DISCRETE AND OSCILLATORY MATRIX MODELS IN CHERN-SIMONS THEORY

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ABSTRACT. We derive discrete and oscillatory Chern-Simons matrix models. The method is based on fundamental properties of the associated orthogonal polynomials. As an application, we show that the discrete model allows to prove and extend the recently found relationship between Chern-Simons theory and q-deformed 2dYM. In addition, the equivalence of the Chern-Simons matrix models gives a complementary view on the equivalence of effective superpotentials in $\mathcal{N} = 1$ gauge theories.

1. INTRODUCTION

Since the seventies, when random matrix theory [1] was successfully employed in gauge theory [2], matrix models have played a rather remarkable role in gauge theory (see [3, 4] for reviews). In this paper, we shall focus on the matrix models that appear in Chern-Simons theory [5, 6] and also discuss the recently found connection with 2dYM [7, 8] as well as $\mathcal{N} = 1$ supersymmetric gauge theories. More precisely, with recent work that discusses the equivalence of effective superpotentials in these theories [9, 10, 11].

Let us briefly recall very basic facts of Chern-Simons gauge theory: In [12], Witten considered a topological gauge theory for a connection on an arbitrary three-manifold M, based on the Chern-Simons action:

$$(1) \qquad S_{\text{CS}}(A) = \frac{k}{4\pi} \int_M \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

with k an integer number. One of the most important aspects of Chern-Simons theory is that it provides a physical approach to three dimensional topology. In particular, it gives three-manifold invariants and knot invariants. For example, the partition function,

$$(2) \qquad Z_k(M) = \int \mathcal{D}A \mathrm{e}^{iS_{\text{CS}}(A)},$$

delivers a topological invariant of M, the so-called Reshetikhin-Turaev-Witten invariant. Recent reviews are [4, 13].

As reviewed in detail in [13], a great deal of interest has been recently focused on the fact that Chern-Simons theory provides large N duals of topological strings. This connection between Chern-Simons theory and topological strings was already pointed out by Witten [14] (see also [15]), and then considerably extended in [16].

Interestingly enough, Mariño has found a tight connection between Chern-Simons theory and random matrix models [5]. More precisely, building upon [17], in [5] it is

shown that the partition function of Chern-Simons theory on S^3 with gauge group $U(N)$ is given by the partition function of the following random matrix model:

$$(3) \qquad Z = \frac{\mathrm{e}^{-\frac{g_s}{12} N (N^2 - 1)}}{N!} \int \prod_{i=1}^{N} \mathrm{e}^{-u_i^2 / 2g_s} \prod_{i<j} \left(2 \sinh \frac{u_i - u_j}{2} \right)^2 \frac{du_i}{2\pi},$$

From the point of view of topological strings, this describes open topological A strings on T^*S^3 with N branes wrapping S^3 [5]. Chern-Simons matrix models have been further considered in [6] and [18]-[26]. Most of these works focus on the relevance to topological strings. In [6], the emphasis is on exact solutions and on the special features of the matrix models. In this paper we shall be further developing these results, while making some connections, at the end, with matrix models in $\mathcal{N} = 1$ suspersymmetric gauge theories.

It should be stressed that the result (3) in [5] is just a particular case of more general models and our results will apply to them as well. Explicit expressions for them will be given in the next section.

As shown in [6], the Stieltjes-Wigert polynomials, a member of the q-deformed orthogonal polynomials family [27], allows to compute, in exact fashion, quantities associated to the matrix model. In the computation, the q-parameter of the polynomials turns out to be naturally identified with the q-parameter of the quantum group invariants associated to the Chern-Simons theory. In [6], the focus was on $U(N)$ Chern-Simons theory on S^3, whose matrix model is also the Stieltjes-Wigert ensemble:

$$(4) \qquad Z = \int [dM] \mathrm{e}^{-\frac{1}{2g_s} \mathrm{Tr}(\log M)^2} .$$

A special property of any of these type of matrix models is that there are infinitely many models that lead to the same partition function [6]. This is a courtesy of their very weakly confining potentials. We shall extend this result, essentially by exploiting results from the moment problem [28, 29], as in [6], in a more explicit way. In some cases, it just suffices to consider well-known facts of orthogonal polynomials with undetermined weight [29]. In others, the ones corresponding to the oscillatory/periodic models, one should further develop -in a simple and natural way-, the results of the moment problem to fully understand the extent of the equivalence between the models.

To summarize, as a first result we have the existence of Chern-Simons matrix models with a discrete weight, which essentially turns out to be a discretization of the weight of the usual models [5, 6]. That is to say, the continuous models in [5, 6] can be castled into an homogenous and exponential lattice, respectively. As we shall see, this discrete model is precisely the q-deformed 2dYM partition function. The advantage of our method is that it can be applied to any Chern-Simons matrix model. Then, we shall deal with the continuous case, and show the manifold possibilities, including losing differentiability of the weight function. From a mathematical point of view, it is remarkable that the potential may be so wild even to reach fractal behavior and still leave the partition function (which in addition is a Chern-Simons partition function) unchanged. Interpretations in terms of gauge theories and quantum groups are discussed in some instances, and we show that the equivalence of potentials of the (generalized) Chern-Simons matrix models introduced, leads to a complementary view of the equivalence of effective superpotentials in $\mathcal{N} = 1$ supersymmetric gauge theories [9, 10, 11]. In the last

section, we comment on the possible physical relevance and some further avenues of research are suggested.

2. Generalized Chern-Simons matrix models

The idea, already introduced in [6], is that Chern-Simons quantities can be obtained from a very precise coarse-grained knowledge of infinitely many different distributions, not just the log-normal. In matrix model language: there are infinitely many, equivalent, matrix models. Remarkable particular cases are:

1) Discrete matrix model. The discretization of the models in [5, 6] is deeply related to the quantum group symmetry of Chern-Simons theory, and leads directly to 2dYM [7, 8].

2) The continuous and periodic models. The main feature is the presence of log-periodic [30, 31][1] and periodic behavior.

These examples underline to what extent the matrix models can be deformed while preserving the Chern-Simons information.

2.1. **Preliminaries.** For future use, we quote here the moments of the weight function associated with the Stieltjes-Wigert polynomials:

$$(5) \qquad \frac{k}{\sqrt{\pi}} \int_0^\infty \mathrm{d}x \, x^n \mathrm{e}^{-k^2 (\log x)^2} = \mathrm{e}^{(n+1)^2/4k^2} .$$

In what follows we will set $q = \mathrm{e}^{-1/2k^2}$. Notice that the above formula is valid as long as $\operatorname{Re} k^2 \geq 0$. For the applications in Chern-Simons theory, we will be interested in the two cases $\operatorname{Re} k^2 = 0$ and $\operatorname{Im} k^2 = 0$.

We now consider the set of weights that satisfy the functional equation:

$$(6) \qquad w(qx) = \sqrt{q} \, x \, w(x),$$

it is easy to see that w also satisfies

$$(7) \qquad w(xq^n) = q^{n^2/2} x \, w(x).$$

Let us normalize w such that

$$(8) \qquad q^{-1/4} \int_0^\infty \mathrm{d}x \, w(x) = 1 .$$

Applying the proof of proposition 2.1 of [35] to this weight, it now follows that it has the same moments as the log-normal weight:

$$(9) \qquad \int_0^\infty \mathrm{d}x \, x^n w(x) = q^{-(n+1)^2/2} .$$

The log-normal distribution (5) is a distinguished case of an undetermined moment problem and a basic result from the theory of moments is that the set of solutions of an indeterminate moment problem contains discrete measures as well as absolutely continuous measures. Indeed, it was Stieltjes in his seminal work [28], the first to show that all the functions in the following family:

$$(10) \qquad f_\vartheta (x) = \mathrm{e}^{-\log^2 x} \left(1 + \vartheta \sin (2\pi \log x)\right), \quad \text{with } \vartheta \in [-1, 1],$$

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[1]This behavior is typical of models with a discrete-scale invariance [31]. It appears in rigorous formulations of fractal geometry [30] and in several condensed-matter physics applications [31]. Connections with quantum group symmetries are discussed in [32]. Recent physical applications involving this behavior include limit-cycles in RG flows (see [33] for example) and critical phenomena in gravitational collapse [34].

have the same moments, the log-normal moments. Thus, the parameter ϑ does not play any role regarding integer moments and the infinitely many functions in the family $g_\vartheta(x)$ all have the same integer moments (and consequently, the same orthogonal polynomials). This case corresponds to $k = 1$. More generally, for an arbitrary k, the function is then:

$$(11) \qquad f_\vartheta(x) = e^{-k^2 \log^2 x}\left(1 + \vartheta \sin\left(2\pi \frac{\log x}{\log q}\right)\right), \quad \text{with } \vartheta \in [-1, 1],$$

and with $q \equiv e^{-1/2k^2}$ as above.

We consider one more class of weight functions, namely the discrete ones [35, 36]:

$$(12) \qquad w(x) = \frac{1}{\sqrt{q}M(c)} \sum_{n=-\infty}^{\infty} c^n q^{\frac{n^2}{2}+n}\delta(x - cq^n),$$

where

$$(13) \qquad M(c) = (-cq\sqrt{q}, -\sqrt{q}/c, q; q)_\infty .$$

From now on we will take q to be real and c an arbitrary real positive constant, $c > 0$. It is not hard to see that w again has the same moments (9): one just uses the Jacobi triple product identity

$$(14) \qquad \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = (x, q/x, q; q)_\infty,$$

and translates the sum.

2.2. **Generic description.** In fact, it is possible to have a more general characterization of the solutions of the log-normal moment problem. Following [36, 35], consider two generic weight functions, $\omega_1(x)$ and $\omega_2(x)$ satisfying the functional equation (6). Since the functional equation determines the log-normal moments, these generic functions possess the log-normal moments. Consider the quotient: $g(x) = \omega_1(x) / \omega_2(x)$, which leads to a function that satisfies a functional equation:

$$(15) \qquad g(x) = g(qx),$$

and thus, two generic functions with log-normal moments differ by a function that satisfies (15). This type of function can be naturally named, as in [36, 35], q-periodic, and it is manifest that any logarithmic oscillatory term (with the proper frequency, as in (11) for example) satisfies this property. This type of function is very natural when q deformations are present.

Thus, any integral constructed from the moments with any of the above weights will give the same result. We can now apply this to matrix models where we have N such integrals. Consider the following matrix model:

$$(16) \qquad Z = \int_0^\infty dx_1 \ldots \int_0^\infty dx_N \, e^{-\frac{1}{2g_s}\sum_{i=1}^N \log^2 x_i} \prod_{k<l}(x_k - x_l)^2 \, P(x_1, \ldots, x_N),$$

where P has a finite Laurent expansion in the x's. We write it as follows:

$$(17) \qquad P(x_1, \ldots, x_N) = \sum_{n_1 \ldots n_N} p_{n_1 \ldots n_N} x_1^{n_1} \ldots x_N^{n_N},$$

for some coefficients $p_{n_1 \ldots n_N}$, where the n_i's can be positive or negative integers. Notice that P is a symmetric Laurent series in the x_i's, since an antisymmetric

piece in any two x_i's would not contribute to the integral. We can expand P in a basis of symmetric polynomials or, alternatively, a basis of Schur functions.

It is now obvious that (16) inherits the invariance under deformation of the weights as discussed above. The Vandermonde interaction itself is a polynomial of degree $N(N-1)$ in x, so the integrand factorizes into a product of integrals over the x_i's, and the weights can be deformed for each integral. Thus, we get that

$$Z = \int_0^\infty dx_1 \ldots \int_0^\infty dx_N \, e^{-\frac{1}{2g_s} \sum_{i=1}^N \log^2 x_i} \prod_{k<l} (x_k - x_l)^2 \, P(x_1, \ldots, x_N)$$

$$(18) \qquad = \int_0^\infty dx_1 \ldots \int_0^\infty dx_N \, w(x_1) \ldots w(x_N) \prod_{k<l} (x_k - x_l)^2 \, P(x_1, \ldots, x_N),$$

where $w(x)$ is any of the above weights, whether continuous or discrete.

Therefore, the following matrix model is also invariant under the same deformations:

$$(19) \qquad Z = \int [dM] e^{-\frac{1}{2g_s} \operatorname{Tr}(\log M)^2} f(M),$$

where $f(M)$ has a finite Laurent expansion in the moments $\operatorname{Tr} M^n$, and M is a Hermitian matrix.

We now apply the above to Chern-Simons theory. Consider the partition function of Chern-Simons theory on a Seifert space $M = X(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n})$. This is obtained by doing surgery on a link in S^3 with $n+1$ components, out of which n are parallel, unlinked unknots, and one has link number 1 with each of the n unknots. The surgery data are p_j/q_j for the unlinked unknots, $j = 1, \ldots, n$, and 0 for the last component. The partition function is[2] [5]:

$$(20) \quad \mathbf{Z}_{\mathrm{CS}}(M) = \frac{(-1)^{|\Delta_+|}}{|\mathcal{W}|\,(2\pi i)^r} \left(\frac{\mathrm{Vol}\,\Lambda_{\mathrm{w}}}{\mathrm{Vol}\,\Lambda_{\mathrm{r}}} \right) \frac{[\mathrm{sign}(P)]^{|\Delta_+|}}{|P|^{r/2}} e^{\frac{\pi i d}{4}\mathrm{sign}(H/P) - \frac{\pi i d y}{12l}\phi}$$

$$\times \sum_{t \in \Lambda_{\mathrm{r}}/H\Lambda_{\mathrm{r}}} \int d\beta \, e^{-\beta^2/2g_s - lt\cdot\beta} \frac{\prod_{i=1}^n \prod_{\alpha>0} 2\sinh\frac{\beta\cdot\alpha}{2p_i}}{\prod_{\alpha>0} \left(2\sinh\frac{\beta\cdot\alpha}{2} \right)^{n-2}} .$$

This expression gives the contribution of the reducible flat connections to the partition functions. Recall that for both S^3 and lens spaces this amounts to the exact partition function. The case $n = 0$ corresponds to the three-sphere S^3 that leads to (3). Thus, for the case of $U(N)$, and focusing on a particular sector of flat connections, we get the following matrix model:

$$(21) \qquad Z_{\mathrm{CS}}(M) = \prod_{i=1}^N \int_{-\infty}^\infty dy_i \, e^{-y_i^2/2g_s - lt_i y_i} \frac{\prod_{j=1}^n \prod_{k<l} 2\sinh\frac{y_k - y_l}{2p_j}}{\prod_{k<l} \left(2\sinh\frac{y_k - y_l}{2} \right)^{n-2}} .$$

We used a different letter to remind ourselves that to obtain the full partition function one needs to include the constant prefactor and sum over t. Notice that because of the integrals, the integrand in the above expression is automatically symmetric in the t_i's.

All we have to do in order to show invariance under deformations of the weight is to show that this is of the form (16). But this should be clear. As in [6], we

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[2]See the Appendix for details on the notation

perform a coordinate transformation

(22) $$y_i = a \log x_i + b,$$

for some constants a and b. We get:

$$
Z_{\mathrm{CS}}(M) = a^N \mathrm{e}^{-\frac{Nb^2}{2g_s} - bl \sum_{i=1}^N t_i} \int_0^\infty \mathrm{d}x_1 \ldots \int_0^\infty \mathrm{d}x_N \, \mathrm{e}^{-\frac{a^2}{2g_s} \sum_{i=1}^N \log^2 x_i} (x_1 \ldots x_N)^{-d}
$$

(23) $$\times \prod_{i=1}^N \frac{1}{x_i^{alt_i}} \prod_{k<l} \left((x_k^a - x_l^a)^{2-n} \prod_{i=1}^n (x_k^{a/p_i} - x_l^{a/p_i}) \right),$$

where

(24) $$d = 1 + \frac{ab}{g_s} + \frac{1}{2} a(N-1)(\sum_{i=1}^n \frac{1}{p_i} + 2 - n) .$$

Obviously, a sufficient condition for this to be of the form (16) is to choose $a = \tilde{a}\, p_1 \ldots p_n$, $b = \tilde{c}\, g_s$, where $\tilde{a}, \tilde{b} \in \mathbb{Z}$. Thus, all these models can be deformed in the way shown above. Basically, any matrix model with an integrand where the periodicity is broken by a Gaussian weight, will have this deformation property. This includes all of the known Chern-Simons matrix models, the most general one being that for torus link invariants [37]. We will discuss this elsewhere [38].

2.3. **Discrete model and 2d qYM.** In the early 70's, Chihara employed the functional equation (6) to show the existence of an infinite number of discrete measures with the same log-normal moments [36]. An example is (12), discussed above. This case is remarkable, since it is completely explicit. Furthermore, it not only leads to a discrete Chern-Simons but it also turns out that this model is immediately relevant to the correspondence between Chern-Simons theory and 2D Yang-Mills [7] as we shall see at the end of this section.

Note that the moments of (12) do not depend on the parameter c. Thus, as in the case of the continuous functions, this example already contains infinitely many functions with the same moments. The discrete case leads, for $U(N)$ and S^3, to the following Chern-Simons matrix model :

(25) $$Z = C_N \int \prod_{i=1}^N dx_i \sum_{n=-\infty}^\infty c^n q^{n+n^2/2} \delta\left(x - cq^n\right) \prod_{i<j} (x_i - x_j)^2$$

$$= \widetilde{C}_N \sum_{n_1=-\infty}^\infty \cdots \sum_{n_N=-\infty}^\infty \prod_{i=1}^N c^{n_i} q^{n_i^2 + n_i/2} \prod_{i<k} (q^{n_j} - q^{n_k})^2 .$$

At this point, it is worth to recall the works on combinatorial quantization [39, 40], whose aim is a mathematically careful quantization of pure Chern-Simons theory. These works *simulate* Chern-Simons theory on a lattice, in such a way that partition functions and correlators of the lattice model coincide with those of the continuous model. Interestingly enough, in [39] it is shown that one can also choose the point of view of constructing noncommutative gauge fields starting from some symmetry algebra placed at the lattice sites. It might be a quantum symmetry algebra, but one can also choose another symmetry algebra. In [39], they choose the gauge symmetry algebra. Doing this, it is found that the gauge theory is indeed reconstructed if the symmetry algebra is endowed with a co-multiplication. That

is to say, that it is extended to a Hopf algebra, essentially. This is a remarkable check of the role of the quantum group symmetry in the discretization.

The previous expression was a discretization of the Stieltjes-Wigert ensemble, the discretization of (3) would be:

$$
(26) \qquad Z = \sum_{u_1,\ldots,u_N=-\infty}^{\infty} \mathrm{e}^{-g_s \sum_{i=1}^{N} u_i^2} \prod_{i<j} \sinh^2 (u_i - u_j) \ .
$$

Note also that, as one can easily see from the observation of the value of the coefficients at each point of the lattice, we do not only have a discretization of the log-normal or the Gaussian, but there is also a sort of modular transformation -like in Poisson resummation for example- since the weights are of the type $q^{n^2} = \mathrm{e}^{-g_s n^2}$, in contrast to the e^{-n^2/g_s} that would come from a direct discretization of the original matrix models. This also seems to be analogous to what happens in combinatorial quantization, where the symmetry at each site of the lattice is a quantum group symmetry.

The formula (26), coming from the partition function of Chern-Simons on S^3, is easily seen to give the partition function of 2dYM on the cylinder with trivial holonomies around the two endpoints of the cylinder [41]. It is also the partition function of q-deformed 2dYM on S^2 [8]. As explained in [7, 26], both facts agree since these partition functions are the same. Thus, our discretization directly proves the equivalence between the Chern-Simons matrix model and 2dYM, including the numerical prefactors, which we did not keep track of in (26). This applies to the more general cases worked out in [7] and [8], and in fact it can be generalized to all the cases where discretization applies (for example, the Seifert homology spheres discussed above). It can also be used in the opposite direction, that is, one starts with a discrete model and rewrites it as a continuous matrix model. This can be helpful to compute large N limits of such models, since the large N techniques are much better understood in the continuous case. We will discuss this in more detail elsewhere [38].

3. Oscillatory behavior in Chern-Simons matrix models

3.1. **Connection with complex dimensions and fractal behavior.** The connection with complex dimensions [30, 31] can now be easily obtained. We consider the Mellin transform of the q-periodic function (15):

$$
(27) \qquad h(s) \equiv \int_0^{\infty} \widetilde{\omega}(x) \, x^s dx.
$$

Taking into account the following property of Mellin transforms:

$$
(28) \qquad \int_0^{\infty} \widetilde{\omega}(qx) \, x^s dx = q^{-s} h(s),
$$

and considering the q-periodic property (15) :

$$
(29) \qquad q^{-s} h(s) = h(s) \Rightarrow s_n = \frac{2\pi i n}{\log q}, \quad n \in \mathbb{Z} \ .
$$

Thus, the Mellin transform of a q periodic function contains infinitely many complex poles. From standard results in Mellin asymptotics, this implies that the q periodic function may be fractal. This depends on the precise form of the function. Namely,

on the residue of the poles. This is a usual situation in Mellin asymptotics with complex poles. If the residues decay fast enough with n, then the function will be differentiable.

Another way to arrive at the same conclusion as above, is to notice that $1 + \vartheta \sin\left(2\pi \frac{\log x}{\log q}\right)$ in (11) is just one possible deformation among many. The parameter ϑ takes its possible values in $[-1, 1]$ just to ensure that the resulting function is positive definite everywhere. Of course, the oscillatory term can be easily generalized. For example, one can consider:

$$(30) \qquad f(x) = e^{-k^2 \log^2 x}\left(1 + \sum_{n=1}^{m} a_n \sin\left(2\pi n \frac{\log x}{\log q}\right)\right),$$

and the integer moments keep their log-normal value.

3.2. **Periodic/Self-similar matrix models.** Note that in (30) we are still limiting the finest scale possible, since the sum stops at $n = m$ and thus the function is not a genuine fractal, albeit the oscillatory pattern is certainly much richer than in (11). But we can consider a full Fourier series and take $m \to \infty$ as long as we restrict the coefficients in order to ensure positiveness of the function. This restriction is actually very mild. Consider for example, $a_n = n^{-\gamma}$ with $\gamma > 1$; then:

$$(31) \qquad f(x) = e^{-k^2 \log^2 x}\left(1 + \lambda \sum_{n=1}^{\infty} \frac{1}{n^\gamma} \sin\left(2\pi b_n \frac{\log x}{\log q}\right)\right),$$

is positive definite for a low enough value of the parameter λ. One of these functions looks as:
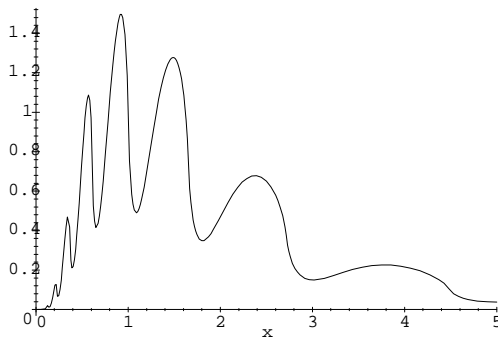

FIGURE 1. $f(x)$ in (31) with $a_n = n^{-2}, b_n = n, k = 1, \lambda = 0.5, m = 200$

It is plain that for some low enough values of γ, the resulting function is fractal [42]. Namely, continuous but differentiable nowhere. An example of such a function is plotted in Figure 2.

Therefore, we have a family of non-polynomial potentials, all of them equivalent in the sense that they give rise to the same partition functions. The other possible relevant quantities, such as the density of states and correlation functions are trivially related [6].
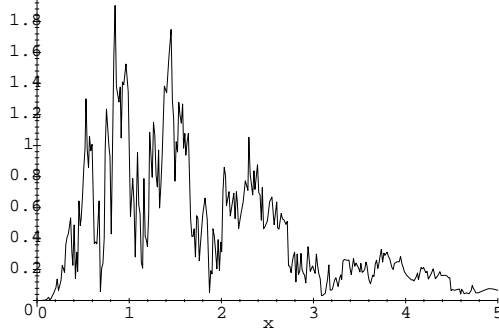
FIGURE 2. $f(x)$ in (31) with $a_n = n^{-1}, b_n = n^2, k = 1, \lambda = 0.5, m = 50$

Going back to the original Chern-Simons matrix model, we are lead to ensembles such as, for example:

$$
(32) \quad Z = \int \prod_{i=1}^{N} e^{-\frac{u_i^2}{2g_s}} \left(1 + \lambda \sum_{n=1}^{\infty} a_n \sin \left(2\pi b_n \frac{u_i + Ng_s}{\log q} \right) \right) du_i \prod_{i<j} \left(2 \sinh \frac{u_i - u_j}{2} \right)^2 ,
$$

with suitable a_n and b_n. At the level of the confining potential we have:

$$
(33) \quad V(u) = u^2 - \ln \left(1 + \lambda \sum_{n=1}^{\infty} a_n \sin \left(2\pi b_n \frac{u + Ng_s}{\log q} \right) \right),
$$

Since $V(x) = -\ln \omega(x)$, the derivative of the potential is the logarithmic derivative of the weight function $V'(x) = -\frac{\omega'(x)}{\omega(x)}$ and thus, non-differentiability of $\omega'(x)$ implies non-differentiability of $V'(x)$, since $\omega(x)$ is a continuous function. Therefore, the potential itself may be a fractal. This possibility depends on the choice of the coefficients a_n and b_n as the previous figures clearly show. A more detailed analysis follows from results in Fourier series [42]. While we shall discuss this mathematical property elsewhere, recall that if we have a Fourier series such as $f(u) = \sum_m a_m \exp(imu)$, where a_m have random or pseudorandom phases, then, if the power spectrum $|a_m|^2$ has the asymptotic form:

$$
(34) \quad |a_m|^2 \sim |m|^{-\beta} \quad \text{as } |m| \to \infty, \text{ where } 1 < \beta \leq 3,
$$

the graphs of $\text{Re} f$ and $\text{Im} f$ are continuous but non-differentiable, with fractal dimension:

$$
(35) \quad D_f = \frac{1}{2} (5 - \beta).
$$

3.3. **Oscillatory behavior of the bare model: density of states.** The way oscillatory behavior appears in the matrix models may seem surprising. Nevertheless, discrete scale invariance is known to be related to q-deformations with q real [32], so its appearance in Chern-Simons matrix models is natural in fact. Furthermore, the oscillatory terms seem to be intimately related to an $SL(2, \mathbb{Z})$ invariance. We shall discuss this elsewhere, but at least we can readily show now that oscillatory

behavior is also a feature even in the original *bare* model (3). For this, we have just to consider the famous expression for the density of states of a Hermitian matrix model [1]:

$$(36) \qquad \rho(x) = \omega(x) \sum_{n=0}^{N-1} P_n^2(x),$$

valid for any N. Some features of this quantity have already been studied as they are useful in the context of topological strings [4]. However, in this case, one is interested in the perturbative expansion in N. This leads one to consider the 't Hooft limit where $\lambda = N g_s$ is kept constant while $N \to \infty$. Since this corresponds to $q \sim 1$, the limit where the q-matrix model approaches a classical one, we lose the oscillations in (36), as we shall see. On the other hand, to give a generic expression (e.g. the analogous of the semi-circle law for the Hermitian Gaussian model) is also not very useful. The problem is that there are apparently no analogues of the Plancherel-Rotach asymptotic formulas [1] and therefore any expression is rather cumbersome. But just to get a feeling of the behavior of the density of states for arbitrary q, we can carry -employing *Mathematica*, for example- exact analytic computations for various values of N and q, to get an idea of the beautiful behavior of the density of states. More precisely, we employ (36) with:

$$(37) \quad P_n(x) \;=\; (-1)^n \, q^{n/2+1/4} \prod_{j=1}^{n} \left(1 - q^j\right)^{-1/2} \sum_{v=0}^{n} \binom{n}{v}_q q^{v^2} \left(-q^{1/2} x\right)^v,$$

$$\omega(x) \;=\; \mathrm{e}^{-k^2 \log^2 x} \text{ with } \binom{n}{v}_q = \frac{\prod_{i=0}^{v-1}\left(1 - q^{n-i}\right)}{\prod_{i=0}^{v-1}\left(1 - q^{i+1}\right)}.$$

That is, the log-normal weight function and the Stieltjes-Wigert polynomials [35]. We also go back to the original variables, employing $x = \mathrm{e}^{u - N g_s}$, and thus, obtain several plots of the density of states of the model (3).

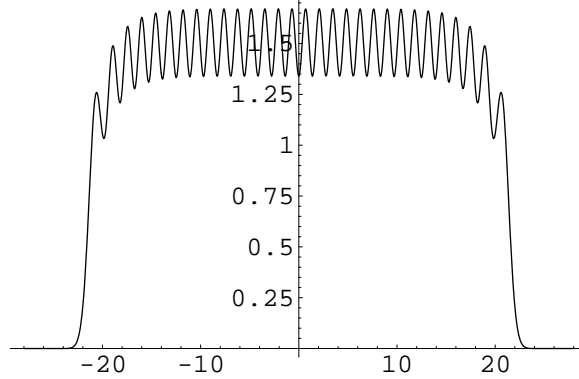
FIGURE 3. $\rho(u)$ of model (3), based on (36)-(37) with $q = 0.5, N = 30$

Note how the compact support is much bigger than that of a Gaussian ensemble, for example. Actually, it is of the order $N g_s$, in comparison with the $\sqrt{N}$ of a Gaussian model. The oscillations that in a Gaussian model are smoothed out, are here manifest, as can be seen in Fig. 3 for example. The number of oscillations

coincides with N. Crystalline behavior is known to be linked with the presence of quantum group symmetries [32].
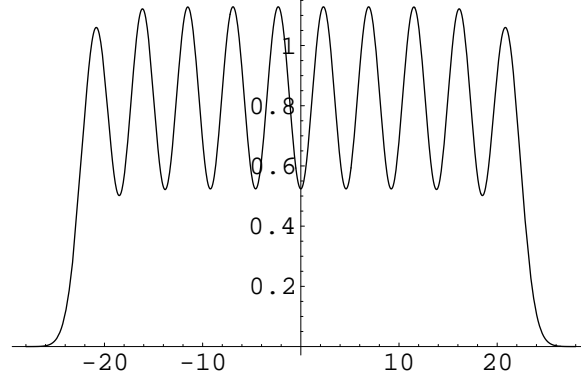
FIGURE 4. $\rho(u)$ of model (3), based on (36)-(37) with $q = 0.3, N = 10$

With lower values of q the oscillations should be more patent and this is exactly what happens in the next plot, Figure 4. Note also how the support grows with q. Finally, we go into a higher size where, in contrast to the Wigner-Dyson case, the amplitude of the oscillatory behavior remains the same, as can be seen in Figure 5. However, note how keeping the same size but taking q towards 1 we are approaching the familiar semi-circle like behavior. The oscillations considerably decrease in amplitude and so does the support of the distribution. It is therefore, as shown in Fig. 6, closer to the Wigner-Dyson case were the oscillations are of smaller amplitude and are essentially smoothed out when N is very large.
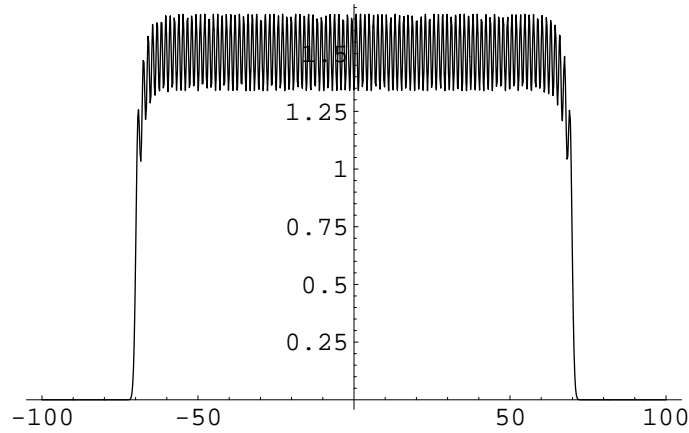
FIGURE 5. $\rho(u)$ of model (3), based on (36)-(37) with $q = 0.5, N = 100$

4. EQUIVALENCE OF EFFECTIVE SUPERPOTENTIALS

Now we would like to further investigate the possible consequences of the equivalence that we have found. It seems that this equivalence in Chern-Simons matrix
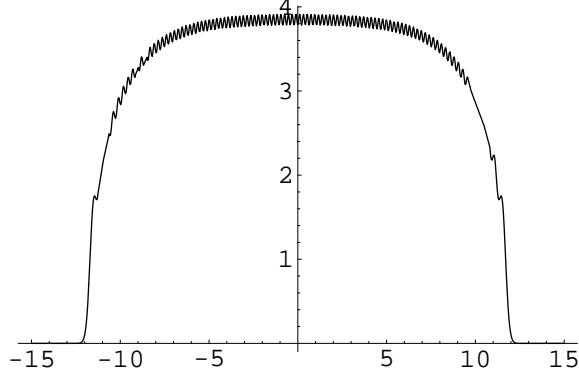
FIGURE 6. $\rho\left(u\right)$ of model (3), based on (36)-(37) with $q = 0.9, N = 100$

models may be related to recent findings in $\mathcal{N} = 1$ gauge theory [9, 10, 11] (see [43] for a review on planar equivalence of related theories). More precisely, these works show that different $\mathcal{N} = 1$ supersymmetric gauge theories can have an effective glueball superpotential with exactly the same functional form. In [10] for example, it is found that the low-energy effective superpotential of an $\mathcal{N} = 1$ $U(N)$ gauge theory with matter in the adjoint and arbitrary even tree-level superpotential has the same functional form as the effective superpotential of a $U(N)$ gauge theory with matter in the fundamental and the same tree-level interactions.

Then, taking into account Dijkgraaf-Vafa [44], the effective superpotential of an $\mathcal{N} = 1$ supersymetric theory is computed by a matrix model whose potential is the tree level superpotential of the gauge theory, then the equivalence of effective superpotentials, which has been derived employing $\mathcal{N} = 1$ techniques -like Konishi anomaly-, implies the equivalence of the free energies of the corresponding matrix models.

Note that equivalence of free energies is also one of the mathematical results obtained in this paper. However, we have discussed Chern-Simons matrix models. Do these models have any relationship or implication for $\mathcal{N} = 1$ theories ? Indeed, we have just to recall that at low energies and in four dimensions, IIA theory compactified on T^*S^3 and with N $D6$ branes wrapping S^3 reduces to $\mathcal{N} = 1$ gauge theory. Therefore, again due to [44], a Hermitian matrix model describing CS theory on S^3 leads to the superpotential of such an $\mathcal{N} = 1$ theory, which is [45]:

$$(38) \qquad W = \sum_n \left(S + 2\pi ni\right) \log \left(S + 2\pi ni\right)^{-N}.$$

This role of the Chern-Simons matrix model is already discussed in [18]. Therefore, this superpotential can be interpreted as the effective superpotential of an $\mathcal{N} = 2$ theory whose tree-level superpotential is the CS matrix model potential, and we have just seen that there are infinitely many choices for this potential -of course, once again, all of them leading to an identical free energy-, so, all of them lead to the same superpotential, namely to (38).

That is to say, (38) as the effective superpotential of an $\mathcal{N} = 2$ theory whose tree-level superpotential is:

$$(39) \qquad\qquad V(M) = k^2 \log^2 M + \log g(M),$$

with a $q-$periodic $g(M) = g(qM)$, and $q = \mathrm{e}^{-1/2k^2} = \mathrm{e}^{-g_s}$ as usual. So, regardless the precise form for $g(M)$, we always end up with a superpotential with the same functional form. Furthermore, recall that the role of (39) can also be played by a discrete potential:

$$(40) \qquad\qquad V(M) = -\log\left(\sum_{n=-\infty}^{\infty} c^n q^{\frac{n^2}{2}+n} \delta(M - cq^n)\right),$$

that, dropping constant terms, comes straightforwardly from (12) and (25).

Finally, just to mention that in [8], a different Hermitian matrix model for Chern-Simons theory was considered:

$$(41) \qquad V(M) = M^2 + \frac{1}{2}\sum_{k=1}^{\infty} a_k \sum_{s=0}^{2k} (-1)^s \binom{2k}{s} \mathrm{Tr}M^s \mathrm{Tr}M^{2k-s}.$$

This cumbersome expression, including double-trace terms, comes from the transformation:

$$(42) \qquad\qquad \prod_{i<j}\left(2\sinh\frac{u_i - u_j}{2}\right)^2 = \prod_{i<j}(u_i - u_j)^2 f(u),$$

where $f(u)$ is the logarithm of the second term in (41) [8]. Since this only comes from manipulations on the repulsion term of the matrix model, it is independent of the weight function. Thus, employing our result, one can consider the generalized weight, instead of just the quadratic one and then apply the method in [8], based on the trick (42). Therefore, even if one wants to focus on these type of Hermitian matrix models, the equivalence result identically holds. That is to say, the addition of periodic terms, such as the previously discussed (33), following the M^2 term, will not modify the role of (41), as happens with $g(M)$ in (39).

Thus, the arguments and results of this Section are the same regardless the Hermitian matrix models considered. Another different issue is the comparison between these Hermitian matrix models, that we shall discuss elsewhere.

To conclude, note that, while the oscillatory terms may seem at first a bit awkward, they may be related with the particular form of (38). Namely, with the existence of infinitely many domain walls [45]. In addition, these oscillatory terms, as mentioned, seem to be related to $SL(2, \mathbb{Z})$, which is a crucial ingredient in the canonical quantization of Chern-Simons theory, and thus in the derivation [5] of the original matrix model.

5. Conclusions and Outlook

From [6] and the present work, it can be said that numerous Chern-Simons quantities can be extracted from a certain *coarse-grained* knowledge of either a Gaussian distribution or a log-normal distribution. This *coarse-grained* knowledge, is nothing else than a suitable combination of the set of positive integer moments of the distribution. This particular combination of the integer moments is the one given by the orthogonal polynomials. But it turns out that the matrix models have such a

fluctuating weight function that this coarse-grained information -the moments- no longer fixes the weight function uniquely, in sharp contrast with polynomial potentials, for example. Therefore, one has the freedom of modifying the weight function, while leaving intact certain amount of information. This reduced information is the only one required by the matrix model, and consequently by Chern-Simons theory. We have shown the rather large extent to which different superpotentials can be equivalent. Mathematically, both the discrete case and the oscillatory -especially the ones with very low differentiability- seem interesting on their own. Note that the most extreme cases are matrix models with fractal potentials that, nevertheless, give rise to physically meaningful objects like Chern-Simons partition functions.

Regarding possible avenues for further research, discrete matrix models have been poorly studied in gauge theory, at least in comparison with continuous models. The technology available in these type of models is already quite formidable and, in addition, many remarkable combinatorial properties of these models are known [46], so their study may prove useful in relationship with concepts relevant to topological string theory, like crystal melting. In addition, as discussed above, our methods provide a direct understanding of why at the level of the matrix models Chern-Simons theory and 2d qYM are equivalent. It also gives a concrete computational scheme where one can straightforwardly extend this equivalence between continuous and discrete models to other cases, namely, to more general manifolds like Seifert homology spheres, and generic torus link invariants. It would be interesting to see whether the discrete model that one gets is still reproduced by a BF-type of theory, and on which manifold. Hopefully, the equivalence between the continuous and the discrete matrix models can be applied to compute large N limits of 2dYM and topological strings on more general manifolds as well. Needless to say, our results apply also to orthogonal and symplectic groups, so one could consider Calabi-Yau orientifolds.

Interestingly enough, it seems that this non-uniqueness also leads to the equivalence of effective superpotentials in $\mathcal{N} = 1$, through a rather different route from [9, 10, 11]. A further understanding of this seems desirable. Another very much related question would be to understand if the precise way in which this non-uniqueness occurs in the matrix models is related to the specific form of the superpotential (38), found in [45].

Appendix A. Notation and Chern-Simons quantities

We give here some information about the Chern-Simons quantities that appear in the text, mainly in (20). For more information, see [5] and references therein. To understand the origin of other quantities in (20), one has to take into account the constructions of Seifert homology spheres from surgery. Seifert homology spheres can be constructed by performing surgery on a link $\mathcal{L}$ in $\mathbf{S}^3$ with $n+1$ components, consisting on n parallel and unlinked unknots together with a single unknot whose linking number with each of the other n unknots is one. The surgery data are p_j/q_j for the unlinked unknots, $j = 1, \cdots, n$, and 0 on the final component. p_j is coprime to q_j for all $j = 1, \cdots, n$, and the p_j's are pairwise coprime. After doing surgery, one obtains the Seifert space $M = X(\frac{p_1}{q_1}, \cdots, \frac{p_n}{q_n})$. This is rational homology sphere whose first homology group $H_1(M, \mathbf{Z})$ has order $|H|$, where

$$(43) \qquad H = P \sum_{j=1}^{n} \frac{q_j}{p_j}, \quad \text{and} \quad P = \prod_{j=1}^{n} p_j.$$

Another topological invariant that will enter the computation is the signature of $\mathcal{L}$, which turns out to be:

$$(44) \qquad \sigma(\mathcal{L}) = \sum_{i=1}^{n} \text{sign}\left(\frac{q_i}{p_i}\right) - \text{sign}\left(\frac{H}{P}\right).$$

For $n = 1, 2$, Seifert homology spheres reduce to lens spaces, and one has that $L(p, q) = X(q/p)$. For $n = 3$, we obtain the Brieskorn homology spheres $\Sigma(p_1, p_2, p_3)$ (in this case the manifold is independent of q_1, q_2, q_3). In particular, $\Sigma(2, 3, 5)$ is the Poincaré homology sphere. Finally, the Seifert manifold $X(\frac{2}{-1}, \frac{m}{(m+1)/2}, \frac{t-m}{1})$, with m odd, can be obtained by integer surgery on a $(2, m)$ torus knot with framing t. Note that in (20) the weight and root lattices of G are denoted by Λ_{w} and Λ_{r}, respectively.

Finally, there is a phase factor in (20) that comes from the framing correction, that guarantees that the resulting invariant is in the canonical framing for the three-manifold M. Its explicit expression is:

$$(45) \qquad \phi = 3\text{sign}\left(\frac{H}{P}\right) + \sum_{i=1}^{n} 12s\left(q_i, p_i\right) - \frac{q_i}{p_i},$$

where $\sigma(\mathcal{L})$ is again the signature of the linking matrix of $\mathcal{L}$ and $s(p, q)$ is the Dedekind sum

$$(46) \qquad s(p, q) = \frac{1}{4q} \sum_{n=1}^{q-1} \cot\left(\frac{\pi n}{q}\right) \cot\left(\frac{\pi n p}{q}\right).$$

References

[1] M.L. Mehta, *Random matrices*, second ed., Academic Press, Boston, MA, 1991.

[2] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, "Planar Diagrams," Commun. Math. Phys. **59**, 35 (1978).

[3] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, "2-D Gravity and random matrices," Phys. Rept. **254**, 1 (1995) arXiv:hep-th/9306153.

[4] M. Mariño, "Les Houches lectures on matrix models and topological strings," arXiv:hep-th/0410165.

[5] M. Mariño, "Chern-Simons theory, matrix integrals, and perturbative three-manifold invariants," Commun. Math. Phys. **253**, 25 (2004) arXiv:hep-th/0207096.

[6] M. Tierz, "Soft matrix models and Chern-Simons partition functions," Mod. Phys. Lett. A **19**, 1365 (2004) arXiv:hep-th/0212128.

[7] S. de Haro, "Chern-Simons theory in lens spaces from 2d Yang-Mills on the cylinder," JHEP **0408**, 041 (2004) arXiv:hep-th/0407139.

[8] M. Aganagic, H. Ooguri, N. Saulina and C. Vafa, "Black holes, q-deformed 2d Yang-Mills, and non-perturbative topological strings," arXiv:hep-th/0411280.

[9] A. Armoni, A. Gorsky and M. Shifman, "An exact relation for N = 1 orientifold field theories with arbitrary Nucl. Phys. B **702**, 37 (2004) arXiv:hep-th/0404247.

[10] R. Argurio, "Equivalence of effective superpotentials," Phys. Rev. D **70**, 055012 (2004) arXiv:hep-th/0405250.

[11] K. Landsteiner, "Konishi anomalies and curves without adjoints," Nucl. Phys. B **700**, 140 (2004) arXiv:hep-th/0406220.

[12] E. Witten, "Quantum field theory and the Jones polynomial," Commun. Math. Phys. **121** (1989) 351.

[13] M. Mariño, "Chern-Simons theory and topological strings," arXiv:hep-th/0406005.

[14] E. Witten, "Chern-Simons gauge theory as a string theory," hep-th/9207094.

[15] V. Periwal, "Topological closed string interpretation of Chern-Simons theory," Phys. Rev. Lett. **71**, 1295 (1993) arXiv:hep-th/9305115; M. R. Douglas, "Chern-Simons-Witten theory as a topological Fermi liquid," hep-th/9403119

[16] R. Gopakumar and C. Vafa, "On the gauge theory/geometry correspondence," Adv. Theor. Math. Phys. **3**, 1415 (1999) arXiv:hep-th/9811131; H. Ooguri and C. Vafa, "Knot invariants and topological strings," Nucl. Phys. B **577**, 419 (2000) arXiv:hep-th/9912123.

[17] L. Rozansky, "A large k asymptotics of Witten's invariant of Seifert manifolds," hep-th/9303099, Commun. Math. Phys. **171** (1995) 279; "A contribution of the trivial connection to Jones polynomial and Witten's invariant of 3-D manifolds. 1, 2," hep-th/9401061, 9403021, Commun. Math. Phys. **175** (1996) 275, 297; "Residue formulas for the large k asymptotics of Witten's invariants of Seifert manifolds: The case of $SU(2)$," hep-th/9412075, Commun. Math. Phys. **178** (1996) 27.

[18] M. Aganagic, A. Klemm, M. Mariño and C. Vafa, "Matrix model as a mirror of Chern-Simons theory," JHEP **0402**, 010 (2004) arXiv:hep-th/0211098.

[19] N. Halmagyi and V. Yasnov, "Chern-Simons matrix models and unoriented strings," JHEP **0402**, 002 (2004) arXiv:hep-th/0305134; "The spectral curve of the lens space matrix model," arXiv:hep-th/0311117.

[20] N. Halmagyi, T. Okuda and V. Yasnov, "Large N duality, Lens spaces and the Chern-Simons matrix model," JHEP **0404**, 014 (2004) arXiv:hep-th/0312145.

[21] V. Yasnov, "Chern-Simons matrix model for local conifold," arXiv:hep-th/0409136.

[22] T. Okuda, "Derivation of Calabi-Yau crystals from Chern-Simons gauge theory," arXiv:hep-th/0409270.

[23] S. Mizoguchi, "On unitary / Hermitian duality in matrix models," arXiv:hep-th/0411049.

[24] A. Gorsky and V. Lysov, "From effective actions to the background geometry," arXiv:hep-th/0411063.

[25] S. de Haro and M. Tierz, "Brownian motion, Chern-Simons theory, and 2d Yang-Mills," Phys.Lett. **B** 601, 201 (2004) arXiv:hep-th/0406093.

[26] S. de Haro, "Chern-Simons theory, 2d Yang-Mills, and lie algebra wanderers," arXiv:hep-th/0412110.

[27] R. Koekoek and R. F. Swarttouw : The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue. Delft University of Technology, Faculty of Information Technology

and Systems, Department of Technical Mathematics and Informatics, Report no. 98-17, 1998. Available on-line at: http://aw.twi.tudelft.nl/˜koekoek/askey.html

[28] T. Stieltjes, Recherches sur les fractions continues, Anns, Sci. Fac. Univ. Toulouse (1894-1895), 8, J1-J122; 9, A5-A47. English version in T.J. Stieltjes , Collected Papers, G. van Dijk (ed.), Vol II, Springer, Berlin 1993.

[29] B. Simon, "The Classical Moment Problem as a Self-Adjoint Finite Difference Operator", Adv. Math. 137 (1998) 82-203; N.I. Akhiezer, *The Classical Moment Problem*, Hafner Publishing Co., New York, 1965; J.A. Shohat and J.D. Tamarkin, *The problem of moments*, Amer. Math. Soc. Surveys, No. 1 (1943).

[30] M.L. Lapidus and M. van Frankenhuysen 2000, *Fractal Geometry and Number Theory (Complex dimensions of fractal strings and zeros of zeta functions)*, Research Monograph, Birkhäuser, Boston.

[31] D. Sornette, "Discrete Scale invariance and Complex Dimensions," *Phys. Rept.* **297** 239 (1998) arXiv:cond-mat/9707012.

[32] M. Tierz, "Quantum group symmetry and discrete scale invariance: spectral aspects," arXiv:hep-th/0308121.

[33] A. LeClair, J. M. Roman and G. Sierra, "Log-periodic behavior of finite size effects in field theories with RG limit Nucl. Phys. B **700**, 407 (2004) arXiv:hep-th/0312141; E. Braaten and H. W. Hammer, Phys. Rev. Lett. **91**, 102002 (2003) arXiv:nucl-th/0303038.

[34] C. Gundlach, "Critical phenomena in gravitational collapse," Phys. Rept. **376**, 339 (2003) arXiv:gr-qc/0210101; M. W. Choptuik, Phys. Rev. Lett. **70**, 9 (1993).

[35] J.S. Christiansen, "The moment problem associated with the Stieltjes-Wigert polynomials," J. Math. Anal. Appl. **277** (2003) 218-245

[36] T.S. Chihara, "A characterization and a class of distribution functions for the Stieltjes-Wigert polynomials," Canad. Math. Bull. **13** (1970) 529; "On generalized Stieltjes-Wigert and related polynomials," J. Comput. Appl. Math. **5** (1979) 291

[37] M. Mariño, unpublished

[38] In preparation

[39] A. Y. Alekseev, H. Grosse and V. Schomerus, Commun. Math. Phys. **172**, 317 (1995) arXiv:hep-th/9403066.

[40] A. Y. Alekseev, H. Grosse and V. Schomerus, Commun. Math. Phys. **174**, 561 (1995) arXiv:hep-th/9408097.

[41] D. J. Gross and A. Matytsin, "Some properties of large N two-dimensional Yang-Mills theory," Nucl. Phys. B **437** (1995) 541 [arXiv:hep-th/9410054].

[42] B. B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman: San Francisco, 1982.

[43] A. Armoni, M. Shifman and G. Veneziano, "From super-Yang-Mills theory to QCD: Planar equivalence and its implications," arXiv:hep-th/0403071.

[44] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," Nucl. Phys. B **644**, 3 (2002) arXiv:hep-th/0206255, "A perturbative window into non-perturbative physics," arXiv:hep-th/0208048.

[45] C. Vafa, "Superstrings and topological strings at large N," J. Math. Phys. **42**, 2798 (2001) arXiv:hep-th/0008142.

[46] K. Johansson, "Discrete orthogonal polynomial ensembles and the Plancherel measure,", Ann. of Math. (2) **153**, 259 (2001) arXiv:math.CO/9906120

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