_i(w) = -\frac{\lambda}{2}\delta E_{ij}G^{jk}\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xk} - \frac{\lambda}{2}\delta E_{ij}\oint_{C_0'}\frac{d\phi^j(z,\bar{z})}{z-w}\phi^x(z,\bar{z}), \tag{9.44}$$

where we recall that $d\phi^\nu(z,\bar{z}) = \partial\phi^\nu(z)dz + \bar{\partial}\phi^\nu(\bar{z})d\bar{z}$. For $\partial\phi_x$ there is no δ_E transformation, so we have

$$\delta\partial\phi_x(w) = \frac{\lambda}{2}\delta E_{ij}\int_{\Sigma'}\frac{d^2z}{z-w}\partial\phi^i(z)\bar{\partial}\phi^j(\bar{z}). \tag{9.45}$$

Thus, we can now substitute this into the stress tensor to get the deformed stress tensor $T = g^{\mu\nu}\partial X_\mu(E)\partial X_\nu(E)$, where $E = g + B$, and if we write $g_{ij} = G_{ij} + \lambda\delta g_{ij}$ then, to first order, $\delta g^{ij} = -G^{ik}\delta g_{kl}G^{lj}$. After a short computation, we find that

$$\begin{aligned} T_E(w) = T_G(w) &- \lambda\delta g_{ij}\phi^x(w,\bar{w})\partial\phi^i(w)\partial\phi^j(\bar{w}) + \lambda\partial\phi_x(w)\delta E_{ij}\int_{\Sigma'}\frac{d^2z}{z-w}\partial\phi^i(z)\bar{\partial}\phi^j(\bar{z}) \\ &+ \lambda\partial\phi^i(w)\left(-\delta E_{ij}G^{jk}\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xk}(z,\bar{z}) - \delta E_{ij}\oint_{C_0'}\frac{\phi^x(z,\bar{z})d\phi^j(z,\bar{z})}{z-w}\right). \end{aligned} \tag{9.46}$$

We could also write the last term in integral form if it is more convenient, using (D.29). This can now be used to compute the T-dual operator $O_{\tilde{E}}$. We will demonstrate this by doing the H-flux/nilfold/T-fold example explicitly.

Example: H-flux, nilfold and T-fold

Let us look at how we can use the above to compute the T-dual to first order. Starting with the H-flux, we have $E_{ij} = G_{ij} + m\phi^x\delta E_{ij} + ...$, where δE_{ij} and the associated deformation operator are

$$\delta E_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{O}_H = m \int\limits_{\Sigma} \phi^x F^-_{yz}. \tag{9.47}$$

Substituting in the results (8.19), (8.20), (8.21), the stress tensor for the H-flux is given by

$$T_H(w) = T_G(w) + m\varepsilon^{\mu\nu\rho}\partial\phi_\mu(w) \int\limits_{\Sigma'} \frac{d^2 z}{z-w} F^-_{\nu\rho}(z,\bar{z}) + \delta_\Lambda T_H(w), \tag{9.48}$$

where

$$\delta_\Lambda T_H(w) = -m\varepsilon^{ij}\partial\phi_i(w) \oint\limits_{C_0'} \frac{\phi^x(z,\bar{z}) d\phi_j(z,\bar{z})}{z-w} \tag{9.49}$$

is a gauge-dependent piece ($\varepsilon^{yz} = -\varepsilon^{zy} = 1$). Let us now compute the T-duality transformation for this in the y-direction, which we recall is obtained via the transformation $\partial\phi_y \to -\partial\phi_y$. Doing so gives

$$\begin{aligned} T_H(w) \to \tilde{T}_H(w) = T_G(w) &- m\partial\phi_x(w) \int\limits_{\Sigma'} \frac{d^2 z}{z-w} F^+_{yz}(z,\bar{z}) \\ &+ m\partial\phi_y(w) \int\limits_{\Sigma'} \frac{d^2 z}{z-w} F^-_{xz}(z,\bar{z}) + m\partial\phi_z(w) \int\limits_{\Sigma'} \frac{d^2 z}{z-w} F^+_{xy}(z,\bar{z}) \\ &+ m\partial\phi_y(w) \oint\limits_{C_0'} \frac{\phi^x(z,\bar{z}) d\phi_z(z,\bar{z})}{z-w} + m\partial\phi_z(w) \oint\limits_{C_0'} \frac{\phi^x(z,\bar{z}) * d\phi_y(z,\bar{z})}{z-w}, \end{aligned} \tag{9.50}$$

and using (D.29) we can extract a non-integrated term from the last term in this expression, so that we get something of the same form as (9.46):

$$\begin{aligned} \tilde{T}_H(w) = T_G(w) &- m\partial\phi_x(w) \int\limits_{\Sigma'} \frac{d^2 z}{z-w} F^+_{yz}(z,\bar{z}) + m\partial\phi_y(w) \int\limits_{\Sigma'} \frac{d^2 z}{z-w} F^-_{xz}(z,\bar{z}) \\ &+ m\partial\phi_z(w) \int\limits_{\Sigma'} \frac{d^2 z}{z-w} F^-_{xy}(z,\bar{z}) + m\partial\phi_y(w) \oint\limits_{C_0'} \frac{\phi^x(z,\bar{z}) d\phi_z(z,\bar{z})}{z-w} \\ &+ m\partial\phi_z(w) \oint\limits_{C_0'} \frac{\phi^x(z,\bar{z}) d\phi_y(z,\bar{z})}{z-w} + 2m\phi^x(w,\bar{w})\partial\phi_y(w)\partial\phi_z(w). \end{aligned} \tag{9.51}$$

Comparing with the general result (9.46), we can read off that this dual background is $\tilde{E}_{ij} = G_{ij} + m\phi^x \delta\tilde{E}_{ij}$, where $\delta\tilde{E}_{ij}$ and its associated deformation operator are

$$\delta\tilde{E}_{ij} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{O}_{\tilde{E}} = -m \int\limits_{\Sigma} \phi^x F^+_{yz} = \mathcal{O}_N. \tag{9.52}$$

We recognise this as the nilfold background (8.36) to first order. We can also verify, using (8.43), (8.44), (8.45), that (9.51) is indeed the stress tensor of the nilfold. Thus, the known duality is recovered to leading order.

For the T-fold, recall that, to first order, the deformation was simply minus that of the H-flux. Thus, since the metric is simply $g_{\mu\nu} = \delta_{\mu\nu} + O(m^2)$, the stress tensor will be

$$T_{T-fold} = (\partial\phi_x + \delta\partial\phi_x)^2 + (\partial\phi_y + \delta\partial\phi_y)^2 + (\partial\phi_z + \delta\partial\phi_z)^2 = T_G - \delta T_H, \tag{9.53}$$

where $\delta T_H = T_H - T_G$. Thus, to confirm the duality, we simply need to dualise the nilfold stress tensor (9.51) and compare with (9.48). The duality is in the z-direction, so we do $\partial\phi_z \to -\partial\phi_z$, which gives

$$\begin{aligned} T_N(w) \to \tilde{T}_N(w) = T_G(w) &- m\partial\phi_x(w) \int\limits_{\Sigma'} \frac{d^2z}{z-w} F^-_{yz}(z,\bar{z}) \\ &+ m\partial\phi_y(w) \int\limits_{\Sigma'} \frac{d^2z}{z-w} F^+_{zx}(z,\bar{z}) - m\partial\phi_z(w) \int\limits_{\Sigma'} \frac{d^2z}{z-w} F^+_{xy}(z,\bar{z}) \\ &+ m\partial\phi_y(w) \oint\limits_{C_0'} \frac{\phi^x(z,\bar{z}) * d\phi_z(z,\bar{z})}{z-w} - m\partial\phi_z(w) \oint\limits_{C_0'} \frac{\phi^x(z,\bar{z}) * d\phi_y(z,\bar{z})}{z-w}, \end{aligned} \tag{9.54}$$

which can be rewritten as

$$\begin{aligned} \tilde{T}_N(w) = T_G(w) &- \delta T_H(w) \\ &- 2m\partial\phi_y(w) \oint \frac{dz}{z-w} \phi^x(z,\bar{z}) \partial\phi_z(z) + 2m\partial\phi_z(w) \oint \frac{dz}{z-w} \phi^x(z,\bar{z}) \partial\phi_y(z) \\ &+ 2m\partial\phi_y \int \frac{d^2z}{z-w} \partial\phi_z(z) \bar{\partial}\phi_x(\bar{z}) - 2m\partial\phi_z(w) \int \frac{d^2z}{z-w} \partial\phi_y(z) \bar{\partial}\phi_x(\bar{z}). \end{aligned} \tag{9.55}$$

Using (D.29), it can easily be shown that the last two rows in (9.55) cancel, and so we do indeed have $\tilde{T}_N = T_{T-fold}$ to first order in m, so the duality is recovered once again.

Note that, whilst in the Buscher procedure the conditions for its application are not strictly met in the case of the T-fold, since the isometry in the z-direction for the nilfold is not globally defined, in the procedure above the same issue does not seem to be present. Aside from the fact that the T-fold background itself is not globally defined under periodic identification of the base, the application of the above procedure is still valid.

9.3 Making use of the doubled formalism

The benefit of the doubled formalism is that T-duality is a symmetry of the sigma model and so many of the complications of the previous section do not arise in this framework. For torus bundles of the kind we have been considering (isometric, non-degenerating), there is an explicit doubled formalism, and the deformations may be understood in terms of a deformation of the doubled metric. As such, the deformation operator $\mathcal{O}$ transforms naturally under $O(d,d;\mathbb{Z})$ and T-duality may be simply understood. It is rare that we have a concrete doubled formalism[5] and so we do not expect to learn anything new, but it is useful to see how a doubled formalism may be put to good use when one is available.

The doubled action is given by[6] (7.74) and can be written as $S = S_0 + \Delta S_E$, where S_0 is the action for the flat doubled torus, and we expect that the associated deformation operators satisfy

$$e^{iQ}\mathcal{O}_E(E)e^{-iQ} = \mathcal{O}_{E'}(E'), \tag{9.56}$$

so the deformation operators of dual backgrounds are indeed T-dual in the doubled formalism. This is easiest to understand by looking at an example, so let us show this explicitly for the nilfold/H-flux case. Starting with the nilfold, with metric (8.36), we have the deformation

$$\Delta S_N[X] = \frac{1}{2}\int_\Sigma \delta\mathcal{H}_{IJ}(X)\partial\mathbb{X}^I\bar{\partial}\mathbb{X}^J, \qquad \delta\mathcal{H}_{IJ}(X) = \begin{pmatrix} 0 & -mX & 0 & 0 \\ -mX & (mX)^2 & 0 & 0 \\ 0 & 0 & (mX)^2 & mX \\ 0 & 0 & mX & 0 \end{pmatrix}. \tag{9.57}$$

[5]Identity structure manifolds seem to be the exception.

[6]We explicitly discuss the case where only the fibres of the torus bundle are doubled. The generalisation to cases where all directions are doubled including the associated WZW term in the doubled action is expected to be straightforward.

Now, let us do the same for the H-flux. In this case, we have the deformation operator

$$\Delta S_H[X] = \frac{1}{2}\int_\Sigma \delta\mathcal{H}'_{IJ}(X)\partial\mathbb{X}^I\bar{\partial}\mathbb{X}^J, \qquad \delta\mathcal{H}'_{IJ}(X) = \begin{pmatrix} (mX)^2 & 0 & 0 & mX \\ 0 & (mX)^2 & -mX & 0 \\ 0 & -mX & 0 & 0 \\ mX & 0 & 0 & 0 \end{pmatrix}. \tag{9.58}$$

Thus, we have the marginal operators

$$\mathcal{O}_N[\phi^x] = \frac{1}{2}\int_\Sigma \Big(-m\phi^x(\partial\phi_y\bar{\partial}\phi_z + \partial\phi_z\bar{\partial}\phi_y - \partial\tilde{\phi}_y\bar{\partial}\tilde{\phi}_z - \partial\tilde{\phi}_z\bar{\partial}\tilde{\phi}_y) + (m\phi^x)^2(\partial\tilde{\phi}_y\bar{\partial}\tilde{\phi}_y + \partial\phi_z\bar{\partial}\phi_z)\Big)\,, \tag{9.59}$$

$$\mathcal{O}_H[\phi^x] = \frac{1}{2}\int_\Sigma \Big(-m\phi^x(\partial\tilde{\phi}_y\bar{\partial}\phi_z + \partial\phi_z\bar{\partial}\tilde{\phi}_y - \partial\phi_y\bar{\partial}\tilde{\phi}_z - \partial\tilde{\phi}_z\bar{\partial}\phi_y) + (m\phi^x)^2(\partial\phi_y\bar{\partial}\phi_y + \partial\phi_z\bar{\partial}\phi_z)\Big)\,, \tag{9.60}$$

and we can see that, under the duality transformation $\phi^y \leftrightarrow \tilde{\phi}^y$, we do indeed have $e^{iQ}\mathcal{O}_N e^{-iQ} = \mathcal{O}_H$, as expected. The utility of the doubled formalism is that it gives a duality-covariant parameterisation of this limited space of backgrounds.

Note that the above result is non-perturbative in m. Thus, for these specific cases where we have a doubled formalism which is explicitly $O(d,d)$ covariant, it is easy to recover the full duality to all orders. For more general backgrounds where we do not have a doubled formalism, we will need to use the methods of section 9.2.2.

Chapter 10

Fermion Deformations

Everything we have done so far in this thesis has been for bosons. We now turn our attention to fermion deformations. We will look at deformations for both the flat torus case and cases where the background has explicit coordinate dependence. We will find that the bosonic deformations we derived above obtain fermionic corrections in the non-CFT case. We will also look at the T-duality between the nilfold and H-flux and show that, even with the fermion deformations included, we are able to recover the duality to first order. We will also explore how picture changing fits into this deformation story, and we will see that this puts operator deformations on a slightly less certain footing in the supersymmetric context. We work solely with fermions in the NS-NS sector throughout, and we will only look at $N = 1$ supersymmetry. The content of this chapter is based on unpublished work by the author.

10.1 Fermion deformations on a flat torus

As we did for the bosonic case, we will initially look at the deformation on a flat torus background in (2.1). We will find it useful to adopt the superspace configuration initially, since the construction is reminiscent of the purely bosonic case. Much of the calculations will go through similarly to the bosonic case, but there are subtle differences that must be addressed with care, as we will see. Initially, we will proceed naively, assuming that the methods used for the bosonic case generalise easily to the fermionic case. Later on, in section 10.6, we will show that one must be careful because the deformation operator can be written in different pictures, and this changes how we think of operator deformations in general.

10.1.1 Backgrounds without B-field

To start with, let us look at the deformation of the fermion on a flat torus for general metric, but B-field $B = 0$. It is easiest to do this in superspace, i.e. we put the toroidal CFT on a supermanifold with anticommuting coordinates $\theta, \bar{\theta}$, such that

$$\theta^2 = \bar{\theta}^2 = \{\theta, \bar{\theta}\} = 0, \tag{10.1}$$

where integration is defined by

$$\int d^2\theta \theta\bar{\theta} = 1, \tag{10.2}$$

with all other integrals zero (up to equivalence via the relations (10.1)). We define the superderivatives

$$D = \partial_\theta + \theta\partial_z, \qquad \bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}, \tag{10.3}$$

which satisfy

$$D^2 = \partial_z, \qquad \bar{D}^2 = \partial_{\bar{z}}, \qquad \{D, \bar{D}\} = 0. \tag{10.4}$$

The superconformal-invariant action is given by

$$S = \frac{1}{2} \int\limits_{\mathcal{S}} d^2z d^2\theta D\mathbf{X}^\mu(\mathbf{z})\bar{D}\mathbf{X}_\mu(\bar{\mathbf{z}}), \tag{10.5}$$

where we recall that we have $dz \wedge d\bar{z}/2\pi i$, the α' factors are absorbed into the fields, $\mathcal{S}$ is the super Riemann surface and

$$\mathbf{X}^\mu(\mathbf{z}, \bar{\mathbf{z}}) = \sqrt{2}X^\mu(z, \bar{z}) + i\theta\psi^\mu(z) + i\bar{\theta}\bar{\psi}^\mu(\bar{z}), \tag{10.6}$$

where $\mathbf{z} = (z, \theta)$ are the super-coordinates on the worldsheet. Note that the theta integral can be done explicitly since the integral simply extracts the coefficient of $\theta\bar{\theta}$. Doing so, we get (when there is no B-field)

$$S = \frac{1}{2} \int\limits_{\Sigma} d^2z \left(2\partial X^\mu(z)\bar{\partial}X_\mu(\bar{z}) + \psi^\mu\bar{\partial}\psi_\mu(z) + \bar{\psi}^\mu\partial\bar{\psi}_\mu(\bar{z})\right), \tag{10.7}$$

which is the usual $N = 1$ superstring action. The equation of motion can be written as

$$D\bar{D}X^\mu(\mathbf{z}, \bar{\mathbf{z}}) = 0, \tag{10.8}$$

and the OPE is

$$\mathbf{X}^\mu(\mathbf{z}_1,\bar{\mathbf{z}_1})\mathbf{X}^\nu(\mathbf{z}_2,\bar{\mathbf{z}_2}) \sim -g^{\mu\nu}\log|z_1-z_2-\theta_1\theta_2|^2, \tag{10.9}$$

or if we take ψ^μ on its own,

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{g^{\mu\nu}}{z-w}, \tag{10.10}$$

and similarly for $\bar{\psi}$. From this, we can compute

$$D\mathbf{X}^\mu(\mathbf{z}_1)D\mathbf{X}^\nu(\mathbf{z}_2) \sim \frac{g^{\mu\nu}}{z_1-z_2-\theta_1\theta_2}, \tag{10.11}$$

$$D\mathbf{X}^\mu(\mathbf{z}_1)\mathbf{X}^\nu(\mathbf{z}_2,\bar{\mathbf{z}}_2) \sim \frac{g^{\mu\nu}(\theta_2-\theta_1)}{z_1-z_2-\theta_1\theta_2}, \tag{10.12}$$

$$\mathbf{X}^\mu(\mathbf{z}_1,\bar{\mathbf{z}}_1)D\mathbf{X}^\nu(\mathbf{z}_2) \sim \frac{g^{\mu\nu}(\theta_1+\theta_2)}{z_1-z_2-\theta_1\theta_2}. \tag{10.13}$$

Now, we would like to use the above setup to compute the deformation of ψ^μ. Firstly, given everything we have done so far, a natural question to ask would be whether there is a universal coordinate argument we can make for fermions. Given that the superspace construction looks rather similar to the bosonic construction, it may seem as though there should be, but in fact there is not. We will explain why in section 10.3, but for now let us move on and compute the deformation using the parallel transport formalism of chapter 7.

As we did in the bosonic case, we can look at the marginal operator $\mathcal{O}$ and use this to compute the deformation of $D\mathbf{X}$. Consider the case where we have a metric deformation $g \to g+\delta g$. In the bosonic case, the deformation operator was simply given by the deformation of the action, and this deformation was generated by the graviton vertex operator. In this case, we would expect the same idea to apply. However, recall that we now have the added complication of different picture numbers. In this chapter, our calculations will only involve the deformation operator corresponding to the $(0,0)$ picture graviton vertex operator, which is in fact given by a simple deformation of the superspace action, i.e.

$$\mathcal{O} = \frac{1}{2}\int\limits_\Sigma d^2\mathbf{z}\delta g_{\mu\nu}D\mathbf{X}^\mu(\mathbf{z})\bar{D}\mathbf{X}^\nu(\bar{\mathbf{z}}). \tag{10.14}$$

Note that, were we to integrate out the fermionic coordinates, we would return to the bosonic deformation operator, since the fermionic parts vanish on-shell. In section 10.6, we will explain in more detail how this is the $(0,0)$ picture number deformation operator, but it is possible to construct deformation operators in other pictures as well, such as the canonical one.

Now, considering the deformation of $D\mathbf{X}_\mu(\mathbf{w})$, where $\mathbf{w} = (w, \phi)$, we have the OPE

$$\frac{1}{2}\int_{\Sigma'} d^2\mathbf{z}\delta g_{\nu\rho} D\mathbf{X}^\nu(\mathbf{z})\bar{D}\mathbf{X}^\rho(\bar{\mathbf{z}})D\mathbf{X}_\mu(\mathbf{w}) \sim -\frac{1}{2}\int_{\Sigma'} d^2\mathbf{z}\frac{\delta g_{\mu\rho}\bar{D}\mathbf{X}^\rho(\bar{\mathbf{z}})}{z - w - \theta\phi}. \tag{10.15}$$

Now, recall that in the bosonic case we compared the results of the $\mathcal{O}$ calculation with the mode expansion of $\partial X_\mu(g + \delta g)$, i.e. the deformed operator. Here, we would like to do something similar. What should we compare the above to? Since it is $D\mathbf{X}(w)$ which we are deforming, we need to know the mode expansion of this operator. We have

$$D\mathbf{X}_\mu(\mathbf{w}) = i\psi_\mu(w) + \sqrt{2}\phi\partial X_\mu(w), \tag{10.16}$$

where, since we are in the NS-NS sector, the mode expansions are

$$\psi_\mu(w) = \sum_n g_{\mu\nu}\psi^\nu_{n+1/2}w^{-n-1}, \qquad \bar{\psi}_\mu(w) = \sum_n g_{\mu\nu}\bar{\psi}^\nu_{n+1/2}w^{-n-1}, \tag{10.17}$$

where the modes obey the anticommutation relations

$$\{\psi^\mu_{n+1/2}, \psi^\nu_{m-1/2}\} = \{\bar{\psi}^\mu_{n+1/2}, \bar{\psi}^\nu_{m-1/2}\} = g^{\mu\nu}\delta_{n+m,0}. \tag{10.18}$$

Thus,

$$\sqrt{2}\phi\partial X_\mu(w) + i\psi_\mu(w) = \sum_n g_{\mu\nu}\left(-i\phi\alpha^\nu_n + \psi^\nu_{n+1/2}\right)w^{-n-1}. \tag{10.19}$$

Going back to the $\mathcal{O}$ calculation, we can expand

$$\frac{1}{w - (z - \theta\phi)} = \sum_{n\geq 0} w^{-n-1}(z - \theta\phi)^n = \sum_{n\geq 0} w^{-n-1}(z^n - n\theta\phi z^{n-1}), \tag{10.20}$$

where the last equality follows from the fact that θ, ϕ are fermionic. Thus, (10.15) can be written as

$$\frac{\delta g_{\mu\nu}}{2}\sum_{n\geq 0} w^{-n-1}\int_{\Sigma'} d^2\mathbf{z}(z^n - n\theta\phi z^{n-1})\left(i\bar{\psi}^\nu(\bar{z}) + \sqrt{2}\bar{\theta}\bar{\partial}X^\nu(\bar{z})\right), \tag{10.21}$$

and doing the fermionic integral gives

$$\frac{\delta g_{\mu\nu}}{2}\sum_{n\geq 0} w^{-n-1}\int_{\Sigma'} d^2z z^{-n-1}\phi\bar{\partial}X(\bar{z})^\nu = -\frac{\delta g_{\mu\nu}}{2}\sum_{n\geq 0}\phi w^{-n-1}\oint_{C'} d\bar{z}\bar{z}^n\bar{\partial}X(\bar{z})^\nu, \quad (10.22)$$

where in the last step we have integrated out a ∂_z to go to the boundary. Thus, comparing this to (10.19), we see that we recover the usual bosonic transformation of the α_n modes, and additionally the fermionic modes are unchanged, i.e.

$$\delta_{\mathcal{O}}\psi_\mu = 0. \quad (10.23)$$

Given our discussion above regarding the picture number, this is as expected. If we assume that the δ_E deformation acts as usual, we overall have that

$$(\delta_{\mathcal{O}} + \delta_E)\psi_\mu = \psi_\mu + \frac{1}{2}\delta g_{\mu\nu}\psi^\nu. \quad (10.24)$$

Now, recall that the transformation of ∂X_μ truncated at first order in δg. At higher orders, the $\delta_{\mathcal{O}}$ and δ_E transformations cancelled out with each other (the details of this calculation at second order are in appendix E). However, for fermions this will clearly not be the case, since $\delta_{\mathcal{O}}\psi_\mu = 0$. In fact, if we apply δ_E again we get

$$\delta_E^2\psi_\mu = -\frac{1}{4}\delta g_{\mu\nu}g^{\nu\rho}\delta g_{\rho\sigma}\psi^\sigma. \quad (10.25)$$

We can iterate this to obtain the finite transformation. Since we have to work to arbitrarily high order here, it takes a bit more work compared to the bosonic case, especially since we cannot use a universality argument. Note that the finite result is not a simple exponential transformation, since δ_E acts on all metric and vielbein factors. For example, in (10.25), δ_E would act on $g^{\nu\rho}$ and ψ^σ, and overall we would find that

$$\delta_E^3\psi_\mu = \frac{3}{8}\left[\left(\delta g g^{-1}\right)^3\right]_\mu^{\ \nu}\psi_\nu, \quad (10.26)$$

where we have used the shorthand

$$\left[\left(\delta g g^{-1}\right)^3\right]_\mu^{\ \nu} = \delta g_{\mu\rho}g^{\rho\sigma}\delta g_{\sigma\tau}g^{\tau\theta}\delta g_{\theta\alpha}g^{\alpha\nu}, \quad (10.27)$$

and similarly for $(\delta g g^{-1})^n$. Recalling that the operator δ_E acts via an exponential operator, we have

$$\left(1+\delta_E+\frac{1}{2!}\delta_E^2+\frac{1}{3!}\delta_E^3\right)\psi_\mu = \left(\delta_\mu{}^\nu + \frac{1}{2}\left[\left(\delta g g^{-1}\right)\right]_\mu{}^\nu - \frac{1}{8}\left[\left(\delta g g^{-1}\right)^2\right]_\mu{}^\nu + \frac{1}{16}\left[\left(\delta g g^{-1}\right)^3\right]_\mu{}^\nu\right)\psi_\nu, \tag{10.28}$$

which we notice looks like a $(1+\delta g g^{-1})^{1/2}$ expansion (to third order). In fact, we can prove inductively that this is the case. Taking as our hypothesis that

$$\frac{1}{n!}\delta_E^n\psi_\mu = \frac{1\cdot 3\cdot \ldots \cdot (2n-3)}{n!2^n}\left[-\left(\delta g g^{-1}\right)^n\right]_\mu{}^\nu \psi_\nu, \tag{10.29}$$

it is straightforward to apply δ_E to this and prove the hypothesis for all n. Thus, the finite transformation is

$$\psi_\mu(g') = \left[\left(1+(g'-g)g^{-1}\right)^{1/2}\right]_\mu{}^\nu \psi_\nu(g), \tag{10.30}$$

where

$$\left[\left(1+(g'-g)g^{-1}\right)^{1/2}\right]_\mu{}^\nu \equiv \sum_{n\geq 0}\frac{1\cdot 3\cdot \ldots \cdot (2n-3)}{n!2^n}\left[-\left((g'-g)g^{-1}\right)^n\right]_\mu{}^\nu. \tag{10.31}$$

Note that $\bar{\psi}_\mu$ will have the obvious analogous transformation. We can verify that this transformation is correct by looking at the OPEs

$$\begin{aligned}\psi_\mu(g')(z)\psi_\nu(g')(w) &= \left[\left(1+(g'-g)g^{-1}\right)^{1/2}\right]_\mu{}^\rho \left[\left(1+(g'-g)g^{-1}\right)^{1/2}\right]_\nu{}^\sigma \psi_\rho(g)(z)\psi_\sigma(g)(w)\\ &\sim \frac{1}{z-w}\left[\left(1+(g'-g)g^{-1}\right)^{1/2}\right]_\mu{}^\rho \left[\left(1+(g'-g)g^{-1}\right)^{1/2}\right]_\nu{}^\sigma g_{\rho\sigma},\end{aligned} \tag{10.32}$$

and using the expansion (10.31) it can be shown that

$$\left[\left(1+(g'-g)g^{-1}\right)^{1/2}\right]_\mu{}^\rho \left[\left(1+(g'-g)g^{-1}\right)^{1/2}\right]_\nu{}^\sigma g_{\rho\sigma} = (g+(g'-g))_{\mu\nu} = g'_{\mu\nu}, \tag{10.33}$$

and so we recover the correct OPEs, as required. Note the finite transformation can also be written in the simpler form,

$$\psi_\mu(g') = \left(g'g^{-1}\right)_\mu{}^\nu \psi_\nu(g). \tag{10.34}$$

Superconformal symmetry

Now, how does the above square with the superconformal symmetry? The superconformal transformations are

$$\sqrt{2}\Delta X^\mu = \varepsilon\psi^\mu + \bar{\varepsilon}\bar{\psi}^\mu, \tag{10.35}$$

$$\frac{1}{\sqrt{2}}\Delta\psi^\mu = -\varepsilon\partial X^\mu, \tag{10.36}$$

$$\frac{1}{\sqrt{2}}\Delta\bar{\psi}^\mu = -\bar{\varepsilon}\bar{\partial} X^\mu, \tag{10.37}$$

where $\varepsilon, \bar{\varepsilon}$ are anticommuting parameters, and the currents which generate these transformations are

$$j^\varepsilon = \varepsilon T_F, \qquad \bar{j}^{\bar{\varepsilon}} = \bar{\varepsilon}\bar{T}_F, \tag{10.38}$$

where $\bar{T}_F$ is the antiholomorphic supercurrent. It is possible to use the above results to relate SUSY transformations at different backgrounds. Before we do so, it will be useful to distinguish the point of enhanced symmetry as we did for the boson. Therefore, we define[1]

$$\eta^\mu := \psi^\mu(G), \tag{10.39}$$

similarly to how we defined $\phi^\mu = X^\mu(G)$ for bosons.

As an example, let us look at the circle, since this is easy to do to all orders. Let us suppose we start from the self-dual radius and we deform to a radius $R = 1 + \delta R$, where δR may not be small. In this case, we have, for example,[2]

$$\Delta_R^{\varepsilon_R}\psi(R) = -\sqrt{2}\varepsilon_R\partial X(R). \tag{10.40}$$

Then, using (4.30), we can write this as

$$\begin{aligned}\Delta_R^{\varepsilon_R}\psi(R) =& -\sqrt{2}\varepsilon_R\left(\partial\phi + \left(\delta R + \frac{1}{2}\delta R^2\right)(\partial\phi - \bar{\partial}\phi)\right)\\ =&\frac{\varepsilon_R}{1+\delta R+\frac{1}{2}\delta R^2}\left(\frac{1}{\varepsilon}\Delta^\varepsilon\eta + \left(\delta R + \frac{1}{2}\delta R^2\right)\left(\frac{\Delta^\varepsilon\eta}{\varepsilon} - \frac{\Delta^\varepsilon\bar{\eta}}{\bar{\varepsilon}}\right)\right)\\ =&\frac{\varepsilon_R}{\varepsilon}\Delta^\varepsilon\eta - \frac{(1+\delta R)^2-1}{(1+\delta R)^2+1}\frac{\varepsilon_R}{\bar{\varepsilon}}\Delta^\varepsilon\bar{\eta},\end{aligned} \tag{10.41}$$

[1] Note that, since ψ is not universal, there is no ambiguity or confusion in defining η directly instead of indirectly through something like $\partial\eta$.

[2] As usual, when we write ∂X in the $d = 1$ case we mean ∂X_μ, i.e. indices lowered. When we write ψ, we mean ψ^μ, i.e. indices *raised*.

where we have used an arbitrary SUSY transformation at the self-dual radius ε. If we choose $\varepsilon_R = \varepsilon$, we get

$$\Delta^{\varepsilon}_{R}\psi = \Delta^{\varepsilon}\eta - \frac{R^2-1}{R^2+1}\frac{\varepsilon}{\bar{\varepsilon}}\Delta^{\varepsilon}\bar{\eta}. \tag{10.42}$$

Similarly, we have

$$\Delta^{\varepsilon}_{R}\bar{\psi}(R) = \Delta^{\varepsilon}\bar{\eta} - \frac{R^2-1}{R^2+1}\frac{\varepsilon}{\bar{\varepsilon}}\Delta^{\varepsilon}\eta. \tag{10.43}$$

We can do the same for the SUSY transformation of X, which gives

$$\Delta^{\varepsilon}_{R}X(R) = \frac{1}{R}\Delta^{\varepsilon}\phi. \tag{10.44}$$

10.2 General toroidal backgrounds

In the case when we have a non-zero B-field, there are further complications. Note that we are still dealing with the constant background case (i.e. still in the realm of CFTs) and we will come to the non-CFT case later. Firstly, the action is simply

$$S = \frac{1}{2}\int_{\mathcal{S}} d^2\mathbf{z} E_{\mu\nu}(\mathbf{X}) D\mathbf{X}^{\mu}(\mathbf{z})\bar{D}\mathbf{X}^{\nu}(\bar{\mathbf{z}}), \tag{10.45}$$

which in this case can be written as

$$S = \int_{\Sigma} d^2 z E_{\mu\nu}\partial X^{\mu}(z)\bar{\partial}X^{\nu}(\bar{z}) + \frac{1}{2}\int_{\Sigma} d^2 z g_{\mu\nu}(\psi^{\mu}\bar{\partial}\psi^{\nu}(z) + \bar{\psi}^{\mu}\partial\bar{\psi}^{\nu}(\bar{z})), \tag{10.46}$$

i.e. there is still no mixing between the fermionic and bosonic parts (we will see that this is not the case when E has non-trivial coordinate dependence). Thus, the $\mathcal{O}$ deformation is as before for ψ_μ, i.e. $\delta_{\mathcal{O}}\psi_\mu = 0$.

For δ_E, recall that, in the bosonic case, we used the doubled geometry to derive the transformation in the case when $B \neq 0$. In [77], a similar construction was demonstrated for fermions. We will not go into the details here, but essentially it was done using a superspace approach. Using this formulation, a sigma model like (3.52) could be constructed for the doubled superfield and the form was largely the same as the bosonic case, but with superderivatives instead. Thus, the δ_E transformation follows in the same way as it did for the bosonic case and we have

$$\delta_E\psi_\mu = \frac{1}{2}\delta E_{\mu\nu}\psi^{\nu}, \tag{10.47}$$

and this can be iterated to obtain the transformation to any order desired. For example, at second order we have

$$\delta_E^2\psi_\mu = \frac{1}{2}\delta E_{\mu\nu}\delta_E(\psi^\nu) = -\frac{1}{4}\delta E_{\mu\nu}g^{\nu\rho}\delta E_{\rho\sigma}\psi^\sigma. \tag{10.48}$$

Following the same method as for the $B = 0$ case above, we find that the finite transformation is given by

$$\psi_\mu(E') = \left[\left(1 + (E' - E)g^{-1}\right)^{1/2}\right]_\mu^{\ \nu}\psi_\nu(E). \tag{10.49}$$

In what follows, we will only be interested in the first order transformation. A question one could ask is whether we can successfully recover the Buscher rules using the stress tensor or the supercurrent T_F. As we have seen in chapter 9, we can use the stress tensor to derive T-dual backgrounds, and this is equivalent to using the Buscher rules for a flat background. Deriving this in the supersymmetric case can also be viewed as a consistency check for the fermion deformations. We will look at the supercurrent T_F and recover the Buscher rules to first order. Recall that $T_F = i\sqrt{2}\psi^\mu\partial X_\mu$ and consider the background

$$E = 1 + \delta E, \tag{10.50}$$

and the T-dual background

$$\tilde{E} = 1 + \delta\tilde{E}. \tag{10.51}$$

We will show that the T-dual of $T_F(\tilde{E})$ is indeed $T_F(E)$. Firstly, using (2.47) and expanding the metric and B-field to first order, we obtain the first order Buscher rules as

$$\begin{aligned}\delta\tilde{g}_{xx} = -\delta g_{xx}, \qquad \delta\tilde{g}_{xi} = -\delta B_{xi}, \qquad \delta\tilde{g}_{ij} = \delta g_{ij},\\ \delta\tilde{B}_{xi} = -\delta g_{xi}, \qquad \delta\tilde{B}_{ij} = \delta B_{ij},\end{aligned} \tag{10.52}$$

where we have spacetime coordinates $x^\mu = (x, x^i)$ and we are taking the T-dual in the x-direction. Then, using (4.30),(10.47) and (10.52), we can write $\psi_\mu(\tilde{E}), \partial X_\mu(\tilde{E})$ in terms of objects at the self-dual point as

$$\psi_\mu(\tilde{E}) = \begin{pmatrix}\psi_x(\tilde{E})\\ \psi_i(\tilde{E})\end{pmatrix} = \begin{pmatrix}\eta_x - \frac{1}{2}\delta E_{xx}\eta_x - \frac{1}{2}\delta E_{xi}\eta_i\\ \eta_i + \frac{1}{2}\delta E_{ij}\eta_j + \frac{1}{2}\delta E_{ix}^T\eta_x\end{pmatrix}, \tag{10.53}$$

$$\partial X_\mu(\tilde{E}) = \begin{pmatrix} \partial X(\tilde{E}) \\ \partial X_i(\tilde{E}) \end{pmatrix} = \begin{pmatrix} \partial\phi_x - \frac{1}{2}\delta E_{xx}(\partial\phi_x - \bar{\partial}\phi_x) - \frac{1}{2}\delta E_{xi}(\partial\phi_i - \bar{\partial}\phi_i) \\ \partial\phi_i + \frac{1}{2}\delta E_{ij}(\partial\phi_j - \bar{\partial}\phi_j) + \frac{1}{2}\delta E^T_{ix}(\partial\phi_x - \bar{\partial}\phi_x) \end{pmatrix}. \quad (10.54)$$

Now, $T_F(E)$ is given by

$$T_F(E) = \delta^{\mu\nu}\eta_\mu\partial\phi_\nu + \frac{1}{2}\delta E_{\mu\nu}\eta^\mu(\partial\phi^\nu - \bar{\partial}\phi^\nu) + \frac{1}{2}\delta E_{\mu\nu}\partial\phi^\mu\eta^\nu - \delta g_{\mu\nu}\partial\phi^\nu. \quad (10.55)$$

Using this general form, we substitute (10.53), (10.54) into $T_F(\tilde{E})$ and apply the T-duality transformation $(\eta_x, \partial\phi_x) \to -(\eta_x, \partial\phi_x)$ to get

$$\begin{aligned} T_F(\tilde{E}) \to & \delta^{\mu\nu}\eta_\mu\eta_\nu - \frac{1}{2}\delta g_{xx}\eta_x(\partial\phi_x + \bar{\partial}\phi_x) + \frac{1}{2}\delta B_{xi}\eta_x(\partial\phi_i - \bar{\partial}X_i) + \frac{1}{2}\delta g_{xi}\eta_x(\partial\phi_i - \bar{\partial}\phi_i) \\ & - \frac{1}{2}\delta B_{ix}\eta_i(\partial\phi_x + \bar{\partial}\phi_x) - \frac{1}{2}\delta g_{xi}\eta_i(\partial\phi_x + \bar{\partial}\phi_x) + \frac{1}{2}\delta E_{ij}\eta_i(\partial\phi_j - \bar{\partial}\phi_j) \\ & - \frac{1}{2}\delta g_{xx}\eta_x\partial\phi_x + \frac{1}{2}\delta B_{xi}\partial\phi_x\eta_i + \frac{1}{2}\delta g_{xi}\partial\phi_x\eta_i + \frac{1}{2}\delta B_{xi}\partial\phi_i\eta_x - \frac{1}{2}\delta g_{ix}\partial\phi_i\eta_x \\ & + \frac{1}{2}\delta E_{ij}\partial\phi_i\eta_j + \delta g_{xx}\eta_x\partial\phi_i - \delta B_{xi}(\eta_x\partial\phi_i + \eta_i\partial\phi_x) - \delta g_{ij}\eta_i\partial\phi_j. \end{aligned} \quad (10.56)$$

Collecting the terms together, this can be shown to equal (10.55), so we recover the correct T-duality transformation to first order.

Before moving onto non-CFT cases, let us briefly explore the possibility of a universal coordinate construction like that utilised for the boson on a flat torus. In particular, let us see why a similar construction does not seem possible for fermions.

10.3 Universal coordinates for fermions

Recall that the basis for a universal coordinate construction for bosons was the background-independent canonical commutation relations. For fermions, we have canonical anticommutation relations given by

$$\{\psi^\mu(\sigma), \psi^\nu(\sigma')\} = 2\pi g^{\mu\nu}\delta(\sigma - \sigma'), \quad (10.57)$$

and similarly for $\bar{\psi}$. Note that we have explicit dependence on the metric here, and so it does not seem that we can immediately generalise the bosonic case. However, we might still hope that we can use the superspace formalism to construct universal objects. To do this, we need to find a 'conjugate supermomentum' $\mathbf{\Pi}_\mu$. We could try to find such an object the canonical way using the action, but it is easier to use our knowledge of the bosonic case. Recall that we have

$$X'^{\mu}(z,\bar{z}) = -ig^{\mu\nu}\left(z\partial X_\nu(z) - \bar{z}\bar{\partial}X_\nu(\bar{z})\right),$$
$$\Pi_\mu(z,\bar{z}) = izE^T_{\mu\nu}\partial X^\nu(z) + i\bar{z}E_{\mu\nu}\bar{\partial}X^\nu(\bar{z}). \quad (10.58)$$

We can construct analogues for $\mathbf{X}', \mathbf{\Pi}$ by replacing $\partial, \bar{\partial}$ with $D, \bar{D}$, i.e.

$$\mathbf{X}'^{\mu}(\mathbf{z},\bar{\mathbf{z}}) = -ig^{\mu\nu}\left(zD\mathbf{X}_\nu(\mathbf{z}) - \bar{z}\bar{D}\mathbf{X}_\nu(\bar{\mathbf{z}})\right),$$
$$\mathbf{\Pi}_\mu(\mathbf{z},\bar{\mathbf{z}}) = izE^T_{\mu\nu}D\mathbf{X}^\nu(\mathbf{z}) + i\bar{\mathbf{z}}E_{\mu\nu}\bar{D}\mathbf{X}^\nu(\bar{\mathbf{z}}). \quad (10.59)$$

In terms of modes, these can be written as

$$\mathbf{X}'^{\mu}(\mathbf{z},\bar{\mathbf{z}}) = \sum_n \left(\psi^\mu_{n+1/2} - \theta\alpha^\mu_n\right) z^{-n} - \sum_n \left(\bar{\psi}^\mu_{n+1/2} - \bar{\theta}\bar{\alpha}^\mu_n\right)\bar{z}^{-n},$$
$$\mathbf{\Pi}_\mu(\mathbf{z},\bar{\mathbf{z}}) = E^T_{\mu\nu}\sum_n \left(\theta\alpha^\nu_n - \psi^\nu_{n+1/2}\right) z^{-n} + E_{\mu\nu}\sum_n \left(\bar{\theta}\bar{\alpha}^\nu_n - \bar{\psi}^\nu_{n+1/2}\right)\bar{z}^{-n}. \quad (10.60)$$

Note that, whilst $\mathbf{X}$ is bosonic, these objects are fermionic. Let us look at the commutation/anticommutation relations of these objects to see if we can consistently identify any of them as universal. Note that we will switch to real coordinates (σ,τ) (in Euclidean signature) here. Firstly, after some short algebra, we find that[3]

$$[\mathbf{\Pi}_\mu(\sigma), \mathbf{X}^\nu(\sigma')] = \left(\theta\left(\delta_\mu{}^\nu - B_{\mu\rho}g^{\rho\nu}\right) + \bar{\theta}\left(\delta_\mu{}^\nu + B_{\nu\rho}g^{\rho\nu}\right)\right)2\pi i\delta(\sigma-\sigma'), \quad (10.62)$$

where we have suppressed the $\tau,\theta,\bar{\theta}$ dependence for clarity. What we notice here is that the $\theta,\bar{\theta}$ dependence prevents the background-dependent terms from cancelling. Thus, from this commutator it seems as though we cannot identify $\mathbf{X},\mathbf{\Pi}$ as universal. Even if we look at the anticommutator between $\mathbf{X}'$ and $\mathbf{\Pi}$, we find that

$$\{\mathbf{\Pi}_\mu(\sigma), \mathbf{X}'^\nu(\sigma')\} = \left(-\left(\delta_\mu{}^\nu - B_{\mu\rho}g^{\rho\nu}\right)e^{\tau-i\sigma} + \left(\delta_\mu{}^\nu + B_{\mu\rho}g^{\rho\nu}\right)e^{\tau+i\sigma'}\right), \quad (10.63)$$

where the exponential factors come from $\{\psi^\mu_{n+1/2}, \psi^\nu_{m+1/2}\} = g^{\mu\nu}\delta_{n+m+1,0}$. Thus, once again we see that we do not get a background independent result.

The above is not to say that a universality argument is impossible for fermions, but that it is at least not as straightforward as the bosonic case. In any case, the fermion

[3] Where we have used

$$\sum_n e^{in(\sigma-\sigma')} = 2\pi\delta(\sigma-\sigma') \quad (10.61)$$

transformation is simple enough that we did not need such an argument to compute its deformation to arbitrary order.

Let us now move away from CFTs and turn to the case where there is coordinate dependence in the background deformation.

10.4 General NLSMs

We now come to the case of general metric and B-field, so there can be some coordinate dependence in the background. It will be easiest to integrate out the fermions in the action (10.45) before we compute the deformation.

We expand the background $E(\mathbf{X})$ as

$$E_{\mu\nu}(\mathbf{X}) = E_{\mu\nu}(X) + \frac{1}{\sqrt{2}}(i\theta\psi^\rho + i\bar{\theta}\bar{\psi}^\rho)\partial_\rho E_{\mu\nu}(X) - \frac{1}{2}\theta\bar{\theta}\psi^\rho\bar{\psi}^\sigma\partial_\rho\partial_\sigma E_{\mu\nu}. \tag{10.64}$$

Substituting this in, we obtain the action

$$S = \int_\Sigma \left\{ E_{\mu\nu}(X)\partial X^\mu \partial X^\nu + \frac{1}{2}g_{\mu\nu}(X)\left(\psi^\mu \bar{D}\psi^\nu + \bar{\psi}^\mu D\bar{\psi}^\nu\right) + \frac{1}{2}R_{\mu\nu\rho\sigma}(X)\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma \right\}, \tag{10.65}$$

where

$$\bar{D}\psi^\nu = \bar{\partial}\psi^\nu + \left({\Gamma^\nu}_{\rho\sigma}(X) + g^{\nu\tau}(X)\partial_\tau B_{\rho\sigma}(X)\right)\bar{\partial}X^\rho\psi^\sigma, \tag{10.66}$$

$$D\psi^\nu = \partial\bar{\psi}^\nu + \left({\Gamma^\nu}_{\rho\sigma}(X) - g^{\nu\tau}(X)\partial_\tau B_{\rho\sigma}(X)\right)\partial X^\rho\bar{\psi}^\sigma. \tag{10.67}$$

What we see now is that there are terms which mix fermions and bosons. Thus, we expect the deformations of the fields to be significantly different to the CFT case. In particular, the $\delta_{\mathcal{O}}\partial X$ transformation will now differ from the purely bosonic case and will contain fermionic corrections. Let us look at how ∂X and ψ now transform. Note that, when we look at the general case below, we will not explicitly compute the contributions from the X-dependence in the metric and B-field, since the precise effect these have on the deformations must be computed on a case-by-case basis. When we come to the H-flux/nilfold example, we will compute the full deformation, with all X-dependence taken into account explicitly.

10.4.1 ∂X_μ deformation

We will focus on the new terms arising due to the covariant derivatives $D, \bar{D}$. For ∂X_μ, the term with $D\psi$ will be the one of relevance. We compute the OPE

$$
\begin{aligned}
&\frac{1}{2}\int_{\Sigma'} d^2z g_{\nu\rho}(X(z,\bar{z}))\bar{\psi}^\nu(\bar{z})\left({\Gamma^\rho}_{\sigma\tau}(X(z,\bar{z})) - g^{\rho\theta}(X(z,\bar{z}))\partial_\theta B_{\sigma\tau}(X(z,\bar{z}))\right)\partial X^\sigma(z)\bar{\psi}^\tau(\bar{z})\partial X_\mu(w) \\
\sim &-\frac{1}{4}\int_{\Sigma'}\frac{d^2z}{(z-w)^2}g_{\nu\rho}(X(z,\bar{z}))\bar{\psi}^\nu(\bar{z})\left({\Gamma^\rho}_{\mu\sigma}(X(z,\bar{z})) - g^{\rho\tau}(X(z,\bar{z}))\partial_\tau B_{\mu\sigma}(X(z,\bar{z}))\right)\bar{\psi}^\sigma(\bar{z}) + ... \\
= &-\frac{1}{4}\oint_{C'}\frac{d\bar{z}}{z-w}g_{\nu\rho}(X(z,\bar{z}))\bar{\psi}^\nu\left({\Gamma^\rho}_{\mu\sigma}(X(z,\bar{z})) - g^{\rho\tau}(X(z,\bar{z}))\partial_\tau B_{\mu\sigma}(X(z,\bar{z}))\right)\bar{\psi}^\sigma(\bar{z}) + ...,
\end{aligned}
\tag{10.68}
$$

where the ... denotes terms that would arise due to the X-dependence in the background objects. Note that there may also be some extra terms from the curvature term in the action, though for the nilfold the curvature vanishes at first order in m. Thus, the deformation of ∂X_μ can be summarised as

$$\delta\partial X_\mu = \delta_B\partial X_\mu + \delta_F\partial X_\mu, \tag{10.69}$$

where the bosonic deformation $\delta_B\partial X_\mu$ is as usual, and

$$\delta_F\partial X_\mu = -\frac{1}{4}g_{\nu\rho}(X)\bar{\psi}^\nu\left({\Gamma^\rho}_{\mu\sigma}(X) - g^{\rho\tau}(X)\partial_\tau B_{\mu\sigma}(X)\right)\bar{\psi}^\sigma + ..., \tag{10.70}$$

where the ..., in addition to the terms discussed above, also includes the terms from the potential X-dependence in the Riemann tensor. A similar results holds for $\bar{\partial}X_\mu$. Let us now look at the fermion deformation.

10.4.2 ψ_μ deformation

In the constant case, we saw that there was no $\mathcal{O}$ contribution to the fermion deformation, and this could be seen in a number of ways, one of which was through the superspace construction that we employed. Another way we can see this is that, in the constant case, once we take the OPE, we schematically get

$$\int \bar{\partial}\psi, \tag{10.71}$$

and since the flat equation of motion is $\bar{\partial}\psi = 0$, this vanishes. What we will now find is that, due to the X-dependence, this is no longer the case, since the equation of motion is more complicated. Once again, we will just look at the terms which give new contributions compared to the constant case. We have

$$
\begin{aligned}
&\frac{1}{2}\int_{\Sigma'} d^2z g_{\nu\rho}(X)\psi^\nu(z)\left(\Gamma^\rho{}_{\sigma\tau}(X) + g^{\rho\theta}(X)\partial_\theta B_{\sigma\tau}(X)\right)\bar{\partial}X^\sigma(\bar{z})\psi^\tau(z)\psi_\mu(w) \\
\sim&\frac{1}{2}\int_{\Sigma'} \frac{d^2z}{z-w} g_{\nu\rho}(X)\left(\Gamma^\rho{}_{\sigma\tau}(X) + g^{\rho\theta}(X)\partial_\theta B_{\sigma\tau}(X)\right)\left(\psi^\nu(z)\bar{\partial}X^\sigma(\bar{z})\delta^\tau_\mu - \psi^\tau(z)\bar{\partial}X^\sigma(\bar{z})\delta^\nu_\mu\right) \\
=&\frac{1}{2}\int_{\Sigma'} \frac{d^2z}{z-w}\psi^\nu(z)\bar{\partial}X^\rho(\bar{z})(\partial_\mu g_{\nu\rho}(X) - \partial_\nu \delta g_{\rho\mu}(X) + 2\partial_\mu B_{\nu\rho}(X)),
\end{aligned}
\tag{10.72}
$$

where we have used that

$$
g_{\nu\rho}\Gamma^\rho{}_{\sigma\tau} = \frac{1}{2}\left(\partial_\sigma g_{\nu\tau} + \partial_\tau g_{\nu\sigma} - \partial_\nu g_{\sigma\tau}\right), \tag{10.73}
$$

as well as the fact that

$$
:\psi_\mu\psi_\nu::\psi_\rho: \;\sim -:\psi_\nu\psi_\mu::\psi_\rho:. \tag{10.74}
$$

Now, since we cannot take out a derivative in (10.72) like we could in the ∂X_μ case, we cannot immediately deduce the deformation of ψ_μ as a whole, but we can write the mode deformation as

$$
\delta(g_{\mu\nu}\psi^\nu_{n+1/2}) = -\frac{1}{2}\int_{\Sigma'} d^2z z^n \psi^\nu(z)\bar{\partial}X^\rho(\bar{z})(\partial_\mu g_{\nu\rho}(X) - \partial_\nu\delta g_{\rho\mu}(X) + 2\partial_\mu B_{\nu\rho}(X)). \tag{10.75}
$$

Thus, we see that, in general, there is also an $\mathcal{O}$ contribution to the fermion deformation, and it involves both the fermionic and bosonic fields. To get a better feel for this, let us look at the specific cases of the H-flux and nilfold. In particular, we will use these deformations to demonstrate that the T-duality holds as expected.

10.5 Example: The nilfold & T^3 with H-flux

We will first derive the deformations of the fields using the results derived above. We will then look at the stress tensor as well as the fermionic supercurrents, and we will show that these objects transform as expected under T-duality. Note that we will only work to first order in the parameter m throughout. This will make the calculations particularly simple since, to first order, there are no extra terms resulting from the X-dependence (i.e. the '...' terms above are zero here to first order), so we can directly use the results above.

10.5.1 Nilfold

Using the nilfold background (8.36), we can compute the first order Christoffel symbols as

$$\Gamma^x{}_{yz} = \frac{1}{2}m, \qquad \Gamma^y{}_{zx} = -\frac{1}{2}m, \qquad \Gamma^z{}_{xy} = -\frac{1}{2}m, \tag{10.76}$$

with all else zero up to symmetry. Also, as mentioned above, to first order, the Riemann tensor R vanishes.

Now, note that, although the metric does indeed have X-dependence, the term in the action

$$\frac{1}{2}\int\limits_\Sigma g_{\mu\nu}(X)\left(\psi^\mu \bar{D}\psi^\nu + \bar{\psi}^\mu D\bar{\psi}^\nu\right) \tag{10.77}$$

in fact does *not* have any explicit X-dependence to first order in m. This is because the Christoffel symbols are at least order m, and their $O(m)$ parts are constant, so the $O(m)$ part of (10.77) is given by the product of the $O(m)$ part of the Christoffel symbols and the $O(1)$ part of the metric, both of which are constant. Thus, we can simply use the results derived above, and we find that the full transformations to order

m are[4]

$$\delta\partial\phi_x(w) = -\frac{1}{2}m\int\limits_{\Sigma'}\frac{d^2z}{z-w}\left(\partial\phi_y(z)\bar{\partial}\phi_z(\bar{z}) - \partial\phi_z(z)\bar{\partial}\phi_y(\bar{z})\right), \tag{10.78}$$

$$\begin{aligned}\delta\partial\phi_y(w) = \frac{1}{2}m\int\limits_{\Sigma'}\frac{d^2z}{z-w}(\partial\phi_x(z)\bar{\partial}\phi_z(\bar{z}) - \partial\phi_z(z)\bar{\partial}\phi_x(\bar{z})) + \frac{1}{2}m\oint\limits_{C'}\frac{\phi^x(z,\bar{z})d\phi_z(z,\bar{z})}{z-w} \\ - \frac{1}{4}m\bar{\eta}_x(\bar{w})\bar{\eta}_z(\bar{w}),\end{aligned} \tag{10.79}$$

$$\begin{aligned}\delta\partial\phi_z(w) = \frac{1}{2}m\int\limits_{\Sigma'}\frac{d^2z}{z-w}(\partial\phi_x(z)\bar{\partial}\phi_y(\bar{z}) - \partial\phi_y(z)\bar{\partial}\phi_x(\bar{z})) + \frac{1}{2}m\oint\limits_{C'}\frac{\phi^x(z,\bar{z})d\phi^y(z,\bar{z})}{z-w} \\ - \frac{1}{4}m\bar{\eta}_x(\bar{w})\bar{\eta}_y(\bar{w}),\end{aligned} \tag{10.80}$$

$$\delta\eta_x(w) = -\frac{1}{2}m\int\limits_{\Sigma'}\frac{d^2z}{z-w}\left(\eta_y(z)\bar{\partial}\phi_z(\bar{z}) + \eta_z(z)\bar{\partial}\phi_y(\bar{z})\right), \tag{10.81}$$

$$\delta\eta_y(w) = -\frac{1}{2}m\left(\phi^x(w,\bar{w})\eta_z(w) - \int\limits_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar{\partial}\phi_z(\bar{z})\right), \tag{10.82}$$

$$\delta\eta_z(w) = -\frac{1}{2}m\left(\phi^x(w,\bar{w})\eta_y(w) - \int\limits_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar{\partial}\phi_y(\bar{z})\right). \tag{10.83}$$

It is interesting to note that, even here, the holomorphic and antiholomorphic fermion fields do not interact in any way. This could also have been anticipated from the action, since the only term mixing them was the curvature term, which vanishes in the examples considered here. The holomorphic bosonic fields, on the other hand, *do* interact with the antiholomorphic fermion fields.

10.5.2 T^3 with H-flux

For the H-flux, notice that, in the new contributions that arose in section 10.4, the B-field only came in through ∂B terms, which, for the H-flux, is always constant.

[4] Note that we once again use η to refer to the point where $E = G$, though, as in the bosonic case, we are going to the covering space where the x-direction is decompactified, so the starting point is not a T^3 strictly speaking.

Thus, as with the nilfold, we can directly take the results of section 10.4. We get

$$\delta\partial\phi_x(w) = \frac{1}{2}m\int\limits_{\Sigma'}\frac{d^2z}{z-w}\left(\partial\phi_y(z)\bar{\partial}\phi_z(\bar{z}) - \partial\phi_z(z)\bar{\partial}\phi_y(\bar{w})\right), \tag{10.84}$$

$$\delta\partial\phi_y(w) = -\frac{1}{2}m\int\limits_{\Sigma'}\frac{d^2z}{z-w}(\partial\phi_x(z)\bar{\partial}\phi_z(\bar{z}) - \partial\phi_z(z)\bar{\partial}\phi_x(\bar{z})) - \frac{1}{2}m\oint\limits_{C'}\frac{\phi^x(z,\bar{z})d\phi_z(z,\bar{z})}{z-w} + \frac{1}{4}m\bar{\eta}_x(\bar{w})\bar{\eta}_z(\bar{w}), \tag{10.85}$$

$$\delta\partial\phi_z(w) = \frac{1}{2}m\int\limits_{\Sigma'}\frac{d^2z}{z-w}(\partial\phi_x(z)\bar{\partial}\phi_y(\bar{z}) - \partial\phi_y(z)\bar{\partial}\phi_x(\bar{z})) + \frac{1}{2}m\oint\limits_{C'}\frac{\phi^x(z,\bar{z})d\phi_y(z,\bar{z})}{z-w} - \frac{1}{4}m\bar{\eta}_x(\bar{w})\bar{\eta}_y(\bar{w}), \tag{10.86}$$

$$\delta\eta_x(w) = \frac{1}{2}m\int\limits_{\Sigma'}\frac{d^2z}{z-w}\left(\eta_y(z)\bar{\partial}\phi_z(\bar{z}) - \eta_z(z)\bar{\partial}\phi_y(\bar{z})\right), \tag{10.87}$$

$$\delta\eta_y(w) = \frac{1}{2}m\left(\phi^x(w,\bar{w})\eta_z(w) - \int\limits_{\Sigma'}\frac{d^2z}{z-w}\psi_x(z)\bar{\partial}\phi_z(\bar{z})\right), \tag{10.88}$$

$$\delta\eta_z(w) = -\frac{1}{2}m\left(\phi^x(w,\bar{w})\eta_y(w) - \int\limits_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar{\partial}\phi_y(\bar{z})\right). \tag{10.89}$$

Let us now look at how the T-duality can be recovered. This will essentially be an extension of what we did in chapter 9.

10.5.3 T-duality

In the bosonic case, we were only interested in the stress tensor as far as the T-duality was concerned. Here, we shall additionally be interested in the fermionic supercurrent, i.e. we will look at the objects

$$T_B = -g^{\mu\nu}\partial X_\mu\partial X_\nu - \frac{1}{2}g^{\mu\nu}\psi_\mu\partial\psi_\nu, \qquad T_F = i\sqrt{2}g^{\mu\nu}\psi_\mu\partial X_\nu, \tag{10.90}$$

and the corresponding antiholomorphic objects. We will look at T-duality in the y-direction, which, in terms of the objects at the self-dual point, is given by

$$(\partial\phi_y, \eta_y) \to -(\partial\phi_y, \eta_y). \tag{10.91}$$

Substituting all of the above results in, we have

$$
\begin{aligned}
T_B^N(w) = {} & T_G(w) + m\partial\phi_x(w)\int_{\Sigma'}\frac{d^2z}{z-w}F^+_{yz}(z,\bar z) - m\partial\phi_y(w)\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xz}(z,\bar z) \\
& - m\partial\phi_z(w)\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xy}(z,\bar z) - m\partial\phi_y(w)\oint_{C_0'}\frac{\phi^x(z,\bar z)d\phi_z(z,\bar z)}{z-w} \\
& - m\partial\phi_z(w)\oint_{C_0'} -2m\phi^x(w,\bar w)\partial\phi_y(w)\partial\phi_z(w)\frac{\phi^x(z,\bar z)d\phi_y(z,\bar z)}{z-w} \\
& + \frac{1}{2}m\Big(\partial\phi_y(w)\bar\eta_x(\bar w)\bar\eta_z(\bar w) + \partial\phi_z(w)\bar\eta_x(\bar w)\bar\eta_y(\bar w)\Big) \\
& + \frac{1}{4}m\left(\eta_x(w)\int_{\Sigma'}\frac{d^2z}{(z-w)^2}(\eta_y(z)\bar\partial\phi_z(\bar z) + \eta_z(z)\bar\partial\phi_y(\bar z))\right. \\
& - \partial\eta_x(w)\int_{\Sigma'}\frac{d^2z}{z-w}(\eta_y(z)\bar\partial\phi_z(\bar z) + \eta_z(z)\bar\partial\phi_y(\bar z)) \\
& - \eta_y(w)\int_{\Sigma'}\frac{d^2z}{(z-w)^2}\eta_x(z)\bar\partial\phi_z(\bar z) + \partial\eta_y(w)\int_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar\partial\phi_z(\bar z) \\
& \left. -\eta_z(w)\int_{\Sigma'}\frac{d^2z}{(z-w)^2}\eta_x(z)\bar\partial\phi_y(\bar z) + \partial\eta_z(w)\int_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar\partial\phi_y(\bar z)\right),
\end{aligned}
\tag{10.92}
$$

$$
\begin{aligned}
T_B^H(w) = {} & T_G(w) - m\partial\phi_x(w)\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{yz}(z,\bar z) + m\partial\phi_y(w)\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xz}(z,\bar z) \\
& - m\partial\phi_z(w)\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xy}(z,\bar z) + m\partial\phi_y(w)\oint_{C_0'}\frac{\phi^x(z,\bar z)d\phi_z(z,\bar z)}{z-w} \\
& - m\partial\phi_z(w)\oint_{C_0'}\frac{\phi^x(z,\bar z)d\phi_y(z,\bar z)}{z-w} - \frac{1}{2}m\Big(\partial\phi_y(w)\bar\eta_x(\bar w)\bar\eta_z(\bar w) - \partial\phi_z(w)\bar\eta_x(\bar w)\bar\eta_y(\bar w)\Big) \\
& - \frac{1}{4}m\left(\eta_x(w)\int_{\Sigma'}\frac{d^2z}{(z-w)^2}(\eta_y(z)\bar\partial\phi_z(\bar z) - \eta_z(z)\bar\partial\phi_y(\bar z))\right. \\
& - \partial\eta_x(w)\int_{\Sigma'}\frac{d^2z}{z-w}(\eta_y(z)\bar\partial\phi_z(\bar z) - \eta_z(z)\bar\partial\phi_y(\bar z)) \\
& - \eta_y(w)\int_{\Sigma'}\frac{d^2z}{(z-w)^2}\eta_x(z)\bar\partial\phi_z(\bar z) + \partial\eta_y(w)\int_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar\partial\phi_z(\bar z) \\
& \left. +\eta_z(w)\int_{\Sigma'}\frac{d^2z}{(z-w)^2}\eta_x(z)\bar\partial\phi_y(\bar z) - \partial\eta_z(w)\int_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar\partial\phi_y(\bar z)\right),
\end{aligned}
\tag{10.93}
$$

$$
\begin{aligned}
\frac{1}{i\sqrt{2}}T_F^N(w) &= \frac{1}{i\sqrt{2}}T_F^G(w) + m\phi^x(w,\bar{w})(\eta_y(w)\partial\phi_z(w) + \eta_z(w)\partial\phi_y(w)) \\
&+ \frac{1}{2}m\left[-\eta_x(w)\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{yz}(z,\bar{z})\right. \\
&+ \eta_y(w)\left(\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xz}(z,\bar{z}) + \oint_{C'}\frac{\phi^x(z,\bar{z})d\phi_z(z,\bar{z})}{z-w} - \frac{1}{2}\bar{\eta}_x(\bar{w})\bar{\eta}_z(\bar{w})\right) \\
&+ \eta_z(w)\left(\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xy}(z,\bar{z}) + \oint_{C'}\frac{\phi^x(z,\bar{z})d\phi_y(z,\bar{z})}{z-w} - \frac{1}{2}\bar{\eta}_x(\bar{w})\bar{\eta}_y(\bar{w})\right) \\
&- \partial\phi_x(w)\int_{\Sigma'}\frac{d^2z}{z-w}(\eta_y(z)\bar{\partial}\phi_z(\bar{z}) + \eta_z(z)\bar{\partial}\phi_y(\bar{z})) \\
&- \partial\phi_y(w)\left(\phi^x(w,\bar{w})\eta_z(w) - \int_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar{\partial}\phi_z(\bar{z})\right) \\
&\left.-\partial\phi_z(w)\left(\phi^x(w,\bar{w})\eta_y(w) - \int_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar{\partial}\phi_y(\bar{z})\right)\right],
\end{aligned} \tag{10.94}
$$

$$
\begin{aligned}
\frac{1}{i\sqrt{2}}T_F^H(w) &= \frac{1}{i\sqrt{2}}T_F^G(w) + \frac{1}{2}m\left[-\eta_x(w)\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{yz}(z,\bar{z})\right. \\
&- \eta_y(w)\left(\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xz}(z,\bar{z}) + \oint_{C'}\frac{\phi^x(z,\bar{z})d\phi_z(z,\bar{z})}{z-w} - \frac{1}{2}\bar{\eta}_x(\bar{w})\bar{\eta}_z(\bar{w})\right) \\
&+ \eta_z(w)\left(\int_{\Sigma'}\frac{d^2z}{z-w}F^-_{xy}(z,\bar{z}) + \oint_{C'}\frac{\phi^x(z,\bar{z})d\phi_y(z,\bar{z})}{z-w} - \frac{1}{2}\bar{\eta}_x(\bar{w})\bar{\eta}_y(\bar{w})\right) \\
&+ \partial\phi_x(w)\int_{\Sigma'}\frac{d^2z}{z-w}(\eta_y(z)\bar{\partial}\phi_z(\bar{z}) - \eta_z(z)\bar{\partial}\phi_y(\bar{z})) \\
&+ \partial\phi_y(w)\left(\phi^x(w,\bar{w})\eta_z(w) - \int_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar{\partial}\phi_z(\bar{z})\right) \\
&\left.-\partial\phi_z(w)\left(\phi^x(w,\bar{w})\eta_y(w) - \int_{\Sigma'}\frac{d^2z}{z-w}\eta_x(z)\bar{\partial}\phi_y(\bar{z})\right)\right],
\end{aligned} \tag{10.95}
$$

where

$$
T_G = -(\partial\phi_x^2 + \partial\phi_y^2 + \partial\phi_z^2) - \frac{1}{2}(\eta_x\partial\eta_x + \eta_y\partial\eta_y + \eta_z\partial\eta_z), \tag{10.96}
$$

$$
\frac{1}{i\sqrt{2}}T_G^F = \eta_x\partial\phi_x + \eta_y\partial\phi_y + \eta_z\partial\phi_z. \tag{10.97}
$$

Then, using

$$\phi^x(w,\bar{w})\partial\phi_y(w) = \int_{\Sigma'} \frac{d^2z}{z-w}\partial\phi_y(z)\bar{\partial}\phi_x(\bar{z}) - \oint_{C'} \frac{dz}{z-w}\phi^x(z,\bar{z})\partial\phi_y(z), \tag{10.98}$$

it is not too hard to show that, under the T-duality transformation (10.91),

$$\begin{aligned} T_B^N &\leftrightarrow T_B^H, \\ T_F^N &\leftrightarrow T_F^H. \end{aligned} \tag{10.99}$$

Thus, even with the fermionic contributions, the expected T-duality still holds when we take the worldsheet interactions into account.

10.6 The deformation operator and picture changing

The deformation operator we have been using above has been derived from the worldsheet sigma model. As such, for the case of constant background, we found that there was no $\delta_{\mathcal{O}}$ contribution to the fermion deformation since the on-shell fermion deformation operator vanishes. Recall that the Type II graviton vertex operator in the $(0,0)$ picture takes the form

$$V_{(0,0)}(k,\epsilon,z,\bar{z}) = \epsilon_{\mu\nu}(k)V^{\mu}_{(0)}(k,z)\bar{V}^{\nu}_{(0)}(k,\bar{z}), \tag{10.100}$$

where

$$V^{\mu}_{(0)}(k,z) = \sqrt{2}\left(i\partial X^{\mu}(z) + \frac{1}{2}(k\cdot\psi)\psi^{\mu}(z)\right)e^{ik\cdot X_L(z)}. \tag{10.101}$$

As we can see, in the limit $k\to 0$, this reduces to the usual bosonic graviton operator, and so the deformation operator would be the standard metric deformation operator that we have been using throughout. However, a natural question now arises: what happens when we consider the graviton in other pictures aside from the $(0,0)$ one? The other common picture is the canonical $(-1,-1)$ picture, where the graviton vertex operator is given by

$$V_{(-1,-1)}(k,\epsilon,z,\bar{z}) = \epsilon_{\mu\nu}(k)V^{\mu}_{(-1)}(k,z)V^{\nu}_{(-1)}(k,\bar{z}), \tag{10.102}$$

where ϵ is a symmetric polarisation tensor and

$$V^{\mu}_{(-1)}(k,z) = e^{-\varphi(z)}\psi^{\mu}(z)e^{ik\cdot X_L(z)}, \tag{10.103}$$

where φ is a bosonic ghost field obeying the OPE

$$\varphi(z)\varphi(w) \sim -\log(z-w). \tag{10.104}$$

If we now construct our deformation operator using this vertex operator instead of the usual deformation of the action, then we have

$$\mathcal{O}_{(-1,-1)} = \delta g_{\mu\nu}\int_{\Sigma} d^2z\psi^{\mu}(z)\bar{\psi}^{\nu}(\bar{z}), \tag{10.105}$$

where we ignore the ghost fields since we will not focus on them here, and we assume we are working within the realm of toroidal CFTs, i.e. $g, \delta g$ are constant. Then, we have the contraction

$$\mathcal{O}_{(-1,-1)}\psi_{\mu}(w) \sim -\delta g_{\mu\nu}\int_{\Sigma'} \frac{d^2z}{z-w}\bar{\psi}^{\nu}(\bar{z}), \tag{10.106}$$

from which we derive

$$\delta_{\mathcal{O}_{(-1,-1)}}(g_{\mu\nu}\psi^{\nu}_{n+1/2}) = -\delta g_{\mu\nu}\int_{\Sigma'} d^2z z^n\bar{\psi}^{\nu}_{n+1/2}. \tag{10.107}$$

Thus, in the canonical picture, the $\mathcal{O}$ deformation of ψ_{μ} is non-zero, whereas we would have

$$\delta_{\mathcal{O}_{(-1,-1)}}\partial X_{\mu} = 0. \tag{10.108}$$

The δ_E transformation is presumably independent of picture, since it only depends on the change in the background.

One could take this further and compute the deformations for a general toroidal background (i.e. non-zero B-field), and then look at non-constant backgrounds, T-duality, etc., but we will not do so here. The main point that we want to make here is that the deformation of an operator in the supersymmetric context is picture-dependent. Of course, this is fine since, in a correlation function, the operators would be in whatever pictures are necessary to have the correct overall picture number. To compute the deformation, we would then insert the deformation operator in the $(0,0)$ picture (to preserve picture number), and we could switch to deformation operators in different pictures using the picture changing operator, as long as the overall picture number of the correlator is correct. Indeed, taking the picture raising operator to be

$$P^{+}(z) = 2T_F(z)e^{\varphi(z)} + ..., \tag{10.109}$$

where the +... represents ghost terms which are unimportant here, we find that

$$\oint_{|z-w|=1} dz P^{+}(z)V_{(-1)}(w) = 2V_{(0)}(w). \tag{10.110}$$

Thus, we can use the appropriate picture changing operators to switch between different deformation operators, since they are constructed from the graviton vertex operators. If the goal is to compute the deformation of a correlator, then which picture one uses to compute the deformations should not make any difference to the final result.

What is interesting to note is that the deformation (10.107) can be obtained from (10.21) by simply setting $\theta = \bar{\theta} = 0$ instead of integrating out θ. Looking at (10.16), we see that this will leave only the fermion fields and comparing coefficients with (10.21) as we would normally do will reproduce (10.107). This is reminiscent of arguments made in the literature[5] where it was claimed that the canonical picture could be obtained by taking a section of the super Riemann surface where $\theta = \bar{\theta} = 0$. In general, it seems that the best way to think of the deformation operator is as follows. We start with the integration over the Riemann surface with coordinates $z, \bar{z}$ only, i.e.

$$\mathcal{O} = \frac{1}{2}\delta g_{\mu\nu} \int_{\Sigma} d^2 z D\mathbf{X}^{\mu}(\mathbf{z})\bar{D}\mathbf{X}^{\nu}(\bar{\mathbf{z}}). \tag{10.111}$$

Then, the way we deal with the θ dependence will determine the picture. If we integrate out θ then we obtain the 0 picture, which is what we have used in most of the above calculations. If we set $\theta = 0$ then we are in the -1, or canonical, picture.

Thus, we see that, in the supersymmetric setting, deformations of individual operators are much subtler. Also, it is harder to isolate the deformations from the correlation function under consideration. Given that we usually need to preserve picture number, if we assume that we always insert the picture zero operator then we can always choose to work with this operator only, but we must bear in mind that what we mean by an operator deformation is picture-dependent.

Clearly, there are many subtle details regarding this picture changing story, and it would be interesting to study this in more detail, and indeed the SUSY operator

[5] See [78, 79] for example.

deformation story more generally. For example, we have not studied the ghost fields at all, and it may be that the way they behave is significant. We will not go any further in this thesis, but, hopefully, what we have presented here will inspire future research in this direction. Evidently, our understanding of operator deformations is far from complete, and there are many avenues yet to be explored.

Chapter 11

Conclusion

The goal of this thesis has been to shed light on the subject of operator deformations in string theory, and to gain a new perspective on T-duality. To some extent, we have succeeded in making both subjects better understood and opened up new avenues for exploration. We started by considering the idea of using universal coordinates to deduce operator deformations and to compute T-duality using the stress tensor [15]. Such ideas had gone largely unnoticed in the literature, and we brought them into a modern context. We also developed these ideas further by applying them to torus bundle target spaces and framing universal coordinates in the context of the $\hat{\Gamma}$ connection on the space of toroidal CFTs. More generally, our motivation was to reframe and place in a contemporary setting the operator algebra arguments around stringy symmetries which appeared in the older literature [18, 15].

Another of our aims was to provide a framework in which to discuss T-duality in a wide range of cases that did not rely solely on questions of the existence of isometries of the target space theory. One motivation for a different approach is the desire to better understand under what conditions a T-dual description of a background exists, given that there are cases where global isometries are not a feature of the background yet something akin to T-duality appears possible. That the operator algebra approach provides a language to discuss T-duality that is not reliant on target space concepts is particularly appealing. To this end, we clarified some of the issues surrounding the operator approach to T-duality that were not, to our knowledge, addressed in the older literature. In particular, the non-uniqueness of the T-duality charge and the role isometries played in simplifying the discussion were clarified. Though the duality calculations in this thesis may sometimes have seemed more complicated than their Buscher procedure counterparts, the formalism discussed here is expected to apply to a more general class of backgrounds and seems to be a more fundamental way of understanding T-duality.

We sketched out a general framework in which issues of T-duality rest on the construction of a connection and a path γ in moduli space between a background of enhanced symmetry, in which the duality is manifest, and the background in question. This combines the ideas of [27, 15] with the studies of connections on the space of string backgrounds given in [20, 43, 22]. For on-shell considerations, the connection is on the state space of CFTs, but this can be generalised to more general sigma models, as we discussed in chapter 5. From a string field theory perspective, this provides a way of discussing off-shell physics. This provides a different starting point for T-duality than the traditional Buscher construction and one that may admit concrete discussions of non-isometric generalisation from the perspective of the full worldsheet quantum theory. We briefly discussed non-isometric T-duality and in particular we turned to an idea in [15] where coordinate dependence in the background was reframed in terms of Fourier expansions. We developed this idea slightly and described some of the issues with it, such as the non-commutativity of taking the Fourier expansion of a function of the target space coordinates and pulling objects back to the worldsheet. Though lacking somewhat as a valid approach to non-isometric T-duality, there is perhaps some potential in the idea and it may be that the issues discussed can be ironed out. We also looked at the possibility of directly using the T-duality charge to compute the T-duality of $\phi^x(\sigma)$. We found that the simple $\phi^x(\sigma) \to \widetilde{\phi}^x(\sigma)$ transformation may not be too naive, though again there are complications here. It would be good to study this issue in the context of an exact solution (or to leading order in α'), as is discussed in [54–56] or more recently the class of backgrounds found in [52, 80]. As a simpler and more tractable case, it would be interesting to study the duality on orbifolds where some part of the enhanced gauge symmetry is broken by the orbifold action. This might give more perspective on mirror symmetry in K3 and Calabi-Yau manifolds via their orbifold limits. For similar reasons, it would be interesting to see to what extent the construction considered here could be generalised to torus bundles with degenerating fibres. A good example to study in more detail might be the $SU(2)$ example mentioned briefly in section 6.3, though this may present a significant computational challenge.

Somewhat surprisingly, the doubled algebra was shown to arise from the zero modes of the commutation relations of the torus bundle directly, where the central extensions played an important role. The doubled geometry arose in this context as an effective classical description of the quantum theory. The distinction between the doubled geometry of [12] and the centrally extended algebras that appeared here is subtle and deserves further investigation. Similarly, it would be interesting to know what significance the more general algebras discussed in section 6.11 have. They are clearly related to the parallelizable flux compactifications [62, 32], but may have wider applicability. In the same way that the contraction of the doubled algebra generated by the vector fields (3.12) arose from the zero modes of the operator algebra (6.87), it would be interesting to see if there are non-parallelisable cases where the operator algebra of section 6.11.2 can be used to find a concrete proposal for the corresponding doubled geometry.

We also considered a general formalism for constructing the deformations of operators. As mentioned, a description of how to deform a generic operator at a point in the space of backgrounds, for both CFTs and more general spaces of QFTs, was lacking in the literature, since the focus has primarily been on correlation functions and the stress tensor. In particular, our formalism explicitly takes into account the target space tensor structure of the operator of interest, and we showed how this affects the deformation. One of our main goals was to show how our formalism extends to more general QFTs, and this was demonstrated by looking at the H-flux and nilfold. For marginal deformations of free CFTs, the deformation preserves the subspace spanned by ∂X and $\bar{\partial} X$. For the non-CFT cases, we found that the X-dependence made a significant difference to the deformation of $\partial\phi_\mu$, mixing in terms of the form $\phi^x \partial\phi_\nu$ and $\phi^x \bar{\partial}\phi_\nu$, as well as the $\bar{\partial}\phi_\nu$ terms seen in trivial torus bundles. More general deformation operators would lead to a more general mixing of the operator basis.

We also described how this method would work at higher orders, and the second order result for the flat torus deformation of ∂X_μ is demonstrated explicitly in appendix E. Our construction should be applicable to a wide variety of contexts, and is a significant step forward in operator deformations. Of course, there are issues, the main one being the computational difficulty of some of the calculations. As we saw for the torus bundle deformations, getting the first order results was already somewhat involved. Of course, given that we are dealing with interacting worldsheet theories here, we expect the calculations to be difficult, and it may be more of a problem with the context rather than the approach. However, a question that would be interesting to consider is whether or not there is a specific connection that makes these calculations easier. We saw that the $\hat{\Gamma}$ connection was the natural choice for CFTs and it usually made the calculations simpler. If we take the H-flux for example, is there a connection which makes the deformations particularly simple? The $c, \bar{c}$ connections described in [22] were natural connections to use in a general context, but it may be that, practically, using specific connections relevant to one's needs is a more useful way to do these deformation calculations. An idea related to this is that of a general universal coordinate construction. As we saw, the $\hat{\Gamma}$ connection was essentially equivalent to the universal coordinates in the flat torus case, but this did not extend to more general cases. It would be interesting to see if there is a connection which would allow us to construct some analogous universal objects for the H-flux case (or nilfold). If such a connection exists, the objects invariant under deformation wrt this connection would be considered universal, and ideally we could use these objects to obtain the deformations to all orders relatively easily.

We also discussed T-duality in the context of these torus bundle backgrounds away from the adiabatic limit, and we recovered the expected duality. This was a direct generalisation of the ideas employed in [15] and has the potential to be applicable to a much wider class of backgrounds. Computationally, using this approach for a completely general background seems like a difficult task beyond first order, though in principle it at least extends the applicability of T-duality beyond the context of the Buscher rules and leads to a different perspective on the duality and its relationship with symmetry enhancement.

One of the benefits of the approach explored here is that it should be applicable to a very large class of sigma models. Of particular interest would be to explore whether these techniques can be applied to effective worldsheet theories, where specific quantum effects (such as worldsheet instantons) have been incorporated. A specific example is the KK-monopole/NS5-brane duality. Here, worldsheet instantons localise the solutions, breaking the global worldsheet symmetry associated with a target space isometry [53, 54]. Following the notation of [52], the NS5-brane background is given by

$$ds_{10}^2 = V(x^i)ds^2(\mathbb{R}^4) + ds^2(\mathbb{R}^{1,5}), \tag{11.1}$$

where the x^i are coordinates on the transverse space $\mathbb{R}^4$, $ds^2(\mathbb{R}^4)$ and $ds^2(\mathbb{R}^{1,5})$ are the standard flat metrics, the function $V(x^i)$ is a harmonic function and the H-flux is given by $H_{ijk} = -\epsilon_{ijkl}\delta^{lm}\partial_m V$.

In the case where $V = V(r)$[1], where $r = |(x^1, x^2, x^3)|$, V is given by

$$V(r) = \frac{1}{g^2} + \frac{1}{2r}. \tag{11.2}$$

[1]In [52], they are mainly interested in the case where $V = V(\tau)$, i.e. V only depends on a single coordinate τ. This is because this case can be interpreted as a T^3 with H-flux fibred over a line, which is relevant in constructing certain types of hyperkahler manifolds. For this case, the T-duality is relatively easy to study since it reduces to the T-duality chain of the H-flux, which is well-known [12].

As explained in [54], this *smeared* NS5-brane is localised in the θ direction when worldsheet instantons are taken into account. This results in the replacement $V(r) \to V(r,\theta)$, where

$$V(r,\theta) = \frac{1}{g^2} + \frac{1}{2r}\left(1 + \sum_{k\geq 1}\sum_{\pm} e^{-kr\pm ik\theta}\right). \tag{11.3}$$

Recall that we found that the T-duality automorphism did not have a well-defined action on the worldsheet coordinate X, but that it *did* have a well-defined action on exponentials of the form e^{inX_L}, for integers n. Therefore, it may be that the T-dual of the *localised* NS5-brane can be computed explicitly using the T-duality methods discussed in this thesis[2].

Specifically, one could start with an *effective* action, based on the instanton-corrected potential (11.3), imagine tuning θ to be at the self-dual radius[3] and then perform a T-duality automorphism on this effective sigma model to recover the KK-monopole background. The analysis of [54] predicts

$$V(r,\theta) = \frac{1}{g^2} + \frac{1}{2r}\left(1 + \sum_{k\geq 1}\sum_{\pm} m(k)e^{-kr\pm ik\theta}\right), \tag{11.6}$$

where the $m(k)$ are unknown constants. In principle, the results of [1] predict a relationship between the $m(k)$ and their counterparts $\tilde{m}(k)$ in the KK-monopole solution.

It would be interesting to apply the construction developed in this thesis to these localised backgrounds. Usually, one performs the duality with the smeared backgrounds and then localises by incorporating the instanton corrections, but is it possible to incorporate the instanton effects directly? The idea would be to apply the construction

[2]We note that the localising instantons are not arising from non-perturbative effects in λ. Reinserting the α' dependence [55]: $g = R/\sqrt{\alpha'}$, $r = Rr'/\alpha'$, $\theta = R\theta'$ and $ds^2 = -ds'^2/\alpha'$, giving

$$V = \frac{\alpha'}{R^2}\left(1 + \frac{R}{2r'}\right). \tag{11.4}$$

To view the NS5-brane as a deformation of the flat background, we introduce a parameter λ and so rescale $R \to \lambda R$, which gives

$$V \to \frac{\alpha'}{(\lambda R)^2}\left(1 + \frac{\lambda R}{2r'}\right). \tag{11.5}$$

In these coordinates, the instanton corrections become $e^{-\frac{kr'R}{\alpha'}}$ and we see that, with λ included, the instanton corrections go like $e^{-\frac{kr'R\lambda}{\alpha'}}$, i.e. they are analytic in λ.

[3]There is a tension here. The effective action describes the large volume supergravity limit. At the self-dual radius, we would need to include other α' corrections to the supergravity. As such, this discussion is only illustrative.

above to the quantum effective worldsheet action in which these instanton effects have already been incorporated.

Recall that we made a brief note about topology change in these off-shell deformations. Essentially, we ignored this by always going to a convenient covering space where such issues were not present. It would be good to look at topological issues in these deformations more closely.

In the last chapter, we extended our formalism for operator deformations to the supersymmetric case, where we looked at $N = 1$ supersymmetry in the NS-NS sector. We discussed the possibility of constructing universal coordinates for fermions as we did for bosons in the flat torus case, and we saw that such a construction did not seem possible. Our investigation was not conclusive however, so it would be interesting to see if this really is the case or if there is a universal coordinate construction for fermions. If such a construction exists, it is most likely much more complicated to describe compared to the bosonic case. Even without such a construction, we were able to derive deformations of fermions for flat toroidal target spaces, as well as the H-flux and nilfold. In these latter cases, we found that $\partial\phi_\mu$ also received extra contributions in its deformation from fermions. We once again showed that the T-duality between these backgrounds holds as expected.

It would be interesting to see if this supersymmetric case could be pushed further to include more complicated cases such as Ramond flux. Since there is no sigma model construction for turning on Ramond flux, we would have to rely directly on the vertex operators corresponding to Ramond flux instead of the action. Throughout this thesis, we have used the action to construct the deformation operator, but in section 10.6, we briefly described how we could construct deformation operators starting from the vertex operators. Thus, it may be possible to extend the ideas presented there to Ramond vertex operators. In fact, this vertex operator approach potentially opens up a much more general construction of this deformation formalism where we do not need to know the action of the theory. Indeed, we do not even need the theory to have a sigma model formulation. This would also potentially allow deformations much more general than metric deformations. This would place the formalism in a more contemporary setting where we are relying solely on the operators of the theory as opposed to always having to start with a worldsheet sigma model.

Another interesting direction is further investigation of non-abelian T-duality [81, 82]. It would be interesting to see whether, using the formalism in this thesis, the status of non-abelian T-duality could be further clarified. The perspective that the enhanced symmetry group should have some off-shell significance and is broken by a choice of vacuum may be useful here. In the cases we have considered, torus bundles - for which the action of the unbroken $\mathbb{Z}_2$ symmetry is clear - play a central role. To make progress on the general question of non-abelian duality from the perspective advocated here, one would need a better understanding of the relationship, if any, between the enhanced symmetry group and the non-abelian isometries of the target space.

Finally, we note that there is also a close connection with the constructions discussed here and the linear sigma model approach used to prove mirror symmetry [3, 4]. There, the starting point is an 'off-shell' model, in that it does not describe a genuine string background. The model then flows to a string background under renormalization, constrained by the superpotential. It would be interesting to explore this connection further.

The predominant theme of this thesis has largely been one of a unification of ideas. Ideas and formalisms such as universal coordinates, parallel transport, CFT algebras and doubled geometry have been brought together in an attempt to further our understanding of worldsheet sigma models. Though the reader may perhaps be familiar with some of the concepts described here, it is unlikely that they are familiar with all of them, and we hope that by collating them in the novel manner in which we have done in this thesis, we have furthered our understanding of operator deformations and T-duality in string theory.

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Appendix A

Vertex Operators and WZW

This appendix contain some comments and observations that may be known but, to our knowledge, have not appeared in the literature.

A.1 More general operators

In this appendix we extend the results of chapter 6 to more general operators, with a particular focus on those operators which may be thought of as the building blocks of vertex operators. Of particular interest are the vertex operators, for which we shall need a better understanding of how operators $e^{ik_\mu \phi_L^\mu}$ and $\partial^n \phi^\mu$ transform under the particular automorphism in question for general k_μ and n. In general, the action of an $SU(2)$ automorphism will transform an operator of conformal weight h into a linear combination of other operators of the same weight. In particular, though the transformation of $\partial\phi_\mu(z)$ is straightforward, the transformation of $\phi_L^\mu(z)$ is anything but (see section 6.9), and so the transformation of operators of the form $e^{in\phi_L^\mu(z)}$ needs careful consideration. We will use OPEs instead of commutation relations here, as opposed to most of our calculations in chapter 6, as they are easier to work with for exponential calculations. Therefore, we will need to use Euclidean signature. Additionally, we will work in $d = 1$ to keep things simple.

The transformation of $\partial^n \phi$ was given in (4.47) and was straightforward to deduce. As mentioned earlier, [15] compute the transformation of exponentials using a point-splitting argument, but we will take an inductive approach.

The transformation of $e^{2i\phi_L(z)}$ under the T-duality automorphism was considered in section 4.3. In order to better understand the transformation of $e^{in\phi_L(z)}$ for $n \in \mathbb{Z}$, let us next look at the transformation of $e^{i\phi_L(z)}$. Using the OPE (4.45), we have

$$[Q, e^{i\phi_L(w)}] = \frac{\pi}{2} e^{-i\phi_L(w)}, \qquad [Q^{(2)}, e^{i\phi_L(w)}] = \frac{\pi^2}{4} e^{i\phi_L(w)}, \tag{A.1}$$

where the notation for nested commutators is given in (6.69). There is a clear repeating pattern, oscillating between $e^{i\phi_L}$ and $e^{-i\phi_L}$, and it is not hard to guess the general term. Thus, we obtain the result

$$e^{iQ} e^{i\phi_L(z)} e^{-iQ} = i e^{-i\phi_L(z)}. \tag{A.2}$$

This transformation seems at odds with the general expectation $\phi_L(z) \to -\phi_L(z)$ that we have seen in the massless vertex operators, suggesting instead $\phi_L(z) \to -\phi_L(z) + \pi/2$ (though this is not true either). In fact, as we will now show, we have

$$e^{in\phi_L(z)} \to \begin{cases} ie^{-in\phi_L(z)}, & \text{n odd} \\ e^{-in\phi_L(z)}, & \text{n even}. \end{cases} \tag{A.3}$$

We see that, in the n odd case, there is an extra factor of i compared to expectations. It is in fact not the case that we can simply look at transformations such as (A.3) and deduce a transformation of ϕ_L. We can prove (A.3) via an inductive argument, which goes as follows. Define composite operators $A_n^{\pm}$ by

$$e^{\pm i2\phi_L(z)} = e^{\pm i2\phi_L(w)} \sum_{n=0}^{\infty} \frac{1}{n!} A_n^{\pm}(w). \tag{A.4}$$

Then, the first step is to show that

$$A_n^{\pm} = E_n \pm O_n, \tag{A.5}$$

where E_n is even under T-duality and O_n is odd. We can prove this via induction. We have:

$$\begin{aligned} A_n^+ =& 2i\partial\phi A_{n-1}^+ + \partial A_{n-1}^+ \\ =& 2i\partial\phi(E_{n-1} + O_{n-1}) + \partial E_{n-1} + \partial O_{n-1} \\ =& : E_n + O_n, \end{aligned} \tag{A.6}$$

where

$$E_n = 2i\partial\phi O_{n-1} + \partial E_{n-1}, \tag{A.7}$$
$$O_n = 2i\partial\phi E_{n-1} + \partial O_{n-1}. \tag{A.8}$$

Doing the same for A_n^-, we find that

$$A_n^- = E_n - O_n, \tag{A.9}$$

as required. Finally, we note that the result is clearly true for $n = 1$, and hence we have proven (A.5).
The next step is to consider the following commutator:

$$\begin{aligned}
[Q, \cos(n\phi_L)] =& \frac{1}{8}\oint dz(e^{i2\phi_L(z)} + e^{-i2\phi_L(z)})(e^{in\phi_L(w)} + e^{-in\phi_L(w)}) \\
\sim& \frac{\pi}{4}\oint dz(z-w)^{-n}\left(e^{-i(n-2)\phi_L(w)}\sum_m \frac{1}{m!}A_m^+(w) + e^{(n-2)\phi_L(w)}\sum_m \frac{1}{m!}A_m^-(w)\right) \\
=& \frac{\pi}{4}\left(e^{-i(n-2)\phi_L(w)}\frac{1}{(n-1)!}A_{n-1}^+(w) + e^{i(n-2)\phi_L(w)}\frac{1}{(n-1)!}A_{n-1}^-(w)\right) \\
=& \frac{1}{2(n-1)!}\left(E_n(w)\cos((n-2)\phi_L(w)) - iO_n(w)\sin((n-2)\phi_L(w))\right).
\end{aligned} \tag{A.10}$$

Now, what we are leading up to is the result

$$\cos(n\phi_L) \to \begin{cases} i\cos(n\phi_L), & \text{n odd} \\ \cos(n\phi_L), & \text{n even}, \end{cases} \tag{A.11}$$

$$\sin(n\phi_L) \to \begin{cases} -i\sin(n\phi_L), & \text{n odd} \\ -\sin(n\phi_L), & \text{n even}, \end{cases} \tag{A.12}$$

which is equivalent to (A.3). Once again, we will use induction, this time via (A.10). First, note that

$$[Q, e^{iQ}Pe^{-iQ}] = e^{iQ}[Q, P]e^{-iQ}, \tag{A.13}$$

where P is any operator. Using this in (A.10) with the known transformations of E_n, O_n and the induction hypothesis on $\cos((n-2)\phi_L), \sin((n-2)\phi_L)$ (i.e. the RHS of (A.10)), we deduce that (A.11) is indeed true (we have already shown that it is true for $n = 1, 2$). The same process leads to the proof of (A.12), and hence we have shown that (A.3) is true, as required.

Note that we could also do exactly the same with the sine charge and it would just give different phases for the exponential transformations. We will just state the results:

$$e^{ni\phi_L} \xrightarrow{Q^s} \begin{cases} (-1)^{\frac{n+1}{2}} e^{-ni\phi_L}, & \text{n odd,} \\ (-1)^{\frac{n}{2}} e^{-ni\phi_L}, & \text{n even.} \end{cases} \tag{A.14}$$

This leads to

$$\cos(n\phi_L) \xrightarrow{Q^s} \begin{cases} (-1)^{\frac{n+1}{2}} i \sin(n\phi_L), & \text{n odd,} \\ (-1)^{\frac{n}{2}} \cos(n\phi_L), & \text{n even,} \end{cases} \tag{A.15}$$

$$\sin(n\phi_L) \xrightarrow{Q^s} \begin{cases} (-1)^{\frac{n+1}{2}} i \cos(n\phi_L), & \text{n odd,} \\ (-1)^{\frac{n}{2}+1} \sin(n\phi_L), & \text{n even.} \end{cases} \tag{A.16}$$

We can see that, as opposed to the cosine charge, we have sines and cosines transforming into each other. However, as we saw in 4.3.5, it does not matter which charge we use since they are equivalent up to $U(1)$ gauge transformations.

A.2 The WZW formulation

To better understand what is going on, we turn to the WZW formulation of the model at the self-dual radius.

We can formulate the T-duality of circle compactifications at the SDR using the $SU(2)$ WZW model. The massless $(1,0)$ currents define a level 1 $\widehat{su(2)}$ affine lie algebra. If we define

$$J^1 = \cos(2\phi_L), \qquad J^2 = \sin(2\phi_L), \qquad J^3 = i\partial\phi, \tag{A.17}$$

then, as with all conformal primaries, we can expand them in modes as

$$J^i(z) = \sum_{n=-\infty}^{\infty} J^i_n z^{-n-1}. \tag{A.18}$$

These obey the level 1 current algebra

$$[J^i_n, J^j_m] = 2\pi \left(\frac{n}{2} \delta_{n+m} \delta^{ij} + i\epsilon^{ijk} J^k_{n+m} \right). \tag{A.19}$$

In this formulation, the (cosine) charge takes the simple form

$$Q = \frac{1}{2} \oint dz \, \cos(2\phi_L(z)) = \frac{1}{2} J^1_0, \tag{A.20}$$

or J_0^2 for the sine charge. It can then be verified that the effect of the charge on the modes $J^i(z) \to e^{iQ}J^i(z)e^{-iQ}$ is

$$J_n^1 \to J_n^1, \qquad J_n^2 \to -J_n^2, \qquad J_n^3 \to -J_n^3. \tag{A.21}$$

This should reproduce all of the T-duality transformation results that we derived earlier. We will verify this using the states corresponding to the relevant vertex operators. The states at level 1 can be constructed from the highest weight states with level 1 by acting on them with the modes J_n^i, $n < 0$ (note that the level of a weight in the case of $SU(2)$ is given by the sum of its Dynkin labels). The states that are generated from a single highest weight state $|\lambda\rangle$ form the irreducible module L_λ. In the case of $SU(2)$, it turns out that there are 2 such modules, $L_{[1,0]}$ and $L_{[0,1]}$, where the subscripts are the Dynkin labels of the corresponding highest weight states (see [83] for a review). We start with the module $L_{[1,0]}$, for which the highest weight state is simply the vacuum. The generic state is then

$$|\lambda'\rangle = J_{-n}^i ... J_{-m}^j |0\rangle. \tag{A.22}$$

For the $L_{[0,1]}$ module, the highest weight state is $|0'\rangle := e^{i\phi_L(0)}|0\rangle$, and the generic state is

$$|\lambda'\rangle = J_{-n}^i ... J_{-m}^j |0'\rangle. \tag{A.23}$$

Note that, from the point of view of the WZW model, it is clear to see that each mass level has either 'odd' or 'even' exponentials, but not both (i.e. $e^{in\phi_L}$ where n is odd or even). The vertex operators with odd and even exponentials belong to different modules, so they do not mix under the action of $SU(2)$. It is only via the WZW formulation that this becomes clear. In particular, the action of T-duality on one of these states is

$$e^{iQ}J_{-n}^i ... J_{-m}^j |\lambda\rangle = e^{iQ}J_{-n}^i e^{-iQ} ... e^{iQ} J_{-m}^j e^{-iQ} e^{iQ} |\lambda\rangle, \tag{A.24}$$

where $|\lambda\rangle$ is one of the highest weight states. If we recall also that, under T-duality with the cosine charge,

$$e^{i\phi_L} \to ie^{-i\phi_L}, \tag{A.25}$$

we see that the action of T-duality simply amounts to sign changes and the possible factor of i coming from the transformation of the $|0'\rangle$ state,

$$e^{-iQ}|0'\rangle = i|0'\rangle. \tag{A.26}$$

There is no mixing between the two modules[1]. This can also be seen from the fact that the states at each grade (i.e. each conformal dimension) form representations of $SU(2)$.

A.3 An example

Let us now look at a specific example to see explicitly the equivalence between these two formulations of the bosonic string at the SDR. We will look at the case of the operators

$$\partial^2\phi(z), \qquad \partial\phi(z)\partial\phi(z), \qquad \partial\phi(z)e^{\pm 2i\phi_L(z)}, \tag{A.29}$$

which are used to build massive vertex operators. We can order these by their eigenvalues of $J_0^3 = \oint J^3$. The state with the largest eigenvalue is the highest weight state, and we can obtain all other states by acting with $J_0^- = \oint J^-$. For example, we have

$$[J_0^3, \partial\phi(w)e^{2i\phi_L(w)}] = \partial\phi(w)e^{2i\phi_L(w)}, \tag{A.30}$$

i.e. the J_0^3 eigenvalue is $+1$ in this case. Repeating for the other states, we find that the eigenvalues for $\partial^2\phi, \partial\phi^2, \partial\phi e^{-2i\phi_L}$ are $0, 0, -1$ respectively. Thus, $\partial\phi e^{2i\phi_L}$ corresponds to the highest weight state. Acting with J_0^-, we obtain

$$[J_0^-, \partial\phi(w)e^{2i\phi_L(w)}] = \oint dz J^-(z)\partial\phi(w)e^{2i\phi_L(w)} = \partial^2\phi(w). \tag{A.31}$$

Similarly, lowering again gives $\partial\phi e^{-2i\phi_L}$. Thus, these three states are in a triplet of $SU(2)$ and $\partial\phi^2$ is in a singlet. We can also determine the action of T-duality straightforwardly using the state-operator correspondence. The states corresponding

[1] We can also write

$$e^{i\phi_L}|0\rangle \to ie^{-i\phi_L}|0\rangle = iJ_0^- e^{i\phi_L}|0\rangle, \tag{A.27}$$

where $J^\pm = J^1 \pm iJ^2$, and we can expand J^- to get

$$iJ^-(z)e^{i\phi_L}|0\rangle = i\sum_n : \frac{J_n^-}{z^{n+1}} e^{i\phi_L} : |0\rangle \xrightarrow{z\to 0} iJ_{-1}^- e^{ix_L}|0\rangle, \tag{A.28}$$

where $x_L = \frac{1}{2}(x - \tilde{x})$, and we have used the fact that the normal ordering means we do not have to worry about terms in J^- which do not commute with $e^{i\phi_L}$.

to the above operators are

$$\partial\phi(z)e^{\pm 2i\phi_L(z)} \leftrightarrow J^3_{-1}J^{\pm}_{-1}|0\rangle \equiv |2,\pm 1\rangle_{(2)}, \tag{A.32}$$

$$\partial^2\phi(z) \leftrightarrow J^-_{-2}|0\rangle \equiv |2,0\rangle_{(2)}, \tag{A.33}$$

$$\partial\phi^2(z) \leftrightarrow J^3_{-1}J^3_{-1}|0\rangle \equiv |2,1\rangle_{(0)}, \tag{A.34}$$

where the numerical labels are, respectively, the L_0 eigenvalue, the J^3_0 eigenvalue and the $SU(2)$ representation, and normal ordering is implicit. By direct calculation or by using the action of T-duality on the modes given in (A.21), we see that the states transform in exactly the same way as the operators would imply, i.e. we recover the transformation

$$\partial^2\phi \to -\partial^2\phi, \qquad \partial\phi^2 \to \partial\phi^2, \qquad \partial\phi e^{\pm 2i\phi_L} \to -\partial\phi e^{\mp 2i\phi_L}. \tag{A.35}$$

Appendix B

Elliptic Monodromy

We present an example of a simple torus bundle with a geometric twist over the base circle from the elliptic conjugacy class of $SL(2;\mathbb{Z})$. Specifically, we will look at the monodromy

$$f = \begin{pmatrix} 0 & \pi/2 \\ -\pi/2 & 0 \end{pmatrix} \implies e^{fx} = \begin{pmatrix} \cos\left(\frac{\pi x}{2}\right) & \sin\left(\frac{\pi x}{2}\right) \\ -\sin\left(\frac{\pi x}{2}\right) & \cos\left(\frac{\pi x}{2}\right) \end{pmatrix}. \tag{B.1}$$

Then, the generators of the algebra are given by

$$\mathcal{Z}_a = (e^{-fX})_a{}^{\mu}\Pi_\mu = \begin{pmatrix} \cos\left(\frac{\pi X}{2}\right)\Pi_y - \sin\left(\frac{\pi X}{2}\right)\Pi_z \\ \sin\left(\frac{\pi X}{2}\right)\Pi_y + \cos\left(\frac{\pi X}{2}\right)\Pi_z \end{pmatrix}, \tag{B.2}$$

$$\mathcal{X}^a = (e^{f^T X})^a{}_{\mu}\mathcal{X}'^{\mu} = \begin{pmatrix} \cos\left(\frac{\pi X}{2}\right)Y' - \sin\left(\frac{\pi X}{2}\right)Z' \\ \sin\left(\frac{\pi X}{2}\right)Y' + \cos\left(\frac{\pi X}{2}\right)Z' \end{pmatrix}, \tag{B.3}$$

and

$$\mathcal{Z}_x = \Pi_x, \qquad \mathcal{X}^x = X'. \tag{B.4}$$

The only commutator which is not trivially satisfied is $[\mathcal{X}^y, \mathcal{Z}_z]$ (and those related to it). This is

$$\begin{aligned}
[\mathcal{X}^y(\sigma), \mathcal{Z}_z(\sigma')] =& \left[\cos\left(\frac{\pi X}{2}\right) Y'(\sigma) - \sin\left(\frac{\pi X}{2}\right) Z'(\sigma), \sin\left(\frac{\pi X}{2}\right) \Pi_y(\sigma') + \cos\left(\frac{\pi X}{2}\right) \Pi_z(\sigma')\right] \\
=& 2\pi i\delta'(\sigma-\sigma')\left(\cos\left(\frac{\pi X(\sigma)}{2}\right)\sin\left(\frac{\pi X(\sigma')}{2}\right) - \sin\left(\frac{\pi X(\sigma)}{2}\right)\cos\left(\frac{\pi X(\sigma')}{2}\right)\right) \\
=& -2\pi i\delta(\sigma-\sigma')\frac{\pi X'(\sigma)}{2}\left(\cos^2\left(\frac{\pi X(\sigma)}{2}\right) + \sin^2\left(\frac{\pi X(\sigma)}{2}\right)\right) \\
=& (-2\pi i\delta(\sigma-\sigma'))\left(\frac{\pi}{2}\mathcal{X}^x(\sigma)\right),
\end{aligned} \tag{B.5}$$

where in the third line we have 'integrated by parts'. We write the final line in such a way as to make explicit the expected algebra of the elliptic monodromy. The other commutators follow similarly and we thus obtain the full doubled algebra

$$\begin{aligned}
&[\mathcal{Z}_x(\sigma), \mathcal{Z}_z(\sigma')] = (-2\pi i\delta(\sigma-\sigma'))\left(\frac{\pi}{2}\mathcal{Z}_y(\sigma)\right), \qquad [\mathcal{Z}_x(\sigma), \mathcal{Z}_y(\sigma')] = (-2\pi i\delta(\sigma-\sigma'))\left(-\frac{\pi}{2}\mathcal{Z}_z(\sigma)\right), \\
&[\mathcal{Z}_x(\sigma), \mathcal{X}^y(\sigma')] = (-2\pi i\delta(\sigma-\sigma'))\left(-\frac{\pi}{2}\mathcal{X}^z(\sigma)\right), \qquad [\mathcal{Z}_x(\sigma), \mathcal{X}^z(\sigma')] = (-2\pi i\delta(\sigma-\sigma'))\left(\frac{\pi}{2}\mathcal{X}^y(\sigma)\right), \\
&[\mathcal{X}^y(\sigma), \mathcal{Z}_z(\sigma')] = -[\mathcal{X}^z(\sigma), \mathcal{Z}_y(\sigma')] = (-2\pi i\delta(\sigma-\sigma'))\left(-\frac{\pi}{2}\mathcal{X}^x(\sigma)\right),
\end{aligned} \tag{B.6}$$

and the central extension

$$[\mathcal{Z}_a(\sigma), \mathcal{X}^b(\sigma')] = 2\pi i\delta_a{}^b\delta'(\sigma-\sigma'), \tag{B.7}$$

agreeing with the doubled geometry [12]. This background is interesting because it is a genuine string theory background that can be obtained from the supergravity construction as a minimum of the potential [29]. Normally, such minima only satisfy the supergravity equations of motion, but in this case the minimum is equivalent to a toroidal orbifold and is therefore a solution of the full string theory equations of motion.

Appendix C

Details of commutation relations in section 6.11

We present here the details of the calculations of the commutation relations of the doubled algebra and the associators in section 6.11.1. We considered there the most general geometric flux compactifications.

C.1 Commutation relations

We compute the commutation relations of the generators (6.98). Firstly, for the '$[\mathcal{Z}, \mathcal{Z}]$' commutators, we have

$$\begin{aligned}
[\mathcal{Z}_a(\sigma), \mathcal{Z}_b(\sigma')] =& [e_a{}^\mu(\sigma)(\Pi_\mu(\sigma) - B_{\mu\nu}(\sigma)X'^\nu(\sigma)), e_b{}^\rho(\sigma')(\Pi_\rho(\sigma') - B_{\rho\sigma}(\sigma')X'^\sigma(\sigma'))] \\
=& e_b{}^\rho(\sigma')\partial_\rho e_a{}^\mu(\sigma) e_\mu{}^e(\sigma)\mathcal{Z}_e(\sigma)2\pi i\delta(\sigma - \sigma') \\
& - e_a{}^\mu(\sigma)\partial_\mu e_b{}^\rho(\sigma') e_\rho{}^e(\sigma')\mathcal{Z}_e(\sigma')2\pi i\delta(\sigma - \sigma') \\
& + 2\pi i\delta(\sigma - \sigma') e_a{}^\mu(\sigma) e_b{}^\rho(\sigma') X'^\nu(\sigma')(-\partial_\mu B_{\nu\rho} - \partial_\rho B_{\mu\nu} - \partial_\nu B_{\rho\mu}) \\
=& 2 e_a{}^\mu e_b{}^\nu \partial_{[\mu} e_{\nu]}{}^c \mathcal{Z}_c(\sigma) 2\pi i\delta(\sigma - \sigma') \\
& - 2\pi i\delta(\sigma - \sigma')(K_{abc}\mathcal{X}^c(\sigma')) \\
=& - f^c_{ab}\mathcal{Z}_c(\sigma)2\pi i\delta(\sigma - \sigma') - K_{abc}\mathcal{X}^c(\sigma')2\pi i\delta(\sigma - \sigma'),
\end{aligned} \tag{C.1}$$

where, for example, we have used $[\Pi_\rho(\sigma'), e_a{}^\mu(\sigma)] = -2\pi i\delta(\sigma - \sigma')\partial_\rho e_a{}^\mu(\sigma)$.

Next, for the '$[\mathcal{Z},\mathcal{X}]$' commutators, we have

$$\begin{aligned}
[\mathcal{Z}_a(\sigma),\mathcal{X}^b(\sigma')] =&[{e_a}^\mu(\sigma)\left(\Pi_\mu(\sigma)-B_{\mu\nu}(\sigma)X'^\nu(\sigma)\right), {e^b}_\rho(\sigma')X'^\rho(\sigma')]\\
=&-{e_a}^\mu(\sigma)\partial_\mu {e^b}_\rho(\sigma')X'^\rho(\sigma')2\pi i\delta(\sigma-\sigma')+{e_a}^\mu(\sigma){e^b}_\mu(\sigma')2\pi i\delta'(\sigma-\sigma')\\
=&-{e_a}^\mu(\sigma)\partial_\mu {e^b}_\rho(\sigma')X'^\rho(\sigma')2\pi i\delta(\sigma-\sigma')\\
&+{e_a}^\mu(\sigma)\left({e^b}_\mu(\sigma)+(\sigma'-\sigma){e^b}_\mu{}'(\sigma)+O((\sigma-\sigma')^2)\right)2\pi i\delta'(\sigma-\sigma')\\
=&f^b_{ac}\mathcal{X}^c(\sigma)2\pi i\delta(\sigma-\sigma')+\delta^b_a 2\pi i\delta'(\sigma-\sigma')+O((\sigma-\sigma')^2),
\end{aligned}\tag{C.2}$$

where the higher order terms are proportional to (for $n\geq 2$)

$$\begin{aligned}
\delta'(\sigma-\sigma')(\sigma-\sigma')^n =&\delta'(\sigma-\sigma')\sum_{m=0}^{n}(\sigma)^{n-m}(-\sigma')^m\begin{pmatrix}n\\m\end{pmatrix}\\
=&\sum_{m=0}^{n}\begin{pmatrix}n\\m\end{pmatrix}(\sigma)^{n-m}(-1)^m\left[(\sigma)^m\delta'(\sigma-\sigma')+m(\sigma)^{m-1}\delta(\sigma-\sigma')\right]\\
=&(\sigma-\sigma)^n\delta'(\sigma-\sigma')+n(\sigma-\sigma)^{n-1}\delta(\sigma-\sigma')=0,
\end{aligned}\tag{C.3}$$

and so

$$[\mathcal{Z}_a(\sigma),\mathcal{X}^b(\sigma')]=f^b_{ac}\mathcal{X}^c(\sigma)2\pi i\delta(\sigma-\sigma')+{\delta_a}^b 2\pi i\delta'(\sigma-\sigma').\tag{C.4}$$

The final '$[\mathcal{X},\mathcal{X}]=0$' commutator follows trivially. We thus obtain the algebra (6.100), as claimed.

C.2 Associativity

Here we verify the nested commutators (6.101) and (6.104) required to compute the associators. For '$[\mathcal{Z},[\mathcal{Z},\mathcal{Z}]]$', we have

$$\begin{aligned}
[\mathcal{Z}_a(\sigma),[\mathcal{Z}_b(\sigma'),\mathcal{Z}_c(\sigma'')]] =&[\mathcal{Z}_a(\sigma),-{f_{bc}}^d\mathcal{Z}_d(\sigma'')-K_{bcd}\mathcal{X}^d(\sigma'')]2\pi i\delta(\sigma'-\sigma'')\\
=&-4\pi^2\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')\Big\{{f_{bc}}^d{f_{ad}}^e\mathcal{Z}_e(\sigma'')+{f_{bc}}^dK_{ade}\mathcal{X}^e(\sigma'')\\
&-{f_{ae}}^dK_{bcd}\mathcal{X}^e(\sigma'')\Big\}+4\pi^2\delta'(\sigma-\sigma'')\delta(\sigma'-\sigma'')K_{bcd}\delta^d_a,
\end{aligned}\tag{C.5}$$

as claimed. Note that, after (6.102), it is claimed that the central extension terms vanish when we add the cyclic permutations via delta function manipulations. Specifically, we use

$$\begin{aligned}\delta'(\sigma-\sigma'')\delta(\sigma'-\sigma'') =&\partial_\sigma\left(\delta(\sigma-\sigma'')\delta(\sigma'-\sigma'')\right)\\ =&\partial_\sigma\left(\delta(\sigma'-\sigma)\delta(\sigma''-\sigma)\right)\\ =&\delta'(\sigma-\sigma')\delta(\sigma''-\sigma)+\delta(\sigma-\sigma')\delta'(\sigma-\sigma''). \end{aligned} \tag{C.6}$$

Substituting this into (6.102), we see that the central extension term does indeed vanish.

For the '$[\mathcal{Z},[\mathcal{Z},\mathcal{X}]]$' nested commutator, we have

$$\begin{aligned}[\mathcal{Z}_b(\sigma'),[\mathcal{Z}_c(\sigma''),\mathcal{X}^a(\sigma)]] =&[\mathcal{Z}_b(\sigma'),{f_{cd}}^a\mathcal{X}^d(\sigma)2\pi i\delta(\sigma''-\sigma)+\delta^a_c 2\pi i\delta'(\sigma''-\sigma)]\\ =&-4\pi^2\delta(\sigma'-\sigma)\delta(\sigma''-\sigma)\left({f_{cd}}^a{f_{be}}^d\mathcal{X}^e(\sigma)\right)\\ &+4\pi^2\delta'(\sigma-\sigma')\delta(\sigma''-\sigma){f_{cb}}^a. \end{aligned} \tag{C.7}$$

Other calculations required to compute the associators proceed along similar lines.

Appendix D

H-flux Mode Transformation

Given everything that we have done, we should be in a position to compute the deformation of the H-flux modes, at least to first order. We will go through the calculation of $\delta\alpha_n^y$ in detail. Note that here we use the formalism of chapter 7.

D.1 $\delta_{\mathcal{O}}$ calculation

Taking the OPE of $\partial\phi_y(w)$ with the deformation operator and integrating by parts gives

$$-\frac{1}{2}m\left(\oint\limits_{C'} d\bar{z}\frac{\phi^x(z,\bar{z})\bar{\partial}\phi_z(\bar{z})}{z-w}+\int\limits_{\Sigma'} d^2z\frac{\partial\phi_x(z)\bar{\partial}\phi_z(\bar{z})}{z-w}\right), \tag{D.1}$$

where $C' = C'_0 \cup C'_w$. Let us deal with the contour integrals first. Expanding in terms of modes, we have:

$$\frac{im}{2\sqrt{2}}\sum_{\substack{k\geq 0\\ p}}\bar{\alpha}_p^z w^{-k-1}\oint\limits_{C'} d\bar{z}\,\phi^x(z,\bar{z})z^k\bar{z}^{-p-1}, \tag{D.2}$$

where

$$\phi^x(z,\bar{z}) = x - \frac{i}{\sqrt{2}}(\alpha_0^x\log z+\bar{\alpha}_0^x\log\bar{z}) + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}(\alpha_n^x z^{-n}+\bar{\alpha}_n^x\bar{z}^{-n}). \tag{D.3}$$

Let us compute the kinds of integrals that appear here. The most involved ones are those with logs, which generally look like

$$\oint\limits_{C'} d\bar{z}z^k\bar{z}^{-p-1}\log z, \tag{D.4}$$

and if we are on a contour where $|z| = r$ then this simplifies to

$$- r^{-2p} \oint\limits_{C'} dz z^{k+p-1} \log z. \tag{D.5}$$

Let us first deal with the circle around $z = 0$. We will take the contour to have radius ϵ and take the limit $\epsilon \to 0$ at the end. There is a branch point at $z = 0$, but, as discussed in chapter 8, the choice of branch simply amounts to a choice of gauge. Therefore, we will still consider a circle around $z = 0$, bearing in mind that there is a gauge-dependent piece. Setting $z = \epsilon e^{i\theta}$, the integral becomes

$$- \frac{\epsilon^{k-p}}{2\pi} \int\limits_0^{2\pi} d\theta e^{i(k+p)\theta} (\log \epsilon + i\theta) = \begin{cases} \frac{\epsilon^{k-p}}{2\pi i (k+p)^2}, & k+p \neq 0, \\ -\epsilon^{2k}(\log \epsilon + i\pi), & k+p = 0. \end{cases} \tag{D.6}$$

In the above integral, the terms which come from the $i\theta$ part of the integral are the branch-dependent part, and if we neglect these terms we get

$$- \frac{\epsilon^{k-p}}{2\pi} \int\limits_0^{2\pi} d\theta e^{i(k+p)\theta} (\log \epsilon + i\theta) = -\delta_{k+p,0} \epsilon^{2k} \log \epsilon + ..., \tag{D.7}$$

where ... represents the branch-dependent contributions. We also need to check whether there is any contribution from the $z = w$ boundary. There is no branch cut here, so we can just use a circular contour with radius $\epsilon \to 0$, so we have

$$\frac{\epsilon}{2\pi} \int\limits_0^{2\pi} d\theta e^{i\theta} w^k \bar{w}^{-p-1} \left(1 + \frac{\epsilon}{w} e^{i\theta}\right)^k \left(1 + \frac{\epsilon}{\bar{w}} e^{-i\theta}\right)^{-p-1} \log(w + \epsilon e^{i\theta}), \tag{D.8}$$

which we can see vanishes in the limit $\epsilon \to 0$, so there is no contribution here. Similarly,

$$\oint\limits_{C'} d\bar{z} z^k \bar{z}^{-p-1} \log \bar{z} = -\delta_{k+p,0} \epsilon^{2k} \log \epsilon - ..., \tag{D.9}$$

where the minus before the ... indicates that the branch-dependent contributions are minus those of (D.7).

The other type of integral which we are interested in is

$$\oint\limits_{C'} d\bar{z} z^{k-n} \bar{z}^{-p-1}. \tag{D.10}$$

This is fairly straightforward and we simply need to compute the contributions from contours around $z = 0, w$ as usual. We will state the results. From $z = 0$, we get $-\epsilon^{k-n-p}\delta_{k-n+p,0}$. From $z = w$, we once again find that there is no contribution when we take the radius to zero.

Thus, we now have all of the integrals we need to compute (D.2). Substituting everything in and neglecting branch-dependent contributions, we have[1]

$$-\frac{im}{2\sqrt{2}}\sum_{k\geq 0}\sum_{p} w^{-k-1}\bar{\alpha}^z_p \lim_{\epsilon\to 0}\left[x\epsilon^{k-p}\delta_{k+p,0} + \frac{i}{\sqrt{2}}\sum_n \frac{1}{n}\epsilon^{k-n-p}\left(\alpha^x_n\delta_{k-n+p,0} + \bar{\alpha}^x_n\delta_{k+n+p,0}\right)\right]. \tag{D.11}$$

A note on evaluating contour integrals

The above calculation is perfectly valid, but it is not the only way of evaluating the contour integral. To illustrate the idea, we will look at the much simpler case of the circle CFT of radius R. Here, as we saw earlier, if we wish to deform $\partial X(R)$ to the circle of radius $R + \delta R$, the $\delta_{\mathcal{O}}$ part of the deformation involves taking the OPE with the marginal operator, and we end up with an integral proportional to

$$\oint_{C_0^\epsilon} \frac{dz}{z^2(w-z)}\bar{\partial}X(\bar{z}), \tag{D.12}$$

as well as an integral around w which is unimportant here. Now, the usual way of evaluating this would be to expand $\bar{\partial}X(\bar{z})$ and $(w-z)^{-1}$ in powers of z, using the fact that $|z| = \epsilon$, and then compute the resulting integrals. Doing this, we get

$$\oint_{C_0^\epsilon} \frac{dz}{z^2(w-z)}\bar{\partial}X(\bar{z}) = -\frac{i}{\sqrt{2}}\sum_{m\geq 0}\bar{\alpha}_{-m}\epsilon^{-2(-m+1)}w^{-m-1}, \tag{D.13}$$

and, together with the δ_E transformation, we would compare coefficients to get the mode transformations for α_n, $n \geq 0$, and then argue that this must extend to all n to preserve commutation relations. However, alternatively, we could treat the contour as a contour around the singularity at *infinity*. This then changes the expansion of $(w-z)^{-1}$, since

$$\frac{1}{w-z} = \sum_{m\geq 0} w^{-m-1}z^m, \qquad |z| < |w|, \tag{D.14}$$

[1]Where the $n = 0$ case is understood as the limit $\lim_{n\to 0}\frac{\epsilon^{k-n-p}}{n} = -\epsilon^{k-p}\log\epsilon$.

$$\frac{1}{w-z} = -\sum_{m<0} w^{-m-1} z^m, \qquad |z| > |w|. \tag{D.15}$$

Thus, if we set $y = 1/z$ and use

$$\frac{dz}{w-z} = -\frac{dy}{y(wy-1)}, \tag{D.16}$$

we instead find that

$$\oint_{C_0^\epsilon} \frac{dz}{z^2(w-z)} \bar{\partial} X(\bar{z}) = -\frac{i}{\sqrt{2}} \sum_{m\geq 0,n} \bar{\alpha}_n \epsilon^{-2(n+1)} w^m \oint_{|y|=1/\epsilon} dy y^{-n+m}, \tag{D.17}$$

which, after evaluating and relabelling, gives

$$-\frac{i}{\sqrt{2}} \sum_{m<0} \bar{\alpha}_{-m} \epsilon^{-2(-m+1)} w^{-m-1}, \tag{D.18}$$

i.e. the same as (D.13), but with $m < 0$. We can now compare coefficients as we would normally do and obtain the mode transformation for α_n, $n < 0$, and clearly this agrees with what we would get if we did it the 'canonical' way, as we would hope. For the circle case, apart from giving a way to directly compute the mode expansions for *all* modes (instead of only half and then inferring the other half), there is no particular benefit here. However, for the H-flux case, it makes the prescription much clearer since we have 2d integrals where the region of integration includes both $|z| > |w|$ and $|z| < |w|$. We come to this calculation now.

The integral over the worldsheet

Let us evaluate the second term in (D.1). First, we look at the region $\epsilon < |z| < |w|$, where we have

$$-\frac{1}{2} m \int_{\substack{\Sigma' \\ |z|<|w|}} \frac{d^2 z}{z-w} \partial \phi_x(z) \bar{\partial} \phi_z(\bar{z}) = -\frac{m}{4} \sum_{\substack{n,p \\ k\geq 0}} \alpha_n^x \bar{\alpha}_p^z w^{-k-1} \int_{\substack{\Sigma' \\ |z|<|w|}} d^2 z z^{k-n-1} \bar{z}^{-p-1}. \tag{D.19}$$

This is straightforward once we set $z = re^{i\theta}$, so we will simply state the results. We get

$$\frac{m}{8} \sum_{\substack{p \\ k\geq 0}} \frac{\alpha_{k+p}^x \bar{\alpha}_p^z}{p} w^{-k-1} \left(|w|^{-2p} - \epsilon^{-2p} \right), \tag{D.20}$$

where the $p=0$ term is understood in terms of a limit in a similar way to (D.11). For $|z|>|w|$, to deal with the singularity at infinity, we will regularise the integral by integrating over the region $|w|<|z|<1/\epsilon$. The calculation is very similar and we get

$$\frac{m}{8}\sum_{\substack{p\\k<0}}\frac{\alpha^x_{k+p}\bar{\alpha}^z_p}{p}w^{-k-1}\left(|w|^{-2p}-\epsilon^{2p}\right). \tag{D.21}$$

Thus, up to divergences in the $\epsilon\to 0$ limit, we get the same result in both regions $|z|<|w|$ and $|z|>|w|$, except for the range of the summation variable k. As we mentioned when discussing the circle case above, we want this to be the case so that our results are the same whichever region we look at and whichever method we use.

Now, we would like to combine (D.11) and (D.20) to obtain $\delta_{\mathcal{O}}\alpha^y_n$. As discussed above, we have different prescriptions depending on whether $n\geq 0$ or $n<0$. For $n\geq 0$, which is the case we usually focus on, we choose the contour integral representation where the contour is $|z|=\epsilon$, and we take the part of the 2d integral where $|z|<|w|$. To read off the deformation of α^y_n, as usual we must take the coefficient of w^{-n-1} in (D.11), (D.20). However, note that the coefficient in (D.20) has w-dependence. To deal with this, we note that, when we are extracting the w^{-n-1} coefficient, formally what we are doing is multiplying by w^n and doing the contour integral $\oint_{|w|=\eta}dw$, for some constant η. When the coefficient is independent of w, this amounts to simply reading off the coefficient and the result is independent of η. However, in the case of (D.20), we find that extracting the w^{-n-1} coefficient amounts to setting $|w|=\eta$. Thus, we obtain

$$\delta_{\mathcal{O}}\alpha^y_n = x\mathcal{A}_n\bar{\alpha}^z_{-n}+\sum_p\mathcal{B}_{np}\bar{\alpha}^z_p\alpha^x_{n+p}+\sum_p\mathcal{C}_{np}\bar{\alpha}^z_p\bar{\alpha}^x_{-n-p}, \qquad n\geq 0, \tag{D.22}$$

where

$$\mathcal{A}_n=-\frac{im}{2\sqrt{2}}\lim_{\epsilon\to 0}\epsilon^{2n}, \qquad \mathcal{B}_{np}=\frac{m}{4}\lim_{\epsilon\to 0}\left(\frac{\epsilon^{-2p}}{n+p}+\frac{\eta^{-2p}-\epsilon^{-2p}}{2p}\right), \qquad \mathcal{C}_{np}=-\frac{m}{4}\lim_{\epsilon\to 0}\frac{\epsilon^{2n}}{n+p}.$$

Given that the above is for $n\geq 0$, we can take the limits for $\mathcal{A}_n$ and $\mathcal{C}_{np}$ since these have no divergences and, extending the result to $n<0$, we get

$$\delta_{\mathcal{O}}\alpha^y_n=\sum_p\mathcal{B}_{np}\bar{\alpha}^z_p\alpha^x_{n+p}, \qquad n\neq 0, \tag{D.23}$$

$$\delta_{\mathcal{O}}\alpha^y_0=-\frac{imx}{2\sqrt{2}}\bar{\alpha}^z_0+\sum_p\mathcal{B}_{0p}\bar{\alpha}^z_p\alpha^x_p-\frac{m}{4}\sum_p\frac{1}{p}\bar{\alpha}^z_p\bar{\alpha}^x_{-p}. \tag{D.24}$$

If we wanted to compute the $n < 0$ case directly, we could use the corresponding integral results for $|z| > |w|$. As discussed above, both methods should give the same results.

Some comments are in order:

- We have left the limits in $\mathcal{B}_{np}$ unresolved in the above deformation. This is because the way we deal with these limits will depend on the connection we choose, i.e. the way we choose to regularise divergences. Recall that, in the circle case, we found from (7.44) that there were no divergent terms and we could take the limit $\epsilon \to 0$ for all terms, which resulted in $\delta_{\mathcal{O}}\partial X = 0$ for the c and $\bar{c}$ connections. However, here we find that, due to the X-dependence, we now *do* have non-zero terms in the limit $\epsilon \to 0$.

- Note that this result depends on $|w| = \eta$, as opposed to the CFT case. Given that we no longer have conformal symmetry, the distance from the origin of the operator we are deforming does indeed have an effect on the transformation.

- Finally, note that, in the adiabatic limit where $X \to x$, the above result reduces to the expected result for a flat torus, i.e. the $\mathcal{O}$ deformation is given solely by the $\mathcal{A}_n$ term.

D.2 δ_E calculation

From earlier considerations, we know that the δ_E transformation is given by

$$\delta_E \partial\phi_y = \frac{1}{2} m \phi^x \partial\phi_z. \tag{D.25}$$

In order to extract the transformation of the modes, we can rewrite this as an integral by noting that[2]

$$0 = \int\limits_{\Sigma'} d^2 z \bar{\partial}\left(\frac{1}{z-w}\right)\phi^x(z,\bar{z})\partial\phi_z(z) = \oint\limits_{C'} \frac{dz}{z-w}\phi^x(z,\bar{z})\partial\phi_z(z) - \int\limits_{\Sigma'} \frac{d^2 z}{z-w}\bar{\partial}\phi_x(\bar{z})\partial\phi_z(z). \tag{D.26}$$

[2]Where we have used

$$\frac{\partial}{\partial\bar{z}}\left(\frac{1}{z-w}\right) = \delta^2(z-w),$$

and integrated by parts.

Now, let us look specifically at the contour integral around the point w. If we expand $\phi^x(z,\bar{z}), \partial\phi_z(z)$ around w, this is

$$\oint_{C'_w} \frac{dz}{z-w} \Big(\phi^x(w,\bar{w}) + ((z-w)\partial\phi_x(w) + ...) + ((\bar{z}-\bar{w})\bar{\partial}\phi_x(\bar{w}) + ...)\Big) \times \Big(\partial\phi_z(w) + (z-w)\partial^2\phi_z(w) + ...\Big), \tag{D.27}$$

and on the contour $|z-w| = \epsilon$ we can set $\bar{z}-\bar{w} = \epsilon^2/(z-w)$, which gives infinitely many non-zero terms in the integral above. However, in the limit $\epsilon \to 0$ only one term survives, so we end up with

$$\oint_{C'_w} \frac{dz}{z-w}\phi^x(z,\bar{z})\partial\phi_z(z) = \phi^x(w,\bar{w})\partial\phi_z(w), \tag{D.28}$$

and so, going back to (D.26), we find that

$$\frac{1}{2}m\phi^x(w,\bar{w})\partial\phi_z(w) = \frac{1}{2}m\int_{\Sigma'} \frac{d^2z}{z-w}\bar{\partial}\phi_x(\bar{z})\partial\phi_z(z) - \frac{1}{2}m\oint_{C'_0} \frac{dz}{z-w}\phi^x(z,\bar{z})\partial\phi_z(z). \tag{D.29}$$

Now we simply have to compute these integrals. The second integral was already dealt with in the $\mathcal{O}$ case. The first is very similar to what we had in the $\mathcal{O}$ case, but not quite the same. However, the details are the same in essence, so we will simply state the final result. We find:

$$\delta_E\alpha^y_n = \frac{imx}{2\sqrt{2}}\alpha^z_n + \sum_p \mathcal{D}_{np}\alpha^z_p\bar{\alpha}^x_{p-n} + \sum_p \mathcal{E}_{np}\alpha^z_p\alpha^x_{n-p}, \tag{D.30}$$

where

$$\mathcal{D}_{np} = -\frac{m}{4}\lim_{\epsilon\to 0}\left(\frac{\eta^{-2(p-n)} + \epsilon^{-2(p-n)}}{2(p-n)}\right), \qquad \mathcal{E}_{np} = -\frac{m}{4}\frac{1}{n-p}, \tag{D.31}$$

where we have also used the same notation as above for cases where the denominator seems to vanish, i.e. these correspond to log terms.[3] Note that the first and last terms in (D.30) are independent of ϵ, which is reassuring given the results we have derived previously for the flat torus case. Once again, the adiabatic limit, i.e. the term proportional to x, gives us what we would expect.

[3]For the terms where there is no power of ϵ in the numerator, we do as follows. Take $\frac{1}{n-p}$ as an example. Look at $\frac{\epsilon^{n-p-k}}{n-p}$ in the limit $n-p \to 0$ and then set $k = n-p$.

Appendix E

Higher Order Deformations

E.1 Multiple $\mathcal{O}$ insertions

Our prescription for deforming an operator in chapter 7 involved the insertion of a deformation operator $\mathcal{O} = \int O$, which was then integrated over the worldsheet with discs removed around insertion points. The choice of connection gave a prescription for how these discs are defined. However, if we have multiple insertions of $\mathcal{O}$ then more information is needed. Here, we give a prescription that reproduces the expected results.

Suppose we have some operator $\mathcal{A}(w, \bar{w})$ which we wish to deform (with target space indices suppressed). As given in section 7.1.3 for ∂X, we define the $\mathcal{O}^n$ operator insertion as

$$\int\limits_{\Sigma_n} d^2 z_n ... \int\limits_{\Sigma_1} d^2 z_1 O(z_n, \bar{z}_n) ... O(z_1, \bar{z}_1) \mathcal{A}(w, \bar{w}), \tag{E.1}$$

where we recall that the domain of integration Σ_i is

$$\Sigma_i = \{ z_i \in \mathbb{C} | \quad |z_i| \geq \epsilon, |z_i - w| > 0, |z_i - z_j| \geq \epsilon \quad \forall j > i \}. \tag{E.2}$$

Once we have this prescription, we can then take OPEs between the various operators and explicitly compute the integral to any order that is desired. As an example, we will look at the $(\mathcal{O} + \delta_E)^2$ calculation for ∂X_μ for a CFT deformation $g \to g + \delta g$.

E.2 $(\mathcal{O}+\delta_E)^2$ for ∂X_μ

For simplicity, we will suppose that there are no B-field deformations involved. We know that, given a deformation $g \to g+\delta g$, if we have operator insertion

$$\mathcal{O} = \delta g_{\mu\nu} \int\limits_{\Sigma} \partial X^\mu \bar{\partial} X^\nu, \tag{E.3}$$

as well as the δ_E operator (7.55), we get the deformation

$$\delta \partial X_\mu = \frac{1}{2}\delta g_{\mu\nu}(\partial X^\nu - \bar{\partial} X^\nu), \tag{E.4}$$

and this is in fact the full transformation. We know this because we already have the full transformation from other methods, such as universal coordinates. If this is the case, it should be that all contributions from higher power insertions of $\mathcal{O}$ and δ_E cancel, e.g. at second order in δg, we expect

$$(\delta_{\mathcal{O}^2} + \delta_{\mathcal{O}}\delta_E + \delta_E\delta_{\mathcal{O}} + \delta_E^2)\partial X_\mu = 0. \tag{E.5}$$

Similar results should hold for higher orders[1]. Let us verify the second order result.

E.2.1 $\mathcal{O}^2$

We will start with $\mathcal{O}^2$ since this is the most involved calculation. We have

$$\int\limits_{\Sigma_2} d^2 z_2 \int\limits_{\Sigma_1} d^2 z_1 O(z_2,\bar{z}_2) O(z_1,\bar{z}_1) \partial X_\mu(w), \tag{E.6}$$

where $O = \delta g_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu$ and

$$\Sigma_1 = \{z_1 \in \mathbb{C} | \quad |z_1| > \epsilon, |z_1 - z_2| > \epsilon, |z_1 - w| > 0\}, \tag{E.7}$$

$$\Sigma_2 = \{z_2 \in \mathbb{C} | \quad |z_2| > \epsilon, |z_2 - w| > 0\}. \tag{E.8}$$

[1]Note that this is only true for ∂X_μ and is *not* true for ∂X^μ. This is because the ∂X^μ transformation involves the inverse metric, which induces corrections to all orders in δg. This is the same reason why the transformation of the modes α_n^μ involves corrections to all orders, but $g_{\mu\nu}\alpha_n^\nu$ truncates at first order in δg, as explained in section 7.1.3

The second order contribution is given by the following contractions:

$$\delta g_{\nu\rho}\delta g_{\sigma\tau}\left(\int\limits_{\Sigma_2} d^2z_2 \int\limits_{\Sigma_1} d^2z_1 \partial X^\nu(z_2)\bar{\partial}X^\rho(\bar{z}_2)\partial X^\sigma(z_1)\bar{\partial}X^\tau(\bar{z}_1)\partial X_\mu(w)\right.$$
$$\left. + \int\limits_{\Sigma_2} d^2z_2 \int\limits_{\Sigma_1} d^2z_1 \partial X^\nu(z_2)\bar{\partial}X^\rho(\bar{z}_2)\partial X^\sigma(z_1)\bar{\partial}X^\tau(\bar{z}_1)\partial X_\mu(w)\right). \tag{E.9}$$

Let us look at the first contraction. This is

$$\frac{1}{4}\delta g_{\nu\rho}\delta g_{\mu\sigma}g^{\rho\sigma}\int\limits_{\Sigma_2} d^2z_2 \int\limits_{\Sigma_1} d^2z_1 \frac{1}{(\bar{z}_2-\bar{z}_1)^2(z_1-w)^2}\partial X^\nu(z_2). \tag{E.10}$$

For ease of notation, we will ignore the metric factors for now and focus solely on the integral, which is

$$\int\limits_{\Sigma_2} d^2z_2 \int\limits_{\Sigma_1} d^2z_1 \frac{\partial X^\nu(z_2)}{(\bar{z}_2-\bar{z}_1)^2(z_1-w)^2} = \int\limits_{\Sigma_2} d^2z_2 \oint\limits_{\partial\Sigma_1} d\bar{z}_1 \frac{\partial X^\nu(z_2)}{(\bar{z}_2-\bar{z}_1)^2(z_1-w)}, \tag{E.11}$$

where we have used that $\frac{\partial}{\partial z_1}\left(\frac{1}{\bar{z}_1-\bar{z}_2}\right)=0$, since Σ_1 excludes a disc around z_2. Let us now consider the integral

$$\sum_i \oint\limits_{\Gamma_i} d\bar{z}_1 \frac{1}{(\bar{z}_1-\bar{z}_2)^2(z_1-w)}, \tag{E.12}$$

where $\bigcup_i \Gamma_i = \partial\Sigma_1$. We have boundaries around $z_1 = 0, w, z_2$. Let us call the boundaries $\Gamma_0, \Gamma_w, \Gamma_2$ respectively, and let us consider each of these in turn.

$\Gamma_0 = \partial D_0$

We take $z_1 = \epsilon e^{i\theta}$ and $s := e^{i\theta}$, which results in the integral

$$-\frac{1}{\bar{z}_2^2}\oint\limits_{|s|=1} \frac{ds}{\left(s-\frac{\epsilon}{\bar{z}_2}\right)^2\left(s-\frac{w}{\epsilon}\right)}, \tag{E.13}$$

which has a pole at $s=\frac{\epsilon}{\bar{z}_2}$ if we take $|w|, |z_i| > \epsilon$, which we do. Evaluating the residue, we obtain the result

$$\oint\limits_{|z_1|=\epsilon} d\bar{z}_1 \frac{1}{(\bar{z}_1-\bar{z}_2)^2(z_1-w)} = -\frac{\epsilon^2}{(w\bar{z}_2-\epsilon^2)^2}. \tag{E.14}$$

$\Gamma_w = \partial D_w$

Let $z_1 = w + \epsilon e^{i\theta}$. Then, the contour integral becomes

$$-i\int_0^{2\pi} d\theta \frac{e^{-2i\theta}}{(\bar{z}_2 - \bar{w} - \epsilon e^{-i\theta})^2} = -\frac{1}{(\bar{z}_2 - \bar{w})^2} \oint_{|s|=1} \frac{ds}{s\left(s - \frac{\epsilon}{\bar{z}_2 - \bar{w}}\right)^2}, \tag{E.15}$$

where $s = e^{i\theta}$ has been used in the second equality. The integrand has poles at both $s = 0$ and $s = \epsilon(\bar{z}_2 - \bar{w})^{-1}$. Doing the residue calculation, we find that we get equal and opposite contributions from each pole, and so we get zero contribution from this boundary.

$\Gamma_1 = \partial D_{z_2}$

Once again we make a substitution by taking $z_1 = z_2 + \epsilon e^{i\theta}, s = e^{i\theta}$, which gives

$$-\frac{1}{\epsilon^2} \oint_{|s|=1} \frac{ds}{s + (z_2 - w)\epsilon^{-1}}, \tag{E.16}$$

which has no poles since $|w|, |z_i| > \epsilon$ and $|w - z_i| > \epsilon$, so this boundary gives zero contribution.

Thus, overall we have

$$\sum_i \oint_{\Gamma_i} d\bar{z}_1 \frac{1}{(\bar{z}_1 - \bar{z}_2)^2 (z_1 - w)} = -\frac{\epsilon^2}{(w\bar{z}_2 - \epsilon^2)^2}. \tag{E.17}$$

Going back to (E.11), we now wish to do the z_2 integral, i.e. we compute

$$\int_{\Sigma_2} d^2 z_2 \partial X^\nu(z_2) \oint_{\partial\Sigma_1} d\bar{z}_1 \frac{1}{(\bar{z}_1 - \bar{z}_2)^2 (z_1 - w)} = \frac{\epsilon^2}{w} \oint_{\partial\Sigma_2} dz_2 \frac{\partial X^\nu(z_2)}{w\bar{z}_2 - \epsilon^2}, \tag{E.18}$$

where we have used $-\frac{\epsilon^2}{(w\bar{z}_2 - \epsilon^2)^2} = \frac{\partial}{\partial \bar{z}_2} \frac{\epsilon^2}{w(w\bar{z}_2 - \epsilon^2)}$ and integrated out the derivative. Using the mode expansion of $\partial X^\nu(z_2)$ and letting $z_2 = \epsilon e^{i\theta}$, this becomes

$$\frac{\epsilon}{w\sqrt{2}} \sum_n \alpha_n^\nu \int_0^{2\pi} d\theta \frac{\epsilon^{-n} e^{-in\theta}}{we^{-i\theta} - \epsilon} = \frac{i}{\sqrt{2}} \sum_n \alpha_n^\nu \epsilon^{-n} \oint_{|s|=1} ds \frac{1}{s^n (s - \frac{w}{\epsilon})}, \tag{E.19}$$

where we have set $s = e^{i\theta}$ in the second equality. The integrand has a single pole at $s = 0$ with residue $-(w/\epsilon)^{-n}$, so overall we have

$$\int_{\Sigma_2} d^2 z_2 \partial X^\nu(z_2) \oint_{\partial\Sigma_1} d\bar{z}_1 \frac{1}{(\bar{z}_1-\bar{z}_2)^2(z_1-w)} = -\frac{i}{\sqrt{2}} \sum_n w^{-n-1}\alpha_n^\nu. \tag{E.20}$$

This completes the calculation of the first contraction in (E.9). The calculation for the second one involves similar integrals and we will give some brief details. The integral is

$$\frac{1}{4}\delta g_{\mu\nu}\delta g_{\rho\sigma} g^{\nu\sigma} \int_{\Sigma_2} d^2 z_2 \int_{\Sigma_1} d^2 z_1 \frac{1}{(\bar{z}_2-\bar{z}_1)^2(z_2-w)^2} \partial X^\rho(z_1) =: \frac{1}{4}\delta g_{\mu\nu}\delta g_{\rho\sigma} g^{\nu\sigma}\, \mathcal{I}^\rho, \tag{E.21}$$

and so the integral $\mathcal{I}^\rho$ that we are interested in calculating is

$$\mathcal{I}^\rho = -\int_{\Sigma_2} d^2 z_2 \oint_{\partial\Sigma_1} dz_1 \frac{\partial X^\rho(z_1)}{(\bar{z}_1-\bar{z}_2)(z_2-w)^2} = \frac{i}{\sqrt{2}} \sum_n \alpha_n^\rho \int_{\Sigma_2} \frac{d^2 z_2}{(z_2-w)^2} \oint_{\partial\Sigma_1} dz_1 \frac{z_1^{-n-1}}{\bar{z}_1-\bar{z}_2}, \tag{E.22}$$

where now we must include $\partial X^\rho(z_1)$ from the outset since it has z_1 dependence. We are thus interested in the integral

$$\sum_i \oint_{\Gamma_i} \frac{z_1^{-n-1}}{\bar{z}_1-\bar{z}_2}. \tag{E.23}$$

As before, let us look at each boundary in turn.

Γ_0

Using $|z_1| = \epsilon$, we have $\bar{z}_1 - \bar{z}_2 = \frac{\epsilon^2 - z_1\bar{z}_2}{z_2}$, and substituting this in gives

$$-\frac{1}{\bar{z}_2} \oint_{|z_1|=\epsilon} dz_1 \frac{z_1^{-n}}{z_1-\epsilon^2/\bar{z}_2} = -\frac{1}{\bar{z}_2}\left[\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial z_1^{n-1}}\left(z_1 - \frac{\epsilon^2}{\bar{z}_2}\right)^{-1}\right]_{z_1=0} - \frac{1}{\bar{z}_2}\left(\frac{\epsilon^2}{\bar{z}_2}\right)^{-n} = 0, \tag{E.24}$$

so we get zero contribution from this boundary.

Γ_w

We set $z_1 = w + \epsilon e^{i\theta}$ as well as $s = e^{i\theta}$, which gives

$$\frac{\epsilon^{-n}}{\bar{w} - \bar{z}_1} \oint_{|s|=1} ds \frac{s}{\left(s + \frac{w}{\epsilon}\right)^{n+1} \left(s + \frac{\epsilon}{\bar{w} - \bar{z}_2}\right)}, \tag{E.25}$$

which has a pole at $s = -\frac{\epsilon}{\bar{w}-\bar{z}_2}$, and evaluating the integral and taking the limit $\epsilon \to 0$, we find that this also vanishes (recall that our prescription is such that we always take the circle around w to vanish), so once again there is no contribution.

Γ_1

We have $|z_1 - z_2| = \epsilon$, so the integral simply becomes

$$\epsilon^2 \oint_{|z_1 - z_2| = \epsilon} dz_1 \frac{z_1 - z_2}{z_1^{n+1}}, \tag{E.26}$$

and setting $z_1 = z_2 + \epsilon s$, where $s = e^{i\theta}$, we get

$$\epsilon^{-n-1} \oint_{|s|=1} ds \frac{s}{\left(s + \frac{z_2}{\epsilon}\right)^{n+1}} = 0, \tag{E.27}$$

since there are no poles inside the unit circle. Thus, all boundaries give zero, so we conclude that

$$\int_{\Sigma_1} d^2 z_1 \int_{\Sigma_2} d^2 z_2 \frac{1}{(\bar{z}_1 - \bar{z}_2)^2 (z_2 - w)^2} \partial X^\rho(z_1) = 0, \tag{E.28}$$

i.e. the second term in (E.9) vanishes, and the only contribution is from the first term. Therefore, overall we have

$$\delta_{\mathcal{O}^2} \partial X_\mu = \frac{1}{4} \delta g_{\mu\nu} g^{\nu\rho} \delta g_{\rho\sigma} \partial X^\sigma. \tag{E.29}$$

Note that this is precisely what we would get if we applied $\delta_{\mathcal{O}}$ in a naive sequential way, i.e. if we said

$$\delta_{\mathcal{O}^2} \partial X_\mu = \frac{1}{2} \delta_{\mathcal{O}} \left(\delta g_{\mu\nu} \partial X^\nu\right) = \frac{1}{4} \delta g_{\mu\nu} g^{\nu\rho} \delta g_{\rho\sigma} \partial X^\sigma, \tag{E.30}$$

and so it seems as though we can simply say $\delta_{\mathcal{O}^n} = (\delta_{\mathcal{O}})^n$. We will see shortly that this is indeed how we claim the higher order transformations work.

E.2.2 ${\delta_E}^2$

This is fairly straightforward. The first action gives $\frac{1}{2}\delta g_{\mu\nu} e^\nu_a \partial X^a$, and so overall, after the second action, we get

$$\delta_E^2 \partial X_\mu = -\frac{1}{4}\delta g_{\mu\nu} g^{\nu\rho} \delta g_{\rho\sigma} \partial X^\sigma = -\delta_{\mathcal{O}}^2 \partial X_\mu, \tag{E.31}$$

and so we have $(\delta_{\mathcal{O}}^2 + \delta_E^2)\partial X_\mu = 0$.

E.2.3 $\mathcal{O}\delta_E + \delta_E \mathcal{O}$

This is again straightforward, so we will simply state the results. We have

$$\delta_E \delta_{\mathcal{O}} \partial X_\mu = -\delta_{\mathcal{O}} \delta_E \partial X_\mu = \frac{1}{4}\delta g_{\mu\nu} g^{\nu\rho} \delta g_{\rho\sigma} \bar{\partial} X^\sigma. \tag{E.32}$$

Thus, we find that $(\delta_{\mathcal{O}} + \delta_E)^2 \partial X_\mu = 0$, as expected. We expect that $(\delta_{\mathcal{O}} + \delta_E)^n \partial X_\mu = 0$ holds for all $n > 1$.

For other operators the story will be different, and each operator must be dealt with on a case-by-case basis. For example, if we are looking at the stress tensor T, we should find that there are non-zero contributions only to order δg^2, and so we expect

$$(\delta_{\mathcal{O}} + \delta_E)^n T = 0, \qquad n \geq 3. \tag{E.33}$$

Of course, we could also just substitute the transformation for ∂X into T instead of deriving it from scratch, and this should give the same result.

E.3 An operational approach to higher order $\mathcal{O}$ insertions

The above calculations suggest a way of 'operationalising' the $\mathcal{O}$ insertions at higher order. What we saw was that, for the $\delta_{\mathcal{O}^2} \partial X_\mu$ calculation, the only double contraction which gave a contribution was the one which corresponded to the order of integration, i.e. the contraction schematically of the form

$$\int_{\Sigma_2}\int_{\Sigma_1} O_2 O_1 \partial X_\mu . \tag{E.34}$$

We postulate that this generalises to higher powers, i.e. at order n, the only contraction of relevance is the following:

$$\int\limits_{\Sigma_n} \dots \int\limits_{\Sigma_1} [O_n[O_{n-1}[\dots[O_2[O_1, \partial X_\mu]]\dots]], \tag{E.35}$$

i.e. we contract in the order in which we compute the integrals (we use commutator notation for clarity, as explained in section 7.1.3). Thus, we would first contract ∂X_μ with O_1, then contract the result with O_2, and so on. This provides a way of making the application of multiple $\mathcal{O}$ operators more systematic, since this can intuitively be understood as sequentially applying the operator insertions. This allows us to write the deformation to all orders as

$$\delta\partial X_\mu = \exp(\delta_{\mathcal{O}} + \delta_E)\partial X_\mu. \tag{E.36}$$

Note that we are not giving a mathematical or physical proof that this approach works, we are simply saying that this is a prescription which is intuitive and seems to agree with known results. Also, although we have specifically looked at ∂X_μ here, we expect this approach to work for any operator.

Appendix F

Level Matching

For a variety of reasons, it is important that, in all of our discussions on operator deformations, we still have level matching, or rotational invariance. One reason is that, in [23], the variational formula that is postulated is averaged over all angular variables.

F.1 Level matching for the circle

First we do the circle case to illustrate how it works in a standard CFT context. For a circle of radius R deformed to $R + \delta R$, we have deformation operator

$$\mathcal{O} = \lambda \int_{\Sigma} \partial X \bar{\partial} X, \tag{F.1}$$

where $\lambda = \delta g/R^2 = (2R\delta R + \delta R^2)/R^2$. To show level matching, we must show that

$$[\mathcal{O}, L_0] = [\mathcal{O}, \bar{L}_0], \tag{F.2}$$

where

$$L_0 = \oint_{|z|=1} dz z T(z), \qquad \bar{L}_0 = \oint_{|z|=1} d\bar{z}\bar{z} T(\bar{z}), \tag{F.3}$$

where both integrals are independent of the radius of the contour. In other words, we must show that $[\mathcal{O}, L_0]$ is invariant under holomorphic $\leftrightarrow$ antiholomorphic. We have

$$\begin{aligned} [\mathcal{O}, L_0] =& \lambda \int_{\Sigma} d^2 z \oint_{C_z} dw \partial X(z) \bar{\partial} X(\bar{z}) g^{-1} w \partial X(w) \partial X(w) \\ \sim & -\lambda \int_{\Sigma} d^2 z \oint_{C_z} \frac{dw}{(z-w)^2} \bar{\partial} X(\bar{z}) w \partial X(w), \end{aligned} \tag{F.4}$$

and doing the contour integral and integrating by parts gives

$$\lambda \oint\limits_{C_0', C_w'} d\bar{z} z \partial X(z) \bar{\partial} X(\bar{z}). \tag{F.5}$$

The integral around C_w can easily be seen to vanish when we take the limit $\epsilon \to 0$, so we are left with

$$\lambda \oint\limits_{C_0} d\bar{z} z \partial X(z) \bar{\partial} X(z), \tag{F.6}$$

and using $z d\bar{z} = -\bar{z} dz$ when $|z|$ is constant, we see that this is invariant under $z \leftrightarrow \bar{z}$, and so we do indeed have $[\mathcal{O}, L_0^-] = 0$, as required.

F.2 Level matching for the H-flux

Now we come to the more complicated case of the H-flux, although we will see that level matching is still preserved. Using the results (8.19), (8.20), (8.21), we have

$$\begin{aligned}
[\mathcal{O}, L_0] &= m \int\limits_{\Sigma} d^2 z \oint\limits_{C_z} dw \phi^x(z,\bar{z}) F^-_{yz}(z,\bar{z}) \Big(\partial\phi_x^2(w) + \partial\phi_y^2(w) + \partial\phi_z^2(w) \Big) \\
&\sim m \int\limits_{\Sigma} d^2 z \oint\limits_{C_z} dw \frac{w}{z-w} \left(F^-_{yz}(z,\bar{z}) \partial\phi_x(w) - \frac{\phi^x(z,\bar{z})}{z-w} (\bar{\partial}\phi_z(\bar{z}) \partial\phi_y(w) - \bar{\partial}\phi_y(\bar{z}) \partial\phi_z(w)) \right) \\
&= -m \int\limits_{\Sigma} d^2 z \Big(z \partial\phi_x(z) F^-_{yz}(z,\bar{z}) + \phi^x(z,\bar{z}) F^-_{yz}(z,\bar{z}) + z \phi^x(z,\bar{z}) \partial F^-_{yz}(z,\bar{z}) \Big),
\end{aligned} \tag{F.7}$$

where in the last step we have done the w contour integral and used that $\partial\bar{\partial}\phi^y = \partial\bar{\partial}\phi^z = 0$. The last term in the final equality above can be written as

$$\begin{aligned}
&- m \int\limits_{\Sigma} d^2 z \Big(\partial_z (z \phi^x(z,\bar{z}) F^-_{yz}(z,\bar{z})) - \phi^x(z,\bar{z}) F^-_{yz}(z,\bar{z}) - z \partial\phi_x(z) F^-_{yz}(z,\bar{z}) \Big) \\
&= m \oint\limits_{C_0', C_w'} d\bar{z} \phi^x(z,\bar{z}) F^-_{yz}(z,\bar{z}) + m \int\limits_{\Sigma} d^2 z \left(\phi^x(z,\bar{z}) + z \partial\phi_x(z) \right) F^-_{yz}(z,\bar{z}).
\end{aligned} \tag{F.8}$$

By the same argument as for the circle, the contour integral around w vanishes, and so overall we have

$$[\mathcal{O}, L_0] = m \oint\limits_{C_0'} d\bar{z} z \phi^x(z,\bar{z}) \left(\partial\phi_y(z) \bar{\partial}\phi_z(\bar{z}) - \partial\phi_z(z) \bar{\partial}\phi_y(\bar{z}) \right), \tag{F.9}$$

which is indeed invariant under $z \leftrightarrow \bar{z}$, so we still have $[\mathcal{O}, L_0^-] = 0$ for the H-flux.