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The spectrum of simplicial volume

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Abstract New constructions in group homology allow us to manufacture high-dimensional manifolds with controlled simplicial volume. We prove that for every dimension bigger than 3 the set of simplicial volumes of orientable closed connected manifolds is dense in $\mathbb{R}_{\geq 0}$. In dimension 4 we prove that every non-negative rational number is the simplicial volume of some orientable closed connected 4-manifold. Our group theoretic results relate stable commutator length to the l^1-semi-norm of certain singular homology classes in degree 2. The output of these results is translated into manifold constructions using cross-products and Thom realisation.

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1 Introduction

The simplicial volume $\|M\|$ of an orientable closed connected (occ) manifold M is a homotopy invariant that captures the complexity of representing fundamental classes by singular cycles with real coefficients (see Sect. 2 for a precise definition and basic terminology). Simplicial volume is known to be positive in the presence of enough negative curvature [35,46,48,56] and known to vanish in the presence of enough amenability [13,35,44,59]. Moreover, it provides a topological lower bound for the minimal Riemannian volume (suitably normalised) in the case of smooth manifolds [35].

Until now, for large dimensions d, very little was known about the precise structure of the set $\mathrm{SV}(d) \subset \mathbb{R}_{\geq 0}$ of simplicial volumes of occ d-manifolds. The set $\mathrm{SV}(d)$ is countable and closed under addition (Remark 2.3). However, the set of simplicial volumes is fully understood only in dimensions 2 and 3 with $\mathrm{SV}(2) = \mathbb{N}[4]$ (Example 2.4) and $\mathrm{SV}(3) = \mathbb{N}[\frac{\mathrm{vol}(M)}{v} \mid M]$, where M ranges over all complete finite-volume hyperbolic 3-manifolds with toroidal boundary and where $v > 0$ is a constant (Example 2.5).

This reveals that there is a *gap* of simplicial volume in dimensions 2 and 3: For $d \in \{2, 3\}$ there is a constant $C_d > 0$ such that the simplicial volume of an occ d-manifold either vanishes or is at least C_d. It was an open question [53, p. 550] whether such a gap exists in higher dimensions. For example, until now the lowest known simplicial volume of an occ 4-manifold has been 24 [4] (Example 2.6).

In the present paper, we show that dimensions 2 and 3 are the *only* dimensions with such a gap.

Theorem A (No-gap; Sect. 8.2) *Let $d \geq 4$ be an integer. For every $\epsilon > 0$ there is an orientable closed connected d-manifold M such that $0 < \|M\| \leq \epsilon$. Hence, the set of simplicial volumes of orientable closed connected d-manifolds is dense in $\mathbb{R}_{\geq 0}$.*

In dimension 4, we get the following refinement of Theorem A.

Theorem B (Rational realisation; Sect. 8.3) *For every $q \in \mathbb{Q}_{\geq 0}$ there is an orientable closed connected 4-manifold M_q with $\|M_q\| = q$.*

Method

We first compute the l^1-semi-norm of certain integral 2-classes in finitely presented groups by relating these semi-norms to stable commutator length.

To formulate this connection, we recall some definitions. For a group G and a class $\alpha \in H_d(G; \mathbb{R})$, the *$l^1$-semi-norm $\|\alpha\|_1$ of α* is the semi-norm induced by the l^1-norm of chains in the singular chain complex of any model of BG

(Sect. 2.1). The class α is *integral* if it lies in the image under the change of coefficients map map induced by $\mathbb{Z} \to \mathbb{R}$.

For an element $g \in [G, G]$ in the commutator subgroup of G, the *commutator length* $\mathrm{cl}_G\, g$ of g is the minimal number of commutators in G needed to express g as their product. The *stable commutator length (scl)* of g is the limit $\mathrm{scl}_G\, g := \lim_{n\to\infty} \mathrm{cl}_G(g^n)/n$. Stable commutator length is now well-understood for many classes of groups thanks largely to Calegari and others [18].

Theorem C (Corollary 6.16) *Let G be a finitely presented group with $H_2(G; \mathbb{R}) \cong 0$ and let $g \in [G, G]$ be an element of infinite order. Then there is a finitely presented group $D(G, g)$ and an integral class $\alpha_g \in H_2(D(G, g); \mathbb{R})$ such that*

$$\|\alpha_g\|_1 = 8 \cdot \mathrm{scl}_G\, g.$$

We apply Theorem C to the universal central extension E of Thompson's group T. Recall that T is the group of piecewise linear homeomorphisms of the circle with dyadic breakpoints and whose slopes are integer powers of 2. In Propostion 5.1, we show that the universal central extension E of T is a finitely presented group with $H_2(E; \mathbb{R}) \cong 0$ and that every non-negative rational number may be realised by the stable commutator length of some element in E. Using Theorem C this shows:

Theorem D (Corollary 6.17) *For every $q \in \mathbb{Q}_{\geq 0}$ there is a finitely presented group G_q and an integral class $\alpha_q \in H_2(G_q; \mathbb{R})$ such that $\|\alpha_q\|_1 = q$. In particular, for every $\epsilon > 0$ there is a finitely presented group G_ϵ and an integral class $\alpha_\epsilon \in H_2(G_\epsilon; \mathbb{R})$ such that $0 < \|\alpha_\epsilon\|_1 \leq \epsilon$.*

We can now take cross-products in homology to obtain integral classes in degree greater than 3 with crude norm control. An application of a normed version of Thom realisation (Theorem 8.1) proves Theorem A.

In dimension 4, we refine this construction by taking products with surfaces and using an exact computation of the product norm. This generalises a result of Bucher [4]. Theorem B will follow from these computations.

Theorem E (Corollary 7.2) *Let G be a group, let $\alpha \in H_2(G; \mathbb{R})$, and let Γ_g be the fundamental group of the oriented closed connected surface Σ_g of genus $g \geq 2$ with fundamental class $[\Sigma_g]_\mathbb{R} \in H_2(\Gamma_g; \mathbb{R})$. Then the l^1-semi-norm of $\alpha \times [\Sigma_g]_\mathbb{R} \in H_4(G \times \Gamma_g; \mathbb{R})$ satisfies*

$$\left\|\alpha \times [\Sigma_g]_\mathbb{R}\right\|_1 = 6 \cdot (g-1) \cdot \|\alpha\|_1.$$

In particular, we establish the following connection between stable commutator length and simplicial volume in dimension 4:

Theorem F (Corollary 8.3) *Let G be a finitely presented group that satisfies $H_2(G;\mathbb{R}) \cong 0$ and let $g \in [G,G]$ be an element in the commutator subgroup. Then there is an orientable closed connected 4-manifold M_g with*

$$\|M_g\| = 48 \cdot \mathrm{scl}_G\, g.$$

Organisation of this article

Sections 2, 3, and 4 recall basic properties and known results on simplicial volume, bounded cohomology and stable commutator length, respectively.

In Sect. 5 we compute scl on the universal central extension E of Thompson's group T (Proposition 5.1). This will be used in Sect. 6 to construct integral 2-classes with controlled l^1-semi-norms. There we also show Theorems C and D.

In Sect. 7 we get the refinement for dimension 4 in group homology: We compute the l^1-semi-norm of cross-products of general 2-classes with certain Euler-extremal 2-classes (Theorem 7.1). As a corollary we obtain Theorem E.

All manifolds constructed in this article will arise via a suitable version of Thom's realisation theorem in Sect. 8. This allows us to manufacture manifolds with controlled simplicial volume and to prove Theorems A, B, and F.

A discussion of related problems may be found in Sect. 8.4.

2 Simplicial volume

We recall the l^1-semi-norm on homology and simplicial volume and establish some notation. In particular, we collect basic properties related to classes in degree 2.

2.1 The l^1-semi-norm and simplicial volume

The notion of simplicial volume of manifolds is based on the l^1-semi-norm on singular homology. More precisely: Let X be a topological space and let $d \in \mathbb{N}$. Then the *l^1-semi-norm* on $H_d(X;\mathbb{R})$ is

$$\begin{aligned} \|\cdot\|_1 \colon H_d(X;\mathbb{R}) &\to \mathbb{R}_{\geq 0} \\ \alpha &\mapsto \inf\bigl\{|c|_1 \bigm| c \in C_d(X;\mathbb{R}),\ \partial c = 0,\ [c] = \alpha\bigr\}; \end{aligned}$$

here, $C_d(X;\mathbb{R})$ is the singular chain module of X in degree d with $\mathbb{R}$-coefficients and $|\cdot|_1$ denotes the l^1-norm on $C_d(X;\mathbb{R})$ associated with the basis of singular simplices. More generally, if $A \subset X$ is a subspace, one can also consider the *relative l^1-semi-norm* on $H_d(X, A; \mathbb{R})$ induced by the l^1-semi-norm on $C_d(X; \mathbb{R})$.

The l^1-semi-norm is a functorial semi-norm in the sense of Gromov [36, p. 302]:

Remark 2.1 If $f : X \to Y$ is continuous, $d \in \mathbb{N}$, and $\alpha \in H_d(X; \mathbb{R})$, then

$$\left\| H_d(f; \mathbb{R})(\alpha) \right\|_1 \le \|\alpha\|_1.$$

Definition 2.2 (Simplicial volume [35]) Let M be an oriented closed connected d-dimensional manifold. Then the *simplicial volume of M* is defined by

$$\|M\| := \left\| [M]_{\mathbb{R}} \right\|_1,$$

where $[M]_{\mathbb{R}} \in H_d(M; \mathbb{R})$ denotes the $\mathbb{R}$-fundamental class of M.

More generally, if $(M, \partial M)$ is an oriented compact connected d-manifold with boundary, then one defines the *relative simplicial volume of $(M, \partial M)$* by

$$\|M, \partial M\| := \left\| [M, \partial M]_{\mathbb{R}} \right\|_1,$$

where $[M, \partial M]_{\mathbb{R}} \in H_d(M, \partial M; \mathbb{R})$ denotes the relative $\mathbb{R}$-fundamental class of $(M, \partial M)$.

Because the definition of simplicial volume is independent of the chosen orientation, we will also speak of the simplicial volume of orient*able* manifolds.

On the one hand, simplicial volume clearly is a topological invariant of (orientable) compact manifolds that is compatible with mapping degrees. On the other hand, simplicial volume is related in a non-trivial way to Riemannian volume, e.g., in the case of hyperbolic manifolds [35,56]. Therefore, simplicial volume is a useful invariant in the study of rigidity properties of manifolds.

Basic examples of simplicial volumes are listed in Examples 2.4, 2.5, and 2.6. In addition to geometric arguments, a key tool for working with simplicial volume is bounded cohomology (see Proposition 3.4 below).

2.2 Simplicial volume in low dimensions and gaps

We collect the low-dimensional examples of simplicial volume as stated in the introduction. Recall that for $d \in \mathbb{N}$ we define $\mathrm{SV}(d) \subset \mathbb{R}_{\ge 0}$ via

$$\mathrm{SV}(d) := \left\{ \|M\| \;\middle|\; M \text{ is an orientable closed connected } d\text{-manifold} \right\}.$$

Remark 2.3 As there are only countably many homotopy types of orientable closed connected (occ) manifolds [49], the set $\mathrm{SV}(d)$ is countable for every $d \in \mathbb{N}$.

The set $\mathrm{SV}(d)$ is also closed under addition. For $d \geq 3$, this follows from the additivity of simplicial volume under connected sums [35][31, Corollary 7.7] and for $d = 2$ this follows from the explicit computation of $\mathrm{SV}(2)$ as seen in Example 2.4.

Clearly, $\mathrm{SV}(0) = \{1\}$ (the only relevant manifold being a single point) and $\mathrm{SV}(1) = \{0\}$ (the only manifold being the circle).

Example 2.4 (Dimension 2) For an orientable closed connected surface Σ_g of genus $g \geq 1$ we have $\|\Sigma_g\| = 2 \cdot \left|\chi(\Sigma_g)\right| = 4 \cdot (g-1)$ [7,35][31, Corollary 7.5]. Hence,

$$\mathrm{SV}(2) = \{0, 4, 8, \ldots\} = \mathbb{N}[4].$$

We observe that the *gap* in simplicial volume of dimension 2 is 4.

Example 2.5 (Dimension 3) We have [35,54][31, Corollary 7.8]

$$\mathrm{SV}(3) = \mathbb{N}\left[\frac{\mathrm{vol}(M)}{v_3} \;\middle|\; \begin{array}{l} M \text{ is a complete hyperbolic 3-manifold} \\ \text{with toroidal boundary and finite volume} \end{array}\right]$$

and where v_3 is the maximal volume of an ideal simplex in $\mathbb{H}^3$. This shows that there is a gap of simplicial volume in dimension 3, namely $w/v_3 \approx 0.928\ldots$, where w is the volume of the Weeks manifold [38]. Moreover, the set $\mathrm{SV}(3)$ has countably many accumulation points (because the set of hypbolic volumes has the order type ω^ω [56]).

Example 2.6 (Dimension 4) The smallest known Riemannian volume $\mathrm{vol}(M)$ of an occ *hyperbolic* 4-manifold is $64 \cdot \pi^2/3$ [24]. In view of the computation of the simplicial volume of hyperbolic manifolds [35,56][31, Chapter 7.3] this means that the smallest known simplicial volume of a hyperbolic occ 4-manifold is $\frac{64\cdot\pi^2}{3\cdot v_4} \in [700, 800]$ where v_4 is the maximal volume of an ideal 4-simplex in $\mathbb{H}^4$.

If Σ_g, Σ_h are orientable closed connected surfaces of genus $g, h \geq 1$, respectively, then Bucher [4] showed that $\|\Sigma_g \times \Sigma_h\| = \frac{3}{2} \cdot \|\Sigma_g\| \cdot \|\Sigma_h\|$. Hence, $\|\Sigma_2 \times \Sigma_2\| = 24$. This has been the smallest known non-trival simplicial volume of a 4-manifold. More general surface bundles over surfaces do not yield lower estimates [2,43]. Also the recent computations/estimates for mapping tori in dimension 4 [3] do not produce improved gap bounds.

2.3 The l^1-semi-norm in degree 2

As classes in degree 2 will play an important role in our constructions, we collect some basic properties concerning the l^1-semi-norm in degree 2.

Proposition 2.7 (l^1-semi-norm in degree 2; [25, Proposition 2.4]) *Let X be a topological space and let $\alpha \in H_2(X; \mathbb{R})$. Then*

$$\begin{aligned}\|\alpha\|_1 = \inf\Biggl\{\sum_{j=1}^{k} |a_j| \cdot \|\Sigma_{(j)}\| \;\Bigg|\; & k \in \mathbb{N}, a_1, \dots, a_k \in \mathbb{R}\setminus\{0\}, \\ & \Sigma_{(1)}, \dots, \Sigma_{(k)} \text{ oriented closed connected surfaces,} \\ & f_1 \colon \Sigma_{(1)} \to X, \dots, f_k \colon \Sigma_{(k)} \to X \text{ continuous} \\ & \text{with } \sum_{j=1}^{k} a_j \cdot H_2(f_j; \mathbb{R})[\Sigma_{(j)}]_{\mathbb{R}} = \alpha \text{ in } H_2(X; \mathbb{R})\Biggr\}.\end{aligned}$$

Remark 2.8 Let X be a path-connected topological space, let $\alpha \in H_2(X; \mathbb{Z})$, and let $\alpha_{\mathbb{R}} \in H_2(X; \mathbb{R})$ be the image of α under the change of coefficients map $\mathbb{Z} \to \mathbb{R}$. Then the description of $\|\alpha_{\mathbb{R}}\|_1$ from Proposition 2.7 simplifies as follows: We have

$$\|\alpha_{\mathbb{R}}\|_1 = \inf_{(f,\Sigma)\in\Sigma(\alpha)} \frac{\|\Sigma\|}{|n(f, \Sigma)|} = \inf_{(f,\Sigma)\in\Sigma(\alpha)} \frac{4 \cdot \bigl(g(\Sigma) - 1\bigr)}{|n(f, \Sigma)|},$$

where $\Sigma(\alpha)$ is the class of all pairs (f, Σ) consisting of an oriented closed connected surface Σ of genus $g(\Sigma) \geq 1$ and a continuous map $f \colon \Sigma \to X$ with $H_2(f; \mathbb{Z})[\Sigma] = n(f, \Sigma) \cdot \alpha$ in $H_2(X; \mathbb{Z})$ for some integer $n(f, \Sigma) \in \mathbb{Z}$.

In Sect. 6, we will relate l^1-semi-norms of relative classes in degree 2 to filling invariants and stable commutator length.

2.4 Simplicial volume of products

We recall basic results on l^1-semi-norms of homological cross-products.

Proposition 2.9 *Let X, Y be topological spaces, let $m, n \in \mathbb{N}$, and let $\alpha \in H_m(X; \mathbb{R})$, $\beta \in H_n(Y; \mathbb{R})$. Then the cross-product $\alpha \times \beta \in H_{m+n}(X \times Y; \mathbb{R})$ satisfies*

$$\|\alpha\|_1 \cdot \|\beta\|_1 \leq \|\alpha \times \beta\|_1 \leq \binom{m+n}{m} \cdot \|\alpha\|_1 \cdot \|\beta\|_1.$$

Proof The lower estimate follows from the duality principle (Proposition 3.4) and an explicit description of the cohomological cross-product (in bounded cohomology), the upper estimate follows from an explicit description of the homological cross-product [35][7, Theorem F.2.5] (this classical argument works also for general homology classes, not only for fundamental classes of manifolds). □

However, in general, it seems to be a hard problem to compute the exact values of l^1-semi-norms of products. One of the few known cases are products of two orientable closed connected surfaces, whose simplicial volumes have been computed by Bucher:

Theorem 2.10 ([4, Corollary 3]) *Let Σ_g, Σ_h be orientable closed connected surfaces of genus $g, h \in \mathbb{N}_{\geq 1}$. Then*

$$\|\Sigma_g \times \Sigma_h\| = \frac{3}{2} \cdot \|\Sigma_g\| \cdot \|\Sigma_h\| = 24 \cdot (g-1) \cdot (h-1).$$

We will generalise this theorem in Sect. 7. For now, let us note that in combination with the description of the l^1-semi-norm in degree 2 in terms of surfaces, we obtain the following general, improved, upper bound:

Corollary 2.11 *Let X and Y be path-connected topological spaces and let $\alpha \in H_2(X;\mathbb{R})$, $\beta \in H_2(Y;\mathbb{R})$. Then*

$$\|\alpha \times \beta\|_1 \leq \frac{3}{2} \cdot \|\alpha\|_1 \cdot \|\beta\|_1.$$

Proof We use the description of $\|\alpha\|_1$ and $\|\beta\|_1$ from Proposition 2.7. Let

$$\alpha = \sum_{i=1}^{k} a_i \cdot H_2(f_i;\mathbb{R})[\Sigma_{(i)}]_{\mathbb{R}} \quad \text{and} \quad \beta = \sum_{j=1}^{m} b_j \cdot H_2(g_j;\mathbb{R})[\Pi_{(j)}]_{\mathbb{R}}$$

be surface presentations of α and β as in Proposition 2.7. Then

$$H_4(f_i \times g_j;\mathbb{R})[\Sigma_{(i)} \times \Pi_{(j)}]_{\mathbb{R}} = H_2(f_i;\mathbb{R})[\Sigma_{(i)}]_{\mathbb{R}} \times H_2(g_j;\mathbb{R})[\Pi_{(j)}]_{\mathbb{R}},$$

and so $\alpha \times \beta = \sum_{i=1}^{k} \sum_{j=1}^{m} a_i \cdot b_j \cdot H_2(f_i \times g_j;\mathbb{R})[\Sigma_{(i)} \times \Pi_{(j)}]_{\mathbb{R}}$. Therefore, by applying the triangle inequality, functoriality (Remark 2.1), and Theorem 2.10, we have

$$\|\alpha \times \beta\|_1 \leq \sum_{i=1}^{k} \sum_{j=1}^{m} |a_i| \cdot |b_j| \cdot H_2(f_i \times g_j;\mathbb{R})[\Sigma_{(i)} \times \Pi_{(j)}]_{\mathbb{R}}$$

$$\leq \sum_{i=1}^{k}\sum_{j=1}^{m} |a_j| \cdot |b_j| \cdot \frac{3}{2} \cdot \|\Sigma_{(i)}\| \cdot \|\Pi_{(j)}\|$$
$$= \frac{3}{2} \cdot \sum_{i=1}^{k} |a_j| \cdot \|\Sigma_{(i)}\| \cdot \sum_{j=1}^{m} |b_j| \cdot \|\Pi_{(j)}\|.$$

By Proposition 2.7, taking the infimum over all such surface presentations of α and β, we obtain $\|\alpha \times \beta\|_1 \leq 3/2 \cdot \|\alpha\|_1 \cdot \|\beta\|_1$. □

3 Bounded cohomology

Bounded cohomology of discrete groups and topological spaces was first systematically studied by Gromov [35]. Gromov established the fundamental properties of bounded cohomology using so-called multicomplexes. Later, Ivanov developed a more algebraic framework via resolutions [44,45].

The reference to this introduction is the recent book by Frigerio [31]. Having applications to stable commutator length and the l^1-semi-norm in mind we will only define bounded cohomology for trivial real and integer coefficients.

Sections 3.1, 3.2 and 3.3 discuss the (relationships between) bounded cohomology of groups and topological spaces. In Sect. 3.4 we state the duality principle, which allows us to compute the l^1-semi-norm. In Sect. 3.6 we define the Euler class.

3.1 Bounded cohomology of groups

Let V be $\mathbb{R}$ or $\mathbb{Z}$ and let G be a group. We will define the bounded cohomology $H_b^n(G; V)$ of G using the *homogeneous resolution*. There is also an *inhomogeneous resolution*, which is useful in low dimensions. We use this resolution only in Sect. 5 for central extensions and refer to the literature [31, Chapter 1.7] for the definition.

Let $C^n(G; V) := \operatorname{map}(G^{n+1}, V)$ be the set of set-theoretic maps from G^{n+1} to V. The group G acts on $C^n(G; V)$ via $g \cdot \phi(g_0, \ldots, g_n) = \phi(g^{-1} \cdot g_0, \ldots, g^{-1} \cdot g_n)$. We denote by $C^n(G; V)^G$ the subset of elements in $C^n(G; V)$ that are invariant under this action. Let $\| \cdot \|_\infty$ be the l^∞-norm on $C^n(G; V)$ and let $C_b^n(G; V)$ be the corresponding subspaces of *bounded* functions.

Define the simplicial coboundary maps $\delta^n : C^n(G; V) \to C^{n+1}(G; V)$ via

$$\delta^n(\alpha)(g_0, \ldots, g_{n+1}) := \sum_{i=0}^{n+1} (-1)^i \cdot \alpha(g_0, \ldots, \widehat{g_i}, \ldots, g_{n+1})$$

where $\alpha(g_0, \ldots, \widehat{g_i}, \ldots, g_{n+1})$ means that the i-th coordinate is omitted. Then δ^n restricts to a map $C_b^n(G; V) \to C_b^{n+1}(G; V)$. The cohomology of the cochain complex $(C^\bullet(G; V)^G, \delta^\bullet)$ is the *group cohomology of G with coefficients in V* and denoted by $H^\bullet(G; V)$. Similarly, the cohomology of the cochain complex $(C_b^\bullet(G; V)^G, \delta^\bullet)$ is the *bounded cohomology of G with coefficients in V* and denoted by $H_b^\bullet(G; V)$. The embedding $C_b^\bullet(G; \mathbb{R}) \hookrightarrow C^\bullet(G; \mathbb{R})$ induces a map $c^\bullet \colon H_b^\bullet(G; V) \to H^\bullet(G; V)$, the *comparison map*.

Bounded cohomology carries additional structure, the semi-norm induced by $\|\cdot\|_\infty$: For an element $\alpha \in H_b^n(G; V)$, we set

$$\|\alpha\| := \inf\bigl\{\|\beta\|_\infty \bigm| \beta \in C_b^n(G; V),\ \delta^n\beta = 0,\ [\beta] = \alpha \in H_b^n(G; V)\bigr\}.$$

Bounded cohomology is functorial in both the group and the coefficients.

3.2 Bounded cohomology of spaces

Let X be a topological space and let $S_n(X)$ be the set of singular n-simplices in X. Moreover, let $C^n(X; V)$ be the set of maps from $S_n(X)$ to V. For an element $\alpha \in C^n(X; V)$ we set

$$\|\alpha\|_\infty := \sup\bigl\{|\alpha(\sigma)| \bigm| \sigma \in S_n(X)\bigr\} \in [0, \infty]$$

and let $C_b^n(X; V) \subset C^n(X; V)$ be the subset of elements that are bounded with respect to this norm. Let $\delta^n : C_b^n(X; V) \to C_b^{n+1}(X; V)$ be the restriction of the singular coboundary map to bounded cochains. Then the *bounded cohomology $H_b^\bullet(X; V)$ of X with coefficients in V* is the cohomology of the complex $(C_b^\bullet(X; V), \delta^\bullet)$ and denoted by $H_b^\bullet(X; V)$. For $\alpha \in H_b^n(X; V)$ we define

$$\|\alpha\|_\infty = \inf\bigl\{\|\beta\|_\infty \bigm| \beta \in C_b^n(X; V),\ \delta^n\beta = 0, [\beta] = \alpha \in H_b^n(X; V)\}$$

and observe that $\|\cdot\|_\infty$ is a semi-norm on $H_b^n(X; V)$. The bounded cohomology of spaces is also functorial in both spaces and coefficients.

3.3 Relationship between bounded cohomology of groups and spaces

Analogously to ordinary group cohomology, bounded cohomology of groups may also be computed using classifying spaces (and thus, we will freely switch between these descriptions).

Theorem 3.1 ([31, Theorem 5.5]) *Let X be a model of the classifying space BG of the group G. Then $H_b^\bullet(X;\mathbb{R})$ is canonically isometrically isomorphic to $H_b^\bullet(G;\mathbb{R})$.*

Remarkably, this statement holds true much more generally: every topological space with the correct fundamental group can be used to compute bounded cohomology of groups; moreover, bounded cohomology ignores amenable kernels [35]:

Theorem 3.2 ([31, Theorem 5.8][45]) *Let X be a path-connected space. Then $H_b^\bullet(X;\mathbb{R})$ is canonically isometrically isomorphic to $H_b^\bullet(\pi_1(X);\mathbb{R})$.*

Theorem 3.3 (Mapping theorem [31, Corollary 5.11][45]) *Let $f\colon X \to Y$ be a continuous map between path-connected topological spaces. If the induced homomorphism $\pi_1(f)\colon \pi_1(X) \to \pi_1(Y)$ is surjective and has amenable kernel, then $H_b^\bullet(f;\mathbb{R})\colon H_b^\bullet(Y;\mathbb{R}) \to H_b^\bullet(X;\mathbb{R})$ is an isometric isomorphism.*

3.4 Duality

Bounded cohomology of groups and spaces may be used to compute the l^1-semi-norm of homology classes. For what follows, let $\langle\cdot,\cdot\rangle\colon H_b^n(X;V) \times H_n(X;V) \to V$ be the map given by evaluation of cochains on chains.

Proposition 3.4 (Duality principle [31, Lemma 6.1]) *Let X be a topological space and let $\alpha \in H_n(X;\mathbb{R})$. Then*

$$\|\alpha\|_1 = \sup\bigl\{\langle\beta,\alpha\rangle \bigm| \beta \in H_b^n(X;\mathbb{R}),\ \|\beta\|_\infty \le 1\bigr\}.$$

Moreover, the supremum is achieved.

Cocycles $\beta \in C_b^n(X,\mathbb{R})$ that satisfy $\|\beta\|_\infty = 1$ and $\langle[\beta],\alpha\rangle = \|\alpha\|_1$ are called *extremal for α*.

Corollary 3.5 (Mapping theorem for the l^1-semi-norm) *Let $f\colon X \to Y$ be a continuous map between path-connected topological spaces. If the induced homomorphism $\pi_1(f)\colon \pi_1(X) \to \pi_1(Y)$ is surjective and has amenable kernel, then $H_\bullet(f;\mathbb{R})\colon H_\bullet(X;\mathbb{R}) \to H_\bullet(Y;\mathbb{R})$ is isometric with respect to the l^1-semi-norm.*

Proof We only need to combine the duality principle (Proposition 3.4) with the mapping theorem in bounded cohomology (Theorem 3.3). $\square$

3.5 Alternating cochains

Recall that V denotes $\mathbb{R}$ or $\mathbb{Z}$ and let $\alpha \in C_b^n(G; V)$ be a bounded homogeneous cochain. We say that α is *alternating*, if for every $g_0, \dots, g_n \in G$ and every permutation $\sigma \in S_{n+1}$ we have that

$$\alpha(g_{\sigma(0)}, \dots, g_{\sigma(n)}) = \operatorname{sign}(\sigma) \cdot \alpha(g_0, \dots, g_n).$$

Every (bounded) cochain $\alpha \in C_b^n(G; \mathbb{R})$ has an associated *alternating* (bounded) cochain $\operatorname{alt}^n(\alpha)$ defined via

$$\operatorname{alt}^n(\alpha)(g_0, \dots, g_n) := \frac{1}{(n+1)!} \cdot \sum_{\sigma \in S_{n+1}} \operatorname{sign}(\sigma) \cdot \alpha(g_{\sigma(0)}, \dots, g_{\sigma(n)}).$$

Observe that $\|\operatorname{alt}^n(\alpha)\|_\infty \leq \|\alpha\|_\infty$. The subcomplex of alternating cochains is denoted by $C^n_{b,\mathrm{alt}}(G; V)$. It is well-known that one can compute real bounded cohomology using alternating cochains:

Proposition 3.6 ([31, Proposition 4.26]) *Let G be a group. The complex $C^\bullet_{b,\mathrm{alt}}(G, \mathbb{R})$ isometrically computes the bounded cohomology with real coefficients. Moreover, for every $\alpha \in C_b^n(G, \mathbb{R})$ the cocycle* $\operatorname{alt}_b^n(\alpha)$ *represents the same class as α in $H_b^n(G; \mathbb{R})$.*

3.6 Euler class and the orientation cocycle

We describe the Euler class associated to a circle action. For details we refer to the literature [12,34].

For three points $x_1, x_2, x_3 \in S^1$ on the circle let $\operatorname{Or}(x_1, x_2, x_3) \in \{-1, 0, 1\}$ be the (respective) circular order. The group $\operatorname{Homeo}^+(S^1)$ of orientation preserving homeomorphisms on the circle preserves $\operatorname{Or} \in C_b^2(S^1; \mathbb{Z}) \subset C^2(S^1; \mathbb{Z})$ and satisfies a (homogeneous) cocycle condition. Hence, Or induces a (bounded) cocycle on $\operatorname{Homeo}^+(S^1)$: For $\xi \in S^1$, the map

$$(g_1, g_2, g_3) \mapsto \operatorname{Or}(g_1.\xi, g_2.\xi, g_3.\xi)$$

is a cocycle in $C_b^2(\operatorname{Homeo}^+(S^1); \mathbb{Z})$ and the bounded cohomology class is independent of the choice of the point ξ. It turns out that this class is divisible by 2, i.e., there is a cocycle $\operatorname{Eu} \in C_b^2(\operatorname{Homeo}^+(S^1); \mathbb{Z})$, called *Euler cocycle*, with $-2 \cdot [\operatorname{Eu}] = [\operatorname{Or}]$ in $H_b^2(\operatorname{Homeo}^+(S^1); \mathbb{Z})$.

Remark 3.7 Let $H := \operatorname{Homeo}^+(S^1)$. For Euler classes (and the orientation cocycle) we will use the following notation: Capital letters (Eu) denote cocycles and lower case letters (eu) denote classes. The classes $\operatorname{eu}^{\mathbb{Z}} \in H^2(H; \mathbb{Z})$,

$\mathrm{eu}^{\mathbb{R}} \in H^2(H;\mathbb{R})$, $\mathrm{eu}_b^{\mathbb{Z}} \in H_b^2(H;\mathbb{Z})$ and $\mathrm{eu}_b^{\mathbb{R}} \in H_b^2(H;\mathbb{R})$ are the ones represented by Eu in the corresponding cohomology groups. If a group G acts on the circle by $\rho\colon G \to \mathrm{Homeo}^+(S^1)$, and α is a class or a cocycle defined on $\mathrm{Homeo}^+(S^1)$ then $\rho^*\alpha$ will be the pullback of α via ρ. If $\Gamma < \mathrm{Homeo}^+(S^1)$ is a subgroup of $\mathrm{Homeo}^+(S^1)$, then we will denote the restriction of a class or a cocycle α to Γ by ${}^{\Gamma}\alpha$. Hence, for example, ${}^{\Gamma}\mathrm{eu}_b^{\mathbb{R}} \in H_b^2(\Gamma;\mathbb{R})$ denotes the restriction of the real bounded Euler class to Γ.

Let G be a group with a circle action $\rho\colon G \to \mathrm{Homeo}^+(S^1)$. Then $\rho^*\mathrm{eu}^{\mathbb{Z}} \in H^2(G;\mathbb{Z})$ is called the *Euler class associated to the action* ρ. The Euler class induces a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1$$

of G by $\mathbb{Z}$, the associated *Euler extension* $\widetilde{G}$. This group has the following explicit description. It is the group defined on the set $\mathbb{Z}\times G$ with multiplication $(z,g)\cdot(z',g') = (z+z'+\rho^*\mathrm{Eu}(1,g,g\cdot g'), g\cdot g')$. Euler extensions are useful for constructing groups with controlled stable commutator length; see Section 5.

We note that $\rho^*\mathrm{Or}^{\mathbb{R}}$ is extremal (in the sense of Proposition 3.4) for surface groups:

Example 3.8 Let $g \in \mathbb{N}_{\geq 2}$ and let Σ_g be an oriented closed connected surface of genus $g \in \mathbb{N}_{\geq 2}$. Recall that $\|[\Sigma_g]_{\mathbb{R}}\|_1 = \|\Sigma_g\| = 4g-4$, where $[\Sigma_g]_{\mathbb{R}} \in H_2(\Gamma_g;\mathbb{Z})$ denotes the fundamental class and $\Gamma_g = \pi_1(\Sigma_g)$. Then Γ_g induces an action on its boundary. By identifying $\partial\Gamma_g \cong S^1$, we obtain a circle action $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ and

$$\begin{aligned}\left\langle [\rho^*\mathrm{Or}]^{\mathbb{R}}, [\Sigma_g]_{\mathbb{R}}\right\rangle &= -2\cdot\left\langle \rho^*\mathrm{eu}_b^{\mathbb{R}}, [\Sigma_g]_{\mathbb{R}}\right\rangle \\ &= -2\cdot\chi(\Sigma_g) = 4\cdot g - 4 = \|\Sigma_g\|,\end{aligned}$$

i.e., $\rho^*\mathrm{Or} \in C_b^2(\Gamma_g;\mathbb{R})$ is an extremal cocycle for the fundamental class $[\Sigma_g]_{\mathbb{R}}$. Indeed, it is the renormalised volume cocycle of ideal simplices in $\mathbb{H}^2$; see [4].

4 Stable commutator length

In recent years the topic of stable commutator length (scl) has seen a vast developemet thanks largely to Calegari et al. [18]. In this section, we will only give a brief overview of scl. The definition and basic properties will be given in Sect. 4.1. A useful tool to compute scl is Bavard's duality theorem, described in Sect. 4.2. We discuss examples and general properties of scl in Sect. 4.3.

4.1 Definition and basic properties

For a group G let G' be its commutator subgroup. The *commutator length* $\mathrm{cl}_G\, g$ of an element $g \in G'$ is defined as

$$\mathrm{cl}_G\, g := \min\bigl\{n \in \mathbb{N} \bigm| \exists_{x_1,\dots,x_n,y_1,\dots,y_n\in G} \quad g = [x_1, y_1]\cdots[x_n, y_n]\bigr\},$$

where $[x, y] := xyx^{-1}y^{-1}$. The *stable commutator length of g in G* is defined as

$$\mathrm{scl}_G\, g := \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n}.$$

If $g_1, \dots, g_m \in G$ are such that $g_1 \cdots g_m \in G'$, we will call $g_1 + \cdots + g_m$ a *chain* and define the corresponding *(stable) commutator length on chains* by

$$\mathrm{cl}_G(g_1 + \cdots + g_m) := \min_{t_1,\dots,t_m\in G} \mathrm{cl}_G(t_1 g_1 t_1^{-1} \cdots t_m g_m t_m^{-1}), \text{ and}$$
$$\mathrm{scl}_G(g_1 + \cdots + g_m) := \lim_{n\to\infty} \frac{\mathrm{cl}_G(g_1^n + \cdots + g_m^n)}{n}.$$

If $\varphi\colon G \to H$ is a group homomorphism, then $\mathrm{scl}_G\, g \geq \mathrm{scl}_H\, \varphi(g)$ for all $g \in G'$; the analogous result holds for chains. In particular, scl is invariant under automorphisms, whence under conjugation. Thus, scl on single chains agrees with the usual definition of stable commutator length.

Stable commutator length has the following geometric interpretation: If X is a connected topological space and $\gamma\colon S^1 \to X$ is a loop, then the stable commutator length of the associated element $[\gamma] \in \pi_1(X)$ measures the least complexity of the surface needed to bound γ (we will not use this interpretation in this paper). In Section 6.2, we will describe yet another interpretation of scl, namely as a topological stable-filling invariant.

4.2 Bavard's duality theorem and bounded cohomology

Let G be a group. A map $\phi\colon G \to \mathbb{R}$ is called a *quasimorphism* if there is a constant $C > 0$ such that

$$\forall_{g,h\in G} \quad \bigl|\phi(g) + \phi(h) - \phi(g \cdot h)\bigr| \leq C.$$

The smallest such C is called the *defect* of ϕ and is denoted by $D(\phi)$. A quasimorphism ϕ is *homogeneous* if in addition we have that $\phi(g^n) = n \cdot \phi(g)$ for all $g \in G, n \in \mathbb{Z}$. Every quasimorphism $\phi\colon G \to \mathbb{R}$ is in bounded distance to a unique homogeneous quasimorphism $\bar{\phi}\colon G \to \mathbb{R}$, defined by setting

$$\bar{\phi}(g) := \lim_{n\to\infty} \frac{\phi(g^n)}{n}$$

for every $g \in G$. Moreover, it is well-known that $D(\bar{\phi}) \leq 2 \cdot D(\phi)$ [18, Lemma 2.58]. Analogously to the duality principle (Proposition 3.4) we may compute scl using homogeneous quasimorphisms:

Theorem 4.1 (Bavard's duality theorem [1]) *Let G be a group and let $g \in G'$. Then*

$$\mathrm{scl}_G\, g = \sup_{\phi} \frac{|\phi(g)|}{2 \cdot D(\phi)},$$

where the supremum is taken over all homogeneous quasimorphisms $\phi\colon G \to \mathbb{R}$. Moreover, this supremum is achieved by an extremal *quasimorphism.*

Remark 4.2 (Homogeneous) quasimorphisms are intimately related to second bounded real cohomology. Using the inhomogeneous resolution, it can be seen that the kernel of $c_G^2\colon H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ corresponds to the space of homogeneous quasimorphisms modulo $\mathrm{Hom}(G,\mathbb{R})$. It follows then from Bavard's dualtiy theorem that the comparison map $c_G^2\colon H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$ is injective if and only if scl_G vanishes on G.

It is well known that $H_b^2(G;\mathbb{R})$ vanishes if G is abelian [35]. Thus every homogeneous quasimorphism on an abelian group is an honest homomorphism.

4.3 Examples

We collect some known results for stable commutator length.

In Sects. 6 and 8 we will promote scl in a finitely presented group G to the simplicial volume of manifolds in higher dimension. For this we need to assert that $H_2(G;\mathbb{R})$ vanishes. Thus, we will have a particular emphasis on this condition in the examples.

4.3.1 Vanishing

An element $g \in G'$ may satisfy that $\mathrm{scl}_G\, g = 0$ for "trivial" reasons, such as if g is torsion or if g is conjugate to its inverse. There are many classes of groups where—besides these trivial reasons—stable commutator length vanishes on the whole group. Recall that this is equivalent to the injectivity of the comparison map $c_G^2\colon H_b^2(G;\mathbb{R}) \to H^2(G;\mathbb{R})$. Examples include:

- Amenable groups: This follows from the vanishing of $H_b^2(G;\mathbb{R})$ for every amenable group G by a result of Trauber [35],
- irreducible lattices in semisimple Lie groups of rank at least 2 [9], and
- subgroups of the group $\mathrm{PL}^+(I)$ of piecewise linear transformations of the interval [15].

4.3.2 Non-abelian free groups

In contrast, Duncan and Howie [29] showed that every element $g \in F' \setminus \{e\}$ in the commutator subgroup of a non-abelian free group F satisfies $\mathrm{scl}_F\, g \geq 1/2$. In a sequence of papers [19,21] Calegari showed that stable commutator length is *rational* in free groups and that every rational number mod 1 is realised as the stable commutator length of some element in the free group. Moreover, he gave an explicit, polynomial time algorithm to compute stable commutator length in free groups. This revealed a surprising distribution of those values. We note that these results generalise to free products of cyclic groups [57] and that all these groups G satisfy $H_2(G;\mathbb{R}) \cong 0$.

4.3.3 Gaps and groups of non-positive curvature

A group G has a *gap in* scl if there is a constant $C > 0$ such that for every group element g, we have $\mathrm{scl}_G\, g \geq C$ unless $\mathrm{scl}_G\, g = 0$ for "trivial" reasons such as torsion or if g is conjugate to its inverse.

In the previous example, we already have seen that non-abelian free groups have a gap in stable commutator length of $1/2$. This result has recently been generalised to right-angled Artin groups [40]. Many classes of non-positively curved groups have a gap in scl, though this gap may not be uniform in the whole class of groups. Prominent examples include hyperbolic groups [20], mapping class groups [10], free products of torsion-free groups [22] and amalgamated free products [23,28,40].

4.3.4 Hyperelliptic mapping class groups

Let $g \in \mathbb{N}$, let $\iota \in \mathcal{M}_g$ be the mapping class of a hyperelliptic involution of the orientable closed connected surface Σ_g of genus g, and let

$$\mathcal{H}_g := \left\{ x \in \mathcal{M}_g \;\middle|\; \iota \cdot x \cdot \iota^{-1} = x \right\} \subset \mathcal{M}_g$$

be the *hyperelliptic mapping class group of* Σ_g. The group $\mathcal{H}_g$ is finitely presented [8] and satisfies $H_2(\mathcal{H}_g;\mathbb{R}) \cong 0$ [47, Corollary 3.3][6, Theorem 1.1][11]. We now let $g \geq 2$. Let $t \in \mathcal{H}_g$ be a Dehn twist about a ι-invariant non-separating curve on Σ_g. Then we have

$$0 < \frac{1}{4 \cdot (2 \cdot g + 1)} \leq \operatorname{scl}_{\mathcal{H}_g} t \leq \frac{1}{2 \cdot (2 \cdot g + 3 + 1/g)};$$

the first estimate is a computation by Monden [50, Theorem 1.2] (similar estimates also appear in the work of Endo and Kotschick [30, proof of Corollary 8]), the second estimate is due to Calegari et al. [26, Theorem 1.7].

5 The universal central extension of Thompson's group T

Thompson's group T was introduced in 1965 by Richard Thompson as the first example of an infinite but finitely presented simple group. It is the subgroup of $\mathrm{PL}^+(S^1)$ which maps dyadic rationals to dyadic rationals, with dyadic breakpoints and where each derivative—if defined—is an integer power of 2 (here, we identify $\mathbb{R}/\mathbb{Z} \cong S^1$) [27].

Stable commutator length on Thompsons's group T vanishes [18, Chapter 5], but interesting values for stable commutator length arise on the central extensions of T and its generalisations associated to the Euler class [60] (for the definition of the Euler extension, see Sect. 3.6).

In this section, we extend these results about stable commutator length on these extensions to the *universal* central extension E of T. For a perfect group G the universal central extension $\widetilde{G}$ is the unique group that is a Schur covering group of G. It satisfies that $H_1(\widetilde{G}; \mathbb{Z}) \cong 0$ and $H_2(\widetilde{G}; \mathbb{Z}) \cong 0$ and there is an explicit construction of $\tilde{G}$ in terms of G and $H^2(\tilde{G}; \mathbb{Z})$ (which we recall during the proof of Proposition 5.1).

Proposition 5.1 *The universal central extension E of Thompson's group T is finitely presented and satisfies that $H_1(E; \mathbb{Z}) \cong 0 \cong H_2(E; \mathbb{Z})$. For every non-negative rational number $q \in \mathbb{Q}_{\geq 0}$, there is an element $e_q \in E$ with $\operatorname{scl}_E e_q = q$.*

In Sect. 5.1, we recall results of Zhuang [60], which describe stable commutator length on the central extension $\widetilde{T}$ of T associated to the Euler class. Using information on the (bounded) 2-cohomology of Thompson's group T (Sect. 5.2), we reduce stable commutator length on E to stable commutator length on $\widetilde{T}$ and show Proposition 5.1 (Sect. 5.3).

5.1 The Euler central extension of Thompson's group T

We recall the connection between stable commutator length and rotation number. This connection has been established by Barge and Ghys [5] and has been used by Zhuang [60] to construct finitely presented groups with transcendental stable commutator length.

Theorem 5.2 ([5,60]) *Let $\widetilde{T}$ be the central extension of Thompson's group T associated to the Euler class ${}^T\mathrm{eu}^{\mathbb{Z}} \in H^2(T;\mathbb{Z})$. Then there is a homogeneous quasimorphism* $\mathrm{rot}\colon \widetilde{T} \to \mathbb{R}$ *of defect* 1*, called* rotation number*, that generates the space of homogeneous quasimorphisms. Hence, for all $\widetilde{t} \in \widetilde{T}$,*

$$\mathrm{scl}_{\widetilde{T}}\,\widetilde{t} = \frac{\left|\mathrm{rot}(\widetilde{t})\right|}{2}.$$

The rotation number is well studied and has a geometric meaning [12]. Hence, one obtains the full spectrum of stable commutator length for $\widetilde{T}$.

Corollary 5.3 ([18]) *Let $\widetilde{T}$ be the central extension of Thompson's group T by the Euler class. Then the image of stable commutator length on $\widetilde{T}$ is $\mathbb{Q}_{\geq 0}$.*

Proof Ghys and Sergiescu [37] showed that the rotation number on $\widetilde{T}$ is rational. Moreover, it is well known that every rational number is realised as such a rotation number. To see this observe that for every integer $n \in \mathbb{N}$ there is an element $t_n \in T$ with a periodic orbit of size n that cyclically permutes the elements of this orbit. An appropriate lift $\widetilde{t}_n$ of this element to $\widetilde{T}$ will satisfy $\mathrm{rot}\,\widetilde{t}_n = 1/n$. By taking powers of such elements we may realise every rational as a rotation number in $\widetilde{T}$. □

However, Ghys and Sergiescu [37] showed that $H_2(\widetilde{T};\mathbb{Z}) \cong \mathbb{Z}$. Thus, we cannot apply Theorem C to the group $\widetilde{T}$.

5.2 (Bounded) 2-cohomology of Thompson's group *T*

The cohomology of Thompson's group T was computed by Ghys and Sergiescu [37]. Ghys and Sergiescu showed that the 2-cohomology $H^2(T;\mathbb{Z})$ is generated by the Euler class ${}^T\mathrm{eu}^{\mathbb{Z}}$ (see Sect. 3.6) and another class α, which has the following combinatorial description.

For a function $\phi\colon S^1 \to \mathbb{R}$ that admits limits on both sides at every point in S^1, let $\phi(x_+)$ be the right and let $\phi(x_-)$ be the left limit at $x \in S^1$. In this case, set $\Delta\phi(x) := \phi(x_+) - \phi(x_-)$. Moreover, for an element $u \in T$ we denote by $u'_r(x)$ the right derivative of u at $x \in S^1$, i.e., $u'_r(x) = u'(x_+)$.

Definition 5.4 (*Discrete Godbillon–Vey cocycle* [37, Theorem E]) The *discrete Godbillon–Vey cocycle* $\overline{\mathrm{gv}}\colon T \times T \to \mathbb{Z}$ is defined as

$$\overline{\mathrm{gv}}(u, v) := \sum_{x \in S^1} \begin{vmatrix} \log_2(v)'_r & \log_2(u \circ v)'_r \\ \Delta \log_2(v)'_r & \Delta \log_2(u \circ v)'_r \end{vmatrix} (x)$$

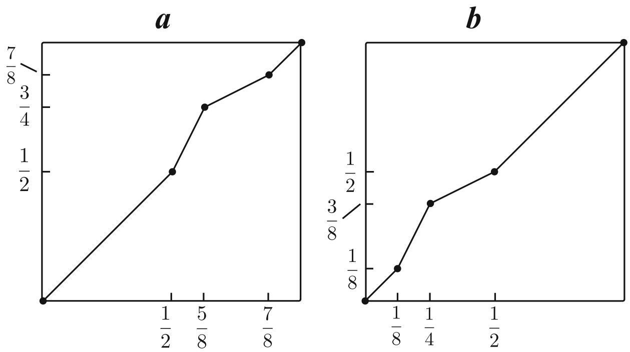

Fig. 1 The generators a (left) and b (right)

where the (finite) sum runs over all $x \in S^1$ that are breakpoints of v, u or $u \circ v$.

The map $\overline{\mathrm{gv}}$ is an *inhomogeneous cocycle*. In this section only we will use inhomogeneous cocycles as they are better to work with in the context of central extensions; the precise definition can for instance be found in Frigerio's book [31, Chapter 1.7].

Theorem 5.5 ([37, Corollary C, Theorem E]) *Thompson's group T satisfies that $H^2(T; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Free generators are the Euler class ${}^{T}\mathrm{eu}^{\mathbb{Z}}$ and a class α. This class satisfies that $2 \cdot \alpha = [\overline{\mathrm{gv}}] \in H^2(G; \mathbb{Z})$, where $\overline{\mathrm{gv}}$ is the discrete Godbillon-Vey cocycle (Definition 5.4).*

For what follows we will also need to compute the *bounded* cohomology of T in degree 2.

Proposition 5.6 *The class $\alpha \in H^2(T; \mathbb{Z})$ from Theorem 5.5 cannot be represented by a bounded cocycle, i.e., α is* not *in the image of the comparison map $H_b^2(T; \mathbb{Z}) \to H^2(T; \mathbb{Z})$. In particular, we have that*

$$H_b^2(T; \mathbb{R}) \cong \mathbb{R},$$

generated by the Euler class.

Proof Note that it is enough to show the unboundedness statement for $[\overline{\mathrm{gv}}]$ as $2 \cdot \alpha = [\overline{\mathrm{gv}}]$ (Theorem 5.5). We will show the proposition by evaluating $\overline{\mathrm{gv}}$ on the subgroup $\mathbb{Z}^2 \cong \langle a, b \rangle_T \subset T$, where a and b are the elements depicted in Fig. 1.

Claim 5.7 *The cocycle $\overline{\mathrm{gv}}$ restricts on $\langle a, b \rangle_T$ to a cocycle representing a non-trivial element of $H^2(\mathbb{Z}^2; \mathbb{Z}) \cong \mathbb{Z}$.*

Proof of Claim 5.7 This claim is implicitly stated in the work of Ghys and Sergiescu [37, proof of Lemma 4.6]. For the convenience of the reader we provide an explicit proof here.

Observe that $(i, i') \mapsto \overline{\mathrm{gv}}(a^i, a^{i'})$ is a (inhomogeneous) 2-cocycle on $\mathbb{Z}$. Since $H^2(\mathbb{Z};\mathbb{Z}) \cong 0$, we see that there is a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $\overline{\mathrm{gv}}(a^i, a^{i'}) = f(i) + f(i') - f(i+i')$ for all $i, i' \in \mathbb{Z}$. Similarly, we see that there is a function $g : \mathbb{Z} \to \mathbb{Z}$ such that $\overline{\mathrm{gv}}(b^j, b^{j'}) = g(j) + g(j') - g(j+j')$. Observe that we have

$$\begin{vmatrix} \log_2(a^i)'_r & \log_2(a^{i+i'})'_r \\ \Delta\log_2(a^i)'_r & \Delta\log_2(a^{i+i'})'_r \end{vmatrix}(x) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

for any $x \in [0, 1/2)$ and that

$$\begin{vmatrix} \log_2(a^i)'_r & \log_2(a^{i+i'})'_r \\ \Delta\log_2(a^i)'_r & \Delta\log_2(a^{i+i'})'_r \end{vmatrix}(1/2) = \begin{vmatrix} i & i+i' \\ i & i+i' \end{vmatrix} = 0.$$

This way we see that

$$\begin{aligned} f(i) + f(i') - f(i+i') &= \overline{\mathrm{gv}}(a^i, a^{i'}) \\ &= \sum_{x \in (1/2, 1]} \begin{vmatrix} \log_2(a^i)'_r & \log_2(a^{i+i'})'_r \\ \Delta\log_2(a^i)'_r & \Delta\log_2(a^{i+i'})'_r \end{vmatrix}(x) \\ &= \sum_{x \in (1/2, 1]} \begin{vmatrix} \log_2(a^i b^j)'_r & \log_2(a^{i+i'} b^{j+j'})'_r \\ \Delta\log_2(a^i b^j)'_r & \Delta\log_2(a^{i+i'} b^{j+j'})'_r \end{vmatrix}(x) \end{aligned}$$

for all $i, j, i', j' \in \mathbb{Z}$. Similarly, we see that

$$\begin{aligned} g(j) + g(j') - g(j+j') &= \overline{\mathrm{gv}}(b^j, b^{j'}) \\ &= \sum_{x \in (0, 1/2)} \begin{vmatrix} \log_2(b^j)'_r & \log_2(b^{j+j'})'_r \\ \Delta\log_2(b^j)'_r & \Delta\log_2(b^{j+j'})'_r \end{vmatrix}(x) \\ &= \sum_{x \in (0, 1/2)} \begin{vmatrix} \log_2(a^i b^j)'_r & \log_2(a^{i+i'} b^{j+j'})'_r \\ \Delta\log_2(a^i b^j)'_r & \Delta\log_2(a^{i+i'} b^{j+j'})'_r \end{vmatrix}(x). \end{aligned}$$

We moreover calculate

$$\begin{aligned} \begin{vmatrix} \log_2(a^i b^j)'_r & \log_2(a^{i+i'} b^{j+j'})'_r \\ \Delta\log_2(a^i b^j)'_r & \Delta\log_2(a^{i+i'} b^{j+j'})'_r \end{vmatrix}(1/2) &= \begin{vmatrix} i & (i+i') \\ (i-j) & (i+i'-j-j') \end{vmatrix} \\ &= -\begin{vmatrix} i & i' \\ j & j' \end{vmatrix}. \end{aligned}$$

Putting the above calculations together, we can now compute the restriction of $\overline{\mathrm{gv}}$ to $\langle a, b\rangle_T$. For all $i, j, i', j' \in \mathbb{Z}$ we see that

$$\begin{aligned} \overline{\mathrm{gv}}(a^i b^j, a^{i'} b^{j'}) &= \sum_{x \in S^1} \begin{vmatrix} \log_2(a^i b^j)'_r & \log_2(a^{i+i'} b^{j+j'})'_r \\ \Delta \log_2(a^i b^j)'_r & \Delta \log_2(a^{i+i'} b^{j+j'})'_r \end{vmatrix} (x) \\ &= - \begin{vmatrix} i & i' \\ j & j' \end{vmatrix} + \delta^1 f_0\big((i, j), (i', j')\big) + \delta^1 g_0\big((i, j), (i', j')\big), \end{aligned}$$

where $f_0(i, j) := f(i)$ and $g_0(i, j) := g(j)$. Hence, evaluating $\overline{\mathrm{gv}}$ on a fundamental cycle shows that $\overline{\mathrm{gv}}$ restricted to $\langle a, b\rangle_T$ represents twice a generator of $H^2(\mathbb{Z}^2; \mathbb{Z}) \cong \mathbb{Z}$. This proves Claim 5.7. $\square$

It is well-known that non-trivial elements of $H^2(\mathbb{Z}^2; \mathbb{Z})$ cannot be represented by a bounded cocycle ($\mathbb{Z}^2$ is amenable). Hence, also $[\overline{\mathrm{gv}}]$ cannot be represented by a bounded cocycle, which proves the first part of Proposition 5.6.

Stable commutator length vanishes on T (Example 4.3.1) and so the comparison map $c_T^2 \colon H_b^2(T; \mathbb{R}) \to H^2(T; \mathbb{R})$ is injective (Remark 4.2). We now assume for a contradiction that $\lambda \cdot {}^T\mathrm{eu}^{\mathbb{R}} + \mu \cdot \alpha \in H^2(T; \mathbb{R})$ lies in the image of the comparison map $c_T^2 \colon H_b^2(T; \mathbb{R}) \to H^2(T; \mathbb{R})$ and $\mu \neq 0$. As ${}^T\mathrm{eu}^{\mathbb{R}}$ is bounded and $\langle a, b\rangle_T$ is amenable, ${}^T\mathrm{eu}^{\mathbb{R}}$ restricts to a trivial class on $\langle a, b\rangle_T$. Thus $\lambda \cdot {}^T\mathrm{eu}^{\mathbb{R}} + \mu \cdot \alpha$ restricts to $\mu \cdot \alpha$ on $\langle a, b\rangle_T$ and generates $H^2(\mathbb{Z}^2; \mathbb{R})$ (by Claim 5.7). This is a contradiction as these classes are not bounded. Hence, the only classes in the image of c_T^2 are multiples of ${}^T\mathrm{eu}^{\mathbb{R}}$. We conclude that

$$H_b^2(T; \mathbb{R}) \cong \mathbb{R},$$

generated by the Euler class. This completes the proof of Proposition 5.6. $\square$

5.3 Proof of Proposition 5.1

We will now prove Proposition 5.1 by explicitly computing the quasimorphisms on E and then invoking Bavard's duality theorem. We note that there is an alternative proof using diagrams by applying Gromov's mapping theorem. A variation of this may be found in the forthcoming paper [42, Section 3.3].

Proof of Proposition 5.1 As E arises as a group extension of finitely presented groups it is itself finitely presented. The group T is simple [27] and thus in particular perfect. The universal central extension E of a perfect group T always satisfies that $H_1(E; \mathbb{Z}) \cong 0 \cong H_2(E; \mathbb{Z})$ [58, Chapter 6.9].

It remains to show that every rational number $q \in \mathbb{Q}$ is the stable commutator length of some element $e_q \in E$. For this note that E may be explicitly described as the group on the set $\mathbb{Z}^2 \times T$ with multiplication

$$\left(\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right), t\right) \times \left(\left(\begin{smallmatrix} i' \\ j' \end{smallmatrix}\right), t'\right) \mapsto \left(\left(\begin{smallmatrix} i+i'+{}^T\mathrm{Eu}(t,t') \\ j+j'+\mathrm{A}(t,t') \end{smallmatrix}\right), t \cdot t'\right),$$

where ${}^T\,\mathrm{Eu}$ (resp. A) is an inhomogeneous cocycle representing ${}^T\mathrm{eu}^{\mathbb{Z}} \in H^2(T;\mathbb{Z})$ (resp. $\alpha \in H^2(T;\mathbb{Z})$). Similarly $\widetilde{T}$ may be described as the set $\mathbb{Z} \times T$ with group multiplication $(i, t) \times (i', t') \mapsto (i + i' +^T \mathrm{Eu}(t, t'), t \cdot t')$.

Claim 5.8 *For every* $(i_0, t_0) \in \widetilde{T}$*, we have that* $\mathrm{scl}_{\widetilde{T}}(i_0, t_0) = \mathrm{scl}_E\big(\left(\begin{smallmatrix} i_0 \\ 0 \end{smallmatrix}\right), t_0\big)$.

Proof of Claim 5.8 The homomorphism $\kappa\colon E \to \widetilde{T}$ defined by $\kappa\colon \big(\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right), t\big) \mapsto (i, t)$ shows by monotonicity of scl that $\mathrm{scl}_E\big(\left(\begin{smallmatrix} i_0 \\ 0 \end{smallmatrix}\right), t_0\big) \geq \mathrm{scl}_{\widetilde{T}}\, \kappa\big(\left(\begin{smallmatrix} i_0 \\ 0 \end{smallmatrix}\right), t_0\big) = \mathrm{scl}_{\widetilde{T}}(i_0, t_0)$. We now prove the converse inequality.

Let $\phi\colon E \to \mathbb{R}$ be a homogeneous extremal quasimorphism to the element $\big(\left(\begin{smallmatrix} i_0 \\ 0 \end{smallmatrix}\right), t_0\big)$. Then ϕ restricted to the centre of E is a homogeneous quasimorphism on an abelian group, and thus a homomorphism; see Remark 4.2. Thus there are constants $\lambda_{\mathrm{Eu}}, \lambda_{\mathrm{A}} \in \mathbb{R}$ such that $\phi\big(\left(\begin{smallmatrix} i \\ j \end{smallmatrix}\right), \mathrm{id}\big) = \lambda_{\mathrm{Eu}} \cdot i + \lambda_{\mathrm{A}} \cdot j$ for all $i, j \in \mathbb{Z}$.

We will first show that $\lambda_{\mathrm{A}} = 0$. For every element z in the centre and an element $e \in E$, the group $\langle z, e\rangle$ generated by z and e is abelian and hence ϕ restricts to a homomorphism on $\langle z, e\rangle$ by again using Remark 4.2. Hence we have $\phi(z \cdot e) = \phi(z) + \phi(e)$ for all z in the centre and $e \in E$. We define $\Delta \in C^2(T;\mathbb{R})$ as $\Delta(t, t') := \delta^1\phi\big(\big(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), t\big), \big(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), t'\big)\big)$ for all $t, t' \in T$. Then Δ is uniformly bounded, because ϕ is a quasimorphism. We compute

$$\begin{aligned}\Delta(t, t') &= \phi\big(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), t\big) + \phi\big(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), t'\big) - \phi\big(\big(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), t\big) \cdot \big(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), t'\big)\big) \\ &= \delta^1\psi(t, t') - \lambda_{\mathrm{Eu}} \cdot^T \mathrm{Eu}(t, t') - \lambda_{\mathrm{A}}\mathrm{A}(t, t')\end{aligned}$$

for all $t, t' \in T$ and for $\psi\colon T \to \mathbb{R}$ defined via $\psi\colon t \mapsto \phi(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), t)$. Thus

$$\lambda_{\mathrm{A}} \cdot \mathrm{A} - \delta^1\psi = -\lambda_{\mathrm{Eu}} \cdot^T \mathrm{Eu} - \Delta$$

and hence $\lambda_{\mathrm{A}} \cdot \mathrm{A} - \delta^1\psi$ defines a bounded cocycle as the right hand side is uniformly bounded. If $\lambda_{\mathrm{A}} \neq 0$, then this would imply that α may be represented by a bounded cocycle, which would contradict Proposition 5.6. Thus $\lambda_{\mathrm{A}} = 0$.

Define a quasimorphism $\phi_{\widetilde{T}}$ on $\widetilde{T}$ by setting $\phi_{\widetilde{T}}(i, t) = \phi(\left(\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right), t)$ and observe that $\phi_{\widetilde{T}}$ is homogeneous as well and that $\phi_{\widetilde{T}}(i_0, t_0) = \phi(\left(\begin{smallmatrix} i_0 \\ 0 \end{smallmatrix}\right), t_0))$. For all $i, i', j, j' \in \mathbb{Z}$, $t, t' \in T$ we compute that

$$\begin{aligned}\delta^1\phi\Big(\Big(\Big(\begin{smallmatrix} i \\ j \end{smallmatrix}\Big), t\Big), \Big(\Big(\begin{smallmatrix} i' \\ j' \end{smallmatrix}\Big), t'\Big)\Big) &= \delta^1\psi(t, t') - \lambda^T_{\mathrm{Eu}} \operatorname{Eu}(t, t') \\ &= \delta^1\phi_{\widetilde{T}}\big((i, t), (i', t')\big)\end{aligned}$$

and thus $D(\phi) = D(\phi_{\widetilde{T}})$.

Using Bavard's duality theorem we compute

$$\operatorname{scl}_{\widetilde{T}}(i_0, t_0) \geq \frac{\phi_{\widetilde{T}}(i_0, t_0)}{2D(\phi_{\widetilde{T}})} = \frac{\phi(\left(\begin{smallmatrix} i_0 \\ 0 \end{smallmatrix}\right), t_0)}{2D(\phi)} = \operatorname{scl}_E(\left(\begin{smallmatrix} i_0 \\ 0 \end{smallmatrix}\right), t_0).$$

This proves the other inequality and thus finishes the proof of Claim 5.8. □

We may now finish the proof of Proposition 5.1. Every $q \in \mathbb{Q}_{\geq 0}$ is the stable commutator length of some element $t_q \in \widetilde{T}$ by Corollary 5.3. Using Claim 5.8, we may construct an element $e_q \in E$ with $\operatorname{scl}_E e_q = \operatorname{scl}_{\widetilde{T}} t_q = q$. □

6 Fillings

Stable commutator length can be interpreted as a homological filling norm (Sect. 6.2). After recalling the basic notions and properties, we will use this interpretation to compute the l^1-semi-norm of classes related to decomposable relators and thus prove Theorem C (Sect. 6.3). This will allow us to establish the group-theoretic version of the no-gap theorem (Theorem D); for the proof of Theorems C and D, we will only need the filling norm in dimension 2, as already considered by Bavard [1] and Calegari [18, Chapter 2.5/2.6]. Moreover, we will explain how in the higher-dimensional case the simplicial volume of manifolds can also be viewed as a filling norm (Sect. 6.4).

6.1 Stable filling norms

We first recall the stable filling norm for the bar complex. We will then extend this notion to topological spaces and higher degrees. For a group G, the bar complex $C_\bullet(G; \mathbb{R})$ (computing $H_\bullet(G; \mathbb{R})$) has the following form in low degrees: We have $C_1(G; \mathbb{R}) = \mathbb{R}[G]$ and $\partial_1 = 0$ as well as $C_2(G; \mathbb{R}) = \mathbb{R}[G]^2$ and

$$\begin{aligned}\partial_2 \colon C_2(G; \mathbb{R}) &\to C_1(G; \mathbb{R}) \\ G \times G \ni (g, h) &\mapsto g + h - g \cdot h.\end{aligned}$$

Moreover, the chain modules of $C_\bullet(G; \mathbb{R})$ are endowed with the l^1-norm corresponding to the bar bases.

Definition 6.1 ((*Stable*) *filling norm*) Let G be a group.

- If $c \in \partial C_2(G;\mathbb{R})$, the *filling norm of* c is defined as

$$\operatorname{fill}_G c := \inf\bigl\{|b|_1 \bigm| b \in C_2(G;\mathbb{R}),\ \partial b = c\bigr\}.$$

- Let $m \in \mathbb{N}$ and let $r_1,\dots,r_m \in G'$. The *stable filling norm of* $r_1+\dots+r_m$ is defined as

$$\operatorname{sfill}_G(r_1+\dots+r_m) := \lim_{n\to\infty}\frac{1}{n}\cdot \operatorname{fill}_G(r_1^n+\dots+r_m^n).$$

Notice that the limit in the definition of the stable filling norm indeed exists [18, p. 34].

For the generalisation to topological spaces, we replace group elements by loops (or maps from simplicial spheres) and we replace taking powers of group elements by composition with self-maps of spheres of the corresponding degree.

Definition 6.2 (*Topological* (*stable*) *filling norms*) Let $d \in \mathbb{N}$, let X be a topological space, and let $\sigma\colon \partial\Delta^d \to X$ be continuous.

- If $c \in \partial(C_d(X;\mathbb{R}))$, the *filling norm of* c is defined as

$$\operatorname{fill}_X c := \inf\bigl\{|b|_1 \bigm| b \in C_d(X;\mathbb{R}),\ \partial b = c\bigr\}.$$

- The *filling norm of* σ is then defined as

$$\operatorname{fill}_X \sigma := \operatorname{fill}_X c_\sigma = \inf\bigl\{|b|_1 \bigm| b \in C_d(X;\mathbb{R}),\ \partial b = c_\sigma\bigr\},$$

  where $c_\sigma := C_{d-1}(\sigma;\mathbb{R})(\partial\, \mathrm{id}_{\Delta^d}) \in C_{d-1}(X;\mathbb{R})$ is the canonical singular cycle associated with σ.
- The *stable filling norm of* σ is defined as

$$\operatorname{sfill}_X \sigma := \lim_{n\to\infty}\frac{\operatorname{fill}_X \sigma[n]}{n},$$

  where for $n \in \mathbb{N}$, we write $w_n\colon \partial\Delta^d \to \partial\Delta^d$ for "the" standard self-map of $\partial\Delta^d \cong S^{d-1}$ of degree n and $\sigma[n] := \sigma\circ w_n$.
- If $m \in \mathbb{N}$ and $\sigma_1,\dots,\sigma_m\colon \partial\Delta^d \to X$ are continuous maps, then we define

$$\operatorname{fill}_X(\sigma_1+\dots+\sigma_m) := \operatorname{fill}_X(c_{\sigma_1}+\dots+c_{\sigma_m})$$
$$\operatorname{sfill}_X(\sigma_1+\dots+\sigma_n) := \lim_{n\to\infty}\frac{1}{n}\cdot\operatorname{fill}_X\bigl(\sigma_1[n]+\dots+\sigma_m[n]\bigr)$$

Remark 6.3 (Existence of the stabilisation limit) The limits in the situation of the definition above indeed exist: For notational convenience, we only prove the existence in the case of $\operatorname{sfill}_X \sigma$; the general case can be proved in the same way (with additional indices). The argument is similar to the one for the stable filling norm in the bar complex. The only complication is that, in order to compare different "powers", we will need to use the uniform boundary condition for $C_\bullet(\partial\Delta^d;\mathbb{R})$.

Because $\pi_1(\partial\Delta^d)$ is amenable, there exists a constant $K \in \mathbb{R}_{>0}$ with the following property [32,51]: For every $z \in \partial(C_d(\partial\Delta^d;\mathbb{R}))$ there is a $b \in C_d(\partial\Delta^d;\mathbb{R})$ with

$$z = \partial b \quad \text{and} \quad |b|_1 \leq K \cdot |z|_1.$$

If $n, m \in \mathbb{N}$, then the chains $c_{w_n} + c_{w_m}$ and $c_{w_{n+m}}$ are homologous in the complex $C_\bullet(\partial\Delta^d;\mathbb{R})$ (because $\deg w_n + \deg w_m = n + m = \deg w_{n+m}$). Thus, there exists a chain $b_{n,m} \in C_d(\partial\Delta^d;\mathbb{R})$ such that

$$c_{w_{n+m}} - c_{w_n} - c_{w_m} = \partial b_{n,m} \quad \text{and} \quad |b_{n,m}|_1 \leq K \cdot 3 \cdot (d+1).$$

Hence, for every continuous map $\sigma\colon \partial\Delta^d \to X$ we obtain

$$\begin{aligned} c_{\sigma[n+m]} - c_{\sigma[n]} - c_{\sigma[m]} &= \partial C_d(\sigma;\mathbb{R})(b_{n,m}) \quad \text{and} \\ \bigl|C_d(\sigma;\mathbb{R})(b_{n,m})\bigr|_1 &\leq |b_{n,m}|_1 \leq K \cdot 3 \cdot (d+1), \end{aligned}$$

and so

$$\operatorname{fill}_X \sigma[n+m] \leq \operatorname{fill}_X \sigma[n] + \operatorname{fill}_X \sigma[m] + K \cdot 3 \cdot (d+1).$$

Now elementary analysis shows that the limit $\lim_{n\to\infty} 1/n \cdot \operatorname{fill}_X \sigma[n]$ does exist.

Remark 6.4 (Change of the self-maps) The map w_n is only unique up to homotopy, but homotopic choices for w_n lead to the same stable filling norm; this can be seen using the uniform boundary condition as in the proof of the existence of the stable filling limits (Remark 6.3). Therefore, this ambiguity will be of no consequence for us.

Remark 6.5 (Change of the singular models) In the situation of Definition 6.2, we could choose other singular cycle models of σ than c_σ: If $c' \in C_{d-1}(\partial\Delta^d;\mathbb{R})$ is a fundamental cycle of $\partial\Delta^d$, if $b' \in C_d(\partial\Delta^d;\mathbb{R})$ satisfies $\partial b' = c' - c$, and if $c'_\sigma := C_{d-1}(\sigma;\mathbb{R})(c')$, then

$$\begin{aligned} \operatorname{sfill}_X \sigma &= \lim_{n\to\infty} \frac{1}{n} \cdot \inf\bigl\{|b|_1 \bigm| b \in C_d(X;\mathbb{R}),\ \partial b = c_{\sigma[n]}\bigr\} \\ &= \lim_{n\to\infty} \frac{1}{n} \cdot \inf\bigl\{\bigl|b + C_d(\sigma[n];\mathbb{R}))(b')\bigr|_1 \bigm| b \in C_d(X;\mathbb{R}),\ \partial b = c_{\sigma[n]}\bigr\} \end{aligned}$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \inf\bigl\{|b|_1 \bigm| b \in C_d(X;\mathbb{R}),\ \partial b = c'_{\sigma[n]}\bigr\}.$$

For the second equality, we use that $|C_d(\sigma[n];\mathbb{R})(b')|_1 \le |b'|_1$ holds for all $n \in \mathbb{N}$ (so that the difference in norm is negligible when taking $n \to \infty$).

Remark 6.6 (Bar filling vs. topological filling) Let G be a group, let $m \in \mathbb{N}$, and let $r_1, \dots, r_m \in G'$. If X is a model of BG and $\sigma_1, \dots, \sigma_m \colon \partial\Delta^2 \to X$ are loops representing $r_1, \dots, r_m$, respectively, then

$$\mathrm{sfill}_G(r_1 + \cdots + r_m) = \mathrm{sfill}_X(\sigma_1 + \cdots + \sigma_m).$$

This can be seen as follows: The standard constructions produce chain maps $\varphi\colon C_\bullet(G;\mathbb{R}) \to C_\bullet(X;\mathbb{R})$ (choosing paths in $\widetilde{X}$ for each group element and inductively filling the simplices) and $\psi\colon C_\bullet(X;\mathbb{R}) \to C_\bullet(G;\mathbb{R})$ (choosing a set-theoretic fundamental domain D for the deck transformation action on $\widetilde{X}$ and looking at the translates of D that contain the vertices of the lifted simplices) with the following properties:

- $\psi \circ \varphi = \mathrm{id}_{C_\bullet(G;\mathbb{R})}$,
- $\varphi \circ \psi \simeq \mathrm{id}_{C_\bullet(X;\mathbb{R})}$ through a chain homotopy h that is bounded in every degree,
- $\|\varphi\| \le 1$ and $\|\psi\| \le 1$.

The first and third conditions easily imply that

$$\mathrm{fill}_G(r_1 + \cdots + r_m) \le \mathrm{fill}_X(\sigma_1 + \cdots + \sigma_m);$$

moreover, we have $\psi(\sigma_j[n]) = r_j^n$ for all $j \in \{1, \dots, m\}$ and all $n \in \mathbb{N}$. Hence,

$$\mathrm{sfill}_G(r_1 + \cdots + r_m) \le \mathrm{sfill}_X(\sigma_1 + \cdots + \sigma_m).$$

Conversely, if $b \in C_2(G;\mathbb{R})$ satisfies $\partial b = r_1 + \cdots + r_m$, then

$$\overline{b} := \varphi(b) + h(\sigma_1 + \cdots + \sigma_m)$$

satisfies

$$\begin{aligned}\partial\overline{b} &= \partial\varphi(b) + \partial h(\sigma_1 + \cdots + \sigma_m)\\ &= \varphi(\partial b) + \sigma_1 + \cdots + \sigma_m - \varphi \circ \psi(\sigma_1 + \cdots + \sigma_m)\\ &= \varphi(r_1 + \cdots + r_m) + \sigma_1 + \cdots + \sigma_m - \varphi(r_1 + \cdots + r_m)\\ &= \sigma_1 + \cdots + \sigma_m.\end{aligned}$$

Thus, $\operatorname{fill}_X(\sigma_1 + \cdots + \sigma_m) \leq |\overline{b}|_1 \leq |b|_1 + \|h\| \cdot m$. Taking the infimum over all such b results in

$$\operatorname{fill}_X(\sigma_1 + \cdots + \sigma_m) \leq \operatorname{fill}_G(r_1 + \cdots + r_m) + \|h\| \cdot m.$$

Passing to the stabilisation limit, we obtain

$$\begin{aligned}
\operatorname{sfill}_X(\sigma_1 + \cdots + \sigma_m) &= \lim_{n\to\infty} \frac{1}{n} \cdot \operatorname{fill}_X\bigl(\sigma_1[n] + \cdots + \sigma_m[n]\bigr) \\
&\leq \lim_{n\to\infty} \frac{1}{n} \cdot \bigl(\operatorname{fill}_G(r_1^n + \cdots + r_m^n) + \|h\| \cdot m\bigr) \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot \operatorname{fill}_G(r_1^n + \cdots + r_m^n) \\
&= \operatorname{sfill}_G(r_1 + \cdots + r_m).
\end{aligned}$$

6.2 Stable commutator length as filling invariant

The fact that every commutator consists of four pieces has the following generalisation in terms of filling norms:

Lemma 6.7 (scl as filling invariant) *Let G be a group, let $m \in \mathbb{N}$, and let $r_1, \dots, r_m \in G'$. Then*

$$\operatorname{scl}_G(r_1 + \cdots + r_m) = \frac{1}{4} \cdot \operatorname{sfill}_G(r_1 + \cdots + r_m).$$

Proof In the case of a single relator, this observation goes back to Bavard [1, Proposition 3.2][18, Lemma 2.69]. Calegari extended this equality to the case of linear combinations [18, Lemma 2.77]. □

Furthermore, as Calegari [17] puts it: "One can interpret stable commutator length as the infimum of the L^1 norm (suitably normalized) on chains representing a certain (relative) class in group homology." We will prove this statement in Corollary 6.9 as a special case of the following generalisation:

Proposition 6.8 (Relative ℓ^1-semi-norm as filling invariant) *Let Z be a CW-complex, let $m \in \mathbb{N}_{>0}$, $d \in \mathbb{N}_{\geq 2}$, let $\partial Z \subset Z$ be a subspace that is homeomorphic to $\coprod_m \partial\Delta^d$ and such that the inclusions $\sigma_1, \dots, \sigma_m \colon \partial\Delta^d \to Z$ of the m components of ∂Z into Z are π_1-injective (this is automatic if $d \geq 3$).*

1. *If $\beta \in H_d(Z, \partial Z; \mathbb{R})$ with $\partial\beta = [\partial Z]_{\mathbb{R}}$, then*

$$\|\beta\|_1 \geq \operatorname{sfill}_Z(\sigma_1 + \cdots + \sigma_m).$$

2. *If the connecting homomorphism* $\partial\colon H_d(Z, \partial Z; \mathbb{R}) \to H_{d-1}(\partial Z; \mathbb{R})$ *is an isomorphism and if* $\beta \in H_d(Z, \partial Z; \mathbb{R})$ *is the class with* $\partial\beta = [\partial Z]_{\mathbb{R}}$*, then*

$$\|\beta\|_1 = \operatorname{sfill}_Z(\sigma_1 + \cdots + \sigma_m).$$

Proof We first show the estimate $\operatorname{sfill}_Z(\sigma_1 + \cdots + \sigma_m) \leq \|\beta\|_1$: Clearly, the group $\pi_1(\partial\Delta^d)$ is amenable; because $\sigma_1, \ldots, \sigma_m$ are π_1-injective, the equivalence theorem [35][14, Corollary 6] ensures that for every $\epsilon \in \mathbb{R}_{>0}$ there exists a relative cycle $c \in C_d(Z; \mathbb{R})$ representing β in $H_d(Z, \partial Z; \mathbb{R})$ with

$$|c|_1 \leq \|\beta\|_1 + \epsilon \quad \text{and} \quad |\partial c|_1 \leq \epsilon. \tag{1}$$

Moreover, because the fundamental group of the m components of ∂Z are amenable, there exists a constant $K \in \mathbb{R}_{>0}$ implementing the uniform boundary condition [51]: For every $z \in \partial(C_d(\partial Z; \mathbb{R}))$ there is a $b \in C_d(\partial Z; \mathbb{R})$ with

$$z = \partial b \quad \text{and} \quad |b|_1 \leq K \cdot |z|_1.$$

Let $\epsilon \in \mathbb{R}_{>0}$, let c be a relative cycle as in (1), and let $n \in \mathbb{N}$ with $n \geq 1/\epsilon$. Then $z := \partial c - 1/n \cdot (c_{\sigma_1[n]} + \cdots + c_{\sigma_m[n]}) \in C_{d-1}(\partial Z; \mathbb{R})$ is a boundary (both summands are fundamental cycles of ∂Z). Hence, there exists a $b \in C_d(\partial Z; \mathbb{R})$ with

$$\begin{aligned}
\partial b &= \partial c - \frac{1}{n} \cdot (c_{\sigma_1[n]} + \cdots + c_{\sigma_m[n]}) \quad \text{and} \\
|b|_1 &\leq K \cdot \Bigl|\partial c - \frac{1}{n} \cdot (c_{\sigma_1[n]} + \cdots + c_{\sigma_m[n]})\Bigr|_1 \leq K \cdot \Bigl(\epsilon + m \cdot \frac{d+1}{n}\Bigr) \\
&\leq K \cdot \bigl(1 + m \cdot (d+1)\bigr) \cdot \epsilon.
\end{aligned}$$

The chain $n \cdot (c - b) \in C_d(Z; \mathbb{R})$ then witnesses that

$$\begin{aligned}
\frac{1}{n} \cdot \operatorname{fill}_Z\bigl(\sigma_1[n] + \cdots + \sigma_m[n]\bigr) &\leq \frac{1}{n} \cdot n \cdot |c - b|_1 = |c - b|_1 \\
&\leq \|\beta\|_1 + \epsilon + K \cdot \bigl(1 + m \cdot (d+1)\bigr) \cdot \epsilon.
\end{aligned}$$

Taking first $n \to \infty$ and then $\epsilon \to 0$ shows that $\operatorname{sfill}_Z(\sigma_1 + \cdots + \sigma_m) \leq \|\beta\|_1$.

Conversely, we will now prove that $\|\beta\|_1 \leq \operatorname{sfill}_Z(\sigma_1 + \cdots + \sigma_m)$ under the additional assumption that $\partial\colon H_d(Z, \partial Z; \mathbb{R}) \to H_{d-1}(\partial Z; \mathbb{R})$ is an isomorphism: Let $n \in \mathbb{N}_{>0}$ and let $b \in C_d(Z; \mathbb{R})$ with $\partial b = c_{\sigma_1[n]} + \cdots + c_{\sigma_m[n]}$. In particular, b is a relative cycle for $(Z, \partial Z)$; moreover, $1/n \cdot b$ represents β (because $\partial(1/n \cdot b) = 1/n \cdot (c_{\sigma_1[n]} + \cdots + c_{\sigma_m[n]})$ is a fundamental cycle of ∂Z). Hence,

$$\|\beta\|_1 \leq \frac{1}{n} \cdot |b|_1.$$

Taking first the infimum over all such b and then $n \to \infty$ yields the desired estimate $\|\beta\|_1 \leq \operatorname{sfill}_Z(\sigma_1 + \cdots + \sigma_m)$. □

Corollary 6.9 (scl as relative l^1-semi-norm) *Let G be a group that satisfies $H_2(G;\mathbb{R}) \cong 0$, let $m \in \mathbb{N}$, let $r_1, \ldots, r_m \in G'$ be elements of infinite order, and let X be a model of BG. Let*

$$\begin{aligned} Z &:= X \cup_{r_1,\ldots,r_m} \Big(\coprod_m S^1 \times [0,1]\Big) \\ &:= X \cup_{\coprod_{j=1}^m \gamma_j \text{ on } \coprod_m S^1 \times \{0\}} \Big(\coprod_m S^1 \times [0,1]\Big) \end{aligned}$$

be the mapping cylinder associated with (loops $\gamma_1, \ldots, \gamma_r$ in X representing) the elements $r_1, \ldots, r_m$, and let $\partial Z := \coprod_m S^1 \times \{1\} \subset Z$. Then there exists a unique relative homology class $\beta \in H_2(Z, \partial Z; \mathbb{R})$ whose boundary class $\partial\beta$ is the fundamental class of $\coprod_m S^1$; the class β satisfies

$$\|\beta\|_1 = 4 \cdot \operatorname{scl}_G(r_1 + \cdots + r_m).$$

Proof The long exact homology sequence of the pair $(Z, \partial Z)$ shows that the connecting homomorphism $\partial\colon H_2(Z, \partial Z; \mathbb{R}) \to H_1(\partial Z; \mathbb{R})$ is an isomorphism (by hypothesis, $H_2(Z;\mathbb{R}) \cong H_2(X;\mathbb{R}) \cong H_2(G;\mathbb{R}) \cong 0$, and the inclusion $\partial Z \hookrightarrow Z$ induces the trivial homomorphism on $H_1(\,\cdot\,;\mathbb{R})$ because $r_1, \ldots, r_m$ are in the commutator subgroup of G). This shows the existence of β.

Because $r_1, \ldots, r_m \in G'$ all have infinite order, the corresponding inclusions $\sigma_1, \ldots, \sigma_m \colon \partial\Delta^2 \to Z$ of the components of ∂Z into Z are π_1-injective.

Applying Proposition 6.8 (using $S^1 \cong \partial\Delta^2$), we obtain

$$\|\beta\|_1 = \operatorname{sfill}_Z(\sigma_1 + \cdots + \sigma_m).$$

In combination with Remark 6.6 and Lemma 6.7, this shows that

$$\|\beta\|_1 = \operatorname{sfill}_{F(S)}(r_1 + \cdots + r_m) = 4 \cdot \operatorname{scl}_S(r_1 + \cdots + r_m). \qquad \square$$

Corollary 6.10 (scl as relative l^1-semi-norm; free groups) *Let S be a set, let $m \in \mathbb{N}$, let $r_1, \dots, r_m \in F(S)'$ be non-trivial, let*

$$
\begin{aligned}
Z &:= \Bigl(\bigvee_S S^1\Bigr) \cup_{r_1,\dots,r_m} \Bigl(\coprod_m S^1 \times [0,1]\Bigr) \\
&:= \Bigl(\bigvee_S S^1\Bigr) \cup_{\coprod_{j=1}^m \gamma_j \text{ on } \coprod_m S^1 \times \{0\}} \Bigl(\coprod_m S^1 \times [0,1]\Bigr)
\end{aligned}
$$

be the mapping cylinder associated with (loops $\gamma_1, \dots, \gamma_r$ representing) $r_1, \dots, r_m$, and let $\partial Z := \coprod_m S^1 \times \{1\} \subset Z$. Moreover, let $\beta \in H_2(Z, \partial Z; \mathbb{R})$ be the relative homology class whose boundary class $\partial\beta$ is the fundamental class of $\coprod_m S^1$. Then

$$\|\beta\|_1 = 4 \cdot \mathrm{scl}_S(r_1 + \dots + r_m).$$

Proof Clearly, $\bigvee_S S^1$ is a model of $BF(S)$ and $H_2(F(S);\mathbb{R}) \cong 0$. Therefore, we can apply Corollary 6.9. $\square$

6.3 Decomposable relators

The filling view allows us to compute the l^1-semi-norm for certain classes in degree 2 associated to "decomposable relators" in terms of stable commutator length. Let us first describe these homology classes:

Setup 6.11 (Decomposable relators I) Let G_1 and G_2 be groups that satisfy $H_2(G_1;\mathbb{R}) \cong 0$ and $H_2(G_2;\mathbb{R}) \cong 0$ and let $r_1 \in G_1'$, $r_2 \in G_2'$ be elements of infinite order. We then consider the glued group

$$D(G_1, G_2, r_1, r_2) := (G_1 * G_2)/\langle r_1 \cdot r_2 \rangle^{\triangleleft} \cong G_1 *_{\mathbb{Z}} G_2,$$

where the amalgamation homomorphisms $\mathbb{Z} \to G_1$ and $\mathbb{Z} \to G_2$ are given by r_1 and r_2^{-1}, respectively.

Associated with this situation, there is a canonical homology class $\alpha \in H_2(D(G_1, G_2, r_1, r_2);\mathbb{R})$: Let X_1 and X_2 be classifying spaces for G_1 and G_2, respectively. We consider the cylinder spaces

$$
\begin{aligned}
Z_1 &:= X_1 \cup_{r_1 \text{ on } S^1 \times \{0\}} \bigl(S^1 \times [0,1]\bigr) \\
Z_2 &:= X_2 \cup_{r_2 \text{ on } S^1 \times \{0\}} \bigl(S^1 \times [0,1]\bigr)
\end{aligned}
$$

for the relators r_1 and r_2, respectively. Then

$$P := Z_1 \cup_{(z,1) \sim (\overline{z},1)} Z_2$$

is a CW-complex such that the canonical maps $Z_1 \to P$ and $Z_2 \to P$ induce an isomorphism $\pi_1(P) \cong D(G_1, G_2, r_1, r_2) =: G$.

Let $\beta_1 \in H_2(Z_1, S^1 \times \{1\}; \mathbb{R})$ and $\beta_2 \in H_2(Z_2, S^1 \times \{1\}; \mathbb{R})$ be the relative classes whose boundaries are fundamental classes of $S^1 \times \{1\}$ (corresponding to the relators r_1 and r_2, respectively). Moreover, let $\widetilde{\alpha} \in H_2(P; \mathbb{R})$ be the class obtained by glueing β_1 and β_2. Then, we define $\alpha \in H_2(G; \mathbb{R})$ as the image of $\widetilde{\alpha}$ under the classifying map $P \to BG$ (which is induced by the canonical maps $Z_1 \to P$ and $Z_2 \to P$).

Remark 6.12 (Integrality of the canonical class) In the situation of Setup 6.11, the canonical homology class α is integral: It suffices to show that $\widetilde{\alpha} \in H_2(P; \mathbb{R})$ is integral. Comparing the long exact sequences of $(Z_1, \partial Z_1)$ with $\mathbb{Z}$- and $\mathbb{R}$-coefficients shows that $\beta_1 \in H_2(Z_1, \partial Z_1; \mathbb{R})$ is an integral class. Analogously, β_2 is integral. Thus, also the glued class $\widetilde{\alpha}$ is integral.

Setup 6.13 (Decomposable relators II) Let G_1 be a group with $H_2(G_1; \mathbb{R}) \cong 0$ and let $r_1, r_2 \in G_1'$ be elements of infinite order. We then consider the group

$$T(G_1, r_1, r_2) := \bigl(G_1 * \langle t \rangle\bigr) / \langle r_1 \cdot t \cdot r_2 \cdot t^{-1} \rangle^{\triangleleft},$$

where t is a fresh generator of $\langle t \rangle \cong \mathbb{Z}$.

Also here, there is a canonical homology class $\alpha \in H_2(T(G_1, r_1, r_2); \mathbb{R})$, which is defined as follows: Let X_1 be a model of BG_1 and let

$$Z := X_1 \cup_{r_1, r_2} \bigl(S^1 \times [0, 1] \sqcup S^1 \times [0, 1]\bigr)$$

be the cylinder space associated with r_1 and r_2. Let $\beta \in H_2(Z, \partial Z; \mathbb{R})$ be the relative "fundamental" class as in Corollary 6.9. Glueing the two cylindrical ends of Z by an orientation reversing homeomorphism leads to a CW-complex P such that $\pi_1(P) \cong T(G_1, r_1, r_2) =: G$ in the obvious way (the additional generator t corresponds to the loop $\{1\} \times ([0, 1] \sqcup_{s \sim s} [0, 1])$ in the looped cylinder. Let $\widetilde{\alpha} \in H_2(P; \mathbb{R})$ be the class obtained by glueing β to itself via the cylinder. Then we define $\alpha \in H_2(G; \mathbb{R})$ as the image of $\widetilde{\alpha}$ under the classifying map $P \to BG$ (which is induced by the canonical map $X_1 \to P$ and the cylinder loop).

Theorem 6.14 (Decomposable relators) *Let G_1 be a group with $H_2(G_1; \mathbb{R}) \cong 0$ and let $r_1 \in G_1'$ be an element of infinite order.*

1. *Let G_2 also be a group with $H_2(G_2; \mathbb{R}) \cong 0$, let $r_2 \in G_2'$ be an element of infinite order, and let $\alpha \in H_2(D(G_1, G_2, r_1, r_2); \mathbb{R})$ be the canonical homology class (Setup 6.11). Then*

$$\|\alpha\|_1 = 4 \cdot (\mathrm{scl}_{G_1} r_1 + \mathrm{scl}_{G_2} r_2) = 4 \cdot \Bigl(\mathrm{scl}_{G_1 * G_2}(r_1 \cdot r_2) - \frac{1}{2}\Bigr).$$

2. *Let $r_2 \in G_1'$ be an element of infinite order, let $\alpha \in H_2(T(G_1, r_1, r_2); \mathbb{R})$ be the canonical class (Setup 6.13), and let t be the fresh letter in $T(G_1, r_1, r_2)$. Then*

$$\|\alpha\|_1 = 4 \cdot \mathrm{scl}_{G_1}(r_1 + r_2) = 4 \cdot \Big(\mathrm{scl}_{G_1 * \langle t \rangle}(r_1 \cdot t \cdot r_2 \cdot t^{-1}) - \frac{1}{2}\Big).$$

Proof In both cases, we will use that the stable commutator length and the canonical CW-complexes of decomposable relators can be expressed in terms of the stable commutator lengths and cylinder complexes of the sub-relators.

Ad 1 It is known that [18, Proposition 2.99]

$$\mathrm{scl}_{G_1 * G_2}(r_1 \cdot r_2) = \mathrm{scl}_{G_1} r_1 + \mathrm{scl}_{G_2} r_2 + \frac{1}{2}.$$

We will now show that $\|\alpha\|_1$ equals $4 \cdot (\mathrm{scl}_{G_1} r_1 + \mathrm{scl}_{G_2} r_2)$. In the following, we will use the notation from Setup 6.11. By construction, we have

$$\alpha = H_2(c; \mathbb{R})(\widetilde{\alpha}),$$

where $c\colon P \to BD(G_1, G_2, r_1, r_2)$ is the classifying map of P. The mapping theorem (Corollary 3.5) shows that $\|\alpha\|_1 = \|\widetilde{\alpha}\|_1$. Therefore, it suffices to compute $\|\widetilde{\alpha}\|_1$.

Because r_1 and r_2 have infinite order, the inclusions of $S^1 \times \{1\}$ into Z_1 and Z_2, respectively, are π_1-injective. As $\pi_1(S^1 \times \{1\}) \cong \mathbb{Z}$ is amenable, the amenable glueing theorem [14, Section 6] shows that

$$\|\widetilde{\alpha}\|_1 = \|\beta_1\|_1 + \|\beta_2\|_1;$$

the proofs of Bucher et al. carry over from the manifold case to this setting, because they established the necessary tools in bounded cohomology in this full generality. Moreover, we know that (Corollary 6.9)

$$\|\beta_1\|_1 = 4 \cdot \mathrm{scl}_{G_1} r_1 \quad \text{and} \quad \|\beta_2\|_1 = 4 \cdot \mathrm{scl}_{G_2} r_2.$$

Putting it all together, we obtain $\|\alpha\|_1 = \|\widetilde{\alpha}\|_1 = 4 \cdot (\mathrm{scl}_{G_1} r_1 + \mathrm{scl}_{G_2} r_2)$, as claimed.

Ad 2 We argue in a similar way as in the first part: In this situation, it is known that [18, Theorem 2.101]

$$\mathrm{scl}_{G_1 * \langle t \rangle}(r_1 \cdot t \cdot r_2 \cdot t^{-1}) = \mathrm{scl}_{G_1}(r_1 + r_2) + \frac{1}{2}.$$

We will now use the notation from Setup 6.13. The classifying map $c\colon P \to BT(G_1, r_1, r_2)$ maps $\widetilde{\alpha}$ to the canonical class $\alpha \in H_2(G; \mathbb{R})$ and the mapping theorem (Corollary 3.5) shows that $\|\alpha\|_1 = \|\widetilde{\alpha}\|_1$.

Because r_1 and r_2 have infinite order, we can again use the amenable glueing theorem to deduce that

$$\|\widetilde{\alpha}\|_1 = \|\beta\|_1.$$

Moreover, Corollary 6.9 shows that

$$\|\beta\|_1 = 4 \cdot \mathrm{scl}_{G_1}(r_1 + r_2).$$

Therefore, we obtain $\|\alpha\|_1 = \|\widetilde{\alpha}_1\|_1 = 4 \cdot \mathrm{scl}_{G_1}(r_1 + r_2)$. □

In the case that G_1 and G_2 are free groups, statements of this type are also contained in arguments of Calegari [16, p. 2004].

Unfortunately, there does not seem to be an easy way to remove the condition $H_2(G_1; \mathbb{R}) \cong 0$ in Theorem 6.14: Without this condition, we have too much ambiguity in the proof of Proposition 6.8 to ensure integrality and norm control simultaneously.

6.4 Simplicial volume as filling invariant

We mention that also simplicial volume of higher-dimensional manifolds admits a description as a filling invariant (this result will not be used in the rest of the paper):

Theorem 6.15 (Simplicial volume as filling invariant) *Let $d \in \mathbb{N}_{\geq 2}$, let M be an orientable closed connected d-manifold, let $\tau\colon \Delta^d \to M$ be an embedding of the standard d-simplex into M (i.e., τ is a homeomorphism onto its image), let $\Delta := \tau(\Delta^d) \subset M$, and let $\sigma := \tau|_{\partial\Delta^d}\colon \partial\Delta^d \to M$.*

1. *Then $\|M \setminus \Delta^\circ, \partial\Delta\| = \mathrm{sfill}_{M\setminus\Delta^\circ}\,\sigma$.*
2. *If $d \geq 3$, then $\|M\| = \mathrm{sfill}_{M\setminus\Delta^\circ}\,\sigma$.*
3. *If $d = 2$ and $M \not\cong S^2$, then $\|M\| = \mathrm{sfill}_{M\setminus\Delta^\circ}\,\sigma - 2$.*

Proof Ad 1 This is a special case of Proposition 6.8: We consider $Z := M \setminus \Delta^\circ$. The map $\sigma\colon \partial\Delta^d \to M \setminus \Delta^\circ = Z$ is π_1-injective (if $d \geq 3$, then $\pi_1(\partial\Delta^d)$ is trivial; if $d = 2$, this holds by the classification of surfaces and the assumption $M \not\cong S^2$). In view of Poincaré duality, the hypothesis on $H_d(Z, \partial Z; \mathbb{R})$ is satisfied. We therefore can apply Proposition 6.8 to obtain

$$\|Z, \partial Z\| = \big\|[Z, \partial Z]_{\mathbb{R}}\big\|_1 = \mathrm{sfill}_Z\,\sigma.$$

Ad 2 In view of the first part, it suffices to show that $\|M\| = \|M \setminus \Delta^\circ, \partial\Delta\|$. The amenable glueing theorem for simplicial volume [14,33,35] shows that

$$\|M\| = \|M \setminus \Delta^\circ, \partial\Delta\| + \|\Delta^d, \partial\Delta^d\|$$

(because $d \geq 3$, both inclusions $\partial\Delta \to M \setminus \Delta^\circ$ and $\partial\Delta \to \Delta$ are π_1-injective). Moreover, $\|\Delta^d, \partial\Delta^d\| = 0$ [35]. Hence, we obtain $\|M\| = \|M \setminus \Delta^\circ, \partial\Delta\|$.

Ad 3 Again, in view of the first part, it suffices to show that $\|M\| = \|M \setminus \Delta^\circ, \partial\Delta\| - 2$. In this, two-dimensional, case, this equality follows from the classification of compact surfaces and the computation of the (relative) simplicial volume of compact surfaces in terms of their genus [35,56]. □

6.5 Proof of Theorems C and D

Theorem C is a special case of Theorem 6.14 with the decomposable relators of Setup 6.11. Let G be a group that satisfies $H_2(G; \mathbb{R}) \cong 0$ and let $r \in G'$ be an element of infinite order. Then we define the *double* $D(G, r)$ of G and r by setting

$$D(G, r) := D(G, G, r, r) = (G_{\text{left}} \star G_{\text{right}})/\langle r_{\text{left}} \cdot r_{\text{right}} \rangle^{\triangleleft},$$

where G_{left} and G_{right} are isomorphic copies of G and $r_{\text{right}} \in G_{\text{right}}$, $r_{\text{left}} \in G_{\text{left}}$ are the elements corresponding to $r \in G$. Observe that if G is finitely presented, then so is $D(G, r)$. As in Setup 6.11 there is a canonical integral class $\alpha \in H_2(D(G, r); \mathbb{R})$.

Corollary 6.16 (Theorem C) *Let G be a group with $H_2(G; \mathbb{R}) \cong 0$ and let $r \in G'$ be of infinite order. Then the canonical integral class $\alpha \in H_2(D(G, r); \mathbb{R})$ satisfies*

$$\|\alpha\|_1 = 8 \cdot \operatorname{scl}_G r.$$

Proof This is an immediate corollary of Theorem 6.14 (1). □

Applying Theorem C to the universal central extension E of Thompson's group T, we deduce Theorem D:

Corollary 6.17 (Theorem D) *For every $q \in \mathbb{Q}_{\geq 0}$, there is a finitely presented group G_q and an integral class $\alpha_q \in H_2(G_q; \mathbb{R})$ such that $\|\alpha_q\|_1 = q$. In particular, for every $\epsilon > 0$ there is a finitely presented group G_ϵ and an integral class $\alpha_\epsilon \in H_2(G_\epsilon; \mathbb{R})$ such that $0 < \|\alpha_\epsilon\|_1 \leq \epsilon$.*

Proof Let $q \in \mathbb{Q}_{\geq 0}$. For $q = 0$ we can take the zero class of the trivial group (or any integral 2-class in any finitely presented amenable group).

For $q > 0$, let $r_q \in E$ be an element in the universal central extension E of Thompson's group T with $\mathrm{scl}_E\, r_q = q/8$. Proposition 5.1 asserts that such an element exists, that E is finitely presented and that $H_2(E;\mathbb{Z}) \cong 0$ and hence $H_2(E;\mathbb{R}) \cong 0$. As $\mathrm{scl}_E\, r_q > 0$ the element $r_q \in E$ has infinite order.

Let $G_q := D(E, r_q)$ be the double and let $\alpha_q \in H_2(G_q;\mathbb{R})$ be the associated integral 2-class. Theorem C shows that

$$\|\alpha_q\|_1 = 8 \cdot \mathrm{scl}_E\, r_q = q. \qquad \square$$

We note that one can prove the second part of Theorem D also via previously known examples of stable commutator length (Example 4.3.4).

7 The l^1-semi-norm of products with surfaces: Proof of Theorem E

Bucher [4] computed the simplicial volume of the product of two surfaces (see Theorem 2.10). We will use her techniques to generalise this statement to the product of more general 2-classes. This will allow us to construct integral 4-classes whose l^1-semi-norm can be expressed in terms of the l^1-semi-norm of 2-classes. Theorem E will be a corollary (Corollary 7.2) of these constructions.

Theorem 7.1 *Let G and Γ be groups, let $\alpha \in H_2(G;\mathbb{R})$, and $\beta \in H_2(\Gamma;\mathbb{R})$. Furthermore, let $\rho\colon \Gamma \to \mathrm{Homeo}^+(S^1)$ be a circle action of Γ. Assume furthermore that $\rho^*\mathrm{Or} \in C_b^2(\Gamma;\mathbb{R})$ is an extremal cocycle for β; see Sect. 3.6. Then the class $\alpha \times \beta \in H_4(G \times \Gamma;\mathbb{R})$ satisfies*

$$\|\alpha \times \beta\|_1 = \frac{3}{2} \cdot \|\alpha\|_1 \cdot \|\beta\|_1.$$

Theorem 7.1 is a strict generalisation of Bucher's result [4] and our proof follows the outline of Bucher's work. Recall that for $g \geq 2$ we denote the oriented closed connected surface of genus g by Σ_g, its fundamental group by Γ_g, and its fundamental class by $[\Sigma_g]_\mathbb{R} \in H_2(\Sigma_g;\mathbb{R}) \cong H_2(\Gamma_g;\mathbb{R})$. Fix a hyperbolic structure on Σ_g and let $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(S^1)$ be the corresponding action on the boundary $\partial\Gamma_g \cong S^1$. The cocycle $\rho^*\mathrm{Or} \in C_b^2(\Gamma_g;\mathbb{R})$ is extremal to $[\Sigma_g]_\mathbb{R}$ (see Sect. 3.6) and satisfies

$$\big\langle [\rho^*\mathrm{Or}], [\Sigma_g]_\mathbb{R} \big\rangle = \|\Sigma_g\| = 4 \cdot g - 4.$$

Therefore, we obtain the following immediate corollary to Theorem 7.1:

Corollary 7.2 (Theorem E) *Let $g \geq 2$, let G be a group, let $\alpha \in H_2(G;\mathbb{R})$ and let Γ_g and $[\Sigma_g]_\mathbb{R} \in H_2(\Gamma_g;\mathbb{R})$ be as above. Then the class $\alpha \times [\Sigma_g]_\mathbb{R} \in$*

$H_4(G \times \Gamma_g; \mathbb{R})$ *satisfies*

$$\left\|\alpha \times [\Sigma_g]_{\mathbb{R}}\right\|_1 = \frac{3}{2} \cdot \|\alpha\|_1 \cdot \|\Sigma_g\| = 6 \cdot (g-1) \cdot \|\alpha\|_1.$$

Proof of Theorem 7.1 In this proof, all cocycles will be given in the homogeneous resolution. The upper bound holds for all classes in degree 2 (Corollary 2.11). For the lower bound we will use duality (Proposition 3.4):

Let ${}^{\Gamma}\mathrm{or}_b^{\mathbb{R}} := [\rho^*\mathrm{Or}]^{\mathbb{R}} = -2 \cdot {}^{\Gamma}\mathrm{eu}_b^{\mathbb{R}} \in H_b^2(\Gamma; \mathbb{R})$ be the orientation class for the given action $\rho \colon \Gamma \to \mathrm{Homeo}^+(S^1)$. Moreover, let $\omega \in C_b^2(G; \mathbb{R})$ be an extremal cocycle for $\alpha \in H_2(G; \mathbb{R})$ in the homogeneous resolution; see Proposition 3.4. By possibly replacing ω by $\mathrm{alt}_b^2(\omega)$, we may assume that ω is alternating; see Sect. 3.5. By assumption, $\|\omega\|_\infty \le 1$ and

$$\big\langle[\omega], \alpha\big\rangle = \|\alpha\|_1 \quad \text{and} \quad \langle{}^{\Gamma}\mathrm{or}_b^{\mathbb{R}}, \beta\rangle = \|\beta\|_1.$$

The cross-product $\omega \times \rho^*\mathrm{Or} \in C_b^4(G \times \Gamma; \mathbb{R})$ of ω and $\rho^*\mathrm{Or}$ is defined via

$$\omega \times \rho^*\mathrm{Or} \colon \big((g_1, \gamma_1), \dots, (g_5, \gamma_5)\big) \mapsto \omega(g_1, g_2, g_3) \cdot \rho^*\mathrm{Or}(\gamma_3, \gamma_4, \gamma_5)$$

and satisfies

$$\big\langle[\omega \times \rho^*\mathrm{Or}], \alpha \times \beta\big\rangle = \|\alpha\|_1 \cdot \|\beta\|_1.$$

This recovers the estimate $\|\alpha\|_1 \cdot \|\beta\|_1 \le \|\alpha \times \beta\|_1$ as seen in Proposition 2.9.

Claim 7.3 *Let* $\Theta := \mathrm{alt}_b^4(\omega \times \rho^*\mathrm{Or})$ *be the associated alternating cocycle of* $\omega \times \rho^*\mathrm{Or}$*; see Sect. 3.5. Then* $\|\Theta\|_\infty \le 2/3$.

Once Claim 7.3 is established, we can argue as follows: Recall that Θ and $\omega \times \rho^*\mathrm{Or}$ represent the same class in $H_b^4(G \times \Gamma; \mathbb{R})$ by Proposition 3.6. Hence, $\langle\Theta, \alpha \times \beta\rangle = \|\alpha\|_1 \cdot \|\beta\|_1$. Moreover by the claim we have that $\|\frac{3}{2} \cdot \Theta\|_\infty \le 1$ and by duality we conclude that $\frac{3}{2} \cdot \|\alpha\|_1 \cdot \|\beta\|_1 \le \|\alpha \times \beta\|_1$. Putting both estimates together, we will obtain

$$\frac{3}{2} \cdot \|\alpha\|_1 \cdot \|\beta\|_1 = \|\alpha \times \beta\|_1,$$

as claimed in Theorem 7.1. To complete the proof, it thus only remains to show Claim 7.3. □

Proof of Claim 7.3 We will follow the outline of Bucher's proof [4, Proposition 7], quoting parts of the proof verbatim. Let $g_0, \dots, g_4 \in G$ and $\gamma_0, \dots, \gamma_4 \in \Gamma$. Moreover, let $\xi \in S^1$ be a point to define Or and set

$x_i := \rho(\gamma_i).\xi$ for all $i \in \{0,\dots,4\}$. We will give upper bounds to $|\Theta((g_0,\gamma_0),\dots,(g_4,\gamma_4))|$ in different cases. By construction,

$$\Theta\big((g_0,\gamma_0),\dots,(g_4,\gamma_4)\big)$$

may be written as

$$\frac{1}{|S_5|}\cdot\sum_{\sigma\in S_5}\operatorname{sign}(\sigma)\cdot\omega(g_{\sigma(0)},g_{\sigma(1)},g_{\sigma(2)})\cdot\operatorname{Or}(x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)}),\qquad(2)$$

where Or is the orientation; see Sect. 3.6. Every permutation $\sigma\in S_5$ may be written uniquely as $\sigma=(0\;1\;2\;3\;4)^k\circ\upsilon$, where $k\in\{0,\dots,4\}$ and $\upsilon\in S_5$ is a permutation with $\upsilon(2)=0$. Using that both Or and ω are alternating we obtain that

$$\Theta\big((g_0,\gamma_0),\dots,(g_4,\gamma_4)\big)=\frac{1}{30}\cdot\sum_{k=0}^{4}A(k),\qquad(3)$$

where

$$\begin{aligned}A(k)=\;&\omega(g_{\tau(0)},g_{\tau(1)},g_{\tau(2)})\cdot\operatorname{Or}(x_{\tau(0)},x_{\tau(3)},x_{\tau(4)})\\&+\omega(g_{\tau(0)},g_{\tau(3)},g_{\tau(4)})\cdot\operatorname{Or}(x_{\tau(0)},x_{\tau(1)},x_{\tau(2)})\\&-\omega(g_{\tau(0)},g_{\tau(1)},g_{\tau(3)})\cdot\operatorname{Or}(x_{\tau(0)},x_{\tau(2)},x_{\tau(4)})\\&-\omega(g_{\tau(0)},g_{\tau(2)},g_{\tau(4)})\cdot\operatorname{Or}(x_{\tau(0)},x_{\tau(1)},x_{\tau(3)})\\&+\omega(g_{\tau(0)},g_{\tau(1)},g_{\tau(4)})\cdot\operatorname{Or}(x_{\tau(0)},x_{\tau(2)},x_{\tau(3)})\\&+\omega(g_{\tau(0)},g_{\tau(2)},g_{\tau(3)})\cdot\operatorname{Or}(x_{\tau(0)},x_{\tau(1)},x_{\tau(4)})\end{aligned}$$

for $\tau=(0\;1\;2\;3\;4)^k$. Observe also that we may assume that the x_i are cyclically ordered, as Θ is alternating.

In what follows we will estimate the terms $A(k)$, depending on the relative position of the x_i:

- All $x_0,\dots,x_4$ are distinct. As all Or-terms in $A(k)$ equal 1 we have

$$\begin{aligned}A(k)&=\omega(g_{\tau(0)},g_{\tau(1)},g_{\tau(2)})+\omega(g_{\tau(0)},g_{\tau(3)},g_{\tau(4)})\\&\quad-\omega(g_{\tau(0)},g_{\tau(1)},g_{\tau(3)})-\omega(g_{\tau(0)},g_{\tau(2)},g_{\tau(4)})\\&\quad+\omega(g_{\tau(0)},g_{\tau(1)},g_{\tau(4)})+\omega(g_{\tau(0)},g_{\tau(2)},g_{\tau(3)})\\&=\omega(g_{\tau(2)},g_{\tau(3)},g_{\tau(4)})+\omega(g_{\tau(0)},g_{\tau(1)},g_{\tau(2)})\\&\quad-\omega(g_{\tau(0)},g_{\tau(1)},g_{\tau(3)})+\omega(g_{\tau(0)},g_{\tau(1)},g_{\tau(4)})\end{aligned}$$

for $\tau = (0\ 1\ 2\ 3\ 4)^k$, where in the last equation we used that

$$\begin{aligned}
0 &= \delta^2\omega(g_{\tau(0)}, g_{\tau(2)}, g_{\tau(3)}, g_{\tau(4)}) \\
&= \omega(g_{\tau(2)}, g_{\tau(3)}, g_{\tau(4)}) - \omega(g_{\tau(0)}, g_{\tau(3)}, g_{\tau(4)}) \\
&\quad + \omega(g_{\tau(0)}, g_{\tau(2)}, g_{\tau(4)}) - \omega(g_{\tau(0)}, g_{\tau(2)}, g_{\tau(3)})
\end{aligned}$$

by the cocycle condition. In particular, we see that $|A(k)| \le 4$ as $\|\omega\|_\infty \le 1$. Hence,

$$\left|\Theta((g_1, \gamma_1), \dots, (g_5, \gamma_5))\right| \le \frac{1}{30} \cdot \sum_{k=0}^{4} |A(k)| \le \frac{20}{30} = \frac{2}{3}.$$

- Two of the x_i are identical and the others are distinct. Without loss of generality assume that $x_0 = x_1$. Observe that in this case $\mathrm{Or}(x_i, x_j, x_k) = 0$ whenever two of the x_i, x_j, x_k are equal to x_0 or x_1. We will estimate $|A(k)|$ in different cases:
  - $k = 0$: Then $A(0) = \omega(g_0, g_1, g_2) - \omega(g_0, g_1, g_3) + \omega(g_0, g_1, g_4)$ and hence $|A(0)| \le 3$.
  - $k = 1$: Then $A(1) = \omega(g_1, g_4, g_0) - \omega(g_1, g_3, g_0) + \omega(g_1, g_2, g_0)$ and hence $|A(1)| \le 3$.
  - $k = 2$: Then

    $$\begin{aligned}
    A(2) = \omega(g_2, g_0, g_1) &- \omega(g_2, g_3, g_0) - \omega(g_2, g_4, g_1) \\
    &+ \omega(g_2, g_3, g_1) + \omega(g_2, g_4, g_0).
    \end{aligned}$$

    By the cocycle condition, it follows that

    $$\begin{aligned}
    0 &= \delta^2\omega(g_2, g_4, g_0, g_1) \\
    &= \omega(g_4, g_0, g_1) - \omega(g_2, g_0, g_1) + \omega(g_2, g_4, g_1) - \omega(g_2, g_4, g_0).
    \end{aligned}$$

    Therefore, $A(2) = \omega(g_4, g_0, g_1) - \omega(g_2, g_3, g_0) + \omega(g_2, g_3, g_1)$ and so $|A(2)| \le 3$.
  - $k = 3$: Then

    $$\begin{aligned}
    A(3) = \omega(g_3, g_4, g_0) &+ \omega(g_3, g_1, g_2) - \omega(g_3, g_4, g_1) \\
    &- \omega(g_3, g_0, g_2) + \omega(g_3, g_0, g_1)
    \end{aligned}$$

    and hence $|A(3)| \le 5$.
  - $k = 4$: Then

    $$A(4) = \omega(g_4, g_0, g_1) - \omega(g_4, g_0, g_2) - \omega(g_4, g_1, g_3)$$

$$+ \omega(g_4, g_0, g_3) + \omega(g_4, g_1, g_2)$$

and hence $|A(4)| \leq 5$.

Putting things together we see that

$$\left|\Theta((g_1, \gamma_1), \ldots, (g_5, \gamma_5))\right| \leq \frac{1}{30} \sum_{k=0}^{4} |A(k)| \leq \frac{19}{30} < \frac{2}{3}.$$

- Three of the x_i are identical, the other ones are different. As Θ is alternating we may assume that $x_0 = x_1 = x_2$. A permutation $\sigma \in S_5$ for which the $\mathrm{Or}(x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$ term in Eq. (2) is non-trivial has to map exactly one of the elements in $\{2, 3, 4\}$ to one of the elements $\{0, 1, 2\}$, and has to map the remaining two elements of $\{2, 3, 4\}$ to $\{3, 4\}$. We then have two more choices for $\sigma(0)$ and $\sigma(1)$. We compute that the total number of such permutations is 36. For all other permutations, the Or-term in Eq. (2) will vanish. We may then estimate

$$\left|\Theta((g_0, \gamma_0), \ldots, (g_4, \gamma_4))\right| \leq \frac{36}{5!} = \frac{36}{120} < \frac{2}{3}.$$

- Two pairs are identical and one element is different from these pairs. Assume without loss of generality that $x_0 = x_1$, $x_2 = x_3$. A permutation $\sigma \in S_5$ for which the $\mathrm{Or}(x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$-term in Equation (2) is non-trivial has to map each of $\{2, 3, 4\}$ to different sets $\{0, 1\}$, $\{2, 3\}$, and $\{5\}$. Moreover, there are two choices for the two elements that get mapped to the sets with two elements. Again, there are two more choices for $\sigma(0)$ and $\sigma(1)$. We compute that there are a total of 48 such permutations. For all other permutations the Or-term in Eq. (2) will vanish. We may then estimate

$$\left|\Theta((g_0, \gamma_0), \ldots, (g_4, \gamma_4))\right| \leq \frac{48}{5!} = \frac{48}{120} < \frac{2}{3}.$$

- If more than three of the x_i are identical or if exactly three of the x_i are identical and the two remaining x_i are identical, then the Or-term in Eq. (2) always vanishes and we get that

$$\left|\Theta((g_0, \gamma_0), \ldots, (g_4, \gamma_4))\right| = 0 < \frac{2}{3}.$$

In summary, in each case we have seen that

$$\left|\Theta((g_0, \gamma_0), \ldots, (g_4, \gamma_4))\right| \leq \frac{2}{3}$$

and hence $\|\Theta\|_\infty \leq \frac{2}{3}$. This finishes the proof of Claim 7.3 (and also the proof of Theorem 7.1). $\square$

8 Manufacturing manifolds with controlled simplicial volumes

The computation of ℓ^1-semi-norms of 2-classes in group homology allows us to construct manifolds with controlled simplicial volume.

This construction will involve a normed version of Thom's realisation theorem, which we recall in Sect. 8.1. Theorem A is proven in Sect. 8.2 and the theorems for dimension 4 (Theorems B and F) are proven in Sect. 8.3. Finally, in Sect. 8.4 we discuss related problems and further research topics.

8.1 Thom's realisation theorem

In order to turn classes in group homology into manifolds with controlled simplicial volume, we will use the following *normed* version of Thom's realisation theorem:

Theorem 8.1 (Normed Thom realisation) *For each $d \in \mathbb{N}_{\geq 4}$, there exists a constant $K_d \in \mathbb{N}_{>0}$ with the following property: If G is a finitely presented group (with model X of BG) and $\alpha \in H_d(X;\mathbb{R})$ is an integral homology class, then there is an oriented closed connected (smooth) d-manifold M, a continuous map $f\colon M \to X$ and a number $m \in \{1,\dots,K_d\}$ with*

$$H_d(f;\mathbb{R})[M]_\mathbb{R} = m \cdot \alpha \quad \textit{and} \quad \|M\| = m \cdot \|\alpha\|_1.$$

Moreover, one can choose $K_4 = 1$ and $K_5 = 1$.

Proof Everything except for the condition on the simplicial volume is contained in Thom's classical realisation theorems [52, Theorems III.3, III.4]. (Thom's original theorems apply to X because every singular homology class of X is supported on a finite subcomplex; as G is finitely presented, we can choose the subcomplex in such a way that the inclusion into X induces a π_1-isomorphism.) One can then apply surgery to obtain a manifold representation of α, where $f\colon M \to X$ in addition is a π_1-isomorphism [25, (proof of) Theorem 3.1] (this will not touch the multiplier m). Therefore, the mapping theorem for the l^1-semi-norm (Corollary 3.5) shows that

$$\|M\| = \left\|H_d(f;\mathbb{R})[M]_\mathbb{R}\right\|_1 = \|m\cdot\alpha\|_1 = m\cdot\|\alpha\|_1. \qquad \square$$

8.2 No gaps in higher dimensions: Proof of Theorem A

We promote the computations of l^1-semi-norms in degree 2 to higher dimensions using cross-products. The manifolds will then be provided by the normed Thom realisation (Theorem 8.1).

Proof of Theorem A Let $d \in \mathbb{N}_{\geq 4}$. We fix an oriented closed connected hyperbolic $(d-2)$-manifold N_d; in particular, $\|N_d\| > 0$. Moreover, let $K_d \in \mathbb{N}$ be the constant provided by Thom's realisation theorem (Theorem 8.1).

Let $\epsilon \in \mathbb{R}_{>0}$. By Theorem D, there exists a finitely presented group G and an integral class $\alpha \in H_2(G;\mathbb{R})$ with $0 < \|\alpha\|_1 \leq \epsilon$. Let X be a model of BG. Then the product class

$$\alpha' := \alpha \times [N_d]_{\mathbb{R}} \in H_d(X \times N_d;\mathbb{R})$$

is integral and satisfies (by Proposition 2.9)

$$0 < \|\alpha\|_1 \cdot \|N_d\| \leq \|\alpha'\|_1 \leq \binom{d}{2} \cdot \|N_d\| \cdot \epsilon.$$

The normed version of Thom's realisation theorem (Theorem 8.1) provides an orientable closed connected d-manifold M and a number $m \in \{1,\ldots,K_d\}$ with

$$\|M\| = m \cdot \|\alpha'\|_1.$$

We conclude that

$$0 < \|M\| \leq \binom{d}{2} \cdot \|N_d\| \cdot K_d \cdot \epsilon.$$

As the constants on the right hand side just depend on d, this shows that there is no gap at 0 in $\mathrm{SV}(d)$, the set of simplicial volumes of orientable closed connected d-manifolds. By additivity (Remark 2.3), the set $\mathrm{SV}(d)$ is also dense in $\mathbb{R}_{\geq 0}$. □

8.3 Dimension 4: Proofs of Theorems B and F

In dimension 4, we have more control on the l^1-norm of integral 4-classes in group homology (Theorem E). This allows us to prove Theorems B and F.

Proposition 8.2 *Let G be a finitely presented group and let $\alpha \in H_2(G;\mathbb{R})$ be an integral class. Then there exists an orientable closed connected 4-manifold M_α with*

$$\|M_\alpha\| = 6 \cdot \|\alpha\|_1.$$

Proof We proceed as in the proof of Theorem A and consider the product class

$$\alpha' := \alpha \times [\Sigma_2]_{\mathbb{R}} \in H_4(G \times \Gamma_2; \mathbb{R})$$

of α with the fundamental class $[\Sigma_2]_{\mathbb{R}}$ of a surface of genus 2. Observe that α' is also integral. Then the normed Thom realisation (Theorem 8.1) shows that there exists an orientable closed connected 4-manifold M_α with $\|M_\alpha\| = \|\alpha'\|_1$. We now apply the norm computation from Corollary 7.2 and obtain

$$\|M_\alpha\| = \|\alpha'\|_1 = 6 \cdot \|\alpha\|_1. \qquad \square$$

Proof of Theorem B We only need to combine Theorem D (which allows to realise any non-negative rational number as l^1-semi-norm of an integral 2-class of a finitely presented group) with Proposition 8.2. □

Moreover, we can summarise the relation between stable commutator length and simplicial volumes in dimension 4 as follows:

Corollary 8.3 (Theorem F, dimension 4, exact values via scl) *Let G be a finitely presented group with $H_2(G;\mathbb{R}) \cong 0$ and let $g \in G'$ be an element in the commutator subgroup. Then there exists an orientable closed connected 4-manifold M_g with*

$$\|M_g\| = 48 \cdot \operatorname{scl}_G g.$$

Proof We may assume without loss of generality that $r = g$ has infinite order (otherwise we can just take $M = S^4$). We again consider the doubled group $D(G, r)$ (as in Corollary 6.16) and the canonical homology class $\alpha \in H_2(D(G, r);\mathbb{R})$, which is integral (Remark 6.12); as G is finitely presented, also $D(G, r)$ is finitely presented. Applying Corollary 6.16 shows that

$$\|\alpha\|_1 = 8 \cdot \operatorname{scl}_G r.$$

In combination with Proposition 8.2, we therefore obtain an orientable closed connected 4-manifold M with

$$\|M\| = 6 \cdot \|\alpha\|_1 = 48 \cdot \operatorname{scl}_G r. \qquad \square$$

Remark 8.4 The concrete example manifolds in the proof of Theorem B, in general, might have different fundamental group; however, by construction, their first Betti numbers are uniformly bounded: For each $q \in \mathbb{Q}$, we have

$$b_1(M_q; \mathbb{Q}) \leq \operatorname{rk} \pi_1(M_q) = \operatorname{rk} \pi_1\big(D(E, e_q) \times \Gamma_2\big) \leq 2 \cdot \operatorname{rk} E + 4.$$

8.4 Related problems

The techniques of this paper may be adopted to construct 4-manifolds with transcendental simplicial volume. Theorem F reduces this problem to finding an appropriate finitely presented group with transcendental stable commutator length. By constructing such groups explicitly, we could show that there are 4-manifolds with arbitrarily small transcendental simplicial volume [42, Theorem A]. Moreover, we could also show that the set of simplicial volumes is contained in the (countable) set $\mathrm{RC}^{\geq 0}$ of non-negative right-computable numbers [42, Theorem B]. It is unkown which real numbers arise as the stable commutator length of elements in the class of finitely presented groups. However, it is known [39] that for the class of recursively presented groups this set is exactly $\mathrm{RC}^{\geq 0}$.

Our techniques for manufacturing manifolds with controlled simplicial volumes are based on group-theoretic methods and not on genuine manifold-geometric constructions. One might wonder whether Theorem A also holds under additional topological or geometric conditions such as asphericity or curvature conditions.

Originally, we set out to study simplicial volume of one-relator groups and its relation with stable commutator length. However, we then realised that some of the techniques applied in a much broader context (with a weak homological condition). We discuss a connection between the l^1-semi-norm of the relator-class in one-relator groups with the stable commutator length of the relator in the free group in a separate article [41].

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