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Erratum: The boostless bootstrap: amplitudes without Lorentz boosts

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We provide a correction to the four-particle test in the presence of broken Lorentz boosts, presented in [1]. Our arguments require an extra assumption about the form of the four-particle amplitudes, which must not depend on the mixed off-diagonal brackets (ij) (see (5)). This condition is equivalent to restricting the corresponding Lagrangians to those that are functions of Lorentz covariant fields (6), with the breaking of boosts implemented through time derivatives.

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The factorization theorem. Let us begin by correcting the statement of Theorem 2.1 from [1].

Theorem 1 *(Factorization Theorem) Singularities of codimension* 1 *in* 4*-particle amplitudes may appear at vanishing energies* ($E_i = 0$) *or else are at most simple poles in the momenta. Each singularity of the latter type is in one-to-one correspondence with an exchange diagram, in the limit when the exchanged particle I goes* ***on-shell****. The residue of each pole factorises into a product of three-particle amplitudes:*

$$\lim_{s=0} (s\mathcal{A}_4) = \mathcal{A}_3(1,2,-I) \times \mathcal{A}_3(3,4,I) \tag{1}$$

where s *is the propagator of the intermediate particle, and* $s \to 0$ *corresponds to the intermediate particle going on-shell.*

S-matrix singularities at $E_i = 0$ do not appear in Lorentz invariant theories, as they would clearly violate Lorentz invariance. More generally, such singularities *cannot* appear if the Lagrangian is local and can be written solely in terms of $X_{\mu_1\mu_2\ldots}, \eta_{\mu\nu}, \epsilon_{\mu\nu\sigma\rho}, \partial_\mu$ and ∂_t (where $X_{\mu_1\mu_2\ldots}$ collectively denotes Lorentz covariant fields). This is because the factor $1/E_i$ is generated only when some of the tensor field indices are spatial indices. In that case the associated polarization tensor $e^{\pm S}$ has a vanishing temporal component and must take the form

$$e^{+S}_{\alpha_i\dot{\alpha}_i}(\mathbf{k}) = \prod_{i=1}^{S} \frac{(\epsilon.\tilde{\lambda})_{\alpha_i}\tilde{\lambda}_{\dot{\alpha}_i}}{2k}, \quad e^{-S}_{\alpha_i\dot{\alpha}_i}(\mathbf{k}) = \prod_{i=1}^{S} \frac{\tilde{\lambda}_{\alpha_i}(\epsilon.\lambda)_{\dot{\alpha}_i}}{2k}. \tag{2}$$

We see that $e^{\pm s}(\mathbf{k})$ has a singularity at $E_k \equiv k = 0$, which might therefore appear also in the helicity amplitude by virtue of the relation

$$\mathcal{A}_4 = e^{h_1,\mu_1} e^{h_2,\mu_2} e^{h_3,\mu_3} e^{h_4,\mu_4} A_{4,\mu_1\mu_2\mu_3\mu_4}, \tag{3}$$

where $A_{4,\mu_1\mu_2\mu_3\mu_4}$ is the *covariant amplitude*, which only has singularities when an exchanged particle goes on-shell.

The four-particle amplitude ansatz. In the absence of Lorentz boost symmetry, we must identify a set of SO(3)-invariant variables that are sufficient to fully determine the on-shell data for the scattering of four particles. In addition to the four external helicities, we must use some of the brackets $\langle ij\rangle$, $[ij]$ and (ij), which constitute a complete list of invariants of mass dimension 1. However, not all of these are independent: all but one of the off-diagonal (ij) brackets can be determined in terms of the other brackets and the energies by using momentum conservation.[1] Therefore, any SO(3) invariant can be written in terms of $\langle ij\rangle$, $[ij]$, E_i and just one of the off-diagonal (ij). It must be emphasized that without at least one (ij) bracket we would be unable to fully determine the kinematic data in the general case. This means that in boost-breaking theories, four-particle amplitudes could depend on one of the (ij)'s and this dependence cannot be eliminated by application of bracket identities.

[1] We verified this via algebraic manipulation in Mathematica. The code is available upon request.

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A general form of a four-particle amplitude consistent with the factorisation theorem is therefore

$$\mathcal{A}_4(1^{h_1}2^{h_2}3^{h_3}4^{h_4}) = (E_1E_2E_3E_4)^{-S}\,\frac{g\left(\{\langle ij\rangle\},\{[ij]\},s,t,u,\{E_i\},(ab)\right)}{stu}, \tag{4}$$

where g is a polynomial and S is the maximum spin of particles in the theory.

However, if we did allow for a generic, explicit dependence of the amplitude on the off-diagonal (ij) brackets, such an ansatz would be too general to be amenable to constraints from the four-particle factorisation test. Instead, we restrict our analysis to the case where $\mathcal{A}_4$ does not depend on the off-diagonal brackets. Our ansatz is therefore

$$\mathcal{A}_4(1^{h_1}2^{h_2}3^{h_3}4^{h_4})_{\text{restricted}} = (E_1E_2E_3E_4)^{-S}\,\frac{g\left(\{\langle ij\rangle\},\{[ij]\},s,t,u,\{E_i\}\right)}{stu}. \tag{5}$$

In this case all the results of [1] follow.

Lagrangian interpretation of the ansatz. There is a special class of Lagrangians for which four-particle amplitudes are functions of $\langle ij\rangle$, $[ij]$ and E_i only. These Lagrangians take the form

$$\mathcal{L} = \mathcal{L}\left[X_{\mu_1\mu_2\ldots},\eta_{\mu\nu},\epsilon_{\mu\nu\sigma\rho},\partial_\mu,\partial_t\right], \tag{6}$$

where $X_{\mu_1\mu_2\ldots}$ collectively denotes Lorentz covariant fields. If a physical four-particle amplitude can be written solely in terms of $\langle ij\rangle$, $[ij]$ and E_i, then there exists a Lagrangian of the form (6) which generates this amplitude. Such a Lagrangian can be constructed as follows: first, write down a Lorentz-invariant Lagrangian that generates the four-particle amplitude with the energy dependence stripped off, and then insert time derivatives acting on appropriate fields to reinstate the desired energy dependence of the amplitude. Suppose, on the other hand, that a four-particle amplitude in some theory cannot be written without at least one round bracket (ij) (which, as we remarked, cannot be determined solely in terms of the $\langle ij\rangle$, $[ij]$ and E_i). Then the corresponding Lagrangian must depend on some objects other than the ones listed in (6). For example, the Larangian could be constructed out of SO(3) covariant fields rather than Lorentz covariant ones.

Framid EFT. As an example of a four-particle amplitude that cannot be written in the form (5), but instead has an explicit dependence on off-diagonal (ij) and therefore is not captured by the analysis in [1], let us consider the Framid EFT [2] which arises from the spontaneous breaking of Poincaré symmetry to an unbroken subgroup of translations and rotations. Indeed, the Framid degrees of freedom are the Goldstone modes of broken Lorentz boosts. With respect to the unbroken SO(3) symmetry, the Framid consistents of three degrees of freedom: a massless tranverse vector and a massless longitudinal scalar with speeds c_T and c_L. Taking $c_L = c_T$, in which the scalar and vector modes have identical propagation speeds as we have been assuming in this work, the Framid Lagrangian up to cubic order in fields takes the form [2]

$$\mathcal{L} = \frac{M_1^2}{2}\left(\dot{\eta}_i^2 - c_L^2\partial_i\eta_j\partial_i\eta_j\right) + M_1^2\left(c_L^2-1\right)\eta_i\partial_i\eta_j\dot{\eta}_j + \mathcal{O}\left(\eta^4\right). \tag{7}$$

After defining rescaled fields $\chi_i = c_L M_1 \eta_i$ and replacing t with the rescaled time coordinate $t' = t/c_L$, we obtain

$$\mathcal{L} = \frac{1}{2}\left(\dot{\vec{\chi}}^2 - \partial_i\chi_j\partial_i\chi_j\right) + \left(\frac{c_L^2-1}{c_L^2 M_1}\right)\chi_i\partial_i\chi_j\dot{\chi}_j + \mathcal{O}\left(\chi^4\right) . \tag{8}$$

Using the above Lagrangian (and rescaled coordinates), we computed the four-particle amplitude $\mathcal{A}_4(1^0 2^+ 3^0 4^-)$ from tree-level exchange to verify and illustrate that it has an explicit dependence on one of the off-diagonal (ij), which cannot be eliminated. The result is the simplest, albeit still quite lengthy, if we allow for the dependence on (42), in which case the amplitude reads as follows:

$$\begin{aligned}\mathcal{A}_4(1^0 2^+ 3^0 4^-) = {}& \frac{1}{4e_4}\left(\frac{c_L^2-1}{c_L^2 M_1}\right)^2 \times \\ & \times\Bigg\{\frac{1}{s}\Big[F_{(1,a)}(E_1,E_2,E_3,E_4;s,t)(42)^2 \\ &\qquad +F_{(1,b)}(E_1,E_2,E_3,E_4;s,t)[23]\langle 34\rangle(42) \\ &\qquad +F_{(1,c)}(E_1,E_2,E_3,E_4;s,t)[23]^2\langle 34\rangle^2\Big] \\ & +\frac{1}{t}\Big[F_{(2,a)}(E_1,E_2,E_3,E_4;s,t)(42)^2 \\ &\qquad +F_{(2,b)}(E_1,E_2,E_3,E_4;s,t)[23]\langle 34\rangle(42)\Big] \\ & +\frac{1}{u}\Big[F_{(1,a)}(E_3,E_2,E_1,E_4;u,t)(42)^2 \\ &\qquad -F_{(1,b)}(E_3,E_2,E_1,E_4;u,t)[23]\langle 34\rangle(42) \\ &\qquad +F_{(1,c)}(E_3,E_2,E_1,E_4;u,t)[23]^2\langle 34\rangle^2\Big]\Bigg\}. \end{aligned} \tag{9}$$

The Framid of [2] is therefore one of the instances where the constraints of [1] on three-particle boost-violating amplitudes do not apply. In the above, we used $e_4 \equiv E_1E_2E_3E_4$ and the functions $F_{(i,x)}$ are:

$$F_{(1,a)}(E_1,E_2,E_3,E_4;s,t) = -4e_4E_{12}^2 - 2sE_{12}E_{23}f + s^2(E_1-E_2)(E_3-E_4) - \frac{s^2}{t}g, \tag{10}$$

$$F_{(1,b)}(E_1,E_2,E_3,E_4;s,t) = 12e_4E_{12} + \frac{2s}{t}(E_2-E_4)g + 3sE_{24}f, \tag{11}$$

$$F_{(1,c)}(E_1,E_2,E_3,E_4;s,t) = -9e_4 + \frac{4E_2E_4}{t}g, \tag{12}$$

$$\begin{aligned}F_{(2,a)}(E_1,E_2,E_3,E_4;s,t) = {}& 4e_4(E_1^2+E_1E_3+E_3^2) + t^2f - st(E_1-E_3)(E_2-E_4) \\ & +sE_1E_3(E_1-E_3)(E_2-E_4) \\ & -tE_1E_3\left(-E_{13}^2 + E_2E_3\left(1+\frac{2E_4}{E_1}\right) + E_1E_4\left(1+\frac{2E_2}{E_3}\right)\right),\end{aligned} \tag{13}$$

$$F_{(2,b)}(E_1,E_2,E_3,E_4;s,t) = 2(E_1-E_3)(E_2^2+E_2E_4+E_4^2)(t-E_1E_3), \tag{14}$$

where we used

$$f = E_1E_4 + E_2E_3, \tag{15}$$

$$g = 4e_4 + \frac{1}{2}E_1E_3(2E_1 + E_2)(2E_3 + E_4) + sf, \tag{16}$$

$$E_{ij} = E_i + E_j. \tag{17}$$

For completeness, we also list all on-shell, three-particle amplitudes for the framid, in the case of equal speeds $c_L = c_T$. We find

$$A_3(1^+2^+3^+) = 0, \tag{18}$$

$$A_3(1^+2^+3^-) = \sqrt{2}g\,(E_1 - E_2)\,\frac{[12]^3}{[23][31]}, \tag{19}$$

$$A_3(1^+2^+3^0) = g[12]^2, \tag{20}$$

$$A_3(1^+2^-3^0) = \frac{1}{2}g\langle 21\rangle^2, \tag{21}$$

$$A_3(1^+2^03^0) = -\frac{1}{\sqrt{2}}g\,(E_1 + 2E_2)\,\frac{[12][31]}{[23]}, \tag{22}$$

$$A_3(1^02^03^0) = 2g\,(E_1E_2 + E_2E_3 + E_3E_1)\,, \tag{23}$$

where

$$g = \frac{c_L^2 - 1}{c_L^2 M_1}. \tag{24}$$

References

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