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Identification and Estimation of Categorical Random Coefficient Models*

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This paper proposes a linear categorical random coefficient model, in which the random coefficients follow parametric categorical distributions. The distributional parameters are identified based on a linear recurrence structure of moments of the random coefficients. A Generalized Method of Moments estimator is proposed, and its finite sample properties are examined using Monte Carlo simulations. The utility of the proposed method is illustrated by estimating the distribution of returns to education in the U.S. by gender and educational levels. We find that rising heterogeneity between educational groups is mainly due to the increasing returns to education for those with postsecondary education, whereas within group heterogeneity has been rising mostly in the case of individuals with high school or less education.

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1 Introduction

Random coefficient models have been used extensively in time series, cross-section and panel regressions. Nicholls and Pagan (1985) consider the estimation of first and second moments of the random coefficient β_i and the error term u_i, in a linear regression model. In the seminal work, Beran and Hall (1992) establish the conditions of identifying and estimating the distribution of β_i and u_i non-parametrically. The baseline linear univariate regression in Beran and Hall (1992) has been extended in non-parametric framework by Beran (1993); Beran and Millar (1994); Beran, Feuerverger, and Hall (1996); Hoderlein, Klemelä, and Mammen (2010); Hoderlein, Holzmann, and Meister (2017) and Breunig and Hoderlein (2018), to just name a few. Hsiao and Pesaran (2008) survey random coefficient models in linear panel data models.

In some econometric applications, Hausman (1981); Hausman and Newey (1995); Foster and Hahn (2000) for examples, the main interest is to estimate the consumer surplus distribution based on a linear demand system where the coefficient associated with the price is random. In such settings, the distribution of the random coefficients is needed when computing the consumer surplus function, and the non-parametric estimation is more general, flexible and suitable for the purpose. On the other hand, parametric models may be favored in applications in which the implied economic meaning of the distribution of the random coefficients is of interests. Examples include estimation of the return to education (Lemieux, 2006b,c) and the labor supply equation (Bick, Blandin, and Rogerson, 2022).

In this paper, we consider a linear regression model with a random coefficient β_i that is assumed to follow a categorical distribution, i.e. β_i has a discrete support $\{b_1, b_2, \cdots, b_K\}$, and $\beta_i = b_k$ with probability π_k. The discretization of the support of the random coefficient β_i naturally corresponds to the interpretation that each individual belongs to a certain category, or group, k with probability π_k. Compared to a non-parametric distribution with continuous support, assuming a categorical distribution allows us not only to model the heterogeneous responses across individuals but also to interpret the results with sharper economic meaning. As we will illustrate in the empirical application in Section 6, it is hard to clearly interpret the distribution of returns to education without imposing some form of parametric restrictions.

In addition, with the categorical distribution imposed, the identification and estimation of the distribution of β_i do not rely on identically distributed error terms u_i and regressors $\mathbf{w}_i$, as shown in Section 2 and 3. Heterogeneously generated errors can be allowed, which is important in many empirical applications. To the best of our knowledge, this is the first identification result in linear random coefficient model without a strict IID setting.

The identification of the distribution of β_i is established in this paper based on the identification of the moments of β_i, which coincides with the identification condition in Beran and Hall (1992) that the distribution of β_i is uniquely determined by its moments, assumed to exist up to an arbitrary order. Since under our setup the distribution of β_i is parametrically specified, the moments of β_i exist and can be derived explicitly. The parameters of the assumed categorical distribution can then be uniquely determined by a system of equations in terms of the moments, as in Theorem 2.

The parameters of the categorical distribution are then estimated consistently by the generalized method of moments (GMM).

The proposed method is illustrated by providing estimates of the distribution of returns to education in the U.S. by gender and educational levels, using the May and outgoing Rotation Group (ORG) supplements of the Current Population Survey (CPS) data. Comparing the estimates obtained over the sub-periods 1973-75 and 2001-03, we find that rising between group heterogeneity is largely due to rising returns to education in the case of individuals with postsecondary education, whilst within group heterogeneity has been rising in the case of individuals with high school or less education.

Related Literature: This paper draws mainly upon the literature of random coefficient models. As already mentioned, the main body of the recent literature is focused on non-parametric identification and estimation. Following Beran and Hall (1992), Beran (1993) and Beran and Millar (1994) extend the model to a linear semi-parametric model with a multivariate setup and propose a minimum distance estimator for the unknown distribution. Foster and Hahn (2000) extend the identification results in Beran and Hall (1992) and apply the minimum distance estimator to a gasoline consumption data to estimate the consumer surplus function. Beran, Feuerverger, and Hall (1996) and Hoderlein, Klemelä, and Mammen (2010) propose kernel density estimators based on the Radon inverse transformation in linear models.

In addition to linear models, Ichimura and Thompson (1998) and Gautier and Kitamura (2013) incorporate the random coefficients in binary choice models. Gautier and Hoderlein (2015) and Hoderlein, Holzmann, and Meister (2017) consider triangular models with random coefficients allowing for causal inference. Matzkin (2012) and Masten (2018) discuss the identification of random coefficients in simultaneous equation models. Breunig and Hoderlein (2018) propose a general specification test in a variety of random coefficient models. Random coefficients are also widely studied in panel data models, for example Hsiao and Pesaran (2008) and Arellano and Bonhomme (2012)

The rest of the paper is organized as follows: Section 2 establishes the main identification results. The GMM estimation procedure is proposed and discussed in Section 3. An extension to a multivariate setting is considered in Section 4. Small sample properties of the proposed estimator are investigated in Section 5, using Monte Carlo techniques under different regressor and error distributions. Section 6 presents and discusses our empirical application to return to education. Section 7 provides some concluding remarks and suggestions for future work. Technical proofs are given in Appendix A.1.

Notations: Largest and smallest eigenvalues of the $p \times p$ matrix $\mathbf{A} = (a_{ij})$ are denoted by $\lambda_{\max}(\mathbf{A})$ and $\lambda_{\min}(\mathbf{A})$, respectively, its spectral norm by $\|\mathbf{A}\| = \lambda_{\max}^{1/2}(\mathbf{A}'\mathbf{A})$, $\mathbf{A} \succ 0$ means that $\mathbf{A}$ is positive definite, $\text{vech}(\mathbf{A})$ denotes the vectorization of distinct elements of $\mathbf{A}$, $\mathbf{0}$ denotes zero matrix (or vector). For $\mathbf{a} \in \mathbb{R}^p$, $\text{diag}(\mathbf{a})$ represents the diagonal matrix with diagonal elements of $a_1, a_2, \cdots, a_p$. For random variables (or vectors) u and v, $u \perp v$ represents u is independent of v. We use c (C) to denote some small (large) positive constants. For a differentiable real-valued function $f(\boldsymbol{\theta})$, $\nabla_{\boldsymbol{\theta}} f(\boldsymbol{\theta})$ denotes the gradient vector. Operator $\to_p$ denotes convergence in probability, and

$\rightarrow_d$ convergence in distribution. The symbols $O(1)$, and $O_p(1)$ denote asymptotically bounded deterministic and random sequences, respectively.

2 Categorical random coefficient model

We suppose the single cross-section observations, $\{y_i, x_i, \mathbf{z}_i\}_{i=1}^n$, follow the categorical random coefficient model

$$y_i = x_i \beta_i + \mathbf{z}_i' \boldsymbol{\gamma} + u_i, \tag{2.1}$$

where $y_i, x_i \in \mathbb{R}$, $\mathbf{z}_i \in \mathbb{R}^{p_z}$, and $\beta_i \in \{b_1, b_2, \cdots, b_K\}$ admits the following K-categorical distribution,

$$\beta_i = \begin{cases} b_1, & \text{w.p. } \pi_1, \\ b_2, & \text{w.p. } \pi_2, \\ \vdots & \vdots \\ b_K, & \text{w.p. } \pi_K, \end{cases} \tag{2.2}$$

w.p. denotes "with probability", $\pi_k \in (0,1)$, $\sum_{k=1}^K \pi_k = 1$, $b_1 < b_2 < \cdots < b_K$, $\boldsymbol{\gamma} \in \mathbb{R}^{p_z}$ is homogeneous and $\mathbf{z}_i$ could include an intercept term as its first element. It is assumed that $\beta_i \perp \mathbf{w}_i = (x_i, \mathbf{z}_i')'$, and the idiosyncratic errors u_i are independently distributed with mean 0.

Remark 1 *The model can be extended to allow $\mathbf{x}_i, \boldsymbol{\beta}_i \in \mathbb{R}^p$, with $\boldsymbol{\beta}_i$ following a multivariate categorical distribution, though with more complicated notations. We will consider possible extensions in Section 4.*

Remark 2 *The number of categories K is assumed to be fixed and known. Conditions $\sum_{k=1}^K \pi_k = 1$, $b_1 < b_2 < \cdots < b_K$, and $\pi_k \in (0,1)$ together are sufficient for the existence of K categories. For example, if $b_k = b_{k'}$, then we can merge categories k and k', and the number of categories reduces to $K-1$. Similarly, if $\pi_k = 0$ for some k, then category k can be deleted, and the number of categories is again reduced to $K-1$. Information criteria can be used to determine K, but this will not be pursued in this paper. Model specification tests could also be considered. See, for examples, Andrews (2001) and Breunig and Hoderlein (2018).*

In the rest of this section, we focus on the model (2.1) and establish the conditions under which the distribution of β_i is identified.

2.1 Identifying the moments of β_i

Assumption 1 *(a) (i) u_i is distributed independently of $\mathbf{w}_i = (x_i, \mathbf{z}_i')'$ and β_i. (ii) $\sup_i \mathrm{E}\left(|u_i^r|\right) < C$, $r = 1, 2, \cdots, 2K-1$. (iii) $n^{-1}\sum_{i=1}^n u_i^4 = O_p(1)$.*

(b) *(i) Let* $\mathbf{Q}_{n,ww} = n^{-1}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{w}_i'$, *and* $\mathbf{q}_{n,wy} = n^{-1}\sum_{i=1}^{n}\mathbf{w}_i y_i$. *Then* $\|\mathrm{E}(\mathbf{Q}_{n,ww})\| < C < \infty$, *and* $\|\mathrm{E}(\mathbf{q}_{n,wy})\| < C < \infty$, *and there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$,

$$0 < c < \lambda_{\min}(\mathbf{Q}_{n,ww}) < \lambda_{\max}(\mathbf{Q}_{n,ww}) < C < \infty.$$

(ii) $\sup_i \mathrm{E}(\|\mathbf{w}_i\|^r) < C < \infty$, $r = 1, 2, \cdots, 4K-2$.
(iii) $n^{-1}\sum_{i=1}^{n}\|\mathbf{w}_i\|^4 = O_p(1)$.

(c) $\|\mathbf{Q}_{n,ww} - E(\mathbf{Q}_{n,ww})\| = O_p\left(n^{-1/2}\right)$, $\|\mathbf{q}_{n,wy} - E(\mathbf{q}_{n,wy})\| = O_p\left(n^{-1/2}\right)$, *and*

$$E(\mathbf{Q}_{n,ww}) = n^{-1}\sum_{i=1}^{n}\mathrm{E}\left(\mathbf{w}_i\mathbf{w}_i'\right) \succ 0.$$

(d) $\|E(\mathbf{Q}_{n,ww}) - \mathbf{Q}_{ww}\| = O\left(n^{-1/2}\right)$, $\|E(\mathbf{q}_{n,wy}) - \mathbf{q}_{wy}\| = O\left(n^{-1/2}\right)$, *and* $\mathbf{Q}_{ww} \succ 0$.

Remark 3 *Part (a) of Assumption 1 relaxes the assumption that* u_i *is identically distributed, and allows for heterogeneously generated errors. For identification of the distribution of* β_i, *we require* u_i *to be distributed independently of* $\mathbf{w}_i$ *and* β_i, *which rules out conditional heteroskedasticity. However, estimation and inference involving* $\mathrm{E}(\beta_i)$ *and* $\boldsymbol{\gamma}$ *can be carried out in presence of conditionally error heteroskedastic, as shown in Theorem 3. Parts (c) and (d) of Assumption 1 relax the condition that* $\mathbf{w}_i$ *is identically distributed across* i. *As we proceed, only* β_i, *whose distribution is of interest, is assumed to be IID across* i, *and it is not required for* $\mathbf{w}_i$ *and* u_i *to be identically distributed over* i.

Remark 4 *The high level conditions in Assumption 1, concerning the convergence in probability of averages such as* $Q_{n,ww} = n^{-1}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{w}_i'$, *can be verified under weak cross-sectional dependence. Let* $f_i = f(\mathbf{w}_i, \beta_i, u_i)$ *be a generic function of* $\mathbf{w}_i$, β_i *and* u_i.[1] *Assume that* $\sup_i \mathrm{E}\left(f_i^2\right) < C$, *and* $\sup_j \sum_{i=1}^{n}|\mathrm{cov}(f_i, f_j)| < C$, *for some fixed* $C < \infty$. *Then,*

$$\mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} f_i\right) \leq \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|\mathrm{cov}(f_i, f_j)| \leq \frac{1}{n}\sup_j\sum_{i=1}^{n}|\mathrm{cov}(f_i, f_j)| \leq \frac{C}{n}.$$

By Chebyshev's inequality, for any $\varepsilon > 0$, *we have* $M_\varepsilon > \sqrt{C/\varepsilon}$ *such that*

$$\Pr\left(\sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}[f_i - \mathrm{E}(f_i)]\right| > M_\varepsilon\right) \leq \frac{n\mathrm{var}\left(n^{-1}\sum_{i=1}^{n} f_i\right)}{C}\varepsilon \leq \varepsilon,$$

i.e. $n^{-1}\sum_{i=1}^{n}[f_i - \mathrm{E}(f_i)] = O_p\left(n^{-1/2}\right)$.

Denote $\boldsymbol{\phi}_i = (\beta_i, \boldsymbol{\gamma}')'$ and $\boldsymbol{\phi} = \mathrm{E}(\boldsymbol{\phi}_i) = (\mathrm{E}(\beta_i), \boldsymbol{\gamma}')'$. Consider the moment condition,

$$\mathrm{E}(\mathbf{w}_i y_i) = \mathrm{E}\left(\mathbf{w}_i\mathbf{w}_i'\right)\boldsymbol{\phi}, \tag{2.3}$$

[1] ω_i is assumed to be a scalar, and we can apply the analysis element-by-element to a matrix, for example $\mathbf{w}_i\mathbf{w}_i'$.

and sum (2.3) over i

$$\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(\mathbf{w}_i y_i\right)=\left[\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(\mathbf{w}_i\mathbf{w}_i'\right)\right]\boldsymbol{\phi}. \tag{2.4}$$

Let $n\to\infty$, then $\boldsymbol{\phi}$ is identified by

$$\boldsymbol{\phi}=\mathbf{Q}_{ww}^{-1}\mathbf{q}_{wy}, \tag{2.5}$$

under Assumption 1.

Assumption 2 *Let $\tilde{y}_i=y_i-\mathbf{z}_i'\boldsymbol{\gamma}$.*

(a) $\left|n^{-1}\sum_{i=1}^{n}\mathrm{E}\left(\tilde{y}_i^r x_i^s\right)-\rho_{r,s}\right|=O\left(n^{-1/2}\right)$, *and* $\left|\rho_{r,s}\right|<\infty$, *for* $r,s=0,1,\cdots,2K-1$.

(b) $\left|n^{-1}\sum_{i=1}^{n}\mathrm{E}\left(u_i^r\right)-\sigma_r\right|=O\left(n^{-1/2}\right)$, *and* $|\sigma_r|<\infty$, *for* $r=2,3,\cdots,2K-1$.

(c) $n^{-1}\sum_{i=1}^{n}\left[\mathrm{var}(x_i^r)-\left(\rho_{0,2r}-\rho_{0,r}^2\right)\right]=O\left(n^{-1/2}\right)$ *where* $\rho_{0,2r}-\rho_{0,r}^2>0$, *for* $r=2,3,\cdots,2K-1$.

Theorem 1 *Under Assumptions 1 and 2,* $\mathrm{E}\left(\beta_i^r\right)$ *and* σ_r*,* $r=2,3,\cdots,2K-1$ *are identified.*

Proof. For $r=2,\cdots,2K-1$,

$$\mathrm{E}\left(\tilde{y}_i^r\right)=\mathrm{E}\left(x_i^r\right)\mathrm{E}\left(\beta_i^r\right)+\mathrm{E}\left(u_i^r\right)+\sum_{q=2}^{r-1}\binom{r}{q}\mathrm{E}\left(x_i^{r-q}\right)\mathrm{E}\left(u_i^q\right)\mathrm{E}\left(\beta_i^{r-q}\right), \tag{2.6}$$

$$\mathrm{E}\left(\tilde{y}_i^r x_i^r\right)=\mathrm{E}\left(x_i^{2r}\right)\mathrm{E}\left(\beta_i^r\right)+\mathrm{E}\left(x_i^r\right)\mathrm{E}\left(u_i^r\right)+\sum_{q=2}^{r-1}\binom{r}{q}\mathrm{E}\left(x_i^{2r-q}\right)\mathrm{E}\left(u_i^q\right)\mathrm{E}\left(\beta_i^{r-q}\right). \tag{2.7}$$

where $\binom{r}{q}=\frac{r!}{q!(r-q)!}$ are binomial coefficients, for non-negative integers $q\leq r$, denotes the binomial coefficients.

Sum over i, then by parts (a) and (b) of Assumption 2,

$$\rho_{0,r}\mathrm{E}\left(\beta_i^r\right)+\sigma_r=\rho_{r,0}-\sum_{q=2}^{r-1}\binom{r}{q}\rho_{0,r-q}\sigma_q\mathrm{E}\left(\beta_i^{r-q}\right), \tag{2.8}$$

$$\rho_{0,2r}\mathrm{E}\left(\beta_i^r\right)+\rho_{0,r}\sigma_r=\rho_{r,r}-\sum_{q=2}^{r-1}\binom{r}{q}\rho_{0,2r-q}\sigma_q\mathrm{E}\left(\beta_i^{r-q}\right). \tag{2.9}$$

Derivation details are relegated to Appendix A.1. By part (c) of 2, the matrix $\begin{pmatrix}\rho_{0,r} & 1\\ \rho_{0,2r} & \rho_{0,r}\end{pmatrix}$ is invertible for $r=2,3,\cdots,2K-1$. As a result, we can sequentially solve (2.8) and (2.9) for $\mathrm{E}\left(\beta_i^r\right)$ and σ_r, for $r=2,3,\cdots,2K-1$. ∎

2.2 Identifying the distribution of β_i

Beran and Hall (1992, Theorem 2.1, pp. 1972) prove the identification of the distribution of the random coefficient, β_i, in a canonical model without covariates, z_i, under the condition that the distribution of β_i is uniquely determined by its moments. We show the identification of moments of β_i holds more generally when x_i and u_i are not identically distributed and the distribution of β_i is identified if it follows a categorical distribution. Note that under (2.2),

$$\mathrm{E}\left(\beta_i^r\right) = \sum_{k=1}^{K} \pi_k b_k^r,\ r = 0, 1, 2, \cdots, 2K-1, \tag{2.10}$$

with $\mathrm{E}\left(\beta_i^r\right)$ identified under Assumption 1. To identify $\boldsymbol{\pi} = (\pi_1, \pi_2, ..., \pi_K)'$ and $\mathbf{b} = (b_1, b_2, ..., b_K)'$, we need to verify that the system of $2K$ equations in (2.10) has a unique solution if $b_1 < b_2 < \cdots < b_K$, and $\pi_k \in (0, 1)$. In the proof, we construct a linear recurrence relation and make use of the corresponding characteristic polynomial.

Theorem 2 *Consider the random coefficient regression model* (2.1)*, suppose that Assumptions 1 and 2 hold. Then* $\boldsymbol{\theta} = (\boldsymbol{\pi}', \mathbf{b}')'$ *is identified subject to* $b_1 < b_2 < \cdots < b_K$ *and* $\pi_k \in (0, 1)$*, for all* $k = 1, 2, \cdots, K$.

Proof. We motivate the key idea of the proof in the special case where $K = 2$, and relegate the proof of the general case to the Appendix A.1. Let $b_1 = \beta_L$, $b_2 = \beta_H$, $\pi_1 = \pi$ and $\pi_2 = 1 - \pi$. Note that

$$\mathrm{E}\left(\beta_i\right) = \pi\beta_L + (1-\pi)\,\beta_H, \tag{2.11}$$

$$\mathrm{E}\left(\beta_i^2\right) = \pi\beta_L^2 + (1-\pi)\,\beta_H^2, \tag{2.12}$$

$$\mathrm{E}\left(\beta_i^3\right) = \pi\beta_L^3 + (1-\pi)\,\beta_H^3, \tag{2.13}$$

and $\mathrm{E}\left(\beta_i^k\right)$, $k = 1, 2, 3$ are identified. (π, β_L, β_H) can be identified if the system of equations (2.11) to (2.13), has a unique solution. By (2.11),

$$\pi = \frac{\beta_H - \mathrm{E}\left(\beta_i\right)}{\beta_H - \beta_L},\ \text{and}\ 1 - \pi = \frac{\mathrm{E}\left(\beta_i\right) - \beta_L}{\beta_H - \beta_L}. \tag{2.14}$$

Plug (2.14) into (2.12) and (2.13),

$$\mathrm{E}\left(\beta_i\right)\left(\beta_L + \beta_H\right) - \beta_L\beta_H = \mathrm{E}\left(\beta_i^2\right), \tag{2.15}$$

$$\mathrm{E}\left(\beta_i^2\right)\left(\beta_L + \beta_H\right) - \mathrm{E}\left(\beta_i\right)\beta_L\beta_H = \mathrm{E}\left(\beta_i^3\right). \tag{2.16}$$

Denote $\beta_{L+H} = \beta_L + \beta_H$ and $\beta_{LH} = \beta_L\beta_H$, and write (2.15) and (2.16) in matrix form,

$$\mathbf{M}\mathbf{D}\mathbf{b}^* = \mathbf{m}, \tag{2.17}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & \mathrm{E}\left(\beta_i\right) \\ \mathrm{E}\left(\beta_i\right) & \mathrm{E}\left(\beta_i^2\right) \end{pmatrix}, \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{b}^* = \begin{pmatrix} \beta_{LH} \\ \beta_{L+H} \end{pmatrix}, \text{ and } \mathbf{m} = \begin{pmatrix} \mathrm{E}\left(\beta_i^2\right) \\ \mathrm{E}\left(\beta_i^3\right) \end{pmatrix}.$$

Under the conditions $0 < \pi < 1$ and $\beta_H > \beta_L$,

$$\det(\mathbf{M}) = \text{var}(\beta_i) = \mathrm{E}\left(\beta_i^2\right) - \mathrm{E}(\beta_i)^2 = \pi(1-\pi)(\beta_H - \beta_L)^2 > 0.$$

As a result, we can solve (2.17) for β_{L+H} and β_{LH} as

$$\beta_{L+H} = \frac{\mathrm{E}\left(\beta_i^3\right) - \mathrm{E}(\beta_i)\,\mathrm{E}\left(\beta_i^2\right)}{\text{var}(\beta_i)}, \tag{2.18}$$

$$\beta_{LH} = \frac{\mathrm{E}(\beta_i)\,\mathrm{E}\left(\beta_i^3\right) - \mathrm{E}\left(\beta_i^2\right)^2}{\text{var}(\beta_i)}. \tag{2.19}$$

β_L and β_H are solutions to the quadratic equation,

$$\beta^2 - \beta_{L+H}\beta + \beta_{LH} = 0. \tag{2.20}$$

We can verify that $\Delta = \beta_{L+H}^2 - 4\beta_{LH} > 0$ by direct calculation using (2.18) and (2.19). Simplifying Δ in terms of $\mathrm{E}\left(\beta_i^k\right)$ and then plugging in (2.11), (2.12) and (2.13),

$$\begin{aligned}\Delta =& \frac{\left[\mathrm{E}\left(\beta_i^3\right) - \mathrm{E}(\beta_i)\,\mathrm{E}\left(\beta_i^2\right)\right]^2 - 4\text{var}(\beta_i)\left[\mathrm{E}(\beta_i)\,\mathrm{E}\left(\beta_i^3\right) - \mathrm{E}\left(\beta_i^2\right)^2\right]}{[\text{var}(\beta_i)]^2} \\ &= (\beta_H - \beta_L)^2 > 0.\end{aligned}$$

Then, we obtain the unique solutions,

$$\beta_L = \frac{1}{2}\left(\beta_{L+H} - \sqrt{\beta_{L+H}^2 - 4\beta_{LH}}\right), \tag{2.21}$$

$$\beta_H = \frac{1}{2}\left(\beta_{L+H} + \sqrt{\beta_{L+H}^2 - 4\beta_{LH}}\right), \tag{2.22}$$

and π can be determined by (2.14) correspondingly. ■

Remark 5 *The key identifying assumption in (2) is the assumed existence of the strict ordinal relation $b_1 < b_2 < \cdots < b_K$ so that b_k and $b_{k'}$ are not symmetric for $k \neq k'$, and $0 < \pi_k < 1$ so that the distribution of β_i does not degenerate. When $K = 2$, the conditions $b_1 < b_2 < \cdots < b_K$, and $\pi_k \in (0, 1)$, are equivalent to $\text{var}(\beta_i) = \pi_1(1-\pi_1)(b_2 - b_1)^2 > 0$. In other words, not surprisingly, the categorical distribution of β_i are identified only if $\text{var}(\beta_i) > 0$.*

In practice, a test for $\mathbb{H}_0 : \text{var}(\beta_i) = 0$ is possible, by noting that $\text{var}(\beta_i) = 0$ is equivalent to

$$\kappa^2 = \frac{\mathrm{E}(\beta_i)^2}{\mathrm{E}\left(\beta_i^2\right)} = 1,$$

where κ^2 is well-defined as long as $\beta_i \not\equiv 0$. One important advantage of basing the test of slope homogeneity on κ^2 rather than on var$(\beta_i) = 0$, is that κ^2 is scale-invariant. $\mathrm{E}(\beta_i)$ and $\mathrm{E}\left(\beta_i^2\right)$ are identified as in Section 2.1, whose consistent estimation does not require $\text{var}(\beta_i) > 0$. Consequently, *in principle it is possible to test slope homogeneity by testing $\mathbb{H}_0 : \kappa^2 = 1$. However, the problem becomes much more complicated when there are more than two categories and/or there are more than one regressor under consideration. A full treatment of testing slope homogeneity in such general settings is beyond the scope of the present paper.*

Remark 6 *Note that in the special case of the proof of Theorem 2 where $K = 2$, $\beta_{L+H} = \beta_L + \beta_H$ and $\beta_{LH} = \beta_L \beta_H$ corresponds to the b_1^* and b_2^* and* (2.17) *is the same as* (A.1.6) *when $K = 2$. The special case illustrates the procedure of identification: identify $(b_k^*)_{k=1}^K$ by the moments of β_i, then solve for $(b_k)_{k=1}^K$ and finally identify $(\pi_k)_{k=1}^K$.*

3 Estimation

In this section, we propose a generalized method of moments estimator for the distributional parameters of β_i. To reduce the complexity of the moment equations, we first obtain a $\sqrt{n}$-consistent estimator of γ and consider the estimation of the distribution of β_i by replacing γ by $\hat{\gamma}$.

3.1 Estimation of γ

Let $\boldsymbol{\phi} = (\mathrm{E}(\beta_i), \boldsymbol{\gamma}')'$, $v_i = \beta_i - \mathrm{E}(\beta_i)$ and using the notation in Assumption 1, (2.1) can be written as

$$y_i = \mathbf{w}_i'\boldsymbol{\phi} + \xi_i, \tag{3.1}$$

where $\xi_i = u_i + x_i v_i$. Then $\boldsymbol{\phi}$ can be estimated consistently by $\hat{\boldsymbol{\phi}} = \mathbf{Q}_{n,ww}^{-1}\mathbf{q}_{n,wy}$ where $\mathbf{Q}_{n,ww}$ and $\mathbf{q}_{n,wy}$ are defined in Assumption 1.

Assumption 3 $\left\| n^{-1}\sum_{i=1}^n \mathrm{E}\left(\mathbf{w}_i\mathbf{w}_i'\xi_i^2\right) - \mathbf{V}_{w\xi}\right\| = O\left(n^{-1/2}\right)$, $\mathbf{V}_{w\xi} \succ 0$, *and*

$$\left\| \frac{1}{n}\sum_{i=1}^n \mathbf{w}_i\mathbf{w}_i'\xi_i^2 - \frac{1}{n}\sum_{i=1}^n \mathrm{E}\left(\mathbf{w}_i\mathbf{w}_i'\xi_i^2\right)\right\| = O_p\left(n^{-1/2}\right). \tag{3.2}$$

Remark 7 *As in the case of Assumption 1, the high level condition* (3.2) *can be shown to hold under weak cross-sectional dependence, assuming that elements of $\mathbf{w}_i\mathbf{w}_i'\xi_i^2$ are cross-sectionally weakly correlated over i. See Remark 4.*

Theorem 3 *Under Assumption 1, $\hat{\boldsymbol{\phi}}$ is a consistent estimator for $\boldsymbol{\phi}$. In addition, under Assumptions 1 and 3, as $n \to \infty$,*

$$\sqrt{n}\left(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right) \to_d N\left(\mathbf{0}, \mathbf{V}_\phi\right), \tag{3.3}$$

where $\mathbf{V}_\phi = \mathbf{Q}_{ww}^{-1}\mathbf{V}_{w\xi}\mathbf{Q}_{ww}^{-1}$. $\mathbf{V}_\phi$ is consistently estimated by

$$\hat{\mathbf{V}}_\phi = \mathbf{Q}_{n,ww}^{-1}\hat{\mathbf{V}}_{w\xi}\mathbf{Q}_{n,ww}^{-1} \to_p \mathbf{V}_\phi,$$

as $n \to \infty$, where $\hat{\mathbf{V}}_{w\xi} = n^{-1}\sum_{i=1}^n \mathbf{w}_i\mathbf{w}_i'\hat{\xi}_i^2$, and $\hat{\xi}_i = y_i - \mathbf{w}_i'\hat{\boldsymbol{\phi}}$.

The proof of Theorem 3 is provided in Appendix A.1.

3.2 Estimation of the distribution of β_i

Denote the moments of β_i on the right-hand side of (2.10) by

$$\mathbf{m}_\beta = (m_1, m_2, ..., m_{2K-1})' = [\mathrm{E}\,(\beta_i^r)]_{r=1}^{2K-1} \in \Theta_m \subset \left\{\mathbf{m}_\beta \in \mathbb{R}^{2K-1} : m_r \geq 0,\ r \text{ is even}\right\},$$

and note that

$$\mathbf{m}_\beta = \begin{pmatrix} m_1 \\ m_2 \\ \vdots \\ m_{2K-1} \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & \cdots & b_K \\ b_1^2 & b_2^2 & \cdots & b_K^2 \\ \vdots & \vdots & \vdots & \vdots \\ b_1^{2K-1} & b_2^{2K-1} & \cdots & b_K^{2K-1} \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_K \end{pmatrix}, \tag{3.4}$$

so in general we can write $\mathbf{m}_\beta \triangleq h\,(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\boldsymbol{\pi}', \mathbf{b}')' \in \Theta$, and $\boldsymbol{\theta}$ can be uniquely determined in terms of $\mathbf{m}_\beta$ by Theorem 2. To estimate $\boldsymbol{\theta}$, we consider moment conditions following a similar procedure as in Section 2, and propose a generalized method of moments (GMM) estimator.

We consider the following moment conditions

$$\mathrm{E}\,(\tilde{y}_i^r) = \sum_{q=0}^{r} \binom{r}{q} \mathrm{E}\left(x_i^{r-q}\right) \mathrm{E}\,(u_i^q)\, m_{r-q},$$

and

$$\mathrm{E}\,(\tilde{y}_i^r x_i^{s_r}) = \sum_{q=0}^{r} \binom{r}{q} \mathrm{E}\left(x_i^{r-q+s_r}\right) \mathrm{E}\,(u_i^q)\, m_{r-q}, \tag{3.5}$$

where $\mathrm{E}\,(u_i) = 0$, $\tilde{y}_i = y_i - \mathbf{z}_i'\boldsymbol{\gamma}$, $r = 1, 2, ..., 2K-1$, and $s_r = 0, 1, \cdots, S-r$, where S is a user-specific tuning parameter, chosen such that the highest order moments of x_i included is at most S, where $S > 2K-1$. [2]

Let $\sigma_0 = 1$ and $\sigma_1 = 0$ such that σ_r is well-defined for $r = 0, 1, \cdots, 2K-1$. Sum (3.5) over i and rearrange terms,

$$\begin{aligned} 0 &= \sum_{q=0}^{r} \binom{r}{q} \left[\frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\left(x_i^{r-q+s_r}\right) \mathrm{E}\,(u_i^q)\right] m_{r-q} - \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\,(\tilde{y}_i^r x_i^{s_r}) \\ &= \sum_{q=0}^{r} \binom{r}{q} \left[\frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\left(x_i^{r-q+s_r}\right)\right] \sigma_q m_{r-q} - \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\,(\tilde{y}_i^r x_i^{s_r}) + \delta_n^{(r,s_r)}, \end{aligned} \tag{3.6}$$

where

$$\delta_n^{(r,s_r)} = \sum_{q=0}^{r} \binom{r}{q} \left[\frac{1}{n}\sum_{i=1}^{n} \mathrm{E}\left(x_i^{r-q+s_r}\right) \left[\mathrm{E}\,(u_i^q) - \sigma_q\right]\right] m_{r-q} = O\left(n^{-1/2}\right),$$

as shown in the proof of Theorem 1.

[2]For identification, we require the moments of x_i to exist up to order $4K-2$. S can take values between $2K$ to $4K-2$. In practice, the choice of S affects the trade-off between bias and efficiency.

Taking $n \to \infty$ in (3.6),

$$\sum_{q=0}^{r} \binom{r}{q} \rho_{0,r-q+s_r} \sigma_q m_{r-q} - \rho_{r,s_r} = 0, \tag{3.7}$$

by Assumption 2. We stack the left-hand side of (3.7) over $r = 1, 2, ..., 2K-1$, and $s_r = 0, 1, \cdots, S-r$ and transform $\mathbf{m}_\beta = h(\boldsymbol{\theta})$ to get $\mathbf{g}_0(\boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\gamma})$.

To implement the GMM estimation we replace $\tilde{y}_i$, by $\hat{\tilde{y}}_i = y_i - \mathbf{z}_i'\hat{\boldsymbol{\gamma}}$, and ρ_{r,s_r} by $n^{-1}\sum_{i=1}^{n} \hat{\tilde{y}}_i^r x_i^{s_r}$. Noting that $\mathbf{m}_\beta = h(\boldsymbol{\theta})$, denote the sample version of the left-hand side of (3.7) by

$$\hat{g}_n^{(r,s_r)}(\boldsymbol{\theta}, \boldsymbol{\sigma}, \hat{\boldsymbol{\gamma}}) = \frac{1}{n}\sum_{i=1}^{n} \hat{g}_i^{(r,s_r)}(\boldsymbol{\theta}, \boldsymbol{\sigma}, \hat{\boldsymbol{\gamma}}), \tag{3.8}$$

where

$$\hat{g}_i^{(r,s_r)}(\boldsymbol{\theta}, \boldsymbol{\sigma}, \hat{\boldsymbol{\gamma}}) = \sum_{q=0}^{r} \binom{r}{q} x_i^{r-q+s_r} \sigma_q [h(\boldsymbol{\theta})]_{r-q} - \hat{\tilde{y}}_i^r x_i^{s_r},$$

and $\boldsymbol{\sigma} = (\sigma_2, \sigma_3, \cdots, \sigma_{2K-1})'$. Stack the equations in (3.8), over $r = 0, 1, ..., 2K-1$ and $s_r = 0, 1, \cdots, S-r$ $(S > 2K-1)$, in vector notations we have

$$\hat{\mathbf{g}}_n(\boldsymbol{\theta}, \boldsymbol{\sigma}, \hat{\boldsymbol{\gamma}}) = \frac{1}{n}\sum_{i=1}^{n} \hat{\mathbf{g}}_i(\boldsymbol{\theta}, \boldsymbol{\sigma}, \hat{\boldsymbol{\gamma}}). \tag{3.9}$$

Given $\hat{\boldsymbol{\gamma}}$, the GMM estimator of $(\boldsymbol{\theta}', \boldsymbol{\sigma}')'$ is now computed as

$$\left(\hat{\boldsymbol{\theta}}', \hat{\boldsymbol{\sigma}}'\right)' = \arg\min_{\boldsymbol{\theta}\in\Theta, \boldsymbol{\sigma}\in\mathcal{S}} \hat{\Phi}_n(\boldsymbol{\theta}, \boldsymbol{\sigma}, \hat{\boldsymbol{\gamma}}),$$

where $\hat{\Phi}_n = \hat{\mathbf{g}}_n(\boldsymbol{\theta}, \boldsymbol{\sigma}, \hat{\boldsymbol{\gamma}})' \mathbf{A}_n \hat{\mathbf{g}}_n(\boldsymbol{\theta}, \boldsymbol{\sigma}, \hat{\boldsymbol{\gamma}})$, and $\mathbf{A}_n$ is a positive definite matrix. We follow the GMM literature using the following choice of $\mathbf{A}_n$,

$$\hat{\mathbf{A}}_n = \left[\frac{1}{n}\sum_{i=1}^{n} \hat{\mathbf{g}}_i\left(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\sigma}}, \hat{\boldsymbol{\gamma}}\right) \hat{\mathbf{g}}_i\left(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\sigma}}, \hat{\boldsymbol{\gamma}}\right)' - \bar{\mathbf{g}}_n \bar{\mathbf{g}}_n'\right]^{-1}, \tag{3.10}$$

where $\bar{\mathbf{g}}_n = \frac{1}{n}\sum_{i=1}^{n} \hat{\mathbf{g}}_i\left(\tilde{\boldsymbol{\theta}}, \tilde{\boldsymbol{\sigma}}, \hat{\boldsymbol{\gamma}}\right)$, and $\tilde{\boldsymbol{\theta}}$ and $\tilde{\boldsymbol{\sigma}}$ are preliminary consistent estimators obtained by implementing the GMM estimator with $\mathbf{A}_n$ equal to the identity matrix.

Assumption 4 *Denote the true values of $\boldsymbol{\theta}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ by $\boldsymbol{\theta}_0$, $\boldsymbol{\sigma}_0$ and $\boldsymbol{\gamma}_0$.*

(a) Θ *and* $\mathcal{S}$ *are compact.* $\boldsymbol{\theta}_0 \in \mathrm{int}(\Theta)$ *and* $\boldsymbol{\sigma}_0 \in \mathrm{int}(\mathcal{S})$.

(b) $\mathbf{A}_n \to_p \mathbf{A}$ *as* $n \to \infty$, *where* $\mathbf{A}$ *is some positive definite matrix.*

(c)

$$\frac{1}{n}\sum_{i=1}^{n} \left[\hat{\tilde{y}}_i^r x_i^{s_r} - \mathrm{E}\left(\tilde{y}_i^r x_i^{s_r}\right)\right] = O_p\left(n^{-1/2}\right),$$

for $r = 0, 1, 2, \cdots, 2K-1$, $s_r = 0, 1, \cdots, S-r$, *and* $S > 2K-1$.

Remark 8 *Parts (a) and (b) of Assumption 4 are standard regularity conditions in the GMM literature. Part (c) together with Assumption 2 are high-level regularity conditions which allow us to generalize the usual IID assumption and nest the IID data generation process as a special case. The sample analogue terms in (c) include* $\hat{\tilde{y}}_i = y_i - \mathbf{z}_i'\hat{\boldsymbol{\gamma}}$, *instead of the infeasible* $\tilde{y}_i = y_i - \mathbf{z}_i'\boldsymbol{\gamma}$. *The* $\sqrt{n}$*-consistency of* $\hat{\boldsymbol{\gamma}}$ *shown in Theorem 3 ensures that replacing* $\tilde{y}_i$ *by* $\hat{\tilde{y}}_i$ *does not alter the convergence rate.*

Theorem 4 *Let* $\boldsymbol{\eta} = \left(\boldsymbol{\theta}', \boldsymbol{\sigma}'\right)'$ *and* $\boldsymbol{\eta}_0 = \left(\boldsymbol{\theta}_0', \boldsymbol{\sigma}_0'\right)'$. *Under Assumptions 1, 2, and 4,* $\hat{\boldsymbol{\eta}} \to_p \boldsymbol{\eta}_0$ *as* $n \to \infty$.

The proof of Theorem 4 is provided in Appendix A.1.

Assumption 5 *Follow the notations as in Assumption 4 and in addition denote* $\mathbf{G}(\boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\gamma}) = \nabla_{(\boldsymbol{\theta}', \boldsymbol{\sigma}')'}\mathbf{g}_0(\boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\gamma})$, $\mathbf{G}_0 = \mathbf{G}(\boldsymbol{\theta}_0, \boldsymbol{\sigma}_0, \boldsymbol{\gamma}_0)$, $\mathbf{G}_\gamma(\boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\gamma}) = \nabla_{\boldsymbol{\gamma}}\mathbf{g}_0(\boldsymbol{\theta}, \boldsymbol{\sigma}, \boldsymbol{\gamma})$, $\mathbf{G}_{0,\gamma} = \mathbf{G}_\gamma(\boldsymbol{\theta}_0, \boldsymbol{\sigma}_0, \boldsymbol{\gamma}_0)$.

(a) $\sqrt{n}\hat{\mathbf{g}}_n(\boldsymbol{\theta}_0, \boldsymbol{\sigma}_0, \boldsymbol{\gamma}_0) \to_d \boldsymbol{\zeta} \sim N(0, \mathbf{V})$ *as* $n \to \infty$.

(b) $\mathbf{G}_0'\mathbf{A}\mathbf{G}_0 \succ 0$.

Remark 9 *In Assumption 5, parts (a) is the high level condition required to ensure the asymptotic normality of* $\hat{\mathbf{g}}_n(\boldsymbol{\theta}_0, \boldsymbol{\sigma}_0, \boldsymbol{\gamma}_0)$, *which can be verified by Lindeberg central limit theorem under low-level regularity conditions. Part (c) of Assumption 5 represents the full-rank condition on* $\mathbf{G}_0$, *required for identification of* $\boldsymbol{\theta}_0$ *and* $\boldsymbol{\sigma}_0$.

By Theorem 3, we have $\sqrt{n}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}) \to_d \zeta_\gamma \sim N(0, V_\gamma)$. The following theorem shows the asymptotic normality of the GMM estimator $\hat{\boldsymbol{\eta}}$.

Theorem 5 *Under Assumptions 1, 3, 4 and 5,*

$$\sqrt{n}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0) \to_d \left(\mathbf{G}_0'\mathbf{A}\mathbf{G}_0\right)^{-1}\mathbf{G}_0'\mathbf{A}\left(\boldsymbol{\zeta} + \mathbf{G}_{0,\gamma}\boldsymbol{\zeta}_\gamma\right),$$

as $n \to \infty$.

The proof of Theorem 5 is provided in Appendix A.1.

Remark 10 *In practice, we estimate the variance of the asymptotic distribution of* $\hat{\eta}$ *by*

$$\hat{\mathbf{V}}_\eta = \left(\hat{\mathbf{G}}'\hat{\mathbf{A}}_n\hat{\mathbf{G}}\right)^{-1}\hat{\mathbf{G}}'\hat{\mathbf{A}}_n\hat{\mathbf{V}}_\zeta\hat{\mathbf{A}}_n'\hat{\mathbf{G}}\left(\hat{\mathbf{G}}'\hat{\mathbf{A}}_n\hat{\mathbf{G}}\right)^{-1}, \tag{3.11}$$

where $\hat{\mathbf{G}} = \nabla_{(\boldsymbol{\sigma}', \boldsymbol{\theta}')'}\hat{\mathbf{g}}_n\left(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\sigma}}, \hat{\gamma}\right)$, $\hat{\mathbf{A}}_n$ *is given by* (3.10), *and*

$$\hat{\mathbf{V}}_\zeta = \frac{1}{n}\sum_{i=1}^n \boldsymbol{\psi}_{n,i}\boldsymbol{\psi}_{n,i}',$$

where

$$\boldsymbol{\psi}_{n,i} = \hat{\mathbf{g}}_i\left(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\sigma}}, \hat{\gamma}\right) + \nabla_{\boldsymbol{\gamma}}\hat{\mathbf{g}}_n\left(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\sigma}}, \hat{\gamma}\right)\mathbf{L}\mathbf{Q}_{n,ww}\left(\mathbf{w}_i\hat{\xi}_i\right),$$

and $\mathbf{L} = \begin{pmatrix}\mathbf{0}_{p_z\times 1} & \mathbf{I}_{p_z}\end{pmatrix}$ *is the loading matrix that selects* $\boldsymbol{\gamma}$ *out of* $\boldsymbol{\phi}$*.*

4 Multiple regressors with random coefficients

One important extension of the regression model (2.1) is to allow for multiple regressors with random coefficients having categorical distribution. With this in mind consider

$$y_i = \mathbf{x}_i'\boldsymbol{\beta}_i + \mathbf{z}_i'\boldsymbol{\gamma} + u_i, \tag{4.1}$$

where the $p \times 1$ vector of random coefficients, $\boldsymbol{\beta}_i \in \mathbb{R}^p$ follows the multivariate distribution[3]

$$\Pr\left(\beta_{i1} = b_{1k_1}, \beta_{i2} = b_{2k_2}, \cdots, \beta_{ip} = b_{pk_p}\right) = \pi_{k_1,k_2,\cdots,k_p}, \tag{4.2}$$

with $k_j \in \{1, 2, \cdots, K\}$, $b_{j1} < b_{j2} < \cdots < b_{jK}$, and

$$\sum_{k_1,k_2,\cdots,k_p \in \{1,2,\cdots,K\}} \pi_{k_1,k_2,\cdots,k_p} = 1.$$

As in Section 2, $\boldsymbol{\gamma} \in \mathbb{R}^{p_z}$, $\mathbf{w}_i = \left(\mathbf{x}_i', \mathbf{z}_i'\right)'$, $\boldsymbol{\beta}_i \perp \mathbf{w}_i$, $u_i \perp \mathbf{w}_i$, and u_i are independently distributed over i with mean 0.

Example 1 *Consider the simple case with* $p = 2$ *and* $K = 2$*. For* $j = 1, 2$*, denote two categories as* $\{L, H\}$*. The probabilities of four possible combinations of realized* $\boldsymbol{\beta}_i$ *is summarized in Table 1, where* $\pi_{LL} + \pi_{LH} + \pi_{HL} + \pi_{HH} = 1$*.*

Table 1: Distribution of $\boldsymbol{\beta}_i$ with $p = 2$ and $K = 2$

	$k_2 = L$	$k_2 = H$
$k_1 = L$	$\pi_{LL} = \Pr\left(\beta_{i1} = b_{1L}, \beta_{i2} = b_{2L}\right)$	$\pi_{LH} = \Pr\left(\beta_{i1} = b_{1L}, \beta_{i2} = b_{2H}\right)$
$k_1 = H$	$\pi_{HL} = \Pr\left(\beta_{i1} = b_{1H}, \beta_{i2} = b_{2L}\right)$	$\pi_{HH} = \Pr\left(\beta_{i1} = b_{1H}, \beta_{i2} = b_{2H}\right)$

We first identify the moments of $\boldsymbol{\beta}_i$. As in Section 2, $\boldsymbol{\phi} = \left(\mathrm{E}\left(\boldsymbol{\beta}_i\right)', \boldsymbol{\gamma}'\right)'$ is identified by

$$\boldsymbol{\phi} = \mathbf{Q}_{ww}^{-1}\mathbf{q}_{wy}, \tag{4.3}$$

under Assumption 1. We now consider the identification of the higher order moments of $\boldsymbol{\beta}_i$ up to the finite order $2K - 1$.

[3] We assume the number of categories K is homogeneous across $j = 1, 2, \cdots, p$. This is for notational simplicity, and can be readily generalized to allow for $K_j \neq K_{j'}$ without affecting the main results.

Since $\boldsymbol{\gamma}$ is identified as in (4.3), we treat it as known and let $\tilde{y}_i^r = y_i - \mathbf{z}_i' \boldsymbol{\gamma}$. For $r = 2, 3, \cdots, 2K-1$, consider the moment conditions

$$\begin{aligned} \mathrm{E}\left(\tilde{y}_i^r\right) &= \mathrm{E}\left[\left(\mathbf{x}_i'\boldsymbol{\beta}_i + u_i\right)^r\right] \\ &= \mathrm{E}\left[\left(\mathbf{x}_i'\boldsymbol{\beta}_i\right)^r\right] + \mathrm{E}\left(u_i^r\right) + \sum_{s=2}^{r-1}\binom{r}{s}\mathrm{E}\left[\left(\mathbf{x}_i'\boldsymbol{\beta}_i\right)^{r-s}\right]\mathrm{E}\left(u_i^s\right). \end{aligned} \tag{4.4}$$

Note that $\mathbf{x}_i'\boldsymbol{\beta}_i = \sum_{j=1}^p \beta_{ij} x_{ij}$, and

$$\mathrm{E}\left[\left(\sum_{j=1}^p \beta_{ij} x_{ij}\right)^r\right] = \sum_{\sum_{j=1}^p q_j = r} \binom{r}{\mathbf{q}} \mathrm{E}\left(\prod_{j=1}^p x_{ij}^{q_j}\right) \mathrm{E}\left(\prod_{j=1}^p \beta_{ij}^{q_j}\right),$$

where $\binom{r}{\mathbf{q}} = \frac{r!}{q_1! q_2! \cdots q_p!}$, for non-negative integers r, q_1, $\cdots$, q_p with $r = \sum_{j=1}^p q_j$, denotes the multinomial coefficients. We stack $\prod_{j=1}^p x_{ij}^{q_j}$ with $\mathbf{q} \in \left\{\mathbf{q} \in \{0, 1, \cdots r\}^p : \sum_{j=1}^p q_j = r\right\}$ in a vector form by denoting [4]

$$\boldsymbol{\tau}_r\left(\mathbf{x}_i\right) = \left[\varphi\left(\mathbf{x}_i, \mathbf{q}_1\right), \varphi\left(\mathbf{x}_i, \mathbf{q}_2\right), \cdots, \varphi\left(\mathbf{x}_i, \mathbf{q}_{\nu_r}\right)\right]',$$

where $\varphi\left(\mathbf{x}_i, \mathbf{q}\right) = \prod_{j=1}^p x_{ij}^{q_j}$ and $\nu_r = \binom{r+p-1}{p-1}$ is the number of distinct monomials of degree r on the variables $x_{i1}, x_{i2}, \cdots, x_{ip}$. Similarly,

$$\boldsymbol{\tau}_r\left(\boldsymbol{\beta}_i\right) = \left[\varphi\left(\boldsymbol{\beta}_i, \mathbf{q}_1\right), \varphi\left(\boldsymbol{\beta}_i, \mathbf{q}_2\right), \cdots, \varphi\left(\boldsymbol{\beta}_i, \mathbf{q}_{\nu_r}\right)\right]',$$

where $\varphi\left(\boldsymbol{\beta}_i, \mathbf{q}\right) = \prod_{j=1}^p \beta_{ij}^{q_j}$.

Example 2 *Consider $p = 2$ and $r = 2$, we have*

$$\begin{aligned} \boldsymbol{\tau}_2\left(\mathbf{x}_i\right) &= \left(x_{i1}^2, x_{i1}x_{i2}, x_{i2}^2\right)', \\ \boldsymbol{\tau}_2\left(\boldsymbol{\beta}_i\right) &= \left(\beta_{i1}^2, \beta_{i1}\beta_{i2}, \beta_{i2}^2\right)', \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}\left[\left(x_{i1}\beta_{i1} + x_{i2}\beta_{i2}\right)^2\right] &= \mathrm{E}\left(x_{i1}^2\right)\mathrm{E}\left(\beta_{i1}^2\right) + 2\mathrm{E}\left(x_{i1}x_{i2}\right)\mathrm{E}\left(\beta_{i1}\beta_{i2}\right) + \mathrm{E}\left(x_{i2}^2\right)\mathrm{E}\left(\beta_{i2}^2\right) \\ &= \left[\mathrm{E}\left(x_{i1}^2\right), \mathrm{E}\left(x_{i1}x_{i2}\right), \mathrm{E}\left(x_{i2}^2\right)\right] \mathrm{diag}\left[(1,2,1)'\right] \left[\mathrm{E}\left(\beta_{i1}^2\right), \mathrm{E}\left(\beta_{i1}\beta_{i2}\right), \mathrm{E}\left(\beta_{i2}^2\right)\right]' \\ &= \mathrm{E}\left[\boldsymbol{\tau}_2\left(\mathbf{x}_i\right)\right]' \boldsymbol{\Lambda}_2 \mathrm{E}\left[\boldsymbol{\tau}_2\left(\boldsymbol{\beta}_i\right)\right], \end{aligned}$$

where $\boldsymbol{\Lambda}_2 = \mathrm{diag}\left[(1,2,1)'\right]$.

[4]For $\mathbf{x} \in \mathbb{R}^p$, note that $\boldsymbol{\tau}_0\left(\mathbf{x}\right) = 1$, $\boldsymbol{\tau}_1\left(\mathbf{x}\right) = \mathbf{x}$ and $\boldsymbol{\tau}_2\left(\mathbf{x}\right) = \mathrm{vech}\left(\mathbf{x}\mathbf{x}'\right)$.

Then the moment condition (4.4) can be written as

$$
\begin{aligned}
\mathrm{E}\left(\tilde{y}_i^r\right) = {} & \mathrm{E}\left[\boldsymbol{\tau}_r\left(\mathbf{x}_i\right)\right]' \boldsymbol{\Lambda}_r \mathrm{E}\left[\boldsymbol{\tau}_r\left(\boldsymbol{\beta}_i\right)\right] + \mathrm{E}\left(u_i^r\right) \\
& + \sum_{s=2}^{r-1} \binom{r}{s} \mathrm{E}\left[\boldsymbol{\tau}_{r-s}\left(\mathbf{x}_i\right)\right]' \boldsymbol{\Lambda}_{r-s} \mathrm{E}\left[\boldsymbol{\tau}_{r-s}\left(\boldsymbol{\beta}_i\right)\right] \mathrm{E}\left(u_i^s\right),
\end{aligned}
\tag{4.5}
$$

where $\boldsymbol{\Lambda}_r = \operatorname{diag}\left[\left[\binom{r}{\mathbf{q}}\right]_{\sum_{j=1}^p q_j = r}\right]$ is the $\nu_r \times \nu_r$ diagonal matrix of multinomial coefficients. We further consider the moment conditions

$$
\begin{aligned}
\mathrm{E}\left(\tilde{y}_i^r \boldsymbol{\tau}_r\left(\mathbf{x}_i\right)\right) = {} & \mathrm{E}\left[\boldsymbol{\tau}_r\left(\mathbf{x}_i\right) \boldsymbol{\tau}_r\left(\mathbf{x}_i\right)'\right] \boldsymbol{\Lambda}_r \mathrm{E}\left[\boldsymbol{\tau}_r\left(\boldsymbol{\beta}_i\right)\right] + \mathrm{E}\left[\boldsymbol{\tau}_r\left(\mathbf{x}_i\right)\right] \mathrm{E}\left(u_i^r\right) \\
& + \sum_{s=2}^{r-1} \binom{r}{s} \mathrm{E}\left[\boldsymbol{\tau}_r\left(\mathbf{x}_i\right) \boldsymbol{\tau}_{r-s}\left(\mathbf{x}_i\right)'\right] \boldsymbol{\Lambda}_{r-s} \mathrm{E}\left[\boldsymbol{\tau}_{r-s}\left(\boldsymbol{\beta}_i\right)\right] \mathrm{E}\left(u_i^s\right),
\end{aligned}
\tag{4.6}
$$

$r = 2, 3, \cdots, 2K-1$. (4.5) and (4.6) reduce to (2.6) and (2.7) when $p = 1$.

Assumption 6

(a) $\left\|n^{-1} \sum_{i=1}^n \mathrm{E}\left(\tilde{y}_i^r \boldsymbol{\tau}_s\left(\mathbf{x}_i\right)\right) - \boldsymbol{\rho}_{r,s}\right\| = O\left(n^{-1/2}\right)$, *and* $\left\|\boldsymbol{\rho}_{r,s}\right\| < \infty$, $r, s = 0, 1, \cdots, 2K-1$.

(b) $\left\|n^{-1} \sum_{i=1}^n \mathrm{E}\left[\boldsymbol{\tau}_r\left(\mathbf{x}_i\right) \boldsymbol{\tau}_s\left(\mathbf{x}_i\right)'\right] - \boldsymbol{\Xi}_{r,s}\right\| = O\left(n^{-1/2}\right)$, *and* $\left\|\boldsymbol{\Xi}_{r,s}\right\| < \infty$, $r, s = 0, 1, \cdots, 2K-1$.

(c) $\left|n^{-1} \sum_{i=1}^n \mathrm{E}\left(u_i^r\right) - \sigma_r\right| = O\left(n^{-1/2}\right)$, *and* $\left|\sigma_r\right| < \infty$ *for* $r = 2, 3, \cdots, 2K-1$.

(d) $\left\|n^{-1} \sum_{i=1}^n \left[\operatorname{var}\left(\boldsymbol{\tau}_r\left(\mathbf{x_i}\right)\right) - \left(\boldsymbol{\Xi}_{r,r} - \boldsymbol{\rho}_{0,r} \boldsymbol{\rho}_{0,r}'\right)\right]\right\| = O(n^{-1/2})$, *where* $\boldsymbol{\Xi}_{r,r} - \boldsymbol{\rho}_{0,r} \boldsymbol{\rho}_{0,r}' \succ 0$ *for* $r = 2, 3 \cdots, 2K-1$.

Theorem 6 *For any* $\mathbf{q} \in \left\{\mathbf{q} \in \{0, 1, \cdots r\}^p : \sum_{j=1}^p q_j = r\right\}$ *and* $r = 2, 3, \cdots, 2K-1$, $\mathrm{E}\left(\prod_{j=1}^p \beta_{ij}^{q_j}\right)$ *and* σ_r *are identified under Assumptions 1 and 6.*

Proof. For $r = 2, 3, \cdots, 2K-1$, sum (4.5) and (4.6) over i, go through the same steps as in the proof of Theorem 1, then by Assumptions 6(a) to (c), we have (for $n \to \infty$)

$$
\boldsymbol{\rho}_{r,0}' \boldsymbol{\Lambda}_r \mathrm{E}\left[\boldsymbol{\tau}_r\left(\boldsymbol{\beta}_i\right)\right] + \sigma_r = \boldsymbol{\rho}_{r,0} - \sum_{s=2}^{r-1} \binom{r}{s} \boldsymbol{\rho}_{0,r-s} \boldsymbol{\Lambda}_{r-s} \mathrm{E}\left[\boldsymbol{\tau}_{r-s}\left(\boldsymbol{\beta}_i\right)\right] \sigma_s, \tag{4.7}
$$

$$
\boldsymbol{\Xi}_{r,r} \boldsymbol{\Lambda}_r \mathrm{E}\left[\boldsymbol{\tau}_r\left(\boldsymbol{\beta}_i\right)\right] + \boldsymbol{\rho}_{0,r} \sigma_r = \boldsymbol{\rho}_{r,r} - \sum_{s=2}^{r-1} \binom{r}{s} \boldsymbol{\Xi}_{r,r-s} \boldsymbol{\Lambda}_{r-s} \mathrm{E}\left[\boldsymbol{\tau}_{r-s}\left(\boldsymbol{\beta}_i\right)\right] \sigma_s. \tag{4.8}
$$

Note that

$$
\mathbf{M}_r = \begin{pmatrix} \boldsymbol{\Xi}_{r,r} & \boldsymbol{\rho}_{0,r} \\ \boldsymbol{\rho}_{0,r}' & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\Lambda}_r & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix},
$$

is invertible since $\det\left(\mathbf{M}_r\right) = \det\left(\boldsymbol{\Xi}_{r,r} - \boldsymbol{\rho}_{0,r} \boldsymbol{\rho}_{0,r}'\right) \det\left(\boldsymbol{\Lambda}_r\right) > 0$, for $r = 2, 3, \cdots, R$, by Assumption 6(d). As a result, we can sequentially solve (4.7) and (4.8) for $\mathrm{E}\left[\boldsymbol{\tau}_r\left(\boldsymbol{\beta}_i\right)\right]$ and σ_r, for $r = 2, 3, \cdots, 2K-1$. ■

We now move from the moments of $\boldsymbol{\beta}_i$ to the distribution of $\boldsymbol{\beta}_i$. We first focus on the identification of the marginal probabilities obtained from (4.2) by averaging out the effects of the other coefficients except for β_{ij}, namely we initially focus on identification of $\lambda_{jk} = \Pr\left(\beta_{ij} = b_{jk}\right)$, for $k = 1, 2, \cdots, K$, and $j = 1, 2, \cdots, p$.

Remark 11 *Focusing on the marginal distribution of β_i is similar to focusing on estimation of partial derivatives in the context of non-parametric estimation, where the curse of dimensionality applies. Consider the estimation of regressing y_i on $\mathbf{x}_i = (x_{i1}, x_{i2}, \cdots, x_{ip})'$,*

$$y_i = F\left(x_{i1}, x_{i2}, \cdots .x_{ip}\right) + u_i.$$

Then if $F\left(x_1, x_{i2}, \cdots, x_{ip}\right)$ is a homogeneous function (of degree $1/\mu$), then

$$y_i = \sum_{j=1}^{p} \left(\mu \frac{\partial F\left(\cdot\right)}{\partial x_{ij}}\right) x_{ij} + u_i,$$

and under certain conditions we can treat $\mu \frac{\partial F(\cdot)}{\partial x_{ij}} \equiv \beta_{ij}$.

By Theorem 6, $\mathrm{E}\left(\beta_{ij}^r\right)$ is identified for $r = 1, 2, \cdots, 2K-1$ under Assumptions 1 and 6. By (4.2), we have equations

$$\mathrm{E}\left(\beta_{ij}^r\right) = \sum_{k=1}^{K} \lambda_{jk} b_{jk}^r, \tag{4.9}$$

$r = 0, 1, \cdots, 2K-1$, which is of the same form as (2.10) and (3.4). To identify $\boldsymbol{\lambda}_j = (\lambda_{j1}, \lambda_{j2}, \cdots, \lambda_{jK})'$ and $\mathbf{b}_j = (b_{j1}, b_{j2}, \cdots, b_{jK})'$, we can verify the system of $2K$ equations in (4.9) has a unique solution if $b_{j1} < b_{j2} < \cdots < b_{jK}$ and $\lambda_{jk} \in (0, 1)$. The following corollary is a direct application of Theorem 2.

Corollary 7 *Consider the model* (4.1) *and suppose that Assumptions 1 and 6 hold. Then the parameters $\boldsymbol{\theta}_j = \left(\boldsymbol{\lambda}_j', \mathbf{b}_j'\right)'$ of the marginal distribution of β_i with respect to β_{ij} is identified subject to $b_{j1} < b_{j2} < \cdots < b_{jK}$ and $\lambda_{jk} \in (0, 1)$ for $j = 1, 2, \cdots, p$.*

The problem of identification and estimation of the joint distribution of $\boldsymbol{\beta}_i$ is subject to the curse of dimensionality. We have $K^p - 1$ probability weights, $\pi_{k_1, k_2, \cdots, k_p}$, to be identified in addition to the pK categorical coefficients b_{ij} that are identified by Corollary 7. The number of parameters increases rapidly with p. Even in the simplest case with $K = 2$, the total number of unknown parameters is $2p + 2^p - 1$, which grows exponentially.

Note that the marginal probabilities λ_{jk} are related to the joint distribution by

$$\lambda_{jk} = \sum_{k_1, \cdots, k_{j-1}, k_{j+1}, \cdots, k_p \in \{1, 2, \cdots, K\}} \pi_{k_1, k_2, \cdots, k_{j-1}, k, k_{j+1}, \cdots, k_p}, \tag{4.10}$$

$k = 1, 2, \cdots, K$ and $j = 1, 2, \cdots, p$. The number of linearly independent equations in (4.10) is $pK - (p-1)$.

Example 3 *Consider the same setup as in Example 1 with $p = 2$ and $K = 2$. The marginal probabilities are obtained by*

$$\begin{aligned}
&\lambda_{1L} = \Pr(\beta_{i1} = b_{1L}) = \pi_{LL} + \pi_{LH}, \quad \lambda_{1H} = \Pr(\beta_{i1} = b_{1H}) = 1 - \lambda_{1L} = \pi_{HL} + \pi_{HH}, \\
&\lambda_{2L} = \Pr(\beta_{i2} = b_{2L}) = \pi_{LL} + \pi_{HL}, \quad \lambda_{2H} = \Pr(\beta_{i2} = b_{2H}) = 1 - \lambda_{2L} = \pi_{LH} + \pi_{HH}. \qquad (4.11)
\end{aligned}$$

Note that any equation in (4.11) *can be expressed as a linear combination of other three equations, for example $\lambda_{2H} = \lambda_{1L} + \lambda_{1H} - \lambda_{2L}$.*

The equations corresponding to the cross-moments, $\mathrm{E}\left(\prod_{j=1}^{p} \beta_{ij}^{q_j}\right)$, are

$$\mathrm{E}\left(\prod_{j=1}^{p} \beta_{ij}^{q_j}\right) = \sum_{k_1,k_2,\cdots,k_p \in \{1,2,\cdots,K\}} \left(\prod_{j=1}^{p} b_{jk_j}^{q_j}\right) \pi_{k_1,k_2,\cdots,k_p}, \qquad (4.12)$$

for $\mathbf{q} \in \left\{\mathbf{q} \in \{0, 1, \cdots r-1\}^p : \sum_{j=1}^{p} q_j = r\right\}$, $r = 2, \cdots, 2K-1$. The linear system (4.12) has

$$\sum_{r=1}^{2K-1} \binom{r+p-1}{p-1} - p(2K-1)$$

equations. Then the total number of equations in (4.10) and (4.12) that can be utilized to identify joint probabilities is $C_r = \sum_{r=1}^{2K-1} \binom{r+p-1}{p-1} - pK$, which is smaller than the number of joint probabilities $K^p - 1$ for large p. When $K = 2$, $C_r < K^p - 1$ for $p \geq 7$.

Identification and estimation of the joint distribution of $\boldsymbol{\beta}_i$ in the general setting will not be pursued in this paper due to the curse of dimensionality. Instead, we consider special cases, that are empirically relevant, in which identification of the joint distribution of $\boldsymbol{\beta}_i$ can be readily established. We first consider small p and K, in particular $p = 2$ and $K = 2$ as in Example 1.

Example 4 Consider the same setup as in Example 1 with $p = 2$ and $K = 2$. In addition to (4.11), consider the cross-moment,

$$\mathrm{E}(\beta_{i1}\beta_{i2}) = b_{1L}b_{2L}\pi_{LL} + b_{1L}b_{2H}\pi_{LH} + b_{1H}b_{2L}\pi_{HL} + b_{1H}b_{2H}\pi_{HH}. \qquad (4.13)$$

Writing (4.11) and (4.13) in matrix form, we have

$$\mathbf{B}\boldsymbol{\pi} = \boldsymbol{\lambda},$$

where

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ b_{1L}b_{2L} & b_{1L}b_{2H} & b_{1H}b_{2L} & b_{1H}b_{2H} \end{pmatrix}, \ \boldsymbol{\pi} = \begin{pmatrix} \pi_{LL} \\ \pi_{LH} \\ \pi_{HL} \\ \pi_{HH} \end{pmatrix}, \ \boldsymbol{\lambda} = \begin{pmatrix} \lambda_{1L} \\ \lambda_{1H} \\ \lambda_{2L} \\ \mathrm{E}(\beta_{i1}\beta_{i2}) \end{pmatrix}.$$

Note that $\mathrm{E}(\beta_{i1}\beta_{i2})$ is identified by Theorem 6, and b_{jk_j} and λ_{jk_j} are identified by Corollary 7, and matrix $\mathbf{B}$ is invertible given that $b_{1L} < b_{1H}$ and $b_{2L} < b_{2H}$. (See Appendix A.1). As a result, the joint probabilities, $\boldsymbol{\pi}$, are identified.

Remark 12 *The argument in Example 4 is applicable for identification of the joint distribution of* $(\beta_{ij}, \beta_{i,j'})'$ *for* $j \neq j'$ *when* $p > 2$ *and* $K = 2$.

5 Finite sample properties using Monte Carlo experiments

We examine the finite sample performance of the categorical coefficient estimator proposed in Section 3 by Monte Carlo experiments.

5.1 Data generating processes

We generate y_i as

$$y_i = \alpha + x_i\beta_i + z_{i1}\gamma_1 + z_{i2}\gamma_2 + u_i, \text{ for } i = 1, 2, ..., n, \tag{5.1}$$

with β_i distributed as in (2.2) with $K = 2$, and the parameters π, β_L and β_H.

We draw β_i for each individual i independently by setting $\beta_i = \beta_L$ with probability π and $\beta_i = \beta_H$ with probability $1 - \pi$, through a sequence of independent Bernoulli draws. We consider two sets of parameters in all DGPs, denoted as *high variance* and *low variance* parametrization, respectively,

$$(\pi, \beta_L, \beta_H, \mathrm{E}(\beta_i), \mathrm{var}(\beta_i)) = \begin{cases} (0.5, 1, 2, 1.5, 0.25) & (\textit{high variance}) \\ (0.3, 0.5, 1.345, 1.0915, 0.15) & (\textit{low variance}) \end{cases}. \tag{5.2}$$

$\beta_H/\beta_L = 2$ for the *high variance* parametrization, and $\beta_H/\beta_L = 2.69$, for the *low variance* parametrization, which is motivated by the estimates in our empirical illustration in Section 6.[5] The values of $\mathrm{E}(\beta_i)$ and $\mathrm{var}(\beta_i)$ are obtained noting that $\mathrm{E}(\beta_i) = \pi\beta_L + (1-\pi)\beta_H$, and $\mathrm{var}(\beta_i) = \pi(1-\pi)(\beta_H - \beta_L)^2$. The remaining parameters are set as $\alpha = 0.25$, and $\boldsymbol{\gamma} = (1, 1)'$, across DGPs.

We generate the regressors and the error terms as follows.

DGP 1 (Baseline) We first generate $\tilde{x}_i \sim \mathrm{IID}\chi^2(2)$, and then set $x_i = (\tilde{x}_i - 2)/2$ so that x_i has 0 mean and unit variance. The additional regressors, z_{ij}, for $j = 1, 2$ with homogeneous slopes are generated as

$$z_{i1} = x_i + v_{i1} \text{ and } z_{i2} = z_{i1} + v_{i2},$$

with $v_{ij} \sim \mathrm{IID}\ N(0, 1)$, for $j = 1, 2$. This ensures that the regressors are sufficiently correlated. The error term, u_i, is generated as $u_i = \sigma_i\varepsilon_i$, where σ_i^2 are generated as $0.5(1 + \mathrm{IID}\chi^2(1))$, and $\varepsilon_i \sim \mathrm{IID}N(0, 1)$. Note that ε_i and σ_i^2 are generated independently, and $E(u_i^2) = 1$.

[5] The estimates for β_H/β_L in our empirical analysis range from 1.66 to 2.73.

DGP 2 (Categorical x) This setup deviates from the baseline DGP, and allows the distribution of x_i to differ across i. Accordingly, we generate $x_i = (\tilde{x}_{1i} - 2)/2$ where $\tilde{x}_{1i} \sim \text{IID}\chi^2(2)$ for $i = 1, 2, \cdots, \lfloor n/2 \rfloor$, and $x_i = (\tilde{x}_{2i} - 2)/4$ where $\tilde{x}_{2i} \sim \text{IID}\chi^2(4)$, for $i = \lfloor n/2 \rfloor + 1, \cdots, n$. The additional regressors, z_{ij}, for $j = 1, 2$ with homogeneous slopes are generated as

$$z_{i1} = x_i + v_{i1} \text{ and } z_{i2} = z_{i1} + v_{i2},$$

with $v_{ij} \sim \text{IID } N(0,1)$, for $j = 1, 2$. The error term u_i is generated the same as in DGP 1.

DGP 3 (Categorical u) We generate x_i and $\mathbf{z}_i$ the same as in DGP 1, but allow the error term u_i to have a heterogeneous distribution over i. For $i = 1, 2, \cdots, \lfloor n/2 \rfloor$, we set $u_i = \sigma_i \varepsilon_i$, where $\sigma_i^2 \sim \text{IID}\chi^2(2)$ and $\varepsilon_i \sim \text{IID}N(0,1)$, and for $i = \lfloor n/2 \rfloor + 1, \cdots, n$, we set $u_i = (\tilde{u}_i - 2)/2$, where $\tilde{u}_i \sim \text{IID}\chi^2(2)$.

We investigate the finite sample performance of the estimator proposed in Section 3 across DGP 1 to 3 with *low variance* and *high variance* scenarios.[6]

5.2 Summary of the MC results

For each sample size $n = 500$, $1,000$, $2,000$, $5,000$, $10,000$ and $100,000$ we run $5,000$ replications of experiments for DGP 1 (baseline), DGP 2 (categorical x) and DGP 3 (categorical u) with *high variance* and *low variance* parametrization, as set out in (5.2).

We first investigate the finite sample performance of $\hat{\boldsymbol{\phi}}$, as an estimator of $\boldsymbol{\phi} = (\text{E}(\beta_i), \boldsymbol{\gamma}')'$. Bias, root mean squared errors (RMSE) for estimation of $\text{E}(\beta_i)$, γ_1 and γ_2, as well as size of testing of the null values at the 5 per cent nominal value are reported in Table 2. In addition, we plot the associated empirical power functions in Figure 1 and 2, for cases of high and low $var(\beta_i)$. The results show that $\hat{\boldsymbol{\phi}}$ has very good small sample properties with small bias and RMSEs, with size very close to the nominal value of 5 per cent across all DGPs and parametrization, even when sample size is relatively small. The power of the test increases steadily as the sample size increases.

Then, we turn to the GMM estimator for the distributional parameters of β_i proposed in Section 3.2. The bias, RMSE, and the test size based on the asymptotic distribution given in Theorem 5, for π, β_L and β_H, are reported in Table 3. The empirical power functions are reported in Figure 3 and 4. The reported results are based on $S = 4$, where S $(> 2K - 1 = 3)$ denotes the highest order of moments of x_i included in estimation.[7]

The upper panel of this table reports the results of the high variance and the lower panel for the low variance parametrization, as set out in (5.2). For all parameters and under all DGPs, the

[6] We can consider a DGP with conditional heteroskedasticity, in which we follow the baseline DGP and generate the error term as $u_i = x_i \varepsilon_i$, where $\varepsilon_i \sim N(0,1)$. The least square estimator for $\boldsymbol{\phi}$ is valid in this setup in terms of estimation and inference, whereas the GMM estimator for the distributional parameters $\boldsymbol{\theta}$ breaks down, which is to be expected since we can only identify the first moment of β_i under conditional heteroskedasticity. The results are available on request.

[7] We also tried estimation based on a larger number of moments (using $S = 5$ and $S = 6$). In the case of current Monte Carlo results, adding more moments does not seem to add much to the precision of the estimates and could be counter-productive when n is not sufficiently large. The results are available on request.

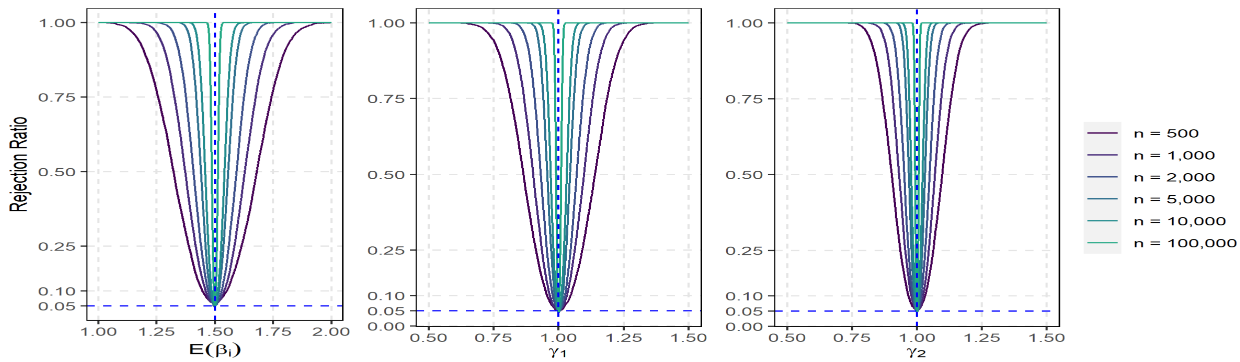
Table 2: Bias, RMSE and size of the least square estimator $\hat{\boldsymbol{\phi}}$

	DGP	Baseline			Categorical x			Categorical u		
	Sample size n	Bias	RMSE	Size	Bias	RMSE	Size	Bias	RMSE	Size
				high variance: $\text{var}(\beta_i) = 0.25$						
$\text{E}(\beta_i) = 1.5$	500	0.0001	0.0932	0.0628	-0.0010	0.0936	0.0626	-0.0007	0.1052	0.0624
	1,000	0.0009	0.0668	0.0600	0.0011	0.0670	0.0592	-0.0002	0.0736	0.0558
	2,000	0.0004	0.0478	0.0566	0.0001	0.0470	0.0516	0.0000	0.0519	0.0510
	5,000	-0.0012	0.0298	0.0510	-0.0003	0.0301	0.0594	-0.0019	0.0332	0.0496
	10,000	0.0000	0.0213	0.0516	0.0000	0.0209	0.0492	-0.0002	0.0236	0.0486
	100,000	0.0000	0.0068	0.0534	0.0001	0.0069	0.0556	-0.0001	0.0074	0.0518
$\gamma_1 = 1$	500	0.0007	0.0708	0.0512	-0.0003	0.0695	0.0488	-0.0015	0.0848	0.0582
	1,000	-0.0009	0.0501	0.0514	-0.0015	0.0487	0.0468	0.0010	0.0588	0.0508
	2,000	0.0005	0.0354	0.0556	-0.0003	0.0354	0.0526	0.0004	0.0420	0.0494
	5,000	0.0002	0.0222	0.0472	-0.0005	0.0222	0.0508	0.0005	0.0268	0.0490
	10,000	0.0004	0.0158	0.0496	0.0000	0.0158	0.0536	0.0001	0.0189	0.0518
	100,000	-0.0001	0.0050	0.0456	0.0000	0.0050	0.0522	0.0000	0.0060	0.0498
$\gamma_2 = 1$	500	-0.0003	0.0502	0.0554	0.0004	0.0492	0.0494	0.0008	0.0598	0.0554
	1,000	0.0006	0.0357	0.0554	0.0011	0.0347	0.0540	-0.0003	0.0417	0.0546
	2,000	-0.0002	0.0250	0.0556	0.0000	0.0249	0.0516	-0.0001	0.0299	0.0544
	5,000	0.0000	0.0155	0.0480	0.0004	0.0159	0.0546	0.0003	0.0187	0.0506
	10,000	0.0000	0.0111	0.0504	0.0000	0.0111	0.0518	0.0003	0.0134	0.0506
	100,000	0.0001	0.0035	0.0462	0.0000	0.0035	0.0498	0.0000	0.0042	0.0538
				low variance: $\text{var}(\beta_i) = 0.15$						
$\text{E}(\beta_i) = 1.0915$	500	0.0011	0.0829	0.0592	-0.0006	0.0840	0.0580	0.0003	0.0948	0.0604
	1,000	0.0015	0.0590	0.0560	0.0015	0.0603	0.0552	0.0003	0.0670	0.0524
	2,000	-0.0002	0.0419	0.0546	-0.0003	0.0422	0.0486	-0.0006	0.0472	0.0498
	5,000	-0.0006	0.0262	0.0518	0.0002	0.0268	0.0530	-0.0013	0.0300	0.0498
	10,000	-0.0002	0.0187	0.0514	0.0000	0.0188	0.0490	-0.0004	0.0213	0.0514
	100,000	0.0001	0.0059	0.0516	0.0001	0.0061	0.0520	-0.0001	0.0065	0.0442
$\gamma_1 = 1$	500	0.0007	0.0679	0.0538	-0.0004	0.0669	0.0506	-0.0015	0.0819	0.0538
	1,000	-0.0006	0.0484	0.0518	-0.0016	0.0472	0.0498	0.0013	0.0572	0.0504
	2,000	0.0004	0.0339	0.0494	-0.0002	0.0339	0.0558	0.0003	0.0407	0.0518
	5,000	0.0002	0.0214	0.0470	-0.0004	0.0214	0.0504	0.0004	0.0259	0.0498
	10,000	0.0003	0.0152	0.0490	0.0001	0.0151	0.0570	0.0001	0.0184	0.0548
	100,000	-0.0001	0.0048	0.0470	-0.0001	0.0048	0.0542	0.0000	0.0058	0.0528
$\gamma_2 = 1$	500	-0.0001	0.0481	0.0542	0.0005	0.0479	0.0512	0.0010	0.0579	0.0546
	1,000	0.0003	0.0344	0.0546	0.0010	0.0335	0.0536	-0.0006	0.0405	0.0506
	2,000	-0.0003	0.0240	0.0506	0.0001	0.0239	0.0524	-0.0002	0.0290	0.0524
	5,000	-0.0001	0.0150	0.0490	0.0003	0.0154	0.0552	0.0002	0.0182	0.0506
	10,000	0.0000	0.0108	0.0532	-0.0001	0.0107	0.0506	0.0003	0.0131	0.0544
	100,000	0.0001	0.0034	0.0508	0.0000	0.0034	0.0522	0.0000	0.0041	0.0514

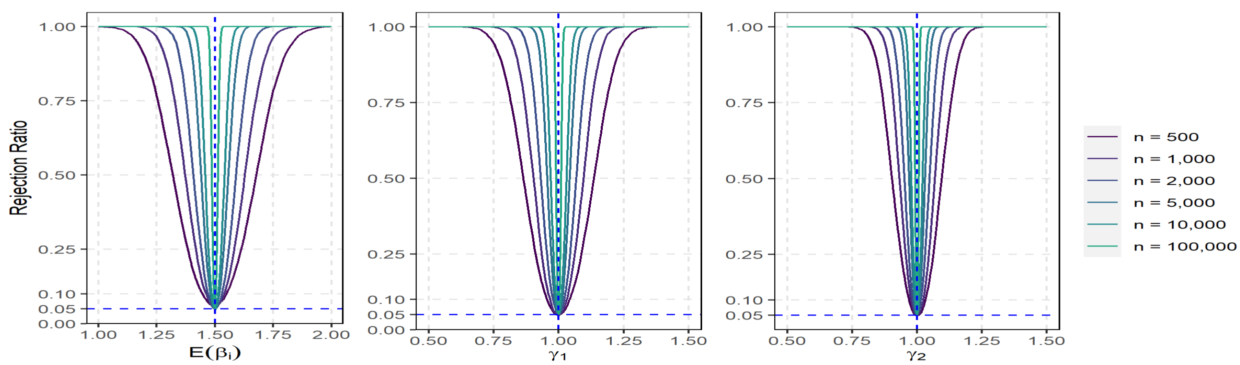
Notes: The data generating process is (5.1). *high variance* and *low variance* parametrization are described in (5.2). "Baseline", "Categorical x" and "Categorical u" refer to DGP 1 to 3 as in Section 5.1. Generically, bias, RMSE and size are calculated by $R^{-1}\sum_{r=1}^{R}\left(\hat{\theta}^{(r)} - \theta_0\right)$, $\sqrt{R^{-1}\sum_{r=1}^{R}\left(\hat{\theta}^{(r)} - \theta_0\right)^2}$, and $R^{-1}\sum_{r=1}^{R}\mathbf{1}\left[\left|\hat{\theta}^{(r)} - \theta_0\right|/\hat{\sigma}_{\hat{\theta}}^{(r)} > \text{cv}_{0.05}\right]$, respectively, for true parameter θ_0, its estimate $\hat{\theta}^{(r)}$, the estimated standard error of $\hat{\theta}^{(r)}$, $\hat{\sigma}_{\hat{\theta}}^{(r)}$, and the critical value $\text{cv}_{0.05} = \Phi^{-1}(0.975)$ across $R = 5,000$ replications, where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution.

Figure 1: Empirical power functions for the least square estimator $\hat{\boldsymbol{\phi}}$ with the *high variance* parametrization ($\mathrm{var}(\beta_i) = 0.25$)

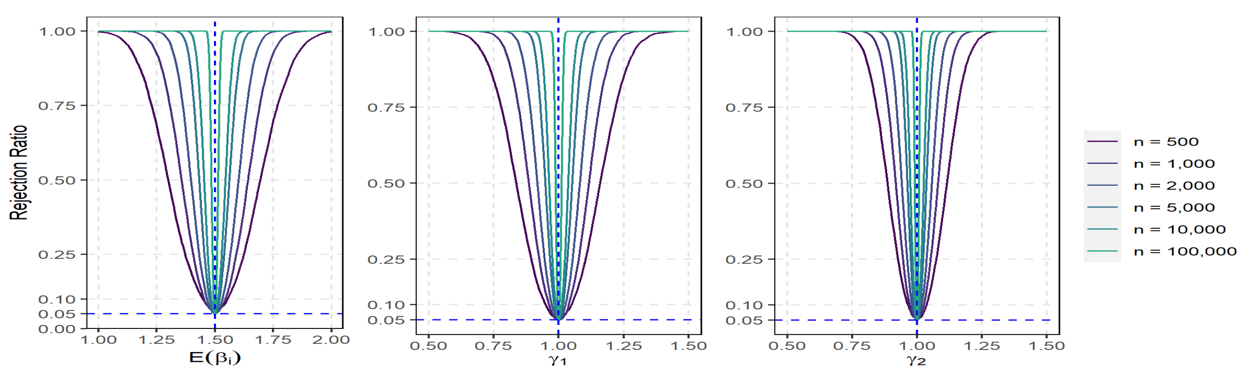
(a) Baseline



(b) Categorical x



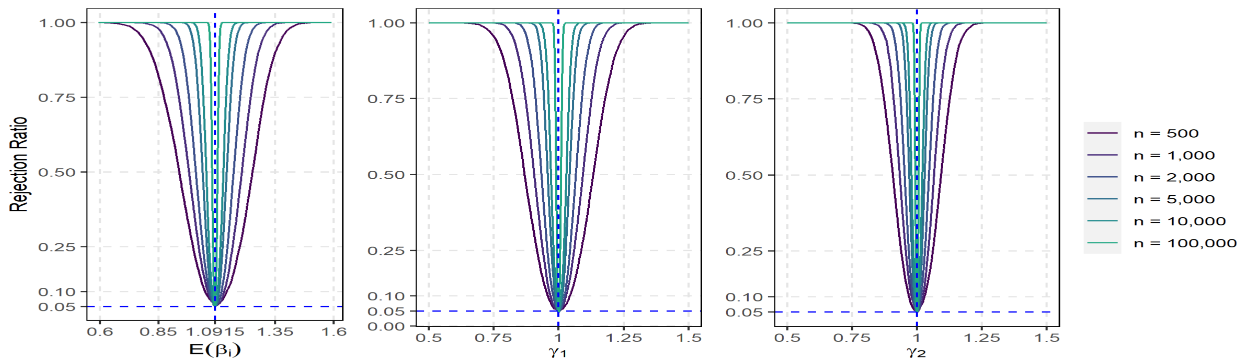
(c) Categorical u



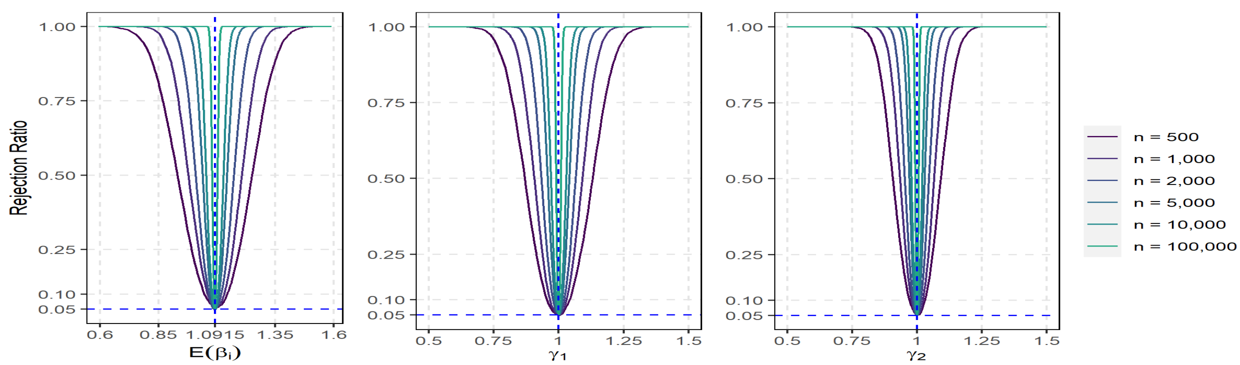
Notes: The data generating process is (5.1) with *high variance* parametrization that is described in (5.2). "Baseline", "Categorical x" and "Categorical u" refer to DGP 1 to 3 as in Section 5.1. Generically, power is calculated by $R^{-1}\sum_{r=1}^{R} \mathbf{1}\left[\left|\hat{\theta}^{(r)} - \theta_\delta\right| / \hat{\sigma}_{\hat{\theta}}^{(r)} > \mathrm{cv}_{0.05}\right]$, for θ_δ in a symmetric neighborhood of the true parameter θ_0, the estimate $\hat{\theta}^{(r)}$, the estimated standard error of $\hat{\theta}^{(r)}$, $\hat{\sigma}_{\hat{\theta}}^{(r)}$, and the critical value $\mathrm{cv}_{0.05} = \Phi^{-1}(0.975)$ across $R = 5,000$ replications, where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution.

Figure 2: Empirical power functions for the least square estimator $\hat{\boldsymbol{\phi}}$ with the *low variance* parametrization ($\text{var}(\beta_i) = 0.15$)

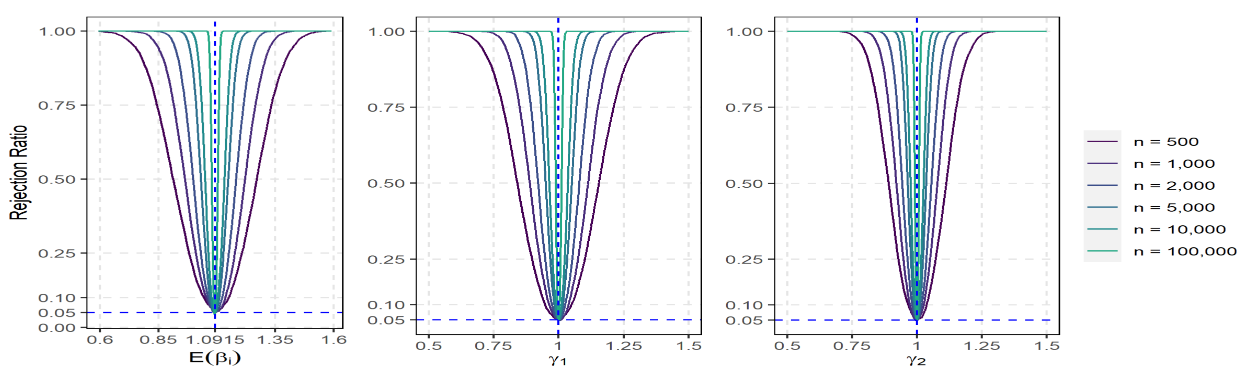
(a) Baseline



(b) Categorical x



(c) Categorical u



Notes: The data generating process is (5.1) with *low variance* parametrization that is described in (5.2). "Baseline", "Categorical x" and "Categorical u" refer to DGP 1 to 3 as in Section 5.1. Generically, power is calculated by $R^{-1}\sum_{r=1}^{R}\mathbf{1}\left[\left|\hat{\theta}^{(r)} - \theta_\delta\right|/\hat{\sigma}_{\hat{\theta}}^{(r)} > \text{cv}_{0.05}\right]$, for θ_δ in a symmetric neighborhood of the true parameter θ_0, the estimate $\hat{\theta}^{(r)}$, the estimated standard error of $\hat{\theta}^{(r)}$, $\hat{\sigma}_{\hat{\theta}}^{(r)}$, and the critical value $\text{cv}_{0.05} = \Phi^{-1}(0.975)$ across $R = 5,000$ replications, where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution.

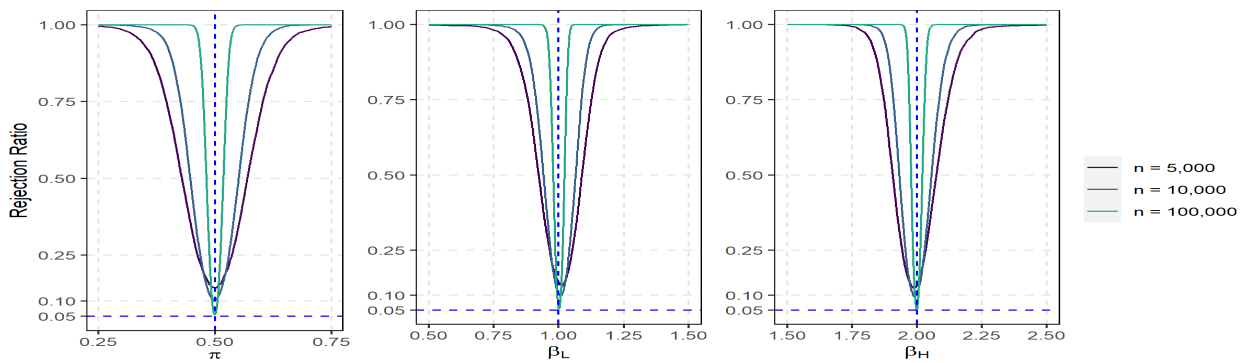
Table 3: Bias, RMSE and size of the GMM estimator for distributional parameters of β

DGP		Baseline			Categorical x			Categorical u		
	Sample size n	Bias	RMSE	Size	Bias	RMSE	Size	Bias	RMSE	Size
high variance: $\text{var}(\beta_i) = 0.25$										
$\pi = 0.5$	500	-0.0180	0.2317	0.3741	-0.0234	0.2384	0.3678	-0.0503	0.2778	0.4193
	1,000	-0.0111	0.1688	0.2948	-0.0185	0.1769	0.2981	-0.0383	0.2160	0.3233
	2,000	-0.0033	0.1170	0.2536	-0.0069	0.1233	0.2376	-0.0215	0.1544	0.2353
	5,000	-0.0009	0.0570	0.1434	-0.0029	0.0677	0.1586	-0.0103	0.0924	0.1506
	10,000	-0.0013	0.0334	0.1004	-0.0010	0.0414	0.1112	-0.0068	0.0625	0.1050
	100,000	0.0000	0.0096	0.0560	0.0001	0.0114	0.0610	-0.0007	0.0191	0.0506
$\beta_L = 1$	500	-0.0801	0.6132	0.3487	-0.0882	0.6297	0.3217	-0.2159	0.7586	0.2806
	1,000	-0.0320	0.3980	0.2976	-0.0362	0.4285	0.2767	-0.0967	0.4372	0.2355
	2,000	-0.0123	0.2213	0.2504	-0.0151	0.2274	0.2370	-0.0399	0.2490	0.1993
	5,000	0.0008	0.0834	0.1426	-0.0020	0.0988	0.1504	-0.0113	0.1152	0.1242
	10,000	0.0011	0.0419	0.1058	0.0008	0.0535	0.1060	-0.0048	0.0708	0.0908
	100,000	0.0007	0.0114	0.0568	0.0006	0.0135	0.0666	-0.0003	0.0202	0.0526
$\beta_H = 2$	500	-0.0150	0.4973	0.2212	-0.0216	0.5816	0.2186	-0.0766	0.7697	0.2509
	1,000	-0.0128	0.2149	0.2049	-0.0198	0.3216	0.2083	-0.0298	0.4376	0.2214
	2,000	-0.0076	0.1352	0.1941	-0.0123	0.1574	0.1828	-0.0255	0.2427	0.1801
	5,000	-0.0059	0.0661	0.1322	-0.0060	0.0735	0.1434	-0.0137	0.1104	0.1380
	10,000	-0.0044	0.0382	0.1036	-0.0032	0.0463	0.1050	-0.0094	0.0747	0.1066
	100,000	-0.0005	0.0114	0.0538	-0.0003	0.0135	0.0620	-0.0013	0.0234	0.0594
low variance: $\text{var}(\beta_i) = 0.15$										
$\pi = 0.3$	500	0.0192	0.2553	0.3758	0.0368	0.2763	0.3865	0.0686	0.3249	0.4867
	1,000	-0.0051	0.1863	0.2835	0.0023	0.2070	0.3094	0.0225	0.2742	0.4136
	2,000	-0.0080	0.1229	0.2149	-0.0074	0.1437	0.2305	-0.0067	0.2043	0.3109
	5,000	-0.0064	0.0658	0.1388	-0.0079	0.0806	0.1527	-0.0095	0.1316	0.1881
	10,000	-0.0041	0.0439	0.1046	-0.0029	0.0537	0.1088	-0.0080	0.0938	0.1374
	100,000	-0.0004	0.0128	0.0538	-0.0003	0.0161	0.0662	-0.0021	0.0301	0.0662
$\beta_L = 0.5$	500	-0.1045	0.6164	0.3074	-0.1252	0.7517	0.2888	-0.2593	1.1597	0.2791
	1,000	-0.0727	0.3800	0.2415	-0.0779	0.4447	0.2352	-0.2177	0.8076	0.2254
	2,000	-0.0270	0.2208	0.2125	-0.0412	0.2308	0.1934	-0.1243	0.4530	0.1730
	5,000	-0.0045	0.1366	0.1330	-0.0133	0.1237	0.1303	-0.0451	0.2113	0.1170
	10,000	-0.0026	0.0616	0.0974	-0.0037	0.0785	0.0994	-0.0235	0.1398	0.0902
	100,000	0.0004	0.0182	0.0610	0.0004	0.0228	0.0634	-0.0034	0.0405	0.0520
$\beta_H = 1.345$	500	0.0450	0.6491	0.1800	0.0758	0.7301	0.1847	0.0819	1.1904	0.2171
	1,000	0.0059	0.3481	0.1715	0.0135	0.3602	0.1781	0.0165	0.6553	0.2369
	2,000	-0.0087	0.1567	0.1555	-0.0042	0.1899	0.1589	-0.0073	0.2811	0.2105
	5,000	-0.0074	0.0537	0.1114	-0.0079	0.0628	0.1247	-0.0098	0.1104	0.1527
	10,000	-0.0050	0.0359	0.0980	-0.0039	0.0410	0.0908	-0.0076	0.0730	0.1180
	100,000	-0.0006	0.0105	0.0580	-0.0006	0.0126	0.0590	-0.0020	0.0228	0.0634

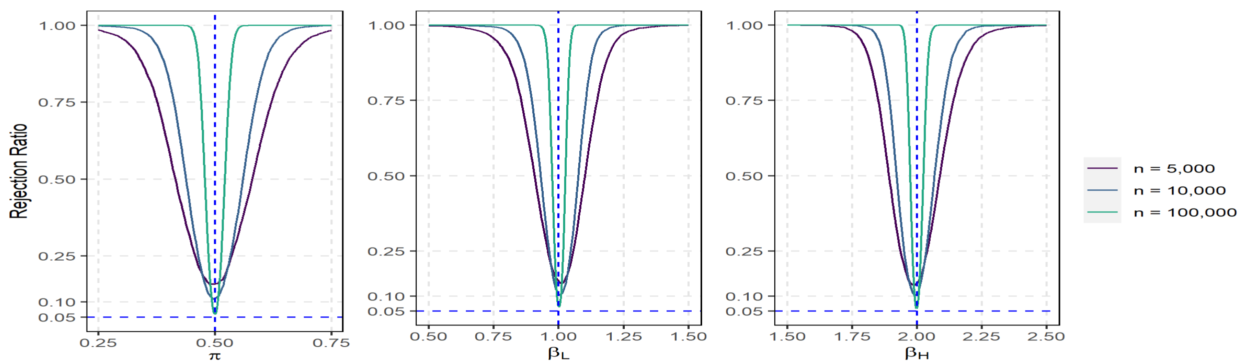
Notes: The data generating process is (5.1). *high variance* and *low variance* parametrization are described in (5.2). "Baseline", "Categorical x" and "Categorical u" refer to DGP 1 to 3 as in Section 5.1. Generically, bias, RMSE and size are calculated by $R^{-1}\sum_{r=1}^{R}\left(\hat{\theta}^{(r)} - \theta_0\right)$, $\sqrt{R^{-1}\sum_{r=1}^{R}\left(\hat{\theta}^{(r)} - \theta_0\right)^2}$, and $R^{-1}\sum_{r=1}^{R}\mathbf{1}\left[\left|\hat{\theta}^{(r)} - \theta_0\right| / \hat{\sigma}_{\hat{\theta}}^{(r)} > \text{cv}_{0.05}\right]$, respectively, for true parameter θ_0, its estimate $\hat{\theta}^{(r)}$, the estimated standard error of $\hat{\theta}^{(r)}$, $\hat{\sigma}_{\hat{\theta}}^{(r)}$, and the critical value $\text{cv}_{0.05} = \Phi^{-1}(0.975)$ across $R = 5,000$ replications, where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution.

Figure 3: Empirical power functions for the GMM estimator of distributional parameters of β with the *high variance* parametrization$(\text{var}(\beta_i) = 0.25)$

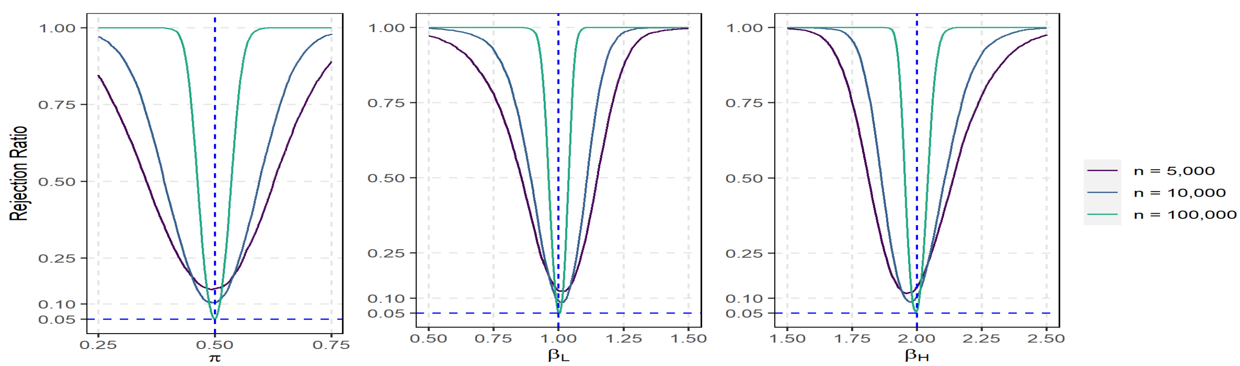
(a) Baseline



(b) Categorical x



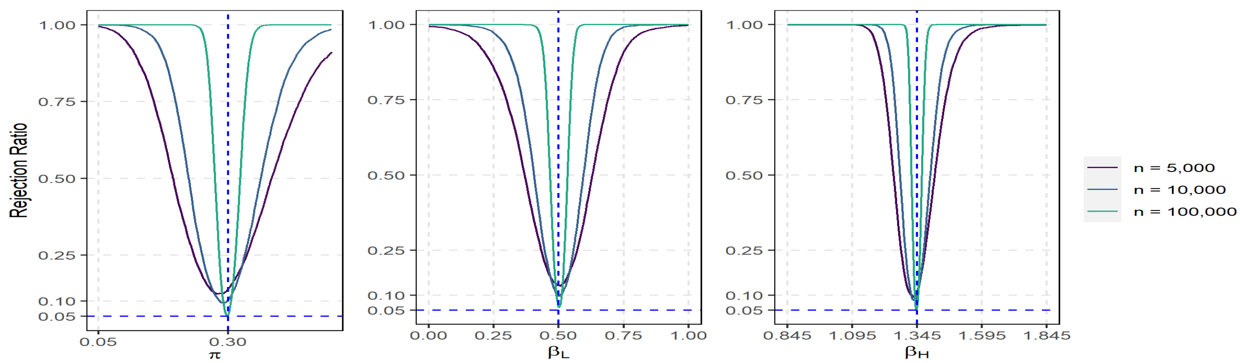
(c) Categorical u



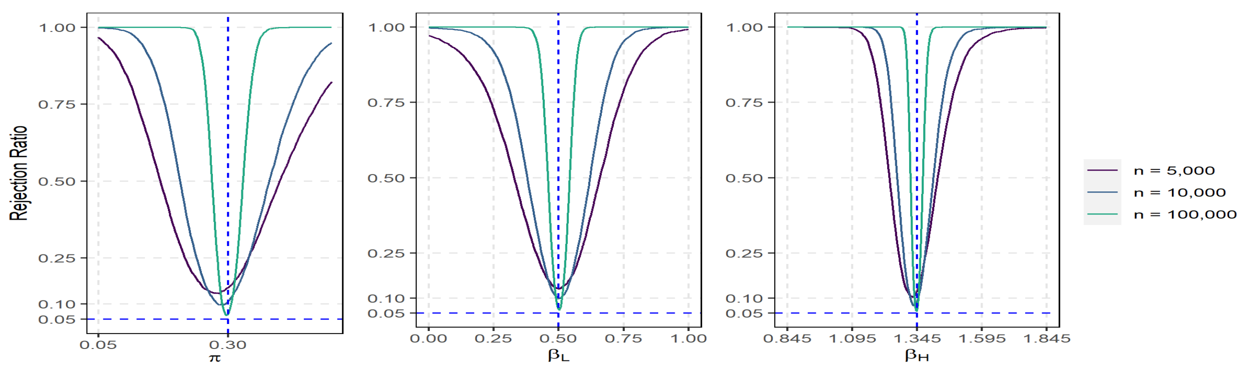
Notes: The data generating process is (5.1) with *high variance* parametrization that is described in (5.2). "Baseline", "Categorical x" and "Categorical u" refer to DGP 1 to 3 as in Section 5.1. The model is estimated with $S = 4$, the highest order of moments of x_i used in estimation. Generically, power is calculated by $R^{-1} \sum_{r=1}^{R} \mathbf{1}\left[\left|\hat{\theta}^{(r)} - \theta_\delta\right| / \hat{\sigma}_{\hat{\theta}}^{(r)} > \text{cv}_{0.05}\right]$, for θ_δ in a symmetric neighborhood of the true parameter θ_0, the estimate $\hat{\theta}^{(r)}$, the estimated standard error of $\hat{\theta}^{(r)}$, $\hat{\sigma}_{\hat{\theta}}^{(r)}$, and the critical value $\text{cv}_{0.05} = \Phi^{-1}(0.975)$ across $R = 5,000$ replications, where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution.

Figure 4: Empirical power functions for the GMM estimator of distributional parameters of β with the *low variance* parametrization $(\text{var}(\beta_i) = 0.15)$

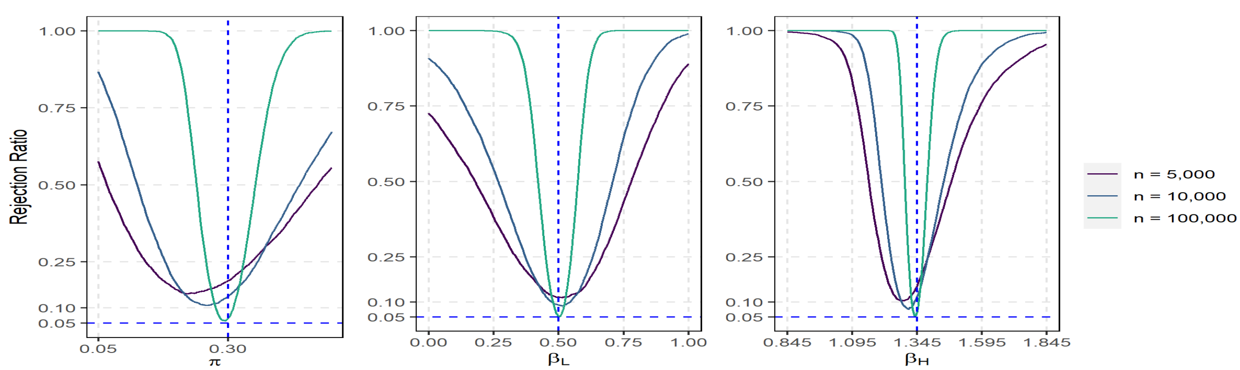
(a) Baseline



(b) Categorical x



(c) Categorical u



Notes: The data generating process is (5.1) with *low variance* parametrization that is described in (5.2). "Baseline", "Categorical x" and "Categorical u" refer to DGP 1 to 3 as in Section 5.1. The model is estimated with $S = 4$, the highest order of moments of x_i used in estimation. Generically, power is calculated by $R^{-1}\sum_{r=1}^{R}\mathbf{1}\left[\left|\hat{\theta}^{(r)} - \theta_\delta\right| / \hat{\sigma}_{\hat{\theta}}^{(r)} > \text{cv}_{0.05}\right]$, for θ_δ in a symmetric neighborhood of the true parameter θ_0, the estimate $\hat{\theta}^{(r)}$, the estimated standard error of $\hat{\theta}^{(r)}$, $\hat{\sigma}_{\hat{\theta}}^{(r)}$, and the critical value $\text{cv}_{0.05} = \Phi^{-1}(0.975)$ across $R = 5,000$ replications, where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution.

bias and RMSE decline steadily with the sample size as predicted by Theorem 4, and confirm the robustness of the GMM estimates to the heterogeneity in the regressor and the error processes. But for a given sample size, the relative precision of the estimates depends on the variability of β_i, as characterized by the true value of $var(\beta_i)$. The precision of the estimates with *high variance* parametrization is relatively higher than that with *low variance* parametrization. This is to be expected since, unlike $E(\beta_i)$, the distributional parameters are only identified if $var(\beta_i) > 0$. As shown in (2.18) and (2.19) for the current case of $K = 2$, $var(\beta_i)$ is in the denominator when we recover the distributional parameters from the moments of β_i. When $\text{var}(\beta_i)$ is small, estimation errors in the moments of β_i can be amplified in the estimation of π, β_L and β_H. On the other hand, the larger the variance the more precisely π, β_H and β_L can be estimated for a given n. The size and power also depends on the parametrization. With both *high variance* and *low variance* parametrization, we can achieve correct size and reasonable power when n is quite large ($n = 100,000$). We plot the empirical power functions for $n \geq 5,000$ for π, β_H and β_L since the size is far above 5 per cent for smaller values of n, and power comparisons are not meaningful in such cases.

6 Heterogeneous return to education: An empirical application

Since the pioneering work by Becker (1962, 1964) on the effects of investments in human capital, estimating returns to education has been one of the focal points of labor economics research. In his pioneering contribution Mincer (1974) models the logarithm of earnings as a function of years of education and years of potential labor market experience (age minus years of education minus six), which can be written in a generic form:

$$\log \text{wage}_i = \alpha_i + \beta_i \text{edu}_i + \phi(\mathbf{z}_i) + \varepsilon_i, \tag{6.1}$$

as in Heckman, Humphries, and Veramendi (2018, Equation (1)), where $\mathbf{z}_i$ includes the labor market experience and other relevant control variables. The above wage equation, also known as the "Mincer equation", has become of the workhorse of the empirical works on estimating the return to education. In the most widely used specification of the Mincer equation (6.1),

$$\phi(\mathbf{z}_i) = \rho_1 \text{exper}_i + \rho_2 \text{exper}_i^2 + \tilde{\mathbf{z}}_i' \tilde{\boldsymbol{\gamma}},$$

where $\tilde{\mathbf{z}}_i$ is the vector of control variables other than potential labor market experience.

Along with the advancement of empirical research on this topic, there has been a growing awareness of the importance of heterogeneity in individual cognitive and non-cognitive abilities (Heckman, 2001) and their significance for explaining the observed heterogeneity in return to education. Accordingly, it is important to allow the parameters of the wage equation to differ across individuals. In equation (6.1) we allow α_i and β_i to differ across individuals, but assume that $\phi(\mathbf{z}_i)$ can be approximated as non-linear functions of experience and other control variables with homogeneous coefficients.

Specifically, following Lemieux (2006b,c) we also allow for time variations in the parameters of the wage equation and consider the following categorical coefficient model over a given cross-section sample indexed by t:[8]

$$\log \text{wage}_{it} = \alpha_{it} + \beta_{it}\text{edu}_{it} + \rho_{1t}\text{exper}_{it} + \rho_{2t}\text{exper}^2_{it} + \tilde{\mathbf{z}}'_{it}\tilde{\boldsymbol{\gamma}}_t + \varepsilon_{it}, \tag{6.2}$$

where the return to education follows the categorical distribution,

$$\beta_{it} = \begin{cases} b_{tL} & \text{w.p. } \pi_t, \\ b_{tH} & \text{w.p. } 1 - \pi_t, \end{cases}$$

and $\tilde{\mathbf{z}}_{it}$ includes gender, martial status and race. $\alpha_{it} = \alpha_t + \delta_{it}$ where δ_{it} is mean 0 random variable assumed to be distributed independently of edu_{it} and $\mathbf{z}_{it} = \left(\text{exper}_{it}, \text{exper}^2_{it}, \tilde{\mathbf{z}}'_t\right)'$. Let $u_{it} = \varepsilon_{it} + \delta_{it}$, and write (6.2) as

$$\log \text{wage}_{it} = \alpha_t + \beta_{it}\text{edu}_{it} + \rho_{1t}\text{exper}_{it} + \rho_{2t}\text{exper}^2_{it} + \tilde{\mathbf{z}}'_{it}\tilde{\boldsymbol{\gamma}}_t + u_{it}. \tag{6.3}$$

The correlation between α_{it} and edu_{it} in (6.1) is the source of "ability bias" (Griliches, 1977). Given the pure cross-sectional nature of our analysis, we do not allow for the endogeneity from "ability bias" or dynamics. To allow for non-zero correlations between α_{it}, edu_{it} and $\mathbf{z}_{it}$, a panel data approach is required, which has its own challenges, as education and experience variables tend to very slow moving (if at all) for many individuals in the panel. Time delays between changes in education and experience, and the wage outcomes also further complicate the interpretation of the mean estimates of β_{it} which we shall be reporting. To partially address the possible dynamic spillover effects, we provide estimates of the distribution of β_{it} using cross-sectional data from two different sample periods, and investigate the extent to which the distribution of return to education has changed over time, by gender and the level of educational achievements.[9]

We estimate the categorical distribution of the return to education in (6.3) using the May and Outgoing Rotation Group (ORG) supplements of the Current Population Survey (CPS) data, as in Lemieux (2006b,c).[10] We pool observations from 1973 to 1975 for the first sample period, $t = \{1973 - 1975\}$ and observations from 2001 to 2003 for the second sample period, $t = \{2001 - 2003\}$. Following Lemieux (2006b), we consider sub-samples of those with less than 12 years of education, "high school or less", and those with more than 12 years of education, "postsecondary education", as well as the combined sample. We also present results by gender. The summary statistics are reported in Table 4. As to be expected, the mean log wages are higher for those with postsecondary

[8]Some investigators have suggested including higher powers of the experience variable in the wage equation. Lemieux (2006a), for example, proposes using a quartic rather than a quadratic function. As a robustness check we also provide estimation results with quartic experience specification in Appendix A.2.

[9]Time variations in return to education has also been investigated in the literature as a possible explanation of increasing wage inequality in the U.S. See, for example, the papers by Lemieux (2006b,c).

[10]The data is retrieved from `https://www.openicpsr.org/openicpsr/project/116216/version/V1/view`.

education (for male and female), with the number of years of schooling and experience rising by about one year across the two sub-period samples. There are also important differences across male and female, and the two educational groupings, which we hope to capture in our estimation.

We treat the cross-section observations in the two sample periods, $t = \{1973 - 1975\}$ and $\{2001 - 2003\}$, as *repeated* cross-sections, rather than a panel data since the data in these two periods do not cover the same individuals, and represent random samples from the population of wage earners in two periods. It should also be noted that sample sizes (n_t), although quite large, are much larger during $\{2001 - 2003\}$, which could be a factor when we come to compare estimates from the two sample periods. For example, for both male and female $n_{73-75} = 111,632$ as compared to $n_{01-03} = 511,819$, a difference which becomes more pronounced when we consider the number observations in postsecondary/female category - which rises from $12,882$ for the first period to $100,007$ in the second period.

We report mean and standard deviations of the return to education (β_{it}) (denoted by s.d.($\hat{\beta}_{it}$)), as well as estimates of π_t, $\beta_{L,t}$ and $\beta_{H,t}$ for $t = \{1973 - 1975\}$ and $\{2001 - 2003\}$. For a given π_t, the ratio $\beta_{H,t}/\beta_{L,t}$ provides a measure of within group heterogeneity and allows us to augment information on changes in mean with changes in the distribution of return of education. The estimates for the distribution of the return to education (β_{it}) are summarized in Table 5, with the estimation results for control variables (such as experience, experienced squared, and other individual specific characteristic) reported in Table A.1 of Appendix A.2.

As can be seen from Table 5, s.d. $(\beta_{it}) > 0$ for all sub-samples over the two sample periods, except for the high school or less group during the first period. As our theoretical analysis show, in such cases $\beta_{L,t} = \beta_{H,t}$, and π_t are not identified. These entries are shown by n/a. The estimates of s.d. (β_{it}) are all strictly positive for other samples, and allows us to estimate the ratio $\beta_{H,t}/\beta_{L,t}$, which measures the within group heterogeneity of return to education. The estimates of $\beta_{H,t}/\beta_{L,t}$, lie between 1.66 to 2.73, with the high estimate obtained for females with high school of less education during $\{2001 - 03\}$, and the low estimate is obtained for females with postsecondary education during $\{2001 - 03\}$.

As our theory suggests the mean estimates of return to education, $E(\beta_{it})$, are very precisely estimated and inferences involving them tend to be robust to conditional error heteroskedasticity. The results in Table 5 show that estimates of $E(\beta_{it})$ have increased over the two sample periods $t = \{1973 - 75\}$ to $t = \{2001 - 03\}$, regardless of gender or educational grouping. The postsecondary educational group show larger increases in the estimates of $E(\beta_{it})$ as compared to those with high school or less. Estimates of $\mathrm{E}(\beta_{it})$ increases by 35 per cent for the postsecondary group while the estimates of mean return to education rises only by around 5 per cent in the case of those with high school or less. This result holds for both gender. Comparing the mean returns across the two educational groups, we find that mean return to education of individuals with postsecondary education is 43 per cent higher than those with high school or less in the $\{1973 - 75\}$ period, but this gap increases to 84 per cent in the second period, $\{2001 - 03\}$. Similar patterns are observed in the sub-samples by gender. The estimates suggest rising between group heterogeneity, which is

Table 4: Summary Statistics of the May and Outgoing Rotation Group (ORG) supplements of the Current Population Survey (CPS) data across two periods, 1973 - 75 and 2001 - 03, by years of education and gender

	1973 - 75			2001 - 03		
	High School or Less	Postsecondary Education	All	High School or Less	Postsecondary Education	All
			Both male and female			
`log wage`	1.59	1.94	1.69	1.47	1.88	1.71
	(0.50)	*(0.53)*	*(0.53)*	*(0.47)*	*(0.57)*	*(0.57)*
`edu.`	10.64	15.21	12.02	11.29	14.96	13.41
	(2.11)	*(1.65)*	*(2.89)*	*(1.68)*	*(1.82)*	*(2.53)*
`expr.`	20.10	13.69	18.17	20.67	18.91	19.65
	(14.44)	*(11.41)*	*(13.91)*	*(12.95)*	*(11.17)*	*(11.98)*
`marriage`	0.67	0.70	0.68	0.52	0.60	0.57
	(0.47)	*(0.46)*	*(0.47)*	*(0.50)*	*(0.49)*	*(0.50)*
`nonwhite`	0.11	0.08	0.10	0.15	0.14	0.15
	(0.32)	*(0.27)*	*(0.30)*	*(0.36)*	*(0.35)*	*(0.35)*
n	77,899	33,733	111,632	216,136	295,683	511,819
			Male			
`log wage`	1.76	2.07	1.86	1.57	2.00	1.81
	(0.48)	*(0.53)*	*(0.52)*	*(0.48)*	*(0.58)*	*(0.58)*
`edu.`	10.44	15.29	12.00	11.19	15.02	13.31
	(2.26)	*(1.69)*	*(3.08)*	*(1.82)*	*(1.84)*	*(2.64)*
`expr.`	20.35	14.00	18.32	20.02	19.22	19.58
	(14.49)	*(11.06)*	*(13.81)*	*(12.75)*	*(11.08)*	*(11.86)*
`marriage`	0.73	0.76	0.74	0.53	0.64	0.59
	(0.44)	*(0.43)*	*(0.44)*	*(0.50)*	*(0.48)*	*(0.49)*
`nonwhite`	0.10	0.06	0.09	0.14	0.13	0.13
	(0.30)	*(0.24)*	*(0.29)*	*(0.34)*	*(0.33)*	*(0.34)*
n	44,299	20,851	65,150	116,129	144,138	260,267
			Female			
`log wage`	1.35	1.71	1.45	1.77	1.36	1.61
	(0.41)	*(0.47)*	*(0.46)*	*(0.54)*	*(0.43)*	*(0.54)*
`edu.`	10.89	15.08	12.05	14.90	11.42	13.52
	(1.87)	*(1.59)*	*(2.60)*	*(1.79)*	*(1.49)*	*(2.40)*
`expr.`	19.78	13.19	17.96	18.61	21.41	19.73
	(14.36)	*(11.94)*	*(14.04)*	*(11.24)*	*(13.13)*	*(12.11)*
`marriage`	0.60	0.60	0.60	0.56	0.51	0.54
	(0.49)	*(0.49)*	*(0.49)*	*(0.50)*	*(0.50)*	*(0.50)*
`nonwhite`	0.13	0.10	0.12	0.15	0.17	0.16
	(0.33)	*(0.30)*	*(0.33)*	*(0.36)*	*(0.38)*	*(0.37)*
n	33,600	12,882	46,482	151,545	100,007	251,552

Notes: "Postsecondary Education" stands for the sub-sample with years of education higher than 12 and "High School or Less" stands for sub-sample with years of education less than or equal to 12). `edu.` and `exper.` are in years. `marriage` and `nonwhite` are dummy variables. n is the sample size. We report mean and standard deviation (in parentheses) of each variable. The data is from the May and Outgoing Rotation Group (ORG) supplements of the Current Population Survey (CPS) data retrived from `https://www.openicpsr.org/openicpsr/project/116216/version/V1/view`.

Table 5: Estimates of the distribution of the return to education across two periods, 1973 - 75 and 2001 - 03, by years of education and gender

	High School or Less		Postsecondary Edu.		All	
	1973 - 75	2001 - 03	1973 - 75	2001 - 03	1973 - 75	2001 - 03
			Both Male and Female			
π	n/a	0.5585	0.2632	0.1927	0.2836	0.2842
β_L	0.0614	0.0402	0.0490	0.0690	0.0461	0.0590
β_H	0.0614	0.0957	0.1019	0.1311	0.0869	0.1217
β_H/β_L	1.0000	2.3800	2.0775	1.9003	1.8870	2.0629
$\text{E}(\beta_i)$	0.0614	0.0647	0.0880	0.1191	0.0754	0.1039
$\text{s.d.}(\beta_i)$	0.0000	0.0276	0.0233	0.0245	0.0184	0.0283
n	77,899	216,136	33,733	295,683	111,632	511,819
			Male			
π	n/a	0.4364	0.3011	0.1864	0.2689	0.2934
β_L	0.0637	0.0373	0.0408	0.0573	0.0435	0.0516
β_H	0.0637	0.0877	0.0912	0.1244	0.0806	0.1162
β_H/β_L	1.0000	2.3540	2.2337	2.1701	1.8532	2.2497
$\text{E}(\beta_i)$	0.0637	0.0657	0.0761	0.1119	0.0707	0.0972
$\text{s.d.}(\beta_i)$	0.0000	0.0250	0.0231	0.0261	0.0165	0.0294
n	44,299	116,129	20,851	144,138	65,150	260,267
			Female			
π	0.2919	0.5904	0.2360	0.2268	0.2034	0.2779
β_L	0.0367	0.0380	0.0662	0.0830	0.0473	0.0685
β_H	0.0668	0.1039	0.1201	0.1376	0.0929	0.1282
β_H/β_L	1.8211	2.7326	1.8159	1.6564	1.9633	1.8707
$\text{E}(\beta_i)$	0.0580	0.0650	0.1074	0.1252	0.0837	0.1116
$\text{s.d.}(\beta_i)$	0.0137	0.0324	0.0229	0.0228	0.0184	0.0267
n	33,600	100,007	12,882	151,545	46,482	251,552

Notes: This table reports the estimates of the distribution of β_i with the quadratic in experience specification (6.2), using $S = 4$ order moments of edu_i. "Postsecondary Edu." stands for the sub-sample with years of education higher than 12 and "High School or Less" stands for those with years of education less than or equal to 12. $\text{s.d.}(\beta_i)$ corresponds to the square root of estimated $\text{var}(\beta_i)$. n is the sample size. "n/a" is inserted when the estimates show homogeneity of β_i and π is not identified and cannot be estimated.

mainly due to the increasing returns to education for the postsecondary group.

Turning to within group heterogeneity, we focus on the estimates of $\beta_{H,t}/\beta_{L,t}$ and first note that over the two periods, within group heterogeneity has been rising mainly in the case of those with high school or less, for both male and female. For the combined male and female samples and the male sub-sample, there is little evidence of within group heterogeneity for the first period $\{1973-75\}$. However, for the second period $\{2001-03\}$ we find a sizeable degree of within group heterogeneity where $\beta_{H,t}/\beta_{L,t}$ is estimated to be around 2.4, with $\text{s.d.}(\beta_{it}) \approx 0.03$. For the female sub-sample with high school or less, estimates of $\beta_{H,t}/\beta_{L,t}$ increases from 1.82 for the first sample period to 2.73 for the second sample period, that corresponds to a commensurate rise in $\text{s.d.}(\beta_i)$

from 0.014 to 0.032. The pattern of within group heterogeneity is very different for those with postsecondary educational. For this group we in fact observe a slight decline in the estimates of $\beta_{H,t}/\beta_{L,t}$ by gender and over two sample periods.

Overall, our between and within estimates of mean return to education are in line with the evidence of rising wage inequality documented in the literature (Corak, 2013).

7 Conclusion

In this paper we consider random coefficient models for repeated cross-sections in which the random coefficients follow categorical distributions. Identification is established using moments of the random coefficients in terms of the moments of the underlying observations. We propose two-step generalized method of moments to estimate the parameters of the categorical distributions. The consistency and asymptotic normality of the GMM estimators are established without the IID assumption typically assumed in the literature. Small sample properties of the proposed estimator are investigated by means of Monte Carlo experiments and shown to be robust to heterogeneously generated regressors and errors. In the empirical application, we apply the model to study the evolution of returns to education over two sub-periods, also considered in the literature by Lemieux (2006b). Our estimates show that mean (ex post) returns to education have risen over the periods from 1973 - 75 to 2001 - 2003 mainly in the case of individuals with postsecondary education, and this result is robust by gender. We find evidence of within group heterogeneity in the case of high school or less educational group as compared to those with postsecondary education.

In our model specification, the number of categories, K, is treated as a tuning parameter and assumed to be known. An information criterion, as in Bonhomme and Manresa (2015) and Su, Shi, and Phillips (2016), to determine K could be considered. Further investigation of models with multiple regressors subject to parameter heterogeneity is also required. These and other related issues are topics for future research.

Appendix

A.1 Proofs

We include proofs and technical details in this section.

Proof of Theorem 1. Sum (2.6) over i and rearrange terms,

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(x_i^r\right)\right)\mathrm{E}\left(\beta_i^r\right)+\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(u_i^r\right)$$
$$=\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(\tilde{y}_i^r\right)-\sum_{q=2}^{r-1}\binom{r}{q}\left(\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(x_i^{r-q}\right)\mathrm{E}\left(u_i^q\right)\right)\mathrm{E}\left(\beta_i^{r-q}\right). \quad \text{(A.1.1)}$$

Note that

$$\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(x_i^{r-q}\right)\mathrm{E}\left(u_i^q\right)=\left(\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(x_i^{r-q}\right)\right)\sigma_q+\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(x_i^{r-q}\right)\left(\mathrm{E}\left(u_i^q\right)-\sigma_r\right),$$

and

$$\left|\frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(x_i^{r-q}\right)\left(\mathrm{E}\left(u_i^q\right)-\sigma_r\right)\right|\le\sup_i\left|\mathrm{E}\left(x_i^{r-q}\right)\right|\left|\frac{1}{n}\sum_{i=1}^{n}\left(\mathrm{E}\left(u_i^q\right)-\sigma_r\right)\right|=O(n^{-1/2}),$$

by Assumption 1(b) and 2(b), then by taking $n\to\infty$ on both sides of (A.1.1), we have (2.8). Similar steps for (2.7) give (2.9). ■

Proof of Theorem 2.

Let $m_r=\mathrm{E}\left(\beta_i^r\right)$, $r=1,2,\cdots,2K-1$, which are taken as known. We show that

$$m_r=\sum_{k=1}^{K}\pi_k b_k^r, \quad \text{(A.1.2)}$$

$r=0,1,2,\cdots,2K-1$, has a unique solution $\boldsymbol{\theta}=(\boldsymbol{\pi}',\mathbf{b}')'$, with $b_1<b_2<\cdots<b_K$ and $\pi_k\in(0,1)$ imposed.

Let

$$q\left(\lambda\right)=\prod_{k=1}^{K}\left(\lambda-b_k\right)=\lambda^K+\left(-1\right)^1 b_1^*\lambda^{K-1}+\cdots+\left(-1\right)^K b_K^*, \quad \text{(A.1.3)}$$

be the polynomial with K distinct roots b_1, b_2, $\cdots$, b_K. Note that for each k, $(b_k^r)_{r=0}^{2K-1}$ satisfies the linear homogeneous recurrence relation,

$$b_k^{K+r}=b_1^*b_k^{K+r-1}+\left(-1\right)^1 b_2^*b_k^{K+r-2}+\cdots+\left(-1\right)^{K-1}b_K^*b_k^r, \quad \text{(A.1.4)}$$

for $r=0,1,\cdots K-1$, since q is the characteristic polynomial of the linear recurrence relation

(A.1.4) and b_k is a root of q (Rosen, 2006, Chapter 5.2). $(m_r)_{r=0}^{2K-1}$ is a linear combination of $(b_1^r)_{r=0}^{2K-1}$, $(b_2^r)_{r=0}^{2K-1}$, $\cdots$, $(b_K^r)_{r=0}^{2K-1}$ by (A.1.2), then $(m_r)_{r=0}^{2K-1}$ also satisfies the linear recurrence relation (A.1.4), i.e.,

$$m_{K+r} = b_1^* m_{K+r-1} + (-1)^1 b_2^* m_{K+r-2} + \cdots + (-1)^{K-1} b_K^* m_r, \tag{A.1.5}$$

for $r = 0, 1, \cdots, K-1$. (A.1.5) is a linear system of K equations in terms of $(b_k^*)_{k=1}^K$. In matrix form,

$$\mathbf{M}\mathbf{D}\mathbf{b}^* = \mathbf{m}, \tag{A.1.6}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & m_1 & \cdots & m_{K-1} \\ m_1 & m_2 & \cdots & m_K \\ \vdots & \vdots & \ddots & \vdots \\ m_{K-1} & m_K & \cdots & m_{2K-2} \end{pmatrix},$$

$\mathbf{D} = \operatorname{diag}\left((-1)^{K-1}, (-1)^{K-2}, \cdots, 1\right)$, $\mathbf{b}^* = \left(b_K^*, b_{K-1}^*, \cdots, b_1^*\right)'$, and $\mathbf{m} = (m_K, m_{K+1}, \cdots, m_{2K-1})'$.

Denote $\boldsymbol{\psi}_k = \left(1, b_k, b_k^2 \cdots, b_k^{K-1}\right)'$ and $\boldsymbol{\Psi} = (\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \cdots, \boldsymbol{\psi}_K)$. Then

$$\mathbf{M}_k = \begin{pmatrix} 1 & b_k & \cdots & b_k^{K-1} \\ b_k & b_k^2 & \cdots & b_k^K \\ \vdots & \vdots & \ddots & \vdots \\ b_k^{K-1} & b_k^K & \cdots & b_k^{2K-2} \end{pmatrix} = \boldsymbol{\psi}_k \boldsymbol{\psi}_k',$$

and $\mathbf{M} = \sum_{k=1}^K \pi_k \mathbf{M}_k = \boldsymbol{\Psi} \operatorname{diag}(\boldsymbol{\pi}) \boldsymbol{\Psi}'$. Note that $\boldsymbol{\Psi}'$ is a Vandermonde matrix then $\det(\Psi) = \prod_{1 \le k < k' \le K} (b_{k'} - b_k) > 0$ since $b_1 < b_2 < \cdots < b_K$.

$$\begin{aligned} \det(\mathbf{MD}) &= \det\left(\boldsymbol{\Psi} \operatorname{diag}(\boldsymbol{\pi}) \boldsymbol{\Psi}'\right) \det(\mathbf{D}) \\ &= \left(\prod_{1 \le k < k' \le K} (b_{k'} - b_k)\right)^2 \left(\prod_{k=1}^K \pi_k\right) \left((-1)^{\frac{1}{2}K(K-1)}\right) \neq 0, \end{aligned}$$

since $\pi_k \in (0, 1)$ for any k. Then we can identify $(b_k^*)_{k=1}^K$ by $(m_r)_{r=0}^{2K-1}$ in (A.1.6), and hence the characteristic polynomial is determined, and we can identify $(b_k)_{k=1}^K$ by (A.1.3).

Since both $(b_k)_{k=1}^K$ and $(m_r)_{r=1}^{2K-1}$ are identified, the first K equations of (A.1.2) is

$$\boldsymbol{\Psi}' \boldsymbol{\pi} = (1, m_1, m_2, \cdots, m_{K-1})',$$

and $\boldsymbol{\pi}$ is identified by inverting the Vandermonde matrix $\boldsymbol{\Psi}'$, which completes the proof. ■

Proof of Theorem 3. From (3.1), we have

$$\frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i y_i = \frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i\mathbf{w}_i'\boldsymbol{\phi} + \frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i\xi_i,$$

where $\xi_i = u_i + x_i v_i$, which can be written equivalently as

$$\mathbf{q}_{n,wy} = \mathbf{Q}_{n,ww}\boldsymbol{\phi} + \frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_i\xi_i.$$

Taking expectations of both sides and rearrange terms, we have

$$\boldsymbol{\phi} = \mathrm{E}\left(\mathbf{Q}_{n,ww}\right)^{-1}\mathrm{E}\left(\mathbf{q}_{n,wy}\right).$$

Consider

$$\begin{aligned}
\hat{\boldsymbol{\phi}} - \boldsymbol{\phi} &= \mathbf{Q}_{n,ww}^{-1}\mathbf{q}_{n,wy} - \mathrm{E}\left(\mathbf{Q}_{n,ww}\right)^{-1}\mathrm{E}\left(\mathbf{q}_{n,wy}\right) \\
&= \left[\mathbf{Q}_{n,ww}^{-1} - E\left(\mathbf{Q}_{n,ww}\right)^{-1} + E\left(\mathbf{Q}_{n,ww}\right)^{-1}\right]\left[\mathbf{q}_{n,wy} - \mathrm{E}\left(\mathbf{q}_{n,wy}\right) + \mathrm{E}\left(\mathbf{q}_{n,wy}\right)\right] - \mathrm{E}\left(\mathbf{Q}_{n,ww}\right)^{-1}\mathrm{E}\left(\mathbf{q}_{n,wy}\right) \\
&= \left[\mathbf{Q}_{n,ww}^{-1} - E\left(\mathbf{Q}_{n,ww}\right)^{-1}\right]\left[\mathbf{q}_{n,wy} - \mathrm{E}\left(\mathbf{q}_{n,wy}\right)\right] + \left[\mathbf{Q}_{n,ww}^{-1} - \mathrm{E}\left(\mathbf{Q}_{n,ww}\right)^{-1}\right]\mathrm{E}\left(\mathbf{q}_{n,wy}\right) \\
&\qquad + \mathrm{E}\left(\mathbf{Q}_{n,ww}\right)^{-1}\left[\mathbf{q}_{n,wy} - \mathrm{E}\left(\mathbf{q}_{n,wy}\right)\right].
\end{aligned}$$

Then,

$$\begin{aligned}
\left\|\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right\| \leq{}& \left\|\mathbf{Q}_{n,ww}^{-1} - \mathrm{E}\left(\mathbf{Q}_{n,ww}\right)^{-1}\right\|\left\|\mathbf{q}_{n,wy} - \mathrm{E}\left(\mathbf{q}_{n,wy}\right)\right\| + \left\|\mathbf{Q}_{n,ww}^{-1} - \mathrm{E}\left(\mathbf{Q}_{n,ww}\right)^{-1}\right\|\left\|\mathrm{E}\left(\mathbf{q}_{n,wy}\right)\right\| \\
&+ \left\|\mathrm{E}\left(\mathbf{Q}_{n,ww}\right)^{-1}\right\|\left\|\mathbf{q}_{n,wy} - \mathrm{E}\left(\mathbf{q}_{n,wy}\right)\right\|.
\end{aligned}$$

By Assumption 1(c), we have $\left\|\mathbf{Q}_{n,ww}^{-1} - \mathrm{E}\left(\mathbf{Q}_{n,ww}\right)^{-1}\right\| = O_p\left(n^{-1/2}\right)$, $\left\|\mathbf{q}_{n,wy} - \mathrm{E}\left(\mathbf{q}_{n,wy}\right)\right\| = O_p\left(n^{-1/2}\right)$, and by Assumption 1(b), $\left\|\mathrm{E}\left(\mathbf{q}_{n,wy}\right)\right\|$ and $\left\|\mathrm{E}\left(\mathbf{Q}_{n,ww}\right)^{-1}\right\|$ are bounded. Thus,

$$\left\|\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right\| = O_p\left(n^{-1/2}\right).$$

To establish the asymptotic distribution of $\hat{\boldsymbol{\phi}}$, we first note that

$$\sqrt{n}\left(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right) = \mathbf{Q}_{n,ww}^{-1}\left(n^{-1/2}\sum_{i=1}^{n}\mathbf{w}_i\xi_i\right).$$

By Assumption 3, we have

$$\mathrm{var}\left(n^{-1/2}\sum_{i=1}^{n}\mathbf{w}_i\xi_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{var}\left(\mathbf{w}_i\xi_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{E}\left(\mathbf{w}_i\mathbf{w}_i'\xi_i^2\right) \to \mathbf{V}_{w\xi} \succ 0.$$

Note that $\xi_i = u_i + x_i v_i$, and $\mathbf{w}_i$ is distributed independently of u_i and v_i. Then

$$\mathbf{w}_i \xi_i = \mathbf{w}_i \left(u_i + x_i v_i\right) = \mathbf{w}_i u_i + \left(\mathbf{w}_i x_i\right) v_i,$$

and by Minkowski's inequality, for $r = 2 + \delta$ with $0 < \delta < 1$,

$$\left[E \left\|\mathbf{w}_i \xi_i\right\|^r\right]^{1/r} \leq \left[E \left\|\mathbf{w}_i u_i\right\|^r\right]^{1/r} + \left[E \left\|\left(\mathbf{w}_i x_i\right) v_i\right\|^r\right]^{1/r}.$$

Due to the independence of u_i and v_i from $\mathbf{w}_i$, we have

$$\mathrm{E}\left(\left\|\mathbf{w}_i u_i\right\|^r\right) \leq E \left\|\mathbf{w}_i\right\|^r E \left\|u_i\right\|^r, \text{ and } E \left\|\left(\mathbf{w}_i x_i'\right) v_i\right\|^r \leq E \left\|\mathbf{w}_i x_i\right\|^r E \left\|v_i\right\|^r.$$

Also, $E \left\|\mathbf{w}_i x_i\right\|^r \leq E \left\|\left(x_i^2, x_i \mathbf{z}_i'\right)'\right\|^r \leq E \left\|\mathbf{w}_i \mathbf{w}_i'\right\|^r \leq E \left\|\mathbf{w}_i\right\|^{2r}$, where $2 < r < 3$, and hence $2r < 6$. By Assumptions 1(a.ii) and 1(b.ii), we have $\sup_i \mathrm{E}\left(\left\|\mathbf{w}_i\right\|^6\right) < C$, $\sup_i \mathrm{E}\left(\left\|u_i\right\|^3\right) < C$, and $\mathrm{E}\left(\left\|v_i\right\|^3\right) \leq \max_{1 \leq k \leq K} \left|b_k - \mathrm{E}\left(\beta_i\right)\right|^3 < C$. Then, we verified that $\sup_i \mathrm{E}\left(\left\|\mathbf{w}_i u_i\right\|^r\right) < C$, and $E \left\|\left(\mathbf{w}_i x_i'\right) v_i\right\|^r < C$, and hence the Lyapunov condition that $\sup_i \mathrm{E}\left(\left\|\mathbf{w}_i \xi_i\right\|^r\right) < C$, where $r = 2 + \delta \in (2, 3)$. By the central limit theorem for independent but not necessarily identically distributed random vectors (see Pesaran (2015, Theorem 18) or Hansen (2022, Theorem 6.5)), we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbf{w}_i \xi_i \rightarrow_d N(\mathbf{0}, \mathbf{V}_{w\xi}),$$

as $n \rightarrow \infty$, and by Assumption 1 and continuous mapping theorem,

$$\sqrt{n}(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}) \rightarrow_d N\left(\mathbf{0}, \mathbf{Q}_{ww}^{-1} \mathbf{V}_{w\xi} \mathbf{Q}_{ww}^{-1}\right).$$

We then turn to the consistent estimation of the variance matrix. By Assumption 3, we have

$$\begin{aligned}
\left\|\hat{\mathbf{V}}_{w\xi} - \mathbf{V}_{w\xi}\right\| &= \left\|\frac{1}{n} \sum_{i=1}^{n} \mathbf{w}_i \mathbf{w}_i' \hat{\xi}_i^2 - \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left(\mathbf{w}_i \mathbf{w}_i \xi_i^2\right) + \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left(\mathbf{w}_i \mathbf{w}_i \xi_i^2\right) - \mathbf{V}_{w\xi}\right\| \\
&\leq \left\|\frac{1}{n} \sum_{i=1}^{n} \mathbf{w}_i \mathbf{w}_i' \xi_i^2 - \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left(\mathbf{w}_i \mathbf{w}_i \xi_i^2\right)\right\| + \left\|\frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left(\mathbf{w}_i \mathbf{w}_i \xi_i^2\right) - \mathbf{V}_{w\xi}\right\| \\
&\quad + \left\|\frac{1}{n} \sum_{i=1}^{n} \mathbf{w}_i \mathbf{w}_i' \left(\hat{\xi}_i^2 - \xi_i^2\right)\right\| \\
&\leq \frac{1}{n} \sum_{i=1}^{n} \left\|\mathbf{w}_i\right\|^2 \left|\hat{\xi}_i^2 - \xi_i^2\right| + O_p(n^{-1/2}). \qquad \text{(A.1.7)}
\end{aligned}$$

Note that $\hat{\xi}_i = \xi_i - \left(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right)' \mathbf{w}_i$, then

$$\begin{aligned}\left|\hat{\xi}_i^2 - \xi_i^2\right| &\leq 2\left|\xi_i \mathbf{w}_i' \left(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right)\right| + \left(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right)' \left(\mathbf{w}_i \mathbf{w}_i'\right) \left(\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right) \\ &\leq 2\left|\xi_i\right| \left\|\mathbf{w}_i\right\| \left\|\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right\| + \left\|\mathbf{w}_i\right\|^2 \left\|\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right\|^2. \end{aligned} \tag{A.1.8}$$

Combine (A.1.7) and (A.1.8), we have

$$\left\|\hat{\mathbf{V}}_{w\xi} - \mathbf{V}_{w\xi}\right\| \leq 2\left(\frac{1}{n}\sum_{i=1}^n \left\|\mathbf{w}_i\right\|^3 \left|\xi_i\right|\right)\left\|\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right\| + \left(\frac{1}{n}\sum_{i=1}^n \left\|\mathbf{w}_i\right\|^4\right)\left\|\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right\|^2. \tag{A.1.9}$$

We showed that $\left\|\hat{\boldsymbol{\phi}} - \boldsymbol{\phi}\right\| = O_p\left(n^{-1/2}\right)$. By Hölder's inequality,

$$\frac{1}{n}\sum_{i=1}^n \left\|\mathbf{w}_i\right\|^3 \left|\xi_i\right| \leq \left(\frac{1}{n}\sum_{i=1}^n \left\|\mathbf{w}_i\right\|^4\right)^{3/4}\left(\frac{1}{n}\sum_{i=1}^n \xi_i^4\right)^{1/4}. \tag{A.1.10}$$

By Assumption 1(b.iii), $n^{-1}\sum_{i=1}^n \left\|\mathbf{w}_i\right\|^4 = O_p(1)$. By Minkowski inequality,

$$\begin{aligned}\left(\frac{1}{n}\sum_{i=1}^n \xi_i^4\right)^{1/4} &= \left(\frac{1}{n}\sum_{i=1}^n \left(u_i + x_i v_i\right)^4\right)^{1/4} \leq \left(\frac{1}{n}\sum_{i=1}^n u_i^4\right)^{1/4} + \left(\frac{1}{n}\sum_{i=1}^n x_i^4 v_i^4\right)^{1/4} \\ &\leq \left(\frac{1}{n}\sum_{i=1}^n u_i^4\right)^{1/4} + \max_k \left\{\left|b_k - \mathrm{E}\left(\beta_i\right)\right|\right\}\left(\frac{1}{n}\sum_{i=1}^n x_i^4\right)^{1/4} \\ &\leq O_p(1),\end{aligned}$$

where the last inequality is from Assumptions 1(a.iii) and (b.iii) that $n^{-1}\sum_{i=1}^n u_i^4 = O_p(1)$, and $n^{-1}\sum_{i=1}^n x_i^4 \leq n^{-1}\sum_{i=1}^n \left\|\mathbf{w}_i\right\|^4 = O_p(1)$. Then we verified in (A.1.10) that

$$\frac{1}{n}\sum_{i=1}^n \left\|\mathbf{w}_i\right\|^3 \left|\xi_i\right| \leq O_p(1).$$

Then by (A.1.9) we have

$$\left\|\hat{\mathbf{V}}_{w\xi} - \mathbf{V}_{w\xi}\right\| = O_p\left(n^{-1/2}\right).$$

■

Proof of Theorem 4. Denote

$$\Phi_0\left(\boldsymbol{\theta}, \boldsymbol{\sigma}, \gamma\right) = \mathbf{g}_0\left(\boldsymbol{\theta}, \boldsymbol{\sigma}, \gamma\right)' \mathbf{A}\mathbf{g}_0\left(\boldsymbol{\theta}, \boldsymbol{\sigma}, \gamma\right),$$

where we stack the left-hand side of (3.7) and transform $\mathbf{m}_\beta = h\left(\boldsymbol{\theta}\right)$ to get $\mathbf{g}_0\left(\boldsymbol{\theta}, \boldsymbol{\sigma}, \gamma\right)$. We suppress and the argument $\hat{\gamma}$ and denote $\boldsymbol{\eta} = \left(\boldsymbol{\theta}', \boldsymbol{\sigma}'\right)'$ for notation simplicity and proceed by verifying the conditions of Newey and McFadden (1994, Theorem 2.1). Theorem 2 provides the identification results which together with the positive definiteness of $\mathbf{A}$ verifies that $\Phi_0(\boldsymbol{\eta}, \boldsymbol{\gamma})$ is uniquely minimized to 0 at $\boldsymbol{\eta}_0$. The compactness of the parameter space holds by Assumption 4(a). Note that $\mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma})$ is a polynomial in $\boldsymbol{\eta}$, which is continuous in $\boldsymbol{\eta}$. $\mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma})$ is bounded on $\Theta \times \mathcal{S}$. We proceed by verify the uniform convergence condition. The additive terms in $\hat{\mathbf{g}}_n(\boldsymbol{\eta}, \hat{\gamma}) - \mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma})$ are of the form $H_{n,1}h^{(r,q)}(\boldsymbol{\eta})$ or $H_{n,2}$, where

$$
\begin{aligned}
|H_{n,1}| &= \left| \frac{1}{n} \sum_{i=1}^{n} x_i^{r-q+s_r} - \rho_{0,r-q+s_r} \right| \\
&\leq \left| \frac{1}{n} \sum_{i=1}^{n} x_i^{r-q+s_r} - \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left(x_i^{r-q+s_r}\right) \right| + \left| \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left(x_i^{r-q+s_r}\right) - \rho_{0,r-q+s_r} \right| \\
&= O_p\left(n^{-1/2}\right),
\end{aligned}
$$

$h^{(r,q\})}(\boldsymbol{\eta})$ is a polynomial in $\boldsymbol{\eta}$, and

$$
\begin{aligned}
|H_{n,2}| &= \left| \frac{1}{n} \sum_{i=1}^{n} \hat{\tilde{y}}_i^r x_i^{s_r} - \rho_{r,s_r} \right| \\
&\leq \left| \frac{1}{n} \sum_{i=1}^{n} \hat{\tilde{y}}_i^r x_i^{s_r} - \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left(\tilde{y}_i^r x_i^{s_r}\right) \right| + \left| \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left(\tilde{y}_i^r x_i^{s_r}\right) - \rho_{r,s_r} \right| \\
&= O_p\left(n^{-1/2}\right).
\end{aligned}
$$

$H_{n,1} = O_p\left(n^{-1/2}\right)$ and $H_{n,2} = O_p\left(n^{-1/2}\right)$ are due to Assumption 2(a) and 4(c).

By the compactness of $\Theta \times \mathcal{S}$, $\sup_{\boldsymbol{\eta} \in \Theta \times \mathcal{S}} h^{(r,q)}(\boldsymbol{\eta}) < C < \infty$ for some positive constant C. By triangle inequality, we have

$$
\sup_{\boldsymbol{\eta} \in \Theta \times \mathcal{S}} \left\| \hat{\mathbf{g}}_n(\boldsymbol{\eta}, \hat{\gamma}) - \mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma}) \right\| \to_p 0, \tag{A.1.11}
$$

as $n \to \infty$. Following the proof of Newey and McFadden (1994, Theorem 2.1),

$$
\begin{aligned}
&\left| \hat{\Phi}_n(\boldsymbol{\eta},\ \hat{\gamma}) - \Phi_0(\boldsymbol{\eta}, \boldsymbol{\gamma}) \right| \\
&\leq \left| [\hat{\mathbf{g}}_n(\boldsymbol{\eta}, \hat{\gamma}) - \mathbf{g}_0(\boldsymbol{\eta},\ \boldsymbol{\gamma})]' \mathbf{A}_n [\hat{\mathbf{g}}_n(\boldsymbol{\eta}, \hat{\gamma}) - \mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma})] \right| + \left| \mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma})' \left(\mathbf{A}_n + \mathbf{A}_n'\right) [\hat{\mathbf{g}}_n(\boldsymbol{\eta}, \hat{\gamma}) - \mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma})] \right| \\
&\qquad + \left| \mathbf{g}_0(\boldsymbol{\eta},\ \boldsymbol{\gamma})' (\mathbf{A}_n - \mathbf{A}) \mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma}) \right| \\
&\leq \left\| \hat{\mathbf{g}}_n(\boldsymbol{\eta},\ \hat{\gamma}) - \mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma}) \right\|^2 \|\mathbf{A}_n\| + 2 \|\mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma})\| \left\| \hat{\mathbf{g}}_n(\boldsymbol{\eta}, \hat{\gamma}) - \mathbf{g}_0(\boldsymbol{\eta},\ \boldsymbol{\gamma}) \right\| \|\mathbf{A}_n\| + \|\mathbf{g}_0(\boldsymbol{\eta}, \boldsymbol{\gamma})\|^2 \|\mathbf{A}_n - \mathbf{A}\|.
\end{aligned}
$$

By (A.1.11) and the boundedness of $\mathbf{g}_0$, $\sup_{\boldsymbol{\eta} \in \eta} \left| \hat{\Phi}_n(\ \boldsymbol{\eta}, \hat{\gamma}) - \Phi_n(\ \boldsymbol{\eta}, \boldsymbol{\gamma}) \right| \to_p 0$, which completes the proof. ■

Proof of Theorem 5. We denote $\boldsymbol{\eta} = \left(\boldsymbol{\theta}', \boldsymbol{\sigma}'\right)'$ for notation simplicity The first-order condition, $\nabla_{\boldsymbol{\eta}} \hat{\mathbf{g}}_n(\hat{\boldsymbol{\eta}}, \hat{\gamma}) \mathbf{A}_n \hat{\mathbf{g}}_n(\hat{\boldsymbol{\eta}}, \hat{\gamma}) = \mathbf{0}$, holds with probability 1. Denote $\hat{\mathbf{G}}(\boldsymbol{\eta}, \boldsymbol{\gamma}) = \nabla_{\boldsymbol{\eta}} \hat{\mathbf{g}}_n(\boldsymbol{\eta}, \boldsymbol{\gamma})$ and expand

$\hat{\mathbf{g}}_n(\hat{\boldsymbol{\eta}}, \hat{\gamma})$ in the first-order condition around $\boldsymbol{\eta}_0$, we have

$$\begin{aligned}\sqrt{n}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0) &= -\left[\hat{\mathbf{G}}(\hat{\boldsymbol{\eta}}, \hat{\gamma})' \mathbf{A}_n \hat{\mathbf{G}}(\bar{\boldsymbol{\eta}}, \hat{\gamma})\right]^{-1} \hat{\mathbf{G}}(\hat{\boldsymbol{\eta}},\ \hat{\gamma})' \mathbf{A}_n \left(\sqrt{n}\hat{\mathbf{g}}_n(\boldsymbol{\eta}_0, \hat{\gamma})\right) \\ &= -\left[\hat{\mathbf{G}}(\hat{\boldsymbol{\eta}}, \hat{\gamma})' \mathbf{A}_n \hat{\mathbf{G}}(\bar{\boldsymbol{\eta}}, \hat{\gamma})\right]^{-1} \hat{\mathbf{G}}(\hat{\boldsymbol{\eta}},\ \hat{\gamma})' \mathbf{A}_n \left[\sqrt{n}\hat{\mathbf{g}}_n(\boldsymbol{\eta}_0, \gamma_0) + \nabla_\gamma \hat{\mathbf{g}}_n(\boldsymbol{\eta}_0, \bar{\gamma})\sqrt{n}(\hat{\gamma} - \gamma_0)\right],\end{aligned}$$

where $\bar{\boldsymbol{\eta}}$ and $\bar{\gamma}$ are between $\hat{\boldsymbol{\eta}}$ and $\boldsymbol{\eta}_0$; and $\hat{\gamma}$ and γ_0, respectively. Note that by term-by-term convergence, we have $\hat{\mathbf{G}}(\hat{\boldsymbol{\eta}}, \hat{\gamma}), \hat{\mathbf{G}}(\bar{\boldsymbol{\eta}}, \hat{\gamma}) \to_p \mathbf{G}_0$ and $\nabla_{\boldsymbol{\gamma}} \hat{\mathbf{g}}_n(\boldsymbol{\eta}_0, \bar{\gamma}) \to_p \nabla_{\boldsymbol{\gamma}} \mathbf{g}_0(\boldsymbol{\eta}_{0,}\gamma_0) = \mathbf{G}_{0,\gamma}$. By Assumption 4(b), $\mathbf{A}_n \to_p \mathbf{A}$. By Assumption 5(a) and (b) and Slutsky theorem,

$$\sqrt{n}(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}_0) \to_d \left(\mathbf{G}_0' \mathbf{A} \mathbf{G}_0\right)^{-1} \mathbf{G}_0' \mathbf{A}\left(\boldsymbol{\zeta} + \mathbf{G}_{0,\gamma}\boldsymbol{\zeta}_\gamma\right),$$

which completes the proof. ■

Further details for Example 4. We need to verify the invertibility of the matrix

$$\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ b_{1L}b_{2L} & b_{1L}b_{2H} & b_{1H}b_{2L} & b_{1H}b_{2H} \end{pmatrix}.$$

The span of first three rows of $\mathbf{B}$ is

$$\mathcal{S} = \left\{(\alpha_1 + \alpha_3, \alpha_1, \alpha_2 + \alpha_3, \alpha_3)' : \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\right\}.$$

$(b_{1L}b_{2L}, b_{1L}b_{2H}, b_{1H}b_{2L}, b_{1H}b_{2H})' \notin \mathcal{S}$ is equivalent to $b_{1H}b_{2H} - b_{1H}b_{2L} \neq b_{1L}b_{2H} - b_{1L}b_{2L}$. This can be verified by

$$(b_{1H}b_{2H} - b_{1H}b_{2L}) - (b_{1L}b_{2H} - b_{1L}b_{2L}) = (b_{1H} - b_{1L})(b_{2H} - b_{2L}) > 0,$$

given that $b_{1L} < b_{1H}$ and $b_{2L} < b_{2H}$ hold. ■

A.2 Additional empirical results

In this section, we provide additional results for the empirical application. In addition to the quadratic in experience in Section 6, we further consider the following quartic in experience specification,

$$\log \text{wage}_i = \alpha + \beta_i \text{edu}_i + \rho_1 \text{exper}_i + \rho_2 \text{exper}_i^2 + \rho_3 \text{exper}_i^3 + \rho_4 \text{exper}_i^4 + \tilde{\mathbf{z}}_i' \tilde{\gamma} + u_i, \tag{A.2.1}$$

where

$$\beta_i = \begin{cases} b_L & \text{w.p. } \pi, \\ b_H & \text{w.p. } 1 - \pi. \end{cases}$$

Table A.2 and A.3 report the estimates of the distributional parameters of β_i and the estimates of $\boldsymbol{\gamma}$ with the specification (A.2.1).

The estimates of parameter of interests with specification (A.2.1) are almost the same as that with quadratic in experience specification (6.3) reported in Table 5. The qualitative analysis and conclusion discussed in Section 6 remain robust to adding higher order powers of exper_i in the regressions.

Table A.1: Estimates of γ associated with control variables $\mathbf{z}_i$ with specification (6.2) across two periods, 1973 - 75 and 2001 - 03, by years of education and gender, which complements Table 5

	High School or Less		Postsecondary Edu.		All	
	1973 - 75	2001 - 03	1973 - 75	2001 - 03	1973 - 75	2001 - 03
			Both male and female			
`exper.`	0.0305	0.0319	0.0415	0.0354	0.0310	0.0321
	(0.0004)	*(0.0002)*	*(0.0008)*	*(0.0003)*	*(0.0003)*	*(0.0002)*
`exper.`^{2} ($\times 10^2$)	-0.0490	-0.0505	-0.0826	-0.0652	-0.0499	-0.0537
	(0.0009)	*(0.0005)*	*(0.0022)*	*(0.0007)*	*(0.0008)*	*(0.0005)*
`marriage`	0.1120	0.0751	0.0886	0.0770	0.1085	0.0818
	(0.0036)	*(0.0020)*	*(0.0059)*	*(0.0020)*	*(0.0031)*	*(0.0014)*
`nonwhite`	-0.0922	-0.0775	-0.0424	-0.0571	-0.0715	-0.0667
	(0.0047)	*(0.0024)*	*(0.0088)*	*(0.0025)*	*(0.0042)*	*(0.0018)*
`gender`	0.4157	0.2298	0.2962	0.2023	0.3892	0.2167
	(0.0029)	*(0.0017)*	*(0.0050)*	*(0.0018)*	*(0.0025)*	*(0.0013)*
n	77,899	216,136	33,733	295,683	111,632	511,819
			Male			
`exper.`	0.0369	0.0366	0.0516	0.0405	0.0389	0.0371
	(0.0005)	*(0.0003)*	*(0.0011)*	*(0.0005)*	*(0.0005)*	*(0.0003)*
`exper.`^{2} ($\times 10^2$)	-0.0589	-0.0589	-0.1016	-0.0752	-0.0635	-0.0629
	(0.0012)	*(0.0008)*	*(0.0029)*	*(0.0011)*	*(0.0010)*	*(0.0007)*
`marriage`	0.1940	0.1123	0.1497	0.1344	0.1828	0.1316
	(0.0053)	*(0.0028)*	*(0.0085)*	*(0.0031)*	*(0.0045)*	*(0.0021)*
`nonwhite`	-0.1241	-0.1165	-0.1172	-0.1010	-0.1178	-0.1093
	(0.0065)	*(0.0035)*	*(0.0127)*	*(0.0039)*	*(0.0058)*	*(0.0027)*
n	44,299	116,129	20,851	144,138	65,150	260,267
			Female			
`exper.`	0.0223	0.0265	0.0271	0.0313	0.0208	0.0272
	(0.0006)	*(0.0003)*	*(0.0011)*	*(0.0004)*	*(0.0005)*	*(0.0003)*
`exper.`^{2} ($\times 10^2$)	-0.0376	-0.0411	-0.0564	-0.0576	-0.0338	-0.0450
	(0.0013)	*(0.0008)*	*(0.0030)*	*(0.0010)*	*(0.0012)*	*(0.0006)*
`marriage`	0.0115	0.0317	-0.0005	0.0262	0.0118	0.0322
	(0.0048)	*(0.0028)*	*(0.0079)*	*(0.0026)*	*(0.0041)*	*(0.0019)*
`nonwhite`	-0.0581	-0.0441	0.0395	-0.0236	-0.0202	-0.0315
	(0.0065)	*(0.0033)*	*(0.0117)*	*(0.0033)*	*(0.0058)*	*(0.0024)*
n	33,600	100,007	12,882	151,545	46,482	251,552

Notes: This table reports the estimates of γ in (6.2). "Postsecondary Edu." stands for the sub-sample with years of education higher than 12 and "High School or Less" stands for those with years of education less than or equal to 12. The standard error of estimates of coefficients associated with control variables are estimated based on Theorem 3 and reported in parentheses. n is the sample size.

Table A.2: Estimates of the distribution of the return to education with specification (A.2.1) across two periods, 1973 - 75 and 2001 - 03, by years of education and gender

	High School or Less		Postsecondary Edu.		All	
	1973 - 75	2001 - 03	1973 - 75	2001 - 03	1973 - 75	2001 - 03
			Both Male and Female			
π	n/a	0.5766	0.2698	0.1988	0.2928	0.2940
β_L	0.0623	0.0421	0.0491	0.0692	0.0481	0.0604
β_H	0.0623	0.0978	0.1021	0.1309	0.0874	0.1221
β_H/β_L	1.0000	2.3265	2.0807	1.8905	1.8170	2.0206
$\mathrm{E}(\beta_i)$	0.0623	0.0657	0.0878	0.1186	0.0759	0.1040
$\mathrm{s.d.}(\beta_i)$	0.0000	0.0276	0.0235	0.0246	0.0179	0.0281
n	77,899	216,136	33,733	295,683	111,632	511,819
			Male			
π	n/a	0.4647	0.3006	0.1938	0.2682	0.2998
β_L	0.0650	0.0401	0.0407	0.0580	0.0453	0.0535
β_H	0.0650	0.0908	0.0912	0.1245	0.0808	0.1171
β_H/β_L	1.0000	2.2652	2.2383	2.1453	1.7838	2.1876
$\mathrm{E}(\beta_i)$	0.0650	0.0672	0.0760	0.1116	0.0713	0.0980
$\mathrm{s.d.}(\beta_i)$	0.0000	0.0253	0.0231	0.0263	0.0157	0.0291
n	44,299	116,129	20,851	144,138	65,150	260,267
			Female			
π	0.2400	0.6061	0.2666	0.2314	0.2206	0.2984
β_L	0.0354	0.0389	0.0672	0.0827	0.0502	0.0697
β_H	0.0661	0.1049	0.1211	0.1370	0.0936	0.1280
β_H/β_L	1.8652	2.6943	1.8014	1.6559	1.8639	1.8367
$\mathrm{E}(\beta_i)$	0.0587	0.0649	0.1068	0.1244	0.0840	0.1106
$\mathrm{s.d.}(\beta_i)$	0.0131	0.0322	0.0238	0.0229	0.0180	0.0267
n	33,600	100,007	12,882	151,545	46,482	251,552

Notes: This table reports the estimates of the distribution of β_i with the quartic in experience specification (A.2.1), using $S = 4$ order moments of edu_i. "Postsecondary Edu." stands for the sub-sample with years of education higher than 12 and "High School or Less" stands for those with years of education less than or equal to 12. $\mathrm{s.d.}(\beta_i)$ corresponds to the square root of estimated $\mathrm{var}(\beta_i)$. n is the sample size. "n/a" is inserted when the estimates show homogeneity of β_i and π is not identified and cannot be estimated.

Table A.3: Estimates of $\boldsymbol{\gamma}$ associated with control variables $\mathbf{z}_i$ with specification (A.2.1) across two periods, 1973 - 75 and 2001 - 03, by years of education and gender, which complements Table A.2

	High School or Less		Postsecondary Edu.		All	
	1973 - 75	2001 - 03	1973 - 75	2001 - 03	1973 - 75	2001 - 03
			Both male and female			
`exper.`	0.0769	0.0526	0.0817	0.0763	0.0757	0.0603
	(0.0015)	*(0.0009)*	*(0.0029)*	*(0.0012)*	*(0.0013)*	*(0.0007)*
`exper.`^{2}	-0.0040	-0.0020	-0.0045	-0.0039	-0.0038	-0.0024
	(0.0001)	*(0.0001)*	*(0.0003)*	*(0.0001)*	*(0.0001)*	*(0.0001)*
`exper.`^{3} $(\times 10^5)$	9.2470	3.4329	11.2100	8.9370	8.3625	3.6521
	(0.4146)	*(0.2882)*	*(1.2538)*	*(0.4460)*	*(0.3677)*	*(0.2412)*
`exper.`^{4} $(\times 10^5)$	-0.0768	-0.0236	-0.1074	-0.0777	-0.0654	-0.0169
	(0.0043)	*(0.0031)*	*(0.0158)*	*(0.0054)*	*(0.0039)*	*(0.0027)*
`marriage`	0.0819	0.0700	0.0728	0.0674	0.0799	0.0718
	(0.0037)	*(0.0020)*	*(0.0060)*	*(0.0020)*	*(0.0031)*	*(0.0014)*
`nonwhite`	-0.1052	-0.0808	-0.0486	-0.0613	-0.0855	-0.0719
	(0.0046)	*(0.0024)*	*(0.0088)*	*(0.0025)*	*(0.0041)*	*(0.0018)*
`gender`	0.4146	0.2272	0.2933	0.2008	0.3854	0.2150
	(0.0029)	*(0.0017)*	*(0.0049)*	*(0.0018)*	*(0.0025)*	*(0.0013)*
n	77,899	216,136	33,733	295,683	111,632	511,819
			Male			
`exper.`	0.0823	0.0620	0.0859	0.0780	0.0825	0.0664
	(0.0020)	*(0.0012)*	*(0.0040)*	*(0.0018)*	*(0.0017)*	*(0.0010)*
`exper.`^{2} $(\times 10^2)$	-0.0039	-0.0024	-0.0041	-0.0036	-0.0037	-0.0025
	(0.0002)	*(0.0001)*	*(0.0004)*	*(0.0002)*	*(0.0001)*	*(0.0001)*
`exper.`^{3} $(\times 10^5)$	8.2014	4.3686	9.2747	7.3170	7.4306	3.6749
	(0.5321)	*(0.3864)*	*(1.7422)*	*(0.6709)*	*(0.4700)*	*(0.3241)*
`exper.`^{4} $(\times 10^5)$	-0.0650	-0.0314	-0.0880	-0.0582	-0.0552	-0.0161
	(0.0054)	*(0.0042)*	*(0.0223)*	*(0.0081)*	*(0.0049)*	*(0.0036)*
`marriage`	0.1493	0.1052	0.1310	0.1234	0.1421	0.1192
	(0.0056)	*(0.0029)*	*(0.0088)*	*(0.0031)*	*(0.0048)*	*(0.0021)*
`nonwhite`	-0.1362	-0.1191	-0.1214	-0.1040	-0.1309	-0.1136
	(0.0064)	*(0.0035)*	*(0.0126)*	*(0.0039)*	*(0.0057)*	*(0.0027)*
n	44,299	116,129	20,851	144,138	65,150	260,267
			Female			
`exper.`	0.0713	0.0455	0.0911	0.0782	0.0729	0.0568
	(0.0022)	*(0.0013)*	*(0.0040)*	*(0.0016)*	*(0.0019)*	*(0.0011)*
`exper.`^{2} $(\times 10^2)$	-0.0044	-0.0018	-0.0067	-0.0045	-0.0045	-0.0025
	(0.0002)	*(0.0001)*	*(0.0004)*	*(0.0002)*	*(0.0002)*	*(0.0001)*
`exper.`^{3} $(\times 10^5)$	11.0325	3.4767	19.6859	11.2858	11.3406	4.4944
	(0.6649)	*(0.4360)*	*(1.7412)*	*(0.5915)*	*(0.6095)*	*(0.3682)*
`exper.`^{4} $(\times 10^5)$	-0.0974	-0.0264	-0.1979	-0.1046	-0.0969	-0.0272
	(0.0071)	*(0.0048)*	*(0.0216)*	*(0.0071)*	*(0.0066)*	*(0.0042)*
`marriage`	-0.0078	0.0278	-0.0175	0.0168	-0.0082	0.0234
	(0.0048)	*(0.0028)*	*(0.0080)*	*(0.0026)*	*(0.0041)*	*(0.0020)*
`nonwhite`	-0.0714	-0.0479	0.0276	-0.0291	-0.0356	-0.0375
	(0.0065)	*(0.0033)*	*(0.0117)*	*(0.0033)*	*(0.0057)*	*(0.0024)*
n	33,600	100,007	12,882	151,545	46,482	251,552

Notes: This table reports the estimates of $\boldsymbol{\gamma}$ in (A.2.1). "Postsecondary Edu." stands for the sub-sample with years of education higher than 12 and "High School or Less" stands for those with years of education less than or equal to 12. The standard error of estimates of coefficients associated with control variables are estimated based on Theorem 3 and reported in parentheses. n is the sample size.

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