TREND TO EQUILIBRIUM FOR THE BECKER-DÖRING EQUATIONS: AN ANALOGUE OF CERCIGNANI'S CONJECTURE

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Abstract. We investigate the rate of convergence to equilibrium for subcritical solutions to the Becker-Döring equations with physically relevant coagulation and fragmentation coefficients and mild assumptions on the given initial data. Using a discrete version of the log-Sobolev inequality with weights we show that in the case where the coagulation coefficient grows linearly and the detailed balance coefficients are of typical form, one can obtain a linear functional inequality for the dissipation of the relative free energy. This results in showing Cercignani's conjecture for the Becker-Döring equations and consequently in an exponential rate of convergence to equilibrium. We also show that for all other typical cases one can obtain an 'almost' Cercignani's conjecture that results in an algebraic rate of convergence to equilibrium.

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1. Introduction

1.1. **The Becker-Döring Equations.** The Becker-Döring equations are a fundamental set of equations which describe the kinetics of a first order phase transition. Amongst the phenomena to which they are relevant one can find crystallisation [20], nucleation of polymers [12], vapour condensation, aggregation of lipids [23] and phase separation in alloys [34]. For more general reviews of nucleation theory see for instance [30, 25].

The Becker-Döring equations give the time evolution of the size distribution of clusters of a certain substance. Denoting by $\{c_i(t)\}_{i\in\mathbb{N}}$, the density of clusters of size i at time $t \geqslant 0$ (i.e. the density of clusters that are composed of i particles), the equations read

$$\frac{\mathrm{d}}{\mathrm{d}t}c_i(t) = W_{i-1}(t) - W_i(t), \qquad i \in \mathbb{N}\setminus\{1\}, \tag{1.1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}c_1(t) = -W_1(t) - \sum_{k=1}^{\infty} W_k(t), \tag{1.1b}$$

where

$$W_i(t) := a_i\, c_1(t)c_i(t) - b_{i+1}\, c_{i+1}(t) \qquad i \in \mathbb{N}. \tag{1.2}$$

and a_i, b_i, assumed to be strictly positive, are the *coagulation and fragmentation coefficients*. They determine, respectively, the rate at which clusters of size i combine with clusters of size 1 to create a cluster of size $i+1$, and the rate at which clusters of size $i+1$ splits into clusters of size i and 1. This corresponds to the basic assumption of the underlying model: if we represent symbolically by $\{i\}$ the chemical species of clusters of size i, then the only (relevant) chemical reactions that take place are

$$\{i\} + \{1\} \rightleftharpoons \{i+1\}.$$

The quantity $W_i(t)$ defined in (1.2) represents the *net rate* of the reaction $\{i\} + \{1\} \rightleftharpoons \{i+1\}$, and under the above set of equations it is easy to see that the *density*, or *mass*, of the solution, defined by

$$\varrho := \sum_{i=1}^{\infty} i c_i(0) = \sum_{i=1}^{\infty} i c_i(t) \tag{1.3}$$

is formally conserved under time evolution. The original equations proposed by Becker and Döring [5] were similar to (1.1), with the slight change that the density of one particle c_1, usually called the monomer density, was assumed to be constant. The current version, motivated by the conservation of total density, was first discussed in [7] and [28] and is widely used in classical nucleation theory.

Much like in other kinetic equations, the study of a state of equilibrium and the convergence to it is a fundamental question in the study of the Becker-Döring equations. Defining the *detailed balance coefficients* Q_i recursively by

$$Q_1 = 1, \quad Q_{i+1} = \frac{a_i}{b_{i+1}} Q_i \quad i \in \mathbb{N} \tag{1.4}$$

one can see that for a given $z \geqslant 0$ the sequence

$$c_i = Q_i z^i \tag{1.5}$$

is formally an equilibrium of (1.1). However, depending on the coagulation and fragmentation coefficients a_i and b_i, many of these formal equilibria do not have a finite mass. The largest $z_s \geqslant 0$, possibly $z_s = +\infty$, for which

$$\sum_{i=1}^{\infty} iQ_i z^i < +\infty \quad \text{for all } 0 \leqslant z < z_s$$

is called the *critical monomer density*, or sometimes the monomer saturation density. The *critical mass* (or, again, saturation mass) is then defined by

$$\varrho_s := \sum_{i=1}^{\infty} iQ_i z_s^i \in [0, +\infty]. \tag{1.6}$$

It is important to note that both z_s and ϱ_s are uniquely determined by a_i and b_i and that $\{Q_i z^i\}_{i\in\mathbb{N}}$ is a finite-mass equilibrium only for $0 \leqslant z < z_s$, with the possibility for the equality $z = z_s$ only when $\varrho_s < +\infty$. Additionally, it is easy to see that for a given finite mass $\varrho \leqslant \varrho_s$ there exists a unique $\overline{z} \geqslant 0$ such that

$$\varrho = \sum_{i=1}^{\infty} iQ_i \overline{z}^i,$$

giving us a candidate for the asymptotic equilibrium state of (1.1) under a given initial data. These are in fact the only finite-mass equilibria (see [3]), and $\overline{z}$ defined above is called the *equilibrium monomer density* for a given mass ρ.

A finite mass solution is called *subcritical* when its mass ϱ, is strictly less than ϱ_s. It is called *critical* if $\varrho = \varrho_s$ and *supercritical* if $\varrho > \varrho_s$, assuming $\varrho_s < +\infty$. In this paper we will only concern ourselves with subcritical solutions. Thus, to avoid triviality we always assume that $z_s > 0$.

The critical density ϱ_s, if finite, marks a change in the behaviour of equilibrium states: if $\varrho < \varrho_s$ then a unique equilibrium state with mass ϱ exists, while if $\varrho > \varrho_s$ no such equilibrium can occur and a phase transition phenomenon takes place — reflected in the fact that the excess density $\varrho - \varrho_s$ is concentrated in larger and larger clusters as time progresses.

1.2. **Previous Results.** Let us briefly review existing results on the mathematical theory of the Becker-Döring equations, which has advanced much since the first rigorous works on the topic [1, 3]. In [3] the authors showed (among other things) existence and uniqueness of a global solution to (1.1) when

$$a_i \leqslant C_1 i, \quad b_i \leqslant C_2 i, \qquad \sum_{i=1}^{\infty} i^{1+\epsilon} c_i(0) < +\infty, \tag{1.7}$$

for some constants $C_1, C_2, \epsilon > 0$. As expected, under the above assumptions the unique solution conserves mass (this is, (1.3) holds rigorously). This basic existence theory is applicable to all solutions we consider in this work.

The asymptotic behaviour of solutions to (1.1) is one of the most interesting aspects of the equation. Supercritical behaviour, while not dealt with in this work, has a particularly interesting link to late-stage coarsening and has been studied extensively in [27, 31, 15, 24], with several questions still open. Asymptotic approximations of such solutions have been developed in [17, 18, 22].

Regarding the subcritical regime, it was proved in [1, 3] that solutions with subcritical mass ϱ approach the unique equilibrium with this mass (determined by (1.3)). A fundamental quantity in understanding this approach is the *free energy*, $H(\boldsymbol{c})$, defined for any nonnegative sequence $\boldsymbol{c} = \{c_i\}_{i\in\mathbb{N}}$ by

$$H(\boldsymbol{c}) := \sum_{i=1}^{\infty} c_i \left(\log \frac{c_i}{Q_i} - 1 \right), \tag{1.8}$$

whenever the sum converges. It can be shown that $H(\boldsymbol{c}(t))$ decreases along solutions $\boldsymbol{c} = \boldsymbol{c}(t)$ to the Becker-Döring equations; in fact, for a (strictly positive, suitably decaying for large i) solution $\boldsymbol{c}(t) = \{c_i(t)\}_{i\in\mathbb{N}}$ of (1.1) we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} H(\boldsymbol{c}(t)) &= -D(\boldsymbol{c}(t)) \\ &:= -\sum_{i=1}^{\infty} a_i Q_i \left(\frac{c_1 c_i}{Q_i} - \frac{c_{i+1}}{Q_{i+1}} \right) \left(\log \frac{c_1 c_i}{Q_i} - \log \frac{c_{i+1}}{Q_{i+1}} \right) \leqslant 0. \end{aligned} \tag{1.9}$$

This free energy is motivated by physical considerations and constitutes a Lyapunov functional for our equation. Since it does not have a definite sign we define a more natural candidate to measure the distance of $\boldsymbol{c}(t) = \{c_i(t)\}_{i\in\mathbb{N}}$ to the equilibrium. Using the notation

$$(\mathcal{Q}_z)_i = Q_i z^i$$

and denoting by $\mathcal{Q} = \mathcal{Q}_{\overline{z}}$, we can define the *relative* free energy as

$$H(\boldsymbol{c}|\mathcal{Q}) := \sum_{i=1}^{\infty} c_i \left(\log \frac{c_i}{\overline{z}^i Q_i} - 1 \right) + \sum_{i=1}^{\infty} \overline{z}^i Q_i = H(\boldsymbol{c}) - \log \overline{z} \sum_{i=1}^{\infty} i c_i + \sum_{i=1}^{\infty} \overline{z}^i Q_i. \tag{1.10}$$

The relative free energy has the same time derivative as the free energy, and thus the same monotonicity property

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\boldsymbol{c}(t)|\mathcal{Q}) = -D(\boldsymbol{c}(t)) \qquad \forall t \geqslant 0,$$

where the *free energy dissipation* D is defined in (1.9). The relative free energy also satisfies

- $H(\boldsymbol{c}|\mathcal{Q}) \geqslant 0$, as can be seen by writing

$$H(\boldsymbol{c}|\mathcal{Q}) = \sum_{i=1}^{\infty} \mathcal{Q}_i \varphi \left(\frac{c_i}{\mathcal{Q}_i} \right), \qquad \text{with } \varphi(r) := r \log r - r + 1 \geqslant 0 \tag{1.11}$$

- $H(\boldsymbol{c}|\mathcal{Q}) = 0$ if and only if $c_i = \mathcal{Q}_i = Q_i \overline{z}^i$ for any $i \in \mathbb{N}$, which is readily seen from (1.11).

This hints that $H(\boldsymbol{c}|\mathcal{Q})$ is the right 'distance' to investigate. Indeed, while strictly speaking $H(\boldsymbol{c}|\mathcal{Q})$ is not a distance, it does control the ℓ^1 distance between $\boldsymbol{c}$ and $\mathcal{Q}$ by the celebrated Csiszár-Kullback inequality[1], which in our case translates to

$$\|\boldsymbol{c} - \mathcal{Q}\|_{\ell^1(\mathbb{N})} = \sum_{i=1}^{\infty} |c_i - \mathcal{Q}_i| \leqslant \sqrt{2\varrho H(\boldsymbol{c}|\mathcal{Q})}. \tag{1.12}$$

(See also [19, Corollary 2.2] for a version involving the ℓ^1 distance with weight i.) The issue of estimating the rate of convergence to equilibrium of subcritical solutions is the main concern of this paper. The first result in this direction was the work [19] by Jabin and Niethammer, where they investigated the possibility of applying

[1]Sometimes called Pinsker or Kullback-Pinsker inequality.

the so-called *entropy method* to the Becker-Döring equation. This consists roughly in looking for functional inequalities between a suitable Lyapunov functional of the equation (generally called the entropy; it corresponds to the relative free energy in our case) and its dissipation, so that one obtains a differential inequality that estimates the rate of convergence to equilibrium. In the case of the Becker-Döring equation, it was proved in [19] that there exists a constant $C > 0$, depending only on the fixed parameters of the problem and the initial data, such that

$$D(\boldsymbol{c}) \geqslant C \frac{H(\boldsymbol{c}|\mathcal{Q})}{(\log H(\boldsymbol{c}|\mathcal{Q}))^2}, \tag{1.13}$$

for all nonnegative sequences $\boldsymbol{c}$ with subcritical mass ϱ, satisfying $\epsilon \leqslant c_1 \leqslant z_s - \epsilon$ for some $\epsilon > 0$ and

$$\sum_{i=1}^{\infty} e^{\mu i} c_i =: M^{\exp} < +\infty. \tag{1.14}$$

The constant C depends on ϵ and $M^{\exp}$. This result applies under resonable conditions on the coefficients a_i and b_i; in particular, it applies to the coefficients (1.23) and (1.25), which we give as examples below. If we consider now a solution $\boldsymbol{c} = \boldsymbol{c}(t)$ to (1.1), we may apply the inequality (1.13) to $\boldsymbol{c}(t)$ as long as $\boldsymbol{c}(t)$ satisfies the appropriate conditions, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t} H(\boldsymbol{c}(t)|\mathcal{Q}) = -D(\boldsymbol{c}(t)) \leqslant -C \frac{H(\boldsymbol{c}(t)|\mathcal{Q})}{(\log H(\boldsymbol{c}(t)|\mathcal{Q}))^2}.$$

Adding to this some additional considerations for the times t for which the inequality (1.13) is not applicable to $\boldsymbol{c}(t)$, one can deduce that $H(\boldsymbol{c}(t)|\mathcal{Q})$ is (roughly) bounded above by the solution of the above differential inequality, namely that

$$H(\boldsymbol{c}(t)|\mathcal{Q}) \leqslant H(\boldsymbol{c}(0)|\mathcal{Q}) e^{-K t^{\frac{1}{3}}}$$

for some $K > 0$. Using inequality (1.12), this gives an almost-exponential rate of convergence to equilibrium for subcritical solutions in the $\ell^1(\mathbb{N})$ norm.

The question remained open of whether the convergence is in fact exponential or not. Recently this has been answered positively in [11] by two of the authors of the present paper, through a different approach involving a detailed study of the spectrum of the linearisation of equation (1.1) around a subcritical equilibrium. This is an approach with a strong analogy to results in the theory of the Boltzmann equation; we refer to [11, 32, 33] for more details on this parallel. The idea of the argument is to use the inequality (1.13) when one is far from equilibrium. Then, once we have reached a region which is close enough to equilibrium, the linearised regime is dominant and one can use the spectral study of the linearised operator in order to show that the convergence is in fact exponential. The outcome of this strategy is the following: for many interesting coefficients (including (1.23) and (1.25)), subcritical solutions $\boldsymbol{c} = \boldsymbol{c}(t)$ to (1.1) with

$$\sum_{i=1}^{\infty} e^{\mu i} c_i(0) =: M^{\exp} < +\infty \quad \text{for some } \mu > 0$$

satisfy that

$$\sum_{i=1}^{\infty} e^{\mu' i} |c_i(t) - \mathcal{Q}_i| \leqslant C e^{-\lambda t} \quad \text{for } t \geqslant 0$$

for some $0 < \mu' < \mu$, $C > 0$ and $\lambda > 0$ which depend on the parameters of the problem and on $M^{\exp}$. In fact, μ' and C only depend on the initial data $\boldsymbol{c}(0)$ through its mass and the value of $M^{\exp}$; λ depends only on the coefficients and the initial mass and can be estimated explicitly. The value of λ is bounded above by (and can be taken very close to) the size of the spectral gap of the linearised operator. Recently Murray and Pego [21] have used this spectral gap and developed the local estimates of the linearised operator in order to obtain convergence to equilibrium at a polynomial rate with milder conditions on the decay of the initial data. These results, like those in [11], are local in nature and require the use of some global estimate such as (1.13) in order to provide global rates of convergence to equilibrium.

1.3. **Main Results.** Our main goal in this work is to complete the picture of convergence to equilibrium by investigating modified and improved versions of the inequality (1.13). We show optimal inequalities and settle the question of whether full exponential convergence can be obtained through a *linear* inequality of the form

$$D(\boldsymbol{c}) \geqslant KH(\boldsymbol{c}|\mathcal{Q}),$$

for some constant $K > 0$. In analogy to the Boltzmann equation, we refer to the question of whether such K exists along solutions to (1.1) as *Cercignani's conjecture for the Becker-Döring equations.* In fact, we show that under relatively mild conditions on the initial data, typical coagulation and fragmentation coefficients (covering the physically relevant situations, see Section 1.4) admit an "almost" Cercignani conjecture for the energy dissipation, i.e. an inequality bounding below $D(\boldsymbol{c})$ by a power of $H(\boldsymbol{c}|\mathcal{Q})$, yielding an explicit rate of convergence to equilibrium. Surprisingly, we also find a relevant case ($a_i \sim i$ for all i) for which the conjecture is actually valid.

We will often require the following assumptions on the coagulation and fragmentation coefficients. Some of these are similar to those in [19], and always include physically relevant coefficients as those described in Section 1.4. We recall that we always assume $a_i, b_i > 0$ for all $i \in \mathbb{N}$, and that the detailed balance coefficients Q_i were defined in (1.4) — given a_i one can determine b_i through Q_i, and vice versa.

Hypothesis 1. $0 < z_s < +\infty$.

Hypothesis 2. *For all* $i \in \mathbb{N}$, $Q_i = z_s^{1-i}\alpha_i$, *where* $\{\alpha_i\}_{i\in\mathbb{N}}$ *is a non-increasing positive sequence with* $\alpha_1 = 1$ *and* $\lim_{i\to\infty} \frac{\alpha_{i+1}}{\alpha_i} = 1$.

Hypothesis 3. $a_i = O(i^\gamma)$ *for some* $0 \leqslant \gamma \leqslant 1$, *i.e. there exist* $C_1, C_2 > 0$ *such that*

$$C_1 i^\gamma \leqslant a_i \leqslant C_2 i^\gamma \quad \text{for all } i \in \mathbb{N}.$$

Hypothesis 2 on the form of Q_i is given as a compromise that allows us to give simple quantitative estimates of the constants in our theorems while allowing for the most commonly used types of coefficients. As one can see from the proofs, this assumption may be relaxed at the price of obtaining more involved estimates for our constants, particularly the logarithmic Sobolev constant in Proposition 3.4.

In most of our estimates a crucial role will be played by the *lower free energy dissipation*, $\overline{D}(\boldsymbol{c})$, defined for a given non-negative sequence $\boldsymbol{c}$ by

$$\overline{D}(\boldsymbol{c}) = \sum_{i=1}^{\infty} a_i Q_i \left(\sqrt{\frac{c_1 c_i}{Q_i}} - \sqrt{\frac{c_{i+1}}{Q_{i+1}}} \right)^2 \tag{1.15}$$

At this point one notices that the elementary inequality $(x-y)(\log x - \log y) \geqslant 4\left(\sqrt{x}-\sqrt{y}\right)^2$ when $x, y > 0$ implies that

$$D(\boldsymbol{c}) \geqslant 4\overline{D}(\boldsymbol{c})$$

for any non-negative sequence $\boldsymbol{c}$. Thus, any lower bound that is obtained for $\overline{D}(\boldsymbol{c})$ will transfer immediately to $D(\boldsymbol{c})$.

We now state our main result on general functional inequalities for the free energy dissipation, from which later we conclude a quantitative rate of convergence to equilibrium. It can be divided into two parts: functional inequalities when c_1 is not too small and not too far from $\overline{z}$, and inequalities in the case where c_1 escapes the above region.

Theorem 1.1. *Let $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$ satisfy Hypotheses 1-3 and let $\boldsymbol{c} = \{c_i\}_{i\in\mathbb{N}}$ be an arbitrary positive sequence with finite total density $0 < \varrho < \varrho_s$.*

(i) ***(Estimate for $a_i \sim i$.)*** *Assume that $\gamma = 1$ and that there exist $\delta > 0$ such that*

$$\delta < c_1 < z_s - \delta. \tag{1.16}$$

Then there exists $K > 0$ depending only on δ, ϱ and the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 2}$, such that

$$\overline{D}(\boldsymbol{c}) \geqslant KH(\boldsymbol{c}|\mathcal{Q}). \tag{1.17}$$

(ii) ***(Estimate for $a_i \sim i^\gamma$ with $\gamma < 1$.)*** *Assume that $0 \leqslant \gamma < 1$ and that c_1 satisfies (1.16) for some $\delta > 0$. If, in addition, there exists $\beta > 1$ with*

$$M_\beta(\boldsymbol{c}) = \sum_{i=1}^{\infty} i^\beta c_i < +\infty \tag{1.18}$$

then there exists $K > 0$ depending only on δ, ϱ, $M_\beta(\boldsymbol{c})$ and the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 2}$, such that

$$\overline{D}(\boldsymbol{c}) \geqslant KH(c|\mathcal{Q})^{\frac{\beta-\gamma}{\beta-1}}. \tag{1.19}$$

(iii) ***(Estimate for c_1 far from equilibrium.)*** *Assume that $\gamma = 1$, or that $0 \leqslant \gamma < 1$ and (1.18) holds for some $\beta > 1$. Assume also that for some $\delta > 0$*

$$c_1 \leqslant \delta$$

or that

$$c_1 \geqslant z_s - \delta$$

(i.e., c_1 is outside of the range given in (1.16)). Then if $\delta > 0$ is small enough (depending only on ϱ and $\{Q_i\}_{i\geqslant 1}$), there exists $\varepsilon > 0$ depending only on δ, ϱ and the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 2}$ if $\gamma = 1$ (and additionally on $M_\beta(\boldsymbol{c})$ if $\gamma < 1$) such that

$$\overline{D}(\boldsymbol{c}) \geqslant \varepsilon. \tag{1.20}$$

The constants K and ε can be estimated explicitly in all cases.

We emphasise that all constants in the above theorem depend only on ϱ, the coefficients $\{a_i\}_{i\in\mathbb{N}}$, $\{b_i\}_{i\in\mathbb{N}}$, and the additional bounds δ or M_β (notice that ϱ_s is determined by the coefficients alone). The case (ii) of Theorem 1.1 is optimal in the following sense:

Theorem 1.2. *Call X_ϱ the set of nonnegative sequences $\boldsymbol{c} = \{c_i\}_{i\in\mathbb{N}}$ with mass ϱ (i.e., such that $\sum_{i=1}^{\infty} ic_i = \varrho$). Then, there exist $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$ that satisfy Hypotheses 1-3 with $\gamma < 1$ such that*

$$\inf_{X_\varrho} \frac{D(\boldsymbol{c})}{H(\boldsymbol{c}|\mathcal{Q})} = 0.$$

for any $\varrho < \varrho_s$.

In other words, this shows that a linear inequality as that of Theorem 1.1 (i) cannot hold if $a_i \sim i^\gamma$ with $\gamma < 1$.

The idea behind the proof of Theorem 1.1 is to use a discrete logarithmic Sobolev inequality with weights, motivated by works of Bobkov and Götze [6] and Barthe and Roberto [4], to show part (i). As the conditions for the validity of the log-Sobolev inequality are *not* satisfied under the conditions of part (ii), a simple interpolation is used to show the desired result in that case. Part (iii) is proved by two estimates: The case where c_1 is too large follows an idea of Jabin and Niethammer, and is essentially stated already in [19], while the case where c_1 is too small seems to be a new result which we provide.

From Theorem 1.1 one can conclude in a straightforward way the following theorem, our main result on the rate of convergence to equilibrium:

Theorem 1.3. *Let $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$ satisfy Hypotheses 1–3 with $0 \leqslant \gamma \leqslant 1$, and let $\boldsymbol{c} = \boldsymbol{c}(t) = \{c_i(t)\}_{i\in\mathbb{N}}$ be a solution to the Becker-Döring equations with mass $\varrho \in (0, \varrho_s)$.*

(i) ***(Rate for $a_i \sim i$.)*** *If $\gamma = 1$ then there exists a constant $K > 0$ depending only on δ, ϱ and the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 2}$, and a constant $C > 0$ depending only on $H(\boldsymbol{c}(0)|\mathcal{Q})$, ϱ and the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 2}$ such that*

$$H(\boldsymbol{c}(t)|\mathcal{Q}) \leqslant Ce^{-Kt} \qquad \text{for } t \geqslant 0.$$

(ii) ***(Rate for $a_i \sim i^\gamma$, $\gamma < 1$.)*** *If $\gamma < 1$ and $M_\beta(\boldsymbol{c}(0)) < +\infty$ for some $\beta \geqslant 2 - \gamma$ then there exists a constant $K > 0$ depending only on M_β, δ, ϱ and the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 2}$, and a constant $C > 0$ depending only on $H(\boldsymbol{c}(0)|\mathcal{Q})$, M_β, δ, ϱ and the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 2}$ such that*

$$H(\boldsymbol{c}(t)|\mathcal{Q}) \leqslant \frac{1}{\left(C + \frac{1-\gamma}{\beta-1}Kt\right)^{\frac{\beta-1}{1-\gamma}}} \qquad \text{for } t \geqslant 0.$$

The constants K and C can be estimated explicitly.

In order to deduce Theorem 1.3 we use the inequalities in Theorem 1.1 when they are applicable. Of course, the assumption that $c_1(t)$ is in the 'good' region given by (1.16) becomes eventually true, since $c_1(t)$ is known to converge to $\overline{z}$. More explicitly, one can apply the Csiszár-Kullback inequality (1.12) to obtain that for any $t > t_0$ we have

$$\overline{z} - H(\boldsymbol{c}(t_0)|\mathcal{Q}) \leqslant c_1(t) \leqslant \overline{z} + H(\boldsymbol{c}(t_0)|\mathcal{Q}), \qquad t \geqslant t_0.$$

If $H(\boldsymbol{c}(t_0)|\mathcal{Q})$ is small enough this implies (1.16). For times t such that $c_1(t)$ is outside this 'good' region we use the inequality in Theorem 1.1 (iii); details are given in Section 4.

There are several improvements in these theorems with respect to the existing theory. One of them is that they apply to more general initial conditions, removing the need for a finite exponential moment present in [11, 19]. Another one is that they answer the question of whether one can obtain a linear inequality such as (1.17) (i.e., whether the equivalent of Cercignani's conjecture holds), making clear the link to discrete logarithmic Sobolev inequalities. It does hold in the case $a_i \sim i$, which is physically relevant for example in modelling polymer chains [18, 23]. As a result, the statement for $a_i \sim i$ is quite strong: it gives full exponential convergence, with explicit constants in terms of the parameters, with no restriction on the initial data except that of subcritical mass. Point (ii) in 1.3 also relaxes the requirements on the initial data, at the price of obtaining a slower convergence than that of [11]; we do not know whether this rate is optimal for initial conditions with polynomially decaying tails (so that $M_\beta < \infty$ for some $\beta > 1$, but $M_{\beta'} = +\infty$ for some $\beta' > \beta$). Recently, Murray and Pego [21] investigated this rate of convergence, concluding an algebraic rate of decay as well. It would be interesting to verify the optimality of this result by determining whether the corresponding linearised operator admits a spectral gap in ℓ^1 spaces with polynomial weights (in ℓ^1 spaces with exponential weights, the answer is positive and an estimate of the spectral gap can be found in [11]). We believe that no such spectral gap exists for $0 \leqslant \gamma < 1$, i.e. that the algebraic rate of convergence is optimal even for close to equilibrium initial data.

One may wonder if the method presented here can be used to reach an inequality like Jabin and Niethammer's (1.13) under the additional condition of an exponential moment. The answer is indeed positive:

Theorem 1.4. *Let $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$ satisfy Hypothesis 1–3 with $0 \leqslant \gamma < 1$.*

(i) ***(Functional inequality.)*** *Let $\mathbf{c} = \{c_i\}_{i\in\mathbb{N}}$ be an arbitrary positive sequence with mass $\varrho \in (0, \varrho_s)$ for which there exists $\mu > 0$ such that*

$$M_\mu^{\exp}(\boldsymbol{c}) := \sum_{i=1}^{\infty} e^{\mu i} c_i < +\infty. \tag{1.21}$$

Then there exist $K_1, K_2, \varepsilon > 0$ depending only on $M_\mu^{\exp}(\boldsymbol{c})$, δ, ϱ and the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 2}$ such that

$$\overline{D}(\boldsymbol{c}) \geqslant \min\left(\frac{K_1 H(\boldsymbol{c}|\mathcal{Q})}{\left|\log\left(K_2 H(\boldsymbol{c}|\mathcal{Q})\right)\right|^{1-\gamma}}, \varepsilon \right). \tag{1.22}$$

Moreover, K_1, K_2 and ε can be given explicitly.

(ii) ***(Rate of convergence.)*** *If $\mathbf{c}(t) = \{c_i(t)\}_{i\in\mathbb{N}}$ is a solution to the Becker-Döring equations with mass $0 < \varrho < \varrho_s$ such that there exists $\mu > 0$ with*

$$M_\mu^{\exp}(\boldsymbol{c}(0)) := \sum_{i=1}^{\infty} e^{\mu i} c_i(0) < +\infty,$$

then there exists a constant $K > 0$ depending only on $M_\mu^{\exp}(\boldsymbol{c}(0))$, δ, ϱ and the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 2}$, and a constant $C > 0$ depending only on $H(\boldsymbol{c}(0)|\mathcal{Q})$, $M_\mu^{\exp}(\boldsymbol{c}(0))$, δ, ϱ and the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 2}$ such that

$$H(\boldsymbol{c}(t)|\mathcal{Q}) \leqslant C e^{-K t^{\frac{1}{2-\gamma}}}.$$

Moreover, K and C can be given explicitly.

1.4. **Typical Coefficients.** The above results are valid for coagulation and fragmentation coefficients satisfying Hypothesis 1 – 3. To motivate our choice of assumptions, we briefly recall here some physically motivated coagulation and fragmentation coefficients found in the literature.

Common model coefficients appearing in the theory of density-conserving phase transitions (see [24, 26]) are given by

$$a_i = i^\gamma, \qquad b_i = a_i \left(z_s + \frac{q}{i^{1-\mu}} \right) \qquad \text{for all } i \geqslant 1, \tag{1.23}$$

for some $0 < \gamma \leqslant 1$, $z_s > 0$, $q > 0$ and $0 < \mu < 1$. These coefficients may be derived from simple assumptions on the mechanism of the reactions taking place; we take particular values from [24]:

$$\begin{aligned} \gamma &= 1/3, \quad \mu = 2/3 \qquad \text{(diffusion-limited kinetics in 3-D)}, \\ \gamma &= 0, \quad \mu = 1/2 \qquad \text{(diffusion-limited kinetics in 2-D)}, \\ \gamma &= 2/3, \quad \mu = 2/3 \qquad \text{(interface-reaction-limited kinetics in 3-D)}, \\ \gamma &= 1/2, \quad \mu = 1/2 \qquad \text{(interface-reaction-limited kinetics in 2-D)}. \end{aligned} \tag{1.24}$$

The case $\gamma = 1$ appears for example in modelling polymer chains, where the binding energy increases by a constant each time a monomer is added.

A different kind of reasoning, based on a statistical mechanics argument involving the binding energy of clusters, results in the coefficients results in the coefficients

$$a_i = i^\gamma, \qquad b_i = z_s (i-1)^\gamma \exp\left(\sigma i^\mu - \sigma (i-1)^\mu\right), \qquad i \in \mathbb{N}, \tag{1.25}$$

for appropriate constants γ, μ and where $\sigma > 0$ is related to the surface tension of the aggregates. The values of μ and γ for various situations are still those in (1.24).

As already mentioned, the choice $\gamma = 1$ corresponds to the physically relevant example in modelling polymer chains (for instance for proteins aggregating in a cubic phase of lipid bilayers, [18, 23]).

The behaviour of (1.23) and (1.25) is similar: observe that for large i we have $i^\mu - (i-1)^\mu \sim \mu i^{\mu-1}$, so the fragmentation coefficients become roughly

$$b_i \sim z_{\mathrm{s}}\, a_i \exp\left(\sigma \mu i^{\mu-1}\right) \sim a_i \left(z_{\mathrm{s}} + \frac{z_{\mathrm{s}} \sigma \mu}{i^{1-\mu}}\right),$$

which is like (1.23) with $q = z_{\mathrm{s}} \sigma \mu$. Moreover, for both classes of coefficients, we can write (by definition of Q_i)

$$Q_i = \frac{a_1 a_2 \dots a_{i-1}}{b_2 b_3 \dots b_i} = z_s^{1-i} \alpha_i, \tag{1.26}$$

where $\{\alpha_i\}_{i\in\mathbb{N}}$ is non-increasing and satisfies

$$\lim_{i\to\infty} \frac{\alpha_{i+1}}{\alpha_i} = 1.$$

In other words, Hypotheses 1–3 hold true for both models.

1.5. **Application to General Coagulation and Fragmentation Models.** The Becker-Döring equations are the simplest form of a coagulation and fragmentation process, assuming that the only relevant reactions are governed by monomers. Other models take into account the fact that clusters of size i and size j, for $i, j \in \mathbb{N}$, may interact. A discrete model—similar to the Becker-Döring equations (1.1)—can be formulated, now with coagulation and fragmentation coefficients of the form $a_{i,j}, b_{i,j}$ (see Section 5.1). Together with an assumption of detailed balance, one can once again find equilibria to the process and inquire about the rate of convergence to them. Our study of the Becker-Döring equations allows us to give a quantitative answer (though not optimal) for this question. We leave the detailed description of the model we have in mind for Section 5.1. For such a model, using the same notion of free relative energy we will show that

Theorem 1.5 (Asymptotic behaviour of the coagulation-fragmentation system)**.** *Let $\{a_{i,j}\}_{i,j\in\mathbb{N}}, \{b_{i,j}\}_{i,j\in\mathbb{N}}$ be the coagulation and fragmentation coefficients for equation (5.1), and assume that the detailed balance condition (5.6) holds. Assume that*

$$a_{i,j} = i^\gamma + j^\gamma, \tag{1.27}$$

for some $0 \leqslant \gamma < 1$ *and that* $\{Q_i\}_{i\in\mathbb{N}}$ *satisfies Hypothesis 2. Assume in addition that* $M_k(\boldsymbol{c}(0)) < +\infty$ *for some* $k \in \mathbb{N}$, $k > 1$. *Then*

$$H(\boldsymbol{c}(t)|\mathcal{Q}) \leqslant \frac{1}{(C_1 + C_2 \log t)^{\frac{k-1}{1-\gamma}}} \qquad t \geqslant 0, \tag{1.28}$$

where $C_1, C_2 > 0$ *are constants depending only on* $H(\boldsymbol{c}(0)|\mathcal{Q}), z_s, \varrho, \{\alpha_i\}_{i\in\mathbb{N}}, k, \gamma$ *and* $M_k(\boldsymbol{c}(0))$.

1.6. **Organisation of the Paper.** The structure of the paper is as follows: In Section 2 we will present our main technical tool, a discrete version of the log-Sobolev inequality with weights. Section 3 contains the proof of Theorem 1.1 and uses Section 2 to show the first part of the theorem. We also show in this section that this method is optimal and that Cercignani's conjecture cannot hold when $\gamma < 1$, proving Theorem 1.2 and explore the additional inequality that appears under the assumption of a finite exponential moment. Section 4 deals with the consequences of our functional inequalities for the solutions to the Becker-Döring equation and contains the proof of Theorem 1.3 and part (ii) of Theorem 1.4. In Section 5 we provide the proof of Theorem 1.5 and remark on the difficulties of obtaining stronger results in this general setting. Lastly, we give an appendix where proofs to some technical lemmas can be found.

2. A Discrete Weighted Logarithmic Sobolev Inequality

One of the key ingredients in proving Cercignani's conjecture for the Becker-Döring equations in the terms of Theorem 1.1 is a discrete log-Sobolev inequality with weights. The theory presented here follows closely the work of Bobkov and Götze in [6], and that of Barthe and Roberto in [4], and can be seen as a discrete version of the aforementioned papers. It is worth noting that a discrete version is explicitly mentioned in [4, Section 4], with a remark that the arguments in [4] can be adapted to prove it. Indeed, our proof is essentially an adaptation of the one in [6], and we give it in this section for the sake of completeness (and since we have not been able to find an explicit proof in the discrete case). Some further technical details are postponed to Appendix A.

2.1. **The Main Log-Sobolev Inequality.** We start with some basic definitions:

Definition 2.1. We say that $\boldsymbol{\mu} \in P(\mathbb{N})$ if $\boldsymbol{\mu} = \{\mu_i\}_{i\in\mathbb{N}}$ is a non-negative sequence such that

$$\sum_{i=1}^{\infty} \mu_i = 1.$$

For any non-negative sequence $\boldsymbol{g} = \{g_i\}_{i\in\mathbb{N}}$ with $\sum_{i=1}^{\infty} \mu_i g_i < +\infty$, we define its *entropy* with respect to $\boldsymbol{\mu}$ as

$$\mathrm{Ent}_\mu(\boldsymbol{g}) = \sum_{i=1}^{\infty} \mu_i g_i \log \frac{g_i}{\sum_{i=1}^{\infty} \mu_i g_i}. \tag{2.1}$$

Definition 2.2. Given $\boldsymbol{\mu} \in P(\mathbb{N})$ and positive sequence $\boldsymbol{\nu} = \{\nu_i\}_{i\in\mathbb{N}}$ (not necessarily normalised) we say that $\boldsymbol{\nu}$ admits a log-Sobolev inequality with respect to $\boldsymbol{\mu}$ with constant $0 < C_{\mathrm{LS}} < +\infty$ if for any sequence $\boldsymbol{f} = \{f_i\}_{i\in\mathbb{N}}$

$$\mathrm{Ent}_\mu\left(\boldsymbol{f}^2\right) \leqslant C_{\mathrm{LS}} \sum_{i=1}^{\infty} \nu_i \left(f_{i+1} - f_i\right)^2, \tag{2.2}$$

where $\boldsymbol{f}^2 = \{f_i^2\}_{i\in\mathbb{N}}$.

In what follows we will always assume that $\boldsymbol{\mu} \in P(\mathbb{N})$. Denoting by

$$\Psi(x) = |x| \log(1 + |x|)$$

the main theorem, and its simplified corollary, that we will prove in this Section are:

Theorem 2.3. *The following two conditions are equivalent:*

(i) $\boldsymbol{\nu}$ *admits a log-Sobolev inequality with respect to* $\boldsymbol{\mu}$ *with constant* C_{LS}.

(ii) For any $m \in \mathbb{N}$ *such that*

$$\max\left(\sum_{i=1}^{m-1} \mu_i, \sum_{i=m+1}^{\infty} \mu_i\right) < \frac{2}{3}$$

we have that

$$B_1 = \sup_{k \geqslant m} \frac{\sum_{i=1}^{k} \frac{1}{\nu_i}}{\Psi^{-1}\left(\frac{1}{\sum_{i=k+1}^{\infty} \mu_i}\right)} < +\infty. \tag{2.3}$$

Moreover, if (ii) is valid then one can choose

$$C_{\mathrm{LS}} = 40(B_2 + 4B_1), \qquad \textit{where} \quad B_2 = \frac{\sum_{i=1}^{m-1} \frac{1}{\nu_i}}{\Psi^{-1}\left(\frac{1}{\sum_{i=1}^{m-1} \mu_i}\right)}. \tag{2.4}$$

A somehow more tractable consequence is the following.

Corollary 2.4. *The following two conditions are equivalent:*

(i) $\boldsymbol{\nu}$ *admits a log-Sobolev inequality with respect to* $\boldsymbol{\mu}$ *with constant* C_{LS}.

(ii) For any $m \in \mathbb{N}$ *such that*

$$\max\left(\sum_{i=1}^{m-1} \mu_i, \sum_{i=m+1}^{\infty} \mu_i\right) < \frac{2}{3}$$

we have that

$$D_1 = \sup_{k \geqslant m} \left(-\sum_{i=k+1}^{\infty} \mu_i \log\left(\sum_{i=k+1}^{\infty} \mu_i\right)\right)\left(\sum_{i=1}^{k} \frac{1}{\nu_i}\right) < \infty. \tag{2.5}$$

Moreover, if (ii) is valid then one can choose

$$C_{\mathrm{LS}} = 120(D_2 + 4D_1), \tag{2.6}$$

where $D_2 = \left(-\sum_{i=1}^{m-1} \mu_i \log\left(\sum_{i=1}^{m-1} \mu_i\right)\right)\left(\sum_{i=1}^{m-1} \frac{1}{\nu_i}\right)$.

Remark 2.5. One can clearly see that if

$$\sup_{k \geqslant 1} \left(-\sum_{i=k+1}^{\infty} \mu_i \log\left(\sum_{i=k+1}^{\infty} \mu_i\right)\right)\left(\sum_{i=1}^{k} \frac{1}{\nu_i}\right) < \infty$$

then one has a log-Sobolev inequality of $\boldsymbol{\nu}$ with respect to $\boldsymbol{\mu}$. However, the introduction of the 'approximate median' m allows us to have an explicit estimation on the log-Sobolev constant C_{LS}.

The rest of the Section is dedicated to the proof of the above results and will be divided in various steps - each one corresponding to a subsection.

2.2. **A Reformulation as a Poincaré Inequality in Orlicz Spaces.** As in the work of Bobkov and Götze in [6], a key argument in the proof of Theorem 2.3 and Corollary 2.4 is to recast the log-Sobolev inequality as a Poincaré inequality in the Orlicz space associated to Ψ. We start with the definition:

Definition 2.6. Given $\boldsymbol{\mu} \in P(\mathbb{N})$ and a Young Function, $\Sigma : [0,+\infty) \to [0,+\infty)$, i.e. a convex function such that

$$\frac{\Sigma(x)}{x} \underset{x\to+\infty}{\longrightarrow} +\infty, \qquad \frac{\Sigma(x)}{x} \underset{x\to 0}{\longrightarrow} 0,$$

we define the Orlicz space $L_\Sigma^{(\mu)}$ as the space of all sequences $\boldsymbol{f}$ such that there exists $k>0$ with

$$\sum_{i=1}^{\infty} \mu_i \Sigma\left(\frac{|f_i|}{k}\right) < \infty.$$

In that case we define

$$\|\boldsymbol{f}\|_{L_\Sigma^{(\mu)}} = \inf_{k>0} \left\{ \sum_{i=1}^{\infty} \mu_i \Sigma\left(\frac{|f_i|}{k}\right) \leqslant 1 \right\}.$$

In what follows we will drop the superscript μ from the Orlicz space of Ψ and its norm. Additionally we denote by $\Phi(x) = \Psi(x^2)$ and notice that:

$$\begin{aligned} \|\boldsymbol{f}^2\|_{L_\Psi} &= \inf_{k>0} \left\{ \sum_{i=1}^{\infty} \mu_i \Psi\left(\frac{f^2}{k}\right) \leqslant 1 \right\} \\ &= \left(\inf_{\sqrt{k}>0} \left\{ \sum_{i=1}^{\infty} \mu_i \Phi\left(\frac{|f|}{\sqrt{k}}\right) \leqslant 1 \right\} \right)^2 = \|\boldsymbol{f}\|_{L_\Phi}^2 . \end{aligned} \tag{2.7}$$

We have then the following version of Rothaus's Lemma:

Lemma 2.7. *Given $\boldsymbol{\mu} \in P(\mathbb{N})$ and a sequence $\boldsymbol{f} = \{f_i\}_{i\in\mathbb{N}}$ we set*

$$\mathcal{L}(\boldsymbol{f}) = \sup_{\alpha\in\mathbb{R}} \mathrm{Ent}_\mu\left((\boldsymbol{f}+\alpha)^2\right) \tag{2.8}$$

where $\boldsymbol{f}+\alpha = \{f_i+\alpha\}_{i\in\mathbb{N}}$. Then,

$$\mathrm{Ent}_\mu(\boldsymbol{f}^2) \leqslant \mathcal{L}(\boldsymbol{f}) \leqslant \mathrm{Ent}_\mu(\boldsymbol{f}^2) + 2\sum_{i=1}^{\infty} \mu_i f_i^2. \tag{2.9}$$

Remark 2.8. This Lemma is an adaptation of the appropriate Lemma in [29]. We leave the proof of it to Appendix A.

We have then the following equivalent formulation of the log-Sobolev inequality:

Proposition 2.9. *The following conditions are equivalent:*

(i) $\boldsymbol{\nu}$ admits a log-Sobolev inequality with respect to $\boldsymbol{\mu}$ with constant C_{LS}.

(ii) For any sequence $\boldsymbol{f}$

$$\mathcal{L}(\boldsymbol{f}) \leqslant C_{\mathrm{LS}} \sum_{i=1}^{\infty} \nu_i \left(f_{i+1} - f_i\right)^2 . \tag{2.10}$$

(iii) For any sequence $\boldsymbol{f}$

$$\|\boldsymbol{f} - \langle \boldsymbol{f} \rangle\|_{L_\Phi}^2 \leqslant \lambda \sum_{i=1}^{\infty} \nu_i \left(f_{i+1} - f_i\right)^2 . \tag{2.11}$$

where $\langle \boldsymbol{f} \rangle = \sum_{i=1}^{\infty} \mu_i f_i$.
Moreover, if (i) or (ii) are valid one can choose $\lambda = \frac{3}{2} C_{\mathrm{LS}}$. *If (iii) is valid one can choose* $C_{\mathrm{LS}} = 5\lambda$.

The proof of the proposition relies on the following lemma:

Lemma 2.10. *For any sequence* $\boldsymbol{f}$ *one has that*

$$\frac{2}{3} \| \boldsymbol{f} - \langle \boldsymbol{f} \rangle \|_{L_\Phi}^2 \leqslant \mathcal{L}(\boldsymbol{f}) \leqslant 5 \| \boldsymbol{f} - \langle \boldsymbol{f} \rangle \|_{L_\Phi}^2 \tag{2.12}$$

Proof. We start by noticing that we may assume that $\langle \boldsymbol{f} \rangle = 0$ as well as $\| \boldsymbol{f} - \langle \boldsymbol{f} \rangle \|_{L_\Phi} = 1$. This is true as $\mathcal{L}$ is invariant under translations and

$$\mathrm{Ent}_\mu(\alpha \boldsymbol{f}) = \alpha\, \mathrm{Ent}_\mu(\boldsymbol{f}).$$

Using Lemma 2.7, we find that

$$\begin{aligned}
\mathcal{L}(\boldsymbol{f}) \leqslant \mathrm{Ent}_\mu(\boldsymbol{f}^2) + 2 \sum_{i=1}^{\infty} \mu_i f_i^2 &= \sum_{i=1}^{\infty} \mu_i f_i^2 \log\left(f_i^2\right) + 2 \sum_{i=1}^{\infty} \mu_i f_i^2 \\
&\quad - \left(\sum_{i=1}^{\infty} \mu_i f_i^2 \right) \log \left(\sum_{i=1}^{\infty} \mu_i f_i^2 \right) \\
&\leqslant \sum_{i=1}^{\infty} \mu_i \Phi(f_i) + h \left(\sum_{i=1}^{\infty} \mu_i f_i^2 \right),
\end{aligned}$$

where $h(x) = 2x - x \log x$ for $x > 0$. As h is an increasing function on $[0, e]$ and

$$\| \boldsymbol{f} \|_{L^1_\mu} \leqslant \| \boldsymbol{f} \|_{L^2_\mu} \leqslant \sqrt{\frac{3}{2}} \| \boldsymbol{f} \|_{L_\Phi},$$

(see Lemma A.2 in Appendix A) we have that

$$\| \boldsymbol{f} \|_{L^2_\mu}^2 \leqslant 2.$$

Thus, as

$$\sum_{i=1}^{\infty} \mu_i \Phi(f_i) = \sum_{i=1}^{\infty} \mu_i \Phi \left(\frac{f_i}{\| f \|_{L_\Phi}} \right) \leqslant 1,$$

we find that

$$\mathcal{L}(\boldsymbol{f}) \leqslant 1 + h(2) \leqslant 5,$$

proving the right hand side inequality of (2.12). To show the left hand side inequality we assume that $\mathcal{L}(\boldsymbol{f}) = 2$. By the definition of $\mathcal{L}$ and the fact that

$$\| \boldsymbol{f} - \langle \boldsymbol{f} \rangle \|_{L^2_\mu}^2 = \frac{1}{2} \lim_{|a| \to \infty} \mathrm{Ent}_\mu \left((\boldsymbol{f} + a)^2 \right)$$

(see Lemma A.3 in Appendix A) we know that

$$\| \boldsymbol{f} \|_{L^2_\mu}^2 \leqslant \frac{1}{2} \mathcal{L}(\boldsymbol{f}) = 1.$$

This implies that

$$\begin{aligned}
\sum_{i=1}^{\infty} \mu_i \Phi(f_i) \leqslant 1 + \sum_{i=1}^{\infty} \mu_i f_i^2 \log f_i^2 &= 1 + \mathrm{Ent}_\mu(\boldsymbol{f}^2) + \| \boldsymbol{f} \|_{L^2_\mu}^2 \log \left(\| \boldsymbol{f} \|_{L^2_\mu}^2 \right) \\
&\leqslant 1 + \mathcal{L}(\boldsymbol{f}) = 3,
\end{aligned}$$

where we have used the fact that $x \log(1 + x) \leqslant 1 + x \log x$ when $x > 0$.

Since for any $a \geqslant 1$, $\Phi\left(\frac{x}{\sqrt{a}}\right) = \frac{x^2}{a^2}\log\left(1+\frac{x^2}{a^2}\right) \leqslant \frac{1}{a^2}\Phi(x)$, the above implies that

$$\sum_{i=1}^{\infty} \mu_i \Phi\left(\frac{f_i}{\sqrt{3}}\right) \leqslant 1$$

and as such, by the definition of $\|\cdot\|_{L_\Phi}$, we conclude that

$$\|\boldsymbol{f}\|_{L_\Phi}^2 \leqslant 3 = \frac{3}{2}\mathcal{L}(\boldsymbol{f}),$$

and the proof is complete. □

Proof of Proposition 2.9. The equivalence of (ii) and (iii) is immediate following Lemma 2.10, which also proves the desired connection between C_{LS} and λ. To show that (i) implies (ii) we notice that as the right hand side of (2.2) is invariant under translation. Taking the supremum over all possible translations results in (ii). The fact that (ii) implies (i) is immediate as $\mathrm{Ent}_\mu(\boldsymbol{f}^2) \leqslant \mathcal{L}(\boldsymbol{f})$. □

2.3. **Discrete Hardy Inequalities.** The above observation that the log-Sobolev inequality with weights is actually a form of a Poincaré inequality brings to mind another inequality with weights that is closely connected to the Poincaré inequality - Hardy inequality. In its discrete form, we have that

Lemma 2.11. *Let $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ two sequences of positive numbers and let $m \in \mathbb{N}$. Then, the following two conditions are equivalent:*

(i) There exists a finite constant $A_{1,m} \geqslant 0$ such that

$$\sum_{i=m}^{\infty} \mu_i \left(\sum_{j=m}^{i} f_j\right)^2 \leqslant A_{1,m} \sum_{i=m}^{\infty} \nu_i f_i^2,$$

for any sequence $\boldsymbol{f}$.

(ii) The following holds:

$$B_{1,m} = \sup_{k \geqslant m} \left(\sum_{i=k}^{\infty} \mu_i\right)\left(\sum_{i=m}^{k} \frac{1}{\nu_i}\right) < \infty.$$

Moreover, if any of the conditions holds than $B_{1,m} \leqslant A_{1,m} \leqslant 4B_{1,m}$.

The proof for the case $m = 1$ can be found in [11], and the general case follows by the same method of proof.

Corollary 2.12. *Let*

$$B_m^{(1)} = \sup_{k \geqslant m} \left(\sum_{i=k+1}^{\infty} \mu_i\right)\left(\sum_{i=m}^{k} \frac{1}{\nu_i}\right).$$

Then for any sequence $\boldsymbol{f}$ such that $f_m = 0$ we have that

$$\sum_{i=m}^{\infty} \mu_i f_i^2 \leqslant A_m^{(1)} \sum_{i=m}^{\infty} \nu_i \left(f_{i+1} - f_i\right)^2, \tag{2.13}$$

if and only if $B_m^{(1)} < \infty$. In that case $B_m^{(1)} \leqslant A_m^{(1)} \leqslant 4B_m^{(1)}$. Additionally,

$$B_{1,m} \leqslant B_m^{(1)} \leqslant B_{1,m+1}.$$

Proof. This follows immediately from Lemma 2.11 applied to the sequence $g_i = f_{i+1} - f_i$ and a simple translation argument. □

Besides the above, we will also need to have a Hardy-type inequality for sums up to a fixed integer m.

Lemma 2.13. *Let $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ two sequences of positive numbers and let $m \in \mathbb{N}$. Then, for any sequence $\boldsymbol{f}$ such that $f_m = 0$ we have that if there exists $A > 0$ such that*

$$\sum_{i=1}^{m-1} \mu_i f_i^2 \leqslant A \sum_{i=1}^{m-1} \nu_i \left(f_{i+1} - f_i\right)^2, \tag{2.14}$$

then $b_{2,m} \leqslant A$ where

$$b_{2,m} = \sup_{k \leqslant m-1} \sum_{i=1}^{k} \mu_i \left(\sum_{j=k}^{m-1} \frac{1}{\nu_j}\right).$$

Moreover, one can always choose

$$A = B_{2,m} = \sum_{i=1}^{m-1} \mu_i \left(\sum_{j=i}^{m-1} \frac{1}{\nu_j}\right).$$

Proof. We start by noticing that for any $1 \leqslant i \leqslant m-1$ we have that

$$\begin{aligned} f_i^2 = \left[\sum_{j=i}^{m-1} \left(f_{j+1} - f_j\right)\right]^2 &\leqslant \left(\sum_{j=i}^{m-1} \frac{1}{\nu_j}\right) \left(\sum_{j=i}^{m-1} \nu_j \left(f_{j+1} - f_j\right)^2\right) \\ &\leqslant \left(\sum_{j=i}^{m-1} \frac{1}{\nu_j}\right) \left(\sum_{j=1}^{m-1} \nu_j \left(f_{j+1} - f_j\right)^2\right). \end{aligned}$$

Thus

$$\sum_{i=1}^{m-1} \mu_i f_i^2 \leqslant \left[\sum_{i=1}^{m-1} \mu_i \left(\sum_{j=i}^{m-1} \frac{1}{\nu_j}\right)\right] \left(\sum_{j=1}^{m-1} \nu_j \left(f_{j+1} - f_j\right)^2\right) = B_{2,m} \sum_{j=1}^{m-1} \nu_j \left(f_{j+1} - f_j\right)^2,$$

completing the second statement. Next, for any $j \leqslant m-1$ we denote by

$$\sigma_j = \sum_{i=j}^{m-1} \frac{1}{\nu_i}.$$

Fix $k \leqslant m-1$ and define $\boldsymbol{f}^{(k)}$ to be such that $f_i^{(k)} = \sigma_k$ when $i \leqslant k$ and $f_i^{(k)} = \sigma_i$ when $i > k$. We have that

$$\sum_{i=1}^{m-1} \nu_i \left(f_{i+1}^{(k)} - f_i^{(k)}\right)^2 = \sum_{i=k}^{m-1} \nu_i \left(f_{i+1}^{(k)} - f_i^{(k)}\right)^2 = \sum_{i=k}^{m-1} \frac{1}{\nu_i} = \sigma_k.$$

On the other hand

$$\sum_{i=1}^{m-1} \mu_i \left(f_i^{(k)}\right)^2 \geqslant \sum_{i=1}^{k} \mu_i \left(f_i^{(k)}\right)^2 = \sigma_k^2 \left(\sum_{i=1}^{k} \mu_i\right).$$

As (2.14) is valid we see that $A \geqslant \left(\sum_{i=k}^{m-1} \frac{1}{\nu_i}\right) \left(\sum_{i=1}^{k} \mu_i\right)$ for all k. This completes the proof. $\square$

2.4. **Proof of the Main Inequality.** The last ingredient we need in order to prove Theorem 2.3 is the following lemma:

Lemma 2.14. *The following conditions are equivalent:*

(i) $\boldsymbol{\nu}$ *admits a log-Sobolev inequality with respect to* $\boldsymbol{\mu}$ *with constant* C_{LS}.

(ii) There exists $\eta > 0$ *such that, for any sequence* $\boldsymbol{f} = \{f_i\}$ *such that* $f_m = 0$ *with* $m \in \mathbb{N}$ *satisfying*

$$\max\left(\sum_{i=1}^{m-1} \mu_i, \sum_{i=m+1}^{\infty} \mu_i\right) < \frac{2}{3}$$

we have that

$$\left\|\left(\boldsymbol{f}^{(0)}\right)^2\right\|_{L_\Psi} + \left\|\left(\boldsymbol{f}^{(1)}\right)^2\right\|_{L_\Psi} \leqslant \eta \sum_{i=1}^{\infty} \nu_i \left(f_{i+1} - f_i\right)^2,$$

where $\boldsymbol{f}^{(0)} = \boldsymbol{f}\mathbb{1}_{i<m}$ *and* $\boldsymbol{f}^{(1)} = \boldsymbol{f}\mathbb{1}_{i>m}$.

Moreover, if condition (ii) *is valid one can choose* $C_{\mathrm{LS}} = 40\eta$.

Proof. Using Proposition 2.9 we notice that it is enough for us to show the equivalence of conditions (ii) of our theorem and that of Proposition 2.9.

Assume, to begin with, that (ii) of Proposition 2.9 is valid. As was shown in the aforementioned theorem, this implies that

$$\|\boldsymbol{f} - \langle \boldsymbol{f}\rangle\|_{L_\Phi}^2 \leqslant \frac{3C_{\mathrm{LS}}}{2} \sum_{i=1}^{\infty} \nu_i \left(f_{i+1} - f_i\right)^2. \tag{2.15}$$

Due to the conditions on $\boldsymbol{f}$ and the definition of $\boldsymbol{f}^{(0)}$ and $\boldsymbol{f}^{(1)}$ one has that

$$\|\langle \boldsymbol{f}^{(0)}\rangle\|_{L_\Phi} \leqslant |\langle \boldsymbol{f}^{(0)}\rangle| \leqslant \|\boldsymbol{f}^{(0)}\|_{L^2_\mu} \sqrt{\sum_{i=1}^{m-1} \mu_i}$$

$$\|\langle \boldsymbol{f}^{(1)}\rangle\|_{L_\Phi} \leqslant |\langle \boldsymbol{f}^{(1)}\rangle| \leqslant \|\boldsymbol{f}^{(1)}\|_{L^2_\mu} \sqrt{\sum_{i=m+1}^{\infty} \mu_i}$$

(see Lemma A.4 in Appendix A). Thus

$$\|\boldsymbol{f}^{(0)}\|_{L_\Phi} \leqslant \|\boldsymbol{f}^{(0)} - \langle \boldsymbol{f}^{(0)}\rangle\|_{L_\Phi} + \|\langle \boldsymbol{f}^{(0)}\rangle\|_{L_\Phi} \leqslant \|\boldsymbol{f}^{(0)} - \langle \boldsymbol{f}^{(0)}\rangle\|_{L_\Phi} + \sqrt{\frac{3}{2}\sum_{i=1}^{m-1} \mu_i}\, \|\boldsymbol{f}^{(0)}\|_{L_\Phi},$$

implying that

$$\|\boldsymbol{f}^{(0)}\|_{L_\Phi} \leqslant \frac{1}{1 - \sqrt{\frac{3}{2}\sum_{i=1}^{m-1} \mu_i}} \|\boldsymbol{f}^{(0)} - \langle \boldsymbol{f}^{(0)}\rangle\|_{L_\Phi},$$

and similarly

$$\|\boldsymbol{f}^{(1)}\|_{L_\Phi} \leqslant \frac{1}{1 - \sqrt{\frac{3}{2}\sum_{i=m+1}^{\infty} \mu_i}} \|\boldsymbol{f}^{(1)} - \langle \boldsymbol{f}^{(1)}\rangle\|_{L_\Phi}.$$

We can conclude, by applying (2.15) to $\boldsymbol{f}^{(0)}$ and $\boldsymbol{f}^{(1)}$, that

$$\|\boldsymbol{f}^{(0)}\|_{L_\Phi}^2 \leqslant \frac{3C_{\mathrm{LS}}}{2\left(1-\sqrt{\frac{3}{2}\sum_{i=1}^{m-1}\mu_i}\right)^2}\sum_{i=1}^{m-1}\nu_i\left(f_{i+1}-f_i\right)^2$$

$$\text{and} \qquad \|\boldsymbol{f}^{(1)}\|_{L_\Phi}^2 \leqslant \frac{3C_{\mathrm{LS}}}{2\left(1-\sqrt{\frac{3}{2}\sum_{i=m+1}^{\infty}\mu_i}\right)^2}\sum_{i=m}^{\infty}\nu_i\left(f_{i+1}-f_i\right)^2.$$

The result now follows from (2.7).

To show the converse, we use the translation invariance of (ii) from Proposition 2.9 to assume that $f_m=0$. As such we have that $\boldsymbol{f}=\boldsymbol{f}^{(0)}+\boldsymbol{f}^{(1)}$. Moreover,

$$\begin{aligned}
\|\boldsymbol{f}-\langle\boldsymbol{f}\rangle\|_{L_\Phi}^2 &\leqslant \left(\left\|\boldsymbol{f}^{(0)}-\langle\boldsymbol{f}^{(0)}\rangle\right\|_{L_\Phi}+\left\|\boldsymbol{f}^{(1)}-\langle\boldsymbol{f}^{(1)}\rangle\right\|_{L_\Phi}\right)^2\\
&\leqslant \left(\left(1+\sqrt{\frac{3}{2}}\sqrt{\sum_{i=1}^{m-1}\mu_i}\right)\|\boldsymbol{f}^{(0)}\|_{L_\Phi}+\left(1+\sqrt{\frac{3}{2}}\sqrt{\sum_{i=m+1}^{\infty}\mu_i}\right)\|\boldsymbol{f}^{(1)}\|_{L_\Phi}\right)^2\\
&\leqslant 2\left(1+\sqrt{\frac{3}{2}}\sqrt{\sum_{i=1}^{m-1}\mu_i}\right)^2\|\boldsymbol{f}^{(0)}\|_{L_\Phi}^2+2\left(1+\sqrt{\frac{3}{2}}\sqrt{\sum_{i=m+1}^{\infty}\mu_i}\right)^2\|\boldsymbol{f}^{(1)}\|_{L_\Phi}^2\\
&\leqslant 2\eta\max\left(\left(1+\sqrt{\frac{3}{2}}\sqrt{\sum_{i=1}^{m-1}\mu_i}\right)^2,\left(1+\sqrt{\frac{3}{2}}\sqrt{\sum_{i=m+1}^{\infty}\mu_i}\right)^2\right)\sum_{i=1}^{\infty}\nu_i\left(f_{i+1}-f_i\right)^2
\end{aligned}$$

where we again used (2.7). This shows the desired result due to Proposition 2.9. □

Proof of Theorem 2.3. Our main tool will be the above Lemma 2.14. It is known that

$$\|\boldsymbol{f}^2\|_{L_\Psi}=\sup\left\{\sum_{i=1}^{\infty}\mu_i f_i^2 g_i\ ;\ \sum_{i=1}^{\infty}\mu_i\Xi(g_i)\leqslant 1\right\},$$

where Ξ is the Young complement of Ψ. Using Corollary 2.12 we know that if $f_m=0$ then

$$\sum_{i=m}^{\infty}\mu_i f_i^2 g_i\leqslant C_{\mathrm{LS}}\sum_{i=m}^{\infty}\nu_i\left(f_{i+1}-f_i\right)^2$$

if and only if

$$B=\sup_{k\geqslant m}\left(\sum_{i=k+1}^{\infty}g_i\mu_i\right)\left(\sum_{i=1}^{k}\frac{1}{\nu_i}\right)<\infty.$$

Taking supremum over all appropriate $\boldsymbol{g}=\{g_i\}$, we find that

$$\left\|\boldsymbol{f}^2\mathbb{1}_{i>m}\right\|_{L_\Psi}\leqslant C_{\mathrm{LS}}\sum_{i=m}^{\infty}\nu_i\left(f_{i+1}-f_i\right)^2 \tag{2.16}$$

if and only if

$$B=\sup_{k\geqslant m}\left\|\mathbb{1}_{[k+1,\infty)}\right\|_{L_\Psi}\sum_{i=1}^{k}\frac{1}{\nu_i}<\infty.$$

As

$$\left\|\mathbb{1}_{[k+1,\infty)}\right\|_{L_\Psi} = \inf_{\alpha>0}\left\{\sum_{i=k+1}^{\infty}\mu_i\Psi\left(\frac{1}{\alpha}\right)\leqslant 1\right\} = \inf_{\alpha>0}\left\{\Psi\left(\frac{1}{\alpha}\right)\leqslant\frac{1}{\sum_{i=k+1}^{\infty}\mu_i}\right\}$$
$$= \frac{1}{\Psi^{-1}\left(\frac{1}{\sum_{i=k+1}^{\infty}\mu_i}\right)}$$

we find that (2.16) is equivalent to $B_1<\infty$, showing that (i) implies (ii). Conversely, using Lemma 2.13 we find that if $f_m=0$ then

$$\sum_{i=1}^{m-1}\mu_i f_i^2 g_i \leqslant \left[\sum_{i=1}^{m-1}\mu_i g_i\left(\sum_{j=i}^{m-1}\frac{1}{\nu_j}\right)\right]\sum_{i=1}^{m-1}\nu_i\left(f_{i+1}-f_i\right)^2$$
$$\leqslant\left[\left(\sum_{i=1}^{m-1}\mu_i g_i\right)\left(\sum_{j=1}^{m-1}\frac{1}{\nu_j}\right)\right]\sum_{i=1}^{m-1}\nu_i\left(f_{i+1}-f_i\right)^2$$

and again, by taking supremum over the appropriate $\boldsymbol{g}$, we find that

$$\left\|\boldsymbol{f}^2\mathbb{1}_{i<m}\right\|_{L_\Psi}\leqslant B_2\sum_{i=1}^{m-1}\nu_i\left(f_{i+1}-f_i\right)^2. \tag{2.17}$$

Thus, if $\boldsymbol{f}=\{f_i\}$ is a sequence such that $f_m=0$, and if in addition $B_1<\infty$ we have that

$$\left\|\left(\boldsymbol{f}^{(0)}\right)^2\right\|_{L_\Psi}+\left\|\left(\boldsymbol{f}^{(1)}\right)^2\right\|_{L_\Psi}\leqslant B_2\sum_{i=1}^{m-1}\nu_i\left(f_{i+1}-f_i\right)^2+4B_1\sum_{i=m}^{\infty}\nu_i\left(f_{i+1}-f_i\right)^2$$
$$\leqslant\left(B_2+4B_1\right)\sum_{i=1}^{\infty}\nu_i\left(f_{i+1}-f_i\right)^2,$$

where we have used Corollary 2.12. We conclude, using Lemma 2.14, that if $B_1<\infty$ then $\boldsymbol{\nu}$ admits a log-Sobolev inequality with respect to $\boldsymbol{\mu}$ with constant C_{LS} that can be chosen to be $C_{\text{LS}}=40(B_1+4B_2)$. $\square$

We are only left with the proof of Corollary 2.4. The proof relies on the following technical lemma, whose proof is left to Appendix A:

Lemma 2.15. *For any $t\geqslant\frac{3}{2}$ one has that*

$$\frac{1}{3}\frac{t}{\log t}\leqslant\Psi^{-1}(t)\leqslant 2\frac{t}{\log t}.$$

Proof of Corollary 2.4. Due to the choice of m and Lemma 2.15 we know that $\Psi^{-1}(t)$ and $\frac{t}{\log t}$ are equivalent for our choice of

$$t=\frac{1}{\sum_{i=m+1}^{\infty}\mu_i}.$$

This shows the desired equivalence using Theorem 2.3. As for the last estimation, it follows immediately from the fact that

$$B_i\leqslant 3D_i,$$

for $i=1,2$. $\square$

Now that we have achieved a necessary and sufficient condition to the validity of a discrete log-Sobolev inequality with weight, we will proceed to see how it can be used to prove Theorem 1.1.

3. Energy Dissipation Inequalities

3.1. The Log-Sobolev Inequality and the Becker-Döring Equations. Motivated by our previous section, the first step in trying to show the validity of Cercignani's conjecture would be to relate the energy dissipation, $D(\boldsymbol{c})$, and a term that resembles the right hand side of (2.2). Recall that, for any non-negative sequence $\boldsymbol{c} = \{c_i\}$ we defined

$$D(\boldsymbol{c}) = \sum_{i=1}^{\infty} a_i Q_i \, \Theta\left(\frac{c_1 c_i}{Q_i}, \frac{c_{i+1}}{Q_{i+1}}\right)$$

with $\Theta(x,y) := (x-y)(\log x - \log y)$, and

$$\overline{D}(\boldsymbol{c}) = \sum_{i=1}^{\infty} a_i Q_i \left(\sqrt{\frac{c_1 c_i}{Q_i}} - \sqrt{\frac{c_{i+1}}{Q_{i+1}}}\right)^2.$$

We have the following properties:

Lemma 3.1. *For any non-negative sequence $\boldsymbol{c}$, the following holds*

(i) We have that

$$4\overline{D}(\boldsymbol{c}) \leqslant D(\boldsymbol{c}) \tag{3.1}$$

(ii) For any $z > 0$ we can rewrite $D(\boldsymbol{c})$ as

$$D(\boldsymbol{c}) = \sum_{i=1}^{\infty} a_i Q_i z^{i+1} \Theta\left(\frac{c_1 c_i}{Q_i z^{i+1}}, \frac{c_{i+1}}{Q_{i+1} z^{i+1}}\right) \tag{3.2}$$

(recalling $\Theta(x,y) := (x-y)(\log x - \log y)$), and

$$\overline{D}(\boldsymbol{c}) = \sum_{i=1}^{\infty} a_i Q_i z^{i+1} \left(\sqrt{\frac{c_1 c_i}{Q_i z^{i+1}}} - \sqrt{\frac{c_{i+1}}{Q_{i+1} z^{i+1}}}\right)^2 \tag{3.3}$$

Proof. (i) is an immediate consequence of the inequality

$$\Theta(x,y) = (x-y)\left(\log x - \log y\right) \geqslant 4\left(\sqrt{x} - \sqrt{y}\right)^2$$

and (ii) is immediate from the homogeneity of the expressions involved. $\square$

Property (ii) of the above lemma gives an indication of how we may be able to find a connection between $\overline{D}(\boldsymbol{c})$ and the relative entropy between $\boldsymbol{c}$ and *some* equilibrium, by appropriately choosing z. Similar to the work of Jabin and Niethammer [19], another equilibrium state that will play an important role in what is to follow is

$$\tilde{\mathcal{Q}} = \mathcal{Q}_{c_1} = \{Q_i c_1^i\}_{i\geqslant 1}.$$

Indeed, it is the *only* possible equilibrium under which the right hand side of (3.3) attains a form that is suitable for the log-Sobolev theory developed in the previous section. From (3.3) we find, after cancelling c_1, that

$$\overline{D}(\boldsymbol{c}) = \sum_{i=1}^{\infty} a_i \tilde{\mathcal{Q}}_i \tilde{\mathcal{Q}}_1 \left(\sqrt{\frac{c_i}{\tilde{\mathcal{Q}}_i}} - \sqrt{\frac{c_{i+1}}{\tilde{\mathcal{Q}}_{i+1}}}\right)^2 \tag{3.4}$$

This enables us to finally link $\overline{D}(\boldsymbol{c})$ to $H(\boldsymbol{c}|\mathcal{Q})$:

Proposition 3.2. *For given coagulation and detailed balance coefficients, $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$, and a given positive sequence $\boldsymbol{c}$ with finite mass ϱ and such that*

$$\sum_{i=1}^{\infty} \tilde{\mathcal{Q}}_i < +\infty, \quad \sum_{i=1}^{\infty} a_i \tilde{\mathcal{Q}}_i < +\infty$$

(recall $\mathcal{Q}_i := Q_i c_1^i$ for $i \geqslant 1$), we define the following measures

$$\mu_i = \frac{\tilde{\mathcal{Q}}_i}{\sum_{i=1}^{\infty} \tilde{\mathcal{Q}}_i}, \qquad \nu_i := \frac{a_i \tilde{\mathcal{Q}}_i}{\sum_{j=1}^{\infty} a_j \tilde{\mathcal{Q}}_j}, \qquad i \in \mathbb{N}. \tag{3.5}$$

Then, if $\boldsymbol{\nu}$ admits a log-Sobolev inequality with respect to $\boldsymbol{\mu}$ with constant C_{LS} we have that

$$\overline{D}(\boldsymbol{c}) \geqslant \frac{c_1^3 \left(\sum_{i=1}^{\infty} a_i \tilde{\mathcal{Q}}_i\right)}{C_{\mathrm{LS}} \left(\sum_{i=1}^{\infty} \tilde{\mathcal{Q}}_i\right) \left(c_1^2 + 2\left(\sum_{i=1}^{\infty} c_i\right)\left(\sum_{i=1}^{\infty} \tilde{\mathcal{Q}}_i\right)\right)} H(\boldsymbol{c}|\mathcal{Q}) \tag{3.6}$$

Proof. Denote by $f_i = \sqrt{\frac{c_i}{\tilde{\mathcal{Q}}_i}}$. Since $\boldsymbol{\nu}$ admits a log-Sobolev inequality with respect to $\boldsymbol{\mu}$ with constant C_{LS} we have that

$$\overline{D}(\boldsymbol{c}) = \left(\sum_{i=1}^{\infty} a_i \tilde{\mathcal{Q}}_i \tilde{\mathcal{Q}}_1\right) \sum_{i=1}^{\infty} \nu_i \left(f_{i+1} - f_i\right)^2 \geqslant \frac{c_1 \left(\sum_{i=1}^{\infty} a_i \tilde{\mathcal{Q}}_i\right)}{C_{\mathrm{LS}}} \mathrm{Ent}_{\mu}\left(\boldsymbol{f}^2\right). \tag{3.7}$$

Next, we notice that

$$\begin{aligned} \left(\sum_{i=1}^{\infty} \tilde{\mathcal{Q}}_i\right) \mathrm{Ent}_{\mu}(\boldsymbol{f}^2) &= \sum_{i=1}^{\infty} c_i \log \frac{c_i}{\tilde{\mathcal{Q}}_i} - \left(\sum_{i=1}^{\infty} c_i\right)\left(\log \sum_{i=1}^{\infty} c_i - \log \sum_{i=1}^{\infty} \tilde{\mathcal{Q}}_i\right) \\ &= H(\boldsymbol{c}|\tilde{\mathcal{Q}}) + \sum_{i=1}^{\infty} c_i - \sum_{i=1}^{\infty} \tilde{\mathcal{Q}}_i - \left(\sum_{i=1}^{\infty} c_i\right)\left(\log \sum_{i=1}^{\infty} c_i - \log \sum_{i=1}^{\infty} \tilde{\mathcal{Q}}_i\right) \\ &= H(\boldsymbol{c}|\tilde{\mathcal{Q}}) - \left(\sum_{i=1}^{\infty} \tilde{\mathcal{Q}}_i\right) \Lambda\left(\frac{\sum_{i=1}^{\infty} c_i}{\sum_{i=1}^{\infty} \tilde{\mathcal{Q}}_i}\right), \end{aligned} \tag{3.8}$$

where $\Lambda(x) = x \log x - x + 1$. We now use the fact that $\mathcal{Q}$ minimises the relative entropy to the set of equilibria to bound the first term,

$$H(\boldsymbol{c}|\tilde{\mathcal{Q}}) \geqslant H(\boldsymbol{c}|\mathcal{Q}) \tag{3.9}$$

(see Lemma B.1 in Appendix B). The only remaining bound is to show that the term with the negative sign at the end of (3.8) is in fact bounded by $\mathrm{Ent}_{\mu}(\boldsymbol{f}^2)$. For this we will use the following Csiszár-Kullback inequality:

$$\mathrm{Ent}_{\mu}(\boldsymbol{f}^2) \geqslant \frac{1}{2\langle \boldsymbol{f}^2 \rangle} \left(\sum_{i=1}^{\infty} |f_i^2 - \langle \boldsymbol{f}^2 \rangle| \mu_i\right)^2, \tag{3.10}$$

where

$$\langle \boldsymbol{f}^2 \rangle := \sum_{i=1}^{\infty} f_i^2 \mu_i.$$

With (3.10) we find that in our particular setting

$$\begin{aligned}\mathrm{Ent}_\mu(\boldsymbol{f}^2) &\geqslant \frac{\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i}{2\sum_{i=1}^\infty c_i}\left(\sum_{i=1}^\infty \left|\frac{c_i}{\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i} - \frac{\tilde{\mathcal{Q}}_i\left(\sum_{i=1}^\infty c_i\right)}{\left(\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i\right)^2}\right|\right)^2 \\ &= \frac{\sum_{i=1}^\infty c_i}{2\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i}\left(\sum_{i=1}^\infty \left|\frac{c_i}{\sum_{i=1}^\infty c_i} - \frac{\tilde{\mathcal{Q}}_i}{\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i}\right|\right)^2\end{aligned}$$

and keeping only the first term in the last sum we get

$$\mathrm{Ent}_\mu(\boldsymbol{f}^2) \geqslant \frac{\sum_{i=1}^\infty c_i}{2\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i}\left|\frac{c_1}{\sum_{i=1}^\infty c_i} - \frac{\tilde{\mathcal{Q}}_1}{\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i}\right|^2 = \frac{c_1^2}{2\sum_{i=1}^\infty c_i \sum_{i=1}^\infty \tilde{\mathcal{Q}}_i}\left(1 - \frac{\sum_{i=1}^\infty c_i}{\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i}\right)^2$$

Continuing from (3.8) and using (3.9), the above inequality and the fact that

$$\Lambda(x) \leqslant (x-1)^2$$

show that

$$\begin{aligned}\left(\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i\right)\mathrm{Ent}_\mu\left(\boldsymbol{f}^2\right) &\geqslant H(\boldsymbol{c}|\mathcal{Q}) - \left(\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i\right)\left(\frac{\sum_{i=1}^\infty c_i}{\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i} - 1\right)^2 \\ &\geqslant H(\boldsymbol{c}|\mathcal{Q}) - \frac{2}{c_1^2}\left(\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i\right)^2\left(\sum_{i=1}^\infty c_i\right)\mathrm{Ent}_\mu\left(\boldsymbol{f}^2\right).\end{aligned}$$

Thus,

$$H(\boldsymbol{c}|\mathcal{Q}) \leqslant \left(\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i\right)\left(1 + \frac{2}{c_1^2}\left(\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i\right)\left(\sum_{i=1}^\infty c_i\right)\right)\mathrm{Ent}_\mu\left(\boldsymbol{f}^2\right).$$

Combining the above with (3.7) completes the proof. □

3.2. Main Inequality for c_1 'Close' to Equilibrium.

On the basis of Proposition 3.2, one obtains the following

Proposition 3.3. *Assume the conditions of Proposition 3.2 and the additional condition that $c_1 < z_*$ for some $0 < z_* < z_s$. Calling*

$$\varrho_* := \sum_{i=1}^\infty i Q_i z_*^i < \infty$$

we have that

$$\overline{D}(\boldsymbol{c}) \geqslant \frac{a_1 z_*^2 c_1^2}{C_{\mathrm{LS}}\left(z_* + \varrho_*\right)\left(z_*^2 + 2\varrho(z_* + \varrho_*)\right)} H(\boldsymbol{c}|\mathcal{Q}). \tag{3.11}$$

In particular, if $0 < \delta < c_1 < z_s - \delta$ for some $\delta > 0$,

$$\overline{D}(\boldsymbol{c}) \geqslant \lambda H(\boldsymbol{c}|\mathcal{Q})$$

for some constant $\lambda > 0$ which depends only on δ, ρ, a_1 and $\{Q_i\}_{i\geqslant 1}$.

Proof. This follows immediately from (3.6) and the estimates

$$\sum_{i=1}^\infty \tilde{\mathcal{Q}}_i = \sum_{i=1}^\infty Q_i c_1^i \leqslant c_1\left(1 + \frac{1}{z_*}\sum_{i=2}^\infty Q_i z_*^i\right) < c_1\left(1 + \frac{\varrho_*}{z_*}\right),$$

$$\sum_{i=1}^{\infty} c_i \leqslant \sum_{i=1}^{\infty} i c_i = \varrho,$$

together with $\sum_{i=1}^{\infty} a_i \tilde{\mathcal{Q}}_i \geqslant a_1 c_1$. □

Proposition 3.3 shows us that as long as c_1 is bounded away from 0 and z_s, Cercignani's conjecture will follow immediately from a log-Sobolev inequality for $\boldsymbol{\nu}$ with respect to $\boldsymbol{\mu}$ (which were defined in Proposition 3.2). Our next result shows that this is indeed true for subcritical masses, under reasonable conditions on the coefficients:

Proposition 3.4. *Let $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$ satisfy Hypothesis 1-3 with $\gamma = 1$ and let $\boldsymbol{c} = \{c_i\}_{i\in\mathbb{N}}$ be an arbitrary positive sequence with finite total density $\varrho < \varrho_s < +\infty$. Assume that there exists $\delta > 0$ such that*

$$c_1 \leqslant z_s - \delta.$$

Then, the measure $\boldsymbol{\nu}$ admits a log-Sobolev inequality with respect to the measure $\boldsymbol{\mu}$ with constant

$$C_{LS} = \frac{60 z_s^3}{\delta^3} C\left(\frac{z_s-\delta}{z_s}\right)\left(4 + 2e \sup_k \left|\log\left(\alpha_{k+1}^{\frac{1}{k+1}}\right)\right| + e\log\frac{z_s}{\delta}\right) \tag{3.12}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ were defined in Proposition 3.2 and

$$C(\eta) = 1 + \sup_{k\geqslant 3}\left(k\left(1+\log\left(\frac{k}{2}\right)\right)\eta^{\frac{k}{2}}\right) + \frac{2\eta}{1-\eta}$$

for $\eta < 1$.

Proof. We just need to estimate the constant given in Corollary 2.4. As mentioned in the introduction, we can assume without loss of generality that $a_i = i$. We denote by

$$\eta = \frac{c_1}{z_s} \leqslant \frac{z_s - \delta}{z_s} =: \eta_1 < 1.$$

As

$$\tilde{\mathcal{Q}}_i = \alpha_i z_s^{1-i} c_1^i \leqslant z_s \alpha_i \eta^i$$

we find that due to the monotonicity of $\{\alpha_i\}_{i\in\mathbb{N}}$

$$z_s \alpha_{k+1}\eta^{k+1} = \tilde{\mathcal{Q}}_{k+1} \leqslant \sum_{i=k+1}^{\infty} \tilde{\mathcal{Q}}_i \leqslant z_s \eta^{k+1}\sum_{i=1}^{\infty}\alpha_{i+k}\eta^{i-1} \leqslant \frac{z_s\alpha_{k+1}\eta^{k+1}}{1-\eta}.$$

As such

$$\alpha_{k+1}(1-\eta)\eta^k \leqslant \sum_{i=k+1}^{\infty}\mu_i \leqslant \alpha_{k+1}\frac{\eta^k}{1-\eta},$$

implying that

$$-\sum_{i=k+1}^{\infty}\mu_i \log\left(\sum_{i=k+1}^{\infty}\mu_i\right) \leqslant \frac{\alpha_{k+1}\eta^k}{1-\eta}\left(k\log\left(\frac{1}{\eta}\right) - \log\left(\alpha_{k+1}(1-\eta)\right)\right). \tag{3.13}$$

Next, we notice that as

$$\sum_{i=1}^{\infty} i y^i = \frac{y}{(1-y)^2},$$

one has that

$$z_s\eta \leqslant \sum_{i=1}^{\infty} i\alpha_i z_s \eta^i = \sum_{i=1}^{\infty} a_i\tilde{\mathcal{Q}}_i \leqslant z_s\frac{\eta}{(1-\eta)^2},$$

from which we find that

$$i\alpha_i(1-\eta)^2\eta^{i-1} \leqslant \nu_i \leqslant i\alpha_i\eta^{i-1}.$$

We notice that for $k \geqslant 3$ the monotonicity of $\{\alpha_i\}_{i\in\mathbb{N}}$ implies that

$$k\alpha_k\eta^k \sum_{i=1}^{k} \frac{1}{i\alpha_i}\left(\frac{1}{\eta}\right)^i = 1 + \sum_{i=1}^{k-1} \frac{k\alpha_k}{i\alpha_i}\eta^{k-i}$$

$$\leqslant 1 + \sum_{i=1}^{k-1} \frac{k}{i}\eta^{k-i} = 1 + \sum_{i=1}^{\left[\frac{k}{2}\right]} \frac{k}{i}\eta^{k-i} + \sum_{i=\left[\frac{k}{2}\right]+1}^{k-1} \frac{k}{i}\eta^{k-i} \leqslant 1 + k\eta_1^{\frac{k}{2}} \sum_{i=1}^{\left[\frac{k}{2}\right]} \frac{1}{i} + \frac{k}{\left[\frac{k}{2}\right]+1}\sum_{j=1}^{\infty}\eta_1^j$$

$$\leqslant 1 + k\left(1+\log\left(\frac{k}{2}\right)\right)\eta_1^{\frac{k}{2}} + \frac{2\eta_1}{1-\eta_1}.$$

Using the definition of $C(\eta)$ and the fact that $C(\eta) > 1+\eta$ we find that for all $k \in \mathbb{N}$

$$k\alpha_k\eta^k \sum_{i=1}^{k} \frac{1}{i\alpha_i}\left(\frac{1}{\eta}\right)^i \leqslant C(\eta_1).$$

and as such

$$\sum_{i=1}^{k} \frac{1}{\nu_i} \leqslant C(\eta_1)\frac{\eta}{(1-\eta)^2}\frac{1}{k\alpha_k}\left(\frac{1}{\eta}\right)^k \tag{3.14}$$

Combining the above with (3.13) yields the bound

$$\left(-\sum_{i=k+1}^{\infty}\mu_i \log\left(\sum_{i=k+1}^{\infty}\mu_i\right)\right)\left(\sum_{i=1}^{k}\frac{1}{\nu_i}\right)$$

$$\leqslant C(\eta_1)\frac{\alpha_{k+1}}{\alpha_k}\frac{\eta}{(1-\eta)^3}\left(\log\left(\frac{1}{\eta}\right) - \frac{1}{k}\log\left(\alpha_{k+1}(1-\eta)\right)\right).$$

Thus, with the notation of Corollary 2.4

$$D_1 \leqslant \frac{C(\eta_1)}{(1-\eta_1)^3}\left(\sup_{0\leqslant x\leqslant 1}\left(-\eta\log(\eta)\right) + \eta_1\sup_k \frac{k+1}{k}\left|\log\left(\alpha_{k+1}^{\frac{1}{k+1}}\right)\right| + \eta_1\log\left(\frac{1}{1-\eta_1}\right)\right)$$

$$\leqslant \frac{C(\eta_1)}{(1-\eta_1)^3}\left(\frac{1}{e} + 2\eta_1\sup_k\left|\log\left(\alpha_{k+1}^{\frac{1}{k+1}}\right)\right| + \eta_1\log\left(\frac{1}{1-\eta_1}\right)\right),$$

As m, defined in Corollary 2.4, is always finite we conclude using the same Corollary that $\boldsymbol{\nu}$ admits a log-Sobolev inequality with respect to $\boldsymbol{\mu}$. However, in order to estimate the constant C_{LS} we still need to estimate the constant D_2 in the case where $m > 1$ (otherwise, $D_2 = 0$).

Since

$$\sum_{i=m}^{\infty}\mu_i \leqslant \frac{\alpha_m}{1-\eta}\eta^{m-1}$$

the requirement that $\sum_{i=1}^{m-1}\mu_i < \frac{2}{3}$ implies that

$$\frac{1}{\alpha_{m-1}\eta^{m-1}} \leqslant \frac{\alpha_m}{\alpha_{m-1}}\frac{3}{(1-\eta)} \leqslant \frac{3}{(1-\eta)}.$$

Using the above along with the fact that $m > 1$ and inequality (3.14) shows that

$$\sum_{i=1}^{m-1}\frac{1}{\nu_i} \leqslant 3C(\eta_1)\frac{\eta_1}{(1-\eta_1)^3}\frac{1}{m-1} \leqslant 3C(\eta_1)\frac{\eta_1}{(1-\eta_1)^3}.$$

We can conclude that

$$\left(-\sum_{i=m-1}^{\infty}\mu_i\log\left(\sum_{i=m-1}^{\infty}\mu_i\right)\right)\left(\sum_{i=1}^{m-1}\frac{1}{\nu_i}\right)\leqslant 3\sup_{0\leqslant x\leqslant 1}\left(-x\log x\right)C(\eta_1)\frac{\eta_1}{(1-\eta_1)^3} \tag{3.15}$$

from which we conclude that

$$D_2\leqslant\frac{3}{e}C(\eta_1)\frac{\eta_1}{(1-\eta_1)^3}$$

which completes the proof, as the result follows directly from Corollary 2.4. □

We finally have all the tools to prove part (i) of Theorem 1.1:

Proof of part (i) *of Theorem 1.1.* The result follows immediately from Corollary 3.3, Proposition 3.4 and condition (1.16). □

The last part of this section will be devoted to the proof of part (ii) of Theorem 1.1. For that we will need the following lemma:

Lemma 3.5. *For any $\beta\geqslant 0$, any non-negative sequence $\boldsymbol{c}$ and positive sequence $\{Q_i\}_{i\geqslant 1}$ it holds that*

$$\sum_{i=1}^{\infty}i^{\beta}Q_i\left(\sqrt{\frac{c_1c_i}{Q_i}}-\sqrt{\frac{c_{i+1}}{Q_{i+1}}}\right)^2\leqslant 2\left(c_1+\sup_j\frac{Q_j}{Q_{j+1}}\right)\sum_{i=1}^{\infty}i^{\beta}c_i. \tag{3.16}$$

Proof. The proof is a direct consequence of the inequality $(a+b)^2\leqslant 2(a^2+b^2)$:

$$\begin{aligned}\sum_{i=1}^{\infty}i^{\beta}Q_i\left(\sqrt{\frac{c_1c_i}{Q_i}}-\sqrt{\frac{c_{i+1}}{Q_{i+1}}}\right)^2&\leqslant 2c_1\sum_{i=1}^{\infty}i^{\beta}c_i+2\sum_{i=1}^{\infty}i^{\beta}\frac{Q_i}{Q_{i+1}}c_{i+i}\\&\leqslant 2\left(c_1+\sup_j\frac{Q_j}{Q_{j+1}}\right)\sum_{i=1}^{\infty}i^{\beta}c_i.\end{aligned}$$ □

Proof of part (ii) *of Theorem 1.1.* We denote by $\overline{D}_{\gamma}(\boldsymbol{c})$ the lower free energy dissipation of $\boldsymbol{c}$ associated to the coagulation coefficient $a_i=i^{\gamma}$. According to part (i) of Theorem 1.1, there exists $K>0$ that depends only on δ, z_{s}, ϱ and $\{\alpha_i\}_{i\in\mathbb{N}}$ such that

$$\overline{D}_1(\boldsymbol{c})\geqslant KH(\boldsymbol{c}|\mathcal{Q}).$$

Using interpolation between γ and β we find that

$$\overline{D}_1(c)\leqslant\overline{D}_{\gamma}^{\frac{\beta-1}{\beta-\gamma}}(c)\overline{D}_{\beta}^{\frac{1-\gamma}{\beta-\gamma}}(c)\leqslant 2^{\frac{1-\gamma}{\beta-\gamma}}\overline{D}_{\gamma}^{\frac{\beta-1}{\beta-\gamma}}(c)\left(z_s+\frac{1}{z_s}\sup_j\frac{\alpha_j}{\alpha_{j+1}}\right)^{\frac{1-\gamma}{\beta-\gamma}}M_{\beta}^{\frac{1-\gamma}{\beta-\gamma}} \tag{3.17}$$

where we have used Lemma 3.5, the upper bound on c_1 and Hypothesis 2. Therefore

$$D(\boldsymbol{c})\geqslant\overline{D}_{\gamma}(\boldsymbol{c})\geqslant\left(\frac{z_sK^{\frac{\beta-\gamma}{1-\gamma}}}{2\left(z_s^2+\sup_j\frac{\alpha_j}{\alpha_{j+1}}\right)M_{\beta}}\right)^{\frac{1-\gamma}{\beta-1}}H(\boldsymbol{c}|\mathcal{Q})^{\frac{\beta-\gamma}{\beta-1}} \tag{3.18}$$

and the proof is now complete. □

This concludes the part of the proof of Theorem 1.1 that relied on the log-Sobolev inequality. In the next subsection we will address the question of what happens when c_1 escapes the 'good region' delimited by (1.16).

3.3. **Energy Dissipation Estimate when c_1 is 'Far' From Equilibrium.** The goal of this subsection is to show that when c_1 is far from equilibrium, in the aforementioned sense, then while we may lose our desired inequality between $\overline{D}(\boldsymbol{c})$ and $H(\boldsymbol{c}|\mathcal{Q})$, the energy dissipation becomes *uniformly large* - forcing the free energy to decrease (and as a consequence, the distance between c_1 and $\overline{z}$ decreases as well).

The next proposition, dealing with the case when c_1 is 'too large', is an adaptation of a theorem from [19].

Proposition 3.6. *Let $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$ be the coagulation and detailed balance coefficients for the Becker-Döring equations. Assume that* $\inf_i a_i > 0$ *and*

$$\lim_{i\to\infty} \frac{Q_{i+1}}{Q_i} = \frac{1}{z_s}.$$

Let $\boldsymbol{c} = \{c_i\}$ be a non-negative sequence with finite total density $\varrho < \varrho_s$. Then, if

$$c_1 > \overline{z} + \delta$$

for any $\delta > 0$, we have that

$$\overline{D}(\boldsymbol{c}) > \varepsilon_1,$$

for a fixed constant ε_1 that depends only on $\{Q_i\}_{i\in\mathbb{N}}, \overline{z}, z_s$ and δ.

Proof. Without loss of generality we may assume that $\overline{z} + \delta < z_s$. Denoting by $u_i = \frac{c_i}{Q_i}$ we notice that

$$\overline{D}(\boldsymbol{c}) = \sum_{i=1}^{\infty} a_i Q_i \left(\sqrt{c_1\, u_i} - \sqrt{u_{i+1}}\right)^2.$$

Let $\lambda < 1$ be such that $\lambda c_1 = \overline{z} + \frac{\delta}{2}$ and let $i_0 \in \mathbb{N}$ be the first index such that

$$u_{i+1} < \lambda c_1 u_i.$$

This index exists, else, for any $i \in \mathbb{N}$ we have

$$u_{i+1} \geqslant \lambda c_1 u_i \geqslant (\lambda c_1)^i c_1 \tag{3.19}$$

and thus

$$\varrho = \sum_{i=1}^{\infty} i c_i \geqslant c_1 + c_1 \sum_{i=2}^{\infty} i Q_i (\lambda c_1)^{i-1} \geqslant \sum_{i=1}^{\infty} i Q_i \left(\overline{z} + \frac{\delta}{2}\right)^i,$$

which is a contradiction.

Due to the positivity of each term in the sum consisting of the lower free energy dissipation, we conclude that

$$\overline{D}(\boldsymbol{c}) \geqslant a_{i_0} Q_{i_0} \left(1 - \sqrt{\lambda}\right)^2 c_1 u_{i_0} \geqslant a_{i_0} Q_{i_0} \lambda^{i_0 - 1} c_1^{i_0+1} \left(1 - \sqrt{\lambda}\right)^2 \tag{3.20}$$

where we have used the fact that up to $i_0 - 1$ we have inequality (3.19).

As we know that there exists $C > 0$, depending only on $\{Q_i\}_{i\in\mathbb{N}}, \overline{z}, z_s$ and δ such that

$$\sum_{i=i_0+1}^{\infty} i c_1 (\lambda c_1)^{i-1} Q_i \leqslant C Q_{i_0} (\lambda c_1)^{i_0} c_1$$

(see Lemma B.2 in Appendix B), we conclude that, using (3.19) again,

$$C Q_{i_0} (\lambda c_1)^{i_0} c_1 \geqslant \tilde{\varrho} - \sum_{i=1}^{i_0} i Q_i (\lambda c_1)^{i-1} c_1 \geqslant \tilde{\varrho} - \sum_{i=1}^{i_0} i c_i \geqslant \tilde{\varrho} - \varrho,$$

where $\tilde{\varrho} = \sum_{i=1}^{\infty} iQ_i \left(\lambda c_1\right)^{i-1} c_1$. We can estimate the difference $\varrho - \tilde{\varrho}$ as

$$\tilde{\varrho} - \varrho \geqslant \sum_{i=1}^{\infty} iQ_i \left(\left(\overline{z} + \frac{\delta}{2} \right)^i - \overline{z}^i \right) \geqslant \left(\sum_{i=1}^{\infty} i^2 Q_i \overline{z}^{i-1} \right) \frac{\delta}{2}.$$

In conclusion, there exists a universal constant $C_1 > 0$, depending only on $\{Q_i\}_{i\in\mathbb{N}}, \overline{z}, z_s$ and δ, and not on i_0, c_1 or λ, such that

$$Q_{i_0} \left(\lambda c_1\right)^{i_0} c_1 > C_1.$$

Recalling (3.20) and using the fact that $\lambda = \dfrac{\overline{z} + \frac{\delta}{2}}{c_1} < \dfrac{\overline{z} + \frac{\delta}{2}}{\overline{z} + \delta}$ we find that:

$$\overline{D}(\boldsymbol{c}) \geqslant C_1\, a_{i_0} \frac{(1-\sqrt{\lambda})^2}{\lambda} \geqslant C_1 \inf_{i\geqslant 1} a_i \frac{\left(\sqrt{\overline{z}+\delta} - \sqrt{\overline{z} + \frac{\delta}{2}}\right)^2}{\overline{z} + \frac{\delta}{2}},$$

completing the proof. □

Next, we present a new lower bound estimate for the energy dissipation in the case where c_1 is 'too small'.

Lemma 3.7. *Let $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$ be the coagulation and detailed balance coefficients for the Becker-Döring equations. Assume that*

$$\overline{Q} = \sup_i \frac{Q_i}{Q_{i+1}} < +\infty \quad \underline{Q} = \inf_i \frac{Q_i}{Q_{i+1}} < +\infty$$

$$\overline{a} = \sup_i \frac{a_i}{a_{i+1}} < +\infty \quad \underline{a} = \inf_i \frac{a_i}{a_{i+1}} < +\infty,$$

and let $\boldsymbol{c}$ be a non-negative sequence such that

$$c_1 < \delta$$

for some $\delta > 0$. Then,

$$\overline{D}(\boldsymbol{c}) \geqslant \underline{Q}\underline{a} \left(\sum_{i=1}^{\infty} a_i c_i - a_1 \delta \right) - 2\sqrt{\delta}\sqrt{\overline{Q}\overline{a}} \left(\sum_{i=1}^{\infty} a_i c_i \right).$$

Proof. Expanding the square, one has

$$\overline{D}(\boldsymbol{c}) = c_1 \sum_{i=1}^{\infty} a_i c_i + \sum_{i=1}^{\infty} a_i \frac{Q_i}{Q_{i+1}} c_{i+1} - 2\sqrt{c_1} \sum_{i=1}^{\infty} a_i \sqrt{\frac{Q_i}{Q_{i+1}}} \sqrt{c_i c_{i+1}}$$

so that

$$\begin{aligned}\overline{D}(\boldsymbol{c}) \geqslant \underline{Q}\underline{a} \left(\sum_{i=2}^{\infty} a_i c_i \right) - 2\sqrt{c_1}\sqrt{\overline{Q}\overline{a}} \sqrt{\sum_{i=2}^{\infty} a_i c_i} \sqrt{\sum_{i=1}^{\infty} a_i c_i} \\ \geqslant \underline{Q}\underline{a} \left(\sum_{i=1}^{\infty} a_i c_i - a_1 \delta \right) - 2\sqrt{\delta}\sqrt{\overline{Q}\overline{a}} \left(\sum_{i=1}^{\infty} a_i c_i \right),\end{aligned}$$

which is the desired result. □

Proposition 3.8. *Let $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$ be the coagulation and detailed balance coefficients for the Becker-Döring equations. Assume that*

$$\overline{Q} = \sup_i \frac{Q_i}{Q_{i+1}} < +\infty \quad \underline{Q} = \inf_i \frac{Q_i}{Q_{i+1}} < +\infty.$$

Let $\boldsymbol{c}$ be a non-negative sequence with finite total density ϱ. Then:

(i) *If $a_i = i$ then there exists a $\delta_1 > 0$, depending only on $\overline{Q}, \underline{Q}$ and ϱ such that if $c_1 < \delta_1$ then*

$$\overline{D}(\boldsymbol{c}) \geqslant \frac{\underline{Q}\varrho}{4}.$$

(ii) *If $a_i = i^\gamma$ for $\gamma < 1$ and there exists $\beta > 1$ such that $M_\beta < +\infty$, then there exists $\delta_1 > 0$, depending only on $\overline{Q}, \underline{Q}, \varrho$ and M_β such that if $c_1 < \delta_1$ then*

$$\overline{D}(\boldsymbol{c}) \geqslant \frac{\underline{Q}\varrho^{\frac{\beta-\gamma}{\beta-1}}}{4M_\beta^{\frac{1-\gamma}{\beta-1}}}.$$

Proof. Both (i) and (ii) will follow immediately from Lemma 3.7 and a suitable choice of δ_1 . Indeed, for (i) we notice that

$$\underline{Q}\underline{a}\left(\sum_{i=1}^{\infty} a_i c_i - a_1\delta\right) - 2\sqrt{\delta}\sqrt{\overline{Q}\overline{a}}\left(\sum_{i=1}^{\infty} a_i c_i\right) = \frac{\underline{Q}}{2}(\varrho-\delta) - 2\sqrt{\delta}\sqrt{\overline{Q}}\varrho,$$

where we have used the notations of Lemma 3.7. As the above is less than $\frac{\underline{Q}\varrho}{2}$ and converges to it as δ goes to zero, we can find δ_1 that satisfies the desired result.

For (ii) we notice that the following interpolation estimate

$$\varrho = \sum_{i=1}^{\infty} i c_i \leqslant \left(\sum_{i=1}^{\infty} i^\gamma c_i\right)^{\frac{\beta-1}{\beta-\gamma}} (M_\beta)^{\frac{1-\gamma}{\beta-\gamma}}$$

along with the fact that $\sum_{i=1}^{\infty} i^\gamma c_i \leqslant \varrho$ implies that

$$\underline{Q}\underline{a}\left(\sum_{i=1}^{\infty} a_i c_i - a_1\delta\right) - 2\sqrt{\delta}\sqrt{\overline{Q}\overline{a}}\left(\sum_{i=1}^{\infty} a_i c_i\right) \geqslant \frac{\underline{Q}}{2}\left(\frac{\varrho^{\frac{\beta-\gamma}{\beta-1}}}{M_\beta^{\frac{1-\gamma}{\beta-1}}} - \delta\right) - 2\sqrt{\delta}\sqrt{\overline{Q}}\varrho,$$

from which the result follows. $\square$

We are finally ready to complete the proof of Theorem 1.1:

Proof of part (iii) of Theorem 1.1. This follows immediately from Propositions 3.6 and 3.8. $\square$

Now that we have our general functional inequality at hand one may wonder how sharp is this method of using the log-Sobolev inequality? Perhaps we were too coarse in our estimation, and Cercignani's conjecture *is* valid in the case $a_i = i^\gamma$ with $\gamma < 1$ under the restrictions of Theorem 1.1. The answer, surprisingly, is that the result is optimal, as we shall see in the next subsection.

3.4. **Optimality of the Results.** This subsection is devoted to showing that unlike the case $a_i = i$, the case $a_i = i^\gamma$ when $\gamma < 1$ does not satisfy Cercignani's Conjecture, even if c_1 is bounded appropriately. This is stated in Theorem 1.2.

Proof of Theorem 1.2. We start by choosing $a_i = i^\gamma$, $\gamma < 1$, and $Q_i = e^{-\lambda(i-1)}$ $(i \geqslant 1)$ for some $\lambda \geqslant 0$. We will show the desired result by constructing a family of non-negative sequences, $\left\{\boldsymbol{c}^{(\varepsilon)}\right\}_{\varepsilon>0}$ with a fixed mass ϱ such that

$$\lim_{\varepsilon\to 0} \frac{D\left(\boldsymbol{c}^{(\varepsilon)}\right)}{H\left(\boldsymbol{c}^{(\varepsilon)}|\mathcal{Q}\right)} = 0.$$

Let $\xi > 0$ be such that

$$\frac{\varrho}{2} = \sum_{i=1}^{\infty} i e^{\lambda} e^{-\xi i} = \frac{e^{\lambda-\xi}}{\left(1-e^{-\xi}\right)^2}.$$

Consider the sequence $\boldsymbol{c}^{(\varepsilon)} = \left\{c_i^{(\varepsilon)}\right\}$ given by

$$c_i^{(\varepsilon)} = e^{\lambda} e^{-\xi i} + A_\varepsilon e^{-\varepsilon i}, \qquad i \in \mathbb{N}$$

where $0 < \varepsilon$ is small and A_ε is chosen such that the mass of the sequence $\boldsymbol{c}^{(\varepsilon)}$ is ϱ, i.e. $A_\varepsilon = \frac{\varrho}{2} e^{\varepsilon}\left(1-e^{-\varepsilon}\right)^2$. Next, as $\frac{Q_i}{Q_{i+1}} = e^{\lambda}$ for any $i \geqslant 1$, we see that

$$\begin{aligned}\frac{Q_i}{Q_{i+1}} c_{i+1}^{(\varepsilon)} - c_1^{(\varepsilon)} c_i^{(\varepsilon)} &= e^{2\lambda} e^{-\xi(i+1)} + A_\varepsilon e^{\lambda} e^{-\varepsilon(i+1)} - e^{2\lambda} e^{-\xi(i+1)} - A_\varepsilon e^{\lambda}\left(e^{-\xi i-\varepsilon} + e^{-\varepsilon i-\xi}\right) - A_\varepsilon^2 e^{-\varepsilon(i+1)} \\ &= A_\varepsilon e^{\lambda} e^{-\varepsilon(i+1)}\left(1 - e^{-(\xi-\varepsilon)} - e^{-(\xi-\varepsilon)i} - A_\varepsilon e^{-\lambda}\right) > 0\end{aligned}$$

for ε small enough depending only on λ, ξ and ϱ but not on i. Additionally, one can easily verify that

$$\frac{Q_i c_{i+1}^{(\varepsilon)}}{Q_{i+1} c_1^{(\varepsilon)} c_i^{(\varepsilon)}} \leqslant e^{\lambda}\left(1 + \frac{1}{A_\varepsilon}\right).$$

As such, setting $B_{z,\gamma} = \sum_{i=1}^{\infty} i^\gamma e^{-zi}$ for any $z > 0$, we find that

$$\begin{aligned} D\left(\boldsymbol{c}^{(\varepsilon)}\right) &= \sum_{i=1}^{\infty} i^\gamma \left(\frac{Q_i}{Q_{i+1}} c_{i+1}^{(\varepsilon)} - c_i^{(\varepsilon)}\right) \log\left(\frac{Q_i c_{i+1}^{(\varepsilon)}}{Q_{i+1} c_1^{(\varepsilon)} c_i^{(\varepsilon)}}\right) \\ &\leqslant A_\varepsilon e^{\lambda} B_{\varepsilon,\gamma} \log\left(e^{\lambda}\left(1+\frac{1}{A_\varepsilon}\right)\right)\left(\left(1 - A_\varepsilon e^{-\lambda}\right) e^{-\varepsilon} - e^{-\xi}\right) \\ &\quad - A_\varepsilon e^{\lambda} B_{\xi,\gamma} \log\left(e^{\lambda-\varepsilon}\left(1+\frac{1}{A_\varepsilon}\right)\right). \end{aligned} \tag{3.21}$$

As $A_\varepsilon \approx \frac{\varrho}{2}\varepsilon^2$ when ε approaches zero, and $B_{\varepsilon,\gamma}$ is of order $\varepsilon^{-(1+\gamma)}$ (see Lemma B.3 in Appendix B) we conclude that

$$\lim_{\varepsilon\to 0} D\left(\boldsymbol{c}^{(\varepsilon)}\right) = 0.$$

Lastly, we turn our attention to the relative free energy. We start by denoting by $\overline{\xi} > 0$ the unique parameter for which

$$\varrho = e^{\lambda} \sum_{i=1}^{\infty} i e^{-\overline{\xi} i}.$$

Clearly, $\overline{\xi} < \xi$ and the associated equilibrium with mass ϱ is $\mathcal{Q}_i = e^{\lambda} e^{-\overline{\xi} i}$. Since, for any fixed $i \geqslant 1$, it holds

$$\lim_{\varepsilon\to 0} c_i^{(\varepsilon)} = c_i^{(0)} = e^{\lambda} e^{-\xi i}$$

using Fatou's lemma we can conclude that

$$\liminf_{\varepsilon\to 0} H\left(c^{(\varepsilon)}|\mathcal{Q}\right) \geqslant H\left(\boldsymbol{c}^{(0)}|\mathcal{Q}\right) > 0$$

as $\boldsymbol{c}^{(0)} \neq \mathcal{Q}$. □

Remark 3.9. We notice the following:

- In the example we provided $z_s = e^{\lambda} < +\infty$ but $\varrho_s = +\infty$. This, however, is not a great obstacle as all our proofs rely on some *positive* distance from z_s and ϱ_s, and can be reformulated accordingly.
- The constructed sequence $\boldsymbol{c}^{(\varepsilon)}$ satisfies

$$\sup_{\varepsilon} \sum_{i=1}^{\infty} i^{\beta}\, c_i^{(\varepsilon)} = +\infty$$

for any $\beta > 1$. Thus, the conclusion of part (ii) of Theorem 1.1 does not apply to it. Actually, one can easily check that $\lim_{\varepsilon\to 0} \frac{D(\boldsymbol{c}^{(\varepsilon)})}{\left(H(\boldsymbol{c}^{(\varepsilon)}|\mathcal{Q})\right)^s} = 0$ for any $s > 0$.

3.5. **Inequalities with Exponential Moments.** Up to now, we have avoided using exponential moments in any of our functional inequalities. In this section we will show that when $0 \leqslant \gamma < 1$, under the additional assumption of a bounded exponential moment, one can obtain an improved functional inequality between $\overline{D}(\boldsymbol{c})$ and $H(\boldsymbol{c}|\mathcal{Q})$, extending the result given by Jabin and Niethammer in [19]. The key idea in this section is to avoid using the interpolation inequality (3.17) and replace it with one that involves an exponential weight.

Proposition 3.10. *Let $\boldsymbol{f}$ be a non-negative sequence and let $0 \leqslant \gamma < 1$. Assume that there exists $\mu \in (0, 4\log 2)$ such that*

$$\sum_{i=1}^{\infty} e^{\mu i} f_i = M_\mu^{\exp}(\boldsymbol{f}) < +\infty.$$

Then,

$$M_\gamma(\boldsymbol{f}) \geqslant \frac{M_1(\boldsymbol{f})}{2\left(\frac{2}{\mu}\log\left(\frac{4M_\mu^{\exp}(\boldsymbol{f})}{\mu e M_1(\boldsymbol{f})}\right)\right)^{1-\gamma}} \tag{3.22}$$

where $M_\alpha(\boldsymbol{f})$ denotes the $\alpha-$moment of $\boldsymbol{f}$ and $M_\mu^{\exp}(\boldsymbol{f})$ is the exponential moment defined in (1.14).

Proof. For simplicity, we will use the notation of M_1 and $M_\mu^{\exp}$ instead of $M_1(\boldsymbol{f})$ and $M_\mu^{\exp}(\boldsymbol{f})$. We start with the simple inequality

$$\begin{aligned} M_1 = \sum_{i=1}^{\infty} i f_i &= \sum_{i=1}^{N} i^{1-\gamma} i^{\gamma} f_i + \sum_{i=N+1}^{\infty} i e^{-\frac{\mu i}{2}} e^{-\frac{\mu i}{2}} e^{\mu i} f_i \\ &\leqslant N^{1-\gamma} M_\gamma + \frac{2e^{-\frac{\mu(N+1)}{2}}}{\mu e} M_\mu^{\exp}, \qquad \forall N \in \mathbb{N} \end{aligned} \tag{3.23}$$

where we used the fact that $\sup_{x\geqslant 0} x e^{-\lambda x} = \frac{1}{\lambda e}$ for any $\lambda > 0$. Our goal will be to choose a particular N to plug in the inequality above to conclude the desired result. Again, using the supremum of $g(x) = xe^{-\lambda x}$, we conclude that

$$M_1 \leqslant \frac{1}{\mu\, e} M_\mu^{\exp}.$$

As $\mu < 4\log 2$ we find that

$$M_1 < \frac{4M_\mu^{\exp}}{\mu e^{1+\frac{\mu}{2}}}.$$

from which we conclude that $N = \left[\frac{2}{\mu}\log\left(\frac{4M_\mu^{\exp}}{\mu e M_1}\right)\right] \geqslant 1$. Plugging this N into (3.23) we see that $e^{-\frac{\mu(N+1)}{2}} \leqslant \frac{\mu e M_1}{4M_\mu^{\exp}}$, and as such

$$M_\gamma \geqslant N^{\gamma-1}\frac{M_1}{2}$$

and the result follows. □

With this proposition at hand, we are prepared to show part (i) of Theorem 1.4.

Proof of part (i) *of Theorem 1.4.* Without loss of generality we may assume that $\mu \in (0, 4\log 2)$. Introduce the sequence $\boldsymbol{f} = \{f_i\}$ where

$$f_i = Q_i\left(\sqrt{\frac{c_1 c_i}{Q_i}} - \sqrt{\frac{c_{i+1}}{Q_{i+1}}}\right)^2, \qquad i \geqslant 1.$$

Following the same proof as presented in Lemma 3.5 we find that

$$M_\mu^{\exp}(\boldsymbol{f}) \leqslant 2\left(c_1 + z_s \sup_j \frac{\alpha_j}{\alpha_{j+1}}\right) M_\mu^{\exp}(\boldsymbol{c}).$$

Thus, using the simple fact that $M_\alpha(\boldsymbol{f}) = \overline{D}_\alpha(\boldsymbol{c})$, for any $\alpha > 0$, together with Proposition 3.10 and parts (i) and (iii) of Theorem 1.1 yield the desired functional inequality. □

4. Rate of Convergence to Equilibrium

In this section we will use all the information we gathered so far to prove Theorems 1.3 and part (ii) of Theorem 1.4, giving an explicit rate of convergence to equilibrium for the Becker-Döring equations.

The convergence result in Theorem 1.3 is a consequence of Theorem 1.1. To use the functional inequality established there, we need first to invoke uniform (and explicit) upper bounds on moments $M_\beta(\boldsymbol{c}(t))$ (see (1.18)). This is provided by the following (see [10]):

Proposition 4.1. *Let $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$ satisfy Hypotheses 1–3 with $0 \leqslant \gamma \leqslant 1$, and let $\boldsymbol{c}(t) = \{c_i(t)\}_{i\in\mathbb{N}}$ be a solution to the Becker-Döring equations with mass $\varrho \in (0, \varrho_s)$. Let $\beta \geqslant 2-\gamma$ be such that*

$$M_\beta(\boldsymbol{c}(0)) = \sum_{i=1}^\infty i^\beta c_i(0) < \infty.$$

There exists a constant $C > 0$ depending only on β, $M_\beta(0)$, the initial relative free energy $H(\boldsymbol{c}(0)|\mathcal{Q})$, the coefficients $\{a_i\}_{i\geqslant 1}$, $\{b_i\}_{i\geqslant 1}$ and the mass ϱ such that

$$M_\beta(\boldsymbol{c}(t)) = \sum_{i=1}^\infty i^\beta c_i(t) \leqslant C \qquad \textit{for all } t \geqslant 0.$$

Using such an estimate, the proof is easily derived from Theorem 1.1 and part (i) of Theorem 1.4, yet we provide a proof here for the sake of completeness and to show that we can find all the constants explicitly.

Proof of Theorem 1.3. Combining Theorem 1.1 and Proposition 4.1 we conclude the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}H(\boldsymbol{c}(t)|\mathcal{Q}) \leqslant \begin{cases} -\min\left(KH(\boldsymbol{c}(t)|\mathcal{Q}), \varepsilon\right) & \gamma = 1. \\ -\min\left(KH(\boldsymbol{c}(t)|\mathcal{Q})^{\frac{\beta-\gamma}{\beta-1}}, \varepsilon\right) & 0 \leqslant \gamma < 1, \end{cases} \tag{4.1}$$

for appropriate K and ε. We claim that there exists $t_0 \geqslant 0$ such that for all $t \geqslant t_0$

$$H(c(t)|\mathcal{Q}) \leqslant \begin{cases} \frac{\varepsilon}{K} & \gamma = 1 \\ \left(\frac{\varepsilon}{K}\right)^{\frac{\beta-1}{\beta-\gamma}} & 0 \leqslant \gamma < 1. \end{cases} \tag{4.2}$$

Indeed, if $H(\boldsymbol{c}(t))|\mathcal{Q})$ is larger than the appropriate constants in $[0,t]$ then

$$\frac{\mathrm{d}}{\mathrm{d}s}H(\boldsymbol{c}(s)|\mathcal{Q}) \leqslant -\varepsilon \quad \forall s \in (0,t),$$

implying that

$$H(\boldsymbol{c}(t)|\mathcal{Q}) \leqslant H(\boldsymbol{c}(0)|\mathcal{Q}) - \varepsilon t.$$

We define

$$t_0 = \begin{cases} \min\left(0, \frac{H(\boldsymbol{c}(0)|\mathcal{Q}) - \frac{\varepsilon}{K}}{\varepsilon}\right) & \gamma = 1 \\ \min\left(0, \frac{H(\boldsymbol{c}(0)|\mathcal{Q}) - \left(\frac{\varepsilon}{K}\right)^{\frac{\beta-1}{\beta-\gamma}}}{\varepsilon}\right) & 0 \leqslant \gamma < 1. \end{cases}$$

and find that $H(\boldsymbol{c}(t_0)|\mathcal{Q})$ satisfies the appropriate inequality in (4.2). As $H(\boldsymbol{c}(t)|\mathcal{Q})$ is decreasing, we conclude that (4.2) is valid for any $t \geqslant t_0$.

With this in hand, along with (4.1), we have that for all $t \geqslant t_0$:

$$H(\boldsymbol{c}(t)|\mathcal{Q}) \leqslant \begin{cases} H(\boldsymbol{c}(t_0)|\mathcal{Q})e^{-K(t-t_0)} & \gamma = 1 \\ \frac{1}{\left(H(\boldsymbol{c}(t_0)|\mathcal{Q})^{\frac{\gamma-1}{\beta-1}} + \frac{1-\gamma}{\beta-1}K(t-t_0)\right)^{\frac{\beta-1}{1-\gamma}}} & 0 \leqslant \gamma < 1. \end{cases}$$

As

$$H(\boldsymbol{c}(t_0)|\mathcal{Q}) = \begin{cases} \min\left(H(\boldsymbol{c}(0)|\mathcal{Q}), \frac{\varepsilon}{K}\right) & \gamma = 1 \\ \min\left(H(\boldsymbol{c}(0)|\mathcal{Q}), \left(\frac{\varepsilon}{K}\right)^{\frac{\beta-1}{\beta-\gamma}}\right) & 0 \leqslant \gamma < 1, \end{cases}$$

and t_0 is given explicitly we conclude that

$$C\left(H(\boldsymbol{c}(0)|\mathcal{Q})\right) = \begin{cases} H(\boldsymbol{c}(0)|\mathcal{Q}) & \gamma = 1, t_0 = 0 \\ \frac{\varepsilon}{K}e^{K\frac{H(\boldsymbol{c}(0)|\mathcal{Q}) - \frac{\varepsilon}{K}}{\varepsilon}} & \gamma = 1, t_0 > 0 \\ H(\boldsymbol{c}(0)|\mathcal{Q}) & 0 \leqslant \gamma < 1, t_0 = 0 \\ \left(\frac{\varepsilon}{K}\right)^{\frac{\gamma-1}{\beta-\gamma}} - \frac{1-\gamma}{\beta-1}K\frac{H(\boldsymbol{c}(0)|\mathcal{Q}) - \left(\frac{\varepsilon}{K}\right)^{\frac{\beta-1}{\beta-\gamma}}}{\varepsilon} & 0 \leqslant \gamma < 1, t_0 > 0, \end{cases}$$

completing the proof. □

Proof of part (ii) of Theorem 1.4. This follows form part (i) of Theorem 1.4 by the same methods used in the above proof and the fact that

$$\sup_{t \geqslant 0} M_{\mu'}^{\exp}(\boldsymbol{c}(t)) < +\infty$$

for some $0 < \mu' < \mu$ (a known result from [19]). □

5. Consequences for General Coagulation and Fragmentation Models

In this final Section we illustrate how the functional inequalities investigated in Section 3 provide new insights on the behaviour of solutions to general discrete coagulation-fragmentation models.

5.1. General Discrete Coagulation-Fragmentation Equation.

The Becker-Döring equations (1.1) are derived under the assumption that the only relevant reactions taking place are those between monomers and clusters of any size. One can obtain a more general model by taking into account reactions between clusters of any size. Keeping the notation of the introduction, this means that we consider reactions of the type

$$\{i\} + \{j\} \rightleftharpoons \{i+j\}$$

for any positive integer sizes i and j. We assume their coagulation rate (i.e., the reaction from left to right) is determined by a coefficient we call $a_{i,j}$, and their fragmentation rate (the reaction from right to left) by a coefficient called $b_{i,j}$. These coefficients are always assumed to be nonnegative (as before) and symmetric in i, j (that is, $a_{i,j} = a_{j,i}$ and $b_{i,j} = b_{j,i}$ for all i, j). The corresponding to eq. (1.1) is then

$$\frac{\mathrm{d}}{\mathrm{d}t} c_i(t) = \frac{1}{2} \sum_{j=1}^{i-1} W_{j,i-j}(t) - \sum_{j=1}^{\infty} W_{i,j}(t), \qquad i \in \mathbb{N}. \tag{5.1}$$

where

$$W_{i,j}(t) := a_{i,j}\, c_i(t) c_j(t) - b_{i,j}\, c_{i+j}(t) \qquad i \in \mathbb{N}. \tag{5.2}$$

The system (1.1) is then a particular case of (5.1) obtained by choosing $a_{i,j}$, $b_{i,j}$ as

$$a_{i,j} = b_{i,j} = 0 \quad \text{ when } \min\{i,j\} \geqslant 2, \tag{5.3}$$

$$a_{1,1} := 2a_1, \qquad a_{i,1} = a_{1,i} = a_i \quad \text{ for } i \geqslant 2, \tag{5.4}$$

$$b_{1,1} := 2b_2, \qquad b_{i,1} = b_{1,i} = b_{i+1} \quad \text{ for } i \geqslant 2. \tag{5.5}$$

The mathematical theory of this full system is much less complete than that of (1.1). Well-posedness of mass-conserving solutions has been studied in [2], and there are a number of works on asymptotic behaviour, for instance [8, 9, 13, 14], but it is still not fully understood. To start with, it is unclear whether equilibria of (5.1) are unique or not (when they exist). A common physical condition imposed on the coefficients $a_{i,j}$, $b_{i,j}$ which avoids this problem is that of *detailed balance*: we say it holds when there exists a sequence $\{Q_i\}_{i \geqslant 1}$ of strictly positive numbers such that

$$a_{i,j} Q_i Q_j = b_{i,j} Q_{i+j} \quad \text{ for any } i, j, \tag{5.6}$$

where we always further assume without loss of generality that $Q_1 = 1$. This is the analogue of (1.4), but in this case it needs to be imposed as a condition since numbers Q_i satisfying (5.6) cannot always be found (unlike in the Becker-Döring case). If we assume (5.6) then equilibria (5.1) exist and have the same form (1.5) as in the Becker-Döring case, and a similar phase transition in the long-time behaviour has been rigorously proved in some cases (see [8, 9, 13, 14] for more details). However, even with detailed balance the long-time behaviour is in general not understood except in particular cases. If clusters larger than a given size N do not react among themselves (that is, if $a_{i,j} = b_{i,j} = 0$ whenever $\min\{i,j\} > N$) the system is known as the *generalised Becker-Döring system*, and has been studied in [8, 16]. For coefficients $a_{i,j}$ given by

$$a_{i,j} = i^\gamma j^\eta + i^\eta j^\gamma \quad \text{ for any } i, j, \tag{5.7}$$

with $\eta \leqslant 0 \leqslant \gamma$ and $\gamma + \eta \leqslant 1$, the asymptotic behaviour was identified in [9] and a constructive (though probably far from optimal) rate of convergence to equilibrium was given. Very little is known about the asymptotic behaviour for coefficients of the type (5.7) with $\gamma, \eta > 0$ and $\gamma + \eta \leqslant 1$. In this case the size of $a_{i,i}$ is larger than that of $a_{i,1}$ and the system (5.1) may behave quite differently from (1.1).

A natural question is whether any of the functional inequalities investigated in this paper can shed new light on the behaviour of solutions to (5.1). Assuming the detailed balance condition (5.6), along a solution $\boldsymbol{c}(t) = \{c_i(t)\}_{i \geqslant 1}$ to (5.1) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H(\boldsymbol{c}(t)) &= -D_{\mathrm{CF}}(\boldsymbol{c}(t)) \\
&:= -\frac{1}{2}\sum_{i,j=1}^{\infty} a_{i,j} Q_i Q_j \left(\frac{c_i c_j}{Q_i Q_j} - \frac{c_{i+j}}{Q_{i+j}}\right)\left(\log \frac{c_i c_j}{Q_i Q_j} - \log \frac{c_{i+j}}{Q_{i+j}}\right) \\
&\leqslant -\sum_{i=1}^{\infty} a_i Q_i \left(\frac{c_i c_1}{Q_i} - \frac{c_{i+1}}{Q_{i+1}}\right)\left(\log \frac{c_i c_1}{Q_i} - \log \frac{c_{i+1}}{Q_{i+1}}\right) = D(\boldsymbol{c}(t)) \leqslant 0
\end{aligned} \tag{5.8}$$

(see [9] for a rigorous proof) where a_i are defined by (5.4) for any $i \geqslant 1$. Hence the free energy is also a Lyapunov functional for (5.1), and it dissipates at a *faster* rate than for the Becker-Döring equations (since more types of reactions are allowed). As such, it is reasonable to think that the inequalities from Section 3 can be useful also in this case. This turns out to be true, and some improvements can be made on existing results. However, it also turns out that our results are not able to extend the range of possible coefficients for which convergence to a particular subcritical equilibrium can be proved; we cannot give any new results for coefficients such as (5.7) with $\gamma, \eta > 0$ and $\gamma + \eta \leqslant 1$.

5.2. **Proof of Theorem 1.5.** We now give the proof of our main result concerning the above model (5.1). One of the main obstacles in applying directly our results to equation (5.1) is that, unlike for the Becker-Döring equations, the moments of solutions to the general coagulation and fragmentation system are not known to be bounded (i.e. Proposition 4.1 is not available for (5.1)). One can for example say the following about integer moments (this result can easily be extended to non-integer powers by interpolation, and was known from the early works in the topic [13, 14]). From this point onward we will assume that

$$a_{i,j} = i^{\gamma} j^{\eta} + i^{\eta} j^{\gamma} \quad \text{for } i, j \in \mathbb{N}, \tag{5.9}$$

with $\eta \leqslant \gamma$ and $0 \leqslant \lambda := \gamma + \eta \leqslant 1$.

Lemma 5.1. *Let $k \in \mathbb{N}$ and let $\boldsymbol{c} = \boldsymbol{c}(t) = \{c_i(t)\}_{i \in \mathbb{N}}$ be a solution with mass ϱ to the coagulation and fragmentation system (5.1) with coefficients satisfying (5.9). Then*

$$M_k(\boldsymbol{c}(t)) \leqslant \begin{cases} \left(M_k(\boldsymbol{c}(0)) + \frac{1-\lambda}{k-1}\left(2^k - 2\right)\varrho^{\frac{1-\gamma}{k-1}} t\right)^{\frac{k-1}{1-\lambda}} & \text{if} \quad 0 < \lambda < 1 \\ M_k(\boldsymbol{c}(0)) \exp\left(2\left(2^k - 2\right)\varrho t\right) & \text{if} \quad \lambda = 1 \end{cases} \tag{5.10}$$

where $M_p(\boldsymbol{c}(t)) := \sum_{i=1}^{\infty} i^p c_i(t)$ for any $p \geqslant 0$, $t \geqslant 0$.

Proof. We give a formal proof for completeness; a rigorous one can be obtained by standard approximation methods, and can be found in [2]. To simplify the notation and since $\boldsymbol{c}(t)$ is fixed, we denote $M_j(t) = M_j(\boldsymbol{c}(t))$ for any $j \geqslant 1$, $t \geqslant 0$. One can check the following weak formula for the integral of the right hand side of (5.1) against a test sequence $\{\phi(i)\}_i$:

$$\sum_{i=1}^{\infty}\phi(i)\left(\frac{1}{2}\sum_{j=1}^{i-1}W_{i-j,j}-\sum_{j=1}^{\infty}W_{i,j}\right)=\frac{1}{2}\sum_{i,j}^{\infty}\left(\phi(i+j)-\phi(i)-\phi(j)\right)W_{i,j}.$$

Applying this to $\phi(i):=i^k$, neglecting the negative contribution of the fragmentation terms and using the binomial formula one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}M_k(t)\leqslant\sum_{l=1}^{k-1}\binom{k}{l}M_{l+\gamma}(t)M_{k-l+\eta}(t)\qquad\forall t\geqslant 0.$$

Next, we use the interpolation

$$M_\delta(t)\leqslant M_1^{\frac{k-\delta}{k-1}}(t)M_k^{\frac{\delta-1}{k-1}}(t)$$

where $1<\delta<k$, to find that

$$M_{l+\gamma}(t)M_{k-l+\eta}(t)\leqslant M_1(t)^{\frac{k-\lambda}{k-1}}M_k(t)^{\frac{k+\lambda-2}{k-1}}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}M_k(t)\leqslant\left(2^k-2\right)\varrho^{\frac{k-\lambda}{k-1}}M_k^{\frac{k+\lambda-2}{k-1}}(t)\qquad\forall t\geqslant 0$$

and the result follows from this differential inequality. □

With the above at hand, we are now able to prove our main result about the rate of convergence to equilibrium in the general setting of coagulation and fragmentation equations:

Proof of Theorem 1.5. Assume for the moment that $a_{i,j}$ is of the form (5.7), in order to see why the proof only works for coefficients of the form (1.27).

Fix $\delta>0$ such that $0<\delta<\overline{z}<z_{\mathrm{s}}-\delta$. We use the observation (5.8) that $D_{\mathrm{CF}}(\boldsymbol{c}(t))\geqslant D(\boldsymbol{c}(t))$ at all times $t\geqslant 0$ (defining $\{a_i\}_{i\in\mathbb{N}}$ by (5.4)). Using Theorem 1.1 (actually, its more detailed forms in equation (3.18) and Proposition 3.8) we obtain the following:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H(\boldsymbol{c}(t)|\mathcal{Q})&=-D_{\mathrm{CF}}(\boldsymbol{c}(t))\leqslant -D(\boldsymbol{c}(t))\\&\leqslant\begin{cases}-CM_k(\boldsymbol{c}(t))^{\frac{\gamma-1}{k-1}}H(\boldsymbol{c}(t)|\mathcal{Q})^{\frac{k-\gamma}{k-1}} & \text{if } \delta<c_1(t)<z_{\mathrm{s}}-\delta\\ -CM_k(\boldsymbol{c}(t))^{\frac{\gamma-1}{k-1}} & \text{if } c_1(t)<\delta \text{ or } c_1(t)\geqslant z_{\mathrm{s}}-\delta.\end{cases}\\&\leqslant -C_0M_k(\boldsymbol{c}(t))^{\frac{\gamma-1}{k-1}}H(\boldsymbol{c}(t)|\mathcal{Q})^{\frac{k-\gamma}{k-1}}\end{aligned}$$

for some constant $C_0>0$ that depends also on $H(\boldsymbol{c}(0)|\mathcal{Q})$. Using Lemma 5.1 this implies

$$\frac{\mathrm{d}}{\mathrm{d}t}H(\boldsymbol{c}(t)|\mathcal{Q})\leqslant-\frac{C_0}{\left(M_k(\boldsymbol{c}(0))+\frac{1-\lambda}{k-1}(2^k-2)\varrho^{\frac{k-\lambda}{k-1}}t\right)^{\frac{1-\gamma}{1-\lambda}}}H(\boldsymbol{c}(t)|\mathcal{Q})^{\frac{k-\gamma}{k-1}}\qquad t\geqslant 0.$$

This implies decay of $H(\boldsymbol{c}(t))$ only when $\lambda=\gamma$, that is, when $\eta=0$ (since $\lambda=\gamma+\eta$). Solving the differential inequality yields the result. □

Remark 5.2. The same decay rate was obtained in [9] by means of the particular case of inequality (1.19) for $\beta=2-\gamma$. Here we obtain slightly different decay rates by assuming higher moments of the initial data $\boldsymbol{c}(0)$ are finite, but the method does not seem to give a better decay than a power of $\log t$ in any case.

Remark 5.3. It seems to the authors that the inequality we use in the proof of Theorem 1.5 is not optimal, and could be improved to deal with the case

$$a_{i,j} = i^\gamma j^\eta + i^\eta j^\gamma,$$

with a resulting convergence rate that would depend on $\lambda = \gamma + \eta$.

Appendix A. Additional Computations for the Theory of the Discrete Log-Sobolev Inequality With Weights

We have collected here technical Lemmas from Subsection 2 that we felt would have encumbered it.

Lemma A.1. *For any sequence* $\boldsymbol{f}$*, we have*

$$\operatorname{Ent}_\mu(\boldsymbol{f}^2) \leqslant \mathcal{L}(\boldsymbol{f}) \leqslant \operatorname{Ent}_\mu(\boldsymbol{f}^2) + 2\sum_{i=1}^{\infty} \mu_i f_i^2.$$

Proof. From the definition of $\mathcal{L}$ the inequality

$$\operatorname{Ent}_\mu(\boldsymbol{f}^2) \leqslant \mathcal{L}(\boldsymbol{f})$$

it trivial. We thus consider the right hand side inequality. For a given sequence $\boldsymbol{f}$ and any $\alpha \in \mathbb{R}$ we define

$$\begin{aligned} G_\alpha(t) &= \sum_{i=1}^{\infty} \mu_i \left(tf_i + \alpha\right)^2 \log\left(\frac{\left(tf_i+\alpha\right)^2}{\sum_{i=1}^{\infty} \mu_i \left(tf_i+\alpha\right)^2}\right) \\ &= 2\sum_{i=1}^{\infty} \mu_i \left(tf_i + \alpha\right)^2 \log|tf_i+\alpha| - \left(\sum_{i=1}^{\infty} \mu_i \left(tf_i+\alpha\right)^2\right) \log\left(\sum_{i=1}^{\infty} \mu_i \left(tf_i+\alpha\right)^2\right), \end{aligned}$$

and notice that

$$G_0(t) = t^2 \operatorname{Ent}_\mu(\boldsymbol{f}^2).$$

Next, we define $g(t) = G_0(t) + 2t^2 \sum_{i=1}^{\infty} \mu_i f_i^2$ and notice that the inequality we want to prove is equivalent to

$$G_\alpha(1) \leqslant g(1).$$

for any $\alpha \in \mathbb{R}$. Clearly $G_\alpha(t) \leqslant g(t)$ when $t = 0$. Differentiating G we find that

$$\begin{aligned} G'_\alpha(t) &= 4\sum_{i=1}^{\infty} \mu_i f_i \left|tf_i+\alpha\right| \log\left(tf_i+\alpha\right) + 2\sum_{i=1}^{\infty} \mu_i f_i \left(tf_i+\alpha\right) \\ &\qquad - 2\left(\sum_{i=1}^{\infty} \mu_i f_i \left(tf_i+\alpha\right)\right) \log\left(\sum_{i=1}^{\infty} \mu_i \left(tf_i+\alpha\right)^2\right) - 2\sum_{i=1}^{\infty} \mu_i f_i \left(tf_i+\alpha\right) \\ &= 4\sum_{i=1}^{\infty} \mu_i f_i \left(tf_i+\alpha\right) \log|tf_i+\alpha| - 2\left(\sum_{i=1}^{\infty} \mu_i f_i \left(tf_i+\alpha\right)\right) \log\left(\sum_{i=1}^{\infty} \mu_i \left(tf_i+\alpha\right)^2\right) \end{aligned}$$

which satisfies $G'_\alpha(0) = 0$ for any $\boldsymbol{f}$ and α, implying that $G'_\alpha(0) = g'(0) = 0$. As G is defined for any $t \in [0,1]$ we see that it is enough to show that when defined,

$$G''_\alpha(t) \leqslant g''(t)$$

for any α. Indeed,

$$\begin{aligned}
G_\alpha''(t) &= 4\sum_{i=1}^\infty \mu_i f_i^2 \log|tf_i+\alpha| + 4\sum_{i=1}^\infty \mu_i f_i^2 - 2\sum_{i=1}^\infty \mu_i f_i^2 \log\left(\sum_{i=1}^\infty \mu_i\left(tf_i+\alpha\right)^2\right) \\
&\quad - 4\frac{\left(\sum_{i=1}^\infty \mu_i f_i\left(tf_i+\alpha\right)\right)^2}{\sum_{i=1}^\infty \mu_i\left(tf_i+\alpha\right)^2} \\
&= 2\sum_{i=1}^\infty \mu_i f_i^2 \log\left(\frac{\left(tf_i+\alpha\right)^2}{\sum_{i=1}^\infty \mu_i\left(tf_i+\alpha\right)^2}\right) + 4\sum_{i=1}^\infty \mu_i f_i^2 - 4\frac{\left(\sum_{i=1}^\infty \mu_i f_i\left(tf_i+\alpha\right)\right)^2}{\sum_{i=1}^\infty \mu_i\left(tf_i+\alpha\right)^2}
\end{aligned}$$

As

$$\operatorname{Ent}_\mu(\boldsymbol{f}^2) = \sup\left\{\sum_{i=1}^\infty \mu_i f_i^2 \log h_i \;;\; \sum_{i=1}^\infty \mu_i h_i = 1\right\}$$

we see that by choosing $h_i = \frac{(tf_i+\alpha)^2}{\sum_{i=1}^\infty \mu_i (tf_i+\alpha)^2}$

$$G_\alpha''(t) \leqslant 2\operatorname{Ent}_\mu(\boldsymbol{f}^2) + 4\sum_{i=1}^\infty \mu_i f_i^2 = g''(t),$$

completing the proof. $\square$

Lemma A.2. *For all $\boldsymbol{f}\in L_\Phi$ we have that*

$$\|\boldsymbol{f}\|_{L^1_\mu} \leqslant \|\boldsymbol{f}\|_{L^2_\mu} \leqslant \sqrt{\frac{3}{2}}\,\|\boldsymbol{f}\|_{L_\Phi}. \tag{A.1}$$

Proof. The inequality

$$\|\boldsymbol{f}\|_{L^1_\mu} \leqslant \|\boldsymbol{f}\|_{L^2_\mu}$$

is immediate as $\boldsymbol{\mu}$ is a probability measure. To show the last inequality we may assume that $\|\boldsymbol{f}\|_{L_\Phi} = 1$. Due to Fatou's Lemma we know that if $k_n \underset{n\to\infty}{\longrightarrow} k > 0$ then

$$\sum_{i=1}^\infty \mu_i \Phi\left(\frac{|f_i|}{k}\right) \leqslant \liminf_{n\to\infty} \sum_{i=1}^\infty \mu_i \Phi\left(\frac{|f_i|}{k_n}\right),$$

implying that if $\|f\|_{L_\Phi} > 0$ then

$$\sum_{i=1}^\infty \mu_i \Phi\left(\frac{|f_i|}{\|f\|_{L_\Phi}}\right) \leqslant 1.$$

In our case, since $\Psi(x)$ is convex we find that

$$1 \geqslant \sum_{i=1}^\infty \mu_i \Phi(f_i) = \sum_{i=1}^\infty \mu_i \Psi(f_i^2) \geqslant \Psi\left(\sum_{i=1}^\infty \mu_i f_i^2\right) = \Psi\left(\|\boldsymbol{f}\|^2_{L^2_\mu}\right).$$

As Ψ is increasing and $\Psi(1.5) > 1$ we conclude that

$$\|\boldsymbol{f}\|^2_{L^2_\mu} < \frac{3}{2},$$

yielding the desired result. $\square$

Lemma A.3. *Let $\boldsymbol{f}\in L_\Phi$. Then*

$$\|\boldsymbol{f} - \langle \boldsymbol{f}\rangle\|^2_{L^2_\mu} = \frac{1}{2}\lim_{|a|\to\infty} \operatorname{Ent}_\mu\left((\boldsymbol{f}+a)^2\right) \tag{A.2}$$

Proof. We start by noticing that

$$\mathrm{Ent}_\mu\left((\boldsymbol{f}+a)^2\right)=\sum_{i=1}^{\infty}\mu_i\left(f_i^2+2af_i+a^2\right)\log\left(\frac{\left(1+\frac{f_i}{a}\right)^2}{\sum_{i=1}^{\infty}\mu_i\left(1+\frac{f_i}{a}\right)^2}\right),$$

and continue by assuming that f_i is uniformly bounded, from which the result will follow with an application of an appropriate convergence theorem. There exists a_0 such that if $|a|>|a_0|$ we have that $\left|\frac{f_i}{a}\right|<\frac{1}{2}$ uniformly in i. As on $\left[-\frac{1}{2},\frac{1}{2}\right]$ we have that there exists $C>0$ such that

$$\left|\log(1+x)-x+\frac{x^2}{2}\right|\leqslant Cx^3.$$

we conclude that

$$\log\left(1+2\frac{f_i}{a}+\frac{f_i^2}{a^2}\right)=\left(2\frac{f_i}{a}+\frac{f_i^2}{a^2}\right)-2\frac{f_i^2}{a^2}+\frac{E_{1,i}}{a^3}=2\frac{f_i}{a}-\frac{f_i^2}{a^2}+\frac{E_{1,i}}{a^3}$$

and

$$\log\left(1+2\frac{\langle\boldsymbol{f}\rangle}{a}+\frac{\|\boldsymbol{f}\|_{L^2_\mu}^2}{a^2}\right)=2\frac{\langle\boldsymbol{f}\rangle}{a}+\frac{\|\boldsymbol{f}\|_{L^2_\mu}^2}{a^2}-2\frac{\langle\boldsymbol{f}\rangle^2}{a^2}+\frac{E_{2,i}}{a^3},$$

where $E_{1,i},E_{2,i}$ are uniformly bounded in i. This implies that

$$\begin{aligned}\mathrm{Ent}_\mu\left((\boldsymbol{f}+a)^2\right)=\sum_{i=1}^{\infty}\mu_i\left(f_i^2+2af_i+a^2\right)&\left(2\frac{f_i}{a}-2\frac{\langle\boldsymbol{f}\rangle}{a}-\frac{f_i^2}{a^2}-\frac{\|\boldsymbol{f}\|_{L^2_\mu}^2}{a^2}+2\frac{\langle\boldsymbol{f}\rangle^2}{a^2}\right)\\&+\frac{1}{a}\sum_{i=1}^{\infty}\mu_i\left(1+2\frac{f_i}{a}+\frac{f_i^2}{a^2}\right)\left(E_{1,i}-E_{2,i}\right).\end{aligned}$$

The last term clearly goes to zero as $|a|$ goes to infinity, so we are only left to deal with the first expression.

$$\begin{aligned}\sum_{i=1}^{\infty}\mu_i\left(f_i^2+2af_i+a^2\right)\left(2\frac{f_i}{a}-2\frac{\langle\boldsymbol{f}\rangle}{a}-\frac{f_i^2}{a^2}-\frac{\|\boldsymbol{f}\|_{L^2_\mu}^2}{a^2}+2\frac{\langle\boldsymbol{f}\rangle^2}{a^2}\right)&=4\|\boldsymbol{f}\|_{L^2_\mu}^2-4\langle\boldsymbol{f}\rangle^2\\+2a\langle\boldsymbol{f}\rangle-2a\langle\boldsymbol{f}\rangle-\|\boldsymbol{f}\|_{L^2_\mu}^2-\|\boldsymbol{f}\|_{L^2_\mu}^2+2\langle\boldsymbol{f}\rangle^2+\frac{E_3}{a}&\\&=2\left(\|\boldsymbol{f}\|_{L^2_\mu}^2-\langle\boldsymbol{f}\rangle^2\right)+\frac{E_3}{a}.\end{aligned}$$

This completes the proof as $\|\boldsymbol{f}-\langle\boldsymbol{f}\rangle\|_{L^2_\mu}^2=\|\boldsymbol{f}\|_{L^2_\mu}^2-\langle\boldsymbol{f}\rangle^2$. □

Lemma A.4. *Let $\boldsymbol{f}$ be a sequence such that $f_m=0$ for some $m\in\mathbb{N}$. Denote by $\boldsymbol{f}^{(0)}=\boldsymbol{f}\mathbb{1}_{i<m}$ and $\boldsymbol{f}^{(1)}=\boldsymbol{f}\mathbb{1}_{i>m}$. Then*

$$\begin{aligned}\left\|\langle\boldsymbol{f}^{(0)}\rangle\right\|_{L_\Phi}\leqslant\left|\langle\boldsymbol{f}^{(0)}\rangle\right|\leqslant\left\|\boldsymbol{f}^{(0)}\right\|_{L^2_\mu}\sqrt{\sum_{i=1}^{m-1}\mu_i}\\\left\|\langle\boldsymbol{f}^{(1)}\rangle\right\|_{L_\Phi}\leqslant\left|\langle\boldsymbol{f}^{(1)}\rangle\right|\leqslant\left\|\boldsymbol{f}^{(1)}\right\|_{L^2_\mu}\sqrt{\sum_{i=m+1}^{\infty}\mu_i}\end{aligned}\tag{A.3}$$

Proof. We start by noticing that for any constant sequence $\boldsymbol{f}=\alpha$ one have

$$\begin{aligned}\|\alpha\|_{L_\Phi} &= \inf_{k>0}\left\{\sum_{i=1}^{\infty}\mu_i\Phi\left(\frac{|\alpha|}{k}\right)\leqslant 1\right\} = \inf_{k>0}\left\{\Phi\left(\frac{|\alpha|}{k}\right)\leqslant 1\right\}\\ &= \frac{|\alpha|}{\Phi^{-1}(1)}\leqslant |\alpha|\,,\end{aligned}$$

as long as $\Phi(1)<1$ which is valid in our case. Next we notice that

$$\left|\langle \boldsymbol{f}^{(0)}\rangle\right| \leqslant \sum_{i=1}^{m-1}\mu_i\left|f_i\right| \leqslant \sqrt{\sum_{i=1}^{m-1}\mu_i f_i^2}\sqrt{\sum_{i=1}^{m-1}\mu_i} = \left\|\boldsymbol{f}^{(0)}\right\|_{L^2_\mu}\sqrt{\sum_{i=1}^{m-1}\mu_i}.$$

This yields the first inequality and similar arguments yield the second inequality. ☐

Remark A.5. As was shown in the proof of Lemma A.4 one can actually improve the bounds in (A.3) by a factor of $\Psi^{-1}(1)$.

Lemma A.6. *For any $t\geqslant\frac{3}{2}$ one has that*

$$\frac{1}{3}\frac{t}{\log t}\leqslant \Psi^{-1}(t)\leqslant 2\frac{t}{\log t}. \tag{A.4}$$

Proof. We start by noticing that

$$\begin{aligned}\Psi\left(\frac{1}{3}\frac{t}{\log t}\right) = \frac{1}{3}\frac{t}{\log t}\log\left(1+\frac{1}{3}\frac{t}{\log t}\right) &\leqslant \frac{1}{3}\frac{t}{\log t}\log\left(1+\frac{t}{\log\left(\frac{27}{8}\right)}\right)\\ &\leqslant \frac{1}{3}\frac{t}{\log t}\log\left(1+t\right).\end{aligned}$$

Thus, one notices that if

$$1+t\leqslant t^3$$

when $t\geqslant\frac{3}{2}$, we have that $\Psi\left(\frac{1}{3}\frac{t}{\log t}\right)\leqslant t$, yielding the left hand side of (A.4). This is indeed the case as $g(t)=t^3-t-1$ is increasing on $\left[\frac{1}{\sqrt{3}},\infty\right)$ and $g\left(\frac{3}{2}\right)>0$.
For the converse we notice that

$$\Psi\left(2\frac{t}{\log t}\right) = 2\frac{t}{\log t}\log\left(1+2\frac{t}{\log t}\right)\geqslant t$$

if and only if

$$1+2\frac{t}{\log t}\geqslant\sqrt{t}.$$

Considering the function $g(x)=\frac{x}{\log x}$ for $x>1$ we see that it obtains a minimum at $x=e$. Thus, for any $x>1$ $g(x)\geqslant e>1$. We conclude that for $t>\frac{3}{2}$

$$2\frac{t}{\log t}=\sqrt{t}g(\sqrt{t})\geqslant\sqrt{t},$$

showing the desired result. ☐

Appendix B. Additional Useful Computations

Lemma B.1. *For a given coagulation and detailed balance coefficients, $\{a_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}$, and a given positive sequence $\boldsymbol{c}$ with finite mass ϱ, we have that for any $z>0$*

$$H(\boldsymbol{c}|\mathcal{Q}) \leqslant H(\boldsymbol{c}|\mathcal{Q}_z),$$

where $\mathcal{Q} = Q_{\overline{z}}$.

Proof. We have that

$$H(\boldsymbol{c}|\mathcal{Q}_z) = \sum_{i=1}^{\infty} c_i \left(\log\left(\frac{c_1}{Q_i z^i}\right) - 1\right) + \sum_{i=1}^{\infty} Q_i z^i$$

implying that

$$H(\boldsymbol{c}|\mathcal{Q}_{z_1}) - H(\boldsymbol{c}|\mathcal{Q}_{z_2}) = \sum_{i=1}^{\infty} i c_i \log\left(\frac{z_2}{z_1}\right) + \sum_{i=1}^{\infty} Q_i \left(z_1^i - z_2^i\right).$$

In particular, if $z_2 = \overline{z}$ we have that for any $z>0$

$$H(\boldsymbol{c}|\mathcal{Q}_z) = H(\boldsymbol{c}|\mathcal{Q}) + \varrho \log\left(\frac{\overline{z}}{z}\right) + \sum_{i=1}^{\infty} Q_i \left(z^i - \overline{z}^i\right)$$

$$= H(\boldsymbol{c}|\mathcal{Q}) + \sum_{i=1}^{\infty} i Q_i \overline{z}^i \log\left(\frac{\overline{z}}{z}\right) + \sum_{i=1}^{\infty} Q_i z^i \left(1 - \left(\frac{\overline{z}^i}{z}\right)\right)$$

$$= H(\boldsymbol{c}|\mathcal{Q}) + \sum_{i=1}^{\infty} Q_i z^i \left(\left(\frac{\overline{z}}{z}\right)^i \log\left(\left(\frac{\overline{z}}{z}\right)^i\right) - \left(\frac{\overline{z}}{z}\right)^i + 1\right)$$

$$= H(\boldsymbol{c}|\mathcal{Q}) + \sum_{i=1}^{\infty} Q_i z^i \Lambda\left(\frac{(\mathcal{Q}_z)_i}{\mathcal{Q}_i}\right),$$

where $\Lambda(x) = x\log x - x + 1 > 0$ when $x>0$. This completes the proof. $\square$

Lemma B.2. *Let $\{Q_i\}_{i\in\mathbb{N}}$ be a non-negative sequence such that $\lim_{i\to\infty} \frac{Q_{i+1}}{Q_i} = \frac{1}{r}$ for some $r>0$. Assume that $0 < x < r_1 < r$. Then*

$$\sum_{i=i_0+1}^{\infty} i Q_i x^{i-1} \leqslant C Q_{i_0} x^{i_0},$$

where C is a constant depending only on $\{Q_i\}_{i\in\mathbb{N}}$ and r_1.

Proof. Define $\beta_i = \frac{Q_{i+1}}{Q_i}$. We have that $\lim_{i\to\infty} \beta_i = \frac{1}{r}$, and as such we fan find $l \in \mathbb{N}$ such that for all $i > l$

$$\Lambda_1 = \sup_{i>l} \beta_i < \frac{1}{r_1}.$$

Denote $\Lambda_2 = \sup_{i\leqslant l} \beta_i$. As for any $i > i_0$

$$Q_i = \left(\prod_{j=i_0}^{i-1} \beta_j\right) Q_{i_0}$$

we see that

$$\sum_{i=i_0+1}^{\infty} iQ_i x^{i-1} = Q_{i_0}x^{i_0}\sum_{i=i_0+1}^{\infty} i\left(\prod_{j=i_0}^{i-1}\beta_j\right)x^{i-i_0-1}$$
$$\leqslant Q_{i_0}x^{i_0}\left(\Lambda_2\sum_{j=0}^{l-i_0} i\left(\Lambda_2 r_1\right)^j + \Lambda_1\sum_{j=l+1-i_0}^{\infty} i\left(\Lambda_1 r_1\right)^j\right)$$
$$\leqslant Q_{i_0}x^{i_0}\left(\Lambda_2\sum_{j=0}^{l} j\left(\Lambda_2 r_1\right)^j + \Lambda_1\sum_{j=0}^{\infty} j\left(\Lambda_1 r_1\right)^j\right),$$

completing the proof as l, Λ_1 and Λ_2 depend solely on $\{Q_i\}_{i\in\mathbb{N}}$ □

Lemma B.3. *Let $\varepsilon > 0$ and $\gamma > 0$. Denote by*

$$B_{\varepsilon,\gamma} = \sum_{i=1}^{\infty} i^{\gamma} e^{-\varepsilon i}.$$

Then $\varepsilon^{1+\gamma}B_{\varepsilon,\gamma}$ is of order 1 when ε goes to zero.

Proof. We start by noticing that the function $g_{\varepsilon,\gamma}(x) = x^{\gamma}e^{-\varepsilon x}$ is increasing in $\left[0, \frac{\gamma}{\varepsilon}\right]$ and decreasing in $\left[\frac{\gamma}{\varepsilon}, \infty\right)$. As such

$$B_{\varepsilon,\gamma} \geqslant \sum_{i=\left[\frac{\gamma}{\varepsilon}\right]+1}^{\infty} i^{\gamma}e^{-\varepsilon i} \geqslant \int_{\left[\frac{\gamma}{\varepsilon}\right]+1}^{\infty} x^{\gamma}e^{-\varepsilon x}\,\mathrm{d}x$$
$$= \varepsilon^{-(1+\gamma)}\int_{\varepsilon\left(\left[\frac{\gamma}{\varepsilon}\right]+1\right)}^{\infty} y^{\gamma}e^{-y}\,\mathrm{d}y \geqslant \varepsilon^{-(1+\gamma)}\int_{\varepsilon}^{\infty} y^{\gamma}e^{-y}\,\mathrm{d}y,$$

showing the lower bound. For the upper bound we notice that

$$B_{\varepsilon,\gamma} \leqslant \sup_{x\geqslant 0} g_{\frac{\varepsilon}{e},\gamma}(x)\sum_{i=1}^{\infty} e^{-\frac{\varepsilon}{2}i} = \left(\frac{2\gamma}{\varepsilon}\right)^{\gamma} e^{-\gamma}\frac{e^{-\frac{\varepsilon}{2}}}{1-e^{-\frac{\varepsilon}{2}}}$$

which completes the proof since $\sup_{\varepsilon>0}\dfrac{\varepsilon\, e^{-\frac{\varepsilon}{2}}}{1-e^{-\frac{\varepsilon}{2}}} < +\infty$. □

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