UNIVERSITY OF CAMBRIDGE

Contributions to mixing and hypocoercivity in kinetic models

Helge Gerhard Walter Dietert

Queens' College

11 July 2016

This dissertation is submitted for the degree of Doctor of Philosophy.

Summary

Contributions to mixing and hypocoercivity in kinetic models
Helge Gerhard Walter Dietert

The main results of my work contribute to the mathematical study of a stability mechanism common to both the Vlasov–Poisson equation and the Kuramoto equation. These kinetic models come from very different areas of physics: the Vlasov–Poisson equation models plasmas and the Kuramoto equation models synchronisation behaviour.

The stability was first described by Landau in 1946 and is a subtle behaviour, because the damping only happens in a suitably weak sense. In fact, the models are not dissipative and cannot be stable in a strong topology. Instead, the so-called Landau damping happens through phase mixing. My contributions include a simplified linear analysis for the Vlasov–Poisson equation around the spatially homogeneous state. For the Kuramoto equation, I cover the linear analysis around general stationary states and show nonlinear stability results with algebraic and exponential decay. Moreover, I show how the mean-field estimate by Dobrushin can be improved around the incoherent state.

In addition, I study how a kinetic system can reach a thermal equilibrium. This is modelled by adding a dissipative term, which by itself drives the system to a local equilibrium. In hypocoercivity theory, the complementary effect of the transport operator is used to show exponential decay to a global equilibrium. In particular, I show how a probabilistic treatment can complement the standard hypocoercivity theory, which constructs equivalent norms, and I discuss the necessity of the geometric control condition for the spatially degenerate kinetic Fokker–Planck equation.

Finally, I study the possible discretisation of the velocity variable for kinetic equations. For the numerical stability, Hermite functions are a suitable choice, because their differentiation matrix is skew-symmetric. However, so far a fast expansion algorithm has been lacking and this is addressed in this work.

Declaration

This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except where specified in the text. This dissertation is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other university.

Helge Gerhard Walter Dietert
11 July 2016

Acknowledgements

for Edith

I am very grateful for all the support and encouragement which brought me to this state. Together with my undergraduate education, I spend eight years in Cambridge and it has always been an amazing and inspiring time.

This would not have been possible without the encouragement from my school, in particular Dr May, and the support of my parents and brother, who helped me to move to England and to study in Cambridge. In all the time, Queens' College and the mathematics department provided a supporting environment. For this I would like to thank all the supervisors, tutors and lectures, but also all the staff and the administration.

For the work on my PhD, I am delighted to thank my supervisors Clément Mouhot and Arieh Iserles for their inspiration, encouragement, support and guidance. It has been a great pleasure to be part of the Cambridge Centre for Analysis (CCA) and I would like to thank James Norris for supervising my long mini-project. During the work, the kinetic theory group, the numerical analysis group and the fellow graduate students always provided a friendly atmosphere.

I would also like to thank Bastien Fernandez, David Gérard-Varet and Giambattista Giacomin who have read my first preprint and invited me to Paris, which was very encouraging for me. Moreover, it let to a very inspiring collaboration.

Also I owe many thanks to Mihalis Dafermos and Frédéric Rousset for examining this thesis.

Finally, I am grateful to the Queens' College Boat Club (QCBC) and the Cambridge 99 boat club, from where I received the energy to keep working.

Declaration on collaboration and publication of results

Chapter 1 is my own review of the current literature and the results of Chapters 2 to 6.

Chapter 2 is my own review on the mean-field limit.

Chapter 3 reviews the common approach in Landau damping (Section 3.1) and the theory of the Volterra equation (Section 3.2). The application (Section 3.3) seems to be new.

Chapter 4 contains my original results, which are mostly published in [46].

Chapter 5 is the result of a collaboration with Bastien Fernandez and David Gérard-Varet from Université Paris 7 Denis Diderot and follows the preprint [45].

Chapter 6 is my own work and has not been published.

Chapter 7 contains Section 7.2, which is the result of a collaboration with Josephine Evans and Thomas Holding and follows the preprint [44]. The counter-example in Section 7.3 is my own work and has not been published.

Chapter 8 is my own work under the supervision of Arieh Iserles and has not been published.

Contents

1 Kinetic models with Landau damping

This chapter introduces the setup for the results on Landau damping, i.e. the stability through phase mixing, and reviews the related literature. We start with a brief formal discussion of the stability mechanism, Section 1.1. The mechanism is common to the two studied systems, coming from different areas of physics. The Vlasov–Poisson equation, Section 1.2, models plasmas and was the model in which Landau originally observed the damping. The Kuramoto equation, Section 1.3, models synchronisation phenomena and is a focus of this work. Both models are derived as mean-field limits. We finish the chapter by reviewing the mathematical approaches and the results of the thesis on phase-mixing in Section 1.4.

1.1 Convergence through phase mixing

In order to introduce the stability created by phase mixing, we start with a formal discussion of a toy model without any interaction and discuss how the stability emerges from the collective behaviour.

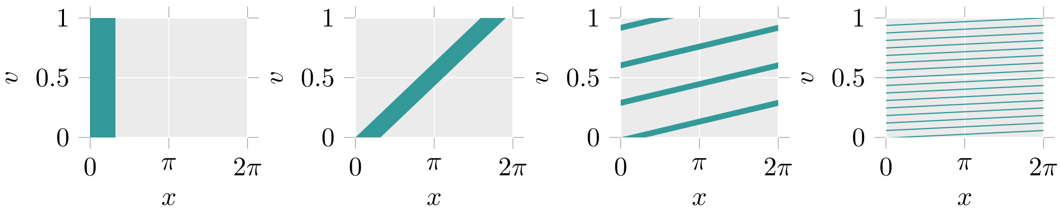
We start with a particle system of N particles, where each particle, labelled $i = 1, \dots, N$, is described by a position P_i in the phase space Γ, which often consists of the spatial position and velocity, but can also include the size and the spin of the particles. The evolution is described by a vector field A so that

$$\frac{\mathrm{d}}{\mathrm{d}t} P_i(t) = A(t, P_i(t)).$$

The trajectories of a particle are given by the maps $T_{t,s}$, where $T_{t,s}(P)$ is the position of a particle at time t given that the particle was at position P at time s. With the vector field A, an equivalent description is the ODE

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} T_{t,s}(P) = A(t, T_{t,s}(P)), \\ T_{s,s}(P) = P. \end{cases}$$

In order to understand a large population of particles, it is helpful to focus on their distribution, i.e. the *empirical measure* $f(t, \cdot) = N^{-1} \sum_{i=1}^{N} \delta_{P_i(t)}$. The empirical measure describes the collective behaviour of the particle system and its evolution over time is given



(a) Initial configuration **(b)** At time $t = 5$ **(c)** At time $t = 20$ **(d)** At time $t = 100$

Figure 1.1 – Evolution of an initial distribution under the free transport in $\mathbb{T} \times [0, 1]$.

by the push-forward of $T_{t,s}$, i.e.

$$(T_{t,s})_* f(s, \cdot) = f(t, \cdot).$$

The evolution then solves the PDE

$$\partial_t f(t, P) + \nabla_P \cdot [A(t, P) f(t, P)] = 0,$$

which is the Liouville equation for the density of one particle.

As a toy model, we consider the case where the phase space $\Gamma = \mathbb{T} \times [0, 1]$ consists of a spatial variable x on the torus $\mathbb{T} = [0, 2\pi)$ and a velocity variable v in $[0, 1]$. For the vector field A we consider the simple case of free transport

$$A(t, x, v) = \begin{pmatrix} v \\ 0 \end{pmatrix},$$

so that the density f evolves according to

$$\partial_t f(t, x, v) + \partial_x [v f(t, x, v)] = 0.$$

The solution starting from the initial data f_{in} at time $t = 0$ is given by

$$f(t, x, v) = f_{\text{in}}(x - tv, v)$$

and is, for localised initial data $f_{\text{in}}(x, v) = 1_{x \in [0,1]}$, shown in Figure 1.1. The figure shows that the initial data is distributed in very fine filaments and converges to the uniform state after averaging, i.e. in a weak sense. On the other hand, all L^p norms are exactly conserved and thus no form of convergence occurs in these norms.

For the mathematical analysis, we can describe the evolution by the semigroup e^{tL} so that

$$f(t, x, v) = (\mathrm{e}^{tL} f_{\text{in}})(x, v).$$

For the toy model with the free transport operator the generator L takes the form $L = -v\partial_x$.

The stability of such a semigroup is naturally studied by the spectral properties of the generator. The spectrum is obtained by taking the Fourier transform in x, i.e. to represent the function by the Fourier coefficients

$$\tilde{f}_l(v) = (\mathcal{F}_{x\to l} f)_l(v) = \int_{x\in\mathbb{T}} \mathrm{e}^{-\mathrm{i}lx} f(x,v)\mathrm{d}x$$

for $l \in \mathbb{Z}$. In this representation the generator is given by $\tilde{L} = -\mathrm{i}vl$.

Here we see that the spatial modes decouple and the generator acts as a simple multiplication operator. Thus in usual function spaces, e.g. $L^2(\Gamma)$, the generator has a continuous spectrum along the imaginary axis so that the spectral analysis shows that there is no decay in the strong topology.

If we further take the Fourier transform in v, i.e.

$$\hat{f}_l(\xi) = (\mathcal{F}f)_l(\xi) = \int_{(x,v)\in\mathbb{T}\times\mathbb{R}} \mathrm{e}^{-\mathrm{i}lx-\mathrm{i}\xi v} f(x,v)\mathrm{d}x\mathrm{d}v,$$

then the evolution becomes again a transport equation

$$\partial_t \hat{f}_l(t,\xi) = l\partial_\xi \hat{f}_l(t,\xi),$$

so that the solution after a time t is simply a shift as

$$\hat{f}_l(t,\xi) = (\hat{f}_{\mathrm{in}})_l(t,\xi+lt).$$

By the Plancherel identity, the solution after time t tested against a test function h satisfies

$$\langle f(t), h\rangle = \langle \hat{f}(t), \hat{h}\rangle = \sum_{l\in\mathbb{Z}} \int_{\xi\in\mathbb{R}} (\hat{f}_{\mathrm{in}})_l(\xi+lt)\, \overline{\hat{h}_l(\xi)}\, \mathrm{d}\xi.$$

For regular functions f_{in} and h, the Fourier transforms $(\hat{f}_{\mathrm{in}})_l(\xi)$ and $\hat{h}_l(\xi)$ are localised around $\xi \approx 0$ and decay as $|\xi| \to \infty$. For spatial modes $l \neq 0$, the two factors are shifted apart as $t \to \infty$ and thus the integral converges to 0. Therefore, the inner product converges to the spatial average

$$\int_{\xi\in\mathbb{R}} (\hat{f}_{\mathrm{in}})_0(\xi)\, \overline{\hat{h}_0(\xi)}\, \mathrm{d}\xi = 2\pi \int_{v\in\mathbb{R}} \left(\int_{x\in\mathbb{T}} f_{\mathrm{in}}(x,v) \frac{\mathrm{d}x}{2\pi} \right) \left(\int_{x\in\mathbb{T}} h(x,v) \frac{\mathrm{d}x}{2\pi} \right) \mathrm{d}v$$

and we have found stability. For $l \neq 0$, the speed of decay of the product

$$\int_{\xi\in\mathbb{R}} (\hat{f}_{\mathrm{in}})_l(\xi+lt)\, \overline{\hat{h}_l(\xi)}\, \mathrm{d}\xi$$

depends on the strength of localisation, i.e. how fast $(\hat{f}_{\mathrm{in}})_l(\xi)$ and $\hat{h}_l(\xi)$ decay as $|\xi| \to \infty$. Finally, the decay in Fourier variables characterises the regularity of a function.

Very heuristically, the regularity implies the weak decay, because for a regular function close to any particle is another particle with a slightly different velocity. Due to the inhomogeneity in the velocity, this means that trajectories will diverge and by ergodicity spread over the whole space.

Going back to the spectral study, consider the map $\lambda \to \langle (L-\lambda)^{-1} f_{\mathrm{in}}, h\rangle$, the resolvent applied to f_{in} and tested against h. This is an analytic function for λ in the resolvent set of the operator L. If f_{in} and h are analytic, we find, however, that the map has an analytic continuation beyond the continuous spectrum, so that the continuous spectrum seems to disappear. Formulated as Laplace transform, this has already been used in the first paper on this class of decay by Landau [87] in 1946.

An intuitive picture of this setup is the study of particles around a central body, like the rings of Saturn consisting of small particles. Here we can describe the position of a particle by its angle $x \in \mathbb{T}$ and radial distance $r \in [R_-, R_+]$ within a positive range. Assuming a circular orbit, the radius stays fixed and the frequency ω satisfies $\omega^2 r^3 = \mu$, where μ is a physical constant depending on the gravitation. Thus, we can understand the radial distance r as velocity variable and have the same phenomenon of phase mixing, cf. Figure 1.2. This gives an explanation of how such a ring structure can form and be stable without interaction. The discussion of the rings of Saturn was also the topic of Maxwell's famous Adam's price essay [54].

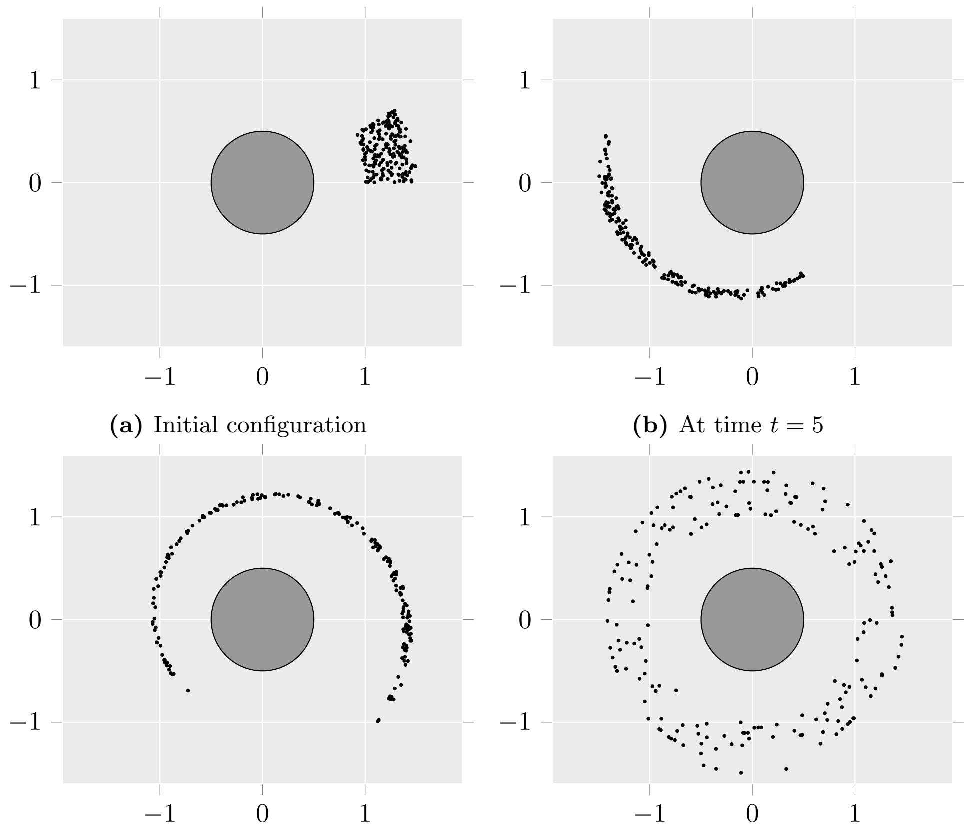
A brief discussion from the physical viewpoint is also given in the book by Reichl [135]. As another intuitive example, she describes an oil-film on water, which does not diffuse but can be mixed under a suitable velocity field to appear homogeneous.

In the whole discussion, the system is still time-reversible and the decay comes purely from the assumed regularity and the phase mixing. Thus a regular initial system tested against a regular function is decaying, which introduces the sense of time-irreversibility. In physical applications, regular initial data can be justified by regularising effects like noise or collisions. For systems including such additional effects, the stability through phase mixing can show damping on much smaller time-scales, where these other effects can be neglected. This faster decay belongs to the paradigm of the so-called violent damping [152, 159].

1.2 Vlasov–Poisson equation: Understanding collisionless plasmas

1.2.1 Physical modelling

A plasma consists of freely moving particles which are charged so that they exert a long range interaction through electromagnetic fields. A typical example is an electron gas, but ionised atoms and molecules can also act as charged particles. In the universe, a plasma is a very common state of matter in space [34, 141].



(a) Initial configuration

(b) At time $t = 5$

(c) At time $t = 10$

(d) At time $t = 100$

Figure 1.2 – Phase mixing on an orbital system. Particles are assumed to have circular orbits whose frequency is given by $\omega = r^{-3/2}$.

The physical description was developed in the early 20th century and is nowadays treated in many textbooks [94, 118, 142]. For our work, we focus on the so-called kinetic description, where we work with the density of particles over their position and velocity. This kind of description was pioneered for gases by Maxwell [107] and Boltzmann [20] at the end of the 19th century and was very controversial at that time [57].

In this work, we further restrict ourselves to systems with one species of particles, which are described by classical mechanics with an electrostatic interaction. While the results can easily be extended to several species, a treatment of relativistic effects and of the full Maxwell equations changes the mathematical structure significantly so that the previous studies cannot be extended directly.

Starting from a *microscopic description*, we have a large number of particles $i = 1, \dots, N$ whose phase space position $P_i = (q_i, p_i)$ evolves as

$$\frac{\partial q_i}{\partial t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial q_i}$$

with an Hamiltonian $H = H(P_1, \dots, P_N)$. In the physical modelling, we restrict to a symmetric interaction potential $N^{-1}W$ and an external potential V so that the Hamiltonian is given by

$$H(P_1, \dots, P_N) = \frac{1}{2m} \sum_{i=1}^{N} p_i^2 + \frac{1}{N} \sum_{1 \le i < j \le N} W(q_i, q_j) + \sum_{i=1}^{N} V(q_i),$$

where m is the mass of a single particle.

The scaling ensures that the different terms are of the same order. Physically, the scaling can be motivated by hypothetically splitting every particle into several smaller particles, where the overall mass and charge is kept constant. In this context, the charge Q and mass m of a particle behaves like N^{-1} giving the claimed scaling from the electrostatic interaction potential $W(q_i, q_j) = (4\pi\epsilon_0)^{-1} Q^2 |q_i - q_j|^{-1}$, where ϵ_0 is the electrostatic constant.

In practise the number of particles is very large. For the order of magnitude recall the Avogadro constant $6 \times 10^{23}\,\mathrm{mol}^{-1}$. We therefore try to understand the system statistically by a distribution over the phase space Γ of a single particle.

A possible derivation of such a description follows the paradigm of statistical physics and uses *ensembles* pioneered by Gibbs [56] in 1902.

In this, we do not claim that we know the exact state $(P_1, \dots, P_N) \in \Gamma^N$ of the whole system, but rather know a distribution F^N of the microstate $(P_1, \dots, P_N)$ over the complete phase space Γ^N. The particles are thought to be indistinguishable, which we include by imposing that F^N is symmetric. The distribution of a typical particle is then given by the first marginal

$$f(P) = f^{1,N}(P) = \int_{\Gamma^{N-1}} F^N(P, P_2, \dots, P_N)\, \mathrm{d}P_2 \dots \mathrm{d}P_N,$$

which we treat as adequate description of our system.

Each microstate evolves independently by Hamilton's equation of motion, so that the density F^N evolves by the *Liouville equation*

$$\partial_t F^N + \nabla \cdot (\dot{P} F^N) = 0,$$

where $\dot{P}$ is the evolution of the microstate. By the Hamiltonian structure, Liouville's theorem implies $\nabla \cdot \dot{P} = 0$, which shows that

$$\partial_t F^N + \{F^N, H\} = 0 \tag{1.1}$$

with the Poisson bracket

$$\{F^N, H\} = \sum_{i=1}^{N} \left(\nabla_{q_i} F^N \cdot \nabla_{p_i} H^N - \nabla_{p_i} F^N \cdot \nabla_{q_i} H^N \right),$$

where the coordinate P_i of the particle i consists of its position q_i and its momentum p_i.

In probability theory this density evolution is also called master or Kolmogorov equation and as such can easily be adapted to include noise terms due to background interactions.

The dynamic of F^N still includes the full complexity and, for a reduction, we look at the marginals $f^{k,N}$ defined by

$$f^{k,N}(P_1, \ldots, P_k) = \int_{\Gamma^{N-k}} F^N(P_1, \ldots, P_k, P_{k+1}, \ldots, P_N) \, \mathrm{d}P_{k+1} \ldots \mathrm{d}P_N.$$

Taking the marginals in (1.1) implies that the marginals satisfy

$$\begin{aligned} &\partial_t f^{k,N}(P_1, \ldots, P_k) + \{f^{k,N}, H^k\}(P_1, \ldots, P_k) \\ &\quad = \sum_{i=1}^{k} \nabla_{p_i} f^{k,N}(P_1, \ldots, P_k) \cdot \left(\frac{N-k}{N} \right) \nabla_{q_i} \int_{\Gamma} W(q_i, q) f^{k+1,N}(P_1, \ldots, P_k, P) \, \mathrm{d}q \mathrm{d}p, \end{aligned} \tag{1.2}$$

where $P = (q, p)$ in the last integral and H^k is the reduced Hamiltonian

$$H^k(P_1, \ldots, P_k) = \frac{1}{2m} \sum_{i=1}^{k} p_i^2 + \frac{1}{N} \sum_{1 \le i < j \le k} W(q_i, q_j) + \sum_{i=1}^{k} V(q_i).$$

This is the famous BBGKY hierarchy [118, 135, 142] (named after Bogoliubov, Born, Green, Kirkwood, and Yvon), which relates the evolution of the k-th marginal to the $(k+1)$-th marginal. Hence for the solution of the first marginal, all other moments are needed. In order to find a reduced description of the first marginal $f = f^{1,N}$, we therefore need to make further assumptions. The simplest case was introduced by Vlasov [161] for a plasma and assumes that the particles are uncorrelated, i.e. he replaces $f^{2,N}$ by $f^{1,N} \otimes f^{1,N}$. This assumption was pioneered by Boltzmann as molecular chaos in the kinetic theory of gases and is believed to hold in the limit $N \to \infty$. In this regime, the limit preserves the time-reversibility and can be described as Hamiltonian system [111].

Using the electrostatic interaction potential described by the Poisson equation, the resulting equation is called Vlasov–Poisson equation and, for particles of mass m and charge e, is given by

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) - \dfrac{e}{m} \nabla \phi(t,x) \cdot \nabla_v f(t,x,v) = 0, \\ \Delta \phi(t,x) = -\dfrac{\rho(t,x)}{\epsilon_0}, \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v) \,\mathrm{d}v - \rho_\infty(x) \end{cases} \tag{1.3}$$

describing a density $f(x,v)$ with respect to the spatial position x and velocity v, where ρ_∞ is a background charge and ϵ_0 is a physical constant, the permittivity. A possible source of such a background charge are ionised atoms which are approximately static on the considered time-scale.

For dilute plasma this approximation works very well for suitable configurations and is successfully used for experimental results. For the electrostatic interaction, two further approximations of $f^{2,N}$ are known in the physics literature: The Landau equation [88] and the so-called Lenard-Balescu equation [8, 92]. See also the textbooks [118, 142]. The modelling in these approaches already creates a preferred direction of time and the resulting equations have the structure of a Fokker–Planck equation. In particular, the Hamiltonian structure has not been preserved. Finally, collisions can be included in the model by adding a Boltzmann collision operator, cf. [142, Section 4.6], which again breaks time-reversibility.

Another approach follows the Klimontovich equation [82]. This approach starts with the empirical measure μ^N of the particle distribution $P_1, \dots, P_N$ given as

$$\mu^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{P_i}.$$

The empirical measure of a solution of the microscopic evolution satisfies the Klimontovich equation, which is the PDE

$$\begin{cases} \partial_t \mu(t,q,p) + \dfrac{p}{m} \cdot \nabla_q \mu(t,q,p) - \nabla_q V(t,q) \cdot \nabla_p \mu(t,q,p) = 0, \\ V(t,q) = \displaystyle\int_\Gamma W(q,q') \mu(t,q',p') \mathrm{d}q' \mathrm{d}p' \end{cases} \tag{1.4}$$

understood weakly against C^1 functions for particles of mass m. The fact that the empirical distribution of a solution satisfies (1.4) can be directly verified and, for sufficiently nice interaction potentials, this PDE determines the solution uniquely.

Formally, the Klimontovich equation (1.4) equals the Vlasov equation. However, their physical interpretation is different: the Vlasov equation describes the evolution of a density, while the Klimontovich equation describes the exact evolution of a configuration expressed as empirical measure. In physical discussions, we then take some form of averaging and argue that in the limit $N \to \infty$ a suitable version of the law of large numbers holds, so that the fluctuations can be neglected and we arrive again at the Vlasov equation [118, 142].

Another viewpoint assumes that the Klimontovich equation (1.4) is stable in a sufficiently fine topology of measures. This implies that the evolution of a particle distribution can be well-approximated by a smooth solution, which is initially close to the particle system. In numerical analysis this justifies the *meta-particle approach*, where we reduce the particle dynamics to a particle dynamics with fewer particles.

Related to this is the idea of pulverisation, which is a thought experiment, where we split the particles in smaller particles keeping the overall mass and charge constant. Assuming that the system with more particles results in a similar behaviour macroscopically, we expect stability in a suitable sense, so that the system can be described by the smooth solution.

This limiting behaviour $N \to \infty$ is well-understood for Lipschitz interactions, see Chapter 2, but is not proven for the singular Poisson interaction. Towards this, Hauray and Jabin [69, 70] showed results for slightly irregular potentials.

From a physical viewpoint, boundary conditions from a barrier or a confining potential are natural. However, a confining potential complicates the analysis significantly, so that we focus on periodic boundary condition. By rescaling, we can assume without loss of generality that the space is the standard d-dimensional torus $\mathbb{T}^d = [0, 2\pi)^d$ and we take an interaction of the form $W(x,y) = W(x-y)$. The evolution then takes the form

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + F(t,x) \cdot \nabla_v f(t,x,v) = 0, \\ F(t,x) = -\nabla_x \left[W * \rho(t,\cdot)\right](x), \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \end{cases} \tag{1.5}$$

where the star $*$ denotes the convolution over the space variable x. For a Poisson potential, there exists a constant C_W such that W is characterised by

$$\hat{W}_k = \int_{x \in \mathbb{T}^d} W(x) \mathrm{e}^{-\mathrm{i}k\cdot x} \mathrm{d}x = \begin{cases} 0 & \text{if } k = 0, \\ k^{-2} C_W & \text{otherwise.} \end{cases}$$

For a careful discussion of these periodic boundary conditions, see the introduction of Mouhot and Villani [114].

Compared to the spatially homogeneous plasma over $\mathbb{R}^d$, this setup has the advantage that, apart from the zero mode, all modes have a strictly positive size $|k| \geq 1$, which physically excludes very slow varying perturbations.

Even though the derivation was done for plasmas, another application of the Vlasov–Poisson equation is the dynamics of galaxies, where stars interact through Newtonian gravity. This was pioneered by Jeans [77, 78] in 1915, much before the work by Vlasov, and the equation is also called collisionless Boltzmann equation in this context. The possible application of Landau damping was discussed by Lynden-Bell [99]. For a brief discussion of the modelling, we again refer to the introduction of Mouhot and Villani [114].

In contrast to many equations in mathematical physics, the existence and uniqueness of solutions is well understood. Some reviews are given in [60, 136]. We have results on the existence of weak and classical solutions [5, 97, 125, 129, 140] and uniqueness [98].

1.2.2 Linear study

Already in his paper from 1938, Vlasov [161] looked at the linearisation around a stationary state f_{st}. We assume a spatially homogeneous basis configuration $f_{\mathrm{st}}(x,v) = f_{\mathrm{st}}(v)$ and overall neutrality, so that the electric potential vanishes in the stationary state. Considering a perturbation, again denoted by f, its evolution is on the linear level governed by

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + F(t,x) \cdot \nabla_v f_{\mathrm{st}}(t,v) = 0 \\ F(t,x) = -\nabla_x \left[W * \rho(t,\cdot)\right](x), \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^d} f(t,x,v) \mathrm{d}v \end{cases} \tag{1.6}$$

with the interaction potential $W(x,y) = W(x-y)$. Looking for normal modes, Vlasov took the ansatz $f(t,x,v) = c(v)\mathrm{e}^{\mathrm{i}k\cdot x - \mathrm{i}\omega t}$. The linearised equation becomes

$$\begin{cases} (-\mathrm{i}\omega + \mathrm{i}k \cdot v) f(t,x,v) + F(t,x) \cdot \nabla_v f_{\mathrm{st}}(v) = 0, \\ F(t,x) = -\mathrm{i}k\, \mathrm{e}^{\mathrm{i}k\cdot x - \mathrm{i}\omega t}\, \hat{W}_k \displaystyle\int_{\mathbb{R}^d} c(v) \mathrm{d}v, \end{cases}$$

where

$$\hat{W}_k = \int_{x\in\mathbb{T}^d} W(x) \mathrm{e}^{-\mathrm{i}k\cdot x} \mathrm{d}x.$$

Hence formally, c needs to take the form

$$c(v) = \frac{\hat{W}_k \int_{\mathbb{R}^d} c(\tilde{v})\, \mathrm{d}\tilde{v}\, k \cdot \nabla_v f_{\mathrm{st}}(v)}{k \cdot v - \omega}$$

and this is a valid solution if

$$1 = \hat{W}_k \int_{\mathbb{R}^d} \frac{k \cdot \nabla_v f_{\mathrm{st}}(v)}{k \cdot v - \omega} \mathrm{d}v. \tag{1.7}$$

With the electrostatic interaction, $\hat{W}_k$ takes the form $k^{-2}\omega_p^2$, where ω_p is the plasma frequency depending on the physical parameter of the plasma.

The dispersion relation (1.7) correctly describes the existence of growing modes, however, for real ω the factor $(k \cdot v - \omega)^{-1}$ becomes singular and the meaning for the asymptotic behaviour becomes unclear. In 1946, Landau [87] argued that the problem should instead be considered as Cauchy problem, which he formally solved using the Fourier and Laplace transform.

As a first step, take the Fourier transform in space

$$\tilde{f}_k(t,v) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}k\cdot x} f(t,x,v)\mathrm{d}x.$$

The linearised equation then becomes

$$\begin{cases} \partial_t \tilde{f}_k(t,v) + \mathrm{i}k\cdot v\tilde{f}_k(t,v) + \tilde{F}_k(t)\cdot\nabla_v f_{\mathrm{st}}(v) = 0, \\ \tilde{F}_k(t) = -\mathrm{i}k\hat{W}_k\tilde{\rho}_k, \\ \tilde{\rho}_k(t) = \displaystyle\int_{\mathbb{R}^d} \tilde{f}_k(t,v)\mathrm{d}v. \end{cases} \tag{1.8}$$

Note that the spatial modes k completely decouple and that each mode can be solved separately. This is a key feature of spatially homogeneous states, which makes their analysis feasible.

In order to solve the initial value problem, recall the *Laplace transform* $\mathcal{L}$ of a function $F:\mathbb{R}^+ \mapsto \mathbb{C}$ as

$$(\mathcal{L}F)(a) := \int_0^\infty \mathrm{e}^{-at}F(t)\,\mathrm{d}t$$

for all $a\in\mathbb{C}$ for which the integral is converging [48]. Taking the Laplace transform in time t, we then find

$$\begin{cases} -(\tilde{f}_{\mathrm{in}})_k(v) + a(\mathcal{L}\tilde{f}_k)(a,v) + \mathrm{i}k\cdot v(\mathcal{L}\tilde{f}_k)(a,v) + (\mathcal{L}\tilde{F}_k)(a)\cdot\nabla_v f_{\mathrm{st}}(v) = 0, \\ (\mathcal{L}\tilde{F}_k)(a) = \mathrm{i}k\hat{W}_k(\mathcal{L}\tilde{\rho}_k)(a), \\ (\mathcal{L}\tilde{\rho}_k)(a) = \displaystyle\int_{\mathbb{R}^d}(\mathcal{L}\tilde{f}_k)(a,v)\,\mathrm{d}v, \end{cases}$$

where f_{in} are the initial data for f.

Solving the first equation for $\mathcal{L}\tilde{f}_k$ gives

$$(\mathcal{L}\tilde{f}_k)(a) = \frac{(\tilde{f}_{\mathrm{in}})_k(v) - (\mathcal{L}\tilde{F}_k)(a)\cdot\nabla_v f_{\mathrm{st}}(v)}{a+\mathrm{i}k\cdot v}.$$

Using the second equation for $\mathcal{L}\tilde{F}_k$ and integrating over v shows

$$(\mathcal{L}\tilde{\rho}_k)(a) = \int_{\mathbb{R}^d}\frac{(\tilde{f}_{\mathrm{in}})_k(v)}{a+\mathrm{i}k\cdot v}\mathrm{d}v + (\mathcal{L}\tilde{\rho}_k)(a)\int_{\mathbb{R}^d}\frac{\mathrm{i}\hat{W}_k k\cdot\nabla_v f_{\mathrm{st}}(v)}{a+\mathrm{i}k\cdot v}\mathrm{d}v.$$

Hence we can explicitly solve the equation for $(\mathcal{L}\tilde{\rho}_k)$ as

$$(\mathcal{L}\tilde{\rho}_k)(a) = \left(1-\hat{W}_k\int_{\mathbb{R}^d}\frac{k\cdot\nabla_v f_{\mathrm{st}}(v)}{k\cdot v-\mathrm{i}a}\mathrm{d}v\right)^{-1}\int_{\mathbb{R}^d}\frac{(f_{\mathrm{in}})_k(v)}{a+\mathrm{i}k\cdot v}\mathrm{d}v,$$

where the first term reassembles the dispersion relation (1.7) with the identification of the rate as $a = -\mathrm{i}\omega$.

Assuming exponential growth bounds, this expression gives the correct Laplace transform for $\Re a$ large enough. For sufficiently nice stationary solutions f_{st}, the part

$$L(a) := \hat{W}_k \int_{\mathbb{R}^d} \frac{k \cdot \nabla_v f_{\mathrm{st}}(v)}{k \cdot v - \mathrm{i}a} \mathrm{d}v$$

gives an analytic function over $\Re a > 0$. Landau then noted, that for analytic functions $f_{\mathrm{st}}(v)$ with sufficient decay, the function L can be analytically continued beyond $\Re a = 0$. Likewise, for sufficiently nice initial data, the second term in the expression for $\mathcal{L}\tilde{\rho}_k$ can be analytically extended.

We assume that $\mathcal{L}\tilde{\rho}_K$ can be analytically continued as meremorphic function to the region $\{a \in \mathbb{C} : \Re a \geq -a_0\}$ with possible poles $b_1, \ldots, b_M$ at the points where $1 = L(a)$. Formally taking the inverse Laplace transform (also called *Dunford formula* in the context of semigroups), the solution is

$$\rho_k(t) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty+\sigma_0}^{\mathrm{i}\infty+\sigma_0} (\mathcal{L}\tilde{\rho}_k)(a)\mathrm{e}^{at}\mathrm{d}a$$

for a sufficiently large constant σ_0. Formally, deforming the contour along $\Re a = -a_0$, we find an integral along $\Re a = -a_0$, which decays as $\mathrm{e}^{-a_0 t}$, and contributions from the poles $i = 1, \ldots, M$, which evolve as $\mathrm{e}^{b_i t}$.

In the case of a Maxwellian distribution, Landau showed that all poles are decaying. Hence he concluded that the observable spatial density (or equivalently the electric potential) is damped.

In general, for a function f, the mapping to

$$(\mathcal{H}f)(s) = \int_{-\infty}^{\infty} \frac{f(u)}{u - s} \mathrm{d}u$$

is called the Hilbert transform. Landau defined the analytic continuation from the upper half plane by changing the contour such that the pole $u = s$ is above the contour of integration, which is called Landau prescription in this context, cf. Figure 1.3.

This class of integrals had been well-studied in the mathematical analysis [115, 151]. The result allowed Backus [7] to show that the above reasoning can be made rigorous in order to show the claimed linear stability.

In 1959, Penrose [128] also studied the limiting behaviour of $L(a)$ as $\Re a \to 0 + 0$. For this he uses the Plemelj formula [115] stated in its basic form as:

Lemma 1.1. *If $f \in L^1$ is a Lipschitz continuous function, then*

$$\lim_{\lambda \to 0+0} \int_{\mathbb{R}} \frac{f(x)}{x - \mathrm{i}\lambda} \mathrm{d}x = \mathrm{PV} \int \frac{f(x)}{x} \mathrm{d}x + \mathrm{i}\pi f(0). \tag{1.9}$$

Here PV denotes the principal value and in fact the convergence can be quantified.

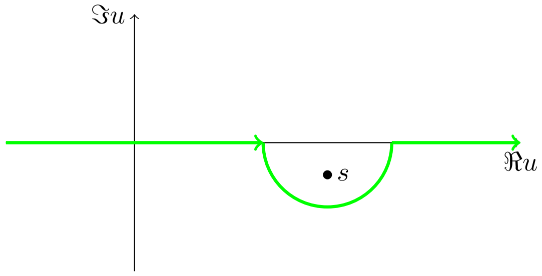


Figure 1.3 – Landau's prescription for changing the contour of the integral in the Hilbert transform $\int_{-\infty}^{\infty} \frac{f(u)}{u-s} \mathrm{d}u$ if $\Im s \leq 0$ in order to find the analytic continuation from the upper half plane. Note that in terms of the Laplace transform u is identified with $\mathrm{i}\omega$ and integral is along the imaginary axis. The figure has been adapted from [87].

From this Penrose found an explicit expression for $L(a)$ for $\Re a = 0$, which is the boundary value of the analytic function L. Moreover, by the Riemann-Lebesgue lemma, $|L(a)| \to 0$ as $|a| \to \infty$ for $\Re a > 0$. Therefore, $x \to L(\mathrm{i}x)$ defines a complex curve starting and ending at 0. By the argument principle, the number of solutions to $L(a) = 1$ for $\Re a > 0$ equals the winding number of the curve around 1. Hence we can use the curve to determine the number of unstable modes. In particular, we can use this to identify stable configurations exactly.

In the electrostatic case with $\hat{W}_k = k^{-2}\omega_p^2$, we find that the dispersion relation takes the form

$$k^2 = Z_{\hat{e}}(a/|k|)$$

with

$$Z_{\hat{e}}(a) = \omega_p^2 \int_{\mathbb{R}^d} \frac{\hat{e} \cdot \nabla_v f_{\mathrm{st}}(v)}{\hat{e} \cdot v - \mathrm{i}a} \mathrm{d}v,$$

where $\hat{e}$ is the unit vector $k/|k|$. Hence, we have unstable modes if and only if there exists a with $\Re a > 0$ such that $Z_{\hat{e}}(a) \in (0, \infty)$. Identifying the existence of such solutions by the argument principle gives the *Penrose criterion*, which states the stability in terms of the minima of the marginal velocity distribution along $\hat{e}$.

Proposition 1.2 (Penrose criterion)**.** *The configuration f_{st} has a growing mode in the direction $\hat{e}$ if and only if the marginal density*

$$h(u) = \int_{\mathbb{R}^d} \delta_{v \cdot \hat{e} = u} f_{\mathrm{st}}(v) \mathrm{d}v$$

has a relative minimum $x \in \mathbb{R}$ with

$$\mathrm{PV} \int_{-\infty}^{\infty} \frac{h'(u)}{u - x} \mathrm{d}u > 0.$$

For the stability analysis, Penrose noted that the electric field or potential satisfies a Volterra equation for which the stability is well-understood. In this work, I will follow this approach, which is discussed in Chapter 3.

In 1955, van Kampen [153] also showed that the behaviour can be recovered by a normal mode analysis when the singularity is resolved properly. In this, he finds singular eigenmodes for every $\omega \in \mathbb{R}$, in which he expands the solution. By understanding the dispersion, he then recovers the stability, where he already draws the connection to resonance states in scattering theory for the decaying modes. Case [29] and Backus [7] clarified this approach in 1959 and 1960, respectively.

From the scaling of $k^2 = Z_{\hat{e}}(a/|k|)$, we see that modes with small $|k|$ decay with a slower rate and $k = 0$ corresponds to the overall density, which is not decaying. This motivates the use of the torus, allowing to prove uniform decay. Some indications of the behaviour over the whole line are given in [11, 58, 59].

The analysis was rigorously revisited in the 1980s, focusing on the observed decay rate by Maslov and Fedoryuk [106] and Degond [41]. Degond worked on the torus and showed the decay for the result tested against for test functions, when the initial data and test function have a uniformly bounded continuation in a strip around the real axis.

From a physical viewpoint, the understanding is that the energy of the electric potential is transferred to the kinetic energy of the particles [40]. A naive interpretation follows the surfer picture, in which particles with a slower speed than the wave speed are accelerated, while particles with a higher speed are decelerated. With a unimodal velocity distribution, there are more slower particles and thus energy is taken out of the wave. However, such a naive reasoning is wrong and a more careful analysis is needed [51, 142].

1.2.3 Nonlinear behaviour

In his paper from 1959, Backus [7] pointed out, that the linearisation is questionable, because the damping creates very fine filaments and we are ignoring a term taking a velocity derivative. Moreover, the ignored nonlinear term is the highest order term in v, which normally determines the behaviour of a PDE. Nevertheless, the predicted damping behaviour was experimentally verified in the early 1960s [100, 101, 166].

In fact Mouhot and Villani [114] showed in their celebrated paper from 2011, that linear stability implies nonlinear stability on the torus for regular enough perturbations (Gevrey regularity). Villani [157] also produced lecture notes, which nicely review the result and its context. Bedrossian, Masmoudi and Mouhot [13] later also simplified the proof.

A major obstacle are the *plasma echos* discovered by Gould, O'Neil, Malmberg and Wharton [64, 102, 103] in 1967 and 1968. If one perturbs a stable plasma, then the perturbation will be damped by Landau damping, in particular the electric field will decay. A second perturbation after time T will also be damped, but then after a further time T another excitation, the echo, can be observed.

For a simple explanation [94], consider the free transport from Section 1.1 on the torus. An initial perturbation $\cos(x)g(v)$ will evolve after time t to

$$f(t,x,v) = \cos(x - tv)g(v),$$

which appears to be damped due to the phase factor $\mathrm{e}^{\pm itv}$. A second perturbation at time T introduces a perturbation of the form $\cos(2x)$. For such a perturbation, the solution will be

$$\begin{aligned} f(t+T,x,v) &= \cos(2(x-tv))\cos(x-(t+T)v)g(v) \\ &= \frac{1}{2}\left[\cos(3(x-tv)-Tv) + \cos(x-(t-T)v)\right]g(v), \end{aligned}$$

where we see that the second term $\cos(x-(t-T)v)$ creates an echo at the further time $t = T$.

The question for the full nonlinear behaviour then becomes, whether the echos can replicate themselves creating a nonlinear instability. In order to exclude such a behaviour, the current results need a strong regularity assumption on the perturbation.

Already in 1957, Bernstein, Greene and Kruskal [16] realised that the nonlinear behaviour allows a large class of wave solutions. Their idea is to prescribe an electric potential and a distribution for the untrapped energy levels of the Hamiltonian. Then they show that they can construct a valid solution by choosing a distribution for the trapped energy levels. This construction is nowadays standard in textbooks [118, 142] and Lin and Zeng [95, 96] showed that this implies a minimal regularity of H^σ for $\sigma > 3/2$ of the perturbations for the damping to be possible.

For the analysis, the simplified Vlasov–HMF model (Hamiltonian Mean Field model) has been devised, where the interaction W is simplified. On the torus, we take the interaction potential W such that only finitely many modes $\hat{W}_k$ are nonzero and suppose the evolution as in (1.5), cf. [9]. This greatly reduces the effect of echos and allowed stability results with Sobolev regularity [52].

1.3 Kuramoto equation: Understanding synchronisation

1.3.1 Modelling and applications

The Kuramoto model was originally devised as simple model of globally coupled oscillators [83, 84] and is a special case of the earlier Winfree model [164]. The aim of the model is to understand synchronisation behaviour. Common historical examples are the synchronisation of pendulum clocks on the same wall observed by Huygens in the 17th century [126] or the synchronised flashing of fireflies [130].

The model describes the behaviour of N oscillators, where each oscillator $i = 1, \dots, N$ has its own natural frequency ω_i and its state is described by its phase angle $\theta_i \in \mathbb{T}$. The evolution is then modelled by the following set of first order equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_i = \omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i) \quad \text{for } i = 1, \ldots, N,$$

where K parametrises the coupling strength.

This simple model still captures the phase transition to synchrony in collective systems and has gained substantial attention with many applications, e.g. in Chemistry and Biology, cf. the reviews [2, 130, 147]. It can be thought as weak coupling limit of oscillators, where the interaction does not significantly change the shape of the limit cycle. As a first order system, it is not Hamiltonian and even after changing ω it is not time reversible. However, it can be embedded in a Hamiltonian system [165].

The coupling can be expressed through a global *order parameter*

$$\eta = \frac{1}{N}\sum_{i=1}^{N} \mathrm{e}^{\mathrm{i}\theta_i},$$

which also measures how much a population is synchronised. Using the order parameter, the evolution can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_i = \omega_i + \frac{K}{2\mathrm{i}}\left[\eta \mathrm{e}^{-\mathrm{i}\theta_i} - \overline{\eta}\mathrm{e}^{\mathrm{i}\theta_i}\right].$$

If the oscillator system rotates with a global frequency $\bar{\omega}$, we can factor out this rotation by taking $\theta \to \theta - \bar{\omega}t$ and $\omega_i \to \omega_i - \bar{\omega}$ without changing the synchronisation behaviour. Therefore, in studying a stable rotating state, we can without loss of generality choose a suitable rotating frame, where the system becomes stationary.

The early studies by Kuramoto [83, 84] focused on the self-consistency of possible stationary solutions. For studies of the asymptotic behaviour, the particle system seems intractable, so that we again focus on the mean-field limit $N \to \infty$, where we treat each oscillator $i = 1, \ldots, N$ as particle with spatial position $\theta_i \in \mathbb{T}$ and velocity $\omega_i \in \mathbb{R}$. In the context of the Kuramoto equation, the mean-field limit has first been suggested by Sakaguchi [138] in the case with noise and by Mirollo and Strogatz [148] in the case without noise. The system is described by a distribution f over the phase space $\Gamma = \mathbb{T} \times \mathbb{R}$, which evolves as

$$\begin{cases} \partial_t f(t,\theta,\omega) + \partial_\theta \left[\left(\omega + \frac{K}{2\mathrm{i}}(\eta(t)\,\mathrm{e}^{-\mathrm{i}\theta} - \overline{\eta(t)}\,\mathrm{e}^{\mathrm{i}\theta})\right) f(t,\theta,\omega)\right] = 0, \\ \eta(t) = \int_{(\theta,\omega)\in\Gamma} \mathrm{e}^{\mathrm{i}\theta} f(t,\theta,\omega)\mathrm{d}\theta\mathrm{d}\omega. \end{cases} \tag{1.10}$$

Here we keep the velocity ω as parameter of the density, but note that it stays constant. In particular the velocity marginal

$$g(\omega) = \int_{\theta \in \mathbb{T}} f(\theta, \omega) \mathrm{d}\theta$$

is constant in time.

In 2005, Lancellotti [86] noted that the mean-field theory applies, which rigorously justifies the limit and shows well-posedness of the PDE, see Chapter 2. For applications of the Kuramoto model, the number of oscillators can become very small, which motivated many numerical experiments about the fluctuations observed in finite particle system. In particular, it has been suggested to look at the second marginal in the BBGKY hierarchy in order to further understand the fluctuations [71].

1.3.2 Self-consistency equation and stationary states

The spatially homogeneous state

$$f(\theta, \omega) = \frac{1}{2\pi} g(\omega)$$

with the velocity marginal $g(\omega) = \int_{\theta \in \mathbb{T}} f(\theta, \omega) \mathrm{d}\theta$ is a stationary state of the Kuramoto equation (1.10) with order parameter $\eta = 0$. It corresponds to a completely incoherent distribution.

For a possible synchronised state, we can assume without loss of generality, that we have chosen the observation frame such that $\eta \in (0, 1]$ and is constant in time. Looking for a stationary state f_{pls} of (1.10), oscillators with $|\omega| > K\eta$ must be distributed proportional to

$$\frac{\mathrm{d}\theta}{|\omega - K\eta \sin\theta|}.$$

In order to compute the normalising constant and the Fourier transform l, consider $\omega > K\eta$ and $l \in \mathbb{N}$. For this choice, compute the integral with the substitution $z = \mathrm{e}^{-\mathrm{i}\theta}$ and Cauchy's residue theorem

$$\begin{aligned}\int_{\theta \in \mathbb{T}} \mathrm{e}^{-\mathrm{i}l\theta} \frac{\mathrm{d}\theta}{|\omega - K\eta \sin\theta|} &= \int_{|z|=1} \frac{z^l \, \mathrm{d}z}{-\mathrm{i}\omega z + \frac{K\eta}{2}(1 - z^2)} \\ &= \frac{2\pi \left(\beta\left(\frac{\omega}{K\eta}\right)\right)^l}{\sqrt{\omega^2 - (K\eta)^2}},\end{aligned}$$

where $\beta(x)$ is the root of $z^2 + 2\mathrm{i}xz - 1$ with modulus less than 1. The same can be done for $\omega < -K\eta$. Fixing the normalisation of the conditional distribution for $|\omega| > K\eta$ thus shows that the conditional distribution is

$$\frac{\sqrt{\omega^2 - (K\eta)^2}}{|\omega - K\eta \sin\theta|} \mathrm{d}\theta,$$

whose Fourier series for $l \in \mathbb{N}$ is

$$\int_{\theta\in\mathbb{T}} \mathrm{e}^{-\mathrm{i}l\theta} \frac{\sqrt{\omega^2-(K\eta)^2}}{|\omega-K\eta\sin\theta|}\,\mathrm{d}\theta = \left(\beta\left(\frac{\omega}{K\eta}\right)\right)^l,$$

where

$$\beta(x) = -\mathrm{i}x\left(1-\sqrt{1-\frac{1}{x^2}}\right) \quad \text{for } |x|>1.$$

For oscillators with $|\omega| \le K\eta$, the evolution of the phase angle θ has a stable fixed point at $\theta = \arcsin(\omega/(K\eta))$ and an unstable fixed point at $\theta = \pi - \arcsin(\omega/(K\eta))$. Hence these oscillators are phase locked and, with the combination of unlocked oscillators, such a state is called *partially locked*.

A general partially locked state therefore takes the form

$$f_{\mathrm{pls}}(\theta,\omega) = \begin{cases} \left(\alpha(\omega)\delta_{\arcsin(\omega/(K\eta))}(\theta) + (1-\alpha(\omega))\delta_{\pi-\arcsin(\omega/(K\eta))}(\theta)\right) g(\omega) & \text{if } |\omega| \le K\eta \\ \dfrac{\sqrt{\omega^2-(K\eta)^2}}{2\pi|\omega-K\eta\sin\theta|} g(\omega) & \text{if } |\omega| > K\eta \end{cases}$$

for an arbitrary measurable function $\alpha : [-K\eta, K\eta] \mapsto [0,1]$, which describes the proportion of mass at the stable fixed point.

The Fourier transform in x, can be easily expressed for $l \in \mathbb{N}$ as

$$\begin{aligned} (\tilde{f}_{\mathrm{pls}})_l(\omega) &= \int_{\theta\in\mathbb{T}} \mathrm{e}^{-\mathrm{i}l\theta} f_{\mathrm{pls}}(\theta,\omega)\mathrm{d}\theta \\ &= \begin{cases} \left(\alpha(\omega)\beta_+^l\left(\dfrac{\omega}{K\eta}\right) + (1-\alpha(\omega))\beta_-^l\left(\dfrac{\omega}{K\eta}\right)\right) g(\omega) & \text{if } |\omega| \le K\eta, \\ \beta^l\left(\dfrac{\omega}{K\eta}\right) g(\omega) & \text{if } |\omega| > K\eta, \end{cases} \end{aligned}$$

where $\beta_\pm(x)$ are again the roots of $z^2 + 2\mathrm{i}xz - 1$, explicitly given as

$$\beta_\pm(x) = -\mathrm{i}x \pm \sqrt{1-x^2}.$$

In the study of stability of partially locked states, we assume that all oscillators are concentrated at the stable fixed point and denote such a state by f_{st}. In this context, we also take $\beta_+ = \beta$.

The structure of the partially locked state f_{st} can be observed as asymptotic behaviour in a large discrete system. As an example, we take 1000 oscillators whose natural frequencies are drawn from the standard Gaussian distribution and choose a sufficiently large coupling $K = 1.75$ to induce a synchronisation. The order parameter quickly converges to a constant value and, waiting a long time $t = 1000$, the distribution of the oscillators is close to the partially locked state f_{st}. The position of the oscillators in the phase space (θ,ω) is shown in Figure 1.4. The distribution shows that the oscillators with $|\omega| < K|\eta|$ are at the stable fixed-point and that the oscillators with $|\omega| > K|\eta|$ are distributed over θ. Looking at their time-evolution, we can also observe that the oscillators with $|\omega| < K|\eta|$ are locked, i.e. not moving relative to the overall synchronisation, while the other oscillators are still moving.

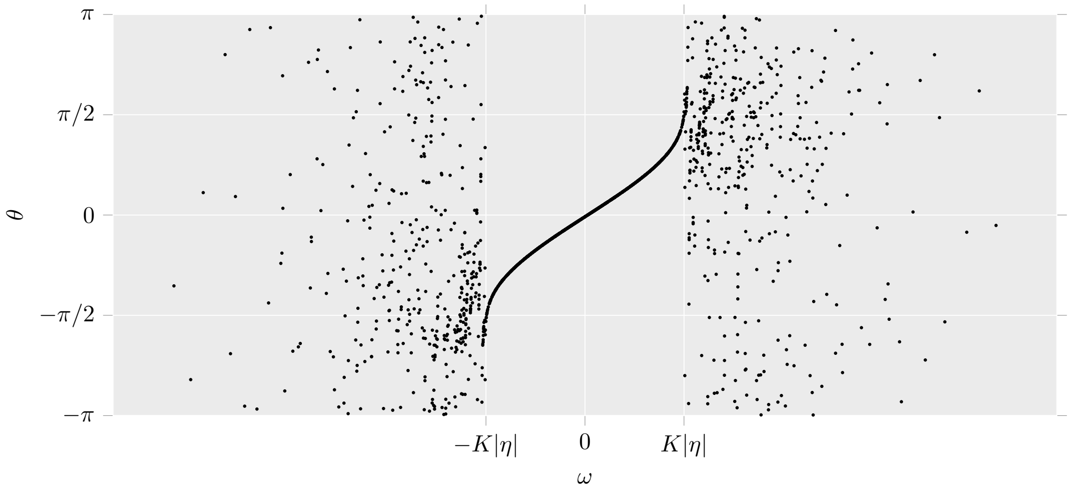


Figure 1.4 – Phase diagram of a discrete simulation ($N = 1000$) after a long time $t = 1000$, where the order parameter has converged to a fixed value. The resulting configuration has been rotated so that the order parameter is real ($\eta \approx 0.48$). The natural frequencies are drawn initially form a standard Gaussian distribution and the coupling $K = 1.75$ has been chosen to ensure synchronisation.

For a stationary solution, the partially locked state f_{pls} must recreate the imposed order parameter η. Recalling that η is chosen to be real, the calculation of the Fourier series shows that this condition is

$$\begin{aligned}\eta &= \int_{|\omega|>K\eta} \beta\left(\frac{\omega}{K\eta}\right) g(\omega)\mathrm{d}\omega + \int_{|\omega|\leq K\eta} \left(\alpha(\omega)\beta_+\left(\frac{\omega}{K\eta}\right) + (1-\alpha(\omega))\beta_-\left(\frac{\omega}{K\eta}\right)\right) g(\omega)\mathrm{d}\omega \\ &= -\int_{\omega\in\mathbb{R}} \frac{\mathrm{i}\omega}{K\eta} g(\omega)\mathrm{d}\omega + \int_{|\omega|>K\eta} \frac{\mathrm{i}\omega}{K\eta}\sqrt{1-\left(\frac{K\eta}{\omega}\right)^2}\, g(\omega)\mathrm{d}\omega \\ &\qquad + \int_{|\omega|\leq K\eta} (2\alpha(\omega)-1)\sqrt{1-\left(\frac{\omega}{K\eta}\right)^2}\, g(\omega)\mathrm{d}\omega.\end{aligned}$$

Taking the real part requires

$$\eta = \int_{|\omega|\leq K\eta} (2\alpha(\omega)-1)\sqrt{1-\left(\frac{\omega}{K\eta}\right)^2}\, g(\omega)\mathrm{d}\omega,$$

where we see that only the locked oscillators at the stable fixed point produce the field. The imaginary part imposes

$$0 = -\int_{\omega\in\mathbb{R}} \frac{\omega}{K\eta} g(\omega)\mathrm{d}\omega + \int_{|\omega|>K\eta} \frac{\omega}{K\eta}\sqrt{1-\left(\frac{\omega}{K\eta}\right)^2}\, g(\omega)\mathrm{d}\omega.$$

On an intuitive level, the condition from the real part means that we need to have enough locked oscillators in order to produce the strength of the order parameter and the condition from the imaginary part means that the centre is not drifting away. Here the unlocked oscillators are not contributing to the strength of the order parameter.

In the case of a symmetric distribution with $g(\omega) = g(-\omega)$, the condition from the imaginary part is always satisfied. In fact, for a unimodal distribution[1], which is symmetric around its maximum, the condition from the imaginary part has only a solution if the frame is chosen such that the maximum is at 0. Physically, it means that a partially locked state can only exists with the mean-frequency of such a distribution.

For a symmetric distribution, the condition on the real part can be written as

$$1 = F(\eta)$$

with

$$F(\eta) = K \int_{|x|\leq 1} [2\alpha(K\eta x) - 1]\sqrt{1-x^2}\, g(K\eta x)\,\mathrm{d}x.$$

For an absolutely continuous velocity marginal, F varies continuously with respect to η and using that g is a probability density, we find that $F(1) < 1$.

On the other hand, for symmetric unimodal distributions, F is monotone decreasing, so that a nontrivial solution exists if and only if $F(0) > 1$. For an absolutely continuous velocity distribution g and $\alpha \equiv 1$ we find the limiting value

$$F(0) = \frac{K\pi}{2} g(0),$$

which can be understood as a condition on K for the existence of synchronised states. From a physical point of view, it was conjectured from the early studies that these states are the stable configuration, which bifurcate from the incoherent state as K increases [147].

For general distributions, the picture can be more complicated with several possible solutions, see e.g. the discussion by Omel'chenko and Wolfrum [120].

1.3.3 Neutral stability and Landau damping

One of the first questions studied was the stability of the incoherent state on the linear level [148, 149]. Using the Fourier transform

$$\tilde{f}_l(t,\omega) = \int_{\mathbb{T}} \mathrm{e}^{-\mathrm{i}l\theta} f(t,\theta,\omega)\mathrm{d}\theta$$

[1] a distribution with a density with exactly one relative maximum

for $l \in \mathbb{Z}$, the evolution becomes

$$\begin{cases} \partial_t \tilde{f}_l(t,\omega) + \mathrm{i} l \tilde{f}_l(t,\omega) + \frac{Kl}{2}\Big(\eta(t)\tilde{f}_{l+1}(t,\omega) - \overline{\eta(t)}\tilde{f}_{l-1}(t,\omega)\Big) = 0, \\ \eta(t) = \int_{\omega\in\mathbb{R}} \tilde{f}_{-1}(t,\omega)\mathrm{d}\omega. \end{cases} \tag{1.11}$$

The zeroth mode is the constant velocity marginal g, so that a perturbation is characterised by the restriction $(\tilde{f}_l(t,\omega))_{l\geq 1}$. The time-evolution of the perturbation is then given by

$$\begin{cases} \partial_t \tilde{f}_1(t,\omega) + \mathrm{i}\tilde{f}_1(t,\omega) + \frac{K}{2}\Big(\eta(t)\tilde{f}_2(t,\omega) - \overline{\eta(t)}\tilde{g}(\omega)\Big) = 0, \\ \partial_t \tilde{f}_l(t,\omega) + \mathrm{i} l \tilde{f}_l(t,\omega) + \frac{Kl}{2}\Big(\eta(t)\tilde{f}_{l+1}(t,\omega) - \overline{\eta(t)}\tilde{f}_{l-1}(t,\omega)\Big) = 0 \quad \text{for } l \geq 2, \\ \eta(t) = \overline{\int_{\omega\in\mathbb{R}} \tilde{f}_1(t,\omega)\mathrm{d}\omega}. \end{cases}$$

As f is real, we have $\tilde{f}_{-l} = \overline{\tilde{f}_l}$, so that the study for $l \geq 1$ suffices to characterise the evolution.

On the linear level, we see that all modes decouple. The first mode evolves as

$$\begin{cases} \partial_t \tilde{f}_1(t,\omega) + \mathrm{i}\tilde{f}_1(t,\omega) - \frac{K}{2}\overline{\eta(t)}\tilde{g}(\omega) = 0, \\ \eta(t) = \overline{\int_{\omega\in\mathbb{R}} \tilde{f}_1(t,\omega)\mathrm{d}\omega}, \end{cases}$$

where $\tilde{g}$ is the Fourier transform of the velocity marginal g, and for $l \geq 2$ the evolution is

$$\partial_t \tilde{f}_l(t,\omega) + \mathrm{i} l \tilde{f}_l(t,\omega) = 0.$$

In 1991 Mirollo and Strogatz [148] did a formal spectral analysis, where they find a continuous spectrum along the imaginary axis. For a symmetric, unimodal, absolutely continuous velocity distribution, they find that there are no poles for couplings below the critical coupling $K_c = 2(K\pi g(0))^{-1}$, which they called *neutral stability*. In fact, together with Matthews [149], they noticed in 1992 that the structure is the same as the linear Landau damping of the Vlasov–Poisson equation for one mode (1.6) and concluded that the observed stability in numerical experiments follows Landau damping.

In the early 1990s, the system was also studied in the case of white noise acting on θ [138], which creates a dissipative term in strong topology. Using this dissipation, the nonlinearity can be handled and the nonlinear stability was obtained. Even the bifurcation behaviour at the onset of instability could be proven using a center-manifold reduction [22, 38]. Depending on the velocity distribution and the interaction, they have found a range of different behaviours, especially in the case of distributions with two relative maxima [1, 21, 36, 37]. Crawford [38] noted that the asymptotic state in the bifurcation behaviour converges to a limit as the strength of the noise is decreased to zero, but could not justify the validity of his approach then.

Exploiting the structure of the nonlinearity, we can in fact see

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\sum_{l=1}^{\infty}\frac{1}{l}|\tilde{f}_l(t,\omega)|^2 g^{-1}(\omega)\mathrm{d}\omega = K|\eta(t)|^2,$$

showing that a perturbation in this norm grows and the system is not dissipative. Using the relative entropy, a similar results has been shown by Nordenfelt [119]. In particular, he concludes that the perturbed system cannot relax to the unperturbed system in $L^2(\Gamma)$. Therefore, the observed stability must be measured in a suitable weak sense, like the Landau damping in Vlasov–Poisson.

For the stability of the partially locked states, Mirollo and Strogatz [109] considered the case where all locked oscillators are at the stable fixed point. For symmetric unimodal distributions, they then show again neutral stability.

In case all oscillators are phase locked, no phase-mixing occurs and strong convergence results can be shown. However, this work focuses on the Landau damping, so that we just refer to [28] as a starting point. The nonlinear studies are a major part of this thesis and are reviewed in Section 1.4.

1.3.4 Ott–Antonsen ansatz

For the analysis of the asymptotic behaviour, a reduced system has been proposed, the so-called Ott–Antonsen ansatz [122]. It consists of two steps: The first step reduces the system to the PDE of one mode and is always applicable. The second step assumes a rational velocity distribution in order to reduce the system to a finite dimensional ODE.

The first mode reduction starts with the observation, that the stable stationary state takes in Fourier the form

$$\tilde{f}_l(\omega) = \alpha^l(\omega)g(\omega)$$

for $l \in \mathbb{N}$ and a function $\alpha : \mathbb{R} \mapsto \mathbb{C}$. In fact, we can verify that

$$\tilde{f}_l(t,\omega) = \alpha^l(t,\omega)g(\omega)$$

is a valid solution to the Kuramoto equation (1.11) if

$$\begin{cases} \partial_t\alpha(t,\omega) + \mathrm{i}\omega\alpha(t,\omega) + \dfrac{K}{2}\left(\eta(t)\alpha^2(t,\omega) - \overline{\eta(t)}\right) = 0, \\ \overline{\eta(t)} = \displaystyle\int_{\mathbb{R}}\alpha(t,\omega)g(\omega)\mathrm{d}\omega. \end{cases} \tag{1.12}$$

For a further reduction, we assume that g is analytic in the lower half plane $\{\omega \in \mathbb{C} : \Im\omega \le 0\}$ with sufficient decay and poles $\omega_1, \dots, \omega_M$ with corresponding residues $2\pi\mathrm{i}\rho_1, \dots, 2\pi\mathrm{i}\rho_M$.

Assuming that α is analytic in the lower half plane, we find by Cauchy's residue theorem

$$\overline{\eta(t)} = -\sum_{i=1}^{M} \alpha(\omega_i)\rho_i.$$

Hence with $\alpha_i(t) = \alpha(t, \omega_i)$, we have found a close relation for $(\alpha_i(t))_{i=1}^M$ as

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_i + \mathrm{i}\omega_i\alpha_i(t) + \frac{K}{2}\left(\sum_{i=1}^{M} \alpha_i(t)\rho_i - \overline{\alpha_i(t)\rho_i}\,\alpha_i^2(t)\right) = 0 \quad \text{for } i = 1, \dots, M. \tag{1.13}$$

For the Cauchy distribution proportional to $(1+\omega^2)^{-1}$ and the rational distribution proportional to $(1+\omega^4)^{-1}$, Ott and Antonsen [121, 122] and Ott, Hunt and Antonsen [123] used this reduction to deduce the stability of the partially locked states with all locked oscillators at the stable fixed point. In fact, as we will see in Section 5.2, imposing for the partially locked state that α is analytic in the lower half plane is equivalent to imposing that all locked oscillators are at the stable fixed point. With this reduction Martens et al. [105] studied the dynamic behaviour for a bi-Cauchy distribution and the dependence on the parameter in detail. In particular, they recover the standing waves, which were found by Crawford [38] by a bifurcation argument around the incoherent state.

Using only the first reduction, Omel'chenko and Wolfrum [120] performed this for a general linear stability analysis of stationary states. In their analysis, they still observe a continuous spectrum along the imaginary axis and only discuss the existence of growing modes.

1.4 Results and perspective

The Landau damping of a plasma implies the orbital stability of the state. This orbital stability can be proven by the construction of suitable Lyapunov functions. In the Vlasov–Poisson equation, an early example is the stability for spatially homogeneous distributions with isotropic and monotone velocity dependence [73, 104]. This approach has been particularly successful in the gravitational case of the Vlasov–Poisson equation, where we refer to the nice review by Rein [136] and the very recent work by Lemou [89], Lemou, Luz and Méhats [90] and Lemou, Méhats and Raphaël [91].

For a perturbative approach, the linear analysis is done with norms capturing the weak decay by phase mixing. The understanding is that the solution with regular initial data is relaxing towards the stationary state after testing against regular test functions, where the speed depends on the assumed regularity. In particular, the order parameter and the electric field modes are obtained by integrating against an analytic function and thus show the decay. As this controls the nonlinearity, we can hope that the interactions decay over time.

Very briefly, this is the result Mouhot and Villani [114] were able to show. The regularity is measured by factoring out the free transport, i.e. they considered the problem in the so-called *gliding frame*

$$h(t,x,v) = f(t,x-tv,v).$$

In this frame, they showed that the solution converges and that the norms stay bounded. Going back to the laboratory frame, this then shows that the solution f relaxes weakly towards the homogeneous configuration.

On the linear level, this means that the generator L has a continuous spectrum along the imaginary axis, however, taking ϕ and ψ as sufficiently nice functions, the function

$$\lambda \to \langle (L-\lambda)^{-1}\phi, \psi\rangle$$

has an analytic continuation beyond the imaginary axis. Exactly this was used by Landau [87] and later in the formulation with test functions by Degond [41].

Already in 1955, Van Kampen [153] draws the analogy to the quantum scattering problem, where such continuations are also considered. In this context the resulting poles are called *resonances* and, physically, these states are considered meta-stable with a slow decay rate (Fermi's golden rule) [108, 134]. This correspondence might seem natural as the Schrödinger equation, like the Vlasov–Poisson equation, is conservative and time-reversible. In 1989, Crawford and Hislop [39, 72] also used the technique of *spectral deformation* developed in quantum mechanics to understand Landau damping in the Vlasov–Poisson equation on a linear level.

The relation to quantum scattering was also used in 1989 by Caglioti and Maffei [26] in order to show the existence of states which exhibit Landau damping for the Vlasov–Poisson equation. This was later revisited by Hwang and Velázquez [74].

In the 1950s and 1960s, the school around Gel'fand [55] developed a mathematical theory of *rigged Hilbert spaces*, where operators are applied to a dense subset $\mathcal{C}$ and considered as mapping to $\mathcal{C}'$, i.e. tested against elements of $\mathcal{C}$. For the evolution problems, such a topological view, creates an arrow of time, which is documented for the Schrödinger equation [17].

This viewpoint was taken by Chiba [31] and Chiba and Nishikawa [33] in order to understand the stability and the bifurcation of the incoherent state in the Kuramoto equation. In the study of the linear Vlasov–Poisson equation, this was implicitly done by Degond [41].

The basic linear behaviour was well-understood for the Vlasov–Poisson equation around the homogeneous case, but for the nonlinear analysis [13, 114] the results needed to be quantified in the functional setting. Following the general framework of Volterra equations, I show in Chapter 3 how such estimates on the decay of the order parameter can be obtained in a robust way. In particular, we can also handle algebraic decay easily, simplifying for example this linear part in the treatment of the Vlasov-HMF model by Faou and Rousset [52].

For the Kuramoto equation, Chiba [31][2] first studied the damping around the incoherent state and its bifurcation through a center-manifold reduction. In his study, he only considered

[2]Note that a preprint already appeared in 2010 on the arXiv `1008.0249`.

a Gaussian and rational velocity distribution and constructed complicated functional spaces specialised to the velocity distribution. In the autumn of 2014, the preprints by Fernandez, Gérard-Varet and Giacomin [53] (arXiv `1410.6066`, 22 Oct 2014), myself [46] (arXiv `1411.3752`, 13 Nov 2014) and Benedetto, Caglioti and Montemagno [14] (arXiv `1412.1923`, 5 Dec 2014) appeared on the arXiv and study the behaviour of the incoherent state. Benedetto, Caglioti and Montemagno [14] showed exponential damping for analytic perturbations for a small enough coupling constant. Fernandez, Gérard-Varet and Giacomin [53] linked the linear stability to the nonlinear stability under perturbations of Sobolev regularity with algebraic damping, following Faou and Rousset [52]. My work is presented in Chapter 4 and I noticed that the spatial modes decouple so that more adapted norms can be used. This allowed me to prove a global stability result, nonlinear stability from linear stability under Sobolev and analytic regularity with the appropriate rate and a center-manifold reduction to show the bifurcation behaviour.

Together with Bastien Fernandez and David Gérard-Varet, we adapted the norms to investigate the stability of inhomogeneous states. We achieved to find an appropriate linear stability criterion and to show the nonlinear stability, which is presented in Chapter 4 and [45].

Despite this progress, many questions about Landau damping remain unanswered. Staying with the basic Kuramoto equation, a question following naturally is the study of standing waves, which are time-periodic solutions.

Considering other models with Landau damping, our result on the stability of an inhomogeneous state through Landau damping is, to the best of my knowledge, the first such result. Therefore, it would be very interesting for future work to extend this result to other models. A first step could be the study of modifications of the Kuramoto equation, e.g. with inertia or with higher harmonic coupling, and the study of the Vlasov-HMF equation. A big goal in this direction is the understanding of more realistic plasmas with a confinement.

A related question would be whether the stability result of inhomogeneous states can be shown under Sobolev regularity with algebraic decay. This could be in particular of interest as some heuristic arguments [10] suggest that the decay is generically only algebraic around inhomogeneous data.

As a very small step towards different geometries, Chapter 6 contains a linear study for the Vlasov–Poisson equation in a slightly more complex space geometry S^3. This considered space is the three dimensional sphere S^3, which could in the long run be of interest for early universe models explaining the cosmic microwave background (CMB) [163].

Another profound open question is the validity and derivation of the Vlasov–Poisson equation. Here the hope could be to show the validity around stable states. Along this line, Section 4.10 contains some basic result for the much easier Kuramoto equation. Physically, one would also like to replace the interaction with the full Maxwell equation, where many questions remain open.

Finally, we remark that another appearance of such damping is the 2d Euler equation, where it is called inviscid damping. As a starting point, we refer to the recent work by Bedrossian and Masmoudi [12], who were inspired by Mouhot and Villani [114] and managed to show nonlinear damping around a shear flow.

2 Mean-field limit

For a Lipschitz continuous interaction, the mean-field limit was rigorously understood as a stability result by Braun and Hepp [23], Dobrushin [47] and Neunzert [117]. Later, Spohn [144, 145] clarified the relation to the BBGKY hierarchy, which relates to the propagation of chaos, see also [63]. The results are well-reviewed in the physics literature [93, 143], as well as in the mathematical literature [61, 62].

This applies to the Kuramoto equation (1.10) and we review the basic setup mostly following Neunzert [117] in order to give a precise uniqueness and existence statement. Moreover, it shows easily the propagation of regularity. Finally, it shows how the stability results can be applied to finite particle systems over finite time ranges.

The application of this theory in the context of the Kuramoto equation was first reported by Lancellotti [86] and reviewed in [28]. A similar result using moments was proven by Chiba [32].

The key idea is to note that the empirical measure

$$\mu^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{P_i}$$

of a particle distribution satisfies the Vlasov equation

$$\partial_t \mu(t,P) + \nabla_P \cdot (A[\mu(t,\cdot)](P)\mu(t,P)) = 0 \tag{2.1}$$

if the particles $1,\dots,N$ have positions P_i in Γ and evolve as

$$\frac{\mathrm{d}}{\mathrm{d}t}P_i(t) = A[\mu^N(t,\cdot)](P_i(t)),$$

where $A[\mu^N(t,\cdot)]$ is the vector field created by the current configuration.

For an interaction $F(P,P')$, we define the vector field by

$$A[\mu](P) = \int_\Gamma F(P,P')\mathrm{d}\mu,$$

which becomes for the empirical measure

$$A[\mu^N](P) = \frac{1}{N}\sum_{i=1}^{N}F(P,P')$$

exactly modelling a binary interaction.

In the Kuramoto equation with phase space $\Gamma = \mathbb{T} \times \mathbb{R}$, the vector field takes the form

$$A[\mu](\theta,\omega) = \begin{pmatrix} \omega + \frac{K\mathrm{i}}{2}\left(\eta \mathrm{e}^{\mathrm{i}\theta} - \overline{\eta}\mathrm{e}^{-\mathrm{i}\theta}\right) \\ 0 \end{pmatrix}$$

for a point $P = (\theta, \omega)$ and

$$\eta = \int_{(\theta,\omega)\in\Gamma} \mathrm{e}^{\mathrm{i}\theta} \mathrm{d}\mu(\theta,\omega).$$

The mean-field limit can now be understood as the stability problem for the Vlasov equation. We assume that, in the limit $N \to \infty$, the initial configuration μ^N converges weakly to a limiting distribution μ. The mean-field limit then claims that, at later times t, the limit of $\mu^N(t)$ is the solution of the Vlasov equation with initial data μ. Formulated differently, it means that instead of considering a particle system we can consider a similar smooth distribution, because at later times the evolved particle system is still similar to the evolved smooth distribution.

In this sense, we do not distinguish between a probability measure μ on Γ and a (distributional) density f on Γ, which corresponds to a probability measure. As we always work with Polish spaces, both notions are equivalent by Riesz representation theorem and are used interchangeably. Furthermore, we denote the set of probability measures as $\mathcal{M}$ or $\mathcal{M}(\Gamma)$.

For establishing the stability, we need a suitable distance between probability measures, which is sufficiently fine to include weak convergence so that the empirical measure μ^N can converge to a smooth measure μ.

A suitable choice is the Wasserstein distance [4, 160] used by Dobrushin [47], which is defined between two measure μ and ν over Γ as minimal transportation cost

$$\mathcal{W}_1(\mu,\nu) = \inf_{\pi \in N(\mu,\nu)} \int_{\Gamma\times\Gamma} \|x-y\| \mathrm{d}\pi(x,y),$$

where $N(\mu,\nu)$ are all probability measures π over $\Gamma \times \Gamma$ with first marginal μ and second marginal ν. If Γ is compact or with suitable moment conditions, it can be shown that $\mathcal{W}_1$ induces the weak topology. By the Kantorovich duality, an equivalent formulation used by Neunzert [117] is as dual Lipschitz metric

$$d_{DL}(\mu,\nu) = \sup_{\phi \in \mathrm{Lip}(\Gamma)} \left(\int_{x\in\Gamma} \phi(x)\mathrm{d}\mu(x) - \int_{x\in\Gamma} \phi(x)\mathrm{d}\nu(x) \right),$$

where $\mathrm{Lip}(\Gamma)$ are all Lipschitz functions on Γ with Lipschitz constant at most 1. In fact the duality shows $d_{DL} = \mathcal{W}_1$.

In order to avoid moment conditions in the case of noncompact Γ, Dobrushin and Neunzert changed the distance by changing the distance on Γ to $\min(1, \|x-y\|)$ between $x, y \in \Gamma$ and by restricting the range of the Lipschitz functions to $[0, 1]$, respectively. By the Kantorovich duality, both formulations are again equivalent and we arrive at the distance

$$
\begin{aligned}
d(\mu,\nu) &= \inf_{\pi\in N(\mu,\nu)} \int_{\Gamma\times\Gamma} \min\big(1, \|x-y\|\big)\,\mathrm{d}\pi(x,y) \\
&= \sup_{\phi:\Gamma\mapsto[0,1],\phi\in\mathrm{Lip}(\Gamma)} \left(\int_{x\in\Gamma} \phi(x)\,\mathrm{d}\mu(x) - \int_{x\in\Gamma} \phi(x)\,\mathrm{d}\nu(x)\right).
\end{aligned}
$$

On the vector field A we impose the existence of finite constants C_d and C_Γ such that for all points $P, Q \in \Gamma$ and probability measures μ and ν over Γ it holds that

$$
\|A[\mu](P) - A[\nu](P)\| \leq C_d\, d(\mu,\nu)
$$

and

$$
\|A[\mu](P) - A[\mu](Q)\| \leq C_\Gamma\, \|P-Q\|.
$$

We note that for bounded Lipschitz interaction kernels F, we can always find such constants and over compact Γ without the modification of the Wasserstein distance, we can take the Lipschitz constant of the interaction kernel as C_d and C_Γ.

For the solutions, we assume without loss of generality that the initial data are given at time $t = 0$ and we solve for $\mathbb{R}^+$. By the assumptions on the vector field A, we have a unique global solution for every particle system and the empirical measure $\mu(t,\cdot)$ for the solution at time t is in the following solution space.

Definition 2.1. *Let $C_\mathcal{M}$ be the solution space $C_w(\mathbb{R}^+, \mathcal{M}(\Gamma))$, which consists of the families of weakly continuous probability measures on Γ, i.e. $f \in C_\mathcal{M}$ is a family $\{f(t,\cdot) \in \mathcal{M}(\mathbb{R}) : t \in \mathbb{R}^+\}$ such that for every $h \in C_b(\Gamma)$ the function $t \mapsto \int_\Gamma h(P)\mathrm{d}f(t,P)$ is continuous.*

We call $f \in C_\mathcal{M}$ a solution to the Vlasov equation (2.1) *with initial data $f_{\mathrm{in}}(\cdot) = f(0,\cdot)$ if for all test functions $h \in C_0^1(\mathbb{R}^+ \times \Gamma)$ it holds that*

$$
\int_\Gamma h(0,P)\mathrm{d}f_{\mathrm{in}}(P) + \int_{\mathbb{R}^+}\int_\Gamma \left[V[f(t,\cdot)](P)\cdot\nabla_P h(t,P) + \partial_t h(t,P)\right]\mathrm{d}f(t,P)\,\mathrm{d}t = 0.
$$

For such a solution $f \in C_\mathcal{M}$, the resulting vector field $A[f(t,\cdot)](P)$ is Lipschitz continuous with respect to P with the global constant C_Γ and is continuous in time. Therefore, the trajectories $T_{t,s}[f] : \Gamma \mapsto \Gamma$ can be globally defined as $T_{t,s}[f](Q) = P(t)$, where $P : \mathbb{R}^+ \mapsto \Gamma$ is the solution to the initial value problem

$$
\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}t}P(t) = A[f(t,\cdot)](P(t)), \\
P(s) = Q.
\end{cases}
$$

In the particle model $t \to T_{t,s}[f^N](P_i(s))$ is the position of the i-th particle at time t given its position $P_i(s)$ at time s. The maps $T_{t,s}(P) : \Gamma \mapsto \Gamma$ are invertible with inverse $T_{s,t}$.

By standard arguments for the scalar transport equation, the following lemma holds.

Lemma 2.2. *Let $f \in \mathcal{M}(\Gamma)$. Then $f \in C_{\mathcal{M}}$ is a solution to the Vlasov equation with initial data $f_{\mathrm{in}}(\cdot) = f(0, \cdot)$ if and only if $f(t, \cdot)$ is the push-forward of the initial data along the trajectories, i.e. $f(t, \cdot) = (T_{t,0}[f])_*(f_{\mathrm{in}})$, which is defined by*

$$\int_A \mathrm{d}f(t, P) = \int_{(T_{t,0}[\rho.])^{-1}(A)} \mathrm{d}f_{\mathrm{in}}(P)$$

for every Borel set A of Γ or equivalently by

$$\int_\Gamma g(P)\mathrm{d}f(t, P) = \int_\Gamma g(T_{t,0}(P))\, \mathrm{d}f_{\mathrm{in}}(P)$$

for every measurable function g on Γ.

The stability can be understood by the following estimate, which is often called Dobrushin estimate.

Lemma 2.3. *Let $f, g \in C_{\mathcal{M}}$ be two solutions with initial data f_{in} and g_{in}. Then for $t \in \mathbb{R}^+$ holds the estimate*

$$d(f(t, \cdot), g(t, \cdot)) \leq e^{(C_d + C_\Gamma)t} d(f_{\mathrm{in}}, g_{\mathrm{in}}).$$

Proof. We work with the Wasserstein formulation of the distance, but note that the proof can be written equivalently using the dual Lipschitz formulation.

For the initial data there exists an optimal coupling $\pi_{\mathrm{in}} \in N(f_{\mathrm{in}}, g_{\mathrm{in}})$, so that

$$d(f_{\mathrm{in}}, g_{\mathrm{in}}) = \int_{\Gamma\times\Gamma} \min\left(1, \|x - y\|\right) \mathrm{d}\pi_{\mathrm{in}}(x, y),$$

see e.g. [4, 160]. Lemma 2.2 shows that, by the trajectories $T_{t,0}[f]$ and $T_{t,0}[g]$, we can define a coupling $\pi_t = (T_{t,0}[f], T_{t,0}[g])_* \pi_{\mathrm{in}}$ between $f(t, \cdot)$ and $g(t, \cdot)$.

This coupling gives the following bound on the distance

$$\begin{aligned} d(f(t, \cdot), g(t, \cdot)) &\leq \int_{\Gamma\times\Gamma} \min\left(1, \|x - y\|\right) \mathrm{d}\pi_t(x, y) \\ &= \int_{\Gamma\times\Gamma} \min\left(1, \|T_{t,0}[f](x) - T_{t,0}[g](y)\|\right) \mathrm{d}\pi_{\mathrm{in}}(x, y) \\ &\leq \int_{\Gamma\times\Gamma} \min\left(1, \|T_{t,0}[f](x) - T_{t,0}[f](y)\|\right) \mathrm{d}\pi_{\mathrm{in}}(x, y) \\ &\quad + \int_{\Gamma\times\Gamma} \min\left(1, \|T_{t,0}[f](y) - T_{t,0}[g](y)\|\right) \mathrm{d}\pi_{\mathrm{in}}(x, y). \end{aligned}$$

By the definition of the trajectories and the Lipschitz continuity of the velocity field, we have the estimate

$$\|T_{t,0}[f](x) - T_{t,0}[f](y)\| \leq \mathrm{e}^{tC_\Gamma} |x - y|$$

so that the first term can be bounded as

$$\int_{\Gamma\times\Gamma} \min\big(1, \|T_{t,0}[f](x) - T_{t,0}[f](y)\|\big)\,\mathrm{d}\pi_{\mathrm{in}}(x,y) \leq \mathrm{e}^{tC_\Gamma} d(f_{\mathrm{in}}, g_{\mathrm{in}}).$$

For the second term, we fix y and look at

$$\delta_y(t) = \|T_{t,0}[f](y) - T_{t,0}[g](y)\|.$$

By definition of the trajectories we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\delta_y(t) &\leq \|A[f(t,\cdot)](T_{t,0}[f](y)) - A[g(t,\cdot)](T_{t,0}[g](y))\|\\ &\leq \|A[f(t,\cdot)](T_{t,0}[f](y)) - A[f(t,\cdot)](T_{t,0}[g](y))\|\\ &\quad + \|A[f(t,\cdot)](T_{t,0}[g](y)) - A[g(t,\cdot)](T_{t,0}[g](y))\|\\ &\leq C_\Gamma\,\delta_y(t) + C_d\,d(f(t,\cdot), g(t,\cdot)).\end{aligned}$$

Hence

$$\delta_y(t) \leq \int_0^t \mathrm{e}^{C_\Gamma(t-s)} C_d\,d(f(s,\cdot), g(s,\cdot))\,\mathrm{d}s.$$

Overall, we therefore find for the distance

$$d(f(t,\cdot), g(t,\cdot)) \leq \mathrm{e}^{tC_\Gamma} d(f_{\mathrm{in}}, g_{\mathrm{in}}) + \int_0^t \mathrm{e}^{C_\Gamma(t-s)} C_d\,d(f(s,\cdot), g(s,\cdot))\,\mathrm{d}s.$$

By Gronwall, this implies the claimed bound. □

This shows the claimed well-posedness.

Theorem 2.4. *The Vlasov equation* (2.1) *is well-posed in $C_{\mathcal{M}}$. In particular, for every $f_{\mathrm{in}} \in \mathcal{M}$ exists a unique solution $f \in C_{\mathcal{M}}$ with initial data f_{in}.*

Proof. The previous estimate shows uniqueness. For existence, we can use a Picard-Lindelöf type argument with the mapping $T : C_{\mathcal{M}} \mapsto C_{\mathcal{M}}$ defined by

$$(Tf)(t,\cdot) = (T_{t,0}[f])_* f_{\mathrm{in}},$$

which can be shown to be a contraction by the previous estimate.

Alternatively, the initial data can be approximated by empirical measures. For these initial data, we can define a solution and these solutions have a cluster point by compactness arguments. Such a cluster point is a solution. □

As indicated in the beginning, this proves the mean-field limit if the initial empirical measure μ^N converges weakly to f_{in}. By the basic sampling theorem from statistics, this holds for example if the initial configurations are drawn independently from the distribution f_{in}.

In the case of the Kuramoto equation, the vector field A is smooth with uniformly bounded derivatives. Hence the trajectories $T_{t,s}$ are also smooth as a function of Γ. Thus Lemma 2.2 shows propagation of regularity.

Lemma 2.5. *Let $f \in C_{\mathcal{M}}$ be the solution to the Kuramoto equation with initial data f_{in}. If f_{in} has a density g_{in}, then for every $t \in \mathbb{R}^+$ the measure $f(t,\cdot)$ has a density $g(t,\cdot)$ given by the flow*

$$g(t,P) = g_{\text{in}}(t, T_{0,t}[f](P)) \operatorname{div}(T_{0,t}[f]).$$

For a Sobolev norm H^s with $s \in \mathbb{N}$, suppose that $\|f_{\text{in}}\|_{H^s} < \infty$, then for every compact $\Delta \subset \mathbb{R}^+$

$$\sup_{t\in\Delta} \|f(t,\cdot)\|_{H^s} < \infty$$

and $\|f(t,\cdot)\|_{H^s}$ is continuous in time.

Proof. Use the solution formula from Definition 2.1 and the smoothness of the trajectories. $\square$

Similarly, it directly shows that all solutions leave the velocity marginal invariant.

Lemma 2.6. *Let $f \in C_{\mathcal{M}}$ be a solution to the Kuramoto equation and $g \in \mathcal{M}(\mathbb{R})$ be the velocity marginal of the initial datum f_{in}. Then g is for all $t \in \mathbb{R}^+$ the velocity marginal of $f(t,\cdot)$.*

Proof. By Lemma 2.2, the solution is $f(t,\cdot) = (T_{t,0}[f])_*(f_{\text{in}})$ and the trajectories leave the velocity marginals invariant. $\square$

3 Volterra integral equations for linear stability studies

3.1 Duhamel reduction

The success of Landau's approach in solving the linearised Vlasov–Poisson equation crucially depends on the ability to close the evolution on the decoupled density modes. In the Kuramoto equation, we can similarly find a closed equation for the order parameter under the linear evolution. In both cases the evolution over time can be expressed as Volterra equation, which can be derived in a general framework by the Duhamel principle.

For the general framework, let $y(t)$ be the state of the system at time t. For example, in the case of the Vlasov–Poisson equation we study each spatial mode separately and for one mode l we would take $y(t) = \tilde{f}_l(t, \cdot)$. In the Kuramoto equation we take y to be the first spatial mode $\tilde{f}_1$ for the study around the incoherent state and the full density for the study around partially locked states.

In the studied mean-field limits, the system y evolves according to a first-order transport equation whose coefficients depend through a *generalised macroscopic density* η on the current state y. We assume that η depends linearly on y and our aim is to close the evolution for η. In the example of the Vlasov–Poisson equation, this are the density modes and for the Kuramoto equation the dynamic is reduced to the order parameter.

Considering y as perturbation of a stationary state, the linearised evolution is given by the linear generator L as

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = Ly.$$

The operator L can be split in two parts $L = L_1 + L_2$, where L_1 corresponds to the evolution of y due to the transport of particles as in the unperturbed system and L_2 corresponds to the deviation of the unperturbed particle profile due to the field created by the perturbation. The effect of the field created by the perturbation on the evolution of the perturbation is of quadratic order and therefore ignored.

Here L_1 is the part creating the mixing and the weak decay and L_2 takes the effect of the interaction into account, which typically works against damping.

As the coefficients of the transport equation only depend on y through the generalised macroscopic density η, we have that $L_2 y$ only depends on $\eta[y]$, i.e. there exists $\bar{L}_2$ such that

$$L_2 y = \bar{L}_2 \eta[y].$$

By Duhamel's principle the solution can be expressed as

$$y(t) = \mathrm{e}^{tL_1} y_{\mathrm{in}} + \int_0^t \mathrm{e}^{(t-s)L_1} L_2 y(s) \mathrm{d}s,$$

where $y_{\mathrm{in}} = y(0)$ are the initial data. This implies for the order parameter $\eta(t) = \eta[y(t)]$

$$\eta(t) = \eta\left[\mathrm{e}^{tL_1} y_{\mathrm{in}}\right] + \int_0^t \eta\left[\mathrm{e}^{(t-s)L_1} \bar{L}_2 y(s)\right] \mathrm{d}s,$$

which is the claimed Volterra equation. Let X be the space for the generalised macroscopic density at a given time. Then the Volterra equation states that $\eta : \mathbb{R}^+ \mapsto X$ satisfies

$$\eta(t) + (k * \eta)(t) = F(t) \qquad \text{for } t \in \mathbb{R}^+,$$

where $F : \mathbb{R}^+ \mapsto X$ is the forcing given by

$$F(t) = \eta\left[\mathrm{e}^{tL_1} y_{\mathrm{in}}\right]$$

and $k * \eta$ denotes the convolution over $\mathbb{R}^+$, i.e.

$$(k * \eta)(t) = \int_0^t k(t-s)\eta(s)\mathrm{d}s,$$

and $k : \mathbb{R}^+ \mapsto M(X, X)$ is the kernel with $M(X, X)$ being the space of linear maps from X to itself and is given by

$$k(t)\nu = -\eta\left[\mathrm{e}^{tL_1} \bar{L}_2 \nu\right] \qquad \text{for } \nu \in X.$$

For a scalar valued generalised macroscopic density η, i.e. $X = \mathbb{C}$, the kernel is just a factor, i.e. $k : \mathbb{R}^+ \mapsto \mathbb{C}$. For the more general case $X = \mathbb{C}^d$, we can express k as matrix valued function, i.e. $k : \mathbb{R}^+ \mapsto \mathbb{C}^{d \times d}$.

Under the Laplace transform in time, this reduction yields an algebraic equation, as the convolution integral becomes a multiplication. As discussed in Section 1.2, this reduction was used and solved by Landau for the Vlasov–Poisson equation.

For the Vlasov–Poisson equation around a spatially homogeneous state, the Volterra equation for the density $\tilde{\rho}_l$ for a spatial mode l has the forcing

$$F(t) = \int_{\mathbb{R}^d} (\tilde{f}_{\mathrm{in}})_l(v) \mathrm{e}^{-\mathrm{i}l \cdot vt} \mathrm{d}v = (\hat{f}_{\mathrm{in}})_l(lt)$$

with the Fourier transform $\hat{f}_{\text{in}}$ in v as

$$(\hat{f}_{\text{in}})_l(\xi) = \int_{\mathbb{R}^d} (\tilde{f}_{\text{in}})_l(v) \mathrm{e}^{-\mathrm{i}\xi\cdot v} \mathrm{d}v.$$

Assuming that the interaction $k * \eta$ can be controlled, this shows that the density mode has the same decay and regularity as $\hat{f}_{\text{in}}$. Indeed, the remaining chapter shows that the interaction can be controlled apart from possible eigenmodes. The same discussion also holds for the Kuramoto equation, cf. Sections 4.3 and 5.4.

In the case of the Vlasov–Poisson equation around a spatially homogeneous configuration, this viewpoint was already suggested by Penrose [128] in 1960. In this setting, we study each mode separately and the density is a scalar. We remark that some authors also formulate the linear analysis in terms of the electric potential or electric field. As the modes only differ by constant factors, this is equivalent. For perturbations around general configurations, Maslov and Fedoryuk [106] noted that the full electric field satisfies such a Volterra equation.

In the Kuramoto equation, this reduction was noted in the first paper of Landau damping around the incoherent state by Strogatz, Mirollo and Matthews [149] in 1992. Around general stationary states, this operator splitting has been used by Mirollo and Strogatz [109] and Omel'chenko and Wolfrum [120] on the level of the resolvent.

3.2 Finite-dimensional Volterra equations

For the Vlasov–Poisson equation, Penrose [128] noted that the general theory of scalar Volterra equations can be applied to give a rigorous proof of linear stability. He refers to the basic Paley–Wiener theorem [124][1]. We will present in this section the core results and note that the results extend to finite-dimensional Volterra equations. We will present the results in this framework following the very nice presentation of the book by Gripenberg, Londen and Staffans [65].

Thus we study the equation

$$y(t) + (k * y)(t) = F(t) \qquad \text{for } t \in \mathbb{R}^+ \tag{3.1}$$

for $y : \mathbb{R}^+ \mapsto \mathbb{C}^d$, where $F : \mathbb{R}^+ \mapsto \mathbb{C}^d$ is the forcing and $k : \mathbb{R}^+ \mapsto \mathbb{C}^{d\times d}$ is the matrix-valued kernel. Throughout we will assume that d is finite.

The convolution is a very well-behaved associative operation. Its size can be controlled by the Young inequality.

Lemma 3.1. *Let $X = \mathbb{R}$ or $X = \mathbb{R}^+$. If $a \in L^1(X)$ and $b \in L^p(X)$ for $p \in [1, \infty]$, then $a * b \in L^p(X)$ with*

$$\|a * b\|_{L^p(X)} \le \|a\|_{L^1(X)} \|b\|_{L^p(X)}.$$

[1] This Paley–Wiener theorem is unrelated to the more famous Paley–Wiener in Fourier analysis.

The proof is a direct application of the Cauchy-Schwarz inequality, cf. Theorem 2.2 of Chapter 2 of [65], which also contains more cases.

For the solution, we define the resolvent r which plays a similar role as that of a fundamental solution. Stating Theorem 3.1 of Chapter 2 of [65], we have:

Lemma 3.2. *Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{C}^{d\times d})$. Then*

$$r + k * r = r + r * k = k$$

has a unique solution in $L^1_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{C}^{d\times d})$. The solution r is called the resolvent *of k and depends continuously on k in $L^1_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{C}^{d\times d})$.*

Remark 3.3. In the case $d = 1$, where r and k are scalar valued, we have $r * k = k * r$. In general this, however, does not hold for matrix value r and k.

Proof sketch. Assuming two solutions r and $\bar{r}$, we have by associativity that

$$r = k - k * r = k - (\bar{r} + \bar{r} * k) * r = k - \bar{r} * r - \bar{r} * (k * r) = k - \bar{r} * r - \bar{r} * (k - r) = \bar{r},$$

which shows the uniqueness.

By the uniqueness, it suffices to construct a solution over $[0, T]$ for an arbitrary $T \in \mathbb{R}^+$. Moreover, note that the relation for r and k holds if and only if it holds for r^a and k^a, where

$$r^a(t) = \mathrm{e}^{-at} r(t) \text{ and } k^a(t) = \mathrm{e}^{-at} k(t).$$

For suitable large a, we can assume $\|k^a\|_{L^1([0,T])} < 1$ and then a solution is

$$r^a(t) = \sum_{j=1}^{\infty} (-1)^{j-1} (k^a)^{*j}. \qquad \square$$

The resolvent can be used to solve the Volterra equation. For this we quote Theorem 3.5 of Chapter 2 of [65]:

Lemma 3.4. *Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{C}^{d\times d})$. For any $F \in L^1_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{C}^{d\times d})$, the equation* (3.1) *has a unique solution $y \in L^1_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{C}^{d\times d})$, which is given by*

$$y(t) = F(t) - (r * F)(t).$$

Proof sketch. If y is a solution, then

$$F = y + k * y = y + (r + r * k) * y = y + r * y + r * (F - y) = y + r * F$$

showing the claimed form of the solution. Using the relation between r and k, we can directly verify that the given formula is indeed a solution. $\square$

Taking the Laplace transform of (3.1), we find the relation

$$(\mathcal{L}y)(z)\,[1 + (\mathcal{L}k)(z)] = (\mathcal{L}F)(z).$$

This suggests that we have eigenmodes at $1 + (\mathcal{L}k)(z) = 0$, and otherwise we transfer the decay of the forcing F to the solution y.

The growth of a solution of the Volterra equation can be characterised by the integrability. If the resolvent satisfies $r \in L^1(\mathbb{R}^+, \mathbb{C}^{d\times d})$, then the Young inequality (Lemma 3.1) shows that the solution y is controlled by the forcing F as

$$\|y\|_{L^p(\mathbb{R}^+)} \le \left(1 + \|r\|_{L^1(\mathbb{R}^+)}\right) \|F\|_{L^p(\mathbb{R}^+)}$$

for $p \in [1,\infty]$. Indeed if such an inequality holds for $p = 1$ or $p = \infty$, we must have $r \in L^1(\mathbb{R}^+, \mathbb{C}^{d\times d})$.

With the additional condition $k \in L^1(\mathbb{R}^+, \mathbb{C}^{d\times d})$, we can characterise the case $r \in L^1(\mathbb{R}^+, \mathbb{C}^{d\times d})$ precisely (quoting from [65], originally from [124]):

Theorem 3.5 (Half-line Paley–Wiener)**.** *Let $k \in L^1(\mathbb{R}^+, \mathbb{C}^{d\times d})$. Then the resolvent r satisfying $r + k * r = r + r * k = k$ is in $L^1(\mathbb{R}^+, \mathbb{C}^{d\times d})$ if and only if*

$$\det(\mathrm{Id} + \mathcal{L}k(z)) \neq 0 \qquad \textit{for all } \Re z \ge 0.$$

Taking the Laplace transform of the relation defining the resolvent, shows

$$[\mathrm{Id} + \mathcal{L}k(z)]\,(\mathcal{L}r)(z) = (\mathcal{L}r)(z)\,[\mathrm{Id} + \mathcal{L}k(z)] = (\mathcal{L}k)(z).$$

This shows that the condition $\det(\mathrm{Id} + \mathcal{L}k(z)) \neq 0$ is necessary.

Assuming the condition $\det(\mathrm{Id} + \mathcal{L}k(z)) \neq 0$, the idea is to use the inverse Laplace transform along $\Re z = 0$. Along this line, we can interpret this as Fourier transform. For this we quote Theorem 4.3 of Chapter 2 of [65], where the convolution is taken over $\mathbb{R}$ and $\mathcal{F}$ denotes the Fourier transform.

Theorem 3.6 (Whole-line Paley–Wiener)**.** *Let $k \in L^1(\mathbb{R}, \mathbb{C}^{d\times d})$. Then with the convolution over $\mathbb{R}$ the equation*

$$r + k * r = r + r * k = k$$

has a solution $r \in L^1(\mathbb{R}, \mathbb{C}^{d\times d})$ if and only if

$$\det(\mathrm{Id} + \mathcal{F}k(x)) \neq 0 \qquad \textit{for } x \in \mathbb{R}.$$

The idea of the proof is to split $k \in L^1(\mathbb{R}, \mathbb{C}^{d\times d})$ into small pieces $(k_n)_n$, where we can find a solution $(r_n)_n$ by the series expression.

Proof sketch of Theorem 3.5 using Theorem 3.6. Given the kernel $k \in L^1(\mathbb{R}^+)$, extend it as

$\tilde{k}$ to $\mathbb{R}$ by setting $\tilde{k}(x) = 0$ for $x < 0$. Then $\tilde{k} \in L^1(\mathbb{R})$ and $(\mathcal{F}\tilde{k})(x) = (\mathcal{L}k)(\mathrm{i}x)$ so that $\tilde{k}$ satisfies the assumption of Theorem 3.6. Hence there exists $\tilde{r} \in L^1(\mathbb{R})$ solving

$$\tilde{r} + \tilde{k} * \tilde{r} = \tilde{r} + \tilde{r} * \tilde{k} = \tilde{k}$$

over $\mathbb{R}$. If $\tilde{r}$ is vanishing over $(-\infty, 0)$, then the convolution integrals over $\mathbb{R}$ reduce to the convolution integrals over $\mathbb{R}^+$ and the restriction $r = \tilde{r}|_{\mathbb{R}^+}$ is the sought solution.

Imposing that $\tilde{r}$ vanishes for $(-\infty, 0)$ is exactly imposing that $1 + \mathcal{L}k(a) \neq 0$ for $\Re a > 0$. Indeed the Fourier transform of $\tilde{r}$ is given by

$$(\mathcal{F}\tilde{r})(\omega) = \frac{\mathcal{F}\tilde{k}(\omega)}{1 + \mathcal{F}\tilde{k}(\omega)} = \frac{\mathcal{L}k(-\mathrm{i}\omega)}{1 + \mathcal{L}k(-\mathrm{i}\omega)}.$$

Hence the condition on the half plane is equivalent to the condition that $\mathcal{F}\tilde{r}$ is bounded and integrable in the lower half plane. By a Paley–Wiener theorem of Fourier analysis [137, Theorem 19.2], this holds if and only if $\tilde{r}$ vanishes on $(-\infty, 0)$.

For a direct prove, split $\tilde{r}$ as $\tilde{r} = \tilde{r}_- + \tilde{r}_+$ where $\tilde{r}_-(x) = \tilde{r}_+(-x) = 0$ for $x > 0$. Then

$$(\mathcal{F}\tilde{r}_-)(\omega) = \frac{\mathcal{L}k(-\mathrm{i}\omega)}{1 + \mathcal{L}k(-\mathrm{i}\omega)} - (\mathcal{F}\tilde{r}_+)(\omega).$$

By the assumption the RHS is a bounded and analytic function of ω in the lower half plane, while the LHS is a bounded analytic function on the upper half plane. Hence both sides can be extended as bounded entire function and thus must be zero. $\square$

In general, we can precisely identify the modes for which the condition fails.

Theorem 3.7. *Let $k \in L^1(\mathbb{R}^+)$ and suppose*

$$\det(\mathrm{Id} + \mathcal{L}k(\mathrm{i}x)) \neq 0 \qquad \text{for } x \in \mathbb{R}.$$

Then the solution to the Volterra equation (3.1) *is given by*

$$y(t) = F(t) - \int_0^\infty q(t-s)F(s)\,\mathrm{d}s$$

with

$$q(t) = r_s(t) + \sum_{i=1}^{n} \sum_{j=0}^{p_i - 1} b_{i,j} t^j \mathrm{e}^{\lambda_i t},$$

where q is vanishing for negative arguments and r_s satisfies $\|r_s\|_{L^1(\mathbb{R})} < \infty$ and $\lambda_1, \dots, \lambda_n$ are the finitely many zeros of $\det(\mathrm{Id} + (\mathcal{L}k)(z))$ in $\Re z > 0$ with multiplicities $p_1, \dots, p_n$ and $b_{i,j}$ depends on the residues of $[\mathrm{Id} + (\mathcal{L}k)(z)]^{-1}$ at λ_i.

In $\Re z \geq 0$ the Laplace transform $\mathcal{L}k$ is analytic and by the Riemann-Lebesgue lemma $\mathcal{L}k(z) \to 0$ as $|z| \to \infty$ for $\Re z \geq 0$. Hence we can only have finitely many zeros. The remaining part follows from the proof of Theorem 3.5 identifying the poles, cf. Theorem 2.1 of Chapter 7 of [65]).

For possible growth, which can be algebraic, if the characteristic equation $\det(\mathrm{Id}+k(z)) = 0$ has solutions on the critical line $\Re z = 0$, we refer to Section 7.3 of [65]. In the homogeneous Vlasov–Poisson case, also the paper by Backus [7] contains some direct computations and he also considered conditions slightly different than $k \in L^1(\mathbb{R})$. However, for our systems this condition is extremely weak compared to the assumptions needed to control the nonlinearity.

There is no general bound for the norm $\|r\|_{L^1(\mathbb{R}^+)}$ of the resolvent by Theorem 3.6. However, with mild extra conditions, the proof is constructive with an explicit bound, which is stated as Theorem 6.1 of Chapter 2.1 in [65].

Theorem 3.8. *Let $k \in L^1(\mathbb{R})$ with*

$$\sup_{\omega\in\mathbb{R}} |(\mathrm{Id} + (\mathcal{F}k)(x))^{-1}| = q < \infty,$$

where $\mathcal{F}k$ is the Fourier transform of k and let T, δ satisfying

$$\int_{|s|\geq T} |k(t)|\,\mathrm{d}t \leq \frac{1}{12q}$$

and

$$\sup_{0<s<\delta} \int_{-\infty}^{\infty} |k(t) - k(t-s)|\,\mathrm{d}t \leq \frac{1}{4}$$

then the resolvent r of k satisfies

$$\|r\|_{L^1(\mathbb{R})} \leq \big(8\lceil 6qT\|k\|_{L^1(\mathbb{R})}\rceil \lceil 8\|k\|_{L^1(\mathbb{R})}/\delta\rceil + 6\big)\, q\|k\|_{L^1(\mathbb{R})}.$$

Here $\lceil a \rceil$ is the smallest integer $\geq a$. For the application in the Vlasov–Poisson equation, we formulate the following direct corollary.

Corollary 3.9. *Given constants $\nu > 0$, $\mu > 0$, C_1, C_L, C_D there exists a constant C_R such that the resolvent $r \in L^1(\mathbb{R}^+)$ satisfies*

$$\|r\|_{L^1(\mathbb{R}^+)} \leq q\, C_R,$$

where

$$q = \sup_{y\in\mathbb{R}} \left| [1 + (\mathcal{L}k)(\mathrm{i}y)]^{-1} \right|,$$

for all kernels $k \in L^1(\mathbb{R}^+)$ satisfying

$$\|k\|_{L^1(\mathbb{R}^+)} \leq C_1, \quad \int_T^{\infty} |k(t)|\,\mathrm{d}t \leq C_L T\, \mathrm{e}^{-\nu T}, \quad \int_0^{\infty} |k(t+s) - k(t)|\,\mathrm{d}t \leq C_D s^{\mu}$$

and

$$(\mathcal{L}k)(z) \neq -1 \qquad \textit{for } \Re z \geq 0.$$

In order to discuss the decay, it is helpful to introduce the weighted spaces $L^p(X, \phi)$ by the norm

$$\|f\|_{L^p(X,\phi)} = \|f\phi\|_{L^p(X)} = \begin{cases} \left(\int_{x\in X} |f(x)\phi(x)|^p \mathrm{d}x\right)^{1/p} & \text{for } p \in [1, \infty), \\ \operatorname{ess\,sup}_{x\in X} |f(x)|\phi(x) & \text{for } p = \infty, \end{cases}$$

where $\phi : X \mapsto \mathbb{R}^+$ is a weight function. In particular, we use the exponential weight $\exp_a : x \to \mathrm{e}^{ax}$. By noting that $\exp_a(t)k(t)$ and $\exp_a(t)r(t)$ satisfy the same relation as r and k, we can shift the results in the complex plane. For example for Theorem 3.5, the result becomes that $r \in L^1(\mathbb{R}^+, \exp_a)$ if and only if $\det(\mathrm{Id} + \mathcal{L}k(z)) \neq 0$ for $\Re z \geq -a$ under $k \in L^1(\mathbb{R}^+, \exp_a)$. Likewise, the Young inequality then shows that the solution y is controlled by

$$\|y\|_{L^p(\mathbb{R}^+,\exp_a)} \leq (1 + \|r\|_{L^1(\mathbb{R}^+,\exp_a)})\|F\|_{L^p(\mathbb{R}^+,\exp_a)},$$

which expresses exponential decay.

For functions with bounds in Sobolev norms, the Fourier transform decays algebraically. For this case, we introduce for $A \geq 0$ and $b \geq 0$ the weight

$$p_{A,b}(x) = (A + x)^b \qquad \text{for } x \geq 0$$

with the short-hand $p_b = p_{1,b}$.

The weight p_b is sub-multiplicative over $\mathbb{R}^+$, i.e. for $x, y \in \mathbb{R}^+$ there holds $p_b(x + y) \leq p_b(x)p_b(y)$. This implies that the Young inequality can be adapted as

$$\|a * b\|_{L^p(\mathbb{R}^+,p_b)} \leq \|a\|_{L^1(\mathbb{R}^+,p_b)}\|b\|_{L^p(\mathbb{R}^+,p_b)}.$$

Moreover, we find its asymptotic rate as

$$\lim_{t\to\infty} \frac{\log(p_b(t))}{t} = 0.$$

This means that the weight grows subexponentially so that the stability depends on the poles in the region $\Re z \geq 0$. Applying Corollary 4.7 of Chapter 4 of [65], gives the precise statement:

Theorem 3.10. *Let $k \in L^1(\mathbb{R}^+, \mathbb{C}^{d\times d}, p_b)$. Then the resolvent r is in $L^1(\mathbb{R}^+, \mathbb{C}^{d\times d}, p_b)$ if and only if*

$$\det(\mathrm{Id} + \mathcal{L}k(z)) \neq 0 \qquad \textit{for } \Re z \geq 0.$$

The basic statement is also called Gel'fand's theorem and proved non-constructively by Banach algebra techniques.

3.3 Linear analysis for the Vlasov–Poisson equation in flat space

For the Vlasov–Poisson equation on the torus as introduced in Section 1.2, each spatial mode $l \in \mathbb{Z}^d$ decouples on the linear level. In the linearised evolution the potential

$$\phi_l = \hat{W}_l \tilde{\rho}_l = \hat{W}_l \int_{v \in \mathbb{R}^d} \tilde{f}_l(v) \,\mathrm{d}v$$

satisfies the Volterra equation

$$\phi_l(t) + (k_l * \phi_l)(t) = F_l(t)$$

with the kernel

$$k_l(t) = \hat{f}_{\mathrm{st}}(lt)\, \hat{W}_l |l|^2 t$$

and forcing

$$F_l(t) = \hat{W}_l (\hat{f}_{\mathrm{in}})_l(lt),$$

where we use the Fourier transform in both variables x and v denoted by $\hat{\cdot}$ (instead of $\tilde{\cdot}$ for the transform only in the spatial variable), i.e.

$$\hat{f}_{\mathrm{st}}(\xi) = \int_{v \in \mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\xi\cdot v} f_{\mathrm{st}}(v) \,\mathrm{d}v$$

and

$$(\hat{f}_{\mathrm{in}})_l(\xi) = \int_{\mathbb{T}^d \times \mathbb{R}^d} \mathrm{e}^{-\mathrm{i}l\cdot x - \mathrm{i}\xi\cdot v} f_{\mathrm{in}}(x,v) \,\mathrm{d}x \,\mathrm{d}v.$$

For each mode, we can therefore use the Paley–Wiener theorem to obtain the linear stability, which gives a rigorous proof of linear Landau damping. For the study of the Vlasov-HMF equation with finitely many $\hat{W}_l$, the non-quantitative control versions of Theorem 3.5 for exponential decay and Theorem 3.10 for algebraic decay yield a uniform control of the linear evolution. This is analogous to the handling in the Kuramoto equation around the incoherent state, cf. Section 4.3.

For the full Vlasov–Poisson equation, a quantitative version is needed. In this we just suppose that there exists a constant C_W and $\gamma \geq 0$ such that the coefficients satisfy

$$|\hat{W}_l| \leq \frac{C_W}{|l|^{1+\gamma}} \qquad \text{for } l \in \mathbb{Z}^d. \tag{3.2}$$

Mouhot and Villani [114] and Bedrossian, Masmoudi and Mouhot [13] introduce the stability condition **(L)** quantifying the needed regularity of the spatially homogeneous background distribution f_{st}. Emphasising that the bound on the resolvent is only needed along the critical line, we state the condition slightly different as condition **(L′)**.

Definition 3.11. *A velocity distribution f_{st} satisfies the* stability condition $(\mathbf{L}')$ *with rate $\lambda > 0$, if there exists constants C_0, $\bar{\lambda} > \lambda$, q and an integer $M > d/2$ such that*

$$\left\| \hat{f}_{\mathrm{st}}(\xi)\, \mathrm{e}^{\bar{\lambda}\langle \xi \rangle} \right\|_{H^M(\mathbb{R}^d)} \le C_0$$

and for all $l \in \mathbb{Z}^d$ and $\Re z \ge -\lambda$

$$\mathcal{M}(z,l) \neq -1$$

and for all $l \in \mathbb{Z}^d$ and $\Re z = -\lambda$

$$|1 + \mathcal{M}(z,l)|^{-1} \ge q$$

where

$$\mathcal{M}(z,l) = \mathcal{L}k_l(|l|z) = \int_0^\infty \hat{f}_{\mathrm{st}}(lt)\, \hat{W}_l |l|^2 t\, \mathrm{e}^{-z|l|t}\, \mathrm{d}t.$$

Here $\langle \xi \rangle$ denotes the regularised weight

$$\langle \xi \rangle = \sqrt{1 + \xi^2}.$$

Compared to the original condition $(\mathbf{L})$, the bound C_0 on $\hat{f}_{\mathrm{st}}$ is formulated in an equivalent norm in order to avoid more notation. Furthermore, the condition on the poles is stated in our convention of the Laplace transform. Finally, $(\mathbf{L})$ states that $|1 + \mathcal{M}(z,l)|^{-1} \ge q$ holds for all $\Re z \ge -\lambda$, which is equivalent by the maximum principle of analytic functions.

The stability condition scales the Laplace transform depending on the mode in order to use that perturbations decay under the free transport with rate $\lambda|l|$ in the spatial mode l, i.e. faster in larger spatial modes. The stability condition captures this decay.

Theorem 3.12. *Let $f_{\mathrm{st}}(v)$ satisfies the stability condition $(\mathbf{L}')$, then there exists an explicit constant $C_R = C_R(C_0, \bar{\lambda}, \lambda, q)$ such that the Volterra kernel k_l for the potential of every spatial mode $l \in \mathbb{Z}^d$ has a resolvent r_l satisfying*

$$\left\| r_l(t)\, \mathrm{e}^{\lambda |l| t} \right\|_{L^1(\mathbb{R})} \le C_R.$$

For the proof we note the following immediate consequence of the Sobolev inequality.

Lemma 3.13. *Let f_{st} satisfy the stability condition $(\mathbf{L}')$. Then there exist constants $C_\infty, C_2, \alpha > 0$ such that for all $z \in \mathbb{R}^d$*

$$\left| \hat{f}_{\mathrm{st}}(z) \mathrm{e}^{\bar{\lambda}\langle z \rangle} \right| \le C_\infty$$

and for $y \in \mathbb{R}^d$

$$\left| \hat{f}_{\mathrm{st}}(y+z) \mathrm{e}^{\bar{\lambda}\langle y+z \rangle} - \hat{f}_{\mathrm{st}}(z) \mathrm{e}^{\bar{\lambda}\langle z \rangle} \right| \le C_2 |y|^\alpha.$$

In order to prove Theorem 3.12, we now just need to check the bounds on the scaled kernel $k_l k(t) \mathrm{e}^{\lambda |l| t}$.

Proof of Theorem 3.12. Using the previous lemma, we find

$$\begin{aligned}\left\|k_l(t)\mathrm{e}^{\lambda|l|t}\right\|_{L^1(\mathbb{R})^+} &\le C_\infty \int_0^\infty \mathrm{e}^{-\bar\lambda\langle lt\rangle}\hat W_l|l|^2 t\,\mathrm{e}^{\lambda|l|t}\,\mathrm{d}t\\ &\le C_\infty C_W|l|^{1-\gamma}\int_0^\infty \mathrm{e}^{-(\bar\lambda-\lambda)|l|t}t\,\mathrm{d}t\\ &\le C_\infty C_W|l|^{-1-\gamma}|\bar\lambda-\lambda|^{-2},\end{aligned}$$

which is uniformly bounded over $l\in\mathbb{Z}^d$ using (3.2) with $\gamma\ge -1$.

Similarly

$$\int_T^\infty |k_l(t)|\mathrm{e}^{\lambda|l|t}\,\mathrm{d}t \le C_\infty C_W|l|^{1-\gamma}\int_T^\infty \mathrm{e}^{-(\bar\lambda-\lambda)|l|t}t\,\mathrm{d}t$$

which gives the required uniform bound.

For the variation, use the previous lemma to find for $s,t\in\mathbb{R}^+$

$$\begin{aligned}\left|\hat f_{\mathrm{st}}(l(t+s))\mathrm{e}^{\lambda|l|(t+s)}-\hat f_{\mathrm{st}}(lt)\mathrm{e}^{\lambda|l|t}\right| &\le \mathrm{e}^{-(\bar\lambda-\lambda)|l|t}\left|\hat f_{\mathrm{st}}(l(t+s))\mathrm{e}^{\bar\lambda\langle lt\rangle+\lambda s}-\hat f_{\mathrm{st}}(lt)\mathrm{e}^{\bar\lambda\langle lt\rangle}\right|\\ &\le \mathrm{e}^{-(\bar\lambda-\lambda)|l|t}\left[C_2|ls|^\alpha + C_\infty\left|1-\mathrm{e}^{\bar\lambda\langle lt\rangle+\lambda|l|s-\bar\lambda\langle l(t+s)\rangle}\right|\right]\\ &\le C_3\mathrm{e}^{-(\bar\lambda-\lambda)|l|t}|l|s^\alpha\end{aligned}$$

for a constant C_3 and reducing α to $\alpha\le 1$ if needed.

Hence we find

$$\begin{aligned}&\int_0^\infty \left|k_l(t+s)\mathrm{e}^{\lambda|l|(t+s)}-k_l(t)\mathrm{e}^{\lambda|l|t}\right|\mathrm{d}t\\ &\quad\le C_W|l|^{1-\gamma}\int_0^\infty\left|(t+s)\hat f_{\mathrm{st}}(l(t+s))\mathrm{e}^{\lambda|l|(t+s)}-t\hat f_{\mathrm{st}}(lt)\mathrm{e}^{\lambda|l|t}\right|\mathrm{d}t\\ &\quad\le C_W|l|^{1-\gamma}\int_0^\infty\left[s\left|\hat f_{\mathrm{st}}(l(t+s))\mathrm{e}^{\lambda|l|(t+s)}\right|+t\left|\hat f_{\mathrm{st}}(l(t+s))\mathrm{e}^{\lambda|l|(t+s)}-\hat f_{\mathrm{st}}(lt)\mathrm{e}^{\lambda|l|t}\right|\right]\mathrm{d}t\\ &\quad\le C_4|l|^{-\gamma}s^\alpha\end{aligned}$$

for a constant C_4. Hence the theorem follows from Corollary 3.9. □

In this approach to the linear analysis we focus on the resolvent which only depends on the stationary state. This is the result of Theorem 3.12 during which we exclude possible poles. With the resulting bound on the resolvent we can then directly conclude stability results for a large class of norms by an adapted Young inequality. This is in contrast to the work by Mouhot and Villani [114] and Bedrossian, Masmoudi and Mouhot [13], where the results are directly obtained from the Fourier transform of the solution.

As possible weights w_l for mode l, impose that there exists a constant C_M such that

$$w_l(s+t)\le C_M\mathrm{e}^{\lambda|l|s}w_l(t). \tag{3.3}$$

Adapting the Young inequality, we immediately find the following lemma.

Lemma 3.14. *If the weight w satisfies (3.3) then for $p \in [1, \infty]$ and $l \in \mathbb{Z}^d$*

$$\|w_l(t)(r_l * F_l)(t)\|_{L^p(\mathbb{R}^+)} \le C_M \|r_l(t)\mathrm{e}^{\lambda |l| t}\|_{L^1(\mathbb{R}^+)} \|w_l(t) F_l(t)\|_{L^p(\mathbb{R}^+)}.$$

Proof. For $p < \infty$ we use Jensen's inequality to find

$$\begin{aligned}\int_{t=0}^{\infty} w_l(t)^p \left| \int_0^t F_l(t-s) r_l(s) \mathrm{d}s \right|^p \mathrm{d}t &\le C_M^p \int_{t=0}^{\infty} \left| \int_0^t w_l(t-s) F_l(t-s) \mathrm{e}^{\lambda |l| s} r_l(s) \mathrm{d}s \right|^p \mathrm{d}t \\ &\le \|r_l(t)\, \mathrm{e}^{\lambda |l| t}\|_{L^1(\mathbb{R}^+)} \int_0^{\infty} \int_0^t |w_l(t-s)|^p |F_l(t-s)|^p |r_l(s)| \mathrm{e}^{\lambda |l| s} \mathrm{d}s \, \mathrm{d}t\end{aligned}$$

from which the inequality follows by a change of variable. For $p = \infty$, we can use the direct estimate. □

Bedrossian, Masmoudi and Mouhot [13] use the following weight for Gevrey regularity

$$w_l(t) = A_l(l, lt) = \mathrm{e}^{\lambda(t) \langle l, lt \rangle^s} \langle l, lt \rangle^s$$

for $s \in [0, 1]$. If $\lambda(t)$ is decreasing, we see that the weight satisfies (3.3) as follow:

- for $s < 1$ any $\lambda > 0$ works with any $\lambda(0)$,
- for $s = 1$, we need $\lambda > \lambda(t)$ eventually.

Remark 3.15. This shows that there is a distinction of the obtainable rate between the cases $s < 1$ and $s = 1$.

Now their linear estimate, Lemma 4.1 in [13], becomes an easy corollary of Theorem 3.12 and Lemma 3.14.

Corollary 3.16. *If the stability condition* $(\mathbf{L}')$ *is satisfied, then the solution ϕ_l to the Volterra equation from the linear evolution satisfies*

$$\int_0^{\infty} w_l(t)^2 |\phi_l(t)|^2 \mathrm{d}t \le C \int_0^{\infty} w_l(t)^2 |F_l(t)|^2 \mathrm{d}t$$

for an explicit constant C.

4 Stability of the incoherent state in the Kuramoto equation

This chapter studies the behaviour of the Kuramoto equation around the incoherent state and is mostly published in [46].

4.1 Overview

The structure of the Kuramoto equation can best be understood by taking the Fourier transform in both variables. The transformed density $\hat{f}$ is explicitely given by

$$\hat{f}_l(\xi) = \int_{\mathbb{T}\times\mathbb{R}} \mathrm{e}^{-\mathrm{i}l\theta-\mathrm{i}\xi\omega} f(\theta,\omega)\,\mathrm{d}\theta\mathrm{d}\omega.$$

For the transformed density $\hat{f}$, the evolution equation (1.10) imposes the time evolution

$$\begin{cases} \partial_t \hat{f}_l(t,\xi) = l\partial_\xi \hat{f}_l(t,\xi) + \dfrac{Kl}{2}\left(\overline{\eta(t)}\hat{f}_{l-1}(t,\xi) - \eta(t)\hat{f}_{l+1}(t,\xi)\right), \\ \eta(t) = \hat{f}_{-1}(0) = \overline{\hat{f}_1(0)}. \end{cases}$$

Since f is real, $\hat{f}_{-l}(-\xi) = \overline{\hat{f}_l(\xi)}$, so that it suffices to consider $l \geq 0$.

The spatially homogeneous distribution is characterised by vanishing non-zero moments, so that the stability of the incoherent state is equivalent to the decay of $\hat{f}$ restricted to $l \geq 1$. The restriction is denoted by u which satisfies

$$\begin{cases} \partial_t u_1(t,\xi) = \partial_\xi u_1(t,\xi) + \dfrac{K}{2}\left[\overline{\eta(t)}\,\hat{g}(\xi) - \eta(t)\,u_2(t,\xi)\right], \\ \partial_t u_l(t,\xi) = l\partial_\xi u_l(t,\xi) + \dfrac{Kl}{2}\left[\overline{\eta(t)}\,u_{l-1}(t,\xi) - \eta(t)\,u_{l+1}(t,\xi)\right] \text{ for } l \geq 2, \\ \eta(t) = \overline{u_1(t,0)}, \end{cases} \tag{4.1}$$

where $\hat{g}$ is the Fourier transform of the velocity marginal.

The interaction at (t,l,ξ) is given by $u_1(t,0) = \overline{\eta(t)}$ and $u_{l\pm1}(t,\xi)$. This localisation of the interaction means that the constant mode $l = 0$ with $\hat{g} = \hat{f}_0$ decouples the positive modes $l > 0$ and the negative modes $l < 0$.

In this formulation the free transport always moves $\xi \mapsto u_l(t,\xi)$ to the left. Hence with

$M \geq 0$, the region $\mathbb{N} \times \{\xi : \xi \geq -M\}$ is its own domain of dependence. This already suggests to use norms focusing on $\xi \geq 0$ in order to show convergence.

For the global energy estimate, we note that the interaction is skew-Hermitian except for the first mode. Thus we consider the energy functional

$$I(t) = \int_{\xi=0}^{\infty} \sum_{l\geq 1} \frac{1}{l} |u_l(t,\xi)|^2 \phi(\xi) \mathrm{d}\xi, \tag{4.2}$$

where ϕ is an increasing weight. To prove the first result of this chapter, we balance the gain of the first linear interaction term with the decay by the increasing weight.

Theorem 4.1. *For a velocity distribution $g \in \mathcal{M}(\mathbb{R})$ suppose $\int_0^\infty |\hat{g}(\xi)|\mathrm{d}\xi < \infty$ and let*

$$K_{ec} = \frac{2}{\int_{\xi=0}^{\infty} |\hat{g}(\xi)|\mathrm{d}\xi}.$$

If the coupling constant K satisfies $K < K_{ec}$, then there exists a finite constant $c > 0$ and a bounded increasing weight $\phi \in C^1(\mathbb{R}^+)$ with $\phi(0) = 1$, so that for a solution to the Kuramoto equation with velocity marginal g the energy functional $I(t)$ defined by Equation (4.2) *satisfies*

$$I(t) + c\int_0^t |\eta(s)|^2 \mathrm{d}s \leq I(0)$$

for all $t \in \mathbb{R}^+$. In particular this shows that I is non-increasing and the order parameter η satisfies

$$\int_0^\infty |\eta(s)|^2 \mathrm{d}s \leq c^{-1} I(0) < \infty.$$

If, moreover, the initial distribution f_{in} is in $L^2(\Gamma)$, then $I(0) \leq \|\phi\|_\infty \|f_{\mathrm{in}}\|_2^2$.

The linear evolution of u is determined by

$$\begin{cases} \partial_t u_1(t,\xi) = \partial_\xi u_1(t,\xi) + \dfrac{K}{2} u_1(t,0)\hat{g}(\xi), \\ \partial_t u_l(t,\xi) = l\partial_\xi u_l(t,\xi) \qquad \text{for } l \geq 2. \end{cases} \tag{4.3}$$

Hence under the linear evolution, the order parameter satisfies the Volterra equation

$$\eta(t) + (k * \eta)(t) = \overline{(u_{\mathrm{in}})_1(t)} \tag{4.4}$$

with the convolution kernel

$$k(t) = -\frac{K}{2}\hat{g}(t). \tag{4.5}$$

From the discussion in Chapter 3, we can immediately express the stability by the resolvent r of the Volterra equation. Recall the weights used to measure the decay, $\exp_a(x) = \mathrm{e}^{ax}$ and $p_{A,b}(x) = (A + x)^b$ with $p_b = p_{1,b}$.

Theorem 4.2. *If for $a \in \mathbb{R}$ we have $\hat{g} \in L^1(\mathbb{R}^+, \exp_a)$, then $r \in L^1(\mathbb{R}^+, \exp_a)$ is equivalent to $(\mathcal{L}k)(z) \neq -1$ for all $z \in \mathbb{C}$ with $\Re z \geq -a$.*

If for $b \geq 0$ we have $\hat{g} \in L^1(\mathbb{R}^+, p_b)$, then $r \in L^1(\mathbb{R}^+, p_b)$ is equivalent to $(\mathcal{L}k)(z) \neq -1$ for all $z \in \mathbb{C}$ with $\Re z \geq 0$.

Hence the linear evolution of the order parameter is stable if $(\mathcal{L}k)(z) \neq -1$ for all $z \in \mathbb{C}$ with $\Re z \geq 0$. By bounding the absolute value, we see that if $K < K_{ec}$ the system is linearly stable. In case of the Gaussian or the Cauchy distribution, $\hat{g}$ is always positive so that K_{ec} equals the critical coupling.

Like in the Vlasov equation [128], this condition can be related to a Penrose criterion for the complex boundary, which also visualises how often the condition fails. Under this we can also relate the stability to the known condition [148, 149].

For small perturbations, my second main result is the nonlinear stability of the incoherent state. In a first version we propagate control in

$$\sup_{l \geq 1} \sup_{\xi \in \mathbb{R}} |u_l(t, \xi)| \mathrm{e}^{a(\xi + tl/2)}.$$

Theorem 4.3. *Let $g \in \mathcal{M}(\mathbb{R})$ and $a > 0$ such that $\hat{g}$ satisfies the exponential stability condition of rate a in Theorem 4.2, i.e. $\hat{g} \in L^1(\mathbb{R}^+, \exp_a)$ and $(\mathcal{L}k)(z) \neq -1$ for all $z \in \mathbb{C}$ with $\Re z \geq -a$. Then there exists a $\delta > 0$ and finite c_1 such that for initial data $f_{\mathrm{in}} \in \mathcal{M}(\Gamma)$ with velocity marginal g, Fourier transform u_{in} and $M_{\mathrm{in}} := \sup_{l \in \mathbb{N}} \sup_{\xi \in \mathbb{R}} |(u_{\mathrm{in}})_l(\xi)| \mathrm{e}^{a\xi} \leq \delta$ the Fourier transform u of the solution to the Kuramoto equation with initial data f_{in} satisfies*

$$\sup_{t \in \mathbb{R}^+} \sup_{l \geq 1} \sup_{\xi \in \mathbb{R}} |u_l(t, \xi)| \mathrm{e}^{a(\xi + tl/2)} \leq (1 + c_1) M_{\mathrm{in}}.$$

In particular, this shows that the order parameter $\eta(t) = u_1(t, 0)$ decays as $O(\mathrm{e}^{-at/2})$.

For the algebraic decay we propagate control in

$$\sup_{l \geq 1} \sup_{\xi \in \mathbb{R}^+} |u_l(t, \xi)| \frac{(1 + \xi + t)^b}{(1 + t)^{\alpha(l-1)}}.$$

Theorem 4.4. *Let $g \in \mathcal{M}(\mathbb{R})$ and $b > 1$ such that $\hat{g}$ satisfies the algebraic stability condition of parameter b in Theorem 4.2, i.e. $\hat{g} \in L^1(\mathbb{R}^+, p_b)$ and $(\mathcal{L}k)(z) \neq -1$ for all $z \in \mathbb{C}$ with $\Re z \geq 0$. For $\alpha > 0$ satisfying $b > 1 + \alpha$, there exists $\delta > 0$ and finite γ_1 such that for initial data $f_{\mathrm{in}} \in \mathcal{M}(\Gamma)$ with velocity marginal g, Fourier transform u_{in} and $M_{\mathrm{in}} := \sup_{l \in \mathbb{N}} \sup_{\xi \in \mathbb{R}^+} |(u_{\mathrm{in}})_l(\xi)| (1 + \xi)^b \leq \delta$ the Fourier transform u of the solution to the Kuramoto equation with initial data f_{in} satisfies*

$$\sup_{t \in \mathbb{R}^+} \sup_{l \geq 1} \sup_{\xi \in \mathbb{R}^+} |u_l(t, \xi)| \frac{(1 + \xi + t)^b}{(1 + t)^{\alpha(l-1)}} \leq (1 + \gamma_1) M_{\mathrm{in}}.$$

In particular, this shows that the order parameter $\eta(t) = u_1(t, 0)$ decays as $O(t^{-b})$.

Note that the exponential and algebraic controls are one-sided and thus do not control the L^2 or Sobolev norm in the original space. This explains why we can have decay with these norms, even though there is no decay of perturbations in the L^2 norm and the linear system is only neutrally stable [148]. For the application, however, the Sobolev norm $\mathcal{W}^{b,1}$ controls M_{in} and $g \in \mathcal{W}^{b,1}$ ensures the required decay of $\hat{g}$.

For all $b > 1$, we can choose a suitable α such that the algebraic stability result holds. On the linear level, the damping holds for all $b \geq 0$ and the threshold $b = 1$ is a technical result of the nonlinear estimate.

If the stability condition fails, we can identify growing modes in the Volterra equation, which is the direct reformulation of Theorem 3.7.

Theorem 4.5. *For $a \in \mathbb{R}$ suppose that $\hat{g} \in L^1(\mathbb{R}^+, \exp_a)$ and that the convolution kernel k defined by Equation (4.5) satisfies $(\mathcal{L}k)(z) \neq -1$ for $z \in \mathbb{C}$ with $\Re z = -a$. Then the solution to the Volterra equation (4.4) is given by*

$$\eta(t) = (u_{\mathrm{in}})_1(t) - \int_0^\infty q(t-s)\,(u_{\mathrm{in}})_1(s)\,\mathrm{d}s$$

with

$$q(t) = r_s(t) + \sum_{i=1}^{n} \sum_{j=0}^{p_i - 1} b_{i,j} t^j \mathrm{e}^{\lambda_i t}$$

where q is vanishing for negative arguments and r_s satisfies $\|r_s\|_{L^1(\mathbb{R}, \exp_a)} < \infty$ and $\lambda_1, \ldots, \lambda_n$ are the finitely many zeros of $1 + (\mathcal{L}k)(z)$ in $\Re z > -a$ with multiplicities $p_1, \ldots, p_n$ and $b_{i,j}$ depends on the residues of $[1 + (\mathcal{L}k)(z)]^{-1}$ at λ_i.

This says that the linear evolution is governed by eigenmodes $t^j \mathrm{e}^{\lambda_i t}$ for $i = 1, \ldots, n$ and $j = 0, \ldots, p_i - 1$ and a remaining stable part r_s. The previous linear stability theorem is included in this formulation as $r = r_s$.

If $\hat{g} \in L^1(\mathbb{R}^+, \exp_a)$ for $a > 0$, this shows that if the stability condition fails, there exists an unstable mode. Without extra assumption, for every $a < 0$ we have $\hat{g} \in L^1(\mathbb{R}^+, \exp_a)$ because g is a probability distribution. Hence, unless we are at the critical case, we have unstable modes if the stability condition fails. Therefore, the linear stability condition is sharp.

The additional restriction $(\mathcal{L}k)(z) \neq -1$ along the line $\Re z = -a$ is very weak, because, if $\hat{g} \in L^1(\mathbb{R}^+, \exp_a)$, then $(\mathcal{L}k)(z)$ is analytic for $\Re z > -a$ and by the Riemann-Lebesgue lemma vanishing as $|z| \to \infty$. Hence we can choose a smaller a' such that $|a - a'|$ is arbitrarily small and the theorem applies.

The eigenmodes of the Volterra equation can be related to eigenmodes $z_{\lambda,j}$ of the linear evolution (Equation (4.3)) on $C(\mathbb{N} \times \mathbb{R})$. These eigenmodes are vanishing except in the first spatial modes where they satisfy $(z_{\lambda,j})_1 \in L^\infty(\mathbb{R}^+, \exp_a)$.

For the center-unstable manifold reduction we therefore look for solutions in

$$\mathcal{Z}^a = \{u \in C(\mathbb{N} \times \mathbb{R}) : \|u\|_{\mathcal{Z}^a} < \infty\}$$

with

$$\|u\|_{\mathcal{Z}^a} = \sup_{l\geq 1} \|u_l\|_{L^\infty(\mathbb{R},\exp_a)} = \sup_{l\geq 1}\sup_{\xi\in\mathbb{R}} \mathrm{e}^{a\xi}|u_l(\xi)|.$$

The nonlinearity is not controlled in $\mathcal{Z}^a$, but within

$$\mathcal{Y}^a = \{u \in C(\mathbb{N}\times\mathbb{R}) : \|u\|_{\mathcal{Y}^a} < \infty\}$$

with

$$\|u\|_{\mathcal{Y}^a} = \sup_{l\geq 1} l^{-1}\|u_l\|_{L^\infty(\mathbb{R},\exp_a)} = \sup_{l\geq 1}\sup_{\xi\in\mathbb{R}} \frac{\mathrm{e}^{a\xi}}{l}|u_l(\xi)|.$$

By a spectral analysis we can find a continuous projection P_{cu} from $\mathcal{Y}^a$ to $\mathcal{Z}^a_{cu} := \langle z_{\lambda_i,j} : i = 1,\dots,n \text{ and } j = 0,\dots,p_i-1\rangle$ with complementary projection P_s mapping $\mathcal{Z}^a$ to $\mathcal{Z}^a_s$ and $\mathcal{Y}^a$ to $\mathcal{Y}^a_s$. The image of P_s is invariant under the linear evolution and decaying with rate a. In fact the higher modes decay quicker in this norm, so that the solution to the linear evolution with forcing in $\mathcal{Y}^a_s$ is within $\mathcal{Z}^a_s$.

Hence the theory of center-unstable manifold reduction [68, 154, 155] applies and we can reduce the dynamics for the local behaviour. This is my third main result and here we also consider K as variable in order to discuss the asymptotic result for couplings K close to a fixed coupling K_c.

Theorem 4.6. *Given $a > 0$ and $g \in \mathcal{M}(\mathbb{R})$ with $\hat{g} \in L^1(\mathbb{R}^+,\exp_a)$. Let K_c be a coupling constant such that Theorem 4.5 applies with center-unstable modes $\lambda_1,\dots,\lambda_n$ satisfying $\Re\lambda_i \geq 0$ for $i = 1,\dots,n$. For $k \in \mathbb{N}$ there exist $\psi \in C^k(\mathcal{Z}^a_{cu}\times\mathbb{R},\mathcal{Z}^a_s)$ and $\delta > 0$ such that for $|\epsilon| \leq \delta$ the manifold*

$$\mathcal{M}_\epsilon = \{y + \psi(y,\epsilon) : y \in \mathcal{Z}^a_{cu}\}$$

is invariant and exponentially attractive under the nonlinear evolution with coupling constant $K = K_c + \epsilon$ in the region $\{u \in \mathcal{Z}^a : \|u\|_{\mathcal{Z}^a} \leq \delta\}$. Moreover, the derivatives of ψ can explicitly be computed at 0.

Hence we can reduce the dynamics around the spatially homogeneous distribution to the finite dimensional dynamics of $\mathcal{M}_\epsilon$. We demonstrate the application by recovering Chiba's bifurcation result for the Gaussian distribution [31]. In particular, it shows the nonlinear stability of the partially locked states close to the bifurcation.

The key point for the center-unstable manifold reduction is the use of the regularity in the natural frequency ω. This allows the use of the weighted space $\mathcal{Z}^a$ on which the linear generator does not have a continuous spectrum along the imaginary axis, as seen in earlier linear analysis [148]. Instead the linear evolution decays with rate al in the lth spatial mode except in the first mode $l = 1$, where a discrete spectrum can arise. Thus the effect of using $\mathcal{Z}^a$ is very similar to adding white noise [138] of strength D to the system and in the case $D > 0$ the stability and center manifold reduction have been done [22, 38, 148].

We remark that the use of $\mathcal{Z}^a$ is compatible with adding noise and thus for sufficiently regular velocity distribution the center-unstable manifold reduction can be used to study the bifurcation behaviour with the noise strength D as additional parameter in the limit $D \to 0$. The behaviour of the amplitude equations in the center manifold reduction in this limit is reviewed in [147, Section 11] and this explains why taking the limit $D \to 0$ gave the correct behaviour for regular distributions [38]. Note that even in this case the order parameter can decay faster than rate D through the Landau damping mechanism.

In order to understand the norm $\mathcal{Z}^a$, which is also the measure for the perturbations in Theorem 4.3, we relate the norm to the analytic continuation in the strip $\{z \in \mathbb{C} : -a \leq \Im z \leq 0\}$ of the complex plane. Here we can demonstrate explicitly that for suitable velocity marginals the partially locked states have finite norm in $\mathcal{Z}^a$ and are small perturbations if their order parameter η is small. In particular, this shows that partially locked states with a small η, which are observed as asymptotic state for a coupling constant K just above K_c, are exponential damped if the coupling K is below K_c, cf. Theorem 4.3. This is particularly interesting, because these partially locked states are not regular in general, e.g. they do not even have a density.

The exponential stability of the incoherent state and the convergence in $\mathcal{Z}^a$ to the reduced manifold do not imply convergence in L^2. Nevertheless, we can relate it to weak convergence for sufficiently nice test functions because the Fourier transform is bounded.

In case of the algebraic stability (Theorem 4.4), the convergence is very weak but we can conclude $\int_0^\infty |\eta(t)| \mathrm{d}t < \infty$. Going back to the original equation, this shows control in the gliding frame and weak convergence of the phase marginal.

The used exponential norm can also be used to discuss the convergence to the Ott–Antonsen reduction from Section 1.3.4, see Section 4.9.

The linear stability can also be used to show direct results on a finite particle model showing a better control on the order parameter compared to the normal mean-field theory, see Section 4.10.

4.2 Solutions in Fourier space and uniqueness

The discussion of the mean-field limit (Chapter 2) shows that the Kuramoto equation has a unique solution in $C_{\mathcal{M}}$ for any initial distribution $f_{\mathrm{in}} \in \mathcal{M}(\Gamma)$. Taking the Fourier transform, we find the following theorem for the evolution of the restriction.

Theorem 4.7. *Suppose $f_{\mathrm{in}} \in \mathcal{M}(\Gamma)$ with velocity marginal $g \in \mathcal{M}(\mathbb{R})$. Let $f \in C_{\mathcal{M}}$ be the solution to the Kuramoto equation. Then the Fourier transform u is in $C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$, is* *bounded uniformly by* 1 *and satisfies*

$$
\begin{aligned}
0 = &\sum_{l\in\mathbb{Z}} \int_{\mathbb{R}} (u_{\mathrm{in}})_l(\xi) h_l(0,\xi) \mathrm{d}\xi \\
&+ \int_{\mathbb{R}^+} \int_{\mathbb{R}} \frac{K}{2} \hat{g}(\xi) \overline{\eta(t)} h_1(t,\xi) + \sum_{l\geq 2} \frac{Kl}{2} u_{l-1}(t,\xi) \overline{\eta(t)} h_l(t,\xi) \\
&\quad + \sum_{l\in\mathbb{N}} \left[-\frac{Kl}{2} u_{l+1}(t,\xi)\eta(t) h_l(t,\xi) + u_l(t,\xi) \left[\partial_t h_l(t,x) - l \partial_\xi h_l(t,\xi)\right] \right] \mathrm{d}\xi \mathrm{d}t
\end{aligned}
\tag{4.6}
$$

for all $h \in C_0^1(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$, *where* $\eta(t) = \overline{u_1(t,0)}$ *and* $\hat{g}$ *is the Fourier transform of* g.

Proof. For every $t \in \mathbb{R}^+$, the distribution $f(t,\cdot)$ is a probability measure, so that $u_l(t,\xi)$ is bounded by 1 and continuous with respect to ξ. Since $f(t,\cdot)$ is weakly continuous, u is also continuous with respect to time.

By Lemma 2.6 the velocity marginal is always g so that Plancherel's formula shows that Equation (4.6) holds for all $h \in \mathscr{S}(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$. By the density of $\mathscr{S}$ the result extends to all $h \in C_0^1(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$. □

The free transport is a well-posed linear problem, so that Duhamel's principle on the free transport holds.

Lemma 4.8. *Let* $u_{\mathrm{in}} \in C(\mathbb{N}\times\mathbb{R})$, $\hat{g} \in C(\mathbb{R})$ *and* $u \in C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$. *For* $t \in \mathbb{R}^+$ *and* $\xi \in \mathbb{R}$ *define*

$$
v_1(t,\xi) = (u_{\mathrm{in}})_1(\xi + t) + \frac{K}{2} \int_0^t \left[\overline{\eta(s)} \hat{g}(\xi + (t-s)) - \eta(s) u_2(s, \xi + (t-s)) \right] \mathrm{d}s
$$

and for $l \geq 2$

$$
\begin{aligned}
v_l(t,\xi) = &(u_{\mathrm{in}})_l(\xi + lt) \\
&+ \frac{Kl}{2} \int_0^t \left[\overline{\eta(s)} u_{l-1}(s, \xi + l(t-s)) - \eta(s) u_{l+1}(s, \xi + l(t-s)) \right] \mathrm{d}s
\end{aligned}
$$

where $\eta(t) = \overline{u_1(t,0)}$. *If* u *satisfies Equation* (4.6) *for all* $h \in C_0^1(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$, *then* $u \equiv v$.

Proof. In the weak formulation (4.6) consider all terms except the free transport as forcing. Then the problem has a unique solution given by v. □

For the stability and bifurcation result we construct solutions u to Equation (4.6). By the following theorem these correspond to the Fourier transform of the mean-field solution.

Theorem 4.9. *Let* $u_{\mathrm{in}} \in C(\mathbb{N}\times\mathbb{R})$ *and* $\hat{g} \in C(\mathbb{R})$. *Suppose* $u, \tilde{u} \in C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$ *satisfy Equation* (4.6) *for every* $h \in C_0^1(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$. *If for* $T \in \mathbb{R}^+$ *there exists* $M \in \mathbb{R}^+$ *with* $a, \beta \geq 0$ *satisfying*

$$
\sup_{\xi\in\mathbb{R}} |\hat{g}(\xi)| \min(1, \mathrm{e}^{a\xi}) \leq M
$$

and

$$\sup_{t\in[0,T]}\sup_{l\in\mathbb{N}}\sup_{\xi\in\mathbb{R}} \mathrm{e}^{-\beta l}\min(1,\mathrm{e}^{a\xi})\max(|u_l(t,\xi)|,|\tilde{u}_l(t,\xi)|)\leq M,$$

then $u(t,\cdot)\equiv\tilde{u}(t,\cdot)$ *for* $t\in[0,T]$.

Proof. For $\gamma>0$ consider the difference measure e defined by

$$e(t)=\sup_{l\in\mathbb{N}}\sup_{\xi\in\mathbb{R}}\mathrm{e}^{-\gamma tl}\mathrm{e}^{-\beta l}\min(1,\mathrm{e}^{a\xi})|u_l(t,\xi)-\tilde{u}_l(t,\xi)|.$$

Then by Lemma 4.8 for $t\in[0,T]$, $l\in\mathbb{N}$ and $\xi\in\mathbb{R}$ we have

$$\begin{aligned}|u_l(t,\xi)-\tilde{u}_l(t,\xi)|\mathrm{e}^{-\gamma tl}\mathrm{e}^{-\beta l}\min(1,\mathrm{e}^{a\xi}) &\leq 2KMl\int_0^t \mathrm{e}^{2\beta+\gamma s}e(s)\mathrm{e}^{-\gamma l(t-s)}\mathrm{d}s\\ &\leq \frac{2KM}{\gamma}\mathrm{e}^{2\beta+\gamma t}\sup_{s\in[0,t]}e(s).\end{aligned}$$

Choose $\gamma>0$ and $t^*>0$ so that $(2KM/\gamma)\mathrm{e}^{2\beta+\gamma t^*}<1/2$. Then the above shows for $t\in[0,t^*]$

$$e(t)\leq\frac{1}{2}\sup_{s\in[0,t^*]}e(s).$$

Therefore, we must have $e(t)=0$ for $t\leq t^*$, i.e. the solutions agree. Since we can repeat the argument in steps of t^* up to time T, this shows uniqueness up to time T. □

By Theorem 4.7 the Fourier transform u of a solution to the Kuramoto equation satisfies the assumptions of the previous uniqueness theorem for every a and β.

Theorem 4.10. *The restriction of the Fourier transform u to $C(\mathbb{R}^+\times\mathbb{N}\times\mathbb{R}^+)$, i.e. $\xi\geq 0$, satisfies an appropriate weak PDE (Theorem 4.7) to which the Duhamel formula (Lemma 4.8) applies and the uniqueness holds for the restriction $\xi\geq 0$ in Theorem 4.9.*

Proof. Characteristics of the full equation are never entering the restricted region and the nonlinear interaction in the restricted region is determined by the region itself. Hence the restriction satisfies the appropriate restriction of the weak PDE Theorem 4.7 whose solution is given by the Duhamel formula Lemma 4.8. The proof of the uniqueness theorem holds as before when restricting ξ to $\mathbb{R}^+$ everywhere. □

4.3 Linear analysis

As in the mean-field limit, the linear evolution given by Equation (4.3) is understood as PDE for continuous solutions in the same weak sense, i.e. tested against C_0^1 functions. We denote its evolution operator by e^{tL}, i.e. $t,l,\xi\mapsto(e^{tL}u_{\text{in}})_l(\xi)$ is the weak solution to Equation (4.3) in $C(\mathbb{R}^+\times\mathbb{N}\times\mathbb{R})$ with initial data u_{in} given at time $t=0$.

Lemma 4.11. *Let $\hat{g} \in C(\mathbb{R})$. Then the evolution operator e^{tL} is well-defined from continuous initial data to the unique continuous solution. For $u_{\mathrm{in}} \in C(\mathbb{N} \times \mathbb{R})$ let $\nu(t) = \overline{(\mathrm{e}^{tL} u_{\mathrm{in}})_1(0)}$. Then ν satisfies the Volterra equation* (4.4) *with the convolution kernel from Equation* (4.5) *and*

$$(\mathrm{e}^{tL} u_{\mathrm{in}})_1(\xi) = (u_{\mathrm{in}})_1(\xi + t) + \int_0^t \frac{K}{2} \overline{\nu(s)} \hat{g}(\xi + t - s) \mathrm{d}s \tag{4.7}$$

and for $l \geq 2$

$$(e^{tL} u_{\mathrm{in}})_l(\xi) = u_{\mathrm{in}}(l, \xi + tl). \tag{4.8}$$

Proof. The free transport has a unique weak solution, so that, as in Lemma 4.8, Duhamel's formula implies

$$(\mathrm{e}^{tL} u_{\mathrm{in}})_1(\xi) = (u_{\mathrm{in}})_1(\xi + t) + \frac{K}{2} \int_0^t (\mathrm{e}^{sL} u_{\mathrm{in}})_1(0) \hat{g}(\xi + (t - s)) \mathrm{d}s \tag{4.9}$$

and for $l \geq 2$

$$(\mathrm{e}^{tL} u_{\mathrm{in}})_l(\xi) = (u_{\mathrm{in}})_l(\xi + lt).$$

Taking $\xi = 0$ in Equation (4.9) shows that $\nu(t) = (\mathrm{e}^{tL} u_{\mathrm{in}})_1(0)$ must satisfy Equation (4.4). By the general theory of Volterra equation (Lemma 3.4), Equation (4.4) has a unique solution and the solution is continuous. Hence the evolution operator e^{tL} is well-defined and given by the Volterra equation (4.4) and Equations (4.7) and (4.8). □

For the polynomial weight, we find that we can also propagate a control with the weight $p_{A,b}$ for any $A \geq 1$ by a resolvent $p_{1,b}$. For the completeness of the statement, we also state the result for the exponential weight.

Lemma 4.12. *Let $r, f \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$. For $a \in \mathbb{R}$*

$$\|r * f\|_{L^\infty(\mathbb{R}^+, \exp_a)} \leq \|r\|_{L^1(\mathbb{R}^+, \exp_a)} \|f\|_{L^\infty(\mathbb{R}^+, \exp_a)}$$

and for $A \geq 1$ and $b \geq 0$

$$\|r * f\|_{L^\infty(\mathbb{R}^+, p_{A,b})} \leq \|r\|_{L^1(\mathbb{R}^+, p_b)} \|f\|_{L^\infty(\mathbb{R}^+, p_{A,b})}.$$

Proof. The exponential weight is Young's inequality Lemma 3.1. For $A \geq 1$ and $b \geq 0$ and $t \in \mathbb{R}^+$

$$\begin{aligned} |(r * f)(t)|(A + t)^b &\leq \int_0^t |r(t - s)|(1 + t - s)^b |f(s)|(A + s)^b \mathrm{d}s \\ &\leq \|f\|_{L^\infty(\mathbb{R}^+, p_{A,b})} \int_0^t |r(t - s)|(1 + t - s)^b \mathrm{d}s \end{aligned}$$

where we used that $A + t \leq (1 + t - s)(A + s)$ holds for $s, t \in \mathbb{R}^+$, because $A \geq 1$. Taking the supremum shows the second claim. □

With the decay of the order parameter, we can go back to the linear evolution, in order to propagate the decay with the rate expected from the free transport.

Lemma 4.13. *Let $u_{\mathrm{in}} \in C(\mathbb{N} \times \mathbb{R})$. Then for $a \in \mathbb{R}$*

$$\sup_{t\geq 0} \mathrm{e}^{at} \|(\mathrm{e}^{tL} u_{\mathrm{in}})_1\|_{L^\infty(\mathbb{R}^+,\exp_a)} \leq c_1 \|(u_{\mathrm{in}})_1\|_{L^\infty(\mathbb{R}^+,\exp_a)}$$

with $c_1 = 1 + (1 + \|r\|_{L^1(\mathbb{R}^+,\exp_a)})(K/2)\|g\|_{L^1(\mathbb{R}^+,\exp_a)}$ and for $A \geq 1$ and $b \geq 0$

$$\sup_{t\geq 0} \sup_{\xi \geq 0} |(\mathrm{e}^{tL} u_{\mathrm{in}})_1(\xi)|(A+\xi+t)^b \leq \gamma_1 \|(u_{\mathrm{in}})_1\|_{L^\infty(\mathbb{R}^+,p_{A,b})}$$

with $\gamma_1 = 1 + (1 + \|r\|_{L^1(\mathbb{R}^+,p_b)})(K/2)\|g\|_{L^1(\mathbb{R}^+,p_b)}$.

Proof. By the previous lemma and the explicit formula for ν, Equation (4.7) shows for $\xi \in \mathbb{R}$

$$\begin{aligned} &|(\mathrm{e}^{tL} u_{\mathrm{in}})_1(\xi)| e^{a(t+\xi)} \leq \|(u_{\mathrm{in}})_1\|_{L^\infty(\mathbb{R}^+,\exp_a)} \\ &+ \int_0^t \frac{K}{2}(1 + \|r\|_{L^1(\mathbb{R}^+,\exp_a)}) \|(u_{\mathrm{in}})_1\|_{L^\infty(\mathbb{R}^+,\exp_a)} \mathrm{e}^{a(\xi+t-s)} |\hat{g}(\xi+t-s)| \mathrm{d}s, \end{aligned}$$

which proves the claim.

In the algebraic case we find for $\xi \geq 0$

$$\begin{aligned} &|(\mathrm{e}^{tL} u_{\mathrm{in}})_1(\xi)|(A+\xi+t)^b \leq \|(u_{\mathrm{in}})_1\|_{L^\infty(\mathbb{R}^+,p_{A,b})} \\ &+ \int_0^t \frac{K}{2}(1 + \|r\|_{L^1(\mathbb{R}^+,p_b)}) \|(u_{\mathrm{in}})_1\|_{L^\infty(\mathbb{R}^+,p_{A,b})} \frac{(A+\xi+t)^b}{(A+s)^b} |\hat{g}(\xi+t-s)| \mathrm{d}s, \end{aligned}$$

which implies the result as $(A+\xi+t)/(A+s) \leq (1+\xi+t-s)$. □

The stability can therefore be characterised by the weighted L^1 norm of the resolvent. In turn this can be characterised nicely by Theorem 3.5 and Theorem 3.10. This gives exactly the statement of Theorem 4.2, which is thus proved.

The stability criterion $1 + \mathcal{L}k(z) \neq 0$ for $\Re z \geq 0$ can be understood as imposing that $\mathcal{L}\hat{g}(\{z : \Re z \geq 0\})$ does not contain $2/K$. Since $\mathcal{L}\hat{g}$ is analytic, the argument principle shows that this is equivalent to imposing that $\mathcal{L}\hat{g}(i\mathbb{R})$ does not encircle $2/K$.

For $\Re z > 0$ we find by Fubini

$$\mathcal{L}\hat{g}(z) = \int_0^\infty e^{-z\xi} \hat{g}(\xi) \mathrm{d}\xi = \int_{-\infty}^\infty g(\omega) \frac{1}{z - \mathrm{i}\omega} \mathrm{d}\omega.$$

By continuity of $\mathcal{L}\hat{g}$ we thus find

$$\mathcal{L}\hat{g}(ix) = \lim_{\lambda \to 0+0} \int_{-\infty}^\infty \frac{g(\omega)}{\mathrm{i}(x-\omega)+\lambda} \mathrm{d}\omega = \mathrm{i}\,\mathrm{PV} \int_{-\infty}^\infty \frac{g(x+\omega)}{\omega} \mathrm{d}\omega + \pi g(x),$$

where the last equality is Plemelj formula for continuous g and PV denotes the principal value integral.

The curve $\mathcal{L}\hat{g}(\mathrm{i}\mathbb{R})$ starts and ends at 0, is bounded, and goes through the right half plane where it crosses the real axis by continuity of the principal value integral. If $g > 0$, this shows that it crosses the real axis at a positive value. Thus there exists a critical value K_c such that for $0 \leq K < K_c$ the system is linearly stable and for $K_c + \delta > K > K_c$ with some $\delta > 0$ the system is unstable.

For the Gaussian distribution, we can plot the boundary explicitly (cf. Figure 4.1) and note that for $K < 2/(\pi g(0))$ the solution is stable while for $K > 2/(\pi g(0))$ there is exactly one root of $(1 + \mathcal{L}k)(z)$ for $\Re z \geq 0$.

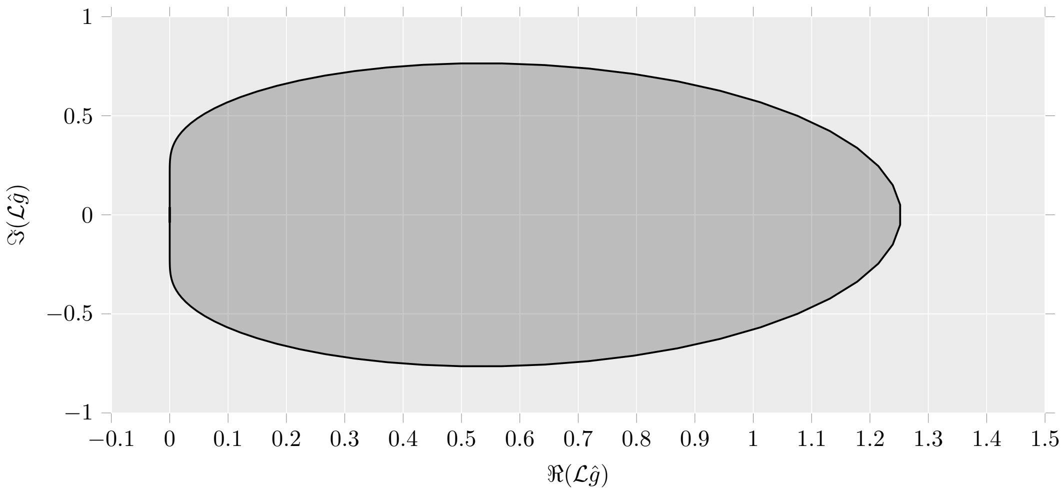


Figure 4.1 – The black line shows $\mathcal{L}\hat{g}(\mathrm{i}\mathbb{R})$ with the enclosed shaded area $\mathcal{L}\hat{g}(\{z : \Re z > 0\})$ for the Gaussian distribution $g(\omega) = (2\pi)^{-1/2}\mathrm{e}^{-\omega^2/2}$.

Already for two Gaussians with variance 1 centred at $\omega = \pm 1.5$, we see a more interesting behaviour (cf. Figure 4.2). Here we see that at the critical coupling two roots of $(1 + \mathcal{L}k)(z)$ appear and that for sufficiently large K this reduces to one root.

A sufficient condition for stability is

$$K < \frac{2}{\pi \|g\|_\infty}.$$

In this case $\Re(\mathcal{L}\hat{g}(\mathrm{i}x)) < 2/K$ so that $\mathcal{L}\hat{g}(\mathrm{i}\mathbb{R})$ cannot encircle $2/K$ and the solution is stable. In case of a symmetric distribution with a maximum at 0, this condition is sharp for the first instability because then $(\mathcal{L}\hat{g})(0) = \pi g(0)$.

Comparing the linear stability condition to possible stationary solutions, we can understand the stability condition on the real part $\pi g(x)$ as critical mass density to form an instability while the imaginary part ensures that the order parameter has no drift.

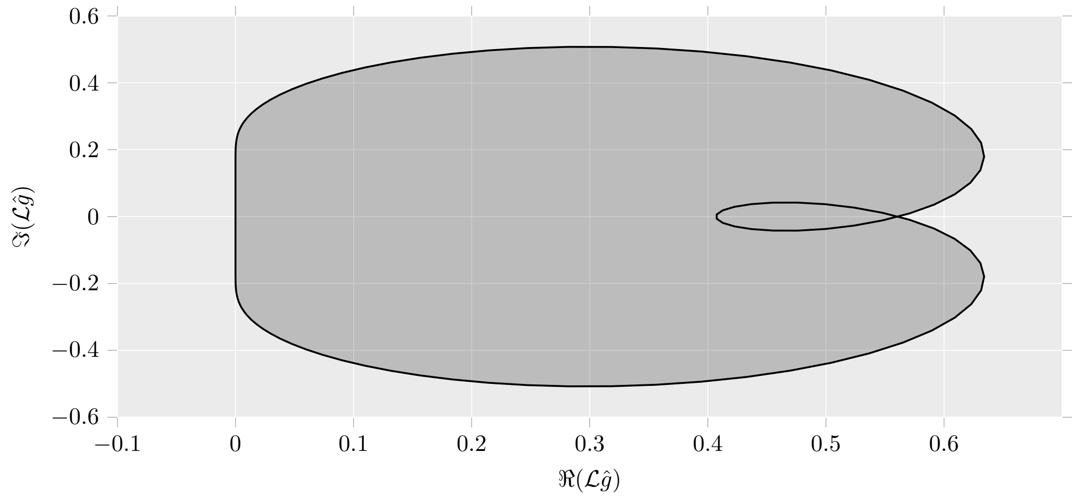


Figure 4.2 – The black line shows $\mathcal{L}\hat{g}(\mathrm{i}\mathbb{R})$ with the enclosed shaded area $\mathcal{L}\hat{g}(\{z : \Re z > 0\})$ for two Gaussian distributions with variance 1 and centred at $\omega = \pm 1.5$.

For $\Re z > 0$, the eigenmode equation

$$\frac{2}{K} = \int_{-\infty}^{\infty} \frac{g(\omega)}{z - \mathrm{i}\omega} \mathrm{d}\omega$$

is known in literature. In particular, Omel'chenko and Wolfrum [120] consider the generalised Kuramoto equation with a phase delay α (Sakaguchi-Kuramoto model [139]). In the finite model the oscillator evolve as

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_i = \omega_i + \frac{K}{N}\sum_{i=1}^{N} \sin(\theta_j - \theta_i - \alpha) \quad \text{for } i = 1, \dots, N.$$

This only adds the factor $\mathrm{e}^{\mathrm{i}\alpha}$ to the order parameter in the evolution equation. Thus all the results on the Kuramoto equation still hold and the stability condition becomes

$$\frac{2\mathrm{e}^{\mathrm{i}\alpha}}{K} = \int_{-\infty}^{\infty} \frac{g(\omega)}{z - \mathrm{i}\omega} \mathrm{d}\omega.$$

Omel'chenko and Wolfrum [120] noted that by varying α the whole shape of the Penrose diagram can be explored and find a very rich behaviour.

4.4 Energy estimate for global stability

With the weight l^{-1} of the spatial mode $l \in \mathbb{N}$, the nonlinear interaction in the evolution (4.1) is skew-Hermitian apart from the interaction of the first mode $l = 1$ with the constant mode $l = 0$. Recalling, that the free transport moves u to the left, we introduced the energy functional I in (4.2), which we repeat here as

$$I(t) = \int_{\xi=0}^{\infty} \sum_{l\geq 1} \frac{1}{l} |u_l(t,\xi)|^2 \phi(\xi) \mathrm{d}\xi,$$

for an increasing weight ϕ, which we scale such that $\phi(0) = 1$.

Formally, we find

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) = -\sum_{l\geq 1} |u_l(t,0)|^2 - \int_{\xi=0}^{\infty} \sum_{l\geq 1} |u_l(t,\xi)|^2 \phi'(\xi) \mathrm{d}\xi + 2\Re \left[\int_0^{\infty} \overline{\eta(t) \frac{K\hat{g}(\xi)}{2}} \overline{u_1(t,\xi)} \phi(\xi) \mathrm{d}\xi \right].$$

The last term can be bounded as

$$2\Re \left[\int_0^{\infty} \overline{\eta(t) \frac{K\hat{g}(\xi)}{2}} \overline{u_1(t,\xi)} \phi(\xi) \mathrm{d}\xi \right] \leq \int_0^{\infty} \left[|u_1(t,\xi)|^2 \phi'(\xi) + |\eta(t)|^2 \left| \frac{K\hat{g}(\xi)}{2} \right|^2 \frac{\phi^2(\xi)}{\phi'(\xi)} \right] \mathrm{d}\xi.$$

Hence, we can control the functional I formally as

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leq (\alpha - 1) |\eta(t)|^2$$

with

$$\alpha = \frac{K^2}{4} \int_{\xi=0}^{\infty} \frac{|\hat{g}(\xi)|^2}{\phi'(\xi)} \phi^2(\xi) \mathrm{d}\xi.$$

This can now be used for the global nonlinear stability. As a first step, we use an approximate procedure to overcome the non-differentiability of u.

Lemma 4.14. *Let $\phi \in C^1(\mathbb{R}^+)$ be a bounded increasing weight with $\phi(0) = 1$ whose derivative ϕ' is bounded and satisfies $\phi'(\xi) > 0$ for $\xi \in \mathbb{R}^+$.*

For the Fourier transform $u \in C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$ of a solution to the Kuramoto equation with initial density $f_{\text{in}} \in L^2(\Gamma)$ define the functional $I(t)$ by Equation (4.2). Then for all $t \in \mathbb{R}^+$

$$I(t) \leq I(0) + (\alpha - 1) \int_0^t |\eta(s)|^2 \mathrm{d}s$$

where

$$\alpha = \frac{K^2}{4} \int_{\xi=0}^{\infty} \frac{|\hat{g}(\xi)|^2}{\phi'(\xi)} \phi^2(\xi) \mathrm{d}\xi.$$

Proof. In order to overcome the non-differentiability, let $\chi \in C_c^{\infty}(\mathbb{R})$ be a non-negative function with $\int_{x\in\mathbb{R}} \chi(x) \mathrm{d}x = 1$ and let $\chi_\delta(x) = \delta^{-1} \chi(x/\delta)$ for $\delta > 0$. Define the mollified u_δ by

$$(u_\delta)_l(t,\xi) = \int_{x\in\mathbb{R}} \chi_\delta(\xi - x) u_l(t,x) \mathrm{d}x.$$

Since u is bounded by 1, also u_δ is bounded by 1 and has a bounded derivative with respect to ξ. Moreover, $\|(u_\delta)_l(t,\cdot)\|_2 \leq \|u_l(t,\cdot)\|_2$.

By Lemma 4.8

$$(u_\delta)_1(t,\xi) = (u_{\mathrm{in},\delta})_1(\xi + l) \\ + \frac{K}{2}\int_0^t \left[\overline{\eta(s)}\,\hat{g}_\delta(\xi + t - s) - \eta(s)\,(u_\delta)_2(s,\xi + t - s)\right] \mathrm{d}s$$

and for $l \geq 2$

$$(u_\delta)_l(t,\xi) = (u_{\mathrm{in},\delta})_l(\xi + tl) \\ + \frac{Kl}{2}\int_0^t \left[\overline{\eta(s)}\,(u_\delta)_{l-1}(s,\xi + (t-s)l) - \eta(s)\,(u_\delta)_{l+1}(s,\xi + (t-s)l)\right] \mathrm{d}s,$$

where $\hat{g}_\delta$ and $u_{\mathrm{in},\delta}$ are the mollifications of $\hat{g}$ and u_{in}, respectively. Hence u_δ is also continuously differentiable with respect to $t \in \mathbb{R}^+$ and satisfies classically

$$\partial_t (u_\delta)_1(t,\xi) = \partial_\xi (u_\delta)_1(t,\xi) + \frac{K}{2}\left[\overline{\eta(t)}\,\hat{g}_\delta(\xi) - \eta(t)\,(u_\delta)_2(t,\xi)\right]$$

and for $l \geq 2$

$$\partial_t (u_\delta)_l(t,\xi) = l\partial_\xi (u_\delta)_l(t,\xi) + \frac{Kl}{2}\left[\overline{\eta(t)}\,(u_\delta)_{l-1}(t,\xi) - \eta(t)\,(u_\delta)_{l+1}(t,\xi)\right].$$

For $M \in \mathbb{N}$ and $N \in \mathbb{R}^+$ define

$$I_{\delta,M,N}(t) = \sum_{l=1}^{M} \int_{\xi=0}^{N} \frac{1}{l} |(u_\delta)_l(t,\xi)|^2 \phi(\xi) \mathrm{d}\xi.$$

Since u_δ and its derivative are bounded in the integration region, we can differentiate under the integral sign to find

$$
\begin{aligned}
\partial_t I_{\delta,M,N}(t) &= \sum_{l=1}^{M} \int_{\xi=0}^{N} \frac{1}{l} \partial_t |(u_\delta)_l(t,\xi)|^2 \phi(\xi) \mathrm{d}\xi \\
&= -\sum_{l=1}^{M} \int_{\xi=0}^{N} |(u_\delta)_l(t,\xi)|^2 \phi'(\xi) \mathrm{d}\xi + \sum_{l=1}^{M} \left[|(u_\delta)_l(t,\xi)|^2 \phi(\xi) \right]_{\xi=0}^{N} \\
&\quad - \int_{\xi=0}^{N} K \Re \left[\eta(t) (u_\delta)_{M+1}(t,\xi) \overline{(u_\delta)_M(t,\xi)} \right] \phi(\xi) \mathrm{d}\xi \\
&\quad + \int_{\xi=0}^{N} K \, \Re \left[\overline{\eta(t)} \hat{g}_\delta(\xi) \overline{(u_\delta)_1(t,\xi)} \right] \phi(\xi) \mathrm{d}\xi \\
&\leq \int_{\xi=0}^{N} \left(-|(u_\delta)_1(t,\xi)|^2 \phi'(\xi) + K \, \Re \left[\overline{\eta(t)} \hat{g}_\delta(\xi) \overline{(u_\delta)_1(t,\xi)} \right] \phi(\xi) \right) \mathrm{d}\xi \\
&\quad - |(u_\delta)_1(t,0)|^2 + \|\phi\|_\infty \sum_{l=1}^{M} |(u_\delta)_l(t,N)|^2 \\
&\quad + K \|\phi\|_\infty \|u_M(t,\cdot)\|_2 \|u_{M+1}(t,\cdot)\|_2 .
\end{aligned}
$$

The first integral can be controlled as

$$
\begin{aligned}
&\int_{\xi=0}^{N} \left(-|(u_\delta)_1(t,\xi)|^2 \phi'(\xi) + K \, \Re \left[\overline{\eta(t)} \hat{g}_\delta(\xi) \overline{(u_\delta)_1(t,\xi)} \right] \phi(\xi) \right) \mathrm{d}\xi \\
&\qquad = - \int_{\xi=0}^{N} \left| (u_\delta)_1(t,\xi) - \frac{K\overline{\eta(t)}}{2} \frac{\hat{g}_\delta(\xi)}{\phi'(\xi)} \phi(\xi) \right|^2 \phi'(\xi) \mathrm{d}\xi \\
&\qquad\quad + |\eta(t)|^2 \frac{K^2}{4} \int_{\xi=0}^{N} \frac{|\hat{g}_\delta(\xi)|^2}{\phi'(\xi)} \phi^2(\xi) \mathrm{d}\xi .
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_{\delta,M,N}(t) &= I_{\delta,M,N}(0) + \int_0^t \partial_t I_{\delta,M,N}(s) \mathrm{d}s \\
&\leq I_{\delta,M,N}(0) + \int_0^t \left[\alpha_{\delta,N} |\eta(s)|^2 - |(u_\delta)_1(s,0)|^2 \right] \mathrm{d}s \\
&\quad + \int_0^t \left[\|\phi\|_\infty \sum_{l=1}^{M} |(u_\delta)_l(s,N)|^2 + K \|\phi\|_\infty \|u_M(s,\cdot)\|_2 \|u_{M+1}(s,\cdot)\|_2 \right] \mathrm{d}s,
\end{aligned}
$$

where

$$
\alpha_{\delta,N} = \frac{K^2}{4} \int_{\xi=0}^{N} \frac{|\hat{g}_\delta(\xi)|^2}{\phi'(\xi)} \phi^2(\xi) \mathrm{d}\xi .
$$

Taking $\delta \to 0$ by dominated convergence

$$\begin{aligned} I_{M,N}(t) \le I_{M,N}(0) + \int_0^t (\alpha - 1)|\eta(s)|^2 \\ + \int_0^t \left[\|\phi\|_\infty \sum_{l=1}^{M} |u_l(s,N)|^2 + K\|\phi\|_\infty \|u_M(s,\cdot)\|_2 \|u_{M+1}(s,\cdot)\|_2 \right] \mathrm{d}s, \end{aligned}$$

where

$$I_{M,N}(t) = \sum_{l=1}^{M} \int_{\xi=0}^{N} \frac{1}{l} |u_l(t,\xi)|^2 \phi(\xi) \mathrm{d}\xi.$$

By dominated and monotone convergence and the Riemann-Lebesgue lemma, we can take $N \to \infty$. By Lemma 2.5 and Plancherel's identity, the L^2 norm of u is uniformly bounded. Hence by taking $M \to \infty$, we arrive at the result by dominated convergence. □

The global stability result, Theorem 4.1, follows from this lemma by finding a suitable weight ϕ such that $\alpha < 1$. For this we want to minimise the integral defining α. Formally, the Euler-Lagrange equation is

$$0 = |\hat{g}(\xi)| + \partial_\xi \left[|\hat{g}(\xi)| \left(\frac{\phi(\xi)}{\phi'(\xi)} \right) \right].$$

With $\phi(0) = 1$ the solution formally is

$$\phi(\xi) = \frac{A}{A - \int_0^\xi |\hat{g}(\zeta)| \mathrm{d}\zeta}$$

for a constant A. If $A > \int_0^\infty |\hat{g}(\xi)| \mathrm{d}\xi$, this indeed defines a bounded increasing weight and we find

$$\frac{K^2}{4} \int_0^\infty \frac{|\hat{g}(\xi)|^2}{\phi'(\xi)} \phi^2(\xi) \mathrm{d}\xi = \frac{K^2}{4} A \int_0^\infty |\hat{g}(\xi)| \mathrm{d}\xi.$$

Thus if $K < K_{ec}$, we have the required control for a suitable A.

Proof of Theorem 4.1. Lemma 4.14 requires the additional condition $\phi'(\xi) > 0$ for all $\xi \in \mathbb{R}^+$. Therefore, we slightly modify the weight formally found before.

For $A > \int_0^\infty |\hat{g}(\xi)| \mathrm{d}\xi$ and $\gamma > 0$ the integral

$$\frac{K^2}{4} \int_0^\infty A \frac{|\hat{g}(\xi)|^2}{|\hat{g}(\xi)| + \mathrm{e}^{-\gamma \xi}} \mathrm{d}\xi$$

tends to

$$\frac{K^2}{4} \left(\int_0^\infty |\hat{g}(\xi)| \mathrm{d}\xi \right)^2$$

as $A \to \int_0^\infty |\hat{g}(\xi)| \mathrm{d}\xi$ and $\gamma \to \infty$. Since $K < K_{ec}$, we can thus find $\bar{\gamma} > 0$ and $\bar{A} >$

$\bar\gamma^{-1} + \int_0^\infty |\hat g(\xi)|\mathrm{d}\xi$ such that

$$\frac{K^2}{4}\int_0^\infty \bar A \frac{|\hat g(\xi)|^2}{|\hat g(\xi)| + \mathrm{e}^{-\bar\gamma\xi}}\mathrm{d}\xi < 1.$$

With $\bar A$ and $\bar\gamma$ define the weight ϕ by

$$\phi(\xi) = \frac{\bar A}{\bar A - \int_0^\xi \left(|\hat g(\zeta)| + \mathrm{e}^{-\bar\gamma\zeta}\right)\mathrm{d}\zeta}.$$

The integrand is continuous so that $\phi \in C^1(\mathbb{R}^+)$ and $\phi'(\xi) > 0$ for all $\xi \in \mathbb{R}^+$. Moreover, ϕ is bounded and $\phi(0) = 1$.

Thus ϕ is a valid weight for the theorem and by Lemma 4.14 a solution with initial density $f_{\mathrm{in}} \in L^2(\Gamma)$ satisfies the required bound

$$I(t) + c\int_0^t |\eta(s)|^2\mathrm{d}s \le I(0)$$

with

$$c = 1 - \alpha = 1 - \frac{K^2}{4}\int_0^\infty \bar A\frac{|\hat g(\xi)|^2}{|\hat g(\xi)| + \mathrm{e}^{-\bar\gamma\xi}}\mathrm{d}\xi > 0.$$

For general initial data, mollify the initial density f_{in} as $f_{\mathrm{in},\epsilon} = f_{\mathrm{in}} * \chi_\epsilon$ where χ_ϵ is the scaled Gaussian distribution in phase space Γ. Then Lemma 4.14 applies to the mollified solution. By the well-posedness of the problem (Theorem 2.4), the mollified Fourier transform converges pointwise to the original Fourier transform as $\epsilon \to 0$. Using Fatou's lemma, we can take the limit and conclude the result.

For the initial bound, use Plancherel to note

$$I(0) \le \|\phi\|_\infty \sum_{l\ge 1}\int_{\xi\in\mathbb{R}^+} |u_{\mathrm{in}}(l,\xi)|^2\mathrm{d}\xi \le \|\phi\|_\infty \|f\|_2^2. \qquad \square$$

The energy functional is an L^2 norm over the ξ-variable of u and as such, we can only expect a control on the order parameter in L^2 over time by the free transport. Thus this global result does not pointwise show $\eta(t) \to 0$ as $t \to \infty$.

For a pointwise control, we can, however, note that $\partial_\xi u$ satisfies the same equation as u, i.e.

$$\begin{cases} \partial_t(\partial_\xi u_1(t,\xi)) = \partial_\xi(\partial_\xi u_1(t,\xi)) + \dfrac{K}{2}\left[\overline{\eta(t)}\left(\partial_\xi \hat g(\xi)\right) - \eta(t)\left(\partial_\xi u_2(t,\xi)\right)\right], \\ \partial_t(\partial_\xi u_l(t,\xi)) = l\partial_\xi(\partial_\xi u_l(t,\xi)) + \dfrac{Kl}{2}\left[\overline{\eta(t)}\left(\partial_\xi u_{l-1}(t,\xi)\right) - \eta(t)\left(\partial_\xi u_{l+1}(t,\xi)\right)\right] \text{ for } l \ge 2. \end{cases}$$

Therefore, we can consider the adapted energy functional

$$I(t) = \int_{\xi=0}^\infty \sum_{l\ge1}\frac{1}{l}|u_l(t,\xi)|^2\phi(\xi)\mathrm{d}\xi + \epsilon\int_{\xi=0}^\infty\sum_{l\ge1}\frac{1}{l}|\partial_\xi u_l(t,\xi)|^2\psi(\xi)\mathrm{d}\xi,$$

where ϵ is a positive constant and ψ is a weight with the same properties as ϕ. By the same argument we now find

$$I(t) \le I(0) + (\alpha + \epsilon\beta - 1) \int_0^t |\eta(s)|^2 \mathrm{d}s - \int_0^t |u_2(s,0)|^2 \mathrm{d}s - \epsilon \int_0^t |\partial_\xi u_1(s,0)|^2 \mathrm{d}s,$$

where

$$\beta = \frac{K^2}{4} \int_{\xi=0}^{\infty} \frac{|\partial_\xi \hat{g}(\xi)|^2}{\psi'(\xi)} \psi^2(\xi) \mathrm{d}\xi.$$

As the order parameter $\eta(t)$ satisfies

$$\partial_t \overline{\eta(t)} = \partial_\xi u_1(t,0) + \frac{K}{2} \left[\eta(t)\, \hat{g}(0) - \overline{\eta(t)}\, u_2(t,0) \right],$$

we can in the stable case $K < K_{ec}$ choose a suitable ψ and small enough $\epsilon > 0$, so that we can control

$$\int_0^\infty \left[|\eta(t)|^2 + \epsilon |\partial_t \eta(t)|^2 \right] \mathrm{d}t.$$

For a frequency distribution with enough moments so that u is differentiable enough with respect to τ, this means that a solution can be controlled as

$$\int_0^\infty \left[|\eta(t)|^2 + \epsilon |\partial_t \eta(t)|^2 \right] \mathrm{d}t < C I(0)$$

for a constant C. Hence by the Sobolev embedding theorem, we have a pointwise control, i.e. $\eta(t) \to 0$ as $t \to \infty$.

4.5 Nonlinear stability

In Fourier variables the nonlinear interaction is given by $R : C(\mathbb{N} \times \mathbb{R}) \mapsto C(\mathbb{N} \times \mathbb{R})$ defined as

$$\begin{cases} R(v)(1,\xi) = -\dfrac{K}{2} \overline{v_1(0)} v_2(\xi), \\ R(v)(l,\xi) = \dfrac{Kl}{2} \left[v_1(0) v_{l-1}(\xi) - \overline{v_1(0)} v_{l+1}(\xi) \right] \text{ for } l \ge 2 \end{cases}$$

for $v \in C(\mathbb{N} \times \mathbb{R})$. Here we see that the nonlinearity is not bounded as it increases with the spatial mode l. In the case of exponential convergence, we can match this with the faster decay of the linear operator in higher modes. In the algebraic case, we compensate for this in the norm by the factor $(1+t)^{-\alpha(l-1)}$.

We characterise the solution by Duhamel's principle on the linear evolution.

Lemma 4.15. *Let $\hat{g} \in C(\mathbb{R})$ and e^{tL} be the linear evolution from Lemma 4.11. For $u_{\mathrm{in}} \in$*

$C(\mathbb{N}\times\mathbb{R})$ define the map $T : C(\mathbb{R}^+ \times \mathbb{N}\times\mathbb{R}) \mapsto C(\mathbb{R}^+ \times \mathbb{N}\times\mathbb{R})$ by

$$(Tu)_l(t,\xi) = (\mathrm{e}^{tL}u_{\mathrm{in}})_l(\xi) + \int_0^t (\mathrm{e}^{(t-s)L}R(u(s,\cdot))_l(\xi)\,\mathrm{d}s.$$

Then $u \in C(\mathbb{R}^+ \times \mathbb{N}\times\mathbb{R})$ satisfies Equation (4.6) for all $h \in C_0^1(\mathbb{R}^+ \times \mathbb{N}\times\mathbb{R})$ if and only if $Tu = u$.

Proof. For u consider the nonlinear terms as forcing in Equation (4.6). The linear problem has a unique solution given by Tu. Hence u is a solution if and only if $Tu = u$. □

For the exponential and algebraic stability we want to propagate the norms given in the introduction using Lemma 4.15. However, we do not know a priori if these norms stay finite. Therefore, we construct a solution using Banach fixed point theorem and by the uniqueness (Theorem 4.9) this must be the mean-field solution.

Proof of Theorem 4.3. Define on $C(\mathbb{R}^+ \times \mathbb{N}\times\mathbb{R})$ the norm $\|\cdot\|_e$ by

$$\|u\|_e = \sup_{t\in\mathbb{R}^+}\sup_{l\ge 1}\sup_{\xi\in\mathbb{R}} |u_l(t,\xi)|\mathrm{e}^{a(\xi+tl/2)}$$

for $u \in C(\mathbb{R}^+ \times \mathbb{N}\times\mathbb{R})$.

By the assumption, Lemma 4.13 applies with a finite constant c_1. This shows for $u \in C(\mathbb{R}^+ \times \mathbb{N}\times\mathbb{R})$ and $t \in \mathbb{R}^+, \xi \in \mathbb{R}$

$$\begin{aligned}\mathrm{e}^{a(\xi+t/2)}|(Tu)_1(t,\xi)| &\le c_1\mathrm{e}^{-at/2}M_{\mathrm{in}} + \frac{Kc_1\|u\|_e^2}{2}\int_0^t \mathrm{e}^{-a(t-s)/2}\mathrm{d}s\\ &\le c_1M_{\mathrm{in}} + \frac{Kc_1}{a}\|u\|_e^2\end{aligned}$$

and by the explicit form (Lemma 4.11) for $l \ge 2$

$$\begin{aligned}\mathrm{e}^{a(\xi+tl/2)}|(Tu)_l(t,\xi)| &\le \mathrm{e}^{-atl/2}M_{\mathrm{in}} + Kl\|u\|_e^2\int_0^t \mathrm{e}^{-al(t-s)/2}\mathrm{d}s\\ &\le M_{\mathrm{in}} + \frac{2K}{a}\|u\|_e^2.\end{aligned}$$

Hence

$$\|Tu\|_e \le c_1M_{\mathrm{in}} + c_2\|u\|_e^2$$

where $c_2 = \max(c_1,2)K/a$. Therefore, there exists $\delta > 0$ such that, if $M_{\mathrm{in}} \le \delta$ and $\|u\|_e \le (1+c_1)M_{\mathrm{in}}$, then $\|Tu\|_e \le (1+c_1)M_{\mathrm{in}}$.

If $u, \tilde{u} \in C(\mathbb{R}^+ \times \mathbb{N}\times\mathbb{R})$ satisfy $\|u\|_e, \|\tilde{u}\|_e \le (1+c_1)\delta$, then as before we find

$$\|Tu - T\tilde{u}\|_e \le \frac{2K(1+c_1)\delta}{a}\max(c_1,2)\|u-\tilde{u}\|_e.$$

Hence we can choose δ small enough such that T is a contraction on $\{u \in C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R}) : \|u\|_e \leq (1 + c_1)M_{\text{in}}\}$ for $M_{\text{in}} \leq \delta$.

Thus for $M_{\text{in}} \leq \delta$, there exists a unique fixed point u with $\|u\|_e \leq (1 + c_1)M_{\text{in}}$. By Theorem 4.9, this u must be equal to the Fourier transform of the solution to the Kuramoto equation. □

The structure for the proof of algebraic stability is very similar to the previous proof of exponential estimate.

Proof of Theorem 4.4. Define on $C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$ the norm $\|\cdot\|_a$ by

$$\|u\|_a = \sup_{t\in\mathbb{R}^+} \sup_{l\geq 1} \sup_{\xi\in\mathbb{R}^+} |u_l(t,\xi)| \frac{(1+\xi+t)^b}{(1+t)^{\alpha(l-1)}}$$

for $u \in C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$.

By the assumption, Lemma 4.13 applies with a finite constant γ_1. This shows for $u \in C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$ and $t \in \mathbb{R}^+, \xi \in \mathbb{R}$

$$\begin{aligned} |(Tu)_1(t,\xi)|(1+\xi+t)^b &\leq \gamma_1 M_{\text{in}} + \frac{K\|u\|_a^2}{2} \int_0^t \gamma_1 (1+s)^{\alpha-b} \mathrm{d}s \\ &\leq \gamma_1 M_{\text{in}} + \frac{\gamma_1 K}{2(b-1-\alpha)} \|u\|_a^2. \end{aligned}$$

For modes $l \geq 2$ the explicit formula Lemma 4.11 shows

$$|(Tu)_l(t,\xi)| \frac{(1+\xi+t)^b}{(1+t)^{\alpha(l-1)}} \leq M_{\text{in}} + K\|u\|_a^2 l \int_0^t \frac{(1+s)^{\alpha l-b}}{(1+t)^{\alpha(l-1)}} \mathrm{d}s.$$

For $l \in \mathbb{N}$ with $\alpha l > b - 1$ we can bound

$$l \int_0^t \frac{(1+s)^{\alpha l - b}}{(1+t)^{\alpha(l-1)}} \mathrm{d}s \leq \frac{l}{\alpha l - b + 1}$$

by a finite constant independent of l.

For $l \in \mathbb{N}$ with $l \geq 2$ and $\alpha l - b \leq 1$ the integral $\int_0^t (1+s)^{\alpha l - b} \mathrm{d}s$ grows at most logarithmically, so that it can be bounded by $(1+t)^{\alpha(l-1)}$. Hence there exists a finite γ_2 satisfying

$$\|Tu\|_a \leq \gamma_1 M_{\text{in}} + \gamma_2 \|u\|_a^2.$$

Thus there exists $\delta > 0$ such that $M_{\text{in}} \leq \delta$ implies that $\|Tu\|_a \leq (1 + \gamma_1)M_{\text{in}}$ holds, if $\|u\|_a \leq (1 + \gamma_1)M_{\text{in}}$.

For two functions $u, \tilde{u} \in C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$ with $\|u\|_a, \|\tilde{u}\|_a \leq (1 + \gamma_1)\delta$, we find for $t \in \mathbb{R}^+$

and $\xi \in \mathbb{R}^+$ as before

$$\begin{aligned}|(Tu - T\tilde{u})_1(t,\xi)|(1+\xi+t)^b &\leq K\gamma_1(1+\gamma_1)\delta\|u-\tilde{u}\|_a \int_0^t (1+s)^{\alpha-b}\mathrm{d}s \\ &\leq \frac{K\gamma_1(1+\gamma_1)\delta}{b-1-\alpha}\|u-\tilde{u}\|_a\end{aligned}$$

and for $l \geq 2$

$$|(Tu - T\tilde{u})_l(t,\xi)|\frac{(1+\xi+t)^b}{(1+t)^{\alpha(l-1)}} \leq 2K(1+\gamma_1)\delta\|u-\tilde{u}\|_a l \int_0^t \frac{(1+s)^{\alpha l-b}}{(1+t)^{\alpha(l-1)}}\mathrm{d}s.$$

By the same argument, the factors with l are uniformly bounded over $l \geq 2$, so that, for small enough $\delta > 0$, the map T is a contraction on $\{u \in C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R}) : \|u\|_a \leq (1+\gamma_1)M_{\mathrm{in}}\}$ when $M_{\mathrm{in}} \leq \delta$. Hence there exists a unique fixed point $Tu = u$ with $\|u\|_a \leq (1+\gamma_1)M_{\mathrm{in}}$. For every $T > 0$, the restricted uniqueness (Theorem 4.9 and Theorem 4.10) shows that this solution must be equal to the Fourier transform of the mean-field solution to the Kuramoto equation up to time T. Hence u always equals the Fourier transform of the solution and the bound on u proves the theorem. □

4.6 Bifurcation analysis

4.6.1 Linear eigenmodes and spectral decomposition

By the study of the Volterra equation, we can easily find the eigenmodes for the order parameter if the system is unstable or critical. Going back to the linear evolution of u, we can use this to characterise the stable subspace.

For the characterisation of the stable subspace we use the functional α. Abstractly, it is the corresponding element in the kernel of the adjoint L^*. For this work we can, however, characterise it explicitly.

Lemma 4.16. *For $a \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$ with $\Re\lambda > -a$ define the continuous functional $\alpha_{\lambda,j}$ on $\mathcal{Y}^a$ by*

$$\alpha_{\lambda,j}(u) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^j [\mathcal{L}u_1](z)\bigg|_{z=\lambda} = \int_0^\infty u_1(t)(-t)^j \mathrm{e}^{-\lambda t}\mathrm{d}t.$$

Proof. Since $|u(1,\xi)| \leq \mathrm{e}^{-a\xi}\|u\|_{\mathcal{Y}^a}$, the functional is well-defined, continuous and, for $j \geq 1$, the derivative can be computed by differentiating under the integral sign. □

Lemma 4.17. *Assume $a \in \mathbb{R}$ and $\hat{g} \in C(\mathbb{R})$ as in Theorem 4.5 and let $\lambda_1, \ldots, \lambda_n$ be the roots of multiplicity $p_1, \ldots, p_n$ as in Theorem 4.5. Define the closed subspace*

$$\mathcal{Y}^a_s = \{u \in \mathcal{Y}^a : \alpha_{\lambda_i,j}(u) = 0 \textit{ for } i = 1, \ldots, n \textit{ and } j = 0, \ldots, p_i - 1\}.$$

Then $\mathcal{Y}_s^a$ *is invariant under the linear evolution* e^{tL} *and* $u \in \mathcal{Y}^a$ *is in* $\mathcal{Y}_s^a$ *if and only if for* $\nu(t) = (\mathrm{e}^{tL}u)(1,0)$ *holds* $\nu \in L^\infty(\mathbb{R}^+, \exp_a)$. *In this case*

$$\sup_{t\geq 0} \mathrm{e}^{at} \|(\mathrm{e}^{tL}u)_1\|_{L^\infty(\mathbb{R}^+,\exp_a)} \leq c_1 \|u_1\|_{L^\infty(\mathbb{R}^+,\exp_a)} \tag{4.10}$$

holds with $c_1 = 1 + (1 + \|r_s\|_{L^1(\mathbb{R}^+,\exp_a)})(K/2)\|g\|_{L^1(\mathbb{R}^+,\exp_a)} < \infty$.

Proof. If $u \in \mathcal{Y}_s^a$, the solution of the Volterra equation for ν in Theorem 4.5 simplifies to

$$\nu(t) = u_1(t) - (r_s * u_1)(t).$$

Since $r_s \in L^1(\mathbb{R}, \exp_a)$, this shows $\nu \in L^\infty(\mathbb{R}^+, \exp_a)$ and, as in Lemma 4.13, Equation (4.10) follows with the given constant c_1.

Conversely, assume $\nu \in L^\infty(\mathbb{R}^+, \exp_a)$. Then at $z \in \mathbb{C}$ with $\Re z > -a$ the Laplace transforms of ν, k and u_1 are analytic and, by the Volterra equation (4.4), satisfy

$$(\mathcal{L}\nu)(z) + (\mathcal{L}k)(z)(\mathcal{L}\nu)(z) = [\mathcal{L}u_1](z).$$

Hence the LHS has roots $\lambda_1, \ldots, \lambda_n$ of multiplicities $p_1, \ldots, p_n$. Thus the RHS must have the same roots, i.e. $\alpha_{\lambda_i,j}(u) = 0$ for $i = 1, \ldots, n$ and $j = 0, \ldots, p_i - 1$, which shows $u \in \mathcal{Y}_s^a$.

The growth condition $\nu \in L^\infty(\mathbb{R}^+, \exp_a)$ under the linear evolution is invariant, so that $\mathcal{Y}_s^a$ is invariant under the linear evolution. □

We can use this to control the decay in $\mathcal{Z}_s^a = \mathcal{Y}_s^a \cap \mathcal{Z}^a$ under the linear evolution and the forcing term from a Duhamel equation.

Lemma 4.18. *Assume the hypothesis of Lemma 4.17 with* $a \geq 0$. *Then for* $u \in \mathcal{Z}_s^a$ *we have for* $t \in \mathbb{R}^+$

$$\|\mathrm{e}^{tL}u\|_{\mathcal{Z}^a} \leq c_1 \mathrm{e}^{-at} \|u\|_{\mathcal{Z}^a}.$$

For $0 \leq \mu < a$ *consider* $F \in C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R})$ *with* $F(t,\cdot) \in \mathcal{Y}_s^a$ *for all* $t \in \mathbb{R}^+$ *and norm*

$$\|F\|_{\mathcal{Y}^a,-\mu} := \sup_{t\in\mathbb{R}^+} \mathrm{e}^{-\mu t} \|F(t,\cdot)\|_{\mathcal{Y}^a} < \infty.$$

Then

$$v = \int_0^\infty \mathrm{e}^{tL} F(t,\cdot,\cdot)\mathrm{d}t$$

is well-defined and $v \in \mathcal{Z}_s^a$ *with*

$$\|v\|_{\mathcal{Z}^a} \leq c(\mu)\|F\|_{\mathcal{Y}^a,-\mu}$$

where $c(\mu) = (a-\mu)^{-1} c_1$.

Remark 4.19. The forcing F is measured in the weaker norm $\mathcal{Y}^a$ and we are able to recover the control in the stronger norm $\mathcal{Z}^a$.

Proof. Since u is in $\mathcal{Z}^a_s$, we find by Lemma 4.17 for $t \in \mathbb{R}^+$ and $\xi \in \mathbb{R}$

$$\mathrm{e}^{a\xi}|(\mathrm{e}^{tL}u)_1(\xi)| \leq c_1 \mathrm{e}^{-at}\|u\|_{\mathcal{Z}^a}$$

and by (4.8) for $l \geq 2$

$$\mathrm{e}^{a\xi}|(\mathrm{e}^{tL}u)_l(\xi)| \leq \mathrm{e}^{-alt}\|u\|_{\mathcal{Z}^a},$$

which shows the first claim.

For the second part note that for every $t \in \mathbb{R}^+$ the integrand $\mathrm{e}^{tL}F(t,\cdot,\cdot)$ is continuous. By Lemma 4.17 we can control for $\xi \in \mathbb{R}$

$$\mathrm{e}^{a\xi}|[\mathrm{e}^{tL}F(t,\cdot)]_1(\xi)| \leq c_1\mathrm{e}^{-at}\|F(t,\cdot)\|_{\mathcal{Y}^a} \leq c_1\mathrm{e}^{(\mu-a)t}\|F\|_{\mathcal{Y}^a,\mu}$$

and for $l \geq 2$ by Equation (4.8)

$$\mathrm{e}^{a\xi}|[\mathrm{e}^{tL}F(t,\cdot)]_l(\xi)| \leq l\mathrm{e}^{(\mu-al)t}\|F\|_{\mathcal{Y}^a,\mu}.$$

These bounds are uniformly integrable as

$$\int_0^\infty c_1\mathrm{e}^{(\mu-a)t}\mathrm{d}t = \frac{c_1}{a-\mu}$$

and for $l \geq 2$

$$l\int_0^\infty \mathrm{e}^{(\mu-al)t}\mathrm{d}t = \frac{l}{al-\mu} = \frac{1}{a-\mu/l}.$$

Hence the integral is well-defined and defines a continuous function $v \in C(\mathbb{N}\times\mathbb{R})$. Moreover, it shows the claimed control of $\|v\|_{\mathcal{Z}^a}$. $\square$

In order to find the eigenmodes in $\mathcal{Z}^a$ and the projection to them, we consider the linear generator L on $\mathcal{Z}^a$.

Lemma 4.20. *Assume the hypothesis of Lemma 4.17 with $a > 0$. Define the closed operator $L : D(L) \subset \mathcal{Z}^a \mapsto \mathcal{Y}^a$ with $D(L) = \{u \in C^1(\mathbb{N}\times\mathbb{R}) : \|u\|_{\mathcal{Z}^a} < \infty\}$ by*

$$(Lu)_1(\xi) = \partial_\xi u_1(\xi) + \frac{K}{2}u_1(0)\hat{g}(\xi)$$

and for $l \geq 2$ by

$$(Lu)_l(\xi) = l\partial_\xi u_l(\xi).$$

Then in $\{\lambda : \Re\lambda > -a\}$ the spectrum of L equals $\sigma = \{\lambda_1, \ldots, \lambda_n\}$. For $\lambda \notin \sigma$ and $\Re\lambda > -a$

the resolvent $(L-\lambda)^{-1}$ is given by

$$[(L-\lambda)^{-1}v]_1(\xi) = -\int_\xi^\infty \mathrm{e}^{-\lambda(\zeta-\xi)} \left(v_1(\zeta) + \frac{K}{2} \frac{[\mathcal{L}v_1](\lambda)}{1+(\mathcal{L}k)(\lambda)} \hat{g}(\zeta) \right) \mathrm{d}\zeta \tag{4.11}$$

for $\xi \in \mathbb{R}$ where k is the convolution kernel from Equation (4.5) *and for $l \geq 2$*

$$[(L-\lambda)^{-1}v]_l(\xi) = -\frac{1}{l} \int_\xi^\infty \mathrm{e}^{-(\lambda/l)(\zeta-\xi)} v(l,\zeta) \mathrm{d}\zeta. \tag{4.12}$$

Proof. By classical analysis the differentiation and the uniform limit can be interchanged so that L is closed.

For the resolvent fix $\lambda \in \mathbb{C}$ with $\Re\lambda > -a$ and $\lambda \notin \sigma$ and $v \in \mathcal{Y}^a$. We now look for solutions of $(L-\lambda)u = v$ in $u \in D(L) \subset \mathcal{Z}^a$.

For $l \geq 2$ this implies

$$l\partial_\xi u_l(\xi) - \lambda u_l(\xi) = v_l(\xi) \;\Rightarrow\; \partial_\xi \left(\mathrm{e}^{-(\lambda/l)\xi} u_l(\xi) \right) = \frac{1}{l} v_l(\xi) \mathrm{e}^{-(\lambda/l)\xi}.$$

Since $\Re\lambda > -a$ this shows that u is uniquely given by Equation (4.12).

For $l = 1$ we likewise find that u must satisfy

$$u_1(\xi) = -\int_\xi^\infty \mathrm{e}^{-\lambda(\zeta-\xi)} \left(v_1(\zeta) - \frac{K}{2} u_1(0) \hat{g}(\zeta) \right) \mathrm{d}\zeta.$$

Hence by taking $\xi = 0$ we find that u must satisfy

$$u_1(0)\left(1 + (\mathcal{L}k)(\lambda)\right) = -[\mathcal{L}v_1](\lambda).$$

Since $\lambda \notin \sigma$, this determines $u_1(0)$ uniquely and the solution is given by Equation (4.11). Moreover, the found solution satisfies $\|u\|_{\mathcal{Z}^a} < \infty$. Hence such a λ is not in the spectrum of L and the resolvent map takes the given form.

If $\lambda \in \sigma$, the above shows that $L - \lambda$ is not injective, so that λ is in the spectrum. □

Equation (4.11) shows that at a root λ of multiplicity p the residue of the resolvent map $(L-\lambda)^{-1}$ is in the space spanned by $\langle z_{\lambda,j} : j = 0, \ldots, p-1 \rangle$ where for $\xi \in \mathbb{R}$

$$(z_{\lambda,j})_1(\xi) = \frac{K}{2} \int_\xi^\infty \mathrm{e}^{-\lambda(\zeta-\xi)} (\xi - \zeta)^j \hat{g}(\zeta) \, \mathrm{d}\zeta$$

and for $l \geq 2$

$$(z_{\lambda,j})_l(\xi) = 0.$$

Since we assume $\hat{g} \in L^1(\mathbb{R}, \exp_a)$, we always have $z_{\lambda,j} \in D(L) \subset \mathcal{Z}^a$.

Lemma 4.21. *Assume the hypothesis of Lemma 4.17 with $a > 0$. Then for a root λ with $\Re\lambda > -a$ of $1 + (\mathcal{L}k)(z)$ with multiplicity p, the elements $z_{\lambda,0}, \ldots, z_{\lambda,p-1}$ are generalised* *eigenvectors of L with eigenvalue λ.*

Proof. Note that for $j = 0, \dots, p-1$

$$(z_{\lambda,j})_1(0) = -\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^j (\mathcal{L}k)(z)\Big|_{z=\lambda} = \begin{cases} 1 & j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then by direct computation

$$Lz_{\lambda,j} = \begin{cases} \lambda z_{\lambda,j} & j = 0, \\ \lambda z_{\lambda,j} + j z_{\lambda,j-1} & \text{otherwise.} \end{cases}$$ □

Using holomorphic functional calculus, we can construct the required projection.

Lemma 4.22. *Assume the hypothesis of Lemma 4.17 with $a > 0$. There exists a continuous projection $P_{cu} : \mathcal{Y}^a \mapsto \mathcal{Z}^a_{cu}$ such that the complementary projection $P_s = I - P_{cu}$ maps to $\mathcal{Y}^a_s$.*

Proof. Take γ a contour in $\{z \in \mathbb{C} : \Re z > -a\}$ encircling all roots $\lambda_1, \dots, \lambda_n$ once and define

$$P_{cu}(u) = \frac{-1}{2\pi \mathrm{i}} \oint_\gamma (L - \lambda)^{-1}(u)\,\mathrm{d}\lambda.$$

Then P_{cu} is a continuous projection. Since its value is given by the residues, Equation (4.11) shows that P_{cu} maps into $\mathcal{Z}^a_{cu}$.

On the image of the complementary projection P_s the resolvent map $(L-\lambda)^{-1}$ is continuous for $\Re\lambda > -a$. By Equation (4.11) this can only be true if for all roots $\lambda_1, \dots, \lambda_n$ with multiplicities $p_1, \dots, p_n$ also $\mathcal{L}u_1$ has a root of matching multiplicity, i.e. $\alpha_{\lambda_i,j}$ must be vanishing for $i = 1, \dots, n$ and $j = 0, \dots, p_i - 1$. Hence the image is within $\mathcal{Y}^a_s$. □

4.6.2 Center-manifold reduction

Since the nonlinearity R is smooth from $\mathcal{Z}^a$ to $\mathcal{Y}^a$, the center-unstable manifold reduction follows from the decay on the stable part (Lemma 4.18) and the continuous projections (Lemma 4.22). The center-manifold reduction in Banach spaces is discussed in [155, Chapter 2], which can be adapted to our case with a weak PDE as Duhamel's principle holds again. Moreover, instead of a center manifold reduction, we do a center-unstable manifold reduction [154, Section 1.5], as this implies convergence to the reduced manifold. The statements are also discussed in [68].

In order to understand the bifurcation behaviour, the center-unstable manifold is constructed with $\epsilon = K - K_c$ as additional parameter. Furthermore, we need to localise the non-linearity around the incoherent state. This localisation might affect the resulting reduced manifold, which is not unique. Moreover, we choose the localisation to preserve the rotation symmetry of the problem so that the resulting manifold has the same symmetry.

Hence we study the solution in $\tilde{\mathcal{Z}}^a = \mathcal{Z}^a \times \mathbb{R}$ and $\tilde{\mathcal{Y}}^a = \mathcal{Y}^a \times \mathbb{R}$ with norms $\|(u,\epsilon)\|_{\tilde{\mathcal{Z}}^a} = \max(\|u\|_{\mathcal{Z}^a}, |\epsilon|)$ and $\|(u,\epsilon)\|_{\tilde{\mathcal{Y}}^a} = \max(\|u\|_{\mathcal{Y}^a}, |\epsilon|)$. In these define the linear evolution operator $\mathrm{e}^{t\tilde{L}}$ as

$$\mathrm{e}^{t\tilde{L}}\tilde{u} = \mathrm{e}^{t\tilde{L}}(u,\epsilon) = (\mathrm{e}^{tL}u, \epsilon).$$

The center-unstable space $\tilde{Z}^a_{cu}$ is now spanned by $(0,1)$ and $(z_{\lambda_i,j}, 0)$ for $i = 1,\dots,n$ and $j = 0,\dots,p_i - 1$. The projection $\tilde{P}_{cu}$ is defined as $\tilde{P}_{cu}(u,\epsilon) = (P_{cu}u, \epsilon)$ with P_{cu} from Lemma 4.22. The complementary projection $\tilde{P}_s$ maps into the stable part $\tilde{\mathcal{Y}}^a_s = \mathcal{Y}^a_s \times \{0\}$.

For the center-unstable manifold reduction the used properties of the linear evolution are collected in the following lemma.

Lemma 4.23. *Assume the hypothesis of Theorem 4.6. Then there exists a continuous projection $\tilde{P}_{cu} : \mathcal{Y}^a \mapsto \tilde{\mathcal{Z}}^a_{cu}$ with complementary projection $\tilde{P}_s = I - \tilde{P}_{cu}$ mapping $\tilde{\mathcal{Y}}^a$ to $\tilde{\mathcal{Y}}^a_s$. The subspaces $\tilde{\mathcal{Z}}^a_{cu}$ and $\tilde{\mathcal{Y}}^a_s$ are invariant under the linear evolution. On $\tilde{\mathcal{Z}}^a_{cu}$ the linear evolution acts as finite-dimensional matrix exponential whose generator has spectrum $\lambda_1,\dots,\lambda_n$. On $\tilde{\mathcal{Z}}^a_s = \tilde{P}_s\tilde{\mathcal{Z}}^a$ there exists a constant c_1 such that for $t \in \mathbb{R}^+$ and $u \in \tilde{\mathcal{Z}}^a_s$*

$$\|\mathrm{e}^{t\tilde{L}}u\|_{\tilde{\mathcal{Z}}^a} \le c_1 \mathrm{e}^{-at}\|u\|_{\tilde{\mathcal{Z}}^a}$$

and there exists a continuous function $c : [0,a) \mapsto \mathbb{R}^+$ such that for $\mu \in [0,a)$ and $F \in C(\mathbb{R}^+ \times \mathbb{N} \times \mathbb{R} \times \mathbb{R})$ with $F(t,\cdot) \in \tilde{\mathcal{Y}}^a_s$ for $t \in \mathbb{R}^+$ and norm

$$\|F\|_{\mathcal{Y}^a,-\mu} := \sup_{t\in\mathbb{R}^+} \mathrm{e}^{-\mu t}\|F(t,\cdot)\|_{\tilde{\mathcal{Y}}^a} < \infty,$$

the integral

$$v = \int_0^\infty \mathrm{e}^{t\tilde{L}}F(t,\cdot)\,\mathrm{d}t$$

is well-defined and $v \in \tilde{\mathcal{Z}}^a_s$ with

$$\|v\|_{\tilde{\mathcal{Z}}^a} \le c(\mu)\|F\|_{\tilde{\mathcal{Y}}^a,-\mu}.$$

Proof. The linear evolution on ϵ is constant so that the properties follow from the study of e^{tL}, i.e. Lemmas 4.18, 4.21 and 4.22. □

In the extended dynamics for $\tilde{u}$ in $\tilde{\mathcal{Z}}^a$, the nonlinearity takes the form $\tilde{R} : \tilde{\mathcal{Z}}^a \mapsto \tilde{Y}^a$ given by

$$\tilde{R}(\tilde{u}) = (N(\tilde{u}), 0),$$

where $N : \tilde{\mathcal{Z}}^a \mapsto \mathcal{Y}^a$ is for $\tilde{u} = (u,\epsilon)$ defined by

$$(N(\tilde{u}))_1(\xi) = \frac{\epsilon}{2}u_1(0)\hat{g}(\xi) - \frac{(K_c+\epsilon)}{2}\overline{u_1(0)}u_2(\xi)$$

for $\xi \in \mathbb{R}$ and for $l \leq 2$

$$(N(\tilde{u}))_l(\xi) = \frac{(K_c+\epsilon)l}{2}\left(u_1(0)u_{l-1}(\xi) - \overline{u_1(0)}u_{l+1}(\xi)\right).$$

For the localisation let $\chi : \mathbb{R}^+ \mapsto [0,1]$ be a smooth function with $\chi(x) = 1$ for $x \leq 1$ and $\chi(x) = 0$ for $x \geq 2$. Then let $\chi_\delta(x) = \chi(x/\delta)$ and $s_\delta(x) = x\chi_\delta(|x|)$. With this define the localised nonlinearity $\tilde{R}_\delta : \tilde{\mathcal{Z}}^a \mapsto \tilde{\mathcal{Y}}^a$ by

$$\tilde{R}_\delta(\tilde{u}) = (N_\delta(\tilde{u}), 0)$$

where $N_\delta : \tilde{\mathcal{Z}}^a \mapsto \mathcal{Y}^a$ is for $\tilde{u} = (u, \epsilon)$ defined by

$$(N_\delta(\tilde{u}))_1(\xi) = \frac{s_\delta(\epsilon)}{2}s_\delta(u_1(0))\hat{g}(\xi) - \frac{(K_c+s_\delta(\epsilon))}{2}s_\delta(\overline{u_1(0)})s_\delta(u_2(\xi))$$

for $\xi \in \mathbb{R}$ and for $l \geq 2$

$$(N_\delta(\tilde{u}))_l(\xi) = \frac{(K_c+s_\delta(\epsilon))l}{2}\left(s_\delta(u_1(0))s_\delta(u_{l-1}(\xi)) - s_\delta(\overline{u_1(0)})s_\delta(u_{l+1}(\xi))\right).$$

Then for $\delta > 0$, the localised nonlinearity $\tilde{R}_\delta$ agrees with the original nonlinearity of the mean-field equation for $\|\tilde{u}\|_{\tilde{\mathcal{Z}}^a} \leq \delta$, is in $C^\infty(\tilde{\mathcal{Z}}^a, \tilde{\mathcal{Y}}^a)$ with $\tilde{R}_\delta(0) = D\tilde{R}_\delta(0) = 0$ and is bounded and Lipschitz continuous with constants tending to 0 as $\delta \to 0$.

We now study the evolution PDE

$$\begin{cases} \partial_t u_1(t,\xi) = \partial_\xi u_1(t,\xi) + \dfrac{K_c}{2}u_1(t,0)\hat{g}(\xi) + (N_\delta(\tilde{u}(t,\cdot)))_1(\xi), \\ \partial_t u_l(t,\xi) = l\partial_\xi u_l(t,\xi) + (N_\delta(\tilde{u}(t,\cdot))_l)(\xi) \qquad \text{for } l \geq 2, \\ \partial_t \epsilon(t) = 0, \end{cases} \tag{4.13}$$

for $\tilde{u} = (u, \epsilon) \in C(\Delta \times \mathbb{N} \times \mathbb{R} \times \mathbb{R})$ understood weakly against test functions in C_0^1 and where $\Delta \subset \mathbb{R}$ is a time interval.

Lemma 4.24. *A solution $\tilde{u}$ of Equation* (4.13) *satisfies Duhamel's formula, i.e. for $t_0, t_1 \in \Delta$ with $t_0 \leq t_1$ the following holds*

$$\tilde{u}(t_1) = \mathrm{e}^{(t_1-t_0)\tilde{L}}\tilde{u}(t_0) + \int_{t_0}^{t_1} \mathrm{e}^{(t_1-s)\tilde{L}}\tilde{R}_\delta(\tilde{u}(s))\,\mathrm{d}s.$$

Proof. Consider $\tilde{R}_\delta(\tilde{u})$ as forcing. Then the linear problem has a unique solution given by the formula. □

Lemma 4.25. *Let $\tilde{u}$ be a solution of Equation* (4.13)*. Given $\tilde{u}(t_0) \in \tilde{\mathcal{Z}}^a$ the solution is uniquely determined at later times $t \geq t_0$ and $\|\tilde{u}(t)\|_{\tilde{\mathcal{Z}}^a}$ is uniformly bounded for compact intervals of later times. If δ is small enough, there exists, given $\tilde{u}(t_0) \in \tilde{\mathcal{Z}}^a$, a global solution* *for* $t \geq t_0$*.*

Proof. The projections commute with the linear evolution in the Duhamel formula and are bounded. Since $\tilde{R}_\delta$ is bounded, this shows with the control of the linear evolution (Lemma 4.23) that $\|\tilde{u}(t)\|_{\tilde{\mathcal{Z}}^a}$ is uniformly bounded for compact time-intervals of later times. Furthermore, $\tilde{R}_\delta$ is Lipschitz so that the Duhamel formulation gives a Gronwall control showing uniqueness.

If δ is small enough, then the Picard iteration

$$\tilde{u} \mapsto \mathrm{e}^{(t-t_0)\tilde{L}}\tilde{u}(t_0) + \int_{t_0}^{t} \mathrm{e}^{(t-s)\tilde{L}}\tilde{R}_\delta(\tilde{u}(s))\,\mathrm{d}s$$

is a contraction, so that a solution exists. □

With this we can construct the reduced manifold. We want to capture the evolution in the reduced manifold where the stable part has decayed, which we can characterise by the growth as $t \to -\infty$.

Theorem 4.26. *Assume the hypothesis of Theorem 4.6. There exists* $\mu \in (0, a)$ *and* δ_b *such that for* $\delta \leq \delta_b$ *there exists* $\tilde{\psi} \in C^k(\tilde{\mathcal{Z}}^a_{cu}, \tilde{\mathcal{Z}}^a_s)$ *such that*

$$\begin{aligned}\mathcal{M} = \{\tilde{u}_0 \in \tilde{\mathcal{Z}}^a : \exists \tilde{u} \textit{ satisfying Equation } (4.13) \textit{ with } \tilde{u}(0) = \tilde{u}_0 \\ \textit{and } \sup_{t\in\mathbb{R}^-} \mathrm{e}^{\mu t}\|\tilde{u}(t)\|_{\tilde{\mathcal{Z}}^a} < \infty\} \\ = \{v + \tilde{\psi}(v) : v \in \tilde{\mathcal{Z}}^a_{cu}\}\end{aligned}$$

which is the reduced manifold. Under the evolution of Equation (4.13) *the reduced manifold is invariant and exponentially attractive. Moreover, the Taylor series of* $\tilde{\psi}$ *is explicitly computable at* 0 *starting with*

$$\begin{aligned}&\tilde{\psi}|_0 = 0,\\ &D\tilde{\psi}|_0 = 0,\\ &D^2\tilde{\psi}|_0(h_1, h_2) = \int_{-\infty}^{0} \mathrm{e}^{-t\tilde{L}}\tilde{P}_s D^2[\tilde{R}_\delta|_0(\mathrm{e}^{t\tilde{L}}h_1, \mathrm{e}^{t\tilde{L}}h_2)]\mathrm{d}t.\end{aligned}$$

The growth condition already suggests that this captures the dominant behaviour and $\mathcal{M}$ is attractive.

Proof. Since we have a Duhamel formula, the proofs of Theorem 1 and 2 of [155] work. We further adapt it, as in Section 1.5 of [154], to the center-unstable case. Parts of the results are also stated as Theorem 2.9 and 3.22 of Section 2 of [68].

The key point is to note that, by the Duhamel formulation, a solution with the growth bound is a fixed point of the map T given by

$$(T\tilde{u})(t) = \mathrm{e}^{t\tilde{L}}\tilde{P}_{cu}\tilde{u}_0 - \int_t^0 \mathrm{e}^{(t-s)\tilde{L}}\tilde{P}_{cu}\tilde{R}_\delta(\tilde{u}(s))\mathrm{d}s + \int_{-\infty}^{t} \mathrm{e}^{(t-s)\tilde{L}}\tilde{P}_s\tilde{R}_\delta(\tilde{u}(s))\mathrm{d}s$$

for $t \in \mathbb{R}^-$ on the space $C(\mathbb{R}^- \times \mathbb{N} \times \mathbb{R} \times \mathbb{R})$ with norm $\|\tilde{u}\| = \sup_{t\in\mathbb{R}^-} \mathrm{e}^{\mu t}\|\tilde{u}(t,\cdot)\|_{\tilde{\mathcal{Z}}^a}$. Here note that on $\tilde{\mathcal{Z}}^a_{cu}$ the linear evolution $\mathrm{e}^{t\tilde{L}}$ restricts to a finite dimensional matrix exponential, which has a well-defined inverse $\mathrm{e}^{-t\tilde{L}}$.

For small enough δ, the map T is a contraction with a unique fixed point depending on $\tilde{P}_{cu}\tilde{u}_0$ so that there exists $\tilde{\psi} : \tilde{\mathcal{Z}}^a_{cu} \mapsto \tilde{\mathcal{Z}}^a_s$ such that the unique solution is $\tilde{P}_{cu}\tilde{u}_0 + \tilde{\psi}(\tilde{P}_{cu}\tilde{u}_0)$ [155, Theorem 1]. Hence the set $\mathcal{M}$ is a reduced manifold with the given form. Moreover, the growth condition is invariant under the evolution.

By a Fibre contraction argument ([155, Theorem 2] and [154, Section 1.3]), for small enough δ, the map $\tilde{\psi}$ has the claimed regularity and the derivatives can be computed explicitly at 0 with the given form.

As discussed in [154, Section 1.5] and [68, Theorem 3.22 of Chapter 2] a similar fixed point argument shows the exponential attractiveness. □

Hence, with a sufficient localisation, we can prove Theorem 4.6.

Proof of Theorem 4.6. Apply Theorem 4.26 with a small enough δ. Then for $|\epsilon| \leq \delta$ and $\|u\| \leq \delta$ the nonlinearity of $\tilde{R}_\delta$ agrees with the nonlinearity of the Kuramoto equation, i.e. a solution to the localised evolution is also a solution to the original problem in the restricted region.

The map $\tilde{\psi}$ maps into $\tilde{\mathcal{Z}}^a_s$ and thus has the form $\tilde{\psi}(u,\epsilon) = (\psi(u,\epsilon), 0)$ for ψ mapping into $\mathcal{Z}^a_s$. As ϵ is constant, the reduced manifold must have the claimed form and the result follows from the uniqueness (Theorem 4.9). □

Remark 4.27. The Kuramoto equation has the rotation symmetry, i.e. if u is a solution to the Kuramoto equation, then $\bar{u}_l(t,\xi) = \mathrm{e}^{\mathrm{i}l\alpha}u_l(t,\xi)$ is again a solution. This symmetry is also satisfied by the localised nonlinearity and thus by the reduced manifold.

4.6.3 Example for Gaussian velocity distribution

The center-manifold reduction can be used to determine the bifurcation behaviour by studying the evolution on the reduced manifold. The resulting behaviour crucially depends on the velocity distribution and [38] contains several examples based on the center-manifold reduction with noise, which at this point is very similar. The Penrose diagrams (cf. Figures 4.1 and 4.2) already show the dimension of the reduced manifold as the covering number, e.g. this shows for the example of Figure 4.2 that at the critical coupling two eigenmodes appear.

As an example we repeat Chiba's analysis [31] for the bifurcation of the Gaussian distribution. In this example the density is given by

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\omega^2/2},$$

and the Fourier transform is

$$\hat{g}(\xi) = \mathrm{e}^{-\xi^2/2}.$$

From the discussion in Section 4.3, we see that the critical coupling is

$$K_c = \frac{2}{\pi g(0)} = \frac{4}{\sqrt{2\pi}}$$

and that at the critical coupling K_c there exists a single eigenvalue at $\lambda = 0$ for small enough a.

Recall, that the eigenvector $z_{0,0}$ is given by

$$(z_{0,0})_1(\xi) = \frac{K_c}{2}\int_\xi^\infty \hat{g}(\zeta)\mathrm{d}\zeta.$$

Hence $(z_{0,0})_1(0) = 1$ and

$$\begin{aligned}\alpha_{0,0}(z_{0,0}) &= \frac{K_c}{2}\int_0^\infty \int_\xi^\infty \hat{g}(\zeta)\mathrm{d}\zeta\mathrm{d}\xi \\ &= \frac{K_c}{2},\end{aligned}$$

so that

$$P_{cu}(u) = \frac{2}{K_c}\alpha_{0,0}(u)z_{0,0}.$$

Now apply Theorem 4.6 with sufficiently small $a > 0$ such that $\lambda_1 = 0$ is the only eigenmode and with $k \geq 3$. Then for $\tilde{u} = (u, \epsilon) \in \tilde{\mathcal{Z}}^a_{cu}$

$$\psi(\tilde{u}) = \frac{1}{2}b(\tilde{u}) + O(\|\tilde{u}\|^3),$$

where

$$b(\tilde{u}) = D^2\psi|_0(\tilde{u},\tilde{u}) = \int_{-\infty}^0 \mathrm{e}^{-t\tilde{L}}\tilde{P}_s[D^2N|_0(\tilde{u},\tilde{u})]\,\mathrm{d}t.$$

For this we find that only the first two spatial modes are non-vanishing with

$$(b(\tilde{u}))_1(\xi) = O(|\epsilon|\|u\|)$$

and

$$(b(\tilde{u}))_2(\xi) = \int_{-\infty}^0 2K_c u_1(\xi - 2t)u_1(0)\,\mathrm{d}t.$$

Hence we can compute

$$(N(u+\psi(u,\epsilon)))_1(\xi) = \frac{\epsilon}{2}u_1(0)\hat{g}(\xi) - \frac{K_c}{2}\overline{u_1(0)}\frac{(b(\tilde{u}))_2(\xi)}{2} + O(|\epsilon|^2\|u\|, \|\tilde{u}\|^4).$$

As $\tilde{u} = u_1(0)z_{0,0}$ we find

$$\begin{aligned}\int_0^\infty (b(\tilde{u}))_2(\xi)\mathrm{d}\xi &= (u_1(0))^2 \int_0^\infty \int_{-\infty}^0 2K_c(z_{0,0})_1(\xi-2t)(z_{0,0})_1(0)\,\mathrm{d}t\,\mathrm{d}\xi \\ &= K_c^2(u_1(0))^2 \int_{\xi=0}^\infty \int_{t=0}^\infty \int_{\zeta=\xi+2t}^\infty \hat{g}(\zeta)\,\mathrm{d}\zeta\,\mathrm{d}t\,\mathrm{d}\xi \\ &= K_c^2(u_1(0))^2 \int_{\zeta=0}^\infty \frac{\zeta^2}{4}\hat{g}(\zeta)\,\mathrm{d}\zeta \\ &= \frac{\sqrt{2\pi}}{8}.\end{aligned}$$

Hence we find

$$\alpha_{0,0}(N(u+\psi(u,\epsilon))) = \frac{\sqrt{2\pi}}{4}\epsilon u_1(0) - \frac{1}{\pi}|u_1(0)|^2 u_1(0) + O(|\epsilon|^2\|u\|, \|\tilde{u}\|^4).$$

On the reduced manifold the solution is $\beta(t)z_{0,0} + \psi(\beta(t)z_{0,0},\epsilon)$, which evolves by the previous computation as

$$\frac{K_c}{2}\partial_t\beta = \frac{\sqrt{2\pi}}{4}\epsilon\beta - \frac{1}{\pi}|\beta|^2\beta + O(|\epsilon|^2\|u\|, \|\tilde{u}\|^4).$$

For small enough $\epsilon \le 0$, it follows that $\beta = 0$ is a stable fixed point, i.e. the incoherent state is stable. For small enough $\epsilon > 0$, the zero solution is unstable, but the set $|\beta| = \beta_c$ are stable attractors where

$$\beta_c = \sqrt{\frac{\pi}{4}\sqrt{2\pi}}\sqrt{\epsilon} + O(\epsilon).$$

Hence we have proved the claimed bifurcation from the incoherent state and have shown that for small enough ϵ the appearing stationary states are nonlinearly stable. Since $(z_{0,0})_1(0) = 1$, the value of β agrees with the order parameter so that we have found the known result.

4.7 Boundedness and convergence in the exponential norms

The bifurcation analysis (Section 4.6) shows the nonlinear stability of states with non-vanishing order parameter. In particular, these states have finite small norm in $\mathcal{Z}^a$. From the discussion on the stationary states, we expect them to be the partially locked states f_{st} from Section 1.3.2 with all mass at the stable fixed point. In this section, we show how the exponential norms in Fourier can be understood by an analytic extension. Furthermore, we show that $f_{\mathrm{st}} \in \mathcal{Z}^a$ for suitable velocity marginals g.

The exponential norm $L^\infty(\mathbb{R}, \exp_a)$ can be related to an analytic extension in a strip $\{z : -a < \Im z \le 0\}$.

Lemma 4.28. *Let $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ with Fourier transform $\hat{f} \in L^1(\mathbb{R})$ satisfying $\hat{f} \in L^\infty(\mathbb{R}, \exp_a)$. Then f has an analytic continuation to $\{z : -a < \Im z \le 0\}$.*

Proof. For $z \in \mathbb{R}$ the Fourier inversion formula shows

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}z\xi} \hat{f}(\xi) \mathrm{d}\xi.$$

By the assumed supremum bound on $\hat{f}$ for $\xi \geq 0$ and the bound $|\hat{f}(\xi)| \leq \|f\|_1$ for $\xi \leq 0$, this definition extends to the given range and has a complex derivative given by differentiating under the integral sign. □

Lemma 4.29. *Let $f \in L^1(\mathbb{R})$ and suppose f has an analytic continuation to $\{z : -a \leq \Im z \leq 0\}$ such that $|f(z)| \to 0$ uniformly as $|\Re z| \to \infty$ and $f(\cdot - \mathrm{i}a) \in L^1(\mathbb{R})$. Then the Fourier transform $\hat{f}$ is bounded as*

$$\|\hat{f}\|_{L^\infty(\mathbb{R}, \exp_a)} \leq \|f(\cdot - \mathrm{i}a)\|_{L^1(\mathbb{R})}.$$

Proof. By the assumed decay we can deform the contour of integration as

$$\hat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi y} f(y) \mathrm{d}y = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi(y - \mathrm{i}a)} f(y - \mathrm{i}a) \mathrm{d}y = \mathrm{e}^{-a\xi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi y} f(y - \mathrm{i}a) \mathrm{d}y.$$

Bounding the integral with the L^1 norm of f shows the claimed result. □

For a stationary state f_{st} with all locked oscillator at the stable fixed point, we recall from Section 1.3.2 its spatial Fourier transform $\tilde{f}_{\mathrm{st}}$ as

$$\tilde{f}_l(\omega) = g(\omega) \left(\beta \left(\frac{\omega}{K\eta} \right) \right)^l,$$

where we again assume without loss of generality that $\eta \in [0, 1]$ and β is given by

$$\beta(z) = \begin{cases} -\mathrm{i}z \left(1 - \sqrt{1 - \frac{1}{z^2}} \right) & \text{for } |z| \geq 1, \\ -\mathrm{i}z + \sqrt{1 - z^2} & \text{for } |z| < 1. \end{cases}$$

The factor β has a very nice analytic continuation in the needed region, the lower half plane.

Lemma 4.30. *The function β has an analytic continuation given by the above formula in the lower half plane $\{z \in \mathbb{C} : \Im z \leq 0\}$. It is bounded in this region by 1 and for $|z| \geq \sqrt{2}$*

$$|\beta(z)| \leq \frac{1}{|z|}.$$

Moreover, for any $a > 0$

$$\sup_{x \in \mathbb{R}} |\beta(x - \mathrm{i}a)| < 1.$$

Proof. The above formula defines analytic functions in $|z| < 1$ by $-\mathrm{i}z + \sqrt{1-z^2}$ and in $|z| > 1$ by $-\mathrm{i}z(1 - \sqrt{1-z^{-2}})$, as the argument of the square root always has a non-negative real part. Moreover, it is the solution to the quadratic equation $y^2 + 2\mathrm{i}yz - 1 = 0$ for y, whose roots are separated unless $z = \pm 1$. At $z = -1$, we find

$$\lim_{z \to -\mathrm{i}, |z|<1} \beta(z) = -1 + \sqrt{2} = \lim_{z \to -\mathrm{i}, |z|>1} \beta(z),$$

so that the branches match up in the lower half plane. Hence β is analytic in the claimed region.

For $|z| \geq \sqrt{2}$, we can use the series expansion to find

$$|\beta(z)| \leq \frac{1}{2|z|} \sum_{i=0}^{\infty} |z|^{-2i} \leq \frac{1}{|z|}.$$

Moreover, along $\Im z = 0$, we can immediately check that β is bounded by 1. Thus the maximum principle shows the remaining claimed bounds. □

This bound with the form of the stable partially locked states immediately shows the following theorem.

Theorem 4.31. *Suppose a velocity marginal with density $g \in L^1(\mathbb{R})$, where g has an analytic continuation to $\{z : -a \leq \Im z \leq 0\}$ such that $|g(z)| \to 0$ uniformly as $|\Re z| \to \infty$ and $g(\cdot - \mathrm{i}a) \in L^1(\mathbb{R})$. Then a partially locked state f_{st} with all locked oscillators at the stable fixed-point has a Fourier transform $u \in \mathcal{Z}^a$. If furthermore $|\eta| \leq a/(\sqrt{2}K)$, then*

$$\|u\|_{\mathcal{Z}^a} \leq \frac{K|\eta|}{2} \|g(\cdot - ia)\|_{L^1(\mathbb{R})}.$$

Proof. By the above discussion we find

$$\|u_l\|_{L^\infty(\mathbb{R}, \exp_a)} \leq \sup_{\omega \in \mathbb{R}} \left| \beta\left(\frac{\omega - \mathrm{i}a}{K|\eta|}\right) \right|^l \|g(\cdot - \mathrm{i}a)\|_{L^1(\mathbb{R})} \leq \frac{K|\eta|}{a} \|g(\cdot - \mathrm{i}a)\|_{L^1(\mathbb{R})}.$$

Hence the previous lemma shows the claimed control. □

4.8 Implied convergence

We can relate convergence in $\mathcal{Z}^a$ as found in the center-unstable manifold reduction and the exponential stability with weak convergence.

Theorem 4.32. *Let $f \in C_{\mathcal{M}}$ be a solution to the Kuramoto equation with initial data $f_{\mathrm{in}} \in \mathcal{M}(\Gamma)$ and Fourier transform u. If $u(t, \cdot) \to v$ in $\mathcal{Z}^a$ as $t \to \infty$, where v is the Fourier transform of $f_\infty \in \mathcal{M}(\Gamma)$, then for every $\phi \in H^4(\Gamma)$ the integral $\int_\Gamma \phi(\theta, \omega) f(t, \theta, \omega) \mathrm{d}\theta \mathrm{d}\omega$ converges to $\int_\Gamma \phi(\theta, \omega) f_\infty(\theta, \omega) \mathrm{d}\theta \mathrm{d}\omega$ as $t \to \infty$.*

Proof. Since the velocity marginal is conserved, without loss of generality assume $\int_{\theta\in\mathbb{T}} \phi(\theta,\omega)\mathrm{d}\theta = 0$ for all $\omega \in \mathbb{R}$ and ϕ to be real-valued. Then by Plancherel theorem

$$\int_\Gamma \phi(\theta,\omega) f(t,\theta,\omega)\mathrm{d}\theta\mathrm{d}\omega = 2\Re\left(\sum_{l\geq 1}\int_{\xi\in\mathbb{R}} u_l(t,\xi)\overline{\hat{\phi}_l(\xi)}\mathrm{d}\xi\right),$$

where $\hat{\phi}$ is the Fourier transform of ϕ. Since $\phi \in H^4$, we have $\hat{\phi} \in L^1(\mathbb{N}\times\mathbb{R})$. Hence for every $\epsilon > 0$, there exists some M such that

$$\sum_{l\geq 1}\int_{-\infty}^{M} |\hat{\phi}_l(\xi)|\mathrm{d}\xi \leq \frac{\epsilon}{8}.$$

Since u and v correspond to probability measures, they are bounded by 1 so that

$$\left|\sum_{l\geq 1}\int_{-\infty}^{M} (u_l(t,\xi) - v_l(\xi))\,\overline{\hat{\phi}_l(\xi)}\,\mathrm{d}\xi\right| \leq \frac{\epsilon}{4}.$$

For $\xi \geq M$ we have a control by $\mathcal{Z}^a$

$$\left|\sum_{l\geq 1}\int_{M}^{\infty} (u_l(t,\xi) - v_l(\xi))\,\overline{\hat{\phi}_l(\xi)}\,\mathrm{d}\xi\right| \leq \mathrm{e}^{-aM}\|u-v\|_{\mathcal{Z}^a}\|\hat{\phi}\|_1.$$

Hence for large enough t we have $|\rho_t(\phi) - \rho_\infty(\phi)| \leq \epsilon$. Since ϵ is arbitrary, this shows the claimed convergence. □

In the stability theorem 4.4 we only control a norm in half of the Fourier coefficients which also looses control for higher spatial modes. However, this controls the order parameter η and in particular it shows $\int_0^\infty |\eta(t)|\mathrm{d}t < \infty$. With this knowledge we can go back to the original equation to deduce convergence properties. As an example we show a convergence in the gliding frame. This case is particularly interesting, because other results usually prove Landau damping by convergence in the gliding frame, e.g. [14, 52, 53, 114].

In the gliding frame the position of each oscillator is corrected by the effect of the free transport, i.e. if f is the solution of the Kuramoto equation, then the density h in the gliding frame is given by

$$h(t,\theta,\omega) = f(t,\theta + t\omega,\omega).$$

If $f \in C^1(\mathbb{R}^+\times\Gamma)$, then f satisfies classically

$$\partial_t f(t,\theta,\omega) + \partial_\theta\left[\omega + \left(\frac{K}{2\mathrm{i}}\left(\eta(t)\,\mathrm{e}^{-\mathrm{i}\theta} - \overline{\eta(t)}\,\mathrm{e}^{\mathrm{i}\theta}\right)\right) f(t,\theta)\right] = 0$$

so that then $h \in C^1(\mathbb{R}^+ \times \Gamma)$ and h satisfies

$$\partial_t h(t,\theta,\omega) + \partial_\theta \left[\frac{K}{2\mathrm{i}} \left(\eta(t) \,\mathrm{e}^{-\mathrm{i}(\theta+t\omega)} - \overline{\eta(t)} \,\mathrm{e}^{\mathrm{i}(\theta+t\omega)} \right) h(t,\theta,\omega) \right] = 0. \tag{4.14}$$

Theorem 4.33. *Suppose $f_{in} \in C^2(\Gamma)$ and let f be the solution of the Kuramoto equation with initial data f_{in}. Then f has a gliding frame density $h \in C^2(\mathbb{R}^+ \times \Gamma)$ and for every $t \in \mathbb{R}^+$ and $\omega \in \mathbb{R}$ holds*

$$\int_{\mathbb{T}} (h(t,\theta,\omega))^2 \mathrm{d}\theta \le \mathrm{e}^{K \int_0^t |\eta(s)| \mathrm{d}s} \int_{\mathbb{T}} (f_{\mathrm{in}}(\theta,\omega))^2 \mathrm{d}\theta$$

and

$$\begin{aligned} &\int_{\mathbb{T}} (\partial_\theta h(t,\theta,\omega))^2 \mathrm{d}\theta \\ &\le e^{3K \int_0^T |\eta(s)| \mathrm{d}s} \left[\int_{\mathbb{T}} (\partial_\theta f_{in}(\theta,\omega))^2 \mathrm{d}\theta + K \int_0^t |\eta(s)| \mathrm{d}s \sup_{s\in[0,t]} \int_{\mathbb{T}} (h(t,\theta,\omega))^2 \mathrm{d}\theta \right]. \end{aligned}$$

If $\int_0^\infty |\eta(t)| \mathrm{d}t < \infty$, then for every fixed $\omega \in \mathbb{R}$ the function $h(t,\cdot,\omega)$ converges in $L^2(\mathbb{T})$ as $t \to \infty$.

If additionally $f_{\mathrm{in}} \in L^2(\Gamma)$ and $\partial_\theta f_{\mathrm{in}} \in L^2(\Gamma)$, then $h(t,\cdot,\cdot)$ converges in $L^2(\Gamma)$ as $t \to \infty$.

Proof. Along the evolution the regularity is propagated [86], so that $f \in C^2(\mathbb{R}^+ \times \Gamma)$. Hence $h \in C^2(\mathbb{R}^+ \times \Gamma)$ and h satisfies Equation (4.14).

For the bounds, fix ω and let

$$z(t) = \int_{\theta\in\mathbb{T}} (h(t,\theta,\omega))^2 \mathrm{d}\theta.$$

By the assumed regularity, we can differentiate under the integral sign to find

$$\frac{\mathrm{d}}{\mathrm{d}t} z(t) = \int_{\theta\in\mathbb{T}} \frac{K}{2} \left(\eta(t) \,\mathrm{e}^{-\mathrm{i}(\theta+t\omega)} + \overline{\eta(t)} \,\mathrm{e}^{\mathrm{i}(\theta+t\omega)} \right) (h(t,\theta,\omega))^2 \mathrm{d}\theta \le K|\eta(t)| z(t).$$

Hence Gronwall's inequality shows the first part

$$z(t) \le \mathrm{e}^{K \int_0^t |\eta(s)| \mathrm{d}s} z(0) = \mathrm{e}^{K \int_0^t |\eta(s)| \mathrm{d}s} \int_{\mathbb{T}} (f_{\mathrm{in}}(\theta,\omega))^2 \mathrm{d}\theta.$$

Similar, we consider

$$y(t) = \int_{\theta\in\mathbb{T}} (\partial_\theta h(t,\theta,\omega))^2 \mathrm{d}\theta$$

and find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} y(t) &= \frac{K}{2} \int_{\theta\in\mathbb{T}} \left[(h(t,\theta,\omega))^2 + 3(\partial_\theta h(t,\theta,\omega))^2 \right] \left(\eta(t) \,\mathrm{e}^{-\mathrm{i}(\theta+t\omega)} + \overline{\eta(t)} \,\mathrm{e}^{\mathrm{i}(\theta+t\omega)} \right) \mathrm{d}\theta \\ &\le K|\eta(t)| z(t) + 3K|\eta(t)| y(t). \end{aligned}$$

Then Gronwall's inequality shows the claimed bound.

If $\int_0^t |\eta(t)|\mathrm{d}t < \infty$, these bounds show that there exist constants $C_1, C_2 \in \mathbb{R}$ such that for every $\omega \in \mathbb{R}$

$$\int_{\theta\in\mathbb{T}} (h(t,\theta,\omega))^2 \mathrm{d}\theta \le C_1 \int_{\theta\in\mathbb{T}} (f_{\mathrm{in}}(\theta,\omega))^2 \mathrm{d}\theta$$

and

$$\int_{\theta\in\mathbb{T}} (\partial_\theta h(t,\theta,\omega))^2 \mathrm{d}\theta \le C_2 \int_{\theta\in\mathbb{T}} \left[(f_{\mathrm{in}}(\theta,\omega))^2 + (\partial_\theta f_{\mathrm{in}}(\theta,\omega))^2\right] \mathrm{d}\theta.$$

In particular the bounds are finite as the integral is over a compact domain and we assume $f_{\mathrm{in}} \in C^2(\Gamma)$.

Now consider for $t \ge s \ge 0$

$$D(t,s) = \int_{\theta\in\mathbb{T}} (h(t,\theta,\omega) - h(s,\theta,\omega))^2 \mathrm{d}\theta.$$

By the assumed regularity we can differentiate under the integral sign and find

$$\begin{aligned}\partial_t D(t,s) &= \int_{\theta\in\mathbb{T}} 2[\partial_\theta h(t,\theta,\omega) - \partial_\theta h(s,\theta,\omega)] \frac{K}{2\mathrm{i}} \left(\eta(t)\,\mathrm{e}^{-\mathrm{i}(\theta+t\omega)} - \overline{\eta(t)}\,\mathrm{e}^{\mathrm{i}(\theta+t\omega)}\right) h(t,\theta,\omega)\mathrm{d}\theta \\ &\le 2K|\eta(t)|(\sqrt{y(t)} + \sqrt{y(s)})\sqrt{z(t)}.\end{aligned}$$

Hence

$$D(t,s) \le C \int_s^t |\eta(\bar{s})|\mathrm{d}\bar{s}$$

for a constant C. As $\int_0^\infty |\eta(t)|\mathrm{d}t < \infty$, this shows the claimed convergence in $L^2(\mathbb{T})$ as $t \to \infty$.

If $f_{\mathrm{in}} \in L^2(\Gamma)$ and $\partial_\theta f_{\mathrm{in}} \in L^2(\Gamma)$, we can integrate the previous inequality over ω to show the convergence of $h(t,\cdot,\cdot)$ in $L^2(\Gamma)$ as $t \to \infty$. □

The convergence in the gliding frame implies for example weak convergence in the normal setup.

Theorem 4.34. *Suppose the assumption of Theorem 4.33 with $\int_0^\infty |\eta(t)|\mathrm{d}t < \infty$ and $f_{\mathrm{in}} \in L^2(\Gamma)$ and $\partial_\theta f_{\mathrm{in}} \in L^2(\Gamma)$. Then for every $\phi \in L^2(\Gamma)$*

$$\int_{\mathbb{R}} \int_{\mathbb{T}} f(t,\theta,\omega)\phi(\theta,\omega)\mathrm{d}\theta\mathrm{d}\omega \to \int_{\mathbb{R}} \int_{\mathbb{T}} \frac{g(\omega)}{2\pi}\phi(\theta,\omega)\mathrm{d}\theta\mathrm{d}\omega \qquad \text{as } t \to \infty.$$

Proof. By the previous theorem, the gliding frame density converges in $L^2(\Gamma)$. Let h_∞ be the limit, i.e.

$$h(t,\cdot,\cdot) \to h_\infty(\cdot,\cdot)$$

holds in $L^2(\Gamma)$ as $t \to \infty$.

The inner product can be expressed in terms of h as

$$\int_{\mathbb{R}} \int_{\mathbb{T}} f(t,\theta,\omega)\phi(\theta,\omega)\mathrm{d}\theta\mathrm{d}\omega = \int_{\mathbb{R}} \int_{\mathbb{T}} h(t,\theta,\omega)\phi(\theta + t\omega,\omega)\mathrm{d}\theta\mathrm{d}\omega.$$

Given $\epsilon > 0$, we then have for large enough t by the convergence in L^2

$$\left| \int_{\mathbb{R}} \int_{\mathbb{T}} f(t,\theta,\omega)\phi(\theta,\omega)\mathrm{d}\theta\mathrm{d}\omega - \int_{\mathbb{R}} \int_{\mathbb{T}} h_\infty(\theta,\omega)\phi(\theta + t\omega,\omega)\mathrm{d}\theta\mathrm{d}\omega \right| \le \frac{\epsilon}{2}.$$

Let $\hat{h}_\infty$ and $\hat{\phi}$ be the Fourier transform in both variables θ and ω. Then the Plancherel theorem shows

$$\int_{\mathbb{R}} \int_{\mathbb{T}} h_\infty(\theta,\omega)\phi(\theta + t\omega,\omega)\mathrm{d}\theta\mathrm{d}\omega = \sum_{k\in\mathbb{Z}} \int_{\xi\in\mathbb{R}} (\hat{h}_\infty)_k(\xi)\hat{\phi}_k(\xi - kt)\mathrm{d}\xi.$$

The velocity marginal g is constant so that $(\hat{h}_\infty)_0(\xi) = \hat{g}(\xi)$, which shows

$$\int_{\xi\in\mathbb{R}} (\hat{h}_\infty)_0(\xi)\hat{\phi}_0(\xi)\mathrm{d}\xi = \int_{\omega\in\mathbb{R}} \int_{\theta\in\mathbb{T}} \frac{g(\omega)}{2\pi}\phi(\theta,\omega)\mathrm{d}\theta\mathrm{d}\omega.$$

For $k \neq 0$ we find that

$$\left| \int_{\xi\in\mathbb{R}} (\hat{h}_\infty)_k(\xi)\hat{\phi}_k(\xi - kt)\mathrm{d}\xi \right| \le \|(\hat{h}_\infty)_k\|_2 \|\hat{\phi}_k\|_2$$

and

$$\int_{\xi\in\mathbb{R}} (\hat{h}_\infty)_k(\xi)\hat{\phi}_k(\xi - kt)\mathrm{d}\xi \to 0$$

as $t \to \infty$.

Since $\hat{h} \in L^2$ and $\hat{\phi} \in L^2$, we have the bound

$$\sum_{k\in\mathbb{Z}} \|(\hat{h}_\infty)_k\|_2 \|\hat{\phi}_k\|_2 < \infty,$$

so that dominated convergence shows as $t \to \infty$

$$\sum_{k\in\mathbb{Z}} \int_{\xi\in\mathbb{R}} (\hat{h}_\infty)_k(\xi)\hat{\phi}_k(\xi - kt)\mathrm{d}\xi \to \int_{\xi\in\mathbb{R}} (\hat{h}_\infty)_0(\xi)\hat{\phi}_0(\xi)\mathrm{d}\xi,$$

which is the claimed limit. □

4.9 Spatial mode reduction (Ott–Antonsen ansatz)

In this short section, we remark how the exponential norms can be utilised to show the spatial mode reduction as discussed in Section 1.3.4. We recall that the reduced dynamics impose that the distribution f has the form

$$\tilde{f}_l(\omega) = \alpha^l(\omega)g(\omega)$$

for $l \in \mathbb{N}$ and some function α.

In order to measure the defect of a solution f to the reduced dynamics, we introduce

$$w_{n,m}(t,\xi) = (\hat{f}_{n+m}(t,\cdot) * \hat{g})(\xi) - (\hat{f}_n(t,\cdot) * \hat{f}_m(t,\cdot))(\xi)$$

for $n, m \in \mathbb{N}$. By the symmetry of the convolution, we have the boundary conditions

$$w_{n,0} \equiv w_{0,m} \equiv 0$$

for $n, m \in \mathbb{N}$. Otherwise the evolution equation becomes

$$\begin{aligned}\partial_t w_{n,m}(t,\xi) =& (n+m)\partial_\xi w_{n,m}(t,\xi) \\ & - \frac{Kn}{2}\left[\overline{\eta(t)} w_{n-1,m}(t,\xi) - \eta(t) w_{n+1,m}(t,\xi)\right] \\ & - \frac{Km}{2}\left[\overline{\eta(t)} w_{n,m-1}(t,\xi) - \eta(t) w_{n,m+1}(t,\xi)\right],\end{aligned}$$

where $\eta(t)$ is the order parameter. Since $|\eta| \le 1$, we can choose a large enough rate a such that

$$\sup_{n,m\in\mathbb{N}} \|w_{n,m}\|_{L^\infty(\mathbb{R},\exp_a)} = \sup_{n,m\in\mathbb{N}} \sup_{\xi\in\mathbb{R}} \mathrm{e}^{a\xi} |w_{n,m}(t,\xi)|$$

is exponentially decaying to zero. Similarly, the decay can be observed in the L^2 norms used in Chapter 5.

This gives an explicit way of convergence to the reduced manifold with high regularity assumptions. A similar result is by Ott and Antonsen [121] and Ott, Hunt and Antonsen [123], where they solved the equation in the analytic continuation for ω and obtain less explicit convergence estimates. Even though this reduces the dynamics, the resulting equation is still a PDE without dissipation, e.g. a spectral analysis in strong topology still shows a continuous spectrum along the imaginary axis [120].

4.10 Direct stability results for particle systems

The mean-field theory (Chapter 2) does not take advantage of the structure of the equation and just uses a crude Lipschitz bound for the stability (Lemma 2.3). We recall from Chapter 2 that the dual bonded Lipschitz distance d between two solutions can diverge with rate at most $C_D = \sqrt{2}\max(1,K) + 2K$, cf. [86], i.e. for two solutions f and h to the Kuramoto equation, we have

$$d(f(t,\cdot), h(t,\cdot)) \le \mathrm{e}^{C_D t} d(f_{\mathrm{in}}, h_{\mathrm{in}}).$$

Even though this estimate gives a quantitative control, the factor $\mathrm{e}^{C_D t}$ can become very large for the considered time frame. Taking advantage of the structure of the equation, we obtain a control of the order parameter with a slower time growth.

We obtain the result by controlling a suitable distance between a smooth solution f^∞, e.g. the incoherent state or a solution converging to it, and a rough approximation f^N, e.g. the empirical measure of a finite particle system. Assuming an initial closeness, we propagate the control in time by using that f^N and f^∞ are both solutions to the mean-field limit Kuramoto equation.

As the distance to the empirical measure must be finite, we use the norm

$$d_{\alpha,\beta}(f^N, f^\infty) = \sup_{l\geq 0}\sup_{\xi\geq 0} \mathrm{e}^{-\alpha\xi-\beta l}|(\hat{f}^N)_l(\xi) - (\hat{f}^\infty)_l(\xi)|$$

for $\alpha, \beta > 0$. This norm is very weak and has the opposite weight in Fourier than the weight characterising analytic functions.

In order to find the expected rate of divergence, consider the free transport e^{tL_f}, whose generator is in this representation $L_f = -l\partial_\xi$. After time t, we have the following control in the first mode

$$|(\mathrm{e}^{-tl\partial_\xi}\hat{f}^N)_1(\xi) - (\mathrm{e}^{-tl\partial_\xi}\hat{f}^\infty)_1(\xi)| \leq \mathrm{e}^{\alpha t}\mathrm{e}^{\beta} d_{\alpha,\beta}(f^N, f^\infty).$$

In particular this shows that under the free transport, the possible difference of the order parameter grows with rate α, i.e. we would have

$$\left|\eta\left[\mathrm{e}^{tL_f} f^N\right] - \eta\left[\mathrm{e}^{tL_f} f^\infty\right]\right| \leq \mathrm{e}^{\alpha t}\mathrm{e}^{\beta} d_{\alpha,\beta}(f^N, f^\infty).$$

Very roughly, we can indeed prove that the difference of the order parameter can be controlled with this rate over finite time intervals for states close enough to a stable incoherent state.

This improves the control of the order parameter for finite particle systems around the incoherent state, in particular, because $d_{\alpha,\beta}$ can be controlled by the stronger adapted Wasserstein distance d from Chapter 2.

More precisely, we find directly from the dual Lipschitz formulation

$$\begin{aligned}|(\hat{f}^N)_l(\xi) - (\hat{f}^\infty)_l(\xi)| &= \left|\int_\Gamma \mathrm{e}^{-\mathrm{i}l\theta-\mathrm{i}\xi\omega} f^N(\theta,\omega)\mathrm{d}\theta\mathrm{d}\omega - \int_\Gamma \mathrm{e}^{-\mathrm{i}l\theta-\mathrm{i}\xi\omega} f^\infty(\theta,\omega)\mathrm{d}\theta\mathrm{d}\omega\right| \\ &\leq 2\sqrt{2}\sqrt{l^2+\xi^2}\, d(f^N, f^\infty).\end{aligned}$$

In order to relate this control to $d_{\alpha,\beta}$, we study

$$(l,\xi) \to \mathrm{e}^{-\alpha\xi-\beta l}\sqrt{l^2+\xi^2}$$

as function for $(l,\xi) \in \mathbb{R}^+ \times \mathbb{R}^+$. On the boundary $l = 0$ and $\xi = 0$, it obtains the maxima

$$\frac{1}{\mathrm{e}\,\alpha} \quad \text{and} \quad \frac{1}{\mathrm{e}\,\beta}$$

and as $\|(l,\xi)\| \to \infty$ the weight vanishes. In the interior we have the local extremum at $(l,\xi) = (\beta,\alpha)/(\alpha^2+\beta^2)$ with the value

$$\frac{1}{\mathrm{e}\sqrt{\alpha^2+\beta^2}}.$$

Therefore,

$$\sup_{l\geq 0}\sup_{\xi\geq 0} 2\sqrt{2}\sqrt{l^2+\xi^2}\,\mathrm{e}^{-\alpha\xi-\beta l} = \frac{2\sqrt{2}}{\mathrm{e}\min(\alpha,\beta)}$$

which implies

$$d_{\alpha,\beta}(f^N,f^\infty) \leq \frac{2\sqrt{2}}{\mathrm{e}\min(\alpha,\beta)} d(f^N,f^\infty).$$

Hence if $\alpha,\beta > 0$, then an empirical measure f^N close in weak topology to the continuous measure f^∞, is also close in $d_{\alpha,\beta}$ distance. Conversely, the $d_{\alpha,\beta}$ distance can only become small if $\alpha,\beta > 0$, because the Fourier transform of the empirical measure is not decaying.

Applied to the order parameter, the result of this section now shows that, for initial data close to the incoherent state and finite time range $t \in [0,T]$, we can control the order parameter as

$$\left|\eta\left[f^N(t,\cdot)\right] - \eta\left[f^\infty(t,\cdot)\right]\right| \leq C\frac{2\sqrt{2}}{\mathrm{e}\min(\alpha,\beta)}\mathrm{e}^{\alpha t} d(f^N,f^\infty)$$

with an explicit constant C and any $\alpha,\beta > 0$ (the choice of α and β changes however the required closeness to the incoherent state). Compared to the standard estimate (Lemma 2.3)

$$\left|\eta\left[f^N(t,\cdot)\right] - \eta\left[f^\infty(t,\cdot)\right]\right| \leq 2\sqrt{2}\,\mathrm{e}^{C_D t} d(f^N,f^\infty),$$

this improves the quantitative control in this setting.

For the control of the difference, we first consider the linearisation L_{f^∞} around the incoherent state with velocity distribution g^∞. Explicitly, in the given Fourier representation, we recall it as

$$\begin{cases} (L_{f^\infty}u)_1(\xi) = \partial_\xi u_1(\xi) + \dfrac{K}{2}u_1(0)\hat{g}^\infty(\xi), \\ (L_{f^\infty}u)_l(\xi) = l\partial_\xi u_l(\xi) \qquad \text{for } l \geq 2. \end{cases}$$

As we assume that the evolution will be close to the incoherent state, we assume that L_{f^∞} is stable, i.e. $\hat{g} \in L^1(\mathbb{R}^+)$ and

$$1 - \frac{K}{2}\mathcal{L}\hat{g}^\infty(\lambda) \neq 0 \quad \text{for } \Re\lambda \geq 0.$$

Recalling Theorem 4.2, this shows that the resolvent of the associated Volterra equation has finite $L^1(\mathbb{R})$ norm. We can adapt Lemma 4.13 to the situation of a decreasing weight, showing that the linear evolution of L_{f^∞} can be controlled with the rate of the free transport.

Lemma 4.35. *Let g^∞ be a stable velocity distribution with $\hat{g} \in L^1(\mathbb{R}^+)$. Then there exists a* *constant c_1 such that for any weight $w : \mathbb{N} \times \mathbb{R}^+ \mapsto [0,1]$, which is decreasing in the second variable ξ, the semigroup $\mathrm{e}^{tL_{f^\infty}}$ satisfies for $t \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^+$*

$$w_1(\xi+t)|(\mathrm{e}^{tL_{f^\infty}}u)_1(\xi)| \le c_1 \sup_{\xi\ge 0} w_1(\xi)u_1(\xi)$$

and for $l \ge 2$

$$w_l(\xi+t)|(\mathrm{e}^{tL_{f^\infty}}u)_l(\xi)| \le \sup_{\xi\ge 0} w_l(\xi)u_l(\xi).$$

Proof. Let $\nu(t) = (\mathrm{e}^{tL_{f^\infty}}u)_1(0)$ be the order parameter under the linear evolution. By Theorem 4.2, there exists a resolvent $r \in L^1(\mathbb{R}^+)$ such that

$$\nu(t) = u_1(t) - \int_0^t u_1(s)r(t-s)\mathrm{d}s.$$

Hence we find

$$\begin{aligned} w_1(t)|\nu(t)| &\le \left[\sup_{\xi\ge 0} w_1(\xi)|u_1(\xi)|\right]\left[1+\int_0^t \frac{w_1(t)}{w_1(s)}|r(t-s)|\mathrm{d}s\right] \\ &\le \left[\sup_{\xi\ge 0} w_1(\xi)|u_1(\xi)|\right]\left[1+\|r\|_{L^1(\mathbb{R}^+)}\right]. \end{aligned}$$

Thus we can control

$$\begin{aligned} &w_1(\xi+t)|(\mathrm{e}^{tL_{f^\infty}}u)_l(\xi)| \\ &\quad\le \left[\sup_{\xi\ge 0} w_1(\xi)|u_1(\xi)|\right]\left[1+\int_0^t \frac{K}{2}(1+\|r\|_{L^1(\mathbb{R}^+)})\frac{w_1(\xi+t)}{w_1(s)}|\hat{g}^\infty(\xi+t-s)|\mathrm{d}s\right] \\ &\quad\le \left[\sup_{\xi\ge 0} w_1(\xi)|u_1(\xi)|\right]\left[1+(1+\|r\|_{L^1(\mathbb{R}^+)})\|\hat{g}^\infty\|_{L^1(\mathbb{R}^+)}\right]. \end{aligned}$$

The control for $l \ge 2$ follows directly from the explicit solution. □

We first consider the evolution close to the stationary incoherent state.

Theorem 4.36. *Let f^∞ be the stable incoherent state with velocity marginal g satisfying $\hat{g} \in L^1(\mathbb{R})$ and let c_1 be the corresponding constant from Lemma 4.35. Then for $\alpha, \beta, A, \lambda > 0$ a solution f^N to the Kuramoto equation with*

$$\delta := d_{\alpha,\beta}(f^N_{\mathrm{in}}, f^\infty_{\mathrm{in}})$$

is controlled as

$$\sup_{\xi\ge 0} \mathrm{e}^{-\alpha\xi}|\hat{f}^N_1(t,\xi)| \le \mathrm{e}^{\beta}\mathrm{e}^{\lambda t}\mathrm{e}^{\alpha t}(c_1+A)\delta$$

for times $t \in [0,T]$, where T is such that

$$\delta < \frac{A}{\frac{Kc_1(1-e^{-\lambda T})}{2\lambda}(c_1+A) + Kc_1 \mathrm{e}^{2\beta+1} T \mathrm{e}^{2\alpha T} \mathrm{e}^{\lambda T} (c_1+A)^2}.$$

In particular, this controls the order parameter as

$$|\eta(t)| \leq \mathrm{e}^{\beta} \mathrm{e}^{(\lambda+\alpha)t}(c_1+A)\delta.$$

In the case $\lambda = 0$ the condition becomes

$$\delta < \frac{A}{\frac{Kc_1 T}{2}(c_1+A) + Kc_1 \mathrm{e}^{2\beta+1} T \mathrm{e}^{2\alpha T} (c_1+A)^2}.$$

Remark 4.37. The condition on δ is always satisfied if $T \to 0$ and the condition can equivalently formulated by restricting the time range.

The proof is based on the following lemma controlling the growth.

Lemma 4.38. *With the setup of Theorem 4.36 and with $\gamma, \lambda > 0$ define*

$$e(t) = \mathrm{e}^{-\lambda t} \sup_{l \geq 1} \sup_{\xi \geq 0} e^{-\beta l} \mathrm{e}^{-\alpha(\xi + lt)} \mathrm{e}^{-\gamma t(l-1)} |\hat{f}_l^N(t,\xi) - \hat{f}_l^\infty(t,\xi)|,$$

Then $t \to e(t)$ is continuous and the growth is controlled as

$$e(t) \leq c_1\delta + \frac{Kc_1(1-\mathrm{e}^{-\lambda t})}{2\lambda}\delta \left[\sup_{s\in[0,t)} e(s)\right] + \frac{Kc_1}{\gamma}\mathrm{e}^{2\beta}\mathrm{e}^{2\alpha t}\mathrm{e}^{\gamma t}\mathrm{e}^{\lambda t}\left[\sup_{s\in[0,t)} e(s)\right]^2.$$

Note that we here restrict to $l \geq 1$ as the $l = 0$ mode is preserved. As in the stability proof, we allow some loss over time in the higher modes.

Proof. The result is obtained by studying the difference

$$v_l(t,\xi) = \hat{f}_l^N(t,\xi) - \hat{f}_l^\infty(t,\xi). \tag{4.15}$$

Since the weight is decreasing and $|v_l(t,\xi)| \leq 2$, the continuity of the Fourier transform implies that $t \to e(t)$ is continuous. Moreover, by mollification of the initial data and propagation of regularity (Lemma 2.5), the following a priori estimates give the claimed control.

The evolution of the difference v is given by

$$\partial_t v = L_{f^\infty} v + R(v),$$

where R collects the remaining terms, i.e.

$$(Rv)_1(\xi) = \frac{K}{2}\left[v_1(0)(\hat{g}^N(\xi) - \hat{g}^\infty(\xi)) - \overline{v_1(0)}v_2(\xi)\right]$$

and for $l \geq 2$

$$(Rv)_l(\xi) = \frac{Kl}{2}\left[v_1(0)v_{l-1}(\xi) - \overline{v_1(0)}v_{l+1}(\xi)\right].$$

The remaining term can be bounded as

$$\sup_{\xi\geq 0} \mathrm{e}^{-\alpha\xi}|(Rv)_1(\xi)| \leq \frac{K}{2}|v_1(0)|\Big(\delta + \|v_2\|_{L^\infty(\mathbb{R}^+,\exp_{-\alpha})}\Big)$$

and for $l \geq 2$

$$\sup_{\xi\geq 0} \mathrm{e}^{-\alpha\xi}|(Rv)_l(\xi)| \leq \frac{Kl}{2}|v_1(0)|\Big(\|v_{l-1}\|_{L^\infty(\mathbb{R}^+,\exp_{-\alpha})} + \|v_{l+1}\|_{L^\infty(\mathbb{R}^+,\exp_{-\alpha})}\Big).$$

Hence we find by Duhamel for $l \geq 2$

$$\begin{aligned}
\mathrm{e}^{-\lambda t}\mathrm{e}^{-\beta l}\mathrm{e}^{-\alpha(\xi+lt)}\mathrm{e}^{-\gamma t(l-1)}|v_l(\xi)| &\leq \mathrm{e}^{-\gamma(l-1)t-\lambda t}\delta + \int_0^t Kl\mathrm{e}^{2\beta}\mathrm{e}^{2\alpha s}\mathrm{e}^{\gamma t}\mathrm{e}^{-\gamma l(t-s)}\mathrm{e}^{-\lambda t+2\lambda s}e(s)^2\mathrm{d}s \\
&\leq \mathrm{e}^{-\gamma(l-1)t-\lambda t}\delta + \frac{Kl}{\gamma l + 2\lambda + 2\alpha}\mathrm{e}^{2\beta}\mathrm{e}^{2\alpha t}\mathrm{e}^{\gamma t}\mathrm{e}^{\lambda t}\left[\sup_{s<t} e(s)\right]^2
\end{aligned}$$

and for $l = 1$

$$\begin{aligned}
&\mathrm{e}^{-\lambda t}\mathrm{e}^{-\beta}\mathrm{e}^{-\alpha(\xi+t)}|v_1(\xi)| \\
&\quad\leq c_1 e^{-\lambda t}\delta + \int_0^t \frac{Kc_1}{2}\mathrm{e}^{\alpha s+\lambda s}e(s)\left[\mathrm{e}^{-\lambda t}\mathrm{e}^{-\alpha s}\delta + \mathrm{e}^{-\lambda(t-s)}\mathrm{e}^{2\beta}\mathrm{e}^{\alpha s}\mathrm{e}^{\gamma s}e(s)\right]\mathrm{d}s \\
&\quad\leq c_1\mathrm{e}^{-\lambda t}\delta + \frac{Kc_1(1-\mathrm{e}^{-\lambda t})}{2\lambda}\delta\left[\sup_{s<t} e(s)\right] + \frac{Kc_1\mathrm{e}^{2\beta}\mathrm{e}^{2\alpha t}\mathrm{e}^{\gamma t}\mathrm{e}^{\lambda t}}{4\lambda + 4\alpha + 2\gamma}\left[\sup_{s<t} e(s)\right]^2.
\end{aligned}$$

□

Remark 4.39. A finer estimate looking more carefully at the linear term is

$$e(t) \leq c_1\delta + \frac{Kc_1}{2}\delta\int_0^t \mathrm{e}^{-\lambda(t-s)}e(s)\mathrm{d}s + \frac{Kc_1}{\gamma}\mathrm{e}^{2\beta}\mathrm{e}^{2\alpha t}\mathrm{e}^{\gamma t}\mathrm{e}^{\lambda t}\left[\sup_{s\in[0,t)} e(s)\right]^2.$$

In particular, ignoring the quadratic term, we can obtain a stricter result on the decay, showing the continuity of the resolvent on the level of the Volterra equation. Moreover, the obtained constants can be optimised.

With this estimate, we can now easily obtain the proof of Theorem 4.36.

Proof of Theorem 4.36. Assume for $\gamma, T > 0$ holds

$$(c_1 + A)\delta > c_1\delta + \frac{Kc_1(1 - \mathrm{e}^{-\lambda T})}{2\lambda}\delta(c_1 + A)\delta + \frac{Kc_1}{\gamma}\mathrm{e}^{2\beta}\mathrm{e}^{2\alpha T}\mathrm{e}^{\gamma T}\mathrm{e}^{\lambda T}(c_1 + A)^2\delta^2.$$

Then the previous Lemma 4.38 shows that for $t \in [0, T]$ holds

$$e(t) \le (c_1 + A)\delta.$$

Here γ is a free parameter and the optimal choice for controlling the first spatial mode $l = 1$ is $\gamma = T^{-1}$, which then gives the claimed condition. $\square$

The next step is to consider the case when f^∞ is a smooth solution converging to the incoherent state through Landau damping. In this case the evolution of the difference

$$v_l(t, \xi) = \hat{f}_l^N(t, \xi) - \hat{f}_l^\infty(t, \xi)$$

evolves according to

$$\partial_t v_l = l\partial_\xi v_l + \frac{Kl}{2}\left[u_1^N(0)v_{l-1} + v_1(0)u_{l-1}^\infty - (\overline{u_1^N(0)}v_{l+1} + \overline{v_1(0)}u_{l+1}^\infty)\right],$$

where u^N and u^∞ are the Fourier transform of f^N and f^∞, respectively, with the understanding that $v_0 = \hat{g}^N - \hat{g}^\infty$ and $u_0 = \hat{g}^\infty$.

Again we separate the linear part as

$$\partial_t v = L_{f^\infty} v + Rv$$

where

$$(Rv)_1(\xi) = \frac{K}{2}\left[u_1^N(0)v_0 - \overline{u_1^N(0)}v_2 - \overline{v_1(0)}u_2^\infty\right]$$

and for $l \ge 2$

$$(Rv)_l(\xi) = \frac{Kl}{2}\left[u_1^N(0)v_{l-1} + v_1(0)u_{l-1}^\infty - (\overline{u_1^N(0)}v_{l+1} + \overline{v_1(0)}u_{l+1}^\infty)\right].$$

Using Duhamel's principle with L_{f^∞}, we can again control the growth, cf. Lemma 4.35.

Lemma 4.40. *Assume a damped solution f^∞ to the Kuramoto equation such that the semigroup generated by L_{f^∞} is stable as in Lemma 4.35 and that with $a > 0$ holds*

$$\sup_{l\ge1}\sup_{\xi\ge0}|u_l^\infty(t,\xi)| \le d^\infty(t) \le \mathrm{e}^{-at}\delta^\infty.$$

Let $\alpha, \beta, \lambda, \gamma > 0$ and consider another solution f^N with $d(f_{\mathrm{in}}^N, f_{\mathrm{in}}^\infty) = \delta$, i.e. initially the *difference $v = u^N - u^\infty$ in Fourier of f^N and f^∞ satisfies*

$$\sup_{l\geq 0}\sup_{\xi\geq 0} \mathrm{e}^{-\beta l}\mathrm{e}^{-\alpha\xi}|(v_{\mathrm{in}})_l(\xi)| = \delta.$$

Then for

$$e(t) = \mathrm{e}^{-\lambda t}\sup_{l\geq 1}\sup_{\xi\geq 0} \mathrm{e}^{-\beta l}\mathrm{e}^{-\alpha(\xi+lt)}\mathrm{e}^{-\gamma t(l-1)}|v_l(\xi)|$$

we have the control

$$e(t) \leq c_1\delta + c_1^\infty\delta^\infty\delta + c_L^\infty\delta^\infty\left[\sup_{s\in[0,t)} e(s)\right] + c_L\delta\left[\sup_{s\in[0,t)} e(s)\right] + c_Q\left[\sup_{s\in[0,t)} e(s)\right]^2$$

with

$$\begin{aligned}
c_1^\infty &= \frac{Kc_1\mathrm{e}^{-\beta}}{2(\alpha+a)},\\
c_L^\infty &= \max\left(\frac{c_1}{2}, 2\right)\frac{K}{\gamma}\mathrm{e}^{\beta+\alpha t+\gamma t},\\
c_L &= \frac{Kc_1}{2}\frac{\left(1-\mathrm{e}^{-\lambda t}\right)}{\lambda},\\
c_Q &= \max\left(\frac{c_1}{2}, 1\right)\frac{K}{\gamma}\mathrm{e}^{2\beta}\mathrm{e}^{2\alpha t}\mathrm{e}^{\lambda t}\mathrm{e}^{\gamma t}.
\end{aligned}$$

Proof. The remaining term Rv can be bounded as

$$\sup_{\xi\geq 0}\mathrm{e}^{-\alpha\xi}|(Rv)_1(\xi)| \leq \frac{K}{2}\left[(d^\infty(t) + |v_1(0)|)(\delta + \|v_2\|_{L^\infty(\mathbb{R}^+,\exp_{-\alpha})}) + |v_1(0)|d^\infty(t)\right]$$

and for $l \geq 2$

$$\begin{aligned}
\sup_{\xi\geq 0}\mathrm{e}^{-\alpha\xi}|(Rv)_l(\xi)| \leq \frac{Kl}{2}\Big[&(d^\infty(t) + |v_1(0)|)\left(\|v_{l-1}\|_{L^\infty(\mathbb{R}^+,\exp_{-\alpha})} + \|v_{l+1}\|_{L^\infty(\mathbb{R}^+,\exp_{-\alpha})}\right)\\
&+ 2|v_1(0)|d^\infty(t)\Big].
\end{aligned}$$

Use the Duhamel formula splitting to find

$$
\begin{aligned}
&\mathrm{e}^{-\lambda t}\mathrm{e}^{-\beta}\mathrm{e}^{-\alpha(\xi+t)}|v_1(t,\xi)| \\
&\le c_1\mathrm{e}^{-\lambda t}\delta + \frac{Kc_1}{2}\int_0^t \mathrm{e}^{-\lambda t-\beta-\alpha s}\Big[\big(d^\infty(s)+\mathrm{e}^{\lambda s+\beta+\alpha s}e(s)\big)\big(\delta+\mathrm{e}^{\lambda s+2\beta+2\alpha s+\gamma s}e(s)\big) \\
&\qquad\qquad + \mathrm{e}^{\lambda s+\beta+\alpha s}e(s)d^\infty(s)\Big]\mathrm{d}s \\
&\le c_1\mathrm{e}^{-\lambda t}\delta + \frac{Kc_1}{2}\delta\int_0^t d^\infty(s)\mathrm{e}^{-\lambda t-\beta-\alpha s}\mathrm{d}s \\
&\quad + \frac{Kc_1}{2}\left[\sup_{s<t} e(s)\right]\left[\delta\int_0^t \mathrm{e}^{-\lambda(t-s)}\mathrm{d}s + \int_0^t \mathrm{e}^{-\lambda(t-s)}(1+\mathrm{e}^{\beta+\alpha s+\gamma s})d^\infty(s)\mathrm{d}s\right] \\
&\quad + \frac{Kc_1}{2}\left[\sup_{s<t} e(s)\right]^2\int_0^t \mathrm{e}^{\lambda t-2\lambda(t-s)+2\beta+2\alpha s+\gamma s}\mathrm{d}s
\end{aligned}
$$

and for $l \ge 2$

$$
\begin{aligned}
&\mathrm{e}^{-\lambda t}\mathrm{e}^{-\beta l}\mathrm{e}^{-\alpha(\xi+lt)}\mathrm{e}^{-\gamma t(l-1)}|v_l(t,\xi)| \\
&\le \mathrm{e}^{-\lambda t}\mathrm{e}^{-\gamma t(l-1)}\delta + Kl\int_0^t (d^\infty(s)+\mathrm{e}^{\lambda s+\beta+\alpha s}e(s))\mathrm{e}^{-\lambda(t-s)+\beta+\alpha s+\gamma t-\gamma l(t-s)}e(s)\mathrm{d}s \\
&\quad + Kl\int_0^t \mathrm{e}^{\lambda s+\beta+\alpha s}e(s)\mathrm{e}^{-\lambda t-\beta l-\alpha lt-\gamma(l-1)t}d^\infty(s)\mathrm{d}s \\
&\le \mathrm{e}^{-\lambda t}\mathrm{e}^{-\gamma t(l-1)}\delta + 2Kl\left[\sup_{s<t} e(s)\right]\int_0^t \mathrm{e}^{-\lambda(t-s)+\beta+\alpha s+\gamma t-\gamma l(t-s)}d^\infty(s)\mathrm{d}s \\
&\quad + Kl\left[\sup_{s<t} e(s)\right]^2\int_0^t \mathrm{e}^{-\lambda t+2\lambda s+2\beta+2\alpha s+\gamma t-\gamma l(t-s)}\mathrm{d}s.
\end{aligned}
$$

Using the bound on $d^\infty(s)$ then shows the claimed estimate. □

This allows to control the behaviour around a sufficiently damped solution.

Theorem 4.41. *Assume a damped solution f^∞ such that the semigroup generated by L_{f^∞} is stable as in Lemma 4.35 and that with $a > 0$ holds*

$$\sup_{l\ge 1}\sup_{\xi\ge 0}|u_l^\infty(t,\xi)| \le d^\infty(t) \le \mathrm{e}^{-at}\delta^\infty.$$

Let $\alpha,\beta > 0$ and consider another solution f^N with initial difference

$$d_{\alpha,\beta}(f^N_{\mathrm{in}}, f^\infty_{\mathrm{in}}) = \delta.$$

Then for $A,\lambda > 0$ and a time range T satisfying

$$c_L^\infty\delta^\infty < 1$$

and

$$A > \frac{c_L^\infty \delta^\infty (c_1 + c_1^\infty \delta^\infty)}{1 - c_L^\infty \delta^\infty}$$

and

$$\delta < \frac{A - c_L^\infty \delta^\infty (c_1 + c_1^\infty \delta^\infty + A)}{c_L (c_1 + c_1^\infty \delta^\infty + A) + c_Q (c_1 + c_1^\infty \delta^\infty + A)^2}$$

the solution is controlled as

$$\sup_{\xi \geq 0} \mathrm{e}^{-a\xi} |v_1^N(t,\xi)| \leq \mathrm{e}^{\beta} \mathrm{e}^{\lambda t} \mathrm{e}^{\alpha t} (c_1 + c_1^\infty \delta^\infty + A)\delta,$$

where c_1^∞ is as in Lemma 4.40 and

$$\begin{aligned}
c_L^\infty &= \max\left(\frac{c_1}{2}, 2\right) \frac{K}{\gamma} \mathrm{e}^{\beta + \alpha T + \gamma T}, \\
c_L &= \frac{K c_1}{2} \frac{\left(1 - \mathrm{e}^{-\lambda T}\right)}{\lambda}, \\
c_Q &= \max\left(\frac{c_1}{2}, 1\right) K T \mathrm{e}^{2\beta} \mathrm{e}^{2\alpha t} \mathrm{e}^{\lambda t}.
\end{aligned}$$

Remark 4.42. By choosing the time-range small enough, we can make c_L^∞ arbitrary small so that the conditions are satisfied for any given A and δ.

Proof. The proof is as in Theorem 4.36, where we again choose $\gamma = T^{-1}$ to optimise and propagate control through Lemma 4.40. □

Finally, we remark that as in the nonlinear stability (Theorem 4.4), we can also handle the distance in a suitable Sobolev norm. In this case we assume, that we control the smooth solution as

$$|u_l^\infty(t,\xi)| \frac{(1+\xi+t)^b}{(1+t)^{a(l-1)}} \leq \delta^\infty$$

by Theorem 4.4. The initial distance to f^N is measured by

$$\delta = \sup_{l \geq 0} \sup_{\xi \geq 0} \frac{|(v_{\mathrm{in}})_l(\xi)|}{(1+\xi)^\alpha \max(1,l)^\beta},$$

where v is again the difference of the Fourier transform of f^N and f^∞. In this distance we assume that $\alpha, \beta \geq 1$, which implies that it is controlled by the adapted Wasserstein distance $d(f_{\mathrm{in}}^N, f_{\mathrm{in}}^\infty)$.

In this setup, we propagate, similar to Lemma 4.40, the following distance

$$e(t) = \sup_{l \geq 1} \sup_{\xi \geq 0} \frac{|v_l(t,\xi)|}{(1+\xi+lt)^\alpha l^\beta (1+t)^{\gamma(l-1)}}.$$

Lemma 4.43. *Let $1+\alpha > b$ and $\gamma \geq a$. Then the distance $e(t)$ satisfies*

$$e(t) \leq c_1\delta + c_1^\infty \delta^\infty \delta + c_L^\infty \delta^\infty \left[\sup_{s\in[0,t)} e(s)\right] + c_L\delta \left[\sup_{s\in[0,t)} e(s)\right] + c_Q \left[\sup_{s\in[0,t)} e(s)\right]^2$$

with

$$\begin{aligned}
c_1^\infty &= \frac{Kc_1}{2(1-b)}\left[(1+t)^{1-b} - 1\right],\\
c_L^\infty &= \max\left(K\left[\left(\frac{3}{2}\right)^\beta + 1\right]\frac{(1+t)^{\alpha-b+\gamma+1}}{\gamma}, \frac{Kc_1}{2}\left[\frac{(1+t)^{a-b+1}}{a+\alpha-b+1} + \frac{(1+t)^{\alpha+\gamma-b+1}-1}{\alpha+\gamma-b+1}\right]\right),\\
c_L &= \frac{Kc_1}{1+\alpha}\left[(1+t)^{1+\alpha} - 1\right],\\
c_Q &= \max\left(K\left(\frac{3}{2}\right)^\beta \frac{(1+t)^{2\alpha+\gamma-1}}{\gamma}, \frac{Kc_1}{2^{1-\beta}}\frac{\left[(1+t)^{2\alpha+\gamma+1}-1\right]}{2\alpha+\gamma}\right).
\end{aligned}$$

Proof. We again use Duhamel. For $l = 1$ we find

$$\begin{aligned}
\frac{|v_1(t,\xi)|}{(1+\xi+t)^\alpha} \leq c_1\delta &+ \frac{Kc_1}{2}\int_0^t \left(\frac{\delta^\infty}{(1+s)^b} + e(s)(1+s)^\alpha\right)\left(\delta + 2^\beta(1+s)^{\alpha+\gamma}e(s)\right)\mathrm{d}s\\
&+ \frac{Kc_1}{2}\int_0^t e(s)(1+s)^\alpha \delta^\infty \frac{(1+s)^{a-b}}{(1+t)^\alpha}\mathrm{d}s.
\end{aligned}$$

and for $l \geq 2$

$$\begin{aligned}
&\frac{v_l(t,\xi)}{(1+\xi+lt)^\alpha l^\beta (1+t)^{\gamma(l-1)}}\\
&\leq \frac{\delta}{(1+t)^{\gamma(l-1)}} + \frac{Kl}{2}\int_0^t \left(\frac{\delta^\infty}{(1+s)^b} + e(s)(1+s)^\alpha\right) 2e(s)\left(\frac{l+1}{l}\right)^\beta \frac{(1+s)^{\gamma l+\alpha}}{(1+t)^{\gamma(l-1)}}\mathrm{d}s\\
&\quad + \frac{Kl}{2}\int_0^t e(s)(1+s)^\alpha 2\delta^\infty \frac{(1+s)^{al-b}}{l^\beta(1+t)^{\alpha+\gamma(l-1)}}\mathrm{d}s.
\end{aligned}$$

Bounding the integrals and taking the supremum then shows the claimed bound. □

As in the exponential case, we can use this to find an control over finite time ranges.

We finish this section with a brief discussion. As u^N and u^∞ are the Fourier transform of probability measures, they are trivially bounded by 1. As the weight is decreasing, we could use this bound for large l and ξ, e.g. in order to break the nonlinearity or to restrict to a finite range of l. As the case for large l implies the exponential growth this could allow some truly algebraic control. Also the constants are often bounded by the bound for large l so that using this control can improve the estimates. Another improvement could be the use of an L^2 type norm in order to take advantage of the divergence structure, similar to the ideas in Chapter 5.

Finally, for applications, it would be interesting to compare the results with the expected distance of a particle system approximating a smooth state, i.e. if we create initial data f_{in}^N by sampling a smooth distribution f_{in}^{∞}, we know that in the Wasserstein distance $d(f_{\mathrm{in}}^{N}, f_{\mathrm{in}}^{\infty}) \to 0$ as $N \to \infty$. However, for a better comparison, we are interested in the precise convergence behaviour in the different distances introduced in this section. With this, we could optimise the predicted control of the order parameter and compare the approaches more accurately.

5 Stability of partially locked states in the Kuramoto model

The work in this chapter has been done in collaboration with Bastien Fernandez and David Gérard-Varet from Université Paris 7 Denis Diderot and follows the joint preprint [45]. In this work, we want to understand the stability of partially locked states.

5.1 Overview

In many cases the Kuramoto equation seems to converge to a synchronised stationary state for a large enough coupling constant. The structure of such states has been discussed in Section 1.3.2 and as asymptotic states, we expect that all locked oscillators are at the stable fixed-point. These states are denoted by f_{st} and numerical simulations suggest that these states can be stable and capture the asymptotic behaviour. In particular, for a symmetric unimodal distribution, Mirollo and Strogatz [109] showed the absence of growing modes with a continuous spectrum along the imaginary axis, see also [120]. These states are irregular and nevertheless the stability of the unlocked oscillators only happens through phase mixing in a weak sense.

The starting point of this work is the bifurcation analysis in my previous work (Chapter 4 and [46]), which included the stability of these states around the incoherent state, see also [31].

Thus the idea is to understand the stability again with one-sided exponential norms in Fourier variables. However, the transport operator L_1 has no easy explicit solution in contrast to the free transport operator. We compensate this by using ℓ^2 norms and resolvent estimates, which take advantage of the divergence structure like in Section 4.4. In devising a suitable norm, a key requirement is a possible control of the order parameter. With the Sobolev embedding theorem, this inspired the use of the following weighted norms for the Fourier transform $\hat{f}$. For $a > 0$ and $k \in \mathbb{R}$ define

$$\|h\|_a = \left(\int_{\mathbb{R}} \mathrm{e}^{2a\xi} \left(|h(\xi)|^2 + |h'(\xi)|^2 \right) \mathrm{d}\xi \right)^{1/2}$$

for $h : \mathbb{R} \mapsto \mathbb{C}$ and

$$\|u\|_{a,k} = \left(\sum_{l\in\mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} l^{2k} \left(|u_l(\xi)|^2 + |\partial_\xi u_l(\xi)|^2 \right) \mathrm{d}\xi \right)^{1/2}.$$

for $u : \mathbb{N} \times \mathbb{R} \mapsto \mathbb{C}$ and let $\mathcal{X}_{a,k}$ be the corresponding Hilbert space.

By the Gearhart-Prüss theorem, we can conclude exponential decay under the transport operator in $\mathcal{X}_{a,-1}$ and $\mathcal{X}_{a,0}$ by the resolvent estimates. Adding the finite-rank operator L_2 to find the complete linearised dynamics, we still have exponential decay apart from possible eigenmodes.

In fact, the stationary states have a rotation symmetry, which means that the linearised equation has always an eigenvalue 0. Therefore, a stability result must mean that a perturbed state asymptotically converges to a possibly rotated state. This can be understood through a center-manifold reduction or by polar coordinates. Our main result exactly shows this for the full nonlinear dynamics in the absence of additional linear modes.

During the writeup F. Rousset pointed out to us, that a similar use of weighted spaces was done by Pego and Weinstein [127] in order to understand the asymptotic stability of solitary waves in the generalised Korteweg–de Vries (KdV) equation.

An additional difficulty is the handling of the regularity loss by the nonlinearity, which must be compensated by the regularisation of the linear semigroup. In this setup, we have a solution in $\mathcal{X}_{a,0}$ and a forcing in $\mathcal{X}_{a,-1}$ by the nonlinearity.

Typically, like in the work by Pego and Weinstein [127] and in the case of regularising noise (heat equation), one shows for the semigroup e^{tL} of the linear evolution an estimate of the form

$$\|\mathrm{e}^{tL} u\|_{a,0} \le C t^{-1/2} \mathrm{e}^{-at} \|u\|_{a,-1}.$$

By the Duhamel formula, this then controls the norm at a later time t, which we can use to close the argument.

However, in our case we can only expect that

$$\|\mathrm{e}^{tL} u\|_{a,0} \le C t^{-1} \mathrm{e}^{-at} \|u\|_{a,-1},$$

which is not integrable so that this approach does not work. In order to see this, take L to be the free transport operator, where we find

$$\|(\mathrm{e}^{tL} u)_l\|_a = \mathrm{e}^{-alt} \|(u)_l\|_a.$$

Hence we find

$$\|\mathrm{e}^{tL}\|_{\mathcal{X}_{a,-1} \mapsto \mathcal{X}_{a,0}} = \max_{l\in\mathbb{N}} l \mathrm{e}^{-alt},$$

which behaves in the claimed way.

In order to overcome the difficulty, we consider estimates in L^2 in time, where we can close the estimates assuming that the nonlinearity is small. By using additional energy estimates, we can conclude a pointwise bound in time justifying the assumed smallness and yielding the result.

Theorem 5.1. *Consider a partially locked state f_{st} with velocity marginal g. Let $a > 0$ be such that $\|\hat{g}\|_a < \infty$ and let $a > a' > 0$ be such that no eigenmode with $\Re\lambda \geq -a'$ exists apart from the eigenmode corresponding to the rotation symmetry.*

Then there exists $\epsilon > 0$ such that for every probability measure $f_{\mathrm{in}} \in \mathcal{M}(\Gamma)$ with velocity marginal g satisfying

$$\|\hat{f}_{\mathrm{in}} - \hat{f}_{\mathrm{st}}\|_{a,0} < \epsilon,$$

there exists $\Theta_\infty \in \mathbb{T}$ so that the solution f to the Kuramoto equation has the asymptotic behaviour

$$\|\hat{f}(t,\cdot) - \widehat{R_{\Theta_\infty} f}_{\mathrm{st}}\|_{a,0} = O(\mathrm{e}^{-a't}).$$

Here the hat $\hat{\cdot}$ denotes the Fourier transform as in Section 1.3, which we identified with their restriction to $\mathbb{N} \times \mathbb{R}$, and R_Θ is the rotated measure, i.e.

$$(R_\Theta f)(\theta, \omega) = f(\theta + \Theta, \omega).$$

The possible eigenmodes can be understood by perturbing the transport operator L_1 with L_2, the effect of the perturbed order parameter on the stationary state. This can be understood as in Chapter 3 and the resolvent equation. However, L_2 is not linear over $\mathbb{C}$ as it involves the order parameter η and its conjugate $\overline{\eta}$. We can handle this by formally separating η and $\overline{\eta}$, which yields a 2×2 matrix condition, see Section 5.4

Interestingly, the stability condition exactly matches the stability in the Ott–Antonsen ansatz, see Section 5.6.2.

5.2 Stationary states in $\mathcal{X}_{a,k}$

The exponential decay of the Fourier transform used in $\mathcal{X}_{a,k}$ can be related to the analytic continuation of the original function. Using L^2 norms, we can prove the equivalence of these notions.

Lemma 5.2. *Let f be a complex valued Radon measure on $\mathbb{R}$ and fix $a > 0$. Then the map $\xi \to \hat{f}(\xi)\mathrm{e}^{a\xi} \in L^2(\mathbb{R})$ if and only if there exists $F : \mathbb{C} \to \mathbb{C}$ such that*

(i) F is analytic in the strip $\{z \in \mathbb{C} : \Im z \in (-a, 0)\}$,

(ii) For all $\epsilon > 0$ holds

$$\sup_{y \in [-a,-\epsilon]} \|F(\cdot + \mathrm{i}y)\|_{L^2(\mathbb{R})} < +\infty$$

and for $y \in [-a, -\epsilon]$ holds

$$\|F(\cdot + \mathrm{i}y)\|_{L^2(\mathbb{R})} = \|\mathrm{e}^{-y\cdot}\hat{f}(\cdot)\|_{L^2(\mathbb{R})},$$

(iii) In $\mathcal{S}'(\mathbb{R})$ holds

$$\lim_{y\to 0-} F(\cdot + \mathrm{i}y) = f(\cdot).$$

Proof. Assume that $f \in \mathcal{S}'(\mathbb{R}, \mathbb{C})$ is such that $\xi \to \hat{f}(\xi)\mathrm{e}^{a\xi} \in L^2(\mathbb{R})$ for some $a > 0$ and let F be defined by

$$F(z) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}z\xi}\hat{f}(\xi)\mathrm{d}\xi.$$

Item (i) is a simple consequence of holomorphy under integral sign. Item (ii) is a basic application of Plancherel isometry.

To prove (iii), we first observe that Plancherel Theorem implies the following relation

$$\langle F(\cdot + \mathrm{i}y), \phi\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-y\xi}\hat{f}(\xi)\overline{\hat{\phi}(\xi)}\,\mathrm{d}\xi,$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. A simple application of the dominated convergence theorem then yields

$$\lim_{y\to 0-} \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-y\xi}\hat{f}(\xi)\overline{\hat{\phi}(\xi)}\mathrm{d}\xi = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{\phi}(\xi)}\mathrm{d}\xi = \langle f, \phi\rangle,$$

as desired.

Conversely, assume that items (i) to (iii) are fulfilled. By (iii) and the continuity of the Fourier transform in $\mathcal{S}'$, we have

$$\lim_{y\to 0-} \widehat{F(\cdot + \mathrm{i}y)} = \hat{f},$$

in $\mathcal{S}'$. Now (ii) implies that we can write for almost every ξ

$$\widehat{F(\cdot + \mathrm{i}y)}(\xi) = \mathrm{e}^{-y\xi}\int_{\mathbb{R}+\mathrm{i}y} \mathrm{e}^{-\mathrm{i}x\xi}F(x)\mathrm{d}x,$$

where the integral is to be understood in semi-convergence sense. Proceeding as in [137], holomorphy of F yields

$$\int_{\mathbb{R}+\mathrm{i}y} \mathrm{e}^{-\mathrm{i}x\xi}F(x)\mathrm{d}x = \int_{\mathbb{R}-\mathrm{i}a} \mathrm{e}^{-\mathrm{i}x\xi}F(x)\mathrm{d}x = \mathrm{e}^{-a\xi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}x\xi}F(x - \mathrm{i}a)\mathrm{d}x = \mathrm{e}^{-a\xi}u(\xi),$$

where $u \in L^2(\mathbb{R})$ as the Fourier transform of $F(\cdot - \mathrm{i}a) \in L^2(\mathbb{R})$. Therefore, we get

$$\begin{aligned}\hat{f} &= \lim_{y \to 0-} \mathrm{e}^{-y\xi}\mathrm{e}^{-a\xi}u \qquad \text{in } \mathcal{S}'(\mathbb{R}) \\ &= \lim_{y \to 0-} \mathrm{e}^{-y\xi}\mathrm{e}^{-a\xi}u \qquad \text{in } \mathcal{D}'(\mathbb{R}) \\ &= \mathrm{e}^{-a\xi}u\end{aligned}$$

from where the conclusion follows using $u \in L^2(\mathbb{R})$. $\square$

Recall from Section 1.3, that a possible partially locked state f_{pls} with a given velocity marginal g and order parameter $\eta \in [0,1]$ has the form

$$f_{\mathrm{pls}}(\theta,\omega) = \begin{cases} \big(\alpha(\omega)\delta_{\arcsin(\omega/(K\eta))}(\theta) + (1-\alpha(\omega))\delta_{\pi-\arcsin(\omega/(K\eta))}(\theta)\big)\, g(\omega) & \text{if } |\omega| \le K\eta, \\ \dfrac{\sqrt{\omega^2-(K\eta)^2}}{2\pi|\omega - K\eta\sin\theta|}g(\omega) & \text{if } |\omega| > K\eta \end{cases} \tag{5.1}$$

for a measurable function $\alpha : [-K\eta, K\eta] \mapsto [0,1]$. Assuming the regularity $\|\hat{g}\|_a < \infty$ on the velocity marginal, we then find that $\|\hat{f}_{\mathrm{pls}}\|_{a,k} < \infty$ if and only if $\alpha \equiv 1$, i.e. all locked oscillators are at the stable fixed point. Recall that we denoted this stable state by f_{st}.

Proposition 5.3. *Assume that $g \in L^1(\mathbb{R})$ and that $\|\hat{g}\|_a < +\infty$ for some $a > 0$. Then, for every $\eta \in [0,1]$, the probability measure f_{pls} defined by Equation* (5.1) *satisfies $\|\hat{f}_{\mathrm{pls}}\|_{a,k} < +\infty$ for some $k \in \mathbb{Z}$, if and only if $\alpha \equiv 1$. Moreover, $\|\hat{f}_{\mathrm{pls}}\|_{a,k} < +\infty$ if and only if $\|\hat{f}_{\mathrm{pls}}\|_{a,k'} < +\infty$ for all $k' \neq k$.*

Proof. The proof is split into two parts. The first part shows that $\alpha \equiv 1$ implies the finite norm for any $k \in \mathbb{Z}$. The second part then assures that a finite norm for any $k \in \mathbb{Z}$ implies $\alpha \equiv 1$, which yields the statement.

Proof that $\|\hat{f}_s\|_{a,k} < \infty$. The previous lemma shows that g has an analytic continuation in the strip $\{z \in \mathbb{C} : \Im z \in [-a, 0]\}$ and β is analytic in the lower half plane, cf. Lemma 4.30. The norm can therefore be expressed as

$$\|\hat{f}_s\|_{a,k} = \sum_{l \in \mathbb{N}} l^{2k} \int_{\mathbb{R}} (1 + |\omega - \mathrm{i}a|^2)|g(\omega - \mathrm{i}a)|^2 \left|\beta\left(\frac{\omega - \mathrm{i}a}{K\eta}\right)\right|^{2l} \mathrm{d}\omega.$$

By Lemma 4.30, we have

$$\sup_{\omega\in\mathbb{R}} \left|\beta\left(\frac{\omega - \mathrm{i}a}{K\eta}\right)\right| < 1,$$

and the assumed regularity on the velocity marginal

$$\|\hat{g}\|_a = \int_{\mathbb{R}} (1 + |\omega - \mathrm{i}a|^2)|g(\omega - \mathrm{i}a)|^2 \mathrm{d}\omega.$$

Using the bounds in the integral expression shows the claimed finiteness.

Proof of uniqueness. The proof proceeds by contradiction and relies on the following statement, whose proof is given below.

Lemma 5.4. *Let $f \in L^1(\mathbb{R})$ be such that $f(x) = 0$ for $|x| > \delta$ for some $\delta > 0$, and $\xi \to \hat{f}(\xi)\mathrm{e}^{a\xi} \in L^2(\mathbb{R})$ for some $a > 0$. Then $f = 0$ a.e.*

Now, in addition to f_{st}, assume the existence of a PLS f_{pls} with order parameter η, and $\alpha \not\equiv 1$ and $\|\hat{f}_{\mathrm{pls}}\|_{a,k} < +\infty$. Then expression (5.1) implies that the first Fourier coefficient of the difference $h = f_{\mathrm{pls}} - f_{\mathrm{st}}$ satisfies

- $\tilde{h}_1 \in L^1(\mathbb{R})$,
- $\tilde{h}_1(\omega) = 0$ for $|\omega| > K\eta$,
- the Fourier transform satisfies $\xi \to \hat{h}_1(\xi)\mathrm{e}^{a\xi} \in L^2(\mathbb{R})$.

However, Lemma 5.4 asserts that $h = 0$; hence the contradiction. □

Proof of Lemma 5.4. Consider the shifted f such that $\mathrm{supp}(f) \subset [-2\delta, 0]$. We have $|\hat{f}(z)| \leq \|f\|_1$ for $\Im(z) \geq 0$. Now, introduce the map h by

$$h(z) = \hat{f}\left(\frac{\mathrm{i} - \mathrm{i}z}{1+z}\right).$$

Then h is holomorphic in the unit ball $D = \{z : |z| < 1\}$ and continuous in $\overline{D} \setminus \{-1\}$. Moreover, h is bounded within $\overline{D} \setminus \{-1\}$ by $\|f\|_1$.

Up to dividing by z^n, we can assume w.l.o.g., that $h(0) \neq 0$. By contradiction, assume that f is not identically 0. The mapping $z \mapsto \log|h(z)|$ is subharmonic (see e.g. [137, Theorem 15.19]), which yields

$$\log|h(0)| \leq \frac{1}{2\pi} \lim_{r\to 1-} \int_{\mathbb{T}} \log|h(r\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta = \frac{1}{2\pi} \int_{\mathbb{T}} \log|h(\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta,$$

where the equality follows from Lebesgue's dominated convergence theorem based on that the quantity $|h(\mathrm{e}^{r\mathrm{i}\theta})|$ is bounded above. Hence the negative part $\log_-|h(e^{\mathrm{i}\cdot})|$ must be integrable over $\mathbb{T}$, i.e.

$$\int_{\mathbb{T}} \log_-|h(\mathrm{e}^{\mathrm{i}\theta})|\mathrm{d}\theta = 2\int_{\mathbb{R}} \frac{\log_-|\hat{f}(x)|}{1+x^2}\mathrm{d}x < +\infty,$$

where we have used a change of variable.

Now, let $A = \{x \in \mathbb{R}^+ : \mathrm{e}^{ax}|\hat{f}(x)| > 1\}$. We must have $\mathrm{Leb}(A) < +\infty$, otherwise we would have $\|f\|_a = +\infty$. Moreover,

$$\int_{\mathbb{R}} \frac{\log_-|\hat{f}(x)|}{1+x^2}\mathrm{d}x \geq \int_{\mathbb{R}^+} 1_{x\notin A}\frac{\log_-|\hat{f}(x)|}{1+x^2}\mathrm{d}x \geq \int_{\mathbb{R}^+} 1_{x\notin A}\frac{ax}{2(1+x^2)}\mathrm{d}x.$$

Now, using that $\mathrm{Leb}(A) < +\infty$, we get

$$\int_{\mathbb{R}^+} 1_{x\in A} \frac{ax}{1+x^2} \mathrm{d}x \leq C \,\mathrm{Leb}(A) < +\infty,$$

for some $C \in \mathbb{R}^+$. However, the integral $\int_{\mathbb{R}^+} \frac{ax}{1+x^2} \mathrm{d}x$ diverges. Therefore, the integral $\int_{\mathbb{R}^+} 1_{x\notin A} \frac{ax}{2(1+x^2)} \mathrm{d}x$ also diverges, and this contradicts the fact that $\theta \to \log_- |h(\mathrm{e}^{\mathrm{i}\theta})|$ is integrable. □

Finally, we remark that the convergence to the partially locked state in these norms implies weak convergence.

Lemma 5.5. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of probability measures on the cylinder with frequency marginal g and let f be with the same property. We have*

$$\lim_{n\to\infty} \|\hat{f}_n - \hat{f}\|_{a,0} = 0 \implies \lim_{n\to\infty} f_n = f,$$

where convergence here is understood in the weak sense.

Proof. The sequence of frequency marginals $\int_{\mathbb{T}} f_n(\theta, \omega) \mathrm{d}\theta$ is tight, because it is constant. Hence, the sequence $\{f_n\}_{n\in\mathbb{N}}$ itself is tight.

Let f' be any accumulation point of $\{f_n\}_{n\in\mathbb{N}}$ and let $\{n_i\}$ be the corresponding subsequence. Convergence in weak topology implies

$$\lim_{i\to\infty} (\hat{f}_{n_i})_l(\xi) = \hat{f}'_l(\xi), \qquad \forall (l, \xi).$$

However, the convergence $\|\hat{f}_n - \hat{f}\|_{a,0} \to 0$ implies that every $(\hat{f}_n)_l$ converges in $H^1([-m, m])$, for every $m \in \mathbb{R}^+$. By the Sobolev embedding $H^1([-m, m]) \hookrightarrow C_0([-m, m])$, this implies

$$\lim_{n\to\infty} (\hat{f}_n)_l(\xi) = \hat{f}_l(\xi), \qquad \forall (l, \xi).$$

Since the Fourier transform is one-to-one, we must have $f' = f$ for every accumulation point f'. Hence $\lim_{n\to\infty} f_n = f$. □

5.3 Cauchy problem in $\mathcal{X}_{a,k}$

Using energy estimates with the divergence structure, we can directly prove that the Cauchy problem in the norm $\mathcal{X}_{a,0}$ is well-posed.

Proposition 5.6. *Let $f_{\mathrm{in}} \in \mathcal{M}(\Gamma)$ with velocity marginal g satisfying $\|\hat{f}_{\mathrm{in}}\|_{a,0} < \infty$ and $\|\hat{g}\|_a < \infty$ for some $a > 0$. Then the solution to the Kuramoto equation with initial data f_{in} satisfies*

$$\sup_{t\in[0,T]} \|\hat{f}(t)\|_{a,0} < \infty$$

for all $T > 0$. Moreover, the map $t \to \hat{f}(t)$ is strongly continuous in $\mathcal{X}_{a,0}$.

We identified the evolution of the Fourier transform $u(t) = \hat{f}|_{\mathbb{N}\times\mathbb{R}}$ in Theorem 4.7, which we recall as

$$\partial_t u_l(\xi) = l\partial_\xi u_l(\xi) + \frac{Kl}{2}\left(u_1(0)u_{l-1}(\xi) - \overline{u_1(0)}u_{l+1}(\xi)\right), \ \forall (l,\xi) \in \mathbb{N}\times\mathbb{R}, \tag{5.2}$$

with the identification $u_0 = \hat{g}$. Using energy estimates, we can construct solutions in $\mathcal{X}_{a,0}$ with the claimed property.

Lemma 5.7. *Assume that $\|\hat{g}\|_a < +\infty$ for some $a > 0$. For every u_{in} in $\mathcal{X}_{a,0}$, there exists a unique weak solution u of Equation* (5.2) *that satisfies $u(0) = u_{\text{in}}$ and*

$$u \in L^\infty(0,T,\mathcal{X}_{a,0}) \cap L^2(0,T,\mathcal{X}_{a,1/2})$$

for all $T > 0$. Moreover, u belongs to $C(\mathbb{R}^+,\mathcal{X}_{a,0})$.

This shows the main result with the uniqueness from Theorem 4.9.

Proof of Proposition 5.6. Let $f = f(t)$ be a solution of the Kuramoto equation with initial data f_{in} and let $u_f := \hat{f}|_{\mathbb{N}\times\mathbb{R}}$. Clearly, u_f solves (5.2), and as the Fourier transform of a measure and it is uniformly bounded in (l,ξ). Let now u be the solution of (5.2) given by Lemma 5.7, with initial data u_{in}. Then, u_f and u both satisfy the assumption for the uniqueness theorem 4.9 for any $\beta > 0$, which implies $u_f = u$. □

Proof of Lemma 5.7. The proof of existence proceeds via an approximation scheme and a standard compactness argument based on Aubin-Lions Lemma. We start with the a priori estimates that are crucial for the limit processes. (NB: these estimates are well-defined for those $\{u_l(\xi)\}$ that are finite vectors of smooth functions with compact support).

The first estimate is obtained by testing (5.2) against $\mathrm{e}^{2a\xi}l^{-1}\overline{u_l(\xi)}$. After integration in ξ, summation in l, and taking the real part, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}\frac{|u_l(\xi)|^2}{l}\mathrm{d}\xi + a\sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}|u_l(\xi)|^2\mathrm{d}\xi \\
&= -K\Re\left(\sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}u_1(0)u_{l-1}(\xi)\overline{u_l(\xi)}\mathrm{d}\xi - \sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}\overline{u_1(0)}u_{l+1}(\xi)\overline{u_l(\xi)}\mathrm{d}\xi\right) \\
&= -K\Re\left(u_1(0)\int_{\mathbb{R}}\mathrm{e}^{2a\xi}\hat{g}(\xi)u_1(\xi)\mathrm{d}\xi\right),
\end{aligned}$$

where the last equality follows from a change $l \mapsto l+1$ of index in the first sum. Using the Cauchy-Schwarz inequality, it follows that we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}\frac{|u_l(\xi)|^2}{l}\mathrm{d}\xi + a\sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}|u_l(\xi)|^2\mathrm{d}\xi \le K|u_1(0)|\|\hat{g}\|_a\left(\int_{\mathbb{R}}\mathrm{e}^{2a\xi}|u_1(\xi)|^2\mathrm{d}\xi\right)^{1/2}.$$

Proceeding similarly for the derivative $\partial_\xi u_l(\xi)$ and combining the resulting inequality with the one here then yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{a,-1/2}^2 + 2a\|u\|_{a,0}^2 \leq 2\sqrt{2}K|u_1(0)|\|\hat{g}\|_a\|u_1\|_a,$$

Now, using the Sobolev embedding $H^1(-1,1)) \hookrightarrow C([-1,1])$, we infer

$$|u_1(0)| \leq C\|u\|_{a,-1/2}, \tag{5.3}$$

for some $C \in \mathbb{R}^+$. We also have $\|u_1\|_a \leq \|u\|_{a,-1/2}$ and the Gronwall's Lemma and the assumption $\|\hat{g}\|_a < +\infty$ imply the existence of $C_1 \in \mathbb{R}^+$ such that

$$\|u(t)\|_{a,-1/2}^2 + 2a\int_0^t \|u(s)\|_{a,0}^2 \mathrm{d}s \leq \mathrm{e}^{C_1 t}\|u_{\mathrm{in}}\|_{a,-1/2}^2, \ \forall t \in \mathbb{R}^+.$$

In particular, (5.3) implies

$$\sup_{t\in[0,T]} |u_1(t,0)| < +\infty, \ \forall T \in \mathbb{R}^+,$$

provided that $\|u_{\mathrm{in}}\|_{a,0} < +\infty$.

With this control on $|u_1(t,0)|$ provided, we can now pass to the estimate on $\|u\|_{a,0}$. To that goal, we test (5.2) against $\mathrm{e}^{a\xi}\overline{u_l(\xi)}$. Proceeding as before, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\in\mathbb{N}}\int_{\mathbb{R}} \mathrm{e}^{2a\xi}|u_l(\xi)|^2\mathrm{d}\xi + a\sum_{l\in\mathbb{N}}\int_{\mathbb{R}} \mathrm{e}^{2a\xi} l|u_l(\xi)|^2\mathrm{d}\xi \\
&= -K\Re\left(\sum_{l\in\mathbb{N}}\int_{\mathbb{R}} \mathrm{e}^{2a\xi} l u_1(0)u_{l-1}(\xi)\overline{u_l(\xi)}\mathrm{d}\xi - \sum_{l\in\mathbb{N}}\int_{\mathbb{R}} \mathrm{e}^{2a\xi} l\overline{u_1(0)}u_{l+1}(\xi)\overline{u_l(\xi)}\mathrm{d}\xi\right) \\
&= -K\Re\left(u_1(0)\sum_{l\in\mathbb{N}}\int_{\mathbb{R}} \mathrm{e}^{2a\xi} u_{l-1}(\xi)\overline{u_l(\xi)}\mathrm{d}\xi\right) \\
&\leq K|u_1(0)|\left(\|\hat{g}\|_a\left(\int_{\mathbb{R}} \mathrm{e}^{2a\xi}|u_1(\xi)|^2\mathrm{d}\xi\right)^{1/2} + \sum_{l\in\mathbb{N}}\int_{\mathbb{R}} \mathrm{e}^{2a\xi}|u_l(\xi)|^2\mathrm{d}\xi\right)
\end{aligned}$$

Repeating the argument for the derivative $\partial_\xi u_l(\xi)$ then yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{a,0}^2 + 2a\|u\|_{a,1/2}^2 \leq 2K\left(\sqrt{2}|u_1(0)|\|\hat{g}\|_a\|u_1\|_a + |u_1(0)|\|u\|_{a,0}^2\right).$$

Finally, we use on one hand the bound (5.3) with $\|u\|_{a,0}$ instead of $\|u\|_{a,-1/2}$ and the inequality $\|u_1\|_a \leq \|u\|_{a,0}$, and on the other hand the first estimate on $\sup_{t\in[0,T]}|u_1(t,0)|$, to conclude the existence of $C_T < +\infty$ (growing at most exponentially with $T \in \mathbb{R}^+$) such that the following inequality holds

$$\|u(t)\|_{a,0}^2 + 2a\int_0^t \|u(s)\|_{a,\frac{1}{2}}^2 \mathrm{d}s \leq \mathrm{e}^{C_T t}\|u_{\mathrm{in}}\|_{a,0}^2, \ \forall t \in [0,T]. \tag{5.4}$$

This estimate allows one to construct a global weak solution using standard arguments. For instance, one can consider a sequence of approximate systems, by projecting equation (5.2) onto a finite number of modes:

$$\partial_t u_l^n(\xi) = l\partial_\xi u_l^n(\xi) + \mathbb{P}_n \frac{Kl}{2}\left(u_1^n(0)u_{l-1}^n(\xi) - \overline{u_1^n(0)}u_{l+1}^n(\xi)\right), \ \forall (l,\xi) \in \{1,\dots,n\}\times\mathbb{R}. \tag{5.5}$$

Here, $\mathbb{P}_n$ is the projection onto modes $l \in \{1,\cdots,n\}$. The approximate initial data $u^n(0) := u_{\mathrm{in}}^n$ is taken smooth, zero for $l > n$ and $|\xi| > n$, and such that it converges to u_{in} in $\mathcal{X}_{a,0}$. For any given n, (5.5) is a simple transport equation with a smooth semilinear term and a smooth and compactly supported initial data. The existence of a local in time solution u^n is well-known [133]. The solution is smooth and compactly supported, with $\mathrm{supp}(u(t)) \subset \{1,\cdots,n\}\times[-n(1+t), n-t]$. Moreover, the previous a priori estimates extend straightforwardly to this approximate equation

$$\|u^n(t)\|_{a,0}^2 + 2a\int_0^t \|u^n(s)\|_{a,\frac{1}{2}}^2 \mathrm{d}s \leq \mathrm{e}^{C_T t}\|u_{\mathrm{in}}^n\|_{a,0}^2 \leq C'\mathrm{e}^{C_T t}, \ \forall t \in [0,T],$$

for any T less than the maximal time of existence T^n. It follows in particular that T^n is infinite. Indeed, assume a contrario that T^n is finite. As u^n is compactly supported, the previous bound implies that u_n belongs to $L^\infty((0,T^n)\times\{1,\dots,n\}\times\mathbb{R})$. This prevents blow up of the solution in finite time, and we get a contradiction.

Let $T > 0$. From the bound on $(u^n)_{n\in\mathbb{N}}$ in $L^\infty(0,T,\mathcal{X}_{a,0})$, one can obtain a bound on the sequence $(\partial_t u^n)_{n\in\mathbb{N}}$, using equation (5.5). More precisely, the quantity $h_l^n(t,\xi) := \frac{u_l^n(t,\xi)}{l}$ is such that

$$(\partial_t h^n)_{n\in\mathbb{N}} \text{ is bounded in } L^\infty(0,T,l^2(\mathbb{N},L^2(\mathrm{e}^{2a\xi}\mathrm{d}\xi)).$$

Thus, $(u_1^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T,H^1(-1,1))$ and the time derivative $(\partial_t u_1^n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T,L^2(-1,1))$. By Aubin-Lions Lemma, one obtains the strong convergence of a subsequence of $(u_1^n(\cdot,0))_{n\in\mathbb{N}}$ in $L^\infty(0,T)$. Together with the weak compactness of $(u^n)_{n\in\mathbb{N}}$ in $L^\infty(0,T,\mathcal{X}_{a,0})\cap L^2(0,T,\mathcal{X}_{a,\frac{1}{2}})$, this allows to take the limit $n\to+\infty$ in (5.5) and yields the existence of a solution u of (5.2).

For the proof of uniqueness, we use an energy estimate for the difference $v = u_2 - u_1$ of two solutions. Proceeding similarly to as for the a priori estimate above, one first obtains

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}\frac{|v_l(\xi)|^2}{l^2}d\xi + a\sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}\frac{|v_l(\xi)|^2}{l}\mathrm{d}\xi =$$
$$-\frac{K}{2}\Re\Bigg((u_1)_1(0)\sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}\frac{v_l(\xi)}{l}\frac{\overline{u_{l+1}(\xi)}}{l+1}d\xi + v_1(0)\sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}(u_2)_{l-1}(\xi)\frac{\overline{v_l(\xi)}}{l}\mathrm{d}\xi$$
$$-\overline{v_1(0)}\sum_{l\in\mathbb{N}}\int_{\mathbb{R}}\mathrm{e}^{2a\xi}(u_2)_{l+1}(\xi)\frac{\overline{v_l(\xi)}}{l}\mathrm{d}\xi\Bigg)$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{a,-1}^2 + 2a\|v\|_{a,-1/2}^2 \leq C'\|v\|_{a,-1}^2,$$

for some $C' \in \mathbb{R}^+$. Applying Gronwall's Lemma, the assumption $v(0) = 0$ implies that $v(t) = 0$ for all $t > 0$ as desired.

To prove continuity, letting $h_l(t,\xi) = \frac{u_l(t,\xi)}{l}$, we first observe that

$$h \in L^\infty\left(0, T, \mathcal{X}_{a,1}\right) \text{ and } \partial_t h \in L^\infty\left(0, T, l^2(\mathbb{N}, L^2(\mathrm{e}^{2a\xi}\mathrm{d}\xi))\right),\ \forall T \in \mathbb{R}^+.$$

From standard functional analysis (see e.g. Theorem 2.1 in [146]), it follows that h is weakly continuous in time with values in $\mathcal{X}_{a,1}$, and thus $u \in C_w(\mathbb{R}^+, \mathcal{X}_{a,0})$. Moreover, since $\mathcal{X}_{a,0}$ is a Hilbert space (hence a uniformly convex space) to obtain strong continuity, it suffices to prove that

$$t \mapsto \|u(t)\|_{a,0}$$

is continuous (see e.g. Proposition 3.32 in [24]). We consider separately the cases $t = 0+$ and $t > 0$.

Right continuity at 0 is rather straightforward. On one hand weak continuity implies

$$\liminf_{t\to 0^+}\|u(t)\|_{a,0} \geq \|u(0)\|_{a,0}.$$

On the other hand, the estimate (5.4) above implies

$$\limsup_{t\to 0^+}\|u(t)\|_{a,0} \leq \|u(0)\|_{a,0}.$$

For $t > 0$, we use the regularization effect induced by the weight. The integral term of (5.4) shows that

$$\|u(\delta)\|_{a,1/2} < +\infty, \text{ for a.e. } \delta \in \mathbb{R}^+.$$

Take any such δ. By mimicking the arguments above, one can construct a solution $\tilde{u} = \{\tilde{u}_l(t,\xi)\}$ of (5.2) over $(\delta, +\infty)$ satisfying

$$\tilde{u} \in L^\infty(\delta, T, \mathcal{X}_{a,\frac{1}{2}}) \cap L^2(\delta, T, \mathcal{X}_{a,1}), \quad \text{for all } T > \delta,$$

with $\tilde{u}(\delta) = u(\delta)$. By invoking the uniqueness of the solution in $L^\infty(\delta, T, \mathcal{X}_{a,0}) \cap L^2(\delta, T, \mathcal{X}_{a,1/2})$, we deduce $\tilde{u} = u$. It follows in particular that $u \in L^2(\delta, T, \mathcal{X}_{a,1})$ for any $T > \delta > 0$. It is then easily seen that (5.2) reads

$$\partial_t u_l(\xi) - l\partial_\xi u_l(\xi) = F_l(\xi)$$

where $F \in L^2(\delta, T, \mathcal{X}_{a,0})$ for any $T > \delta > 0$. Using the explicit formula

$$u_l(t, \xi) = u_l(\delta, \xi) + \int_\delta^t F_l(s, \xi + l(t-s))\mathrm{d}s, \quad t > \delta > 0,$$

one can check that u is continuous at positive times with values in $\mathcal{X}_{a,0}$. □

5.4 Linear analysis

As for the Cauchy problem, the dynamics of a perturbation is studied in Fourier variables. For a stationary state f_{st} with order parameter η_{st}, we consider the evolution of a perturbed system $f_{\mathrm{st}} + u$ in Fourier variables. Throughout the analysis we will take w.l.o.g. $\eta_{\mathrm{st}} \in [0, 1]$ and use u to denote a perturbation in Fourier variables. Focusing again on the spatial modes $l \in \mathbb{N}$ and using the notation $u_0 \equiv 0$, the evolution is governed by

$$\partial_t u = Lu + Qu,$$

where $L = L_1 + L_2$ is the linearisation given by

$$(L_1 u)_l(\xi) = l\left(\partial_\xi u_l(\xi) + \frac{K\eta_{\mathrm{st}}}{2}\left(u_{l-1}(\xi) - u_{l+1}(\xi)\right)\right),$$

and

$$(L_2 u)_l(\xi) = \frac{Kl}{2}\left(u_1(0)(\hat{f}_{\mathrm{st}})_{l-1}(\xi) - \overline{u_1(0)}(\hat{f}_{\mathrm{st}})_{l+1}(\xi)\right),$$

and the nonlinearity

$$(Qu)_l(\xi) = \frac{Kl}{2}\left(u_1(0)u_{l-1}(\xi) - \overline{u_1(0)}u_{l+1}(\xi)\right).$$

Throughout we fix $a > 0$ such that the velocity marginal g satisfies $\|g\|_a < \infty$. For any $k \in \mathbb{N}$ the transport operator L_1 is a closed densely-defined operator on $\mathcal{X}_{a,k}$ with domain

$$D_{a,k} = \{u \in \mathcal{X}_{a,k} \ : \ L_1 u \in \mathcal{X}_{a,k}\}.$$

The density can easily be seen from the fact that $D_{a,k}$ contains all smooth compactly supported functions. As the free transport operator $l\partial_\xi$ generates a strongly continuous semigroup, it is closed and thus is the combined operator L_1.

5.4.1 Transport operator

In the coherent state we could explicitly solve the transport operator and conclude the decay. For the inhomogeneous state, we replace this with the following resolvent estimate. This is the key estimate capturing the decay and the main reason for the choice of the norm.

Lemma 5.8. *Let $k \in \{-1, 0\}$. The resolvent set of L_1 over $\mathcal{X}_{a,k}$ contains the half-plane $\Re(\lambda) > -a$ and we have*

$$\|(\lambda \mathrm{Id} - L_1)^{-1}\|_{\mathcal{X}_{a,-1} \to \mathcal{X}_{a,0}} \le \frac{1}{\min\{a, \Re(\lambda) + a\}}, \ \forall \lambda \in \mathbb{C} \ : \ \Re(\lambda) > -a.$$

Moreover, letting $\lambda_0 = -a + \frac{K\eta_{\mathrm{st}}}{2}$, we have in the half-plane $\Re(\lambda) > \lambda_0$

$$\|(\lambda \mathrm{Id} - L_1)^{-1}\|_{a,k} \le \frac{1}{\Re(\lambda) - \lambda_0}.$$

By the Hille-Yosida theorem L_1 therefore generates a strongly continuous semigroup. In addition, the first estimate of Lemma 5.8 implies that $\|(\lambda \mathrm{Id} - L_1)^{-1}\|_{a,k}$ is uniformly bounded over the half-plane $\Re(\lambda) > -a + \epsilon$ for any $\epsilon > 0$. The Gearhart-Prüss Theorem (see e.g. Corollary 2.2.5 in [116]) then implies that the semigroup e^{tL_1} must be exponentially stable with some rate $b > a - \epsilon$.

Corollary 5.9. *For $k \in \{-1, 0\}$, the operator L_1 generates a strongly continuous semigroup on $\mathcal{X}_{a,k}$. Moreover, for every $b < a$, there exist $C \in \mathbb{R}^+$ such that*

$$\|\mathrm{e}^{tL_1}\|_{a,k} \le C\mathrm{e}^{-bt}, \ \forall t \in \mathbb{R}^+. \tag{5.6}$$

Moreover, the first estimate shows the needed gain of regularity.

Proof of Lemma 5.8. To derive the claimed inequalities, we consider the resolvent equation

$$(\lambda \mathrm{Id} - L_1)u = v.$$

The second estimate is obtained by testing against $\mathrm{e}^{2a\xi} l^{2k} \overline{u_l(\xi)}$, under the assumption $\|u\|_{a,k} < +\infty$. After integration in ξ, summation in l and taking the real part, we obtain

$$\begin{aligned}
&\Re(\lambda) \sum_{l \in \mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} l^{2k} |u_l(\xi)|^2 d\xi + a \sum_{l \in \mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} l^{2k+1} |u_l(\xi)|^2 \mathrm{d}\xi \\
&- \frac{K\eta_{\mathrm{st}}}{2} \Re \left(\sum_{l \in \mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} l^{2k+1} (u_{l-1}(\xi)\overline{u_l(\xi)} - \overline{u_{l+1}(\xi)} u_l(\xi)) \mathrm{d}\xi \right) \le \|u\|_{a,k} \|v\|_{a,k}
\end{aligned}$$

A change $l \mapsto l+1$ of index in the third sum yields, also using $u_0(\xi) = 0$ and simplifying the expression of $(l+1)^{2k+1} - l^{2k+1}$ for $k \in \{-1, 0\}$

$$\begin{aligned}
&(\Re(\lambda) + a) \sum_{l\in\mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} l^{2k} |u_l(\xi)|^2 \mathrm{d}\xi \\
&\leq \frac{K\eta_{\mathrm{st}}}{2} \Re\left(\sum_{l\in\mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} \left((l+1)^{2k+1} - l^{2k+1}\right) u_l(\xi)\overline{u_{l+1}(\xi)} \mathrm{d}\xi \right) + \|u\|_{a,k} \|v\|_{a,k} \\
&\leq \frac{K\eta_{\mathrm{st}}}{2} \|u\|_{a,k}^2 + \|u\|_{a,k} \|v\|_{a,k}.
\end{aligned}$$

As in the proof of Lemma 5.7, one can proceed similarly for the derivative $\partial_\xi u_l(\xi)$ and the second estimate easily follows.

For the first estimate, we proceed similarly for $k = -1/2$ and use the inequality

$$\Re(\lambda) \sum_{l\in\mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} |u_l(\xi)|^2 d\xi \leq \Re(\lambda) \sum_{l\in\mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} l^{-1} |u_l(\xi)|^2 \mathrm{d}\xi, \quad \forall\lambda \ : \ \mathrm{Re}(\lambda) \leq 0,$$

to obtain

$$\min\{a, \Re(\lambda) + a\} \sum_{l\in\mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} |u_l(\xi)|^2 \mathrm{d}\xi \leq \|u\|_{a,0} \|v\|_{a,-1},$$

and then

$$\|u\|_{a,0} \leq \frac{\|v\|_{a,-1}}{\min\{a, \Re(\lambda) + a\}}, \quad \forall\lambda \in \mathbb{C} \ : \ \Re(\lambda) > -a,$$

from where the first estimate follows. Finally, standard arguments (e.g. Galerkin approximation) based on this estimate show that $\lambda\mathrm{Id} - L_1$ is invertible for $\Re(\lambda) > -a$. □

5.4.2 Eigenmodes

The operator L_2 depends only on the order parameter and, as discussed in Section 3.1, we can use Duhamel's principle to derive a Volterra equation for the order parameter. However, the eigenmode analysis works over $\mathbb{C}$ and L_2 is not linear over $\mathbb{C}$.

In order to get a $\mathbb{C}$-linear operator and to investigate its spectral properties, one may consider the real and imaginary parts separately, as in [109, 120]. We use an alternative approach here, based on complex conjugates. Given $u = \{u_l(\xi)\}_{\mathbb{N}\times\mathbb{R}}$ and $v = \{v_l(\xi)\}_{\mathbb{N}\times\mathbb{R}}$ (which is a substitute for $\overline{u}$), let

$$\mathsf{u} = \{\mathsf{u}_l(\xi)\}_{\mathbb{N}\times\mathbb{R}} \text{ where } \mathsf{u}_l(\xi) = \begin{pmatrix} u_l(\xi) \\ v_l(\xi) \end{pmatrix} \in \mathbb{C}^2, \quad \forall (l, \xi) \in \mathbb{N}\times\mathbb{R},$$

and consider the operator $\mathsf{L} = \mathsf{L}_1 + \mathsf{L}_2$ defined by

$$(\mathsf{L}_1\mathsf{u})_l(\xi) = \begin{pmatrix} (L_1 u)_l(\xi) \\ (L_1 v)_l(\xi) \end{pmatrix}$$

and

$$(\mathsf{L}_2\mathsf{u})_l(\xi) = \frac{Kl}{2}\begin{pmatrix} (\hat{f}_{\mathrm{st}})_{l-1}(\xi) & -(\hat{f}_{\mathrm{st}})_{l+1}(\xi) \\ -\overline{(\hat{f}_{\mathrm{st}})_{l+1}(\xi)} & \overline{(\hat{f}_{\mathrm{st}})_{l-1}(\xi)} \end{pmatrix}\begin{pmatrix} u_1(0) \\ v_1(0) \end{pmatrix}.$$

The operators L_i are defined in such a way that when $v_l(\xi) = \overline{u_l(\xi)}$, we have

$$(\mathsf{L}_i\mathsf{u})_l(\xi) = \begin{pmatrix} (L_i u)_l(\xi) \\ \overline{(L_i u)_l(\xi)} \end{pmatrix}, \quad \text{for } i = 1, 2.$$

As L_1 is just the tensor product of L_1, it generates a semigroup with the results from Corollary 5.9. Now L_2 is a bounded $\mathbb{C}$-linear operator, so that L generates a strongly continuous semigroup. For the range of L_2 we also introduce $\hat{p}_k$ for $k \in \mathbb{Z}$ by

$$(\hat{p}_k)_l(\xi) = l(\hat{f}_{\mathrm{st}})_{l+k-1}(\xi), \quad \forall (l,\xi) \in \mathbb{N} \times \mathbb{R}.$$

Using Duhamel's principle as in Section 3.1 then shows that the order parameter η of the perturbation satisfies the Volterra equation

$$\begin{pmatrix} \overline{\eta}(t) \\ \eta(t) \end{pmatrix} + \int_0^t k(t-s)\begin{pmatrix} \overline{\eta}(s) \\ \eta(s) \end{pmatrix} \mathrm{d}s = F(t)$$

with the kernel

$$k(t) = -\frac{K}{2}\begin{pmatrix} (\mathrm{e}^{tL_1}\hat{p}_0)_1(0) & -(\mathrm{e}^{tL_1}\hat{p}_2)_1(0) \\ -\overline{(\mathrm{e}^{tL_1}\hat{p}_2)_1(0)} & \overline{(\mathrm{e}^{tL_1}\hat{p}_0)_1(0)} \end{pmatrix}$$

and the forcing

$$F(t) = \left(\mathrm{e}^{t\mathsf{L}_1}\mathsf{u}_{\mathrm{in}}\right)_1(0).$$

Thus we find an eigenmode $\lambda \in \Re\lambda > -a$ in the complexified version if

$$\det\left(\mathrm{Id} - \frac{K}{2}M(\lambda, \eta_{\mathrm{st}})\right) = 0,$$

where

$$\mathcal{L}k = \frac{K}{2}M(\lambda, \eta_{\mathrm{st}}).$$

By the decay of the transport operator L_1, the matrix M is well-defined and analytic for $\Re\lambda > -a$. Explicitely, it is with the resolvent of L_1 given by

$$M(\lambda, \eta_{\mathrm{st}}) = \begin{pmatrix} ((\lambda\mathrm{Id} - L_1)^{-1}\hat{p}_0)_1(0) & -((\lambda\mathrm{Id} - L_1)^{-1}\hat{p}_2)_1(0) \\ -\overline{((\overline{\lambda}\mathrm{Id} - L_1)^{-1}\hat{p}_2)_1(0)} & \overline{((\overline{\lambda}\mathrm{Id} - L_1)^{-1}\hat{p}_0)_1(0)} \end{pmatrix}. \tag{5.7}$$

In fact, we can show that the complexification does not introduce and spurious eigenmodes. Therefore, it describes the linear stability accurately. In particular, the existence of such an eigenmode means that the order parameter is not decaying under the linear evolution and this is the main observable of the system.

Resolvent of L

As we need to derive the decay and regularisation of the linear semigroup from the resolvent estimates, we establish the eigenmode condition on the level of the resolvent of L. We note, however, that the results can also be obtained by studying the Volterra equation like in Section 4.3.

By the Riemann-Lebesgue lemma, we can characterise the behaviour of M.

Lemma 5.10. *The matrix $M(\lambda, \eta_{\mathrm{st}})$ from (5.7) is analytic in $\Re\lambda > -a$. Moreover, for any $a' < a$ we have $\|M(\lambda, \eta_{\mathrm{st}})\| \to 0$ uniformly as $|\lambda| \to \infty$ in $\Re\lambda > -a'$.*

Proof. As the Laplace transform, the decay Corollary 5.9 shows the claimed analyticity in the region. The decay follows likewise from the decay of the semigroup e^{tL_1} and the Riemann-Lebesgue lemma. □

The spectrum of L can now be described by the following lemma.

Lemma 5.11. *Let $k \in \{-1, 0\}$. In the region $\{\lambda \in \mathbb{C} : \Re\lambda > -a\}$, the operator L on $\mathcal{X}^2_{a,k}$ has only a discrete spectrum, where the eigenvalues are the roots*

$$\det(\mathrm{Id} - \frac{K}{2} M(\lambda, \eta_{\mathrm{st}})) = 0.$$

For $a' < a$ there exists a constant R such that

$$\sup_{\Re\lambda \geq -a',\, |\lambda| > R} \|((-a' + \mathrm{i}y)\mathrm{Id} - \mathsf{L})^{-1}\|_{\mathcal{X}_{a,-1} \to \mathcal{X}_{a,0}} < +\infty.$$

In particular, if $a' < a$ is such that the line $\{\lambda \in \mathbb{C} : \Re\lambda = -a'\}$ does not contain an eigenvalue, then

$$\sup_{y \in \mathbb{R}} \|((-a' + \mathrm{i}y)\mathrm{Id} - \mathsf{L})^{-1}\|_{\mathcal{X}_{a,-1} \to \mathcal{X}_{a,0}} < +\infty.$$

Proof. By Corollary 5.9, the operator L_1 generates a strongly continuous semigroup with decay rate at least a. Hence the resolvent contains the claimed region. The operator L_2 is a bounded finite-rank operator and thus does not change the essential spectrum of L.

For the discrete spectrum, let

$$\mathsf{U} = \{\mathsf{U}_l(\xi)\}_{\mathbb{N} \times \mathbb{R}} \text{ where } \mathsf{U}_l(\xi) = \begin{pmatrix} U_l(\xi) \\ V_l(\xi) \end{pmatrix} \in \mathbb{C}^2,$$

then the resolvent equation $(\lambda\mathrm{Id} - \mathsf{L})\mathsf{u} = \mathsf{U}$ can be written in the region $\Re(\lambda) > -a$

$$\left(\mathrm{Id} - (\lambda\mathrm{Id} - \mathsf{L}_1)^{-1}\mathsf{L}_2\right)\mathsf{u} = (\lambda\mathrm{Id} - \mathsf{L}_1)^{-1}\mathsf{U}.$$

An important property is that the vector $\mathsf{L}_2\mathsf{u}$, and hence $(\lambda\mathrm{Id}-\mathsf{L}_1)^{-1}\mathsf{L}_2\mathsf{u}$, only involves the component $\mathsf{u}_1(0)$ of u. Using also the commutation $L_1\overline{u}=\overline{L_1 u}$, it follows that the component $(l,\xi)=(1,0)$ of the resolvent equation writes

$$\left(\mathrm{Id}-\frac{K}{2}M(\lambda,\eta_{\mathrm{st}})\right)\mathsf{u}_1(0)=(\lambda\mathrm{Id}-\mathsf{L}_1)^{-1}\mathsf{U}_1(0), \tag{5.8}$$

Therefore, in the case where $\mathrm{Id}-\frac{K}{2}M(\lambda,\eta_{\mathrm{st}})$ is invertible, let $\mathsf{u}^*=\{\mathsf{u}_l^*(\xi)\}_{\mathbb{N}\times\mathbb{R}}$ be any vector for which $\mathsf{u}_1^*(0)$ solves (5.8). We infer that the resolvent equation has a solution given by

$$\mathsf{u}=(\lambda\mathrm{Id}-\mathsf{L}_1)^{-1}\left(\mathsf{L}_2\mathsf{u}^*+\mathsf{U}\right),$$

which is unique since $(\lambda\mathrm{Id}-\mathsf{L}_1)^{-1}\mathsf{L}_2\mathsf{u}^*$ only involves the component $\mathsf{u}_1^*(0)$.

For $a'<a$ the inverse of M for $\Re\lambda>-a'$ can be controlled by Lemma 5.10 for large enough $|\lambda|$. Hence the claimed supremum bound follows from Lemma 5.8. Likewise, if $a'<a$ is chosen with no poles on the line $\Re\lambda=-a'$, the solution for $\mathsf{u}_1(0)$ is bounded and the result follows.

On the other hand, if $\det\left(\mathrm{Id}-\frac{K}{2}M(\lambda,\eta_{\mathrm{st}})\right)=0$, let $\mathsf{u}^\dagger$ be with component

$$\mathsf{u}_1^\dagger(0)\in\ker\left(\mathrm{Id}-\frac{K}{2}M(\lambda,\eta_{\mathrm{st}})\right).$$

Using once again that $(\lambda\mathrm{Id}-\mathsf{L}_1)^{-1}\mathsf{L}_2\mathsf{u}^\dagger$ only involves $\mathsf{u}_1^\dagger(0)$, one directly checks that $(\lambda\mathrm{Id}-\mathsf{L}_1)^{-1}\mathsf{L}_2\mathsf{u}^\dagger$ is an eigenvector of L with eigenvalue λ. Consequently, $\det\left(\mathrm{Id}-\frac{K}{2}M(\lambda,\eta_{\mathrm{st}})\right)=0$ if and only if λ is an eigenvalue of L in the half-plane $\Re(\lambda)>-a$. $\square$

Absence of spurious modes

We show that the complexification process does not generate unstable spurious modes, i.e. to any eigenvalue λ with $\Re(\lambda)>0$ of L (resp. non-zero eigenvalue on the imaginary axis), corresponds a diverging (resp. rotating) solution of $\partial_t u=Lu$. We consider the cases $\Im(\lambda)\neq 0$ and $\lambda\in\mathbb{R}$ separately.

Case $\Im(\lambda)\neq 0$. In this case also $\overline{\lambda}$ must be an eigenvalue of L. Choose u with components

$$\mathsf{u}_1(0)=\begin{pmatrix}u_1(0)\\ v_1(0)\end{pmatrix}\in\ker\left(\mathrm{Id}-\frac{K}{2}M(\lambda,\eta_{\mathrm{st}})\right)$$

and $\overline{\mathsf{u}}$ with components

$$\overline{\mathsf{u}}_1(0)=\begin{pmatrix}\overline{v}_1(0)\\ \overline{u}_1(0)\end{pmatrix}\in\ker\left(\mathrm{Id}-\frac{K}{2}M(\overline{\lambda},\eta_{\mathrm{st}})\right).$$

Then the trajectory $t \mapsto \mathsf{U}(t)$, (uniquely) defined by

$$\mathsf{U}(t) = \mathrm{e}^{\lambda t}(\lambda \mathrm{Id} - \mathsf{L}_1)^{-1}\mathsf{L}_2\mathsf{u} + \mathrm{e}^{\overline{\lambda} t}(\overline{\lambda}\mathrm{Id} - \mathsf{L}_1)^{-1}\mathsf{L}_2\overline{\mathsf{u}},$$

is a solution of the equation $\partial_t \mathsf{u} = \mathsf{L}\mathsf{u}$. Moreover, this solution components satisfy $(V(t))_l(\xi) = \overline{(U(t))_l(\xi)}$; hence the definition of L implies that $\{(U(t))_l(\xi)\}_{\mathbb{N}\times\mathbb{R}}$ satisfies the equation $\partial_t u = Lu$.

Case $\lambda \in \mathbb{R}$. In this case, the matrix $\mathrm{Id} - \frac{K}{2}M(\lambda, \eta_{\mathrm{st}})$ must be Hermitian and of the form

$$\rho \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi_0} & \mathrm{e}^{\mathrm{i}\phi_2} \\ \mathrm{e}^{-\mathrm{i}\phi_2} & \mathrm{e}^{-\mathrm{i}\phi_0} \end{pmatrix},$$

for some $\phi_0, \phi_2 \in \mathbb{T}$ and $\rho \in \mathbb{R}^+$. Clearly, as $\rho \neq 0$, the kernel of this matrix is spanned by

$$\begin{pmatrix} \mathrm{e}^{i\phi} \\ \mathrm{e}^{-i\phi} \end{pmatrix}$$

for a given $\phi \in \mathbb{T}$. Letting $\mathsf{u}_1(0)$ be in this kernel and $\mathsf{U}(t) = \mathrm{e}^{\lambda t}(\lambda\mathrm{Id} - \mathsf{L}_1)^{-1}\mathsf{L}_2\mathsf{u}$, we have that the first component $\{(U(t))_l(\xi)\}_{\mathbb{N}\times\mathbb{R}}$ must also satisfy the equation $\partial_t u = Lu$ in this case.

Stability condition

The rotation symmetry R_Θ of the Kuramoto equation expresses as a phase symmetry in Fourier variables, i.e. if $t \mapsto u(t) = \{u_l(t,\xi)\}_{\mathbb{N}\times\mathbb{R}}$ satisfies (5.2), then for every $\Theta \in \mathbb{T}$, the trajectory $t \mapsto \hat{R}_\Theta u(t)$, where $(\hat{R}_\Theta u(t))_{l,\xi} = \mathrm{e}^{\mathrm{i}\Theta l} u_l(t,\xi)$, also solves that equation. As noted in [109], this indifference to phase changes implies that we must have

$$Lu = 0 \quad \text{for } u = \mathrm{D}\hat{R}\hat{f}_{\mathrm{st}},$$

where $\mathrm{D}\hat{R} := \frac{\mathrm{d}\hat{R}_\Theta}{\mathrm{d}\Theta}|_{\Theta=0}$ is the symmetry infinitesimal generator and writes

$$(\mathrm{D}\hat{R}u)_l(\xi) = \mathrm{i}lu_l(\xi).$$

In particular, L has always the eigenvalue 0.

If this is the only eigenvalue with nonnegative real part, we define the partially locked state as linear stable. Indeed if it fails, the order parameter is not decaying, so that the system cannot appear to be stable.

Definition 5.12. *The stationary state f_{st} is stable with rate $a' < a$ if*

$$\det\left(\mathrm{Id} - \frac{K}{2}M(\lambda, \eta_{\mathrm{st}})\right) \neq 0 \qquad \textit{for } \Re\lambda \geq -a' \textit{ and } \lambda \neq 0$$

and

$$\liminf_{\lambda\to 0}\left|\frac{1}{\lambda}\det\left(\mathrm{Id}-\frac{K}{2}M(\lambda,\eta_{\mathrm{st}})\right)\right|>0.$$

This is precisely the sense in the statement of Theorem 5.1 and the second part ensures that the eigenvalue 0 is simple. Indeed as in Lemma 4.21, we would otherwise have a generalised eigenmode not corresponding to the rotation symmetry.

By Lemma 5.10 we can always find a decay rate if there is no other non-negative eigenvalue.

Proposition 5.13. *If f_{st} is such that*

$$\det\left(\mathrm{Id}-\frac{K}{2}M(\lambda,\eta_{\mathrm{st}})\right)\neq 0 \qquad \textit{for } \Re\lambda\geq 0 \textit{ and } \lambda\neq 0$$

and

$$\liminf_{\lambda\to 0}\left|\frac{1}{\lambda}\det\left(\mathrm{Id}-\frac{K}{2}M(\lambda,\eta_{\mathrm{st}})\right)\right|>0,$$

then there exists $a'>0$ such that f_{st} is stable in the sense of Definition 5.12.

Proof. The function

$$\lambda\to\det\left(\mathrm{Id}-\frac{K}{2}M(\lambda,\eta_{\mathrm{st}})\right)$$

is by Lemma 5.10 analytic in $\Re\lambda>-a$ and converges to 1 as $|\lambda|\to\infty$ in $\Re\lambda>-a+\epsilon$ for any $\epsilon>0$. Hence it can only have finitely many roots so that a claimed rate a' exists. □

Using the special form of the stationary states, we can express the components of M as explicit integrals for $\Re\lambda>0$.

Proposition 5.14. *For $k\in\mathbb{N}\cup\{0\}$ and $\Re\lambda>0$ we have*

$$J_k(\lambda,\eta_{\mathrm{st}}):=((\lambda\mathrm{Id}-L_1)^{-1}\hat{p}_k)_1(0)=\int_{\mathbb{R}}\frac{\beta^k\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right)}{\lambda+\mathrm{i}\omega+K\eta_{\mathrm{st}}\beta\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right)}g(\omega)\mathrm{d}\omega.$$

Proof. We use inverse Fourier transforms with respect to ξ. Let

$$(p_k)_l(\omega)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}\omega\xi}(\hat{p}_k)_l(\xi)\mathrm{d}\xi,$$

and let $\check{L}_1$ be the inverse Fourier transform of L_1, i.e. $\widehat{\check{L}_1p_k}=L_1\hat{p}_k$, when passing to Fourier transforms with respect to ξ. We have

$$((\lambda\mathrm{Id}-L_1)^{-1}\hat{p}_k)_1(0)=\int_{\mathbb{R}}((\lambda\mathrm{Id}-\check{L}_1)^{-1}p_k)_1(\omega)\mathrm{d}\omega,$$

provided that $((\lambda\mathrm{Id}-\check{L}_1)^{-1}p_k)_1\in L^1(\mathbb{R})$, and expression (5.1) implies

$$(p_k)_l(\omega)=l(\tilde{f}_{\mathrm{st}})_{l+k-1}(\omega)=l\beta^{l+k-1}\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right)g(\omega).$$

Now, using the expression

$$(\check{L}_1 u)_l(\omega) = l\left(\mathrm{i}\omega u_l(\omega) + \frac{K\eta_{\mathrm{st}}}{2}(u_{l-1}(\omega) - u_{l+1}(\omega))\right),$$

and, twice in a row, the equation $\beta(x)^2 + 2\mathrm{i}x\beta(x) - 1 = 0$ from Section 1.3.2 that defines β, one obtains

$$(\check{L}_1 p_k)_l(\omega) = -\frac{K\eta_{\mathrm{st}}}{2} \cdot \frac{1 + \beta^2\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right)}{\beta\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right)}(p_k)_l(\omega) = -\left(\mathrm{i}\omega + K\eta_{\mathrm{st}}\beta\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right)\right)(p_k)_l(\omega),$$

from where it results that

$$((\lambda\mathrm{Id} - \check{L}_1)^{-1} p_k)_1(\omega) = \frac{\beta^k\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right) g(\omega)}{\lambda + \mathrm{i}\omega + K\eta_{\mathrm{st}}\beta\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right)}.$$

Using the expression of β and $|\beta(\cdot)| \leq 1$, the following inequality holds

$$\left|\frac{\beta^k\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right)}{\lambda + \mathrm{i}\omega + K\eta_{\mathrm{st}}\beta\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right)}\right| \leq \frac{1}{\Re(\lambda)},$$

provided that $\Re(\lambda) > 0$. The integral is therefore convergent and the lemma follows. $\square$

For $\Re(\lambda) = 0$, the integrals have to be understood in a weak sense: it is the limit as $\Re\lambda \to 0+$. This limit exists because of the continuous dependence of the resolvent $(\lambda\mathrm{Id} - L_1)^{-1}$ on λ. In practice, the values on the imaginary axis can be computed as a principal value with correction terms, using Plemelj formula as in [157].

These integral expressions allow an easy numerical computation of $M(\lambda, \eta_{\mathrm{st}})$. Moreover, we can again formulate by the argument principle an easily verifiable stability criterion.

Criterion 5.15. *The stationary state f_{st} is stable if the curve*

$$x \to \frac{1 + \mathrm{i}x}{x} \det\left(\mathrm{Id} - \frac{K}{2} M(\mathrm{i}x, \eta_{\mathrm{st}})\right)$$

for $x \in \mathbb{R}$ has zero winding number around 0.

5.4.3 Spectral projection

From now on assume, that the configuration is stable with rate a', i.e. with $\Re\lambda \geq -a'$ there exists only the zero eigenmode corresponding to the rotation symmetry. In order to separate the direction, we need for $k \in \{-1, 0\}$ a projection operator P_0 on $\mathcal{X}_{a,k}$ such that $LP_0 = 0$ and $P_0 L = 0$, i.e. the range of P_0 is $\ker L$ and P_0 commutes with L.

On the complexified L, we can define such a projection P through holomorphic functional calculus as in Lemma 4.22.

Proposition 5.16. *Let f_{st} be stable with rate a' and $k \in \{-1, 0\}$. Define the spectral projection P_0 on $\mathcal{X}^2_{a,k}$ by*

$$\mathsf{P}_0 : \mathsf{u} \to \frac{-1}{2\pi \mathrm{i}} \oint_{|\lambda| = a'/2} (\lambda \mathrm{Id} - \mathsf{L})^{-1} \mathsf{u} \, \mathrm{d}\lambda.$$

Then P_0 satisfies $\mathsf{L}\mathsf{P}_0 = 0$ and $\mathsf{P}_0\mathsf{L} = 0$. On the complementary subspace $\mathsf{P}_s\mathcal{X}^2_{a,-1}$ with $\mathsf{P}_s = \mathrm{Id} - \mathsf{P}_0$ we have

$$\sup_{\Re\lambda \geq -a'} \|(\lambda \mathrm{Id} - \mathsf{L})^{-1}\|_{\mathsf{P}_s\mathcal{X}^2_{a,-1} \to \mathcal{X}^2_{a,0}} < \infty.$$

Moreover, the subspaces $\mathsf{P}_0\mathcal{X}_{a,k}$ and $\mathsf{P}_s\mathcal{X}_{a,k}$ are invariant under the semigroup $\mathrm{e}^{t\mathsf{L}}$.

Proof. The properties of the projection follow directly from the holomorphic functional calculus and Lemma 5.11. As P_0 and P_s commute with the generator L, the subspaces are invariant. □

Using the adjoint L^*, we can characterise the projection more explicitely. By the rotation symmetry, we recall that

$$\begin{pmatrix} \mathrm{D}\hat{R}\hat{f}_{\mathrm{st}} \\ \overline{\mathrm{D}\hat{R}\hat{f}_{\mathrm{st}}} \end{pmatrix} \in \ker \mathsf{L}$$

and $\ker L$ is one-dimensional. Accordingly, let u^* be in $\ker \mathsf{L}^*$ such that

$$\langle \begin{pmatrix} \mathrm{D}\hat{R}\hat{f}_{\mathrm{st}} \\ \overline{\mathrm{D}\hat{R}\hat{f}_{\mathrm{st}}} \end{pmatrix}, \mathsf{u}^* \rangle = 1$$

using the scalar product on $\mathcal{X}^2_{a,0}$ defined by

$$\langle \mathsf{u}, \mathsf{u}' \rangle = \langle u, u' \rangle_{a,0} + \langle v, v' \rangle_{a,0}, \qquad \text{where } \mathsf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \text{ and } \mathsf{u}' = \begin{pmatrix} u' \\ v' \end{pmatrix}.$$

Lemma 5.17. *The projection P_0 has the form*

$$\mathsf{P}\mathsf{u} = \langle \mathsf{u}, \mathsf{u}^* \rangle \begin{pmatrix} \mathrm{D}\hat{R}\hat{f}_{\mathrm{st}} \\ \overline{\mathrm{D}\hat{R}\hat{f}_{\mathrm{st}}} \end{pmatrix}$$

for $\mathsf{u} \in \mathcal{X}^2_{a,k}$. Moreover, u^* takes the form

$$\mathsf{u}^* = \begin{pmatrix} u^* \\ \overline{u^*} \end{pmatrix}$$

with $u^* \in \mathcal{X}_{a,k}$ for all $k \in \mathbb{N} \cup \{0\}$.

The last property shows that P_0 defines in $\mathcal{X}^2_{a,k}$ for any $k \in \mathbb{Z}$ a continuous projection.

The form of u^* shows that we can define the projection P_0 on $\mathcal{X}_{a,k}$ by

$$P_0 u = 2\Re(\langle u, u^* \rangle_{a,0})\, \mathrm{D}\hat{R}\hat{f}_{\mathrm{st}}$$

with

$$\mathsf{P}_0 u = \begin{pmatrix} P_0 u \\ \overline{P_0 u} \end{pmatrix} \text{ for } \mathsf{u} = \begin{pmatrix} u \\ \overline{u} \end{pmatrix},$$

so that P_0 and $P_s = \mathrm{Id} - P_0$ inherit all the properties from P_0 and P_s.

Proof. By Lemma 5.2, we always have $\mathrm{D}\hat{R}\hat{f}_{\mathrm{st}} \in \mathcal{X}_{a,k}$ and by the rotation symmetry it is in $\ker L$. Furthermore, it shows that u^* is well-defined and we explicitly find

$$\mathsf{P}_0 \mathsf{L} = \mathsf{L}\mathsf{P}_0 = 0,$$

which completely characterises P_0.

For the structure of u^*, let $\begin{pmatrix} u \\ v \end{pmatrix} \in \ker(\mathsf{L}^*)$ be arbitrary. Then $\begin{pmatrix} u + \overline{v} \\ v + \overline{u} \end{pmatrix} = \begin{pmatrix} u + \overline{v} \\ \overline{u + \overline{v}} \end{pmatrix} \in \ker \mathsf{L}^*$, which implies $\mathsf{u}^* = \begin{pmatrix} u^* \\ \overline{u^*} \end{pmatrix}$ as claimed. Indeed, either $u + \overline{v} \neq 0$ and that $\ker(\mathsf{L}^*)$ is one-dimensional implies that we must have $u^* = \lambda(u + \overline{v})$ for some $\lambda \in \mathbb{C}$. Or $u + \overline{v} = 0$ and then $u^* = \lambda i u$ for some $\lambda \in \mathbb{C}$.

For the regularity observe that L^* can be explicitly computed as $\mathsf{L}^* = \mathsf{L}_1^* + \mathsf{L}_2^*$ where

$$\mathsf{L}_1^* \mathsf{u} = \begin{pmatrix} L_1^* u \\ L_1^* v \end{pmatrix}$$

with

$$(L_1^* u)_l = -l\left(\partial_\xi u_l + 2a u_l\right) + \frac{K\eta_{\mathrm{st}}}{2}\left((l+1)u_{l+1} - (l-1)u_{l-1}\right)$$

and

$$(\mathsf{L}_2^* \mathsf{u})_l = \frac{K}{2}\begin{pmatrix} m_u w \delta_{l,1} \\ m_v w \delta_{l,1} \end{pmatrix}$$

with

$$\begin{pmatrix} m_u \\ m_v \end{pmatrix} = \sum_{l \in \mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} l \begin{pmatrix} \overline{(\hat{f}_{\mathrm{st}})_{l-1}(\xi)} & -(\hat{f}_{\mathrm{st}})_{l+1}(\xi) \\ -\overline{(\hat{f}_{\mathrm{st}})_{l+1}(\xi)} & (\hat{f}_{\mathrm{st}})_{l-1}(\xi) \end{pmatrix} \begin{pmatrix} u_l(\xi) \\ v_l(\xi) \end{pmatrix} \mathrm{d}\xi,$$

where we have used the Kronecker symbol and $w : \mathbb{R} \to \mathbb{C}$ is the function such that $\|w\|_a < +\infty$ and

$$\langle w, w' \rangle_a = \overline{w'(0)},\ \forall w' : \mathbb{R} \to \mathbb{C} \ :\ \|w'\|_a < +\infty,$$

where $\langle \cdot, \cdot \rangle_a$ is the scalar product that generates $\|\cdot\|_a$. The existence and uniqueness of w is guaranteed by the Riesz Representation Theorem, using Sobolev embedding to ensure that

$w' \mapsto \overline{w'(0)}$ is a continuous linear functional.

Consequently, u^* must satisfy the equation $(L_1^* u^*)_l = -c^* w \delta_{l,1}$ where

$$c^* = -\sum_{l \in \mathbb{N}} \int_{\mathbb{R}} \mathrm{e}^{2a\xi} l \left(\overline{(\hat{f}_{\mathrm{st}})_{l-1}(\xi)} u_l^*(\xi) - (\hat{f}_{\mathrm{st}})_{l+1}(\xi) \overline{u_l^*(\xi)} \right) d\xi.$$

Letting $x_l^* = -\frac{K\eta_{\mathrm{st}}}{2}(u_{l+1} + u_{l-1}) + c^* w \delta_{l,1}$, the equation can be written

$$-l \left(\partial_\xi u_l^* + 2a u_l^* + \frac{K\eta_{\mathrm{st}}}{2}(u_{l+1}^* - u_{l-1}^*) \right) = x_l^*.$$

We have $x^* \in \mathcal{X}_{a,0}$; hence one can perform similar energy estimates to those in the proof of Lemma 5.8 to obtain

$$\|u^*\|_{a,1/2} \leq C \, \|x^*\|_{a,0},$$

for some $C \in \mathbb{R}^+$. This inequality implies that $x^* \in \mathcal{X}_{a,1/2}$, and therefore

$$\|u^*\|_{a,1} \leq C' \, \|x^*\|_{a,1/2},$$

and the claimed regularity follows using a bootstrap argument. □

5.4.4 Forced linear equation

By the Gearhart-Prüss theorem, Proposition 5.16 shows decay on $P_s \mathcal{X}_{a,k}$ for $k \in \{-1, 0\}$.

Corollary 5.18. *Let $k \in \{-1, 0\}$. There exists a constant C such that*

$$\|\mathrm{e}^{tL} u\|_{a,k} \leq C \mathrm{e}^{-a't} \|u\|_{a,k}$$

for $u \in P_s \mathcal{X}_{a,k}$.

For the regularisation, we introduce norms over time and adapt the Gearhart-Prüss theorem. Given an Hilbert space H with norm $\|\cdot\|_H$, a positive real number γ, and a mapping $w : \mathbb{R}^+ \to H$, consider the norm $\|w\|_{H,\gamma}$ defined by

$$\|w\|_{H,\gamma} = \left(\int_{\mathbb{R}^+} \mathrm{e}^{2\gamma t} \|w(t)\|_H^2 \mathrm{d}t \right)^{1/2}.$$

We have the following statement.

Lemma 5.19. *Let X and Y be Hilbert spaces, where X is continuously embedded in Y. Let A be a densely defined linear operator that generates a semigroup, both on X and on Y. Assume the existence of $\gamma \in \mathbb{R}^+$ such that the resolvent of A over both spaces contains the half-plane $\Re(\lambda) \geq -\gamma$ and satisfies*

$$\sup_{y \in \mathbb{R}} \|((-\gamma + \mathrm{i}y)\mathrm{Id} - A)^{-1}\|_{Y \to X} < C_R,$$

for some $C_R \in \mathbb{R}^+$ *and*

$$\sup_{\Re\lambda \geq -\gamma} \|(\lambda \mathrm{Id} - A)^{-1}\|_{Y \to X} < \infty.$$

Then the unique mild solution $w \in C(\mathbb{R}^+, Y)$ *of the initial value problem*

$$\frac{\mathrm{d}w}{\mathrm{d}t} = Aw + G$$

where the forcing $G : \mathbb{R}^+ \mapsto Y$ *satisfies* $\|G\|_{Y,\gamma} < +\infty$ *and the initial condition* $w(t) = w_{\mathrm{in}}$ *satisfies* $\|w_{\mathrm{in}}\|_X < +\infty$*, has the following properties*

- $w(t) \in X$ *for a.e.* $t \in \mathbb{R}^+$
- $\|w\|_{X,\gamma} \leq C\left(\|w_{\mathrm{in}}\|_X + \|G\|_{Y,\gamma}\right)$

for some $C \in \mathbb{R}^+$*.*

Proof. The mild solution of the initial value problem is characterized by the Duhamel's formula

$$w(t) = \mathrm{e}^{tA} w_{\mathrm{in}} + I(t), \ \forall t \in \mathbb{R}^+, \text{ where } I(t) = \int_0^t \mathrm{e}^{(t-s)A} G(s) \mathrm{d}s.$$

By the Gearhart-Prüss Theorem, the resolvent estimate shows that there exists $\gamma_G > \gamma$ and $C_G \in \mathbb{R}^+$ such that

$$\|e^{tA} w\|_X \leq C_G \mathrm{e}^{-\gamma_G t} \|w\|_X, \ \forall w \in X, t \in \mathbb{R}^+,$$

which yields

$$\|e^{tA} w_{\mathrm{in}}\|_{X,\gamma} \leq \frac{C_G}{(\gamma_G - \gamma)} \|w_{\mathrm{in}}\|_X.$$

Moreover, for $\Re(z) > -\gamma$, the Laplace transform $\mathcal{L}I$ of the integral term I exists as Bochner integral over Y and satisfies

$$(\mathcal{L}I)(z) = (z\mathrm{Id} - A)^{-1} (\mathcal{L}G)(z),$$

where $\mathcal{L}G$ is the Laplace transform of G. On the line $\Re(z) = -\gamma$, the Laplace transform $\mathcal{L}G$ exists as a L^2 function by the Plancherel's Theorem, and we have

$$\int_{\mathbb{R}} \|(\mathcal{L}G)(-\gamma + \mathrm{i}y)\|_Y^2 \mathrm{d}y \leq 2\pi \|G\|_{Y,\gamma}^2.$$

The assumption on the resolvent estimate then implies

$$\int_{\mathbb{R}} \|(\mathcal{L}I)(-\gamma + \mathrm{i}y)\|_X^2 \mathrm{d}y \leq 2\pi C_R \|G\|_{Y,\gamma}^2.$$

Using the Plancherel's Theorem again, this time to $(\mathcal{L}I)(-\gamma + \mathrm{i}\cdot)$, it follows that $I(t) \in X$ for a.e. $t \in \mathbb{R}^+$ and $\|I\|_{X,\gamma}^2 \leq 2\pi C_R \|G\|_{Y,\gamma}^2$. Combined with the estimate on the initial term this shows the claimed result. □

Now, Proposition 5.16 implies that the operator L satisfies the condition of the Lemma with $X = \mathsf{P}_s(\mathcal{X}^2_{a,0})$, $Y = \mathsf{P}_s(\mathcal{X}^2_{a,-1})$ and $\gamma = a'$. The Lemma then yields the following conclusion for the initial value problem

$$\partial_t u = Lu + P_s F(t), \text{ and } u(0) = u_{\text{in}} \in P_s(\mathcal{X}_{a,0}), \tag{5.9}$$

where we use the notation

$$\|u\|_{a,k,b} = \left(\int_{\mathbb{R}^+} \mathrm{e}^{2bt} \|u(t)\|^2_{a,k} \mathrm{d}t \right)^{1/2}.$$

Corollary 5.20. *Let f_{st} be a stable state with rate a'. Then there exist $C > 0$ such that, for every forcing signal satisfying $\|F\|_{a,-1,b} < +\infty$, the initial value problem (5.9) has a unique mild solution $t \to u(t) \in C(\mathbb{R}^+, \mathcal{X}_{a,-1})$ with the following properties*

- *$u(t) \in \mathcal{X}_{a,0}$ for a.e. $t \in \mathbb{R}^+$,*
- *$\|u\|_{a,0,b} \le C \left(\|u_{\text{in}}\|_{a,0} + \|F\|_{a,-1,b} \right)$.*

5.5 Nonlinear stability

The proof of the nonlinear stability result (Theorem 5.1) proceeds through a localisation of the nonlinearity. For a strong enough localisation, we can then show that $\|u\|_{a,0,a'}$ can be made arbitrary small, where u denotes the distance to the (rotated) stationary state.

The control on $\|u\|_{a,0,a'}$ follows from Corollary 5.20 using Duhamel's principle and the smallness of the nonlinearity through the localisation. The handling of the rotation symmetry can be done by a center manifold reduction (Theorem 4.26) replacing the norms

$$\sup_t \mathrm{e}^{\mu t} \|f(t)\|$$

by

$$\left(\int_0^t \mathrm{e}^{2\mu t} \|f(t)\|^2 \mathrm{d}t \right)^{1/2}.$$

By the rotation symmetry, we can explicitly identify the reduced manifold and conclude the result.

However, we can also construct the projection explicitly using polar type coordinates. Let $\hat{f}$ be the Fourier transform of the perturbed system in a sufficiently small neighbourhood of the circle $\{\hat{R}_\Theta \hat{f}_{\text{st}}\}_{\Theta \in \mathbb{T}}$ in $\mathcal{X}_{a,0}$. We can decompose it as

$$\hat{f} = \hat{R}_\Theta \left(\hat{f}_{\text{st}} + u \right), \tag{5.10}$$

where $(\Theta, u) \in \mathbb{T} \times P_s(\mathcal{X}_{a,0})$ is such that $\|u\|_{a,0} \to 0$ when the distance $d\left(\hat{f}, \{\hat{R}_\Theta \hat{f}_{\mathrm{st}}\}_{\Theta\in\mathbb{T}}\right) \to 0$.

To see this, consider the map $F : \mathbb{T} \times \mathcal{X}_{a,0} \to \mathbb{R}$ defined by

$$F(\Theta, \hat{f}) = \Re\langle \hat{R}_{-\Theta}\hat{f} - \hat{f}_{\mathrm{st}}, u^*\rangle_{a,0},$$

which is such that $F(\Theta, \hat{f}) = 0$ if and only if $\hat{f}$ satisfies (5.10).

We compute

$$F(0, \hat{f}_{\mathrm{st}}) = 0 \text{ and } \partial_\Theta F(0, \hat{f}_{\mathrm{st}}) = -\Re\left(\langle D\hat{R}\hat{f}_{\mathrm{st}}, u^*\rangle_{a,0}\right) \neq 0,$$

hence by the implicit function theorem, for $\hat{f}$ close enough to $\hat{f}_{\mathrm{st}}$, there is a smooth $\Theta_0 = \Theta_0(\hat{f})$ near 0 such that $F(\Theta_0(\hat{f}), \hat{f}) = 0$. This proves the claim in the neighbourhood of $\hat{f}_{\mathrm{st}}$, with $u = \hat{R}_{-\Theta_0(\hat{f})}\hat{f} - \hat{f}_{\mathrm{st}}$. To extend that property to a neighbourhood of $\{\hat{R}_\Theta \hat{f}_{\mathrm{st}}\}_{\Theta\in\mathbb{T}}$, notice that letting

$$\Theta_{\min} = \mathrm{argmin}_{\Theta\in\mathbb{T}} \|\hat{f} - \hat{R}_\Theta \hat{f}_{\mathrm{st}}\|_{a,0},$$

the element $\hat{R}_{-\Theta_{\min}}\hat{f}$ is close to $\hat{f}_{\mathrm{st}}$ when $\hat{f}$ is close to the circle. Hence, one can apply the previous argument to $\hat{R}_{-\Theta_{\min}}\hat{f}$ to obtain $\hat{f} = \hat{R}_{\Theta_0(\hat{R}_{-\Theta_{\min}}\hat{f})+\Theta_{\min}}\left(\hat{f}_{\mathrm{st}} + u\right)$ with u as desired.

In this coordinates we can precisely express the decay.

Lemma 5.21. *Let f_{st} be stable with rate a'. Then there exists $\epsilon > 0$ and C such that for initial data f_{in} with decomposition*

$$\hat{f}_{\mathrm{in}} = \hat{R}_{\Theta_{\mathrm{in}}}\left(\hat{f}_{\mathrm{st}} + u\right)$$

with Θ_{in} and $u_{\mathrm{in}} \in P_s\mathcal{X}_{a,0}$ and $\|u\|_{a,0} \leq \epsilon$, the solution f of the Kuramoto equation satisfies

$$\|\hat{f} - \hat{R}_{\Theta(t)}\|_{a,0} \leq C\|u_{\mathrm{in}}\|\mathrm{e}^{-a't}$$

for a function $\Theta : \mathbb{R}^+ \mapsto \mathbb{T}$ exponentially converging with rate a' to a limit.

This lemma in particular proves Theorem 5.1.

Proof. By inserting the expression (5.10) into the equation (5.2) of the Kuramoto dynamics in Fourier variables, one gets after using equivariance

$$\mathrm{D}\hat{R}(\hat{f}_{\mathrm{st}} + u)\frac{\mathrm{d}\Theta}{\mathrm{d}t} + \partial_t u = Lu + Qu.$$

Applying P_0 and P_s respectively, and using $P_0 L = 0$, $P_0 L = LP_0$ and the normalization $\langle \mathrm{D}\hat{R}\hat{f}_{\mathrm{st}}, u^*\rangle = 1/2$, two independent equations result for the variables Θ and u, namely

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \frac{2\Re\langle Qu, u^*\rangle_{a,0}}{1 + 2\Re\langle \mathrm{D}\hat{R}u, u^*\rangle_{a,0}}, \tag{5.11}$$

and

$$\partial_t u = Lu + P_s Q' u \text{ where } Q'u = Qu - \frac{2\Re\langle Qu, u^*\rangle_{a,0}}{1 + 2\Re\langle \mathrm{D}\hat{R}u, u^*\rangle_{a,0}} \mathrm{D}\hat{R}u. \tag{5.12}$$

As intended, the right hand sides of these equations do not depend on the angular variable Θ. Moreover, the Cauchy-Schwarz inequality implies

$$\left|\langle \mathrm{D}\hat{R}u, u^*\rangle_{a,0}\right| \leq 2\|u\|_{a,0}\|u^*\|_{a,1},$$

and the regularity from Lemma 5.17 implies $\|u^*\|_{a,1} < +\infty$. Therefore, these equations are well-defined as long as $\|u\|_{a,0}$ is small enough (so that the denominators do not vanish).

Now, if the (restriction to $\mathbb{N} \times \mathbb{R}$ of the) Fourier transform $\hat{f}_{\text{in}}$ of an initial probability measure f_{in} is sufficiently close to $\hat{f}_{\text{st}}$, then not only the corresponding initial $u_{\text{in}} \in P_s(\mathcal{X}_{a,0})$ is small, but the solution $\hat{f}(t)$ must remain close to $\hat{f}_{\text{st}}$ for $t \in (0, T)$, a sufficiently small time interval, by the continuous dependence in time (Proposition 5.6). Hence, both ansatz (5.10) holds and the equations above are well-defined over $(0, T)$. That these properties holds for all times (provided that $\hat{f}_{\text{in}}$ is taken even closer to $\hat{f}_{\text{st}}$) is a direct consequence of the following statement.

Proposition 5.22. *For a stable state f_{st} with rate a', there exist $\epsilon', C > 0$ such that for all $u_{\text{in}} \in P_s(\mathcal{X}_{a,0})$ satisfying $\|u_{\text{in}}\|_{a,0} < \epsilon'$, equation* (5.12) *has a unique solution $t \to u(t)$ satisfying $u(0) = u_{\text{in}}$ and*

$$\|u(t)\|_{a,0} \leq C\|u_{\text{in}}\|_{a,0}\, \mathrm{e}^{-a't}, \ \forall t \in \mathbb{R}^+.$$

This statement is not as obvious as it may look because the quadratic term Q' maps $\mathcal{X}_{a,0}$ into $\mathcal{X}_{a,-1}$ and is proved below.

In addition to ensuring that both ansatz (5.10) and equations (5.11) and (5.12) are globally well-defined when starting sufficiently close to $\hat{f}_{\text{st}}$, Proposition 5.22 implies that the solution must asymptotically approach the PLS circle. To complete the proof, it remains to show that the solution's angle asymptotically converges $\Theta(t)$. We have

$$|\langle Qu, u^*\rangle_{a,0}| \leq \|Qu\|_{a,-1}\|u^*\|_{a,1},$$

and the definition of Q and the Sobolev embedding yield

$$\|Qu\|_{a,-1} \leq C'\|u\|_{a,-1/2}\|u\|_{a,0} \leq C'\|u\|_{a,0}^2, \tag{5.13}$$

for some $C' \in \mathbb{R}^+$. Hence the driving term in equation (5.11) must also decay exponentially with rate b. Consequently, the following limit exists

$$\Theta_\infty := \lim_{t\to+\infty} \Theta(t) = \Theta(0) + \int_{\mathbb{R}^+} \frac{2\Re\langle Qu(s), u^*\rangle_{a,0}}{1 + 2\Re\langle \mathrm{D}\hat{R}u(s), u^*\rangle_{a,0}} \mathrm{d}s,$$

and we have $\Theta(t)$ with limit Θ_∞ converging exponentially. $\square$

Using the forced linear analysis, the used proposition can be proved.

Proof of Proposition 5.22. Given $\epsilon > 0$, consider the localisation $Q'_\epsilon : \mathcal{X}_{a,0} \to \mathcal{X}_{a,-1}$ as smooth mapping such that

$$Q'_\epsilon u = \begin{cases} Q'u & \text{if } \|u\|_{a,0} \le \epsilon \\ 0 & \text{if } \|u\|_{a,0} \ge 2\epsilon. \end{cases}$$

When ϵ is small enough, the denominator in the expression of Q' remains positive over the ball $\{u \;:\; \|u\|_{a,0} < 2\epsilon\}$; hence Q'_ϵ is globally defined over $\mathcal{X}_{a,0}$ in this case. Moreover, using the inequality (5.13), we infer

$$\|Q'_\epsilon u\|_{a,-1} \le 2\epsilon C_K \|u\|_{a,0}.$$

Adapting the analysis in Section 5.3, one can show for all $u_{\text{in}} \in P_s(\mathcal{X}_{a,0})$, there exists a unique global in time weak solution

$$u \in C([0,T], \mathcal{X}_{a,0}) \cap L^2(0,T,\mathcal{X}_{a,1/2}) \quad \forall T > 0,$$

of

$$\partial_t u = Lu + P_s Q'_\epsilon u \tag{5.14}$$

with $u(0) = u_{\text{in}}$. Moreover, by standard arguments, it coincides with the mild solution of Equation (5.9) with $F = Q'_\epsilon$.

Applying Corollary 5.20, we conclude that the solution of (5.14) satisfies the inequality

$$\|u\|_{a,0,a'} \le \frac{C}{1 - 2\epsilon C C_K} \|u_{\text{in}}\|_{a,0}, \tag{5.15}$$

provided that ϵ is small enough, so that the denominator here is positive.

In order to get an L^∞ bound, we directly perform an estimate on equation (5.14). We get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{a,0}^2 + a\|u\|_{a,1/2}^2 \le C_1\|u\|_{a,0}^2 + C_1'|\langle Q'_\epsilon u, u\rangle_{a,0}|$$

for constants C_1 and C_1'. The second term in the right hand side can controled as follows

$$|\langle Q'u, u\rangle_{a,0}| \le C_2 \chi\left(\frac{\|u\|_{a,0}}{\epsilon}\right)(|u(1,0)| + \|u\|_{a,0})\|u\|_{a,1/2}^2$$

for some $C_2 \in \mathbb{R}^+$ and where $\chi : \mathbb{R}^+ \to [0,1]$ is a smooth function such that

$$\chi(x) = \begin{cases} 1 & \text{if } x \le 1, \\ 0 & \text{if } x \ge 2. \end{cases}$$

For ϵ small enough, this term can be absorbed by the left-hand side, and the following inequality results

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{a,0}^2 \leq C_3\|u\|_{a,0}^2$$

for some $C_3 \in \mathbb{R}^+$ and then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{2a't}\|u\|_{a,0}^2) \leq (2a' + C_3)\mathrm{e}^{2a't}\|u\|_{a,0}^2.$$

The Gronwall's Lemma yields in turn

$$\mathrm{e}^{2a't}\|u(t)\|_{a,0}^2 \leq \|u_{\mathrm{in}}\|_{a,0}^2 \exp\left((2a' + C_3)\int_0^t \mathrm{e}^{2a's}\|u(s)\|_{a,0}^2 \mathrm{d}s\right).$$

Using the bound (5.15), the desired exponential decay follows

$$\sup_{t\in\mathbb{R}^+} \mathrm{e}^{2a't}\|u(t)\|_{a,0} \leq C_4\|u_{\mathrm{in}}\|_{a,0}^2,$$

for the solution of equation (5.14). Finally, by choosing $\|u_{\mathrm{in}}\|_{a,0}$ small enough, this inequality implies in particular that $\|u(t)\|_{a,0} \leq \epsilon$ for all $t \in \mathbb{R}^+$ and hence we have $Q_\epsilon u(t) = Qu(t)$ for all t, i.e. $t \to u(t)$ is actually a solution of (5.12). □

5.6 Analysis of the stability condition

As discussed in Section 5.4.2, the stability criterion Definition 5.12 is equivalent to the linear stability of the circle $\{R_\Theta f_{\mathrm{st}}\}_{\Theta\in\mathbb{T}}$, more precisely that 0 is the only eigenvalue, which is simple, in the half-plane $\Re(\lambda) \geq 0$, and that the rest of the spectrum lies in $\Re(\lambda) \leq -\epsilon$ for some $\epsilon > 0$. Of note, that 0 must always be an eigenvalue is a consequence of the rotation symmetry R_Θ. However, this property can be obtained independently, as in [120], by using the equations

$$\beta^2(z) + 2iz\beta(z) - 1 = 0,$$

and the self-consistency condition

$$K\int_{\mathbb{R}} g(Kr_{\mathrm{s}}\omega)\beta(\omega)d\omega = 1.$$

Indeed, one directly checks with the notation J_k from Proposition 5.14 that

$$\frac{K}{2}\left(J_0(\lambda,\eta_{\mathrm{st}}) + \frac{2\lambda}{K\eta_{\mathrm{st}}}J_1(\lambda,\eta_{\mathrm{st}}) + J_2(\lambda,\eta_{\mathrm{st}})\right) = 1,\ \forall\lambda\ :\ \Re(\lambda) > 0,$$

from where $\det(\mathrm{Id} - \frac{K}{2}M(0,\eta_{\mathrm{st}})) = 0$ immediately follows when taking the limit $\lambda \to 0$ in $\mathbb{R}$.

5.6.1 Even frequency distributions

The stability of partially locked states depends on context and, as for existence, various situations can occur depending on the bifurcation that generates these states. For instance, when the velocity marginal g is an even function, we have $\overline{J_k(\bar{\lambda}, r)} = J_k(\lambda, r)$ for all $\lambda \in \mathbb{C}$ and then

$$\det\left(\mathrm{Id} - \frac{K}{2}M(\lambda, \eta_{\mathrm{st}})\right)$$
$$= \left(1 - \frac{K}{2}\left(J_0(\lambda, \eta_{\mathrm{st}}) - J_2(\lambda, \eta_{\mathrm{st}})\right)\right)\left(1 - \frac{K}{2}\left(J_0(\lambda, \eta_{\mathrm{st}}) + J_2(\lambda, \eta_{\mathrm{st}})\right)\right).$$

Moreover, one can show (we skip the tedious computation for brevity) that

$$J_0(\lambda, \eta_{\mathrm{st}}) - J_2(\lambda, \eta_{\mathrm{st}}) = 2h_c(\lambda) \quad \text{and} \quad J_0(\lambda, \eta_{\mathrm{st}}) + J_2(\lambda, \eta_{\mathrm{st}}) = 2h_s(\lambda),$$

where the functions h_c and h_s are defined in [109] and can be identified as even and odd perturbations. In this way, we can link our stability criterion to the results of [109]. In the case of unimodal g, Proposition 4 in this paper implies that for $K > K_c := \frac{2}{\pi g(0)}$, the factor $1 - Kh_c$ does not vanish over $\Re(\lambda) \geq 0$, while the only zero of $1 - Kh_s$ in $\Re(\lambda) \geq 0$ is $\lambda = 0$. It follows that

$$\det\left(\mathrm{Id} - \frac{K}{2}M(\lambda, \eta_{\mathrm{st}})\right) > 0, \ \forall \lambda \neq 0 \text{ with } \Re(\lambda) \geq 0,$$

for the unique PLS f_{s} which exists for $K > K_c$. To check the second point in the stability condition, use the expression of h_s given in [109], we find

$$\begin{aligned}
h_s'(0) &= \int_{|\omega| \geq K\eta_{\mathrm{st}}} \frac{g(\omega)\mathrm{d}\omega}{\sqrt{\omega^2 - (K\eta_{\mathrm{st}})^2}(|\omega| + \sqrt{\omega^2 - (K\eta_{\mathrm{st}})^2})} - \frac{1}{(K\eta_{\mathrm{st}})^2}\int_{|\omega| \leq K\eta_{\mathrm{st}}} g(\omega)\mathrm{d}\omega \\
&= \frac{2}{K\eta_{\mathrm{st}}}\left(\int_1^{+\infty} \frac{g(K\eta_{\mathrm{st}}x)\mathrm{d}x}{\sqrt{x^2-1}(x + \sqrt{x^2-1})} - \int_0^1 g(K\eta_{\mathrm{st}}x)\mathrm{d}x\right) \\
&> \frac{2}{K\eta_{\mathrm{st}}} g(K\eta_{\mathrm{st}})\left(\int_1^{+\infty} \frac{\mathrm{d}x}{\sqrt{x^2-1}(x + \sqrt{x^2-1})} - \int_0^1 \mathrm{d}x\right),
\end{aligned}$$

where the last inequality coming from the fact that g is unimodal. A simple computation shows that

$$\int_1^{+\infty} \frac{\mathrm{d}x}{\sqrt{x^2-1}(x + \sqrt{x^2-1})} = 1$$

so that $h_s'(0) \neq 0$. Then Theorem 5.1 implies that, when it exists, this stationary solution is always asymptotically stable.

Finally, notice that uniqueness of a PLS circle does not necessarily imply its stability. By considering a three modal distribution counterexamples can be constructed.

5.6.2 Relation to Ott–Antonsen ansatz

Mode reduction

Recall from Section 1.3.4, that the Kuramoto equation has the invariant manifold with distributions f satisfying

$$\tilde{f}_l(\omega) = \alpha^l(\omega)g(\omega), \quad \forall (l,\omega) \in \mathbb{N} \times \mathbb{R}$$

with amplitude function α, which evolves according to (1.12).

Within this manifold the existence of eigenmodes can be studies as before yielding exactly the same linear stability condition. In particular, Omel'chenko and Wolfrum [120] studied this approach in a strong topology and faced the same issue of a continuous spectrum on the imaginary axis. They expressed the linear stability theory through the matrix B defined by

$$\frac{K}{2}M(\lambda, r) = PB(\lambda)P^{-1} \quad \text{where} \quad P = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}$$

and find modes with eigenvalue λ if

$$\det(\mathrm{Id} - B(\lambda)) = 0.$$

Hence the stability condition on the restricted manifold agrees with the stability in the full space. In other words, no loss of generality results in investigating the existence and stability of f_{s} in the OA manifold. The Ott-Antonsen ansatz is perfectly legitimate.

Pole reduction for rational functions

In case of a rational frequency marginal g, the system can be further simplified to an ODE system for the amplitudes on the poles of g, cf. Section 1.3.4. A well-studied example is the bi-Cauchy frequency distribution, which shows a rich behaviour and is well-studied by Martens et al. [105]. In particular, using complex analysis the integrals of the stability condition can be evaluated explicitly.

Surprisingly, this further reduction exactly catches the possible eigenmodes.

Recall from (1.13), that the amplitude α_i on the pole ω_i evolves as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\alpha_i = -\mathrm{i}\omega_i\alpha_i(t) - \dfrac{K}{2}\left(z(t) - \overline{z(t)}\alpha_i^2(t)\right) = 0, \quad \text{for } i = 1, \ldots, M, \\ z(t) = -\displaystyle\sum_{i=1}^{M} \omega_i\alpha_i(t). \end{cases}$$

For a steady state, we assume without loss of generality that the order parameter η_{st} is real. Then at the stationary state $\alpha_i = \beta_i = \beta\left(\frac{\omega}{K\eta_{\mathrm{st}}}\right)$ and we consider a perturbation $(\gamma_i)_{i=1}^{M}$, i.e.

$\beta_i + \gamma_i$ is a solution. It then evolves as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\gamma_i = -\mathrm{i}\omega_i\gamma_i - \dfrac{K}{2}\left[z - \overline{z}\beta_i^2 - 2\beta_i\eta_{\mathrm{st}}\gamma_i\right], \\ z = -\displaystyle\sum_{i=1}^{N}\omega_i\gamma_i. \end{cases}$$

In order to perform a spectral analysis, we complexify again the system, i.e. we consider $\breve{\gamma}_i$ and $\breve{z}$ as formal adjoint evolving independently. Thus we arrive at the system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\gamma_i = -\mathrm{i}\omega_i\gamma_i - \dfrac{K}{2}\left[z - \breve{z}\beta_i^2 - 2\beta_i\eta_{\mathrm{st}}\gamma_i\right], \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\breve{\gamma}_i = \mathrm{i}\omega_i\gamma_i - \dfrac{K}{2}\left[z - \breve{z}\beta_i^2 - 2\beta_i\eta_{\mathrm{st}}\gamma_i\right], \\ z = -\displaystyle\sum_{i=1}^{N}\omega_i\gamma_i \\ \breve{z} = -\displaystyle\sum_{i=1}^{N}\overline{\omega_i}\breve{\gamma}_i \end{cases}$$

Here $(\gamma, \breve{\gamma})$ is an eigenmode with eigenvalue λ if

$$\begin{cases} (\lambda + \mathrm{i}\omega_i - K\eta_{\mathrm{st}}\beta_i)\gamma_i = -\dfrac{K}{2}z + \dfrac{K}{2}\beta_i^2\breve{z}, \\ (\lambda - \mathrm{i}\overline{\omega_i} - K\eta_{\mathrm{st}}\overline{\beta_i})\breve{\gamma}_i = -\dfrac{K}{2}\breve{z} + \dfrac{K}{2}\overline{\beta_i}^2 z, \\ z = -\displaystyle\sum_{i=1}^{N}\omega_i\gamma_i, \\ \breve{z} = -\displaystyle\sum_{i=1}^{N}\overline{\omega_i}\breve{\gamma}_i. \end{cases}$$

Knowing z and $\breve{z}$ determines $\gamma, \breve{\gamma}$. Hence we have an eigenvector if and only if

$$\begin{cases} z = \displaystyle\sum_{i=1}^{N}\omega_i\frac{K}{2}\frac{z - \beta_i^2\breve{z}}{\lambda + \mathrm{i}\omega_i - K\eta_{\mathrm{st}}\beta_i}, \\ \breve{z} = \displaystyle\sum_{i=1}^{N}\overline{\omega_i}\frac{K}{2}\frac{\breve{z} - \beta_i^2 z}{\lambda - \mathrm{i}\overline{\omega_i} - K\eta_{\mathrm{st}}\overline{\beta_i}}. \end{cases}$$

This is equivalent to $(z, \breve{z}) \in \ker N$ with

$$N = \begin{pmatrix} 1 - (K/2)\sum_{i=1}^{N}\omega_i(\lambda + i\omega_i - K\eta_{\mathrm{st}}\beta_i)^{-1} & (K/2)\sum_{i=1}^{N}\omega_i\beta_i^2(\lambda + i\omega_i - K\eta_{\mathrm{st}}\beta_i)^{-1} \\ (K/2)\sum_{i=1}^{N}\overline{\omega_i}\overline{\beta_i}^2(\lambda - i\overline{\omega_i} - K\eta_{\mathrm{st}}\overline{\beta_i})^{-1} & 1 - (K/2)\sum_{i=1}^{N}\overline{\omega_i}(\lambda - i\overline{\omega_i} - K\eta_{\mathrm{st}}\overline{\beta_i})^{-1} \end{pmatrix}.$$

On the other hand for $\Re\lambda > 0$, we find for J_k from Proposition 5.14

$$J_k(\lambda, \eta_{\mathrm{st}}) = -\sum_{i=1}^{M} \omega_i \frac{\beta_i}{\lambda + \mathrm{i}\omega_i + K\eta_{\mathrm{st}}\beta_i}.$$

Hence we find that

$$N = \mathrm{Id} - \frac{K}{2} M$$

and the stability conditions on the finite-dimensional system agrees with the full linear stability.

6 Vlasov–Poisson equation in S^3

The recent paper by Diacu, Ibrahim, Lind and Shen [43] studied the Vlasov–Poisson equation in spaces of constant Gaussian curvature κ in order to model stellar dynamics, e.g. the density of galaxies in galaxy clusters or the density of stars in galaxies. The motivation is, on the one hand, the understanding of possible universe models and, on the other hand, a deeper understanding of the flat case as the limit of the dynamics on the sphere.

This physical motivation comes from the observation of the cosmic microwave background (CMB) created by the plasma of the early universe, see the nice review by Weeks [163] for an introduction to the model and [131] for a discussion with recent observation data. Here the decay of the physical system can be linked formally to the observed fluctuations by the fluctuation-dissipation theorem.

On the mathematical side, another motivation is the understanding of Landau damping and its spectral decay in more general geometries, where this study provides a first step. A notable difference to the configuration on the torus with periodic boundary condition, used by Mouhot and Villani [114], is that all geodesics are closed, whereas on the torus every trajectory with irrational angle will come arbitrarily close to any point.

The work [43] considered positive and negative constant curvature with two space dimensions. For the derivation of the Vlasov–Poisson equation, classical mechanics is used and relativistic effects are neglected. Under these premises, they derived the resulting equation using an extrinsic parametrisation of the space, which works in general dimensions. For their study of Landau damping, they assume that all mass is along a great circle and study the stability of this reduced system.

In this chapter, I will consider the case of positive curvature in three dimensions. In this case, we can scale the space to the unit sphere S^3, which we embed into $\mathbb{R}^4$. This choice allows us to find global nonvanishing vector fields acting as basis for the velocities, which is impossible in S^2 due to the *hairy ball* theorem. Furthermore, on S^3 we can use spherical harmonics as spatial basis. Combining these ingredients allows a convenient description of the dynamics, in which we perform a global linear stability analysis.

6.1 Description of phase space

This section introduces the curved space and the resulting phase space. For further discussions and other descriptions of the setup, we refer to [43] and the references given within.

We parametrise the spatial variable on the manifold

$$S^3 = \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$$

embedded in $\mathbb{R}^4$ and endowed with the induced metric, which is denoted by g_{S^3}.

The velocity of a particle at position q is in the tangent space $T_q S^3$, which is, in the extrinsic coordinates, given by

$$T_q S^3 = \{p \in \mathbb{R}^4 : q \cdot p = 0\}.$$

We introduce the basis vector fields v_1, v_2 and v_3 by

$$v_1 = \begin{pmatrix} y \\ -x \\ w \\ -z \end{pmatrix}, \qquad v_2 = \begin{pmatrix} z \\ -w \\ -x \\ y \end{pmatrix}, \qquad v_3 = \begin{pmatrix} w \\ z \\ -y \\ -x \end{pmatrix}$$

at $q = (x, y, z, w) \in S^3$. From the definition, we can immediately see that at every point $q \in S^3$, the vector fields create an orthonormal basis of $T_q S^3$ under g_{S^3}. This means that a velocity $p \in T_q S^3$ can be expressed by the coefficients (a^1, a^2, a^3) such that

$$p = a^1 v_1 + a^2 v_2 + a^3 v_3.$$

In differential geometry the tangent vector is associated with the corresponding partial derivative operator. Acting on a function $f : S^3 \mapsto \mathbb{R}$, this is, at a point $q = (x, y, z, w) \in S^3$, given by

$$v_1(f) = \partial_{v_1} f = \begin{pmatrix} y \\ -x \\ w \\ -z \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \\ \partial_w \end{pmatrix} = [y\partial_x - x\partial_y + w\partial_z - z\partial_w]f$$

and accordingly for v_2 and v_3. Looking at the commutator, we immediately find

$$[v_1, v_2] = 2v_3, \qquad [v_2, v_3] = 2v_1, \qquad [v_3, v_1] = 2v_2,$$

where we see the curvature of the space. In the extrinsic representation the inherited Levi–Civita connection is

$$\nabla_i v_j = \sum_{k=1}^{3} \epsilon_{ijk} v_k,$$

where ϵ is the standard alternating tensor.

In order to describe the trajectory of a particle starting from a position $q \in S^3$ with velocity $p \in T_q S^3$, we need to prescribe what the velocity is at the new position $q' \in S^3$, because a priori the tangent spaces $T_q S^3$ and $T_{q'} S^3$ are not related.

The induced metric gives a canonical prescription for the new velocity at the new position. Around any point $q \in S^3$, we can introduce so-called normal coordinates, which are as flat as possible at q, i.e. as similar as possible to $\mathbb{R}^3$ around q. In the flat case, the velocity vector is constant under the free transport, so we impose the same at q in the normal coordinates. This procedure defines a well-defined prescription for the trajectory of a particle. It is therefore the natural extension to manifolds and is used as equivalence principle in general relativity to define the laws of physics in curved space-time. In [43] this prescription is obtained following restricted Lagrangian mechanics.

On the manifold S^3, this means that the velocity is parallelly transported and that the trajectory is a geodesic. Using extrinsic coordinates, it means that $\dot{p}$ is normal to the manifold S^3. Intrinsically, an equivalent condition is that along the curve $\nabla_p p = 0$ holds with the Levi–Civita connection ∇.

The particular choice of the basis v_1, v_2, v_3 for the velocity means that the coefficients a^1, a^2, a^3 remain constant, i.e. a free moving particle follows

$$\partial_t q(t) = p(t) = \left[a^1 v_1 + a^2 v_2 + a^3 v_3\right]_{q(t)}.$$

As the extrinsic coordinates of v_1, v_2, v_3 change according to the position, we find by the chain rule that

$$\partial_t p = \sum_{i=1}^{3} a^i \partial_t (v_i|_{q(t)}) = -((a^1)^2 + (a^2)^2 + (a^3)^2) \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = -\|p\|^2 q.$$

Hence the change of velocity is indeed normal to the manifold with the explicit correction in extrinsic coordinates due to curvature as in [43, Proposition 3.1].

Using the induced Levi–Civita connection ∇ gives as expected

$$\nabla_p p = \sum_{i=1}^{3} a^i \nabla_{v_i} \left(\sum_{j=1}^{3} a^j v_j \right) = \sum_{i,j,k} a^i a^j \epsilon_{ijk} v_k = 0,$$

which gives an intrinsic proof.

Given a potential $\phi : S^3 \mapsto \mathbb{R}$ on S^3, it naturally defines a cotangent field $\mathrm{d}\phi$, which can be identified with a vector field $\nabla\phi$ through the metric. With the canonical symplectic form, its effect on the equation of motion is given by

$$\partial_t p = -\nabla\phi,$$

which in the coefficients (a^1, a^2, a^3) reads

$$\partial_t \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} = - \begin{pmatrix} \partial_{v_1}\phi \\ \partial_{v_2}\phi \\ \partial_{v_3}\phi \end{pmatrix}.$$

Thus in these coordinates, the equation of motion of a particle can be described by the following proposition, which matches [43, Proposition 3.1].

Proposition 6.1. *Consider a potential $\phi : \mathbb{R} \times S^3 \mapsto \mathbb{R}$. Then the phase space position of a particle at $(q, p) \in TS^3$ evolves according to*

$$\begin{cases} \partial_t q(t) = \sum_{i=1}^{3} a^i(t) v_i|_{q(t)}, \\ \partial_t a^i(t) = -\partial_{v_i}\phi(t) \qquad \textit{for } i = 1, 2, 3. \end{cases}$$

In the case of a vanishing potential, the coefficients a^1, a^2, a^3 are constant and the equation of motions reduce to

$$q(t) = T^a_t q(0),$$

which, in extrinsic coordinates, is given by the matrix

$$T^a_t = \begin{pmatrix} \cos(t\|a\|) & \frac{a^1}{\|a\|}\sin(t\|a\|) & \frac{a^2}{\|a\|}\sin(t\|a\|) & \frac{a^3}{\|a\|}\sin(t\|a\|) \\ -\frac{a^1}{\|a\|}\sin(t\|a\|) & \cos(t\|a\|) & \frac{a^3}{\|a\|}\sin(t\|a\|) & -\frac{a^2}{\|a\|}\sin(t\|a\|) \\ -\frac{a^2}{\|a\|}\sin(t\|a\|) & -\frac{a^3}{\|a\|}\sin(t\|a\|) & \cos(t\|a\|) & \frac{a^1}{\|a\|}\sin(t\|a\|) \\ -\frac{a^3}{\|a\|}\sin(t\|a\|) & \frac{a^2}{\|a\|}\sin(t\|a\|) & -\frac{a^1}{\|a\|}\sin(t\|a\|) & \cos(t\|a\|) \end{pmatrix}$$

with $\|a\| = \sqrt{(a^1)^2 + (a^2)^2 + (a^3)^2} = \|p\|$.

Using this parametrisation, the resulting Vlasov equation is

$$\partial_t f(t, q, a^1, a^2, a^3) + \sum_{i=1}^{3} a^i(t)\partial_{v_i} f(t, q, a^1, a^2, a^3) - \sum_{i=1}^{3} [\partial_{v_i}\phi(t, q)]\, \partial_{a^i} f(t, q, a^1, a^2, a^3) = 0, \tag{6.1}$$

where $f(t, \cdot, \cdot)$ is a distribution with spatial position q and velocity coefficients a^1, a^2, a^3 supported on $q \in S^3$. In order to close the evolution, we then need to prescribe the potential.

6.2 The Laplace–Beltrami operator on S^3

In flat space the potential ϕ is given by the *Poisson equation*

$$\Delta\phi = c\rho$$

with the density ρ and a physical constant c. The natural generalisation for Riemannian manifolds is the Laplace–Beltrami operator, in the studied case Δ_{S^3}. Generally, with the Levi–Civita connection ∇ it takes the form

$$\Delta_{S^3} f = \sum_{i,j=1}^{3} g_{S^3}^{ij} \nabla_i \nabla_j f,$$

where $g_{S^3}^{ij}$ are the components of the inverse of the metric g_{S^3}.

As the vector fields v_1, v_2, v_3 create normal coordinates at every point through the exponential map, this simplifies to

$$\Delta_{S^3} f = \sum_{i=1}^{3} \partial_{v_i} \partial_{v_i} f = \left\{ \left[\begin{pmatrix} y \\ -x \\ w \\ -z \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \\ \partial_w \end{pmatrix} \right]^2 + \left[\begin{pmatrix} z \\ -w \\ -x \\ y \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \\ \partial_w \end{pmatrix} \right]^2 + \left[\begin{pmatrix} w \\ z \\ -y \\ -x \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \\ \partial_w \end{pmatrix} \right]^2 \right\} f,$$

which emphasises that this is indeed the natural extension. Moreover, this choice ensures that the Gauss law still holds for the resulting field as expected from flat electrostatic theory.

For a general sphere S^d in $\mathbb{R}^{d+1}$, we can relate the Laplacian by introducing polar coordinates (r, p) for $r \in \mathbb{R}^+$ and $p \in S^d$. Then the standard Laplace operator on $\mathbb{R}^{d+1}$ and the Laplace–Beltrami operator on S^d are related by

$$\Delta f = \frac{1}{r^d} \partial_r \left(r^d \partial_r f \right) + \frac{1}{r^2} \Delta_{S^d} f. \tag{6.2}$$

On the sphere S^d the eigenfunctions of the Laplace–Beltrami operator are given by the spherical harmonics $(\mathcal{H}_m(S^d))_{m=0}^{\infty}$, which we briefly review following the book by Axler, Bourdon and Ramey [6].

Let $\mathcal{P}_m(\mathbb{R}^{d+1})$ be the homogeneous polynomials of degree m on $\mathbb{R}^{d+1}$ so that

$$p(rx) = r^m p(x) \qquad \text{for } p \in \mathcal{P}_m(\mathbb{R}^{d+1}) \text{ and } x \in \mathbb{R}^{d+1}.$$

Then define the harmonic polynomials $\mathcal{H}_m(\mathbb{R}^{d+1})$ of degree m by

$$\mathcal{H}_m(\mathbb{R}^{d+1}) = \{p \in \mathcal{P}_m(\mathbb{R}^{d+1}) : \nabla p = 0\}.$$

A main result in the development of harmonic polynomials is that the restriction $\mathcal{H}_M(S^d)$ of $\mathcal{H}_M(\mathbb{R}^{d+1})$ to S^d forms a decomposition of $L^2(S^d)$.

Proposition 6.2. *The spherical harmonic polynomials $\mathcal{H}_m(S^d)$ are an orthogonal decomposition of $L^2(S^d)$, i.e.*

$$L^2(S^d) = \oplus_{m=0}^{\infty} \mathcal{H}_m(S^d)$$

and $\mathcal{H}_m(S^d)$ is the eigenspace of the Laplace–Beltrami operator with eigenvalue $m(m+d-1)$.

Proof. The decomposition follows from the decomposition of an arbitrary polynomial into spherical harmonic polynomials on S^d and the density of polynomials, see [6, Theorem 5.12].

The second part is also well known and follows for example from $\Delta p = 0$ for $p \in \mathcal{H}_m(\mathbb{R}^{d+1})$ and Equation (6.2). □

Moreover, the spherical harmonic polynomials can be explicitly computed, see e.g. [6, Theorem 5.25], and their dimension is

$$\dim \mathcal{H}_m(S^d) = \binom{m+d-1}{d-1} + \binom{m+d-2}{d-1}.$$

They are a natural extension of the Fourier series used on the torus and for $d = 1$, we exactly recover the Fourier basis of the circle.

When solving the Poisson equation $\Delta_{S^d}\phi = c\rho$ (with $c < 0$ meaning repulsion and $c > 0$ meaning attraction), we need to impose

$$\int_{S^d} \rho(q) \mathrm{d}s_{S^d}(q) = 0 \tag{6.3}$$

with the induced surface measure $\mathrm{d}s_{S^d}$ on S^d, since the space is compact. With this condition, the solution is given by

$$\phi = \sum_{m=1}^{\infty} \frac{c}{m(m+d-1)} \rho_m,$$

where $(\rho_m)_{m=0}^{\infty}$ is the decomposition of ρ into $(\mathcal{H}_m)_{m=0}^{\infty}$, i.e. $\rho_m \in \mathcal{H}_m(S^d)$ and

$$\rho = \sum_{m=0}^{\infty} \rho_m,$$

where $\rho_0 \equiv 0$ by (6.2).

Following the setup in Section 1.2, we can consider more general mean-field models described by the coefficients $(\hat{W}_m)_{m\in\mathbb{N}}$ and imposing

$$\phi = \sum_{m=1}^{\infty} \hat{W}_m \rho_m,$$

i.e. the decomposition of ϕ is given by $\phi_m = \hat{W}_m \rho_m$.

Such a prescription completes the Vlasov equation (6.1) with the density

$$\rho(t,q) = \int_{\mathbb{R}^3} f(t,q,a^1,a^2,a^3)\, \mathrm{d}a^1 \mathrm{d}a^2 \mathrm{d}a^3.$$

The resulting potential ϕ can also be expressed through the fundamental solution, which has been explicitly computed in [35].

6.3 Linearised dynamics and mode reduction

We now look at the linearised dynamics around a spatially homogeneous state $f_{\mathrm{st}}(a^1, a^2, a^3)$, whose density is cancelled by a background charge density and which is a stationary state. For a perturbed state the previous prescription then defines a dynamic through the Vlasov equation. For a perturbation f the linearised dynamic is given by

$$\begin{cases} \partial_t f(t, q, a^1, a^2, a^3) + \displaystyle\sum_{i=1}^{3} a^i(t) \partial_{v_i} f(t, q, a^1, a^2, a^3) - \sum_{i=1}^{3} \left[\partial_{v_i} \phi(t, q)\right] \partial_{a^i} f_{\mathrm{st}}(a^1, a^2, a^3) = 0, \\ \phi = \displaystyle\sum_{m=1}^{\infty} \hat{W}_m \rho_m \qquad \text{with } \rho = \sum_{m=1}^{\infty} \rho_m \text{ and } \rho_m \in \mathcal{H}_m(S^3), \\ \rho(t, q) = \displaystyle\int_{\mathbb{R}^3} f(t, q, a^1, a^2, a^3) \mathrm{d}a^1 \mathrm{d}a^2 \mathrm{d}a^3. \end{cases}$$

As the overall mass is conserved, we can assume that for the perturbation holds $\int_{S^3} \rho(t, q) \mathrm{d}s_{S^3}(q) = 0$, so that $\rho_0 \equiv 0$.

We decompose f spatially into spherical harmonics. The choice of the velocity basis implies for $p \in \mathcal{H}_m(S^3)$ that $\partial_{v_i} p \in \mathcal{H}_m(S^3)$. This shows that the different degrees $\mathcal{H}_m(S^3)$ decouple in the linearised evolution.

Let $Y_{m,1}, \ldots, Y_{m,N_m}$ be an orthonormal basis of $\mathcal{H}_m(S^d)$. Then we expand f as

$$f(t, q, a^1, a^2, a^3) = \sum_{m=0}^{\infty} \sum_{l=1}^{N_m} \tilde{f}_{m,l}(t, a^1, a^2, a^3) Y_{m,l}(q).$$

Corresponding to ∂_{v_i}, introduce the matrices $M_{m,i}$ acting on the coefficients in the basis $Y_{m,1}, \ldots, Y_{m,N_m}$, i.e. for all coefficients $(h_l)_{l=1}^{N_m}$ holds

$$\partial_{v_i} \sum_{l=1}^{N_m} h_l Y_{m,l}(q) = \sum_{l=1}^{N_m} (M_{m,i} h)_l Y_{m,l}(q).$$

The linearised dynamic within $\mathcal{H}_m(S^3)$ is given by

$$\begin{cases} \partial_t \tilde{f}_m(t, a^1, a^2, a^3) + \displaystyle\sum_{i=1}^{3} a^i(t) \left[M_{m,i} \tilde{f}_m(t, a^1, a^2, a^3)\right]_l \\ \qquad\qquad\qquad - \displaystyle\sum_{i=1}^{3} \left[M_{m,i} \tilde{\phi}_m(t)\right] \partial_{a^i} f_{\mathrm{st}}(a^1, a^2, a^3) = 0, \\ \tilde{\phi}_m(t) = \hat{W}_m \displaystyle\int_{\mathbb{R}^3} \tilde{f}_m(t, a^1, a^2, a^3) \, \mathrm{d}a^1 \mathrm{d}a^2 \mathrm{d}a^3, \end{cases}$$

where $\tilde{f}_m(t, a^1, a^2, a^3)$ denotes the vector in $\mathbb{R}^{N_m}$ according to the basis $(Y_{m,l})_{l=1}^{N_m}$ and likewise $\tilde{\phi}_m(t)$.

For every degree $m \in \mathbb{N}$, we have finitely many components $\phi_{m,1}, \ldots, \phi_{m,N_m}$ of the potential, which satisfies, as discussed in Section 3.1, a vector-valued Volterra equation

$$\phi_m(t) + \int_0^t k_m(t-s)\phi_m(s)\mathrm{d}s = F_m(t)$$

with the convolution kernel defined by

$$k_m(t)u = \hat{W}_m \int_{\mathbb{R}^3} \mathrm{e}^{t(a^1 M_{m,1} + a^2 M_{m,2} + a^3 M_{m,3})} \left(\sum_{i=1}^{3} M_{m,i} u \, \partial_{a^i} f_{\mathrm{st}}(a) \right) \mathrm{d}a^1 \mathrm{d}a^2 \mathrm{d}a^3$$

for $u \in \mathbb{C}^{N_m}$ and the forcing

$$F_m(t) = \hat{W}_m \int_{\mathbb{R}^3} \mathrm{e}^{t(a^1 M_{m,1} + a^2 M_{m,2} + a^3 M_{m,3})} (\tilde{f}_{\mathrm{in}})_m(t, a^1, a^2, a^3) \, \mathrm{d}a^1 \mathrm{d}a^2 \mathrm{d}a^3 \tag{6.4}$$

with the initial data $\tilde{f}_{\mathrm{in}}$.

For the asymptotic behaviour we can apply the theory of finite-dimensional Volterra equation reviewed in Section 3.2.

For the understanding of Landau damping, we need to understand how the spatial density decays through phase mixing. For a velocity profile $g : \mathbb{R}^3 \mapsto \mathbb{R}$ and $t \in \mathbb{R}$, we introduce the operator

$$S_t f(q) := \int_{\mathbb{R}^3} \mathrm{e}^{t(a^1 \partial_{v_1} + a^2 \partial_{v_2} + a^3 \partial_{v_3})} f(q) g(a) \, \mathrm{d}a$$

for $f : S^3 \mapsto \mathbb{R}$ and $q \in S^3$. If $f \in \mathcal{H}_m(S^d)$, then $S_t f \in \mathcal{H}_m(S^3)$ and in terms of a vector $u \in \mathbb{R}^{N_m}$ expanded in the basis of $\mathcal{H}_m(S^3)$ it becomes

$$S_t^m u = \left(\int_{\mathbb{R}^3} \mathrm{e}^{t(a^1 M_{m,1} + a^2 M_{m,2} + a^3 M_{m,3})} g(a) \, \mathrm{d}a \right) u$$

for a matrix $S_t^m \in \mathbb{R}^{N_m \times N_m}$. The next lemma establishes the decay and recovers faster decay for higher degrees like in the flat case, despite the fact that geodesics are always closed.

Lemma 6.3. *Let $g : \mathbb{R}^3 \mapsto \mathbb{R}$ be a smooth function such that there exist constants r and C_g with*

$$\|\Delta^n g\|_{L^1(\mathbb{R}^3)} \leq C_g r^{2n} (2n)! \qquad \text{for } n \in \mathbb{N}.$$

Then for $p \in [1, \infty]$ and $f \in \mathcal{H}_m(S^3)$ holds

$$\|S_t f\|_{L^p(S^3)} \leq \exp\left(1 - 2\left\lfloor \frac{t\sqrt{m(m+2)}}{2r} \right\rfloor\right) \sqrt{2\left\lfloor \frac{t\sqrt{m(m+2)}}{2r} \right\rfloor} \; C_g \, \|f\|_{L^p(S^3)}.$$

In particular, for every $\bar{r} > r$ there exists a constant $C_{\bar{r}}$, independent of m, such that

$$\|S_t f\|_{L^p(S^3)} \leq C_{\bar{r}} \mathrm{e}^{-mt/\bar{r}} \|f\|_{L^p(S^3)}.$$

Proof. For $f \in \mathcal{H}_m(S^3)$, applying the Laplacian in $a = (a^1, a^2, a^3)$ gives

$$\begin{aligned}\Delta_a \mathrm{e}^{t(a^1\partial_{v_1}+a^2\partial_{v_2}+a^3\partial_{v_3})} f &= \mathrm{e}^{t(a^1\partial_{v_1}+a^2\partial_{v_2}+a^3\partial_{v_3})} t^2[\partial_{v_1}^2 + \partial_{v_2}^2 + \partial_{v_3}^2] f \\ &= t^2 m(m+2) \mathrm{e}^{t(a^1\partial_{v_1}+a^2\partial_{v_2}+a^3\partial_{v_3})} f,\end{aligned}$$

where we noted that $\partial_{v_1}^2 + \partial_{v_2}^2 + \partial_{v_3}^2 = \Delta_{S^3}$ and f is an eigenvector. Therefore, for $n \in \mathbb{N}$ we find by partial integration

$$S_t f = \int_{\mathbb{R}^3} \mathrm{e}^{t(a^1\partial_{v_1}+a^2\partial_{v_2}+a^3\partial_{v_3})} f \frac{\Delta_a^n g(a)}{(t^2 m(m+2))^n} \, \mathrm{d}a.$$

As $\mathrm{e}^{t(a^1\partial_{v_1}+a^2\partial_{v_2}+a^3\partial_{v_3})}$ is just rotating f, we have

$$\|\mathrm{e}^{t(a^1\partial_{v_1}+a^2\partial_{v_2}+a^3\partial_{v_3})} f\|_{L^p(S^3)} = \|f\|_{L^p(S^3)}.$$

Hence we find

$$\|S_t f\|_{L^p(S^3)} \le C_g \frac{r^{2n}(2n)!}{(t^2 m(m+2))^n} \|f\|_{L^p(S^3)}.$$

By Stirling's formula, we can bound the factor as

$$\frac{r^{2n}(2n)!}{(t^2 m(m+2))^n} \le \mathrm{e}\sqrt{2n} \left(\frac{2nr}{\mathrm{e}t\sqrt{m(m+2)}} \right)^{2n}.$$

We choose optimally

$$n = \left\lfloor \frac{t\sqrt{m(m+2)}}{2r} \right\rfloor$$

with $\lfloor x \rfloor$ being the floor of x, i.e. the largest integer not greater than x. Then we find as claimed

$$\frac{r^{2n}(2n)!}{(t^2 m(m+2))^n} \le \sqrt{2 \left\lfloor \frac{t\sqrt{m(m+2)}}{2r} \right\rfloor} \exp\left(1 - 2 \left\lfloor \frac{t\sqrt{m(m+2)}}{2r} \right\rfloor \right).$$

Hence for every $\bar{r} > r$, there exists a constant $C_{\bar{r}}$ such that for every $m \in \mathbb{N}$ and $f \in \mathcal{H}_m(S^3)$ it holds that

$$\|S_t f\|_{L^p(S^3)} \le C_{\bar{r}} \mathrm{e}^{-mt/\bar{r}} \|f\|_{L^p(S^3)}. \qquad \square$$

This shows that the forcing term in (6.4) for such regular initial data is exponentially decaying. Moreover, it allows us to study the decay of the convolution kernel.

Proposition 6.4. *Let $s > 0$ and $f_{\mathrm{st}} : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that there exists a constant C_f with*

$$\|\Delta^n \partial_{v_i} f_{\mathrm{st}}\|_{L^1(\mathbb{R}^3)} \le C_f r^{2n} (2n)! \qquad \text{for } n \in \mathbb{N} \text{ and } i = 1, 2, 3.$$

Then for $\lambda < r^{-1}$ there exists a constant C such that the convolution kernel is controlled as

$$\|k_m\|_{L^1(\mathbb{R}^+,\exp_{m\lambda})} = \int_0^\infty \|k_m(t)\| \, e^{m\lambda t} \, dt \le C \, \hat{W}_m.$$

Proof. Choose $r^{-1} > \bar{r}^{-1} > \lambda$ and apply the previous lemma to the convolution kernel, where we note that $\|M_{m,i}\| \le m$ as $M_{m,i}$ is skew-symmetric with largest eigenvalue m. This shows

$$\int_0^\infty \|k_m(t)\| \, e^{m\lambda t} \, dt \le C_{\bar{r}} \hat{W}_m \int_0^\infty m \, e^{m\lambda t - \bar{r}^{-1} m t} \, dt = \frac{C_{\bar{r}}}{\bar{r}^{-1} - \lambda} \hat{W}_m. \qquad \square$$

Assuming the mild condition $\hat{W}_m \to 0$ as $m \to \infty$, which is satisfied for the Poisson equation, we observe a quantitative Landau damping apart from possible discrete modes.

Proposition 6.5. *Let $s > 0$ and $f_{\mathrm{st}} : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function such that there exist a constant C_f with*

$$\|\Delta^n \partial_{v_i} f_{\mathrm{st}}\|_{L^1(\mathbb{R}^3)} \le C_f r^{2n} (2n)! \qquad \text{for } n \in \mathbb{N} \text{ and } i = 1, 2, 3$$

and assume $\hat{W}_m \to 0$ as $m \to \infty$.

For $\lambda < r^{-1}$, there exist at most finitely many roots of

$$\det(\mathrm{Id} - \mathcal{L} k_m(mz)) = 0$$

in $\Re z \ge \lambda$ and $m \in \mathbb{N}$. In particular, if $\sup_m \hat{W}_m$ is small enough, no such root exists.

If no such root exists, then there exists a global constant C such that for $m \in \mathbb{N}$ the resolvent r_m of the Volterra equation is controlled as

$$\int_0^t \|r_m(t)\| e^{m\lambda t} dt \le C.$$

As in Section 3.3, this shows a quantitative version of Landau damping. As in the torus, we have a minimal nonzero degree by compactness, which shows that a small enough interaction strength implies stability.

Proof. By Proposition 6.4, we have

$$\|k_m\|_{L^1(\mathbb{R}^+,\exp_{m\lambda})} \le C \hat{W}_m.$$

If $\|k_m\|_{L^1(\mathbb{R}^+,\exp_{m\lambda})} < 1$, then no root can exist and the resolvent is bounded by $(1 - \|k_m\|_{L^1(\mathbb{R}^+,\exp_{m\lambda})})^{-1}$. As $\hat{W}_m \to \infty$, this proves the statements of the theorem apart from possible finitely many m.

For a fixed m, Proposition 6.4 shows that

$$\det(\mathrm{Id} - \mathcal{L} k_m(mz))$$

is analytic for $\Re z > r$. Hence only finitely many eigenmodes can exist. If the stability condition holds, the Paley–Wiener theorem 3.5 shows that $\|r_m\|_{L^1(\mathbb{R}^+,\exp_{m\lambda})} < \infty$. Taking the supremum over the finitely many modes shows the result. $\square$

The proof shows that only finitely many eigenmodes can exist and that otherwise the linear evolution is damped. For the verification of the stability of a specific stationary state, the proof also shows that only finitely many degrees m need to be verified. For each m, the basis and the related matrices are easily computable by software packages (e.g. Appendix B of the book by Axler, Bourdon and Ramey [6]) so that the condition can be easily decided by numerically computing the curve $x \to \det(\mathrm{Id} - \mathcal{L}k_m(\mathrm{i}x))$ for the finitely many $m \in \mathbb{N}$.

7 Hypocoercivity

In this chapter, I study how a kinetic equation can reach a smooth equilibrium state through a collision operator and a transport operator. Typically, the collision operator only acts on the velocity variable and thus does not produce any spatial averaging. However, in combination with the transport operator, the kinetic equation can have exponential relaxation towards equilibrium. For this interaction, Villani coined the term *hypocoercivity* [156].

7.1 Structure of hypocoercivity

In a typical kinetic equation with relaxation towards equilibrium, the evolution operator G consists of a transport term T and a relaxation term M, i.e. $G = T + M$.

As an example consider a particle distribution on the d-dimensional torus $\mathbb{T}^d$ and velocities $v \in \mathbb{R}^d$. Under the free transport, the density evolves as

$$\partial_t f = Tf = -v \cdot \nabla_x f.$$

As relaxation mechanism, we add an operator M, which acts on every spatial point separately and models collision. We focus on the linear setting, where M models the effect of random collisions with the background. An example is the Fokker-Planck operator defined by

$$Mf = \nabla_v \cdot (\nabla_v f + 2vf).$$

On its own, the generated semigroup e^{tM} converges at every spatial point to an element of the local null space, which is the span of the unique equilibrium e^{-v^2}. However, the distribution at different spatial points is disconnected, so that for any $g \in \mathcal{M}(\mathbb{T}^d)$, the function $g(x)\mathrm{e}^{-v^2}$ is an equilibrium for M, i.e. it is in the nullspace of M. Nevertheless, we expect that $\mathrm{e}^{tG}f$ converges to the unique equilibrium $c\,\mathrm{e}^{-v^2}$ for a constant c as $t \to \infty$.

Considering f in $L^2(\mathbb{T}^d \times \mathbb{R}^d, \mathcal{F}_\infty^{-1})$ with the equilibrium $\mathcal{F}_\infty(x,v) = \pi^{-1/2}\mathrm{e}^{-v^2}$, we indeed find by Poincaré's inequality that

$$-\langle Mf, f\rangle_{L^2(\mathcal{F}_\infty^{-1})} \geq \tilde{\lambda}\|f - \Pi_l f\|_{L^2(\mathcal{F}_\infty^{-1})}$$

for a constant $\tilde{\lambda} > 0$, where Π_l is the projection to the local equilibrium, i.e.

$$(\Pi_l f)(x,v) = \pi^{-d/2} \left(\int_{\mathbb{R}^d} f(x,\bar{v}) \, \mathrm{d}\bar{v} \right) \mathrm{e}^{-v^2}.$$

However, there exists no constant $\lambda > 0$ such that

$$-\langle Gf, f \rangle_{L^2(\mathcal{F}_\infty^{-1})} \geq \lambda \|f - \Pi_g f\|_{L^2(\mathcal{F}_\infty^{-1})}$$

holds with the global projection Π_g onto $\mathcal{F}_\infty$ conserving the mass, i.e.

$$(\Pi_g f)(x,v) = \pi^{-d/2} (2\pi)^{-d} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} f(\bar{x},\bar{v}) \, \mathrm{d}\bar{x} \mathrm{d}\bar{v} \right) \mathrm{e}^{-v^2}.$$

This holds because for every element f in the nullspace we have

$$-\langle Gf, f \rangle_{L^2(\mathcal{F}_\infty^{-1})} = -\langle Tf, f \rangle_{L^2(\mathcal{F}_\infty^{-1})} = 0,$$

as T is anti-selfadjoint.

This implies that the semigroup is not contracting, i.e. there exists no $\lambda > 0$ such that

$$\|\mathrm{e}^{tG}(f - \Pi_g f)\|_{L^2(\mathcal{F}_\infty^{-1})} \leq \mathrm{e}^{-\lambda t} \|(f - \Pi_g f)\|_{L^2(\mathcal{F}_\infty^{-1})},$$

which can be shown by differentiating the contraction inequality of the semigroup with respect to time [156, Proposition 9].

In contrast, there exist constants C and $\lambda > 0$ such that

$$\|\mathrm{e}^{tG}(f - \Pi_g f)\|_{L^2(\mathcal{F}_\infty^{-1})} \leq C\mathrm{e}^{-\lambda t} \|(f - \Pi_g f)\|_{L^2(\mathcal{F}_\infty^{-1})},$$

which by itself a non-trivial result and is a key problem in the development of the theory of hypocoercivity [42, 49, 113, 156]. This behaviour is called hypocoercivity [156, Definition 12] and as pointed out by Serre in [156, Remark 13] it is equivalent to coercivity in an equivalent norm

$$N(h) := \sup_{t \geq 0} \left(\mathrm{e}^{\lambda t} \|\mathrm{e}^{tG} h\| \right).$$

The transport operator T does not create any relaxation by itself in $L^2(\mathcal{F}_\infty^{-1})$, but in combination with the localised relaxation M it forces global decay. Villani's theory of hypocoercivity [156] is about the construction of suitable equivalent spaces based on commutators of the transport operator and the square root of the relaxation operator M.

On a probabilistic level, we can understand the relaxation in $L^1(\Gamma)$ or $\mathcal{M}(\Gamma)$ with the total variation norm in a similar way to the relaxation of Markov chains. For this let K_t be the fundamental solution of e^{tG}, i.e.

$$(\mathrm{e}^{tG}f)(x,v) = \int_\Gamma K_t(x',v';x,v)f(x',v')\,\mathrm{d}x'\mathrm{d}v',$$

and assume for notational simplicity that $f \equiv 1$ is the stationary state. Moreover, the semigroup preserves mass and is non-negative.

Lemma 7.1. *Let e^{tG} be a mass-conserving semigroup on the distributions over Γ with non-negative fundamental solution K_t and with stationary state 1. If there exists $T > 0$ and $\alpha > 0$ such that*

$$K_T(x,v;x',v') \geq \alpha, \qquad \forall (x,v) \in \Gamma, (x',v') \in \Gamma,$$

then there exist constants C and $\lambda > 0$ such that for non-negative f holds

$$\|\mathrm{e}^{tG}f - \Pi_g f\|_{TV} \leq C\mathrm{e}^{-\lambda t}\|f\|_{TV},$$

where Π_g is the mass-preserving projection to the stationary state 1.

Proof. After time T, split $\mathrm{e}^{tG}f$ as

$$\mathrm{e}^{tG}f = \alpha\,1 + (1-\alpha\|1\|_{TV})f_1,$$

where f_1 is a measure with the same mass as f. As f is non-negative and by the bound on the kernel K_T, we have that f_1 is indeed non-negative. Iterating, we find for $n \in \mathbb{N}$

$$\mathrm{e}^{nTG}f = (1-(1-\alpha\|1\|_{TV})^n)\|1\|_{TV}^{-1}\,1 + (1-\alpha\|1\|_{TV})^n f_n$$

for a non-negative measure f_n with the same mass. Now

$$\|\mathrm{e}^{nTG}f - \Pi_g f\|_{TV} \leq (1-\alpha\|1\|_{TV})^n\|\mathrm{e}^{nTG}f - f_n\|_{TV} \leq 2(1-\alpha\|1\|_{TV})^n\|f\|_{TV},$$

which shows the claimed decay. □

In comparison with Markov chains, we can understand the condition as saying that every state can be reached from any other state (irreducibility) at a uniform time T (aperiodicity).

A possible simple application is a kinetic equation for particles moving freely in a bounded domain, where the particles are either reflected or thermalised at the boundary. For the given configuration, we then only need to check that we can connect any two phase space positions. Another simple application is the degenerated linear Boltzmann equation, where we again just need to find a connecting trajectory.

7.2 Wasserstein contraction for the Fokker-Planck equation on the torus

The work in this section has been done in collaboration with Josephine Evans and Thomas Holding and follows the preprint [44].

7.2.1 Introduction

In this section, we prove contraction properties of the spatially periodic kinetic Fokker-Planck equation in the Wasserstein metric, and show to what extent the probabilistic technique of coupling can be used in such situations. This is of interest, both intrinsically, and in the broader context of analytic and probabilistic methods of proving convergence to equilibrium and contraction properties of Fokker-Planck equations. Recall the Wasserstein distance from optimal transport, which is defined as

$$\mathcal{W}_2(\mu,\nu) = \inf_{\pi\in\Pi_{\mu,\nu}} \left(\int |x-y|^2 \mathrm{d}\pi(x,y) \right)^{1/2},$$

where $\Pi_{\mu,\nu}$ is the set of all couplings between μ and ν.

With the contraction property we mean that for two solutions μ_t, ν_t we always have

$$\mathcal{W}_2(\mu_t,\nu_t) \le C\mathrm{e}^{-\mu t}\mathcal{W}_2(\mu_0,\nu_0)$$

for suitable constants C and $\mu > 0$. By taking ν_t to be the equilibrium state, this immediately shows exponential relaxation.

As mentioned in the introduction, the Fokker-Planck equation has received much attention in the development of hypocoercivity. These works [42, 49, 113, 156], however, do not address the question of convergence or contraction in the Wasserstein metric $\mathcal{W}_2$, as this distance seems inaccessible from these analytic tools; the closest result is by Mischler and Mouhot [110], where $\mathcal{W}_1$ results are obtained by duality and space enlargement methods.

A second analytic approach to the study of the Fokker-Planck equation is the theory of gradient flows [79], in which the Fokker-Planck equation is identified with the steepest descent flow of an entropy functional in the Wasserstein space $\mathcal{W}_2$. The degeneracy in the diffusion of the space variable causes this theory to fail for the kinetic Fokker-Plank equation. For the normal Fokker-Planck equation, Bolley, Gentil and Guillin [18] also developed analytic methods to show dissipation in the Wasserstein distance for non-gradient drifts.

A common probabilistic technique to show contraction or convergence is the construction of a *coupling* between two copies of the stochastic process that realises the desired bound on the metric between the laws. In the spatially homogeneous Fokker-Planck equation, the *synchronisation* coupling, where the infinitesimal motions of the noise are coupled together, gives contraction in Wasserstein metrics when the velocity potential is strongly convex. In the spatially inhomogeneous case with a confining potential, such a straightforward coupling so far has only been used to establish a contraction if the confining potential is quadratic, or a small perturbation thereof, see for example [19]. Establishing contraction in the Wasserstein metric for more general confining potentials is an open problem. In the spatially periodic case, results are even more limited. In this case, the synchronisation coupling does not cause the spatial distance on the torus to decay. In contrast to the analytic setting, where having the spatial variable on the torus makes computations simpler, the spatially periodic case is more difficult in the probabilistic case.

In this section, we study the contraction properties in the Wasserstein metric of the kinetic Fokker-Planck equation with spatial variables on the torus. Despite the simplicity of this equation, this question has not been answered in the literature, to my knowledge. A second goal is to understand what difficulties might explain the lack of previous results. In higher dimensions the different spatial directions decouple so that we can reduce the problem to the one-dimensional case.

The kinetic Fokker-Planck equation describes the law of a particle moving in the phase space $\mathbb{T} \times \mathbb{R}$ whose location in the phase space is (X_t, V_t) and evolves by the stochastic differential equation (SDE)

$$\begin{cases} \mathrm{d}X_t = V_t \mathrm{d}t, \\ \mathrm{d}V_t = -\lambda V_t \mathrm{d}t + \mathrm{d}W_t, \end{cases} \tag{7.1}$$

where $\lambda > 0$ is the relaxation rate in the velocity variable, $\mathrm{d}W_t$ is standard white noise and the spatial variable is in the torus $\mathbb{T} = \mathbb{R}/(2\pi L\mathbb{Z})$ of length $2\pi L$ for $L > 0$.

The corresponding measure μ_t on $\Gamma = \mathbb{T} \times \mathbb{R}$ evolves as

$$\partial_t \mu_t + v\partial_x \mu_t = \partial_v \left[\lambda v \mu_t + \frac{1}{2}\partial_v \mu_t \right], \tag{7.2}$$

where this equation is considered in the weak sense.

We show exponential decay of the distance between two solutions (7.2).

Theorem 7.2. *If $(\mu_t)_{t\geq 0}$ and $(\nu_t)_{t\geq 0}$ are two solutions in $\mathcal{M}(\Gamma)$ to the kinetic Fokker-Planck equation* (7.2)*, then we have*

$$\mathcal{W}_2(\mu_t, \nu_t) \leq \left(\mathrm{e}^{-\lambda t} + c\, \mathrm{e}^{-t/4\lambda^2 L^2} \right) \mathcal{W}_2(\mu_0, \nu_0)$$

for a constant c only depending on L.

The key idea is to study the stochastic trajectory leading to a later time t. After fixing the velocity variable at time t, the spatial variable has enough randomness left to allow such a coupling. This approach is not based on a functional inequality that is integrated over time and in fact the evolution is not a contraction semigroup. We can show the lack of coercivity directly using the explicit solution to the SDE.

Proposition 7.3. *The semigroup of the kinetic Fokker-Planck operator is not a contraction in the Wasserstein distance, i.e. there is no $\lambda > 0$ such that*

$$\mathcal{W}_2(\mu_t, \nu_t) \leq \mathrm{e}^{-\lambda t} \mathcal{W}_2(\mu_0, \nu_0).$$

In order to construct a coupling showing convergence in the Wasserstein distance, random variables (X_t^i, V_t^i) are constructed for $t \in \mathbb{R}^+$ and $i = 1, 2$ such that (X_t^1, V_t^1) has law μ_t and (X_t^2, V_t^2) has law ν_t. Then, for $t \in \mathbb{R}^+$, the coupling $((X_t^1, V_t^1), (X_t^2, V_t^2))$ gives an upper bound of the Wasserstein distance $\mathcal{W}_2(\mu_t, \nu_t)$.

The stochastic differential equation (7.1) motivates us to look at couplings where (X_t^i, V_t^i) are continuous Markov processes with initial distribution μ_0 and ν_0, respectively, and whose transition semigroup is determined by (7.1). For such couplings, we can consider a more restrictive class of couplings.

Definition 7.4 (co-adapted coupling)**.** *The coupling $((X_t^1, V_t^1), (X_t^2, V_t^2))$ is co-adapted if, for $i = 1, 2$, under the filtration $\mathcal{F}$ generated by the coupling $((X_t^1, V_t^1), (X_t^2, V_t^2))$, the process (X_t^i, V_t^i) is a continuous Markov process whose transition semigroup is determined by* (7.1).

This is an important subclass of couplings, which contains many natural couplings. An even more restrictive subclass is the class of Markovian couplings, where additionally the coupling itself is imposed to be Markovian. The existence and obtainable convergence behaviour of co-adapted couplings has already been studied in different cases, e.g. [25, 30, 85]. Note that the co-adapted coupling is equivalent to the condition that the filtration generated by (X_t^i, V_t^i) is immersed in the filtration generated by the coupling, which motivates Kendall [81] to call such couplings *immersed couplings*.

By adapting the reflection/synchronisation coupling, we can still obtain exponential convergence but with a loss in the dependence on the initial data.

Theorem 7.5. *Given initial distributions μ_0 and ν_0, there exists a co-adapted coupling $((X_t^1, V_t^1), (X_t^2, V_t^2))$ such that*

$$\begin{aligned} \mathcal{W}_2(\mu_t, \nu_t) &\leq \left(\mathbb{E}\left[|X_t^1 - X_t^2|_{\mathbb{T}}^2 + (V_t^1 - V_t^2)^2\right]\right)^{1/2} \\ &\leq C\zeta(t)\left(\sqrt{\mathcal{W}_2(\mu_0, \nu_0)} + \mathcal{W}_2(\mu_0, \nu_0)\right), \end{aligned}$$

where

$$\zeta(t) = \begin{cases} \mathrm{e}^{-\min(2\lambda, 1/(2\lambda^2 L^2))t} & \text{if } 4L^2\lambda^3 \neq 1 \\ \mathrm{e}^{-2\lambda t}(1+t) & \text{if } 4L^2\lambda^3 = 1 \end{cases}$$

and C is a constant that depends only on λ and L.

Here, we used the notation $|X_t^1 - X_t^2|_{\mathbb{T}}$ to emphasise that this is the distance on the torus $\mathbb{T}$. In fact, the filtrations generated by (X^1, V^1) and (X^2, V^2) agree. Such a coupling is called an equi-filtration by Kendall [81].

Remark 7.6. This achieves the same exponential decay rate as the non-Markovian argument, except for the case $4L^2\lambda^3 = 1$, when the spatial and velocity decay rates coincide and we have an additional polynomial factor.

In general, the loss in the dependence is necessary.

Theorem 7.7. *Suppose there exists a function $\alpha : \mathbb{R}^+ \mapsto \mathbb{R}^+$ and a constant $\gamma > 0$ such that for all initial distributions μ_0 and ν_0 there exists a co-adapted coupling $((X_t^1, V_t^1), (X_t^2, V_t^2))$ such that*

$$\left(\mathbb{E}\left[|X_t^1 - X_t^2|_{\mathbb{T}}^2 + (V_t^1 - V_t^2)^2\right]\right)^{1/2} \le \alpha(\mathcal{W}_2(\mu_0, \nu_0))\mathrm{e}^{-\gamma t}.$$

Then, there exists a constant C such that for $z \in (0, \pi L]$ we have the following lower bound on the dependence on the initial distance

$$\alpha(z) \ge C\sqrt{z}.$$

The idea is to focus on a drift-corrected position on the torus, which evolves as a Brownian motion. By stopping the Brownian motion at a large distance we can then prove the claimed lower bound.

This shows that a simple hypocoercivity argument on a Markovian coupling cannot work in $\mathcal{W}_2$. More precisely, there cannot exist a semigroup P on the probability measures over $(\mathbb{T} \times \mathbb{R})^{\times 2}$, whose marginals behave like the solution of (7.1) and which satisfies $H(P_t(\pi)) \le cH(\pi)\mathrm{e}^{-\gamma t}$ for $H^2(\pi) = \int[(X^1 - X^2)^2 + (V^1 - V^2)^2]\mathrm{d}\pi(X^1, V^1, X^2, V^2)$. Otherwise, the Markov process associated to P would be a coupling contradicting Theorem 7.7.

7.2.2 Setup

The stochastic differential equation (7.1) has an explicit solution when posed in $\mathbb{R}^2$. For clarity, we will use $\hat{X}$ if we consider X on $\mathbb{R}$ rather than the torus. The explicit solution is

$$\begin{aligned} \hat{X}_t &= \hat{X}_0 + \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda t})V_0 + \int_0^t \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda(t-s)})\,\mathrm{d}W_s, \\ V_t &= \mathrm{e}^{-\lambda t}V_0 + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}W_s, \end{aligned} \tag{7.3}$$

where W_t is the common Brownian motion. In this, we separate the stochastic driving as (A_t, B_t) given by the stochastic integrals

$$\begin{aligned} A_t &= \int_0^t \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda(t-s)})\mathrm{d}W_s, \\ B_t &= \int_0^t \mathrm{e}^{-\lambda(t-s)}\mathrm{d}W_s, \end{aligned}$$

which evolve as a vector in $\mathbb{R}^2$ with the common Brownian motion W_t. By Itō's isometry, (A_t, B_t) is a Gaussian random variable with covariance matrix $\Sigma(t)$ given by

$$\Sigma_{AA}(t) = \frac{1}{\lambda^2}\left[t - \frac{2}{\lambda}(1 - \mathrm{e}^{-\lambda t}) + \frac{1}{2\lambda}(1 - \mathrm{e}^{-2\lambda t})\right], \tag{7.4}$$

$$\Sigma_{AB}(t) = \frac{1}{\lambda^2}\left[(1 - \mathrm{e}^{-\lambda t}) - \frac{1}{2}(1 - \mathrm{e}^{-2\lambda t})\right], \tag{7.5}$$

$$\Sigma_{BB}(t) = \frac{1}{2\lambda}(1 - \mathrm{e}^{-2\lambda t}). \tag{7.6}$$

From this we calculate that the conditional distribution of A_t given B_t is a Gaussian with variance $\Sigma_{AA}(t) - \Sigma_{AB}^2(t)\Sigma_{BB}^{-1}(t)$ and mean $\mu_{A|B}(t,b) = \Sigma_{AB}(t)\Sigma_{BB}^{-1}(t)b$. We write $g_{A|B}$ for the conditional density of A given B and g_B for the marginal density of B. Hence,

$$g(t,a,b) = g_{A|B}(t,a,b)g_B(t,b) \tag{7.7}$$

is the joint density of A and B.

The last part of the setup is the change of variables, which we will need for the Markovian coupling in Section 7.2.4. We define new coordinates (Y, V) by taking the drift away

$$\begin{cases} Y = X + \dfrac{1}{\lambda}V, \\ V = V. \end{cases} \tag{7.8}$$

This change is motivated by the explicit formulas (7.3) from which we see that Y is the limit of X_t as $t \to \infty$ without additional noise. In the new variables, (7.1) becomes

$$\begin{cases} \mathrm{d}Y_t = \dfrac{1}{\lambda}\mathrm{d}W_t, \\ \mathrm{d}V_t = -\lambda V_t \mathrm{d}t + \mathrm{d}W_t, \end{cases}$$

for the common Brownian motion W_t. Note that the motion of Y_t does not explicitly depend upon V_t and that Y_t is a Brownian motion on the torus.

It remains to show that these new coordinates define an equivalent norm on $\mathbb{T} \times \mathbb{R}$. This follows from the triangle inequality as

$$|X^1 - X^2|_{\mathbb{T}} + |V^1 - V^2| \leq |Y^1 - Y^2|_{\mathbb{T}} + \left(1 + \frac{1}{\lambda}\right)|V^1 - V^2|$$

and the other direction is similar. Thus, the two norms are equivalent up to a constant factor that only depends on λ.

7.2.3 Non-Markovian Coupling

We wish to estimate how much the spatial variable will spread out over time. We will then use this to construct a coupling at a fixed time t which exploits the fact that a proportion of the spatial density is distributed uniformly. In order to do this, we give a lemma on the spreading of a Gaussian density wrapped on the torus.

Lemma 7.8. *For $\sigma^2 > 2L^2 \log(3)$ consider the Gaussian density h on $\mathbb{R}$ given by*

$$h(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-x^2/2\sigma^2}$$

and wrap it onto the torus $\mathbb{T}$, i.e. define the density Qh on $\mathbb{T}$ by

$$(Qh)(x) = \sum_{n\in\mathbb{Z}} h(x + 2\pi L n). \tag{7.9}$$

We have the following estimate on the spatial spreading

$$Qh(x) \geq \frac{\beta}{2\pi L},$$

where

$$1 - \beta := \frac{2\mathrm{e}^{-\sigma^2/2L^2}}{1 - \mathrm{e}^{-\sigma^2/2L^2}} \in (0, 1).$$

Proof. We define the Fourier transform of a function on $\mathbb{T}$ to be

$$(\mathcal{F}g)(k) = \int_{\mathbb{T}} \mathrm{e}^{-\mathrm{i}kx/L} g(x)\,\mathrm{d}x,$$

where

$$\int_{\mathbb{T}} g(x)\,\mathrm{d}x = \int_0^{2\pi L} g(x)\,\mathrm{d}x.$$

By the definition of Q, the Fourier transform of Qh is given by

$$\begin{aligned}
(\mathcal{F}Qh)(k) &= \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} h(x + 2\pi L n)\mathrm{e}^{-\mathrm{i}kx/L}\,\mathrm{d}x \\
&= \int_{\mathbb{R}} h(x)\mathrm{e}^{-\mathrm{i}kx/L}\,\mathrm{d}x \\
&= \exp\left(-\frac{k^2\sigma^2}{2L^2}\right),
\end{aligned}$$

where we have used the standard Fourier transformation of a Gaussian.

We deduce from the Fourier series representation that

$$Qh(x) - \frac{\beta}{2\pi L} = \frac{1}{2\pi L} \sum_{|k|\geq 1} \mathrm{e}^{-k^2\sigma^2/2L^2 + \mathrm{i}kx/L} + \frac{1-\beta}{2\pi L}.$$

The lemma claims that this is positive. For this it is sufficient to show that

$$\left| \sum_{|k|\geq 1} \mathrm{e}^{-k^2\sigma^2/2L^2+\mathrm{i}kx/L} \right| \leq 1-\beta.$$

We estimate the left hand side by

$$\left| \sum_{|k|\geq 1} \mathrm{e}^{-k^2\sigma^2/2L^2+\mathrm{i}kx/L} \right| \leq 2\sum_{k\geq 1} \mathrm{e}^{-k\sigma^2/2L^2} = 1-\beta,$$

where the final equality follows from summing the geometric series. □

We can now construct a coupling at time t, allowing to prove exponential decay in the Wasserstein distance.

Lemma 7.9. *Let $t \geq 0$ be large enough so that the variance of $g_{A|B}$ is greater than $2L\log(3)$, and β be such that*

$$(Qg_{A|B})(t,a,b) \geq \frac{\beta}{2\pi L},$$

where $g_{A|B}$ is defined by (7.7) above. Let μ_t and ν_t be the distribution of the solution to the kinetic Fokker-Planck equation (7.2) with deterministic initial data $\mu_0 = \delta_{x_0^1,v_0^1}$ and $\nu_0 = \delta_{x_0^2,v_0^2}$ at time t, respectively. Then there exists a coupling $((X_t^1,V_t^1),(X_t^2,V_t^2))$ between μ_t and ν_t satisfying

$$\mathbb{E}\left[(V_t^1-V_t^2)^2\right] = \mathrm{e}^{-2\lambda t}\left[(v_0^1-v_0^2)^2\right]$$

and

$$\mathbb{E}\left[|X_t^1-X_t^2|_{\mathbb{T}}^2\right] \leq 2(1-\beta)\left[|x_0^1-x_0^2|_{\mathbb{T}}^2 + \frac{1}{\lambda^2}(v_0^1-v_0^2)^2\right].$$

Proof. Let us construct such a coupling. Since we have seen that $g_{A|B}$ is a Gaussian density with variance $\sigma^2 = \Sigma_{AA}(t) - \Sigma_{AB}^2(t)\Sigma_{BB}^{-1}(t)$, we can use Lemma 7.8 to split the distribution $Qg_{A|B}$ as

$$Qg_{A|B}(t,a,b) = \frac{\beta}{2\pi L} + (1-\beta)s(t,a,b).$$

Then, by assumption, s is again a probability density for the variable a on the torus $\mathbb{T}$. We now consider the torus as a subset of $\mathbb{R}$ and then $Qg_{A|B}$ and s are probability density functions supported on $[0,2\pi L]$. Let B be an independent random variable with density $g_B(t,b)$, let Z be an independent uniform random variable over $[0,1]$ and let U be an independent uniform random variable over the torus. Finally, let S be a random variable on $\mathbb{R}$ with density $s(t,\cdot,B)$, viewed as a density function on $\mathbb{R}$, only depending on B.

With this, define the random parts A^1, A^2 of X_t^1, X_t^2 as

$$
\begin{aligned}
A^1 =& 1_{Z\le\beta}\left[U - x_0^1 - \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda t})v_0^1\right] + 1_{\beta>Z}S,\\
A^2 =& 1_{Z\le\beta}\left[U - x_0^2 - \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda t})v_0^2\right] + 1_{\beta>Z}S.
\end{aligned}
$$

We then construct $(\hat{X}_t^1, V_t^1)$ defined by

$$
\begin{aligned}
\hat{X}_t^1 &= x_0^1 + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda t})v_0^1 + A^1,\\
V_t^1 &= \mathrm{e}^{-\lambda t}v_0^1 + B,
\end{aligned}
$$

and $(\hat{X}_t^2, V_t^2)$ defined by

$$
\begin{aligned}
\hat{X}_t^2 &= x_0^2 + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda t})v_0^2 + A^2,\\
V_t^2 &= \mathrm{e}^{-\lambda t}v_0^2 + B.
\end{aligned}
$$

We then construct X_t^i by wrapping $\hat{X}_t^i$ onto the torus, i.e. $X_t^i \in [0, 2\pi L)$ and $X_t^i \equiv \hat{X}_t^i$ mod $2\pi L$. By construction, the pairs (X^i, V^i) have the right laws so they form a valid coupling.

We find

$$
\mathbb{E}\left[(V_t^1 - V_t^2)^2\right] = \mathrm{e}^{-2\lambda t}\left[(v_0^1 - v_0^2)^2\right]
$$

and

$$
\mathbb{E}\left[|X_t^1 - X_t^2|_{\mathbb{T}}^2\right] = (1-\beta)\left[\left|x_0^1 - x_0^2 + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda t})(v_0^1 - v_0^2)\right|_{\mathbb{T}}^2\right]
$$

and we can use Young's inequality to find the claimed control. □

We now put these two lemmas together to prove Theorem 7.2, which states exponential convergence in the Wasserstein $\mathcal{W}_2$ distance.

Proof of Theorem 7.2. We first show that we can reduce the result to the case of deterministic initial conditions. We denote $\mu_t^{x,v}$ to be the law of the solution to the SDE initialized at (x, v). Suppose we know that

$$
\mathcal{W}_2(\mu_t^{x_1,v_1}, \mu_t^{x_1,v_2}) \le \omega(t) d((x_1, v_1), (x_1, v_2)).
$$

Then, given any coupling π of initial measures μ_0, ν_0, we have

$$
\begin{aligned}
\mathcal{W}_2(\mu_t, \nu_t)^2 &\le \int_{(\mathbb{T}\times\mathbb{R})^2} \mathcal{W}_2(\mu_t^{x_1,v_1}, \mu_t^{x_2,v_2})^2 \mathrm{d}\pi((x_1, v_1), (x_2, v_2))\\
&\le \omega(t)^2 \int_{(\mathbb{T}\times\mathbb{R})^2} d((x_1, v_1), (x_2, v_2))^2 \mathrm{d}\pi((x_1, v_1), (x_2, v_2)).
\end{aligned}
$$

Taking an infimum over π shows that this implies

$$\mathcal{W}_2(\mu_t,\nu_t) \le \omega(t)\mathcal{W}_2(\mu_0,\nu_0).$$

Given any initial points $((x_0^1,v_0^1),(x_0^2,v_0^2))$, we let $((X_t^1,V_t^1),(X_t^2,V_t^2))$ be the coupling between μ_t and ν_t from Lemma 7.9. By explicitly calculating the variance of the distribution of $A|B$ using (7.4), (7.5) and (7.6), we see that the variance grows asymptotically as t/λ^2. Hence by Lemma 7.8 we can choose β so that $1-\beta \to 0$ exponentially fast with rate $1/2\lambda^2L^2$. This, combined with the control from the second lemma, shows that

$$\mathbb{E}\left[(V_t^1-V_t^2)^2+|X_t^1-X_t^2|_{\mathbb{T}}^2\right] \le \left(\mathrm{e}^{-2\lambda t}+c\,\mathrm{e}^{-t/2\lambda^2L^2}\right)\left[(v_0^1-v_0^2)^2+|x_0^1-x_0^2|_{\mathbb{T}}^2\right]. \qquad \square$$

The explicit solution also allows to prove that the evolution is not a contraction semigroup.

Proof of Proposition 7.3. We will prove the theorem by contradiction. Suppose $\gamma > 0$ and let $a \neq b$ be two distinct points on the torus. Consider the initial measures

$$\mu_0 = \delta_{x=a}\delta_{v=0}$$

and

$$\nu_0 = \delta_{x=b}\delta_{v=0}.$$

Then the distance is $\mathcal{W}_2(\mu_0,\nu_0) = |a-b|_{\mathbb{T}}$.

At time t the spatial distribution of μ_t and ν_t, interpreted in $\mathbb{R}$, is a Gaussian with variance Σ_{AA}, which, by the explicit formula (7.4), can be bounded as

$$\Sigma_{AA}(t) \le C_A t^2$$

for a constant C_A and $t \le 1$.

Hence, if $d > 0$ and $t \le 1$, the spatial spreading is controlled as

$$\begin{aligned}\mu_t((\mathbb{T}\setminus[a-d,a+d])\times\mathbb{R}) &\le \frac{2\Sigma_{AA}(t)}{d\sqrt{2\pi}}\exp\left(\frac{-d^2}{2\Sigma_{AA}^2(t)}\right)\\ &\le C_1\frac{t^2}{d}\exp\left(-C_2\frac{d^2}{t^4}\right)\end{aligned}$$

for positive constants C_1 and C_2, where we have used the standard tail bound for the Gaussian distribution (see e.g. [112, Lemma 12.9]).

For any $d > 0$ small enough that $a \pm d$ and $b \pm d$ do not wrap around the torus, any coupling between μ_t and ν_t must at least transfer the mass

$$1-\mu_t((\mathbb{T}\setminus[a-d,a+d])\times\mathbb{R})-\nu_t((\mathbb{T}\setminus[b-d,b+d])\times\mathbb{R})$$

between $[a-d,a+d]$ and $[b-d,b+d]$.

Hence the Wasserstein distance is bounded by

$$\mathcal{W}_2^2(\mu_t,\nu_t) \geq (|a-b|_{\mathbb{T}} - 2d)^2 \left(1 - 2C_1 \frac{t^2}{d} \exp\left(-C_2 \frac{d^2}{t^4}\right)\right).$$

Taking $d = |a-b|_{\mathbb{T}} t^{3/2}$ for t sufficiently small, this shows that

$$\mathcal{W}_2^2(\mu_t,\nu_t) \geq |a-b|_{\mathbb{T}}^2 (1-2t^{3/2})^2 \left(1 - \frac{2C_1}{|a-b|_{\mathbb{T}}} \sqrt{t} \exp\left(-\frac{C_2|a-b|_{\mathbb{T}}^2}{t}\right)\right).$$

However, for all small enough positive t, we have

$$(1-2t^{3/2})^2 > \mathrm{e}^{-\gamma t/2}$$

and

$$\left(1 - \frac{2C_1}{|a-b|_{\mathbb{T}}} \sqrt{t} \exp\left(-\frac{C_2|a-b|_{\mathbb{T}}^2}{t}\right)\right) > \mathrm{e}^{-\gamma t/2}$$

contradicting the assumed contraction. For the second estimate we use $\exp(-c/t) \leq (1 + c/t)^{-1} = t/(c+t)$. □

7.2.4 Co-adapted couplings

Existence

For Theorem 7.5, we construct a reflection/synchronisation coupling using the drift-corrected positions Y_t^i. As the positions are on the torus we can use a reflection coupling until Y_t^1 and Y_t^2 agree. Afterwards, we use a synchronisation coupling which keeps $Y_t^1 = Y_t^2$ and reduces the velocity distance.

For a formal definition let $((X_0^1,V_0^1),(X_0^2,V_0^2))$ be a coupling between μ and ν obtaining the Wasserstein distance (the existence of such a coupling is a standard result, see e.g. [158, Theorem 4.1.]).

We then define the evolution of this coupling in two stages. First, define (X_t^1,V_t^1) and (X_t^3,V_t^3) to be strong solutions to (7.1) with initial conditions $((X_0^1,V_0^1)$ and (X_0^2,V_0^2) respectively and driving Brownian motion W_t^1. Then we recall the definition of Y^i from (7.8), and define the stopping time $T := \inf\{t \geq 0 : Y_t^1 = Y_t^3\}$. Then we define a new process W_t^2 by

$$W_t^2 = \begin{cases} -W_t^1 & \text{if } t \leq T, \\ W_t^1 - 2W_T^1 & \text{if } t > T. \end{cases}$$

By the reflection principle, W^2 is a Brownian motion. We use this to define a new solution (X_t^2,V_t^2) to be the strong solution to (7.1) with driving Brownian motion W^2 and initial condition (X_0^2,V_0^2). Note now that $T = \inf\{t \geq 0 : Y_t^1 = Y_t^2\}$.

For the analysis we introduce the notation

$$\begin{aligned} M_t &= Y_t^1 - Y_t^2, \\ Z_t &= V_t^1 - V_t^2. \end{aligned}$$

Then by the construction the evolution is given by

$$\mathrm{d}M_t = \frac{2}{\lambda} 1_{t \leq T} \mathrm{d}W_t^1, \tag{7.10}$$

$$\mathrm{d}Z_t = -\lambda Z_t \mathrm{d}t + 2 \cdot 1_{t \leq T} \mathrm{d}W_t^1, \tag{7.11}$$

where M_t evolves on the torus $\mathbb{T}$.

As a first step we introduce a bound for T.

Lemma 7.10. *The stopping time T satisfies*

$$\mathbb{P}(T > t | M_0) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \exp\left(-\frac{(2k+1)^2}{2\lambda^2 L^2} t \right) \sin\left(\frac{(2k+1)|M_0|_{\mathbb{T}}}{2L} \right). \tag{7.12}$$

Proof. As M_t evolves on the torus, T is the first exit time of a Brownian motion starting at M_0 from the interval $(0, 2\pi L)$. See [112, (7.14-7.15)], from which the claim follows after rescaling to incorporate the $2/\lambda$ factor. □

Remark 7.11. The second expression in (7.12) is obtained by solving the heat equation on $[0, 2\pi L]$ with Dirichlet boundary conditions and initial condition δ_{M_0}.

Lemma 7.12. *There exists a constant C such that for any $t > 0$ the following holds*

$$\mathbb{P}(T > t | M_0) \leq C |M_0|_{\mathbb{T}} (1 + t^{-1/2}) \mathrm{e}^{-t/(2\lambda^2 L^2)}. \tag{7.13}$$

Proof. Using (7.12) and the inequality $\sin(x) \leq x$ for $x \geq 0$, we have

$$\begin{aligned} \mathbb{P}(T > t | M_0) &\leq \frac{4}{\pi} \mathrm{e}^{-t/(2\lambda^2 L^2)} \sum_{k=0}^{\infty} \frac{|M_0|_{\mathbb{T}}}{2L} \frac{2k+1}{2k+1} \mathrm{e}^{-4k^2 t/(2\lambda^2 L^2)} \\ &\leq \frac{2}{\pi L} |M_0|_{\mathbb{T}} \mathrm{e}^{-t/(2\lambda^2 L^2)} \left(1 + \int_0^\infty \mathrm{e}^{-4u^2 t/(2\lambda^2 L^2)} \mathrm{d}u \right) \\ &= \frac{2}{\pi L} |M_0|_{\mathbb{T}} \mathrm{e}^{-t/(2\lambda^2 L^2)} \left(1 + \sqrt{\frac{\pi}{8t/(\lambda^2 L^2)}} \right) \\ &\leq C |M_0|_{\mathbb{T}} (1 + t^{-1/2}) \mathrm{e}^{-t/(2\lambda^2 L^2)}, \end{aligned}$$

where on the second line we have bounded the sum by an integral. □

Using these simple estimates, we now study the convergence rate of the coupling.

Lemma 7.13. *There exists a constants C such that for any $t \geq 0$ we have the bound*

$$\mathbb{E}\left[|M_t|_{\mathbb{T}}^2 + |Z_t|^2 \big| (Z_0, M_0)\right] \leq |Z_0|^2 \mathrm{e}^{-2\lambda t} + \begin{cases} C|M_0|_{\mathbb{T}} \mathrm{e}^{-2\lambda t} & \text{if } 2\lambda < 1/(2\lambda^2 L^2), \\ C|M_0|_{\mathbb{T}}(1+t)\mathrm{e}^{-2\lambda t} & \text{if } 2\lambda = 1/(2\lambda^2 L^2), \\ C|M_0|_{\mathbb{T}} \mathrm{e}^{-t/(2\lambda^2 L^2)} & \text{if } 2\lambda > 1/(2\lambda^2 L^2). \end{cases}$$

Proof. Without loss of generality we may assume that Z_0 and M_0 are deterministic in order to avoid writing the conditional expectation.

Applying Itō's lemma, we find from (7.11) that

$$\mathrm{d}|Z_t|^2 = -2\lambda |Z_t|^2 \mathrm{d}t + 4 \cdot 1_{t \leq T} Z_t \mathrm{d}W_t^1 + 2 \cdot 1_{t \leq T} \mathrm{d}t.$$

After taking expectations we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}|Z_t|^2 = -2\lambda \mathbb{E}|Z_t|^2 + 2\mathbb{P}(t \leq T). \tag{7.14}$$

By explicitly solving (7.14) and using Lemma 7.12, we obtain

$$\begin{aligned} \mathbb{E}|Z_t|^2 &= |Z_0|^2 \mathrm{e}^{-2\lambda t} + 2\mathrm{e}^{-2\lambda t} \int_0^t \mathrm{e}^{2\lambda s} \mathbb{P}(s \leq T)\,\mathrm{d}s \\ &\leq |Z_0|^2 \mathrm{e}^{-2\lambda t} + C|M_0|_{\mathbb{T}} \mathrm{e}^{-2\lambda t} \underbrace{\int_0^t \mathrm{e}^{(2\lambda - 1/(2\lambda^2 L^2))s}(1 + s^{-1/2})\,\mathrm{d}s}_{=:I_t}. \end{aligned}$$

Let us bound I_t. As the integrand is locally integrable, we have for a constant C

$$I_t \leq C\left(1 + \int_0^t \mathrm{e}^{(2\lambda - 1/(2\lambda^2 L^2))s}\,\mathrm{d}s\right).$$

Here the $s^{-1/2}$ term can be bounded by 1 for $s > 1$ and for $s \leq 1$ the additional contribution can be absorbed into the constant. To bound the remaining integral we consider three cases:

- $2\lambda < 1/(2\lambda^2 L^2)$: The integral (and I_t) are uniformly bounded, $I_t \leq C$.
- $2\lambda = 1/(2\lambda^2 L^2)$: The integrand is equal to 1 and $I_t \leq C(1+t)$.
- $2\lambda > 1/(2\lambda^2 L^2)$: The integrand grows and $I_t \leq C(1 + \mathrm{e}^{(2\lambda - 1/(2\lambda^2 L^2))t})$.

In each case we multiply I_t by $\mathrm{e}^{-2\lambda t}$ to obtain the decay rate. In the first two cases this gives the dominant term with $|M_0|_{\mathbb{T}}$ (as opposed to $|Z_0|$) dependence, while in the last case it is lower order than the $\mathrm{e}^{-t/(2\lambda^2 L^2)}$ decay we obtain from $\mathbb{E}|M_t|_{\mathbb{T}}^2$ below.

Next let us consider $\mathbb{E}|M_t|_{\mathbb{T}}^2$. Using the finite diameter of the torus we have the simple estimate

$$\mathbb{E}|M_t|_{\mathbb{T}}^2 \leq \pi^2 L^2\, \mathbb{P}(T > t).$$

We now handle the cases of small and large t separately. For $t \geq 1$ (say), we can use Lemma 7.12, to obtain

$$\mathbb{E}|M_t|_{\mathbb{T}}^2 \leq C|M_0|_{\mathbb{T}}\, \mathrm{e}^{-t/(2\lambda^2 L^2)}.$$

This leaves the case when $t \leq 1$ where (7.13) blows up. We instead use the martingale property of M_t. Without loss of generality we may assume that $M_0 \in [0, \pi L]$. Then as M_t is stopped at T we know that $M_t \in [0, 2\pi L]$ for all $t \geq 0$. Hence, for any $t \geq 0$,

$$\mathbb{E}|M_t|_{\mathbb{T}}^2 \leq \mathbb{E}|M_t|^2 \leq 2\pi L \mathbb{E} M_t = 2\pi L M_0 = 2\pi L |M_0|_{\mathbb{T}}$$

by the martingale property. Combining the $t \leq 1$ and $t \geq 1$ estimates we have

$$\mathbb{E}|M_t|_{\mathbb{T}}^2 \leq C|M_0|_{\mathbb{T}} \mathrm{e}^{-t/(2\lambda^2 L^2)} \quad \text{for } t \geq 0.$$

This together with the bound for $\mathbb{E}|Z_t|^2$ provides the claimed bounds of the lemma and completes its proof. □

Proof of Theorem 7.5. By the equivalence of the norms from (X, V) and (Y, V), we see that

$$\begin{aligned}
&\mathbb{E}\left(|X_t^1 - X_t^2|_{\mathbb{T}}^2 + |V_t^1 - V_t^2|^2\right) \leq \left(1 + \frac{1}{\lambda}\right) \mathbb{E}\left(|M_t|_{\mathbb{T}}^2 + |Z_t|^2\right) \\
&\leq C'\zeta(t)\mathbb{E}(|M_0|_{\mathbb{T}} + |Z_0|^2) \\
&\leq C\zeta(t)\mathbb{E}\left(\left(|X_0^1 - X_0^2|_{\mathbb{T}}^2 + |V_0^1 - V_0^2|^2\right)^{1/2} + \left(|X_0^1 - X_0^2|_{\mathbb{T}}^2 + |V_0^1 - V_0^2|^2\right)\right).
\end{aligned}$$

Here we used Lemma 7.13 to go between the first and second line, and to find the exponentially decreasing term ζ. The constants C and C' come from the constants in equivalence of norms. □

Optimality

In order to show Theorem 7.7, we focus on the drift-corrected positions Y_t^1 and Y_t^2 which behave like time-rescaled Brownian motion on the torus. We prove the following decay bound on their quadratic distance.

Proposition 7.14. *Suppose there exist functions $\alpha : (0, \pi L] \mapsto \mathbb{R}^+$ and $\zeta : [0, \infty) \mapsto \mathbb{R}^+$ with $\zeta \in L^1([0, \infty))$, such that, for any $z \in (0, \pi L]$ there exist two standard Brownian motions W_t^1 and W_t^2 on the torus $\mathbb{T} = \mathbb{R}/(2\pi L\mathbb{Z})$ with respect to a common filtration such that $|W_0^1 - W_0^2| = z$, and for $t \in \mathbb{R}^+$ it holds that*

$$\mathbb{E}\Big[|W_t^1 - W_t^2|_{\mathbb{T}}^2\Big] \leq (\alpha(z))^2 \zeta(t).$$

Then with a constant c only depending on L, the function α satisfies the bound

$$\alpha(z) \geq c\|\zeta\|_{L^1([0,\infty))}^{-1/2} \sqrt{z}.$$

From this Theorem 7.7 follows easily.

Proof of Theorem 7.7. Fix $z \in (0, \pi L]$ and consider the initial distributions $\mu = \delta_{X=0}\delta_{V=0}$ and $\nu = \delta_{X=z}\delta_{V=0}$. Between μ and ν, there is only one coupling and $\mathcal{W}_2(\mu, \nu) = z$.

If there exists a co-adapted coupling $((X_t^1, V_t^1), (X_t^2, V_t^2))$ satisfying the bound, then Y^1_{t/λ^2} and Y^2_{t/λ^2} are Brownian motions on the torus with a common filtration. Moreover,

$$\mathbb{E}[|Y_t^1 - Y_t^2|_{\mathbb{T}}^2] \leq C\,\mathbb{E}[|X_t^1 - X_t^2|_{\mathbb{T}}^2 + |V_t^1 - V_t^2|^2]$$

for a constant C only depending on λ. Hence we can apply Proposition 7.14 to find the claimed lower bound for α. $\square$

For the proof of Proposition 7.14, we first prove the following lemma.

Lemma 7.15. *Given two Brownian motions W_t^1 and W_t^1 on the torus with a common filtration, then there exists a numerical constant c such that*

$$\mathbb{E}[|W_t^1 - W_t^2|_{\mathbb{T}}^2] \geq c\,\mathrm{e}^{-2t/L^2}\mathbb{E}[|W_0^1 - W_0^2|_{\mathbb{T}}^2].$$

Proof. The natural (squared) metric $|x-y|_{\mathbb{T}}^2$ on the torus is not a global smooth function of $x, y \in \mathbb{R}$ as it takes $x, y \bmod 2\pi L$. Therefore we introduce the equivalent metric

$$d_{\mathbb{T}}^2(x,y) = L^2 \sin^2\left(\frac{x-y}{2L}\right),$$

which is a smooth function of $x, y \in \mathbb{R}$. Moreover, the constants of equivalence are independent of L, i.e. there exist numerical constants c_1 and c_2 such that

$$c_1|x-y|_{\mathbb{T}}^2 \leq d_{\mathbb{T}}^2(x,y) \leq c_2|x-y|_{\mathbb{T}}^2.$$

Now consider H_t defined by

$$H_t = L\sin\left(\frac{W_t^1 - W_t^2}{2L}\right)\exp\left(\frac{[W^1 - W^2]_t}{4L^2}\right).$$

As W_t^1 and W_t^2 are Brownian motions, their quadratic variation is controlled as $[W^1 - W^2]_t \leq 4t$. By Itō's lemma

$$\mathrm{d}H_t = \frac{1}{2}\cos\left(\frac{W_t^1 - W_t^2}{2L}\right)\exp\left(\frac{[W^1 - W^2]_t}{4L^2}\right)\mathrm{d}(W^1 - W^2)_t.$$

Therefore we may bound the quadratic variation of H by

$$\begin{aligned}[H]_t &= \int_0^t \frac{1}{4}\cos^2\left(\frac{W_t^1 - W_t^2}{2L}\right)\exp\left(\frac{[W^1 - W^2]_t}{2L^2}\right)\mathrm{d}[W^1 - W^2]_t \\ &\leq \int_0^t \exp\left(\frac{2t}{L^2}\right)\mathrm{d}t \\ &< \infty.\end{aligned}$$

Therefore, as also $|H_0| \leq L$, the local martingale H_t is a true martingale and by Jensen's inequality

$$\mathbb{E}\Big[|H_t|^2\Big] \geq \mathbb{E}\Big[|H_0|^2\Big].$$

Using the equivalence of two metrics, we thus find the required bound

$$\begin{aligned}\mathbb{E}[|W_t^1 - W_t^2|_{\mathbb{T}}^2] &\geq c_2^{-1}\mathbb{E}\left[|H_t|^2\exp\left(-\frac{[W^1 - W^2]_t}{2L^2}\right)\right] \\ &\geq c_2^{-1}\mathbb{E}\left[|H_0|^2\right]\exp\left(-\frac{2t}{L^2}\right) \\ &\geq c_1 c_2^{-1}\mathbb{E}[|W_0^1 - W_0^2|_{\mathbb{T}}^2]\exp\left(-\frac{2t}{L^2}\right).\end{aligned}$$

□

With this lemma we prove the remaining proposition.

Proof of Proposition 7.14. Fix $a \in (0,1)$, let $z \in (0, \pi L]$ be given, and by symmetry assume without loss of generality that $W_0^1 - W_0^2 = |W_0^1 - W_0^2| = z$. Then define the stopping time

$$T = \inf\Big\{t \geq 0 : W_t^1 - W_t^2 \notin (az, \pi L)\Big\}.$$

The distance can be bounded as

$$\mathbb{E}\Big[|W_t^1 - W_t^2|_{\mathbb{T}}^2\Big] \geq \mathbb{P}[T \geq t](az)^2.$$

As ζ is integrable, it must decay along a subsequence of times and thus T must be almost surely finite.

As W_t^1 and W_t^2, considered on $\mathbb{R}$, are continuous martingales, their difference is also a continuous martingale. By the construction of the stopping time, the stopped martingale $(W^1 - W^2)_{t \wedge T}$ is bounded by πL and the optional stopping theorem implies

$$\mathbb{P}\Big[W_T^1 - W_T^2 = \pi L\Big] = \frac{z - az}{\pi L - az}.$$

Since Brownian motions satisfy the strong Markov property, we find

$$
\begin{aligned}
\mathbb{E}\int_0^\infty |W_t^1 - W_t^2|_{\mathbb{T}}^2 \,\mathrm{d}t &\geq \mathbb{E}\int_T^\infty |W_t^1 - W_t^2|_{\mathbb{T}}^2 \,\mathrm{d}t \\
&\geq \mathbb{P}\Big[W_T^1 - W_T^2 = \pi L\Big]\,\mathbb{E}\Big[\int_T^\infty |W_t^1 - W_t^2|_{\mathbb{T}}^2 \,\mathrm{d}t \,\Big|\, W_T^1 - W_T^2 = \pi\Big] \\
&\geq \mathbb{P}\Big[W_T^1 - W_T^2 = \pi L\Big]\, c\,(\pi L)^2 \int_0^\infty \mathrm{e}^{-2t/L^2}\,\mathrm{d}t \\
&\geq \frac{z - az}{\pi L - az} c\,(\pi L)^2 \frac{L^2}{2} \\
&\geq C_a z
\end{aligned}
$$

for a constant C_a only depending on a and L, where the strong Markov property and then Lemma 7.15 are applied on the second line.

On the other hand, integrating the assumed bound gives

$$
\mathbb{E}\int_0^\infty |W_t^1 - W_t^2|_{\mathbb{T}}^2 \,\mathrm{d}t \;\leq\; (\alpha(z))^2 \int_0^\infty \zeta(t)\,\mathrm{d}t \;\leq\; (\alpha(z))^2 \|\zeta\|_{L^1([0,\infty))}.
$$

Hence

$$
C_a z \leq (\alpha(z))^2 \|\zeta\|_{L^1([0,\infty))}
$$

which is the claimed result. □

7.3 Spatially degenerate Fokker-Planck equation

In this section, we study the problem with a spatially degenerate relaxation, i.e. an evolution of the form

$$
\partial_t f = G f = (T + \sigma M) f,
$$

where σ is a non-negative spatial weight. Here T is again the transport operator and M is the relaxation operator. The weight σ models the amount of relaxation through the background and the degenerate case occurs if σ is not bounded below by a positive constant.

The case where M is the linear Boltzmann operator has been studied by Bernard and Salvarani [15] and Han-Kwan and Léautaud [67], where they prove that hypocoercivity is equivalent to the (uniform) geometric control condition. In order to define the condition, we let $(X_{x,v}(t), \Xi_{x,v}(t))$ be the characteristic of T starting from (x,v), i.e. for an Hamiltonian system it is the solution of

$$
\begin{cases}
\dfrac{\mathrm{d}X_{x,v}(t)}{\mathrm{d}t} = \dfrac{\partial H(X_{x,v}(t), \Xi_{x,v}(t))}{\partial \Xi}, \\
\dfrac{\mathrm{d}\Xi_{x,v}(t)}{\mathrm{d}t} = -\dfrac{\partial H(X_{x,v}(t), \Xi_{x,v}(t))}{\partial X}, \\
X_{x,v}(0) = x, \qquad \Xi_{x,v}(0) = v
\end{cases}
$$

with the Hamiltonian H.

Definition 7.16. *The weight function σ satisfies the uniform geometric control condition for the transport operator T, if there exist constants $T^* < \infty$ and $C > 0$ such that*

$$\int_0^{T^*} \sigma(X_{x,v}(s))\,\mathrm{d}s \geq C$$

for a.e. $(x,v) \in \Gamma$.

Intuitively, the condition imposes that every trajectory spends a significant time within the non-degenerate region where σ is non-vanishing.

Together with Frédéric Hérau, Harsha Hutridurga and Clément Mouhot, we established that the geometric control condition is also sufficient in the case that M is the Fokker-Planck operator. In contrast to the linear Boltzmann equation, I could establish that this condition is not necessary.

Intuitively, if the geometric control condition, then for any time T^*, there exists a sequence of initial data such that for a position (x,v) in their support $\int_0^{T^*} \sigma(X_{x,v}(s))\,\mathrm{d}s$ is arbitrary small. In the case of the linear Boltzmann equation, this provides a counter-example of exponential convergence. However, for the Fokker-Planck operator this can fail, as the gradient of the initial data can be more and more increasing.

We consider one space and spatial dimension, both over $\mathbb{R}$, i.e. $\Gamma = \mathbb{R}^2$. As equilibrium state, we impose

$$\mathcal{F}_\infty(x,v) = \pi^{-1}\mathrm{e}^{-x^2-v^2}.$$

We suppose that the transport operator T is generated by the Hamiltonian $H = x^2/2 + v^2/2$. Finally, we take the diffusion operator M as

$$Mf = \nabla_v \cdot (\nabla_v f + 2vf).$$

We finally suppose $\sigma(x) = x^2$, so that the geometric control condition is violated. Nevertheless, we have exponential decay, i.e. the generator G is hypocoercive.

Theorem 7.17. *In $L^2(\mathbb{R}^2, \mathcal{F}_\infty^{-1})$, the semigroup generated by*

$$G = T + \sigma M$$

is exponentially decaying to the equilibrium.

We shall equivalently consider the density h with respect to $\mathcal{F}_\infty$, i.e. $f = h\mathcal{F}_\infty$, which evolves as

$$\partial_t h + v\nabla_x h - x\nabla_v h = x^2\left(\Delta_v h - 2v\nabla_v h\right).$$

Expressed in h, the decay writes:

Theorem 7.18. *Let $A = x\partial_v$ and $B = v\partial_x - x\partial_v$ be the (unbounded) operators in $L^2(\mathbb{R}^2, \mathcal{F}_\infty)$. Then the semigroup $\mathrm{e}^{-t(A^*A+B)}$ is exponentially decaying to its equilibrium.*

We introduce the commutators

$$C_k = [C_{k-1}, B]$$

for $k \geq 1$ with $C_0 = A$.

The previous statement is proven in the framework of hypocoercivity [156, Theorem 24]. A simplified statement is:

Theorem 7.19. *Let H be a Hilbert space with the unbounded operators A and B with $B^* = B$. Let $L = A^*A + B$ and $K = \ker L$. Assume there exists $N_c \in \mathbb{N}$ such that C_{N_c+1} is bounded relative to $\{C_j\}_{0\leq j\leq N_c}$ and $\{C_jA\}_{0\leq j\leq N_c}$ and for $k \in \{0, \dots, N_c\}$*

(i) $[A, C_k]$ is bounded relative to $\{C_j\}_{0\leq j\leq k}$ and $\{C_jA\}_{0\leq j\leq k-1}$,

(ii) $[C_k, A^*]$ is bounded relative to I and $\{C_j\}_{0\leq j\leq k}$.

If further

$$D = \sum_{j=0}^{N_c} C_j^* C_j$$

is coercive, then there exist constants C and $\lambda > 0$ such that for $f \in H$ with $f \in K^\perp$ holds

$$\|\mathrm{e}^{-tL} f\| \leq C\mathrm{e}^{-\lambda t}\|f\|.$$

We note that

$$\begin{aligned}
C_1 &= x\nabla_x - v\nabla_v, \\
C_2 &= -2\left(v\nabla_x + x\nabla_v\right), \\
C_3 &= -4C_1.
\end{aligned}$$

Hence with $N_c = 2$, we can satisfy the boundedness of the commutators.

Lemma 7.20. *With $N_c = 2$, we have that C_{N_c+1} is bounded relative to C_1 and (i) and (ii) of Theorem 7.19 hold.*

Proof. As $C_{N_c+1} = C_3 = -4C_1$, the first part is trivial. For the second part note

$$\begin{aligned}
[A, C_1] &= -2x\nabla_v = -2A, \\
[A, C_2] &= -2C_1, \\
[C_0, A^*] &= 2x^2, \\
[C_1, A^*] &= -2x\nabla_v = -2A, \\
[C_2, A^*] &= -2\left(x\nabla_x - v\nabla_v + 2x^2 + 2v^2\right),
\end{aligned}$$

from which the result follows. □

Therefore, the claimed result follows from the following coercivity estimate, as the constant solution 1 is spanning the kernel of the generator.

Proposition 7.21. *Let D be as defined in Theorem 7.19 with $N_c = 2$. Then there exists $\kappa > 0$ such that*

$$\langle Dh, h\rangle \geq \kappa \|h\|$$

holds for h with $\langle h, 1\rangle = 0$.

We first note that D controls the weighted derivatives.

Lemma 7.22. *In $L^2(\mathbb{R}^2, \mathcal{F}_\infty)$ holds for $h \in L^2(\mathbb{R}^2, \mathcal{F}_\infty)$ that*

$$\|Dh\|^2 \geq \|x\nabla_x h\|^2 + \|v\nabla_v h\|^2 + \|x\nabla_v h\|^2 + \|v\nabla_x h\|^2.$$

Proof. We have

$$\begin{aligned}
\|Dh\|^2 &= \int_{x,v\in\mathbb{R}} \left[(x\nabla_v h)^2 + (x\nabla_x h - v\nabla_v h)^2 + 4(v\nabla_x h + x\nabla_v h)^2\right] \mathcal{F}_\infty(x,v)\,\mathrm{d}x\,\mathrm{d}v \\
&= \|x\nabla_x h\|^2 + \|v\nabla_v h\|^2 + \|x\nabla_v h\|^2 + \|v\nabla_x h\|^2 \\
&\quad + \int_{x,v\in\mathbb{R}} 3(v\nabla_x h + x\nabla_v h)^2 \mathcal{F}_\infty(x,v)\,\mathrm{d}x\,\mathrm{d}v,
\end{aligned}$$

which shows the claim. □

Hence it remains to proof the weighted Poincaré inequality.

Lemma 7.23. *There exists $c > 0$ such that if $\langle h, 1\rangle = 0$ then*

$$\|Dh\|^2 \geq c\|h\|^2$$

for $h \in L^2(\mathbb{R}^2, \mathcal{F}_\infty)$.

For the remaining Poincaré inequality, we use Hermite polynomials $(\mathrm{H}_n)_{n\in\mathbb{N}}$, which form an orthogonal basis of $L^2(\mathbb{R}, \mathrm{e}^{-x^2})$ and satisfy for $n, m \in \mathbb{N}$

$$\begin{gathered}
\int_{x\in\mathbb{R}} \mathrm{H}_n(x)\mathrm{H}_m(x)\mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi}2^n n!\delta_{nm}, \\
\mathrm{H}_n'(x) = 2n\mathrm{H}_{n-1}(x), \\
\mathrm{H}_{n+1}(x) = 2x\mathrm{H}_n(x) - \mathrm{H}_n'(x) = 2x\mathrm{H}_n(x) - 2n\mathrm{H}_{n-1}(x),
\end{gathered}$$

where we use the convention that $\mathrm{H}_{-n} \equiv 0$.

In one dimension, it allows us to show a refined inequality.

Lemma 7.24. *Let $f \in L^2(\mathbb{R}, \mathrm{e}^{-x^2})$ with expansion*

$$f(x) = \sum_{n\in\mathbb{N}} b_n \frac{\mathrm{H}_n(x)}{\pi^{1/4}\sqrt{2^n n!}}.$$

Then for small enough constants $c_e, c_o \in \mathbb{R}^+$ we have

$$\|x\nabla_x f\|^2_{L^2(\mathbb{R},\mathrm{e}^{-x^2})} \geq -c_o\alpha_1|b_1|^2 + (2 - c_e\alpha_2)|b_2|^2 + \sum_{n\geq 3, n\, even} c_e|b_n|^2 + \sum_{n\geq 3, n\, odd} c_o|b_n|^2$$

for

$$\alpha_n = 100\frac{(n+10)}{(n+2)^2}. \tag{7.15}$$

In terms of the expansion, we have

$$\|f\|_{L^2(\mathrm{e}^{-x^2})} = \sum_{n\in\mathbb{N}} |b_n|^2,$$

so that we have shown a Poincaré inequality apart from the term $-c_o\alpha_1|b_1|^2$. Indeed the proof shows that this missing term is necessary.

The key in proving this estimate is the following inequality for the sequence $(\alpha_n)_{n\in\mathbb{N}}$.

Lemma 7.25. *Let $(\alpha_n)_{n\in\mathbb{N}}$ be given by Equation (7.15). Then for small enough $c \in \mathbb{R}^+$ and $n \geq 2$ we have*

$$-n^2\alpha_n + n(n-1)\frac{\alpha_{n-2}}{1 + c\,\alpha_{n-2}} \geq 1.$$

Proof. Using the definition we find

$$\begin{aligned}
&-n^2\alpha_n + n(n-1)\frac{\alpha_{n-2}}{1 + c\,\alpha_{n-2}} - 1 \\
&\quad= \frac{n^4\left(-10000c + 99\right)}{\left(n+2\right)^2\left(100c\left(n+8\right) + n^2\right)} \\
&\quad+ \frac{n^3\left(-180100c + 2396\right) + n^2\left(-801200c - 404\right) + n\left(-3600c - 3200\right) - 3200c}{\left(n+2\right)^2\left(100c\left(n+8\right) + n^2\right)}
\end{aligned}$$

The denominator is always positive and in the second fraction we can bound the numerator for $n \geq 2$ as

$$\begin{aligned}
&n^3\left(-180100c + 2396\right) + n^2\left(-801200c - 404\right) + n\left(-3600c - 3200\right) - 3200c \\
&\geq n^3\left(-582000c + 1394\right)
\end{aligned}$$

For $c \leq 697/291000 \approx 0.0024$, this is non-negative as well as $-10000c + 99$ is non-negative. Hence both fractions are non-negative, which is the claimed result. $\square$

With this we can prove the claimed result.

Proof of Lemma 7.24. From the Hermite polynomials we have $x\nabla_x \mathrm{H}_n(x) = 2nx\mathrm{H}_{n-1}(x) = n\mathrm{H}_n(x) + 2n(n-1)\mathrm{H}_{n-2}(x)$ so that

$$\|x\nabla_x f\|^2 = \sum_{n\in\mathbb{N}} \left| nb_n + \sqrt{(n+1)(n+2)}b_{n+2} \right|^2.$$

For the even coefficients we find

$$\begin{aligned}
&\sum_{n\in\mathbb{N}, n \text{ even}} \left| nb_n + \sqrt{(n+1)(n+2)}b_{n+2} \right|^2 \\
&\qquad = 2|b_2|^2 + \sum_{n\geq 2, n \text{ even}} \left(\left| nb_n\sqrt{1+c_e\alpha_n} + \sqrt{(n+1)(n+2)}\frac{b_{n+2}}{\sqrt{1+c_e\alpha_n}} \right|^2 \right. \\
&\qquad\qquad \left. - c_e\alpha_n n^2|b_n|^2 + \frac{c_e\alpha_n}{1+c_e\alpha_n}(n+1)(n+2)|b_{n+2}|^2 \right) \\
&\qquad \geq (2 - c_e\alpha_2)|b_2|^2 + \sum_{n\geq 4, n \text{ even}} \left(-c_e n^2\alpha_n + \frac{c_e\alpha_{n-2}}{1+c_e\alpha_{n-2}} n(n-1) \right) |b_n|^2 \\
&\qquad \geq (2 - c_e\alpha_2)|b_2|^2 + \sum_{n\geq 4, n \text{ even}} c_e|b_n|^2,
\end{aligned}$$

where the last line follows from the Lemma 7.25 for small enough c_e.

Likewise for the even coefficients we find the lower bound

$$-c_o\alpha_1|b_1|^2 + \sum_{n\geq 3, n \text{ odd}} c_o|b_n|^2.$$

Combining the two cases gives the claimed result. □

To complete the estimate we need to use the mixed terms $x\nabla_v$ and $v\nabla_x$ to gain in the first two coefficients.

Lemma 7.26. *Assume $f \in L^2(\mathbb{R}^2, e^{-x^2-v^2})$ with expansion*

$$f = \sum_{n,m\in\mathbb{N}} b_{n,m} \frac{\mathrm{H}_n(x)}{\pi^{1/4}\sqrt{2^n n!}} \frac{\mathrm{H}_m(v)}{\pi^{1/4}\sqrt{2^m m!}}.$$

Then for $\gamma \geq 0$

$$\|x\nabla_v f\|_2^2 \geq \sum_{m\geq 1} \left(|b_{1,m}|^2 + \frac{\gamma}{1+\gamma}|b_{0,m}|^2 - 2\gamma|b_{2,m}|^2 \right)$$

and

$$\|v\nabla_x f\|_2^2 \geq \sum_{n\geq 1} \left(|b_{n,1}|^2 + \frac{\gamma}{1+\gamma}|b_{n,0}|^2 - 2\gamma|b_{n,2}|^2 \right)$$

Proof. With the convention of $b_{-1,m} = 0$ we find

$$x\nabla_v f = \sum_{n,m\in\mathbb{N}} \sqrt{m+1}\left(\sqrt{n}b_{n-1,m+1} + \sqrt{n+1}b_{n+1,m+1}\right) \frac{\mathrm{H}_n(x)}{\pi^{1/4}\sqrt{2^n n!}} \frac{\mathrm{H}_m(v)}{\pi^{1/4}\sqrt{2^m m!}}.$$

Hence

$$\begin{aligned}\|x\nabla_v f\|^2 &\geq \sum_{m\geq 1}\left(|b_{1,m}|^2 + (b_{0,m} + \sqrt{2}b_{2,m})^2\right)\\ &\geq \sum_{m\geq 1}\left(|b_{1,m}|^2 + \frac{\gamma}{1+\gamma}|b_{0,m}|^2 - 2\gamma|b_{2,m}|^2\right)\end{aligned}$$

and likewise for the other mixed operator. □

We can thus combine the operators to find the following lemma.

Lemma 7.27. *Assume $f \in L^2(\mathbb{R}^2, \mathrm{e}^{-x^2-v^2})$ with expansion*

$$f = \sum_{n,m\in\mathbb{N}} b_{n,m} \frac{\mathrm{H}_n(x)}{\pi^{1/4}\sqrt{2^n n!}} \frac{\mathrm{H}_m(v)}{\pi^{1/4}\sqrt{2^m m!}}.$$

Then there exists a constant $c > 0$ such that

$$\|x\nabla_x f\|^2 + \|x\nabla_v f\|^2 \geq c\sum_{n\in\mathbb{N}}\sum_{m\geq 1}|b_{n,m}|^2$$

and

$$\|v\nabla_v f\|^2 + \|v\nabla_x f\|^2 \geq c\sum_{n\geq 1}\sum_{m\in\mathbb{N}}|b_{n,m}|^2.$$

Proof. Combine Lemmas 7.24 and 7.26 to find

$$\begin{aligned}\|x\nabla_x f\|^2 + \|x\nabla_v f\|^2 \geq \sum_{m\geq 1}\Big(&-c_o\alpha_1|b_{1,m}|^2 + (2 - c_e\alpha_2)|b_{2,m}|^2\\ &+ \sum_{n\geq 3, n\text{even}} c_e|b_{n,m}|^2 + \sum_{n\geq 3, n\text{odd}} c_o|b_{n,m}|^2\\ &+ |b_{1,m}|^2 + \frac{\gamma}{1+\gamma}|b_{0,m}|^2 - 2\gamma|b_{2,m}|^2\Big).\end{aligned}$$

Hence for a suitable combination of small constants c_e, c_o, γ the result follows. The other direction is the same. □

This implies the claimed Poincaré inequality and thus the exponential decay.

Proof of Lemma 7.23. Given $h \in L^2$ with $\langle h, 1\rangle = 0$, expand it as

$$h = \sum_{n,m\in\mathbb{N}} b_{n,m} \frac{\mathrm{H}_n(x)}{\pi^{1/4}\sqrt{2^n n!}} \frac{\mathrm{H}_m(v)}{\pi^{1/4}\sqrt{2^m m!}}.$$

The condition $\langle h, 1\rangle$ then implies that $b_{0,0}$ vanishes and thus Lemma 7.27 shows the inequality for a positive constant. $\square$

8 Expansion into Hermite functions

This chapter studies the expansion into Hermite functions, which form an interesting spectral basis for data on unbounded domains. In particular, for the transport equation, the structure of the differentiation matrix allows a stable semi-discretisation.

8.1 Introduction

8.1.1 Motivation

Consider a general evolution equation

$$\begin{cases} \partial_t f = Af, \\ f(0,\cdot) = f_{\text{in}} \end{cases}$$

with an evolution operator A and initial data f_{in}.

For the numerical solution, most algorithms discretise f by an approximation f^N characterised by coefficients $(a_n)_{n=0}^{N-1}$ as

$$f^N(t,\cdot) = \sum_{n=0}^{N-1} a_n(t)\phi_n(\cdot),$$

where often the elements ϕ_n are fixed in time. In many cases, the partial differential equation (PDE) with the operator A is then reduced to an ordinary differential equation (ODE) for the coefficients $(a_n)_{n=0}^{N-1}$, which is then numerically solved. The resulting ODE is the semi-discretisation

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_n)_{n=0}^{N-1} = \mathsf{A}(a_n)_{n=0}^{N-1},$$

where A is the discretised evolution operator.

Popular choices are finite-differences or finite-elements, in which the discretised evolution operator is localised. Another interesting class are spectral methods, where A is not localised, but the approximation converges geometrically, i.e. much faster.

A crucial requirement in this process is numerical stability, which means that the discretised evolution operator A does not amplify discretisation and rounding errors in an uncontrolled way. In particular, the discretisation cannot have additional modes amplifying noise. For the transport equation $A = \partial_x$, this is a delicate issue, because the evolution operator has the spectrum along the imaginary axis and any positive eigenmode of the discretisation A ruins the stability. It turns out that the crucial property is the anti-selfadjointness of the differential operator, which needs to be shared with the discretisation. See the work by Hairer and Iserles [66] and Iserles [76] for a review.

The most used choice for a spectral method is the expansion into a Fourier series. In this basis, the differentiation matrix is anti-selfadjoint and diagonal. Moreover, using the fast Fourier transform (FFT), the expansion can be computed very fast. However, it assumes periodic boundary conditions. In contrast, many physical systems are modelled over $\mathbb{R}$ without periodic boundary conditions. For example in kinetic equations the velocity variable is normally over $\mathbb{R}$ and in the case of a confining potential the position is also modelled over $\mathbb{R}$.

A solution can be the use of Hermite functions $(\phi_n)_{n\in\mathbb{N}}$, which form an orthonormal basis of $L^2(\mathbb{R})$ and are given by

$$\phi_n(x) = \frac{\mathrm{H}_n(x)\mathrm{e}^{-x^2/2}}{\pi^{1/4}\sqrt{2^n n!}},$$

where $(\mathrm{H}_n)_{n\in\mathbb{N}}$ are the Hermite polynomials. Indeed, the differentiation is

$$\frac{\mathrm{d}\phi_n(x)}{\mathrm{d}x} = -\sqrt{\frac{n+1}{2}}\phi_{n+1}(x) + \sqrt{\frac{n}{2}}\phi_{n-1}(x),$$

which shows that the differentiation matrix is skew-symmetric, i.e. anti-selfadjoint. Moreover, the differentiation matrix is tri-diagonal allowing a fast computation.

Using the orthonormality, a function can be expanded as

$$a_n = \langle f, \phi_n \rangle = \int_{\mathbb{R}} f(x)\phi_n(x)\mathrm{d}x.$$

Naively, we can use a quadrature rule in order to compute the coefficients a_n for $n = 0, \dots, N-1$. However, using N nodes, this takes $O(N^2)$ operations, which is too slow for an iterative use.

Therefore, we consider the task of expanding a given function into Hermite functions as a first step and provide a solution for sufficiently localised functions.

For the application, we note that the basis can also be scaled, i.e. $[\lambda\phi_n(\lambda\cdot)]_{n\in\mathbb{N}}$ is also an orthonormal basis and this scaling is important for a meaningful expansion.

Finally, we remark that another important application is the Schrödinger equation, where the skew-symmetry of the differentiation matrix ensures unitarity. Moreover, using a splitting method, the fast expansion is needed in order to compute the expansion of $V(x)f(x)$, where V is the potential. In this case a localised potential V automatically ensures a localisation of the expanded function.

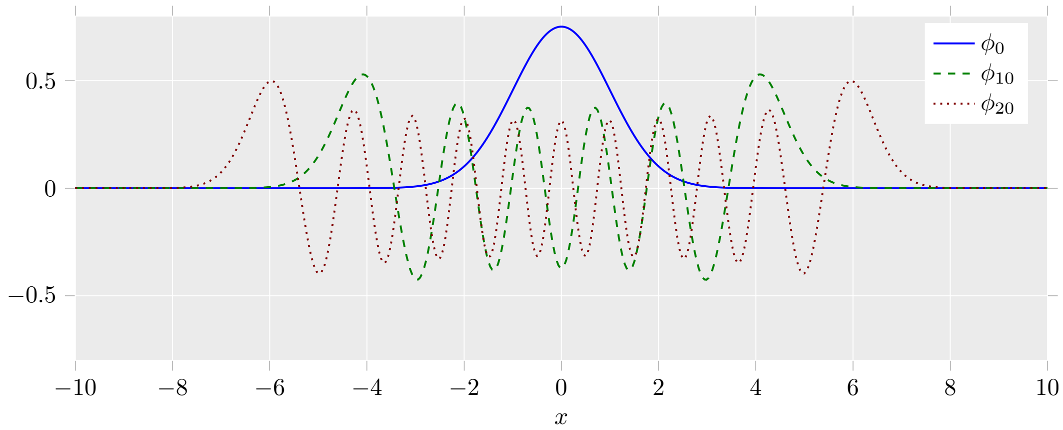


Figure 8.1 – Visualisation of Hermite functions.

8.1.2 Reexpansion

The approach follows the recent progress in the expansion into Jacobi polynomials. The common element is the use of the FFT, often in the form of the discrete cosine transform, in order to compute an expansion into Chebyshev polynomials and then relating the Chebyshev coefficients with the sought Jacobi coefficients.

For the last step, Driscoll and Healy [50] and Potts, Steidl and Tasche [132] used a successive expansion into Chebyshev polynomials with the three-point recurrence relation to compute the coefficients exactly up to rounding errors in $O(N \log^2 N)$. Alpert and Rokhlin [3] and Keiner [80] noticed that the connection coefficients can be well-approximated by a low-degree multipole expansion converting the coefficients in $O(N)$ up to a controlled error. Finally, Cantero and Iserles [27], Iserles [75] and Wang and Huybrechs [162] explored the observation that, for analytic data, only the near diagonal elements are relevant, yielding a fast and simple algorithm.

Normally, the interpolation points are chosen to be the roots of the N-th polynomial so that the Gaussian quadrature ensures a fast convergence, but like the Clenshaw-Curtis quadrature similar distributed nodes work as well. Asymptotically, the roots of any Jacobi polynomial are distributed as $(1-x^2)^{-1/2}\mathrm{d}x$ over $[-1,1]$, which suggests that the reexpansion works.

For the Hermite functions $(\phi_n)_{n\in\mathbb{N}}$, we recall that ϕ_N is oscillating in the region $[-\sqrt{2N}, \sqrt{2N}]$ and decaying exponentially outside this region [150], cf. Figure 8.1. Moreover, the distribution of the roots of ϕ_N is asymptotically proportional to $\sqrt{1-\frac{x^2}{2N}}\mathrm{d}x$ over the region. This motivates our approach to first expand over $[-\sqrt{2N}, \sqrt{2N}]$ and then to reexpand it. Instead of an initial expansion into Chebyshev polynomials, we reexpand into a Fourier series in order to keep closer to the distribution of zeros.

In fact, we will focus on sufficiently localised functions such that $g(x) := f(x)\mathrm{e}^{\alpha x^2}$ for $\alpha \geq 0$ decays to zero fast and expand g into a Fourier series. In the case of $\alpha = 1/2$, we can rearrange the weights, so that it looks like an expansion into Hermite polynomials $(\mathrm{H}_n)_{n\in\mathbb{N}}$.

Section 8.2 shows that an expansion into Fourier series converges fast for a sufficiently localised function, which we call Sombrero phenomenon. Finally, in Section 8.3, we show how to relate the obtained coefficients from the Fourier series to the coefficients of the Hermite expansion.

8.2 Sombrero phenomenon

As the first step, we expand a function f over the interval $[-A, A]$ into a Fourier series and notice that in our application f is not periodic. Common wisdom tells that a Fourier series is not appropriate in this case. While a slow asymptotic decay rate is true, it is misleading, because for sufficiently decaying functions the decay of the Fourier coefficients is much faster up to an error much smaller than machine accuracy.

More formally, this section considers a function $f : [-A, A] \mapsto \mathbb{C}$, which decays towards $\pm A$. Such a function can be represented by the Fourier series

$$f(x) = \sum_{n\in\mathbb{N}} \hat{f}_n \mathrm{e}^{\mathrm{i}\pi nx/A},$$

where

$$\hat{f}_n = \frac{1}{2A} \int_{-A}^{A} \mathrm{e}^{-\mathrm{i}\pi nx/A} f(x)\mathrm{d}x.$$

As an example consider the functions $x \to \sin(x^2)\mathrm{e}^{-x^2}$ and $x \to (1 + x^2)^{-1}\mathrm{e}^{-x^2}$, which are practically zero around possible endpoints ± 6. Their coefficients are shown in Figure 8.2. While the slow asymptotic decay is true, it only appears as Sombrero flat after a much faster decay, which follows the usual geometric decay.

In fact, we can relate the Sombrero flat to the smallness at the endpoints for analytic functions.

Theorem 8.1. *For a function $f : [-A, A] \mapsto \mathbb{C}$ let $r > 0$ such that f has an analytic continuation in*

$$\{z \in \mathbb{C} : \Re z \in [-A, A] \text{ and } \Im z \in [-r, r]\}.$$

Define

$$\delta = \sup_{x\in[-A,A]} \max(|f(x - \mathrm{i}r)|, |f(x + \mathrm{i}r)|)$$

and

$$\epsilon = \sup_{y\in[-r,r]} \max(|f(-A + \mathrm{i}y)|, |f(A + \mathrm{i}y)|),$$

then the Fourier coefficients are controlled by

$$|\hat{f}_n| \leq \delta\, \mathrm{e}^{-\pi|n|r/A} + \frac{\epsilon}{\pi|n|}$$

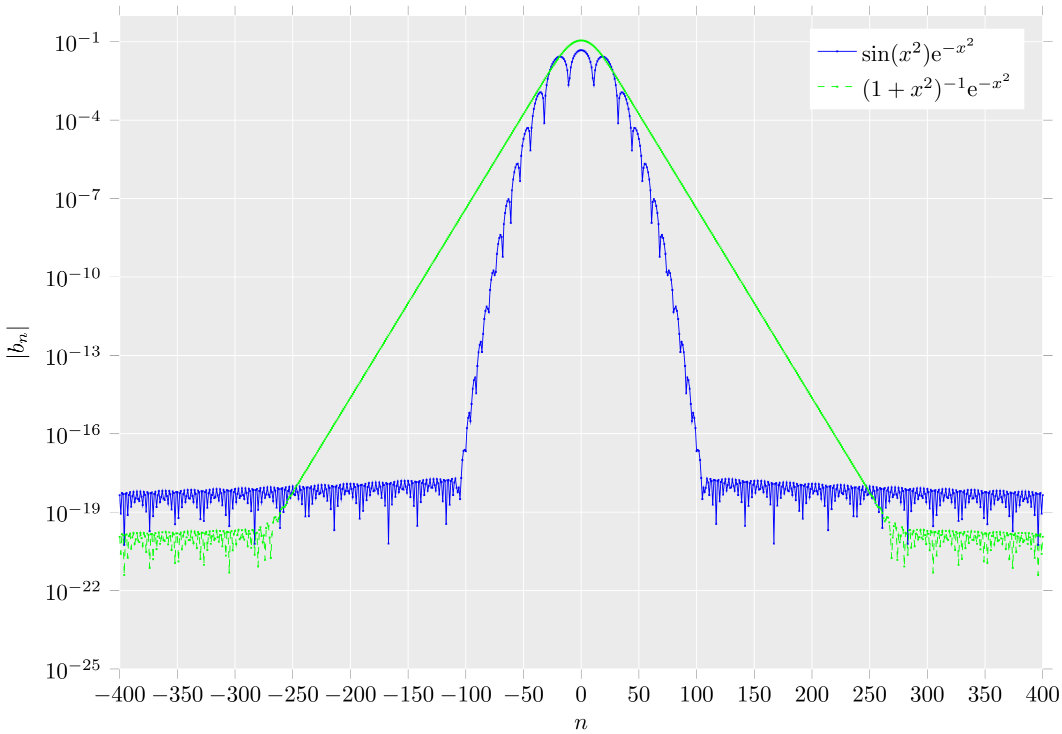


Figure 8.2 – Fourier series expansion over $[-6, 6]$.

for $n \in \mathbb{Z}$ *and* $n \neq 0$.

The first term $\delta e^{-\pi|n|r}$ is exactly the fast decay, while the second term $\epsilon/(\pi|n|)$ is the Sombrero flat.

Proof. Introduce the contours Γ^+ and Γ^- as union of $\Gamma_1^\pm, \Gamma_2^\pm, \Gamma_3^\pm$ as shown in Figure 8.3, i.e.

$$\begin{aligned}
\Gamma_1^\pm &= -A \pm \mathrm{i}[0, r], \\
\Gamma_2^\pm &= \pm \mathrm{i}r + [-A, A], \\
\Gamma_3^\pm &= A \pm \mathrm{i}[0, r].
\end{aligned}$$

If $n > 0$, deform the integral for $\hat{f}_n$ along Γ^- by Cauchy's integral theorem. This shows

$$\hat{f}_n = I_1 + I_2 + I_3,$$

where for $j = 1, 2, 3$

$$I_j = \frac{1}{2A} \int_{\Gamma_j^-} e^{-\mathrm{i}\pi n z/A} f(z) \, \mathrm{d}z.$$

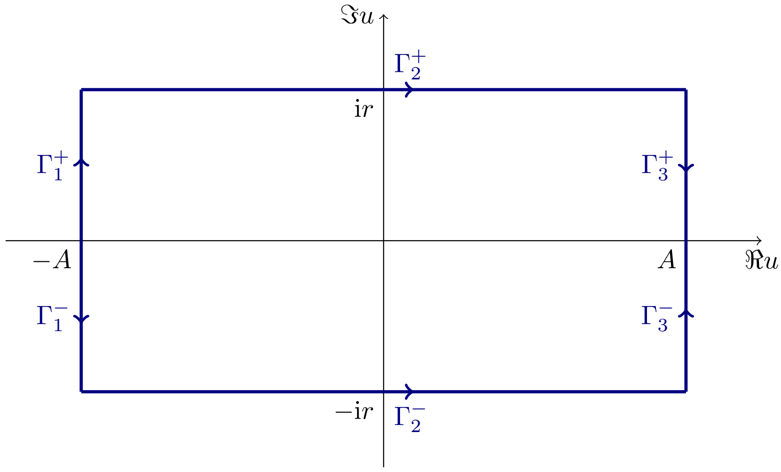


Figure 8.3 – Contours $\Gamma^{\pm}$ for deforming the Fourier integral.

Along I_1 we find

$$\begin{aligned}|I_1| &\leq \frac{1}{2A}\int_0^r \mathrm{e}^{-\pi n y/A}|f(-A-\mathrm{i}y)|\,\mathrm{d}y\\ &\leq \frac{\epsilon}{2\pi n}\end{aligned}$$

and likewise $|I_3| \leq \epsilon/(2\pi n)$.

Finally, for I_2 we find

$$\begin{aligned}|I_2| &\leq \frac{1}{2A}\int_{-A}^{A} \mathrm{e}^{-\pi n r/A}|f(x-\mathrm{i}r)|\,\mathrm{d}x\\ &\leq \delta\,\mathrm{e}^{-\pi n r/A}.\end{aligned}$$

Combining the estimates shows the claim. For $n < 0$, we can perform the same estimates along Γ^+. □

In practise the coefficients are computed using numerical quadrature at N points, which can be done efficiently using FFT. If we use, the quadrature points $(x_n)_{n=0}^{N-1}$ at

$$x_n = \frac{2n - N + 1}{N} \cdot A,$$

then the result $\hat{f}_m^N$ of the numerical quadrature is given by the aliasing of the higher modes. For this note that for all $n = 0, \ldots, N-1$ holds

$$\mathrm{e}^{-\mathrm{i}\pi N x_n/A} = \mathrm{e}^{-\mathrm{i}\pi(1-N)}.$$

Hence the result of the numerical quadrature is

$$\hat{f}_m^N = \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\pi(1-N)k} \hat{f}_{m+kN}.$$

For an efficient usage, we want to ensure that the coefficients converge fast up to a small error. In the case of even N, we use the basis elements with

$$m = 0, \dots, \frac{N}{2}, -\frac{N}{2}+1, \dots, -1$$

and for odd N

$$m = 0, \dots, \frac{N-1}{2}, -\frac{N-1}{2}, \dots, -1.$$

In fact, we can show that the quadrature error is again small.

Theorem 8.2. *Assume the hypothesis of Theorem 8.1. Then for $m \le N/2$ we have*

$$\begin{aligned}
|\hat{f}_m - \hat{f}_m^N| &\le \frac{\epsilon}{\pi}\left[\frac{1}{N+m} + \frac{1}{N-m}\right] + \delta\left[\frac{\mathrm{e}^{-\pi(N+m)r/A}}{1-\mathrm{e}^{-\pi(N+m)r/A}} + \frac{\mathrm{e}^{-\pi(N-m)r/A}}{1-\mathrm{e}^{-\pi(N-m)r/A}}\right] \\
&\le \frac{4\epsilon}{\pi N} + 2\delta \frac{\mathrm{e}^{-\pi N r/(2A)}}{1-\mathrm{e}^{-\pi N r/(2A)}}.
\end{aligned}$$

Proof. From the aliasing, the error can be split as

$$|\hat{f}_m - \hat{f}_m^N| \le \left|\sum_{k=1}^{\infty} \mathrm{e}^{-\mathrm{i}\pi(1-N)k} \hat{f}_{m+kN}\right| + \left|\sum_{k=1}^{\infty} \mathrm{e}^{-\mathrm{i}\pi(1-N)k} \hat{f}_{m-kN}\right|.$$

For the first sum deform the contour along Γ^- as in the proof of Theorem 8.1. We then find for $K \in \mathbb{N}$

$$\begin{aligned}
\left|\sum_{k=1}^{K} \hat{f}_{m+kN}\right| &\le \frac{1}{2A}\int_{\Gamma^-} \left|\mathrm{e}^{-\mathrm{i}\pi(m+N)z/A}\right| \cdot \left|\sum_{k=0}^{K} \mathrm{e}^{[-\mathrm{i}\pi N z/A - \mathrm{i}\pi(1-N)]k}\right| \cdot |f(z)|\,\mathrm{d}z \\
&\le \frac{1}{2A}\int_{\Gamma^-} \left|\mathrm{e}^{-\mathrm{i}\pi(m+N)z/A}\right| \cdot \left|\frac{1-\mathrm{e}^{[-\mathrm{i}\pi N z/A - \mathrm{i}\pi(1-N)](K+1)}}{1-\mathrm{e}^{-\mathrm{i}\pi N z/A - \mathrm{i}\pi(1-N)}}\right| \cdot |f(z)|\,\mathrm{d}z.
\end{aligned}$$

As in the proof of Theorem 8.1, we find along the part Γ_1^-

$$\begin{aligned}
|I_1| &= \left|\frac{1}{2A}\int_{\Gamma_1^-} \sum_{k=1}^{\infty} \mathrm{e}^{-\mathrm{i}\pi(m+kN)z/A} f(z)\,\mathrm{d}z\right| \\
&\le \frac{\epsilon}{2A}\int_{\Gamma^-} \mathrm{e}^{-\pi(m+N)y/A}\, \frac{1+\mathrm{e}^{-\pi N(K+1)y/A}}{1+\mathrm{e}^{-\pi N y/A}}\,\mathrm{d}z \\
&\le \frac{\epsilon}{2\pi(m+N)}
\end{aligned}$$

and likewise for I_3. For the part along Γ_2^-, we find

$$\begin{aligned}|I_1| &= \left| \frac{1}{2A} \int_{\Gamma_2^-} \sum_{k=1}^{\infty} \mathrm{e}^{-\mathrm{i}\pi(m+kN)z/A} f(z)\,\mathrm{d}z \right| \\ &\leq \delta \, \frac{\mathrm{e}^{-\pi(m+N)r/A}}{1-\mathrm{e}^{-\pi(m+N)r/A}}.\end{aligned}$$

As this holds uniformly over K, the bounds hold for the infinite sum. For the second sum, we can do the same analysis over Γ^+, yielding the claimed bound. □

8.3 Reexpansion

The previous section shows that over an interval $[-\sqrt{2N}, \sqrt{2N}]$ a sufficiently decaying function g can be well-approximated by

$$\sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} \mathrm{e}^{\mathrm{i}m\pi x/A} \hat{g}_m^N,$$

where the $\hat{g}_m^N$ are the discretised Fourier coefficients and we assume for notational convenience that N is odd. After expanding $g(x) = f(x)\mathrm{e}^{\alpha x^2}$ for $\alpha \geq 0$, the Hermite expansion can be computed by

$$a_n := \langle f, \phi_n \rangle \approx \langle \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} \mathrm{e}^{\mathrm{i}m\pi x/A} \hat{g}_m^N \mathrm{e}^{-\alpha x^2}, \phi_n \rangle,$$

where the integral outside $[-\sqrt{2N}, \sqrt{2N}]$ is negligible as ϕ_n and f are assumed to be negligible there. This gives a linear system

$$a_n = \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_{n,m}\, \hat{g}_m^N$$

with the connection coefficients $c_{n,m}$. In general, the computation of $(a_n)_{n=0}^{N-1}$ from $\hat{f}_m^N$ takes $O(N^2)$ operations and we would not have gained anything. However, the connection coefficients are highly localised allowing the reduction to $O(kN)$ operators for a small integer k depending on the sought precision.

Proposition 8.3. *Let $\alpha > -1/2$, then for $\alpha \neq 1/2$*

$$c_{n,m} = \langle \mathrm{e}^{\mathrm{i}\pi m x/A} \mathrm{e}^{-\alpha x^2}, \phi_n \rangle = \frac{\pi^{1/4}}{\sqrt{\bar{\alpha}}} \frac{(\sqrt{\gamma})^n}{\sqrt{2^n n!}} \mathrm{H}_n\left(\frac{\mathrm{i}\pi m}{2A\bar{\alpha}\sqrt{\gamma}}\right) \exp\left(-\frac{1}{\bar{\alpha}}\left(\frac{m\pi}{2A}\right)^2\right)$$

with

$$\bar{\alpha} = \alpha + \frac{1}{2} \qquad \text{and} \qquad \gamma = 1 - \frac{1}{\bar{\alpha}} = \frac{2\alpha - 1}{2\alpha + 1}$$

and for $\alpha = 1/2$ we find

$$c_{n,m} = \frac{\pi^{1/4}}{\sqrt{\bar{\alpha}}} \frac{1}{\sqrt{2^n n!}} \left(\frac{\mathrm{i}\pi m}{A}\right)^n \exp\left(-\left(\frac{m\pi}{2A}\right)^2\right).$$

Proof. Recall the generating function for Hermite polynomials

$$\sum_{n\in\mathbb{N}} \frac{\mathrm{H}_n(x)}{n!} t^n = \mathrm{e}^{2xt-t^2}.$$

Hence we find

$$\begin{aligned}\sum_{n\in\mathbb{N}} \frac{2^{n/2} c_{n,m}}{n!} t^n &= \pi^{-1/4} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\pi m x/A} \mathrm{e}^{-\alpha x^2} \mathrm{e}^{2tx-t^2} \mathrm{e}^{-x^2/2} \mathrm{d}x \\ &= \frac{\pi^{1/4}}{\sqrt{\bar{\alpha}}} \exp\left(-\frac{1}{\bar{\alpha}}\left(\frac{\pi m}{2A}\right)^2 + \frac{\mathrm{i}\pi m t}{A\bar{\alpha}} - t^2(1-\frac{1}{\bar{\alpha}})\right).\end{aligned}$$

In the case $\alpha = 1/2$, we have $\gamma = 1 - \frac{1}{\bar{\alpha}} = 0$, so that the claimed expression can be directly read off the equation.

In the case $\gamma \neq 0$, we find again using the generating function of the Hermite polynomials

$$\sum_{n\in\mathbb{N}} \frac{2^{n/2} c_{n,m}}{n!} t^n = \frac{\pi^{1/4}}{\sqrt{\bar{\alpha}}} \exp\left(-\frac{1}{\bar{\alpha}}\left(\frac{\pi m}{2A}\right)^2\right) \sum_{n=0}^{\infty} \mathrm{H}_n\left(\frac{\mathrm{i}\pi m}{2A\bar{\alpha}\sqrt{\gamma}}\right) \frac{(t\sqrt{\gamma})^n}{n!},$$

from which we can again read off the claimed form. □

Now focus on the case $\alpha = 1/2$, which corresponds to expanding the function into Hermite polynomials. For $A = \sqrt{2N}$ with $N = 200$, the connection coefficients are shown in Figure 8.4, where we see that for any given n only a few coefficients are relevant.

For a precise formulation, assume that the coefficients $(\hat{g}_m^N)_{|m|<N/2}$ have the form

$$|\hat{g}_m^N| \leq \delta\, \mathrm{e}^{-r|m|},$$

which can be assured by the results of the previous section. The following proposition then shows that the sum can be truncated with a uniformly controlled error.

Proposition 8.4. *Consider $c_{n,m}$ as given in Proposition 8.3 for $\alpha = 1/2$ as continuous function. Then for $\epsilon > 0$, there exists k such that for every $r \geq 0$ and every $n > 0$, there exists $m_n \geq 0$ such that for $|m - m_n| > k$ and $m \geq 0$ holds*

$$|c_{n,m}\mathrm{e}^{-r|m|}| \leq \epsilon |c_{n,m_n}\mathrm{e}^{-r|m_n|}|.$$

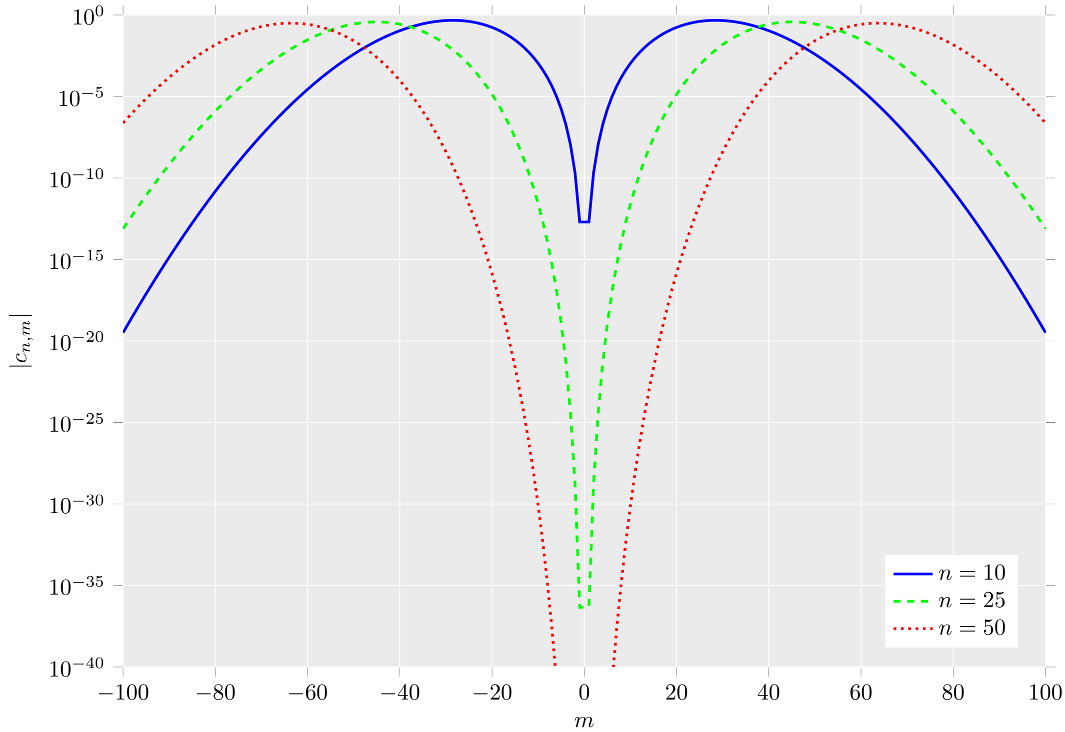


Figure 8.4 – Connection coefficients $c_{n,m}$ with $\alpha = 1/2$ and range $A = \sqrt{2N}$ with $N = 100$ as in Proposition 8.3.

Explicitly, the result holds with

$$\epsilon = \exp\left(-\left(\frac{\pi}{2A}\right)^2 \cdot \frac{k^2}{2}\right).$$

The RHS is exactly the dominant term and determines the generic size of a_n. The LHS then shows that coefficients m further than k away from m_n can be neglected with a uniformly controlled accuracy with respect to the generic size.

Proof. For a given n consider

$$g(m) = \log(|c_{n,m}|\mathrm{e}^{-rm}),$$

which is a strictly concave function over $m \in [0, \infty)$. In fact

$$g''(m) = -\frac{n}{m^2} - \left(\frac{\pi}{2A}\right)^2,$$

which shows that it is strictly concave. Taking m_n to be the unique maximum, we thus find for m with $|m - m_n| \geq k$ that

$$g(m) \leq -\left(\frac{\pi}{2A}\right)^2 \cdot \frac{k^2}{2} + g(m_n),$$

from which the claimed result follows. $\square$

Bibliography

[1] J. A. Acebrón et al. 'Breaking the symmetry in bimodal frequency distributions of globally coupled oscillators'. In: *Phys. Rev. E* 57 (May 1998), pp. 5287–5290. DOI: 10.1103/PhysRevE.57.5287.

[2] J. A. Acebrón et al. 'The Kuramoto model: A simple paradigm for synchronization phenomena'. In: *Reviews of Modern Physics* 77 (Jan. 2005), pp. 137–185. DOI: 10.1103/RevModPhys.77.137.

[3] Bradley K. Alpert and Vladimir Rokhlin. 'A fast algorithm for the evaluation of Legendre expansions'. In: *SIAM J. Sci. Statist. Comput.* 12.1 (1991), pp. 158–179. DOI: 10.1137/0912009.

[4] Luigi Ambrosio and Nicola Gigli. 'A user's guide to optimal transport'. In: *Modelling and optimisation of flows on networks*. Vol. 2062. Lecture Notes in Math. Springer, Heidelberg, 2013, pp. 1–155. DOI: 10.1007/978-3-642-32160-3_1.

[5] A. A. Arsen′ev. 'Existence in the large of a weak solution of Vlasov's system of equations'. In: *Ž. Vyčisl. Mat. i Mat. Fiz.* 15 (1975), pp. 136–147, 276.

[6] Sheldon Axler, Paul Bourdon and Wade Ramey. *Harmonic function theory*. Second. Vol. 137. Graduate Texts in Mathematics. Springer-Verlag, New York, 2001, pp. xii+259. DOI: 10.1007/978-1-4757-8137-3.

[7] George Backus. 'Linearized plasma oscillations in arbitrary electron velocity distributions'. In: *J. Mathematical Phys.* 1 (1960), 178–191; erratum, 559.

[8] R. Balescu. 'Irreversible processes in ionized gases'. In: *Phys. Fluids* 3 (1960), pp. 52–63.

[9] J. Barré et al. 'The Vlasov equation and the Hamiltonian mean-field model'. In: *Physica A Statistical Mechanics and its Applications* 365 (June 2006), pp. 177–183. DOI: 10.1016/j.physa.2006.01.005. eprint: cond-mat/0511070.

[10] Julien Barré, Alain Olivetti and Yoshiyuki Y. Yamaguchi. 'Algebraic damping in the one-dimensional Vlasov equation'. In: *J. Phys. A* 44.40 (2011), pp. 405502, 24. DOI: 10.1088/1751-8113/44/40/405502.

[11] J. Bedrossian, N. Masmoudi and C. Mouhot. 'Landau damping in finite regularity for unconfined systems with screened interactions'. In: *ArXiv e-prints* (Apr. 2016). arXiv: 1604.05783 [math.AP].

[12] Jacob Bedrossian and Nader Masmoudi. 'Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations'. In: *Publ. Math. Inst. Hautes Études Sci.* 122 (2015), pp. 195–300. DOI: 10.1007/s10240-015-0070-4.

[13] Jacob Bedrossian, Nader Masmoudi and Clément Mouhot. 'Landau Damping: Paraproducts and Gevrey Regularity'. In: *Ann. PDE* 2.1 (2016), 2:4. DOI: 10.1007/s40818-016-0008-2.

[14] Dario Benedetto, Emanuele Caglioti and Umberto Montemagno. 'Exponential dephasing of oscillators in the kinetic Kuramoto model'. In: *J. Stat. Phys.* 162.4 (2016), pp. 813–823. DOI: 10.1007/s10955-015-1426-3.

[15] Étienne Bernard and Francesco Salvarani. 'On the exponential decay to equilibrium of the degenerate linear Boltzmann equation'. In: *J. Funct. Anal.* 265.9 (2013), pp. 1934–1954. DOI: 10.1016/j.jfa.2013.06.012.

[16] I. B. Bernstein, J. M. Greene and M. D. Kruskal. 'Exact Nonlinear Plasma Oscillations'. In: *Physical Review* 108 (Nov. 1957), pp. 546–550. DOI: 10.1103/PhysRev.108.546.

[17] A. Böhm. 'Resonance poles and Gamow vectors in the rigged Hilbert space formulation of quantum mechanics'. In: *J. Math. Phys.* 22.12 (1981), pp. 2813–2823. DOI: 10.1063/1.524871.

[18] François Bolley, Ivan Gentil and Arnaud Guillin. 'Convergence to equilibrium in Wasserstein distance for Fokker-Planck equations'. In: *J. Funct. Anal.* 263.8 (2012), pp. 2430–2457. DOI: 10.1016/j.jfa.2012.07.007.

[19] François Bolley, Arnaud Guillin and Florent Malrieu. 'Trend to equilibrium and particle approximation for a weakly selfconsistent Vlasov-Fokker-Planck equation'. In: *M2AN Math. Model. Numer. Anal.* 44.5 (2010), pp. 867–884. DOI: 10.1051/m2an/2010045.

[20] Ludwig Boltzmann. 'Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen'. In: *Wiener Berichte* 66 (1872). Reproduced in *Entropie und Wahrscheinlichkeit*, edited and with an introduction by Dieter Flamm, Verlag Harri Deutsch, 2000, pp. 275–370.

[21] L. L. Bonilla, C. J. Pérez Vicente and R. Spigler. 'Time-periodic phases in populations of nonlinearly coupled oscillators with bimodal frequency distributions'. In: *Phys. D* 113.1 (1998), pp. 79–97. DOI: 10.1016/S0167-2789(97)00187-5.

[22] Luis L. Bonilla, John C. Neu and Renato Spigler. 'Nonlinear stability of incoherence and collective synchronization in a population of coupled oscillators'. In: *J. Statist. Phys.* 67.1-2 (1992), pp. 313–330. DOI: 10.1007/BF01049037.

[23] W. Braun and K. Hepp. 'The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles'. In: *Comm. Math. Phys.* 56.2 (1977), pp. 101–113.

[24] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011, pp. xiv+599.

[25] Krzysztof Burdzy and Wilfrid S. Kendall. 'Efficient Markovian couplings: examples and counterexamples'. In: *Ann. Appl. Probab.* 10.2 (2000), pp. 362–409. DOI: 10.1214/aoap/1019487348.

[26] E. Caglioti and C. Maffei. 'Time asymptotics for solutions of Vlasov-Poisson equation in a circle'. In: *J. Statist. Phys.* 92.1-2 (1998), pp. 301–323. DOI: 10.1023/A:1023055905124.

[27] María José Cantero and Arieh Iserles. 'On rapid computation of expansions in ultraspherical polynomials'. In: *SIAM J. Numer. Anal.* 50.1 (2012), pp. 307–327. DOI: 10.1137/110829568.

[28] José A. Carrillo et al. 'Contractivity of transport distances for the kinetic Kuramoto equation'. In: *J. Stat. Phys.* 156.2 (2014), pp. 395–415. DOI: 10.1007/s10955-014-1005-z.

[29] K. M. Case. 'Plasma Oscillations'. In: *Annals of Physics* 7 (July 1959), pp. 349–364. DOI: 10.1016/0003-4916(59)90029-6.

[30] Mufa Chen. 'Optimal Markovian couplings and applications'. In: *Acta Math. Sinica (N.S.)* 10.3 (1994). A Chinese summary appears in Acta Math. Sinica **3**8 (1995), no. 4, 575, pp. 260–275. DOI: 10.1007/BF02560717.

[31] Hayato Chiba. 'A proof of the Kuramoto conjecture for a bifurcation structure of the infinite-dimensional Kuramoto model'. In: *Ergodic Theory Dynam. Systems* 35.3 (2015), pp. 762–834. DOI: 10.1017/etds.2013.68.

[32] Hayato Chiba. 'Continuous limit and the moments system for the globally coupled phase oscillators'. In: *Discrete Contin. Dyn. Syst.* 33.5 (2013), pp. 1891–1903. DOI: 10.3934/dcds.2013.33.1891.

[33] Hayato Chiba and Isao Nishikawa. 'Center manifold reduction for large populations of globally coupled phase oscillators'. In: *Chaos* 21.4 (2011), pp. 043103, 10. DOI: 10.1063/1.3647317.

[34] A.R. Choudhuri. *The physics of fluids and plasmas: an introduction for astrophysicists.* Cambridge University Press, 1998.

[35] Howard S. Cohl. 'Fundamental solution of Laplace's equation in hyperspherical geometry'. In: *SIGMA Symmetry Integrability Geom. Methods Appl.* 7 (2011), Paper 108, 14. DOI: 10.3842/SIGMA.2011.108.

[36] John Crawford. 'Scaling and Singularities in the Entrainment of Globally Coupled Oscillators'. In: *Phys. Rev. Lett.* 74 (21 May 1995), pp. 4341–4344. DOI: 10.1103/PhysRevLett.74.4341.

[37] John D. Crawford and K. T. R. Davies. 'Synchronization of globally coupled phase oscillators: singularities and scaling for general couplings'. In: *Phys. D* 125.1-2 (1999), pp. 1–46. DOI: 10.1016/S0167-2789(98)00235-8.

[38] John David Crawford. ‘Amplitude expansions for instabilities in populations of globally-coupled oscillators’. In: *J. Statist. Phys.* 74.5-6 (1994), pp. 1047–1084. DOI: 10.1007/BF02188217.

[39] John David Crawford and Peter D. Hislop. ‘Application of the method of spectral deformation to the Vlasov-Poisson system’. In: *Ann. Physics* 189.2 (1989), pp. 265–317. DOI: 10.1016/0003-4916(89)90166-8.

[40] J. Dawson. ‘On Landau Damping’. In: *Physics of Fluids* 4 (July 1961), pp. 869–874. DOI: 10.1063/1.1706419.

[41] Pierre Degond. ‘Spectral theory of the linearized Vlasov-Poisson equation’. In: *Trans. Amer. Math. Soc.* 294.2 (1986), pp. 435–453. DOI: 10.2307/2000192.

[42] L. Desvillettes and C. Villani. ‘On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation’. In: *Comm. Pure Appl. Math.* 54.1 (2001), pp. 1–42. DOI: 10.1002/1097-0312(200101)54:1<1::AID-CPA1>3.0.CO;2-Q.

[43] Florin Diacu et al. ‘The Vlasov–Poisson System for Stellar Dynamics in Spaces of Constant Curvature’. In: *Communications in Mathematical Physics* (2016), pp. 1–37. DOI: 10.1007/s00220-016-2608-9.

[44] H. Dietert, J. Evans and T. Holding. ‘Convergence to equilibrium for the kinetic Fokker-Planck equation on the torus’. In: *ArXiv e-prints* (June 2015). arXiv: 1506.06173 [math.PR].

[45] H. Dietert, B. Fernandez and D. Gérard-Varet. ‘Landau damping to partially locked states in the Kuramoto model’. In: *ArXiv e-prints* (June 2016). arXiv: 1606.04470 [math.AP].

[46] Helge Dietert. ‘Stability and bifurcation for the Kuramoto model’. In: *J. Math. Pures Appl. (9)* 105.4 (2016), pp. 451–489. DOI: 10.1016/j.matpur.2015.11.001.

[47] R. L. Dobrušin. ‘Vlasov equations’. In: *Funktsional. Anal. i Prilozhen.* 13.2 (1979), pp. 48–58, 96.

[48] Gustav Doetsch. *Handbuch der Laplace-Transformation. Band I. Theorie der Laplace-Transformation.* Verlag Birkhäuser, Basel, 1950, p. 581.

[49] Jean Dolbeault, Clément Mouhot and Christian Schmeiser. ‘Hypocoercivity for linear kinetic equations conserving mass’. In: *Trans. Amer. Math. Soc.* 367.6 (2015), pp. 3807–3828. DOI: 10.1090/S0002-9947-2015-06012-7.

[50] James R. Driscoll and Dennis M. Healy Jr. ‘Computing Fourier transforms and convolutions on the 2-sphere’. In: *Adv. in Appl. Math.* 15.2 (1994), pp. 202–250. DOI: 10.1006/aama.1994.1008.

[51] W. E. Drummond. ‘Landau damping’. In: *Physics of Plasmas* 11 (Feb. 2004), pp. 552–560. DOI: 10.1063/1.1628685.

[52] Erwan Faou and Frédéric Rousset. 'Landau damping in Sobolev spaces for the Vlasov-HMF model'. In: *Arch. Ration. Mech. Anal.* 219.2 (2016), pp. 887–902. DOI: 10.1007/s00205-015-0911-9.

[53] Bastien Fernandez, David Gérard-Varet and Giambattista Giacomin. 'Landau Damping in the Kuramoto Model'. In: *Ann. Henri Poincaré* 17.7 (2016), pp. 1793–1823. DOI: 10.1007/s00023-015-0450-9.

[54] Elizabeth Garber. 'Subjects great and small: Maxwell on Saturn's rings and kinetic theory'. In: *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* 366.1871 (2008), pp. 1697–1705. DOI: 10.1098/rsta.2007.2179.

[55] I. M. Gel′fand and N. Ya. Vilenkin. *Generalized functions. Vol. 4*. Applications of harmonic analysis, Translated from the Russian by Amiel Feinstein. Academic Press, New York-London, 1964, pp. xiv+384.

[56] J.W. Gibbs. *Elementary Principles in Statistical Mechanics: Developed with Especial Reference to the Rational Foundations of Thermodynamics*. Elementary Principles in Statistical Mechanics: Developed with Especial Reference to the Rational Foundation of Thermodynamics. C. Scribner's sons, 1902.

[57] D. Giulini and N. Straumann. '".. I didn't reflect much on what I was doing.." How Planck discovered his radiation formula'. In: *eprint arXiv:quant-ph/0010008* (Oct. 2000). eprint: quant-ph/0010008.

[58] Robert Glassey and Jack Schaeffer. 'On time decay rates in Landau damping'. In: *Comm. Partial Differential Equations* 20.3-4 (1995), pp. 647–676. DOI: 10.1080/03605309508821107.

[59] Robert Glassey and Jack Schaeffer. 'Time decay for solutions to the linearized Vlasov equation'. In: *Transport Theory Statist. Phys.* 23.4 (1994), pp. 411–453. DOI: 10.1080/00411459408203873.

[60] Robert T. Glassey. *The Cauchy problem in kinetic theory*. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 1996, pp. xii+241. DOI: 10.1137/1.9781611971477.

[61] F. Golse. 'On the Dynamics of Large Particle Systems in the Mean Field Limit'. In: *ArXiv e-prints* (Jan. 2013). arXiv: 1301.5494 [math.AP].

[62] François Golse. 'The mean-field limit for the dynamics of large particle systems'. In: *Journées "Équations aux Dérivées Partielles"*. Univ. Nantes, Nantes, 2003, Exp. No. IX, 47.

[63] François Golse, Clément Mouhot and Valeria Ricci. 'Empirical measures and Vlasov hierarchies'. In: *Kinet. Relat. Models* 6.4 (2013), pp. 919–943. DOI: 10.3934/krm.2013.6.919.

[64] R. W. Gould, T. M. O'Neil and J. H. Malmberg. 'Plasma Wave Echo'. In: *Physical Review Letters* 19 (July 1967), pp. 219–222. DOI: 10.1103/PhysRevLett.19.219.

[65] G. Gripenberg, S.-O. Londen and O. Staffans. *Volterra integral and functional equations*. Vol. 34. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1990, pp. xxii+701. DOI: 10.1017/CBO9780511662805.

[66] Ernst Hairer and Arieh Iserles. 'Numerical Stability in the Presence of Variable Coefficients'. In: *Found. Comput. Math.* 16.3 (2016), pp. 751–777. DOI: 10.1007/s10208-015-9263-y.

[67] Daniel Han-Kwan and Matthieu Léautaud. 'Geometric analysis of the linear Boltzmann equation I. Trend to equilibrium'. In: *Ann. PDE* 1.1 (2015), Art. 3, 84. DOI: 10.1007/s40818-015-0003-z.

[68] Mariana Haragus and Gérard Iooss. *Local bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems*. Universitext. Springer-Verlag London, Ltd., London; EDP Sciences, Les Ulis, 2011, pp. xii+329. DOI: 10.1007/978-0-85729-112-7.

[69] Maxime Hauray and Pierre-Emmanuel Jabin. 'N-particles approximation of the Vlasov equations with singular potential'. In: *Arch. Ration. Mech. Anal.* 183.3 (2007), pp. 489–524. DOI: 10.1007/s00205-006-0021-9.

[70] Maxime Hauray and Pierre-Emmanuel Jabin. 'Particle approximation of Vlasov equations with singular forces: propagation of chaos'. In: *Ann. Sci. Éc. Norm. Supér. (4)* 48.4 (2015), pp. 891–940.

[71] E. J. Hildebrand, M. A. Buice and C. C. Chow. 'Kinetic Theory of Coupled Oscillators'. In: *Physical Review Letters* 98.5, 054101 (Feb. 2007), p. 054101. DOI: 10.1103/PhysRevLett.98.054101. eprint: nlin/0612029.

[72] Peter D. Hislop and John David Crawford. 'Application of spectral deformation to the Vlasov-Poisson system. II. Mathematical results'. In: *J. Math. Phys.* 30.12 (1989), pp. 2819–2837. DOI: 10.1063/1.528465.

[73] Darryl D. Holm et al. 'Nonlinear stability of fluid and plasma equilibria'. In: *Phys. Rep.* 123.1-2 (1985), p. 116. DOI: 10.1016/0370-1573(85)90028-6.

[74] Hyung Ju Hwang and Juan J. L. Velázquez. 'On the existence of exponentially decreasing solutions of the nonlinear Landau damping problem'. In: *Indiana Univ. Math. J.* 58.6 (2009), pp. 2623–2660. DOI: 10.1512/iumj.2009.58.3835.

[75] Arieh Iserles. 'A fast and simple algorithm for the computation of Legendre coefficients'. In: *Numer. Math.* 117.3 (2011), pp. 529–553. DOI: 10.1007/s00211-010-0352-1.

[76] Arieh Iserles. 'On skew-symmetric differentiation matrices'. In: *IMA J. Numer. Anal.* 34.2 (2014), pp. 435–451. DOI: 10.1093/imanum/drt013.

[77] J. H. Jeans. 'On the theory of star-streaming and the structure of the universe'. In: MNRAS 76 (Dec. 1915), pp. 70–84. DOI: 10.1093/mnras/76.2.70.

[78] J. H. Jeans. 'On the theory of star-streaming and the structure of the universe'. In: MNRAS 76 (May 1916), pp. 552–572. DOI: 10.1093/mnras/76.7.552.

[79] Richard Jordan, David Kinderlehrer and Felix Otto. 'The variational formulation of the Fokker-Planck equation'. In: *SIAM J. Math. Anal.* 29.1 (1998), pp. 1–17. DOI: 10.1137/S0036141096303359.

[80] Jens Keiner. 'Computing with expansions in Gegenbauer polynomials'. In: *SIAM J. Sci. Comput.* 31.3 (2009), pp. 2151–2171. DOI: 10.1137/070703065.

[81] Wilfrid S. Kendall. 'Coupling, local times, immersions'. In: *Bernoulli* 21.2 (2015), pp. 1014–1046. DOI: 10.3150/14-BEJ596.

[82] I.U.L. Klimontovich. *Statistical theory of non-equilibrium processes in a plasma.* [1st English ed.] Translated by H. S. H. Massey and O. M. Blunn. Edited by D. Ter Haar. Oxford, New York: Pergamon Press, 1967.

[83] Y. Kuramoto. *Chemical oscillations, waves, and turbulence.* Vol. 19. Springer Series in Synergetics. Springer-Verlag, Berlin, 1984, pp. viii+156. DOI: 10.1007/978-3-642-69689-3.

[84] Yoshiki Kuramoto. 'Self-entrainment of a population of coupled non-linear oscillators'. In: *International Symposium on Mathematical Problems in Theoretical Physics (Kyoto Univ., Kyoto, 1975).* Springer, Berlin, 1975, 420–422. Lecture Notes in Phys., 39.

[85] Kazumasa Kuwada. 'Characterization of maximal Markovian couplings for diffusion processes'. In: *Electron. J. Probab.* 14 (2009), no. 25, 633–662. DOI: 10.1214/EJP.v14-634.

[86] Carlo Lancellotti. 'On the Vlasov limit for systems of nonlinearly coupled oscillators without noise'. In: *Transport Theory Statist. Phys.* 34.7 (2005), pp. 523–535. DOI: 10.1080/00411450508951152.

[87] L. Landau. 'On the vibrations of the electronic plasma'. In: *Acad. Sci. USSR. J. Phys.* 10 (1946), pp. 25–34.

[88] L.D. Landau. 'Die kinetische Gleichung für den Fall Coulombscher Wechselwirkung'. In: *Physikalische Zeitschrift der Sowjetunion* 10 (1936), pp. 154–164.

[89] M. Lemou. 'Extended Rearrangement inequalities and applications to some quantitative stability results'. In: *ArXiv e-prints* (Sept. 2015). arXiv: 1509.08287 [math.AP].

[90] M. Lemou, A. M. Luz and F. Méhats. 'Nonlinear stability criteria for the HMF Model'. In: *ArXiv e-prints* (Sept. 2015). arXiv: 1509.08637 [math.AP].

[91] Mohammed Lemou, Florian Méhats and Pierre Raphaël. 'Orbital stability of spherical galactic models'. In: *Invent. Math.* 187.1 (2012), pp. 145–194. DOI: 10.1007/s00222-011-0332-9.

[92] Andrew Lenard. 'On Bogoliubov's kinetic equation for a spatially homogeneous plasma'. In: *Ann. Physics* 10 (1960), pp. 390–400.

[93] R.L. Liboff. *Kinetic Theory: Classical, Quantum, and Relativistic Descriptions.* Graduate Texts in Contemporary Physics. Springer New York, 2003.

[94] E. M. Lifschitz and L. P. Pitajewski. *Physical kinetics.* Course of theoretical physics. Butterworth-Heinemann, 1981.

[95] Zhiwu Lin and Chongchun Zeng. 'Small BGK waves and nonlinear Landau damping'. In: *Comm. Math. Phys.* 306.2 (2011), pp. 291–331. DOI: `10.1007/s00220-011-1246-5`.

[96] Zhiwu Lin and Chongchun Zeng. 'Small BGK waves and nonlinear Landau damping (higher dimensions)'. In: *Indiana Univ. Math. J.* 61.5 (2012), pp. 1711–1735. DOI: `10.1512/iumj.2012.61.4738`.

[97] P.-L. Lions and B. Perthame. 'Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system'. In: *Invent. Math.* 105.2 (1991), pp. 415–430. DOI: `10.1007/BF01232273`.

[98] Grégoire Loeper. 'Uniqueness of the solution to the Vlasov-Poisson system with bounded density'. In: *J. Math. Pures Appl. (9)* 86.1 (2006), pp. 68–79. DOI: `10.1016/j.matpur.2006.01.005`.

[99] D. Lynden-Bell. 'Statistical mechanics of violent relaxation in stellar systems'. In: MNRAS 136 (1967), p. 101. DOI: `10.1093/mnras/136.1.101`.

[100] J. H. Malmberg and C. B. Wharton. 'Collisionless Damping of Electrostatic Plasma Waves'. In: *Physical Review Letters* 13 (Aug. 1964), pp. 184–186. DOI: `10.1103/PhysRevLett.13.184`.

[101] J. H. Malmberg and C. B. Wharton. 'Dispersion of Electron Plasma Waves'. In: *Physical Review Letters* 17 (July 1966), pp. 175–178. DOI: `10.1103/PhysRevLett.17.175`.

[102] J. H. Malmberg et al. 'Observation of Plasma Wave Echoes'. In: *Physics of Fluids* 11 (June 1968), pp. 1147–1153. DOI: `10.1063/1.1692075`.

[103] J. H. Malmberg et al. 'Plasma Wave Echo Experiment'. In: *Physical Review Letters* 20 (Jan. 1968), pp. 95–97. DOI: `10.1103/PhysRevLett.20.95`.

[104] C. Marchioro and M. Pulvirenti. 'A note on the nonlinear stability of a spatially symmetric Vlasov-Poisson flow'. In: *Math. Methods Appl. Sci.* 8.2 (1986), pp. 284–288. DOI: `10.1002/mma.1670080119`.

[105] E. A. Martens et al. 'Exact results for the Kuramoto model with a bimodal frequency distribution'. In: *Phys. Rev. E (3)* 79.2 (2009), pp. 026204, 11. DOI: `10.1103/PhysRevE.79.026204`.

[106] V. P. Maslov and M. V. Fedoryuk. 'The linear theory of Landau damping'. In: *Mat. Sb. (N.S.)* 127(169).4 (1985), pp. 445–475, 559.

[107] J. Clerk Maxwell. 'On the Dynamical Theory of Gases'. In: *Philosophical Transactions of the Royal Society of London* 157 (1867), pp. 49–88. DOI: `10.1098/rstl.1867.0004`. eprint: `http://rstl.royalsocietypublishing.org/content/157/49.full.pdf+html`.

[108] M. Merkli and I. M. Sigal. 'A time-dependent theory of quantum resonances'. In: *Comm. Math. Phys.* 201.3 (1999), pp. 549–576. DOI: `10.1007/s002200050568`.

[109] R. Mirollo and S. H. Strogatz. 'The spectrum of the partially locked state for the Kuramoto model'. In: *J. Nonlinear Sci.* 17.4 (2007), pp. 309–347. DOI: `10.1007/s00332-006-0806-x`.

[110] S. Mischler and C. Mouhot. 'Exponential Stability of Slowly Decaying Solutions to the Kinetic-Fokker-Planck Equation'. In: *Arch. Ration. Mech. Anal.* 221.2 (2016), pp. 677–723. DOI: `10.1007/s00205-016-0972-4`.

[111] Philip J. Morrison. 'The Maxwell-Vlasov equations as a continuous Hamiltonian system'. In: *Phys. Lett. A* 80.5-6 (1980), pp. 383–386. DOI: `10.1016/0375-9601(80)90776-8`.

[112] Peter Mörters and Yuval Peres. *Brownian motion.* Cambridge Series in Statistical and Probabilistic Mathematics. With an appendix by Oded Schramm and Wendelin Werner. Cambridge University Press, Cambridge, 2010, pp. xii+403. DOI: `10.1017/CBO9780511750489`.

[113] Clément Mouhot and Lukas Neumann. 'Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus'. In: *Nonlinearity* 19.4 (2006), pp. 969–998. DOI: `10.1088/0951-7715/19/4/011`.

[114] Clément Mouhot and Cédric Villani. 'On Landau damping'. In: *Acta Math.* 207.1 (2011), pp. 29–201. DOI: `10.1007/s11511-011-0068-9`.

[115] N. I. Muskhelishvili. *Singular integral equations. Boundary problems of function theory and their application to mathematical physics.* Translation by J. R. M. Radok. P. Noordhoff N. V., Groningen, 1953, pp. vi+447.

[116] Jan van Neerven. *The asymptotic behaviour of semigroups of linear operators.* Vol. 88. Operator Theory: Advances and Applications. Birkhäuser Verlag, Basel, 1996, pp. xii+237. DOI: `10.1007/978-3-0348-9206-3`.

[117] H. Neunzert. 'An introduction to the nonlinear Boltzmann-Vlasov equation'. In: *Kinetic theories and the Boltzmann equation (Montecatini, 1981).* Vol. 1048. Lecture Notes in Math. Springer, Berlin, 1984, pp. 60–110. DOI: `10.1007/BFb0071878`.

[118] D.R. Nicholson. *Introduction to plasma theory.* Wiley series in plasma physics. Wiley, 1983.

[119] A. Nordenfelt. 'Entropy in the Kuramoto model and its implications for the stability of partially synchronized states'. In: *ArXiv e-prints* (Oct. 2014). arXiv: `1410.2849 [nlin.PS]`.

[120] Oleh E. Omel'chenko and Matthias Wolfrum. 'Bifurcations in the Sakaguchi-Kuramoto model'. In: *Phys. D* 263 (2013), pp. 74–85. DOI: 10.1016/j.physd.2013.08.004.

[121] Edward Ott and Thomas M. Antonsen. 'Long time evolution of phase oscillator systems'. In: *Chaos* 19.2 (2009), pp. 023117, 5. DOI: 10.1063/1.3136851.

[122] Edward Ott and Thomas M. Antonsen. 'Low dimensional behavior of large systems of globally coupled oscillators'. In: *Chaos* 18.3 (2008), pp. 037113, 6. DOI: 10.1063/1.2930766.

[123] Edward Ott, Brian R. Hunt and Thomas M. Antonsen Jr. 'Comment on "Long time evolution of phase oscillator systems" [Chaos 19, 023117 (2009)] [MR2548751]'. In: *Chaos* 21.2 (2011), pp. 025112, 2. DOI: 10.1063/1.3574931.

[124] R.E.A.C. Paley and N. Wiener. *Fourier transforms in the complex domain*. Colloquium Publications - American Mathematical Society. American Mathematical Society, 1934.

[125] Christophe Pallard. 'Space moments of the Vlasov-Poisson system: propagation and regularity'. In: *SIAM J. Math. Anal.* 46.3 (2014), pp. 1754–1770. DOI: 10.1137/120881178.

[126] J. Pantaleone. 'Synchronization of metronomes'. In: *American Journal of Physics* 70 (Oct. 2002), pp. 992–1000. DOI: 10.1119/1.1501118.

[127] Robert L. Pego and Michael I. Weinstein. 'Asymptotic stability of solitary waves'. In: *Comm. Math. Phys.* 164.2 (1994), pp. 305–349.

[128] O. Penrose. 'Electrostatic Instabilities of a Uniform Non-Maxwellian Plasma'. In: *Physics of Fluids* 3 (Mar. 1960), pp. 258–265. DOI: 10.1063/1.1706024.

[129] K. Pfaffelmoser. 'Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data'. In: *J. Differential Equations* 95.2 (1992), pp. 281–303. DOI: 10.1016/0022-0396(92)90033-J.

[130] A. Pikovsky and M. Rosenblum. 'Dynamics of globally coupled oscillators: Progress and perspectives'. In: *Chaos* 25.9, 097616 (Sept. 2015), p. 097616. DOI: 10.1063/1.4922971. arXiv: 1504.06747 [nlin.AO].

[131] Planck Collaboration et al. 'Planck 2013 results. XXVI. Background geometry and topology of the Universe'. In: A&A 571, A26 (Nov. 2014), A26. DOI: 10.1051/0004-6361/201321546. arXiv: 1303.5086.

[132] Daniel Potts, Gabriele Steidl and Manfred Tasche. 'Fast algorithms for discrete polynomial transforms'. In: *Math. Comp.* 67.224 (1998), pp. 1577–1590. DOI: 10.1090/S0025-5718-98-00975-2.

[133] Jeffrey Rauch. *Hyperbolic partial differential equations and geometric optics*. Vol. 133. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012, pp. xx+363. DOI: 10.1090/gsm/133.

[134] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1978, pp. xv+396.

[135] L. E. Reichl. *A modern course in statistical physics.* With a foreword by Ilya Prigogine. University of Texas Press, Austin, Tex., 1980, pp. xv+709.

[136] Gerhard Rein. 'Collisionless kinetic equations from astrophysics—the Vlasov-Poisson system'. In: *Handbook of differential equations: evolutionary equations. Vol. III.* Handb. Differ. Equ. Elsevier/North-Holland, Amsterdam, 2007, pp. 383–476. DOI: 10.1016/S1874-5717(07)80008-9.

[137] Walter Rudin. *Real and complex analysis.* Third. McGraw-Hill Book Co., New York, 1987, pp. xiv+416.

[138] Hidetsugu Sakaguchi. 'Cooperative phenomena in coupled oscillator systems under external fields'. In: *Progr. Theoret. Phys.* 79.1 (1988), pp. 39–46. DOI: 10.1143/PTP.79.39.

[139] Hidetsugu Sakaguchi and Yoshiki Kuramoto. 'A soluble active rotator model showing phase transitions via mutual entrainment'. In: *Progr. Theoret. Phys.* 76.3 (1986), pp. 576–581. DOI: 10.1143/PTP.76.576.

[140] Jack Schaeffer. 'Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions'. In: *Comm. Partial Differential Equations* 16.8-9 (1991), pp. 1313–1335. DOI: 10.1080/03605309108820801.

[141] Boris V. Somov. *Plasma Astrophysics, Part I: Fundamentals and Practice.* Astrophysics and Space Science Library. Springer, 2006.

[142] K.H. Spatschek. *Theoretische Plasmaphysik: eine Einführung.* Teubner Studienbücher: Physik. Teubner, 1990.

[143] H. Spohn. *Large scale dynamics of interacting particles.* Texts and monographs in physics. Springer-Verlag, 1991.

[144] H. Spohn. 'On the Vlasov hierarchy'. In: *Math. Methods Appl. Sci.* 3.4 (1981), pp. 445–455. DOI: 10.1002/mma.1670030131.

[145] Herbert Spohn. 'Boltzmann hierarchy and Boltzmann equation'. In: *Kinetic theories and the Boltzmann equation (Montecatini, 1981).* Vol. 1048. Lecture Notes in Math. Springer, Berlin, 1984, pp. 207–220. DOI: 10.1007/BFb0071883.

[146] Walter A. Strauss. 'On continuity of functions with values in various Banach spaces'. In: *Pacific J. Math.* 19 (1966), pp. 543–551.

[147] Steven H. Strogatz. 'From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators'. In: *Phys. D* 143.1-4 (2000). Bifurcations, patterns and symmetry, pp. 1–20. DOI: 10.1016/S0167-2789(00)00094-4.

[148] Steven H. Strogatz and Renato E. Mirollo. 'Stability of incoherence in a population of coupled oscillators'. In: *J. Statist. Phys.* 63.3-4 (1991), pp. 613–635. DOI: 10.1007/BF01029202.

[149] Steven H. Strogatz, Renato E. Mirollo and Paul C. Matthews. 'Coupled nonlinear oscillators below the synchronization threshold: relaxation by generalized Landau damping'. In: *Phys. Rev. Lett.* 68.18 (1992), pp. 2730–2733. DOI: 10.1103/PhysRevLett.68.2730.

[150] Gábor Szegő. *Orthogonal polynomials.* Fourth. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975, pp. xiii+432.

[151] E.C. Titchmarsh. *Introduction to the theory of Fourier integrals.* The Clarendon Press, 1948.

[152] I. Tristani. 'Landau damping for the linearized Vlasov Poisson equation in a weakly collisional regime'. In: *ArXiv e-prints* (Mar. 2016). arXiv: 1603.07219 [math.AP].

[153] N. G. Van Kampen. 'on the theory of stationary waves in plasmas'. In: *Physica* 21 (1955), pp. 949–963. DOI: 10.1016/S0031-8914(55)93068-8.

[154] A. Vanderbauwhede. 'Centre manifolds, normal forms and elementary bifurcations'. In: *Dynamics reported, Vol. 2.* Vol. 2. Dynam. Report. Ser. Dynam. Systems Appl. Wiley, Chichester, 1989, pp. 89–169.

[155] A. Vanderbauwhede and G. Iooss. 'Center manifold theory in infinite dimensions'. In: *Dynamics reported: expositions in dynamical systems.* Vol. 1. Dynam. Report. Expositions Dynam. Systems (N.S.) Springer, Berlin, 1992, pp. 125–163.

[156] Cédric Villani. 'Hypocoercivity'. In: *Mem. Amer. Math. Soc.* 202.950 (2009), pp. iv+141. DOI: 10.1090/S0065-9266-09-00567-5.

[157] Cédric Villani. *Notes for a course given in Cotonou, Benin, and in CIRM, Luminy, Summer 2010.* Available online at http://cedricvillani.org/wp-content/uploads/2012/08/B13.Landau.pdf. 2010.

[158] Cédric Villani. *Optimal transport.* Vol. 338. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Old and new. Springer-Verlag, Berlin, 2009, pp. xxii+973. DOI: 10.1007/978-3-540-71050-9.

[159] Cédric Villani. 'Particle systems and nonlinear Landau damping'. In: *Physics of Plasmas* 21.3, 030901 (Mar. 2014), p. 030901. DOI: 10.1063/1.4867237.

[160] Cédric Villani. *Topics in optimal transportation.* Vol. 58. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2003, pp. xvi+370. DOI: 10.1007/b12016.

[161] A. A. Vlasov. 'The vibrational properties of an electron gas'. In: *Physics-Uspekhi* 10.6 (1968). Reprint from Zh. Eksp. Teor. Fiz. 8, 291 (1938). Original submitted October 1, 1937, pp. 721–733. DOI: 10.1070/PU1968v010n06ABEH003709.

[162] Haiyong Wang and Daan Huybrechs. *A fast algorithm for the calculation of the Jacobi expansion coefficients*. Tech. rep. TW 645. Celestijnenlaan 200A – B-3001 Heverlee (Belgium): Katholieke Universiteit Leuven, Department of Computer Science, Apr. 2014.

[163] Jeffrey Weeks. 'The Poincaré Dodecahedral Space and the Mystery of the Missing Fluctuations'. In: *Notices of the AMS* (2004).

[164] Arthur T Winfree. 'Biological rhythms and the behavior of populations of coupled oscillators'. In: *Journal of theoretical biology* 16.1 (1967), pp. 15–42.

[165] D. Witthaut and M. Timme. 'Kuramoto dynamics in Hamiltonian systems'. In: Phys. Rev. E 90.3, 032917 (Sept. 2014), p. 032917. DOI: 10.1103/PhysRevE.90.032917. arXiv: 1305.1742 [nlin.CD].

[166] A. Y. Wong, R. W. Motley and N. D'Angelo. 'Landau Damping of Ion Acoustic Waves in Highly Ionized Plasmas'. In: *Physical Review* 133 (Jan. 1964), pp. 436–442. DOI: 10.1103/PhysRev.133.A436.