UNIVERSITY OF CAMBRIDGE

On Gaussian multiplicative chaos and conformal field theory

Guillaume Baverez

Fitzwilliam College

This dissertation is submitted on December 2021 for the degree of Doctor of Philosophy

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except for Chapter 3 which has been written in collaboration with Mo Dick Wong with equal contributions. It is not substantially the same as any that I have submitted, or am concurrently submitting, for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or is being concurrently submitted, for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.

Guillaume Baverez
December 16$^{\text{th}}$, 2021

Abstract

On Gaussian multiplicative chaos and conformal field theory

Guillaume Baverez

This thesis is concerned with conformally invariant stochastic processes in two dimensions and their applications to conformal field theory (CFT). The main probabilistic objects are the Gaussian free field (GFF) and the random geometries associated to it. Especially, we are interested in Gaussian multiplicative chaos (GMC), Schramm-Loewner evolution (SLE) and Liouville CFT, which can be understood as theories of random surfaces.

From the point of view of physics, the idea of a "summing over surfaces" can be traced back to Polyakov's work on bosonic string theory [Pol81]. Indeed, the starting point of string theory is to replace a point particle by a one dimensional manifold (a string), so that one must replace the worldline by a worldsheet, i.e. an embedding of a surface into space-time. The path integral that Polyakov wrote down features a random conformal factor that should be described by the quantisation of the Liouville action. Therefore, this probability measure should describe random fluctuations around the uniform metric.

Polyakov also suggested that the resulting quantum field theory should exhibit conformal invariance. This means that the Hilbert space of the theory should carry a projective unitary representation of the group of local conformal transformations, i.e. a unitary representation of the Virasoro algebra. Since it is an infinite dimensional Lie algebra, this is a huge constraint to put on a system and this led Belavin, Polyakov & Zamolodchikov to give an axiomatic framework for CFT based on the representation theory of the Virasoro algebra [BPZ84]. Here, the game is somehow reversed: one tries to exhibit and classify all theories fitting in this framework. In particular, it is not even clear in the first place that such algebraic structures exist.

In this context, Liouville theory is a success story in the interaction of algebra, geometry and probability. On the one hand, the algebraic point of view was successful in finding a theory fitting in the BPZ framework [DO94, ZZ96, Tes03]. On the other hand, it was unclear that this theory should correspond to the actual path integral written down by Polyakov, let alone the fact that this path integral was not a rigorously defined mathematical object. Only recently was this path integral constructed using a rigorous

probabilistic framework [DKRV16, DRV16, GRV19] and shown to satisfy all the properties predicted by the algebraic formulation [KRV19, KRV20, GKRV20].

The construction of the Liouville path integral relies on Gaussian multiplicative chaos, a theory pioneered by Kahane in the context of turbulence [Kah85], allowing one to exponentiate a logarithmically correlated Gaussian field such as the two-dimensional GFF. The resulting object is a random multifractal measure which has found many applications in modern probability theory [RV14]. In Liouville theory, partition functions and correlation functions are expressed as expected values of observables associated with GMC. The fact that the path integral fits in the BPZ framework has two important consequences. First, the algebraic constraints coming from the BPZ framework give a better understanding of the law of GMC. Second, having a concrete representation of the axiomatic structure allows one to perform additional computations and answer some algebraic questions, such as the convergence of conformal blocks [GKRV20, GRSS20].

Apart from Liouville theory, CFT has a wide scope and is conjectured (in a few cases proved) to describe the scaling limits of many statistical mechanics models at criticality. On the probabilistic side of the story, a major step was performed by Schramm [Sch00] with the introduction of stochastic Loewner evolutions (SLE). He was able to classify all conformally invariant probability measures on paths joining two points on the boundary of a planar domain, therefore describing all possible scaling limits of interfaces of spin clusters of critical models (provided they are conformally invariant in the limit). These measures are indexed by a real parameter $\kappa > 0$ (understood as Planck's constant) and are related to the central charge of the theory (i.e. the universality class of the model considered). Sheffield later showed that SLE is the solution to a problem of conformal welding involving GMC [She16], a deep result which was considerably generalised in [DMS14]. Roughly speaking, Sheffield's result means that Liouville theory is stable under gluing of boundary components, and that the interface curve arising from the gluing is an SLE. Another corollary is the existence of a natural parameterisation of SLE known as the quantum length.

In this thesis, we tried to explore the above mentioned connections between probability, geometry and algebra. In Chapters 2 and 3, we study the asymptotic behaviour of Liouville correlation functions in two specific geometric cases: the once-puncture torus and the four-punctured sphere. This is a purely probabilistic statement, which can be interpreted physically as the factorisation of the partition function on the boundary of the moduli space. The two cases considered constitute the two degeneration paradigms (self-gluing and gluing of disconnected components) and the methods could generalise easily to other moduli spaces of stable curves.

The data of a conformal structure on a surface can be understood as the data of Brownian motion up to reparameterisation. In this context, Liouville Brownian motion

(LBM) is the Brownian motion with the parameterisation induced by GMC viewed as a random conformal factor. The existence of such a process is not clear due to the irregularity of GMC but was carried out in [GRV16, GRV14, RV15, Ber15]. In Chapter 4 we introduce the boundary version of (LBM), which is a Cauchy process reparameterised by GMC. Using Sheffield's result, an interesting consequence is the existence of a diffusion process on SLE, which is a Cauchy process parameterised by quantum length. The subsequent Chapter 5 studies the regularity of the welding homeomorphism of SLE for $\kappa = 4$, which is a critical situation from many points of view.

One advantage of conformal welding is that we can view SLE as a probability measure on the group $\mathrm{Homeo}(\mathbb{S}^1)$ of orientation preserving homeomorphisms of the circle. Thus, it is natural to ask whether SLE is related in a certain sense to the quantisation of $\mathrm{Diff}(\mathbb{S}^1)$. More interestingly, the homogeneous space $\mathbb{S}^1\backslash\mathrm{Diff}(\mathbb{S}^1)$ arises as a coadjoint orbit of $\mathrm{Diff}(\mathbb{S}^1)$ and was shown to possess a two-parametric family of homogeneous Kähler forms [Kir98], for which there is a globally defined potential. Among the various formulae known for this potential, the universal Liouville action of Takhtajan & Teo [TT06] suggests a link between SLE and the geometric quantisation of $\mathbb{S}^1\backslash\mathrm{Diff}(\mathbb{S}^1)$. This connection will be made more precise in an ongoing work described in Section 1.5, where we use the canonical action of $\mathrm{Diff}(\mathbb{S}^1)$ to define a unitary representation of the Virasoro algebra on the L^2-space of SLE endowed with its quantum length. Using conformal welding, this action can be expressed in terms of the so-called universal period mapping of Nag & Sullivan [NS95]. Interestingly, the integration by parts formula from Malliavin calculus can be interpreted in the context of symplectic geometry and the stress-energy tensor emerges in connection with the momentum map for the $\mathrm{Diff}(\mathbb{S}^1)$-action.

L'effort dont j'étais capable, tout l'effort, je l'ai donné en cette œuvre misérable et décousue.

Octave Mirbeau, *Le Calvaire*

Acknowledgements

A number of mathematicians have contributed to shape my mathematical interests, including Gaëtan Borot, Rémi Rhodes, Xin Sun & Vincent Vargas. I address particularly warm thanks to Gaëtan Borot for giving me the opportunity to join his group at HU Berlin as a postdoc.

I am grateful to Juhan Aru, Nathanaël Berestycki, Christopher Bishop, Zhen-Qing Chen, Colin Guillarmou, Ewain Gwynne, Takashi Kumagai, Antti Kupiainen, Eveliina Peltola, Ellen Powell & Eero Saksman for illuminating discussions. I also thank Roland Bauerschmidt & Rémi Rhodes for agreeing to be the examiners of this thesis.

Nathanaël Berestycki deserves a special place in these lines, for the support and counselling he gave me in hard times.

Midway between mathematicians and friends lie the mathematician friends: Simon Becker, Antoine Jego, Baptiste Louf & Thibault Lefeuvre, to name a few. Talking about "real stuff" can become challenging when maths is lurking in the corner.

I have been lucky enough to share a house with Hippolyte P. A. G. Astier & Hugo C. G. Poplimont, *la crème* of housemates who made these Cantabrigian years unforgettable. This crew would of course not be complete without the likes of Romain Alves and Lorena Gazzotti.

Like most of the material in this thesis, the PhD student's clock is multifractal: long periods of void are followed by short, turbulent events. I thank my family and friends[1] for putting up with this unpredictible barometer and supporting me the whole time. Let me just mention the names of Christine & Jean-Louis Moulin and warmly thank them for the roof they put over my head in the midst of the pandemic.

[1] List intentionally left blank.

Contents

Chapter 1

Introduction

This chapter gives some background on various theories that will turn out useful in subsequent chapters. Section 1.1 is an introduction to two-dimensional geometry: Riemann surfaces, uniformisation and Teichmüller theory. In Section 1.2 we move on to $\mathrm{Diff}(\mathbb{S}^1)$ and the universal Teichmüller space, which is an infinite dimensional manifold containing all finite-dimensional Teichmüller spaces. Section 1.3 introduces the probabilistic notions of interest. In Section 1.4 we review various approaches to quantisation and two-dimensional conformal field theory and explain how these approaches are connected to each other.

1.1 Riemann surfaces

> It's called memory cloth. Regularly flexible, but put a current through it... molecules realign, it becomes rigid.
>
> ---
>
> Lucius Fox, *Batman Begins*

In this section we recall the uniformisation theorem of surfaces and give a basic introduction to Teichmüller theory, which can be understood as the space of all Riemann surfaces with a given topology. This material is standard and can be found for instance in [Wol83, Wol85, Mir07a]. We also introduce the Dirichlet energy and the Liouville action, which will play an important role in subsequent probabilistic constructions.

1.1.1 Uniformisation

A *Riemann surface* is a two-dimensional real manifold with holomorphic transition functions; equivalently it is a complex manifold of complex dimension one. The uniformisation theorem states that every simply connected Riemann surface is biholomorphic to either one of the Riemann sphere $\mathbb{S}^2$, the complex plane $\mathbb{C}$ or the unit disc $\mathbb{D}$. In particular, every

Riemann surface admits a Riemannian metric of constant scalar curvature, and the three models above describe positive, flat, and negative curvature respectively.

Every Riemann surface is the quotient of its universal cover by the free, proper and holomorphic action of a discrete group. If the universal cover is the plane, we obtain the tori $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, for τ in the upper-half plane, which are the surfaces of genus 1. The most interesting case is that of surfaces of genus ≥ 2, which are uniformised by the unit disc. Every such surface is the quotient of $\mathbb{D}$ by a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

A *conformal structure* on a (smooth) surface is the data of a Riemannian metric modulo conformal multiplication, i.e. we identify the metrics g and $\widehat{g}$ if there exists $\sigma \in \mathcal{C}^\infty(\Sigma)$ such that $\widehat{g} = e^{2\sigma}g$. From a probabilistic point of view, a conformal structure is the data of Brownian motion modulo time reparameterisation. One remarkable aspect of the uniformisation theorem is the fact that a conformal structure is equivalent to a complex structure. This statement follows from the existence of isothermal coordinates (i.e. coordinates which are conformally equivalent to the standard metric), which can be established by solving the Beltrami equation. Intuitively, this equivalence can be understood as follows. It is known that a complex structure on a surface is equivalent to an *almost complex structure*, which is a smooth section $\mathbf{J}$ of $\mathrm{End}(T\Sigma)$ satisfying $\mathbf{J}^2 = -\mathrm{Id}$ fibrewise. The almost complex structure may be understood as a rotation of angle $\frac{\pi}{2}$ in each tangent space, i.e. it gives the notion of right angles. On the other hand, a conformal structure gives us an inner-product modulo scaling, which precisely allows us to speak of right angles. From this we see that a conformal and a complex structure give the same data.

From uniformisation, every Riemann surface admits a Riemannian metric with constant scalar curvature, which we will refer to as a *geometric structure*. It turns out that each conformal class has a *unique* geometric structure. Under Weyl rescaling $\widehat{g} = e^{2\sigma}g$, the variation of the scalar curvature is given by

$$K_{\widehat{g}} = e^{-2\sigma}(K_g - 2\Delta_g\sigma), \tag{1.1}$$

where $-\Delta_g$ is the Laplace-Beltrami operator. Looking for a geometric structure means solving this equation for $K_{\widehat{g}} = k$ constant, leading to the non-linear PDE

$$\Delta_g\sigma = \frac{1}{2}(ke^{2\sigma} + K_g).$$

This equation is known as the *Liouville equation* and has a unique solution in the conformal class. In genus 1, we obtain the flat structure on the tori. In genus ≥ 2, we obtain the surfaces of constant negative curvature, i.e. the hyperbolic surfaces.

To summarise, every Riemann surface has a unique geometric structure compatible with the complex structure. This unifies two seemingly different but powerful areas of mathematics: complex analysis and Riemannian geometry. This means that there are at least to ways to study a problem dealing with Riemann surfaces, leading to fruitful interactions.

1.1.2 Teichmüller theory

Let Σ be a compact, oriented surface and $\mathrm{Diff}_+(\Sigma)$ be the group of orientation preserving diffeomorphisms of Σ. Let also $\mathrm{Diff}_0(\Sigma)$ be the subgroup of $\mathrm{Diff}_+(\Sigma)$ consisting of those diffeomorphisms which are isotopic to the identity. Recall that two diffeomorphisms ϕ_0, ϕ_1 are *isotopic* if there exists a continuous path $\phi : [0,1] \to \mathrm{Diff}_+(\Sigma)$ such that $\phi(0) = \phi_0$ and $\phi(1) = \phi_1$.

The Teichmüller space $\mathcal{T}_\Sigma$ the space of complex structures on Σ modulo the action of $\mathrm{Diff}_0(\Sigma)$. By the uniformisation theorem, one can also view Teichmüller theory as the study of geometric structures modulo $\mathrm{Diff}_0(\Sigma)$. Thus, there are naturally two approaches to Teichmüller theory, one analytic and one geometric. The case of genus 0 is of no interest since there is only one complex structure on the sphere. The Teichmüller space of a surface of genus 1 is the upper-half plane $\mathbb{H}$, parameterising the tori $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for $\tau \in \mathbb{H}$. In the follwing we assume that the genus is ≥ 2.

1.1.2.1 via complex analysis

A marked Riemann surface is a pair (X, ϕ) where X is a Riemann surface and $\phi : \Sigma \to X$ is a diffeomorphism. We declare two marked Riemann surfaces (X, w) and (X', w') equivalent if $w' \circ w^{-1}$ is isotopic to a biholomorphism. The Teichmüller space $\mathcal{T}_\Sigma$ is the space of equivalence classes of marked Riemann surfaces. A point in $\mathcal{T}_\Sigma$ can be represented by a Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ giving the Riemann surface $\mathbb{H}/\Gamma$.

One way to deform the complex structure on $\mathbb{H}/\Gamma$ is to use quasi-conformal mappings. A homeomorphism w is *quasi-conformal* if it is differentiable almost everywhere and there exists a function μ such that $\mu < 1$ almost everywhere and

$$\frac{\partial w}{\partial \bar{z}} = \mu \frac{\partial w}{\partial z}.$$

This equation is called the *Beltrami equation* and μ is called a *Beltrami differential*, which is a tensor of type $(-1,1)$. More precisely, Beltrami differentials are functions on $\mathbb{H}$ satisfying $\mu(\gamma(z))\frac{\bar{\gamma}'(z)}{\bar{\gamma}(z)} = \mu(z)$ for all $\gamma \in \Gamma$. The space of essentially bounded, Γ-invariant Beltrami differentials is denoted $M(\Gamma)$. It is known that the Beltrami equation has a homeomorphic solution w_μ for each $\mu \in M(\Gamma)_1$, the unit ball of $M(\Gamma)$. The solution is a quasi-conformal homeomorphism of $\mathbb{H}$ and is unique up to Möbius transformations of $\mathbb{H}$. In fact, the solution extends to a homeomorphism of $\bar{\mathbb{H}}$, and the boundary value of w_μ is called a quasi-symmetric homeomorphism of $\mathbb{R}$. A convenient way to normalise the solution is to require that w_μ fixes 0, 1 and ∞. Thus, solving the Beltrami equation gives a new complex structure on Σ, so we have a map $M_1(\Gamma) \to \mathcal{T}_\Sigma$. The Teichmüller space $\mathcal{T}_\Sigma$ inherits a complex structure from $M(\Gamma)$.

Being $(-1,1)$-tensors, Beltrami differentials are dual to tensors of type $(2,0)$, a.k.a. the quadratic differentials. Quadratic differentials satisfy the transformation rule $\phi(\gamma(z))\gamma'(z)^2 = \phi(z)$. The space of integrable holomorphic quadratic differentials is denoted $Q(\Gamma)$, and it is finite dimensional. For a compact surface of genus $\mathbf{g}$, the (complex) dimension is equal to $3\mathbf{g}-3$. Given $\phi \in Q(\Gamma)$ and $\mu \in M(\Gamma)$, the product $\phi\mu$ is a $(1,1)$-form and the natural pairing $(\phi,\mu) \mapsto \int_\Sigma \phi\mu$ is called the *Weil-Petersson pairing*. The kernel of this pairing is denoted $N(\Gamma)$ and the quotient $M(\Gamma)/N(\Gamma)$ is identified with the holomorphic tangent space to $\mathcal{T}_\Sigma$ at the Riemann surface $\mathbb{H}/\Gamma$. Similarly, $Q(\Gamma)$ is the holomorphic cotangent space. The Weil-Petersson pairing introduces a symplectic form ω_{WP} on $\mathcal{T}_\Sigma$, which turns out to be Kähler, turning $\mathcal{T}_\Sigma$ into a Kähler manifold of complex dimension $3\mathbf{g}-3$.

1.1.2.2 via hyperbolic geometry

Now we proceed with the geometric description of $\mathcal{T}_\Sigma$. Let $\mathrm{Met}_{-1}(\Sigma)$ be the space of metrics on Σ with constant scalar curvature -1. The Teichmüller space of Σ is given by $\mathcal{T}_\Sigma = \mathrm{Met}_{-1}(\Sigma)/\mathrm{Diff}_0(\Sigma)$.

There are natural coordinates on $\mathcal{T}_\Sigma$ using hyperbolic geometry, known as the *Fenchel-Nielsen* coordinates. Every homotopy class of a simple closed curve (not isotopic to a point) has a unique geodesic representative. Now, we can pick $3\mathbf{g}-3$ such geodesics $\eta_1, ..., \eta_{3\mathbf{g}-3}$, which are furthermore non-intersecting. Cutting the surface along these geodesics yields a collection of $2\mathbf{g}-2$ hyperbolic pairs pants (a pair of pants is a sphere with three discs removed). The hyperbolic length of η_i is denoted ℓ_i and these lengths constitute half of the coordinates. The other half is given by the Fenchel-Nielsen twist, which is described as follows. Cutting the surface along η_i, we obtain a new hyperbolic surface with two additional boundary circles, which are geodesics of length ℓ_i. Now, rotate one boundary circle by a distance τ_i with respect to the other, and glue the boundaries back together. The hyperbolic metric from the cut surface defines a hyperbolic metric on the glued surface, defining a new point in the Teichmüller space of the original surface. The coordinate functions $(\ell_i, \tau_i)_{1\le i\le 3\mathbf{g}-3}$ are the *Fenchel-Nielsen coordinates*, and $\mathcal{T}_\Sigma$ is isomorphic to $((0,\infty)\times\mathbb{R})^{3\mathbf{g}-3}$ as a smooth manifold. Note that the choice of geodesics is not unique and each such choice gives a system of coordinates. Wolpert gave a remarkable expression for the Weil-Petersson symplectic form in terms of the Fenchel-Nielsen coordinates [Wol85]:

$$\omega_{\mathrm{WP}} = \sum_{i=1}^{3\mathbf{g}-3} \mathrm{d}\ell_i \wedge \mathrm{d}\tau_i.$$

The infinitesimal version of the Fenchel-Nielsen twist gives a vector field on $\mathcal{T}_\Sigma$, which is the Hamiltonian vector field of the hyperbolic length function.

Applying a twist of length $\tau_i = \ell_i$ along the geodesic η_i is called the *Dehn twist* and it is a diffeomorphism not isotopic to the identity. In fact, Dehn twists along simple closed curves generate the group of isotopy classes of diffeomorphisms of Σ, a.k.a. the *mapping class group* Mod_Σ. The *moduli space* is the quotient $\mathcal{M}_\Sigma := \mathcal{T}_\Sigma/\mathrm{Mod}_\Sigma$ and it is a complex orbifold (with singularities arising due to the fixed points of the action). The boundary of $\mathcal{M}_\Sigma$ is described by those surfaces with nodes, which corresponds to shrinking one or more geodesic lengths to 0. A fundamental result is that the Weil-Petersson is Mod_Σ-invariant and descends to a symplectic form on $\mathcal{M}_\Sigma$. The total volume of the associated Liouville volume form is finite. These volumes have been the subject of intense study since the pioneering work of Mirzakhani [Mir07a], in connection with the intersection theory of tautological classes [Mir07b] and the Eynard-Orantin topological recursion [ABO17].

1.1.2.3 via character varieties

For completeness, we include a third approach to Teichmüller theory based on character varieties. This approach is important in order to understand higher Teichmüller theory and the connection of between quantum Liouville theory and the quantisation of moduli spaces of flat connections [TV15].

By the uniformisation theorem, every Riemann surface can be represented as the quotient $\mathbb{H}/\Gamma$ with Γ a certain discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. Homotopy classes of simple closed curves act by isometries, inducing a representation of $\pi_1(\Sigma)$ into $\mathrm{PSL}(2,\mathbb{R})$. On the other hand, representations of the fundamental group of a manifold into a group G are linked to flat G-connections on a G-bundle over that manifold. Indeed, the parallel transport along a curve using a flat connection only depends on the homotopy class of that curve, and holonomies induce a representation of the fundamental group into G.

To make things more precise, introduce the $\mathrm{PSL}(2,\mathbb{C})$-*character variety* of $\pi_1(\Sigma)$:

$$\mathcal{M}_{\mathrm{char}}^{\mathbb{C}} := \mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{C}))/\mathrm{PSL}(2,\mathbb{C}),$$

where the action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{PSL}(2,\mathbb{C}))$ is by conjugation. Then $\mathcal{M}_{\mathrm{char}}^{\mathbb{C}}$ contains a real slice $\mathcal{M}_{\mathrm{char}}^{\mathbb{R}}$, which is made of finitely many connected components. The connected component to the identity, $\mathcal{M}_{\mathrm{char}}^{\mathbb{R},0}$, contains all Fuchsian representations and is identified with the Teichmüller space of Σ. In summary, $\mathcal{T}_\Sigma$ is a connected component of the moduli space of flat $\mathrm{PSL}(2,\mathbb{C})$-connections on Σ.

With this new characterisation of Teichmüller space, one can forget about complex structures and replace $\mathrm{PSL}(2,\mathbb{C})$ with other algebraic groups (say $\mathrm{PSL}(n,\mathbb{C})$). This leads to the so-called *higher Teichmüller theory* [FG06]. This explains the link between Teichmüller spaces and moduli spaces of flat connections.

1.1.3 Dirichlet energy

Let Σ be a Riemann surface with a compatible Riemannian metric g. We assume for a moment that $\partial\Sigma = \varnothing$ Denote by vol_g the associated volume form and $-\Delta_g$ the (non-negative) Laplace Beltrami operator. The Sobolev space $H^1(\Sigma)$ is the subspace of $L^2(\Sigma, \mathrm{vol}_g)$ of functions with (weak) partial derivatives in $L^2(\Sigma, \mathrm{vol}_g)$. The homogeneous space $\dot{H}^1(\Sigma) = H^1(\Sigma)/\mathbb{R}$ is endowed with the following norm called the *Dirichlet energy*:

$$\|f\|^2_{\dot{H}^1(\Sigma)} = \frac{1}{2\pi}\int_\Sigma |\mathrm{d}f|^2_g \mathrm{vol}_g.$$

Here, d denotes the de Rham differential and $|\mathrm{d}f|^2_g$ is the norm of $\mathrm{d}f$ under the identification of $T\Sigma$ with $T^*\Sigma$ given by the metric.

The Dirichlet energy is *conformally invariant*. Given a biholomorphism $\psi : \widetilde{\Sigma} \to \Sigma$, we have $\|f \circ \psi\|_{\dot{H}^1(\widetilde{\Sigma})} = \|f\|_{\dot{H}^1(\Sigma)}$. In other words, precomposition with ψ is a unitary map from $\dot{H}^1(\Sigma)$ to $\dot{H}^1(\widetilde{\Sigma})$. This conformal invariance also justifies why we do not make explicit the dependence of $\dot{H}^1(\Sigma)$ on the metric, since this space only depends on the conformal structure of the surface, and not on the precise value of the conformal factor.

Probabilistically, the Dirichlet energy is the action defining a conformally invariant Gaussian process known as the Gaussian free field (see Section 1.3.1). It turns out that a more natural object to consider is a random field transforming not as a function but as conformal factor. For that matter, one considers the *classical Liouville action*, which is the Dirichlet energy perturbed by a linear term: $\mathbf{S}_L^{\mathrm{cl}}(\sigma; g) := \frac{1}{2\pi}\int_\Sigma (|\mathrm{d}\sigma|^2_g + K_g\sigma)\mathrm{vol}_g$. Geometrically, the minimiser σ_0 of the Liouville action (if it exists) is a conformal factor such that the metric $e^{2\sigma_0}g$ is flat. For other values of the curvature, the Liouville action gets an extra non-linear term, which is the starting point of the probabilistic approach to Liouville CFT (see Section 1.4.4). Anticipating on the quantum theory, we need a further modification of this action. Given $\gamma > 0$ (understood as Planck's constant), one defines

$$\mathbf{S}_L(\sigma; g) := \frac{1}{2\pi}\int_\Sigma \left(|\mathrm{d}\sigma|^2_g + QK_g\sigma\right)\mathrm{vol}_g,$$

where $Q := \frac{2}{\gamma} + \frac{\gamma}{2}$. The term $\frac{\gamma}{2}$ in the definition of Q is small as $\gamma \to 0$ and is known as a "quantum correction". The classical Liouville action is recovered when omitting the quantum correction and taking $\gamma = 2$.

Suppose now that Σ has a non-empty (smooth) boundary homeomorphic to a collection of disjoint circles and consider the subspace $H^1_0(\Sigma) \subset H^1(\Sigma)$ of functions with zero boundary conditions. Otherwise stated, $H^1_0(\Sigma)$ is the Hilbert space completion (with respect to the Dirichlet inner-product) of the space of smooth, compactly supported functions in Σ. We have the well-known orthogonal decomposition

$$\dot{H}^1(\Sigma) = H_0^1(\Sigma) \oplus \mathcal{H} \tag{1.2}$$

where $\mathcal{H}$ is the subspace of $\dot{H}^1(\Sigma)$ of harmonic functions in Σ.

Sobolev functions have a trace on $\partial\Sigma$. The trace operator "loses" 1/2-point of regularity, so that the trace on each boundary circle gives an element of $\dot{H}^{1/2}(\mathbb{S}^1)$. The converse operation consists in harmonic extension, which identifies $\dot{H}^{1/2}(\partial\Sigma)$ with $\mathcal{H}$. Green's formula also tells us that $\|u\|_{\dot{H}^{1/2}(\partial\mathbb{D})} = \frac{1}{2\pi}\int_{\partial\mathbb{D}} u\partial_n u \mathrm{d}\theta$, where ∂_n is the outer normal derivative (w.r.t. the orientation on the Riemann surface Σ) and $\mathrm{d}\theta$ is arclength induced by the metric g.

The Liouville action also gets an extra curvature term in this case:

$$\mathbf{S}_L(\sigma; g) = \frac{1}{2\pi}\int_\Sigma \left(|\mathrm{d}\sigma|_g^2 + QK_g\sigma\right)\mathrm{vol}_g + \frac{1}{\pi}\int_{\partial\Sigma} Qk_g\sigma\mathrm{d}\theta,$$

where k_g denotes the geodesic curvature of $\partial\Sigma$ in the metric g.

The case of the unit disc. One special case of interest is $\Sigma = \mathbb{D}$ with the Euclidean metric (so that $k_g \equiv 1$). The boundary term of the Liouville action on $\mathbb{D}$ is denoted by

$$\mathbf{S}_{\partial\mathbb{D}}(u) = \frac{1}{2\pi}\int_{\mathbb{S}^1}(u\partial_n u + 2Qu)\mathrm{d}\theta, \qquad u \in H^{1/2}(\partial\mathbb{D}). \tag{1.3}$$

Identifying u with its harmonic extension to $\mathbb{D}$, we have $\mathbf{S}_{\partial\mathbb{D}}(u) = \|u\|^2_{\dot{H}^{1/2}(\partial\mathbb{D})} + 2Qu(0)$ by the mean value principle. A similar action $\mathbf{S}_{\partial\mathbb{D}^*}$ can be written in $\mathbb{D}^*$. Their classical counterpart ($Q = 1$) is denoted $\mathbf{S}^{\mathrm{cl}}_{\partial\mathbb{D}}, \mathbf{S}^{\mathrm{cl}}_{\partial\mathbb{D}^*}$.

The space $\dot{H}^{1/2}(\partial\mathbb{D})$ is not just a Hilbert space but carries additional structures. The *Hilbert transform* $\mathbf{H}$ associates to each $u \in \dot{H}^{1/2}(\partial\mathbb{D})$ its harmonic conjugate, i.e. $u + i\mathbf{H}u$ is the boundary value of a holomorphic function in $\mathbb{D}$. Since the Hilbert transform is unique up to constant, we have a well-defined operator $\mathbf{H} : \dot{H}^{1/2}(\partial\mathbb{D}) \to \dot{H}^{1/2}(\partial\mathbb{D})$, which is actually unitary. We also have $\mathbf{H}^2 = -\mathrm{Id}$, i.e. $\mathbf{H}$ is a complex structure. This also gives rise to a symplectic structure Θ defined by $\langle u, v\rangle_{\dot{H}^{1/2}(\partial\mathbb{D})} = \Theta(u, \mathbf{H}v)$. Due to the Cauchy-Riemann equations, we have the simple expression

$$\Theta(u, v) = \frac{1}{2\pi}\int u\mathrm{d}v. \tag{1.4}$$

1.2 $\mathrm{Diff}(\mathbb{S}^1)$ and the universal Teichmüller space

> Here the boundaries meet and all contradictions exist side by side.
>
> Fyodor Dostoevsky, *The brothers Karamazov*

This section introduces the group of orientation preserving diffeomorphisms of the unit circle and its Lie algebra, the space of smooth vector fields on the circle (a.k.a. the Witt algebra). In physics, this group is understood as the group of reparameterisations of the closed string and is therefore central in string theory. The central extension of the Witt algebra is known as the Virasoro algebra and is the symmetry algebra of conformal field theory, so that much of CFT can be understood in terms of its representation theory. Roughly speaking, one could understand CFT as harmonic analysis on $\mathrm{Diff}(\mathbb{S}^1)$.

The group $\mathrm{Diff}(\mathbb{S}^1)$ (modulo Möbius) lies inside the universal Teichmüller space, which is the group of quasi-symmetric homeomorphisms of the unit circle (modulo Möbius), $T(1) := \mathrm{M\ddot{o}b}(\mathbb{S}^1)\backslash \mathrm{QS}(\mathbb{S}^1)$. These homeomorphisms arise as boundary values of the quasi-conformal mappings of $\mathbb{D}$ introduced in Section 1.1.2.1. Via the Beltrami equation, finite dimensional Teichmüller spaces appear as subspaces of $T(1)$ satisfying the desired covariance under the action of a given Fuchsian group, which justifies the terminology "universal". The Weil-Petersson class is a subgroup $T_0(1)$ of $T(1)$ which has been extensively studied by [TT06]. It has a rich geometry of infinite dimensional Kähler manifold.

1.2.1 The Virasoro algebra

1.2.1.1 Definition

The *Witt algebra* $\mathrm{Vect}(\mathbb{S}^1)$ is the Lie algebra of smooth vector fields on the unit circle, i.e. it is the Lie algebra of the group $\mathrm{Diff}(\mathbb{S}^1)$ of orientation preserving diffeomorphisms of the unit circle. Its complexification $\mathrm{Vect}_{\mathbb{C}}(\mathbb{S}^1)$ has generators $\ell_n = ie^{ni\theta}\partial_\theta$, with commutation relations

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}, \qquad n, m \in \mathbb{Z}.$$

Recall that a *two-cocycle* on a Lie algebra $\mathfrak{g}$ is a skew-symmetric pairing ω satisfying the condition

$$\omega(u, [v, w]) + \omega(v, [w, u]) + \omega(w, [u, v]) = 0, \qquad u, v, w \in \mathfrak{g}.$$

Cocycles are important objects in the theory of Lie algebras since they enable one to construct the central extension $\mathfrak{g} \oplus \mathbb{R}\mathbf{c}$, where $\mathbf{c}$ is central and the bracket is otherwise given by $[u, v]_{\text{new}} = [u, v]_{\text{old}} + \omega(u, v)$ for $u, v \in \mathfrak{g}$. The cocycle condition ensures that the new bracket satisfies the Jacobi identity, thus defining an honest Lie bracket.

On the Witt algebra, there is a two-parameter family of cocycles [KY88]:

$$\omega_{a,b}(\ell_n, \ell_m) = (an^3 + bn)\delta_{n,-m}. \tag{1.5}$$

The canonical choice is $a = -b = \frac{1}{12}$, known as the *Virasoro cocycle* ω. In terms of the vector fields, it corresponds to

$$\omega(u\partial_\theta, v\partial_\theta) = -\frac{1}{12}\int_0^{2\pi} u(v' + v''')\frac{\mathrm{d}\theta}{2\pi}.$$

The cocycle can be used to define a one-dimensional central extension of $\mathrm{Vect}_{\mathbb{C}}(\mathbb{S}^1)$, known as the *Virasoro algebra*. It has a central element $\mathbf{c} = c\mathrm{Id}$ and other generators $(L_n)_{n\in\mathbb{Z}}$ with the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}.$$

The complex number c is called the *central charge* and we write $\mathfrak{V}_c$ for the Virasoro algebra of central charge c. It has two subalgebras $\mathfrak{V}_c^+$, $\mathfrak{V}_c^-$ generated by the positive and negative modes respectively.

1.2.1.2 Highest-weight representations

The representation theory of the Virasoro is well-known and we recall some important features. We refer to [Rib14] for a simple presentation of this material and we will also remind the reader of some basic terminology from the theory of Lie algebras. A representation is called *irreducible* if it contains no non-trivial subrepresentation. A *highest-weight vector* (or primary state in physics) of weight $h \in \mathbb{C}$ is a vector v such that $L_n v = 0$ for all $n > 0$ and $L_0 v = hv$. Such a vector generates a *highest-weight module* $\mathcal{V}_{c,h} := \mathcal{U}(\mathfrak{V}_c)v$, where $\mathcal{U}(\cdot)$ denotes the universal enveloping algebra. That is, $\mathcal{V}_{c,h}$ is generated by vectors of the form $L_{-\lambda_p} \cdots L_{-\lambda_2}L_{-\lambda_1}v$ for all integer partitions $\lambda = \lambda_1 \geq \lambda_2 \cdots \lambda_p \geq 1$. These vectors are the *descendants* of v and their *level* is the number partitioned by λ. A simple computation shows that a descendant vector of weight n is an eigenvector of L_0 with eigenvalue $h + n$.

A *Verma module* is a heighest-weight module which is linearly isomorphic to $\mathcal{U}(\mathfrak{V}_c^-)$. A highest-weight representation which is not a Verma module is called *degenerate*. Degenerate representations arise when they contain a vector generating its own representation. Such a vector is then called *singular* or degenerate. It is a simple exercise to look for degenerate vectors at level 2: such a vector must be of the form $aL_{-2} + bL_{-1}^2$ and be annihilated by $\mathfrak{V}_c^+$. The vanishing condition must only be checked for L_1 and L_2, and a repeated application of the Virasoro bracket leads to

$$\begin{cases} 3a + (4h+2)b = 0 \\ (4h + \dfrac{c}{2})a + 6b = 0. \end{cases}$$

The determinant of this system of linear equations is a second degree polynomial in h

$$\Delta = 4(2h+1)^2 + (c-13)(2h+1) + 9. \tag{1.6}$$

Thus, (non-trivial) singular vectors at level 2 exist if and only if $\Delta = 0$, i.e.

$$h = \frac{5 - c \pm \sqrt{(c-25)(c-1)}}{16}.$$

It is customary in CFT to use the "Liouville parameterisation" of the central charge $c = 1 + 6Q^2$ and the highest weight $h = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$, where $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and γ is *a priori* an arbitrary complex number[1]. The parameters Q and α are usually called the *background charge* and *Liouville momentum* respectively. Notice the ambiguity in the definition of α under $\alpha \leftrightarrow 2Q - \alpha$. Using this parameterisation, a module has a singular vector at level $r + s - 1$ for the value

$$\alpha_{r,s} = Q - r\frac{\gamma}{2} - s\frac{2}{\gamma},$$

where r, s are arbitrary positive integers. In particular, a singular vector at level 2 corresponds to $(r,s) = (2,1)$ or $(1,2)$, corresponding to the two roots of (1.6). The operators associated to these values are

$$L_{-2} + \frac{4}{\gamma^2} L_{-1}^2; \qquad L_{-2} + \frac{\gamma^2}{4} L_{-1}^2. \tag{1.7}$$

In physics, it is assumed that representations are irreducible, which means that singular vectors must vanish. That is, the above operators must annihilate the highest-weight vector. This is known as the *BPZ equation.*

To conclude this introduction to the representation theory of the Virasoro algebra, we mention the notion of unitarity, which is necessary for a CFT to have a physical (hence probabilistic) meaning. A highest-weight module is *unitary* if it carries a positive definite Hermitian form such that $L_{-n} = L_n^\dagger$ for all $n \in \mathbb{Z}$ and the central element is self-adjoint. This last condition immediately implies that the central charge must be real, $c \in \mathbb{R}$. Unitarity conditions for highest-weight representations are given thanks to the *Kac determinant formula.* Most importantly, one consequence of this formula is that Verma modules are unitary for $c > 1$. There is a discrete set of values of the central charge in

[1]In physics, one would have $\gamma = 2b$)

the regime $c < 1$ for which some degenerate modules are unitary. They correspond to $c = 1 - \frac{6}{m(m+1)}$ for positive integers m and are known as the *discrete series*. We refer to [Rib14, Equation (2.1.36)] for more details on unitarity conditions.

1.2.2 Kähler geometry of $\mathrm{Diff}(\mathbb{S}^1)$

Given a Lie group G with Lie algebra $\mathfrak{g}$, the *coadjoint representation* is an action of G on the dual Lie algebra $\mathfrak{g}^*$. Infinitesimally, this action is given by the map[2]

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g}^* &\to \mathfrak{g}^* \\ (x, \lambda) &\mapsto \mathrm{ad}_x^* \lambda = -\lambda([x, \cdot\,]). \end{aligned}$$

This expression actually endows the codjoint orbits with an invariant symplectic structure. Thus, one may think of a coadjoint orbit as a classical phase space and this raises the question of whether it can be subject to the framework of geometric quantision as introduced in Section 1.4.2. One outcome of quantisation being a unitary irreducible representation of the group (or a central extension) on the quantum Hilbert space, this suggests a link between coadjoint orbits and irreducible unitary representations. This approach to irreducible unitary representations is known as *Kirillov's orbit method* and gives an actual equivalence between the two objects (orbits and representations) in some special cases (e.g. nilpotent groups).

In the 1980's, Kirillov [KY87] initiated the study of coadjoint orbits of $\mathrm{Diff}(\mathbb{S}^1)$ (see also [Wit88] for an account) and found two such orbits carrying additionally a Kähler structure: these are $\mathbb{S}^1\backslash\mathrm{Diff}(\mathbb{S}^1)$ and $\mathrm{Möb}(\mathbb{S}^1)\backslash\mathrm{Diff}(\mathbb{S}^1)$. The first (resp. second) of these manifolds is the space of diffeomorphisms modulo rotations (resp. Möbius transformations of the disc), identified with the group of diffeomorphisms fixing 1 (resp. $1, -1, i$)[3]. In fact, $\mathbb{S}^1\backslash\mathrm{Diff}(\mathbb{S}^1)$ can be realised as a one-dimensional complex fibre bundle over $\mathrm{Möb}(\mathbb{S}^1)\backslash\mathrm{Diff}(\mathbb{S}^1)$. On $\mathbb{S}^1\backslash\mathrm{Diff}(\mathbb{S}^1)$, there is a two-parameter family of symplectic structures. Due to their invariance, they are determined by their value at the identity, which is nothing but the cocycles $\omega_{a,b}$ of (1.5). This cocycle vanishes on the span of $\ell_{-1}, \ell_0, \ell_1$ if and only if $a = -b$, giving the unique (up to scaling) symplectic structure on $\mathrm{Möb}(\mathbb{S}^1)\backslash\mathrm{Diff}(\mathbb{S}^1)$.

We now turn to the complex structure on these manifolds, which can be described using conformal welding. Let $\eta : \mathbb{S}^1 \to \widehat{\mathbb{C}}$ be a smooth Jordan curve bounding complementary Jordand domains $D \ni 0$ and D^*. By Riemann uniformisation, we can fix a univalent map

[2]The group G acts on itslef by conjugation, $(g, h) \mapsto ghg^{-1}$. The adjoint map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the differential at the identity in the second variable. Differentiating in g at the identity gives $\mathrm{ad}_x = [x, \cdot\,]$, the expression following e.g. from the Baker-Campbell-Hausdorff formula. The dual is $\mathrm{ad}_x^*\lambda := -\lambda(\mathrm{ad}_x(\cdot))$.

[3]The two subgroups $\mathbb{S}^1$ and $\mathrm{Möb}(\mathbb{S}^1)$ are generated by the Virasoro generators ℓ_0 and $\ell_{-1}, \ell_0, \ell_1$ respectively. Other orbits include the quotient of $\mathrm{Diff}(\mathbb{S}^1)$ by the group generated by $\ell_{-n}, \ell_0, \ell_n$ but do not carry a natural Kähler structure. It has been suggested that their quantisation should rather be connected to the discrete series [Wit88]

$f : \mathbb{D} \to D$ (resp. $g : \mathbb{D}^* \to D^*$) fixing 0 (resp. ∞). On $\mathbb{S}^1$, the function $g^{-1} \circ f$ defines an element of $\mathrm{Diff}(\mathbb{S}^1)$. Conversely, given $h \in \mathrm{Diff}(\mathbb{S}^1)$, there always exist univalent functions f, g as above and satisfying $g^{-1} \circ f = h$. These functions are not unique but the ambiguity is resolved as follows [Kir98, TT06]. Given $h \in \mathbb{S}^1 \backslash \mathrm{Diff}(\mathbb{S}^1)$ there exists a unique pair of univalent functions f, g on $\mathbb{D}, \mathbb{D}^*$ such that

1. $g^{-1} \circ f = h$ on $\mathbb{S}^1$.

2. $f(0) = 0$ and $f'(0) = 1$.

3. $g(\infty) = \infty$.

The normalisation $f'(0) = 1$ means that the domain $f(\mathbb{D})$ has unit conformal radius viewed from 0.

Thus, the conformal welding procedure identifies $\mathbb{S}^1 \backslash \mathrm{Diff}(\mathbb{S}^1)$ with the space

$$\mathcal{M}_\infty := \{f \text{ univalent function on } \mathbb{D},\ f(0) = 0,\ f'(0) = 1\}$$

This manifold sits embedded in the complex vector space of holomorphic functions on $\mathbb{D}$, endowing it with a complex structure. Moreover, this complex structure is $\mathrm{Diff}(\mathbb{S}^1)$-invariant and compatible with the symplectic structure [Kir98], so that $\mathbb{S}^1 \backslash \mathrm{Diff}(\mathbb{S}^1)$ is a Kähler manifold. Since it is invariant, the almost complex structure is determined by its value at the identity, where it is given by the Hilbert transform already introduced in Section 1.1.3. Namely, the tangent space to the identity of $\mathbb{S}^1 \backslash \mathrm{Diff}(\mathbb{S}^1)$ is the space of smooth vector fields $v(\theta)\partial_\theta$ with vanishing mean, and $\mathbf{H}$ is characterised by

$$\mathbf{H}(\cos(n\theta)\partial_\theta) = -\sin(n\theta)\partial_\theta; \qquad \mathbf{H}(\sin(n\theta)\partial_\theta) = \cos(n\theta)\partial_\theta, \qquad n \geq 1.$$

The same expression (for $n > 1$) defines the complex structure on $\mathrm{M\ddot{o}b}(\mathbb{S}^1) \backslash \mathrm{Diff}(\mathbb{S}^1)$. Writing $v(\theta) = \sum_{n=2}^\infty v_n e^{ni\theta} + \bar{v}_n e^{-in\theta}$, the Weil-Petersson metric is then given by

$$\|\mathbf{v}\|_{\mathrm{WP}}^2 = \omega(\mathbf{v}, \mathbf{H}\mathbf{v}) = \frac{1}{12} \sum_{n=2}^\infty (n^3 - n)|v_n|^2.$$

The Weil-Petersson metric is the unique (up to scaling) invariant Kähler metric on $\mathbb{S}^1 \backslash \mathrm{Diff}(\mathbb{S}^1)$, and it is degenerate in the fibres of $\mathbb{S}^1 \backslash \mathrm{Diff}(\mathbb{S}^1) \to \mathrm{M\ddot{o}b}(\mathbb{S}^1) \backslash \mathrm{Diff}(\mathbb{S}^1)$. On the other hand, the cocycles $\omega_{a,b}$ for $a + b \neq 0$ define non-degenerate metrics on $\mathbb{S}^1 \backslash \mathrm{Diff}(\mathbb{S}^1)$. For $a = 0$, the metric is sometimes called the *Veling-Kirillov metric* [TT06].

Although $\mathcal{M}_\infty$ has a complicated geometry, it is topologically trivial. Indeed, the expression $f_t(z) = \frac{1}{t} f(tz)$, $t \in [0, 1]$ provides a homotopy from $\mathrm{Id}_{\mathcal{M}_\infty}$ to the constant map $f \mapsto \mathrm{Id}_\mathbb{D}$. Therefore, one can expect to get a globally defined Kähler potential for the Weil-Petersson metric. Indeed, various potentials have been exhibited, both for the

Veling-Kirillov [KY87] and Weil-Petersson metrics [KY88, NS95]. However, we will be particularly interested in a more recent potential found by Takhtajan & Teo, the *universal Liouville action* [TT06]:[4]

$$\mathbf{S}(h)=\mathbf{S}^{\mathrm{cl}}_{\mathbb{D}}(\log|f'|)+\mathbf{S}^{\mathrm{cl}}_{\partial\mathbb{D}^*}(\log|g'|),\qquad h=g^{-1}\circ f\in\mathbb{S}^1\backslash\mathrm{Diff}(\mathbb{S}^1).\tag{1.8}$$

1.2.3 Universal Teichmüller space and the Weil-Petersson class

This section gives a brief introduction to the universal Teichmüller space $T(1)$. A more detailed survey can be found in [Pek95]. We also introduce the Weil-Petersson class as studied extensively by Takhtajan & Teo [TT06].

As seen in Section 1.1, the Teichmüller theory of a surface $\Gamma\backslash\mathbb{D}$ can be described by Γ-invariant Beltrami differentials in the unit ball of $L^\infty(\mathbb{D})$. The universal Teichmüller space is what we obtain when we drop the requirement of Γ-invariance. Solving the Beltrami equation in $\mathbb{D}$ for such a Beltrami differential $\mu\in L^\infty(\mathbb{D})_1$ gives a quasi-conformal homeomorphism w_μ of $\mathbb{D}$. Boundary values of such homeomorphisms are the *quasi-symmetric homeomorphisms* $\mathrm{QS}(\mathbb{S}^1)$, characterised by the property that there exists $C>0$ such that for all θ,t,

$$C^{-1}\leq\left|\frac{h(e^{i(\theta+t)})-h(e^{i\theta})}{h(e^{i\theta})-h(e^{i(\theta-t)})}\right|\leq C.$$

An equivalence relation on $L^\infty(\mathbb{D})_1$ is introduced by declaring $\mu\sim\nu$ if the boundary values of w_μ and w_ν coincide up to a Möbius transformation, i.e. μ and ν induce the same homeomorphism of $\mathbb{S}^1$. The *universal Teichmüller space* is then defined to be $T(1):=L^\infty(\mathbb{D})/\sim$, so that

$$T(1)\simeq\mathrm{Möb}(\mathbb{S}^1)\backslash\mathrm{QS}(\mathbb{S}^1).$$

In particular, $\mathrm{Möb}(\mathbb{S}^1)\backslash\mathrm{Diff}(\mathbb{S}^1)$ embeds into $T(1)$ since smooth homeomorphisms are quasi-symmetric.

As for $\mathrm{Diff}(\mathbb{S}^1)$, quasi-symmetric homeomorphisms have a representation in terms of conformal welding, with the same procedure applying verbatim. This result is sometimes called the fundamental theorem of conformal welding. Jordan curves arising from the welding of quasi-symmetries are known as *quasi-circles* and are typically wild, fractal curves. Away from $\mathrm{QS}(\mathbb{S}^1)$, there is no general theory about existence and uniqueness of a welding curve associated with a homeomorphism. In this respect, Sheffield's quantum zipper theorem [She16] is a striking example of existence and uniqueness (for $\kappa<4$) of conformal welding.

One remarkable property of quasi-symmetries is that they act on $\dot{H}^{1/2}(\partial\mathbb{D})$ by symplectomorphisms, i.e. they preserve the symplectic form Θ introduced in Section 1.1.3:

[4]The definition in [TT06] differs by a constant of 2π

given $h \in T(1)$, we have [NS95]

$$\Theta(u \circ h, v \circ h) = \Theta(u, v), \qquad u, v \in \dot{H}^{1/2}(\partial\mathbb{D}), \tag{1.9}$$

where it is implicitely assumed that we view $u \circ h, v \circ h$ up to constant. Furthermore, $T(1)$ is the largest possible subgroup of $\mathrm{Homeo}(\mathbb{S}^1)$ with this property.

The tangent space to the identity of $T(1)$ can be viewed alternatively as a space of Beltrami differential or of vector fields on the circle. The latter space is known as the Zygmund class and is a rather complicated function space. Still, it is known that the Weil-Petersson metric does not converge on the Zygmund class since (1.8) implies that the metric only converges on $H^{3/2}$-regular vector fields. In fact, $H^{3/2}$-vector fields define a subspace of $T_0T(1)$, inducing an invariant subbundle of the tangent bundle by right translations. This subbundle is integrable and the Weil-Petersson class $T_0(1) \subset T(1)$ is obtained by taking the integral manifold containing the identity [TT06]. The Weil-Petersson class gives a conceptually satisfying picture in which $T_0(1)$ is the "completion" of $\mathrm{Möb}(\mathbb{S}^1)\backslash\mathrm{Diff}(\mathbb{S}^1)$ with respect to the Weil-Petersson metric.

Jordan curves arising as the conformal welding of a Weil-Petersson homeomorphism are called *Weil-Petersson quasi-circles*. Contrary to generic quasi-circles, they are rather nice curves: they have Hausdorff dimension 1 and can be parameterised by arclength. We also have the remarkable intrinsic characterisation of Weil-Petersson homeomorphisms [ST20].

Proposition 1.2.1. *A homeomorphism h is the Weil-Petersson class if and only if h is absolutely continuous and* $\log h' \in \dot{H}^{1/2}$.

1.3 Probabilistic background

Ce qui m'intéresse en ce moment, c'est d'échapper à la mécanique, de savoir si l'inévitable peut avoir une issue.

Albert Camus, *L'Étranger*

1.3.1 White-noise

1.3.1.1 Definitions

Let $(H, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space, with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$. The *isonormal Gaussian process* (or *white noise*) based on H is the centred Gaussian process

$(\xi_u)_{u\in H}$ indexed by H satisfying Itô's isometry:

$$\mathbb{E}[\xi_u\xi_v] = \langle u, v\rangle, \qquad u, v \in H. \tag{1.10}$$

Here, we assumed that ξ is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denoted by $\mathbb{E}$ the associated expectation. This process is easily constructed using a sequence $(\xi_n)_{n\in\mathbb{N}}$ of i.i.d. normal random variables. Set $\xi_{e_n} := \xi_n$ for all $n \in \mathbb{N}$ and extend the process to the whole of H by linearity of $u \mapsto \xi_u$. Conversely, the correlation structure (1.10) implies that $(\xi_{e_n})_{n\in\mathbb{N}}$ are i.i.d. normals and it is an easy exercise to check that the map $u \mapsto \xi_u$ must be linear. The linear span of the random variables $\{\xi_n\}$ is often called a *Gaussian Hilbert space*, with the inner-product obtained from Itô's isometry.

Note that for all $u \in H$, ξ_u is almost surely finite. On the other hand, ξ is almost surely an unbounded linear form on H, since $\limsup \xi_n = \infty$ almost surely as $n \to \infty$. Often however, it is possible to realise ξ as a random distribution. That is, we look for a dense subspace $V \subset H$ (with a finer topology than H) such that the formal sum

$$\xi = \sum_n \xi_n e_n$$

almost surely converges in V', the dual of V. The space V is interpreted as a space of test functions, while V' is a space of distributions. Such a triple $V \subset H \subset V'$ is called a *Gelfand triple*.

The Gaussian nature of white-noise has two important consequences. The first one is *symmetry*. For all unitary transformations $U : H \to H$,

$$(\xi_{Uu})_{u\in H} \overset{\text{law}}{=} (\xi_u)_{u\in H},$$

as follows from $\mathbb{E}[\xi_{Uu}\xi_{Uv}] = \mathbb{E}[\xi_u\xi_v]$. The second one is the *Markov property*. Suppose H_0 is a closed subspace of H. Then we can sample ξ as the independent pair (ξ^0, ϕ), where ξ^0 is a white-noise on H_0 and ϕ is a white-noise $H_0^\perp$.

Let T be a non-negative, self-adjoint operator with dense domain $D(T) \subset H$. Assume that $k_0 := \dim\ker T < \infty$ and that T has a compact resolvent. The positive spectrum of T is written $0 < \lambda_1 \leq \lambda_2 \ldots$. The operator T is positive on $H_0 := (\ker T)^\perp$ and we fix a basis of $(e_n)_{n\geq 1}$ of H_0 such that $Te_n = \lambda_n e_n$ and the e_n's are orthonormal in H. The normalised Gaussian field X associated with T is the centred Gaussian process X indexed by $D(T) \cap H_0$ with covariance kernel $\mathbb{E}[X_u X_v] = \langle u, Tv\rangle$. Formally, $X \overset{\text{law}}{=} \sum_{n=1}^\infty \frac{\alpha_n}{\sqrt{\lambda_n}} e_n$, where α_n are i.i.d. normal random variables. We see that this sum converges in H if and only if $\sum_{n=1}^\infty \lambda_n < \infty$. The unnormalised process is obtained by tensorising with Lebesgue measure on $\ker T$ (so we get an infinite measure).

There is a convenient way to sample X using the heat kernel of T. Given a countable family of independent standard Brownian motions $((B_t^{(n)})_{t\geq 0})_{n\geq 1}$, we set

$$X := \int_0^\infty e^{-\frac{t}{2}T}\sum_{n=1}^\infty e_n \mathrm{d}B_t^{(n)} = \sum_{n=1}^\infty \int_0^\infty e^{-\frac{t}{2}\lambda_n}\mathrm{d}B_t^{(n)} e_n.$$

Indeed, we have that for all $n\geq 1$, $X_{e_n} \overset{\text{law}}{=} \int_0^\infty e^{-\frac{t}{2}\lambda_n}\mathrm{d}B_t^{(n)}$, which is a centred Gaussian of variance $\int_0^\infty e^{-t\lambda_n}\mathrm{d}t = \lambda_n^{-1}$. Moreover, the X_{e_n}'s are independent. If X does not converge H, truncating the above integral at a small parameter $\varepsilon>0$ gives a regularisation $(X_\varepsilon)_{\varepsilon>0}$ of X by elements of H. Morally, we are applying the heat kernel $e^{-\frac{t}{2}T}$ to "infinitesimal white-noises" $\sum_{n=1}^\infty e_n\mathrm{d}B_t^{(n)}$ white-noises, which is why this procedure is usually called the *white-noise regularisation*. The interesting aspect of this regularisation is that the different bits of the integral have independent contributions, so that $(X_\varepsilon)_{\varepsilon>0}$ is a martingale.

1.3.1.2 Gaussian free field

In the context of Section 1.3.1.1, the Gaussian free field (GFF) on Σ is the Gaussian process obtained by taking the Hilbert space $H = L^2(\Sigma, \mathrm{vol}_g)$ and operator $T = -\frac{1}{2\pi}\Delta_g$. By Green's formula, we may also think of the GFF as white-noise on $\dot{H}^1(\Sigma)$. The law of this process is conformally invariant due to the conformal invariance of the Dirichlet energy. In the presence of the boundary, one can take the Laplacian with either Dirichlet or Neumann boundary conditions, leading to different boundary conditions. By the Markov property, a GFF with Neumann (or *free*) boundary conditions is the independent sum of a GFF with Dirichlet boundary conditions and a random harmonic function whose covariance kernel is the resolvent of the Dirichlet-to-Neumann operator.

Suppose that $\partial\Sigma = \varnothing$ and let us exhibit a space of distributions in which the GFF almost surely converges. Let $(e_n)_{n\geq 1}$ be an orthonormal basis of $\dot{H}^1(\Sigma)$ consisting of eigenfunctions of $-\frac{1}{2\pi}\Delta_g$, i.e. $-\frac{1}{2\pi}\Delta_g e_n = \lambda_n e_n$. The GFF is then the formal sum $X = \sum_{n\geq 1}\xi_n e_n$, where $(\xi_n)_{n\geq 1}$ are i.i.d. normal random variables. By Weyl's law, we have $\lambda_n \sim n$ as $n\to\infty$, so that Green's formula yields

$$\|X\|^2_{H^s(\Sigma)} = \sum_{n\geq 1} n^{-s}\frac{\xi_n^2}{\lambda_n}.$$

Now, we have almost surely $\xi_n^2 = o(n^\varepsilon)$ for all $\varepsilon>0$, so that $\|X\|_{H^s(\Sigma)}<\infty$ if $s<0$. On the other hand, $\sum_{n\geq 1}\frac{\xi_n^2}{n} = \infty$ almost surely, so that $X\notin L^2(\Sigma,\mathrm{vol}_g)$. In particular, the GFF is almost surely *not* a function and has to be understood as a distribution.

By definition, the correlation kernel of the GFF is given by Itô's isometry

$$\mathbb{E}[\langle X, f\rangle_{\dot{H}^1(\Sigma)}\langle X, g\rangle_{\dot{H}^1(\Sigma)}] = \langle f, g\rangle_{\dot{H}^1(\Sigma)}, \qquad f,g\in\mathcal{C}^\infty(\Sigma).$$

Formally, this implies that the two-point covariance is the resolvent of the Laplacian:

$$\mathbb{E}[X(x)X(y)] = 2\pi(-\Delta_g)^{-1}(x,y) = \log\frac{1}{\mathrm{dist}_g(x,y)} + O(1),$$

where $O(1)$ is uniformly bounded in (x,y). This means that the GFF falls in the scope of logarithmically correlated fields. These fields appear in several areas of probability theory such as random matrix theory. For the purpose of this thesis, we will stick to the GFF, but several results hold in wider generality.

Since the GFF is not defined pointwise, it is important to be able to regularise it. We record two such regularisation procedures.

Circle averages: Given $x \in \Sigma$ and $\varepsilon > 0$ sufficiently small, we let $C_g(x,\varepsilon)$ be the geodesic circle about $x \in \Sigma$ of ε, and $\theta_{x,\varepsilon}$ the uniform measure on $C_g(x,\varepsilon)$. This measure belongs to $H^{-1}(\Sigma)$, so that $X_\varepsilon(x) := \langle X, \theta_{x,\varepsilon}\rangle_{L^2(\Sigma)}$ is a.s. finite. Taking countable dense collections and applying Kolmogorov's criterion yields a jointly Hölder continuous process $(X_\varepsilon(x))_{x\in\Sigma,\varepsilon>0}$ [HMP10]. Moreover, by the Markov property of the GFF, for all (x,ε_0) such that $X_{\varepsilon_0}(x)$ is defined, the process $(X_{e^{-t}\varepsilon_0}(x))_{t\geq 0}$ is a Brownian motion started at $X_{\varepsilon_0}(x)$. Finally, for $x,y \in \Sigma$ distinct and ε_0 such that $C_g(x,\varepsilon_0)\cap C_g(y,\varepsilon_0) = \varnothing$, the Brownian motions $(X_{e^{-t}\varepsilon_0}(x) - X_{\varepsilon_0}(x))_{t\geq 0}$ and $(X_{e^{-t}\varepsilon_0}(y) - X_{\varepsilon_0}(y))_{t\geq 0}$ are independent.

White-noise regularisation: One can also implement the white-noise regularisation of the previous section, which takes the form

$$X_\varepsilon(x) = \int_\varepsilon^\infty p_{t/2}(x,y)\mathrm{d}\xi(t,y),$$

where $p_t(x,y)$ is the heat kernel and ξ is a space-time white-noise on $\Sigma\times(0,\infty)$. Again, the main advantage of this regularisation is its martingale structure, making some arguments about convergence simpler or trivial.

The two regularisation procedures that we have just described are only two widely used examples. Other possible examples include the convolution by a bump function. The key property is that we obtain fields X_ε a.s. converging (in the sense of distributions) to the GFF. We will always assume that $\mathbb{E}[X_\varepsilon^2(x)] = \log\frac{1}{\varepsilon} + O(1)$ as $\varepsilon \to 0$, for all $x \in \Sigma$. For concreteness, we assume in the sequel that we are working with circle averages.

To conclude this introduction to the GFF, we introduce the notion of *thick points*. For each $x \in \Sigma$, the process $X_{e^{-t}}(x)$ behaves like a Brownian motion, so that almost surely $\frac{X_\varepsilon(x)}{\log\frac{1}{\varepsilon}} \to 0$ as $\varepsilon \to 0$. However, we have $\mathbb{P}(X_\varepsilon(x) \geq \gamma\log\frac{1}{\varepsilon}) \asymp \varepsilon^{\frac{\gamma^2}{2}}$, suggesting that the almost sure Hausdorff dimension of the set

$$\mathcal{T}_\gamma := \left\{x\in\Sigma,\ \lim_{\varepsilon\to 0}\frac{X_\varepsilon(x)}{\log\frac{1}{\varepsilon}} \geq \gamma\right\}$$

is equal to $(2-\frac{\gamma^2}{2})_+$ [HMP10]. Moreover, $\mathcal{T}_\gamma$ is a.s. empty for $\gamma > 2$, but uncountably infinite for $\gamma = 2$.

1.3.1.3 Malliavin calculus and Gaussian integration by parts

Malliavin derivative. We briefly recall how to construct the Malliavin derivative, which is a densely defined operator on $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{S}$ be the collection of random variables F such that there exists $N \in \mathbb{N}$ and a smooth function $f : \mathbb{R}^N \to \mathbb{R}$ such that f and its derivatives have at most polynomial growth at ∞ and

$$F := f(\xi_1, \xi_2, ..., \xi_N) \in L^2(\Omega, \mathcal{F}, \mathbb{P}). \tag{1.11}$$

Such random variables are called *smooth random variables*. The interest of smooth random variables is that (i) they bridge the gap between the original space H and the abstract space Ω; (ii) they are dense in $L^2(\mathbb{P})$, so they form a sufficiently large set of test functions in order that $\mathbb{P}$ be determined by its value on them.

Remark 1. Smooth random variables are analogous to the Schwartz space. They form a locally convex vector space with the topology induced by the countable family of seminorms $|\cdot|_{\alpha,\beta,N}$ defined as follows. Let α, β be multi-indices, $N \in \mathbb{N}$ and define the semi-norm

$$|f|_{\alpha,\beta,N} := \sup_{x\in\mathbb{R}^{2N}} |x^\alpha \partial_\beta f(x)| e^{-\frac{|x|^2}{2}}.$$

By the identification of (1.11), this formula defines a semi-norm on $\mathcal{S}$. The topological dual $\mathcal{S}'$ of $\mathcal{S}$ plays the role of tempered distributions, and we will adopt this terminology. For example, if X is the GFF and $z \in \Sigma$ is some point, the formal expression $X^2(z)$ does not define a tempered distribution, but the normally ordered expression $X^2(z) - \mathbb{E}[X^2(z)]$ does.

The *Malliavin derivative* of F is the H-valued random variable

$$\mathrm{d}F := \sum_{k=1}^N \partial_k f e_k.$$

This operator is closable and the domain of its closure is denoted $\mathcal{W}^{1,2} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$. The Malliavin derivative is simply a formulation of the differential in infinite dimensions. Indeed, writing elements of H as $x = \sum_n \xi_n e_n$, we have the tangent vectors fields $\frac{\partial}{\partial \xi_n}$ and differential forms $\mathrm{d}\xi_n = \langle \frac{\partial}{\partial \xi_n}, \cdot \rangle$, which are obtained by parallel transporting e_n. For $F \in \mathcal{W}^{1,2}$, we have

$$\mathrm{d}F = \sum_n \frac{\partial F}{\partial \xi_n} \mathrm{d}\xi_n.$$

An important result from Malliavin calculus is the integration by parts formula. Recall the Gaussian integration by parts in one dimension:

$$\int_{\mathbb{R}} f'(x)e^{-\frac{x^2}{2}}\frac{\mathrm{d}x}{\sqrt{2\pi}} = \int_{\mathbb{R}} xf(x)e^{-\frac{x^2}{2}}\frac{\mathrm{d}x}{\sqrt{2\pi}},$$

or, if $X \sim \mathcal{N}(0,1)$,

$$\mathbb{E}[f'(X)] = \mathbb{E}[Xf(X)], \tag{1.12}$$

Bootstrapping from the finite dimensional case, we have for all $F \in \mathcal{W}^{1,2}$ and $u \in H$,

$$\mathbb{E}\left[\langle \mathrm{d}F, u\rangle\right] = \mathbb{E}\left[F\xi_u\right]. \tag{1.13}$$

Parallel transporting, we may view $\langle \mathrm{d}F, u\rangle$ as the covariant derivative of F in the direction of the vector field u, and the integration by parts formula computes the adjoint of this derivative (which is a densely defined operator on $L^2(\mathbb{P})$).

There is another way to arrive at this integration by parts formula using Girsanov's theorem (i.e. a change of variables). Let us consider again the one-dimensional case. If $X \sim \mathcal{N}(0,1)$, we have for all $u \in \mathbb{R}$, by a change of variables

$$\begin{aligned}\mathbb{E}[f(X+u)] = \int_{\mathbb{R}} f(x+u)e^{-\frac{x^2}{2}}\frac{\mathrm{d}x}{\sqrt{2\pi}} &= \int_{\mathbb{R}} f(x)e^{-\frac{(x-u)^2}{2}}\frac{\mathrm{d}x}{2\pi} \\ &= \int_{\mathbb{R}} e^{ux-\frac{u^2}{2}}f(x)e^{-\frac{x^2}{2}}\frac{\mathrm{d}x}{\sqrt{2\pi}} \\ &= \mathbb{E}[e^{uX-\frac{u^2}{2}}f(X)].\end{aligned}$$

Taking the derivative of this expression at $u = 0$, one recovers (1.12). From here one easily deduces (1.13) in the infinite dimensional setting using smooth random variables.

Viewing H as a Lie group, the integration by parts formula describes how the Gaussian measure transforms under the infinitesimal action of H on itself. Although there is no translation invariant (or Haar) measure on the non locally compact group H, the Gaussian measure can be characterised as the unique measure satisfying the transformation rule (1.13) under the infinitesimal action of H.

Complex Hilbert space. Formula (1.13) can be generalised to non-constant vector fields in the case of a complex Hilbert space. We assume that there is a linear complex structure $\mathbf{J}$ on H (i.e. $\mathbf{J}^2 = -\mathrm{Id}$) compatible with $\langle\cdot,\cdot\rangle$, and denote by $\Theta(u,v) = \langle \mathbf{J}u, v\rangle$ the corresponding symplectic form. Some background on symplectic geometry is given in Section 1.4.2.

Fix a basis $(e_n)_{n\in\mathbb{Z}\setminus\{0\}}$ of H with linear coordinates $u = \sum_{n=1}^{\infty} x_n e_n + y_n e_{-n}$ such that $\Theta = \sum_{n=1}^{\infty} \mathrm{d}x_n \wedge \mathrm{d}y_n$, i.e. these coordinates are Darboux. The subspace $H_N \simeq \mathbb{R}^{2N}$ spanned by $\{e_1, e_{-1}, \cdots e_N, e_{-N}\}$ is a symplectic subspace of H for each $N \geq 1$, and the

Liouville volume form is Lebesgue measure. Let $f, g \in \mathcal{S}$ identified with functions on $\mathbb{R}^{2N}$ for some sufficiently large N. Since H_N is symplectic, there is a Poisson bracket $\{f, g\}$, and this bracket does not depend on the choice of N. Applying the integration by parts formula (1.26), we have

$$\int_{\mathbb{R}^{2N}} \{f, g\} e^{-\rho} \mathrm{d}x_1 \mathrm{d}y_1 \cdots \mathrm{d}x_N \mathrm{d}y_N = \int_{\mathbb{R}^{2N}} g\{f, \rho\} e^{-\rho} \mathrm{d}x_1 \mathrm{d}y_1 \cdots \mathrm{d}x_N \mathrm{d}y_N,$$

where we have denoted $\rho = \frac{|\cdot|^2}{2}$. By density, the Poisson bracket extends uniquely to a continuous bilinear form $\{\cdot, \cdot\} : \mathcal{W}^{1,2} \times \mathcal{W}^{1,2} \to L^1(\mathbb{P})$, and the integration by parts formula reads

$$\mathbb{E}[\{f, g\}] = \mathbb{E}[g, \{f, \rho\}], \qquad f, g \in \mathcal{W}^{1,2}. \tag{1.14}$$

Although $\rho = \infty$ almost surely, the bracket $\{f, \rho\}$ still makes sense as an element of $\mathcal{W}^{-1,2}$, the dual of $\mathcal{W}^{1,2}$ with respect to the $L^2(\mathbb{P})$ inner-product. Formula (1.14) is a generalisation of (1.13) to all square-integrable Hamiltonian vector fields. One recovers (1.13) by specialising f to coordinate functions. There is also a similar extension $\{\cdot, \cdot\} : \mathcal{S}' \times \mathcal{S} \to \mathcal{S}'$ and the same integration by parts formula.

1.3.2 Gaussian multiplicative chaos

In this section we introduce Gaussian multiplicative chaos, which plays a major role in all subsequent chapters. Here, the point is to state its main properties and convey some intuition and heuristic justifications, detailed studies appearing in the later chapters. Standard references for this material are [Kah85, Ber17, RV14, Ber16].

1.3.2.1 Main construction

Let $\gamma \in (0, 2)$. Since the GFF is a.s. not defined pointwise, the quantity $e^{\gamma X}$ is *a priori* ill-defined. Consider a regularisation of the GFF as in the previous section, and note that $\mathbb{E}[e^{\gamma X_\varepsilon(x)}] = e^{\frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon^2(x)]} \asymp \varepsilon^{-\frac{\gamma^2}{2}}$ as $\varepsilon \to 0$. This suggests to introduce the regularised measure

$$\mathrm{d}M_\varepsilon^\gamma(x) = \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\varepsilon(x)} \mathrm{vol}_g(x). \tag{1.15}$$

The question is whether this family of measures converges as $\varepsilon \to 0$ (and for which topology), and if so whether the limit is trivial and/or depends on the choice of regularisation.

These questions can be answered by the theory of Gaussian multiplicative (GMC) pioneered by Kahane [Kah85]. Typically, the fields that are targetted by this theory are the $\star$-*scale invariant* fields in $\mathbb{R}^d$ (see [RV14] for an account on these fields). These are fields having the correlation kernels

$$K(x,y)=\int_0^1 k\left(\frac{x-y}{u}\right)\frac{\mathrm{d}u}{u},$$

where the function k satisfies $k(0)=1$. Under suitable assumptions on k, the kernel K is of σ-positive type and there is indeed a Gaussian process with this correlation kernel. Letting K_ε be the kernel obtained by cutting the integral at $\varepsilon>0$, we get a family of smooth Gaussian processes $(X_\varepsilon)_{\varepsilon>0}$ approximating X. This regularisation exhibits the same martingale structure as the white-noise regularisation of the GFF.

Given $\gamma>0$, one introduces the measures

$$\mathrm{d}M_\varepsilon^\gamma(x)=e^{\gamma X_\varepsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon^2(x)]}\mathrm{d}\sigma(x),$$

where σ is a reference measure on $\mathbb{R}^d$. In Kahane's theory, σ may be singular with respect to Lebesgue measure (e.g. supported on a fractal subset of $\mathbb{R}^d$), but we will stick to σ =Lebesgue for simplicity. The sequence $(M_\varepsilon^\gamma)_{\varepsilon>0}$ forms a measure-valued martingale and the martingale convergence theorem implies that M_ε^γ a.s. converges as $\varepsilon\to 0$ to a random measure M^γ on $\mathbb{R}^d$. The main result of Kahane's theory is

Theorem 1.3.1. *The measure M^γ is non-trivial if and only if $\gamma<\sqrt{2d}$. For $\gamma\geq\sqrt{2d}$, we have $M^\gamma=0$.*

Outside the realm of $\star$-scale invariant fields, we don't have the martingale structure and the convergence of the measures is not granted. This leads to the questions of existence and universality of GMC. Namely, which class of fields have a well-defined GMC? Given two distinct regularisations of the same field (e.g. white-noise, convolution etc.), do we get the same measure in the limit? For the second question, a rather flexible framework has been proposed in [JS17].

The question of existence has been addressed for a large class of log-correlated fields, with Berestycki's approach [Ber17] now being the standard reference. In general, one obtains the convergence *in probability* of the measures for $\gamma\in(0,\sqrt{2d})$. The proof is considerably simpler in the case $\gamma<\sqrt{d}$, which is known as the "L^2-phase". Indeed, for all Borel sets $f\in\mathcal{C}_c^0(\mathbb{R}^d)$, we have by a standard Gaussian computation

$$\begin{aligned}\mathbb{E}\left[M_\varepsilon^\gamma(f)^2\right]&\leq C\varepsilon^{\gamma^2}\int\mathbb{E}\left[e^{\gamma(X_\varepsilon(x)+X_\varepsilon(y))}\right]f(x)f(y)\mathrm{d}x\mathrm{d}y\\&\leq C\int\frac{f(x)f(y)}{|x-y|^{\gamma^2}}\mathrm{d}x\mathrm{d}y<\infty,\end{aligned}$$

so that we get a uniform bound for the second moment of $M_\varepsilon^\gamma(S)$. In fact, one can similarly show that $M_\varepsilon^\gamma(S)$ is not only bounded in $L^2(\mathbb{P})$ but also Cauchy. From here, a standard argument using a countable dense collection of functions allows one to establish the weak convergence of M_ε^γ. The above argument fails in the case $\gamma \in [\sqrt{d}, \sqrt{2d})$ since the second moment is infinite, but one can still show the convergence with some extra work. This case is both more difficult and more enlightening since it forces one to understand where the measure "lives". It turns out that M^γ assigns full mass to the γ-thick point of the field, and the reason why the second moment blows up is because of the points that are "thicker" than γ but otherwise have zero L^1-contribution (since they are assigned zero mass). This is why $\gamma \in [\sqrt{d}, \sqrt{2d})$ is sometimes referred to as the "L^1-phase".

Remark 2. In the special case of the two-dimensional GFF with the circle-average regularisation, Duplantier & Sheffield independently rediscovered this result [DS11b], and they called the limiting measure the *Liouville measure* due to its connection with Liouville CFT. Their measure is not exactly the same as GMC since it is more naturally defined as a tensor. However, we will stick to the terminology GMC due to the multiple occurrences of Liouville's name. To avoid confusions, we mention that the term "Liouville measure" is used by Kupiainen, Rhodes & Vargas for the path integral of Liouville quantum gravity, and the term "Liouville volume form" is the canonical volume form in symplectic geometry.

1.3.2.2 Basic properties

Multifractal properties. GMC is a multifractal measure that is almost surely singular with respect to Lebesgue measure. In this paragraph, we state some of its multifractal properties. We give heuristic justifications for these results and refer to [RV14] for formal proofs. Since this is not specific to the two-dimensional setting, we take M^γ to be the GMC of a d-dimensional log-correlated field.

The first main property is that it gives full mass to the γ-thick points:

$$\mathbb{P}\left(M^\gamma(\Sigma \setminus \mathcal{T}_\gamma) = 0\right) = 1. \tag{1.16}$$

The fact that $\dim(\mathcal{T}_\gamma) \to 0$ as $\gamma \to \sqrt{2d}$ gives a heuristic justification for the $\gamma = \sqrt{2d}$ threshold: there are no more points to support the measure past this value. A quick way to justify (1.16) is to observe that by Girsanov's theorem, we have for all $\alpha > 0$

$$\begin{aligned}
&\mathbb{E}\left[M_\varepsilon^\gamma\left(\left\{x \in \Sigma,\, |X_\varepsilon(x) - \gamma \log \frac{1}{\varepsilon}| > \alpha \log \frac{1}{\varepsilon}\right\}\right)\right] \\
&= \int_\Sigma \mathbb{E}\left[\varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\varepsilon(x)} \mathbb{1}_{\{|X_\varepsilon(x) - \gamma \log \frac{1}{\varepsilon}| > \alpha \log \frac{1}{\varepsilon}\}}\right] |\mathrm{d}x|^2 \\
&= \int_\Sigma \mathbb{P}\left(|X_\varepsilon(x)| > \alpha \log \frac{1}{\varepsilon}\right) \\
&= O(\varepsilon^{\frac{\alpha^2}{2}}) = o(1).
\end{aligned}$$

The fact that M^γ lives on $\mathcal{T}_\gamma$ is already a sufficient reason to study GMC: this measure encodes interesting properties of the field. Knowing that the measure should eventually live on $\mathcal{T}_\gamma$, it is easy to adapt the L^2-construction of the measure to the L^1-phase: introducing the event $G_\varepsilon^\alpha(x) := \{X_\varepsilon(x) - \alpha \log \frac{1}{\varepsilon} \leq 0\}$, our aim is to estimate for all $f \in \mathcal{C}_c^0(\mathbb{R}^d)$

$$\mathbb{E}\left[M_\varepsilon^\gamma\left(f \mathbb{1}_{G_\varepsilon^\alpha}\right)^2\right] = \varepsilon^{\gamma^2} \int f(x)f(y)\mathbb{E}\left[e^{\gamma(X_\varepsilon(x)+X_\varepsilon(y)}\mathbb{1}_{G_\varepsilon^\alpha(x)}\mathbb{1}_{G_\varepsilon^\alpha(y)}\right] \mathrm{d}x\mathrm{d}y$$

For $x, y \in \mathbb{R}^d$ fixed, the two processes $X_{e^{-t}}(x)$ and $X_{e^{-t}}(y)$ are strongly correlated for $t < \log \frac{1}{\varepsilon}$ and "almost" independent for $t > \log \frac{1}{\varepsilon}$. Thus, a good approximation for the joint process $(X_{e^{-t}}(x), X_{e^{-t}(y)})_{t\geq 0}$ is a single Brownian motion branching at $t = \log \frac{1}{\varepsilon}$. Using this, the Markov property of Brownian motion and the Girsanov transform, one gets uniformly in ε

$$\varepsilon^{\gamma^2}\mathbb{E}\left[e^{\gamma(X_\varepsilon(x)+X_\varepsilon(y))}\mathbb{1}_{G_\varepsilon^\alpha(x)}\mathbb{1}_{G_\varepsilon^\alpha(y)}\right] \lesssim |x-y|^{\frac{1}{2}(\alpha-2\gamma)^2-\gamma^2}.$$

Taking α sufficiently close to γ from above, we get

$$\mathbb{E}\left[M_\varepsilon^\gamma\left(f \mathbb{1}_{G_\varepsilon^\alpha}\right)^2\right] \leq C \int \frac{f(x)f(y)}{|x-y|^{\frac{\gamma^2}{2}-0}} \mathrm{d}x\mathrm{d}y,$$

which is finite provided $\gamma < \sqrt{2d}$. This puts us in the situation of the L^2-phase for the measure $\mathbb{1}_{G_\varepsilon^\alpha} M_\varepsilon^\gamma$, so that we have a weak limit in probability as $\varepsilon \to 0$. On the other hand, $M_\varepsilon^\gamma - \mathbb{1}_{G_\varepsilon^\alpha} M_\varepsilon^\gamma$ converges weakly to 0, so that M_ε^γ converges to the same limit.

GMC measures have a heavy tail at ∞: for a bounded Borel set $A \subset \Sigma$, we have for all $p \in \mathbb{R}$,

$$\mathbb{E}[M^\gamma(A)^p] < \infty \Leftrightarrow p < \frac{\sqrt{2d}}{\gamma^2}.$$

In particular, the second moment diverges as soon as $\gamma \geq \sqrt{d}$, and in general the law of $M^\gamma(A)$ is heavy tailed near ∞. On the contrary, the existence of all negative moments implies that $M^\gamma(A)$ is unlikely to be very close to 0.

The multifractal spectrum of M^γ is the study of the moments $M^\gamma(B_g(x,\varepsilon))$, where $B_g(x,\varepsilon)$ denotes the ball centred at x of small radius $\varepsilon > 0$. Suppose for a moment that the field is defined in a simply connected domain $D \subset \mathbb{R}^2$ and it exactly logarithmically correlated, i.e. $\mathbb{E}[X(x)X(y)] = \log \frac{1}{|x-y|}$. Then we have the exact scale invariance

$$X(\varepsilon \cdot) \overset{\text{law}}{=} X + \Lambda_\varepsilon,$$

where Λ_ε is a normal random variable independent of X. Using this identity in law and by a change of variable, we have

$$M^\gamma(B(x,\varepsilon)) \overset{\text{law}}{=} \varepsilon^{\gamma Q} e^{\gamma \Lambda_\varepsilon} M^\gamma(B(x,1)),$$

where we have set $Q := \frac{\gamma}{2} + \frac{d}{\gamma}$. Hence, for all $p < \frac{\sqrt{2d}}{\gamma^2}$, we have

$$\mathbb{E}[M^\gamma(B(x,\varepsilon))^p] \asymp \varepsilon^{\gamma Q p} \mathbb{E}\left[e^{\gamma p \Lambda_\varepsilon}\right] \asymp \varepsilon^{\gamma Q p - \frac{\gamma^2 p^2}{2}} = \varepsilon^{\zeta(p)}.$$

The function

$$\zeta(p) = \gamma Q p - \frac{\gamma^2 p^2}{2}$$

(defined on $(-\infty, \frac{\sqrt{2d}}{\gamma^2})$) is called the *multifractal spectrum* of M^γ.

We say that the *local dimension* of M^γ at $x \in \Sigma$ is equal to $\delta > 0$ if $M^\gamma(B_g(x,\varepsilon)) \asymp \varepsilon^\delta$ as $\varepsilon \to 0$. In multifractal analysis, one aims to compute the Hausdorff dimension of the set of points $G_\delta \subset \Sigma$ where the local dimension is δ. This is known as the *spectrum of singularities* of the measure. This question is intuitively connected to that of the thick points of the GFF. Indeed, a good proxy is to use

$$M^\gamma(B_g(x,\varepsilon)) \sim \mathrm{vol}_g(B_g(x,\varepsilon)) \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\varepsilon(x)} \sim \varepsilon^{\gamma Q} e^{\gamma X_\varepsilon(x)}.$$

Then we have by a standard Gaussian computation

$$\mathbb{P}\left(M^\gamma(B_g(x,\varepsilon) \asymp \varepsilon^\delta\right) \asymp \mathbb{P}\left(X_\varepsilon(x) \sim (\frac{\delta}{\gamma} - Q)\log\frac{1}{\varepsilon}\right) \asymp \varepsilon^{\frac{1}{2}(Q-\frac{\delta}{\gamma})^2}.$$

This rough estimate can be made precise and we have that almost surely [RV14],

$$\dim G_\delta = d - \frac{1}{2}\left(Q - \frac{\delta}{\gamma}\right)^2, \qquad \delta \in [\gamma(Q-\sqrt{2d}), \gamma(Q+\sqrt{2d})].$$

Frisch-Parisi formula. Another aspect of multifractal analysis is the connection between the multifractal spectrum and the Besov regularity of the measure, known as the *Frisch-Parisi formula* (see [Jaf00] for a review of the validity of this formula in a wider context). The study of the Besov regularity of GMC has been initiated in [JSV19]. Besov spaces $B^s_{p,q}(\mathbb{R}^d)$ are spaces of functions/distributions indexed by $s \in \mathbb{R}$ and $p, q \in [1,\infty]$. They are generalisations of Sobolev and Hölder-Zygmund spaces, for instance $B^s_{2,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ and $B^s_{\infty,\infty}(\mathbb{R}^d) = \mathcal{C}^s(\mathbb{R}^d)$.

Their definition is more easily understood with the help of wavelets. One starts with "mother wavelets" $(\psi^{(i)})_{1 \leq i \leq 2^d - 1}$ such that the functions

$$x \mapsto 2^{dj/2}\psi^{(i)}(2^j x - k), \qquad j \in \mathbb{Z}, k \in \mathbb{Z}^d$$

form an orthonormal basis of $L^2(\mathbb{R}^d)$. It is convenient to index the descendant wavelets by dyadic cubes of the form $\lambda = 2^{-j}(k + [0,1]^d)$ and define $\psi_\lambda^{(i)}(x) := \psi(2^j x - k)$. The wavelet coefficients of a distribution f are then

$$c_\lambda^{(i)} := 2^{dj} \int_{\mathbb{R}^d} \psi_\lambda^{(i)} f |\mathrm{d}x|^d,$$

and we also introduce

$$A_j := \left(\sum_{k \in \mathbb{Z}^d} \left| c_\lambda^{(i)} 2^{(s-\frac{d}{p})j} \right|^p \right)^{1/p}.$$

Then by definition, $f \in B_{p,q}^s(\mathbb{R}^d)$ if and only if the sequence (A_j) belongs to ℓ^q.

This characterisation of Besov spaces gives an intuitive way of computing Besov regularity in terms of the spectrum of singularities, by decomposing A_j with respect to this spectrum. For concreteness, consider the case of a GMC measure M^γ on $[0,1]^d$. Roughly speaking, the support of the wavelet $\psi_\lambda^{(i)}$ concentrates on the cube λ so we use the approximation $c_\lambda^{(i)} \approx 2^{dj} M^\gamma(\lambda)$. On the other hand, the number of cubes λ where $M^\gamma(\lambda) \asymp |\lambda|^\delta$ is of order $|\lambda|^{-\dim G_\delta}$, where $|\lambda|$ denotes the sidelength of λ. Hence, the contribution to A_j of those points where the local dimension is δ is of order $2^{s-\frac{d}{p}} \times 2^{j(\frac{\dim G_\delta}{p} - \delta)}$. Thus, the main contribution to A_j is given by the points which maximise the quantity $\dim G_\delta - \delta p$, over δ in the spectrum of singularities. In other words,

$$A_j \sim 2^{s-\frac{d}{p}} \times 2^{j\frac{\eta(p)}{p}}, \tag{1.17}$$

where

$$\eta(p) := \sup_\delta \dim G_\delta - \delta p$$

is the Legendre transform of $\dim G_\delta$, with δ ranging in the support of the spectrum of singularities. In particular, $(A_j) \in \ell^q$ if and only if $s - \frac{d}{p} + \frac{\eta(p)}{p} < 0$, expressing the Besov regularity of the measure in terms of the Legendre transform of the spectrum of singularities. Using the expression for $\dim G_\delta$, one finds that the maximiser δ^* for the Legendre transform satisfies

$$\frac{\delta^*}{\gamma} = Q - \gamma \min(p, \frac{\sqrt{2d}}{\gamma}).$$

Note that for $p \geq \frac{\sqrt{2d}}{\gamma}$, we have $\delta^* = \gamma(Q - \sqrt{2d})$, which is the lower-bound of the support of the spectrum of singularities. Thus, for all $p, q \in [1, \infty]$, the threshold s^* at which $M^\gamma \in B_{p,q}^s$ is given by $s^* = \gamma Q - \frac{\gamma^2 p}{2}$ for $p \leq \frac{\sqrt{2d}}{\gamma}$ and $s^* = \frac{d}{p} + Q - \sqrt{2d}$ if $p \geq \frac{\sqrt{2d}}{\gamma}$. That is, $M^\gamma \in B_{p,q}^s$ for $s < s^*$ and $M^\gamma \notin B_{p,q}^s$ for $s > s^*$ (the case $s = s^*$ not being treated).

Singularities. Another important topic is what kind of singularities can M^γ integrate, namely we look at the integral

$$\int_{B(0,1)} \frac{\mathrm{d}M^\gamma(x)}{|x|^{\gamma\alpha}} \tag{1.18}$$

and ask about the finiteness of its moments for a given $\alpha > 0$. A detailed study is given in [KRV20, Section 3]. To study the local behaviour of M^γ near the singularity, one writes the field as $X(x) = X_{|x|}(x) + Y(x)$, where $X_r(x)$ is the average of X on the sphere of radius r centred at 0 and Y. This is sometimes called the *radial decomposition* of the field; it appears in several places in the literature and in Chapters 2 and 3. It turns out that $X_{|\cdot|}$ provides a reasonably good approximation of X near the singularity. Since $X_{e^{-t}}(x)$ is a Brownian motion, what we obtain for (1.18) when ignoring Y is Yor's exponential functional of Brownian motion [MY05]

$$Z \overset{\text{law}}{=} \int_0^\infty e^{\gamma(B_t-(Q-\alpha)t)}\mathrm{d}t.$$

Obviously this integral a.s. does not converge for $\alpha \geq Q$, so we assume $\alpha < Q$.

Many exact results are known for Z, and it is easy to evaluate the finiteness of its moments using Laplace's method. The distribution of the maximum of drifted Brownian motion is well known

$$S := \sup_{t\geq 0} B_t - (Q-\alpha)t \overset{\text{law}}{=} \mathrm{Exp}(2(Q-\alpha)),$$

where Exp is an exponential random variable. By the Laplace principle, this integral is intuitively dominated by $e^{\gamma S}$. It is possible to justify this intuition using Williams' path decomposition of drifted Brownian motion (see [KRV20, Lemma 3.1]), and one obtains $\mathbb{E}[Z^p] < \infty \Leftrightarrow \mathbb{E}[S^{\gamma p}] < \infty$. In conclusion, one has the following condition of integrability of singularities for M^γ:

$$\mathbb{E}\left[\left(\int_{B(0,1)} \frac{\mathrm{d}M^\gamma(x)}{|x|^{\gamma\alpha}}\right)^p\right] < \infty \Leftrightarrow p < \frac{2}{\gamma}(Q-\alpha).$$

The case $\alpha = Q$ is critical (therefore interesting) and corresponds to a Brownian motion with no drift. Although the integral does not converge, it is possible to describe in detail the behaviour of the GMC mass with a small ball removed. This will be studied in more details in Chapters 2 and 3.

Conformal covariance. We return to the setting where M^γ is the exponential of the two-dimensional GFF. The GFF possesses the additional property of conformal invariance, which is also reflected in its exponential. To be more precise, under a change of coordinate charts, we have

$$\mathrm{d}M^\gamma(w) = \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^{\gamma Q} \mathrm{d}M^\gamma(z).$$

That is, M^γ almost surely varies like a $(\frac{\gamma Q}{2}, \frac{\gamma Q}{2})$-differential. In other words, it is a section of $K^{\frac{\gamma Q}{2}}$, where $K = \Omega^{1,1}(\Sigma)$ is the bundle of $(1,1)$-forms. We will give an idea of the reason why this is true, a detailed proof can be found in [Ber16]. The change of coordinate is a transition function, i.e. a conformal map $f : \widetilde{D} \to D$ between two simply connected domains of $\mathbb{C}$. Given $x \in \widetilde{D}$, we have that $f(C(x,\varepsilon))$ is roughly $C(f(x), |f'(x)|)$. Thus, if X is the GFF in D, we have $(X \circ f)_\varepsilon(z) \simeq X_{\frac{\varepsilon}{|f'(z)|}}(z)$. If M_ε^γ is the regularised GMC in D, we then have

$$\begin{aligned}
\mathrm{d}(f^* M_\varepsilon^\gamma)(z) &= \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma (X\circ f)_\varepsilon(z)} |f'(z)\mathrm{d}z|^2 \\
&\simeq |f'(z)|^{2+\frac{\gamma^2}{2}} \left(\frac{\varepsilon}{|f'(z)|}\right)^{\frac{\gamma^2}{2}} e^{\gamma X_{\varepsilon/|f'(z)|}(f(z))} |\mathrm{d}z|^2 \\
&= |f'(z)|^{\gamma Q} \mathrm{d}\widetilde{M}^\gamma_{\varepsilon/|f'(z)|}(z),
\end{aligned}$$

where $\widetilde{M}^\gamma$ is the GMC in $\widetilde{D}$. From this we infer that $\mathrm{d}(f^* M^\gamma) = |f'|^{\gamma Q} \mathrm{d}\widetilde{M}^\gamma$ as required. This argument is however not a proof for many reasons, starting from the fact that $|f'|$ varies with z.

The fact that $M^\gamma = e^{\gamma X} |\mathrm{d}z|^2$ varies like a $(\frac{\gamma Q}{2}, \frac{\gamma Q}{2})$-tensor implies that X does not vary trivially. It does not even vary like a usual conformal factor, but rather like a *Q-conformal factor*, i.e. under a change of coordinates $w = w(z)$, we have

$$X(z) = X(w) + Q \log\left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|. \tag{1.19}$$

Equivalently, under a Weyl rescaling of the metric $\widehat{g} = e^{2\sigma} g$, the new Liouville field $\widehat{X}$ is

$$\widehat{X} = X + Q\sigma. \tag{1.20}$$

Thus, the Liouville field is really an affine (non-centred) Gaussian field. Its defining action is not the Dirichlet energy but the *Liouville action*

$$\mathbf{S}(\varphi; g) = \frac{1}{2\pi} \int_\Sigma (|\mathrm{d}\varphi|_g^2 + QK_g\varphi)\mathrm{vol}_g. \tag{1.21}$$

where we recall the transformation law of the curvature (1.1). In general, we will call *Liouville field* a GFF that transforms like a Q-conformal factor as in (1.19).

1.3.2.3 Critical case

We conclude this introduction to the theory of Gaussian multiplicative chaos with a brief presentation of the critical case [DRSV14a, DRSV14b]. A more detailed study will also appear in Chapter 4. In $\mathbb{R}^d$, we have seen that for $\gamma \in (0, \sqrt{2d})$, the GMC of a log-correlated field is a random measure M^γ giving full mass to the set of γ-thick points, which is a set of Hausdorff dimension $d - \frac{\gamma^2}{2}$ encoding points where the field is exceptionally "large". For $\gamma = \sqrt{2d}$, one would expect the putative GMC to live on a set of Hausdorff dimension 0 corresponding to the "maximum" of the field. Therefore, critical GMC is intimately connected to extreme values of log-correlated fields. The latter topic is a rather active field of research in probability theory. In the Gaussian realm, Madaule [Mad15, Theorem 1.1] showed that for a large class of log-correlated fields, we have

$$\lim_{\varepsilon\to 0} \sup_{x\in[0,1]^d} X_\varepsilon(x) - \sqrt{2d}\log\frac{1}{\varepsilon} + \frac{3}{2\sqrt{2d}}\log\log\frac{1}{\varepsilon}$$

exists in distribution and is given by a Gumbel variable with a random parameter involving critical GMC. The leading behaviour of the maximum is expected to be a universal feature of many log-correlated fields (not necessarily Gaussian). Most common examples are the logarithm of the characteristic polynomial of large random matrices, or stochastic models for the Riemann ζ-function.

Going back to critical GMC, the story goes as follows. Similarly to the L^1-phase, one wants to compute second moments after having carefully removed the contribution of "bad points" that are not "seen" by the measure but make the second moment blow up. The first main feature of critical GMC is that the usual regularisation applied at $\gamma = \sqrt{2d}$ converges to 0. Instead, one needs to use the so-called "derivative regularisation":

$$\mathrm{d}M'_\varepsilon = (X_\varepsilon - \sqrt{2d}\mathbb{E}[X_\varepsilon^2])e^{\sqrt{2d}X_\varepsilon - d\mathbb{E}[X_\varepsilon^2]}\mathrm{d}x. \tag{1.22}$$

The prime in the notation means "derivative" and the terminology stems from the fact that $\mathrm{d}M_\varepsilon^\gamma = \frac{\mathrm{d}}{\mathrm{d}\gamma}|_{\gamma=\sqrt{2d}}\mathrm{d}M_\varepsilon^\gamma$.

As for the subcritical case, one wants to interpret the derivative martingale as a Radon-Nykodym derivative which changes the behaviour of the field at the point under consideration. While the exponential term gives a $\sqrt{2d}$-drift, we need to understand how the term X_ε affects the measure. Note however that this is not a positive martingale. This situation is similar to the weighting of standard Brownian motion $(B_t)_{t\geq 0}$ started from $x > 0$ by $\frac{B_t}{x}$. A simple way to get a positive martingale is to consider instead the stopped process $B_{t\wedge\tau}$ where τ is the hitting time of 0, and it is well-known that this reweighing gives a three-dimensional Bessel process. This process has three different characterisations

i It is the Euclidean norm of a three dimensional Brownian motion (hence its name) started from (say) $(x, 0, 0)$.

ii It is the law of Brownian motion started from x and conditioned to stay positive.

iii It satisfies the SDE $\mathrm{d}X_t = \frac{\mathrm{d}t}{X_t} + \mathrm{d}B_t$.

It is therefore not surprising that $3d$-Bessel processes play an important role in the construction of critical GMC, and they also appear in the asymptotic analysis of Liouville correlation functions where a similar phenomenon occurs. The result of [DRSV14a] is that for a large class of fields, M'_ε converges a.s. to a random measure M' as $\varepsilon \to 0$. Moreover, this measure gives full mass to a set of Hausdorff dimension 0 and has no atoms.

The derivative normalisation is only one possible regularisation of the critical measure. The other one is the so-called "Seneta-Heyde" regularisation, which consists in multiplying $M^{\sqrt{2d}}$ by the deterministic prefactor $\sqrt{\log \frac{1}{\varepsilon}}$ rather than the random prefactor $X_\varepsilon - \sqrt{2d}\mathbb{E}[X_\varepsilon^2]$. The result of [DRSV14b] is that the measures $\sqrt{\frac{\pi \log \frac{1}{\varepsilon}}{2}} M_\varepsilon^{\sqrt{2d}}$ converges in probability as $\varepsilon \to 0$ to the same measure M'. This result can be understood as follows: since the Bessel process is Brownian motion conditioned to stay positive, we are conditioning on an event of probability asymptotically equivalent to $\sqrt{\frac{2}{\pi \log \frac{1}{\varepsilon}}}$, which is the quantity by which we need to renormalise.

1.3.3 Schramm-Loewner evolution

It is a wide belief that scaling limits of statistical mechanics models at criticality exhibit conformal invariance, and apart from a handful of landmark results this remains largely conjectural (see e.g. [DS11a] for a review of known results). In particular, interfaces between clusters in these models should scale to random, conformally invariant curves. Schramm was able to classify all continuous curves possessing the conformal invariance property, by introducing the so-called Stochastic Loewner evolutions (now Schramm-Loewner evolutions) or SLE. The classification depends on a single parameter $\kappa \in (0, \infty)$ describing the universality class of the underlying model.

The Loewner evolution describes a family of conformal maps from simply connected domains to the upper-half plane. More precisely, suppose $\eta : [0, \infty] \to \bar{\mathbb{H}}$ is a curve from 0 to ∞ in the upper-half plane such that $\mathbb{H} \setminus \eta(0, \tau]$ is simply connected for each $t > 0$. Then there is a unique conformal map $g_\tau : \mathbb{H} \setminus \eta(0, \tau) \to \mathbb{H}$ with the normalisation $g_\tau(z) = z + o(1)$ as $z \to \infty$. Writing $g_\tau(z) = z + \sum_{n \geq 1} a_n z^{-n}$, the coefficient $a_1 =: \mathrm{hcap}(\eta[0, \tau])$ is called the half-plane capacity of $\eta[0, \tau]$. It is an increasing function of τ, so that we can reparameterise g_τ by half-plane capacity. This leads to the *Loewner differential equation*:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - u_t},$$

where the continuous function u_t is called the *Loewner driving function*.

Schramm showed that the conformally invariant random curves arising as solutions to the Loewner equation are those for which the driving function is a multiple of a Brownian motion, $u_t = \sqrt{\kappa} W_t$. More precisely, the SLE_κ curves are the unique curves satisfying the so-called conformal Markov property. That is, if $\mathcal{F}_t = \sigma(u_s, 0 \leq s \leq t)$ and η_t is the curve at time t, we have

i The conditional law of $(g_t(\eta_{t+s}) - u_t)_{s\geq 0}$ given $\mathcal{F}_t$ is equal to the law of $(\eta_s)_{s\geq 0}$.

ii Scale invariance: $(r\eta_{t/r^2})_{t\geq 0} \overset{\text{law}}{=} (\eta_t)_{t\geq 0}$

By conformal invariance, SLE_κ can be defined in any simply connected domain as a random curve joining two boundary points.

In fact, it is not obvious that feeding Brownian motion to the Loewner equation yields a family of curves, and a large amount of work was put in the proof [RS05]. The basic properties of SLE_κ vary wildly with respect to κ. They are a.s. simple for $\kappa \in (0,4)$, but for $\kappa > 4$, they are self- and boundary-intersecting (though never self-crossing). Moreover, the almost sure Hausdorff dimension of SLE_κ is $(1 + \frac{\kappa}{8}) \vee 2$ [Bef08].

The definition of SLE using the stochastic Loewner equation yields a curve that is parameterised by half-plane capacity, i.e. $\mathrm{hcap}(\mathbb{H} \setminus \eta[0,t]) = 2t$. It turns out that this time parameterisation is not the most natural. A more interesting one is the natural parameterisation introduced by Lawler & Sheffield [LS11], which is the unique non-trivial $(1+\frac{\kappa}{8})$-dimensional measure on the curve satisfying a certain Markov property. In turns out that this measure also coincides with the Minkowski content of the curve [LR15]. Keeping in mind the interpretation of SLE as a scaling limit of lattice models, the analogous parameterisation in the discrete world would simply be the number of edges in the path. This also explains why this parameterisation is coined as "natural". We note that it is a remarkable (and maybe underrated) fact that SLE possesses non-trivial Minkowski content. As a matter of comparison, the $(1 + \frac{\kappa}{8})$-Hausdorff measure is almost surely zero.

Since SLE describes interfaces in a field theory, it is important to relate it to the underlying field. Therefore it does not come as a surprise that there exist several couplings between SLE and the GFF [She16]. One of these couplings interprets SLE as the flow lines of the formal vector field $e^{iX/\chi}$ where X is the GFF and $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$, or as the level lines of X in the limiting case $\kappa = 4$. This point of view led to the rich theory of imaginary geometry [MS16a, MS16b, MS16c, MS17].

The other coupling shows that SLE is the solution to a problem of conformal welding involving GMC: this is the "quantum zipper theorem" of [She16]. This coupling was vastly used and generalised in the mating-of-trees theory of Liouville quantum gravity [DMS14]. Among other things, the coupling with GMC introduces another parameterisation of the curve, called the quantum length of SLE. This parameterisation can be understood as a "quantum" version of the natural parameterisation, since it is a multiplicative chaos on the natural parameterisation [Ben18].

Let us now describe in more detail the conformal welding of which SLE is the solution, for $\gamma \in (0,2)$. The formulation in [She16] is rather complicated and we will follow the simpler description in the loop version. More precise statements for this approach can be found in [AHS21]. Let X_+, X_- be independent Liouville fields in $\mathbb{D}$ and $\mathbb{D}^*$, and W_+, W_- be their traces on the boundary. Let $\mu_+ = e^{\gamma X_+}, \mu_- = e^{\gamma X_-}$ be the corresponding bulk GMCs and $\nu_+ = e^{\frac{\gamma}{2} X_+}, \nu_- = e^{\frac{\gamma}{2} X_-}$ be the boundary GMCs on $\partial\mathbb{D} \simeq \mathbb{S}^1, \partial\mathbb{D}^* \simeq \mathbb{S}^1$. Fix $h \in \mathrm{Homeo}(\mathbb{S}^1)$ such that $h^*\nu_- = \nu_+$. Then h is almost surely the conformal welding homeomorphism of a unique Jordan curve η and η has the law of the SLE_κ loop measure of [Zha17], with $\kappa = \gamma^2$. Moreover, letting $f : \mathbb{D} \to D, g : \mathbb{D}^* \to D^*$ be the welding maps, the pushforwards $f_*\mu_+, g_*\mu_-$ define a tensor on $\mathbb{C}$ which has the law of an independent GMC of a Liouville field. Finally, the pushforwards, $f_*\nu_+, g_*\nu_-$ define the same tensor on η, which is called the *quantum length of SLE*.

To summarise, conformally welding according to boundary GMC length preserves the Liouville field, and the interface curve is described by an *independent* SLE. In other words, this result means that Liouville quantum gravity is stable under the operation of welding according to quantum length.

1.4 Quantum field theory

> Elle est le point lointain et blafardement lumineux où convergent toutes les attentions des affolées et des détraquées.
>
> Catulle Mendès, *Méphistophéla*

Quantum field theory is a vast subject with many approaches and it is not always clear how these approaches are related. This section introduces (quite modestly) some mathematical objects that often appear in this study. We start with the concept of ζ-regularised determinants, which among other things allow one to define the partition function of the GFF. In Section 1.4.2 we introduce the framework of geometric quantisation and explain how it can be related to path integrals for infinite dimensional Kähler manifolds. The following Section 1.4.3 describes the algebro-geomtric formulation of conformal field theory, and Section 1.4.4 gives a concrete realisation of this framework in the probabilistic formulation of the Liouville CFT of [DKRV16, KRV19, KRV20].

1.4.1 ζ-regularised determinants

In general, ζ-regularisation is a process allowing one to assign a finite value to a divergent series. Applied to the spectrum of an operator, it can give a way to speak of its determinant. These renormalised determinants arise in several places in quantum field theory. In our case, we are interested in the Laplace-Beltrami operator on a Riemannian surface, whose determinant is interpreted as the partition function of the GFF.

1.4.1.1 ζ-regularisation

Let T be a positive, self-adjoint operator with compact resolvent and write its spectrum $0 < \lambda_1 \leq \lambda_2 \ldots$. Since $\lambda_n \to \infty$ as $n \to \infty$, we obviously have that $\prod_{n=1}^{\infty} \lambda_n$ is divergent. Assume that λ_n grows at least like some positive power of n. The ζ-function of T is the holomorphic function

$$\zeta_T(s) := \sum_{n \geq 1} \lambda_n^{-s},$$

where the region of convergence is for $\operatorname{Re} s$ sufficiently large. In this region, we also have

$$\zeta_T'(s) = -\sum_{n \geq 1} \log \lambda_n \lambda_n^{-s}.$$

Now, suppose that ζ_T has a meromorphic continuation to $\mathbb{C}$ (still denoted ζ_T) such that ζ_T is regular at 0. The *ζ-renormalised determinant* of T is then defined as

$$\det\nolimits_\zeta(T) := e^{-\zeta_T'(0)}.$$

The interpretation of this value is that evaluating ζ_T' at 0 yields formally

$$\text{“}\ \zeta_T'(0) = -\sum_{n \geq 1} \lambda_n = -\log \det T\ \text{”}.$$

It is interesting to note how $\det_\zeta(T)$ varies under rescaling. Given $\alpha > 0$, we have $\zeta_{\alpha T}(s) = \alpha^{-s} \zeta_T(s)$, hence $\zeta_{\alpha T}'(s) = \alpha^{-s}(\zeta_T'(s) - \log \alpha \zeta_T(s))$. Evaluating at $s = 0$ leads to

$$\det\nolimits_\zeta(\alpha T) = \alpha^{\zeta_T(0)} \det\nolimits_\zeta(T).$$

In a sense, $\zeta_T(0)$ can be understood as a “regularised dimension”.

1.4.1.2 The case of the Laplacian

The above procedure is well-known to work for the Laplace-Beltrami operator $-\Delta_g$ on a Riemann surface Σ, which we assume without boundary for simplicity. We then have $\ker(-\Delta_g) = \mathbb{R}$, which is spanned by the constant function $\mathrm{vol}_g(\Sigma)^{-1/2}$, normal in

$L^2(\Sigma, \mathrm{vol}_g)$. The analysis of $\zeta := \zeta_{-\Delta_g}$ is related to the heat kernel of T. Indeed, using the formula $x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt} t^s \frac{\mathrm{d}t}{t}$, we have

$$\zeta_T(s) = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{n \geq 1} e^{-\lambda_n t} t^s \frac{\mathrm{d}t}{t} = \frac{1}{\Gamma(s)} \int_0^\infty \left(\mathrm{Tr}\left(e^{-tT}\right) - \frac{1}{\mathrm{vol}_g(\Sigma)} \right) t^s \frac{\mathrm{d}t}{t}.$$

That is, ζ_T is the Mellin transform of the heat kernel of T.

Well-known estimates for the heat kernel [HPMS67] yield

$$\mathrm{Tr}\left(e^{t\Delta_g}\right) = \frac{\mathrm{vol}_g(\Sigma)}{4\pi t} + \frac{\chi(\Sigma)}{6} + O(t),$$

so that its Mellin transform satisfies

$$\int_0^\infty \mathrm{Tr}\left(e^{t\Delta_g}\right) t^s \frac{\mathrm{d}t}{t} = \frac{\mathrm{vol}_g(\Sigma)}{4\pi(s-1)} + \frac{\chi(\Sigma)}{6s} + f(s),$$

where f is some analytic function in s. This yields

$$\zeta_{-\Delta_g}(s) = \frac{1}{\Gamma(s)} \left(\frac{\mathrm{vol}_g(\Sigma)}{4\pi(s-1)} + \left(\frac{\chi(\Sigma)}{6} - 1 \right) \frac{1}{s} + f(s) \right).$$

Thus, the only pole of $\zeta_{-\Delta_g}$ is at $s = 1$; in particular it is regular at 0 and $\det_\zeta(-\Delta_g)$ is well-defined. Moreover,

$$\zeta_{-\Delta_g}(0) = \frac{\chi(\Sigma)}{6} - 1.$$

A crucial property of $\det_\zeta(-\Delta_g)$ is the *Polyakov-Alvarez anomaly*. Let $\widehat{g} = e^{2\sigma} g$ be a metric conformally equivalent to g. Then we have [OPS88]

$$\log \frac{\det_\zeta(-\Delta_{\widehat{g}})}{\mathrm{vol}_{\widehat{g}}(\Sigma)} = \log \frac{\det_\zeta(-\Delta_g)}{\mathrm{vol}_g(\Sigma)} - \frac{1}{12\pi} \int_\Sigma (|\mathrm{d}\sigma|_g^2 + K_g \sigma) \mathrm{vol}_g \tag{1.23}$$

In the language of conformal field theory, $\frac{\det_\zeta(-\Delta_g)}{\mathrm{vol}_g(\Sigma)}$ satisfies the Weyl anomaly with central charge -2.

Among other things, $\det_\zeta(-\frac{1}{2\pi}\Delta_g)$ allows us to define the partition function of the GFF, which is the formal integral $\int e^{-\frac{1}{4\pi}\int_\Sigma |\mathrm{d}\sigma|_g^2 \mathrm{vol}_g} D\sigma$. Since the correlation matrix of the GFF is $-\frac{1}{2\pi}\Delta_g$ and by analogy with the finite-dimensional setting, this total mass is interpreted as $\det_\zeta(-\frac{1}{2\pi}\Delta_g)^{-1/2}$ (due to the simple scaling property, we may take $\det_\zeta(-\Delta_g)^{-1/2}$ for simplicity). In fact, since $\|1\|_{L^2(\Sigma,\mathrm{vol}_g)} = \mathrm{vol}_g(\Sigma)^{1/2}$, the partition function is more naturally defined as

$$Z_{\mathrm{GFF}} = \left(\frac{\det_\zeta(-\Delta_g)}{\mathrm{vol}_g(\Sigma)} \right)^{-1/2}, \tag{1.24}$$

which satisfies the Weyl anomaly with central charge 1.

1.4.2 Geometric quantisation

Geometric quantisation is a general framework aiming at the quantisation of a classical phase space. By the latter term, it is usually meant a Poisson manifold $(M, \{\cdot,\cdot\})$. To quantise such a system, one seeks a Hilbert space $\mathcal{H}$ together with an assignment $\mathcal{C}^\infty(M) \ni f \mapsto T_f$, where T_f is a self-adjoint operator on $\mathcal{H}$ and the assignment represents the Poisson bracket

$$[T_f, T_g] = iT_{\{f,g\}}. \tag{1.25}$$

It turns out that the more natural setting to achieve this goal is the one of symplectic geometry (every symplectic manifold is canonically Poisson). For the reader's convenience, we will give some background on this topic.

1.4.2.1 Symplectic geometry

Definitions. A *symplectic manifold* (M, ω) is a smooth manifold M with a non degenerate, closed two-form ω. The non-degenracy of ω requires M to be even dimensional, say $\dim M = 2n$. There is a natural volume form on (M, ω), the *Liouville volume form* $e^\omega = \frac{\omega^{\wedge n}}{n!}$.[5] The symplectic form induces a pairing between the tangent and cotangent bundles, with the assignment $X \mapsto \omega(X, \cdot\,)$ for all vector fields X. The *Hamiltonian vector field* of a function $f \in \mathcal{C}^\infty(M)$ is the vector field X_f characterised by

$$\mathrm{d}f = \omega(X_f, \cdot\,) = -\iota_{X_f}\omega,$$

where $\iota_X\alpha = \alpha(X, \cdots)$ is the contraction of the differential form α with the vector field X. Conversely, a vector field X is *Hamiltonian* if it is the Hamiltonian vector field of some function, i.e. if the differential form $\iota_X\omega$ is exact.

A *Poisson manifold* $(M, \{\cdot,\cdot\})$ is a smooth manifold together with a skew-symmetric bilinear form $\{\cdot,\cdot\}$ on $\mathcal{C}^\infty(M)$ satisfying the Leibniz rule and the Jacobi identity. A symplectic manifold is canonically Poisson with the bracket

$$\{f, g\} := \omega(X_f, X_g),$$

and the map $f \mapsto X_f$ is a Lie algebra homomorphism: $X_{\{f,g\}} = [X_f, X_g]$.

A vector field X is *symplectic* if its flow (ϕ_t) preserves ω, i.e. ϕ_t is a symplectomorphism of M for each t. Equivalently, the differential form $\iota_X\omega$ is closed. Thus, the obstruction for a symplectic vector field to be Hamiltonian is the first de Rham cohomology of M.

Now, suppose that a Lie group G with Lie algebra $\mathfrak{g}$ acts on M. The infinitesimal action of G on M induces a vector field $\rho(\xi) \in \Gamma(TM)$ for each $\xi \in \mathfrak{g}$, and the corresponding

[5] In fact, $e^\omega = \sum_{n\geq 0} \frac{\omega^{\wedge n}}{n!}$ with the convention that the integral of a p-form on a k-dimensional submanifold is 0 if $k \neq p$. Note that e^ω is non-degenerate on symplectic submanifolds of M.

mapping $\mathfrak{g} \to \Gamma(TM)$ is a Lie algebra homomorphism. The action of G is *symplectic* (resp. *Hamiltonian*) if $\rho(\xi)$ is a symplectic (resp. Hamiltonian) vector field for each $\xi \in \mathfrak{g}$.

Connections on line bundles. Let $\mathcal{L} \to M$ be a complex line bundle over a smooth manifold M. A *connection* on $\mathcal{L}$ is an operator $\nabla : \Gamma(\mathcal{L}) \to \Gamma(T^*M \otimes \mathcal{L})$ satisfying the Leibniz rule,

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla s, \qquad f \in \mathcal{C}^\infty(M), s \in \Gamma(\mathcal{L}).$$

Locally, a connection has the form $\nabla = \mathrm{d} + \alpha$ where d is the de Rham differential and α is a one-form. The space of connections is an affine space modelled on $\Omega^1(M)$.

The *curvature* of ∇ is the End($\mathcal{L}$)-valued two-form given for all vector fields X, Y by

$$F_\nabla(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

The connection is *flat* if $F_\nabla = 0$, i.e. if the mapping $X \mapsto \nabla_X$ is a homomorphism of Lie algebras. A *Hermitian metric* on $\mathcal{L}$ is a smoothly varying Hermitian inner-product in the fibres of $\mathcal{L}$. The connection ∇ is *compatible* with h if

$$\mathrm{d}(h(s,t)) = h(\nabla s, t) + h(s, \nabla t), \qquad s, t \in \Gamma(\mathcal{L}).$$

On $\Gamma(\mathcal{L})$ we can introduce the inner-product $\langle s, t\rangle = \int_M h(s,t)e^\omega$.

If M is additionally a complex manifold, the connection splits into its holomorphic and anti-holomorphic components, $\nabla = \nabla^{1,0} + \nabla^{0,1}$. A section s is *holomorphic* if $\nabla^{0,1}s = 0$. The metric is determined by the function $h = h(s,s)$, where s is a holomorphic frame. Conversely, given a holomorphic, Hermitian line bundle $\mathcal{L}$, there is a unique connection on $\mathcal{L}$ compatible with both structures, the *Chern connection*. Its curvature satisfies $F_\nabla = \bar{\partial}\partial \log h$.

Integration by parts. Let (M, ω) be a compact symplectic manifold, and $e^\omega = \frac{\omega^n}{n!}$ the Liouville volume, where $\dim M = 2n$. The Lie derivative along a vector field X is denoted $\mathcal{L}_X$. Recall Cartan's formula

$$\mathcal{L}_X = \mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d}.$$

Proposition 1.4.1. *For all $f, g \in \mathcal{C}^\infty(M)$,*

$$\int_M \{f, g\}e^\omega = 0.$$

From this proposition and the Leibniz rule we get the immediate corollary: for all $f, g, h \in \mathcal{C}^\infty(M)$,

$$\int_M \{f,g\}he^\omega = -\int_M \{f,h\}ge^\omega. \tag{1.26}$$

The proof of Proposition 1.4.1 is based on Liouville's theorem, stating that

$$\mathcal{L}_{X_f} e^{\omega} = 0. \tag{1.27}$$

Liouville's theorem is an easy consequence of Cartan's formula and the closedness of ω:

$$\mathcal{L}_{X_f}\omega = \mathrm{d}(\iota_{X_f}\omega) = \mathrm{d}(\omega(X_f,\cdot)) = -\mathrm{d}^2 f = 0.$$

Then (1.27) follows from $e^{\omega} = \frac{\omega^n}{n!}$. With this in hand, one can prove Proposition 1.4.1. By Liouville's theorem and the Leibniz rule

$$\mathcal{L}_{X_f}(ge^{\omega}) = \mathcal{L}_{X_f} g e^{\omega}.$$

On the other hand, $g\frac{\omega^n}{n!}$ is closed as a top-degree form, so by Cartan's formula, $\mathcal{L}_{X_f}(ge^{\omega}) = \mathrm{d}(\iota_{X_f}(ge^{\omega}))$ is exact. Thus, by Stokes' theorem,

$$0 = \int_M \mathcal{L}_{X_f}(ge^{\omega}) = \int_M \mathcal{L}_{X_f} g e^{\omega} = \int_M \{f,g\}e^{\omega}.$$

1.4.2.2 The main construction

A first idea to achieve (1.25) is to realise $\mathcal{H}$ as a space of functions on M. It turns out that a slight generalisation is needed: namely, we need to take $\mathcal{H}$ as a space of sections of some complex line bundle $\mathcal{L} \to M$ with a Hermitian metric h and a compatible connection ∇ such that its curvature satisfies

$$F_{\nabla} = -i\omega.$$

Recall that the *curvature* of a connection ∇ is the two-form

$$F_{\nabla}(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \qquad X, Y \in \Gamma(TM),$$

and ∇ is said to be *compatible with the metric* if for all sections s,t of $\mathcal{L}$, we have

$$\mathrm{d}h(s,t) = h(\nabla s, t) + h(s, \nabla t).$$

The connection is *flat* if $F_{\nabla} = 0$, so that the assignment $X \mapsto \nabla_X$ is a homomorphism of Lie algebras in this case.

If such a triple $(\mathcal{L}, h, \nabla)$ exists, it is called *prequantum data*. The obstruction to the existence of prequantum data is topological: it exists if and only if $\frac{1}{2\pi}[\omega] \in H^2_{\mathrm{dR}}(M,\mathbb{Z})$. In this case, we set $\mathcal{H}$ to be the Hilbert space completion of the space of smooth sections of $\mathcal{L}$ with respect to the inner-product $(s,t) \mapsto \int_M h(s,t)e^{\omega}$. Moreover, one checks that the assignment $f \mapsto T_f := f - i\nabla_{X_f}$ satisfies the commutation relation (1.25). The self-adjointness follows from the integration by parts formula. Indeed, we have for all smooth sections s, t of $\mathcal{L}$

$$0 = \int_M \mathcal{L}_{X_f} h(s,t) e^{\omega} = \int_M h(\nabla_{X_f} s, t) e^{\omega} + \int_M h(s, \nabla_{X_f} t) e^{\omega},$$

i.e. $(i\nabla_{X_f})^\dagger = i\nabla_{X_f}$.

As the name suggests, prequantisation is only the first step of the programme, because the representation is not irreducible. This is usually phrased by saying that the Hilbert space is "too big". A physical justification of this fact is that wave functions usually depend on half the number of variables in the classical phase space. To cut the number of variables in half, one needs to choose a polarisation, i.e. a Lagrangian distribution $\mathfrak{D}$ of TM. Then we only keep in our Hilbert space those sections which are annihilated by ∇_X for all sections X of $\mathfrak{D}$. There are only a few cases with a natural choice of polarisation, typically cotangent bundles and Kähler manifolds.

Hamiltonian group action. The process of geometric quantisation associates a self-adjoint operator $i\nabla_X$ to each Hamiltonian vector field X. The most interesting Hamiltonian vector fields are those arising from the Hamiltonian action of a Lie group G on M. By definition, this means that the fundamental vector field $\rho(\xi)$ on M induced by $\xi \in \mathfrak{g}$ is Hamiltonian, i.e. there exists a function H_ξ such that $\mathrm{d}H_\xi = -\iota_{\rho(\xi)}\omega$. The functions H_ξ may be chosen such that $\xi \mapsto H_\xi$ is a Lie algebra homomorphism, and a *momentum map* for the G-action is a map $\mu : M \to \mathfrak{g}^*$ satisfying $\mu(\xi) = H_\xi$ for all $\xi \in \mathfrak{g}$.

In particular, G acts on M by symplectomorphisms, so that the condition that ω be closed reduces to [BR87]

$$\omega([\rho(x), \rho(y)], \rho(z)) + \omega([\rho(y), \rho(z)], \rho(x)) + \omega([\rho(z), \rho(x)], \rho(y)) = 0$$

for all $x, y, z \in \mathfrak{g}$. Thus, we get a two-cocycle on $\mathfrak{g}$. The fact that F_∇ is proportional to ω means that the assignment $\xi \mapsto i\nabla_\xi$ is a unitary representation of the central extension of $\mathfrak{g}$ by this cocycle.

In summary, given a Hamiltonian action of G on M, the machinery of geometric quantisation provides a projective unitary representation of G on the quantum Hilbert space. The typical example of this situation is Kirillov's orbit method which which aims to study unitary representations of G via the quantisation of its coadjoint orbits.

1.4.2.3 Kähler manifolds

The programme of geometric quantisation becomes simpler when M is not only symplectic but also Kähler. A *Kähler manifold* is a symplectic manifold (M, ω) together with an integrable almost complex structure $\mathbf{J}$ such that the bilinear form

$$g(X,Y) := \omega(X, \mathbf{J}Y), \qquad X, Y \in \Gamma(TM) \tag{1.28}$$

is definite-positive, i.e. g defines a Riemannian metric on M. Thus, Kähler manifolds are simultaneously symplectic, complex and Riemannian, with the compatibility between the three structures expressed by (1.28).

Due to the identity $\mathbf{J}^2 = -\mathrm{Id}_{TM}$, every complex manifold has a splitting of the complexified tangent bundle $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ into the $\pm i$ eigenspaces of $\mathbf{J}$. This splitting applies to the cotangent bundle by duality and to its exterior algebra, so that we have the decomposition $\Omega^k(M) = \oplus_{p+q=k}\Omega^{p,q}(M)$ of the space of k-forms. In coordinates, a (p,q)-form looks like a linear combination of $\mathrm{d}z_{i_1} \wedge \cdots \mathrm{d}z_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \cdots \mathrm{d}\bar{z}_{j_q}$. The de Rham differential has a splitting $\mathrm{d} = \partial + \bar{\partial}$ where $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$. These are the *Dolbeault operators* (a.k.a. the $(1,0)$ and $(0,1)$ components of d), which satisfy the same axioms as an exterior differential with the additional property that they anticommute. The chain complex associated with $\bar{\partial}$ is called the *Dolbeault cohomology.*

Kähler metrics can be described locally by a single real function ρ called the *Kähler potential*:

$$\frac{i}{2}\partial\bar{\partial}\rho = \omega,$$

where ∂ and $\bar{\partial}$ are the holomorphic and anti-holomorphic parts of the de Rham differential. The existence of Kähler potentials is a remarkable feature of Kähler geometry (for instance it is not possible in general to describe a Riemannian metric by a single function).

Suppose for a moment that ρ is globally defined and let $\mathcal{L} = M \times \mathbb{C}$ be the trivial line bundle with holomorphic structure induced from M. We also consider the metric $h = e^{-\rho/2}$, which turns $\mathcal{L}$ into a holomorphic, Hermitian line bundle. The *Chern connection* ∇ is the unique connection on $\mathcal{L}$ compatible with both the holomorphic and Hermitian structures, and we have

$$\nabla = \mathrm{d} - \frac{1}{2}\partial\rho,$$

i.e. the connection form of ∇ is $\vartheta = -\frac{1}{2}\partial\rho$. Thus, the curvature form of ∇ is

$$F_\nabla = \mathrm{d}\vartheta = \frac{1}{2}\partial\bar{\partial}\rho = -i\omega,$$

so that $(\mathcal{L}, h, \nabla)$ gives us prequantum data, and the prequantum Hilbert space is given by $L^2(M, e^{-\rho/2}e^\omega)$. A natural choice of polarisation is to restrict to holomorphic functions, which we write with a subscript $\mathcal{O}$: $\mathcal{H} = L^2_{\mathcal{O}}(M, e^{-\rho/2}e^\omega)$.

Example 1. Consider the standard space $\mathbb{C}^N$ endowed with the canonical symplectic form $\omega = \frac{i}{2}\sum_{k=1}^N \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k$. A global Kähler potential is given by $\rho(z) = |z|^2$. Carrying out the previous construction yields the *Segal-Bargmann space* $L^2_{\mathcal{O}}(\mathbb{C}^N, e^{-\frac{|z|^2}{2}}|\mathrm{d}z|^2)$. The relevance of this space is the hermiticity relation $\partial_{z_k}^\dagger = z_k$.

In summary, the geometric quantisation of a Kähler manifold is closely related to the measure $e^{-\rho/2}e^{\omega}$, giving a probabilistic interpretation of this procedure. This point of view can be useful in infinite dimensions since probabilistic techniques can be used to address the construction of the path integral with action functional given by the Kähler potential.

1.4.3 Conformal field theory

Let Σ be a compact, oriented surface. In a quantum field theory, we wish that the partition function doesn't depend on the parameterisation of the surface, so we look at metrics up to the action of diffeomorphisms. Now, up to the action of $\mathrm{Diff}_+(\Sigma)$, we can write any Riemannian metric $g \in \mathrm{Met}(\Sigma)/\mathrm{Diff}_+(\Sigma)$ as $e^{2\sigma}g_m$, where $(g_m)_{m\in\mathcal{M}_\Sigma}$ is a family of constant curvature metrics parameterising the moduli space, and σ is a conformal factor. It is useful to think of this space as an infinite dimensional vector bundle over $\mathcal{M}_\Sigma$, where the fibre over $m \in \mathcal{M}_\Sigma$ is $\mathcal{C}^\infty(\Sigma)$. In fact, the Dirichlet energy endows this vector bundle with a natural metric.

In all generality, the partition function of a quantum field theory based on Σ is a function $Z(g_m, \sigma)$ depending on both m and σ. In a conformal field theory, we assume on the contrary that the theory does not depend on the representative of the conformal class. More precisely, the partition function must satisfy the *Weyl anomaly*

$$Z(e^{2\sigma}g) = \exp\left(\frac{c}{24\pi}\int_\Sigma (|\mathrm{d}\sigma|_g^2 + K_g\sigma)\mathrm{vol}_g\right) Z(g),$$

where the parameter $c \in \mathbb{R}$ is called the *central charge*. Notice that the Polyakov-Alvarez anomaly formula (1.23) means that the partition function of the GFF (1.24) satisfies the Weyl anomaly with $c = 1$. The Weyl anomaly encodes the variation of the partition function inside the conformal class into a single one-dimensional object. Thus, the infinite dimensional vector bundle of all metrics is reduced to a one-dimensional vector bundle, i.e. a line bundle over $\mathcal{M}_\Sigma$. Letting $\mathcal{L}$ be the line bundle trivialised by Z_{GFF}, sections of the line bundle $\mathcal{L}^{\otimes c}$ are those functions of the metric satisfying the Weyl anomaly with central charge c.

One is also interested in correlation functions, which should be expected values of "fields" inserted at some marked points (or punctures). *A priori*, correlation functions may depend on a local holomorphic chart around the puncture. Hence, one needs to consider tuples $(\Sigma, p_1, z_1, ..., p_n, z_n)$ where Σ is a Riemann surface, p_i is a puncture and z_i is a local coordinate at p_i satisfying $z_i(p_i) = 0$. For simplicity, let us consider the case $n = 1$. We can realise the tuples (Σ, p, z) as a bundle $\widehat{\mathcal{M}}_\Sigma$ over $\mathcal{M}_\Sigma$, with the fibre modelled on the

$z\mathbb{C}\{z\}$, where $\mathbb{C}\{z\}$ is the space of converging power series in a neighbourhood of 0. For technical reasons, the space $\mathbb{C}\{z\}$ is usually replaced with $\mathbb{C}[[z]]$, the space of formal power series. Axiomatically, correlation functions of primary fields rescale homogeneously under a change of local coordinate

$$\langle \Phi_h(p, z) \rangle = \left| \frac{\mathrm{d}w}{\mathrm{d}z}(p) \right|^{2h} \langle \Phi_h(p, w) \rangle. \tag{1.29}$$

That is, correlation functions of primary fields behave like (h, h)-forms and can be understood as sections of $|\mathcal{T}|^{2h} \otimes \mathcal{L}^{\otimes c}$, where $\mathcal{T}$ is the line bundle whose fibre at (Σ, p) is $T_p^*\Sigma$. This line bundle is sometimes called the *tautological line bundle* [Mir07b] and plays a central role in algebraic geometry. In this definition, the number h is called the *conformal weight* of the primary field Φ_h. The notation in (1.29) is purely formal and does not say whether there are actual "fields" whose "expectation" would satisfy that property.

These first principles dictate what kind of algebro-geometric objects should the partition and correlation functions be. In particular, they put the analytic geometry of the moduli space at the centre stage, a point of view that can be traced back to the work of Friedan & Shenker [FS87]. To see what kind of constraints this is imposing, we need to understand the infinitesimal structure of $\widehat{\mathcal{M}}_\Sigma$. We consider the Teichmüller space $\widehat{\mathcal{T}}_{g,n}$ of n-pointed complex curves of genus g. The tangent space to $\widehat{\mathcal{T}}_{g,n}$ can be described by the so-called *Virasoro uniformisation* [FBZ01, Section 17.3]. Let $(\Sigma, p, z) \in \widehat{\mathcal{T}}_{g,1}$ and let us consider a meromorphic vector field with a possible pole at p. Flowing in the direction of this vector field gives an infinitesimal deformation of the coordinate chart at p. Moreover, if the vector field has a pole at p, we also get an infinitesimal deformation of the complex structure of the surface. Writing the Laurent expansion of the vector field $\sum_{n\in\mathbb{Z}} \xi_n z^n \partial_z$, we get an action of the Witt algebra on $\widehat{\mathcal{T}}_{g,1}$. In the case of multiple punctures, $p_1, ..., p_k$, we get an action of $\oplus_{i=1}^n \mathbb{C}((z_i))\partial_{z_i}$. Moreover, the stabiliser of this action is the space $\mathrm{Vect}(\Sigma \setminus \{p_1, ..., p_n\})$ of meromorphic vector fields on Σ with poles only allowed at the punctures. The statement of Virasoro uniformisation is then:

$$T_{(\Sigma,p_1,z_1,...,p_n,z_n)}\widehat{\mathcal{T}}_{g,n} = \mathrm{Vect}(\Sigma \setminus \{p_1, ..., p_n\}) \backslash \bigoplus_{k=1}^{n} \mathbb{C}((z_i))\partial_{z_i}.$$

Additionally, we have

$$T_{(\Sigma,p_1,...,p_n)}\mathcal{T}_{g,n} = \mathrm{Vect}(\Sigma \setminus \{p_1, ..., p_n\}) \backslash \bigoplus_{k=1}^{n} \mathbb{C}((z_i))\partial_{z_i} / \bigoplus_{k=1}^{n} \mathbb{C}[[z_i]]\partial_{z_i}.$$

This description of the tangent space means that we can define representations of the Witt algebra as differential operators acting on functions on $\widehat{\mathcal{T}}_{g,n}$. To get a non-zero central charge, one considers sections suitable line bundles instead of functions. The different representations of the Virasoro algebra are understood as being attached to each puncture. The space $\mathrm{Vect}(\Sigma \setminus \{p_1, ..., p_n\})$ is an infinite dimensional Lie algebra and the invariance of correlation functions under its action implies an infinite hierarchy of equations known as the *conformal Ward identities*.

In principle, one should be able to represent the variation of correlation functions by the insertion of the *stress-energy tensor*: differentiating in the direction of the vector field $\mathbf{v} = v(z_k)\partial_{z_k}$, one has [Fre07]

$$\delta_{\mathbf{v}}\langle \prod_{i=1}^{n} \Phi_{h_i}(p_i)\rangle = \int \langle T(z_k) \prod_{i=1}^{n} \Phi_{h_i}(p_i)\rangle v(z_k) \mathrm{d}z_k \tag{1.30}$$

where the integral is over a small loop surrounding p_k. The stress-energy tensor allows us to differentiate with respect to the complex structure, so it defines a connection on the bundle over the moduli space in which correlation functions take values. Thus, it behaves locally like a one-form on moduli space. Namely, for each value of the complex structure, $z \mapsto T(z)$ should be a quadratic differential on the underlying surface. However, the conformal anomaly prevents T to be a globally defined quadratic differential on the surface, and its transformation rule is given by [FS87]

$$T(z) = T(w)(\mathrm{d}w/\mathrm{d}z)^2 + \frac{c}{12}\mathcal{S}w(z),$$

where c is the central charge

$$\mathcal{S}w = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2 = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$$

is the *Schwarzian derivative*. Tensors with this transformation property are called *projective connections* and the central charge is the obstruction to their flatness.

The question whether one can define actual fields (in a mathematical sense yet to be clarified) satisfying this algebro-geometric framework is a difficult and important one. The probabilistic formulation of Liouville CFT introduced in the next section is an example where this was proved to be the case.

1.4.4 Liouville conformal field theory

Liouville conformal field theory (LCFT) is the path integral with action given by the Liouville action, the variational formulation of the Liouville equation. Recall that the solution to the Liouville equation is a metric with constant scalar curvature, so that LCFT may be understood as a random perturbation of the uniform metric.

In this section we give some background on the probabilistic formulation of LCFT, which is a vast programme spanning [DKRV16, DRV16, GRV19, KRV19, KRV20, GKRV20]. It would therefore be too ambitious to give a full review. We simply explain how Gaussian multiplicative chaos comes into the picture and how one can recover the axiomatic framework described in the previous section.

1.4.4.1 Construction.

To introduce LCFT, let Σ be a Riemann surface with empty boundary and g be a compatible metric with conformal class $[g] = \{e^{2\sigma}g,\, \sigma \in \mathcal{C}^\infty(\Sigma)\}$. The Liouville action is

$$S_L(\sigma; g) = \frac{1}{2\pi} \int_\Sigma \left(|\mathrm{d}\sigma|_g^2 + QK_g\sigma + 2\pi\mu e^{\gamma\sigma} \right) \mathrm{vol}_g.$$

We have already encountered the quadratic term from this equation in Section 1.1.3, which is the action for a random Q-conformal factor. This Gaussian field X is only defined up to constant, but we fix it by imposing that X has vanishing vol_g-mean. Then we tensorise with Haar (i.e. Lebesgue) measure on $\mathbb{R}$. We want to understand the extra term in the Liouville action as a Radon-Nikodym derivative with respect to the GFF (except that ultimately we will obtain a measure singular with respect to the GFF). Note that the last term features the exponential of the field, which we interpret as its GMC M^γ.

We now assume that g is the uniform metric. The interpretation of the path integral associated with the Liouville action is then

$$\int F(\sigma) e^{-\frac{1}{2}S_L(\sigma;g)} D\sigma = \left(\frac{\det_\zeta(-\Delta_g)}{\mathrm{vol}_g(\Sigma)} \right)^{-1/2} \int_\mathbb{R} e^{-Q\chi(\Sigma)c} \mathbb{E}\left[F(X+c) \exp\left(-\mu e^{\gamma c} M^\gamma(\Sigma) \right) \right] \mathrm{d}c, \tag{1.31}$$

where F is in a function space to be determined. To get to this expression, we have used the Gauss-Bonnet theorem $\int_\Sigma K_g \mathrm{vol}_g = 2\pi\chi(\Sigma)$. We have also used that g is the uniform metric, so that $\int XK_g \mathrm{vol}_g = 0$.

The partition function Z_g of the theory is obtained by plugging $F = 1$. It is finite if and only if $\chi(\Sigma) < 0$, in which case

$$Z_g = \left(\frac{\det_\zeta(-\Delta_g)}{\mathrm{vol}_g(\Sigma)} \right)^{-1/2} \gamma^{-1} \mu^{\frac{Q\chi(\Sigma)}{\gamma}} \Gamma\left(-\frac{Q\chi(\Sigma)}{\gamma} \right) \mathbb{E}\left[M^\gamma(\Sigma)^{\frac{Q\chi(\Sigma)}{\gamma}} \right].$$

Recall that GMC has finite negative moments, so that this is indeed finite. In the case $\chi(\Sigma) \geq 0$, the partition function diverges because of the zero mode c. In the case $\chi(\Sigma) < 0$, the partition function satisfies the Weyl anomaly with central charge $c = 1 + 6Q^2$. To see this, let $\widehat{g} = e^{2\sigma}g$ be a conformally equivalent metric and let $F(\cdot; g)$ be a functional depending on the metric and satisfying $F(X; \widehat{g}) = F(X + Q\sigma; g)$. This transformation property is satisfied by M^γ due to the conformal covariance of GMC. Using the variational formula $K_{\widehat{g}} = e^{-2\sigma}(K_g - 2\Delta_g \sigma)$ and Girsanov's theorem, we obtain

$$\begin{aligned}\mathbb{E}\left[e^{-\frac{1}{4\pi}\int_\Sigma QK_{\widehat{g}}X\mathrm{vol}_{\widehat{g}}}F(X;\widehat{g})\right] &= \mathbb{E}\left[e^{-\frac{Q}{4\pi}\int_\Sigma X(K_g-2\Delta_g\sigma)\mathrm{vol}_g}F(X+Q\sigma;g)\right] \\ &= e^{\frac{Q^2}{4\pi}\int_\Sigma(|\mathrm{d}\sigma|_g^2+2K_g\sigma)\mathrm{vol}_g}\mathbb{E}\left[F(X;g)\right].\end{aligned}$$

Thus, we have a conformal anomaly of $6Q^2$. Combining with the anomaly given by the Polyakov-Alvarez formula, one gets $1+6Q^2$ as required.

1.4.4.2 Correlation functions.

Roughly speaking, correlation functions correspond to certain Laplace transforms of the field. More precisely, we fix weights $\alpha_1, ..., \alpha_n \in \mathbb{R}$ and non-coinciding points $z_1, ...z_n \in \Sigma$, and take $F = \prod_{i=1}^n V_{\alpha_i}(z_i)$ in (1.31), where $V_{\alpha_i}(z_i) = e^{\alpha_i X(z_i)}$ is a so-called *vertex operator*. A renormalisation of this operator is required: similarly to the definition of GMC we may set $V_{\alpha_i,\varepsilon}(z_i) = \varepsilon^{\frac{\alpha_i^2}{2}}e^{\alpha_i X_\varepsilon(z_i)}$, where X_ε is a circle average with respect to the background metric. The transformation of the vertex operator goes as follows: in the metric $\widehat{g} = e^{2\sigma}g$, the field transforms as $X \mapsto X + Q\sigma$ and the ε-average in the metric $\widehat{g}$ corresponds approximately to a $\varepsilon e^{-\sigma}$ average in the metric g. Hence the vertex operator transforms as $\widehat{V}_{\alpha_i}(z_i) \mapsto e^{-\alpha_i(Q-\frac{\alpha_i}{2}\sigma(z_i)}V_{\alpha_i}(z_i)$.

Up to a bounded term in ε, the regularised vertex operator is the Radon-Nykodym derivative $e^{\alpha_i X_\varepsilon(z_i)-\frac{\alpha_i^2}{2}\mathbb{E}[X_\varepsilon^2(z_i)}$ corresponding to the change of measure $X \mapsto X + \alpha_i G_\varepsilon(z_i,\cdot)$, where G is the resolvent of $\frac{-1}{2\pi}\Delta$ and G_ε is the corresponding regularisation. Passing to the limit, the Girsanov transform gives the shift $X \mapsto X + \sum_{i=1}^n \alpha_i G(z_i,\cdot)$ and correlation functions are given up to prefactors by

$$\left\langle \prod_{i=1}^n V_{\alpha_i}(z_i)\right\rangle \propto \prod_{1\leq i<j\leq n} e^{-\alpha_i\alpha_j G(z_i,z_j)}\mathbb{E}\left[\left(\int_\Sigma e^{\gamma\sum_{i=1}^n\alpha_i G(z_i,\cdot)}\mathrm{d}M^\gamma\right)^{\frac{Q\chi(\Sigma)-\sum_{i=1}^n\alpha_i}{\gamma}}\right]. \tag{1.32}$$

These correlation functions are the focus of Chapters 2 and 3 where we study their asymptotic behaviour in certain degenerate geometric limits, such as colliding insertion points.

Because of the conformal covariance of M^γ, the shift $X \mapsto X + \sum_{i=1}^n \alpha_i G(z_i,\cdot)$ can be understood as taking the GMC of X in the metric $\prod_{i=1}^n e^{\frac{\alpha_i}{Q}G(z_i,\cdot)}g$. This metric has conical singularities of order $\frac{\alpha_i}{Q}$ at z_i. For such a surface to have negative curvature away from these conical singularities, the Gauss-Bonnet theorem gives us

$$\chi(\Sigma) - \sum_{i=1}^n \frac{\alpha_i}{Q} < 0.$$

Equivalently, the GMC moment in (1.32) is negative. This constraint is known as the *first*

Seiberg bound. On the other hand, for the GMC expectation to be finite, one requires

$$\alpha_i < Q,$$

which is consistent with the geometric interpretation of a conical singularity of order $\frac{\alpha_i}{Q}$ at z_i: at $\alpha_i = Q$ the angle closes. This condition is the *second Seiberg bound*.

In particular, for these two bounds to be satisfied on the sphere ($\chi(\mathbb{S}^2) = 2$), it is necessary to have $n \geq 3$. Moreover, the three-point correlation function on the sphere is completely determined by conformal invariance, since for each triple of non-coinciding points $(z_1, z_2, z_3) \in \widehat{\mathbb{C}}$ there exists a unique Möbius transformation sending (z_1, z_2, z_3) to $(0, 1, \infty)$. The three-point correlation functions on the sphere are known as the *structure constants* and constitute one building block of the theory, and there exists an exact expression for them known as the *DOZZ formula*. This formula was first conjectured in [DO94, ZZ96] using non-rigorous arguments based on Coulomb gas integrals. A more convincing argument was proposed by Teschner [Tes03], and the formula was finally proved in [KRV20].

1.4.4.3 Differential equations.

The conformal anomaly and the transformation properties of the vertex operators mean that the correlation functions satisfying the axioms of Section 1.4.3. In particular, the vertex operators defined above behave like highest-weight vectors of weight $h_i = \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$, and one can write the corresponding conformal Ward identities. Furthermore, for the values $\alpha = -\frac{\gamma}{2}$ and $\alpha = -\frac{2}{\gamma}$, one obtains a degenerate representation at levels $(2, 1)$ and $(1, 2)$ respectively, so that the BPZ equation (1.7) holds. The stress-energy tensor has an expression as a random field formally given by [GKRV20]

$$T(z) = Q\partial_{zz}^2 X - \left((\partial_z X)^2 - \mathbb{E}[(\partial_z X)^2]\right).$$

For the n-point correlation function on the Riemann sphere, the conformal Ward identities take the following explicit form for primary fields [KRV19]

$$\left\langle T(z) \prod_{i=1}^{n} V_{\alpha_i}(z_i) \right\rangle = \left(\sum_{i=1}^{n} \frac{h_i}{(z - z_i)^2} + \frac{\partial_{z_i}}{z - z_i} \right) \left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \right\rangle.$$

One can read the fact that the $V_{\alpha_i}(z_i)$'s are primary through the holomorphicity of this equation: the positive modes of the Virasoro algebra vanish against this expression.

Similarly, the BPZ equation (1.7) takes the explicit form [KRV19]

$$\left(\frac{4}{\gamma^2}\partial_{zz}^2+\sum_{i=1}^n\frac{h_i}{(z-z_i)^2}+\frac{\partial_{z_i}}{z-z_i}\right)\left\langle V_{-\frac{\gamma}{2}}(z)\prod_{i=1}^n V_{\alpha_i}(z_i)\right\rangle=0,$$

and the same equation holds under the substitution $\frac{\gamma}{2}\leftrightarrow\frac{2}{\gamma}$. We stress the conformal Ward identities and the BPZ equations have very different status: while the first are a pure manifestation of conformal symmetry and only involve first order partial differential operators, the second expresses the degeneracy of the Virasoro module generated by the primary field $V_{-\frac{\gamma}{2}}(z)$ and involves higher order differential operators.

The interesting aspect is that these equations are purely a feature of the representation theory of the Virasoro algebra but allow us to say deep things about multiplicative chaos. The most striking examples are the DOZZ formula, the exact distribution of GMC on the circle [Rem20] and the link between the tail of GMC and the reflection coefficient (two-point function) of LCFT [RV19]. Conversely, multiplicative chaos gives a concrete realisation of this otherwise abstract theory. As a consequence, [GKRV20] were able to prove the convergence of Virasoro conformal blocks for almost every element of the spectrum of LCFT, a highly non-trivial fact that algebraic tools had not been sufficient to tackle.

1.5 Perspectives

Morgen ist die Frage.

Berghain facade

To conclude this introduction, we briefly describe some ongoing work whose goal is to establish several connections between the different theories we have mentioned. On the one, we would like to relate SLE to unitary representations of the Virasoro algebra and the quantisation of coadjoint orbits of $\mathrm{Diff}(\mathbb{S}^1)$. On the other hand, we would also like to make more explicit the link between the probabilistic formulation of Liouville CFT and the quantisation of Teichmüller spaces (viewed as classical phase spaces with the Weil-Petersson metric).

1.5.1 A connection on $\dot{H}^{1/2}(\mathbb{S}^1)$

The group $\mathrm{Diff}(\mathbb{S}^1)$ acts on the space of measures on $\mathbb{S}^1$ by pullback: $\rho\mathrm{d}\theta\mapsto\rho\circ hh'\mathrm{d}\theta$. Given $\gamma\in(0,2)$, we can generalise this action to an action on the bundle of $\frac{\gamma Q}{2}$-tensors by

$$\rho(\mathrm{d}\theta)^{\frac{\gamma Q}{2}}\mapsto\rho\circ h(h')^{\frac{\gamma Q}{2}}(\mathrm{d}\theta)^{\frac{\gamma Q}{2}}.$$

Writing such densities as $\rho = e^{\frac{\gamma}{2}u}$, this action is expressed in terms of u by

$$u \mapsto u \circ h + Q \log h'. \tag{1.33}$$

The stabiliser of 0 is the group of rotations, so we may identify the orbit of 0 with the coadjoint orbit $\mathbb{S}^1 \backslash \mathrm{Diff}(\mathbb{S}^1)$. By Proposition 1.2.1, this action extends to all h in the Weil-Petersson class and $u \in \dot{H}^{1/2}(\mathbb{S}^1)$, so we get a right action of $\mathcal{T}_0(1)$ on $\dot{H}^{1/2}(\mathbb{S}^1)$. Proposition 1.2.1 also implies that the action is transitive, so that we can identify $\dot{H}^{1/2}(\mathbb{S}^1)$ with $\mathcal{T}_0(1)$.

We will write $u.h$ for the action of h on u. The differential of the action in the direction $v \in T_u\dot{H}^{1/2}(\mathbb{S}^1) \simeq \dot{H}^{1/2}(\mathbb{S}^1)$ is given for all u, h by

$$\begin{cases} T_u\dot{H}^{1/2} & \to T_{u.h}\dot{H}^{1/2} \\ v & \mapsto v \circ h. \end{cases}$$

This is nothing but the universal (or KYNS) period mapping of Nag & Sullivan [KY88, NS95, TT06], which is a symplectomorphism of $\dot{H}^{1/2}$ (recall (1.9)). Hence the action of $\mathcal{T}_0(1)$ on $\dot{H}^{1/2}$ is symplectic. In fact, this action is even Hamiltonian since $\dot{H}^{1/2}$ is topologically trivial, and it is easy to show that the corresponding Hamiltonian function is

$$H_{\mathbf{v}}(u) := \int_0^{2\pi} \left(-\frac{1}{2}(\partial_\theta u)^2 + Q\partial_\theta^2 u \right) v \frac{\mathrm{d}\theta}{2\pi}.$$

Let us define the function

$$\mathbf{M}(\theta) = \mathbf{M}_u(\theta) := -\frac{1}{2}\partial_\theta u(\theta)^2 + Q\partial_{\theta\theta}^2 u(\theta), \tag{1.34}$$

so that $H_{\mathbf{v}}(u) = \int_0^{2\pi} \mathbf{M}(\theta)v(\theta)\frac{\mathrm{d}\theta}{2\pi}$. For a fixed u, it is natural to view $\mathbf{M}$ as the linear form $v \mapsto \int_0^{2\pi} \mathbf{M}(\theta)v(\theta)\mathrm{d}\theta$. More intrinsically, $\mathbf{M}$ is the quadratic differential $\mathbf{M}(\theta)(\mathrm{d}\theta)^2$ which is dual to vector fields of the form $\mathbf{v} = v(\theta)\frac{\partial}{\partial\theta}$. Hence $\mathbf{M}$ lives in the dual Lie algebra of $\mathrm{Diff}(\mathbb{S}^1)$. Therefore, the mapping $u \mapsto \mathbf{M}_u$ is nothing but a momentum map for the $\mathrm{Diff}(\mathbb{S}^1)$ action on $\dot{H}^{1/2}(\mathbb{S}^1)$.

Using the integration by parts formula, we have for all smooth random variables F,

$$\mathbb{E}[\{\mathbf{M}(\theta), F\}] = \mathbb{E}[F\{\mathbf{M}(\theta), \mathbf{S}\}] \tag{1.35}$$

where $\mathbf{S}(u) = \frac{1}{2}\|u\|^2_{\dot{H}^{1/2}(\mathbb{S}^1)}$ and the expectation is with respect to the probability measure of the trace of the Gaussian free field on $\mathbb{S}^1$. However, this formula does not hold as such since the quadratic term in (1.34) requires normal ordering. Hence the correct expression is $\mathbf{M}(\theta) = -\frac{1}{2}(\partial_\theta W(\theta)^2 - \mathrm{Var}(\partial_\theta W(\theta)^2)) + Q\partial_{\theta\theta}^2 W$, which makes sense as a tempered distribution (in the sense of Remark 1).

Equation (1.35) expresses the variation of the law of the GFF under a infinitesimal change of conformal structure. It is analogous to the Ward identity and $T(\theta) := \{\mathbf{M}(\theta), \mathbf{S}\}$ is the stress-energy tensor (as suggested also by its expression). Integrating (1.35) against vector fields on the circle gives a picture analogous to (1.30).

Next, we consider an SLE loop η with its quantum length ℓ obtained by conformally welding independent instances of GMC. The welding operation is just a unitary operator from the L^2-space of two independent GMCs to the L^2-space of (η, ℓ). Representing the loop by its welding homeomorphism, there are natural left and right actions of $\mathrm{Diff}(\mathbb{S}^1)$ by post- and pre-composition. This is the same as acting on the GFF in the manner described in (1.33). Finally, an energy identity similar to [VW20, Theorem 3.1] shows that the SLE measure can be interpreted as a path integral with respect to the universal Liouville action of Takhtajan & Teo.

The $\mathrm{Diff}(\mathbb{S}^1)$-action can also be interpreted in a broader geometric setting as an extension of the Fenchel-Nielsen twist and Mirzakhani's $\mathbb{S}^1$-action on the moduli space of Riemann surfaces [Mir07b]. She was considering the moduli space $\widehat{\mathcal{M}}_{g,n}$ of bordered hyperbolic surfaces with a marked point on each boundary circle, with the $\mathbb{S}^1$-action given by rotating the marked point. The analysis of the symplectic reduction of this space is at the basis of the connection she found between Weil-Petersson volume and intersection numbers of tautological classes. More generally, we can decorate Mirzakhani's moduli space with a parameterisation of each boundary circles, and consider the $\mathrm{Diff}(\mathbb{S}^1)$-action. In particular, one can attach a Virasoro module to each boundary circle, and it would be worth exploring the connections of this structure with CFT in more details.

1.5.2 Sugawara construction

The previous construction is a natural way to produce a Virasoro representation since it is based on the realisation of the Witt algebra as differential operators in the direction of fundamental vector fields (more precisely, a connection on a line bundle) over the target manifold. However, in conformal field theory, the Virasoro symmetry is usually realised through the celebrated *Sugawara construction*, in which the Virasoro generators are second order differential operators. This is obviously different from a connection, since it is a first order operators.

The Sugawara construction has been extensively studied from an algebro-geometric point of view [FBZ01], but we are currently establishing a natural probabilistic interpretation in connection with the harmonic analysis on $\mathrm{Diff}(\mathbb{S}^1)$. We sketch this interpretation below in the case of Liouville theory. The Hilbert space is the space $L^2(\mathrm{d}c \otimes \mathbb{P})$, where $\mathrm{d}c$ is Lebesgue measure on $\mathbb{R}$ and $\mathbb{P}$ is the law of the trace on $\mathbb{S}^1$ of the GFF in $\mathbb{S}^2$. In Fourier modes, this field is written $\varphi(e^{i\theta}) = \sum_{n=1} \varphi_n e^{ni\theta} + \varphi_{-n} e^{-ni\theta}$, with $\varphi_{-n} = \bar{\varphi}_n$ and the φ_n are independent complex Gaussian with variance $\frac{1}{2n}$ [GKRV20]. The Malliavin differential on $L^2(\mathbb{P})$ has an adjoint d^*, densely defined on the space of $\dot{H}^{1/2}(\mathbb{S}^1)$-valued, square-integrable random variables. The corresponding Laplacian on functions is $\mathrm{d}^*\mathrm{d}$ and it is an infinite dimensional Ornstein-Uhlenbeck (OU) operator: it generates independent OU processes $(\varphi_n(t))_{n\geq 1}$. In the free Liouville theory (where the cosmological constant is set to zero), one can identify the generator $\mathbf{L}_0$ of the Sugawara construction with the operator $\frac{1}{2}(-\partial_c^2+Q^2)+\mathrm{d}^*\mathrm{d}$.

The identification of $\mathbf{L}_0$ with the generator of a diffusion process has a generalisation to the other modes of the Virasoro representation. We call a holomorphic vector field $\mathbf{v}\in\mathbb{C}[z]\partial_z$ *Markovian* if it generates a family of conformal transformations $f_t:\mathbb{D}\to D_t, z\mapsto z+tv(z)+o(t)$ such that $(D_t)_{t\geq 0}$ is a decreasing family of domains with smooth boundary. Let $\mathbf{P}\varphi$ be the harmonic extension to $\mathbb{D}$ of the field φ above, and set $X:=X_{\mathbb{D}}+\mathbf{P}\varphi$ where $X_{\mathbb{D}}$ is an independent Dirichlet GFF. Finally, define a stochastic process

$$\varphi_t:=(X\circ f_t+Q\log|f_t'|)_{|\mathbb{S}^1},$$

i.e. we pull back to $\mathbb{S}^1$ the values of X on $f_t(\mathbb{S}^1)$. Define $\mathbf{L}_{\mathbf{v}}$ to be the generator of this process. It turns out that there is a unique way to extend the assignment $\mathbf{v}\to\mathbf{L}_{\mathbf{v}}$ from Markovian vector fields to a $\mathbb{C}$-linear (resp. $\mathbb{C}$-antilinear) map $\mathbb{C}(z)\partial_z\ni\mathbf{v}\to\mathbf{L}_{\mathbf{v}}^+$ (resp. $\mathbb{C}(z)\partial_z\ni\mathbf{v}\to\mathbf{L}_{\mathbf{v}}^-$) forming two commuting representations of the Virasoro algebra with central charge $c=1+6Q^2$. Namely, we have

$$[\mathbf{L}_{\mathbf{v}}^{\pm},\mathbf{L}_{\mathbf{w}}^{\pm}]=\mathbf{L}_{[\mathbf{v},\mathbf{w}]}^{\pm}+c\omega(\mathbf{v},\mathbf{w});\qquad[\mathbf{L}_{\mathbf{v}}^{+},\mathbf{L}_{\mathbf{w}}^{-}]=0,$$

for all mereomorphic vector fields $\mathbf{v},\mathbf{w}$, where ω is the Virasoro cocycle introduced in Section 1.2.1. Moreover, this representation is equivalent to the Sugawara construction. We stress that this result is a rather elementary result on the GFF and we find it surprising that it seemed to have escaped from the mathematical literature until now (some related computations appear in [KM13]). In this construction, the Markovian vector fields have a rather deep interpretation: they are self-adjoint with respect to a different inner-product on $\dot{H}^{1/2}(\mathbb{S}^1)$ that is still compatible with the canonical symplectic structure.

To go from the free field theory to Liouville theory, one needs to treat the Liouville potential and show that commutation relations are preserved in a suitable sense. This step is non-trivial since the Liouville potential is a distributional random variable (it is not even in $L^1(\mathbb{P})$ for $\gamma\geq\sqrt{2}$). Relying on results from [GKRV20] and the idea of Markovian deformations of the disc, we are able to define the Liouville operators and make rigorous sense of the formulae appearing in the physics literature [Tes01]

$$\mathbf{L}_{\mathbf{v}}^{\mu}=\mathbf{L}_{\mathbf{v}}+\mu\int_0^{2\pi}e^{\gamma\varphi(e^{i\theta})}\mathrm{Re}(e^{-i\theta}v(\theta))\mathrm{d}\theta.$$

Moreover, we can use the scattering theory of the Liouville Hamiltonian developed in [GKRV20] to show that the scattering operator intertwines the Virasoro representations.

After the oral examination of this thesis, the probabilistic interpretation of the Sugawara construction has triggered some discussions with one of the examiners. Together with the authors of [GKRV20], we plan to use these results to prove that the scattering matrix of the Liouville Hamiltonian is diagonal. Another joint project with these authors is to establish the conformal Ward identities in the formalism of [FBZ01] and show that conformal blocks are well-defined as locally analytic functions on the moduli space of Riemann surfaces). Finally, we are planning to estbalish the BPZ equations in all generality in a joint work with J. Dubédat and G. Remy.

It would be interesting to study other probabilistic realisations of the Sugawara construction. One natural direction is the coupling of LCFT with conformal loop ensemble (CLE), where the conformal symmetry is still not completely understood [AS21]. Another direction is CFT with extended symmetry such as WZW models [Wit84], where the symmetry algebra is the affine Kac-Moody algebra of a semi-simple Lie group G. These models are geometrically more involved than Liouville theory since the target manifold is curved. In these models, the natural thing to do is to qantise the coadjoint orbits of the Kac-Moody group, so we need to define a probability measure on an infinite dimensional (curved) manifold. This would pave the way to the harmonic analysis of loop groups and a proabilistic proof of the celebrated Knizhnik-Zamolodchikov equations.

Chapter 2

One-point function in genus 1

This chapter is adapted from [Bav19].

In the context of the probabilistic formulation of LCFT introduced in 1.4.4, we study the asymptotic behaviour of the one-point correlation function in genus 1, as defined in [DRV16]. The degeneration paradigm considered is the limit $\mathrm{Im}\tau \to \infty$, where $\tau \in \mathbb{H}$ denotes the modulus of the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$.

2.1 Introduction

In theoretical physics, there are two approaches to Conformal Field Theories (CFTs). The first is the Hamiltonian approach: it consists in quantising an action functional and is usually treated with Feynman path integrals. The second is the conformal bootstrap: an abstract machinery used to classify CFTs from the algebraic information encoded by conformal invariance. Liouville CFT arises in the Hamiltonian approach in many fields of theoretical physics, notably in string theory [Pol81, Dav88, DP86]. In the conformal bootstrap, it is the first CFT with continuous spectrum that physicists were able to "solve" [Rib14].

From a mathematical point of view, path integrals are not rigorous, but recently, a rigorous probabilistic framework based on the Gaussian Free Field (GFF) and Gaussian Multiplicative Chaos (GMC) was introduced in order to make sense of the path integral approach to LCFT on any compact Riemann surface [DKRV16, DRV16, GRV19]. The remaining challenge for probabilists is to show that the path integral carries all the representation theoretic aspects predicted by the conformal bootstrap.

A first step was made in this direction when [KRV20] showed that the structure constants of LCFT (see Section 2.1.2) satisfy the so-called DOZZ formula. The term "bootstrapping" refers to the recursive computation of correlation functions from the structure constants, and this paper checks the validity of this recursion in a weakly interacting regime. From a probabilistic point of view, the DOZZ formula is a highly non-trivial integrability result on GMC, and it was soon followed by the results of [Rem20, RZ20a] where similar methods were implemented in order to compute the law of GMC on the unit circle and interval.

2.1.1 Path integral

Let M be either the Riemann sphere $\mathbb{S}^2 \simeq \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ or the torus $\mathbb{T}_\tau \simeq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for some $\tau \in \mathbb{H} := \{\mathrm{Im}\tau > 0\}$. The *Liouville action* with background metric g on M is the map $S_{\mathrm{L}} : \Sigma \to \mathbb{R}$ (where Σ is some function space to be determined) defined by[1]

$$S_{\mathrm{L}}(X; g) = \frac{1}{4\pi} \int_M \left(|\nabla X|^2 + 4\pi\mu e^{\gamma X} g(z) \right) dz, \tag{2.1}$$

where $\mu > 0$ is the cosmological constant (whose value is unimportant for this paper) and $\gamma \in (0, 2)$ is the parameter of the theory. Liouville quantum field theory is the measure formally defined by

$$\langle F \rangle := \int F(X) e^{-S_{\mathrm{L}}(X;g)} DX \tag{2.2}$$

for all continuous functional F. Here, DX should stand for "Lebesgue" measure on Σ. Of course, this does not make sense mathematically but it is possible to interpret the formal measure

$$\frac{1}{Z_{\mathrm{GFF}}} e^{-\frac{1}{4\pi} \int_M |\nabla X|^2 dz} DX \tag{2.3}$$

as a Gaussian probability measure on some Hilbert space (to be determined). The resulting field is called the Gaussian Free Field and the quantity Z_{GFF} is a "normalising constant" turning the measure (2.3) into a probability measure. We will refer to it as the partition function of the GFF (see Section 2.2.1).

As it turns out, the GFF does not live in the space of continuous functions (not even in L^2) but is rather a distribution in the sense of Schwartz. It can be shown that the GFF almost surely lives in the topological dual of the Sobolev space H^1 with respect to the L^2 product. Hence the exponential term $e^{\gamma X} dz$ appearing in the action is not *a priori* well-defined, but it can be made sense of after a regularising procedure based on Kahane's theory of Gaussian Mutiplicative Chaos (GMC) (see Section 2.2.2).

The main observables of the theory are the *vertex operators* $V_\alpha(z_0) = e^{\alpha X(z_0)}$ for any $z_0 \in M$ and $\alpha < Q := \frac{2}{\gamma} + \frac{\gamma}{2}$. The point z_0 is called an *insertion* as it has the interpretation of puncturing M with a conical singularity of order α/Q (see [HMW11] and Appendix 2.B). The coefficient α is called the *Liouville momentum* and $\Delta_\alpha := \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ is called the *conformal dimension*. The vertex operators give rise to the correlation functions $\langle \prod_{n=1}^N V_{\alpha_n}(z_n) \rangle$ which are defined for any pairwise disjoint $z_1, ..., z_N \in M$ and

[1] Usually the Liouville action features an additional curvature term. We omitted it since we will work with a background metric which is flat everywhere except on the unit circle.

$\alpha_1, ..., \alpha_N \in \mathbb{R}$ satisfying the so-called *Seiberg bounds*

$$\sum_{n=1}^{N} \frac{\alpha_n}{Q} - \chi(M) > 0 \qquad\qquad \forall n, \alpha_n < Q, \tag{2.4}$$

where $\chi(M)$ is the Euler characteristic. The Seiberg bounds have a geometric nature: the α_n/Q singularity introduced by $V_{\alpha_n}(z_n)$ is integrable only if $\alpha_n < Q$, hence the second bound in (2.4). On the other hand, Gauss-Bonnet theorem shows that the first bound is equivalent to asking for the total curvature on the surface $M \setminus \{z_1, ..., z_N\}$ with prescribed conical singularities α_n/Q at z_n to be negative. In particular, the correlation function exists only if $N \geq 3$ for the sphere and $N \geq 1$ for the torus.

We now briefly recall the results that will be needed in order to state the main result. Consider the Riemann sphere $\mathbb{S}^2 \simeq \widehat{\mathbb{C}}$ equipped with the metric $g(z) = |z|_+^{-4}$ (with the notation $|z|_+ = \max(1, |z|)$). We will refer to this metric as the *crêpe metric* as it consists in two flat disks glued together (as can be seen from the change of variable $z \mapsto 1/z$). The 3-point function enjoys some conformal covariance under Möbius transformations [DKRV16], implying that we can choose to put the insertions at $0, 1, \infty$. It was shown in [KRV20] that for all $\alpha_1, \alpha_2, \alpha_3$ satisfying the Seiberg bounds, $\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle_{\mathbb{S}^2} = C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ where $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ is the celebrated DOZZ formula (see Appendix 2.A).

Recall that a torus is a curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with $\tau \in \mathbb{H}$. The *moduli group* $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ acts on $\mathbb{H}$ via linear fractional transformation

$$\psi.\tau = \frac{a\tau + b}{c\tau + d}$$

for all $\psi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. The *moduli space* is the quotient $\mathcal{M} := \Gamma \setminus \mathbb{H}$. Two tori with moduli τ, τ' respectively are conformally equivalent if and only if there exists $\psi \in \Gamma$ such that $\tau' = \psi.\tau$. The fundamental domain of $\mathcal{M}$ is the set $\{z \in \mathbb{H},\ \mathrm{Re}(z) \in (-1/2, 1/2]$ and $|z| > 1\} \cup \{e^{i\theta}, \theta \in [\frac{\pi}{3}, \frac{\pi}{2}]\}$ (see Figure 2.1), so that the boundary of the moduli space can be approached by moduli $\tau = \frac{it}{\pi}$ for large t. These correspond to "skinny" tori. From [DRV16] it is possible to define the 1-point correlation function $\langle V_\alpha(0)\rangle_\tau$ with flat background metric for each modulus $\tau \in \mathcal{M}$ and $\alpha \in (0, Q)$,

Using the framework of CFT known as the conformal bootstrap, physicists have argued that all correlation functions on any surface can be derived from the three-point function on the sphere by some topological recursion (see Section 2.1.2). In the case of the one-point function on the torus, the formula involves an integral over some algebraically defined objects that do not yet have a probabilistic representation (see Equation (2.6)). However, these objects have nice asymptotic behaviours as $\mathrm{Im}\tau \to \infty$, explaining why we were able to compute the asymptotic behaviour of the one-point toric function and match it with

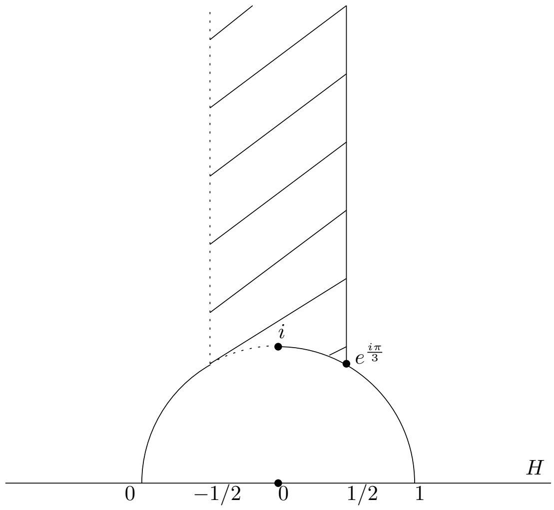

Figure 2.1: The moduli space $\mathcal{M} = \Gamma \setminus \mathbb{H}$ (hashed). The vertical lines are identified, so that it is topologically a sphere with three marked points at $e^{i\pi/3}, i$ and ∞. The interesting boundary point is ∞, and we will approach it using moduli $\tau = \frac{it}{\pi}$ for large t. These correspond to "skinny" tori.

the bootstrap prediction in this limit.

2.1.2 Conformal bootstrap

From the operator theoretic perspective, a quantum field theory is the data of a self-adjoint non-negative Hamiltonian acting on some Hilbert space. In their founding paper, [BPZ84] argued that the Hilbert space of a $2d$ conformal field theory must carry a representation of the Virasoro algebra. This strong constraint on the structure of the Hilbert space led to spectacular integrability results, among which the DOZZ formula from Liouville theory. Although the representation theory of the Virasoro algebra is well-understood mathematically, it is only a conjecture that the path integral of the quantised Liouville action carries the expected algebraic structure. Thus, except for the results of [KRV19, KRV20], all the formulae from the conformal bootstrap are to be considered as predictions and not facts.

In the conformal bootstrap framework, any CFT should be characterised by

1. The *spectrum* of the Hamiltonian $S \subset \mathbb{R}_+$. For each $\alpha \in \mathbb{C}$ such that $\Delta_\alpha \in S$, the field $V_\alpha(\cdot)$ is called a *primary field*. It is important to note that the conformal bootstrap assumes that vertex operators are defined for all $\alpha \in \mathbb{C}$ and not necessarily for α in the "physical region" defined by the Seiberg bounds. The spectrum of Liouville theory is conjectured to be $[\frac{Q^2}{4}, \infty)$, corresponding to momenta $\alpha \in Q + i\mathbb{R}$.

2. The *structure constants*, i.e. the three-point functions $\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle_{\mathbb{S}^2}$. In Liouville CFT, these are given by the DOZZ formula $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ [DO94][ZZ96]. Correlation functions are meromorphic functions of each $\alpha \in \mathbb{C}$.

From the data of the spectrum and the structure constants, the bootstrap machinery gives a way to compute recursively all correlation functions on any Riemann surface of any genus. Thus, "solving" a theory means finding both the spectrum and the structure constants.

The two most simple examples are the 4-point spherical and the 1-point toric correlation function. Given two copies M_1, M_2 of the thrice punctured sphere $\mathbb{S}^2 \setminus \{0, 1, \infty\}$, one can glue together annuli neighbourhoods of punctures in M_1 and M_2 to produce a 4-punctured sphere (see Figure 2.2). Similarly, given one instance of the thrice-punctured sphere, one can glue together annuli neighbourhoods of 0 and ∞ to produce the once-punctured torus.

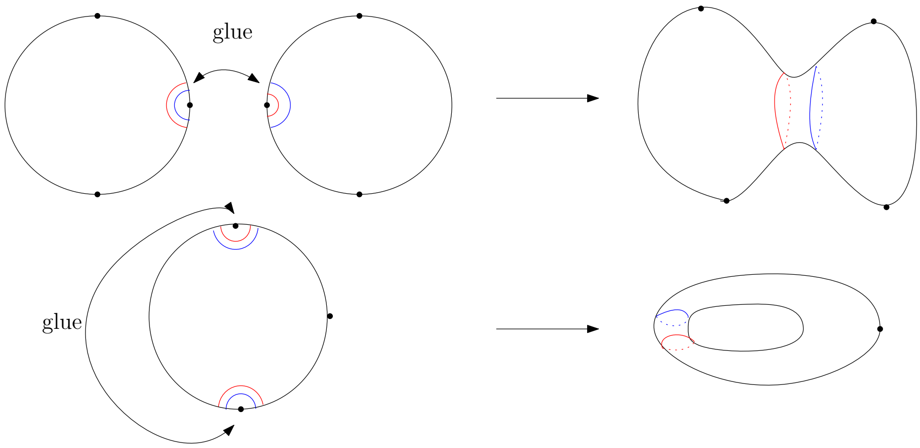

Figure 2.2: Top: On the left, two instances of the thrice-punctured sphere with annuli neighbourhoods to be identified (curves of the same colour are identified). The resulting surface on the right: a sphere with 4 marked points. **Bottom:** Annuli neighbourhoods of the north and south pole are identified to produce a torus with one marked point.

More generally, this procedure gives a way to construct any Riemann surface of genus $\mathbf{g}_1 + \mathbf{g}_2$ and $n_1 + n_2$ punctures by gluing a surface of genus $\mathbf{g}_1$ and $n_1 + 1$ punctures to a surface of genus $\mathbf{g}_2$ and $n_2 + 1$ punctures (see [TV15] for details of this construction). Similarly a surface of genus $\mathbf{g}$ and $n + 2$ punctures gives a surface of genus $\mathbf{g} + 1$ and $n + 2$ punctures after gluing together two distinct punctured neighbourhoods. This gives a recursive procedure to construct any Riemann surface using only instances of the thrice-punctured sphere. This construction is one of the driving ideas behind the fact that three-point functions are building blocks for CFTs.

The two simple examples above are the starting point of the bootstrap programme as they require only one gluing. Physicists have predicted formulae – called the bootstrap equations – that compute these correlation functions using the spectrum and the structure constants. The bootstrap equation for the 4-point function on the sphere is given by[2] [BZ06]

[2] We add the superscript $^{\mathrm{cb}}$ for "conformal bootstrap" and to differentiate it from the path integral correlation function.

$$\begin{aligned}\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}}_{\mathbb{S}^2} = \frac{1}{8\pi}|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)} \\ \times \int_{-\infty}^{\infty} |z|^{2P^2} C_\gamma(\alpha_1,\alpha_2,Q-iP)C_\gamma(Q+iP,\alpha_3,\alpha_4)|\mathcal{F}_P^{\alpha_{1234}}(z)|^2 dP,\end{aligned} \tag{2.5}$$

where $\Delta_i = \frac{\alpha_i}{2}(Q-\frac{\alpha_i}{2})$ $(i=1,2)$ and $\mathcal{F}_P^{\alpha_{1234}}(z) = 1+o(1)$ is the so-called *Virasoro conformal block* – a holomorphic function of z, universal in the sense that it only depends on the α_i's, P and γ.

There is a similar formula to (2.11) for the 1-point toric function [HJS10, Equation (20)], which is the one this paper is concerned about. For a torus of modulus τ, we have

$$\langle V_\alpha(0)\rangle^{\mathrm{cb}}_\tau = \frac{1}{2}\int_{\mathbb{R}} C_\gamma(Q-iP,\alpha,Q+iP)\left|q^{\frac{P^2}{4}}\eta(q)^{-1}\mathcal{H}^\alpha_{\gamma,P}(q)\right|^2 dP, \tag{2.6}$$

where $q=e^{2i\pi\tau}$ is the *nome* and $\eta(\cdot)$ is *Dedekind's êta function*. Here the so-called *elliptic conformal bloc* $\mathcal{H}^\alpha_{\gamma,P}$ admits a power series expansion in q

$$\mathcal{H}^\alpha_{\gamma,P}(q) = \frac{\eta(q)}{q^{1/24}}\left(1+\sum_{n=1}^\infty H^{\alpha,n}_{\gamma,P}q^n\right)$$

and the function in the brackets is holomorphic in q. The elliptic conformal blocks should be understood as a basis of solutions for the one-point toric function, and they are universal in the sense that they depend only on α,γ and P. We will refer to equation (2.6) as the *modular bootstrap*. Again, this formula should be valid *a priori* only for a primary field but we will show that it is true for $\alpha\in(0,Q)$ in the path integral framework when $\mathrm{Im}\tau\to\infty$.

At this stage, let us stress again that equations (2.5) and (2.6) should be understood only as guesses since there is still no mathematical justification for them. In general, one way to establish rigorously the validity of the conformal bootstrap would be to recover its results using the rigorous path integral approach of DKRV. This is usually a hard matter but some works were made in this direction [KRV19, KRV20]. In the first paper, the authors showed the validity of some aspects of the bootstrap approach – namely BPZ equation and Ward identities –, while the second is a proof of the DOZZ formula.

From the point of view of probability, both the conformal blocks and the spectrum are not understood (there is not even a probabilistic interpretation of complex Liouville momenta). As we mentioned earlier, the integral in (2.6) simplifies as $\mathrm{Im}\tau\to\infty$, namely the conformal blocks tend to 1 and the integral freezes at $P=0$, avoiding dealing with complex insertions.

2.1.3 Main result and outline

Suppose $\tau = \frac{it}{\pi}$ with $t > 0$ large, so that $q = e^{-2t}$ is real and small. Recall that the DOZZ formula is meromorphic and symmetric with respect to the real axis, hence

$$C_\gamma(Q+iP,\alpha,Q-iP) \underset{P\to 0}{\sim} P^2\partial^2_{\alpha_1\alpha_3}C_\gamma(Q,\alpha,Q).$$

Taking $\mathcal{H}^\alpha_{\gamma,P}(q) \equiv 1$ uniformly in q as $P \to 0$, equation (2.6) gives in the limit $t \to \infty$

$$\begin{aligned}
\langle V_\alpha(0)\rangle^{\mathrm{cb}}_{\frac{it}{\pi}} &= \frac{|\eta(\frac{it}{\pi})|^{-2}}{2}\int_{\mathbb{R}} C_\gamma(Q-iP,\alpha,Q+iP)\left|q^{\frac{P^2}{4}}\mathcal{H}^\alpha_{\gamma,P}(q)\right|^2 dP\\
&\sim \frac{|\eta(\frac{it}{\pi})|^{-2}}{2}\int_{\mathbb{R}} C_\gamma(Q-iP,\alpha,Q+iP)e^{-\frac{tP^2}{2}}\left|\mathcal{H}^\alpha_{\gamma,P}(q)\right|^2 dP\\
&= \frac{|\eta(\frac{it}{\pi})|^{-2}}{2}t^{-1/2}\int_{\mathbb{R}} C_\gamma\left(Q-i\frac{P}{\sqrt{t}},\alpha,Q+i\frac{P}{\sqrt{t}}\right)e^{-\frac{P^2}{2}}\left|\mathcal{H}^\alpha_{\gamma,\frac{P}{\sqrt{t}}}(q)\right|^2 dP \qquad (2.7)\\
&\sim \frac{|\eta(\frac{it}{\pi})|^{-2}}{2}t^{-3/2}\partial^2_{\alpha_1\alpha_3}C_\gamma(Q,\alpha,Q)\int_{\mathbb{R}} P^2 e^{-\frac{P^2}{2}}dP\\
&\sim \sqrt{\frac{\pi}{2}}\left|\eta\left(\frac{it}{\pi}\right)\right|^{-2} t^{-3/2}\partial^2_{\alpha_1\alpha_3}C_\gamma(Q,\alpha,Q).
\end{aligned}$$

Rewriting this in terms of the modulus, we have in the limit $\mathrm{Im}\tau \to \infty$

$$\langle V_\alpha(0)\rangle^{\mathrm{cb}}_\tau \sim \frac{\sqrt{2}}{\pi}|\eta(\tau)|^{-2}(\mathrm{Im}\tau)^{-3/2}\partial^2_{\alpha_1\alpha_3}C_\gamma(Q,\alpha,Q). \qquad (2.8)$$

There are two noticeable facts about the asymptotic behaviour of $\langle V_\alpha(0)\rangle_{\frac{it}{\pi}}$:

- There is a polynomial decay in $t^{-3/2}$ correcting the exponential term $|\eta(\frac{it}{\pi})|^{-2}$.
- The limit is expressed using the derivative of the DOZZ formula at the critical points $\alpha_1 = \alpha_3 = Q$.

Throughout, we will write $\mathbb{T}_t$ for a torus with modulus $\tau = \frac{it}{\pi}$ and think of t large. Our representation for $\mathbb{T}_t$ is the rectangle $\mathbb{T}_t := (-t,t] \times \mathbb{S}^1$ with edges $\{-t\}\times\mathbb{S}^1$ and $\{t\}\times\mathbb{S}^1$ identified, and equipped with the flat metric. The reason for this choice of parametrisation is that the variable t will appear as the time driving a Brownian motion.

Let $\mathcal{C}_\infty := \mathbb{R}\times\mathbb{S}^1$ be the infinite cylinder. This surface is the image of the twice-punctured sphere $\widehat{\mathbb{C}}\setminus\{0,\infty\}$ under the change of coordinates $\psi : \mathcal{C}_\infty \to \widehat{\mathbb{C}}\setminus\{0,\infty\}$, $z \mapsto e^{-z}$. In the sequel, we will always parametrise the sphere with these coordinates. Of particular interest for us will be the correlation function $\langle V_\lambda(0)V_\alpha(1)V_\lambda(\infty)\rangle_{\mathbb{S}^2}$ for $\lambda,\alpha \in (0,Q)$ and

$\sigma = 2(\lambda - Q) + \alpha > 0$. In the cylinder coordinates, these take the form [KRV20]

$$\langle V_\lambda(0)V_\alpha(1)V_\lambda(\infty)\rangle_{\mathbb{S}^2} = 2\gamma^{-1}\mu^{-\frac{Q\sigma}{\gamma}}\Gamma\left(\frac{Q\sigma}{\gamma}\right)\mathbb{E}\left[\left(\int_{\mathcal{C}_\infty} e^{\gamma((\lambda-Q)|t|+\alpha G(0,t+i\theta))}dM^\gamma(t,\theta)\right)^{-\frac{Q\sigma}{\gamma}}\right] \tag{2.9}$$

where G is Green's function on $\mathcal{C}_\infty$ with zero average on $\{0\}\times\mathbb{S}^1$ and M^γ is the chaos measure associated to a GFF on $\mathcal{C}_\infty$.

The negative drift $\lambda - Q$ is essential in order to make the total GMC mass finite near $\pm\infty$. On the contrary if $\lambda = Q$, the GMC mass is a.s. infinite and the correlation function vanishes. In this critical case, we consider the truncated correlation function

$$\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_t = 2\gamma^{-1}\mu^{-\frac{\alpha}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}\right)\mathbb{E}\left[\left(\int_{\mathcal{C}_t} e^{\gamma\alpha G(0,\cdot)}dM^\gamma(s,\theta)\right)^{-\frac{\alpha}{\gamma}}\right], \tag{2.10}$$

where $\mathcal{C}_t := (-t,t)\times\mathbb{S}^1$.

The truncated correlation function is just the correlation function where we integrate the GMC measure outside a small disc of radius e^{-t} away from the singularities (when seen in the planar coordinates).

As for the torus $\mathbb{T}_t$, the 1-point function is defined by

$$\langle V_\alpha(0)\rangle_{\frac{it}{\pi}} := 2\gamma^{-1}\mu^{-\frac{\alpha}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}\right)\left(\frac{t}{\pi}\right)^{-1/2}|\eta(\frac{it}{\pi})|^{-2}\mathbb{E}\left[\left(\int_{\mathbb{T}_t} e^{\gamma\alpha G_t(0,\cdot)}dM_t^\gamma\right)^{-\frac{\alpha}{\gamma}}\right], \tag{2.11}$$

where G_t is Green's function on $\mathbb{T}_t$.

Our main result, stated as Theorem 2.1.1 below shows that we recover the same polynomial rate and the derivative DOZZ formula when working with the correlation function computed in the path integral framework.

Theorem 2.1.1. *Let $\langle V_\alpha(0)\rangle_{\frac{it}{\pi}}$ be the 1-point toric correlation function given by* (2.11)*. Then*

$$\langle V_\alpha(0)\rangle_{\frac{it}{\pi}} \underset{t\to\infty}{\sim} \frac{3}{4\sqrt{\pi}}\left|\eta(\frac{it}{\pi})\right|^{-2} t^{-3/2}\partial^2_{\alpha_1\alpha_3}C_\gamma(Q,\alpha,Q). \tag{2.12}$$

Corollary 2.1.2. *In the setting of Theorem 1.1, we have for $\tau\in\mathcal{M}$:*

$$\langle V_\alpha(0)\rangle_\tau \underset{\mathrm{Im}\tau\to\infty}{\sim} \frac{3}{4\pi^2}|\eta(\tau)|^{-2}(\mathrm{Im}\tau)^{-3/2}\partial^2_{\alpha_1\alpha_3}C_\gamma(Q,\alpha,Q).$$

Remark 3. The fact that we don't recover the same global overall factor as in equation (2.8) is irrelevant since the correlation functions are defined up to multiplicative factor.

2.1.4 Steps of the proof

There will be two steps in the proof of Theorem 2.1.1. First we will compute the exact asymptotic behaviour of $\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_t$ as $t \to \infty$ (Proposition 2.1.3) and second we will compare $\langle V_\alpha(0)\rangle_{\frac{it}{\pi}}$ to $\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_t$ (Proposition 2.1.4). This is the point of using the cylinder coordinates for the sphere, as we can embed $\mathbb{T}_t$ into $\mathcal{C}_t$. Namely, we will show that negative moments of GMC on $\mathbb{T}_t$ and on $\mathcal{C}_t$ have the same asymptotic behaviour, up to some explicit constant.

Proposition 2.1.3. *For all $\alpha \in (0, Q)$,*

$$\lim_{t\to\infty} t\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_t = \frac{1}{2\pi}\partial^2_{\alpha_1\alpha_3}C_\gamma(Q,\alpha,Q). \tag{2.13}$$

Proposition 2.1.4. *Let X be a GFF on $\mathcal{C}_\infty$ and X_t be a GFF on $\mathbb{T}_t$, i.e. X and X_t have respective covariances G and G_t (Green's function with zero average on $\{0\}\times\mathbb{S}^1$). Let dM^γ and dM^γ_t be the associated chaos measures. Then for all $r > 0$ and $\alpha \in (0, Q)$,*

$$\lim_{t\to\infty} t\mathbb{E}\left[\left(\int_{\mathbb{T}_t} e^{\gamma\alpha G_t(0,z)}dM^\gamma_t(z)\right)^{-r}\right] = \frac{3}{2}\lim_{t\to\infty} t\mathbb{E}\left[\left(\int_{\mathcal{C}_t} e^{\gamma\alpha G(0,z)}dM^\gamma(z)\right)^{-r}\right]. \tag{2.14}$$

We will prove these propositions in Section 2.3. For now, we use Propositions 2.1.3 and 2.1.4 to prove Theorem 2.1.1 and Corollary 2.1.2.

Proof of Theorem 2.1.1. Using Propositions 2.1.3 and 2.1.4, we have

$$\begin{aligned}\langle V_\alpha(0)\rangle_{\frac{it}{\pi}} &= \sqrt{\pi}\frac{2\gamma^{-1}\mu^{-\frac{\alpha}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}\right)}{t^{1/2}|\eta(\frac{it}{\pi})|^2}\mathbb{E}\left[\left(\int_{\mathbb{T}_t} e^{\gamma\alpha G_t(0,\cdot)}dM^\gamma_t\right)^{-\frac{\alpha}{\gamma}}\right]\\ &\underset{t\to\infty}{\sim} \frac{3\sqrt{\pi}}{2}t^{-1/2}|\eta(\frac{it}{\pi})|^{-2}\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_t\\ &\underset{t\to\infty}{\sim} \frac{3}{4\sqrt{\pi}}t^{-3/2}|\eta(\frac{it}{\pi})|^{-2}\partial^2_{\alpha_1\alpha_3}C_\gamma(Q,\alpha,Q).\end{aligned} \tag{2.15}$$

In particular we recover the asymptotic formula of equation (2.7) up to an explicit global multiplicative constant.

□

Proof of Corollary 2.1.2. In this proof and this proof only, we make change the embedding and embed all tori in the square $[0,1]^2$ as in [DRV16]. We only need to compare the negative moments of GMC for tori with moduli τ, τ' such that $\mathrm{Im}\tau = \mathrm{Im}\tau'$ and show that they have the same asymptotic behaviour as $\mathrm{Im}\tau \to \infty$.

Let $\tau \in \mathcal{M}$ with $\mathrm{Im}\tau = \frac{t}{\pi}$. Let G_τ be Green's function on the torus $\mathbb{T}_\tau$ of modulus τ

and set $G_\tau(x) := G_\tau(0,x)$. It is readily seen from [DRV16, Equation (3.4)] that

$$|G_\tau(x) - G_{\frac{it}{\pi}}(x)| = O(e^{-2t})$$

uniformly in $x \in \mathbb{T}_\tau$. Now let dM_τ^γ and $dM_{\frac{it}{\pi}}^\gamma$ be the GMC measures of a GFF on $\mathbb{T}_\tau$ and $\mathbb{T}_{\frac{it}{\pi}}$ respectively. By Kahane's convexity inequality (see Section 2.2.2) we have for all $r > 0$

$$\mathbb{E}\left[\left(\int_{\mathbb{T}_\tau} e^{\gamma\alpha G_\tau(0,\cdot)} dM_\tau^\gamma\right)^{-r}\right] = \mathbb{E}\left[\left(\int_{\mathbb{T}_{\frac{it}{\pi}}} e^{\gamma\alpha G_{\frac{it}{\pi}}(0,\cdot)} dM_{\frac{it}{\pi}}^\gamma\right)^{-r}\right](1 + O(e^{-2t})). \tag{2.16}$$

This concludes the proof. □

The rest of this paper is devoted to proving Propositions 2.1.3, 2.1.4. This will be done is Section 2.3 while Section 2.2 gives the necessary probabilistic background needed for the proofs.

2.2 Background

In this section, we recall the definitions of the basic objects needed to define the correlation functions (2.9) and (2.11) (namely the GFF and GMC) and we give a derivation of the expression of these correlation functions.

2.2.1 Gaussian Free Field

We give a basic introduction to the Gaussian Free Field (GFF) on the complete cylinder $\mathcal{C}_\infty$ and the torus $\mathbb{T}_t$ (we refer the reader to [Dub09, DMS14, DKRV16, DRV16]).

To begin with, let us consider the case of $\mathcal{C}_\infty$ endowed with the flat metric. Let $H_0^1(\mathcal{C}_\infty)$ be the set of functions $f : \mathcal{C}_\infty \to \mathbb{R}$ with weak derivative in $L^2(\mathcal{C}_\infty)$ and such that $\int_0^{2\pi} f(0,\theta)d\theta = 0$. Then the (non-negative) Laplacian $-\frac{1}{2\pi}\Delta$ has a well defined inverse $G : L^2(\mathcal{C}_\infty) \to H_0^1(\mathcal{C}_\infty)$ called the *Green function*. It has a kernel satisfying for all $x \in \mathcal{C}_\infty$

$$\begin{cases} -\dfrac{1}{2\pi}\Delta G(x,\cdot) = \delta_x \\ \displaystyle\int_0^{2\pi} G(x, i\theta)d\theta = 0. \end{cases} \tag{2.17}$$

The GFF on $\mathcal{C}_\infty$ is the Gaussian field X on whose covariance kernel is given by Green's function

$$\mathbb{E}[X(x)X(y)] = G(x,y).$$

This is done at the formal level, since Green's function blows up logarithmically near the diagonal. However, it is possible to show that such a field exists and that it almost surely lives in $H_0^{-1}(\mathcal{C}_\infty)$. Hence the GFF on $\mathcal{C}_\infty$ is a distribution on $\mathcal{C}_\infty$ (and not a function).

We can define $H_0^1(\mathbb{T}_t)$ similarly as the space of functions $f : \mathbb{T}_t \to \mathbb{R}$ with weak derivatives in $L^2(\mathbb{T}_t)$ and vanishing mean on $\mathbb{T}_t$. The Laplacian $-\frac{1}{2\pi}\Delta_t$ on $\mathbb{T}_t$ also has a Green's function $G_t : L^2(\mathbb{T}_t) \to H_0^1(\mathbb{T}_t)$.

As explained in Section 2.1.1, the formal measure $e^{-\frac{1}{4\pi}\int_M |\nabla X|^2} DX$ should be interpreted as a Gaussian measure. To fix ideas, let us consider the case of the torus $\mathbb{T}_t$. Then the map

$$(f, g) \mapsto -\frac{1}{2\pi}\int_{\mathbb{T}_t} \Delta_t f \cdot g =: \langle f, g\rangle_\nabla$$

defines an inner-product on $H_0^1(\mathbb{T}_t)$ that we call the *Dirichlet energy*. We write $\|\cdot\|_\nabla$ for the associated norm. By analogy with the finite dimensional case, we want to interpret the density $e^{-\frac{1}{2}\|X\|_\nabla^2} DX$ as that of a centred Gaussian random variable with covariance kernel given by the inverse of $-\frac{1}{2\pi}\Delta$, i.e. Green's function G_t. This is nothing but the GFF of the previous paragraph. To keep with the analogy with the finite dimensional case, the partition function of the GFF (i.e. the "normalising constant") is given by [Gaw89]

$$Z_{\mathrm{GFF}}(t) := \det(-\Delta_t)^{1/2} = \frac{t}{\pi}|\eta(\frac{it}{\pi})|^2, \tag{2.18}$$

where $\det(-\Delta_t)$ is the zêta regularised determinant of the Laplacian (see [OPS88, Section 1] for a general definition and [Gaw89] p10 for the value on the torus).

The GFF on $\mathbb{T}_t$ can be constructed using an orthonormal basis of $L^2(\mathbb{T}_t)$ of eigenfunctions of $-\Delta_t$. If $(f_n)_{n\geq 0}$ is such a basis with associated eigenvalues $0 = \lambda_0 < \lambda_1, .., \leq \lambda_n ...$, then $(\sqrt{\frac{2\pi}{\lambda_n}} f_n)_{n\geq 1}$ is an orthonormal basis of $H_0^1(\mathbb{T}_t)$ and we set

$$X_t := \sqrt{2\pi}\sum_{n\geq 1}\frac{\alpha_n}{\sqrt{\lambda_n}} f_n,$$

where $(\alpha_n)_{n\geq 1}$ is a sequence of i.i.d. normal random variables. It can be shown that this formal series indeed converges almost surely in $H_0^{-1}(\mathbb{T}_t)$ [DRV16, Section 3.2].

As such, the constant coefficient of the GFF (a.k.a. the *zero mode*) depends on the choice of the background metric, since we impose a vanishing mean in the flat metric. In order to get rid of this dependence, we complete the definition of the field by "sampling" the constant coefficient with Lebesgue measure (see the discussion in [DKRV16, Section 2.2]). Informally, we can interpret the zero mode as a Gaussian random variable with variance $1/\lambda_0 = \infty$ since $\sqrt{\frac{2\pi}{\lambda}}$ times the law of an $\mathcal{N}(0, \lambda^{-1})$ converges vaguely to Lebesgue as $\lambda \to 0$. So we arrive at the field decomposition

$$X = X_t + \frac{c}{\sqrt{t/\pi}}$$

and the final interpretation is that for all continuous functional $F : H_0^{-1}(\mathbb{T}_t) \to \mathbb{R}$, we set[3]

$$\begin{aligned}\int F(X)e^{-\frac{1}{4\pi}\int_{\mathbb{T}_t}|\nabla X|^2}DX &= 2\ \det(-\Delta_t)^{-1/2}\int_{\mathbb{R}}\mathbb{E}\left[F(X_t+\frac{c}{\sqrt{t/\pi}})\right]dc \\ &= 2(\frac{t}{\pi})^{-1/2}|\eta(\frac{it}{\pi})|^{-2}\int_{\mathbb{R}}\mathbb{E}[F(X_t+c)]dc.\end{aligned} \tag{2.19}$$

This formula explains the $t^{-1/2}|\eta(\frac{it}{\pi})|^{-2}$ appearing in the asymptotic formula of Theorem 2.1.1. Applying this to a regularisation of the vertex operator $V_\alpha(0)$ leads to the expression (2.11) of the correlation function $\langle V_\alpha(0)\rangle_{\frac{it}{\pi}}$ [DRV16, Theorem 4.3].

For the torus, the natural eigenbasis of $L^2(\mathbb{T}_t)$ is given by the functions

$$\begin{aligned} f^{ee}_{n,m}(s,\theta) &:= \frac{1}{\sqrt{(1+\mathbb{1}_{n=0})(1+\mathbb{1}_{m=0})\pi t}}\cos(\frac{n\pi s}{t})\cos(m\theta) \\ f^{eo}_{n,m}(s,\theta) &:= \frac{1}{\sqrt{(1+\mathbb{1}_{n=0})\pi t}}\cos(\frac{n\pi s}{t})\sin(m\theta) \\ f^{oe}_{n,m}(s,\theta) &:= \frac{1}{\sqrt{(1+\mathbb{1}_{m=0})\pi t}}\sin(\frac{n\pi s}{t})\cos(m\theta) \\ f^{oo}_{n,m}(s,\theta) &:= \frac{1}{\sqrt{\pi t}}\sin(\frac{n\pi s}{t})\sin(m\theta), \end{aligned} \tag{2.20}$$

and the eigenvalue associated to the eigenfunction $f^{ee,eo,oe,oo}_{m,n}$ is $\lambda_{n,m} := \frac{n^2\pi^2}{t^2} + m^2$. Then we can set

$$X_t := \sqrt{2\pi}\sum_{n,m\neq(0,0)}\frac{\alpha^{ee}_{n,m}}{\sqrt{\lambda_{n,m}}}f^{ee}_{n,m} + \frac{\alpha^{eo}_{n,m}}{\sqrt{\lambda_{n,m}}}f^{eo}_{n,m} + \frac{\alpha^{oe}_{n,m}}{\sqrt{\lambda_{n,m}}}f^{oe}_{n,m} + \frac{\alpha^{oo}_{n,m}}{\sqrt{\lambda_{n,m}}}f^{oo}_{n,m}, \tag{2.21}$$

where $\alpha^{ee,eo,oe,oo}_{n,m}$ are i.i.d. centred normal random variables.

An immediate consequence of this decomposition is that we can sample X_t as follows

1. Sample a GFF X_t^{D} with zero (a.k.a. Dirichlet) boundary conditions[4] on the cylinder $(0,t)\times\mathbb{S}^1$
2. Sample an independent GFF X_t^{N} with free (a.k.a. Neumann) boundary conditions on the cylinder $(0,t)\times\mathbb{S}^1$.
3. For all $(s,\theta)\in(-t,t)\times\mathbb{S}^1$, set $X_t(s,\theta) := \frac{X_t^{\mathrm{N}}(|s|,\theta)+\mathrm{sign}(s)X_t^{\mathrm{D}}(|s|,\theta)}{\sqrt{2}}$.

[3]We add a factor 2 to conform with [KRV20]

[4]We refer the reader to [Ber16, Section 5] for an introduction to different types of boundary conditions.

We call this decomposition the *odd/even decomposition of fields*, which is based on the orthogonal decomposition $H^1_0(\mathcal{C}_t) = H^{1,e}_0(\mathcal{C}_t) \oplus H^{1,o}_0(\mathcal{C}_t)$ where $H^{1,e}_0(\mathcal{C}_t), H^{1,o}_0(\mathcal{C}_t)$ are respectively the subspaces of even and odd functions with respect to $s \in (-t,t)$. The nice property of this decomposition is that we can view the GFF X_t on $\mathbb{T}_t$ as a GFF on $\mathcal{C}_t$ whose odd part is a GFF with zero (Dirichlet) boundary conditions and whose even part is a GFF with free (Neumann) boundary conditions (see [Ber16, Section 5.1] for a discussion of this decomposition).

Let us now introduce the *radial/angular decomposition of fields* [DMS14, KRV20], which is based on the orthogonal decomposition $H^1_0(\mathcal{C}_t) = H^{1,R}_0(\mathcal{C}_t) \oplus H^{1,A}_0(\mathcal{C}_t)$ (for all $t \in (0,\infty]$) where

$$\begin{aligned} H^{1,R}_0(\mathcal{C}_t) &= \{f \in H^1_0(\mathcal{C}_t),\ f(s,\cdot) \text{ is constant on } \mathbb{S}^1 \text{ for all } s \in (-t,t)\} \\ H^{1,A}_0(\mathcal{C}_t) &= \{f \in H^1_0(\mathcal{C}_t),\ \forall s \in (-t,t)\ \int_{\mathbb{S}^1} f(s,\theta)d\theta = 0.\} \end{aligned} \tag{2.22}$$

For a field X on $\mathcal{C}_\infty$, we will write $X_0(t) := \frac{1}{2\pi}\int_{\mathbb{S}^1} X(t,\theta)d\theta$ for its mean on the circle $\{t\} \times \mathbb{S}^1$ for all $t \in \mathbb{R}$. Viewed in the planar coordinates, $X_0(t)$ is the mean value of X on the circle of radius e^{-t} about 0.

Now let X be a GFF on $\mathcal{C}_\infty$, normalised such that $X_0(0) = 0$. Then, from [DKRV17, Lemmata 4.2-3], we can write $X(t,\theta) = B_t + Y(t,\theta)$ with B independent of Y and

1. $B_t = X_0(t)$ has the law of a standard two-sided Brownian motion on $\mathbb{R}$.

2. Y is a log-correlated field with covariance kernel

$$H(t,\theta,t',\theta') := \mathbb{E}[Y(t,\theta)Y(t',\theta')] = \log \frac{e^{-t} \vee e^{-t'}}{|e^{-t+i\theta} - e^{-t'+i\theta'}|}. \tag{2.23}$$

For a GFF X_t on $\mathbb{T}_t$, the radial part is given by the sum of the radial parts of X^{D}_t and X^{N}_t. Hence $(\sqrt{2}B_s)_{0\le s\le t}$ is the independent sum of a Brownian bridge and a standard Brownian motion with its mean subtracted.

2.2.2 Gaussian Multiplicative Chaos

Recall that a GFF X (on $\mathcal{C}_\infty$ or $\mathbb{T}_t$) is only defined as a distribution, so the exponential term $e^{\gamma X}$ is ill-defined *a priori*. However it is possible to make sense of the measure $e^{\gamma X(s,\theta)}dsd\theta$ using a regularising procedure based on Kahane's theory of Gaussian Multiplicative Chaos (GMC) (see [Ber17, RV14] for more detailed reviews). We use the regularisation called the circle average: let $X_\varepsilon(x)$ be a jointly continuous version of the average of the field on the circle of (Euclidean) radius ε about $x \in M$ [Ber16, Section 2]. From [DKRV16, Proposition 2.6] and [DRV16, Proposition 3.8], the sequence of measures

$$dM_\varepsilon^\gamma(x) := e^{\gamma X_\varepsilon(x) - \frac{1}{2}\gamma^2 \mathbb{E}[X_\varepsilon(x)^2]} dx \tag{2.24}$$

converges in probability as $\varepsilon \to 0$ (in the sense of weak convergence of measures) to an almost surely non-trivial measure dM^γ with no atoms, for all $\gamma \in (0,2)$. Moreover, the result of [Ber17, Theorem 1.1] together with universality of the limit (see the discussion in [Ber16] after Theorem 2.1) shows that $M_\varepsilon^\gamma(D) \to M^\gamma(D)$ in L^1 as $\varepsilon \to 0$ for all Borel set D.

An important tool in GMC is Kahane's convexity inequality, which we will use in form of Theorem 2.2.1 below. In this form, this theorem is a consequence of [RV14, Theorem 2.1] (see the discussion after Theorem 2.3 of [RV14]).

Theorem 2.2.1. *[RV14, Theorem 2.1] Let X and Y be two continuous Gaussian fields on $D \subset \mathbb{C}$ such that for all $x, y \in D$*

$$\mathbb{E}[X(x)X(y)] \leq \mathbb{E}[Y(x)Y(y)].$$

Then for all convex function $F : \mathbb{R}_+ \to \mathbb{R}$ with at most polynomial growth at infinity,

$$\mathbb{E}\left[F\left(\int_D e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]} dx\right)\right] \leq \mathbb{E}\left[F\left(\int_D e^{\gamma Y(x) - \frac{\gamma^2}{2}\mathbb{E}[Y(x)^2]} dx\right)\right].$$

In practice, we can apply this result to the GMC measures of log-correlated fields (like the GFF) using the regularising procedure. Suppose X, Y are log-correlated fields with $|\mathbb{E}[X(x)X(y) - \mathbb{E}[Y(x)Y(y)]| \leq \varepsilon$ for all x, y, and write M^γ, N^γ for their respective chaos measure. In particular we have

$$\mathbb{E}[X(x)X(y)] \leq \mathbb{E}[Y(x)Y(y)] + \varepsilon.$$

Notice that the field $Z(x) = Y(x) + \sqrt{\varepsilon}\delta$ – with $\delta \sim \mathcal{N}(0,1)$ independent of everything – has covariance kernel $\mathbb{E}[Y(x)Y(y)] + \varepsilon$. The argument of [RV14] in the discussion following Theorem 2.3 shows that we can apply Kahane's inequality in the limit, and we get:

$$\mathbb{E}[M^\gamma(D)^{-r}] \leq \mathbb{E}[e^{-r\gamma\sqrt{\varepsilon}\delta} N^\gamma(D)^{-r}] = e^{\frac{1}{2}\gamma^2 r^2 \varepsilon}\mathbb{E}[N^\gamma(D)^{-r}].$$

By symmetry of the roles of X and Y, the converse inequality is also true, so that in the end

$$\mathbb{E}[M^\gamma(D)^{-r}] = \mathbb{E}[N^\gamma(D)^{-r}](1 + O_{\varepsilon\to 0}(\varepsilon)).$$

2.2.3 Derivation of the correlation function

Using the GFF and GMC, we give a short derivation of the correlation function $\langle V_\alpha(0)\rangle_{\frac{it}{\pi}}$ on the torus. In [DRV16], this object is constructed so as to satisfy some invariance properties, e.g. the Weyl anomaly (Theorem 4.1) and modular invariance (Theorem 4.6). Hence, as in [KRV20, Section 2.2], we suppose that we have fixed the geometric setting described above (Green's function G_t, representative of the modulus $\tau = \frac{it}{\pi}$) and take the invariance properties as part of the definition.

We renormalise the vertex operator $V_\alpha(0)$ by setting

$$V_{\alpha,\varepsilon}(0) := e^{\alpha X_\varepsilon(0) - \frac{\alpha^2}{2}\mathbb{E}[X_\varepsilon(0)^2]}. \tag{2.25}$$

Applying Girsanov's theorem, then taking $\varepsilon \to 0$ and making the change of variables $u = e^{\gamma c}$ we can set:

$$\begin{aligned}
\langle V_\alpha(0)\rangle_{\frac{it}{\pi}} &:= \lim_{\varepsilon\to 0} 2(\frac{t}{\pi})^{-1/2}|\eta(\frac{it}{\pi})|^{-2}\int_{\mathbb{R}} e^{\alpha c}\mathbb{E}\left[e^{\alpha X_\varepsilon(0)-\frac{\alpha^2}{2}\mathbb{E}[X_\varepsilon(0)^2]}\exp(-\mu e^{\gamma c}M^\gamma(\mathbb{T}_t))\right] dc \\
&= 2(\frac{t}{\pi})^{-1/2}|\eta(\frac{it}{\pi})|^{-2}\int_{\mathbb{R}} e^{\alpha c}\mathbb{E}\left[\exp(-\mu e^{\gamma c}\int_{\mathbb{T}_t} e^{\gamma\alpha G_t(0,\cdot)}dM^\gamma)\right] dc \\
&= 2(\frac{t}{\pi})^{-1/2}|\eta(\frac{it}{\pi})|^{-2}\gamma^{-1}\mu^{-\frac{\alpha}{\gamma}}\Gamma(\frac{\alpha}{\gamma})\mathbb{E}\left[\left(\int_{\mathbb{T}_t} e^{\gamma\alpha G_t(0,\cdot)}dM^\gamma\right)^{-\frac{\alpha}{\gamma}}\right].
\end{aligned} \tag{2.26}$$

Remark 4. At first glance, our choices of renormalisation in (2.24) and (2.25) may seem different than the ones in [DRV16] where the renormalisation factors are $\varepsilon^{\frac{\gamma^2}{2}}$ and $\varepsilon^{\frac{\alpha^2}{2}}$ respectively. However, notice that for the lateral noise Y on the infinite cylinder $\mathcal{C}_\infty$, we have $\mathbb{E}[Y_\varepsilon^2(x)] = \log\frac{1}{\varepsilon} + o(1)$ with $o(1)$ uniform, so we get the same limit by Kahane's inequality. Moreover, our parametrisation is made precisely to have the Green function for the lateral noise Y_t on $\mathbb{T}_t$ converging in a suitable sense to that of Y as $t \to \infty$ (see Section 2.3.2 and in particular (2.42)).

2.3 Proofs

2.3.1 Proof of Proposition 2.1.3

We will start by showing in Section 2.3.1.1 that $t\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_t$ has a limit $t\to\infty$ and find its expression in terms of the derivative DOZZ formula in Section 2.3.1.2. Section 2.3.1.3 gives a heuristic explanation for this limit.

2.3.1.1 Background and notations

Let $g(z) = |z|_+^{-4}$ be the crêpe metric on $\widehat{\mathbb{C}}$. Under the conformal change of coordinates $\psi : \mathcal{C}_\infty \to \widehat{\mathbb{C}}$ defined by $\psi(z) = -\log z$, we get the metric $g_\psi(t,\theta) = e^{-2|t|}$ on the infinite cylinder.

Let $X(t,\theta) = B_t + Y(t,\theta)$ be a GFF on $\mathcal{C}_\infty$. By conformal covariance [GRV19, Equation (3.13)], taking the chaos of X with respect to g_ψ is the same as taking the chaos of $X(t,\theta) - Q|t|$ with respect to Lebesgue measure. Equivalently, this is the same as taking the radial part of the GFF to be the drifted Brownian motion $B_t - Q|t|$. Notice that the angular part is unchanged in this process and we write dN^γ for the GMC measure of Y. We will be interested in the negative moments of GMC. To this end, for all $t < t'$, we introduce the random variable

$$Z_{t,t'}(\lambda) := \int_{[t,t']\times\mathbb{S}^1} e^{\gamma(B_s+(\lambda-Q)|s|+\alpha G(0,s+i\theta))} dN^\gamma(s,\theta), \tag{2.27}$$

where $r > 0$ is fixed throughout the proof and recall $G(\cdot,\cdot)$ is Green's function on $\mathcal{C}_\infty$. For notational convenience, we also define $Z_t(\lambda) := Z_{-t,t}(\lambda)$.

We can see in the expression of $Z_t(\lambda)$ that the Brownian motion has a drift that makes the chaos measure integrable when $|t| \to \infty$. The value of the drift is precisely linked to the strength of the singularity and in vanishes when $\lambda = Q$, causing the mass to explode and the negative moments to vanish, so we have $Z_\infty(Q) := \lim_{t\to\infty} Z_t(Q) = \infty$ a.s., and $\lim_{t\to\infty} \mathbb{E}[Z_t(Q)^{-r}] = 0$ [DKRV16, Theorem 3.2]. On the other hand, $Z_t(\lambda)$ converges a.s. to a positive, finite random variable $Z_\infty(\lambda)$ for all $\lambda \in (Q - \frac{\alpha}{2}, Q)$, and all negative moments of $Z_\infty(\lambda)$ are positive and finite. Furthermore, the DOZZ formula states that for all $\lambda \in (Q - \frac{\alpha}{2}, Q)$, we have

$$C_\gamma(\lambda,\alpha,\lambda) = 2\gamma^{-1}\mu^{-\frac{\alpha}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}\right)\mathbb{E}\left[Z_\infty(\lambda)^{-\frac{\alpha}{\gamma}}\right].$$

The rate at which the negative moments of $Z_t(Q)$ vanish with t was studied in [DKRV17] where it was shown that $t\mathbb{E}[Z_t(Q)^{-r}]$ has a non-trivial limit as $t \to \infty$ (Theorem 2.1 with $k = 2$ and $t = \log\frac{1}{\varepsilon}$). Let us briefly recall what the strategy was, as we will need some ingredients from the proof. What we state from here to equation (2.36) is the idea of the proof of Proposition 3.1 of [DKRV17]. For $b, t > 0$, define the event

$$A_{b,t} := \left\{ \sup_{-t\le s\le t} B_s < b \right\}. \tag{2.28}$$

By independence of the Brownian motions $(B_t)_{t\geq 0}$ and $(B_{-t})_{t\geq 0}$, we have

$$\mathbb{P}(A_{b,t}) = \left(2\int_0^{b/\sqrt{t}} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\right)^2 =: f(b/\sqrt{t})^2, \tag{2.29}$$

with the elementary estimates $f(x) \to 1$ as $x \to \infty$ and

$$f(x) \sim \sqrt{\frac{2}{\pi}}x \text{ as } x \to 0. \tag{2.30}$$

The law of $(b - B_s)_{-t\leq s\leq t}$ converges as $t \to \infty$ to a two-sided, 3-dimensional Bessel process on $\mathbb{R}$ taking the value b at $t = 0$ [DKRV17, Lemma 4.5] and the independence of the left and right processes). Hence the limiting process $(B_s)_{s\in\mathbb{R}}$ goes to $-\infty$ as $|s| \to \infty$ at scale roughly $-\sqrt{|s|}$.

Let $\mathbb{P}_b$ be the law of the GFF on $\mathcal{C}_\infty$ where the radial part is replaced by b minus a 2-sided, 3-dimensional Bessel process taking the value b at $t = 0$. Under $\mathbb{P}_b$, $Z_\infty(Q)$ is a.s. a non-trivial random variable, and $\mathbb{E}_b[Z_\infty(Q)] < \infty$ [DKRV17, Equations (5.5) and (5.6)]. Furthermore the authors show that $\mathbb{E}_b\left[Z_\infty(Q)^{-r}\right] \in (0,\infty)$ and its value is characterised by [DKRV17, Proposition 3.1]:

$$\begin{aligned}
\lim_{t\to\infty} t\mathbb{E}\left[Z_t(Q)^{-r}\right] &= \lim_{t\to\infty}\lim_{b\to\infty} t\mathbb{E}[Z_t(Q)^{-r}\mathbb{1}_{A_{b,t}}] \\
&= \lim_{t\to\infty}\lim_{b\to\infty} tf(b/\sqrt{t})^2\mathbb{E}\left[Z_t(Q)^{-r}\middle| A_{b,t}\right] \\
&= \lim_{b\to\infty}\lim_{t\to\infty} tf(b/\sqrt{t})^2\mathbb{E}\left[Z_t(Q)^{-r}\middle| A_{b,t}\right] \\
&= \frac{2}{\pi}\lim_{b\to\infty} b^2\mathbb{E}_b\left[Z_\infty(Q)^{-r}\right].
\end{aligned} \tag{2.31}$$

The exchange of limits in b and in t is justified by the uniform convergence in b with respect to t. In the last line, the limit in b can be shown to be finite using estimates on hitting probabilities of Bessel processes.

2.3.1.2 Characterisation of the limit

We now turn to the study of the behaviour of $\mathbb{E}[Z_\infty(\lambda)^{-r}]$ as $\lambda \to Q$. From the independence of the left and right radial processes, it suffices to study the one-sided problem and show that $\frac{\mathbb{E}[Z_{0,\infty}(\lambda)^{-r}]}{2(Q-\lambda)}$ has a limit as $\lambda \to Q$ and that this limit coincides with $\lim_{b\to\infty} b\mathbb{E}_b[Z_{0,\infty}(Q)^{-r}]$.

Let $\lambda \in (Q - \frac{\alpha}{2}, Q)$. By the Williams path decomposition (see e.g. [KRV20, Lemma 2.6]), we can sample a Brownian motion in $\mathbb{R}_+$ with drift $\lambda - Q < 0$ as follows:

1. Sample an exponential random variable $M \sim \text{Exp}(2(Q - \lambda))$ (this is the supremum of the process).

2. Conditionally on M, run an independent Brownian motion with drift $Q-\lambda>0$ until its hitting time $T_{\lambda,b}$ of b.

3. Conditionally on $T_{\lambda,b}$, run an independent Brownian motion in $[T_{\lambda,b},\infty)$ with drift $\lambda-Q<0$ started from b and conditioned to stay below b.

By definition, what is meant by Brownian motion with drift $\nu>0$ conditioned to stay positive is the process with generator $\frac{1}{2}\frac{d^2}{dx^2}+\nu\cot(\nu x)\frac{d}{dx}$ [KRV20, Section 12.4]. In the limit $\nu\to 0$, we get the generator $\frac{1}{2}\frac{d^2}{dx^2}+\frac{1}{x}\frac{d}{dx}$ of the $3d$ Bessel process. Thus, on the event that $M=b$, the Williams path decomposition converges in law as $\lambda\to Q$ to the joining of a Brownian motion run until its hitting time of b and a Brownian motion conditioned to stay below b (i.e. b minus a $3d$ Bessel process). Thus, Williams' path decomposition gives a way to make sense of conditioning on the value of the supremum of the radial process, and we can write for all $r>0$,

$$\frac{\mathbb{E}\left[Z_{0,\infty}(\lambda)^{-r}\right]}{2(Q-\lambda)}=\int_0^\infty \mathbb{E}\left[Z_{0,\infty}(\lambda)^{-r}|M=b\right]e^{2b(\lambda-Q)}db.$$

As already seen in Section 2.3.1.1, the properties of the Bessel process imply that, for all $b>0$, $\mathbb{E}[Z_{0,\infty}(Q)^{-r}|M=b]:=\lim\limits_{\lambda\to Q}\mathbb{E}[Z_{0,\infty}(\lambda)^{-r}|M=b]$ exists and is positive. Furthermore, the positivity of the GMC measure implies

$$\mathbb{E}\left[Z_{0,\infty}(\lambda)^{-r}|M=b\right]\leq\mathbb{E}\left[Z_{\tau_{b-1},\tau_b}(\lambda)^{-r}|M=b\right]\leq e^{-r\gamma(b-1)}\mathbb{E}\left[Z_{0,\tau_1}(\lambda)^{-r}|M=1\right], \tag{2.32}$$

where we wrote τ_x for the hitting time of x by the drifted process, and used the Markov property and the stationarity of the lateral noise. From [DKRV17, Lemma 4.4], we know that $\mathbb{E}[Z_{0,\tau_1}(Q)^{-r}|M=b]<\infty$. Actually, this lemma also holds in the case $\lambda<Q$ since it relies on an estimate of $\mathbb{P}(\tau_1<t)$ as $t\to 0$ which gives the same result in the drifted case. This implies that $\mathbb{E}\left[Z_{0,\infty}(\lambda)^{-r}|M=b\right]$ decays exponentially fast as $b\to\infty$. By stochastic domination [KRV20, Section 9.2], $\mathbb{E}[Z_{0,\infty}(\lambda)^{-r}|M=b]$ is also decreasing in λ for all b. It then follows from the dominated convergence theorem that

$$\lim_{\lambda\to Q}\frac{\mathbb{E}\left[Z_{0,\infty}(\lambda)^{-r}\right]}{2(Q-\lambda)}=\int_0^\infty\mathbb{E}\left[Z_{0,\infty}(Q)^{-r}|M=b\right]db. \tag{2.33}$$

To conclude, we must show that this limit coincides with $\lim\limits_{b\to\infty}b\mathbb{E}_b[Z_\infty(Q)^{-r}]$. Under $\mathbb{P}_b$, $(b-B_s)_{s\geq 0}$ is a $3d$-Bessel process started from b, so $(b-B_s)^{-1}$ is a positive continuous local martingale a.s. converging to 0 as $s\to\infty$. Applying the optional stopping theorem, we find that $\mathbb{P}_b(\sigma_x<\infty)=\frac{x}{b}$ for all $x\in(0,b)$, where σ_x is the first hitting time of x by $(b-B_s)_{s\geq 0}$ (this is the well-known fact that the infimum of a Bessel process started from $b>0$ is uniformly distributed in $(0,b)$, see also [MY16, Exercise 2.5] for a more general setting). It follows that under $\mathbb{P}_b$, $M = \sup_{s\geq 0} B_s$ is uniformly distributed in $[0, b]$. Hence

$$\begin{aligned}\lim_{b\to\infty} b\mathbb{E}_b\left[Z_{0,\infty}(Q)^{-r}\right] &= \lim_{b\to\infty}\int_0^b \mathbb{E}_b\left[Z_{0,\infty}(Q)^{-r}|M = b'\right] db' \\ &= \int_0^\infty \mathbb{E}\left[Z_{0,\infty}(Q)^{-r}|M = b\right] db.\end{aligned} \tag{2.34}$$

Thus we find the same limit as in (2.33). Now we go back to the two-sided setting. Since the left and right radial processes are i.i.d. Brownian motions, we can apply the above conditioning to each one of these processes independently, and putting together (2.31), (2.33) and (2.34) yields:

$$\lim_{\lambda\to Q}\frac{\mathbb{E}\left[Z_\infty(\lambda)^{-r}\right]}{4(Q-\lambda)^2} = \lim_{t\to\infty}\frac{\pi t}{2}\mathbb{E}\left[Z_t(Q)^{-r}\right].$$

Plugging this into the expression for the correlation function yields

$$\begin{aligned}\frac{\pi}{2}\lim_{t\to\infty} t\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_t &= 2\gamma^{-1}\mu^{-\frac{\alpha}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}\right)\lim_{t\to\infty}\frac{\pi}{2}t\mathbb{E}\left[Z_t^{-r}\right] \\ &= 2\gamma^{-1}\mu^{-\frac{\alpha}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}\right)\lim_{\lambda\to Q}\frac{\mathbb{E}\left[Z_\infty(\lambda)^{-r}\right]}{4(\lambda-Q)^2} \\ &= \frac{1}{4}\lim_{\lambda\to Q}\frac{C_\gamma(\lambda,\alpha,\lambda)}{(\lambda-Q)^2}.\end{aligned} \tag{2.35}$$

□

2.3.1.3 Heuristic interpretation of the limit

For the record, we give a heuristic interpretation of the result of Proposition 2.1.3. Using the expression of the Radon-Nikodym derivative of the Bessel process with respect to Brownian motion, one can rewrite (2.31) as

$$\lim_{t\to\infty} t\mathbb{E}\left[Z_t^{-r}\right] = \frac{2}{\pi}\lim_{t\to\infty}\mathbb{E}\left[B_tB_{-t}Z_t^{-r}\right] \tag{2.36}$$

and we define the (renormalised) correlation function to be

$${}^R\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_{\mathbb{S}^2} := 2\gamma^{-1}\mu^{-\frac{\alpha}{\gamma}}\Gamma\left(\frac{\alpha}{\gamma}\right)\lim_{t\to\infty}\mathbb{E}\left[B_tB_{-t}Z_t^{-r}\right]. \tag{2.37}$$

We have seen that this correlation function can be expressed using the derivative of DOZZ formula at the critical point $\alpha_1 = \alpha_3 = Q$. The usual interpretation of ${}^R\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_{\mathbb{S}^2}$ is that of a derivative operator. Indeed, the value of B_t in equation (2.37) is the average of the field on the circle of radius e^{-t} about 0, so it is formally $X(0)$

in the limit $t \to \infty$. Still on the formal level, we have the interpretation

$$
\begin{aligned}
{}^R\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle &= \langle X(0)V_Q(0)V_\alpha(1)X(\infty)V_Q(\infty)\rangle_{\mathbb{S}^2} \\
&= \left\langle \frac{d}{d\lambda}V_\lambda(0)_{|\lambda=Q}V_\alpha(1)\frac{d}{d\lambda}V_\lambda(\infty)_{|\lambda=Q}\right\rangle_{\mathbb{S}^2}.
\end{aligned}
\tag{2.38}
$$

This explains why we could expect ${}^R\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_{\mathbb{S}^2}$ to be expressed in terms of the (second) derivative of DOZZ formula at the critical point.

2.3.2 Proof of Proposition 2.1.4

The second item in the proof of Theorem 2.1.1 is the equivalent asymptotic behaviour of $\langle V_\alpha(0)\rangle_{\frac{it}{\pi}}$ and $\langle V_Q(0)V_\alpha(1)V_Q(\infty)\rangle_t$. This will follow from comparisons between Green's function on the infinite cylinder and the torus.

Lemma 2.3.1. *Let X_t be a GFF on the torus $\mathbb{T}_t$ (embedded into $\mathcal{C}_t$) with the normalisation $\int_{\mathbb{S}^1} X_t(0,\theta)d\theta = 0$. Then we can write $X_t(s,\theta) = B_t(s) + Y_t(s,\theta)$ with B_t independent of Y_t and*

1. *For all $s \in (-t,t)$, $B_t(s) = \frac{B^e(|s|)+\mathrm{sign}(s)B^o(|s|)}{\sqrt{2}}$ where $(B^o(s))_{0\le s\le t}$ is standard Brownian bridge and $(B^e(s))_{0\le s\le t}$ is an independent standard Brownian motion.*

2. *Y_t is a* log*-correlated Gaussian field with covariance kernel (recall equation (2.23))*

$$
H_t(s,\theta,s',\theta') = \sum_{n\in\mathbb{Z}} H(s,\theta,s'+2nt,\theta'). \tag{2.39}
$$

Proof. With the choice of normalisation of the Lemma, we can sample X_t simply by setting $X_t := \widetilde{X}_t - \int_{\mathbb{S}^1} X_t(0,\theta)d\theta$ where $\widetilde{X}_t$ is a GFF on $\mathbb{T}_t$ with vanishing mean on $\mathbb{T}_t$. From Section 2.2.1, the radial part of $\widetilde{X}_t$ on $(0,t)\times\mathbb{S}^1$ is $\frac{B^o(s)+B^e(s)}{\sqrt{2}}$ where $(B^o(s))_{0\le s\le t}$ is a standard Brownian bridge and $B^e(s)$ is an independent Brownian motion whose mean has been subtracted. The normalisation of X_t is simply translating B^o along the y axis such that $B^o_t(0) = 0$, so the radial part is the claimed one.

Now we deal with the angular part X_t. From equation (2.23), we have for all $s \in (-t,t)$, $\theta \in \mathbb{S}^1$ and $n \in \mathbb{Z}\setminus\{0\}$

$$
H(0,0,s+2nt,\theta) = \log\frac{1}{|1-e^{-|s+2n|t-i\theta}|} = O_{|n|\to\infty}(e^{-2|n|t}),
$$

implying that the series (2.39) converges absolutely on compact subsets of $\mathcal{C}_t\setminus\{(s,\theta)\}$ for all $(s,\theta)\in\mathcal{C}_t$ (we used the translation invariance of H). In particular, $H_t(s,\theta,\cdot,\cdot)$ defines a function on $\mathbb{T}_t$.

For all $(s,\theta) \in \mathcal{C}_t$, the function $(s',\theta') \mapsto \sum_{n\neq 0} H(s,\theta,s'+2nt,\theta')$ defined on $\mathcal{C}_t$ is an absolutely convergent sum of harmonic functions on $\mathcal{C}_t$ (with respect to the Laplacian on $\mathcal{C}_\infty$), and the second derivatives also converge absolutely. Hence the function is harmonic on $\mathcal{C}_t$. Note also that $H_t(s,\theta,\cdot,\cdot)$ is a sum of angular functions, so it is also angular. Let $\varphi \in \mathcal{C}^\infty(\mathbb{T}_t)$ be an angular function. We can view φ as a $2t$-periodic function on $\mathcal{C}_\infty$ and we have $\langle -\frac{1}{2\pi}\Delta_t H_t(s,\theta,\cdot,\cdot),\varphi\rangle = \langle \frac{-1}{2\pi}\Delta H(s,\theta,\cdot,\cdot),\varphi\rangle = \varphi(s,\theta)$. So by definition H_t is the angular part of Green's function on $\mathbb{T}_t$. $\square$

Proof of Proposition 2.1.4. Let us introduce some notation. Fix $\delta > 0$ and write

$$Z_t := \int_{\mathcal{C}_t} e^{\gamma(B(s)+\alpha G(0,s+i\theta))} dN^\gamma(s,\theta) = U_t + \xi_t, \tag{2.40}$$

where

$$\begin{aligned} U_t &= \int_{\mathcal{C}_{t^{1-\delta}}} e^{\gamma(B(s)+\alpha G(0,s+i\theta))} dN^\gamma(s,\theta) \\ \xi_t &= \int_{(-t,-t^{1-\delta})\cup(t^{1-\delta},t)\times\mathbb{S}^1} e^{\gamma(B(s)+\alpha G(0,s+i\theta))} dN^\gamma(s,\theta) \end{aligned} \tag{2.41}$$

We define also

$$\widetilde{Z}_t := \int_{\mathbb{T}_t} e^{\gamma(B_t(s)+\alpha G_t(0,s+i\theta))} dN_t^\gamma(s,\theta) = \widetilde{U}_t + \widetilde{\xi}_t$$

where $\widetilde{U}_t$ and $\widetilde{\xi}_t$ are defined similarly (here dN_t^γ is the GMC measure of the field Y_t). The term $\widetilde{U}_t$ is the core of the mass while $\widetilde{\xi}_t$ is some error term that we have to control. We will see that $\widetilde{U}_t$ behaves exactly as U_t as $t \to \infty$.

It follows from Lemma 2.3.1 that for all $x, y \in \mathcal{C}_{t^{1-\delta}}$

$$|H_t(x,y) - H(x,y)| = \left|\sum_{n\neq 0} H(x, y+2nt)\right| \leq Ce^{-2t} \tag{2.42}$$

for some constant $C > 0$ independent of t.

Let $b > 0$ and define the event

$$\widetilde{A}_{b,t} := \left\{ \sup_{-t\leq s\leq t} B_t(s) < b \right\}.$$

By Brownian scaling, there exists a function $g : \mathbb{R}_+ \to [0,1]$ such that $\mathbb{P}\left(\widetilde{A}_{b,t}\right) = g(b/\sqrt{t})$. It is clear that $\lim_{x\to\infty} g(x) = 1$ and we will show in Lemma 2.3.2 (at the end of this section) that $g(x) \underset{x\to 0}{\sim} \frac{3}{\pi}x^2$.

On $\widetilde{A}_{b,t}$, the process $(b - B_t(s))_{0\leq s\leq t}$ is absolutely continuous with respect to a 3d-Bessel process started from b. Hence there exists $\nu > 0$ such that the event

$$\left\{\forall s \in (t^{1-\delta},t) \cup (-t,-t^{1-\delta}),\ B_t(s) \leq -t^{1/2-\nu}\right\}$$

occurs with high probability as $t \to \infty$, implying that $\widetilde{\xi}_t \to 0$ in probability conditionally on $\widetilde{A}_{b,t}$ as $t \to \infty$. Similarly, $\xi_t \to 0$ in probability as $t \to \infty$ when conditioned on $A_{b,t}$.

From the previous subsection we know that Z_t conditioned on $A_{b,t}$ has a non-trivial limit Z_∞ as $t \to \infty$, and the negative moments of Z_∞ are finite. Now for each $\varepsilon > 0$, we have

$$\mathbb{E}[U_t^{-r}|A_{b,t}] \geq \mathbb{E}[Z_t^{-r}|A_{b,t}] \geq \mathbb{E}[(U_t + \varepsilon)^{-r} \mathbb{1}_{\xi_t < \varepsilon}|A_{b,t}], \tag{2.43}$$

and taking first $t \to \infty$ then $\varepsilon \to 0$ yields

$$\lim_{t\to\infty} \mathbb{E}[U_t^{-r}|A_{b,t}] = \lim_{t\to\infty} \mathbb{E}[Z_t^{-r}|A_{b,t}].$$

We now turn to the study of $\widetilde{U}_t$. Let $\mathcal{E}_t$ be the Radon-Nikodym derivative of the law of the process $(B_t(s))_{-t^{1-\delta} \leq s \leq t^{1-\delta}}$ (conditioned on $\widetilde{A}_{b,t}$) with respect to that of the process $(B(s))_{-t^{1-\delta} \leq s \leq t^{1-\delta}}$ (conditioned on $A_{b,t}$). From Lemma 2.3.1, this is the Radon-Nikodym derivative of the Brownian bridge B^o in $[0,t]$ stopped at $t^{1-\delta}$ with respect to Brownian motion in $[0, t^{1-\delta}]$. From [MY16, Exercise 9.4], this is explicitly given by $(1-t^{-\delta})^{-1/2} e^{-\frac{(B^o_{t^{1-\delta}})^2}{2(t-t^{1-\delta})}}$, so $\mathcal{E}_t \to 1$ a.s. and in L^1. Thus:

$$\begin{aligned}
\mathbb{E}\left[\widetilde{U}_t^{-r}|\widetilde{A}_{b,t}\right] &= \mathbb{E}\left[\mathcal{E}_t \left(\int_{\mathcal{C}_{t^{1-\delta}}} e^{\gamma(B(s)+\alpha G_t(0,s+i\theta))} dN_t^\gamma(s,\theta)\right)^{-r} \middle| A_{b,t}\right] \\
&= \mathbb{E}\left[\mathcal{E}_t U_t^{-r}|A_{b,t}\right](1 + O(e^{-2t})),
\end{aligned} \tag{2.44}$$

where we have used the estimate (2.42) and Kahane's convexity inequality (Section 2.2.2) to go from Y_t (resp. $G_t(0,\cdot)$) to Y (resp. $G(0,\cdot)$). Hence

$$\lim_{t\to\infty} \mathbb{E}[\widetilde{U}_t^{-r}|\widetilde{A}_{b,t}] = \lim_{t\to\infty} \mathbb{E}[U_t^{-r}|A_{b,t}].$$

Since $\widetilde{\xi}_t \to 0$ in probability conditionally on $\widetilde{A}_{b,t}$, we find using the same argument as in (2.43)

$$\lim_{t\to\infty} \mathbb{E}[\widetilde{Z}_t^{-r}|\widetilde{A}_{b,t}] = \lim_{t\to\infty} \mathbb{E}[\widetilde{U}_t^{-r}|\widetilde{A}_{b,t}] = \lim_{t\to\infty} \mathbb{E}[U_t^{-r}|A_{b,t}] = \lim_{t\to\infty} \mathbb{E}[Z_t^{-r}|A_{b,t}]. \tag{2.45}$$

Finally, we want to take the limit $b \to \infty$ in the above equation and then exchange the order of the limits. This is the argument of [DKRV17] leading to (2.31) but we briefly recall it for completeness. Recall that $\mathbb{E}[Z_t^{-r}|\sup_{0\leq s\leq t} B_s = x] = e^{-\gamma x r} \times O(1)$ as $x \to \infty$, where $O(1)$ is independent of $t > 0$. This is because factorising out the maximum gives a contribution e^{-bxr} on this event (see also (2.32)). Moreover, the law of $\sup_{0\leq s\leq t} B_s$ conditionally on $\{\sup_{0\leq s\leq t} B_s < b\}$ is absolutely continuous with respect to $\frac{\mathbb{1}_{(0,b)}dx}{b}$ (the uniform measure on $(0,b)$), and the density is uniformly bounded in $t > 0$.

Thus, the convergence of $b^2\mathbb{E}[Z_t^{-r}|A_{b,t}]$ as $b \to \infty$ is exponentially fast, with a rate $O(e^{-\gamma br})$ independent of $t > 0$. This uniform convergence enables to exchange limits, and with the estimate (2.30) we find $\lim_{b\to\infty}\lim_{t\to\infty} b^2\mathbb{E}[Z_t^{-r}|A_{b,t}] = \frac{\pi}{2}\lim_{t\to\infty} t\mathbb{E}[Z_t^{-r}]$. The same argument applies to $\widetilde{Z}_t$, and Lemma 2.3.2 then entails:

$$\begin{aligned} \lim_{t\to\infty}\frac{\pi}{3}t\mathbb{E}[\widetilde{Z}_t^{-r}] = \lim_{b\to\infty} b^2 \lim_{t\to\infty}\mathbb{E}\left[\widetilde{Z}_t^{-r}|\widetilde{A}_{b,t}\right] &= \lim_{b\to\infty} b^2 \lim_{t\to\infty}\mathbb{E}\left[Z_t^{-r}|A_{b,t}\right] \\ &= \lim_{t\to\infty}\frac{\pi}{2}t\mathbb{E}\left[Z_t^{-r}\right], \end{aligned} \tag{2.46}$$

i.e. $\lim_{t\to\infty} t\mathbb{E}\left[\widetilde{Z}_t^{-r}\right] = \frac{3}{2}\lim_{t\to\infty} t\mathbb{E}\left[Z_t^{-r}\right]$.

□

We conclude this section by stating and proving Lemma 2.3.2.

Lemma 2.3.2. *Let $(B_t)_{-1\le t\le 1}$ be a standard 2-sided Brownian motion. Then*

$$\mathbb{P}\left(\sup_{-1\le t\le 1} B_t < x \,\middle|\, B_1 = B_{-1}\right) \underset{x\to 0}{\sim} \frac{3}{\pi}x^2,$$

where we abuse notation by writing $\mathbb{P}(\,\cdot\,|B_1 = B_{-1}) = \lim_{\varepsilon\to 0}\mathbb{P}(\,\cdot\,|\,|B_1 - B_{-1}| \le \varepsilon)$.

Proof. For $\varepsilon > 0$ we have

$$\begin{aligned} &\mathbb{P}\left(\sup_{-1\le t\le 1} B_t < x \,\middle|\, |B_1 - B_{-1}| < \varepsilon\right) \\ &\qquad = \mathbb{P}\left(\sup_{-1\le t\le 1} B_t < x\right)\frac{\mathbb{P}\left(|B_1 - B_{-1}| < \varepsilon \,\middle|\, \sup_{-1\le t\le 1} B_t < x\right)}{\mathbb{P}\left(|B_1 - B_{-1}| < \varepsilon\right)}. \end{aligned} \tag{2.47}$$

We have the basic estimate

$$\mathbb{P}(|B_1 - B_{-1}| < \varepsilon) \underset{\varepsilon\to 0}{\sim} 2\varepsilon\int_{\mathbb{R}}\frac{e^{-x^2}}{2\pi}dx = \frac{\varepsilon}{\sqrt{\pi}}.$$

Now we need to estimate the same probability when conditioned on $\left\{\sup_{-1\le t\le 1} B_t < x\right\}$. On this event, the process $(x - B_t)_{-1\le t\le 1}$ has the law of a two-sided Bessel process started from x. At time 1, the density of this Bessel process is the density of $((x+X)^2+Y^2+Z^2))^{1/2}$ where (X, Y, Z) are i.i.d. normal random variables. Let $f_x(\cdot)$ be the density function of this random variable. It is straightforward to check that $f_0(r) = \sqrt{\frac{2}{\pi}}r^2e^{-\frac{r^2}{2}}\mathbb{1}_{u\ge 0}$ and furthermore

$$\int_0^\infty f_0(r)^2dr = \frac{2}{\pi}\int_0^\infty r^4e^{-r^2}dr = \frac{3}{4\sqrt{\pi}}.$$

Now we have the following bounds on f_x (recall $x \geq 0$)

$$\sqrt{\frac{2}{\pi}} r^2 e^{-\frac{(r+x)^2}{2}} \leq f_x(r) \leq \sqrt{\frac{2}{\pi}} r^2 e^{-\frac{(r-x)^2}{2}},$$

so that

$$\int_0^\infty f_x(r)^2 dr = \frac{3}{4\sqrt{\pi}} + o_x(1).$$

From here a straightforward computation shows

$$\lim_{\varepsilon \to 0} \frac{\mathbb{P}\left(|B_1 - B_{-1}| < \varepsilon \,\middle|\, \sup_{-1 \leq t \leq 1} B_t < x \right)}{\mathbb{P}(|B_1 - B_{-1}| < \varepsilon)} = \frac{\int_0^\infty f_0(r)^2 dr}{\int_{\mathbb{R}} \frac{e^{-r^2}}{2\pi} dr} + o_x(1) = \frac{3}{2} + o_x(1).$$

Hence recalling (2.47):

$$\mathbb{P}\left(\sup_{-1 \leq t \leq 1} B_t < x \,\middle|\, B_1 = B_{-1} \right) = \lim_{\varepsilon \to 0} \mathbb{P}\left(\sup_{-1 \leq t \leq 1} B_t < x \,\middle|\, |B_1 - B_{-1}| < \varepsilon \right) \underset{x \to 0}{\sim} \frac{3}{\pi} x^2.$$

□

Let us see how the Lemma is useful. Let $(B_t)_{-1 \leq t \leq 1}$ be standard two-sided Brownian motion. Then the even part $B_t^e := \frac{B_t + B_{-t}}{\sqrt{2}}$ and the odd part $B_t^o := \frac{B_t - B_{-t}}{\sqrt{2}}$ are independent Brownian motions, and $|B_1 - B_{-1}| = \sqrt{2}|B_1^o|$. So conditioning on the event $B_1 = B_{-1}$ is conditioning on $B_1^o = B_{-1}^o$, i.e. taking the odd part to be a Brownian bridge. Hence if $\widetilde{B}_{-1 \leq t \leq 1}$ is the radial part of the GFF on $\mathbb{T}_1$, we have $\mathbb{P}\left(\sup_{-1 \leq t \leq 1} \widetilde{B}_t < x \right) = \mathbb{P}\left(\sup_{-1 \leq t \leq 1} B_t < x \,\middle|\, B_1 = B_{-1} \right)$. The general case follows by Brownian scaling.

2.A The DOZZ formula

The DOZZ formula is the expression of the 3-point correlation function on the sphere $\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\mathbb{S}^2}$. The formula reads

$$\begin{aligned} C_\gamma(\alpha_1, \alpha_2, \alpha_3) = &\left(\pi \mu \left(\frac{\gamma}{2} \right)^{2 - \frac{\gamma^2}{2}} \frac{\Gamma(\gamma^2/4)}{\Gamma(1 - \gamma^2/4)} \right)^{-\frac{\overline{\alpha} - 2Q}{\gamma}} \\ &\times \frac{\Upsilon'_{\frac{\gamma}{2}}(0) \Upsilon_{\frac{\gamma}{2}}(\alpha_1) \Upsilon_{\frac{\gamma}{2}}(\alpha_2) \Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}\left(\frac{\overline{\alpha} - 2Q}{2} \right) \Upsilon_{\frac{\gamma}{2}}\left(\frac{\overline{\alpha}}{2} - \alpha_1 \right) \Upsilon_{\frac{\gamma}{2}}\left(\frac{\overline{\alpha}}{2} - \alpha_2 \right) \Upsilon_{\frac{\gamma}{2}}\left(\frac{\overline{\alpha}}{2} - \alpha_3 \right)}, \end{aligned} \tag{2.48}$$

where $\overline{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$ and $\Upsilon_{\frac{\gamma}{2}}$ is Zamolodchikov's special function. It has the following integral representation for $\mathrm{Re} z \in (0, Q)$

$$\log \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \left(\left(\frac{Q}{2} - z \right)^2 e^{-t} - \frac{\sinh^2\left(\left(\frac{Q}{2} - z \right) \frac{t}{2} \right)}{\sinh\left(\frac{\gamma t}{4} \right) \sinh\left(\frac{t}{\gamma} \right)} \right) \frac{dt}{t}$$

and it extends holomorphically to $\mathbb{C}$.

It satisfies the functional relation $\Upsilon_{\frac{\gamma}{2}}(Q - z) = \Upsilon_{\frac{\gamma}{2}}(z)$ and it has a simple zero at 0 if $\gamma^2 \in \mathbb{R} \setminus \mathbb{Q}$[5]. Thus it also has a simple zero at Q and $\Upsilon'_{\frac{\gamma}{2}}(Q) = -\Upsilon'_{\frac{\gamma}{2}}(0) \neq 0$.

Of great importance in this paper is the derivative DOZZ formula at the critical point $\alpha_1 = Q = \alpha_3$ which has the expression

$$\partial^2_{\alpha_1 \alpha_3} C_\gamma(Q, \alpha, Q) = \left(\pi \mu \left(\frac{\gamma}{2} \right)^{2 - \frac{\gamma^2}{2}} \frac{\Gamma(\gamma^2/4)}{\Gamma(1 - \gamma^2/4)} \right)^{-\frac{\alpha}{\gamma}} \frac{\Upsilon'_{\frac{\gamma}{2}}(0)^3 \Upsilon_{\frac{\gamma}{2}}(\alpha)}{\Upsilon_{\frac{\gamma}{2}}\left(\frac{\alpha}{2} \right)^4}.$$

2.B Conical singularities

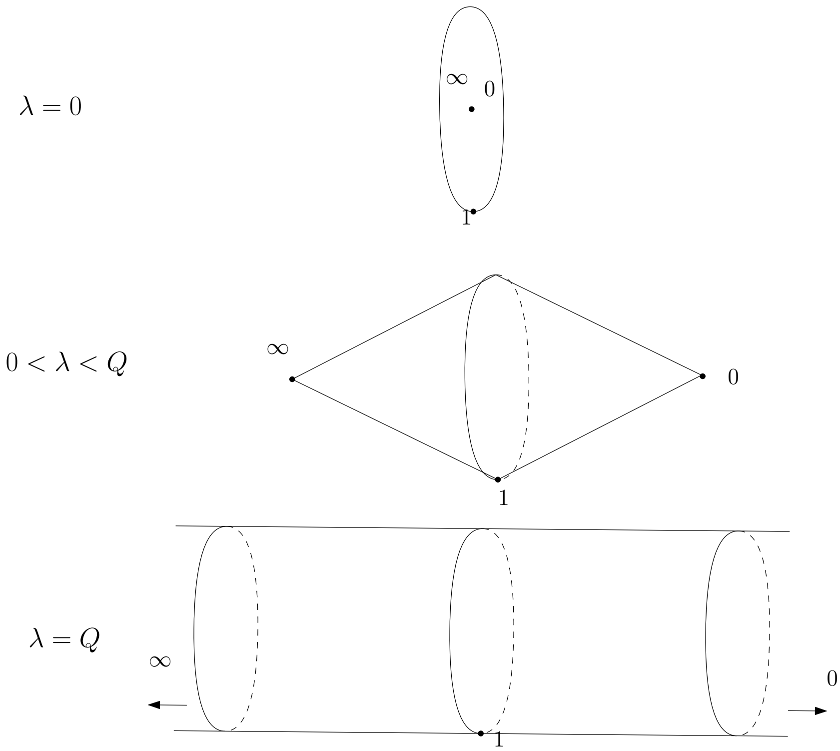

Figure 2.3: Conic degeneration under the insertion of the vertex operators $V_\lambda(0)$ and $V_\lambda(\infty)$. **Top**: For $\lambda = 0$, we have the crêpe metric, i.e. two disks glued together. **Middle**: For $0 < \lambda < Q$, we have two Euclidean cones glued together. **Bottom**: For $\lambda = Q$, the angle of the cones is 0, so we get a bi-infinite cylinder. The limit $\lambda \to Q$ is the setting of the proof of Proposition 2.1.3

[5] This is not really a restriction since the theory is continuous in γ

We study the effect of a change of measure with respect to the Liouville field. Let X be a GFF on $\mathbb{S}^2$[6] with some background metric g and dM_g^γ be the associated chaos measure (regularised in g). Let $\omega \in H_0^1$ be a function such that $e^{\frac{Q}{2}\omega} \in L^1(dM^\gamma)$. Let $\widehat{g} := e^\omega g$ and $dM_{\widehat{g}}^\gamma$ be the chaos of X regularised in $\widehat{g}$. Then for all $r > 0$, applying successively Girsanov's theorem and conformal covariance, we find

$$\mathbb{E}\left[e^{\langle X, \frac{Q}{2}\omega\rangle_\nabla - \frac{Q^2}{8}\|\omega\|_\nabla^2} M^\gamma(\mathbb{S}^2)^{-r}\right] = \mathbb{E}\left[\left(\int_{\mathbb{S}^2} e^{\frac{\gamma Q}{2}\omega} dM_g^\gamma\right)^{-r}\right] = \mathbb{E}\left[M_{\widehat{g}}^\gamma(\mathbb{S}^2)^{-r}\right]. \tag{2.49}$$

In particular, the vertex operator which is formally written $V_\alpha(z) = e^{\alpha X(z) - \frac{\alpha^2}{2}\mathbb{E}[X(z)^2]}$ is a special case of the previous setting with $\omega = \frac{2\alpha}{Q}G(z, \cdot)$. Hence, after regularising, we find that adding a vertex operator is the same as conformally multiplying the metric and set $\widehat{g} = e^{\frac{2\alpha}{Q}G(z,\cdot)}g$, i.e. the new metric satisfies $\log \widehat{g}(z+h) = -\frac{2\alpha}{Q}\log|h| + O_h(1)$ so it has a conical singularity of order α/Q at z.

Another way to see this is to look at the curvature, which reads in the distributional sense

$$K_{\widehat{g}} = e^{-\frac{2\alpha}{Q}G(z,\cdot)}\left(K_g + \frac{4\pi\alpha}{Q}\left(\delta_z - \frac{1}{\mathrm{Vol}_g(\mathbb{S}^2)}\right)\right),$$

where $\mathrm{Vol}_g(\mathbb{S}^2)$ is the volume of the sphere in the metric g. Thus the metric has an atom of curvature at z, meaning it has a conical singularity.

If $\alpha = Q$, the singularity is no longer integrable, so the volume is infinite and the surface has a semi-infinite cylinder. Loosely, we will refer to this situation as a cusp, even though the hyperbolic cusp has finite volume because of the extra log-correction in the metric:

$$\log \widehat{g}(z+h) = -2\log|h| - 2\log\log\frac{1}{|h|} + O(1).$$

The reason for this abuse of terminology is that we are interested in GMC measure. Indeed, suppose $z = 0$ in the sphere coordinates. By conformal covariance, if we use the cylinder coordinates, the log-correction term is the same as shifting the radial part of the GFF from the Brownian motion $(B_s)_{s\geq 0}$ to $(B_s - Q\log(1+s))_{s\geq 0}$. Up to time t, this corresponds to a change of measure given by the exponential martingale $e^{-Q\int_0^t \frac{dB_s}{1+s} - \frac{Q^2}{2}\int_0^t \frac{1}{(1+s)^2}ds}$, which is uniformly integrable since $\int_0^\infty \frac{1}{(1+t)^2}dt < \infty$. So the new field is absolutely continuous with respect to the old one, meaning that GMC does not make a difference between a Euclidean cylinder and a hyperbolic cusp.

[6]We work on the sphere for concreteness but this argument is valid on any compact Riemann surface.

Chapter 3

Four-point function in genus 0

This chapter is adapted from [BW18].

We continue our study of asymptotic behaviour of correlation functions of LCFT in the different context of colliding insertion points. We compute fusion estimates for the four-point correlation function on the Riemann sphere, and find that it is consistent with predictions from the framework of theoretical physics known as the conformal bootstrap. This result fits naturally into the famous KPZ conjecture [KPZ88] which relates the four-point function to the expected density of points around the root of a large random planar map weighted by some statistical mechanics model.

From a purely probabilistic point of view, we give non-trivial results on negative moments of GMC. We give exact formulae based on the DOZZ formula in the Liouville case and asymptotic behaviours in the other cases, with a probabilistic representation of the limit.

Finally, we show how to extend our results to boundary LCFT, treating the cases of the fusion of two boundary or bulk insertions as well as the absorption of a bulk insertion on the boundary.

3.1 Introduction

3.1.1 Path integral

The Liouville action on the Riemann sphere $\mathbb{S}^2 \cong \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the action functional $S_L : \Sigma \to \mathbb{R}$ (where Σ is some function space to be determined) defined by[1]

$$S_{\mathrm{L}}(X) = \frac{1}{4\pi} \int_{\mathbb{S}^2} (|\nabla X|^2 + 4\pi\mu e^{\gamma X} g(z)) d^2 z \tag{3.1}$$

[1]Usually the Liouville action has also a curvature term, which we have omitted here for simplicity. This will not play an important role since we will consider metrics whose curvature concentrates on the unit circle.

where $g(z) = |z|_+^{-4} := (|z| \vee 1)^{-4}$ is the background metric, $\gamma \in (0, 2)$ is the parameter of the theory, and $\mu > 0$ is the cosmological constant (whose value is irrelevant in this paper). Another important parameter is the so-called *background charge* which is defined by $Q := \frac{\gamma}{2} + \frac{2}{\gamma}$. From here, Liouville Conformal Field Theory (LCFT) is the "Gibbs measure" associated to S_L, which is formally defined in the physics literature by

$$\langle F \rangle := \int F(X) e^{-S_{\mathrm{L}}(X)} DX \tag{3.2}$$

for all continuous functional F on Σ. Here DX stands for "Lebesgue measure" on $\mathcal{C}^\infty(\mathbb{S}^2)$, which of course does not make sense mathematically. Nonetheless, it is possible to define (3.2) in a rigorous framework using the Gaussian Free Field (GFF) and Kahane's theory of Gaussian Multiplicative Chaos (GMC) – see [DKRV16] and sections 3.2.1 and 3.2.2 of this paper. Roughly speaking, the GFF X on $\mathbb{S}^2$ is the Gaussian field corresponding to the "Gaussian measure" $e^{-\frac{1}{4\pi}\int_{\mathbb{S}^2} |\nabla X|^2} DX$. We will write $\mathbb{P}$ for its probability measure and $\mathbb{E}$ for the associated expectation. The GFF lives $\mathbb{P}$-a.s. in the topological dual of the Sobolev space $H^1(\mathbb{S}^2, g)$ and is therefore defined as a distribution (in the sense of Schwartz). In this context, GMC is the random measure M^γ on $\mathbb{S}^2$ defined for all $\gamma \in (0, 2)$ and making sense of the exponential of the GFF (which is *a priori* ill-defined). This can be constructed through a regularisation of the field and we will loosely write $dM^\gamma(z) = e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X(z)^2]} g(z) d^2 z$ to refer to the limiting measure, even though X is only defined as a distribution.

The main observables in LCFT are the *vertex operators* $V_\alpha(z) := e^{\alpha X(z)}$, giving rise to the *correlation functions*, which can be thought of as the Laplace transform of the field defined by the measure (3.2):

$$\left\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \right\rangle = \int \prod_{i=1}^N e^{\alpha_i X(z_i)} e^{-S_{\mathrm{L}}(X)} DX \tag{3.3}$$

On the sphere, these are defined for all pairwise disjoint *insertions* $(z_1, ..., z_n) \in \widehat{\mathbb{C}}^N$ and *Liouville momenta* $(\alpha_1, ..., \alpha_n) \in \mathbb{R}_+^N$ satisfying the *Seiberg bounds*

$$\sigma := \sum_{i=1}^N \frac{\alpha_i}{Q} - 2 > 0 \qquad\qquad \forall i,\ \frac{\alpha_i}{Q} < 1 \tag{3.4}$$

In particular, this implies that the correlation function exists only if $N \geq 3$.

For fixed $z_0 \in \widehat{\mathbb{C}}$, the vertex operator $V_\alpha(z_0)$ has a geometric interpretation, as it inserts a conical singularity of order α/Q at z_0 in the physical metric ([Sei90, HMW11], Appendix 2.B). Thus the second Seiberg bound is there to make the singularity integrable around z_0. On the other hand by Gauss-Bonnet theorem, the first bound is equivalent to asking that the surface $\mathbb{S}^2 \setminus \{z_1, ..., z_N\}$ with conical singularities of order α_i/Q at z_i has negative total curvature.

The correlation functions satisfy some conformal covariance under Möbius transformation, namely if ψ is such a map, then [DKRV16]

$$\left\langle \prod_{i=1}^{N} V_{\alpha_i}(\psi(z_i)) \right\rangle = \prod_{i=1}^{N} |\psi'(z_i)|^{-2\Delta_i} \left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \right\rangle$$

where $\Delta_i = \Delta_{\alpha_i} := \frac{\alpha_i}{2}(Q - \frac{\alpha_i}{2})$ is called the *conformal dimension* of $V_{\alpha_i}(\cdot)$. This property implies that the three-point correlation function $\langle \prod_{i=1}^{3} V_{\alpha_i}(z_i) \rangle$ is determined by $\langle V_{\alpha_1}(0) V_{\alpha_2}(1)_{\alpha_3}(\infty) \rangle$ since there is a unique Möbius transformation sending (z_1, z_2, z_3) to $(0, 1, \infty)$. The three-point correlation functions play a central role in the conformal bootstrap approach to CFTs (see Section 3.1.2). For LCFT, they are given by the celebrated DOZZ formula, a proof of which was given for the first time in [KRV20], where the authors rigorously implemented the method known as Teschner's trick [Tes95] (see [DO94, ZZ96] for the original derivation of the formula which uses a different approach).

We now turn to the four-point function. By conformal covariance, we can take the insertions to be at $(z_1, z_2, z_3, z_4) = (0, z, 1, \infty)$ with $z \in \widehat{\mathbb{C}} \setminus \{0, 1, \infty\}$ being the free parameter. In this paper, we will take $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfying the Seiberg bounds and will be concerned about the behaviour of the four-point function as $z \to 0$ (the other fusions being easily deduced from conformal invariance). In the framework of [DKRV16] using the GFF and GMC, the four-point function has the following expression for $|z| \leq 1$:

$$\left\langle \prod_{i=1}^{4} V_{\alpha_i}(z_i) \right\rangle = 2\gamma^{-1} \mu^{-\frac{Q\sigma}{\gamma}} \Gamma\left(\frac{Q\sigma}{\gamma}\right) |z|^{-\alpha_1\alpha_2} |1 - z|^{-\alpha_2\alpha_3} \mathbb{E}\left[\left(\int_{\widehat{\mathbb{C}}} e^{\gamma \sum_{i=1}^{4} \alpha_i G(z_i, \cdot)} dM^\gamma \right)^{-\frac{Q\sigma}{\gamma}} \right] \tag{3.5}$$

where $G = G(\cdot, \cdot)$ is Green's function on $(\mathbb{S}^2, g)$. The main feature of (3.5) is that, up to explicit factors, it is expressed using negative moments of GMC. One of our main results (Theorem 3.1.1) gives the exact asymptotic behaviour of (3.5) as $z \to 0$ using the integrability result of the DOZZ formula. Now the reader will notice that the negative exponent in the definition of (3.5) depends on the α_i's, so the DOZZ formula does not give integrability results for *all* moments of GMC but only for the one corresponding to the Liouville correlation function. However, in our framework, we lose nothing in promoting σ to a free parameter, so we were able to find the asymptotic behaviour of all negative moments (Theorems 3.1.2 and 3.1.3) but only in the Liouville case did we get an exact expression for the limit. In this special case, we were able to confirm a prediction coming from the bootstrap approach to LCFT, which we review now.

3.1.2 Conformal bootstrap

So far, we have described LCFT as a measure μ_L on $\mathcal{D}'(\mathbb{R}\times\mathbb{S}^1)$, the space of distributions on the sphere viewed as the infinite cylinder $\mathbb{R}\times\mathbb{S}^1$ (think of the $\mathbb{R}$ axis as time). This framework is usually referred to as Euclidean field theory in the mathematical physics literature. If this measure satisfies some additional properties known as the Osterwalder-Schrader axioms, a standard procedure enables to construct a Hilbert space $\mathcal{H}$ acted on by a non-negative self-adjoint operator H. Roughly speaking, $\mathcal{H}$ is the space of functionals $F\in L^2(\mu_L)$ which are measurable with respect to the values of the field in the positive part of the cylinder $\mathbb{R}_+\times\mathbb{S}^1$. The inner-product of $F,G\in\mathcal{H}$ is given by $(F,G)\mapsto\langle F\theta G\rangle$, where θG is the time reflection of G. Finally, the Hamiltonian H is the generator of time translations. We refer the reader to Chapter 6 of [GJ81] for a more detailed presentation of this procedure.

In the case of LCFT, this construction was carried out in [Kup16]. Hence, we may view the correlation function (3.5) as the inner-product in $\mathcal{H}$ of the functionals $V_{\alpha_1}(0)V_{\alpha_2}(z)$ and (the reflection of) $V_{\alpha_3}(1)V_{\alpha_4}(\infty)$, provided that these functionals belong to $\mathcal{H}$. Thus, a simple way to express this inner-product would be to diagonalise H and decompose it onto a basis of eigenstates. The difficulty of Liouville theory is that, based on the free field case ($\mu=0$) where one can actually solve the model [Kup16], the spectrum is expected to be purely absolutely continuous $\sigma(H)=[\frac{Q^2}{2},\infty)$, and the eigenstates should exist only in a generalised sense: they are expected to be the vertex operators $(V_{Q+iP})_{P\in\mathbb{R}}$. Notice that these are not *a priori* well-defined in the probabilistic setting since vertex operators are naturally defined inside the Seiberg bounds.

In quantum field theory, the Hilbert space $\mathcal{H}$ must carry a representation of the symmetry algebra: the Virasoro algebra for a conformal field theory. This assumption was the starting point of the founding paper [BPZ84], where the authors argued that correlation functions of a CFT satisfy an infinite hierarchy of identities known as the conformal Ward identities. This strong constraint enabled them to “solve” the so-called minimal models (where the spectrum is discrete), followed later on by Liouville theory (see [Rib14] for an account). Although the representation theory of the Virasoro algebra is well-understood mathematically, the lack of rigour here is that it is not known if the actual Liouville theory (coming from the quantisation of the Liouville action and introduced in the previous subsection) does indeed form a representation of this algebraic structure. Proving this fact in full generality is still a challenge for mathematicians. In any case, this putative algebraic structure led to the following prediction for the expression of (3.5), known as the *conformal bootstrap equation*[2]:

$$\begin{aligned}\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}} &= \frac{1}{8\pi}|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)} \\ &\times \int_{\mathbb{R}} |z|^{\frac{P^2}{2}} C_\gamma(\alpha_1,\alpha_2,Q-iP)C_\gamma(Q+iP,\alpha_3,\alpha_4)|\mathcal{F}^{\alpha_{1234}}_{\gamma,P}(z)|^2 dP\end{aligned} \tag{3.6}$$

where $(z^{\frac{Q^2}{4}-\Delta_1-\Delta_2}\mathcal{F}^{\alpha_{1234}}_{\gamma,P}(\cdot))_{P\in\mathbb{R}}$ are called the four-point conformal block and form a basis of solutions for the conformal Ward identities. $\mathcal{F}^{\alpha_{1234}}_{\gamma,P}$ is expected to be holomorphic in $z\in\mathbb{D}$ (with $\mathcal{F}^{\alpha_{1234}}_{\gamma,P}(0)=1$) and universal in the sense that it depends only on $\gamma,P,\alpha_1,\alpha_2,\alpha_3$ and α_4.

Let us stress again that formula (3.6) is far from having a mathematical justification. A first step towards the validity of the bootstrap programme was made with the proof of the DOZZ formula for the three-point correlation function [KRV19, KRV20]. At this stage, we are still far from having a probabilistic interpretation of formula (3.6) because the eigenstates and the conformal blocks are not properly understood in the path integral approach. However, the fusion estimates of this paper will show that (3.6) is valid in the $z\to 0$ limit.

There is a geometric interpretation of equation (3.6). Indeed, one can produce a four-punctured sphere by gluing together two instances of the thrice-punctured sphere along annuli neighbourhoods of one puncture (see Figure 2.2 and [TV15] for details of this procedure). The bootstrap equation is the CFT counterpart of this gluing procedure since the integrand is a product of DOZZ formulae. We will see in Section 3.1.3 that the factorisation becomes exact in the $z\to 0$ limit. The problem of factorisation of surfaces is an old one and was stressed by Seiberg [Sei90, p336] as the most important open problem in Liouville CFT, at a time when the DOZZ formula was not yet known (nor even guessed). This paper gives a partial answer to the problem since we will show rigorously that the state factorises into two independent states as $z\to 0$.

Finally, let us briefly comment on the place of this work within the existing literature. The recent proof of the DOZZ formula [KRV20] made an extensive use of the BPZ equation, a second order ODE satisfied by the correlation function $z\mapsto\langle V_{-\frac{\gamma}{2}}(z)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle$, which was established in the earlier paper [KRV19] and solved explicitly using hypergeometric functions. The reason why such an equation was expected to hold in the first place is that the representation of the Virasoro algebra associated to the field $V_{-\frac{\gamma}{2}}(\cdot)$ is expected to be degenerate, with a null vector at level two in the Verma module. This drastically simplifies the fusion rule for the fields $V_{-\frac{\gamma}{2}}(z)V_{\alpha_1}(0)$, and using the interpretation of Virasoro generators as differential operators, this leads to the second order BPZ

[2] We add the superscript $^{\mathrm{cb}}$ for "conformal bootstrap", in order to differentiate it with the correlation function given by the path integral.

equation. In this paper on the contrary, we study the general form of the fusion rule, for which the associated representation should not be degenerate in general, thus not leading to a differential equation. To our knowledge, there is no rigorous construction of representations of the Virasoro algebra in Liouville CFT yet, but there are works addressing the question and exploiting null vectors in the context of boundary CFT. For instance, it was shown in [Dub15] that SLE partitions functions can be constructed from highest-weight representations of the Virasoro algebra. In general, some BPZ and Ward-type identities appear in SLE related martingales as the condition making the drift term in Itô's formula vanish [Fri04].

3.1.3 Main results

Let $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be satisfying the Seiberg bounds (3.4). In particular, this implies that either $\alpha_1+\alpha_2 > Q$ or $\alpha_3+\alpha_4 > Q$ (or both), and we assume without loss of generality that $\alpha_3 + \alpha_4 > Q$. Notice that these conditions are equivalent to having the Seiberg bounds being satisfied by (α_1, α_2, Q) (with the exception of the $\alpha_3 = Q$ saturation).

Suppose for now that $\alpha_1 + \alpha_2 \geq Q$. Then equation (3.6) is expected to hold, i.e. we should have

$$\begin{aligned}\langle V_{\alpha_1}(z)V_{\alpha_2}(0)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}} = \frac{1}{8\pi}|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)} \\ \times \int_{\mathbb{R}} |z|^{\frac{P^2}{2}} C_\gamma(\alpha_1, \alpha_2, Q-iP)C_\gamma(Q+iP, \alpha_3, \alpha_4)|\mathcal{F}^{\alpha_{1234}}_{\gamma,P}(z)|^2 dP\end{aligned} \tag{3.7}$$

At the geometrical level, we can produce a four-punctured sphere by gluing together two copies of the thrice-punctured sphere (see Figure 2.2) by picking one puncture on each sphere and identifying together annuli neighbourhoods of these punctures. The form of equation (3.6) reveals this gluing construction: the four-point function is a factorisation of three-point functions.

Assume $\alpha_1 + \alpha_2 > Q$. Taking $\mathcal{F}^{\alpha_{1234}}_{\gamma,P}(z) \equiv 1$ uniformly as $P \to 0$, making the change of variable $P \mapsto P\sqrt{\log \frac{1}{|z|}}$, equation (3.6) gives

$$\begin{aligned} 8\pi|z|^{2(\Delta_1+\Delta_2-\frac{Q^2}{4})}&\langle V_{\alpha_1}(z)V_{\alpha_2}(0)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}} \\ &= \int_{\mathbb{R}} |z|^{\frac{P^2}{2}} C_\gamma(\alpha_1,\alpha_2,Q-iP)C_\gamma(Q+iP,\alpha_3,\alpha_4)|\mathcal{F}^{\alpha_{1234}}_{\gamma,P}(z)|^2 dP \\ &= \frac{1}{\sqrt{\log\frac{1}{|z|}}}\int_{\mathbb{R}} e^{-\frac{P^2}{2}} C_\gamma\left(\alpha_1,\alpha_2,Q-i\frac{P}{\sqrt{\log\frac{1}{|z|}}}\right) \\ &\qquad\qquad \times C_\gamma\left(Q+i\frac{P}{\sqrt{\log\frac{1}{|z|}}},\alpha_3,\alpha_4\right)\left|\mathcal{F}^{\alpha_{1234}}_{\gamma,\frac{P}{\sqrt{\log\frac{1}{|z|}}}}(z)\right|^2 dP \\ &\underset{|z|\to 0}{\sim} \left(\log\frac{1}{|z|}\right)^{-3/2} \partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)\int_{\mathbb{R}} P^2 e^{-\frac{P^2}{2}} dP \\ &= \sqrt{2\pi}\left(\log\frac{1}{|z|}\right)^{-3/2} \partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4) \end{aligned} \tag{3.8}$$

Hence

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}} \underset{z\to 0}{\sim} \frac{|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)}}{4\sqrt{2\pi}\log^{3/2}\frac{1}{|z|}}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4) \tag{3.9}$$

There are two important features in this asymptotic behaviour

- There is a $\left(\log\frac{1}{|z|}\right)^{-3/2}$ term correcting the polynomial rate $|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)}$
- The limit is expressed as a product of two derivative DOZZ formulae. Geometrically speaking, this means that we are sewing two instances of the thrice-punctured spheres, each one presenting a cusp at the $\alpha = Q$ singularity. The fact that we have a product means that we have two "independent" surfaces.

In the case $\alpha_1+\alpha_2 = Q$, the computation of Appendix 2.A shows that:

$$\lim_{P\to 0} C_\gamma(\alpha_1,\alpha_2,Q-iP)C_\gamma(Q+iP,\alpha_3,\alpha_4) = -4\partial_1 C_\gamma(Q,\alpha_3,\alpha_4) \tag{3.10}$$

Going back to the bootstrap equation and noticing that $2(\frac{Q^2}{4}-\Delta_1-\Delta_2) = -\alpha_1\alpha_2$, we can apply the same change of variables as in (3.8), and get in this case

$$\begin{aligned} \langle V_{\alpha_1}(z)V_{\alpha_2}(0)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle^{\mathrm{cb}} &\underset{z\to 0}{\sim} -\frac{|z|^{-\alpha_1\alpha_2}}{2\pi\sqrt{\log\frac{1}{|z|}}}\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)\int_{\mathbb{R}} e^{-\frac{P^2}{2}} dP \\ &= -\frac{1}{\sqrt{2\pi}}\frac{|z|^{-\alpha_1\alpha_2}}{\log^{1/2}\frac{1}{|z|}}\partial_1 C_\gamma(Q,\alpha_3,\alpha_4) \end{aligned} \tag{3.11}$$

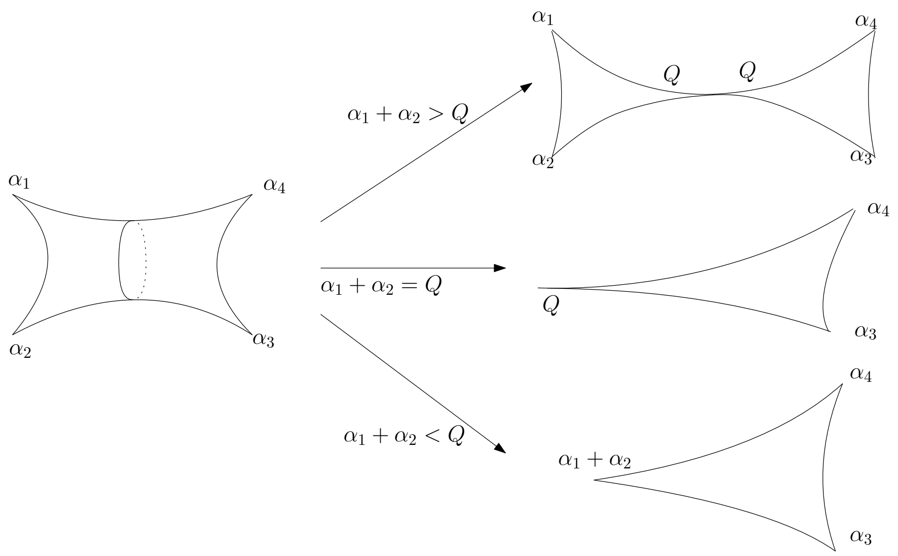

Figure 3.1: The three different regimes depending on the sign of $\alpha_1 + \alpha_2 - Q$. **Up:** Case $\alpha_1 + \alpha_2 > Q$. The surface on the left is a four-punctured sphere with conical singularities of order $(\frac{\alpha_1}{Q}, \frac{\alpha_2}{Q}, \frac{\alpha_3}{Q}, \frac{\alpha_4}{Q})$ at $(0, z, 1, \infty)$. The limiting surface is a pair of thrice-punctured sphere: one with singularities $(\frac{\alpha_1}{Q}, \frac{\alpha_2}{Q}, 1)$ at $(0, 1, \infty)$ (the singularity at ∞ is a cusp), the other with singularities $(1, \frac{\alpha_3}{Q}, \frac{\alpha_4}{Q})$ at $(0, 1, \infty)$. **Middle:** Case $\alpha_1 + \alpha_2 = Q$. The limiting surface is a thrice-punctured sphere with singularities of order $(1, \frac{\alpha_3}{Q}, \frac{\alpha_4}{Q})$ at $(0, 1, \infty)$. **Bottom:** Case $\alpha_1 + \alpha_2 < Q$. The limiting surface is a thrice-punctured sphere with singularities $(\frac{\alpha_1+\alpha_2}{Q}, \frac{\alpha_3}{Q}, \frac{\alpha_4}{Q})$ at $(0, 1, \infty)$.

Again, let us notice two important features of this asymptotic behaviour

- There is a $\left(\log \frac{1}{|z|}\right)^{-1/2}$ correction term to be compared with the power $-3/2$ found in the supercritical case $\alpha_1 + \alpha_2 > Q$ in (3.8). This is explained by the fact that there is only one cusp and one limiting surface (so no extra zero mode).

- The limit is expressed with only one derivative DOZZ block, to be compared with the product found in (3.8). Intuitively, this means that in this critical case $\alpha_1 + \alpha_2 = Q$, we see only one surface with two conical singularities and one cusp.

Finally we turn to the case $\alpha_1 + \alpha_2 < Q$. In this case, equation (3.6) does not hold in this form and there is a need for "discrete corrections" (see [BZ06, Section 8] for a thorough discussion of the phenomenon). This is linked with the fact that the contour of integration in (3.6) includes poles of the DOZZ formula, and the discrete corrections are merely residues. In particular, the leading order as $z \to 0$ is simply

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle^{\mathrm{cb}} \underset{z \to \infty}{\sim} |z|^{-\alpha_1 \alpha_2} C_\gamma(\alpha_1 + \alpha_2, \alpha_3, \alpha_4)$$

so that the geometric interpretation is that the two singularities add up together. This makes sense since $(\alpha_1 + \alpha_2, \alpha_3, \alpha_4)$ satisfies the Seiberg bounds. In this last case, the spectrum is "hidden" behind the discrete leading-order terms. In order to see the spectrum in our probabilistic framework, one would need to push the asymptotic expansion further.

It should be possible to do so using similar techniques as in [KRV20, Section 6] but we restrict ourselves to the leading order for now.

Theorem 3.1.1. *Let* $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ *satisfying the Seiberg bounds and such that* $\alpha_3+\alpha_4 > Q$*. The asymptotic behaviour as* $z\to 0$ *of* $\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle$ *depends on the sign of* $\alpha_1+\alpha_2-Q$ *and is described by the following three cases.*

1. Supercritical case:

   If $\alpha_1+\alpha_2>Q$*, then*

$$\begin{aligned}&\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle\\ &\underset{z\to 0}{\sim}\frac{1}{4\sqrt{2\pi}}\frac{|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)}}{\log^{3/2}\frac{1}{|z|}}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)\end{aligned} \tag{3.12}$$

2. Critical case:

   If $\alpha_1+\alpha_2=Q$*, then*

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle\underset{z\to 0}{\sim}-\frac{1}{\sqrt{2\pi}}\frac{|z|^{-\alpha_1\alpha_2}}{\log^{1/2}\frac{1}{|z|}}\partial_1 C_\gamma(Q,\alpha_3,\alpha_4) \tag{3.13}$$

3. Subcritical case[3]:

   If $\alpha_1+\alpha_2<Q$*, then*

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle\underset{z\to 0}{\sim}|z|^{-\alpha_1\alpha_2}C_\gamma(\alpha_1+\alpha_2,\alpha_3,\alpha_4) \tag{3.14}$$

The different regimes appearing in the statement of Theorem 3.1.1 have a natural geometric explanation (see Figure 3.1 for an illustration of the phenomenon). First, notice that the condition $\alpha_3+\alpha_4-Q>0$ corresponds to having the Seiberg bounds satisfied for (Q,α_3,α_4), except that the first coefficient saturates the second bound. When $\alpha_1+\alpha_2<Q$, the two singularities add up and the limit is non-trivial. When $\alpha_1+\alpha_2=Q$, the second Seiberg bound is saturated and it is natural [DKRV17, Bav19] to expect the factor $(\log\frac{1}{|z|})^{-1/2}\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)$ since the 0^{th} order is trivial in this case. When $\alpha_1+\alpha_2-Q>0$, this also explains the factor $(\log\frac{1}{|z|})^{-1}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)$. The extra $(\log\frac{1}{|z|})^{-1/2}$ term has a more subtle origin. Since both (α_1,α_2,Q) and (Q,α_3,α_4) satisfy the Seiberg bounds, we expect to see the two spheres split and form a disconnected pair of surfaces in the limit. In this limit, the GFF should have two zero modes (given e.g. by the mean on each independent surface). Roughly speaking, upon splitting, the mean on the right surface conditioned on the mean on the total surface is a Gaussian random

[3] This was already proved in [KRV20, Section 6.1] and essentially follows from dominated convergence.

variable with large variance which – when properly rescaled – produces the extra zero mode. This rescaling explains the extra $(\log \frac{1}{|z|})^{-1/2}$ term appearing in (3.12).

Theorem 3.1.1 can be equivalently reformulated in terms of GMC. Since our proof does not depend on the particular choice of $(-\frac{Q\sigma}{\gamma})$-moment in the four-point correlation, we may promote σ to a free parameter and study fusion estimates for arbitrary negative moments of GMC that could be of independent interest. We first record the decay rate in the theorem below.

Theorem 3.1.2. *Let $\kappa > 0$, $\gamma \in (0, 2)$ and $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}_+^4$ be such that the Seiberg bound is satisfied. Also let $(z_1, z_2, z_3, z_4) = (0, z, 1, \infty)$ with $z \in \mathbb{C} \setminus \{0\}$. Then there exists some constant $E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4) > 0$ such that*

$$\lim_{z\to 0} \frac{1}{I^{\gamma,\kappa}_{\alpha_1+\alpha_2}(z)} \mathbb{E}\left[M^\gamma\left(e^{\gamma\sum_{j=1}^4 \alpha_j G(z_j,\cdot)}\right)^{-\kappa}\right] = E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \tag{3.15}$$

where the rate function $I^{\gamma,\kappa}_\alpha$ is given by

$$I^{\gamma,\kappa}_\alpha(z) = \begin{cases} 1 & \alpha - Q < 0, \\ \sqrt{\log \frac{1}{|z|}} & \alpha - Q = 0, \\ |z|^{\frac{(\alpha-Q)^2}{2}} \left(\log \frac{1}{|z|}\right)^{3/2} & \alpha - Q \in (0, \kappa\gamma), \\ |z|^{\frac{(\alpha-Q)^2}{2}} \sqrt{\log \frac{1}{|z|}} & \alpha - Q = \kappa\gamma, \\ |z|^{\frac{(\alpha-Q)^2}{2} - \frac{(\kappa\gamma-(\alpha-Q))^2}{2}} & \alpha - Q > \kappa\gamma. \end{cases}$$

As mentioned in Section 3.1.1, LCFT gives an exact expression for $E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in terms of the DOZZ formula when $\kappa = \frac{\sum_{i=1}^4 \alpha_i - 2Q}{\gamma}$. While this is not the case in general, we can still provide a probabilistic representation of the constant based on the radial/angular decomposition of the GFF on the infinite cylinder (see Section 3.2.1). For this it is useful to introduce the random functional

$$\begin{aligned} F_{a_1,a_2}(u, f(\cdot)) &= e^{-\gamma u} \int_{|x|\geq 1} \frac{dM^\gamma(x)}{|x|^{4-\gamma(a_1+a_2)}|x-1|^{\gamma a_1}} \\ &\quad + \int_{\mathbb{R}_{s\geq 0}\times \mathbb{S}^1_\theta} e^{-\gamma(f(s)-a_1 G(1,e^{-s-i\theta}))} d\widehat{M}^\gamma(s,\theta) \\ &= \int_{\mathcal{C}_\infty} e^{\gamma\left((-u+B_s+(Q-a_2)s)1_{\{s\leq 0\}} - f(s)1_{\{s\geq 0\}} + a_1 G(1,e^{-s-i\theta})\right)} d\widehat{M}^\gamma(s,\theta) \end{aligned} \tag{3.16}$$

where $(B_{-s})_{s\geq 0}$ is a Brownian motion independent of the GMC $d\widehat{M}^\gamma(s,\theta)$ associated with the lateral noise of GFF (see Lemma 3.2.1). We will also write $(\widetilde{\beta}^u_s)_{s\geq 0}$ to denote a $\widetilde{\mathrm{BES}}_u(3)$-process (see Definition 3.2.1).

Theorem 3.1.3. *Let $\alpha_1+\alpha_2-Q\geq 0$. The constant $E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ in (3.15) has the following probabilistic representations.*

- *If $\alpha_1+\alpha_2-Q=0$, then*

$$E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\frac{1}{\kappa\gamma}\sqrt{\frac{2}{\pi}}\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(\tau,\widetilde{\beta}^\tau_\cdot)\right)^{-\kappa}\right] \tag{3.17}$$

  where $\tau\sim\mathrm{Exp}(\kappa\gamma)$.

- *If $\alpha_1+\alpha_2-Q\in(0,\kappa\gamma)$, then*

$$\begin{aligned}E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4)&=\frac{1}{\gamma}\frac{B\left(\frac{\alpha_1+\alpha_2-Q}{\gamma},\kappa-\frac{\alpha_1+\alpha_2-Q}{\gamma}\right)}{(\alpha_1+\alpha_2-Q)(\kappa\gamma-(\alpha_1+\alpha_2-Q))}\sqrt{\frac{2}{\pi}}\\&\times\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(\tau,\widetilde{\beta}^\tau_\cdot)\right)^{-(\kappa-\frac{\alpha_1+\alpha_2-Q}{\gamma})}\right]\mathbb{E}\left[\left(F_{\alpha_2,\alpha_1}(\mathcal{T},\widetilde{\beta}^{\mathcal{T}}_\cdot)\right)^{-\frac{\alpha_1+\alpha_2-Q}{\gamma}}\right]\end{aligned} \tag{3.18}$$

  where $\tau\sim\mathrm{Exp}(\kappa\gamma-(\alpha_1+\alpha_2-Q))$, $\mathcal{T}\sim\mathrm{Exp}(\alpha_1+\alpha_2-Q)$ and $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

- *If $\alpha_1+\alpha_2-Q=\kappa\gamma$, then*

$$E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\frac{1}{\kappa\gamma}\sqrt{\frac{2}{\pi}}\mathbb{E}\left[\left(F_{\alpha_2,\alpha_1}(\mathcal{T},\widetilde{\beta}^{\mathcal{T}}_\cdot)\right)^{-\kappa}\right]. \tag{3.19}$$

  where $\mathcal{T}\sim\mathrm{Exp}(\kappa\gamma)$.

- *If $\alpha_1+\alpha_2-Q>\kappa\gamma$, then*

$$E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\mathbb{E}\left[\left(F_{\alpha_2,\alpha_1}(0,-B^{-(\alpha_1+\alpha_2-Q-\kappa\gamma)}_\cdot)\right)^{-\kappa}\right] \tag{3.20}$$

  where $(B^{-(\alpha_1+\alpha_2-Q-\kappa\gamma)}_s)_{s\geq 0}$ is a Brownian motion with negative drift $-(\alpha_1+\alpha_2-Q-\kappa\gamma)$.

Remark 5. When $\alpha_1+\alpha_2-Q>\kappa\gamma$, we can easily rewrite (3.20) as

$$E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\mathbb{E}\left[\left(\int_{\mathbb{C}}\frac{|x^{-1}|_+^{(\kappa+1)\gamma^2}dM^\gamma(x)}{|x|^{4-\gamma(\alpha_1+\alpha_2)}|x-1|^{\gamma\alpha_2}}\right)^{-\kappa}\right]$$

which is very similar to the subcritical regime $\alpha_1+\alpha_2-Q<0$ where

$$E^\gamma_\kappa(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\mathbb{E}\left[\left(\int_{\mathbb{C}}\frac{dM^\gamma(x)}{|x|^{\gamma(\alpha_1+\alpha_2)}|x-1|^{\gamma\alpha_3}|x|_+^{4-\gamma\sum_{j=1}^4\alpha_j}}\right)^{-\kappa}\right]$$

can be obtained immediately by dominated convergence.

3.1.4 Conjectured link with random planar maps

The result of Theorem 3.1.1 has an interesting counterpart in the world of 2d discretised quantum gravity via the famous KPZ conjecture which was originally formulated in the physics literature by Knizhnik, Polyakov and Zamolodchikov [KPZ88]. Roughly speaking, the authors conjectured that, in some sense, LCFT should be the scaling limit of large random planar maps weighted by some statistical mechanics model.

We start by recalling some facts about planar maps, using the setting of [Kup16] section 1 (see also [DKRV16] section 5.3). A *planar map* is a graph together with an embedding into the sphere such that no two edges cross and viewed up to orientation preserving homeomorphisms.

For concreteness, we will work with triangulations, meaning that all the faces in the map are triangles. Let $\mathcal{T}_{N,3}$ be the set of planar triangulations with N faces and 3 extra marked points (called *roots*). The combinatorics of $\mathcal{T}_{N,3}$ is well known since the work of Tutte [Tut63] and we have

$$\#\mathcal{T}_{N,3} \underset{N\to\infty}{\asymp} N^{-1/2}e^{-\mu_c N}$$

for some $\mu_c > 0$. We mention that a wide class of planar maps fall into the same universality class (e.g. $2p$-angulations), meaning that they scale like $N^{-1/2}e^{-\mu_c N}$ where μ_c depends on the model.

There is a way to conformally embed any triangulation $(\mathbf{t}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ into the sphere by first turning it into a topological manifold and second specifying complex coordinate charts. This endows the triangulation with a structure of Riemann surface with conical singularities at vertices with $n \neq 6$ neighbours, and this embedding is unique if we add the extra requirement that the marked points $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ are sent to $(0, 1, \infty)$ (see e.g. [Kup16]). Concretely, if $\triangle \subset \mathbb{C}$ is an equilateral triangle with unit (Lebesgue) volume, the embedding provides a conformal map $\psi_t : \triangle \to \widehat{\mathbb{C}}$ for each triangle t in the map. For all $a > 0$, we consider the pushforward measure $d\nu_{t,a}(z) = a^2|(\psi_t^{-1})'(z)|^2 dz$ on $\psi_t(\triangle)$, which assigns a mass a^2 to each triangle of $\mathbf{t}$. The collection of $(\nu_{t,a})_{t\in\mathbf{t}}$ defines a measure $\nu_a^{\mathbf{t}}$ on $\widehat{\mathbb{C}}$, and in particular $\nu_a^{\mathbf{t}}(\widehat{\mathbb{C}}) = Na^2$ for all $\mathbf{t} \in \mathcal{T}_{N,3}$.

The model becomes interesting when we choose the triangulation randomly. The simplest example is the case of *pure gravity*, which amounts in sampling the triangulation with respect to the probability measure defined by

$$\mathbb{P}_a(\mathbf{t}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) := \frac{1}{Z_a} e^{-\mu|\mathbf{t}|}$$

where $\mu := (1 + a^2)\mu_c$, $|\mathbf{t}|$ is the number of faces of $\mathbf{t}$ and Z_a is a normalising constant. Notice that $Z_a \to \infty$ as we send $a \to 0$, which means that the measure selects larger and larger maps. When $(\mathbf{t}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is sampled under $\mathbb{P}_a$, the KPZ conjecture states that the random measure $\nu_a = \nu_a^{\mathbf{t}}$ converges in distribution (with respect to the topology of weak convergence of measures) as $a \to 0$ to a random Radon measure ν on $\mathbb{S}^2$. This limiting measure is expected to be given by the Liouville measure (see [DKRV16, Section 3.3] for a definition) and in particular, it should satisfy the property that for all measurable $A \subset \widehat{\mathbb{C}}$,

$$\mathbb{E}\left[\frac{\nu(A)}{\nu(\widehat{\mathbb{C}})}\right] = \int_A f_{\sqrt{8/3},\mu_c}$$

where we have defined the probability density function

$$f_{\gamma,\mu}(z) := \frac{\mu\gamma}{3\gamma - 2Q} \frac{\langle V_\gamma(0)V_\gamma(z)V_\gamma(1)V_\gamma(\infty)\rangle}{C_\gamma(\gamma,\gamma,\gamma)} \tag{3.21}$$

for all $\gamma \in (0,2)$ and $\mu > 0$ (see Appendix 3.A for the derivation of the normalising constant). The critical case of Theorem 3.1.1 is given by $\gamma = \frac{2}{\sqrt{3}}$[4], so that $\gamma = \sqrt{\frac{8}{3}}$ falls into the supercritical case. Thus we have the asymptotic behaviour (note that $\Delta_\gamma = \frac{\gamma}{2} \times \frac{2}{\gamma} = 1$)

$$f_{\gamma,\mu}(z) \underset{z\to 0}{\sim} \frac{\mu\gamma}{2\sqrt{2\pi}(3\gamma - 2Q)} |z|^{\frac{Q^2}{2}-4} \left(\log \frac{1}{|z|}\right)^{-3/2} \frac{(\partial_3 C_\gamma(\gamma,\gamma,Q))^2}{C_\gamma(\gamma,\gamma,\gamma)} \tag{3.22}$$

If we integrate this formula on a small disc of radius ε, we find

$$\int_0^\varepsilon r^{\frac{Q^2}{2}-4} \left(\log \frac{1}{r}\right)^{-3/2} r dr = (Q^2/2-2)^{1/2} \int_{(Q^2/2-2)\log\frac{1}{\varepsilon}}^\infty e^{-u} u^{-3/2} du \underset{\varepsilon\to 0}{\sim} 2\frac{\varepsilon^{\frac{Q^2}{2}-2}}{\sqrt{\log\frac{1}{\varepsilon}}}$$

so that

$$\int_{|z|\le\varepsilon} f_{\gamma,\mu}(z)dz \underset{\varepsilon\to 0}{\sim} \frac{\sqrt{2\pi}\mu\gamma}{3\gamma - 2Q} \frac{(\partial_3 C_\gamma(\gamma,\gamma,Q))^2}{C_\gamma(\gamma,\gamma,\gamma)} \frac{\varepsilon^{\frac{Q^2}{2}-2}}{\sqrt{\log\frac{1}{\varepsilon}}} \tag{3.23}$$

If the conjecture holds true, the asymptotic behaviour (3.23) gives the expected fraction of vertices which are close to 0 in a large planar map. In particular, the exponent of ε is $\frac{Q^2}{2} - 2 = 1/12$ for pure gravity.

Similar conjectures hold for random maps coupled with some statistical mechanics model (e.g. Ising, Potts... see [DKRV16]). The conjectures are essentially the same in each case except that the value of γ and μ may vary (e.g. Ising model corresponds to $\gamma = \sqrt{3}$). However one can still plug the good value of γ in formula (3.23) to conjecture the expected density of points around 0.

[4] We notice that this is a special value of γ from the random maps perspective since it corresponds to the scaling limit of bipolar-oriented maps, see [KMSW19]

3.1.5 Outline

The remainder of this article is organised as follows. In the next section, we provide a summary of GFF and GMC for the construction of Liouville correlation functions, and then explain the main idea of our proofs. Section 3.3 is devoted to the proof of Theorem 3.1.1 (on the four-point correlation) and Theorem 3.1.2 (on the decay of arbitrary negative moments of GMC), while that of Theorem 3.1.3 (on the probabilistic representations of the limiting constants) is treated in Section 3.4. In the appendices we collect the DOZZ formula, discuss our work from the perspective of surfaces with conical singularities and explain how to normalise the four-point correlation to a probability distribution.

Update: Since the first version of this paper was posted online, a proof of the conformal bootstrap formula (3.6) has been proposed [GKRV20] and there has been some progress on the integrability programme of boundary LCFT [RZ20b]. The proof of [GKRV20] relies on conformal symmetry while our methods are purely probabilistic and fall in the scope of logarithmically correlated fields.

3.2 Background

In this section, we recall the mathematical foundation for the Liouville measure (3.2) and the derivation for the 4-point function, and explain the main idea of our approach. To commence with, we quickly review GFF and GMC and mention several facts about them.

3.2.1 Gaussian Free Field

Let $H_0^1(\mathbb{S}^2, g)$ (or simply H_0^1) be the Sobolev space of functions with distributional derivatives in $L^2(\mathbb{S}^2, g)$ and vanishing g-mean. This space is equipped with the norm

$$\|X\|_\nabla^2 := \frac{1}{2\pi}\int_{\mathbb{S}^2} |\nabla X|^2 = -\frac{1}{2\pi}\int_{\mathbb{S}^2} \Delta X \cdot X$$

that we call the *Dirichlet energy*. This interprets the formal measure $\frac{1}{Z_{\mathrm{GFF}}}\int e^{-\frac{1}{2}\|X\|_\nabla^2} DX$ as a Gaussian probability measure on the space H_0^1 (where Z_{GFF} is a "normalising constant" which we will explain at the end of this section). Thus if $(e_n)_{n\geq 1}$ is an orthonormal basis of H_0^1, we define the formal series

$$X = \sum_{n\geq 1} \alpha_n e_n$$

where $(\alpha_n)_{n\geq 1}$ is a sequence of i.i.d. normal random variables. It can be shown that this series converges in H_0^{-1}, the topological dual of H_0^1. In particular, it is not defined as a function but rather as a distribution in the sense of Schwartz. We call this field the Gaussian Free Field (GFF). We write $\mathbb{P}$ for the probability measure of the GFF and $\mathbb{E}$ the associated expectation. The covariance kernel of the GFF is given by Green's function $G := (-\frac{1}{2\pi}\Delta)^{-1}$, i.e. we formally write

$$\mathbb{E}[X(x)X(y)] = G(x,y)$$

where the kernel of Green's function is explicitly given by

$$G(x,y) = \log\frac{1}{|x-y|} + \log|x|_+ + \log|y|_+$$

Thus the "normalising constant" Z_{GFF} that we are looking for should be given by $Z_{\mathrm{GFF}} := (\det(-\frac{1}{2\pi}\Delta))^{1/2}$, which is obtained via zeta-regularisation [OPS88].

There is a convenient choice of basis for H_0^1, which is the family $(\sqrt{\frac{2\pi}{\lambda_n}}\varphi_n)_{n\geq 1}$ where $(\varphi_n)_{n\geq 0}$ is an orthonormal basis of L^2 of eigenfunctions of $-\Delta$ with eigenvalues $0 = \lambda_0 < \lambda_1 \leq ... \leq \lambda_n$.... This gives an L^2 decomposition of the GFF, except that we are missing the zero mode (the coefficient in front of the constant function $\varphi_0 \equiv \mathrm{Vol}_g(\mathbb{S}^2)^{-1/2}$). This should be a Gaussian with infinite variance and we interpret this as Lebesgue measure, since $\sqrt{\frac{2\pi}{\lambda}}$ times the law of a Gaussian random variable with variance λ^{-1} converges vaguely to Lebesgue measure as $\lambda \to 0$. So our final interpretation of the measure $e^{-\frac{1}{2}\|X\|_\nabla^2}DX$ is that we set for all continuous functional $F : H^{-1} \to \mathbb{R}$

$$\int F(X)e^{-\frac{1}{2}\|X\|_\nabla^2}DX = \left(\frac{\det(-\frac{1}{2\pi}\Delta)}{\mathrm{Vol}_g(\mathbb{S}^2)}\right)^{-1/2}\int_{\mathbb{R}}\mathbb{E}[F(X+c)]dc \tag{3.24}$$

Throughout the paper, we will make an extensive use of the so-called *radial/angular* decomposition of the GFF, which is better understood in cylinder coordinates. Let $\mathcal{C}_\infty := \mathbb{R}_s \times \mathbb{S}_\theta^1$ be the complete cylinder. Under the conformal change of coordinates $\psi : z \mapsto -\log z$, the Riemann sphere $(\widehat{\mathbb{C}} \setminus \{0,\infty\}, g)$ endowed with the crêpe metric is mapped to $(\mathcal{C}_\infty, g_\psi)$ with $g_\psi(s,\theta) = e^{-2|s|}$. From now on, we write G for Green's function on $(\mathcal{C}_\infty, g_\psi)$ with vanishing mean on $\{0\} \times \mathbb{S}^1$.

Lemma 3.2.1. *Let $X(s,\theta)$ be a GFF on $\mathcal{C}_\infty$. Then we can write $X(s,\theta) = B_s + Y(s,\theta)$ where*

1. *$(B_s)_{s\in\mathbb{R}}$ is a two-sided Brownian motion. We will call this process the* radial part *of the field.*
2. *Y is a* log*-correlated field with covariance kernel*

$$H(s,\theta,s',\theta') := \mathbb{E}[Y(s,\theta)Y(s',\theta')] = \log\frac{e^{-s}\vee e^{-s'}}{|e^{-s-i\theta} - e^{-s'-i\theta'}|} \tag{3.25}$$

We will call this field the lateral noise *or* angular part *of the field. Notice that the law of Y is translation invariant.*

3. B is independent of Y.

Otherwise stated, Lemma 3.2.1 enables to rewrite Green's function (on the cylinder) as

$$\begin{aligned} G(s,\theta,s',\theta') &= (|s|\wedge|s'|)1_{ss'\geq 0} + H(s,\theta,s',\theta') \\ &= (|s|\wedge|s'|)1_{ss'\geq 0} + H(0,0,s'-s,\theta'-\theta) \\ &= (|s|\wedge|s'|)1_{ss'\geq 0} + G(0,0,s'-s,\theta'-\theta) \end{aligned} \tag{3.26}$$

Remark 6. We will sometimes abuse notations and write the more compact form $G(s+i\theta,s'+i\theta')$ (resp. $H(s+i\theta,s'+i\theta')$) for $G(s,\theta,s',\theta')$ (resp. $H(s+i\theta,s'+i\theta')$).

3.2.2 Gaussian Multiplicative Chaos

Recall that a GFF is only defined as a distribution, so the exponential term $e^{\gamma X}$ is ill-defined *a priori*. However it is possible to make sense of the measure $e^{\gamma X(x)}g(x)d^2x$ using a regularising procedure based on Kahane's theory of Gaussian Multiplicative Chaos (GMC) (see [RV14, Ber17] for more detailed reviews).

We use the regularisation called the circle average. For $\varepsilon>0$, let $X_{g,\varepsilon}$ be the average of X on the geodesic circle of radius ε in the metric g. The field X_ε is continuous, so the measure

$$dM^\gamma_{g,\varepsilon}(x) := e^{\gamma X_{g,\varepsilon}(x)-\frac{1}{2}\gamma^2\mathbb{E}[X_{g,\varepsilon}(x)^2]}d^2x$$

is well-defined for all $\gamma\in(0,2)$, and it is known that the sequence of measures $M^\gamma_{g,\varepsilon}$ converges weakly in probability to a (random) Radon measure M^γ_g with no atoms.

An important property of GMC measure is its conformal covariance under conformal change of metrics (see e.g. equation (3.12) of [GRV19]). Let $\omega\in\mathcal{C}^\infty(\mathbb{S}^2,g)$. Let X be a GFF on $(\mathbb{S}^2,g)$ and $M^\gamma_{\widetilde{g}}$ be the GMC measure obtained when regularising the field with circle averages in the metric $\widetilde{g}:=e^\omega g$. Then $M^\gamma_{\widetilde{g}} = e^{\frac{\gamma Q}{2}}M^\gamma_g$.

Remark 7. For notational convenience, when the regularising metric is the background metric $g(x)=|x|_+^{-4}$ on $\widehat{\mathbb{C}}$, we will drop the subscript and write $M^\gamma = M^\gamma_g$.

Another useful tool of GMC is Kahane's convexity inequality [RV14, Theorem 2.2]

Theorem 3.2.2 (Kahane 1985)**.** *Let X and Y be two continuous Gaussian fields on $D\subset\mathbb{S}^2$ such that for all $x,y\in D$*

$$\mathbb{E}[X(x)X(y)] \leq \mathbb{E}[Y(x)Y(y)]$$

Then for all convex function $F : \mathbb{R}_+ \to \mathbb{R}$ with at most polynomial growth at infinity,

$$\mathbb{E}\left[F\left(\int_D e^{\gamma X(x)-\frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}d^2x\right)\right] \leq \mathbb{E}\left[F\left(\int_D e^{\gamma Y(x)-\frac{\gamma^2}{2}\mathbb{E}[Y(x)^2]}d^2x\right)\right]$$

In practice, one can apply this theorem to the GMC measure log-correlated fields like the GFF after using the regularising procedure.

Now suppose X, Y are log-correlated fields with $|\mathbb{E}[X(x)X(y) - \mathbb{E}[Y(x)Y(y)]| \leq \varepsilon$ and write M^γ, N^γ for their respective chaos measure. In particular we have

$$\mathbb{E}[X(x)X(y)] \leq \mathbb{E}[Y(x)Y(y)] + \varepsilon$$

Notice that the field $Z(x) = Y(x) + \sqrt{\varepsilon}\delta$ – with $\delta \sim \mathcal{N}(0,1)$ independent of everything – has covariance kernel $\mathbb{E}[Y(x)Y(y)] + \varepsilon$. Hence by Kahane's convexity inequality, we have for all $\kappa > 0$

$$\mathbb{E}[M^\gamma(D)^{-\kappa}] \leq \mathbb{E}[e^{-r\gamma\sqrt{\varepsilon}\delta}N^\gamma(D)^{-\kappa}] = e^{\frac{1}{2}\gamma^2 r^2\varepsilon}\mathbb{E}[N^\gamma(D)^{-\kappa}]$$

By the symmetry of the roles played by X and Y, the converse inequality is also true, so

$$\mathbb{E}[M^\gamma(D)^{-\kappa}] = \mathbb{E}[N^\gamma(D)^{-\kappa}](1 + O_{\varepsilon\to 0}(\varepsilon))$$

Similarly, we have for all $c \in \mathbb{R}$,

$$\mathbb{E}\left[\exp(-\mu e^{\gamma c}M^\gamma(D))\right] = \mathbb{E}\left[\exp(-\mu e^{\gamma c}N^\gamma(D))\right](1 + O_{\varepsilon\to 0}(\varepsilon))$$

3.2.3 Derivation of the correlation function

Using the GFF and GMC we are ready to state the definition of the correlation functions on the sphere. For $\varepsilon > 0$, we can regularise the vertex operator $V_{\alpha_i}(z_i)$ by defining $V_{\alpha_i,\varepsilon}(z_i) = e^{\alpha_i X_\varepsilon(z_i) - \frac{\alpha_i^2}{2}\mathbb{E}[X_\varepsilon(z_i)^2]}$. By Cameron-Martin theorem, we have (recall $\sigma = \sum_{i=1}^N \frac{\alpha_i}{Q} - 2 > 0$)

$$\left\langle \prod_{i=1}^N V_{\alpha,\varepsilon}(z_i) \right\rangle = 2e^{C_\varepsilon(\mathbf{z})}\int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}\left[\exp\left(-\mu e^{\gamma c}\int_{\widehat{\mathbb{C}}} e^{\gamma \sum_{i=1}^N \alpha_i G_\varepsilon(z_i,\cdot)}dM^\gamma\right)\right]dc \tag{3.27}$$

where $C_\varepsilon(\mathbf{z}) = \sum_{i<j}\alpha_i\alpha_j G_\varepsilon(z_i, z_j)$. This regularised correlation function (3.27) converges to a positive finite limit as $\varepsilon \to 0$ as long as the Seiberg bounds are satisfied as the GMC measure integrates the singularities around each insertion. We take this limit as our

definition of the correlation function

$$\begin{aligned}\left\langle \prod_{i=1}^{N} V_{\alpha_i}(z_i) \right\rangle &= 2e^{C(\mathbf{z})} \int_{\mathbb{R}} e^{Q\sigma c} \mathbb{E}\left[\exp\left(-\mu e^{\gamma c} \int_{\widehat{\mathbb{C}}} e^{\gamma \sum_{i=1}^{N} \alpha_i G(z_i, \cdot)} dM^{\gamma} \right) \right] dc \\ &= 2e^{C(\mathbf{z})} \gamma^{-1} \mu^{-\frac{Q\sigma}{\gamma}} \Gamma\left(\frac{Q\sigma}{\gamma}\right) \mathbb{E}\left[\left(\int_{\widehat{\mathbb{C}}} e^{\gamma \sum_{i=1}^{N} \alpha_i G(z_i, \cdot)} dM^{\gamma} \right)^{-\frac{Q\sigma}{\gamma}} \right]\end{aligned} \tag{3.28}$$

after making the change of variable $u = e^{\gamma c}$. As can be seen from expression (3.28), the finiteness of the correlation function in our probabilistic formulation is equivalent to the finiteness of the moments of the GMC measure. This holds provided the *extended Seiberg bounds* are satisfied [KRV20]

$$-\frac{Q\sigma}{\gamma} < \frac{4}{\gamma^2} \wedge \min_{1 \le i \le N} (Q - \alpha_i) \qquad \forall i, \alpha_i < Q$$

In particular, if $N = 3$ with insertions at $(0, 1, \infty)$ and Liouville momenta $(\alpha_1, \alpha_2, \alpha_3)$ satisfying the Seiberg bounds, the expression is simply

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle = 2\gamma^{-1} \mu^{-\frac{Q\sigma}{\gamma}} \Gamma\left(\frac{Q\sigma}{\gamma}\right) \mathbb{E}\left[\left(\int_{\widehat{\mathbb{C}}} e^{\gamma(\alpha_1 G(0,\cdot) + \alpha_2 G(1,\cdot) + \alpha_3 G(\infty,\cdot))} dM^{\gamma} \right)^{-\frac{Q\sigma}{\gamma}} \right] \tag{3.29}$$

and this expression equals the DOZZ formula $C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ [KRV20].

As for the four-point correlation function with insertions at $(z_1, z_2, z_3, z_4) = (0, z, 1, \infty)$ with $|z| < 1$, we find

$$\begin{aligned}&\langle V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) \rangle \\ &\quad = \frac{2}{|z|^{\alpha_1 \alpha_2} |1 - z|^{\alpha_2 \alpha_3}} \int_{\mathbb{R}} e^{-Q\sigma c} \mathbb{E}\left[\exp\left(-\mu e^{\gamma c} \int_{\widehat{\mathbb{C}}} e^{\gamma \sum_{i=1}^{4} \alpha_i G(z_i, \cdot)} dM^{\gamma} \right) \right] dc\end{aligned} \tag{3.30}$$

3.2.4 Main idea

We now explain our approach which is inspired by [DKRV17]. By applying the radial/angular decomposition of the GFF as we will see in Section 3.3.1, we can effectively transform our problem to the study of exponential functionals of Brownian motion.

To be more precise consider the following toy model. Let $(B_s^{\lambda})_{s \ge 0}$ be a Brownian motion with drift λ, and suppose $C_1, C_2 > 0$ are two fixed constants. Our goal is to understand the asymptotics of

$$\mathbb{E}\left[\left(C_1 + \int_0^t e^{\gamma B_s^\lambda} ds + C_2 e^{\gamma B_t^\lambda}\right)^{-\kappa}\right] \tag{3.31}$$

as $t \to \infty$. In order to extract the leading order in (3.31), we have to play the game of balancing energy (i.e. asking our drifted Brownian motion $(B_s^\lambda)_s$ to remain small) and entropy (i.e. paying a multiplicative cost given by the probability of such event).

- When $\lambda < 0$, we don't have to do anything because $B_s^\lambda \xrightarrow{s\to\infty} -\infty$ anyway, and

$$\mathbb{E}\left[\left(C_1 + \int_0^t e^{\gamma B_s^\lambda} ds + C_2 e^{\gamma B_t^\lambda}\right)^{-\kappa}\right] \xrightarrow{t\to\infty} \mathbb{E}\left[\left(C_1 + \int_0^\infty e^{\gamma B_s^\lambda} ds\right)^{-\kappa}\right]$$

  by dominated convergence easily.

- When $\lambda = 0$, we should demand our Brownian motion to never exceed an $O(1)$ threshold. On the event that $\{\sup_{s\le t} B_s \le N\}$, $(N - B_s)_{s\le t}$ behaves like a $\mathrm{BES}_N(3)$-process and drifts to $-\infty$, and therefore for suitably chosen $t' \ll t$ we see that

$$C_1 + \int_0^t e^{\gamma B_s^\lambda} ds + C_2 e^{\gamma B_t^\lambda} \approx C_1 + \int_0^{t'} e^{\gamma B_s^\lambda} ds$$

  is expected to be $O(1)$ while the entropy cost is given by

$$\mathbb{P}\left(\sup_{s\le t} B_s \le N\right) \sim \sqrt{\frac{2}{\pi}} \frac{N}{\sqrt{t}} = O\left(t^{-\frac{1}{2}}\right).$$

- When $\lambda \in (0, \kappa\gamma)$, we still demand our drifted Brownian motion B_t^λ to remain below an $O(1)$ threshold, which requires an entropy cost of

$$\mathbb{P}\left(\sup_{s\le t} B_s^\lambda \le N\right) \sim \sqrt{\frac{2}{\pi}} \frac{e^{-\frac{\lambda^2}{2}t}}{\lambda^2 t^{\frac{3}{2}}} N e^{\lambda N} = O\left(e^{-\frac{\lambda^2}{2}t} t^{-\frac{3}{2}}\right).$$

  The structural difference here is that even though B_s^λ is rather negative in the intermediate time interval $s \in [t', t-t']$, the terminal value B_t^λ is typically $O(1)$:

$$\mathbb{P}\left(B_t^\lambda \le x \,\middle|\, \sup_{s\le t} B_s^\lambda \le N\right) \xrightarrow{t\to\infty} e^{-\lambda(N-x)}(1 + \lambda(N-x)), \qquad x \le N.$$

Therefore for the purpose of deriving the renormalised constant, we will have to keep

$$C_1 + \int_0^t e^{\gamma B_s^\lambda} ds + C_2 e^{\gamma B_t^\lambda} \approx C_1 + \int_0^{t'} e^{\gamma B_s^\lambda} ds + \int_{t-t'}^t e^{\gamma B_s^\lambda} ds + e^{\gamma B_t^\lambda} C_2.$$

which is $O(1)$ as $(B_s^\lambda)_{s \le t'}$ and $(B_{t-s}^\lambda - B_t^\lambda)_{s \le t'}$ behave like the negation of two independent BES(3)-processes.

- Moving beyond, we can only ask the B_s^λ not to drift faster than $\lambda - \kappa\gamma$ or else the entropy cost would be too expensive. To proceed we first apply Cameron-Martin theorem to rewrite (3.31) as

$$\begin{aligned}
&\mathbb{E}\left[e^{\kappa\gamma B_t - \frac{\kappa^2\gamma^2}{2}t} \left(C_1 + \int_0^t e^{\gamma B_s^{\lambda-\kappa\gamma}} ds + C_2 e^{\gamma B_t^{\lambda-\kappa\gamma}} \right)^{-\kappa} \right] \\
&\quad = e^{-\kappa\gamma\lambda t + \frac{\kappa^2\gamma^2}{2}t} \mathbb{E}\left[\left(C_1 e^{-\gamma B_t^{\lambda-\kappa\gamma}} + \int_0^t e^{\gamma(B_s^{\lambda-\kappa\gamma} - B_t^{\lambda-\kappa\gamma})} ds + C_2 \right)^{-\kappa} \right]. \qquad (3.32)
\end{aligned}$$

If $\lambda = \kappa\gamma$, there isn't any drift in the expectation. The observation from the case $\lambda = 0$ suggests that we may want to demand $B_{t-s} - B_t$ to not exceed an $O(1)$ threshold for $s \le t$. This would imply again an entropy cost of $O(t^{-\frac{1}{2}})$, and we expect that

$$C_1 e^{-\gamma B_t^{\lambda-\kappa\gamma}} + \int_0^t e^{\gamma(B_s - B_t)} ds + C_2 \approx \int_0^{t'} e^{\gamma(B_s - B_t)} ds + C_2$$

is $O(1)$ because $(B_{t-s} - B_t)_{s \le t'}$ behaves like the negation of a BES(3)-process as before. If $\lambda > \kappa\gamma$, the story is simpler because $B_{t-s}^{\lambda-\kappa\gamma} - B_t^{\lambda-\kappa\gamma}$ may be seen as a Brownian motion with negative drift. Similar to the earlier case where $\lambda < 0$,

$$C_1 e^{-\gamma B_t^{\lambda-\kappa\gamma}} + \int_0^t e^{\gamma(B_s^{\lambda-\kappa\gamma} - B_t^{\lambda-\kappa\gamma})} ds + C_2 \approx \int_0^{t'} e^{\gamma(B_s^{\lambda-\kappa\gamma} - B_t^{\lambda-\kappa\gamma})} ds + C_2.$$

is already $O(1)$ without incurring any further entropy cost.

3.2.5 Path decomposition of BES(3)-processes

Before we proceed to the proofs, we collect Williams' path decomposition theorem [Wil74] for 3-dimensional Bessel processes (abbreviated as BES(3)-processes) which will be helpful when we study the probabilistic representations of the renormalised constant (3.15).

Theorem 3.2.3 (Williams 1974)**.** *Fix $x > 0$, and consider the following independent objects:*

- *$(B_s)_{s\geq 0}$ is a standard Brownian motion (starting from 0).*
- *U is a $\mathrm{Uniform}[0,1]$ random variable.*
- *$(\beta^0_s)_{s\geq 0}$ is a 3-dimensional Bessel process starting from 0.*

Then the process $(\widehat{\beta}^x_s)_{s\geq 0}$ defined by

$$\widehat{\beta}^x_s = \begin{cases} x + B_s & s \leq T_{-x(1-U)}, \\ xU + \beta^0_{s-T_{-x(1-U)}} & s \geq T_{-x(1-U)} \end{cases} \tag{3.33}$$

with

$$T_{-x(1-U)} = \inf\{s > 0 : B_s = -x(1-U)\} = \inf\{s > 0 : x + B_s = xU\}$$

is a 3-dimensional Bessel process starting from x (written as $\mathrm{BES}_x(3)$-process).

In view of Theorem 3.2.3, we introduce the following definition.

Definition 3.2.1. Let $(B_s)_{s\geq 0}$ and $(\beta^0_s)_{s\geq 0}$ be as in Theorem 3.2.3, and $x \geq 0$ an independent random variable. Then the process $(\widetilde{\beta}^x_s)_{s\geq 0}$ defined by

$$\widetilde{\beta}^x_s = \begin{cases} x + B_s & s \leq T_{-x}, \\ \beta^0_{s-T_{-x}} & s \geq T_{-x} \end{cases} \tag{3.34}$$

with

$$T_{-x} = \inf\{s > 0 : x + B_s = 0\}$$

is called a 3-dimensional Bessel process starting from x conditioned on hitting 0, written as $\widetilde{\mathrm{BES}}_x(3)$-process.

3.3 Proof of Theorem 3.1.1

3.3.1 Supercritical case

We set the insertions at $(z_1, z_2, z_3, z_4) := (0, z, 1, \infty)$ with Liouville momenta $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfying the Seiberg bounds, and we write $-\log z = t + i\phi$ with $t > 0$ and $\phi \in [0, 2\pi)$. We assume that both $\alpha_3 + \alpha_4 - Q > 0$ and $\alpha_1 + \alpha_2 - Q > 0$ which corresponds to the supercritical case of Theorem 3.1.1. Notice that this corresponds precisely to having (α_1, α_2, Q) and (Q, α_3, α_4) satisfying the Seiberg bounds (with respectively the 3rd and 1st momenta saturating the second Seiberg bound).

Proof of (3.12). Let $X(s,\theta) = B_s + Y(s,\theta)$ be a GFF on $\mathcal{C}_\infty = \mathbb{R}_s \times \mathbb{S}^1_\theta$. By the conformal covariance of GMC, it is equivalent to study the chaos measure of X with respect to g_ψ or to consider the field $X(s,\theta) + \frac{Q}{2}\log g_\psi(s,\theta) = X(s,\theta) - Q|s|$ and do the regularisation with respect to Lebesgue measure.

From now on, we write $d\widehat{M}^\gamma(s,\theta)$ for GMC measure of the lateral noise with respect to Lebesgue measure on $\mathcal{C}_\infty$ (while $dM^\gamma(x)$ will be used for GMC measure of the entire GFF in spherical coordinates).

We are interested in the total GMC mass

$$\begin{aligned} W_t &:= \int_{\mathcal{C}_\infty} e^{\gamma(B_s+(\alpha_1-Q)s1_{s>0}-(\alpha_4-Q)s1_{s<0}+\alpha_3 G(0,s+i\theta)+\alpha_2 G(t+i\phi,s+i\theta))} d\widehat{M}^\gamma(s,\theta) \\ &= \int_{\mathcal{C}_\infty} e^{\gamma(B_s+(\alpha_1+\alpha_2 1_{s<t}-Q)s1_{s>0}-(\alpha_4-Q)s1_{s<0}+\alpha_3 G(0,s+i\theta)+\alpha_2 G(0,s-t+i(\theta-\phi)))} d\widehat{M}^\gamma(s,\theta) \end{aligned} \tag{3.35}$$

The behaviour of this integral is essentially governed by the radial process. From the expression above, we can see that on the negative real line the process is $(B_{-s}+(\alpha_4-Q)s)_{s\geq 0}$ which is a Brownian motion with negative drift so the integrand is integrable at $s=-\infty$. On the positive real line, the radial process has a positive drift $\alpha_1+\alpha_2-Q$ up to time t, then a negative drift $\alpha_4 - Q$ from t to ∞.

The first step is to apply Cameron-Martin theorem to get rid of the $(\alpha_1+\alpha_2-Q)$ drift term in $[0,t]$, so that for all continuous and bounded function $F:\mathbb{R}\to\mathbb{R}$

$$\mathbb{E}\left[F(W_t)\right] = \mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t - \frac{1}{2}(\alpha_1+\alpha_2-Q)^2 t} F(Z_t)\right] \tag{3.36}$$

where Z_t is the random variable defined by

$$Z_t := \int_{\mathcal{C}_\infty} e^{\gamma(B_s+(\alpha_1-Q)(t-s)1_{s>t}-(\alpha_4-Q)s1_{s<0}+\alpha_2 G(0,t-s+i(\phi-\theta))+\alpha_3 G(0,s+i\theta))} d\widehat{M}^\gamma(s,\theta) \tag{3.37}$$

Hence the correlation function takes the form (recall $t = \log\frac{1}{|z|}$)

$$\begin{aligned} &\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle \\ &\qquad = 2|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)}|1-z|^{-\alpha_2\alpha_3}\int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t}\exp(-\mu e^{\gamma c} Z_t)\right] dc \end{aligned} \tag{3.38}$$

where the exponent for $|z|$ was found by noticing that $\frac{1}{2}(\alpha_1+\alpha_2-Q)^2 - \alpha_1\alpha_2 = 2(\frac{Q^2}{4} - \Delta_1 - \Delta_2)$.

Remark 8. The change of measure (3.36) becomes trivial if $\alpha_1+\alpha_2 = Q$. This is the reason why there is a phase transition at this value and why the case is easier to treat.

Remark 9. From a geometric point of view, the change of measure (3.36) has the effect of changing the background metric from a cone to a cylinder as illustrated in Figure 3.2

(see also Appendix 2.B for links between changes of metrics and changes of probability measures).

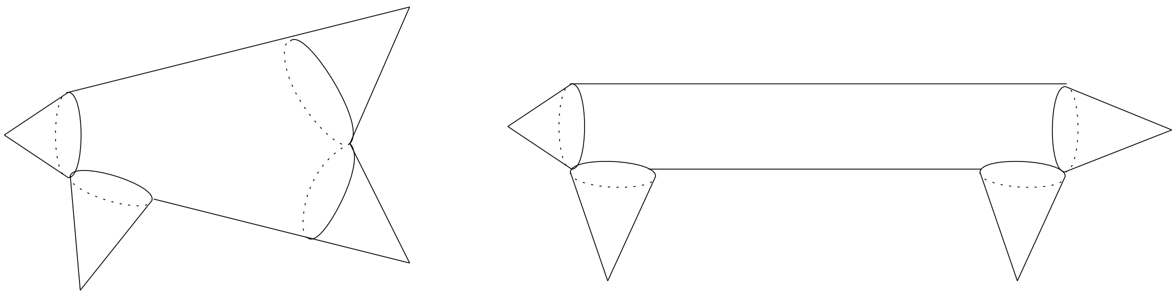

Figure 3.2: Change of measure from the cone to the cylinder

We can sample the radial part $(B_s)_{0\leq s\leq t}$ by the independent sum $B_s = Br_s + \frac{\delta}{\sqrt{t}}s$ where $(Br_s)_{0\leq s\leq t}$ is a standard Brownian bridge and $\delta \sim \mathcal{N}(0,1)$ (see Figure 3.3). We write $(\widetilde{B}_s)_{0\leq s\leq t}$ the process on $\mathbb{R}$ where

1. $(\widetilde{B}_{-s})_{s\geq 0}$ and $(\widetilde{B}_s)_{s\geq t}$ are independent Brownian motions.

2. $(\widetilde{B}_s)_{0\leq s\leq t}$ is a Brownian bridge in $[0,t]$ independent of the two other processes.

Similarly, we write $\widetilde{Z}_t$ for the GMC mass defined similarly as Z_t but with $\widetilde{B}$ instead of B. The result will follow from an analysis of the behaviour of $\widetilde{Z}_t$.

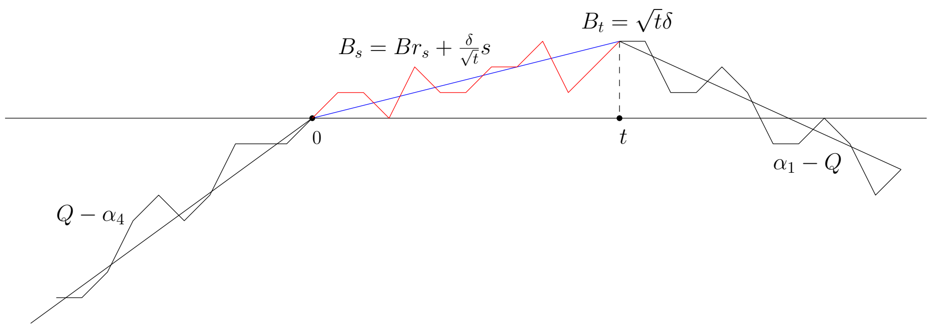

Figure 3.3: The radial process in $(0,t)$ is the independent sum of a Brownian bridge (red) and a random drift (blue).

Let $\eta \in (0,1/2)$. We split $\widetilde{Z}_t$ into three parts and write $\widetilde{Z}_t = \widetilde{L}_t + \widetilde{C}_t + \widetilde{R}_t$ where $\widetilde{L}_t$, $\widetilde{C}_t$ and $\widetilde{R}_t$ are obtained by restricting the domain of integration to $(-\infty, t^{1/2-\eta}) \times \mathbb{S}^1$, $(t^{1/2-\eta}, t - t^{1/2-\eta}) \times \mathbb{S}^1$ and $(t - t^{1/2-\eta}, \infty) \times \mathbb{S}^1$ respectively. We define $Z_t = L_t + C_t + R_t$ similarly. These random variables are the “left”, “central”, and “right” parts of the $\widetilde{Z}_t$ and Z_t.

For $b > 0$, we introduce the event $\widetilde{A}_{b,t} := \left\{ \sup_{0\leq s\leq t} \widetilde{B}_s \leq b \right\}$. This event has probability

$$\mathbb{P}(\widetilde{A}_{b,t}) = 1 - e^{-2b^2/t} =: f(b/\sqrt{t}).$$

Notice that $\lim_{x\to\infty} f(x) = 1$ and $f(x) \underset{x\to 0}{\sim} 2x^2$.

Conditioning on $\widetilde{A}_{b,t}$, the processes $(b-\widetilde{B}_s)_{0\le s\le t/2}$ and $(b-\widetilde{B}_{t-s})_{0\le s\le t/2}$ are absolutely continuous with respect to a $\mathrm{BES}_b(3)$-process. Hence there exists $\eta'>0$ such that with high probability as $t\to\infty$, we have $\sup_{t^{1/2-\eta}\le s\le t-t^{1/2-\eta}} \widetilde{B_s} \le -t^{1/2-\eta'}$. It follows that $\widetilde{C}_t \to 0$ in probability as $t\to\infty$ when conditioned on $\widetilde{A}_{b,t}$.

Let $\mathbb{P}_b$ the law of a field $X(s,\theta) = B_s + Y(s,\theta)$ where

1. Y is a standard lateral noise.

2. $(B_{-s})_{s\ge 0}$ is a standard Brownian motion.

3. $(b-B_s)_{s\ge 0}$ is a $\mathrm{BES}_b(3)$-process independent of $(B_{-s})_{s\ge 0}$.

We now describe the behaviour of $\widetilde{L}_t$ and $\widetilde{R}_t$. On $\widetilde{A}_{b,t}$, the law of the process $(b-\widetilde{B}_s)_{0\le s\le t^{1/2-\eta}}$ is absolutely continuous with respect to that of a $\mathrm{BES}_b(3)$-process, and the Radon-Nikodym derivative tends to 1 a.s. and in L^1 as $t\to\infty$ (see e.g. [MY16, Exercise 9.4]). Hence the pair of processes $((b-\widetilde{B}_s)_{0\le s\le t^{1/2-\eta}}, (b-\widetilde{B}_{t-s})_{0\le s\le t^{1/2-\eta}})$ converges in distribution to a pair of $\mathrm{BES}_b(3)$-processes, and it is clear that these limit processes are independent of each other.

As for the angular part, notice that for all $s < t^{1/2-\eta}$ and $s' > t - t^{1/2-\eta}$, we have for all $\theta,\theta'\in\mathbb{S}^1$,

$$H(s+i\theta, s'+i\theta') = \log\frac{1}{|1-e^{-(s'-s)-i(\theta'-\theta)}|} \le \log\frac{1}{1-e^{-(t-2t^{1/2-\eta})}} = O(e^{-t/2}) \tag{3.39}$$

Now let Y^+, Y^- be independent lateral noises on $\mathcal{C}_\infty$ and define

$$Y'(s,\theta) := Y^+(s,\theta)1_{s<t/2} + Y^-(s,\theta)1_{s\ge t/2}.$$

Let $\widetilde{L}_t^-$ (resp $\widetilde{R}_t^+$) be the random variable defined like $\widetilde{L}_t$ (resp. $\widetilde{R}_t^-$) except we use Y' rather than Y for the lateral noise. Then under $\widetilde{A}_{b,t}$, the pair $(\widetilde{L}_t^-, \widetilde{R}_t^+)$ converges in distribution to a pair of independent random variables (L_∞, R_∞) with

$$L_\infty \overset{\text{law}}{=} \int_{\mathcal{C}_\infty} e^{\gamma(B_s-(\alpha_4-Q)s1_{s\le 0}+\alpha_3 G(0,s+i\theta))} dM^\gamma(s,\theta)$$
$$R_\infty \overset{\text{law}}{=} \int_{\mathcal{C}_\infty} e^{\gamma(B_s-(\alpha_1-Q)s1_{s\le 0}+\alpha_2 G(0,s+i\theta))} dM^\gamma(s,\theta)$$

where the field is sampled from $\mathbb{P}_b$ in both cases.

Using the estimate (3.39) and Kahane's convexity inequality, we have for all $c \in \mathbb{R}$

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-\mu e^{\gamma c}(\widetilde{L}_t + \widetilde{R}_t)\right)|\widetilde{A}_{b,t}\right] &= \mathbb{E}_b\left[\mathcal{E}_t \exp\left(-\mu e^{\gamma c}(\widetilde{L}_t^- + \widetilde{R}_t^+)\right)\right](1 + O(e^{-t/2})) \\
&\underset{t\to\infty}{\to} \mathbb{E}_b\left[\exp(-\mu e^{\gamma c}(L_\infty + R_\infty))\right] \\
&= \mathbb{E}_b\left[\exp(-\mu e^{\gamma c} L_\infty)\right]\mathbb{E}_b\left[\exp(-\mu e^{\gamma c} R_\infty)\right]
\end{aligned}$$

Putting pieces together, we find for all $c \in \mathbb{R}$

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\left[\exp(-\mu e^{\gamma c}\widetilde{Z}_t)|\widetilde{A}_{b,t}\right] &= \lim_{t\to\infty}\mathbb{E}\left[\exp(-\mu e^{\gamma c}\widetilde{L}_t)\exp(-\mu e^{\gamma c}\widetilde{C}_t)\exp(-\mu e^{\gamma c}\widetilde{R}_t)|\widetilde{A}_{b,t}\right] \\
&= \mathbb{E}_b\left[\exp(-\mu e^{\gamma c} L_\infty)\right]\mathbb{E}_b\left[\exp(-\mu e^{\gamma c} R_\infty)\right]
\end{aligned}$$

To conclude we need to relate the behaviour of $\widetilde{Z}_t$ with that of Z_t as $t \to \infty$. To this end we will condition on the value of the drift $\delta \sim \mathcal{N}(0,1)$. For fixed $\delta \in \mathbb{R}$, we have $\frac{\delta}{\sqrt{t}} t^{1/2-\eta} = \delta t^{-\eta}$, and this will be sufficient to show that up to time $t^{1/2-\eta}$, the radial part of the GFF $(B_{t-s} - \frac{\delta}{\sqrt{t}} s)_{0 \le s \le t^{1/2-\eta}}$ does not "feel" the drift and therefore looks like a Brownian motion started from $\sqrt{t}\delta$. More precisely, we have

$$e^{-\gamma|\delta|t^{-\eta}}\widetilde{R}_t \le e^{-\gamma\sqrt{t}\delta} R_t \le e^{\gamma|\delta|t^{-\eta}}\widetilde{R}_t$$

Taking expectations and rescaling δ by $t^{-1/2}$, we get for all $c \in \mathbb{R}$

$$\begin{aligned}
&\sqrt{t}\mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t}\exp(-\mu e^{\gamma c} R_t)|\widetilde{A}_{b,t}\right] \\
&\quad = \int_{\mathbb{R}} e^{(\alpha_1+\alpha_2-Q)\delta}\mathbb{E}\left[\exp(-\mu e^{\gamma(c+\delta+\delta O(t^{-1/2-\eta}))}\widetilde{R}_t)|\widetilde{A}_{b,t}\right]\frac{e^{-\frac{t\delta^2}{2}}}{\sqrt{2\pi}}d\delta \\
&\quad \underset{t\to\infty}{\to} \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{(\alpha_1+\alpha_2-Q)\delta}\mathbb{E}_b[\exp(-\mu e^{\gamma(c+\delta)}R_\infty]d\delta
\end{aligned}$$

where we applied the dominated convergence theorem in the last line.

Remark 10. The take-out message of this computation is that as t gets large the value of the radial part at t is distributed like $\sqrt{t}\delta$, so when properly rescaled, its law converges vaguely to Lebesgue measure. Hence the field in the right part looks like a usual GFF plus a constant which is "distributed" with Lebesgue measure, so δ plays the role of an extra zero mode in the limit. This translates the fact that we see two independent surfaces in the limit.

Recalling the expression of the correlation function (3.38), we make the change of variable $(c,\delta)=(u,v-u)$ (with Jacobian equal to 1) and find

$$
\begin{aligned}
&\sqrt{t}\int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t}\exp(-\mu e^{\gamma c}Z_t)|\widetilde{A}_{b,t}\right]dc\\
&=\sqrt{t}\int_{\mathbb{R}} e^{Q\sigma c}\int_{\mathbb{R}} e^{(\alpha_1+\alpha_2-Q)\sqrt{t}\delta}\mathbb{E}\left[\exp(-\mu e^{\gamma c}Z_t)|\widetilde{A}_{b,t}\right]\frac{e^{-\frac{\delta^2}{2}}}{\sqrt{2\pi}}d\delta dc\\
&\underset{t\to\infty}{\to}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^2} e^{(\alpha_1+\alpha_2-Q)(c+\delta)}e^{(\alpha_3+\alpha_4-Q)c}\mathbb{E}_b\left[\exp(-\mu e^{\gamma c}L_\infty)\right]\mathbb{E}_b\left[\exp(-\mu e^{\gamma(c+\delta)}R_\infty)\right]d\delta dc\\
&=\frac{1}{\sqrt{2\pi}}\left(\int_{\mathbb{R}} e^{(\alpha_3+\alpha_4-Q)u}\mathbb{E}_b\left[\exp(-\mu e^{\gamma u}L_\infty)\right]du\right)\left(\int_{\mathbb{R}} e^{(\alpha_1+\alpha_2-Q)v}\mathbb{E}_b\left[\exp(-\mu e^{\gamma v}R_\infty)\right]dv\right)
\end{aligned}
\tag{3.40}
$$

Thus we have for each $b>0$

$$
\begin{aligned}
&\lim_{t\to\infty} t^{3/2}\int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}[\exp(-\mu e^{\gamma c}Z_t)|\widetilde{A}_{b,t}]\mathbb{P}(\widetilde{A}_{b,t})dc\\
&=\sqrt{\frac{2}{\pi}}b^2\left(\int_{\mathbb{R}} e^{(\alpha_1+\alpha_2-Q)u}\mathbb{E}_b[\exp(-\mu e^{\gamma u}R_\infty)]du\right)\left(\int_{\mathbb{R}} e^{(\alpha_3+\alpha_4-Q)v}\mathbb{E}_b[\exp(-\mu e^{\gamma v}L_\infty)]dv\right)
\end{aligned}
$$

It is shown in [DKRV17] that $b\mathbb{E}_b\left[\exp(-\mu e^{\gamma v}L_\infty)\right]$ has a non-trivial limit as $b\to\infty$ and, exchanging limits, the authors conclude that

$$
\lim_{b\to\infty} b\mathbb{E}_b\left[\exp(-\mu e^{\gamma v}L_\infty)\right]=\lim_{t\to\infty}\sqrt{\frac{\pi t}{2}}\mathbb{E}\left[\exp(-\mu e^{\gamma v}L_t)\right]\tag{3.41}
$$

On the other hand, one can recover the $\mathrm{BES}_b(3)$-process by conditioning a Brownian motion with negative drift to stay below b forever and letting the drift tend to 0. More precisely, if $\tau_{\alpha,b}=\inf\{s\geq 0,\ B_s+(\alpha-Q)s\geq b\}$, then we have $\mathbb{P}(\tau_{\alpha,b}=\infty)\underset{\alpha\to Q^-}{\sim}2(Q-\alpha)b$. Now adding the drift $\alpha-Q$ in the definition of L_∞ gives the correlation function $\frac{1}{2}C_\gamma(\alpha,\alpha_3,\alpha_4)$. In the end (see [Bav19] for details), we have the alternative characterisation of the limit (3.41)

$$
\lim_{b\to\infty} b\int_{\mathbb{R}} e^{(\alpha_3+\alpha_4-Q)v}\mathbb{E}_b\left[\exp(-\mu e^{\gamma v}L_\infty)\right]dv=-\frac{1}{4}\lim_{\alpha\to Q}\frac{C_\gamma(\alpha,\alpha_3,\alpha_4)}{\alpha-Q}=-\frac{1}{4}\partial_1 C_\gamma(\alpha,\alpha_3,\alpha_4)\tag{3.42}
$$

A similar statement holds for the L_∞ term, so we have

$$
\lim_{b\to\infty}\lim_{t\to\infty} t^{3/2}\int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}[\exp(-\mu e^{\gamma c}Z_t)1_{\widetilde{A}_{b,t}}]dc=\frac{1}{8\sqrt{2\pi}}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)
$$

From [DKRV17], the family of functions $\mathbb{E}[\exp(-\mu e^{\gamma c}Z_t)1_{\widetilde{A}_{b,t}}]$ converges uniformly with respect to t as $b \to \infty$, enabling us to exchange limits in b an in t. Hence

$$\begin{aligned}\lim_{t\to\infty} t^{3/2} \int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}[\exp(-\mu e^{\gamma c} Z_t)]dc &= \lim_{b\to\infty}\lim_{t\to\infty} t^{3/2} \int_{\mathbb{R}} e^{Q\sigma c}\mathbb{E}[\exp(-\mu e^{\gamma c} Z_t)1_{\widetilde{A}_{b,t}}]dc \\ &= \frac{1}{8\sqrt{2\pi}}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)\end{aligned}$$

Recall equation (3.38) to find

$$\begin{aligned}&\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle \\ &\quad \underset{z\to 0}{\sim} \frac{1}{4\sqrt{2\pi}}|z|^{2(\frac{Q^2}{4}-\Delta_1-\Delta_2)}|1-z|^{-\alpha_2\alpha_3}(\log\frac{1}{|z|})^{-3/2}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 C_\gamma(Q,\alpha_3,\alpha_4)\end{aligned}$$

□

3.3.2 Critical case

We conclude the proof of Theorem 3.1.1 by proving the asymptotic formula (3.13), i.e. we assume $\alpha_1 + \alpha_2 = Q$.

Proof of (3.13). The analysis of Section 3.3.1 fails only because the limit identified in (3.40) becomes trivial in this case because the triplet (α_1, α_2, Q) violates the first Seiberg bound. Geometrically, the random variable R_t does not have enough mass as $t \to \infty$ in order to produce another surface.

However, the analysis is still valid up to equation (3.36) and the expression of Z_t is the same with this new set of parameters. Consider the same decomposition $Z_t = L_t + C_t + R_t$ and write $\xi_t := C_t + R_t$ with the same $\eta > 0$.

As before, we condition the radial part no to exceed a given value. For $b > 0$, we define the event

$$A_{b,t} := \left\{\sup_{0\le s\le t} B_s \le b\right\}$$

It is well-known that

$$\mathbb{P}(A_{b,t}) = \sqrt{\frac{2}{\pi}}\int_0^{b/\sqrt{t}} e^{-\frac{x^2}{2}}dx =: g(b/\sqrt{t})$$

Notice that $g(x) \underset{x\to\infty}{\to} 1$ and $g(x) \underset{x\to 0}{\sim} \sqrt{\frac{2}{\pi}}x$. The process $(B_s)_{s\ge 0}$ conditioned on $A_{b,t}$ has the law of a $\mathrm{BES}_b(3)$-process. Repeating the argument of the previous subsection, we find that $\xi_t \to 0$ in probability as $t \to \infty$ when conditioned on $A_{b,t}$.

As for the radial part, we have the following estimate for $s < t^{1/2-\eta}$ and $\theta \in \mathbb{S}^1$

$$|H(s+i\theta, t+i\phi)| = \log\frac{1}{|1-e^{-(t-s)-i(\phi-\theta)}|} = O(e^{-t/2})$$

Let $\mathbb{P}_b$ be the law of the field when the radial part $(B_s)_{s\geq 0}$ is conditioned not to exceed b. Applying exactly the same framework as before, we have for all $\kappa > 0$

$$\begin{aligned} \lim_{t\to\infty} \sqrt{t}\mathbb{E}\left[Z_t^{-\kappa}\right] &= \lim_{t\to\infty} \sqrt{t}\mathbb{E}\left[L_t^{-\kappa}\right] \\ &= \sqrt{\frac{2}{\pi}} \lim_{b\to\infty} b\mathbb{E}_b\left[L_t^{-\kappa}\right] \end{aligned} \tag{3.43}$$

So it follows from the result of [Bav19] that

$$\lim_{t\to\infty} \int_{\mathbb{R}} e^{-Q\sigma c}\mathbb{E}\left[\exp\left(-\mu e^{\gamma c}\int_{\widehat{\mathbb{C}}} e^{\gamma\sum_{i=1}^4 \alpha_i G(z_i,\cdot)} dM^\gamma\right)\right] dc = -\frac{1}{2\sqrt{2\pi}}\partial_1 C_\gamma(Q,\alpha_3,\alpha_4) \tag{3.44}$$

which concludes the proof. □

3.3.3 Proof of Theorem 3.1.2

As mentioned in Section 3.1.3, Theorem 3.1.2 follows easily from Theorem 3.1.1 by taking σ to be arbitrary. We will use the notations in Section 3.3.1 and 3.3.2, outlining the differences with the Liouville case and leaving the details to the reader.

Let $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ be such that the Seiberg bound is satisfied. If $\alpha_1+\alpha_2-Q<\kappa\gamma$, the previous analysis applies immediately modulo the obvious substitution $\frac{Q\sigma}{\gamma} \leftrightarrow \kappa$ in the relevant places. If $\alpha_1+\alpha_2-Q \geq \kappa\gamma$, however, we only apply Cameron-Martin to partially offset the positive drift in $[0,t]$ by $\kappa\gamma$, as motivated in Section 3.2.4. This leads to

$$\mathbb{E}\left[W_t^{-\kappa}\right] = e^{-\kappa\gamma(\alpha_1+\alpha_2-Q)t+\frac{\kappa^2\gamma^2}{2}t}\mathbb{E}\left[\left(e^{-\gamma(B_t+(\alpha_1+\alpha_2-Q-\kappa\gamma)t)}\widehat{Z}_t\right)^{-\kappa}\right] \tag{3.45}$$

where W_t is defined in (3.35) and $\widehat{Z}_t$ is defined suitably. Notice that (3.45) is identical to (3.36) when $\alpha_1+\alpha_2-Q=\kappa\gamma$, the analysis of which is similar to that of Section 3.3.2 except that here we consider the event

$$A'_{b,t} := \left\{\sup_{0\leq s\leq t}(B_{t-s}-B_t)\leq b\right\}$$

so that L_t becomes irrelevant in the limit while R_t survives as $t\to\infty$ instead. The case $\alpha_1+\alpha_2-Q>\kappa\gamma$ is straightforward because $e^{-\gamma(B_t+(\alpha_1+\alpha_2-Q-\kappa\gamma)t)}\widehat{Z}_t$ is an integral involving the exponentiation of a two-sided Brownian motion with negative drifts in both directions, and we can even obtain (3.20) by dominated convergence directly.

□

3.4 Proof of Theorem 3.1.3

The rest of this paper is devoted to the proof of Theorem 3.1.3 which gives probabilistic representations for the limits (3.42) and (3.43) for which we do not have exact formulae outside the Liouville case. We will not discuss (3.20) which is basically explained in the last section.

3.4.1 Infinite series representation of $E^{\gamma}_{\kappa}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$

In order to obtain Theorem 3.1.3 we need the following intermediate result.

Lemma 3.4.1. *Fix $h > 0$. When $\alpha_1 + \alpha_2 - Q \in [0, \kappa\gamma]$, the constant $E^{\gamma}_{\kappa}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in (3.15) has the following representations.*

- *If $\alpha_1 + \alpha_2 - Q = 0$, we have*

$$E^{\gamma}_{\kappa}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} nhe^{-\kappa\gamma nh} \mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(nh, \beta^{nh}_{\cdot})\right)^{-\kappa} 1_{\{\min_{s>0} \beta^{nh}_s \leq h\}}\right] \tag{3.46}$$

  where $(\beta^u_s)_{s\geq 0}$ is a $\mathrm{BES}_u(3)$-process.

- *If $\alpha_1 + \alpha_2 - Q \in (0, \kappa\gamma)$,*

$$\begin{aligned} &E^{\gamma}_{\kappa}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) && (3.47)\\ &= \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \frac{nhe^{-(\kappa\gamma-(\alpha_1+\alpha_2-Q))nh}}{(\alpha_1 + \alpha_2 - Q)^2} \mathbb{E}\left[\frac{1_{\{\min_{s>0} \beta^{nh}_{L,s} \leq h\} \cup \{\min_{s>0} \beta^{\mathcal{T}}_{R,s} \leq h\}}}{\left(F_{\alpha_3,\alpha_4}(nh, \beta^{nh}_{L,\cdot}) + F'_{\alpha_2,\alpha_1}(\mathcal{T}, \beta^{\mathcal{T}}_{R,\cdot})\right)^{\kappa}}\right] && (3.48)\end{aligned}$$

  where $(\beta^u_{L,s})_{s\geq 0}$ and $(\beta^{\mathcal{T}}_{R,s})_{s\geq 0}$ are independent $\mathrm{BES}_u(3)$- and $\mathrm{BES}_{\mathcal{T}}(3)$-processes respectively with $\mathcal{T} \sim \mathrm{Gamma}(2, \alpha_1 + \alpha_2 - Q)$, and F' is an independent copy of F.

- *If $\alpha_1 + \alpha_2 - Q = \kappa\gamma$,*

$$E^{\gamma}_{\kappa}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} nhe^{-\kappa\gamma nh} \mathbb{E}\left[\left(F_{\alpha_2,\alpha_1}(nh, \beta^{nh}_{\cdot})\right)^{-\kappa} 1_{\{\min_{s>0} \beta^{nh}_s \leq h\}}\right] \tag{3.49}$$

  where $(\beta^u_s)_{s\geq 0}$ is a $\mathrm{BES}_u(3)$-process.

Proof. For the sake of brevity we only sketch the proof for the case $h = 1$ here and leave the details to the reader. The key idea is the partitioning of

$$A_{n,t} = \left\{\sup_{0\leq s\leq t} B_s \leq n\right\} = \bigcup_{k\leq n} \left\{\sup_{0\leq s\leq t} B_s \in [(k-1), k]\right\} = \bigcup_{k\leq n} \left\{\min_{0\leq s\leq t} k - B_s \in [0, 1]\right\}.$$

When $\alpha_1 + \alpha_2 - Q = 0$, our claim essentially follows from Proposition 3.1 and Lemma 3.2 in [DKRV17], where a dominated convergence argument (see the paragraph after Lemma 3.2 and Section 5.0.3 in that article) implies that the renormalised constant is given by

$$\sum_{n=1}^{\infty} \lim_{t\to\infty} \left(\sqrt{t}\mathbb{E}\left[L_t^{-\kappa} 1_{\{\min_{0\le s\le t} n - B_s \le 1\}} \middle| A_{n,t}\right] \mathbb{P}(A_{n,t}) \right) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} n\mathbb{E}_n \left[L_\infty^{-\kappa} 1_{\{\min_{s\ge 0} n - B_s \le 1\}}\right]$$

which is equivalent to (3.46). The proof of (3.49) is similar.

To apply the same dominated convergence approach to (3.47), we need a control analogous to [DKRV17, equation (3.18)] when $\alpha_1 + \alpha_2 - Q \in (0, \kappa\gamma)$. Indeed the same argument there suggests that

$$t^{3/2}\mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t}(L_t + R_t)^{-\kappa} 1_{\{\sup_{0\le s\le t} B_s \in [(n-1),n]\}}\right] \le Ce^{-(\kappa\gamma-(\alpha_1+\alpha_2-Q))n}$$

for some constant $C > 0$ independent of t and n, and therefore $E_\kappa^\gamma(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ again has an infinite series representation of the form

$$\sum_{n=1}^{\infty} \lim_{t\to\infty} \left(t^{3/2}\mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)B_t}(L_t + R_t)^{-\kappa} 1_{\{\sup_{0\le s\le t} B_s \in [n-1,n]\}}\right]\right). \tag{3.50}$$

Let us highlight several observations.

- For every $n \in \mathbb{N}$, the event $\{\sup_{0\le s\le t} B_s \in [n-1, n]\}$ may be replaced by

$$\underbrace{\left\{ \sup_{0\le s\le t} B_s \le n \right\}}_{=A_{n,t}} \cap \underbrace{\left(\left\{ \min_{0\le s\le t^{1/2-\eta}} n - B_s \le 1 \right\} \cup \left\{ \min_{0\le s\le t^{1/2-\eta}} n - B_t - (B_{t-s} - B_t) \le 1 \right\} \right)}_{=:\overline{A}_{n,t}}$$

  up to a cost of $o(1)$ for neglecting the unlikely event $\left\{\sup_{s\in[t^{1/2-\eta}, t-t^{1/2-\eta}]} B_s \ge n-1\right\}$.

- Similar to the proof of Theorem 3.1.1, if we condition on the event $A_{n,t}$ and $B_t = x$, then

$$(n - B_s)_{0\le s\le t^{1/2-\eta}}, \qquad (n - B_t - (B_{t-s} - B_t))_{0\le s\le t^{1/2-\eta}}$$

  converge in distribution to independent $\mathrm{BES}_n(3)$- and $\mathrm{BES}_{n-x}(3)$-processes $(\beta_{L,s}^n)_{s\ge 0}$ and $(\beta_{R,s}^{n-x})_{s\ge 0}$ respectively. Consequently L_t and R_t converge in distribution to $e^{\gamma n}F_{\alpha_3,\alpha_4}(n, \beta_{L,\cdot}^n)$ and $e^{\gamma n}F'_{\alpha_2,\alpha_1}(n-x, \beta_{R,\cdot}^{n-x})$ respectively.

We now compute

$$\begin{aligned}
&\mathbb{E}\left[1_{A_{n,t}\cap\overline{A}_{n,t}}\middle|(B_s)_{s\in(-\infty,t^{1/2-\eta}]\cup[t-t^{1/2-\eta},\infty)}\right]\\
&\quad=1_{\left\{\min_{0\le s\le t^{1/2-\eta}} n-B_s\le 1\right\}\cup\left\{\min_{0\le s\le t^{1/2-\eta}} n-B_t-(B_{t-s}-B_t)\le 1\right\}}\\
&\qquad\times\mathbb{P}\left(A_{n,t}\middle|(B_s)_{s\in(-\infty,t^{1/2-\eta}]\cup[t-t^{1/2-\eta},\infty)}\right)
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{P}\left(A_{n,t}\middle|(B_s)_{s\in(-\infty,t^{1/2-\eta}]\cup[t-t^{1/2-\eta},\infty)}\right)&=1_{\{\sup_{0\le s\le t^{1/2-\eta}} B_s\le n\}}1_{\{\sup_{0\le s\le t^{1/2-\eta}} B_{t-s}-B_t\le n-B_t\}}\\
&\times\mathbb{P}\left(\sup_{t^{1/2-\eta}\le s\le t-t^{1/2-\eta}} B_s\le n\middle|B_{t^{1/2-\eta}},B_{t-t^{1/2-\eta}}\right)
\end{aligned}$$

and

$$\mathbb{P}\left(\sup_{t^{1/2-\eta}\le s\le t-t^{1/2-\eta}} B_s\le n\middle|B_{t^{1/2-\eta}},B_{t-t^{1/2-\eta}}\right)=1-e^{-\frac{2}{t-2t^{1/2-\eta}}(n-B_{t^{1/2-\eta}})(n-B_t-(B_{t-t^{1/2-\eta}}-B_t))}$$

is asymptotically $\frac{2}{t}(n-B_{t^{1/2-\eta}})(n-B_t-(B_{t-t^{1/2-\eta}}-B_t))$ when t is large. In particular

$$\mathbb{P}\left(A_{n,t}\middle|B_t=x\right)\sim\frac{2}{t}n(n-x)+o(t^{-1}),\qquad t\to\infty.$$

Substituting this into the summand in (3.50), we obtain

$$\begin{aligned}
&\lim_{t\to\infty}t^{3/2}\int_{-\infty}^{n}\mathbb{E}\left[e^{(\alpha_1+\alpha_2-Q)x}(L_t+R_t)^{-\kappa}1_{\overline{A}_{n,t}}\middle|A_{n,t},B_t=x\right]\mathbb{P}(A_{n,t}|B_t=x)\mathbb{P}(B_t\in dx)\\
&=\frac{e^{(\alpha_1+\alpha_2-Q)n}}{\sqrt{2\pi}}\lim_{t\to\infty}t\int_{-\infty}^{n}\mathbb{E}\left[e^{-(\alpha_1+\alpha_2-Q)(n-x)}(L_t+R_t)^{-\kappa}1_{\overline{A}_{n,t}}\middle|A_{n,t},B_t=x\right]\mathbb{P}(A_{n,t}|B_t=x)e^{-\frac{x^2}{2t}}dx\\
&=\frac{2e^{(\alpha_1+\alpha_2-Q)n}}{\sqrt{2\pi}}\int_{-\infty}^{n}\mathbb{E}\left[\frac{e^{-(\alpha_1+\alpha_2-Q)(n-x)}1_{\{\min_{s\ge0}\beta^n_{L,s}\le1\}\cup\{\min_{s\ge0}\beta^{n-x}_{R,s}\le1\}}}{(e^{\gamma n}F_{\alpha_3,\alpha_4}(n,\beta^n_{L,\cdot})+e^{\gamma n}F'_{\alpha_3,\alpha_4}(n-x,\beta^{n-x}_{R,\cdot}))^{\kappa}}\right]n(n-x)dx
\end{aligned}$$

where the last line follows by dominated convergence, and is equal to

$$\sqrt{\frac{2}{\pi}}ne^{-(\kappa\gamma-(\alpha_1+\alpha_2-Q))n}\int_0^{\infty}\mathbb{E}\left[\frac{1_{\{\min_{s\ge0}\beta^n_{L,s}\le1\}\cup\{\min_{s\ge0}\beta^x_{R,s}\le1\}}}{(F_{\alpha_3,\alpha_4}(n,\beta^n_{L,\cdot})+F'_{\alpha_3,\alpha_4}(x,\beta^x_{R,\cdot}))^{\kappa}}\right]xe^{-(\alpha_1+\alpha_2-Q)x}dx$$

so we are done.

□

Remark 11. The careful reader may notice that the proof above when $\alpha_1+\alpha_2-Q\in(0,\kappa\gamma)$ differs slightly from that in Section 3.3.1 where one considers the event $\widetilde{A}_{n,t}=\left\{\sup_{0\le s\le t}\widetilde{B}_s\le n\right\}$ instead of $A_{n,t}=\left\{\sup_{0\le s\le t}B_s\le n\right\}$. The current approach, which addresses the partitioning of probability space instead of factorisation in the first place, may have the drawback that (3.47) does not give a product of two negative moments immediately but it allows for an easier side-by-side comparison with the analysis in [DKRV17].

3.4.2 Proof of Theorem 3.1.3

The infinite series representation in Lemma 3.4.1 is reminiscent of Riemann sums. We now explain how to obtain the simplified expressions in Theorem 3.1.3.

Proof of (3.17) *and* (3.19). We begin with $\alpha_1+\alpha_2-Q=0$. Fix some $N>0$, and without loss of generality choose a sequence of $h\to 0^+$ such that h always divides both N^{-1} and N. Then by Lemma 3.4.1 we have

$$E_\kappa^\gamma(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\sqrt{\frac{2}{\pi}}\sum_{n=1/Nh+1}^{N/h} nhe^{-\kappa\gamma nh}\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(nh,\beta_\cdot^{nh})\right)^{-\kappa}1_{\{\min_{s>0}\beta_s^{nh}\le h\}}\right]+C_N \tag{3.51}$$

for some constant $C_N>0$ which depends on N and the other parameters but not on h, with the property that $\lim_{N\to\infty}C_N=0$.

Recall (3.16) for the definition of the random functional F. By Theorem 3.2.3, we can rewrite the sum in (3.51) as

$$\begin{aligned}
&\sum_{n=1/Nh+1}^{N/h} nhe^{-\kappa\gamma nh}\int_0^{\frac{1}{n}}\mathbb{E}\Bigg[\Bigg(e^{-\gamma nh}\int_{|x|\ge 1}\frac{dM^\gamma(x)}{|x|^{4-\gamma(\alpha_3+\alpha_4)}|x-1|^{\gamma\alpha_3}}\\
&\quad+\int_{\mathbb{R}_{s\ge 0}\times\mathbb{S}^1_\theta} e^{-\gamma((nh+B_s)1_{\{s\le T_{-nh(1-u)}\}}+(nhu+\beta^0_{s-T_{-nh(1-u)}})1_{\{s\ge T_{-nh(1-u)}\}}-\alpha_3G(1,e^{-s-i\theta}))}d\widehat{M}^\gamma(s,\theta)\Bigg)^{-\kappa}\Bigg]du\\
&\overset{x=nh(1-u)}{=}\sum_{n=1/Nh+1}^{N/h} e^{-\kappa\gamma nh}\int_{(n-1)h}^{nh}\mathbb{E}\Bigg[\Bigg(e^{-\gamma nh}\int_{|x|\ge 1}\frac{dM^\gamma(x)}{|x|^{4-\gamma(\alpha_3+\alpha_4)}|x-1|^{\gamma\alpha_3}}\\
&\quad+\int_{\mathbb{R}_{s\ge 0}\times\mathbb{S}^1_\theta} e^{-\gamma((nh+B_s)1_{\{s\le T_{-x}\}}+(nh-x+\beta^0_{s-T_{-x}})1_{\{s\ge T_{-x}\}}-\alpha_3G(1,e^{-s-i\theta}))}d\widehat{M}^\gamma(s,\theta)\Bigg)^{-\kappa}\Bigg]dx\\
&=(1+o(1))\int_{1/N}^{N} e^{-\kappa\gamma x}\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(x,\widetilde{\beta}_\cdot^x)\right)^{-\kappa}\right]dx
\end{aligned}$$

where the $o(1)$ error is with respect to $h\to 0^+$ and comes from the fact that

$$e^{-\gamma nh}=(1+o(1))e^{-\gamma x},\qquad e^{-\gamma(nh-x)}=(1+o(1))$$

uniformly in $h>0$ and $n\in\mathbb{N}$ for all $x\in[(n-1)h,nh]$. The desired formula (3.17) is recovered by sending $h \to 0^+$ and $N \to \infty$. The proof of (3.19) is similar. $\square$

The case where $\alpha_1 + \alpha_2 - Q \in (0, \kappa\gamma)$ is slightly more involved and the following elementary formula will be useful.

Lemma 3.4.2. *Fix $\kappa, \gamma, \lambda > 0$ such that $\lambda < \kappa\gamma$. Let X, Y be independent non-negative random variables and T an independent $\mathrm{Exp}(\lambda)$ random variable. Provided that all the moments below exist, we have*

$$\mathbb{E}\left[\left(X + e^{-\gamma T} Y\right)^{-\kappa}\right] = \frac{\lambda}{\gamma} B\left(\frac{\lambda}{\gamma}, \kappa - \frac{\lambda}{\gamma}\right) \mathbb{E}\left[X^{-(\kappa - \frac{\lambda}{\gamma})}\right] \mathbb{E}\left[Y^{-\frac{\lambda}{\gamma}}\right]. \tag{3.52}$$

where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the beta function.

The proof of Lemma 3.4.2 follows from the same change-of-variable argument in (3.40) and is skipped here. For a sanity check one may quickly verify that both the LHS and RHS of (3.52) converge to $\mathbb{E}[X^{-\kappa}]$ as $\lambda/\gamma \to 0$.

Proof of (3.18). Our starting point is (3.47) from Lemma 3.4.1. It is clear that

$$\begin{aligned}
\mathbb{E}&\left[\frac{1_{\{\min_{s>0} \beta_{L,s}^{nh} \le h\} \cup \{\min_{s>0} \beta_{R,s}^{\mathcal{T}} \le h\}}}{\left(F_{\alpha_3,\alpha_4}(nh, \beta_{L,\cdot}^{nh}) + F'_{\alpha_2,\alpha_1}(\mathcal{T}, \beta_{R,\cdot}^{\mathcal{T}})\right)^{\kappa}}\right] = \mathbb{E}\left[\frac{1_{\{\min_{s>0} \beta_{L,s}^{nh} \le h\}}}{\left(F_{\alpha_3,\alpha_4}(nh, \beta_{L,\cdot}^{nh}) + F'_{\alpha_2,\alpha_1}(\mathcal{T}, \beta_{R,\cdot}^{\mathcal{T}})\right)^{\kappa}}\right] \\
&+ \mathbb{E}\left[\frac{1_{\{\min_{s>0} \beta_{R,s}^{\mathcal{T}} \le h\}}}{\left(F_{\alpha_3,\alpha_4}(nh, \beta_{L,\cdot}^{nh}) + F'_{\alpha_2,\alpha_1}(\mathcal{T}, \beta_{R,\cdot}^{\mathcal{T}})\right)^{\kappa}}\right] - \mathbb{E}\left[\frac{1_{\{\min_{s>0} \beta_{L,s}^{nh} \le h\} \cap \{\min_{s>0} \beta_{R,s}^{\mathcal{T}} \le h\}}}{\left(F_{\alpha_3,\alpha_4}(nh, \beta_{L,\cdot}^{nh}) + F'_{\alpha_2,\alpha_1}(\mathcal{T}, \beta_{R,\cdot}^{\mathcal{T}})\right)^{\kappa}}\right]
\end{aligned}$$

where the last term is $O(h^2)$ and may be safely ignored. Arguing as before, we see that

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \frac{nh e^{-(\kappa\gamma - (\alpha_1 + \alpha_2 - Q))nh}}{(\alpha_1 + \alpha_2 - Q)^2} \mathbb{E}\left[\frac{1_{\{\min_{s>0} \beta_{L,s}^{nh} \le h\}}}{\left(F_{\alpha_3,\alpha_4}(nh, \beta_{L,\cdot}^{nh}) + F'_{\alpha_2,\alpha_1}(\mathcal{T}, \beta_{R,\cdot}^{\mathcal{T}})\right)^{\kappa}}\right] \\
&= \frac{\sqrt{2/\pi}}{(\alpha_1 + \alpha_2 - Q)^2 (\kappa\gamma - (\alpha_1 + \alpha_2 - Q))} \mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(\tau, \widetilde{\beta}_{L,\cdot}^{\tau}) + F'_{\alpha_2,\alpha_1}(\mathcal{T}, \beta_{R,\cdot}^{\mathcal{T}})\right)^{-\kappa}\right] + o(1)
\end{aligned} \tag{3.53}$$

where $\tau \sim \mathrm{Exp}(\kappa\gamma - (\alpha_1 + \alpha_2 - Q))$ and $\mathcal{T} \sim \mathrm{Gamma}(2, \alpha_1 + \alpha_2 - Q)$. Recall that if U is an independent Uniform$[0,1]$ random variable, then $(\mathcal{T}_1, \mathcal{T}_2) := (\mathcal{T}U, \mathcal{T}(1-U))$ is a pair of independent $\mathrm{Exp}(\alpha_1 + \alpha_2 - Q)$ random variables. Combining this fact with Theorem 3.2.3, we obtain

$$F'_{\alpha_2,\alpha_1}(\mathcal{T}, \beta_{R,\cdot}^{\mathcal{T}}) \overset{d}{=} e^{-\gamma \mathcal{T}_1} F'_{\alpha_2,\alpha_1}(\mathcal{T}_2, \widetilde{\beta}_{R,\cdot}^{\mathcal{T}_2})$$

and we can rewrite the expectation in (3.53) as

$$\mathbb{E}\left[\left(F_{\alpha_3,\alpha_4}(\tau,\widetilde{\beta}^{\tau}_{L,\cdot})+e^{-\gamma\mathcal{T}_1}F'_{\alpha_2,\alpha_1}(\mathcal{T}_2,\widetilde{\beta}^{\mathcal{T}_2}_{R,\cdot})\right)^{-\kappa}\right].$$

Similarly, if we let τ_1,τ_2 be independent $\mathrm{Exp}(\kappa\gamma-(\alpha_1+\alpha_2-Q))$, then

$$\begin{aligned}&\sqrt{\frac{2}{\pi}}\sum_{n=1}^{\infty}\frac{nhe^{-(\kappa\gamma-(\alpha_1+\alpha_2-Q))nh}}{(\alpha_1+\alpha_2-Q)^2}\mathbb{E}\left[\frac{1_{\{\min_{s>0}\beta^{\mathcal{T}}_{R,s}\le h\}}}{\left(F_{\alpha_3,\alpha_4}(nh,\beta^{nh}_{L,\cdot})+F'_{\alpha_2,\alpha_1}(\mathcal{T},\beta^{\mathcal{T}}_{R,\cdot})\right)^{\kappa}}\right]\\&=\frac{\sqrt{2/\pi}}{(\alpha_1+\alpha_2-Q)(\kappa\gamma-(\alpha_1+\alpha_2-Q))^2}\mathbb{E}\left[\left(e^{-\gamma\tau_1}F_{\alpha_3,\alpha_4}(\tau_2,\widetilde{\beta}^{\tau_2}_{L,\cdot})+F'_{\alpha_2,\alpha_1}(\mathcal{T}_2,\widetilde{\beta}^{\mathcal{T}_2}_{R,\cdot})\right)^{-\kappa}\right]+o(1).\end{aligned} \tag{3.54}$$

The claim then follows by sending $h\to 0^+$ and applying Lemma 3.4.2 to (3.53) and (3.54). □

3.5 Fusion in boundary Liouville Conformal Field Theory

3.5.1 Boundary Liouville Conformal Field Theory

Boundary LCFT is LCFT on proper simply connected domains $D\subset\mathbb{C}$. We start by a brief review of the theory and refer to [HRV18] for details. Like LCFT on the sphere, the theory is conformally invariant, so by the Riemann uniformisation theorem, it is enough to study it on the upper-half plane $\mathbb{H}:=\{\mathrm{Im}z>0\}$ (the unit disc $\mathbb{D}$ is also a common choice) equipped with some background metric g. In this context, the Liouville action with boundary term is given by[5]

$$S_{\mathrm{L}}(X,g):=\frac{1}{4\pi}\int_{\mathbb{H}}\left(|\nabla X|^2+4\pi\mu e^{\gamma X}g(z)\right)d^2z+\mu_\partial\int_{\mathbb{R}}e^{\frac{\gamma}{2}X}g(x)^{1/2}dx \tag{3.55}$$

where $\mu_\partial>0$ is the boundary cosmological constant. One recognises the Dirichlet energy in the first term of the action, giving rise to a GFF which we take to have Neumann boundary conditions. The GFF is weighted by its bulk GMC mass $M^\gamma(\mathbb{H})$ and its boundary GMC mass $M^\gamma_\partial(\mathbb{R})$, where the boundary GMC is formally

$$dM^\gamma_\partial(x)=e^{\frac{\gamma}{2}X(x)-\frac{\gamma^2}{8}\mathbb{E}[X(x)^2]}g(x)^{1/2}dx$$

and is obtained via a regularisation of the field using semi-circle averages.

As in the sphere case, the observables are the vertex operators $V_\alpha(z)$ for insertions

[5] As in the sphere case, we omit the Ricci and geodesic curvature terms.

$z \in \mathbb{H}$. The main difference is that one can consider insertions on the boundary, which we formally write

$$B_\beta(x) := e^{\frac{\beta}{2}X(x)}$$

for $x \in \mathbb{R}$ and β in a range to be determined. The correlation functions $\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M B_{\beta_j}(x_j)\rangle$ exist iff the Seiberg bounds are satisfied, which in this context are given by

$$\begin{aligned} &\sigma := \sum_{i=1}^N \frac{\alpha_i}{Q} + \sum_{j=1}^M \frac{\beta_j}{2Q} - 1 > 0 \\ &\forall i,\ \alpha_i < Q \\ &\forall j,\ \beta_j < Q \end{aligned} \tag{3.56}$$

If these are satisfied, the correlation function has the following form[6] [HRV18]:

$$\left\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M B_{\beta_j}(x_j) \right\rangle = 2e^{C(\mathbf{z},\mathbf{x})} \int_{\mathbb{R}} e^{Q\sigma c} \mathbb{E}\left[\exp\left(-\mu e^{\gamma c} \int_{\mathbb{H}} e^{\gamma H} dM^\gamma - \mu_\partial e^{\frac{\gamma}{2}c} \int_{\mathbb{R}} e^{\frac{\gamma}{2}H} dM_\partial^\gamma \right) \right] dc \tag{3.57}$$

where H and $C(\mathbf{z},\mathbf{x})$ are the functions defined by

$$\begin{aligned} H &= \sum_{i=1}^N \alpha_i G(z_i, \cdot) + \sum_{j=1}^M \frac{\beta_j}{2} G(x_j, \cdot) \\ C(\mathbf{z},\mathbf{x}) &= \sum_{i<i'} \alpha_i \alpha_{i'} G(z_i, z_i') + \sum_{i,j} \frac{\alpha_i \beta_j}{2} G(z_i, x_j) + \sum_{j<j'} \frac{\beta_j \beta_{j'}}{4} G(x_j, x_j') \end{aligned} \tag{3.58}$$

with G being Green's function with Neumann boundary conditions on $(\mathbb{H}, g)$. Notice that the usual change of variable $u = e^{\gamma c}$ does not give a nicer expression in this case since the exponential term in the expectation is quadratic in $e^{\frac{\gamma}{2}c}$.

Correlation functions are conformally covariant, and if $\psi : \mathbb{H} \to \mathbb{H}$ is a Möbius transformation, then (recall that $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$)

$$\left\langle \prod_{i=1}^N V_{\alpha_i}(\psi(z_i)) \prod_{j=1}^M B_{\beta_j}(\psi(x_j)) \right\rangle = \prod_{i=1}^N |\psi'(z_i)|^{-2\Delta_{\alpha_i}} \prod_{j=1}^M |\psi'(x_j)|^{-\Delta_{\beta_j}} \left\langle \prod_{i=1}^N V_{\alpha_i}(z_i) \prod_{j=1}^M B_{\beta_j}(x_j) \right\rangle$$

Möbius transforms of $\mathbb{H}$ have three real parameters, so when the location of the insertions have less than (or exactly) three real parameters, the correlation functions are determined by conformal invariance, and we have the following structure constants

[6] We chose the prefactor 2 so that the asymptotic behaviour of the bulk 1-point function with $\mu = 0$ coincides with that of [FZZ00, Equation (2.24)].

1. Bulk-boundary two-point function

$$\langle V_\alpha(z)B_\beta(x)\rangle = \frac{R(\alpha,\beta)}{|z-\bar{z}|^{2\Delta_\alpha-\Delta_\beta}|z-x|^{2\Delta_\beta}} \tag{3.59}$$

As a special case of this equation for $\beta = 0$, we have the bulk one-point function

$$\langle V_\alpha(z)\rangle = \frac{U(\alpha)}{|z-\bar{z}|^{2\Delta_\alpha}} \tag{3.60}$$

2. Boundary three-point function

$$\langle B_{\beta_1}(x_1)B_{\beta_2}(x_2)B_{\beta_3}(x_3)\rangle = \frac{c(\beta_1,\beta_2,\beta_3)}{|x_1-x_2|^{\Delta_{\beta_1}+\Delta_{\beta_2}-\Delta_{\beta_3}}|x_2-x_3|^{\Delta_{\beta_2}+\Delta_{\beta_3}-\Delta_{\beta_1}}|x_3-x_1|^{\Delta_{\beta_3}+\Delta_{\beta_1}-\Delta_{\beta_2}}} \tag{3.61}$$

Remark 12. There is also a definition for a boundary two-point function, which we omit here since we will not be needing it for the purpose of this paper. Let us just mention that this object is to the reflection coefficient of [KRV20] what the boundary three-point function is to the DOZZ formula.

The above structure constants are to be understood as meromorphic functions of the parameters and they arise naturally in the bootstrap formalism. Physicists have conjectured exact formulae for the values of these structure constants [FZZ00, PT02], and there are works in progress by Gwynne and Remy establishing the validity of (3.60) and Remy and Zhu addressing (3.61).

3.5.2 Main results

The cases we treat are the fusion on two boundary-insertions, the absorption of a bulk-insertion on the boundary and the fusion of two bulk-insertions.

Theorem 3.5.1 (Boundary four-point)**.** *Let $(\beta_1,\beta_2,\beta_3,\beta_4)$ satisfying the Seiberg bounds and suppose that $\beta_3+\beta_4>Q$. Then the following asymptotic holds:*

1. Supercritical case

If $\beta_1+\beta_2>Q$, then

$$\langle B_{\beta_1}(0)B_{\beta_2}(x)B_{\beta_3}(1)B_{\beta_4}(\infty)\rangle \underset{x\to 0}{\sim} \frac{1}{4\sqrt{\pi}}\frac{|x|^{\frac{Q^2}{4}-\Delta_{\beta_1}-\Delta_{\beta_2}}}{\log^{3/2}\frac{1}{|x|}}\partial_3 c(\beta_1,\beta_2,Q)\partial_1 c(Q,\beta_3,\beta_4) \tag{3.62}$$

2. Critical case

If $\beta_1 + \beta_2 = Q$, then

$$\langle B_{\beta_1}(0)B_{\beta_2}(x)B_{\beta_3}(1)B_{\beta_4}(\infty)\rangle \underset{x\to 0}{\sim} -\frac{1}{\sqrt{\pi}}\frac{|x|^{-\frac{1}{2}\beta_1\beta_2}}{\log^{1/2}\frac{1}{|x|}}\partial_1 c(Q,\beta_3,\beta_4) \tag{3.63}$$

3. Subcritical case

If $\beta_1 + \beta_2 < Q$, then

$$\langle B_{\beta_1}(0)B_{\beta_2}(x)B_{\beta_3}(1)B_{\beta_4}(\infty)\rangle \underset{x\to 0}{\sim} |x|^{-\frac{1}{2}\beta_1\beta_2} c(\beta_1+\beta_2,\beta_3,\beta_4) \tag{3.64}$$

The next theorem is about the fusion in the bulk two-point function.

Theorem 3.5.2 (Bulk two-point: Fusion)**.** *Let $(\alpha_1, \alpha_2, \beta)$ satisfying the Seiberg bounds. Then the following asymptotics hold:*

1. *If $\beta = 0$, then*

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(i+z)\rangle \underset{z\to 0}{\sim} -\frac{2^{-\alpha_1\alpha_2}}{\sqrt{2\pi}}\frac{|z|^{2(\frac{Q^2}{4}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}}{\log^{1/2}\frac{1}{|z|}}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q) \tag{3.65}$$

2. *If $\beta > 0$, then*

(a) Supercritical case

If $\alpha_1 + \alpha_2 > Q$, then

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(i+z)B_\beta(0)\rangle \underset{z\to 0}{\sim} \frac{2^{\Delta_\beta-\frac{Q^2}{2}-\alpha_1\alpha_2}}{4\sqrt{2\pi}}\frac{|z|^{2(\frac{Q^2}{4}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}}{\log^{3/2}\frac{1}{|z|}}\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)\partial_1 R(Q,\beta) \tag{3.66}$$

(b) Critical case

If $\alpha_1 + \alpha_2 = Q$, then

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(i+z)B_\beta(0)\rangle \underset{z\to 0}{\sim} -\frac{2^{\Delta_\beta-\frac{Q^2}{2}-\alpha_1\alpha_2}}{\sqrt{2\pi}}\frac{|z|^{-\alpha_1\alpha_2}}{\log^{1/2}\frac{1}{|z|}}\partial_1 R(Q,\beta) \tag{3.67}$$

(c) Subcritical case

If $\alpha_1 + \alpha_2 < Q$, then

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(i+z)B_\beta(0)\rangle \underset{z\to 0}{\sim} 2^{\Delta_\beta-\frac{Q^2}{2}-\alpha_1\alpha_2}|z|^{-\alpha_1\alpha_2}R(\alpha_1+\alpha_2,\beta) \tag{3.68}$$

Another interesting limit of the bulk two-point function is sending one insertion to the boundary.

Theorem 3.5.3 (Bulk two-point: Absorption)**.** *Let (α_1, α_2) satisfying the Seiberg bounds, and suppose $\alpha_1 > \frac{Q}{2}$. Then the following asymptotic holds:*

1. Supercritical case

   If $\alpha_2 > \frac{Q}{2}$, then

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(z)\rangle \underset{z\to 0}{\sim} \frac{2^{2(\frac{Q^2}{4}-\Delta_{\alpha_1}-\Delta_{\alpha_2})}}{4\sqrt{\pi}} \frac{|z|^{(\alpha_2-\frac{Q}{2})^2}}{\log^{3/2}\frac{1}{|z|}}\partial_2 R(\alpha_1, Q)\partial_2 R(\alpha_2, Q) \tag{3.69}$$

2. Critical case

   If $\alpha_2 = \frac{Q}{2}$, then

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(z)\rangle \underset{z\to 0}{\sim} -\frac{2^{\frac{Q^2}{2}-2\Delta_{\alpha_1}}}{\sqrt{\pi}\log^{1/2}\frac{1}{|z|}}\partial_2 R(\alpha_1, Q) \tag{3.70}$$

3. Subcritical case

   If $\alpha_2 < \frac{Q}{2}$, then

$$\langle V_{\alpha_1}(i)V_{\alpha_2}(z)\rangle \underset{z\to 0}{\sim} \frac{R(\alpha_1, 2\alpha_2)}{2^{2\Delta_{\alpha_1}-\Delta_{2\alpha_2}}} \tag{3.71}$$

We now turn to the bulk-boundary three-point function $\langle V_\alpha(z)B_{\beta_1}(0)B_{\beta_2}(\infty)\rangle$. There is not much to say about the merging of the bulk insertion with a boundary insertion since for all $r > 0$ and $\theta \in (0, \pi)$, the correlation function $\langle V_\alpha(re^{i\theta})B_{\beta_1}(0)B_{\beta_2}(\infty)\rangle$ is deduced from $\langle V_\alpha(e^{i\theta})B_{\beta_1}(0)B_{\beta_2}(\infty)\rangle$ by scaling. The non-trivial parameter we can vary is θ, and the limit $\theta \to 0$ corresponds to the absorption of an bulk insertion on a boundary point which is not an insertion. Thus we will study the correlation function $\langle V_\alpha(z)B_{\beta_1}(1)B_{\beta_2}(\infty)\rangle$ in the limit $z \to 0$. Notice that by Möbius invariance, this is the same as studying the function $\langle V_\alpha(i)B_{\beta_1}(0)B_{\beta_2}(x)\rangle$ in the limit $x \to 0$, i.e. merging the two boundary insertions.

Theorem 3.5.4 (Bulk-boundary three-point)**.** *Let $(\alpha, \beta_1, \beta_2)$ satisfying the Seiberg bounds, and assume that $\beta_1 + \beta_2 > \frac{Q}{2}$. Then the following asymptotic holds*

1. Supercritical case

   If $\alpha > \frac{Q}{2}$, then

$$\langle V_\alpha(z)B_{\beta_1}(1)B_{\beta_2}(\infty)\rangle \underset{z\to 0}{\sim} \frac{2^{\frac{Q^2}{4}-2\Delta_\alpha}}{4\sqrt{\pi}} \frac{|z|^{(\alpha-\frac{Q}{2})^2}}{\log^{3/2}\frac{1}{|z|}}\partial_2 R(\alpha, Q)\partial_1 c(Q, \beta_1, \beta_2) \tag{3.72}$$

2. Critical case

If $\alpha = \frac{Q}{2}$*, then*

$$\langle V_\alpha(z) B_{\beta_1}(1) B_{\beta_2}(\infty)\rangle \underset{z\to 0}{\sim} -\frac{1}{\sqrt{\pi}\log^{1/2}\frac{1}{|z|}}\partial_1 c(Q,\beta_1,\beta_2) \tag{3.73}$$

3. Subcritical case

If $\alpha < \frac{Q}{2}$*, then*

$$\langle V_\alpha(z) V_{\beta_1}(1) V_{\beta_2}(\infty)\rangle \underset{z\to 0}{\sim} c(2\alpha,\beta_1,\beta_2) \tag{3.74}$$

Remark 13. More generally, the fusion rules in the supercritical case are the following:

1. Fusion of boundary-boundary (β_1,β_2)-insertions produces a boundary three-point function $\partial_3 c(\beta_1,\beta_2,Q)$.
2. Absorption of a bulk α-insertion produces a bulk-boundary function $\partial_2 R(\alpha, Q)$.
3. Fusion of bulk-bulk (α_1,α_2)-insertions produces a DOZZ formula $\partial_3 C_\gamma(\alpha_1,\alpha_2,Q)$.

This rule, as well as the rate functions of the above theorems, can be used to compute the asymptotic behaviour of all correlation functions upon fusion of insertions, and express the limit with a lower order correlation function.

As such, we haven't said anything about the fusion of bulk-boundary insertions. This is because it can be seen as a two-step procedure of first absorbing the bulk insertion into the boundary and then merging the boundary insertions. Hence the operation does not produce a structure constant. As an example, consider the correlation function $\langle V_\alpha(z) B_{\beta_1}(0) B_{\beta_2}(1) B_{\beta_3}(\infty)\rangle$ in the limit $z\to 0$, for $(\alpha,\beta_1,\beta_2,\beta_3)$ satisfying the Seiberg bounds, and suppose that both $\beta_3+\beta_4 > Q$ and $2\alpha+\beta_1 > Q$, so that we are in the supercritical case. Then the asymptotic is given by

$$\langle V_\alpha(z) B_{\beta_1}(0) B_{\beta_2}(1) B_{\beta_3}(\infty)\rangle \underset{z\to 0}{\sim} \frac{1}{4\sqrt{\pi}}\frac{|z|^{(\alpha-\frac{Q}{2})^2-\alpha\beta_1}}{\log^{3/2}\frac{1}{|z|}}\frac{\partial}{\partial\beta}\langle V_\alpha(i) B_{\beta_1}(0) B_\beta(\infty)\rangle_{|\beta=Q}\partial_1 c(Q,\beta_2,\beta_3) \tag{3.75}$$

Remark 14. Even though the correlation functions can no longer be expressed in terms of negative moments of GMC (unless $\mu\mu_\partial = 0$), it is still possible to give probabilistic representations of the renormalised constants in the aforementioned theorems by performing the same partitioning-of-probability-space procedure on

$$\mathbb{E}\left[\exp\left(-\mu e^{\gamma c}\int_{\mathbb{H}} e^{\gamma H} dM^\gamma - \mu_\partial e^{\frac{\gamma}{2}c}\int_{\mathbb{R}} e^{\frac{\gamma}{2}H} dM_\partial^\gamma\right)\right]$$

as we did in Section 3.4. We omit the details here.

We now turn to proving Theorems 3.5.1, 3.5.2, 3.5.3 and 3.5.4. We only deal with Theorems 3.5.1 and 3.5.2 since the other cases are similar.

Subcritical cases follow from dominated convergence so we won't treat them. The rest of the proofs are very similar to that of Theorem 3.1.1 so we will be brief.

Proof of Theorem 3.5.1. The setting is the upper-half plane $\mathbb{H}$ equipped with the metric $g(z) = 4|z|_+^{-4}$. We use the same procedure as for the sphere and apply the conformal change of coordinate $\psi : z \mapsto e^{-z/2}$ from the infinite strip $\mathcal{S} := \mathbb{R} \times (0, 2\pi)$ to $\mathbb{H}$. Then Green's function on the strip is given by the even part of Green's function on the cylinder, i.e. if X is a GFF on $\mathbb{R}_s \times (0, 2\pi)_\theta$, we have (recall (3.26))

$$\begin{aligned}\mathbb{E}[X(s,\theta)X(s',\theta')] &= G(\frac{s}{2},\frac{\theta}{2},\frac{s'}{2},\frac{\theta'}{2}) + G(\frac{s}{2},\frac{\theta}{2},\frac{s'}{2},-\frac{\theta'}{2}) \\ &= (|s| \wedge |s'|)1_{ss'\geq 0} + H(\frac{s}{2},\frac{\theta}{2},\frac{s'}{2},\frac{\theta'}{2}) + H(\frac{s}{2},\frac{\theta}{2},\frac{s'}{2},-\frac{\theta'}{2}) \\ &= (|s| \wedge |s'|)1_{ss'\geq 0} + G(0,0,\frac{s'-s}{2},\frac{\theta'-\theta}{2}) + G(0,0,\frac{s'-s}{2},\frac{\theta'+\theta}{2})\end{aligned} \tag{3.76}$$

Hence the field decomposes into the independent sum $X = B+Y$ where $(B_s)_{s\in\mathbb{R}}$ is standard two-sided Brownian motion and Y is a log-correlated field whose covariance kernel is given by the sum of G functions on the right-hand side of the previous equation. It is also clear from the definition that the law of Y is translation invariant. The pullback measure of g on the strip is $g_\psi(s,\theta) = e^{-|s|}$ so we can take the GMC measure of Y with respect to Lebesgue measure on $\mathcal{S}$ and take the drifted process $B_s - \frac{Q}{2}|s|$ for the radial part of the GFF.

First we have to explain how to make sense of boundary (derivative) Q-insertions. A boundary insertion with momentum β at ∞ (on the strip) amounts in adding a positive drift $\frac{\beta}{2}$ to the radial process (on the positive real line), so the total drift vanishes when $\beta = Q$. For $t > 0$, define $\mathbb{H}_t := \mathbb{H} \setminus (e^{-t/2}\mathbb{D})$ (resp. $\mathbb{R}_t := \mathbb{R} \setminus (-e^{-t/2}, e^{-t/2})$) and $\langle B_Q(0)B_{\beta_2}(1)B_{\beta_3}(\infty)\rangle_t$ the correlation function where we integrate the bulk (resp. boundary) GMC measure of (3.57) on $\mathbb{H}_t$ (resp. $\mathbb{R}_t$) instead of $\mathbb{H}$ (resp. $\mathbb{R}$). Viewed in the strip, this is the same as taking $\mathcal{S}_t := (-\infty, t) \times (0, 2\pi)$ and $(-\infty, t) \times \{0, 2\pi\}$ as domains of integration for the bulk and boundary measures.

For fixed $b > 0$, we have

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\leq s\leq t} B_s \leq b\right) \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi t}}b \\ &\mathbb{P}\left(\sup_{t\geq 0} B_s + \frac{1}{2}(\beta - Q)s \leq b\right) \underset{\beta\to Q^-}{\sim} (Q-\beta)b\end{aligned} \tag{3.77}$$

so by previous arguments we have

$$\lim_{t\to\infty}\sqrt{\frac{\pi t}{2}}\langle B_Q(0)B_{\beta_2}(1)B_{\beta_3}(\infty)\rangle_t=\lim_{\beta\to Q^-}\frac{1}{Q-\beta}\langle B_\beta(0)B_{\beta_2}(1)B_{\beta_3}(\infty)\rangle=-\partial_1 c(Q,\beta_2,\beta_3)$$

The critical case (3.63) follows easily from this equality.

Now we turn to the supercritical case. We write $t:=2\log\frac{1}{|x|}$. The radial process has a positive drift $\frac{1}{2}(\beta_1+\beta_2-Q)$ in $(0,t)$, which we kill by Cameron-Martin's theorem (recall (3.36), yielding the Radon-Nikodym derivative $e^{\frac{1}{2}(\beta_1+\beta_2-Q)B_t-\frac{1}{8}(\beta_1+\beta_2-Q)^2t}$. This accounts for the polynomial rate in $|x|$.

Similarly as in Figure 3.3, we condition on value of the process at time t and introduce $B_t=\sqrt{t}\delta$ with $\delta\sim\mathcal{N}(0,1)$ independent of everything. Thus the process in $[0,t]$ is the sum of a random drift $\frac{\delta}{\sqrt{t}}$ and an independent Brownian bridge in $[0,t]$ (see Figure 3.4). Conditioning the Brownian bridge in $(0,t)$ to stay below b, we get a contribution of $\sqrt{\frac{2}{\pi}}t^{-3/2}=\frac{1}{2\sqrt{2\pi}\log^{3/2}\frac{1}{|x|}}$. Taking $t\to\infty$ then $b\to\infty$, the limiting integral on the left is a strip with a β_4-insertions at $-\infty$, a β_3-insertion at 0 and a (derivative) Q-insertion at $+\infty$ (see Figure 3.4), hence the limit is $-\frac{1}{2}\partial_1 c(Q,\beta_3,\beta_4)$ (recall the prefactor 2 in the definition of (3.57)). Similarly the limiting integral on the left is $-\frac{1}{2}\partial_1 c(\beta_1,\beta_2,Q)$, yielding the result.

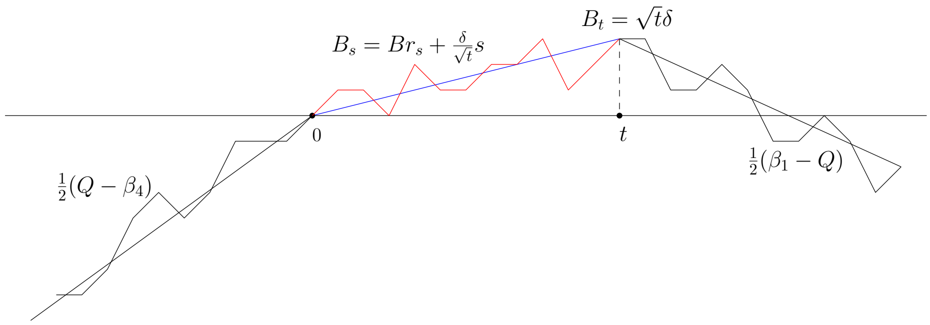

Figure 3.4: The radial process on the strip in $[0,t]$ is the sum of a Brownian bridge (red) and a random independent drift (blue).

□

Proof of Theorem 3.5.2. In this proof, we use the flat disc $(\mathbb{D},dz)$ as set-up, which is mapped to the semi-infinite cylinder $\mathcal{C}_+=\mathbb{R}_+\times\mathbb{S}^1$ equipped with the metric $g(s,\theta)=e^{-2s}$ under the conformal transformation $z\mapsto e^{-z}$. So the GFF decomposes as the sum of a drifted Brownian motion $(B_s-Qs)_{s\geq 0}$ and an independent lateral noise Y from which we take the GMC measure with respect to Lebesgue measure.

We treat the case $\beta>0$ and $\alpha_1+\alpha_2>Q$, the others being similar. Let $t:=\log\frac{1}{|z|}$. With the presence of the insertions, the radial part has a positive drift $\alpha_1+\alpha_2-Q$ in $(0,t)$ and negative drift α_1-Q in (t,∞). Killing the drift in $(0,t)$ with Cameron-Martin's theorem gives the exponent in $|z|$. Conditioning on the value of $B_t=\sqrt{t}\delta$

and conditioning the Brownian bridge not to exceed some $b > 0$ gives a prefactor of $\sqrt{\frac{2}{\pi}}t^{-3/2}$. Taking $t \to \infty$ then $b \to \infty$, we find that the integral on the right is an infinite cylinder with insertions (α_1, α_2, Q) at $(+\infty, 0, -\infty)$, so its value is $-\frac{1}{4}\partial_3 C_\gamma(\alpha_1, \alpha_2, Q)$ (the lateral noise is close to the one used before in this region and can be dealt with using Kahane's convexity inequality). On the other hand, the integral on the left is a semi-infinite cylinder with a Q-insertion at ∞ and a β-insertion on the boundary, so its value is $-\frac{1}{4}\frac{\partial}{\partial\alpha}\langle V_\alpha(i)B_\beta(0)\rangle_{|\alpha=Q}$. □

3.5.3 Links with random planar maps

The above results can be interpreted with respect to the KPZ conjecture on random planar maps with the topology of the disc. For concreteness, let $\mathcal{T}_{n,m}$ be the set of triangulations of the disc with n internal vertices and $m+2$ boundary vertices, with two marked vertices (one internal and one on the boundary). Then it is known [AS02] that there exists $\mu^c, \mu_\partial^c > 0$ such that

$$\#\mathcal{T}_{n,m} \asymp e^{\mu^c n} e^{\mu_\partial^c m} m^{1/2} n^{-5/2}$$

We suppose that for a triangulation $(\mathbf{t}, \mathbf{z}, \mathbf{x})$, we have conformal mapped $\mathbf{t}$ to $\mathbb{H}$ (in the manner of section 3.1.4) and that $\mathbf{z}$ is mapped to i and $\mathbf{x}$ is mapped to 0. For each such triangulation and $a > 0$, we can construct measures $\nu^{\mathbf{t},a}$ (resp. $\nu_\partial^{\mathbf{t},a}$) giving mass a^2 (resp. a) to each triangle (resp. each boundary edge). Now we let $\mu := (1 + a^2)\mu^c$ and $\mu_\partial := (1 + a)\mu_\partial^c$, and sample the triangulations at random with the probability measure

$$\mathbb{P}_a(\mathbf{t}, \mathbf{z}, \mathbf{x}) = \frac{1}{Z_a} e^{-\mu|\mathbf{t}|} e^{-\mu_\partial \ell(\mathbf{t})}$$

where Z_a is the normalising constant and $\ell(\mathbf{t})$ is the boundary length of $\mathbf{t}$. Additionally we choose the internal marked vertex uniformly in the internal vertices of $\mathbf{t}$ and similarly for the boundary marked vertex.

It is conjectured [HRV18] that the pair of random measures $(\nu^{\mathbf{t},a}, \nu_\partial^{\mathbf{t},a})$ converges in distribution to a pair of random measures on $(\mathbb{D}, \partial\mathbb{D})$, and the limit (ν, ν_∂) should be given by (some form of) LQFT on the disc. In particular, it should be the case that for all measurable sets $A \subset \mathbb{H}, B \subset \mathbb{R}$,

$$\begin{aligned} \mathbb{E}\left[\frac{\nu(A)}{\nu(\mathbb{H})}\right] &= \int_A f_{\sqrt{\frac{8}{3}}, \mu^c, \mu_\partial^c}(z) d^2 z \\ \mathbb{E}\left[\frac{\nu_\partial(B)}{\nu_\partial(\mathbb{R})}\right] &= \int_B \lambda_{\sqrt{\frac{8}{3}}, \mu^c, \mu_\partial^c}(x) dx \end{aligned} \tag{3.78}$$

where we define for all $\gamma \in (0,2)$ and $\mu, \mu_\partial > 0$,

$$\begin{aligned} f_{\gamma,\mu,\mu_\partial}(z) &:= \frac{1}{Z}\langle V_\gamma(z)V_\gamma(i)B_\gamma(0)\rangle \\ \lambda_{\gamma,\mu,\mu_\partial}(x) &:= \frac{1}{Z_\partial}\langle B_\gamma(x)V_\gamma(i)B_\gamma(0)\rangle \end{aligned} \tag{3.79}$$

where Z, Z_∂ are normalising constants whose values are discussed in Appendix 3.A.

Similarly to the discussion of section 3.1.4, the result of Theorems 3.5.4 and 3.5.2 gives precise estimates on the expected density of vertices in different settings: internal or boundary vertices around the marked point on the boundary, internal vertices around the internal marked point, and internal vertices around the boundary.

Finally, we mention that one can formulate other conjectures involving different values of γ (e.g. by weighting the measure $\mathbb{P}_a$ by some statistical mechanics model), μ and μ_∂ (e.g. by considering other types of maps).

3.A The normalising constant in (3.21) and (3.79)

We present the computation of the normalising constant for $f_{\gamma,\mu}$ (in a more general setting). The idea is that integrating over the location of a γ-insertion is the same as differentiating with respect to the cosmological constant. We present the main steps and leave the details to the reader.

Let $N \geq 3$ and $z_1, ..., z_N \in \widehat{\mathbb{C}}$ pairwise disjoint and $(\alpha_1, ..., \alpha_N)$ satisfying the Seiberg bounds. For notational convenience, we write $\mathcal{G}(x) := \sum_{i=1}^N \alpha_i G(z_i, x)$ and as usual $\sigma = \sum_{i=1}^N \frac{\alpha_i}{Q} - 2$.

Using Cameron-Martin's theorem to go from the second to third line we find

$$\begin{aligned} &\frac{1}{2} e^{-\sum\limits_{1\leq i<j} \alpha_i\alpha_j G(z_i,z_j)} \int_{\widehat{\mathbb{C}}} \left\langle V_\gamma(z) \prod_{i=1}^N V_{\alpha_i}(z_i) \right\rangle dz \\ &\quad = \int_{\widehat{\mathbb{C}}} e^{\gamma\mathcal{G}(z)} \int_{\mathbb{R}} e^{(Q(\sigma+\frac{\gamma}{Q})c}\mathbb{E}\left[\exp\left(-\mu e^{\gamma c} M^\gamma\left(e^{\gamma(\mathcal{G}+\gamma G(z,\cdot))}\right)\right)\right] dc d^2 z \\ &\quad = \mathbb{E}\left[\int_{\mathbb{R}} e^{Q\sigma c} e^{\gamma c} M^\gamma\left(e^{\gamma\mathcal{G}}\right)\exp\left(-\mu e^{\gamma c} M^\gamma\left(e^{\gamma\mathcal{G}}\right)\right) dc\right] \\ &\quad = -\frac{1}{2} e^{-\sum\limits_{1\leq i<j} \alpha_i\alpha_j G(z_i,z_j)} \frac{\partial}{\partial\mu}\left\langle \prod_{i=1}^N V_{\alpha_i}(z_i)\right\rangle \end{aligned} \tag{3.80}$$

so that in the end

$$\int_{\widehat{\mathbb{C}}} \left\langle V_\gamma(z) \prod_{i=1}^N V_{\alpha_i}(z_i)\right\rangle d^2 z = -\frac{\partial}{\partial\mu}\left\langle \prod_{i=1}^N V_{\alpha_i}(z_i)\right\rangle = \frac{Q\sigma}{\gamma\mu}\left\langle \prod_{i=1}^N V_{\alpha_i}(z_i)\right\rangle \tag{3.81}$$

where we simply used that $\langle\prod_{i=1}^N V_{\alpha_i}(z_i)\rangle$ is equal to $\mu^{-\frac{Q\sigma}{\gamma}}$ times some quantity independent of μ. In particular this yields (3.21) for $N=3$ and $(\alpha_1,\alpha_2,\alpha_3)=(\gamma,\gamma,\gamma)$.

Similarly, in the disc case, we find that for $(\alpha_1,...,\alpha_N,\beta_1,...,\beta_M)$ satisfying the Seiberg bounds, we have

$$\int_{\mathbb{H}} \left\langle V_\gamma(z)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B_{\beta_j}(x_j)\right\rangle d^2z = -\frac{\partial}{\partial\mu}\left\langle\prod_{i=1}^N V_{\alpha_i}(z_i)B_{\beta_j}(x_j)\right\rangle$$

and

$$\int_{\mathbb{R}} \left\langle B_\gamma(x)\prod_{i=1}^N V_{\alpha_i}(z_i)\prod_{j=1}^M B_{\beta_j}(x_j)\right\rangle dx = -\frac{\partial}{\partial\mu_\partial}\left\langle\prod_{i=1}^N V_{\alpha_i}(z_i)B_{\beta_j}(x_j)\right\rangle$$

In general, this does not simplify as nicely as (3.81) but if e.g. $\mu=0$, then we have for instance

$$\int_{\mathbb{R}}\langle B_\gamma(x)V_\gamma(i)B_\gamma(0)\rangle dx = \frac{3\gamma-2Q}{2\gamma\mu}R(\gamma,\gamma)$$

Chapter 4

Liouville-Cauchy process

In this chapter, we define the boundary version of Liouville Brownian motion (γ-LBM) as a time change of the Cauchy process according to Liouville length. Similarly to [GRV16, RV15], we study the qualitative properties of this reparameterisation through an explicit construction of the positive continuous additive functional. The resulting Markov process is called the "Liouville-Cauchy process" (γ-LCP) and is well-defined in both the subcritical and critical regimes ($\gamma \in (0, 2]$). It is fair to say that the methods used in this chapter are not new, and we are still pondering about its publication.

Among other things, the γ-LCP gives a natural notion of local time of γ-LBM on the boundary of the domain. In particular we can use conformal welding to define the "quantum local time" of γ-LBM on an independent $\mathrm{SLE}_{\kappa=\gamma^2}$, together with the natural Markov process on the curve that is invariant for the quantum length.

4.1 Introduction

4.1.1 Overview

4.1.1.1 Background

Liouville quantum gravity was introduced by Polyakov in the context of string theory [Pol81] as a theory of Riemannian surfaces $(\Sigma, \widehat{\sigma})$ endowed with the random metric

$$\widehat{\sigma} = e^{\gamma X}\sigma, \tag{4.1}$$

where σ is a classical solution to Liouville's equation (i.e. a Riemannian metric with constant curvature and geodesic boundary), X is the Gaussian free field (GFF) on (Σ, σ) and $\gamma \in (0, 2]$ is the parameter of the theory. Such an expression does not makes sence literally due to the lack of regularity of the GFF, but the work of Kahane on Gaussian multiplicative chaos (GMC) [Kah85] allows to make sense of (4.1) as a random volume form.

Indeed, Kahane's theory gives a way of constructing random measures by exponentiating logarithmically correlated Gaussian fields, such as the two-dimensional GFF. This was independently rediscovered by Duplantier & Sheffield [DS11b] in the special case of the GFF, where the volume form goes by the name of Liouville measure[1]. The value $\gamma = 2$, which was the one consider by Polyakov in his original paper, is known as "critical" and will be the main focus of this work.

In the case of a bordered surface, the GFF has a trace on the boundary, which is also logarithmically correlated. Exponentiating it endows the boundary with a random length measure. As a conformal field theory, Liouville theory comes with a way of "gluing" surfaces along boundary components in a way that preserves the random structure. First, one isometrically identifies the glued boundary components according to Liouville length. Second, one solves the associated conformal welding problem, and the welding interface is a (form of) Schramm-Loewner evolution (SLE). Although this result is not known in such a generality, the case of simply connected domains is by now well-understood [DMS14], building on the pioneering work of Sheffield [She16]. The critical case of Sheffield's "quantum zipper" theorem was treated by Holden & Powell [HP18].

The boundary version of Liouville theory also exhibits striking integrability properties. For instance, the techniques developed by David, Kupiainen, Rhodes & Vargas [DKRV16, KRV19, KRV20] on the Riemann sphere allowed Remy and Zhu to compute the distribution of the total mass of GMC on the circle [Rem20] and on the interval [RZ20a]. On the other hand, while the DOZZ formula computes *some* moments of GMC observables, it is not sufficient to characterise the distribution of GMC on the Riemann sphere and there is no analogue of the Fyodorov-Bouchaud formula in two dimensions at the moment. Recently [GRSS20], a formula involving some observables of GMC on the circle was proposed for the one-point toric conformal blocks of Liouville CFT, one of the building blocks of the theory. This gives more evidence that boundary GMC plays a fundamental role in the gluing morphism of LCFT, and makes boundary Liouville theory an important subject of study in its own right. At the centre stage is the Sobolev space $H^{1/2}(\partial\Sigma)$, the trace space of $H^1(\Sigma)$, the latter being the space on which the GFF and Brownian motion are based. Similarly, $H^{1/2}(\partial\Sigma)$ is the defining space of the trace of the GFF and the Cauchy process.

Garban, Rhodes & Vargas [GRV16] and independently Berestycki [Ber15] defined "Liouville Brownian motion" (LBM) as the natural diffusion on the random geometry given by (4.1). In short, this consists in time-changing the standard Brownian motion according to the Liouville measure, resulting in a process that leaves the Liouville measure invariant. Further properties of LBM were studied in [GRV14], and [RV15] extended the construction to the critical case. The relevance of this process lies in the fact that it comes with analytic objects such as a heat kernel and a resolvent, the existence of which is not given *a priori*

[1]We will use the words "GMC" and "Liouville measure" in an interchangeable way.

on the irregular geometry of (4.1). Many properties of these objects like the short-time asymptotics of the Liouville heat kernel are connected to the metric features of Liouville quantum gravity, in particular its fractal dimension [DZZ19, DG20]. Although we will not be concerned by this here, we mention that making sense of (4.1) as a random metric and not only a volume form has been achieved only recently [DDDF19, DFG+19, GM19].

The present work addresses the boundary version of LBM in the full range $\gamma \in (0, 2]$: this is a Cauchy process on the boundary, time-changed by the Liouville boundary length, which we call the Liouville-Cauchy process (LCP). The appearance of the Cauchy process is explained by Spitzer's embedding theorem [Spi58], stating that it is obtained by reparameterising the trace of Brownian motion on the boundary by its local time. It is also the Markov process associated to the Dirichlet space $H^{1/2}(\partial\Sigma)$, giving another interpretation of the Cauchy process as the trace of Brownian motion on the boundary. Our construction of this process is similar in spirit to [GRV16, RV15]: it relies on an explicit construction of the positive continuous additive functional (PCAF) of the Liouville measure through a renormalisation procedure (see Theorem 4.1.1). This allows us to get some qualitative information about the time-changed process that are not given *a priori* by the Revuz correspondence. The resulting process is a strong Markov process and we can derive further properties of its heat kernel and resolvent family (see Section 4.3.4). In analogy with Spitzer's representation, the LCP will allow us to define the local time of LBM on $\partial\Sigma$, and we will recover LCP as the trace of LBM reparameterised by local time.

4.1.2 Liouville-Cauchy process and Liouville Brownian motion

4.1.2.1 Cauchy process and Spitzer's embedding

An archetypal example of Dirichlet form is the Dirichlet energy $\mathcal{E}_\nabla$ on $L^2(\bar{\mathbb{D}}, \mathrm{d}z)$, given by

$$\mathcal{E}_\nabla(f, g) := \langle \nabla f, \nabla g \rangle_{L^2(\mathbb{D}, \mathrm{d}z)}$$

whose domain is the Sobolev space $H^1(\mathbb{D}, \mathrm{d}z)$. This form is regular and the associated Markov process is Brownian motion in $\mathbb{D}$ reflected on $\partial\mathbb{D} = \mathbb{S}^1$ [FOT11, Example 4.4.2], which we denote by $B = (B_t)_{t \geq 0}$. It is well-known that $H^1(\mathbb{D}, \mathrm{d}z)$ satisfies a Poincaré inequality in the form of (4.39).

The trace space of $H^1(\mathbb{D}, \mathrm{d}z)$ is the Sobolev space $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta) \subset L^2(\mathbb{S}^1, \mathrm{d}\theta)$ of (Schwartz) distributions with half-derivative in $L^2(\mathbb{S}^1, \mathrm{d}\theta)$. It is also the domain of the Dirichlet form

$$\mathcal{E}(u, v) := \mathcal{E}_\nabla(\mathbf{P}u, \mathbf{P}v), \tag{4.2}$$

where $\mathbf{P} : H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta) \to H^1(\mathbb{D}, \mathrm{d}z)$ denotes the harmonic extension. We have the characterisation (see e.g. [NS95, Section 2])

$$H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta) = \left\{u \in L^2(\mathbb{S}^1, \mathrm{d}\theta),\, \mathcal{E}(u) < \infty\right\}, \tag{4.3}$$

and it is a Hilbert space when endowed with the (squared) norm $\|\cdot\|^2_{L^2(\mathbb{S}^1,\mathrm{d}\theta)} + \mathcal{E}(\cdot)$. Moreover, a Poincaré inequality also holds in $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta)$ in the form of (4.39). In particular, the homogeneous space $\dot{H}^{1/2}(\mathbb{S}^1, \mathrm{d}\theta) := H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta)/\mathbb{R}$ is a Hilbert space when endowed with the (squared) norm $\mathcal{E}(\cdot)$.

The Markov process associated to $(\mathcal{E}, H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta))$ is the symmetric Cauchy process on $\mathbb{S}^1$ [FOT11, Example 6.2.2]. It can also be obtained as a time-change of B by the PCAF of Lebesgue measure $\mathrm{d}\theta$ on $\partial\mathbb{D}$. This measure does not charge any polar set of $\bar{\mathbb{D}}$, so it is Revuz in $H^1(\mathbb{D}, \mathrm{d}z)$, and the PCAF is the local time of B on $\mathbb{S}^1$, denoted L. The Cauchy process is obtained by reparameterising the trace of B on the boundary by its local time. Namely, letting $\tau_s := \inf\{t > 0,\, L_t > s\}$, the process $(B_{\tau_s})_{s\geq 0}$ is a Cauchy process. This representation theorem was first obtained by Spitzer [Spi58].

Of course, in this case the Revuz measure does not have full topological support in $\bar{\mathbb{D}}$, so that L is not strictly increasing, which in turn implies that the paths of the process are not continuous. In particular, the Dirichlet form (4.2) is not local. On the other hand, the quasi-support of $\mathrm{d}\theta$ is the whole unit circle and does coincide with its topological support.

4.1.2.2 Gaussian multiplicative chaos

Let $\gamma \in (0, 2]$ and X be the Gaussian free field (GFF) in $\mathbb{D}$ with free boundary conditions on $\mathbb{S}^1$. We will use a regularisation $(X_\varepsilon)_{0<\varepsilon\leq 1}$ known as the white-noise regularisation and described in Section 4.2.3. The *Gaussian multiplicative chaos* (γ-GMC) measure μ_γ of X (with respect to $\mathrm{d}z$) is the almost surely weak limit of the family of measures

$$\mathrm{d}\mu_{\gamma,\varepsilon}(z) := \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\varepsilon(z)} \mathrm{d}z. \tag{4.4}$$

For the value $\gamma = 2$, the above renormalisation yields zero as a limiting object, but it is possible to change the renormalisation in order to obtain a non-trivial measure μ_2 [DRSV14a]:

$$\mathrm{d}\mu_{2,\varepsilon}(z) := \left(2 \log \frac{1}{\varepsilon} - X_\varepsilon(z)\right) \varepsilon^2 e^{2X_\varepsilon(z)} \mathrm{d}z. \tag{4.5}$$

Again, the convergence $\mu_{2,\varepsilon} \to \mu_2$ as $\varepsilon \to 0$ is almost sure with respect to the topology of weak convergence of measures. The renormalisation procedure (4.5) is called the *derivative renormalisation*, since formally $\mu_2 = -\frac{\mathrm{d}\mu_\gamma}{\mathrm{d}\gamma}_{|\gamma=2}$.

The GFF has a trace on $\mathbb{S}^1$ denoted by W, which is equal in distribution to the isonormal Gaussian process based on $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta)$. This field also has a white-noise regularisation $(W_\varepsilon)_{0<\varepsilon\leq 1}$, and one can define boundary length measures ν_γ as the almost surely weak limit of the family of measures

$$\mathrm{d}\nu_{\gamma,\varepsilon}(e^{i\theta}) = \varepsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}W_\varepsilon(e^{i\theta})}\mathrm{d}\theta, \qquad \gamma \in (0,2);$$
$$\mathrm{d}\nu_{2,\varepsilon}(e^{i\theta}) = \left(\log\frac{1}{\varepsilon} - \frac{1}{2}W_\varepsilon(e^{i\theta})\right)\varepsilon e^{W_\varepsilon(e^{i\theta})}\mathrm{d}\theta.$$

Given $\gamma \in (0,2]$, the measures μ_γ, ν_γ are a.s. finite and we will always assume that $\nu_\gamma(\mathbb{S}^1) = 1$. Since the underlying GFF was only defined up to an additive constant, this can be understood as a way to fix the constant.

Let $\psi : \mathbb{D} \to D$ be a conformal transformation. We can define a GMC measure μ_γ^D using the renormalisation procedure (4.4) (or (4.5) if $\gamma = 2$), where X is now the GFF in $(D, \mathrm{d}z)$, i.e. the isonormal Gaussian process based on $H^1(D, \mathrm{d}z)$. Then we have the following conformal covariance property, also known as the conformal change of coordinate formula [She16]: with $Q := \frac{2}{\gamma} + \frac{\gamma}{2}$, we have

$$\psi^*\mu_\gamma^D = |\psi'|^{\gamma Q}\mu_\gamma. \tag{4.6}$$

4.1.2.3 Liouville Brownian motion

It is known that for all $\gamma \in (0,2]$, almost surely, μ_γ does not charge any polar sets (of Brownian motion). This is relatively straightforward for $\gamma < 2$ but showing it for $\gamma = 2$ constitutes a substantial part of [RV15] (see Section 4.2). Thus, the Revuz correspondence ensures the existence of a PCAF. However, this is a purely abstract statement and does not say anything about the qualitative properties of the time change, e.g. does the time-changed Brownian motion have infinite lifetime? continuous sample paths? Does it get "stuck"?

To address these questions, one needs to show that the PCAF is almost surely a self-homeomorphism of $\mathbb{R}_+$ for all starting points. In [GRV16, RV15], this is done in a constructive way by defining the PCAF through a renormalisation procedure. Namely, one considers the regularised functionals

$$\Phi_t^{\varepsilon,\gamma} := \int_0^t \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_\varepsilon(B_s)}\mathrm{d}s, \qquad \gamma \in (0,2);$$
$$\Phi_t^{\varepsilon,2} := \int_0^t \left(2\log\frac{1}{\varepsilon} - X_\varepsilon(B_s)\right)\varepsilon^2 e^{2X_\varepsilon(B_s)}\mathrm{d}s.$$

Then it is shown that for almost every realisation of X and quasi-all starting points z, the mapping $t \mapsto \Phi_t^{\varepsilon,\gamma}$ converges a.s. under $\mathbb{P}_z$ as $\varepsilon \to 0$. The convergence is uniform on compacts of $\mathbb{R}_+$ and the limiting mapping $t \mapsto \Phi_t^\gamma$ is a self-homeomorphism of $\mathbb{R}_+$. Moreover, Φ^γ coincides with the PCAF of μ_γ in the sense of PCAF equivalence.

The functional Φ^γ is understood as the quadratic variation of the formal martingale

$e^{\frac{\gamma}{2}X(B_t)}\mathrm{d}B_t$, and one can reparameterise this process by quadratic variation. Namely, set $\tau_s^\gamma = (\Phi_\cdot^\gamma)^{-1}(s)$ and define *Liouville Brownian motion* (γ-LBM) by $\mathcal{B}_s^\gamma := B_{\tau_s^\gamma}$.

The 2-LBM is said *critical* since it is based on the critical measure μ_2. The fact that Φ^γ is a homeomorphism for all starting points implies that the time-change is non-degenerate: sample paths of the γ-LBM are continuous with infinite lifetime and never get "stuck". In particular, Φ_γ has full support, i.e. μ_γ has full quasi-support, and one defines the Dirichlet space of γ-LBM in the manner described in (4.40).

4.1.2.4 Liouville-Cauchy process

We will establish analogous statements in the case of the Cauchy process time-changed by the boundary Liouville measure ν_γ. Similarly, these measures a.s. do not charge polar sets of $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta)$, so they admit a PCAF. In order to study this PCAF, we introduce the regularised functionals

$$\begin{aligned} F_t^{\varepsilon,\gamma} &:= \int_0^t \varepsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}W_\varepsilon(C_s)}\mathrm{d}s, \qquad \gamma \in (0,2); \\ F_t^{\varepsilon,2} &:= \int_0^t \left(\log\frac{1}{\varepsilon} - \frac{1}{2}W_\varepsilon(C_s)\right)\varepsilon e^{W_\varepsilon(C_s)}\mathrm{d}s. \end{aligned} \tag{4.7}$$

Here, $(C_t)_{t\geq 0}$ denotes the symmetric Cauchy process on $\mathbb{S}^1$ and we let $\mathbb{P}_x$ be the law of the Cauchy process started from $x \in \mathbb{S}^1$. The next theorem is the main technical result of this paper and its proof is carried out in Section 4.3.

Theorem 4.1.1 (Convergence of the PCAF)**.** *Fix $\gamma \in (0,2]$.*

For almost every realisation of W and quasi-every $x \in \mathbb{S}^1$, $\mathbb{P}_x$-almost surely, the family $(t \mapsto F_t^{\varepsilon,\gamma})_{\varepsilon>0}$ converges as $\varepsilon \to 0$, uniformly on compacts of $\mathbb{R}_+$, and the limiting function $t \mapsto F_t^\gamma$ is a self-homeomorphism of $\mathbb{R}_+$. Moreover, F^γ coincides with the PCAF of ν_γ up to PCAF equivalence.

We will treat the critical and subcritical cases separately. The critical case will be proved in Sections 4.3.1 and 4.3.2, while the subcritical case is carried out in Section 4.3.3. The latter is much simpler to deal with, and it will turn out that the PCAF is in the strict sense in this case.

We discuss a few basic consequences, which will be elaborated upon in Section 4.3.4. Theorem 4.1.1 implies that we can invert the PCAF $\tau_s^\gamma = (F_\cdot^\gamma)^{-1}(s)$ and reparameterise $(C_t)_{t\geq 0}$ accordingly:

$$\mathcal{C}_s^\gamma := C_{\tau_s^\gamma}.$$

From [FOT11, Chapter 6], this defines a strong Markov process on $\mathbb{S}^1$. It comes with a semi-group $(\mathbf{p}_t^\gamma)_{t>0}$ and other analytic objects which will be studied in Section 4.3.4. In particular, it will be shown (Theorem 4.3.15) that the heat kernel is absolutely continuous with respect to the Liouville measure for all $\gamma \in (0,2]$.

Definition 4.1.1 (Liouville-Cauchy process). We call $(\mathcal{C}_s^\gamma)_{s\geq 0}$ the *Liouville-Cauchy process* (γ-LCP) and refer to the 2-LCP as the *critical LCP*. Its Dirichlet space is realised in the manner of (4.40):

$$\begin{cases} H^{1/2}(\mathbb{S}^1,\nu_\gamma) := \left\{u \in L^2(\mathbb{S}^1,\nu_\gamma),\ \exists \widetilde{u} \in H^{1/2}(\mathbb{S}^1,\mathrm{d}\theta) \text{ s.t. } \widetilde{u} = u \quad \nu_\gamma\text{-a.e.}\right\} \\ \mathcal{E}^\gamma(u,v) = \mathcal{E}(\widetilde{u},\widetilde{v}). \end{cases} \tag{4.8}$$

Consider the reflecting Brownian motion $(B_t)_{t\geq 0}$ in $\mathbb{D}$ and the notations of Section 4.1.2.1. We have seen that the Cauchy process is obtained by reparameterising the trace of $(B_t)_{t\geq 0}$ on $\mathbb{S}^1$ by its local time $(L_t)_{t\geq 0}$. Now, the measure μ_γ (viewed as a measure on $\bar{\mathbb{D}}$) has topological support $\bar{\mathbb{D}}$, and is Revuz with full quasi-support in $H^1(\mathbb{D},\mathrm{d}z)$. Thus, it provides a non-degenerate time change of reflecting Brownian motion. Similarly, the measure ν_γ has topological support $\partial\mathbb{D}$, and is also Revuz with full quasi-support by Theorem 4.1.1. Using ν_γ as Revuz measure provides a definition of the local time of reflecting γ-LBM on $\partial\mathbb{D}$.

Definition 4.1.2. The *reflecting γ-LBM in* $\mathbb{D}$ is the Markov process associated to the Dirichlet space $(\mathcal{E}_\nabla^\gamma, H^1(\mathbb{D},\mu_\gamma))$ on $L^2(\bar{\mathbb{D}},\mu_\gamma)$, where

$$H^1(\mathbb{D},\mu_\gamma) := \{f \in L^2(\bar{\mathbb{D}},\mu_\gamma),\ \exists \widetilde{f} \in H^1(\mathbb{D},\mathrm{d}z) \text{ s.t. } \widetilde{f} = f \quad \mu_\gamma\text{-a.e.}\}$$

and $\mathcal{E}_\nabla^\gamma(f,g) = \mathcal{E}_\nabla(\widetilde{f},\widetilde{g})$. The *local time of γ-LBM on $\partial\mathbb{D}$* is the PCAF of Revuz measure ν_γ, denoted $(L_t^\gamma)_{t\geq 0}$.

We also get the Liouville analogue of Spitzer's embedding: let $(\mathcal{B}_t^\gamma)_{t\geq 0}$ be the reflecting γ-LBM in $\mathbb{D}$ with local time $(L_t^\gamma)_{t\geq 0}$ on $\mathbb{S}^1$. Letting $\tau_s^\gamma := \inf\{t>0,\ L_t^\gamma > s\}$, the process $(\mathcal{B}_{\tau_s^\gamma}^\gamma)_{s\geq 0}$ is a γ-LCP.

Remark 15. In [GRV16, GRV14, RV15], only the absorbing LBM is considered. The Dirichlet space of the Euclidean absorbing Brownian motion in $\mathbb{D}$ is $(\mathcal{E}_\nabla, H_0^1(\mathbb{D},\mathrm{d}z))$, where $\mathcal{E}_\nabla$ is considered on $L^2(\mathbb{D},\mathrm{d}z)$ and $H_0^1(\mathbb{D},\mathrm{d}z)$ is the completion of $\mathcal{C}_c^\infty(\mathbb{D})$ with respect to $\mathcal{E}_\nabla$. The Dirichlet space of absorbing LBM is then defined similarly using (4.40).

4.1.3 Schramm-Loewner evolutions

4.1.3.1 Conformal welding

Let $\eta : \mathbb{S}^1 \to \mathbb{C}$ be a Jordan curve with zero area, bounding complementary domains $0 \in D^+ \subset \mathbb{C}$ and $D^- = \widehat{\mathbb{C}} \setminus \bar{D^+}$. Let $\psi_\pm : \mathbb{D}^\pm \to D^\pm$ be uniformising maps fixing 0, where

$\mathbb{D}^+$ is the standard unit disc and $\mathbb{D}^-$ is a copy of $\mathbb{D}^+$ with the opposite orientation. By Carathéodory's theorem, $\psi_\pm$ extends to a homeomorphism $\bar{\mathbb{D}}^\pm \to \bar{D}^\pm$ still denoted $\psi_\pm$, and $h := \psi_-^{-1} \circ \psi_+|_{\mathbb{S}^1}$ is called the *welding homeomorphism* of η. Conversely, given a homeomorphism $h : \partial\mathbb{D}^+ \to \partial\mathbb{D}^-$, we say that the conformal welding problem associated to h has a solution if there exists a triple (η, ψ_+, ψ_-) satisfying the above conditions. See Figure 4.1 for an illustration. Conceptually, what this procedure is achieving is to endow the topological sphere $\mathcal{S} := \bar{\mathbb{D}}^+ \sqcup \bar{\mathbb{D}}^- / \sim_h$ with a structure of Riemann sphere, where $\sim_h$ is the equivalence relation identifying $x \in \partial\mathbb{D}^+$ with $h(x) \in \partial\mathbb{D}^-$. If the solution curve exists and is unique (up to Möbius transformations), then the complex structure is canonically defined.

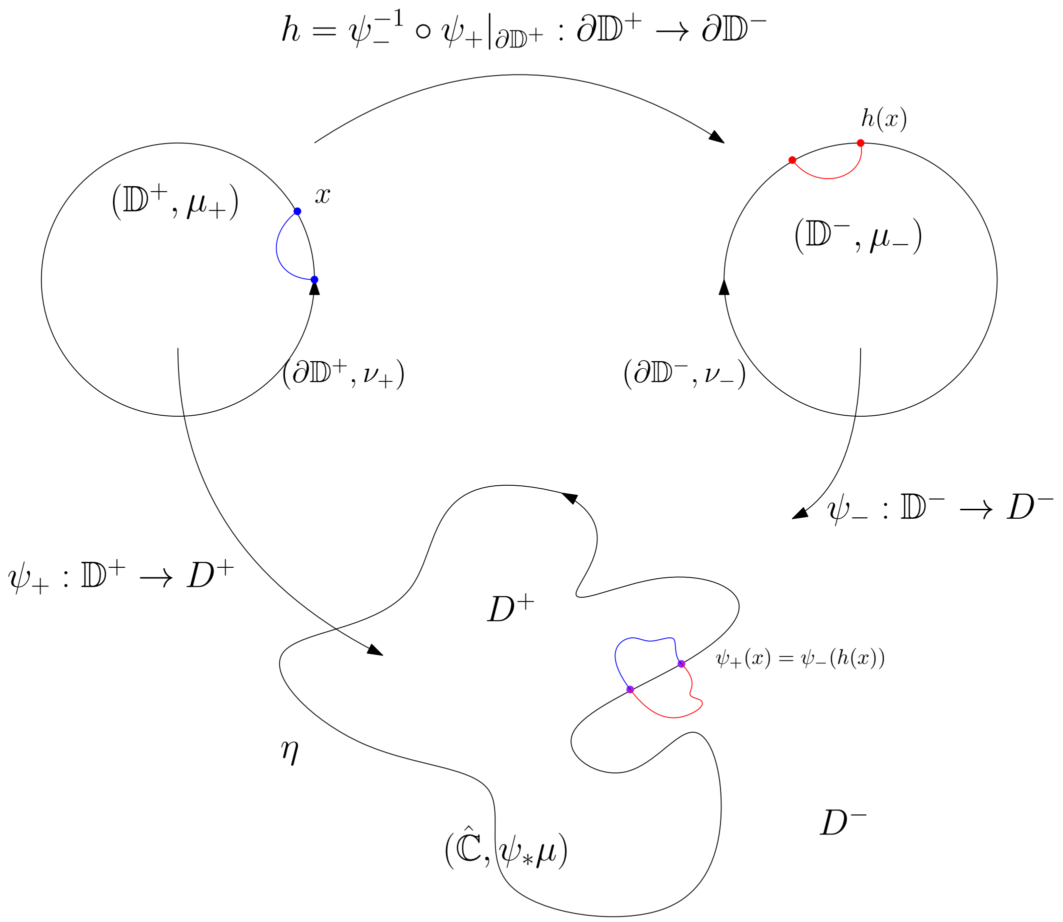

Figure 4.1: Illustration of the conformal welding of two discs into a Riemann sphere using the boundary homeomorphism $h : \partial\mathbb{D}^+ \to \partial\mathbb{D}^-$. The whole procedure endows the topological sphere $\mathcal{S} = \mathbb{D}^+ \sqcup \mathbb{D}^- / \sim_h$ with a complex structure. The blue and red neighbourhoods of $\mathbb{D}^+$ and $\mathbb{D}^-$ define a neighbourhood of $\mathcal{S}$ and the condition $\psi_+|_{\partial\mathbb{D}^+} = \psi_-|_{\partial\mathbb{D}^-} \circ h$ ensures that this neighbourhood is mapped to a neighbourhood of $\widehat{\mathbb{C}}$. In the case of SLE, h is given by isometrically identifying $\partial\mathbb{D}^+$ with $\partial\mathbb{D}^-$ according to the GMC length measures ν_+ and ν_-. The pushforward $\psi_*\mu_\gamma$ is a GMC measure on the sphere, independent of the welding curve η.

It is well-known that not every h is the welding homeomorphism of a Jordan curve, and if it is the solution may not be unique. See [Bis07] for a comprehensive account. A sufficient condition for uniqueness is that η be H^1-removable [Jon95]: recall that a compact set $K \subset \mathbb{C}$ is H^1-*removable* if every $\phi \in H^1(\mathbb{C} \setminus K, \mathrm{d}z) \cap \mathcal{C}^0(\mathbb{C})$ belongs to $H^1(\mathbb{C}, \mathrm{d}z)$. Indeed, it is known that H^1-removability implies conformal removability [Jon95], i.e. every homeomorphism of $\widehat{\mathbb{C}}$ which is conformal off K is a Möbius transformation.

Let $\sigma_\pm := (\psi_\pm)_* \mathrm{d}\theta$ be the harmonic measure on η viewed from 0. Since $\mathcal{E}$ is conformally invariant, it makes sense to tautologically define the Dirichlet spaces

$$\begin{cases} H^{1/2}(\partial D^\pm, \sigma_\pm) := H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta) \circ \psi_\pm^{-1} \\ \mathcal{E}_\pm(u, v) := \mathcal{E}(u \circ \psi_\pm, v \circ \psi_\pm). \end{cases} \tag{4.9}$$

This gives two different notions of the Cauchy process on $\eta = \partial D^\pm$ as the $\psi_\pm$-images of the Cauchy process on $\mathbb{S}^1$. Of fundamental importance in this work is to understand in what sense these processes are "compatible". For instance, it is known that precomposition by h preserves $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta)$ if and only if h is quasi-symmetric, which is also equivalent to η being a quasi-circle [NS95]. It follows in this case that σ_+ (resp. σ_-) is Revuz in $H^{1/2}(\partial D^-, \sigma_-)$ (resp. $H^{1/2}(\partial D^+, \sigma_+)$) with full quasi-support, so that the two Cauchy processes can be expressed as time changes of each other in view of Appendix 4.A.

However, even in this case, the $\psi_\pm$-images of the Cauchy process can have mutually singular clocks (since σ_+ and σ_- are typically singular), and neither of them can claim to be the "natural" Cauchy process on η. Indeed, one would like to define the Cauchy process with respect to a notion of arclength on η arising from the ambient geometry in which η is embedded (namely, the Riemann sphere with its canonical complex structure). Such an arclength fails to be canonically defined if η is not regular enough. This is analogous to the fact that the conformal image of Brownian motion is only a Brownian motion up to time change, and in particular, the clock can become quite different from the Euclidean one when the Brownian motion gets close to the boundary.

Still, in many cases, there are natural measures supported on η that play a role similar to arclength. As will be seen below, this is realised in the case of SLE$_\kappa$ by the "natural parameterisation" [LS11] and the "quantum length" [She16]. The former is nothing but the $(1 + \frac{\kappa}{8})$-dimensional Minkowski content [LR15], while the latter is a multiplicative chaos with respect to the former [Ben18].

4.1.3.2 Schramm-Loewner evolutions

SLEs were introduced by Schramm [Sch00] as the candidate scaling limits of interfaces of clusters of statistical mechanics models at criticality. This picture of SLE is linked to their interpretation as flow lines of the Gaussian free field [MS16a]. Here, we are mainly interested in the other interpretation of SLE as the interface between conformally welded random surfaces. This was first exhibited by Sheffield in his seminal paper [She16], and considerably extended in [DMS14]. See also Berestycki's review [Ber16] for a detailed account of the quantum zipper theorem and [HP18] for the critical case. We recall below the main features of this theory.

Fix $\gamma \in (0,2]$ and let X_+, X_- be independent GFFs in $\mathbb{D}^+$ and $\mathbb{D}^-$, where $\mathbb{D}^+ = \mathbb{D}$ is the unit disc and $\mathbb{D}^-$ is a copy of $\mathbb{D}$ with opposite orientation. We gather these two fields in a single object $X = X_+\mathbb{1}_{\mathbb{D}^+} + X_-\mathbb{1}_{\mathbb{D}^-}$, understood as a GFF on the disjoint union $\mathbb{D}^+ \sqcup \mathbb{D}^-$. Let $\mu_+, \mu_-, \nu_+, \nu_-$ be the bulk and boundary γ-GMC measures constructed with X_+, X_-. Similarly, μ_+ and μ_- are gathered in a single measure μ_γ on $\mathbb{D}^+ \sqcup \mathbb{D}^-$. As before, the normalising constant of the GFFs are chosen such that $\nu_+(\partial\mathbb{D}^+) = \nu_-(\partial\mathbb{D}^-) = 1$. Let $h : \partial\mathbb{D}^+ \to \partial\mathbb{D}^-$ be the homeomorphism of the unit circle fixing 1 that isometrically identifies $\partial\mathbb{D}^+$ with $\partial\mathbb{D}^-$ according to Liouville length, i.e. $\nu_+ = h^*\nu_-$.

It is known that h is the welding homeomorphism of an $\mathrm{SLE}_{\kappa=\gamma^2}$-type curve [She16, HP18]. Moreover, $(\psi_+)_*\nu_+ = (\psi_-)_*\nu_-$ as measures supported on η. This measure is called the *quantum length* of the curve and is denoted ℓ_γ. Last but not least (recall $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$), the field $(X - Q\log|\psi'|) \circ \psi^{-1}$ is absolutely continuous with respect to the GFF on $\widehat{\mathbb{C}}$ (with its structure of Riemann sphere), and it is independent of η. By the conformal change of coordinate formula (4.6), this means that $\psi_*\mu_\gamma$ is absolutely continuous with respect to γ-GMC on the Riemann sphere. In particular, it is Revuz in $H^1(\widehat{\mathbb{C}}, \mathrm{d}z)$ with full quasi-support, and its PCAF defines a Markov process which is absolutely continuous with respect to γ-LBM. This last feature is rather striking: it means that the ψ_+-image of γ-LBM in $\mathbb{D}^+$ coincides in law with γ-LBM in $\widehat{\mathbb{C}}$ up until its hitting time of $\eta = \partial D^+$. This is in sharp contrast with the fact that the ψ_+-image of Euclidean Brownian motion in $\mathbb{D}^+$ becomes singular with respect to Euclidean Brownian motion in $\widehat{\mathbb{C}}$ upon approaching η.

4.1.3.3 LCP on SLE

In this section, we use the previous results on LCP and conformal welding to define a natural Markov process on the SLE curve, interpreted as the trace of LBM on SLE. The generator of this process is a pseudo-differential operator, the Neumann jump operator, depending both on the curve and the quantum length.

Let $\gamma \in (0,2]$ and η an $\mathrm{SLE}_{\kappa=\gamma^2}$ curve as described above by the welding of independent γ-GMC measures ν_+ and ν_- on $\mathbb{S}^1 = \partial\mathbb{D}^\pm$. The curve η splits $\widehat{\mathbb{C}}$ into two simply connected domains D^+, D^-, and we let $\psi_\pm : \mathbb{D}^\pm \to D^\pm$ be uniformising maps. The LCPs in $(\partial\mathbb{D}^\pm, \nu_\pm)$ come with a Dirichlet space $(\mathcal{E}^\gamma_\pm, H^{1/2}(\partial\mathbb{D}^\pm, \nu_\pm))$. Using the conformal maps $\psi_\pm$, we define the Dirichlet spaces $H^{1/2}(\partial D^\pm, \ell)$ by pushing forward $H^{1/2}(\partial\mathbb{D}^\pm, \nu_\pm)$, i.e. with the Dirichlet form $u \mapsto \mathcal{E}^\gamma_\pm(u \circ \psi_\pm)$. The space $H^{1/2}(\partial D^+, \ell) \cap H^{1/2}(\partial D^-, \ell)$ is a Hilbert space with (squared) norm

$$u \mapsto \|u\|^2_{L^2(\ell)} + \mathcal{E}^\gamma_+(u \circ \psi_+) + \mathcal{E}^\gamma_-(u \circ \psi_-).$$

Every $u \in H^{1/2}(\partial D^+, \ell) \cap H^{1/2}(\partial D^-, \ell)$ has a unique harmonic extension to $\widehat{\mathbb{C}}$, which belongs to $H^1(D^+) \cap H^1(D^-)$. Consider the subspace $H^{1/2}(\eta, \ell) \subset H^{1/2}(\partial D^+, \ell) \cap H^{1/2}(\partial D^-, \ell)$ of functions such that this harmonic extension belongs to $H^1(\widehat{\mathbb{C}})$. In fact, for $\gamma < 2$, we have $H^{1/2}(\eta, \ell) = H^{1/2}(\partial D^+, \ell) \cap H^{1/2}(\partial D^-, \ell)$ almost surely, due to the Sobolev removability of SLE_κ for $\kappa < 4$. In general, $H^{1/2}(\eta, \ell)$ is a closed subspace of $H^{1/2}(\partial D^+, \ell) \cap H^{1/2}(\partial D^-, \ell)$, and the Dirichlet form restricts to this space. It has the expression

$$\mathcal{E}^\gamma(u) = \mathcal{E}^\gamma_+(u \circ \psi_+) + \mathcal{E}^\gamma_-(u \circ \psi_-), \qquad u \in H^{1/2}(\eta, \ell).$$

To this Dirichlet form is associated an operator N^γ_η, such that

$$\mathcal{E}^\gamma(u) = \langle u, N^\gamma_\eta u \rangle_{L^2(\eta,\ell)}.$$

which we call the *Neumann jump operator* across η (with respect to the measure ℓ). It is the generator of a Markov process on η, the *Liouville-Cauchy process on SLE.*

4.1.4 Outline

The remainder of this article is devoted to the proof of Theorem 4.1.1 and related results on the heat kernel of the LCP. The proof of convergence essentially boils down to the construction of a multiplicative chaos with respect to the occupation measure of the Cauchy process. Since the critical case is the hardest, we will only prove convergence for this case and state the corresponding results for the subcritical case.

In Section 4.2, we gather some preliminary results on the Cauchy process, log-correlated fields and (critical) Gaussian multiplicative chaos. In the study of critical multiplicative chaos, there is a competition between the "maximum" of the field and the energetic properties of the reference measure. The former is controlled by a result of Madaule (Lemma 4.2.3), while the latter is the content of Lemma 4.2.1.

With these in hand, Section 4.3.1 shows the convergence of the critical PCAF started from a given fixed point, and that this PCAF is a self-homeomorphism of $\mathbb{R}_+$. While the resulting object is closely related to the one of [RV15], our proof strategy differs from time to time and we have found it beneficial to present a self-contained proof. The main difference lies in our definition of good events, which simplifies some computations (see Remark 19).

From here, one can a.s. define the PCAF for a countable dense collection of starting points, and a coupling argument (analogous to [GRV16, RV15]) yields the notion of continuity that is needed to extend the PCAF to quasi-all starting points (Section 4.3.2). The short Section 4.3.3 adapts these results to the subcritical PCAF, with the only difference that it is a PCAF in the strict sense in this case. Lastly, Section 4.3.4 exploits the properties of the PCAF and the results of [FOT11] to define and establish some properties of the LCP. This part is standard and similar to [GRV16, GRV14, RV15].

4.2 Preliminaries

We start by collecting a few preliminary results on Cauchy processes, log-correlated fields and Gaussian multiplicative chaos. Thanks to conformal invariance, the choice of domain is not important and we will work with either the unit disc or the upper-half plane depending on which representation is simpler.

4.2.1 The space $H^{1/2}(\mathbb{R}, \mathrm{d}x)$

In this short subsection, we explain why the space $H^{1/2}(\mathbb{R}, \mathrm{d}x)$ is relevant in the context of boundary Liouville theory and state a few of its properties, mirroring the analytic structure of $H^1(\mathbb{H}, \mathrm{d}z)$. It is not essential for the rest of the article and can be skipped on a first read.

The Sobolev space $H^1(\mathbb{H}, \mathrm{d}z)$ is the domain of the Dirichlet form $\mathcal{E}_\nabla$ on $L^2(\bar{\mathbb{H}}, \mathrm{d}z)$ defined by

$$\mathcal{E}_\nabla(f, g) = \frac{1}{2\pi}\int_{\mathbb{H}} \nabla f \cdot \nabla g \mathrm{d}z.$$

Green's function for the Laplacian with Neumann boundary conditions on $\mathbb{R} = \partial\mathbb{H}$ has a kernel given by

$$(-\Delta)^{-1}(z_1, z_2) = \frac{1}{2\pi}\log\frac{1}{|z_1 - z_2|} + \frac{1}{2\pi}\log\frac{1}{|z_1 - \bar{z}_2|}.$$

The associated heat kernel is the usual Gaussian heat kernel

$$\mathrm{p}_t^{\mathbb{H}}(z_1, z_2) = \frac{1}{2\pi t}\left(e^{-\frac{1}{2t}|z_1-z_2|^2} + e^{-\frac{1}{2t}|z_1-\bar{z}_2|^2}\right), \tag{4.10}$$

which is the fundamental solution to the heat equation

$$\partial_t p_t(z) = \frac{1}{2}\Delta p_t(z), \qquad p_0 = \delta_{z_1}.$$

The trace space of $H^1(\mathbb{H}, \mathrm{d}z)$ is the Sobolev space $H^{1/2}(\mathbb{R}, \mathrm{d}x)$, which is the domain of the Dirichlet form

$$\mathcal{E}(u, v) = \mathcal{E}_\nabla(\mathbf{P}u, \mathbf{P}v) = \frac{1}{2\pi}\langle u, \mathbf{D}v\rangle_{L^2(\mathbb{R}, \mathrm{d}x)}, \tag{4.11}$$

where $\mathbf{D} = -\partial_y \circ \mathbf{P} : H^{1/2}(\mathbb{R}, \mathrm{d}x) \to H^{-1/2}(\mathbb{R}, \mathrm{d}x)$ is the *Dirichlet-to-Neumann* (DtN) operator. In words, the DtN operator associates to a boundary function the normal derivative of its harmonic extension. The last equality in (4.11) is interpreted as the pairing of a distribution with its test function and is a simple application of Green's formula. By conformal invariance of the Dirichlet energy $\mathcal{E}_\nabla$ and of the harmonic extension, the Dirichlet form $\mathcal{E}$ is also conformally invariant. This justifies the fact that we can choose $\mathbb{H}$ as a reference domain without loss of generality.

Green's function for $\mathbf{D}$ is obtained by restricting Green's function for the Laplacian on the real line:

$$\mathbf{D}^{-1}(x_1, x_2) = \mathbf{D}^{-1}(x_1 - x_2) = \frac{1}{\pi} \log \frac{1}{|x_1 - x_2|}.$$

Indeed, take $u \in \mathcal{C}_c^\infty(\mathbb{R})$ and set $\phi(z) := \frac{1}{\pi} \int_{\mathbb{R}} \log \frac{1}{|z-t|} u(t) \mathrm{d}t$. Then ϕ is harmonic in $\mathbb{H}$ and $-\partial_y \phi(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{t}{|z-t|^2} u(t) \mathrm{d}t$. In other words, the normal derivative of Green's function for the Laplacian is the Poisson kernel, so that $-\partial_y \phi$ is harmonic in $\mathbb{H}$ with boundary value given by u. Thus, $\mathbf{D}\phi = u$.

Finally, the heat kernel for $\mathbf{D}$ is the Poisson kernel

$$\mathrm{p}_t(x_1, x_2) = \mathrm{p}_t(x_1 - x_2) = \frac{1}{\pi} \frac{t}{(x_1 - x_2)^2 + t^2}.$$

This is easily seen by noting that for fixed $t > 0$, the harmonic extension of the function $x \mapsto \mathrm{p}_t(x)$ is $(\mathbf{P}\mathrm{p}_t)(x + iy) = \mathrm{p}_{t+y}(x)$. Thus, $\partial_t \mathrm{p}_t(x) = \partial_y \mathbf{P}\mathrm{p}_t(x)|_{y=0}$, i.e. $\partial_t \mathrm{p}_t(x) = -\mathbf{D}\mathrm{p}_t(x)$. Note the explicit formula, for all $x \in \mathbb{R}$ and $0 < s < t$:

$$\int_s^t \mathrm{p}_u(x) \mathrm{d}u = \frac{1}{2\pi} \log \left(\frac{t^2 + x^2}{s^2 + x^2} \right).$$

Green's function is recovered as usual by integrating the heat kernel with respect to time (we introduce a cut-off due to some non-compactness introduced by the zero eigenvalue):

$$\mathbf{D}^{-1}(x) = \frac{1}{\pi} \log \frac{1}{|x|} = \int_0^1 \mathrm{p}_t(x) \mathrm{d}t - \frac{1}{2\pi} \log(1 + x^2).$$

4.2.2 The Cauchy process on the real line

Recall from Section 4.1.2.1 that the Cauchy process on the circle has two interpretations. On the one hand, it is the Markov process based on the Dirichlet space. On the other hand, it is the trace of reflecting Brownian motion in $\mathbb{D}$ reparameterised by its local time on $\partial\mathbb{D}$. It is a pure jump, 1-stable Lévy process $(C_t)_{t \geq 0}$ with càdlàg sample paths. For a given time $t > 0$, the left limit will be denoted C_{t^-}. From now on, we will map the process to the real line, where the transition rates are given by the Poisson kernel

$$\mathrm{p}_t(x) = \frac{1}{\pi}\frac{t}{t^2+x^2}.$$

Lemma 4.2.1 (Energy of the occupation measure)**.** *Let λ_t be the occupation measure of the Cauchy process up to time $t > 0$. Almost surely, for all $p > 2$, we have*

$$\int_{\mathbb{R}^2} \frac{\mathrm{d}\lambda_t(x)\mathrm{d}\lambda_t(y)}{|x-y|\log^p(1+\frac{1}{|x-y|})} < \infty.$$

Proof. By scaling, it suffices to treat the case $t = 1$. Using the expression of the Cauchy distribution (Poisson kernel), we have for all $s \leq 1$:

$$\mathbb{E}\left[\frac{1}{|C_s|\log^p(1+\frac{1}{|C_s|})}\right] = \frac{2}{\pi}\int_0^\infty \frac{1}{x\log^p(1+\frac{1}{x})}\frac{s}{s^2+x^2}\mathrm{d}x$$

Let us split this integral in three terms. First, using $\frac{s}{s^2+x^2} \leq \frac{1}{s}$ and an explicit integration, we have

$$\int_0^{s^{1/4}} \frac{1}{x\log^p(1+\frac{1}{x})}\frac{s}{s^2+x^2}\mathrm{d}x \leq \frac{1}{s}\int_0^{s^{1/4}} \frac{\mathrm{d}x}{x\log^p(1+\frac{1}{x})} \leq \frac{c_1}{s\log^{p-1}(1+\frac{1}{s})}.$$

Second, using $\frac{s}{s^2+x^2} \leq \sqrt{s}$ for $x \geq s^{1/4}$, we have

$$\int_{s^{1/4}}^1 \frac{1}{x\log^p(1+\frac{1}{x})}\frac{s}{s^2+x^2}\mathrm{d}x \leq \sqrt{s}\int_0^1 \frac{\mathrm{d}x}{x\log^p(1+\frac{1}{x})} =: c_2\sqrt{s}.$$

Third, using $\frac{s}{s^2+x^2} \leq \frac{s}{x^2}$ for $x \geq s$, we have

$$\int_1^\infty \frac{1}{x\log^p(1+\frac{1}{x})}\frac{s}{s^2+x^2}\mathrm{d}x \leq s\int_1^\infty \frac{1}{x^3\log^p(1+\frac{1}{x})} =: c_3 s.$$

Thus, using the fact that the Cauchy process is 1-stable,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}^2} \frac{\mathrm{d}\lambda_1(x)\mathrm{d}\lambda_1(y)}{|x-y|\log^p(1+\frac{1}{|x-y|})}\right] \\
&\quad = \int_0^1\int_0^1 \mathbb{E}\left[\frac{1}{|C_u-C_v|\log^p(1+\frac{1}{|C_u-C_v|})}\right]\mathrm{d}u\mathrm{d}v \\
&\quad \leq \int_0^1\int_0^1 \left(\frac{c_1}{|u-v|\log^{p-1}(1+\frac{1}{|u-v|})} + c_2|u-v|^{\frac{1}{2}} + c_3|u-v|\right)\mathrm{d}u\mathrm{d}v < \infty.
\end{aligned}$$

$\square$

Let $\mathbb{P}_x$ be the law of the Cauchy process started from $x \in \mathbb{R}$. Following the standard terminology, we say that $\pi_{x,y}$ is a *successful coupling* between $\mathbb{P}_x$ and $\mathbb{P}_y$ if there exists a $\pi_{x,y}$-a.s. finite time τ such that the marginal paths of $\pi_{x,y}$ coincide after time τ. In this case, τ is called the *coupling time*. We note that the existence of such a coupling follows from general results [SW11]), but we propose an explicit one below, based on Spitzer's embedding.

Lemma 4.2.2 (Coupling of Cauchy processes)**.** *Let $x, y \in \mathbb{R}$ with $x < y$. There exists a coupling $\pi_{x,y}$ between $\mathbb{P}_x$ and $\mathbb{P}_y$ such:*

(i) $\pi_{x,y}$-almost surely, the sample paths C^x and C^y have the same jump times.

(ii) The coupling time τ is stochastically dominated by $\inf\{t \geq 0,\, \widetilde{C}^x_t > \widetilde{C}^y_t\}$, where $\widetilde{C}^x$ and $\widetilde{C}^y$ are independent Cauchy processes started from x and y respectively.

(iii) For each $\varepsilon > 0$, $\pi_{x,y}(\tau \geq \varepsilon) \to 0$ as $|x - y| \to 0$.

Proof. Consider three real Brownian motions B^x, B^y, B^0 started from $x, y, 0$ respectively and coupled as follows. The Brownian motions B^x ans B^y evolve independently before time $T = \inf\{t \geq 0,\, B^x_t = B^y_t\}$ and they are successfully coupled at time T (i.e. $B^x_t = B^y_t$ for all $t \geq T$). Moreover, B^0 is independent of B^x and B^y. The processes $\mathcal{B}^x := B^x + i|B^0|$ and $\mathcal{B}^y := B^y + i|B^0|$ are marginally reflected Brownian motions in the upper-half plane, so we can use them to define Cauchy processes C^x and C^y by Spitzer's embedding. Items (i) and (ii) are then clear by construction. Item (iii) follows by scaling, using the fact that the Cauchy process is a one-stable Lévy process.

□

4.2.3 The trace of the Gaussian free field

The Gaussian free field (GFF) with free boundary conditions in the upper-half plane $\mathbb{H}$ is the Gaussian field X whose covariance kernel is the resolvent of the Laplacian with Neumann boundary conditions on $\partial\mathbb{H} = \mathbb{R}$:

$$\mathbb{E}[X(z_1)X(z_2)] = 2\pi(-\Delta)^{-1}(z_1, z_2) = \log \frac{1}{|z_1 - z_2|} + \log \frac{1}{|z_1 - \bar{z}_2|}, \qquad z_1, z_2 \in \mathbb{H}.$$

Otherwise stated, X is the isonormal Gaussian process based on the Sobolev space $H^1(\mathbb{H}, \mathrm{d}z)$. The GFF has a trace on $\mathbb{R}$, denoted W, and its covariance kernel is given by the restriction of $2\pi(-\Delta^{-1})$ on $\mathbb{R}$. That is, the covariance kernel of W is the Green function of the DtN operator, and W is also the isonormal Gaussian process on $H^{1/2}(\mathbb{R}, \mathrm{d}x)$:

$$\mathbb{E}[W(x)W(y)] = 2\pi\mathbf{D}^{-1}(x, y) = 2\log \frac{1}{|x - y|}, \qquad x, y \in \mathbb{R}. \tag{4.12}$$

Since W is logarithmically correlated on the diagonal, it is not defined pointwise but rather lives $H^{-s}(\mathbb{S}^1, \mathrm{d}\theta)$ for all $s > 0$. We will consider the so-called *white-noise regularisation* of W. Let ξ be a space-time white-noise on $\mathbb{R}_x \times \mathbb{R}_t$ and for all $\varepsilon \in (0,1)$, define the field

$$W_\varepsilon(x) := \sqrt{2\pi} \int_{\mathbb{R}} \int_\varepsilon^1 \mathrm{p}_{s/2}(x-y)\mathrm{d}\xi(y,s). \tag{4.13}$$

Then, one checks that for all $0 < \varepsilon, \delta < 1$,

$$\mathbb{E}[W_\varepsilon(x)W_\delta(y)] = \log \frac{1+(x-y)^2}{(\varepsilon \vee \delta)^2 + (x-y)^2},$$

so that in the limit $\varepsilon \to 0$, $\mathbb{E}[W_0(x)W_0(y)] = \log(1 + \frac{1}{(x-y)^2})$. We remark that this is not exactly the covariance kernel (4.12), but this is not important since one can use e.g. the result of [JSW19] to compare the two. From now on, we will write W for the field with covariance kernel $\mathbb{E}[W(x)W(y)] = \log(1 + \frac{1}{(x-y)^2})$. We will not be considering the bulk field X in the sequel, but we mention that a white-noise regularisation of this field can also be given using the heat kernel $\mathrm{p}_t^{\mathbb{H}}$ from (4.10).

Remark 16. We put the prefactor $\sqrt{2\pi}$ in (4.13) in order to obtain the covariance kernel (4.12) (with the prefactor 2). This choice is made in order to have the range $\gamma \in (0,2]$ and keep explicit the connection with the boundary version of Liouville theory.

We will assume that W is defined on some probability space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ fixed once and for all. The white-noise regularisation provides a natural filtration $\mathcal{F}_W^\varepsilon := \sigma(W_\delta,\ \varepsilon \leq \delta \leq 1)$ for which $(W_\varepsilon)_{1 \geq \varepsilon > 0}$ is a martingale. Note that $\mathbb{E}_W[W_\varepsilon(x)^2] = 2\log\frac{1}{\varepsilon}$ for all $x \in \mathbb{R}$ and $\varepsilon \in (0,1)$, so that $(\frac{1}{\sqrt{2}}W_{e^{-s}}(x))_{s \geq 0}$ has the law of a standard Brownian motion, and we will write $B_s(x) = \frac{1}{\sqrt{2}}W_{e^{-s}(x)}$. Given $x, y \in \mathbb{R}$, the pair $(B_s(x), B_s(y))_{s \geq 0}$ exhibits the following branching structure: for $s \leq s_0 := \log\frac{1}{|x-y|}$, the Brownian motions are strongly correlated and roughly the same, whereas they are weakly correlated for $s \geq s_0$ and evolve roughly independently. More precisely, we have the correlation structure

$$\mathbb{E}[(B_s(x) - B_{s_0}(x))(B_s(y) - B_{s_0}(y))] = \pi \int_s^{s_0} \mathrm{p}_u(x-y)\mathrm{d}u = \frac{1}{2}\log\frac{2s_0^2}{s^2+s_0^2}, \tag{4.14}$$

which is uniformly bounded in both s and s_0.

Remark 17. There are other commonly used regularisations of (4.12), such as semi-circle averages or convolution approximations. In this paper, we will exclusively work with the white-noise regularisation since it yields almost sure convergence of the multiplicative chaos measures thanks to the martingale structure (other setups typically yield convergence in probability).

We will need the following result on the maximum of log-correlated fields, which follows from [Mad15, Theorem 1.1].

Lemma 4.2.3 (Maximum of log-correlated fields)**.** *For all bounded open intervals $I \subset \mathbb{R}$ and $a \in [0, \frac{3}{2})$, $\mathbb{P}_W$-almost surely:*

$$\sup_{s \geq 0} \sup_{x \in I} W_{e^{-s}}(x) - 2s + a \log(1+s) < \infty.$$

4.2.4 Critical Gaussian multiplicative chaos

Let us introduce some notation and terminology. A *gauge function* is a non-decreasing function $f : [0,1) \to \mathbb{R}_+$ satisfying $f(0) = 0$. Given such a function, the f-Hausdorff measure of a Borel set $E \subset \mathbb{R}$ is

$$\mathcal{H}_f(E) := \lim_{\delta \to 0} \inf \sum_i f(|I_i|),$$

where the infimum runs over all coverings $(I_i)_i$ of $E \subset \mathbb{R}$ by open intervals of length less than δ. If $f(t) = t^\alpha$ for some $\alpha \in (0,1]$, we will abuse notation by writing $\mathcal{H}_\alpha = \mathcal{H}_f$. The *Hausdorff dimension* of E is

$$\dim E = \inf\{\alpha > 0,\, \mathcal{H}_\alpha(E) = 0\} = \sup\{\alpha > 0,\, \mathcal{H}_\alpha(E) = \infty\}.$$

Let $\mathcal{I} := [0,1]$. For each $n \in \mathbb{N}$, let $\mathcal{D}_n := \{[k2^{-n}, (k+1)2^{-n}),\, 0 \leq k < 2^n\}$ be the collection of dyadic intervals of length 2^{-n}. Given a gauge function f, the *f-energy* of a measure λ on $\mathcal{I}$ is

$$I_f(\lambda) := \int_{\mathcal{I}^2} \frac{\mathrm{d}\lambda(x)\mathrm{d}\lambda(y)}{f(|x-y|)}.$$

We also write $I_\alpha = I_f$ if $f(t) = t^\alpha$ and call it the α-energy of λ. Frostman's lemma states that $\dim E \geq \alpha$ if and only if there exists a Borel probability measure λ giving full mass to E and with finite α'-energy for all $\alpha' < \alpha$. Similarly, a subset $E \subset \mathbb{S}^1$ is not polar (in $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta)$) if and only if it bears a Borel probability measure with finite log-energy [Bis07, Section 2], where the log-energy is defined with respect to the gauge function $f(t) = \log^{-1}(1 + \frac{1}{t})$.

Let ν be the critical GMC measure on $\mathbb{S}^1$, normalised to be a probability measure. We view ν as a measure on $\mathcal{I}$ in the standard way. Given a gauge function f, we let E_n^f be the union of those $I \subset \mathcal{D}_n$ such that $\nu(I) \geq f(|I|)$ and

$$E^f := \limsup_{n \to \infty} E_n^f.$$

The critical measure ν has a rather wild behaviour. In [BKN+15], it is shown that a.s. for the gauge function $g_\alpha(t) := \exp(-\log^\alpha \frac{1}{t})$, we have $\nu(E^{g_\alpha}) = 1$ for all $\alpha > \frac{1}{2}$. Moreover, $\dim E^{g_\alpha} = 0$ for all $\alpha \in (0,1)$, so ν gives full mass to a set of Hausdorff dimension zero. [BKN+15] also gives a bound on the modulus of continuity of ν, implying that it does not have any atoms. Here and in the sequel, we define the gauge function $\mathbf{f}_k$, for all $k > 0$:

$$\mathbf{f}_k(t) := \log^{-k}\left(1 + \frac{1}{t}\right). \tag{4.15}$$

Then, if $k < \frac{1}{2}$, almost surely, there exists $c > 0$ such that $\nu(I) \leq c\mathbf{f}_k(|I|)$ for all intervals $I \subset \mathcal{I}$. Moreover, the bound $k < \frac{1}{2}$ is the best possible.

We conclude with some important potential theoretic properties of ν. It is known that a.s. ν has finite log-energy:

$$\int_{\mathcal{I}^2} \frac{\mathrm{d}\nu(x)\mathrm{d}\nu(y)}{\mathbf{f}_1(|x-y|)} < \infty.$$

In particular, ν is Revuz and can be associated a PCAF by the Revuz correspondence. On the other hand, almost surely

$$\sup_{x\in\mathcal{I}} \int_{\mathcal{I}} \frac{\mathrm{d}\nu(y)}{\mathbf{f}_1(|x-y|)} = \infty.$$

In other words, the mapping $x \mapsto \int_{\mathcal{I}} \frac{\mathrm{d}\nu(y)}{\mathbf{f}_1(|x-y|)}$ belongs to $L^1(\mathcal{I},\nu)$ but not to $L^\infty(\mathcal{I},\nu)$. In particular, it is not clear that the PCAF of ν is in the strict sense. The next lemma identifies a polar set that will be used as an exceptional set for the PCAF in Section 4.3.2.

Lemma 4.2.4 (Exceptional set). *Almost surely, for all $k < \frac{3}{2}$, $E^{\mathbf{f}_k}$ is polar.*

Proof. Define a homeomorphism of $\mathcal{I}$ by $h(x) := \nu[0,x]$. Let λ be a Borel probability measure giving full mass to $E^{\mathbf{f}_k}$. Then the pushforward measure $h_*\lambda$ gives full mass to $h(E^{\mathbf{f}_k})$, since the modulus of continuity of h is bounded from below by $\mathbf{f}_k$ on $E^{\mathbf{f}_k}$, we have by a change of variables:

$$\begin{aligned}
I_{\mathbf{f}_1}(\lambda) = \int_{\mathcal{I}^2} \log\left(1 + \frac{1}{|x-y|}\right) \mathrm{d}\lambda(x)\mathrm{d}\lambda(y) &= \int_{\mathcal{I}^2} \log\left(1 + \frac{1}{|h(x)-h(y)|}\right) \mathrm{d}h_*\lambda(x)\mathrm{d}h_*\lambda(y) \\
&\geq c\int_{\mathcal{I}^2} \log\left(1 + \frac{1}{\mathbf{f}_k^{-1}(|x-y|)}\right) \mathrm{d}h_*\lambda(x)\mathrm{d}h_*\lambda(y) \\
&= c\int_{\mathcal{I}^2} \frac{\mathrm{d}h_*\lambda(x)\mathrm{d}h_*\lambda(y)}{|x-y|^{1/k}} = c\, I_{1/k}(h_*\lambda).
\end{aligned}$$

That is, the log-energy of λ is bounded from below by the $\frac{1}{k}$-energy of $h_*\lambda$. On the other hand, it is shown in [Bav20, Lemma 3.1] that $\dim h(E^{\mathbf{f}_k}) \leq 1 - \frac{1}{2k}$ for all $k > \frac{1}{2}$. Thus, by Frostman's lemma, any such λ has infinite log-energy as soon as $\frac{1}{k} > 1 - \frac{1}{2k}$, i.e. $k < \frac{3}{2}$. $\square$

4.3 Convergence of the regularised PCAF

In this section, we construct the γ-LCP of Definition 4.1.1 for all $\gamma \in (0,2]$. In fact, the critical case $\gamma = 2$ contains all the difficulty and the subcritical case follows immediately, so we will only prove the critical case and state the corresponding results for $\gamma < 2$ in Section 4.3.3.

In Section 4.3.1, we address the construction of the critical LCP starting from 0, and extend the definition to all $x \in \mathbb{R}$ simultaneously in Section 4.3.2. In particular, this will show that the critical boundary GMC measure on $\mathbb{R}$ has full quasi-support. Mathematically, we need to construct the critical GMC with respect the occupation measure of the Cauchy process. Although there is a systematic way of defining GMC with respect to "non standard" measures in the subcritical regime [Ber17], the critical regime requires more care and the energetic properties of the measure (here, Lemma 4.2.1) play a key part.

4.3.1 The critical PCAF from a fixed starting point

Recall the field W from Section 4.2.3 and its white-noise approximation $(W_\varepsilon)_{0<\varepsilon\leq 1}$ defined on $(\Omega_W, (\mathcal{F}_W^\varepsilon)_{\varepsilon>0}, \mathbb{P}_W)$. Throughout this subsection, $C = (C_t)_{t\geq 0}$ denotes a Cauchy process on $\mathbb{R}$ started from 0, defined on a filtered probability space $(\Omega_C, (\mathcal{F}_C^t)_{t\geq 0}, \mathbb{P}_C)$ and independent of W. We will write $\mathbb{P} := \mathbb{P}_W \otimes \mathbb{P}_C$ for the product measure, i.e. the law of the independent pair (W, C). We also let λ_t be the occupation measure of $(C_s)_{0\leq s\leq t}$. We define the random functionals[2]

$$
\begin{aligned}
F_t^\varepsilon &:= \int_0^t \varepsilon e^{W_\varepsilon(C_s)} \mathrm{d}s, \\
\dot{F}_t^\varepsilon &:= \int_0^t \left(2\log\frac{1}{\varepsilon} - W_\varepsilon(C_s)\right) \varepsilon e^{W_\varepsilon(C_s)} \mathrm{d}s.
\end{aligned}
\tag{4.16}
$$

Remark 18. The dotted notation $\dot{F}_t^\varepsilon$ indicates that we are using the derivative martingale normalisation, in contrast with the "non-derivative" functional F_t^ε which will be shown to converge to 0. We will also use the dot as a subscript to denote the time variable, e.g. $\dot{F}_\cdot^\varepsilon$ denotes the whole function $t \mapsto \dot{F}_t^\varepsilon$.

4.3.1.1 Derivative renormalisation

The goal of this section is to show that $\mathbb{P}$-almost surely, the family of functions $(t \mapsto \dot{F}_t^\varepsilon)$ converges as $\varepsilon \to 0$ and study the qualitative properties of the limiting function. This is the content of the next theorem.

Theorem 4.3.1 (Derivative renormalisation)**.** *$\mathbb{P}$-almost surely, the family $(\dot{F}_\cdot^\varepsilon)_{\varepsilon>0}$ converges as $\varepsilon \to 0$, uniformly on compacts of $\mathbb{R}_+$, to a self-homeomorphism of $\mathbb{R}_+$, denoted $\dot{F}_\cdot$.*

[2] For cosmetic reasons, the definition of $\dot{F}^\varepsilon$ differs from (4.7) by a factor 2.

For each $\beta > 0$ and $a \in [0, \frac{3}{2})$, let us introduce the auxiliary functionals

$$F_t^{\varepsilon,\beta,a} := \int_0^t f_\varepsilon^{\beta,a}(C_s)\mathrm{d}s$$
$$\dot{F}_t^{\varepsilon,\beta,a} := \int_0^t \dot{f}_\varepsilon^{\beta,a}(C_s)\mathrm{d}s,$$

where we have set

$$f_\varepsilon^{\beta,a}(x) := \varepsilon e^{W_\varepsilon(x)}\mathbb{1}_{\{\tau_x^{\beta,a} > \log\frac{1}{\varepsilon}\}}$$
$$\dot{f}_\varepsilon^{\beta,a}(x) := \left(2\log\frac{1}{\varepsilon} - W_\varepsilon(x) + \beta\right)\varepsilon e^{W_\varepsilon(x)}\mathbb{1}_{\{\tau_x^{\beta,a} > \log\frac{1}{\varepsilon}\}}$$
$$\tau_x^{\beta,a} := \inf\left\{s \geq 0,\, W_{e^{-s}}(x) - 2s + a\log(1+s) > \beta\right\}.$$

In order to prove Theorem 4.3.1, we will start by showing that $\dot{F}_t^{\varepsilon,\beta,a}$ satisfies the conclusions of the theorem for each fixed β. This will be broken down into several steps. First, Propositions 4.3.5 and 4.3.6 take care of the L^2-boundedness and convergence of $\dot{F}_t^{\varepsilon,\beta,a}$. Second, we investigate the properties of the limiting function $t \mapsto \dot{F}_t^{\beta,a}$, which follow from the properties of the critical chaos. Continuity is shown in Proposition 4.3.9, strict monotonicity in Proposition 4.3.10, and the limit $\dot{F}_t^{\beta,a} \to \infty$ in Proposition 4.3.11. Finally, standard arguments based on Lemma 4.2.3 will allow us to get Theorem 4.3.1 by taking β sufficiently large.

We start by recording the following useful property, which is a simple variation of [DRSV14a, Proposition 13].

Lemma 4.3.2. *Let $(B_s)_{s\geq 0}$ be a standard Brownian motion and for each $\beta, \alpha, a \geq 0$, define*

$$\tau_\alpha^{\beta,a} := \inf\{s \geq 0,\, \beta - \sqrt{2}(B_s - \alpha s) - a\log(1+s) < 0\}.$$

The process $M_s := e^{\sqrt{2}B_s - s}(\beta - \sqrt{2}B_s + 2s)\mathbb{1}_{\{\tau_{\sqrt{2}}^{\beta,a} > s\}}$ is a non-negative supermartingale. It is a martingale if $a = 0$.

Proof. Here we denote by $\mathbb{P}^0$ the law of standard Brownian motion. Let $s_0 \geq 0$ and put $\mathcal{F}_{s_0} := \sigma(B_s,\, s \leq s_0)$. For $s \geq s_0$, we write $\widetilde{\tau}_\alpha^{\beta,a} = \inf\{s \geq 0,\, \beta - \sqrt{2}(B_{s_0+s} - \alpha(s_0+s)) - a\log(1+s_0+s) < 0\}$. Finally, let $\widetilde{B}_s := B_{s_0+s} - B_{s_0}$, which has the law of a standard

Brownian motion. Then we have by Girsanov's theorem and optional stopping:

$$\begin{aligned}\mathbb{E}^0\left[M_s|\mathcal{F}_{s_0}\right] &= e^{\sqrt{2}B_{s_0}-s_0}\mathbb{1}_{\{\tau^{\beta,a}_{\sqrt{2}}>s_0\}}\mathbb{E}^0\left[e^{\sqrt{2}\widetilde{B}_{s-s_0}-(s-s_0)}\left(\beta-\sqrt{2}B_s+2s\right)\mathbb{1}_{\{\widetilde{\tau}^{\beta,a}_{\sqrt{2}}>s-s_0\}}\middle|\mathcal{F}_{s_0}\right]\\ &\le e^{\sqrt{2}B_{s_0}-s_0}\mathbb{1}_{\{\tau^{\beta,a}_{\sqrt{2}}>s_0\}}\mathbb{E}^0\left[\left(\beta-\sqrt{2}B_{s_0}+2s_0-\sqrt{2}\widetilde{B}_{s-s_0}\right)\mathbb{1}_{\{\widetilde{\tau}^{\beta,0}_0>s-s_0\}}\middle|\mathcal{F}_{s_0}\right]\\ &= e^{\sqrt{2}B_{s_0}-s_0}\mathbb{1}_{\{\tau^{\beta,a}_{\sqrt{2}}>s_0\}}\mathbb{E}^0\left[\left(\beta-\sqrt{2}B_{s_0}+2s_0-\sqrt{2}\widetilde{B}_{(s-s_0)\wedge\tau^{\beta,0}_0}\right)\middle|\mathcal{F}_{s_0}\right]\\ &= M_{s_0}.\end{aligned}$$

Thus, M is a supermartingale. The only inequality in the previous equation is an equality if $a=0$, so M is a martingale in this case. □

Writing $\dot{F}^{\varepsilon,\beta,a}_t = \int_{\mathbb{R}}\dot{f}^{\beta,a}_{\varepsilon}(x)\mathrm{d}\lambda_t(x)$ and noting that $(\dot{f}^{\beta,a}_{e^{-s}}(x))_{s\ge0} \overset{\text{law}}{=} (M_s)_{s\ge0}$ for all $x\in\mathbb{R}$ (with M_s as in the previous lemma), it follows that $(\dot{F}^{\beta,a}_{\varepsilon})_{\varepsilon>0}$ is $\mathbb{P}_C$-almost surely a non-negative supermartingale. An immediate corollary is the following.

Proposition 4.3.3. *For all $t,\beta>0$ and $a\in[0,\frac{3}{2})$, the family $(\dot{F}^{\varepsilon,\beta,a}_t)_{\varepsilon>0}$ converges $\mathbb{P}$-almost surely as $\varepsilon\to0$ to a random variable denoted $\dot{F}^{\beta,a}_t$.*

To show that the limit is non-trivial, we will prove that $(\dot{F}^{\varepsilon,\beta,a}_t)_{\varepsilon>0}$ is L^2-bounded for all $\beta>0, a\in(0,\frac{3}{2})$, and eventually that it converges in L^2. First, we take a small detour and show that the "non-derivative" renormalisation converges almost surely to zero.

Lemma 4.3.4. *For all $t,\beta>0$ and $a\in[0,\frac{3}{2})$, $\mathbb{P}$-almost surely, $F^{\varepsilon,\beta,a}_t\to0$ as $\varepsilon\to0$.*

Proof. Recall that $(W_{e^{-s}}(x))_{s\ge0} \overset{\text{law}}{=} (\sqrt{2}B_s)_{s\ge0}$, where $(B_s)_{s\ge0}$ is a standard Brownian motion, whose law we denote by $\mathbb{P}^0$. It is easy to adapt the proof of Lemma 4.3.2 to see that $(F^{\varepsilon,\beta,a}_t(x))_{\varepsilon>0}$ is $\mathbb{P}_C$-almost surely a non-negative supermartingale, hence that it converges $\mathbb{P}$-almost surely as $\varepsilon\to0$. It remains to show that this limit is 0.

Note that $\{\tau^{\beta,a}_x>t\}\subset\{\tau^{\beta,0}_x>t\}$, so that by Girsanov's theorem

$$\mathbb{E}_W\left[f^{\beta,a}_{\varepsilon}(x)\right]\le\mathbb{E}_W\left[f^{\beta,0}_{\varepsilon}(x)\right]=\mathbb{P}^0\left(\sup_{0\le s\le\log\frac{1}{\varepsilon}}B_s<\frac{\beta}{\sqrt{2}}\right)\underset{\varepsilon\to0}{\sim}\frac{\beta}{\sqrt{\pi\log\frac{1}{\varepsilon}}}.$$

Thus, by definition of the occupation measure, we have

$$\mathbb{E}_W\left[F^{\varepsilon,\beta,a}_t\right]=\int_{\mathbb{R}}\mathbb{E}_W\left[f^{\beta,a}_{\varepsilon}(x)\right]\mathrm{d}\lambda_t(x)\underset{\varepsilon\to0}{\lesssim}\left(\log\frac{1}{\varepsilon}\right)^{-1/2}\underset{\varepsilon\to0}{\to}0.$$

This completes the proof. □

We go back to $\dot{F}^{\varepsilon,\beta,a}_t$ and show convergence in L^2.

Proposition 4.3.5. *For all $t,\beta>0$ and $a\in(0,\frac{3}{2})$, $\mathbb{P}_C$-almost surely, the family $(\dot{F}^{\varepsilon,\beta,a}_t)_{\varepsilon>0}$ is bounded in $L^2(\Omega_W,\mathcal{F}_W,\mathbb{P}_W)$.*

Proof. By definition of the occupation measure λ_t, we have

$$\mathbb{E}_W\left[\left(\dot{F}_t^{\varepsilon,\beta,a}\right)^2\right] = \mathbb{E}_W\left[\left(\int_{\mathbb{R}} \dot{f}_\varepsilon^{\beta,a}(x)\mathrm{d}\lambda_t(x)\right)^2\right] = \int_{\mathbb{R}^2} \mathbb{E}_W\left[\dot{f}_\varepsilon^{\beta,a}(x)\dot{f}_\varepsilon^{\beta,a}(y)\right]\mathrm{d}\lambda_t(x)\mathrm{d}\lambda_t(y).$$

Let us now change variables: we write $s := \log\frac{1}{\varepsilon}$ and relabel $\dot{f}_s^{\beta,a}(x) = \dot{f}_\varepsilon^{\beta,a}(x)$. Recall that $B_s(x) = \frac{1}{\sqrt{2}}W_{e^{-s}}(x)$ is a Brownian motion for all $x \in \mathbb{R}$. For the rest of this proof, we let $\mathbb{P}^0$ be the law of a standard Brownian motion $(B_s)_{s\geq 0}$ and $\mathbb{P}^\alpha$ be the measure under which $(B_s)_{s\geq 0}$ is a Brownian motion with drift $-\alpha$. We also write $\tau^{\beta,a} := \inf\{s \geq 0,\ \sqrt{2}B_s + a\log(1+s) > \beta\}$ and denote $(\mathcal{B}_s^\beta)_{s\geq 0}$ a $3d$-Bessel process started from $\frac{\beta}{\sqrt{2}}$ under $\mathbb{P}^0$. It is a standard fact that the Brownian motion $(B_u)_{0\leq u\leq s}$ conditioned on the event $\{\tau^{\beta,0} > s\}$ is distributed like $(\frac{\beta}{\sqrt{2}} - \mathcal{B}_u^\beta)_{0\leq u\leq s}$.

Suppose first that $s \leq s_0 := \log\frac{1}{|x-y|}$. Using Cauchy-Schwarz and the fact that $\dot{f}_s^{\beta,a}(x) \overset{\text{law}}{=} \dot{f}_s^{\beta,a}(y)$, we have $\mathbb{E}_W\left[\dot{f}_s^{\beta,a}(x)\dot{f}_s^{\beta,a}(y)\right] \leq \mathbb{E}_W\left[\dot{f}_s^{\beta,a}(0)^2\right]$. Let us define the $\mathbb{P}^0$-martingale

$$Z_s := \exp\left(\frac{a}{\sqrt{2}}\int_0^s \frac{\mathrm{d}B_u}{1+u} - \frac{a^2}{4}\int_0^s \frac{\mathrm{d}u}{(1+u)^2}\right), \tag{4.17}$$

which is the Radon-Nikodym derivative of the law of $(B_u + \frac{a}{\sqrt{2}}\log(1+u))_{0\leq u\leq s}$ with respect to $(B_u)_{0\leq u\leq s}$. By Girsanov's theorem (twice), we have:

$$\begin{aligned}
\mathbb{E}_W\left[\dot{f}_s^{\beta,a}(0)^2\right] &= e^{2s}\mathbb{E}^{\sqrt{2}}\left[\left(\beta - \sqrt{2}B_s\right)^2 e^{2\sqrt{2}B_s}\mathbb{1}_{\{\tau^{\beta,a}>s\}}\right] \\
&= e^{s}\mathbb{E}^{0}\left[\left(\beta - \sqrt{2}B_s\right)^2 e^{\sqrt{2}B_s}\mathbb{1}_{\{\tau^{\beta,a}>s\}}\right] \\
&= e^{s}\mathbb{E}^{0}\left[Z_s\left(\beta - \sqrt{2}B_s + a\log(1+s)\right)^2 e^{\sqrt{2}B_s - a\log(1+s)}\mathbb{1}_{\{\tau^{\beta,0}>s\}}\right]
\end{aligned} \tag{4.18}$$

Note that we are already seeing a gain of a factor $(1+s)^{-a}$ compared to conditioning on the event $\{\tau^{\beta,0} > s\}$. Let us expand the square in the expectation and treat the three term separately. A simple computation shows that $\mathbb{E}^0[Z_s^q] = e^{\frac{a^2}{4}q(q-1)}$, so by Hölder's inequality applied with $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\mathbb{E}^0\left[Z_s(\beta - \sqrt{2}B_s)^2 e^{\sqrt{2}B_s}\mathbb{1}_{\{\tau^{\beta,0}>s\}}\right] \leq e^{\frac{a^2}{4}(q-1)}\mathbb{E}^0\left[(\sqrt{2}\mathcal{B}_s^\beta)^{2p}e^{\sqrt{2}p(\frac{\beta}{\sqrt{2}} - \mathcal{B}_s^\beta)}\right]^{1/p}\mathbb{P}^0\left(\tau^{\beta,0} > s\right)^{1/p}.$$

Now, the density function of $\mathcal{B}_s^\beta$ (with respect to Lebesgue on $\mathbb{R}_+$) is given explicitly for all $s, x > 0$ by [GJY03, Appendix A.2]

$$p_s^\beta(x) = \frac{2}{\sqrt{\pi s}}\frac{x}{\beta}e^{-\frac{1}{2s}(\frac{\beta^2}{2}+x^2)}\sinh\left(\frac{\beta x}{\sqrt{2s}}\right) \underset{s\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\frac{x^2}{s^{3/2}},$$

so that

$$\mathbb{E}^0\left[(\mathcal{B}_s^\beta)^{2p}e^{\sqrt{2}p(\frac{\beta}{\sqrt{2}}-\mathcal{B}_s^\beta)}\right]\underset{s\to\infty}{\sim} s^{-3/2}\sqrt{\frac{2}{\pi}}\int_0^\infty x^{2+2p}e^{\sqrt{2}p(\frac{\beta}{\sqrt{2}}-x)}\mathrm{d}x. \tag{4.19}$$

To justify dominated convergence, we have used the fact that the integrand decays uniformly exponentially fast as $x\to\infty$, thanks to the term $e^{-\sqrt{2}px}$ independent of s.

We can treat the other terms in (4.18) similarly, and we obtain the same decay each time up to a logarithmic factor in s. Using the elementary estimate $\mathbb{P}^0(\tau^{\beta,0}>s)\sim\frac{\beta}{\sqrt{\pi s}}$ as $s\to\infty$, our final bound for (4.18) is

$$\mathbb{E}_W\left[\dot{f}_s^{\beta,a}(0)^2\right]\lesssim\frac{e^s}{(1+s)^a}\log^2(1+s)s^{-\frac{3}{2p}}s^{-\frac{1}{2p}}.$$

Taking p arbitrarily close to 1, we obtain for all $k<2+a$:

$$\mathbb{E}_W\left[\dot{f}_s^{\beta,a}(x)\dot{f}_s^{\beta,a}(y)\right]\lesssim s^{-k}e^s. \tag{4.20}$$

Suppose now that $s\geq s_0$. Consider the modified processes $\widetilde{B}_s(x),\widetilde{B}_s(y)$ such that they coincide with $B_s(x),B_s(y)$ up to time $s=s_0$, and then the processes $\widetilde{B}_s(x)-\widetilde{B}_{s_0}(x)$ and $\widetilde{B}_s(y)-\widetilde{B}_{s_0}(y)$ are independent Brownian motions. We denote with a tilde all quantities obtained using $\widetilde{B}$ in place of B. By the martingale property, we have $\mathbb{E}_W[\widetilde{\dot{f}_s^{\beta,a}}(x)\widetilde{\dot{f}_s^{\beta,a}}(y)]=\mathbb{E}_W[\dot{f}_{s_0}^{\beta,a}(x)\dot{f}_{s_0}^{\beta,a}(y)]$. From (4.14), we can couple the law of the pair $(B(x),B(y))$ with the lax of the pair $(\widetilde{B}(x),\widetilde{B}(y))$ such that almost surely $|B_s(x)-\widetilde{B}_s(x)|\lesssim s^{-1/2}$ and $|B_s(y)-\widetilde{B}_s(y)|\lesssim s^{-1/2}$ as $s\to\infty$. Thus,

$$\mathbb{E}_W\left[\dot{f}_s^{\beta,a}(x)\dot{f}_s^{\beta,a}(y)\right]\lesssim\mathbb{E}_W\left[\widetilde{\dot{f}_s^{\beta,a}}(x)\widetilde{\dot{f}_s^{\beta,a}}(y)\right]=\mathbb{E}\left[\dot{f}_{s_0}^{\beta,a}(x)\dot{f}_{s_0}^{\beta,a}(y)\right].$$

as $s\to\infty$. Hence, there is a constant $c>0$ such that for all $x,y\in\mathbb{R}$, $\varepsilon=e^{-s}>0$ and $k<2+a$:

$$\mathbb{E}_W\left[\dot{f}_\varepsilon^{\beta,a}(x)\dot{f}_\varepsilon^{\beta,a}(y)\right]\leq c|x-y|^{-1}\log^{-k}\left(1+\frac{1}{|x-y|}\right). \tag{4.21}$$

By Lemma 4.2.1, we get:

$$\sup_{\varepsilon>0}\mathbb{E}_W\left[\left(\dot{F}_t^{\varepsilon,\beta,a}\right)^2\right]\leq c\int_{\mathbb{R}^2}\frac{\mathrm{d}\lambda_t(x)\mathrm{d}\lambda_t(y)}{|x-y|\log^k(1+\frac{1}{|x-y|})}<\infty$$

□

Remark 19. The estimate (4.19) is analogous to the function H from the proof of [RV15, Proposition 3.8]. Note that they do not add the contribution of the hitting time in the final estimate and they work with $a=0$, explaining the difference by a factor $s^{-\frac{1}{2}-a}$. In particular, our estimate falls directly into the scope of Lemma 4.2.1.

Building on the L^2-bound, we can show that $F_t^{\varepsilon,\beta,a}$ converges in L^2 as $\varepsilon \to 0$.

Proposition 4.3.6 (L^2 convergence)**.** *For all $t, \beta > 0$ and $a \in (0, \frac{3}{2})$, $\mathbb{P}_C$-almost surely, the family $(F_t^{\varepsilon,\beta,a})_{\varepsilon>0}$ is Cauchy in $L^2(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$.*

Proof. We fix a realisation of the Cauchy process and statements are $\mathbb{P}_C$-almost sure. We will use the notations of the proof of Proposition 4.3.5. Given $\varepsilon, \delta > 0$, we have

$$\mathbb{E}_W\left[\left(\dot{F}_t^{\varepsilon,\beta,a} - \dot{F}_t^{\delta,\beta,a}\right)^2\right] = \int_{\mathbb{R}^2} \mathbb{E}_W\left[\left(\dot{f}_\varepsilon^{\beta,a}(x) - \dot{f}_\delta^{\beta,a}(x)\right)\left(\dot{f}_\varepsilon^{\beta,a}(y) - \dot{f}_\delta^{\beta,a}(y)\right)\right] \mathrm{d}\lambda_t(x)\mathrm{d}\lambda_t(y) \tag{4.22}$$

Let $x, y \in \mathbb{R}$ be distinct such that $s_0 := \log \frac{1}{|x-y|} > 0$. Recall that the Brownian motions $(B_s(x))_{s\geq 0}$ and $(B_s(y))_{s\geq 0}$ are asymptotically independent, and that $\mathbb{E}_W[\dot{f}_\varepsilon^{\beta,a}(x)]$ converges to a finite limit as $\varepsilon \to 0$. Thus, by conditioning on $\mathcal{F}_W^{|x-y|}$ and using (4.14) (similarly to the end of the proof of Proposition 4.3.5), we see that both $\lim_{\varepsilon\to 0}\mathbb{E}_W[\dot{f}_\varepsilon^{\beta,a}(x)\dot{f}_\varepsilon^{\beta,a}(y)]$ and $\lim_{\varepsilon,\delta\to 0} \mathbb{E}_W\left[\dot{f}_\varepsilon^{\beta,a}(x)\dot{f}_\delta^{\beta,a}(y)\right]$ exist, are finite and are equal. It follows that, away from the diagonal, the integrand in (4.22) converges pointwise to 0 as $\varepsilon, \delta \to 0$. Since the diagonal has $\lambda_t \otimes \lambda_t$-measure zero $\mathbb{P}_C$-almost surely, it suffices to show that this integrand is uniformly bounded by a $\lambda_t \otimes \lambda_t$-integrable function in order to apply the dominated convergence theorem. This is the content of Proposition 4.3.5, so we are done.

□

4.3.1.2 Seneta-Heyde renormalisation

In this section, we show that $\dot{F}$ can be obtained through the so-called Seneta-Heyde renormalisation, which consists in multiplying F_t^ε by (a multiple of) the deterministic prefactor $\sqrt{\log \frac{1}{\varepsilon}}$. This result is important since it allows the use of Kahane's convexity inequality, a useful tool in proving uniqueness statements and moment estimates. See [DRSV14b, Lemma 16 & Introduction] for a statement and a discussion on the relevance of this inequality.

Theorem 4.3.7 (Seneta-Heyde renormalisation)**.** *$\mathbb{P}_C$-almost surely, the family $(\sqrt{\log \frac{1}{\varepsilon}} F_\cdot^\varepsilon)_{\varepsilon\to 0}$ converges in $\mathbb{P}_W$-probability to $\frac{1}{\sqrt{\pi}}\dot{F}_\cdot$ as $\varepsilon \to 0$, uniformly on compacts of $\mathbb{R}_+$.*

The proof of Theorem 4.3.7 is postponed to the end of Section 4.3.1.3. First, we establish the analogous result for $\dot{F}^{\beta,a}$.

Proposition 4.3.8. *For all $\beta, t > 0$ and $a \in (1, \frac{3}{2})$, $\mathbb{P}_C$-almost surely, the family $\sqrt{\log \frac{1}{\varepsilon}} F_t^{\varepsilon,\beta,a} - \frac{1}{\sqrt{\pi}}\dot{F}_t^{\varepsilon,\beta,a}$ converges to 0 in $L^2(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ as $\varepsilon \to 0$.*

Proof. We use the notations of the proof of Proposition 4.3.5 and make the change of variable $s = \log \frac{1}{\varepsilon}$ as before. We start with the one-point function. Recall that

$\dot{f}_s^{\beta,0}(x)$ is a martingale, so that $\mathbb{E}_W[\dot{f}_s^{\beta,0}(x)] = \mathbb{E}_W[\dot{f}_0^{\beta,0}(x)] = \beta$. On the other hand, $\mathbb{E}_W[f_s^{\beta,0}(x)] = \mathbb{P}^0(\tau^{\beta,0} > s) \sim \frac{\beta}{\sqrt{\pi t}}$ as $s \to \infty$. For the case $a > 0$, let $(\mathcal{B}_u^{\beta,a})_{0 \le u \le s}$ be Brownian motion conditioned on $\{\tau^{\beta,a} > s\}$. Clearly, $\mathbb{E}^0[\mathcal{B}_s^{\beta,a} - \mathcal{B}_s^{\beta,0}] = o(\sqrt{s})$ as $s \to \infty$. Thus,

$$\begin{aligned}
\mathbb{E}_W\left[\sqrt{s} f_s^{\beta,a}(x) - \frac{1}{\sqrt{\pi}} \dot{f}_s^{\beta,a}(x)\right] &= \sqrt{s}\left(\mathbb{P}^0\left(\tau^{\beta,a} > s\right) - \mathbb{P}^0\left(\tau^{\beta,0} > s\right)\right) \\
&\quad + \mathbb{E}_W\left[\sqrt{s} f_s^{\beta,0}(x) - \frac{1}{\sqrt{\pi}} \dot{f}_s^{\beta,0}(x)\right] \\
&\quad + \frac{1}{\sqrt{\pi}}\left(\mathbb{E}^0\left[\mathcal{B}_s^{\beta,0}\right]\right)\left(\mathbb{P}^0\left(\tau^{\beta,0} > s\right) - \mathbb{P}^0\left(\tau^{\beta,a} > s\right)\right) + o(1) \\
&= \left(\mathbb{P}^0\left(\tau^{\beta,a} > s\right) - \mathbb{P}^0\left(\tau^{\beta,0} > s\right)\right)\left(\sqrt{s} - \frac{1}{\sqrt{\pi}}\mathbb{E}^0\left[\mathcal{B}_s^{\beta,0}\right]\right) + o(1) \\
&\underset{s \to \infty}{\longrightarrow} 0.
\end{aligned} \tag{4.23}$$

Let us move on to the two-point function. By definition of the occupation measure, we have:

$$\begin{aligned}
&\mathbb{E}_W\left[\left(\sqrt{s} F_t^{s,\beta,a} - \frac{1}{\sqrt{\pi}} \dot{F}_t^{s,\beta,a}\right)^2\right] \\
&\quad = \int_{\mathbb{R}^2} \mathbb{E}_W\left[\left(\sqrt{s} f_s^{\beta,a}(x) - \frac{1}{\sqrt{\pi}} \dot{f}_s^{\beta,a}(x)\right)\left(\sqrt{s} f_s^{\beta,a}(y) - \frac{1}{\sqrt{\pi}} \dot{f}_s^{\beta,a}(y)\right)\right] \mathrm{d}\lambda_t(x) \mathrm{d}\lambda_t(y)
\end{aligned} \tag{4.24}$$

Let $x, y \in \mathbb{R}$ distinct and $s_0 := \log \frac{1}{|x-y|}$. By conditioning on $\mathcal{F}_W^{|x-y|}$ and using (4.14) and the one-point estimate (4.23), we get that the integrand in (4.24) converges pointwise to 0 away from the diagonal as $s \to \infty$. Since the diagonal has $\lambda_t \otimes \lambda_t$-measure zero $\mathbb{P}_C$-almost surely, it suffices to show that the integrand is uniformly bounded by a $\lambda_t \otimes \lambda_t$-integrable function in order to apply the dominated convergence theorem. This is essentially a variation of Proposition 4.3.5. Using the same notations, we have for $s \le s_0$ and all $p > 1$

$$\begin{aligned}
\mathbb{E}_W\left[\sqrt{s} f_s^{\beta,a}(x) \sqrt{s} f_s^{\beta,a}(y)\right] \le s \mathbb{E}_W\left[\left(f_s^{\beta,a}(0)\right)^2\right] &\lesssim \frac{s e^s}{(1+s)^a} \mathbb{E}^0\left[e^{\sqrt{2} p\left(\frac{\beta}{\sqrt{2}} - \mathcal{B}_s^\beta\right)}\right]^{1/p} \mathbb{P}^0\left(\tau^{\beta,0} > s\right)^{1/p} \\
&\lesssim \frac{s e^s}{(1+s)^a} \times s^{-\frac{3}{2p}} \times s^{-\frac{1}{2p}}.
\end{aligned} \tag{4.25}$$

As in the proof of Proposition 4.3.5, we get $\mathbb{E}_W[f_s^{\beta,a}(x) f_s^{\beta,a}(y)] \lesssim \mathbb{E}_W[f_{s_0}^{\beta,a}(x) f_{s_0}^{\beta,a}(y)]$ as $s \to \infty$. Finally, the cross term in the expansion of the integrand in (4.24) can be treated in the same way, with (4.25) giving the worst contribution. Thus, there is a constant $c > 0$

such that for all $s > 0$ and all $k \in (2, 1+a)$:

$$\mathbb{E}_W\left[\left(\sqrt{s}f_s^{\beta,a}(x) - \frac{1}{\sqrt{\pi}}\dot{f}_s^{\beta,a}(x)\right)\left(\sqrt{s}f_s^{\beta,a}(y) - \frac{1}{\sqrt{\pi}}\dot{f}_s^{\beta,a}(y)\right)\right] \leq \frac{c}{|x-y|\log^k(1+\frac{1}{|x-y|})}.$$

By Lemma 4.2.1, the right-hand-side belongs to $L^1(\mathbb{R}^2, \lambda_t \otimes \lambda_t)$ with $\mathbb{P}_C$-probability one. Thus, pointwise convergence and dominated convergence allow us to conclude that (4.24) converges to 0 as $s \to \infty$.

$\square$

4.3.1.3 The PCAF is a homeomorphism

For all $\beta, \varepsilon > 0$ and $a \in (0, \frac{3}{2})$, the function $t \mapsto \dot{F}_t^{\varepsilon,\beta,a}$ is a self-homeomorphism of $\mathbb{R}_+$, so its differential $\mathrm{d}\dot{F}_\cdot^{\varepsilon,\beta,a}$ is a measure on $\mathbb{R}_+$. From Proposition 4.3.6, it is standard to deduce that $\mathrm{d}\dot{F}_\cdot^{\varepsilon,\beta,a}$ converges $\mathbb{P}$-almost surely to a measure $\mathrm{d}\dot{F}_\cdot^{\beta,a}$ on $\mathbb{R}_+$, with respect to the topology of weak convergence. The goal of this section is to investigate the properties of this measure, in order to show eventually that $\dot{F}_\cdot^{\beta,a}$ is a self-homeomorphism of $\mathbb{R}_+$. Similarly, Proposition 4.3.6 implies that for each $t > 0$ the measure $\mathrm{d}\nu_{\varepsilon,t}^{\beta,a}(x) := \dot{f}_\varepsilon^{\beta,a}(x)\mathrm{d}\lambda_t(x)$ converges $\mathbb{P}$-almost surely weakly as $\varepsilon \to 0$ to a measure $\nu_t^{\beta,a}$ on $\mathbb{R}$ giving full mass to the trace of the Cauchy process up to time t.

Proposition 4.3.9 (Continuity)**.** *For all $t > 0$, $p < \frac{3}{2}$, $\mathbb{P}_C$-almost surely,*

$$\mathbb{E}_W\left[\int_0^t\int_0^t \log^p\left(1 + \frac{1}{|C_u - C_v|}\right)\mathrm{d}\dot{F}_u^{\beta,a}\mathrm{d}\dot{F}_v^{\beta,a}\right] < \infty.$$

In particular, $\dot{F}^{\beta,a}$ does not have any discontinuities with $\mathbb{P}$-probability one.

Proof. Reproducing the proof of Proposition 4.3.5, for all $k \in (2+p, \frac{7}{2})$ there is a constant $c > 0$ such that

$$\mathbb{E}_W\left[\int_0^t\int_0^t \log^p\left(1 + \frac{1}{|C_u - C_v|}\right)\mathrm{d}\dot{F}_u^{\beta,a}\mathrm{d}\dot{F}_v^{\beta,a}\right] \leq c\int_{\mathbb{R}^2}\frac{\mathrm{d}\lambda_t(x)\mathrm{d}\lambda_t(y)}{|x-y|\log^{k-p}(1+\frac{1}{|x-y|})},$$

which is finite $\mathbb{P}_C$-a.s. by Lemma 4.2.1. This implies that $\mathbb{P}$-almost surely, the measure $\nu_t^{\beta,a}$ has no atoms for all $t > 0$. Now, the points of discontinuity of $t \mapsto \dot{F}^{\beta,a}$ correspond to the atoms of $\nu_t^{\beta,a}$, so we deduce that $\dot{F}_\cdot^{\beta,a}$ is continuous. $\square$

Proposition 4.3.10 (Monotonicity)**.** *$\mathbb{P}$-almost surely, the function $t \mapsto \dot{F}_t^{\beta,a}$ is strictly increasing.*

Proof. Let $I \subset \mathbb{R}_+$ be a fixed non-empty open interval. We claim that, with $\mathbb{P}_C$-probability one, the event $\{\int_I \mathrm{d}\dot{F}_t^{\beta,a} > 0\}$ belongs to the tail σ-algebra $\mathcal{T} := \cap_{\varepsilon>0}\mathcal{F}_W^\varepsilon$, which is trivial by Kolmogorov's 0/1-law. This implies that this event has probability 0 or 1. On the other hand, $\mathbb{P}_C$-a.s., we have $\mathbb{E}_W[\int_I \mathrm{d}\dot{F}_t^{\beta,a}] = |I| > 0$ hence $\mathbb{P}(\int_I \mathrm{d}\dot{F}_t^{\beta,a} > 0) = 1$. Taking a countable collection of intervals generating the Borel σ-algebra of $\mathbb{R}_+$ enables to conclude.

Now, to prove the claim, let $0 < \delta < \varepsilon$ and write

$$\begin{aligned}
\mathrm{d}\dot{F}_t^{\delta,\beta,a} = \dot{f}_\delta^{\beta,a}(C_t)\mathrm{d}t &= \left(2\log\frac{1}{\delta} - W_\delta(C_t) + \beta\right)\mathbb{1}_{\{\tau^\beta_{C_t}<\delta\}}\delta e^{W_\delta(C_t)}\mathrm{d}t \\
&= \left(2\log\frac{1}{\varepsilon} - W_\varepsilon(C_t)\right)\mathbb{1}_{\{\tau^\beta_{C_t}<\delta\}}\delta e^{W_\delta(C_t)}\mathrm{d}t \\
&\quad + \varepsilon e^{W_\varepsilon(C_t)}\left(2\log\frac{\varepsilon}{\delta} + W_\varepsilon(C_t) - W_\delta(C_t) + \beta\right)\mathbb{1}_{\{\tau^\beta_{C_t}<\delta\}}\frac{\delta}{\varepsilon}e^{W_\delta(C_t)-W_\varepsilon(C_t)}\mathrm{d}t.
\end{aligned}$$

By Lemma 4.3.4, the first measure in the last line converges $\mathbb{P}-$a.s. weakly to 0 as $\delta \to 0$. On the other hand, up to the prefactor $\varepsilon e^{W_\varepsilon(C_t)}$, the second term is simply the derivative martingale starting the renormalisation at scale ε, so it converges $\mathbb{P}$-a.s. weakly as $\delta \to 0$ to a $\sigma(W_{\varepsilon'}, 0 < \varepsilon' \le \varepsilon)$-measurable random variable. Taking $\varepsilon \to 0$ shows that the event $\{\int_I \mathrm{d}\dot{F}_t^{\beta,a} > 0\}$ is $\mathcal{T}$-measurable. □

Proposition 4.3.11 (Infinite lifetime)**.** *$\mathbb{P}$-almost surely,* $\lim_{t\to\infty} \dot{F}_t^{\beta,a} = \infty$.

Proof. The idea is that the Cauchy process will explore all the state space and each new explored region will contribute roughly independently. More precisely, for all $n \in \mathbb{Z}$, we introduce the stopping times

$$\begin{aligned}
T_n &:= \inf\{t \ge 0,\, C_t \in I_{1/8}(n)\} \\
S_n &:= \inf\{t \ge T_n,\, C_t \notin I_{1/4}(n)\}.
\end{aligned}$$

With $\mathbb{P}_C$-probability one, all these stopping times are finite and $S_n - T_n > 0$. By the strong Markov property of the Cauchy process and the translation invariance of both C and W, the random variables $Z_n := \int_{T_n}^{S_n} \mathrm{d}\dot{F}_t^\beta$ are stochastically bounded from below by $\dot{F}_\tau^{\beta,a}$, where $\tau := \inf\{t \ge 0,\, C_t \notin I_{1/8}(0)\}$. However, they are not independent due to the possible long range correlations of W. From Proposition 4.3.5, we know that $\mathbb{E}_W[\dot{f}_\varepsilon^{\beta,a}(x)\dot{f}_\varepsilon^{\beta,a}(y)] \lesssim |x-y|^{-\alpha}$ for all $\alpha < 1$, so by the strong Markov property of C, there is a constant $c > 0$ such that for all $n \ne m$,

$$\mathbb{E}[Z_n Z_m] \le c|n-m|^{-\alpha}.$$

Thus, $\mathrm{Var}(\sum_{k=-n}^n Z_k) \lesssim n^{2-\alpha}$ for all $\alpha > 1$. Chebyshev's inequality then yields

$$\mathbb{P}\left(\left|\sum_{k=-n}^n Z_k - \mathbb{E}[Z_k]\right| \ge \frac{1}{2}\sum_{k=-n}^n \mathbb{E}[Z_k]\right) \le cn^{-\alpha}.$$

Taking for instance the sequence $n_j = 2^j$ and applying the Borel-Cantelli lemma, we see that $\mathbb{P}$-almost surely, $\sum_{k=-n}^{n} Z_k \gtrsim n$ as $n \to \infty$. Since all the Z_n are positive, we conclude that $\int_{\mathbb{R}_+} \dot{F}_t^{\beta,a} \geq \sum_{n\in\mathbb{Z}} Z_n = \infty$ with $\mathbb{P}$-probability one. □

Remark 20. The above proof makes use of the unboundedness of $\mathbb{R}$. In the compact case of the unit circle, one can easily adapt the ergodic argument of [GRV16, Section 2.9].

4.3.1.4 Concluding the proofs

We have seen that $\mathbb{P}$-almost surely, $\dot{F}_\cdot^{\beta,a}$ is a self-homeomorphism of $\mathbb{R}_+$ for all $\beta > 0$ and $a \in (0, \frac{3}{2})$. From here, we can conclude the proofs of Theorems 4.3.1 and 4.3.7 in a standard way.

Proof of Theorems 4.3.1 and 4.3.7. We already know that the family of measures $(\mathrm{d}\dot{F}_\cdot^{\varepsilon,\beta,a})_{\varepsilon>0}$ converges $\mathbb{P}$-almost surely weakly to $\mathrm{d}\dot{F}_\cdot^{\beta,a}$ as $\varepsilon \to 0$. Combining with the fact that the functions $\dot{F}_\cdot^{\varepsilon,\beta,a}$ and their limit $\dot{F}_\cdot^{\beta,a}$ are continuous, we get that the convergence $\dot{F}_\cdot^{\varepsilon,\beta,a} \to \dot{F}_\cdot^{\beta,a}$ is uniform on compacts of $\mathbb{R}_+$.

The only thing left to do is to compare $\dot{F}_t^\varepsilon$ with $\dot{F}_t^{\varepsilon,\beta,a}$. We have

$$\dot{F}_t^\varepsilon - \dot{F}_t^{\varepsilon,\beta,a} = \int_{\mathbb{R}} \left(2\log\frac{1}{\varepsilon} - W_\varepsilon(x) + \beta\right) \varepsilon e^{W_\varepsilon(x)} (1 - \mathbb{1}_{\{\tau_x^{\beta,a} > \log\frac{1}{\varepsilon}\}}) \mathrm{d}x - \beta \int_{\mathbb{R}} \varepsilon e^{W_\varepsilon(x)} \mathbb{1}_{\{\tau_x^{\beta,a} > \log\frac{1}{\varepsilon}\}} \mathrm{d}x. \tag{4.26}$$

By Lemma 4.3.4, the second term converges $\mathbb{P}$-almost surely to 0 for all $\beta > 0$. In fact, Lemma 4.3.4 implies that the family $(F_\cdot^{\varepsilon,\beta,a})_{\varepsilon>0}$ converges $\mathbb{P}$-almost surely uniformly to 0 on compacts of $\mathbb{R}_+$.

On the other hand, the first term in (4.26) vanishes $\mathbb{P}$-almost surely for sufficiently large (random) β by virtue of Lemma 4.2.3. Thus, $\dot{F}_\cdot^\varepsilon$ converges $\mathbb{P}$-almost surely uniformly on compacts of $\mathbb{R}_+$ as $\varepsilon \to 0$, and the properties of $\dot{F}_\cdot^{\beta,a}$ imply that the limiting function $\dot{F}_\cdot$ is a self-homeomorphism of $\mathbb{R}_+$.

Finally, let us treat the Seneta-Heyde renormalisation. For all $t, \beta > 0$ and $a \in (1, \frac{3}{2})$, we have

$$\sqrt{\log\frac{1}{\varepsilon}} F_t^\varepsilon - \frac{1}{\sqrt{\pi}} \dot{F}_t^\varepsilon = \sqrt{\log\frac{1}{\varepsilon}} (F_t^\varepsilon - F_t^{\varepsilon\beta,a}) + \sqrt{\log\frac{1}{\varepsilon}} F_t^{\varepsilon,\beta,a} - \frac{1}{\sqrt{\pi}} \dot{F}_t^{\varepsilon,\beta,a} + \frac{1}{\sqrt{\pi}} (\dot{F}_t^{\varepsilon,\beta,a} - \dot{F}_t^\varepsilon)$$

In view of Proposition 4.3.8, it suffices to prove that

$$\begin{aligned} &\limsup_{\beta\to\infty} \limsup_{\varepsilon\to 0} \sqrt{\log\frac{1}{\varepsilon}} (F_t^\varepsilon - F_t^{\varepsilon,\beta,a}) = 0 \\ &\limsup_{\beta\to\infty} \limsup_{\varepsilon\to 0} \dot{F}_t^\varepsilon - \dot{F}_t^{\varepsilon,\beta,a} = 0 \end{aligned} \tag{4.27}$$

in $\mathbb{P}$-probability. Denote by $K_t \subset \mathbb{R}$ the support of λ_t, which is bounded $\mathbb{P}_C$-almost surely.

Let

$$S := \sup_{\varepsilon>0} \sup_{x\in K_t} W_\varepsilon(x) - 2\log\frac{1}{\varepsilon} + a\log\log(1+\frac{1}{\varepsilon}),$$

which is finite $\mathbb{P}$-almost surely by Lemma 4.2.3. We have $F_t^\varepsilon \geq F_t^{\varepsilon,\beta,a}$ $\mathbb{P}$-almost surely and $F_t^\varepsilon = F_t^{\varepsilon,\beta,a}$ on the event $\{S \leq \beta\}$. Thus, for all $\delta > 0$, we have

$$\mathbb{P}\left(\sqrt{\log\frac{1}{\varepsilon}}|F_t^\varepsilon - F_t^{\varepsilon,\beta,a}| \geq \delta\right) = \mathbb{P}\left(\sqrt{\log\frac{1}{\varepsilon}}(F_t^\varepsilon - F_t^{\varepsilon,\beta,a}) \geq \delta \,\middle|\, S > \beta\right)\mathbb{P}(S>\beta) \leq \mathbb{P}(S>\beta).$$

Since the latter converges to 0 as $\beta \to \infty$, the first line of (4.27) follows. Going back to (4.26), the second line follows by the same argument. $\square$

Remark 21. The analysis of this section shows that it is possible to construct a critical GMC with respect to any measure of finite f-energy with $f(t) = t\log^p(1+\frac{1}{t})$ and $p > \frac{7}{2}$. This threshold is not sharp: once we have conditioned on the value of $\tau_x^{\beta,0}$, we can further condition on the almost sure behaviour of the Bessel process, so the polylogarithmic term in (4.21) can be improved to stretched exponential. This is done e.g. in [DRSV14a, Pow20, Jeg20], where in the latter paper it is referred to as the "second layer of good events". In our situation, a single layer is sufficient provided we condition on $\tau_x^{\beta,a}$ rather than $\tau_x^{\beta,0}$.

4.3.2 The critical PCAF on $\mathbb{R}$

In this section, we extend our definition of the critical LCP to a Markov process defined for all starting points $x \in \mathbb{R}$. To do so, we will extend $\dot{F}_t$ to all possible starting points and eventually show that the resulting family of mappings coincides with the PCAF of the critical boundary GMC measure on $\mathbb{R}$. We consider the same field W on $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ as in Section 4.2.3, but now the Cauchy process is the Markov process defined on the filtered probability space $(\Omega_C, (\mathcal{F}_C^t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}})$ with $\mathbb{P}_x$ being the law of the Cauchy process started from $x \in \mathbb{R}$.

The results of Section 4.3.1 show that for each $x \in \mathbb{R}$, $\mathbb{P}_W \otimes \mathbb{P}_x$-almost surely, $\dot{F}_\cdot^\varepsilon \to \dot{F}_\cdot$ as $\varepsilon \to 0$ with respect to the topology of uniform convergence on compacts, and the limiting function $t \mapsto \dot{F}_t$ is a self-homeomorphism of $\mathbb{R}_+$. Obviously, this implies that $\mathbb{P}_W$-almost surely, the above convergence holds simultaneously for all dyadic points ζ with $\mathbb{P}_\zeta$-probability one. The goal of this section is two-fold. First, Theorem 4.3.12 shows that $\mathbb{P}_W$-almost surely, the convergence holds simultaneously for quasi-every $x \in \mathbb{R}$ with $\mathbb{P}_x$-probability one. Second, we will identify this functional with the $\mathbb{P}_W$-a.s. PCAF of the critical GMC measure.

Recall that the regularised measures $\mathrm{d}\nu_\varepsilon(x) = (2\log\frac{1}{\varepsilon} - W_\varepsilon(x))\varepsilon e^{W_\varepsilon(x)}\mathrm{d}x$ converge $\mathbb{P}_W$-a.s. as $\varepsilon \to 0$ to the critical GMC measure ν. At the level of the regularisation, we have $\mathbb{P}_W$-a.s. for all $x \in \mathbb{R}$

$$\mathbb{E}_x\left[\dot{F}^{\varepsilon}_t\right] = \int_0^t \mathbb{E}_x\left[\left(2\log\frac{1}{\varepsilon} - W_\varepsilon(C_s)\right)\varepsilon e^{W_\varepsilon(C_s)}\right]\mathrm{d}s = \int_0^t \int_{\mathbb{R}} \mathrm{p}_s(x-y)\mathrm{d}\nu_\varepsilon(y)\mathrm{d}s. \quad (4.28)$$

where $\mathrm{p}_s(x) = \frac{1}{\pi}\frac{s}{x^2+s^2}$ is the Poisson kernel as before. Taking $\varepsilon \to 0$, Theorem 4.3.1 implies that for each fixed $x \in \mathbb{R}$, $\dot{F}^{\varepsilon}_\cdot$ has a limit $\dot{F}_\cdot$ under $\mathbb{P}_x$, and with $\mathbb{P}_W$-a.s. we have

$$\mathbb{E}_x\left[\dot{F}_t\right] = \int_0^t \int_{\mathbb{R}} \mathrm{p}_s(x-y)\mathrm{d}\nu(y)\mathrm{d}s = \frac{1}{2\pi}\int_{\mathbb{R}} \log\left(1+\left(\frac{t}{x-y}\right)^2\right)\mathrm{d}\nu(y). \quad (4.29)$$

From now on, we fix some $k_0 \in (1, \frac{3}{2})$ and define $N_0 := E^{\mathbf{f}_{k_0}}$. By Lemma 4.2.4, N_0 is $\mathbb{P}_W$-a.s. polar for the Cauchy process.

Proposition 4.3.12 (Convergence for q.e. starting points)**.** *The following holds $\mathbb{P}_W$-almost surely, for all $x \in \mathbb{R}$, with $\mathbb{P}_x$-probability one.*

The family $(\dot{F}^{\varepsilon}_\cdot)_{\varepsilon>0}$ converges uniformly on compacts of $(0,\infty)$ to a function $\dot{F}_\cdot$ such that for all $s > 0$, $\dot{F}_\cdot|_{[s,\infty)}$ is continuous, increasing and tends to ∞ as $t \to \infty$. If $x \in \mathbb{R}\setminus N_0$, the convergence $\dot{F}^{\varepsilon}_\cdot \to \dot{F}_\cdot$ is uniform on compacts of $\mathbb{R}_+$ and $\dot{F}_\cdot(x)$ is a self-homeomorphism of $\mathbb{R}_+$.

Proof. The proof has two parts. First, we use Lemma 4.2.2 to extend the definition of $\dot{F}$ to all starting points. Second, we show that $\dot{F}$ is a homeomorphism of $\mathbb{R}_+$ for all $x \in \mathbb{R} \setminus N_0$. In this proof, we fix some almost sure realisation of the field and work $\mathbb{P}_W$-almost surely. Sample paths of $\mathbb{P}_x$ will be denoted by C^x.

Let $x \in \mathbb{R}$ and let us define a coupling $\widetilde{\mathbb{P}}_x$ of the measures $\mathbb{P}_x, (\mathbb{P}_\zeta)_{\zeta\in\Delta}$ such that for each $\zeta \in \Delta$, the marginal distribution of (C^x, C^ζ) is the measure $\pi_{x,\zeta}$ from Lemma 4.2.2. Namely, to sample from $\widetilde{\mathbb{P}}_x$, take independent Brownian motions $B^0, B^x, (B^\zeta)_{\zeta\in\Delta}$ started from $0, x, (\zeta)_{\zeta\in\Delta}$ respectively. Then, let $C^x, (C^\zeta)_{\zeta\in\Delta}$ be the Cauchy processes obtained from the reflecting Brownian motions $B^x + i|B^0|, (B^\zeta + i|B^0|)_{\zeta\in\Delta}$ by Spitzer's embedding. Finally, couple each pair $(C^x, C^\zeta)_{\zeta\in\Delta}$ as in Lemma 4.2.2, so that the final sample (C^x, C^ζ) is distributed according to $\pi_{x,\zeta}$. We let τ_ζ be the coupling time of C^x with C^ζ. Note that all the resulting Cauchy processes share the same points of discontinuity, and eventually they all follow the same trajectory as C^x.

For each $\zeta \in \Delta$, we can consider the PCAF started at ζ using the path C^ζ, which we denote by $\dot{F}_\cdot(\zeta)$. We denote the regularised functionals similarly by $\dot{F}^{\varepsilon}_\cdot(\zeta)$ and $\dot{F}^{\varepsilon}_\cdot(x)$. Since the marginal distribution of C^ζ is $\mathbb{P}_\zeta$, $\dot{F}_t(\zeta)$ has the law of $\dot{F}_t$ under $\mathbb{P}_\zeta$, and the results of Section 4.3.1 imply that the mapping $t \mapsto \dot{F}_t(\zeta)$ is a self-homeomorphism of $\mathbb{R}_+$ simultaneously for all $\zeta \in \Delta$ with $\widetilde{\mathbb{P}}_x$-probability one. Moreover, we have the convergence $\dot{F}^{\varepsilon}_\cdot(\zeta) \to \dot{F}_\cdot(\zeta)$ uniformly on compacts of $\mathbb{R}_+$ as $\varepsilon \to 0$.

Let $\zeta_n \in \Delta$ be a sequence converging to x as $n \to \infty$. By item (iii) of Lemma 4.2.2, up to extracting a subsequence, we can assume that $\widetilde{\mathbb{P}}_x(\tau_{\zeta_n} \to 0 \text{ as } n \to \infty) = 1$. On this event and by definition of the coupling, it holds that $\dot{F}^{\varepsilon}_{\cdot}(x)$ converges uniformly in $\mathbb{P}_W$-probability on compacts of $(0, \infty)$ to a limiting function $\dot{F}_{\cdot}(x)$. Moreover, $\dot{F}_t(x) \to \infty$ as $t \to \infty$ and for all $s > 0$, $\dot{F}_{\cdot}(x)|_{[s,\infty)}$ is continuous and increasing. Furthermore, $\dot{F}_{\cdot}(x)|_{[s,\infty)}$ is measurable with respect to the σ-algebra generated by C^x, hence $\dot{F}_{\cdot}(x)$ is also C^x-measurable. Thus, $\dot{F}_{\cdot}(x)$ does not depend on the approximating sequence and the convergence $\dot{F}^{\varepsilon}_{\cdot}(x) \to \dot{F}_{\cdot}(x)$ holds $\mathbb{P}_x$-almost surely. What is left to show is that the convergence holds not only on compacts of $(0, \infty)$ but on compacts of $\mathbb{R}_+$ provided that $x \notin N_0$.

Let $x \in \mathbb{R} \setminus N_0$ and let us prove that $\dot{F}_t(x) \to 0$ as $t \to 0$. For each $t > 0$, we have from (4.28):

$$\mathbb{E}_x \left[\dot{F}^{\varepsilon}_t \right] = \frac{1}{2\pi} \int_{\mathbb{R}} \log \left(1 + \left(\frac{t}{x-y} \right)^2 \right) \mathrm{d}\nu_\varepsilon(y). \tag{4.30}$$

Note that the Poisson kernel introduces a logarithmic singularity at x. However, for $x \notin N_0$, we know that this singularity is integrable. More precisely, recentring the singularity at 0, we have uniformly in ε,

$$\int_{-t^{1/2}}^{t^{1/2}} \log \left(1 + \frac{t^2}{y^2} \right) \mathrm{d}\nu_\varepsilon(y) \lesssim \int_{-t^{1/2}}^{t^{1/2}} \log \frac{t}{|y|} \mathrm{d}\nu_\varepsilon(y) \lesssim \left(\log \frac{1}{t} \right)^{1-k_0}.$$

Away from the singularity, we have

$$\int_{t^{1/2} \le |y| \le 1} \log \left(1 + \frac{t^2}{y^2} \right) \mathrm{d}\nu_\varepsilon(y) \le \int_{t^{1/2} \le |y| \le 1} \frac{t^2}{y^2} \mathrm{d}\nu_\varepsilon(y) \le t \nu_\varepsilon[-1,1] = O(t),$$

and

$$\int_{|y|>1} \log \left(1 + \frac{t^2}{y^2} \right) \mathrm{d}\nu_\varepsilon(y) \le t^2 \int_{|y|>1} \frac{\mathrm{d}\nu_\varepsilon(y)}{y^2} = O(t^2).$$

Thus, uniformly in ε, we have

$$\mathbb{E}_x \left[\dot{F}^{\varepsilon}_t \right] \underset{t \to 0}{\lesssim} \left(\log \frac{1}{t} \right)^{1-k_0}, \tag{4.31}$$

which tends to 0 as $t \to 0$ since $k_0 > 1$. Thus, by Markov's inequality, $\dot{F}_t \to 0$ as $t \to 0$ almost surely under $\mathbb{P}_x$. Since $\dot{F}^{\varepsilon}_{\cdot}$ is continuous and increasing for each ε and the limit is continuous, we deduce by Dini's theorem that under $\mathbb{P}_x$, the convergence $\dot{F}^{\varepsilon}_{\cdot} \to \dot{F}_{\cdot}$ is uniform on compacts of $\mathbb{R}_+$.

□

We have at our disposal a random homeomorphism $\dot{F}_{\cdot} : \mathbb{R}_+ \to \mathbb{R}_+$ defined $\mathbb{P}_W$-a.s. for quasi-every $x \in \mathbb{R}$ with $\mathbb{P}_x$-probability one. Notice that at this stage we haven't defined $\dot{F}_{\cdot}$ on a common set of $\mathbb{P}_x$-probability one for all $x \in \mathbb{R} \setminus N_0$. However, we will recall the argument of [RV15, Theorem 4.18] allowing to identify $\dot{F}$ with the PCAF of ν.

With $\mathbb{P}_W$-probability one, we have the "pointwise Revuz correspondence" (4.29) holding for all $x \in \mathbb{R} \setminus N_0$. This implies that the Revuz correspondence holds $\mathbb{P}_W$-a.s. in the form (4.42): namely, for all Borel functions f, h, we have

$$\int_{\mathbb{R}} \mathbb{E}_x \left[\int_0^t f(C_s) \mathrm{d}\dot{F}_s \right] h(x) \mathrm{d}x = \int_0^t \int_{\mathbb{R}} f(x) \mathbf{p}_s h(x) \mathrm{d}\nu(x) \mathrm{d}s, \tag{4.32}$$

with $\mathbf{p}_s h(x) = \int_{\mathbb{R}} \mathrm{p}_s(x-y)h(y)\mathrm{d}y$. On the other hand, we know that ν is $\mathbb{P}_W$-a.s. a Revuz measure in $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta)$, so it admits a (unique) PCAF A, with a defining set $\Lambda \subset \Omega_C$ and an exceptional set $N_1 \subset \mathbb{R}$. Moreover, since A satisfies the Revuz correspondence (4.32), we have $\mathbb{P}_x(A_t = \dot{F}_t) = 1$ for all $t > 0$ and $x \in \mathbb{R} \setminus N$, where $N := N_0 \cup N_1$ is polar. Thus, we can modify $\dot{F}$ away from a set of vanishing $\mathbb{P}_x$-probability so that it is defined for all $\omega \in \Lambda$ and (4.32) still holds for $\dot{F}$. Finally, it is clear that this modification of $\dot{F}$ satisfies the additivity property (4.41), so that $\dot{F}$ is now a true PCAF, coinciding with A up to PCAF equivalence. Conversely, any such extension of $\dot{F}$ to a PCAF will satisfy the Revuz correspondence with Revuz measure ν, so that A is the unique such extension.

Corollary 4.3.13. *$\mathbb{P}_W$-almost surely, the random functional $\dot{F}$ extends to a PCAF of Revuz measure ν. This extension is unique and still denoted $\dot{F}$.*

4.3.3 The subcritical PCAF

In this section, we fix some $\gamma \in (0, 2)$ and state the results on the γ-LCP. The underlying field W and Cauchy process C are the same as before. We denote the boundary γ-GMC measure on $\mathbb{R}$ by ν_γ. This measure is $\mathbb{P}_W$-a.s. a Revuz measure, so it admits a PCAF. Define the random functional

$$F_t^{\gamma,\varepsilon} := \int_0^t \varepsilon^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2} W_\varepsilon(C_s)} \mathrm{d}s.$$

The subcritical measure ν_γ is much simpler to deal with than the critical one. Indeed, the local dimension at each point is strictly positive, so that the decay of the size of balls is polynomial. In particular, $\mathbb{P}_W$-almost surely,

$$\sup_{x \in \mathbb{R}} \int_{\mathbb{R}} \frac{\mathrm{d}\nu_\gamma(y)}{\mathbf{f}_1(|x-y|)} < \infty, \tag{4.33}$$

so that the exceptional set of the PCAF of ν_γ is empty in view of [FOT11, Theorem 5.1.6]. We gather these observations in the following theorem, which is the subcritical analogue of Theorem 4.3.12 and Corollary 4.3.13.

Theorem 4.3.14 (Subcritical PCAF)**.** *$\mathbb{P}_W$-almost surely, for all $x \in \mathbb{R}$, with $\mathbb{P}_x$-probability one, the family of mappings $(t \mapsto F_t^{\gamma,\varepsilon})_{\varepsilon>0}$ converges uniformly on compacts of $\mathbb{R}_+$ to a* *random homeomorphism of $\mathbb{R}_+$. Moreover, the limiting functional F^γ uniquely extends to a PCAF of Revuz measure ν_γ. It is a PCAF in the strict sense and it has full support.*

4.3.4 LCP: definition, Dirichlet form and heat kernel

4.3.4.1 Dirichlet form and heat kernel

Let $\gamma \in (0,2]$ and ν_γ be the γ-GMC measure on $\mathbb{R}$. We know that the functionals F^γ from Theorem 4.3.14 and $F^2 := \dot{F}$ from Corollary 4.3.13 are PCAFs of Revuz measure ν_γ, and these PCAFs have full support. Moreover, for all x away from the exceptional set, $t \mapsto F_t^\gamma$ is a homeomorphism of $\mathbb{R}_+$ a.s. under $\mathbb{P}_x$, so it has an inverse $\tau_t^\gamma = (F_\cdot^\gamma)^{-1}(t)$. This allows us to define the γ-LCP and its Dirichlet space [FOT11, Theorem 6.2.1].

Definition 4.3.1. Fix $\gamma \in (0,2]$. Let $(C_t)_{t\geq 0}$ be a Cauchy process defined on $(\Omega_C, (\mathcal{F}_C^t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{S}^1})$ and ν_γ be the Liouville measure with its PCAF F^γ. The *Liouville-Cauchy process* (γ-LCP) is the strong Markov process with state space $(\widetilde{S}_{\nu_\gamma}, \nu_\gamma)$ defined by $\mathcal{C}_t^\gamma := C_{\tau_t^\gamma}$.

Its Dirichlet space $(\mathcal{E}^\gamma, H^{1/2}(\mathbb{R}, \nu_\gamma))$ of the γ-LCP is given by

$$H^{1/2}(\mathbb{S}^1, \nu_\gamma) = \left\{ u \in L^2(\mathbb{R}, \nu_\gamma),\ \exists \widetilde{u} \in H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta) \text{ s.t. } \widetilde{u} = u \quad \nu_\gamma\text{-a.e.} \right\},$$

and $\mathcal{E}^\gamma(u,v) = \mathcal{E}(\widetilde{u}, \widetilde{v})$ independently of the choice of representatives $\widetilde{u}, \widetilde{v} \in H^{1/2}(\mathbb{R}, \mathrm{d}\theta)$.

The strong Markov property of $\mathcal{C}^\gamma$ follows from [FOT11, Chapter 6]. The transition functions

$$\mathbf{p}_t^\gamma f(x) = \mathbb{E}_x \left[f(\mathcal{C}_t^\gamma) \right]$$

form a strongly continuous semi-group on $L^2(\mathbb{S}^1, \nu_\gamma)$. We can also consider the resolvent family $\mathbf{R}_\alpha^\gamma$, for all $\alpha > 0$ [FOT11, Equation (6.2.6)]:

$$\mathbf{R}_\alpha^\gamma f(x) = \mathbb{E}_x \left[\int_0^\infty e^{-\alpha t} f(\mathcal{C}_t^\gamma) \mathrm{d}t \right] = \mathbb{E}_x \left[\int_0^\infty e^{-\alpha F_t^\gamma} f(C_t) \mathrm{d}F_t^\gamma \right]. \tag{4.34}$$

Theorem 4.3.15. *$\mathbb{P}_W$-almost surely, for all $t > 0$ and $x \in \widetilde{S}_{\nu_\gamma}$, the measure $E \mapsto \mathbf{p}_t^\gamma \mathbb{1}_E(x) =: \mathbf{p}_t^\gamma(x, E)$ is absolutely continuous with respect to ν_γ. Thus, there exists a family of jointly measurable, symmetric functions $(\mathrm{p}_t^\gamma(\cdot,\cdot))_{t>0}$ such that for all bounded Borel functions f,*

$$\mathbf{p}_t^\gamma f(x) = \int_{\mathbb{S}^1} \mathrm{p}_t^\gamma(x,y) f(y) \mathrm{d}\nu_\gamma(y).$$

Proof. We only treat the case $\gamma = 2$ and drop the dependence in γ in the notation. According to [FOT11, Theorem 4.2.4], it suffices to prove absolute continuity of the resolvent family $E \mapsto \mathbf{R}_\alpha^\gamma(x, E)$ for all $x \in \widetilde{S}_\nu$ and $\alpha > 0$. Thus, let E be a Borel set such that $\nu(E) = 0$ and let us show that $\mathbf{R}_\alpha^\gamma(x, E) = 0$.

For all Borel functions f and $\delta > 0$, we write

$$\mathbf{R}^{\gamma}_{\delta,\alpha} f(x) := \mathbb{E}_x \left[\int_0^{\delta} e^{-\alpha \dot{F}_t} \mathbb{1}_E(C_t) \mathrm{d}\dot{F}_t \right],$$

so that, by the Markov property,

$$\begin{aligned} \mathbf{R}^{\gamma}_{\alpha} f(x) &= \mathbb{E}_x \left[\int_0^{\delta} e^{-\alpha \dot{F}_t} f(C_t) \mathrm{d}\dot{F}_t \right] + \mathbb{E}_x \left[\int_{\delta}^{\infty} e^{-\alpha \dot{F}_t} f(C_t) \mathrm{d}\dot{F}_t \right] \\ &= \mathbf{R}^{\gamma}_{\delta,\alpha} f(x) + \mathbb{E}_x \left[e^{-\alpha \dot{F}_\delta} \mathbf{R}^{\gamma}_{\alpha} f(C_\delta) \right]. \end{aligned} \tag{4.35}$$

Let us apply this to $f = \mathbb{1}_E$. By definition of $\widetilde{S}_\nu = \mathbb{S}^1 \setminus N_0$ (recall in particular (4.31)), we have uniformly in $x \in \widetilde{S}_\nu$

$$\mathbf{R}^{\gamma}_{\delta,\alpha}(x, E) \le \mathbb{E}_x \left[\dot{F}_\delta \right] \underset{\delta \to 0}{\lesssim} \left(\log \frac{1}{\delta} \right)^{1-k_0}.$$

From [FOT11, Lemma 4.1.1], $\mathbf{R}^{\gamma}_{\alpha}(x, E) = 0$ for ν-a.e. $x \in \mathbb{S}^1$. Since $\nu(\mathbb{S}^1 \setminus \widetilde{S}_\nu) = 0$ and the topological support of ν is $\mathbb{S}^1$, this implies that $\mathbf{R}^{\gamma}_{\alpha}(\cdot, E)$ vanishes on a dense subset of $\mathbb{S}^1$. Hence, the function $x \mapsto \mathbb{E}_x[e^{-\alpha \dot{F}_\delta} \mathbf{R}^{\gamma}_{\alpha}(C_\delta, E)]$ vanishes on a dense set. Now, let F_n be a decreasing sequence of coverings of N_0 such that $\mathrm{Cap}(F_n) \to 0$ as $n \to \infty$. Reproducing the proof of [GRV14, Theorem 2.4], it is easy to deduce from the coupling argument of Proposition 4.3.12 that the function $x \mapsto \mathbb{E}_x[e^{-\alpha \dot{F}_\delta} \mathbf{R}^{\gamma}_{\alpha}(C_\delta, E)]$ is continuous on $\mathbb{S}^1 \setminus F_n$ for all n and all $\delta > 0$. Hence, this function is quasi-continuous for all $\delta > 0$, so it vanishes on $\widetilde{S}_\nu$. Taking $\delta \to 0$ shows that $\mathbf{R}^{\gamma}_{\alpha}(x, E) = 0$ for all $x \in \widetilde{S}_\nu$. □

Theorem 4.3.15 also implies the existence of measurable, symmetric kernels $\mathrm{R}^{\gamma}_{\alpha}(\cdot, \cdot)$ such that for all Borel functions f,

$$\mathbf{R}^{\gamma}_{\alpha} f(x) = \int_{\mathbb{R}} \mathrm{R}^{\gamma}_{\alpha}(x, y) f(y) \mathrm{d}\nu_\gamma(y).$$

Finally, we state two interesting consequences of Theorem 4.3.15 in terms of the path properties of the γ-LCP.

Corollary 4.3.16. *$\mathbb{P}_W$-almost surely, $\mathcal{C}^\gamma$ spends Lebesgue-almost all the time in a set of full ν_γ-measure.*

Corollary 4.3.17. *$\mathbb{P}_W$-almost surely, for all $x \in \widetilde{S}_{\nu_\gamma}$ and $t > 0$, $\mathbb{P}_x$-almost surely, $\mathcal{C}^\gamma_t$ belongs to a set of full ν_γ-measure.*

Finally, in the subcritical regime, we can show the stronger property that $\mathcal{C}^\gamma$ is strong Feller, i.e. that $\mathbf{R}^{\gamma}_{\alpha}$ maps the space of bounded Borel functions into the space of continuous functions.

Theorem 4.3.18. *Assume $\gamma < 2$. $\mathbb{P}_W$-almost surely, $\mathcal{C}^\gamma$ is strong Feller.*

Proof. Let f be a bounded Borel function. Going back to (4.35), we only need to show that $\mathbb{P}_W$-almost surely, $\mathbf{R}^\gamma_{\delta,\alpha} f$ converges uniformly to 0 on $\mathbb{S}^1$ as $\delta \to 0$. We have, for all $x \in \mathbb{S}^1$,

$$\mathbf{R}^\gamma_{\delta,\alpha} f(x) \leq \|f\|_\infty \, \mathbb{E}_x[F^\gamma_\delta] = \frac{\|f\|_\infty}{2\pi} \int_{\mathbb{S}^1} \log\left(1 + \left(\frac{\delta}{x-y}\right)^2\right) \mathrm{d}\nu_\gamma(y),$$

and the result is then a straightforward consequence of (4.33). □

4.3.4.2 LCP on bordered Riemann surfaces

Let (Σ, σ) be a Riemann surface, σ being a Riemannian metric with constant curvature and geodesic boundary. Let v_σ be the volume form in Σ and ℓ be the arclength on $\partial\Sigma$. Write $\partial\Sigma = \sqcup_i \partial_i \Sigma$ and let $\ell_i := \ell|_{\partial_i\Sigma}$. Reflecting Brownian motion in (Σ, σ) is the Markov process associated to the Dirichlet space $H^1(\Sigma, v_\sigma)$ on $L^2(\bar{\Sigma}, v_\sigma)$ as before. The trace space is $H^{1/2}(\partial\Sigma, \ell)$, and the Cauchy process arises similarly by taking ℓ as Revuz measure in $H^1(\Sigma, v_\sigma)$. We obtain the trace of reflecting Brownian motion on $\partial\Sigma$ reparameterised by local time. Note that the process jumps between boundary components according to the excursions of the underlying Brownian motion between boundary components. We could also choose ℓ_i as Revuz measure (for some fixed i): the process would stay in $\partial_i\Sigma$ and be absolutely continuous with respect to the standard Cauchy process on the circle.

Let X be the GFF on (Σ, σ) and μ_γ, ν_γ be the associated chaos measures in Σ and on $\partial\Sigma$ respectively ($\gamma \in (0, 2]$ fixed). As before, μ_γ and ν_γ are Revuz with full quasi-support, so we have a non-degenerate time-change of Brownian motion and of the Cauchy process, associated to the Dirichlet spaces $H^1(\Sigma, \mu_\gamma)$ and $H^{1/2}(\partial\Sigma, \nu_\gamma)$. Moreover, the local time of γ-LBM is the PCAF of ν_γ in $H^1(\Sigma, \mu_\gamma)$, and the γ-LCP can be obtained by taking the trace of γ-LBM on $\partial\Sigma$ and reparameterising by its local time.

4.A Dirichlet forms and Markov processes

In this appendix we collect some technical results on Dirichlet forms. Dirichlet forms are Markovian symmetric forms and are tightly connected to Markov processes, due to the Markovian nature of the semi-group they generate.

4.A.1 Closures of symmetric forms

Following [FOT11, Chapter 1], a *symmetric form* $\mathcal{E}$ on a Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ is a non-negative, symmetric, bilinear form with dense domain $\mathcal{D} \subset H$. $\mathcal{E}$ is *closed* if $\mathcal{D}$ is a

Hilbert space with respect to the inner-product

$$\mathcal{E}_1(u,v) := \langle u,v\rangle_H + \mathcal{E}(u,v).$$

It is *closable* if for all $\mathcal{E}$-Cauchy sequence $(u_n) \in \mathcal{D}$ such that $u_n \to 0$ in H as $n \to \infty$, it holds that $\mathcal{E}(u_n) \to 0$. An *extension* of $\mathcal{E}$ is a symmetric form $\widehat{\mathcal{E}}$ with domain $\widehat{\mathcal{D}} \supset \mathcal{D}$ such that $\widehat{\mathcal{E}}$ coincides with $\mathcal{E}$ on $\mathcal{D}$. A symmetric form $\mathcal{E}$ possesses a closed extension if and only if it is closable.

Let $\mathcal{E}$ be a symmetric form and $\widehat{\mathcal{D}}$ the Hilbert space completion of $\mathcal{D}$ with respect to $\mathcal{E}_1$. The inclusion $\mathcal{D} \hookrightarrow H$ extends to a continuous linear map

$$\iota : \widehat{\mathcal{D}} \to H. \tag{4.36}$$

Moreover, ι is injective if and only if $\mathcal{E}$ is closable. In any case, one can define a closed symmetric form $(\mathcal{E}', \mathcal{D}')$ by

$$\begin{cases} \mathcal{D}' = \iota(\widehat{\mathcal{D}}) \\ \mathcal{E}'(u) = \inf \lim\limits_{n\to\infty} \mathcal{E}(u_n), \end{cases} \tag{4.37}$$

where the infimum is taken over all $\mathcal{E}_1$-Cauchy sequences $u_n \in \mathcal{D}$ such that $u_n \to u$ in H [FST91, Theorem 6.1].

4.A.2 Dirichlet forms

From now on, we will only consider the case $H = L^2(X,m)$ where X is a compact subset of $\mathbb{C}$ and m is a positive, finite Radon measure. A symmetric form $(\mathcal{E},\mathcal{D})$ on $L^2(X,m)$ *Markovian* if for all $u \in \mathcal{D}$, it holds that $u_* := (u \vee 0) \wedge 1 \in \mathcal{D}$ and $\mathcal{E}(u_*) \le \mathcal{E}(u)$. The form is a *pre-Dirichlet form* if it is Markovian and it is a *Dirichlet form* if it is furthermore closed. Given a pre-Dirichlet form, the closed form obtained by applying (4.37) is Markovian, i.e. it is a Dirichlet form [FST91, Theorem 6.1]. A Dirichlet form is *regular* if $\mathcal{C}^0(X) \cap \mathcal{D}$ is dense both in $\mathcal{D}$ for the $\mathcal{E}_1$-topology and in $\mathcal{C}^0(X)$ for the uniform topology. Any subalgebra of $\mathcal{C}^0(X) \cap \mathcal{D}$ satisfying these properties is called a *core* of $(\mathcal{E},\mathcal{D})$.

Let $(\mathcal{E},\mathcal{D})$ be a Dirichlet form on $L^2(X,m)$. The *capacity* of a Borel set $E \subset X$ is

$$\mathrm{Cap}(E) := \inf \mathcal{E}_1(u), \tag{4.38}$$

where the infimum is taken over all functions $u \in \mathcal{D}$ such that $0 \le u \le 1$ and $u = 1$ m-almost everywhere. A set of zero capacity is *polar*[3] and a property is said to hold

[3] We will always assume or prove the absolute continuity condition of [FOT11, Theorem 4.1.2] so that the different notions of exceptional sets (polar, exceptional, of zero capacity) coincide.

quasi-everywhere if it holds away from a polar set. We say that $(\mathcal{E},\mathcal{D})$ satisfies a *Poincaré-Wirtinger inequality* if there exists $c>0$ such that

$$\|u-\bar{u}\|_{L^2(X,m)} \le c\sqrt{\mathcal{E}(u)} \tag{4.39}$$

for all $u \in \mathcal{D}$, where $\bar{u} := \frac{1}{m(X)}\int_X u dm$. In this case, the 0-order capacity coincides with the 1-order capacity, i.e. we can replace $\mathcal{E}_1$ with $\mathcal{E}$ in (4.38). Moreover, the space $\dot{\mathcal{D}} := \mathcal{D}/\mathbb{R}$ becomes a Hilbert space when endowed with the norm $\sqrt{\mathcal{E}(\cdot)}$ and it follows that $\dot{\mathcal{D}} \oplus \mathbb{R} = \mathcal{D}$ coincides with its extended Dirichlet space (see [FOT11, Chapter 1, Section 5] for the definition of extended Dirichlet spaces).

Let $\mathcal{M}=\mathcal{M}(X)$ be the set of positive, finite Radon measures on X, and $\mathcal{M}_0 \subset \mathcal{M}$ be the subset of those measures charging no polar set: elements of $\mathcal{M}_0$ are called *Revuz measures* and they define the same polar sets as m. Given $\mu \in \mathcal{M}_0$, we let S_μ be its topological support and $\widetilde{S}_\mu$ be its *quasi-support* in the sense of [FOT11, Equations 4.6.3-4]. We will not be using the formal definition of the quasi-support, but instead use its probabilistic characterisation given in (4.43) below. It is known that $\mu(S_\mu \setminus \widetilde{S}_\mu)=0$ for all Revuz measures [FST91], but it may happen that $\mathrm{Cap}(S_\mu \setminus \widetilde{S}_\mu) > 0$ [FOT11, Example 5.1.2]: thus, we introduce the subset

$$\mathcal{M}_{00} := \left\{\mu \in \mathcal{M}_0,\ \mathrm{Cap}(S_\mu \setminus \widetilde{S}_\mu) = 0\right\}.$$

The main result of [FST91] may be summarised as follows. Let $(\mathcal{E},\mathcal{D})$ be a regular Dirichlet form on $L^2(X,m)$ with core $\mathcal{C}$ and suppose furthermore that (4.39) holds. Let $\mu \in \mathcal{M}_0$ with $S_\mu = X$. Then the pre-Dirichlet form $(\mathcal{E},\mathcal{C})$ is closable on $L^2(X,\mu)$ if and only if $\mu \in \mathcal{M}_{00}$. Moreover, in this case, the authors give an explicit description of its closure $(\mathcal{E}_\mu,\mathcal{D}_\mu)$ on $L^2(X,\mu)$, see also [FOT11, Theorem 6.2.1]:

$$\begin{cases} \mathcal{D}_\mu := \left\{u \in L^2(X,\mu),\ \exists \widetilde{u} \in \mathcal{D} \text{ s.t. } \widetilde{u} = u \quad \mu\text{-a.e.}\right\} \\ \mathcal{E}_\mu(u,v) = \mathcal{E}(\widetilde{u},\widetilde{v}), \qquad u,v \in \mathcal{D}_\mu. \end{cases} \tag{4.40}$$

The value $\mathcal{E}_\mu(u,v)$ is independent of the choice of representatives $\widetilde{u},\widetilde{v} \in \mathcal{D}$, so the definition makes sense. The main advantage is that (4.40) gives a concrete description of the abstractedly defined form $(\mathcal{E}',\mathcal{D}')$ from (4.37).

4.A.3 Markov processes and the Revuz correspondence

To the Dirichlet form $(\mathcal{D},\mathcal{E})$ on $L^2(X,m)$ is associated a Markov process defined on a filtered probability space $(\Omega,(\mathcal{F}_t)_{t\ge 0},(\mathbb{P}_x)_{x\in X})$ with sample paths $(Z_t)_{t\ge0}$. The space is endowed with shifts of the trajectory $\theta_t : (Z_s)_{s\ge0} \mapsto (Z_{t+s})_{s\ge0}$ for all $t \ge 0$. A *positive continuous additive functional* (PCAF) is an $(\mathcal{F}_t)$-adapted, $[0,\infty]$-valued function $A = A_t(\omega)$ satisfying

$$A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega) \tag{4.41}$$

for all $t, s \geq 0, x \in X \setminus N$ and $\omega \in \Lambda$. Here, $N \subset X$ is a polar set called the *exceptional set* of A and $\Lambda \subset \mathcal{F}_\infty$ is a set of $\mathbb{P}_x$-probability one for all $x \in X \setminus N$, called the *defining set* of A. See [FOT11, Chapter 5] for more details on this definition. A PCAF is said *in the strict sense* if N can be taken empty. A PCAF provides a time change of Z by setting $\check{Z}_t := Z_{\tau_t}$, where $\tau_t := \inf\{s > 0,\, A_s > t\}$ is the right-continuous inverse of A.

The *Revuz correspondence* uniquely associates a PCAF A to each Revuz measure $\mu \in \mathcal{M}_0$ [FOT11, Theorem 5.1.4]. More precisely, under this correspondence, we have for all Borel functions f, g:

$$\int_X \mathbb{E}_x \left[\int_0^t f(Z_s)\mathrm{d}A_s \right] h(x)\mathrm{d}m(x) = \int_0^t \int_X f(x)\mathbf{p}_s h(x)\mathrm{d}\mu(x)\mathrm{d}s, \tag{4.42}$$

where $(\mathbf{p}_t)_{t\geq 0}$ denotes the semi-group of the process.

The quasi-support of μ has an important characterisation in terms of its PCAF:

$$\widetilde{S}_\mu = \{x \in X \setminus N,\, \mathbb{P}_x(R = 0) = 1\}, \tag{4.43}$$

where $R := \inf\{t > 0,\, A_t > 0\}$ is the first time of increase of A [FOT11, Theorem 5.1.5]. We will also refer to $\widetilde{S}_\mu$ as the *support of the PCAF*, and we will say that A has full support if $\mathrm{Cap}(S_\mu \setminus \widetilde{S}_\mu) = 0$, i.e. if $\mu \in \mathcal{M}_{00}$. Thus, in order to check the condition that $\mu \in \mathcal{M}_{00}$, it suffices to show that the PCAF increases instantaneously almost surely under $\mathbb{P}_x$ for quasi-every $x \in X$. This probabilistic reformulation is quite useful since statements involving the fine topology are usually difficult.

Chapter 5

log-regularity of SLE_4

This chapter is adapted from [Bav20].

We prove that the welding homeomorphism of SLE_4 is almost surely log-regular, which is the most natural property that a non-Hölder homeomorphism can have.

5.1 Introduction

5.1.1 Jordan curves

5.1.1.1 Conformal welding

Let $\eta : \mathbb{S}^1 \to \mathbb{C}$ be a Jordan curve, bounding two complementary Jordan domains $\Omega^+, \Omega^- \subset \widehat{\mathbb{C}}$. Without loss of generality, we assume that $0 \in \Omega^+$ and $\infty \in \Omega^-$. Let $\psi_+ : \mathbb{D}^+ \to \Omega^+$ (resp. $\psi_- : \mathbb{D}^- \to \Omega^-$) be a Riemann uniformising map fixing 0 (resp. ∞), where $\mathbb{D}^+$ is the unit disc and $\mathbb{D}^- := \widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}^+$. By Carathéodory's conformal mapping theorem, ψ_+, ψ_- extend continuously to homeomorphisms $\psi_\pm : \bar{\mathbb{D}}^\pm \to \bar{\Omega}^\pm$ and $h := \psi_-^{-1} \circ \psi_+|_{\mathbb{S}^1}$ is a homeomorphism of the circle called the *conformal welding homeomorphism* of η. It is well-known that the mapping $\eta \mapsto h$ is neither injective nor onto: namely, there exist distinct curves (viewed up to Möbius transformations) with the same welding homeomorphism, and not every homeomorphism is the conformal welding of a Jordan curve. At the time of writing, no geometric characterisation of conformal welding homeomorphisms is available in the literature, see [Bis07] for a comprehensive review.

For the curve to be unique, it is sufficient that it is conformally removable (note that the converse is unknown [You18]). Recall that a compact set $K \subset \mathbb{C}$ is *conformally removable* if every homeomorphism of $\widehat{\mathbb{C}}$ which is conformal off K is a Möbius transformation. From the point of view of complex geometry, this means that the conformal maps ψ_+, ψ_- endow the topological sphere $\mathbb{D}^+ \sqcup \mathbb{D}^- / \sim_h$ with a well-defined complex structure, where $\sim_h$ is the equivalence relation identifying $x \in \mathbb{S}^1 = \partial\mathbb{D}^+$ with $h(x) \in \mathbb{S}^1 = \partial\mathbb{D}^-$. Another notion of removability, introduced by Jones [Jon95], is the removability for (continuous) Sobolev functions, or H^1-removability. The set K is H^1-*removable* if any $f \in H^1(\mathbb{C}\setminus\eta, \mathrm{d}z)\cap\mathcal{C}^0(\mathbb{C})$ belongs to $H^1(\mathbb{C}, \mathrm{d}z)$, that is $H^1(\mathbb{C}\setminus\eta, \mathrm{d}z)\cap\mathcal{C}^0(\mathbb{C}) = H^1(\mathbb{C}, \mathrm{d}z)\cap\mathcal{C}^0(\mathbb{C})$. Jones proved that H^1-removability implies conformal removability, but the converse is still an open question.

5.1.1.2 log-regularity

The Hilbert space $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta) \subset L^2(\mathbb{S}^1, \mathrm{d}\theta)$ is the space of traces of $H^1(\mathbb{D}, \mathrm{d}z)$ on $\mathbb{S}^1 = \partial\mathbb{D}$. It is endowed with the norm $\omega \mapsto \|\omega\|^2_{L^2(\mathbb{S}^1,\mathrm{d}\theta)} + \|\omega\|^2_{\partial}$, where the second term denotes the Dirichlet energy of the harmonic extension of ω to $\mathbb{D}$. Negligible sets for $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta)$ are called *polar*, and they are those sets with zero logarithmic capacity. Recall from [Bis07, Section 3] that a Borel set $E \subset \mathbb{S}^1$ has positive logarithmic capacity if and only if there is a Borel probability measure ν on $\mathbb{S}^1$ giving full mass to E and with finite logarithmic energy:

$$\int_{\mathbb{S}^1\times\mathbb{S}^1} \log\frac{2}{|x-y|}\mathrm{d}\nu(x)\mathrm{d}\nu(y) < \infty. \tag{5.1}$$

Following [Bis07], we say that h is log-*regular* if $h(E)$ and $h^{-1}(E)$ have zero Lebesgue measure for all polar sets $E \subset \mathbb{S}^1$. In other words, h is log-regular if the pullback measure $\mu := h^*\mathrm{d}\theta$ does not charge any polar sets of $\mathbb{S}^1$ (and similarly for the pushforward measure), i.e. μ is a Revuz measure. From the theory of Dirichlet forms (see e.g. [FOT11, Theorem 6.2.1]), we can define a Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(\mathbb{S}^1, \mu)$ with domain

$$\mathcal{D} := \{\omega \in L^2(\mathbb{S}^1, \mu),\ \exists\widetilde{\omega} \in H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta) \text{ s.t. } \widetilde{\omega} = \omega \quad \mu\text{-a.e.}\},$$

and the form $\mathcal{E}$ is given unambiguously by $\mathcal{E}(\omega, \omega) = \|\widetilde{\omega}\|^2_{\partial}$. In other words, the injection $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta) \cap \mathcal{C}^0(\mathbb{S}^1) \hookrightarrow L^2(\mathbb{S}^1, \mu)$ extends continuously and injectively to $H^{1/2}(\mathbb{S}^1, d\theta)$, and similarly for $H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta) \circ h$ into $L^2(\mathbb{S}^1, \mathrm{d}\theta)$.

There are two natural measures supported on η: the harmonic measures viewed from 0 and ∞ respectively, which we denote by σ_+ and σ_-. We say that σ_+ (resp. σ_-) is the harmonic measure from the inside (resp. outside) of η. By conformal invariance, $\sigma_\pm$ is simply the pushforward under $\psi_\pm$ of the uniform measure on $\mathbb{S}^1$. Hence, we can understand the log-regularity of h as the statement that traces of $H^1(\Omega^+, \mathrm{d}z)$ form a closed subspace of $L^2(\eta, \sigma_-)$, and vice-versa. So we can initiate a comparison of these spaces of traces in either $L^2(\eta, \sigma_+)$ or $L^2(\eta, \sigma_-)$. For instance, one can introduce the operator $A:\ H^{1/2}(\mathbb{S}^1, \mathrm{d}\theta)^2 \to L^2(\mathbb{S}^1, \mathrm{d}\theta)$, $(\omega_+, \omega_-) \mapsto \omega_+ - \omega_- \circ h$. This operator encodes the "jump" across η of a function in $H^1(\mathbb{C}\setminus\eta, \mathrm{d}z)$ whose traces on each side of η are given by $\omega_+ \circ \psi_+^{-1}$ and $\omega_- \circ \psi_-^{-1}$. In particular, one can expect the kernel of A to contain some information about removability.

5.1.2 Schramm-Loewner Evolution

SLE was introduced by Schramm [Sch00] as the conjectured (and now sometimes proved) scaling limit of interfaces of clusters of statistical mechanics models at criticality. These are random fractal curves joining boundary points of simply connected planar domains, characterised by their conformal invariance and domain Markov properties. To each $\kappa \geq 0$ corresponds a probability measure SLE_κ, whose sample path properties depend heavily on the value of κ. The case $\kappa = 0$ is deterministic and corresponds to the (hyperbolic) geodesic flow, while $\kappa > 0$ describes random fluctuations around it. A phase transition occurs at $\kappa = 4$: the curve is simple for $\kappa \in [0, 4]$ but self- and boundary-intersecting for $\kappa \in (4, 8)$ [RS05, Section 6]. Moreover, the Hausdorff dimension of the SLE_κ trace a.s. equals $\min(1 + \frac{\kappa}{8}, 2)$ [RS05, Bef08].

A few years after Schramm's groundbreaking paper, it was understood that SLE_κ for $\kappa \leq 4$ was the Jordan curve arising from the conformal welding of random surfaces according to their boundary length measure [She16], the latter being an instance of the "Liouville measure" [DS11b]. Although the construction of [DS11b] was independent, the Liouville measure is a special case of the "multiplicative chaos" measures pioneered by Kahane in the 80's [Kah85]. In [She16], Sheffield uses an *a priori* coupling between SLE and the GFF and shows furthermore that the "quantum lengths" measured from each side of the curve coincide, and correspond to the Liouville measure. This is the "quantum zipper" theorem, which also states that slicing a random surface with an independent SLE produces two independent random surfaces. Berestycki's review [Ber16] provides a gentle introduction to these topics and an abundance of complementary details. Subsequently, the quantum zipper was systematically used and generalised in the "mating of trees" approach to Liouville quantum gravity [DMS14]. The critical Liouville measure ($\kappa = 4$) was not constructed when Sheffield's paper was released, but since then Holden & Powell used recent techniques to extend the result to the critical case [HP18].

Another approach to the conformal welding of random surfaces is that of Astala, Kupiainen, Jones & Smirnov [AJKS11]. They use standard complex analysis techniques to show the existence of the welding, but unfortunately the model they consider is not the one that produces SLE. We mention that Aru, Powell, Rohde & Viklund and the author have ongoing (and independent) works aiming at a construction of SLE via conformal welding of multiplicative chaos without using the coupling with the GFF. We stress that this is not an easy problem since it falls outside the scope of standard results from the theory of conformal welding.

Since SLE arises as the interface between conformally welded random surfaces, it is crucial to know that it is conformally removable, as this implies that the complex structure induced on the welded surface is well-defined. It has been known since its introduction that SLE_κ is conformally (and H^1-) removable for $\kappa < 4$ as the boundary of a Hölder domain [RS05, Theorem 5.2]. However, the case $\kappa = 4$ is special as it corresponds to the critical point of the multiplicative chaos measures. At the moment, the only positive result is the one of [MMQ19, Theorem 1.1 & Section 2], saying that the only welding satisfying certain geometric conditions is SLE_4. On the other hand, it is known that SLE_4 is *not* the boundary of a Hölder domain [GMS18, Section 1.3]. To our knowledge, it is unknown whether it satisfies the weaker condition on the modulus of continuity contained in [JS00, Corollary 4]. Motivated by the question of the removability of SLE_4 and the considerations of the previous subsection, it is natural to ask whether the welding homeomorphism is log-regular, which we answer affirmatively.

Theorem 5.1.1. *Almost surely, the welding homeomorphism of SLE_4 is* log-*regular.*

This will be proved in Section 5.3, while Section 5.2 gives the necessary background.

5.2 Background

5.2.1 Gaussian Multiplicative Chaos

Let X be a centred Gaussian field in the unit interval with covariance

$$\mathbb{E}[X(x)X(y)] = 2\log\frac{1}{|x-y|}, \qquad x, y \in (0,1). \tag{5.2}$$

Such a process is *a priori* ill-defined because of the logarithmic divergence on the diagonal, but it can be realised as (the restriction to $(0,1)$ of) the trace on $\mathbb{R}$ of the Gaussian Free Field (GFF) in $\mathbb{H}$ with free boundary conditions. With this procedure, we get a distribution in $(0,1)$ which almost surely belongs to $H^{-s}(0,1)$ for all $s > 0$.

Gaussian Multiplicative Chaos with parameter $\gamma \in (0,2)$ is the random measure μ_γ on $\mathcal{I} := [0,1]$ obtained as the weak limit in probability as $\varepsilon \to 0$ of the family of measures

$$\mathrm{d}\mu_{\gamma,\varepsilon}(x) := e^{\frac{\gamma}{2}X_\varepsilon(x) - \frac{\gamma^2}{8}\mathbb{E}[X_\varepsilon^2(x)]}\mathrm{d}x,$$

where $(X_\varepsilon)_{\varepsilon>0}$ is a suitable regularisation of X at scale ε [Ber17]. This measure is defined only up to multiplicative constant (since the GFF is only defined up to additive constant), but we can fix the constant by requiring it to be a probability measure (this also fixes the constant of the GFF). The point $\gamma = 2$ is critical and the renormalisation procedure above converges to 0 as $\varepsilon \to 0$, but there are several (equivalent) renormalisations that give a non-trivial limit μ_2, e.g.

$$\mathrm{d}\mu_{2,\varepsilon}(x) := \sqrt{\log\frac{1}{\varepsilon}}\,e^{X_\varepsilon(x) - \frac{1}{2}\mathbb{E}[X_\varepsilon^2(x)]}\mathrm{d}x. \tag{5.3}$$

Here, the (deterministic) diverging factor $\sqrt{\log \frac{1}{\varepsilon}}$ compensates for the decay to zero mentioned above. The topology of convergence is the same as in the subcritical case. This renormalisation was considered in [DRSV14b] (the so-called "Seneta-Heyde" renormalisation), but the critical measure can also be obtained by the "derivative martingale" approach [DRSV14a] or as a suitable limit of subcritical measures [APS19]. It is a fact of importance that the limiting measure is universal in the sense that it essentially does not depend on the choice of renormalisation or regularisation of the field [JS17, Pow18]. For concreteness, we will assume the regularisation $(X_\varepsilon)_{\varepsilon>0}$ of [BKN+15]:

$$\frac{1}{2}\mathbb{E}[X_\varepsilon(x)X_\varepsilon(y)] = \begin{cases} \log \dfrac{1}{|x-y|} & \text{if } \varepsilon \leq |x-y| \leq 1 \\ \log \dfrac{1}{\varepsilon} + 1 - \dfrac{|x-y|}{\varepsilon} & \text{if } |x-y| < \varepsilon. \end{cases} \tag{5.4}$$

Because of the exact logarithmic form of the covariance (5.2), the measures μ_γ (for $\gamma \in [0,2]$) satisfy an exact scale invariance property [BKN+15, Appendix A.1]. In particular, for any interval $I \subset \mathcal{I}$, the restriction $\mu_\gamma|_I$ of μ_γ to I satisfies:

$$\mu_\gamma|_I \overset{\text{law}}{=} |I| e^{\frac{\gamma}{2}X_I - \frac{\gamma^2}{8}\mathbb{E}[X_I^2]} \mu_\gamma^I, \tag{5.5}$$

where $X_I \sim \mathcal{N}(0, 2\log\frac{1}{|I|})$ and μ_γ^I is an independent measure with law $\mu_\gamma^I(\cdot) \overset{\text{law}}{=} \mu_\gamma(|I|^{-1}\,\cdot)$.

For the reader's convenience, we recall some basic properties of these measures, highlighting the pathologies arising at the critical point. The behaviour of μ_γ gets wilder as γ increases: almost surely, it gives full mass to a set of Hausdorff dimension $1-\frac{\gamma^2}{4}$, consisting of those points where X is exceptionally large. In the critical case, μ_2 gives full mass to a set of Hausdorff dimension 0, corresponding to the "maximum" of X. This set is still large enough for μ_2 to be non-atomic, see also [BKN+15, Theorem 2] for bounds on the modulus of continuity of μ_2.

As a result, the distribution of $\mu_\gamma(\mathcal{I})$ has a heavy tail near ∞, so that positive moments $\mathbb{E}[\mu_\gamma(\mathcal{I})^p]$ are finite if and only if $p < \frac{4}{\gamma^2}$. In particular, $\mu_2(\mathcal{I})$ does not have a finite expected value, see also [BKN+15, Theorem 1] for precise tail asymptotics. On the other hand, the tail of $\mu_\gamma(\mathcal{I})$ at 0^+ is nice, and $\mathbb{E}[\mu_\gamma(\mathcal{I})^p] < \infty$ for all $p < 0$ and $\gamma \in [0,2]$.

Let $\mu_+^\gamma, \mu_-^\gamma$ be independent GMCs on $\mathcal{I}$ with parameter $\gamma \in [0,2]$. Since a.s. $\mu_\pm^\gamma$ is non-atomic and $\mu_\pm^\gamma(\mathcal{I}) < \infty$, we can define homeomorphisms $h_\pm$ of $\mathcal{I}$ by $h_\pm^\gamma(x) := \mu_\pm^\gamma[0,x]$. We also set $h := h_-^{-1} \circ h_+$. For $\gamma < 2$, $h_\pm$ and $h_\pm^{-1}$ are a.s. Hölder continuous [AJKS11, Theorem 3.7], thus so are h and h^{-1} and in particular they preserve polar sets. Hence h is log-regular in the subcritical case. This property is far from clear in the critical case since h_+ and h_- are a.s. *not* Hölder continuous. The main result of this section, which is proved in Section 5.3, is the following theorem.

Theorem 5.2.1. *For $\gamma = 2$, h is almost surely* log*-regular.*

5.2.2 Applications to SLE_4 and related models

Consider two independent critical GMC measures on $\mathbb{R}$ obtained by exponentiating the trace of the GFF in $\mathbb{H}$. Denote by $h : \mathbb{R} \to \mathbb{R}$ the associated welding homeomorphism. Since countable unions of sets of zero Lebesgue measure have zero Lebesgue measure, we see that h is a.s. log-regular by taking large intervals and applying Theorem 5.2.1.

The construction of SLE using conformal welding is closely related to the above setup and is based on Sheffield's so-called "(γ, α)-quantum wedges" [She16, Section 1.6], see also [Ber16, Section 5.5] and [HP18, Section 2.2] for the critical case. A quantum wedge is essentially a suitably normalised GFF in $\mathbb{H}$ with free boundary conditions and an extra logarithmic singularity at the origin (parametrised by α). One considers two independent (γ, γ)-quantum wedges and there respective boundary Liouville measure $\mu_+^\gamma, \mu_-^\gamma$ on $\mathbb{R}$, and constructs the homeomorphism $h : \mathbb{R}_+ \to \mathbb{R}_-$ characterised by $\mu_+^\gamma[0, x] = \mu_-^\gamma[h(x), 0]$ for all $x \in \mathbb{R}_+$. For $\kappa = \gamma^2 \in (0, 4)$, [She16] proves that solving the conformal welding problem for this model produces an $\mathrm{SLE}_{\kappa=\gamma^2}$ on top of an independent $(\gamma, \gamma - \frac{2}{\gamma})$-quantum wedge. [HP18, Theorem 1.2] extends this result to $\kappa = \gamma^2 = 4$. The extra log-singularity at the origin amounts in conditioning the origin to be a typical point of the Liouville measure, so it does not change any capacity properties of the homeomorphism. Thus, Theorem 5.2.1 implies that the welding homeomorphism of SLE_4 is log-regular, from which Theorem 5.1.1 follows.

Finally, by Möbius invariance, we get similar statements in the disc model. Namely, let μ_+, μ_- be critical GMC measures on $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ (normalised to be probability measures), obtained by exponentiating the trace on $\mathbb{S}^1$ of two independent free boundary GFFs in $\mathbb{D}$. Let $h : \mathbb{S}^1 \to \mathbb{S}^1$ be the associated welding homeomorphism, i.e. $\mu_+[0, \theta] = \mu_-[0, h(\theta)]$ for all $\theta \in \mathbb{R}/\mathbb{Z}$. Theorem 5.2.1 implies that h is almost surely log-regular.

5.2.3 Preliminaries

Recall the setup: μ_+ and μ_- are independent critical GMC measures on $\mathcal{I}$ as defined in Section 5.2.1, $h_\pm(x) = \frac{\mu_\pm[0,x]}{\mu_\pm(\mathcal{I})}$ and $h = h_-^{-1} \circ h_+$. By symmetry, to show the log-regularity of h, it suffices to prove that $|h(E)| = 0$ for all polar sets $E \subset \mathcal{I}$.

It is known that μ_+ is a.s. a Revuz measure [RV15, Section 4] but this is not sufficient to establish that $|h(E)| = 0$ for all E polar. Indeed, it could happen (and it actually does) that there exists some polar set E such that $h_+(E)$ has positive Hausdorff dimension, and then nothing could provide *a priori* $h(E) = h_-^{-1}(h_+(E))$ from having positive Lebesgue measure. To prove Theorem 5.2.1, we will have to analyse better the properties of h_-^{-1} and the sets where μ_+ is exceptionally large.

For each $n \in \mathbb{N}$, we let

$$\mathcal{D}_n := \left\{[k2^{-n}, (k+1)2^{-n}),\, k = 0, ..., 2^n - 1\right\}$$

be the set of all dyadic intervals of length 2^{-n}, and for $x \in \mathcal{I}$, $I_n(x) \in \mathcal{D}_n$ is the dyadic interval containing x. A *gauge function* is a non-decreasing function $f : [0, 1) \to \mathbb{R}_+$ such that $f(0) = 0$. Given such a function, we introduce the set

$$E_n^f := \{I \in \mathcal{D}_n |\, \mu(I) \geq f(|I|)\}$$

and $E^f := \limsup_{n\to\infty} E_n^f$. For $\alpha \geq 0$, we denote by $\mathcal{H}_\alpha$ the α-Hausdorff measure, i.e.

$$\mathcal{H}_\alpha(E) = \lim_{\delta\to 0} \inf \sum_i |I_i|^\alpha,$$

where for a given $\delta > 0$ the infimum runs over all coverings of E by countable collections of open intervals (I_i) with Lebesgue measure $|I_i| \leq \delta$. We denote by $\dim E$ the Hausdorff dimension of a set $E \subset \mathcal{I}$, i.e. $\dim E = \sup\{\alpha \geq 0 \text{ s.t. } \mathcal{H}_\alpha(E) = \infty\} = \inf\{\alpha \geq 0 \text{ s.t. } \mathcal{H}_\alpha(E) = 0\}$.

In [BKN+15], the authors show that μ_+ gives full mass to a set of Hausdorff dimension zero, i.e. they find a gauge function f such that $\mathcal{H}_\alpha(E^f) = 0$ for all $\alpha > 0$ (i.e. $\dim E^f = 0$) and $\mathcal{H}_1(h_+(\mathcal{I} \setminus E)) = 0$ (Theorem 4 & Corollary 24). On the other hand, they give a bound on the modulus of continuity of h_+ (Theorem 2), i.e. they find f such that $E^f = \varnothing$. Such f's are given by $f(u) = C(\log \frac{1}{u})^{-k}$ for $k \in (0, \frac{1}{2})$ and some (random) $C > 0$. To prove Theorem 5.2.1, we need to investigate in more detail the multifractal properties of h_-^{-1} and the behaviour of h_+ on E^{f_k} for $k > \frac{1}{2}$, where here and in the sequel,

$$f_k(u) := \left(\log \frac{1}{u}\right)^{-k}.$$

5.3 Proof of Theorem 5.2.1

5.3.1 Upper-bound on $\dim h_+(E^{f_k})$

The next lemma is a refinement of [BKN+14, Theorem 19 (3)] and its proof follows approximately the same lines.

Lemma 5.3.1. *Fix $k > \frac{1}{2}$. Almost surely, $\dim h_+(E^{f_k}) \leq 1 - \frac{1}{2k}$.*

Proof. Let $\alpha \in (1 - \frac{1}{2k}, 1)$. Fix $1 < \beta < \frac{\alpha}{1-\frac{1}{2k}}$ and $0 < \varepsilon < \frac{1}{\alpha}(\alpha k - (k - \frac{1}{2})\beta)$. This choice of parameters is well-defined and ensures that $\theta := \alpha k - (k - \frac{1}{2})\beta - \alpha\varepsilon > 0$. Also, denote

$G^{f_k} := \liminf_{n\to\infty} (\mathcal{I} \setminus E_n^{f_k})$. This is the set of points $x \in \mathcal{I}$ such that $|h_+(I_n(x))| \leq f_k(x)$ for all sufficiently large n. We first aim at showing that $\mathcal{H}_\alpha(h_+(E^{f_k} \cap G^{f_{k-\varepsilon}})) < \infty$. We have

$$\begin{aligned}\sum_{I\in\mathcal{D}_n} |h_+(I)|^\alpha \mathbb{1}_{\{f_k(|I|)<|h_+(I)|\leq f_{k-\varepsilon}(|I|)\}} &\leq n^{-\alpha(k-\varepsilon)} \sum_{I\in\mathcal{D}_n} \mathbb{1}_{\{f_k(|I|)<|h_+(I)|\leq f_{k-\varepsilon}(|I|)\}} \\ &\leq n^{-\alpha(k-\varepsilon)} \sum_{I\in\mathcal{D}_n} \left(\frac{|h_+(I)|}{f_k(|I|)}\right)^\beta \\ &= n^{-\theta} \sum_{I\in\mathcal{D}_n} \left(n^{1/2}|h_+(I)|\right)^\beta.\end{aligned} \tag{5.6}$$

We need to get a hold on the tail of this last random variable. This is already known for cascades [BKN+14, Lemma 18] and the proof in the case of GMC is a variation of the proof of [BKN+15, Theorem 2] so we will be brief. In the sequel, X denotes the field (5.2) on $(0,1)$ and $\mu = \mu_+$ the associated critical GMC measure. Let $X_{|I|}$ be the regularised field (5.4) and $Y_{|I|} := X_{|I|} - X_1$. Let $\mathcal{D}_n^e \subset \mathcal{D}_n$ be the collection of even intervals, i.e. intervals of the form $[2j2^{-n}, (2j+1)2^{-n})$. We have [BKN+15, Equation (28)]

$$(\mu(I))_{I\in\mathcal{D}_n^e} \overset{\text{law}}{=} \left(|I| \int_I e^{Y_{|I|}-\frac{1}{2}\mathbb{E}[Y_{|I|}^2]} \mathrm{d}\mu_{|I|}\right)_{I\in\mathcal{D}_n^e},$$

where $\mu_{|I|}$ is independent of $Y_{|I|}$ and the restrictions $(\mu_{2^{-n}}|_I)_{I\in\mathcal{D}_n^e}$ form a collection of independent measures. To ease notations, we relabel $Y_{2^{-n}}$ by Y_n and $\mu_{2^{-n}}$ by μ_n.

Fix $q \in (0, \beta^{-1})$ and denote $S_n := \sum_{n\in\mathcal{D}_n^e} (\sqrt{n}\mu(I))^\beta$. It will suffice to get a uniform bound on $\mathbb{E}[S_n^q]$. This will be very similar to step 2 of the proof of [BKN+15, Theorem 2]. First, we rewrite

$$\mathbb{E}\left[S_n^q\right] = \frac{\Gamma(1-q)}{q} \int_0^\infty \lambda^{-q} \left(1 - \mathbb{E}\left[e^{-\lambda S_n}\right]\right) \frac{\mathrm{d}\lambda}{\lambda}. \tag{5.7}$$

Conditionally on Y_n, the random variables $(\mu(I))_{I\in\mathcal{D}_n^e}$ are independent. Moreover, the analysis of [BKN+15] shows that

$$\mathbb{P}\left(\mu(I) \geq t |\, Y_n\right) \leq \frac{\mathcal{C} Z_I}{t}, \tag{5.8}$$

where $Z_I := \int_I e^{Y_n - \frac{1}{2}\mathbb{E}[Y_n^2]} \mathrm{d}x$ and $\mathcal{C}$ is a random variable encapsulating the error. To control this error, [BKN+15] conditions on the event that it is not too large and bounds the probability of the complement. We refer the reader to step 3 of their proof for details and assume for now on that $\mathcal{C}$ is bounded. Using the formula $1 - \mathbb{E}[e^{-\lambda X}] = \int_0^\infty \lambda e^{-\lambda x} \mathbb{P}(X \geq$

$x)\mathrm{d}x$, valid for all non-negative random variables X, (5.8) yields for each $I \in \mathcal{D}_n^e$:

$$1 - \mathbb{E}\left[\exp\left(-\lambda\left(\sqrt{n}\mu(I)\right)^\beta\right)\Big|\, Y_n\right] = \int_0^\infty \lambda e^{-\lambda t}\mathbb{P}\left((\sqrt{n}\mu(I))^\beta \geq t\big|\, Y_n\right)\mathrm{d}t$$
$$\leq \mathcal{C}\sqrt{n}Z_I \int_0^\infty \lambda e^{-\lambda t} t^{-1/\beta}\mathrm{d}t = \widetilde{\mathcal{C}}\lambda^{1/\beta}\sqrt{n}Z_I.$$

where $\widetilde{\mathcal{C}} := \Gamma(1-\frac{1}{\beta})\mathcal{C}$. Thus, by the independence of the measures $(\mu_n|_I)_{I\in\mathcal{D}_n^e}$ and the inequality $e^{-2x} \leq 1-x$ (valid for $x \in [0, \frac{1}{2}]$), we get for sufficiently small $\lambda > 0$

$$\begin{aligned}
1 - \mathbb{E}\left[e^{-\lambda S_n}\right] &= 1 - \mathbb{E}\left[\prod_{I\in\mathcal{D}_n^e}\mathbb{E}\left[\exp\left(-\lambda(\sqrt{n}\mu(I))^\beta\right)\big|\, Y_n\right]\right] \\
&\leq 1 - \mathbb{E}\left[\prod_{I\in\mathcal{D}_n^e}(1 - \widetilde{\mathcal{C}}\lambda^{1/\beta}\sqrt{n}Z_I)\right] \\
&\leq 1 - \mathbb{E}\left[\exp\left(-2\sum_{I\in\mathcal{D}_n^e}\widetilde{\mathcal{C}}\lambda^{1/\beta}\sqrt{n}Z_I\right)\right] \\
&\leq 1 - \mathbb{E}\left[\exp\left(-2\widetilde{\mathcal{C}}\lambda^{1/\beta}\sqrt{n}\int_0^1 e^{Y_n - \frac{1}{2}\mathbb{E}[Y_n^2]}\mathrm{d}x\right)\right]
\end{aligned} \tag{5.9}$$

The last term is the Laplace transform of the Seneta-Heyde renormalised measure (5.3), so we can expect to get a uniform bound in n. Indeed, from step 4 of the proof of [BKN+15, Theorem 2], given $\varepsilon \in (0, 1-\beta q)$, the last line of (5.9) is bounded by $C_\varepsilon\lambda^{\frac{1-\varepsilon}{\beta}}$ for some $C_\varepsilon > 0$ independent of n. Thus, for small $\lambda > 0$ we obtain $1 - \mathbb{E}\left[e^{-\lambda S_n}\right] \leq C_\varepsilon\lambda^{\frac{1-\varepsilon}{\beta}}$. Hence, the integrand in the RHS of (5.7) is $O(\lambda^{\frac{1-\varepsilon}{\beta}-q-1})$ as $\lambda \to 0^+$, which is integrable since $\varepsilon < 1-\beta q$. This proves that $\mathbb{E}[S_n^q]$ is uniformly bounded as $n \to \infty$. By Markov's inequality and the fact that the law of GMC is the same on even and odd intervals, we get:

$$\mathbb{P}\left(\sum_{I\in\mathcal{D}_n}\left(\sqrt{n}\mu(I)\right)^\beta \geq n^{\frac{\theta}{2}}\right) \leq 2\mathbb{P}\left(2S_n \geq n^{\frac{\theta}{2}}\right) \leq 2^{1+q}\mathbb{E}[S_n^q]n^{-\theta q/2},$$

so the Borel-Cantelli lemma implies that there exists an integer $\ell > \frac{2}{\theta}$ such that almost surely for all n sufficiently large:

$$\sum_{I\in\mathcal{D}_{n^\ell}}\left(n^{\ell/2}\mu(I)\right)^\beta \leq n^{-\frac{\ell\theta}{2}}. \tag{5.10}$$

By definition, for each $N \in \mathbb{N}$, the set $h_+(\cup_{n\geq N}E_n^{f_k})$ provides a covering of $h_+(E^{f_k})$. Moreover, given $j^\ell \leq n < (j+1)^\ell$ and $x \in \mathcal{I}$ such that $|h_+(I_n(x))| \geq f_k(2^{-n})$, we have

$$|h_+(I_{j^\ell}(x))| \geq |h_+(I_n(x))| \geq f_k(2^{-n}) \geq f_k(2^{-(j+1)^\ell}) \sim f_k(2^{-j^\ell}).$$

Hence, the set $\cup_{n^\ell \geq N} E_{n^\ell}^{cf_k}$ provides a covering of E^{f_k} for all $N \in \mathbb{N}$ and $c < 1$ (we will use $c = 1$ in the sequel for notational simplicity). Intersecting with $G^{f_{k-\varepsilon}}$ and using equations (5.6), (5.10) as well as $\frac{\ell\theta}{2} > 1$, we get

$$\begin{aligned}\mathcal{H}_\alpha\left(h_+(E^{f_k} \cap G^{f_{k-\varepsilon}})\right) &\leq \sum_{n\in\mathbb{N}} \sum_{I\in\mathcal{D}_{n^\ell}} |h_+(I)|^\alpha \mathbb{1}_{\{f_k(|I|)<h_+(I)\leq f_{k-\varepsilon}(|I|)\}} \\ &\leq \sum_{n\in\mathbb{N}\setminus\{0\}} n^{-\ell\theta} \sum_{I\in\mathcal{D}_{n^\ell}} \left(n^{\ell/2}|h_+(I)|\right)^\beta < \infty.\end{aligned}$$

This shows that $\dim h_+(E^{f_k} \cap G^{f_{k-\varepsilon}}) \leq \alpha$ almost surely.

Note that the above argument can also be applied to show that $\dim h_+(E^{f_{k'}} \cap G^{f_{k'-\varepsilon}}) \leq \alpha$ for all $k' \leq k$ (with the value of ε and δ independent of k'). Hence, we get $\dim h_+(E^{f_k}) \leq \alpha$ as a finite union of sets of the form $h_+(E^{f_{k-j\varepsilon}} \cap G^{f_{k-(j+1)\varepsilon}})$, j integer, all of which of dimension less than or equal to α. This concludes the proof since α can be taken arbitrarily close to $1 - \frac{1}{2k}$. $\square$

5.3.2 Properties of h_-^{-1}

We turn to the properties of h_-^{-1}. We start with an elementary bound on its Hölder regularity.

Lemma 5.3.2. *Almost surely, for all $\alpha < \frac{1}{4}$, h_-^{-1} is α-Hölder continuous.*

In particular, for all $E \subset \mathcal{I}$, $\dim E < \frac{1}{4}$ implies $|h_-^{-1}(E)| = 0$.

Proof. For every $\alpha, p > 0$ and intervals $I \subset \mathcal{I}$, we have by Markov's inequality and the exact scale invariance property (5.5):

$$\mathbb{P}\left(\mu_-(I) \leq |I|^\alpha\right) = \mathbb{P}\left(\mu_-(I)^{-p} \geq |I|^{-\alpha p}\right) \leq \mathbb{E}\left[\mu_-(I)^{-p}\right]|I|^{\alpha p} \leq C|I|^{(\alpha-2)p-p^2}.$$

Hence, for $\alpha > 4$ and $p = 1$, we get $\mathbb{P}(\mu_-(I) \leq |I|^\alpha) \leq C|I|^{\alpha-3} = C|I|^{1+(\alpha-4)}$. Specialising to dyadic intervals, the Borel-Cantelli lemma implies that $|h_-(I)| \geq C|I|^\alpha$ for every arc $I \subset \mathcal{I}$ and some a.s. finite constant $C > 0$, i.e. h_-^{-1} is a.s. α^{-1}-Hölder continuous. $\square$

Now we investigate the multifractal properties of h_-^{-1} in more detail. Lemma 5.3.3 below shows that h_-^{-1} transforms a set of Hausdorff dimension $\frac{1}{2}$ into a set of full Lebesgue measure. Intuitively, this can be deduced from the multifractal analysis of h_- as follows. Let

$$\widetilde{E}_\delta := \left\{x \in \mathcal{I} : \liminf_{n\to\infty} \frac{\log|h_-^{-1}(I_n(x))|}{\log|I_n(x)|} = \delta\right\}$$

and E_δ the analogous set defined for h_- instead of h_-^{-1}. Then we expect to have $h_-^{-1}(\widetilde{E}_\delta) = E_{1/\delta}$ and $\dim \widetilde{E}_\delta = \delta \dim E_{1/\delta}$. To our knowledge, the multifractal analysis of the critical measure has never been written down explicitly, but we can expect $\dim E_\delta = \delta - \frac{\delta^2}{4}$ (hence

$\dim \widetilde{E}_\delta = 1 - \frac{1}{4\delta}$) based on known facts from the subcritical case [RV14, Section 4]. For our purposes, it will be sufficient to give an upper-bound on these dimensions. Notice that these values also explain the Hölder exponent of h_-^{-1} found in Lemma 5.3.2: $E_\delta = \varnothing$ for $\delta > 4$, and the local Hölder exponent of h_-^{-1} should be bounded by $\frac{1}{\delta}$ on $h_-(E_\delta)$.

We look for a set which h_-^{-1} maps to a set of full Lebesgue measure, i.e. we look for δ such that $\delta = \dim \widetilde{E}_\delta$. That is, $0 = \delta^2 - \delta + \frac{1}{4} = (\delta - \frac{1}{2})^2$, i.e. $\delta = \frac{1}{2}$. Precisely, we have:

Lemma 5.3.3. *Almost surely,* $\dim \widetilde{E}_{1/2} \leq \frac{1}{2}$ *and* $|h_-^{-1}(\mathcal{I} \setminus \widetilde{E}_{1/2})| = 0$.

Proof. We denote $E_\delta^{\geq} := \cup_{\delta' \geq \delta} E_{\delta'}$, and their obvious generalisations $E_\delta^{\leq}, \widetilde{E}_\delta^{\geq}, \widetilde{E}_\delta^{\leq}$.

For the first claim, note that for all $\varepsilon > 0$ and $n \in \mathbb{N}$, the fact that $|h_-^{-1}(\mathcal{I})| = 1$ implies $\#\{I \in \mathcal{D}_n, |h_-^{-1}(I)| \geq |I|^{1/2+\varepsilon}\} \leq |I|^{-1/2-\varepsilon}$. Thus, for all $\alpha > \frac{1}{2} + \varepsilon$, we have

$$\mathcal{H}_\alpha(\widetilde{E}_{1/2}^{\leq}) \leq \sum_{n\in\mathbb{N}} \sum_{I \in \mathcal{D}_n} |I|^\alpha \mathbb{1}_{\{|h_-^{-1}(I)| \geq |I|^{1/2+\varepsilon}\}} \leq \sum_{n \in \mathbb{N}} |I|^{\alpha - 1/2 - \varepsilon} < \infty.$$

This implies $\dim \widetilde{E}_{1/2} \leq \dim \widetilde{E}_{1/2}^{\leq} \leq \frac{1}{2} + \varepsilon$, from which the claim follows since $\varepsilon > 0$ was arbitrary.

For the second claim, we start with the following observation. Suppose $\widetilde{I} \in \mathcal{D}_n$ is such that $|h_-^{-1}(\widetilde{I})| \leq |\widetilde{I}|^\delta$. Then there exists $I \in \mathcal{D}_{\lfloor \delta n \rfloor - 1}$ such that $|h_-(I)| \geq |\widetilde{I}| \geq (\frac{1}{4}|I|)^{1/\delta}$. Similarly, if $|h_-^{-1}(\widetilde{I})| \geq |\widetilde{I}|^\delta$, there exists $I \in \mathcal{D}_{\lceil \delta n \rceil + 1}$ such that $|h_-(I)| \leq |\widetilde{I}| \leq (4|I|)^{1/\delta}$. Note that $h_-(I)$ does not cover $\widetilde{I}$, but we can simply add the two dyadic intervals in $\mathcal{D}_n$ directly to the right and to the left of $h_-(I)$. This just has the effect of multiplying everything by a global constant.

Now we get an upper-bound on $\dim E_\delta$. Fix $\delta \in (0, 2)$ and set $\eta := 1 - \frac{\delta}{2} \in (0, 1)$. Let $\alpha > \delta - \frac{\delta^2}{4}$ and $\varepsilon \in (0, 2 - \delta)$. For all $n \in \mathbb{N}$, we have using exact scale invariance (5.5):

$$\begin{aligned}
\mathbb{E}\left[\sum_{I \in \mathcal{D}_n} |I|^\alpha \mathbb{1}_{\{|I|^{\delta+\varepsilon} \leq |h_-(I)| \leq |I|^{\delta+\varepsilon}\}}\right] &\leq \mathbb{E}\left[\sum_{I \in \mathcal{D}_n} |I|^\alpha \left(\frac{\mu_-(I)}{|I|^{\delta+\varepsilon}}\right)^\eta\right] \\
&= |I|^{\alpha - (\delta+\varepsilon)\eta - 1} \mathbb{E}\left[\mu_-(I)^\eta\right] \\
&\leq C|I|^{\alpha - (\delta+\varepsilon)\eta - 1} \times |I|^{2\eta - \eta^2} \\
&\leq C|I|^{\alpha - \varepsilon\eta - (\delta - \delta^2/4)}.
\end{aligned}$$

Summing over $n \in \mathbb{N}$ and taking $\varepsilon > 0$ arbitrarily small, we see that $\mathbb{E}[\mathcal{H}_\alpha(E_\delta)] < \infty$ for all $\alpha > \delta - \frac{\delta^2}{4}$, hence $\dim E_\delta \leq \delta - \frac{\delta^2}{4}$. Moreover, we can write $E_\delta^{\leq}$ as a countable union of sets of zero α-Hausdorff measure, hence $\dim E_\delta^{\leq} \leq \delta - \frac{\delta^2}{4}$. A similar argument shows that $\dim E_\delta^{\geq} \leq \delta - \frac{\delta^2}{4}$ for all $\delta \in (2, 4)$.

Going back to $h_-^{-1}(\widetilde{E}_{1/2})$, let $\varepsilon > 0$, $\delta \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$ and $\alpha \in (\frac{1}{\delta} - \frac{1}{4\delta^2}, 1)$. Using our previous observation, we have

$$\begin{aligned}\mathcal{H}_\alpha\left(h_-^{-1}\left(\widetilde{E}^{\geq}_{\frac{1}{2}+\varepsilon}\right)\right) &\leq \sum_{n\in\mathbb{N}}\sum_{\widetilde{I}\in\mathcal{D}_n}|h_-^{-1}(\widetilde{I})|^\alpha \mathbb{1}_{\{|h_-^{-1}(\widetilde{I})|\leq|\widetilde{I}|^\delta\}}\\ &\leq C\sum_{n\in\mathbb{N}}\sum_{I\in\mathcal{D}_{\lfloor\delta n\rfloor-1}}|I|^\alpha\mathbb{1}_{\{|h_-(I)|\geq(\frac{1}{4}|I|)^{1/\delta}\}}\\ &\leq C\sum_{n\in\mathbb{N}}\sum_{I\in\mathcal{D}_n}|I|^\alpha\mathbb{1}_{\{|h_-(I)|\geq|I|^{1/\delta}\}}.\end{aligned}$$

Here, $C>0$ is a generic constant that may change from one line to the other. By the above, this last quantity is a.s. finite for our choice of α, implying $\mathcal{H}_1(h_-^{-1}(\widetilde{E}^{\geq}_{1/2+\varepsilon}))=0$ almost surely. A similar computation shows that $\mathcal{H}_1(h_-^{-1}(\widetilde{E}^{\leq}_{1/2-\varepsilon}))=0$. Taking a sequence $\varepsilon_n\to 0$, we get $|h_-^{-1}(\mathcal{I}\setminus\widetilde{E}_{1/2})|=0$ as a countable union of sets of zero Lebesgue measure. $\square$

5.3.3 Conclusion of the proof

The next and final lemma gives an upper-bound on the size of $h_+(E^{f_k})\cap\widetilde{E}_{1/2}$.

Lemma 5.3.4. *Fix $k>\frac{1}{2}$. Almost surely,* $\dim(h_+(E^{f_k})\cap\widetilde{E}_{1/2})\leq\frac{1}{2}-\frac{1}{2k}$.

Proof. Let $\alpha,\eta>0$ and fix $\varepsilon,\delta>0$ as in the proof of Lemma 5.3.1 and recall that for all $n\in\mathbb{N}$ we have $M_n:=\#\{I\in\mathcal{D}_n,\,|h_-^{-1}(I)|\geq|I|^{1/2+\eta}\}\leq|I|^{-1/2-\eta}$. Thus, for all $I\in\mathcal{D}_n$, we have $\mathbb{P}(|h_-^{-1}(I)|\geq|I|^{1/2+\eta})=\frac{1}{\#\mathcal{D}_n}\mathbb{E}[M_n]\leq|I|^{1/2-\eta}$. Using this estimate and the fact that h_+ is independent of h_-, we can condition on h_+ to get:

$$\begin{aligned}&\mathbb{E}\left[\sum_{I\in\mathcal{D}_n}|h_+(I)|^\alpha\mathbb{1}_{\{f_k(|I|)<|h_+(I)|\leq f_{k-\varepsilon}(|I|)\}}\mathbb{1}_{\{|h(I)|\geq|h_+(I)|^{1/2+\eta}\}}\,\middle|\,h_+\right]\\ &\quad=\sum_{I\in\mathcal{D}_n}|h_+(I)|^\alpha\mathbb{1}_{\{f_k(|I|)<|h_+(I)|\leq f_{k-\varepsilon}(|I|)\}}\mathbb{P}\left(|h_-^{-1}(h_+(I))|\geq|h_+(I)|^{1/2+\eta}\,\middle|\,h_+\right)\\ &\quad\leq C\sum_{I\in\mathcal{D}_n}|h_+(I)|^{\alpha+\frac{1}{2}-\eta}\mathbb{1}_{\{f_k(|I|)<|h_+(I)|\leq f_{k-\varepsilon}(|I|)\}}.\end{aligned}$$

Suppose $\alpha>\dim(h_+(E^{f_k}))-\frac{1}{2}+\eta$. From the proof of Lemma 5.3.1, there a.s. exists an integer ℓ such that

$$\sum_{n\in\mathbb{N}}\sum_{I\in\mathcal{D}_{n^\ell}}|h_+(I)|^{\alpha+\frac{1}{2}-\eta}\mathbb{1}_{\{f_k(|I|)<|h_+(I)|\leq f_{k-\varepsilon}(|I|)\}}<\infty. \tag{5.11}$$

Moreover, we can cover $h_+(E^{f_k})\cap\widetilde{E}_{1/2}$ with the union over $n\in\mathbb{N}$ of those $I\in\mathcal{D}_{n^\ell}$ such that $|h_+(I)|\in[f_{k-\varepsilon}(|I|),f_k(|I|))$ and $|h(I)|\geq|h_+(I)|^{1/2+\eta}$. We deduce that, almost surely, $\mathbb{E}[\mathcal{H}_\alpha(h_+(E^{f_k})\cap\widetilde{E}_{1/2})|h_+]$ is bounded above by (5.11), hence a.s. $\dim(h_+(E^{f_k})\cap\widetilde{E}_{1/2})\leq\alpha$. Taking η arbitrarily close to 0 enables to take α arbitrarily close to $\dim(h_+(E^{f_k}))-\frac{1}{2}$, so that by Lemma 5.3.1:

$$\dim\left(h_+(E^{f_k}) \cap \widetilde{E}_{1/2}\right) \le \dim h_+(E^{f_k}) - \frac{1}{2} \le \frac{1}{2} - \frac{1}{2k}.$$

□

We now have all the necessary ingredients to conclude the proof of Theorem 5.2.1.

Fix $k > 4$. By definition, the local Hölder regularity of h_-^{-1} on $\widetilde{E}_{1/2}$ is $\frac{1}{2}$, so if $F \subset \widetilde{E}_{1/2}$, we have $\dim h_-^{-1}(F) \le 2 \dim F$. Hence, Lemma 5.3.4 implies

$$\dim\left(h_-^{-1}(h_+(E^{f_k}) \cap \widetilde{E}_{1/2})\right) \le 2\dim(h_+(E^{f_k}) \cap \widetilde{E}_{1/2}) \le 1 - \frac{1}{k} < 1.$$

Moreover, by Lemma 5.3.3:

$$|h_-^{-1}(h_+(E^{f_k}) \cap (\mathcal{I} \setminus \widetilde{E}_{1/2}))| \le |h_-^{-1}(\mathcal{I} \setminus \widetilde{E}_{1/2})| = 0.$$

This proves $|h(E^{f_k})| = 0$ almost surely, so we need only focus on $\mathcal{I} \setminus E^{f_k}$.

Let $F \subset \mathcal{I} \setminus E^{f_k}$ and ν be a Borel probability measure giving full mass to $h_+(F)$. The pullback measure $h_+^*\nu$ gives full mass to F and since we are on $\mathcal{I} \setminus E^{f_k}$, for all $k' \in (4, k)$ there is $C > 0$ such that

$$\int \log \frac{1}{|x-y|} \mathrm{d}h_+^*\nu(x)\mathrm{d}h_+^*\nu(y) \le C \int |x-y|^{-1/k'} \mathrm{d}\nu(x)\mathrm{d}\nu(y).$$

That is, we can bound the log-energy of $h_+^*\nu$ by the $\frac{1}{k'}$-energy of ν. By Frostman's lemma, if $\dim h_+(F) > \frac{1}{k'}$, there exists ν as above with finite $\frac{1}{k'}$-energy, hence $h_+^*\nu$ has finite log-energy and F is not polar by (5.1). Thus, for every polar set $F \subset \mathcal{I} \setminus E^{f_k}$, we have $\dim\, h_+(F) \le \frac{1}{k}$, which further implies $\dim h(F) \le \frac{4}{k} < 1$ by Lemma 5.3.2 and our assumption on k. Hence $|h(F)| = 0$ a.s. for all $F \subset \mathcal{I}$ polar. □

Remark 22. One can expect roughly $n^{k-\frac{1}{2}}$ intervals of length 2^{-n} in $E_n^{f_k}$, so it is natural to expect that the threshold when E^{f_k} becomes polar is $k = \frac{3}{2}$. From the analysis of this section, we have $\dim h(E^{f_{3/2}}) \le \frac{1}{3}$ and it shouldn't be too hard to show that this bound is sharp. Hence we get an upper-bound of $\frac{1}{3}$ for the Hausdorff dimension of the image of a polar set.

Remark 23. It follows easily from our analysis that the criterion of [JS00, Corollary 4] is *not* satisfied by SLE_4. Indeed, [GMS18, Theorem 1.1] implies that the uniformising map ψ_- is bi-Hölder on a set A such that $\dim(A^c) < 1$. Thus, $A \cap h(E^{f_k}) \neq \varnothing$ for some $k > 0$, since $\dim h(E^{f_k}) \to 1$ as $k \to \infty$. Now on $E^{f_k} \cap h^{-1}(A)$, the modulus of continuity of ψ_+ is polylogarithmic, while the Jones-Smirnov criterion is stretched exponential.

Appendix A

Thin points of log-correlated fields

This appendix presents an unpublished result concerning certain exceptional points of log-correlated fields.

We introduce a class of exceptional points of logarithmically correlated Gaussian fields, which we call the "thin points". These points are natural from the point of view of imaginary multiplicative chaos in a similar way that thick points are natural from the point of view of (real) multiplicative chaos. Thin points have the defining property that their regularised process is a Brownian motion conditioned to eventually remain bounded. The almost sure Hausdorff dimension of the thin points is computed.

A.1 Introduction

A.1.1 Overview

Logarithmically correlated fields are a class of stochastic processes on $\mathbb{R}^d$ that have become omnipresent in modern probability theory, and appear in such fields as random matrix theory, stochastic models for the Riemann ζ-function, turbulence or finance. A particularly important example in two dimensions is the Gaussian free field, which exhibits conformal invariance and is known to be the scaling limit of many statistical mechanics models at criticality.

Log-correlated fields have a covariance kernel of the form

$$\mathbb{E}[X(x)X(y)] = \log \frac{1}{|x-y|} + O(1),$$

which justifies the terminology. Here, $O(1)$ is a continuous, uniformly bounded function in x and y. Due to the logarithmic divergence, log-correlated fields are not defined pointwise put are defined as random generalised functions. In this note, we will be concerned with centred Gaussian fields called $\star$-scale invariant, as defined in Section A.1.2.

An interesting feature of log-correlated fields is their extreme value statistics. Namely, considering a regularisation of the field $(X_\varepsilon)_{\varepsilon>0}$ such that $\mathbb{E}[X_\varepsilon^2(x)] = \log\frac{1}{\varepsilon} + O(1)$, one may ask when $X_\varepsilon(x)$ is unusually large. Note that for a fixed $x \in \mathbb{R}^d$, one would typically expects $X_\varepsilon(x)$ to be of order $\sqrt{\log\frac{1}{\varepsilon}}$. For a given $\gamma > 0$, "γ-thick points" are defined to be the points such that $X_\varepsilon(x) \sim \gamma \log\frac{1}{\varepsilon}$ as $\varepsilon \to 0$, and are thus rare points. A trivial Gaussian estimate gives $\mathbb{P}(X_\varepsilon(x) \geq \gamma \log\frac{1}{\varepsilon}) \asymp \varepsilon^{\frac{\gamma^2}{2}}$, so that one can expect the Hausdorff dimension of γ-thick points to be equal to $(d - \frac{\gamma^2}{2})_+$.

In a somewhat orthogonal direction, this note is interested in an other exceptional behaviour of the field: namely, we will define the "β-thin points" to be the points $x \in \mathbb{R}^d$ such that the regularised process $(X_\varepsilon(x))_{\varepsilon>0}$ eventually remains in a bounded interval of length $\frac{\pi}{\beta}$ for $\beta > 0$ (see Section A.1.3). Here, the basic estimate is that $\mathbb{P}(\tau_\beta > \log\frac{1}{\varepsilon}) \asymp \varepsilon^{\frac{\beta^2}{2}}$, where τ_β denotes the exit time of the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ by a standard Brownian motion. Again, this suggests that the Hausdorff dimension of the β-thin points is $(d - \frac{\beta^2}{2})_+$. This is stated as Theorem A.1.1 below and will be proved in Section 2.3. As will explained in Section A.2, thin points are naturally related to imaginary multiplicative chaos, in an analogous way that thick points are related to real multiplicative chaos.

A.1.2 Setup

We consider kernels $K : \mathbb{R}^d \to \mathbb{R}$ of the type

$$K(x) = \int_1^\infty \frac{k(ux)}{u} \mathrm{d}u,$$

where $k \in \mathcal{C}_c^0(\mathbb{R}^d)$ satisfies $k(0) = 1$, compactly supported in $B(0, R)$ for some $R > 0$. We also assume that $|k(x) - k(0)| \leq c|x|^\lambda$ uniformly for some $\lambda > 0$. For each $\varepsilon > 0$, we define the cut-off kernel

$$K_\varepsilon(x) = \int_1^{1/\varepsilon} \frac{k(ux)}{u} \mathrm{d}u.$$

Under the assumptions on K, it is possible to define a family of centred Gaussian fields $(X_\varepsilon)_{\varepsilon>0}$ on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for all ε,

$$\mathbb{E}[X_\varepsilon(x) X_\varepsilon(y)] = K_\varepsilon(x - y).$$

Moreover, for each $0 < \varepsilon < \delta$, the fields $X_\varepsilon - X_\delta$ and X_δ are independent. We will write $\mathcal{F}_\varepsilon$ for the σ-algebra generated by the values of X_ε. We refer to [LRV15] for details on the construction of this family. The process $(X_\varepsilon(x))_{x\in\mathbb{R}^d, \varepsilon>0}$ has a jointly continuous modification, and we will always assume that we work with this version.

Notice that for all $x \in \mathbb{R}^d$, the process $t \mapsto X_{e^{-t}}(x)$ is a standard Brownian motion. We will denote $B_t(x) := X_{e^{-t}}(x)$ this process. Moreover, for all $x, y \in \mathbb{R}^d$ and all

$t \geq t_0 := \log \frac{R}{|x-y|}$, we have

$$\mathbb{E}[(B_t(x) - B_{t_0}(x))(B_t(y) - B_{t_0}(y))] = \int_{R/|x-y|}^{e^t} \frac{k(u(x-y))}{u} \mathrm{d}u$$
$$= \int_R^{e^t|x-y|} \frac{k(u)}{u} \mathrm{d}u = 0.$$

Therefore, the Brownian motions $B_t(x)$ and $B_t(y)$ are independent after time t_0. Up to considering the function $k(\cdot/R)$ instead of k, we can and will assume $R = 1$.

A.1.3 Thin points

Throughout the whole text, we will denote by $\dim T$ the Hausdorff dimension of a set $T \subset \mathbb{R}^d$.

Definition A.1.1. For all $\beta > 0$, the β-*thin points* of X are the set

$$\mathcal{T}_\beta := \left\{ x \in \mathbb{R}^d,\ \limsup_{t\to\infty} B_t(x) - \liminf_{t\to\infty} B_t(x) = \frac{\pi}{\beta} \right\}.$$

Obviously, since $\limsup_{t\to\infty} B_t(x)$ for all $x \in \mathbb{R}^d$, we have $\mathbb{P}(x \in \mathcal{T}_\beta) = 0$ for all $\beta > 0$. However, a standard estimate shows that

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |B_s(x)| < \frac{\pi}{\beta} \right) \underset{t\to\infty}{\asymp} e^{-\frac{\beta^2}{2}t}.$$

This suggests that $\dim \mathcal{T}_\beta = d - \frac{\beta^2}{2}$, which is confirmed by the following theorem, which is our main result. The proof is postponed to Section A.3.

Theorem A.1.1. *Almost surely, for all $\beta > 0$, we have* $\dim \mathcal{T}_\beta = (d - \frac{\beta^2}{2})_+$.

Moreover, $\#\mathcal{T}_{\sqrt{2d}} = \infty$ and $\mathcal{T}_\beta = \varnothing$ for all $\beta > \sqrt{2d}$.

A.2 Link with multiplicative chaos

Due to the logarithmic correlation, the field X only exists as a random distribution and not a function. Therefore, its exponential is not *a priori* well-defined. Making sense of exponentials of logarithmically correlated random fields is the realm of Gaussian multiplicative chaos, which was pioneered by Kahane [Kah85]. In Kahane's original theory, one starts with a parameter $\gamma > 0$ and constructs a random measure μ_γ via the regularisation procedure $\mathrm{d}\mu_{\gamma,\varepsilon}(x) = e^{\gamma X_\varepsilon - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon^2]}\mathrm{d}x$. It is known that this family of measures converges almost surely weakly as $\varepsilon \to 0$ to a non-trivial measure μ_γ, provided $\gamma < \sqrt{2d}$. Formally, we write

$$\mathrm{d}\mu_\gamma = e^{\gamma X - \frac{\gamma^2}{2}\mathbb{E}[X^2]}\mathrm{d}x.$$

The above renormalisation is known to converge to 0 for $\gamma \geq \sqrt{2d}$. The special case $\gamma = \sqrt{2d}$ is known as *critical* and it is possible to modify slightly the renormalisation procedure in order to obtain a non-trivial limiting measure [DRSV14a]. The case $\gamma > \sqrt{2d}$ is known as *supercritical* and requires a different approach.

Gaussian multiplicative chaos (GMC) is now an important object of modern probability theory and appears in several contexts such as turbulence, quantum field theory, random matrix theory, stochastic models for the Riemann ζ-functions or finance. We refer to [Ber17] for a nice introduction to the topic and [RV14] for a review of its applications.

In recent years, there has been growing interest in the imaginary version of GMC, namely when one takes $\gamma = i\beta$ for some real β. The main result in this direction [LRV15] is that the renormalised functions $F_{\beta,\varepsilon} = e^{i\beta X_\varepsilon + \frac{\beta^2}{2}\mathbb{E}[X_\varepsilon^2]}$ converge almost surely to a random distribution F_β as $\varepsilon \to 0$, provided $\beta < \sqrt{d}$ (notice the difference with the real threshold $\gamma < \sqrt{2d}$). Here, we stress that the limiting object is not a complex measure but a complex distribution: that is, F_β lives in the dual of $\mathcal{C}_c^k(\mathbb{R}^d)$ for some $k \geq 1$, but does not live in the dual of $\mathcal{C}^0(\mathbb{R}^d)$. For $\beta \geq \sqrt{d}$, the above renormalisation fails to converge. Several alternative renormalisations exist [LRV15] but the limiting distribution is a complex white-noise, so that all the information about the original field is "lost".

Real GMC encodes interesting aspects of the underlying field. Indeed, the measure μ_γ is known to give full mass to a set of exceptional points of the field called the γ-thick points, defined by

$$\mathcal{T}^\gamma := \left\{ x \in \mathbb{R}^d,\ \lim_{\varepsilon\to 0} \frac{B_t(x)}{t} = \gamma \right\}.$$

Notice that for all $x \in \mathbb{R}^d$, we have obviously $\lim B_t(x)/t = 0$, so that points in $\mathcal{T}^\gamma$ are located where the field is exceptionally large. In fact, it is known that $\dim \mathcal{T}_\gamma = d - \frac{\gamma^2}{2}$. To get an intuition of why this should be the case, consider the event $A_t^\gamma(x) := \{|B_t(x) - \gamma t| > t^{3/4}\}$. Then Girsanov's theorem, we have (with $\varepsilon = e^{-t}$)

$$\begin{aligned}\mathbb{E}\left[\mu_{\gamma,\varepsilon}(\mathbb{1}_{\{A_t^\gamma\}})\right] = \mathbb{E}\left[\int_{[0,1]^d} \mathbb{1}_{\{A_t^\gamma(x)\}} \mathrm{d}\mu_{\gamma,\varepsilon}(x)\right] &= \int_{[0,1]^d} \mathbb{E}\left[e^{\gamma B_t(x) - \frac{\gamma^2}{2}t} \mathbb{1}_{\{A_t^\gamma(x)\}}\right] \mathrm{d}x \\ &= \int_{[0,1]^d} \mathbb{P}\left(A_t^0(x)\right) \mathrm{d}x = o(1)\end{aligned}$$

This heuristic computation shows that μ_γ tends to concentrate on $\mathcal{T}^\gamma$. The natural question is whether similar sets of exceptional points are natural with respect to the imaginary multiplicative chaos F_β. The first thing to realise is that it does not make sense to speak of where F_β "puts the mass" since it is not a complex measure. Moreover, it is easy to see that the support of F_β in the sense of distributions is the whole space.

In the real case, the appearance of thick points is due to the interpretation of $e^{\gamma X - \frac{\gamma^2}{2}\mathbb{E}[X^2]}$ as a Radon-Nykodym derivative adding a γ-drift to the Brownian motion. In the complex case, let us consider the martingale $e^{i\beta B_t+\frac{\beta^2}{2}t}$, where B_t is a standard Brownian motion. Its real part is $\cos(\beta B_t)e^{\frac{\beta^2}{2}t}$. Of course, this is not a positive martingale, so it is not a Radon-Nykodym derivative. However, setting

$$\tau_\beta := \inf\left\{t\geq 0,\, |B_t|>\frac{\pi}{2\beta}\right\},$$

the stopped martingale

$$\cos(\beta B_{t\wedge\tau_\beta})e^{\frac{\beta^2}{2}t\wedge\tau_\beta}$$

is now a positive martingale. Weighted by this martingale, the law of Brownian motion run up to time $t>0$ is the stochastic process driven by the SDE

$$\mathrm{d}X_t = \beta\tan(\beta X_t)\mathrm{d}t + \mathrm{d}B_t,$$

which is the SDE of Brownian motion conditioned to stay in the interval $(-\frac{\pi}{2},\frac{\pi}{2})$ [Pin85]. One can easily read the properties of sample paths: since $\beta\tan(\beta x)\sim(\frac{\pi}{2\beta}-x)^{-1}$ as $x\to\frac{\pi}{2\beta}$ from below, the process $\frac{\pi}{2}-X_t$ looks like a 3-dimensional Bessel process upon approaching 0, and similarly for the process $\frac{\pi}{2}+X_t$.

A.3 Dimension of thin points

In this section, we compute the Hausdorff dimension of $\mathcal{T}_\beta$.

A.3.1 Preliminary estimates

Here we gather some estimates on exit times of Brownian motion from a bounded interval. In this section we denote by $\mathbb{P}_x$ the law of a standard Brownian motion $(B_t)_{t\geq 0}$ started from $x\in\mathbb{R}$, and we write $\mathbb{P}:=\mathbb{P}_0$. We also denote by M_t (resp. m_t) the running maximum (resp. minimum) of B_t, and $\tau_\beta := \inf\{t\geq 0,\, |B_t|>\frac{\pi}{2\beta}\}$

Lemma A.3.1. *There exists $C>0$ such that for all $t\geq 0$,*

$$C^{-1}e^{-\frac{\beta^2}{2}t}\leq \mathbb{P}\left(\tau_\beta>t\right)\leq\mathbb{P}\left(M_t-m_t<\frac{\pi}{\beta}\right)\leq Ce^{-\frac{\beta^2}{2}t}.$$

Proof. Obviously, $\mathbb{P}(\tau_\beta>t)\leq\mathbb{P}(M_t-m_t<\frac{\pi}{\beta})$. We can evaluate $\mathbb{P}(\tau_\beta>t)$ by Lévy's triple law [SP12]: for all $a\in(0,\frac{\pi}{\beta})$, we have

$$\mathbb{P}\left(m_t>a-\frac{\pi}{\beta},M_t<a\right)=\int_{a-\frac{\pi}{\beta}}^{a}\sum_{n\in\mathbb{Z}}\left(e^{-\frac{\left(x+\frac{2n\pi}{\beta}\right)^2}{2t}}-e^{-\frac{\left(x-2a-\frac{2n\pi}{\beta}\right)^2}{2t}}\right)\frac{dx}{\sqrt{2\pi t}}\qquad\text{(A.1)}$$

By Poisson's summation formula, we can rewrite this as

$$\begin{aligned}\mathbb{P}\left(m_t > a - \frac{\pi}{\beta}, M_t < a\right) &= \frac{\beta}{2\pi}\int_{a-\frac{\pi}{\beta}}^{a}\sum_{n\in\mathbb{Z}} e^{-\frac{n^2\beta^2 t}{2}} e^{in\beta x}\left(1-e^{-2in\beta a}\right)dx \\ &= \frac{\beta}{2\pi}\int_{a-\frac{\pi}{\beta}}^{a}\sum_{n\in\mathbb{Z}} e^{-\frac{n^2\beta^2}{2}t} e^{in\beta(x-a)} 2i\sin(n\beta a)\,dx \\ &\underset{t\to\infty}{\sim} \frac{2\beta}{\pi}e^{-\frac{\beta^2 t}{2}}\sin(\beta a)\int_{a-\frac{\pi}{\beta}}^{a}\sin(\beta(a-x))\,dx\end{aligned}$$

In particular, we have

$$C^{-1}e^{-\frac{\beta^2}{2}t} \leq \mathbb{P}(\tau_\beta > t) = \mathbb{P}\left(M_t < \frac{\pi}{2\beta}, m_t > -\frac{\pi}{2\beta}\right) \leq Ce^{-\frac{\beta^2}{2}t}.$$

for some $C > 0$.

Conditionally on $\{M_t - m_t < \frac{\pi}{\beta}\}$, the law of (M_t, m_t) has a density on $(0, \frac{\pi}{\beta}) \times (-\frac{\pi}{\beta}, 0)$, and we have $\mathbb{P}(M_t < \frac{\pi}{2\beta}, m_t > -\frac{\pi}{2\beta} | M_t - m_t < \frac{\pi}{\beta}) \geq p$ for some $p \in (0,1)$ independent of t. Thus,

$$\begin{aligned}\mathbb{P}\left(M_t - m_t < \frac{\pi}{\beta}\right) &\leq \frac{\mathbb{P}\left(M_t - m_t < \frac{\pi}{\beta}, M_t < \frac{\pi}{2\beta}, m_t > -\frac{\pi}{2\beta}\right)}{\mathbb{P}\left(M_t < \frac{\pi}{2\beta}, m_t > -\frac{\pi}{2\beta} | M_t - m_t < \frac{\pi}{\beta}\right)} \\ &\leq \frac{1}{p}\mathbb{P}\left(M_t < \frac{\pi}{2\beta}, m_t > -\frac{\pi}{2\beta}\right) \leq Ce^{-\frac{\beta^2}{2}t}.\end{aligned}$$

□

A.3.2 Upper-bound

In this section we prove the upper-bound on $\dim \mathcal{T}_\beta$.

Proposition A.3.2. *For all $\beta \geq 0$, almost surely,* $\dim \mathcal{T}_\beta \leq (d - \frac{\beta^2}{2})_+$.
Moreover, $\mathcal{T}_\beta = \varnothing$ for all $\beta > \sqrt{2d}$.

We will require a Kolmogorov criterion to estimate the regularity of the joint process $(X_\varepsilon(x))_{x\in\mathbb{R}^d, \varepsilon>0}$. The version given below is not optimal but sufficient for our purposes. Recall that $\lambda > 0$ is such that the kernel function satisfies $|k(x) - k(0)| \leq c|x|^\lambda$.

Lemma A.3.3. *Let $\lambda' \in (0, \lambda)$. Almost surely, there exists a finite $M > 0$ such that for all $x, y \in [0,1]^d$ and $\varepsilon \in (0,1)$, we have*

$$|X_\varepsilon(x) - X_\varepsilon(y)| \leq M\left(\frac{|x-y|}{\varepsilon}\right)^{\lambda'}.$$

Proof. By the assumptions on the function k, we have for all $x, y \in \mathbb{R}^d$ and $\varepsilon > 0$.

$$\mathbb{E}\left[(X_\varepsilon(x) - X_\varepsilon(y))^2\right] = 2\int_{|x-y|}^{|x-y|/\varepsilon} \frac{1-k(u)}{u} \mathrm{d}u \le C\left(\frac{|x-y|}{\varepsilon}\right)^\lambda.$$

By the Kolmogorov criterion, we obtain that for all $\lambda' \in (0, \lambda)$, there exists an a.s. finite $M > 0$ such that for all dyadics $(x, y, \varepsilon) \in [0,1]^d \times [0,1]^d \times (0,1)$, we have

$$|X_\varepsilon(x) - X_\varepsilon(y)| \le M\left(\frac{|x-y|}{\varepsilon}\right)^{\lambda'}. \tag{A.2}$$

Since the process $(X_\varepsilon(x))_{x\in\mathbb{R}^d, \varepsilon>0}$ is jointly continuous, (A.2) extends to all $(x, y, \varepsilon) \in [0,1]^d \times [0,1]^d \times (0,1)$. ☐

Proof. Given $n \in \mathbb{N}$, let $\mathcal{D}_n := e^{-n}\mathbb{Z}^d \cap [0,1]^d$ and $\mathcal{D} := \cup_{n\in\mathbb{N}}\mathcal{D}_n$. In all the proof, $(\eta, \beta', \lambda') \in (0, \frac{1}{2}) \times (0, \beta) \times (0, \lambda)$ are fixed (but arbitrary) parameters. We also fix an almost surely finite $M > 0$ as in Lemma A.3.3.

Let $\mathcal{C}_n \subset \mathcal{D}_n$ be the set of points such that

$$\sup_{\eta n \le s \le (1-\eta)n} B_s(x) - \inf_{\eta n \le s \le (1-\eta)n} B_s(x) < \frac{\pi}{\beta'},$$

and

$$\mathcal{B}_n := \cup_{x\in\mathcal{C}_n} B(x, e^{-n}),$$

where $B(x, r) \subset \mathbb{R}^d$ denotes the ball centred at x of radius r.

Now we show that $\cup_{n\in\mathbb{N}}\mathcal{B}_n$ almost surely covers $\mathcal{T}_\beta$. Let $x \in \mathcal{T}_\beta$. We may fix $n_0 \in \mathbb{N}$ such that

$$\sup_{t\ge\eta n_0} B_t(x) - \inf_{t\ge\eta n_0} B_t(x) < \frac{\pi}{\beta}. \tag{A.3}$$

For all $n \in \mathbb{N}$, fix $x_n \in \mathcal{D}_n$ be such that $|x - x_n| \le e^{-n}$. By Lemma A.3.3, we have for all $t \le (1-\eta)n$,

$$|B_t(x) - B_t(x_n)| \le M\left(e^t|x - x_n|\right)^{\lambda'} \le Me^{-\eta\lambda' n}.$$

Thus, for all $n \ge n_0$ so large that $2Me^{-\eta\lambda' n} \le \frac{\pi}{\beta'} - \frac{\pi}{\beta}$, we have using (A.3)

$$\begin{aligned}
&\sup_{\eta n \le t \le (1-\eta)n} B_t(x_n) - \inf_{\eta n \le t \le (1-\eta)n} B_t(x_n) \\
&\quad \le \sup_{\eta n \le t \le (1-\eta)n} B_t(x) - \inf_{\eta n \le t \le (1-\eta)n} B_t(x) + 2Me^{-\eta\lambda' n} \\
&\quad \le \frac{\pi}{\beta} + 2Me^{-\eta\lambda' n} \le \frac{\pi}{\beta'}.
\end{aligned}$$

Therefore, $x_n \in \mathcal{B}_n$ for all sufficiently large n, so that $\cup_{n\in\mathbb{N}}\mathcal{B}_n$ covers $\mathcal{T}_\beta$ as claimed.

To conclude, we bound the α-Hausdorff content $\mathcal{H}_\alpha$ of $\mathcal{T}_\beta$ using this covering. First, we have for all $\alpha > 0$,

$$\mathcal{H}_\alpha(\mathcal{T}_\beta) \leq \sum_{n\in\mathbb{N}} \sum_{x\in\mathcal{C}_n} \mathrm{diam}(B(x_n, e^{-n}))^\alpha = \sum_{n\in\mathbb{N}} \#\mathcal{C}_n e^{-n\alpha}.$$

On the other hand, we have by Lemma A.3.1

$$\mathbb{E}\left[\#\mathcal{C}_n\right] = \mathbb{E}\left[\sum_{x\in\mathcal{D}_n} \mathbb{1}_{\{x\in\mathcal{C}_n\}}\right] = \#\mathcal{D}_n \mathbb{P}\left(0 \in \mathcal{C}_n\right) \asymp e^{(d-(1-2\eta)\frac{\beta'^2}{2})n}.$$

Assume that $\beta \leq \sqrt{2d}$. For all $\alpha > d - (1-2\eta)\frac{\beta'^2}{2}$, we have

$$\mathbb{E}\left[\mathcal{H}_\alpha(\mathcal{T}_\beta)\right] = \sum_{n\in\mathbb{N}} \mathbb{E}[\#\mathcal{C}_n] e^{-n\alpha} < \infty,$$

In particular $\mathcal{H}_\alpha(\mathcal{T}_\beta) < \infty$ almost surely and $\dim \mathcal{T}_\beta \leq d - (1-2\eta)\frac{\beta'^2}{2}$. The result follows since η (resp. β') can be taken arbitrarily close to 0 (resp. β).

Assume now that $\beta > \sqrt{2d}$ and choose η, β' in such a way that $d - (1-2\eta)\frac{\beta'^2}{2} < 0$. By Markov's inequality, we have $\mathbb{P}(\mathcal{C}_n \neq \varnothing) \leq \mathbb{E}[\#\mathcal{C}_n]$ decays exponentially fast, so the Borel-Cantellli lemma implies that a.s. $\mathcal{C}_n = \varnothing$ for all sufficiently large n. Thus, $\mathcal{T}_\beta = \varnothing$. □

A.3.3 Lower-bound

In this section, we prove the lower-bound. The proof relies on Frostman's lemma, which we now recall. The *α-energy* of a Borel measure ν on $\mathbb{R}^d$ is the quantity

$$I_\alpha(\nu) := \int_{\mathbb{R}^d} \frac{\mathrm{d}\nu(x)\mathrm{d}\nu(y)}{|x-y|^\alpha}.$$

By Frostman's lemma, to show that $\dim \mathcal{T}_\beta \geq d - \frac{\beta^2}{2}$, it suffices to show that there exists a Borel measure ν on $\mathbb{R}^d$ such that $\nu(\mathbb{R}^d \setminus \mathcal{T}_\beta) = 0$ and $I_\alpha(\nu) < \infty$ for all $\alpha < d - \frac{\beta^2}{2}$.

Proposition A.3.4. *For all $\beta \in (0, \sqrt{2d})$, we have* $\dim \mathcal{T}_\beta \geq d - \frac{\beta^2}{2}$.

Throughout this section, we will use the following events, defined for all $0 \leq t_0 \leq t$ and $x \in \mathbb{R}^d$:

$$E^\beta_{t_0,t}(x) := \left\{ \sup_{t_0\leq s\leq t} |B_s(x)| < \frac{\pi}{2\beta} \right\},$$
$$E^\beta_t(x) := \left\{ \sup_{0\leq s\leq t} |B_s(x)| < \frac{\pi}{2\beta} \right\}.$$

Notice that $E_t(x) = E_{t_0}(x) \cap E_{t_0,t}(x)$. When the context is clear, we will write $E_t(x) =$

$E^{\beta}_t(x)$.

To show this proposition, we will require the following two-point estimate.

Lemma A.3.5. *We have for all $x, y \in [0,1]^d$,*

$$\mathbb{P}\left(E_t(x)E_t(y)\right) \asymp |x-y|^{-\frac{\beta^2}{2}} \mathbb{P}\left(E_t(x)\right)\mathbb{P}\left(E_t(y)\right).$$

Proof. Let $t_0 := \log \frac{1}{|x-y|}$ and recall that the two processes $(B_t(x) - B_{t_0}(x))_{t \geq t_0}$ and $(B_t(y) - B_{t_0}(y))_{t \geq t_0}$ are independent Brownian motions. Thus, conditioning on $\mathcal{F}_{|x-y|}$, we have by the Markov property of Brownian motion and Lemma A.3.5,

$$\begin{aligned}
\mathbb{P}\left(E_t(x) \cap E_t(y)\right) &= \mathbb{E}\left[\mathbb{P}\left(E_t(x) \cap E_t(y) | \mathcal{F}_{|x-y|}\right)\right] \\
&= \mathbb{P}\left(E_{t_0}(x) \cap E_{t_0}(y)\right) \mathbb{E}\left[\mathbb{P}\left(E_{t_0,t}(x) \cap E_{t_0,t}(y) | \mathcal{F}_{|x-y|}\right)\right] \\
&\leq \mathbb{P}(E_{t_0}(x))\mathbb{P}\left(E_{t_0,t}(x)\right)\mathbb{P}\left(E_{t_0,t}(y)\right) \\
&= \frac{\mathbb{P}(E_t(x))\mathbb{P}(E_t(y))}{\mathbb{P}(E_{t_0}(y))} \\
&\asymp |x-y|^{-\frac{\beta^2}{2}} \mathbb{P}(E_t(x))\mathbb{P}(E_t(y)).
\end{aligned}$$

□

Proof. We introduce the following subset of $\mathcal{T}_\beta$, called the *β-perfect points*

$$\mathcal{P}_\beta := \left\{ x \in [0,1]^d, \sup_{t \geq 0} |B_t(x)| < \frac{\pi}{2\beta} \right\}.$$

Our goal is to show that almost surely $\dim \mathcal{P}_\beta \geq d - \frac{\beta^2}{2}$.

For each $t \geq 0$, let us define the measure

$$\mathrm{d}\nu_t(x) := \frac{\mathbb{1}_{\{E_t(x)\}}}{\mathbb{P}(E_t(x))} \mathrm{d}x.$$

Notice that

$$\mathbb{E}\left[\nu_t([0,1]^d)\right] = \int_{[0,1]^d} \mathrm{d}x = 1.$$

Let us evaluate the α-energy of ν_t. By Lemma A.3.5, we have

$$\begin{aligned}
\mathbb{E}[I_\alpha(\nu_t)] &= \mathbb{E}\left[\int_{[0,1]^d}\int_{[0,1]^d} \frac{\mathbb{1}_{E_t(x)}\mathbb{1}_{E_t(y)}}{|x-y|^\alpha \mathbb{P}(E_t(x))\mathbb{P}(E_t(y))} \mathrm{d}x\mathrm{d}y\right] \\
&= \int_{[0,1]^d}\int_{[0,1]^d} \frac{\mathbb{P}(E_t(x) \cap E_t(y))}{|x-y|^\alpha \mathbb{P}(E_t(x))\mathbb{P}(E_t(y))} \mathrm{d}x\mathrm{d}y \\
&\leq C \int_{[0,1]^d}\int_{[0,1]^d} |x-y|^{-(\alpha + \frac{\beta^2}{2})} \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

for some constant $C > 0$ independent of t. This last integral is finite provided $\alpha < d - \frac{\beta^2}{2}$, so that $\mathbb{E}[I_\alpha(\nu_t)]$ is uniformly bounded in this case.

From here, a standard argument [HMP10] shows that with positive probability there exists a subsequence ν_{t_n} converging weakly to a measure ν such that $\nu([0,1]^d) \in (0, \infty)$ and $I_\alpha(\nu) < \infty$. On this event, we obviously have $\nu([0,1]^d \setminus \mathcal{P}_\beta) = 0$, so that $\dim \mathcal{T}_\beta \geq \dim \mathcal{P}_\beta \geq d - \frac{\beta^2}{2}$. On the other hand, it is also a standard argument that the dimension of $\mathcal{T}_\beta$ is an almost sure quantity, so that $\mathbb{P}(\dim \mathcal{T}_\beta \geq d - \frac{\beta^2}{2}) = 1$. □

Proposition A.3.6. *Almost surely, $\mathcal{T}_{\sqrt{2d}}$ is uncountable.*

Proof. Consider the modified events

$$\dot{E}_t(x) := \left\{ |B_s(x)| \leq \frac{\pi}{2\beta_s}, \forall 0 \leq s \leq t \right\},$$
$$\dot{E}_{t_0,t}(x) := \left\{ |B_s(x)| \leq \frac{\pi}{2\beta_s}, \forall t_0 \leq s \leq t \right\},$$

where we choose the function $\beta_s = (2d - \frac{5}{1+s})^{1/2}$. Then we have $\mathbb{P}(\dot{E}_t(x)) \leq Ct^3 e^{-dt}$. On the other hand, similarly to the previous proof, we have the two-point estimate

$$\mathbb{P}\left(\dot{E}_t(x) \cap \dot{E}_t(y)\right) \leq C \frac{|x-y|^{-d}}{\log^3(1 + \frac{1}{|x-y|})} \mathbb{P}\left(\dot{E}_t(x)\right) \mathbb{P}\left(\dot{E}_t(y)\right).$$

Now we define the same measure as before, $\mathrm{d}\nu_t = \frac{\mathbb{1}_{\dot{E}_t}}{\mathbb{P}(\dot{E}_t)} \mathrm{d}x$, and we compute its logarithmic energy:

$$\mathbb{E}[I_{\log}(\nu_t)] = \mathbb{E}\left[\int_{[0,1]^d} \int_{[0,1]^d} \log(1 + \frac{1}{|x-y|}) \mathrm{d}\nu_t(x) \mathrm{d}\nu_t(y)\right]$$
$$\leq C \int_{[0,1]^d} \int_{[0,1]^d} \frac{|x-y|^d}{\log^2(1 + \frac{1}{|x-y|})} < \infty.$$

This allows us to define almost surely a measure with finite logarithmic energy giving full mass to $\mathcal{T}_{\sqrt{2d}}$, implying in particular that it is uncountable (in fact we have shown that $\mathcal{T}_{\sqrt{2d}}$ has positive logarithmic capacity). □

Bibliography

[ABO17] Jørgen E. Andersen, Gaëtan Borot, and Nicolas Orantin. Geometric recursion. *arXiv e-prints*, page arXiv:1711.04729, Nov 2017.

[AHS21] Morris Ang, Nina Holden, and Xin Sun. Integrability of SLE via conformal welding of random surfaces. *arXiv e-prints*, page arXiv:2104.09477, April 2021.

[AJKS11] Kari Astala, Peter Jones, Antti Kupiainen, and Eero Saksman. Random conformal weldings. *Acta Math.*, 207(2):203–254, 2011.

[APS19] Juhan Aru, Ellen Powell, and Avelio Sepúlveda. Critical Liouville measure as a limit of subcritical measures. *Electron. Commun. Probab.*, 24:Paper No. 18, 16, 2019.

[AS02] O. Angel and O. Schramm. Uniform infinite planar triangulations. *Communications in Mathematical Physics*, 241:191–213, 2002.

[AS21] Morris Ang and Xin Sun. Integrability of the conformal loop ensemble. *arXiv e-prints*, page arXiv:2107.01788, July 2021.

[Bav19] Guillaume Baverez. Modular bootstrap agrees with the path integral in the large moduli limit. *Electron. J. Probab.*, 24:Paper No. 144, 22, 2019.

[Bav20] Guillaume Baverez. On the log-regularity of SLE_4. *arXiv e-prints*, page arXiv:2004.09462, April 2020.

[Bef08] Vincent Beffara. The dimension of the SLE curves. *Ann. Probab.*, 36(4):1421–1452, 2008.

[Ben18] Stéphane Benoist. Natural parametrization of SLE: the Gaussian free field point of view. *Electron. J. Probab.*, 23:Paper No. 103, 16, 2018.

[Ber15] Nathanaël Berestycki. Diffusion in planar Liouville quantum gravity. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(3):947–964, 2015.

[Ber16] Nathanaël Berestycki. Introduction to the Gaussian Free Field and Liouville Quantum Gravity. *Notes available on the webpage of the author*, 2016.

[Ber17] Nathanaël Berestycki. An elementary approach to Gaussian multiplicative chaos. *Electron. Commun. Probab.*, 22:Paper No. 27, 12, 2017.

[Bis07] Christopher J. Bishop. Conformal welding and Koebe's theorem. *Ann. of Math. (2)*, 166(3):613–656, 2007.

[BKN$^+$14] Julien Barral, Antti Kupiainen, Miika Nikula, Eero Saksman, and Christian Webb. Critical Mandelbrot cascades. *Comm. Math. Phys.*, 325(2):685–711, 2014.

[BKN$^+$15] Julien Barral, Antti Kupiainen, Miika Nikula, Eero Saksman, and Christian Webb. Basic properties of critical lognormal multiplicative chaos. *Ann. Probab.*, 43(5):2205–2249, 2015.

[BPZ84] A. A. Belavin, Alexander M. Polyakov, and A. B. Zamolodchikov. Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory. *Nucl. Phys. B*, 241:333–380, 1984.

[BR87] M.J. Bowick and S.G. Rajeev. The holomorphic geometry of closed bosonic string theory and DiffS^1/S^1. *Nuclear Physics B*, 293:348 – 384, 1987.

[BW18] Guillaume Baverez and Mo Dick Wong. Fusion asymptotics for Liouville correlation functions. *arXiv e-prints*, page arXiv:1807.10207, July 2018.

[BZ06] A. A. Belavin and A. B. Zamolodchikov. Integrals over moduli spaces, ground ring, and four-point function in minimal Liouville gravity. *Theor. Math. Phys.*, 147:729–754, 2006.

[Dav88] F. David. Conformal Field Theories Coupled to 2D Gravity in the Conformal Gauge. *Mod. Phys. Lett. A*, 3:1651, 1988.

[DDDF19] Jian Ding, Julien Dubédat, Alexander Dunlap, and Hugo Falconet. Tightness of Liouville first passage percolation for $\gamma \in (0, 2)$. *arXiv e-prints*, page arXiv:1904.08021, April 2019.

[DFG$^+$19] Julien Dubédat, Hugo Falconet, Ewain Gwynne, Joshua Pfeffer, and Xin Sun. Weak LQG metrics and Liouville first passage percolation. *arXiv e-prints*, page arXiv:1905.00380, May 2019.

[DG20] Jian Ding and Ewain Gwynne. The Fractal Dimension of Liouville Quantum Gravity: Universality, Monotonicity, and Bounds. *Comm. Math. Phys.*, 374(3):1877–1934, 2020.

[DKRV16] François David, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Liouville quantum gravity on the Riemann sphere. *Comm. Math. Phys.*, 342(3):869–907, 2016.

[DKRV17] François David, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Renormalizability of Liouville quantum field theory at the Seiberg bound. *Electron. J. Probab.*, 22:Paper No. 93, 26, 2017.

[DMS14] Bertrand Duplantier, Jason Miller, and Scott Sheffield. Liouville quantum gravity as a mating of trees. *To appear in Astérisque*, page arXiv:1409.7055, Sep 2014.

[DO94] H. Dorn and H.-J. Otto. Two- and three-point functions in Liouville theory. *Nuclear Phys. B*, 429(2):375–388, 1994.

[DP86] Eric D'Hoker and D.H. Phong. Multiloop amplitudes for the bosonic polyakov string. *Nuclear Physics B*, 269(1):205–234, 1986.

[DRSV14a] Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, and Vincent Vargas. Critical Gaussian multiplicative chaos: convergence of the derivative martingale. *Ann. Probab.*, 42(5):1769–1808, 2014.

[DRSV14b] Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, and Vincent Vargas. Renormalization of critical Gaussian multiplicative chaos and KPZ relation. *Comm. Math. Phys.*, 330(1):283–330, 2014.

[DRV16] François David, Rémi Rhodes, and Vincent Vargas. Liouville quantum gravity on complex tori. *J. Math. Phys.*, 57(2):022302, 25, 2016.

[DS11a] Hugo Duminil-Copin and Stanislav Smirnov. Conformal invariance of lattice models. *arXiv e-prints*, page arXiv:1109.1549, September 2011.

[DS11b] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011.

[Dub09] Julien Dubédat. SLE and the free field: partition functions and couplings. *J. Amer. Math. Soc.*, 22(4):995–1054, 2009.

[Dub15] Julien Dubédat. SLE and Virasoro representations: localization. *Comm. Math. Phys.*, 336(2):695–760, 2015.

[DZZ19] Jian Ding, Ofer Zeitouni, and Fuxi Zhang. Heat kernel for Liouville Brownian motion and Liouville graph distance. *Comm. Math. Phys.*, 371(2):561–618, 2019.

[FBZ01] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*, volume 88 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001.

[FG06] Vladimir Fock and Alexander Goncharov. Moduli spaces of local systems and higher teichmüller theory. *Publications Mathématiques de l'IHÉS*, 103:1–211, 2006.

[FOT11] Masatoshi Fukushima, Yoichi Oshima, and Masayoshi Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2011.

[Fre07] Edward Frenkel. Lectures on the Langlands program and conformal field theory. In *Frontiers in number theory, physics, and geometry. II*, pages 387–533. Springer, Berlin, 2007.

[Fri04] Roland Friedrich. On Connections of Conformal Field Theory and Stochastic Lœwner Evolution. *arXiv e-prints*, pages math–ph/0410029, October 2004.

[FS87] Daniel Friedan and Stephen Shenker. The analytic geometry of two-dimensional conformal field theory. *Nuclear Phys. B*, 281(3-4):509–545, 1987.

[FST91] Masatoshi Fukushima, Ken-iti Sato, and Setsuo Taniguchi. On the closable parts of pre-Dirichlet forms and the fine supports of underlying measures. *Osaka J. Math.*, 28(3):517–535, 1991.

[FZZ00] V. Fateev, A. Zamolodchikov, and Al. Zamolodchikov. Boundary Liouville Field Theory I. Boundary State and Boundary Two-point Function. *arXiv e-prints*, pages hep–th/0001012, January 2000.

[Gaw89] Krzysztof Gawedzki. Conformal field theory. In *Séminaire Bourbaki : volume 1988/89, exposés 700-714*, number 177-178 in Astérisque. Société mathématique de France, 1989. talk:704.

[GJ81] J. Glimm and Arthur M. Jaffe. *Quantum physics. A functional integral point of view*. Springer, 1981.

[GJY03] Anja Göing-Jaeschke and Marc Yor. A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349, 2003.

[GKRV20] Colin Guillarmou, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Conformal bootstrap in Liouville Theory. *arXiv e-prints*, page arXiv:2005.11530, May 2020.

[GM19] Ewain Gwynne and Jason Miller. Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$. *arXiv e-prints*, page arXiv:1905.00383, May 2019.

[GMS18] Ewain Gwynne, Jason Miller, and Xin Sun. Almost sure multifractal spectrum of Schramm-Loewner evolution. *Duke Math. J.*, 167(6):1099–1237, 2018.

[GRSS20] Promit Ghosal, Guillaume Remy, Xin Sun, and Yi Sun. Probabilistic conformal blocks for Liouville CFT on the torus. *arXiv e-prints*, page arXiv:2003.03802, March 2020.

[GRV14] Christophe Garban, Rémi Rhodes, and Vincent Vargas. On the heat kernel and the Dirichlet form of Liouville Brownian motion. *Electron. J. Probab.*, 19:no. 96, 25, 2014.

[GRV16] Christophe Garban, Rémi Rhodes, and Vincent Vargas. Liouville Brownian motion. *Ann. Probab.*, 44(4):3076–3110, 2016.

[GRV19] Colin Guillarmou, Rémi Rhodes, and Vincent Vargas. Polyakov's formulation of $2d$ bosonic string theory. *Publ. Math. Inst. Hautes Études Sci.*, 130:111–185, 2019.

[HJS10] Leszek Hadasz, Zbigniew Jasklski, and Paulina Suchanek. Modular bootstrap in liouville field theory. *Physics Letters B*, 685(1):79–85, 2010.

[HMP10] Xiaoyu Hu, Jason Miller, and Yuval Peres. Thick points of the Gaussian free field. *The Annals of Probability*, 38(2):896 – 926, 2010.

[HMW11] Daniel Harlow, J. Maltz, and E. Witten. Analytic continuation of liouville theory. *Journal of High Energy Physics*, 2011:1–105, 2011.

[HP18] Nina Holden and Ellen Powell. Conformal welding for critical Liouville quantum gravity. *arXiv e-prints*, page arXiv:1812.11808, Dec 2018.

[HPMS67] Jr. H. P. McKean and I. M. Singer. Curvature and the eigenvalues of the Laplacian. *Journal of Differential Geometry*, 1(1-2):43 – 69, 1967.

[HRV18] Yichao Huang, Rémi Rhodes, and Vincent Vargas. Liouville quantum gravity on the unit disk. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(3):1694–1730, 2018.

[Jaf00] Stéphane Jaffard. On the Frisch Parisi conjecture. *Journal de Mathématiques Pures et Appliqué es*, 79(6):525–552, 2000.

[Jeg20] Antoine Jego. Critical Brownian multiplicative chaos. *arXiv e-prints*, page arXiv:2005.14610, May 2020.

[Jon95] Peter Jones. On removable sets for Sobolev spaces in the plane. In *Essays on Fourier Analysis in Honor of Elias M. Stein (PMS-42)*, pages 250–267. Princeton University Press, 1995.

[JS00] Peter W. Jones and Stanislav K. Smirnov. Removability theorems for Sobolev functions and quasiconformal maps. *Ark. Mat.*, 38(2):263–279, 2000.

[JS17] Janne Junnila and Eero Saksman. Uniqueness of critical Gaussian chaos. *Electron. J. Probab.*, 22:Paper No. 11, 31, 2017.

[JSV19] Janne Junnila, Eero Saksman, and Lauri Viitasaari. On the regularity of complex multiplicative chaos. *arXiv e-prints*, page arXiv:1905.12027, May 2019.

[JSW19] Janne Junnila, Eero Saksman, and Christian Webb. Decompositions of log-correlated fields with applications. *Ann. Appl. Probab.*, 29(6):3786–3820, 2019.

[Kah85] Jean-Pierre Kahane. Sur le chaos multiplicatif. *Ann. Sci. Math. Québec*, 9(2):105–150, 1985.

[Kir98] A.A. Kirillov. Geometric approach to discrete series of unirreps for vir. *Journal de Mathmatiques Pures et Appliques*, 77(8):735–746, 1998.

[KM13] N.G. Kang and N.G. Makarov. *Gaussian Free Field and Conformal Field Theory*. Asterisque Series. Amer Mathematical Society, 2013.

[KMSW19] Richard Kenyon, Jason Miller, Scott Sheffield, and David B. Wilson. Bipolar orientations on planar maps and SLE_{12}. *The Annals of Probability*, 47(3):1240 – 1269, 2019.

[KPZ88] V. G. Knizhnik, Alexander M. Polyakov, and A. B. Zamolodchikov. Fractal Structure of 2D Quantum Gravity. *Mod. Phys. Lett. A*, 3:819, 1988.

[KRV19] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Local conformal structure of Liouville quantum gravity. *Comm. Math. Phys.*, 371(3):1005–1069, 2019.

[KRV20] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. Integrability of Liouville theory: proof of the DOZZ formula. *Ann. of Math. (2)*, 191(1):81–166, 2020.

[Kup16] Antti Kupiainen. Constructive Liouville Conformal Field Theory. *arXiv e-prints*, page arXiv:1611.05243, November 2016.

[KY87] Alexandre A. Kirillov and D. V. Yur'ev. Kähler geometry of the infinite-dimensional homogeneous space $\mathrm{M} = \mathrm{Diff}_+(\mathrm{S}^1)/\mathrm{Rot}(\mathrm{S}^1)$. *Functional Analysis and Its Applications*, 21:284–294, 1987.

[KY88] A.A. Kirillov and D.V. Yuriev. Representations of the Virasoro algebra by the orbit method. *Journal of Geometry and Physics*, 5(3):351–363, 1988.

[LR15] Gregory F. Lawler and Mohammad A. Rezaei. Minkowski content and natural parameterization for the Schramm-Loewner evolution. *Ann. Probab.*, 43(3):1082–1120, 2015.

[LRV15] Hubert Lacoin, Rémi Rhodes, and Vincent Vargas. Complex Gaussian multiplicative chaos. *Communications in Mathematical Physics*, 337(2):569–632, July 2015.

[LS11] Gregory F. Lawler and Scott Sheffield. A natural parametrization for the Schramm-Loewner evolution. *Ann. Probab.*, 39(5):1896–1937, 2011.

[Mad15] Thomas Madaule. Maximum of a log-correlated Gaussian field. *Ann. Inst. Henri Poincaré Probab. Stat.*, 51(4):1369–1431, 2015.

[Mir07a] Maryam Mirzakhani. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Invent. Math.*, 167(1):179–222, 2007.

[Mir07b] Maryam Mirzakhani. Weil-Petersson volumes and intersection theory on the moduli space of curves. *J. Amer. Math. Soc.*, 20(1):1–23, 2007.

[MMQ19] Oliver McEnteggart, Jason Miller, and Wei Qian. Uniqueness of the welding problem for SLE and Liouville quantum gravity. *Journal of the Institute of Mathematics of Jussieu*, page 127, 2019.

[MS16a] Jason Miller and Scott Sheffield. Imaginary geometry I: interacting SLEs. *Probab. Theory Related Fields*, 164(3-4):553–705, 2016.

[MS16b] Jason Miller and Scott Sheffield. Imaginary geometry II: reversibility of $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ for $\kappa \in (0,4)$. *Ann. Probab.*, 44(3):1647–1722, 2016.

[MS16c] Jason Miller and Scott Sheffield. Imaginary geometry III: reversibility of SLE_κ for $\kappa \in (4,8)$. *Ann. of Math. (2)*, 184(2):455–486, 2016.

[MS17] Jason Miller and Scott Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields*, 169(3-4):729–869, 2017.

[MY05] Hiroyuki Matsumoto and Marc Yor. Exponential functionals of Brownian motion, I: Probabilitylaws at fixed time. *Probability Surveys*, 2(none):312 – 347, 2005.

[MY16] Bastien Mallein and Marc Yor. Exercices sur les temps locaux de semi-martingales continues et les excursions browniennes. *arXiv e-prints*, page arXiv:1606.07118, June 2016.

[NS95] Subhashis Nag and Dennis Sullivan. Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle. *Osaka J. Math.*, 32(1):1–34, 1995.

[OPS88] B Osgood, R Phillips, and P Sarnak. Extremals of determinants of Laplacians. *Journal of Functional Analysis*, 80(1):148–211, 1988.

[Pek95] Osmo Pekonen. Universal Teichmüller space in geometry and physics. *Journal of Geometry and Physics*, 15(3):227–251, 1995.

[Pin85] Ross G. Pinsky. On the Convergence of Diffusion Processes Conditioned to Remain in a Bounded Region for Large Time to Limiting Positive Recurrent Diffusion Processes. *The Annals of Probability*, 13(2):363 – 378, 1985.

[Pol81] A. M. Polyakov. Quantum geometry of bosonic strings. *Phys. Lett. B*, 103(3):207–210, 1981.

[Pow18] Ellen Powell. Critical Gaussian chaos: convergence and uniqueness in the derivative normalisation. *Electron. J. Probab.*, 23:Paper No. 31, 26, 2018.

[Pow20] Ellen Powell. Critical Gaussian multiplicative chaos: a review. *arXiv e-prints*, page arXiv:2006.13767, June 2020.

[PT02] B. Ponsot and J. Teschner. Boundary liouville field theory: boundary three-point function. *Nuclear Physics B*, 622(1):309–327, 2002.

[Rem20] Guillaume Remy. The Fyodorov-Bouchaud formula and Liouville conformal field theory. *Duke Math. J.*, 169(1):177–211, 2020.

[Rib14] Sylvain Ribault. Conformal field theory on the plane. *arXiv e-prints*, page arXiv:1406.4290, June 2014.

[RS05] Steffen Rohde and Oded Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005.

[RV14] Rémi Rhodes and Vincent Vargas. Gaussian multiplicative chaos and applications: a review. *Probab. Surv.*, 11:315–392, 2014.

[RV15] Rémi Rhodes and Vincent Vargas. Liouville Brownian motion at criticality. *Potential Anal.*, 43(2):149–197, 2015.

[RV19] Rmi Rhodes and Vincent Vargas. The tail expansion of Gaussian multiplicative chaos and the Liouville reflection coefficient. *The Annals of Probability*, 47(5):3082 – 3107, 2019.

[RZ20a] Guillaume Remy and Tunan Zhu. The distribution of Gaussian multiplicative chaos on the unit interval. *Ann. Probab.*, 48(2):872–915, 2020.

[RZ20b] Guillaume Remy and Tunan Zhu. Integrability of boundary Liouville conformal field theory. *arXiv e-prints*, page arXiv:2002.05625, February 2020.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[Sei90] Nathan Seiberg. Notes on quantum Liouville theory and quantum gravity. *Prog. Theor. Phys. Suppl.*, 102:319–349, 1990.

[She16] Scott Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.*, 44(5):3474–3545, 2016.

[SP12] Ren L. Schilling and Lothar Partzsch. *Brownian Motion*. De Gruyter, 2012.

[Spi58] Frank Spitzer. Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.*, 87:187–197, 1958.

[ST20] Yuliang Shen and Shuan Tang. Weil-Petersson Teichmüller space II: Smoothness of flow curves of $H^{\frac{3}{2}}$-vector fields. *Adv. Math.*, 359:106891, 25, 2020.

[SW11] René L. Schilling and Jian Wang. On the coupling property of Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 47(4):1147–1159, 2011.

[Tes95] Jörg Teschner. On the liouville three-point function. *Physics Letters B*, 363(1):65–70, 1995.

[Tes01] J Teschner. Liouville theory revisited. *Classical and Quantum Gravity*, 18(23):R153–R222, nov 2001.

[Tes03] J. Teschner. Quantum Liouville theory versus quantized Teichmüller spaces. In *Proceedings of the 35th International Symposium Ahrenshoop on the Theory of Elementary Particles (Berlin-Schmöckwitz, 2002)*, volume 51, pages 865–872, 2003.

[TT06] Leon A. Takhtajan and Lee-Peng Teo. Weil-Petersson metric on the universal Teichmüller space. *Mem. Amer. Math. Soc.*, 183(861):viii+119, 2006.

[Tut63] W. T. Tutte. A census of planar maps. *Canadian Journal of Mathematics*, 15:249271, 1963.

[TV15] J. Teschner and G. Vartanov. Supersymmetric gauge theories, quantization of m-flat, and conformal field theory. *Advances in Theoretical and Mathematical Physics*, 19:1–135, 01 2015.

[VW20] Fredrik Viklund and Yilin Wang. Interplay Between Loewner and Dirichlet Energies via Conformal Welding and Flow-Lines. *Geom. Funct. Anal.*, 30(1):289–321, 2020.

[Wil74] David Williams. Path Decomposition and Continuity of Local Time for One-Dimensional Diffusions, I. *Proceedings of the London Mathematical Society*, s3-28(4):738–768, 06 1974.

[Wit84] Edward Witten. Nonabelian bosonization in two dimensions. *Communications in Mathematical Physics*, 92(4):455 – 472, 1984.

[Wit88] Edward Witten. Coadjoint orbits of the Virasoro group. *Comm. Math. Phys.*, 114(1):1–53, 1988.

[Wol83] Scott Wolpert. On the symplectic geometry of deformations of a hyperbolic surface. *Annals of Mathematics*, 117(2):207–234, 1983.

[Wol85] Scott Wolpert. On the weil-petersson geometry of the moduli space of curves. *American Journal of Mathematics*, 107(4):969–997, 1985.

[You18] Malik Younsi. Removability and non-injectivity of conformal welding. *Ann. Acad. Sci. Fenn. Math.*, 43(1):463–473, 2018.

[Zha17] Dapeng Zhan. SLE Loop Measures. *arXiv e-prints*, page arXiv:1702.08026, February 2017.

[ZZ96] A. Zamolodchikov and Al. Zamolodchikov. Conformal bootstrap in Liouville field theory. *Nuclear Phys. B*, 477(2):577–605, 1996.