Log-Sobolev Inequality for the Continuum Sine-Gordon Model

ROLAND BAUERSCHMIDT
University of Cambridge

THIERRY BODINEAU
École Polytechnique and CNRS

Abstract

We derive a multiscale generalisation of the Bakry-Émery criterion for a measure to satisfy a log-Sobolev inequality. Our criterion relies on the control of an associated PDE well-known in renormalisation theory: the Polchinski equation. It implies the usual Bakry-Émery criterion, but we show that it remains effective for measures that are far from log-concave. Indeed, using our criterion, we prove that the massive continuum sine-Gordon model with $\beta < 6\pi$ satisfies asymptotically optimal log-Sobolev inequalities for Glauber and Kawasaki dynamics. These dynamics can be seen as singular SPDEs recently constructed via regularity structures, but our results are independent of this theory. © 2021 The Authors. *Communications on Pure and Applied Mathematics* published by Wiley Periodicals LLC.

1 Introduction and Results

1.1 Introduction

Log-Sobolev inequalities are strong inequalities with numerous general consequences, including concentration of measure, relaxation and hypercontractivity of stochastic dynamics, transport inequalities, and others. See [4, 47] for a review. They originate from quantum field theory, where log-Sobolev inequalities were first derived for Gaussian measures as a tool to study non-Gaussian measures in infinite dimensions (Euclidean quantum field theories, EQFTs) [26, 32, 55]. As a consequence of a general new approach, we prove log-Sobolev inequalities for the massive sine-Gordon model. This is a fundamental example of a *non-Gaussian* EQFT in two dimensions, and its stochastic dynamics is a prototypical example of a singular SPDE.

As log-Sobolev inequalities provide strong control on the measures they apply to, proving them remains in general a difficult problem even if the equilibrium correlation functions are well understood. This applies especially to strongly correlated measures. For log-concave measures (or measures satisfying a curvature dimension condition), the fundamental Bakry-Émery criterion provides a simple

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and often quite sharp sufficient condition [2, 3]. In its proof, a log-Sobolev inequality for a Markov semigroup is derived by integration of local log-Sobolev inequalities for the *same* Markov semigroup.

Our method also uses local log-Sobolev inequalities, but for a semigroup that is *different* from the one for which the log-Sobolev inequality is proven. Namely, our method uses the time-dependent semigroup driven by the Polchinski equation, a version of the renormalisation semigroup. Unlike the original semigroup, this Polchinski semigroup provides a notion of scale, and hence we effectively obtain a multiscale version of the Bakry-Émery criterion.

The simplest version of our new Polchinski equation criterion for the log-Sobolev inequality is stated in Section 1.2. In Example 1.3, we illustrate that it implies the Bakry-Émery criterion. As an application of the new criterion, demonstrating that it remains effective for measures that are far from log-concave, we prove the following theorem for the continuum sine-Gordon model. For a precise statement of this result and related discussion, we refer to Section 1.3. In Section 1.4, we discuss further directions and related results.

THEOREM 1.1. *The continuum massive sine-Gordon model with $\beta < 6\pi$ satisfies asymptotically optimal log-Sobolev inequalities for Glauber and Kawasaki dynamics (under suitable conditions).*

Throughout this paper, we make the assumption that all functions considered are Borel measurable and that all functions to which derivatives are applied are continuously differentiable of the required order.

1.2 Polchinski Equation and Log-Sobolev Inequality

In this section we state the simplest version of our new criterion for a probability measure to satisfy a log-Sobolev inequality.

Given a linear space $X \subseteq \mathbb{R}^N$ with the induced inner product $(\cdot,\cdot)$, a symmetric matrix A that acts positive definitely on X, and a potential $V_0 : X \to \mathbb{R}$, we consider the probability measure ν_0 with expectation

$$\mathbb{E}_{\nu_0} F \propto \int_X e^{-\frac{1}{2}(\zeta, A\zeta) - V_0(\zeta)} F(\zeta) d\zeta. \tag{1.1}$$

We call the set $\Lambda = \{1, \dots, N\}$ the *index space* and the space X the *field space*; see also Figure 1.1. Let $Q_t = e^{-tA/2}$ be the *heat semigroup* associated with A (acting on elements $\varphi \in X$, i.e., functions $\varphi : \Lambda \to \mathbb{R}$ on the index space), set

$$\dot{C}_t = Q_t^2 = e^{-tA}, \quad C_t = \int_0^t \dot{C}_s \, ds, \tag{1.2}$$

and denote by $\boldsymbol{E}_{C_s}$ the expectation of the Gaussian measure with covariance C_s. For $t > s > 0$, we define the *renormalised potential* V_t, the *renormalisation semigroup* $\boldsymbol{P}_{s,t}$ (acting on functions $F : X \to \mathbb{R}$ on the field space), and the

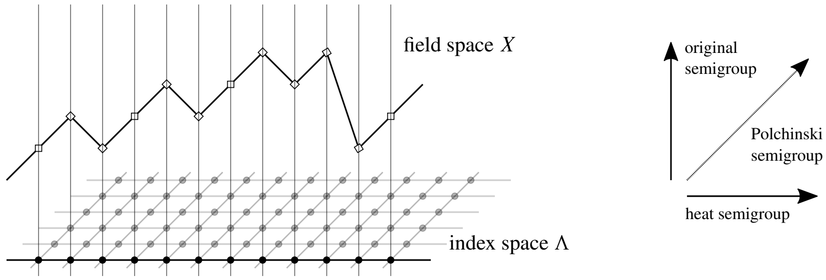

FIGURE 1.1. The heat semigroup Q_t acts on the index space $\Lambda = \{1, \dots, N\}$, i.e., "horizontally." In our primary applications, the index space Λ is identified with a finite approximation to $\mathbb{Z}^d$ or $\mathbb{R}^d$, and A is the Laplacian on Λ. The original semigroup with Dirichlet form $\mathbb{E}_{\nu_0}(\nabla F)^2$ acts on the field space $X \subseteq \mathbb{R}^\Lambda$. It acts "vertically" in the sense that the principal part of its generator is the standard Laplacian on X, i.e., Δ_{id} in the notation (1.11). The Polchinski renormalisation semigroup $\boldsymbol{P}_{s,t}$ also acts on field space X, but it acts "diagonally" in the sense that the principal part of its generator is time dependent and given in terms of the heat kernel as $\Delta_{Q_t^2}$ (see (2.8)).

renormalised measure ν_t by

$$e^{-V_t(\varphi)} = \boldsymbol{E}_{C_t}(e^{-V_0(\varphi+\zeta)}), \tag{1.3}$$

$$\boldsymbol{P}_{s,t}F(\varphi) = e^{V_t(\varphi)}\boldsymbol{E}_{C_t-C_s}(e^{-V_s(\varphi+\zeta)}F(\varphi+\zeta)), \tag{1.4}$$

$$\mathbb{E}_{\nu_t}F = \boldsymbol{P}_{t,\infty}F(0) = e^{V_\infty(0)}\boldsymbol{E}_{C_\infty-C_t}(e^{-V_t(\zeta)}F(\zeta)), \tag{1.5}$$

where $\varphi \in X$, the expectation $\boldsymbol{E}_{C_t}$ applies to ζ, and it is natural to define $\mathbb{E}_{\nu_\infty}F = F(0)$. Essentially equivalently to (1.3), V_t solves the Polchinski equation; see (1.10) below.

In what follows, we will impose the following *ergodicity assumption* on the semigroup $\boldsymbol{P}$: For all bounded smooth functions $F : X \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_{\nu_t} g(\boldsymbol{P}_{0,t}F) \to g(\mathbb{E}_{\nu_0}F) \quad \text{as } t \to \infty. \tag{1.6}$$

Like the ergodicity assumption in the Bakry-Émery theory (see [1,4]), this assumption is *qualitative* and easily seen to be satisfied in all examples of interest.

The following theorem bounds the log-Sobolev constant of the measure ν_0. For its statement, recall that the relative entropy of $F : X \to \mathbb{R}_+$ with respect to ν_0 is given by

$$\mathrm{Ent}_{\nu_0}(F) = \mathbb{E}_{\nu_0}\Phi(F) - \Phi(\mathbb{E}_{\nu_0}F), \quad \Phi(x) = x\log x, \tag{1.7}$$

where $0\log 0 = 0$. We write ∇ for the gradient on X and $(\nabla F)^2 = (\nabla F, \nabla F)$; thus in particular if $X = \mathbb{R}^N$ then $(\nabla F)^2 = \sum_{i=1}^N (\frac{\partial F}{\partial \varphi_i})^2$.

THEOREM 1.2. *In the setup above, assume* (1.6), *let* $\lambda > 0$ *be the smallest eigenvalue of* A, *suppose there are real numbers* $\dot{\mu}_t$ (*possibly negative*) *such that for all* $t \geqslant 0$, *as quadratic forms on* X,

$$Q_t \operatorname{Hess} V_t(\varphi) Q_t \geqslant \dot{\mu}_t \,\mathrm{id} \quad \text{where } Q_t = e^{-tA/2}, \tag{1.8}$$

and define $\mu_t = \int_0^t \dot{\mu}_s \, ds$. *Then* ν_0 *satisfies the log-Sobolev inequality*

$$\mathrm{Ent}_{\nu_0}(F) \leqslant \frac{2}{\gamma} \mathbb{E}_{\nu_0}(\nabla \sqrt{F})^2, \quad \frac{1}{\gamma} = \int_0^\infty e^{-\lambda t - 2\mu_t} \, dt, \tag{1.9}$$

provided the integral is finite.

The proof of Theorem 1.2, given in Section 2, shares significant elements with the celebrated Bakry-Émery argument, but with the crucial difference that it uses the time-dependent Polchinski semigroup (1.4) rather than the original semigroup, associated with the Dirichlet form $\mathbb{E}_{\nu_0}(\nabla F)^2$, to decompose the relative entropy. The above version of our criterion relies on the particular decomposition of the matrix $C_\infty = A^{-1}$ in terms of the heat semigroup $\dot{C}_t = e^{-tA}$. In Section 2, we also consider variations of the criterion that apply to other decompositions.

To apply the theorem, the main task is to verify (1.8). It is not difficult to see that the renormalised potential V_t solves the *Polchinski equation* (see Section 1.4 for its history)

$$\partial_t V_t = \frac{1}{2} \Delta_{\dot{C}_t} V_t - \frac{1}{2} (\nabla V_t)^2_{\dot{C}_t}, \tag{1.10}$$

where we use the notation (and with $w = \mathrm{id}$ if the argument w is omitted)

$$(u, v)_w = \sum_{i,j} w_{ij} u_i v_j, \quad (\nabla F)^2_w = (\nabla F, \nabla F)_w, \quad \Delta_w F = (\nabla, \nabla)_w F. \tag{1.11}$$

In general, verifying (1.8) is a challenging problem because the Polchinski equation is a nonlinear PDE in N dimensions, where in the examples of main interest $N \to \infty$. Nonetheless, we believe that the required estimates are true in many relevant examples, including spin systems near the critical point. In particular, in Section 3, we verify the condition for the continuum sine-Gordon model by analysing the Polchinski equation.

To illustrate our new criterion, we note briefly that (1.8) is not hard to verify for log-concave measures, in which case we recover the Bakry-Émery criterion as a special case.

Example 1.3 (Bakry-Émery criterion). Consider a probability measure ν_0 with expectation

$$\mathbb{E}_{\nu_0} F \propto \int_{\mathbb{R}^N} e^{-H(\zeta)} F(\zeta) d\zeta, \tag{1.12}$$

where Hess $H \geqslant \lambda$ id holds uniformly for some $\lambda > 0$. Equivalently, ν_0 can be written as in (1.1):

$$H(\zeta) = \frac{1}{2}(\zeta, A\zeta) + V_0(\zeta) \quad \text{with } A = \lambda \text{ id and } V_0 \text{ convex.} \tag{1.13}$$

It follows that V_t is convex for all $t \geqslant 0$ (see, e.g., [10, theorem 4.3]). Hence $\mu_t \geqslant 0$ for all t and thus $\gamma \geqslant \lambda$ in (1.9). This is the Bakry-Émery criterion.

We remark that an alternative proof that V_t remains convex for $t > 0$ can be deduced from the maximum principle for symmetric tensors [37, theorem 9.1]. This argument is in fact analogous to the proof that positive Ricci curvature remains positive under the Ricci flow in [37].

Theorem 1.2 can be considered a multiscale version of the Bakry-Émery criterion in which the global convexity assumption $\inf_\varphi \text{Hess}\, V_0(\varphi) \geqslant 0$, which is equivalent to $\inf_{t\geqslant 0} \inf_\varphi \text{Hess}\, V_t(\varphi) \geqslant 0$, is replaced by the assumption (1.8) on the Hessians of the effective potential V_t at each scale t. We emphasise that these Hessians are *not* required to be positive definite; in fact, in the example of the continuum sine-Gordon model that we consider in Section 1.3 below, the effective potential remains nonconvex at all scales $t > 0$. We also emphasise that the application of the heat kernel Q_t to Hess $V_t(\varphi)$ in (1.8) has an important smoothing effect. In particular, for the sine-Gordon model, we will see that this smoothing effect is essential when $\beta > 4\pi$.

Remark 1.4. We have defined the renormalised potential V_t as the convolution solution (1.3) to the Polchinski equation (1.10). Since equivalently $Z_t = e^{-V_t}$ solves the heat equation $\partial_t Z_t = \frac{1}{2}\Delta_{\dot{C}_t} Z_t$, the Polchinski equation has a unique solution under weak conditions. Then one may equivalently solve (1.10) instead of (1.3); for an example for which this is useful, see Section 3.

Remark 1.5. We remark that with the time-dependent metric $g_t = e^{+tA}$ on X and ∇_{g_t} and Δ_{g_t} defined as in Riemannian geometry, i.e., $\nabla_{g_t} = g_t^{-1}\nabla$ and Δ_{g_t} the Laplace-Beltrami operator, one has $\Delta_{\dot{C}_t} = \Delta_{g_t}$ and $(\nabla F)^2_{\dot{C}_t} = (\nabla_{g_t} F)^2_{g_t}$. The condition (1.8) then becomes $\text{Hess}_{g_t} V_t \geqslant \dot{\mu}_t g_t$.

1.3 Continuum Sine-Gordon Model

In Section 3, we apply Theorem 1.2 to prove asymptotically sharp log-Sobolev inequalities for Glauber and Kawasaki dynamics of the massive continuum sine-Gordon model with $\beta < 6\pi$. The massive sine-Gordon model is a fundamental example of a two-dimensional interacting Euclidean quantum field theory, i.e., a non-Gaussian probability measure on $\mathcal{D}'(\mathbb{R}^2)$ sometimes *formally* written as

$$\frac{1}{Z} \exp\Big[-\int_{\mathbb{R}^2} \Big(\frac{1}{2}\varphi(-\Delta\varphi) + \frac{1}{2}m^2\varphi(x)^2 + 2z :\cos(\sqrt{\beta}\varphi(x)): \Big) dx\Big] \prod_{x\in\mathbb{R}^2} d\varphi(x). \tag{1.14}$$

Here Δ is the Laplacian on $\mathbb{R}^2$, and the notation : denotes Wick ordering, i.e., that z is formally multiplied by a divergent constant (making the microscopic potential extremely nonconvex); see (1.15)–(1.16) below for the precise definition that we will use. The Glauber dynamics of the sine-Gordon model (also called dynamical sine-Gordon model) can be realised as a singular SPDE that was recently constructed using the theory of regularity structures. References on the sine-Gordon model are provided further below.

For clarity, we consider the model in a lattice approximation of a two-dimensional torus and prove estimates uniformly in the lattice spacing and in the size of the torus. Therefore, from now on, let $d = 2$, let $\Omega_L = L\mathbb{T}^d$ be the torus of side length $L > 0$, and let $\Omega_{\varepsilon,L} = \Omega_L \cap \varepsilon\mathbb{Z}^d$ be its lattice approximation with mesh size $\varepsilon > 0$ (where we always assume L is a multiple of ε). The continuum sine-Gordon model $\nu_{\varepsilon,L}$ in the lattice approximation is the probability measure on $\mathbb{R}^{\Omega_{\varepsilon,L}}$ with density proportional to $e^{-H_{\varepsilon,L}(\varphi)}$ where $H_{\varepsilon,L}$ is defined for $\varphi : \Omega_{\varepsilon,L} \to \mathbb{R}$ by

$$H_{\varepsilon,L}(\varphi) = \varepsilon^d \sum_{x\in\Omega_{\varepsilon,L}} \left(\frac{1}{2}\varphi_x(-\Delta^\varepsilon\varphi)_x + \frac{1}{2}m^2\varphi_x^2 + 2z_\varepsilon \cos\bigl(\sqrt{\beta}\varphi_x\bigr)\right), \tag{1.15}$$

with *divergent* coupling constant

$$z_\varepsilon = z\varepsilon^{-\beta/4\pi}, \tag{1.16}$$

and where $(\Delta^\varepsilon\varphi)_x = \varepsilon^{-2}\sum_{y\sim x}(\varphi_y - \varphi_x)$ is the discretised Laplacian, i.e., the sum $y \sim x$ is over nearest-neighbour vertices y of x in $\varepsilon\mathbb{Z}^d$. Under suitable assumptions, this normalisation ensures that, for $0 < \beta < 8\pi$, the measures $\nu_{\varepsilon,L}$ converge weakly to a non-Gaussian probability measure on $\mathcal{D}'(\mathbb{R}^2)$ as $\varepsilon \to 0$ and $L \to \infty$; see the discussion after the statement of the theorems below.

Our first theorem is a uniform log-Sobolev inequality for the Glauber dynamics of the massive sine-Gordon measure $\nu_{\varepsilon,L}$ (with dimension always $d = 2$). The Glauber Dirichlet form is given by

$$\boldsymbol{D}_{\varepsilon,L}(F) = \frac{1}{\varepsilon^2}\sum_{x\in\Omega_{\varepsilon,L}} \mathbb{E}_{\nu_{\varepsilon,L}}\left[\left(\frac{\partial F}{\partial\varphi_x}\right)^2\right], \tag{1.17}$$

corresponding to the system of SDEs

$$\frac{\partial}{\partial t}\varphi_x^\varepsilon = (\Delta^\varepsilon\varphi^\varepsilon)_x + m^2\varphi_x^\varepsilon + \varepsilon^{-\beta/4\pi}2z\sqrt{\beta}\sin\bigl(\sqrt{\beta}\varphi_x^\varepsilon\bigr) + \sqrt{2}\dot{W}_x^\varepsilon, \tag{1.18}$$

where $\dot{W}^\varepsilon$ is space-time white noise (with discretised space), i.e., the $(W_x^\varepsilon)_{x\in\Omega_{\varepsilon,L}}$ are independent Brownian motions with quadratic variation $\langle W_x^\varepsilon\rangle(t) = t/\varepsilon^2$.

THEOREM 1.6. *Fix $\beta < 6\pi$, and let $L > 0$, $m > 0$, and $z \in \mathbb{R}$. Then there is $\gamma(\beta, z, m, L) > 0$ independent of $\varepsilon > 0$ such that, for all $F \geqslant 0$,*

$$\mathrm{Ent}_{\nu_{\varepsilon,L}}(F) \leqslant \frac{2}{\gamma(\beta, z, m, L)}\boldsymbol{D}_{\varepsilon,L}(\sqrt{F}). \tag{1.19}$$

Moreover, there is $\delta_\beta > 0$ such that if $Lm \geqslant 1$ and $|z|m^{-2+\beta/4\pi} \leqslant \delta_\beta$, then

$$\gamma(\beta, z, m, L) \geqslant m^2 - O_\beta(m^{\beta/4\pi}|z|), \tag{1.20}$$

where the constant O_β depends on β only (and is thus uniform in $L \geqslant 1/m$).

Our next theorem is a (conservative) Kawasaki version of the previous result. We thus consider the measure $\nu^0_{\varepsilon,L}$ obtained by constraining the mean spin of the measure $\nu_{\varepsilon,L}$ to $\sum_{x\in\Omega_{\varepsilon,L}} \varphi_x = 0$, i.e., $\nu^0_{\varepsilon,L}$ is supported on $\{\varphi : \sum_x \varphi_x = 0\}$. (The same proof also works for arbitrary nonzero mean of φ.) The Dirichlet form for Kawasaki dynamics with invariant measure $\nu^0_{\varepsilon,L}$ is defined by

$$\boldsymbol{D}^0_{\varepsilon,L}(F) = \frac{1}{\varepsilon^4} \sum_{x\sim y\in\Omega_{\varepsilon,L}} \mathbb{E}_{\nu^0_{\varepsilon,L}}\left[\left(\frac{\partial F}{\partial\varphi_x} - \frac{\partial F}{\partial\varphi_y}\right)^2\right]. \tag{1.21}$$

THEOREM 1.7. *Fix $\beta < 6\pi$, and let $L > 0$, $m > 0$, and $z \in \mathbb{R}$. Then there is $\gamma^0(\beta, z, m, L) > 0$ independent of $\varepsilon > 0$ such that, for all $F \geqslant 0$,*

$$\operatorname{Ent}_{\nu^0_{\varepsilon,L}}(F) \leqslant \frac{2}{\gamma^0(\beta, z, m, L)} \boldsymbol{D}^0_{\varepsilon,L}\big(\sqrt{F}\big). \tag{1.22}$$

Moreover, there is $\delta_\beta > 0$ such that if $Lm \geqslant 1$ and $|z|m^{-2+\beta/4\pi} \leqslant \delta_\beta$, then

$$\gamma^0(\beta, z, m, L) \geqslant \frac{(2\pi)^2}{L^2}\left(m^2 + \frac{(2\pi)^2}{L^2} - O_\beta(m^{\beta/4\pi}|z|)\right), \tag{1.23}$$

where the constant O_β depends on β only (and is thus uniform in $L \geqslant 1/m$).

For $z = 0$, the sine-Gordon model degenerates simply to the continuum Gaussian free field with covariance $(-\Delta + m^2)^{-1}$, as $\varepsilon \downarrow 0$, for which the Glauber log-Sobolev constant is m^2 (by [32] or the Bakry-Émery criterion), and similarly in the Kawasaki case. Note that, in this scaling in which the convexity of the Gaussian measure is of order 1, the best lower bound on the Hessian of the interaction term V_0 is of order $-\varepsilon^{-\beta/4\pi}$ if $z \neq 0$ and thus tends to $-\infty$ as $\varepsilon \to 0$. Thus the measure is far out of the scope of the applicability of the Bakry-Émery criterion if $z \neq 0$. Our proof of the above theorems via Theorem 1.2 relies on the smoothing of the effective potential V_t along the flow of the Polchinski equation.

The Glauber dynamics of the sine-Gordon model is considered in [16,36]. Using the theory of regularity structures, it is shown in these references that versions of (1.18) that are regularised in space-time instead of space only converge as $\varepsilon \to 0$ pathwise in a space of distributions on a short noise-dependent time interval. In our setting, it is essential that the noise be white in time for the regularised dynamics to define a Markov process. The question of regularisation in space rather than space-time was considered for the closely related problems of the subcritical continuum φ^4 model and KPZ equation in [34, 35, 66] as well as in [23, 51, 54]. Presumably similar arguments would also apply to the sine-Gordon model but have not been carried out.

Finally, we provide some references on the continuum sine-Gordon model. For $0 < \beta < 8\pi$, at least when the domain is a torus and $z \neq 0$ is small and $m^2 > 0$, it is known that $\nu_{\varepsilon,L} \to \nu$ weakly, where ν is a non-Gaussian measure on $\mathcal{D}'(\mathbb{R}^2)$ with a precise description in terms of renormalised expansions; see [28, 29], [9,56], [14], and [12,20,21] for different approaches. This result is simplest for $\beta < 4\pi$, when in finite volume the continuum sine-Gordon measure is absolutely continuous with respect to the Gaussian free field. For $4\pi \leqslant \beta < 8\pi$, there is an infinite sequence of thresholds at $\beta = 8\pi(1 - 1/2n)$, $n = 1, 2, \dots$, at which the partition function (but not the normalised probability measure) acquires divergent contributions; see [9] for further discussion. The physical meaning of these divergences remains debated [27].

The sine-Gordon model satisfies a very interesting duality with the massive Thirring model, the Coleman correspondence, or bosonization [17]. For restricted values of β, this correspondence has been established in finite volume or with a mass term [8, 18, 29], but in general its proof remains an open problem, most importantly in the formally massless case $m^2 = 0$. In particular, under this correspondence, for the special value $\beta = 4\pi$, the correlation functions of the sine-Gordon model are equivalent to those of free fermions. In general, an important question for the sine-Gordon model that has remained open is the formally massless case $L \to \infty$ and $m^2 \to 0$, in which case correlations decay polynomially if $z = 0$. For $z \neq 0$, it is conjectured that the equilibrium correlation functions have exponential decay for any $\beta < 8\pi$. Closely related results for small β were obtained in [13, 64]. It would be very interesting to understand the dynamical behaviour in this regime.

Our result extends up to the second threshold $\beta < 6\pi$ and makes use of the approach of [14]. It remains a very interesting problem to extend our results to the optimal regime $\beta < 8\pi$. Recent progress in the direction of extending the method of [14] includes [43]. Other recent results for the sine-Gordon model include [40]. For a one-dimensional analogue of the sine-Gordon model, a recent construction using martingales was given in [44].

1.4 More Discussion of Our Approach and of Further Directions

Our approach to the log-Sobolev inequality involves the Polchinski equation (1.10). The Polchinski equation is a continuous version of Wilson's renormalisation group (which typically proceeds in discrete steps) and variations of it go back to [62, 63], while the continuous point of view was first systematically used by Polchinski [59]. See [42] for a review of its history as well as for an account of the important role it has played in recent advances in perturbative quantum field theory. The relation of the Polchinski equation to the Mayer expansion and its iterated versions was investigated in [14], on which we rely for the sine-Gordon model. Ideas related to the Polchinski equation were also used recently in [5] for a simple construction of the continuum φ^4 model in $d = 2, 3$. We also mention that approaches involving aspects of renormalisation have been used for a long time to study dynamics of spin systems, e.g., in the form of block dynamics [45, 50, 65] and more recently in the two-scale approach [22,33,53,57]. Our approach involves infinitely many scales.

The regime of the continuum limit considered in Section 1.3 is known as the *ultraviolet problem* in physics, which for the two-dimensional sine-Gordon model is well-posed for $\beta < 8\pi$. The long-distance behaviour is predicted to be independent of ε. For $\beta < 8\pi$, it can studied as a property of the continuum limit $\varepsilon \to 0$, but it makes sense for all $\beta > 0$ when the regularisation ε is fixed (lattice problem). For $\beta \geqslant \beta_c$ (where the curve $\beta_c(z)$ passes through 8π at $z = 0$; see [24, 25]) and small z and $m^2 = 0$, the scaling limit is known to be Gaussian free field in a suitable sense for the model defined on the torus [19, 25]. This is called the *infrared problem* in physics. However, we emphasise that, while the ultraviolet problem can be translated to a lattice problem, as we do, the scaling of the infrared problem is more delicate than that of the ultraviolet problem. For the sine-Gordon model, in the ultraviolet limit, the microscopic coupling constant is very small, of order $\varepsilon^{2-\beta/4\pi} \ll 1$. For the infrared problem, the microscopic coupling constant is of order 1, and unlikely field configurations play a more important role in understanding the measure (large field problem); see [19, 24, 25]. We studied the spectral gap for the hierarchical version of the infrared problem in [6]. Using Theorem 2.6 and the estimates proved in [6], the results for the spectral gap stated in [6] can be improved to results about the log-Sobolev constant; see Example 2.7.

The next natural class of models that would be interesting to apply Theorem 1.2 to is the φ^4 model. The problem analogous to the one considered for the sine-Gordon model would be the continuum φ^4 model on $\mathbb{R}^d$ where $d = 2, 3$ with sufficiently large mass (ultraviolet problem). On a finite two-dimensional torus, a spectral gap result for the continuum φ^4 model has been shown in [61]. We stress again that the Polchinski equation has also been used in [5] in the construction of the continuum φ^4 model on a torus in $d = 2, 3$. As in the case of the sine-Gordon model, the infrared problem appears more difficult than the ultraviolet problem. For the latter we expect that the log-Sobolev constant of the lattice φ^4 model or the Ising model in $d = 4$ (respectively $d > 4$) scales as $u(-\log u)^z$ (respectively u) as the critical point is approached with distance $u \downarrow 0$. Again, for the hierarchical φ^4 model, we proved the analogous statement for the spectral gap in [6], and the results of this paper can again be used to improve the latter result to prove also an analogous log-Sobolev inequality; again see Example 2.7.

In a different direction, the Bakry-Émery theory has a well-known formulation in the context of manifolds (and beyond). The Polchinski equation is closely related to the Gaussian convolution semigroup $\boldsymbol{E}_{C_t}$ on X and thus to the linear structure of X. However, with the disintegration of the Gaussian measure taking the role of the reverse Ricci flow, there is an interesting resemblence of our construction to those in [48, 52, 58]; see also Remark 1.5.

Finally, we remark that log-Sobolev inequalities are a very useful tool to derive mixing results in general; see, e.g., [49]. It would be very interesting to derive such results in our context.

2 Log-Sobolev Inequality and the Polchinski Equation

In this section we prove Theorem 1.2 and variations of this result that apply in slightly different setups. The proofs share many elements with the Bakry-Émery argument, which we will review.

2.1 The Renormalisation Semigroup

Let $t \in [0,\infty] \mapsto C_t$ be a function of positive semidefinite matrices on $\mathbb{R}^N$ increasing continuously as quadratic forms to a matrix C_∞. More precisely, we assume that $C_t = \int_0^t \dot{C}_s \, ds$ for all t, where $t \mapsto \dot{C}_t$ is a bounded function with values in the space of positive semidefinite matrices that is the derivative of C_t except at isolated points. As before, we denote by $\boldsymbol{E}_{C_t}$ the expectation of the possibly degenerate Gaussian measure with covariance C_t. We consider a probability measure ν_0 with expectation

$$\mathbb{E}_{\nu_0} F \propto \boldsymbol{E}_{C_\infty}(e^{-V_0(\zeta)} F(\zeta)) \tag{2.1}$$

for a potential $V_0 : \mathbb{R}^N \to \mathbb{R}$. For $t > s > 0$, we recall the definitions

$$e^{-V_t(\varphi)} = \boldsymbol{E}_{C_t}(e^{-V_0(\varphi+\zeta)}), \tag{2.2}$$

$$\boldsymbol{P}_{s,t} F(\varphi) = e^{V_t(\varphi)} \boldsymbol{E}_{C_t - C_s}(e^{-V_s(\varphi+\zeta)} F(\varphi+\zeta)), \tag{2.3}$$

$$\mathbb{E}_{\nu_t} F = \boldsymbol{P}_{t,\infty} F(0) = e^{V_\infty(0)} \boldsymbol{E}_{C_\infty - C_t}(e^{-V_t(\zeta)} F(\zeta)), \tag{2.4}$$

where the expectations again apply to ζ. We impose the following *continuity assumption*: For all bounded smooth functions $F : X \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}_{\nu_t} g(\boldsymbol{P}_{0,t} F) \quad \text{is continuous in } t \in [0,+\infty]. \tag{2.5}$$

The assumption (2.5) reduces to (1.6) when C_t is differentiable in t, as in Section 1.2, and it is again clear in all examples of practical interest.

The following proposition collects some properties of the above definitions; we postpone its elementary proof to Section 2.4.

PROPOSITION 2.1. *Let (C_t) be as above, let $V_0 \in C^2$, and assume* (2.5). *Then for every t such that C_t is differentiable, the renormalised potential V defined in* (1.3) *satisfies the Polchinski equation*

$$\partial_t V_t = \frac{1}{2} \Delta_{\dot{C}_t} V_t - \frac{1}{2} (\nabla V_t)^2_{\dot{C}_t}. \tag{2.6}$$

The operators $(\boldsymbol{P}_{s,t})_{s \leqslant t}$ form a time-dependent Markov semigroup with generators $(\boldsymbol{L}_t)$ in the sense that $\boldsymbol{P}_{t,t} = \mathrm{id}$ and $\boldsymbol{P}_{r,t} \boldsymbol{P}_{s,r} = \boldsymbol{P}_{s,t}$ for all $s \leqslant r \leqslant t$, that

$\boldsymbol{P}_{s,t} F \geqslant 0$ *if* $F \geqslant 0$ *and* $\boldsymbol{P}_{s,t} 1 = 1$, *and that for all t at which C_t is differentiable* (*respectively s at which C_s is differentiable*),

$$\frac{\partial}{\partial t} \boldsymbol{P}_{s,t} F = \boldsymbol{L}_t \boldsymbol{P}_{s,t} F, \quad -\frac{\partial}{\partial s} \boldsymbol{P}_{s,t} F = \boldsymbol{P}_{s,t} \boldsymbol{L}_s F \quad (s \leqslant t), \tag{2.7}$$

for all smooth functions F, where $\boldsymbol{L}_t$ acts on a smooth function F by

$$\boldsymbol{L}_t F = \frac{1}{2} \Delta_{\dot{C}_t} F - (\nabla V_t, \nabla F)_{\dot{C}_t}. \tag{2.8}$$

The measures ν_t evolve dual to $(\boldsymbol{P}_{s,t})$ in the sense that

$$\mathbb{E}_{\nu_t} \boldsymbol{P}_{s,t} F = \mathbb{E}_{\nu_s} F \quad (s \leqslant t), \qquad -\frac{\partial}{\partial t} \mathbb{E}_{\nu_t} F = \mathbb{E}_{\nu_t} \boldsymbol{L}_t F. \tag{2.9}$$

Finally, for any smooth function F with values in a compact subset of $(0, \infty)$ *and* $\Phi(x) = x \log x$,

$$\mathbb{E}_{\nu_t} \Phi(\boldsymbol{P}_{0,t} F) \quad \textit{is continuous in } t \in [0, +\infty]. \tag{2.10}$$

Remark 2.2. The Polchinski semigroup operates from the right, that is, $\boldsymbol{P}_{s,t} = \boldsymbol{P}_{r,t} \boldsymbol{P}_{s,r}$ for $s \leqslant r \leqslant t$. Thus it acts on probability densities relative to ν_t: if $\mu_0 = F\, d\nu_0$ is a probability measure, then $\mu_t = \boldsymbol{P}_{0,t} F\, d\nu_t$ is again a probability measure. For a time-independent semigroup $\boldsymbol{T}_{s,t} = \boldsymbol{T}_{t-s}$ that is reversible with respect to the measure ν_0 (as, for example, the original semigroup associated to the Dirichlet form), one has the dual point of view that $\boldsymbol{T}$ describes the evolution of an observable:

$$\mathbb{E}_{\mu_t} G = \int G(\boldsymbol{T}_t F) d\nu_0 = \int (\boldsymbol{T}_t G) F\, d\nu_0 = \mathbb{E}_{\mu_0}(\boldsymbol{T}_t G). \tag{2.11}$$

Such a dual semigroup can be realised in terms of a Markov process (φ_t) as $\boldsymbol{T}_t F(\varphi) = \boldsymbol{E}_{\varphi_0 = \varphi} F(\varphi_t)$. Since the Polchinski semigroup is not reversible and time-dependent, this interpretation does not apply to the Polchinski semigroup. Instead, the Polchinski semigroup $\boldsymbol{P}_{s,t}$ can be realised in terms of an SDE that starts at time t and runs time in the negative direction from t to s. Indeed, set $\varphi_r = \widetilde{\varphi}_{t-r}$ where $\widetilde{\varphi}$ satisfies

$$d\widetilde{\varphi}_r = -\dot{C}_{t-r} \nabla V_{t-r}(\widetilde{\varphi}_r) dr + \sqrt{\dot{C}_{t-r}}\, dB_r, \quad 0 \leqslant r \leqslant t. \tag{2.12}$$

Since $G(r, \varphi) = \boldsymbol{P}_{s,t-r} F(\varphi)$ satisfies $\partial_r G + \boldsymbol{L}_{t-r} G = 0$ for $s < r < t$ by (2.7), Itô's formula and (2.12) imply that $G(r, \widetilde{\varphi}_r) = \boldsymbol{P}_{s,t-r} F(\varphi_{t-r})$ is a martingale for $r \in [s, t]$. This implies

$$\boldsymbol{P}_{s,t} F(\varphi) = \mathbb{E}_{\varphi_t = \varphi} F(\varphi_s). \tag{2.13}$$

Thus if φ_t is distributed according to ν_t by the above backward-in-time evolution φ_s is distributed according to ν_s for $s < t$. Our interpretation of this is that, while the renormalised measures ν_t are supported on increasing smooth (in the index space) configurations as t grows, the backward evolution restores the small-scale fluctuations of ν_0.

For later use we also record the following useful relations for the derivatives of V_t; we will not use these in Section 2. The formulas follow immediately by differentiating (2.2) using (2.3).

PROPOSITION 2.3. *For all $f \in X$ and $t \geqslant s \geqslant 0$,*

$$(f, \nabla V_t) = \boldsymbol{P}_{s,t}(f, \nabla V_s), \tag{2.14}$$

$$\begin{aligned}(f, \operatorname{Hess} V_t f) = {}& \boldsymbol{P}_{s,t}(f, \operatorname{Hess} V_s f) \\ & - \big[\boldsymbol{P}_{s,t}\big((f, \nabla V_s)^2\big) - (\boldsymbol{P}_{s,t}(f, \nabla V_s))^2\big].\end{aligned} \tag{2.15}$$

2.2 Relative Entropy, Markov Semigroups, and the Bakry-Émery Argument

In a time-dependent generalisation, we now review the decomposition of the relative entropy in terms of a semigroup that underlies the Bakry-Émery argument. By approximation (see, e.g., [60, theorem 3.1.13]), to prove a log-Sobolev inequality, it suffices to consider smooth functions $F : X \to \mathbb{R}$ with values in a compact subset of $(0, \infty)$, which we will do from now on.

We consider a curve of probability measures $(\nu_t)_{t \geqslant 0}$ and a corresponding dual time-dependent Markov semigroup $(\boldsymbol{P}_{s,t})$ with generators $(\boldsymbol{L}_t)$ as in Proposition 2.1. Namely, we assume that (2.7) and (2.9) hold, that $\boldsymbol{L}_t$ is of the form (2.8) for some positive semidefinite matrices $\dot{\boldsymbol{C}}_t$ and functions V_t (not necessarily satisfying (2.6)), and also that (2.10) holds. Denoting $F_t = \boldsymbol{P}_{0,t}F$ and $\dot{F}_t = \frac{\partial}{\partial t}F_t$, using first (2.9) and then (2.8), it is then elementary to see that

$$\begin{aligned}-\frac{\partial}{\partial t}\mathbb{E}_{\nu_t}\Phi(F_t) &= \mathbb{E}_{\nu_t}\big(\boldsymbol{L}_t(\Phi(F_t)) - \Phi'(F_t)\dot{F}_t\big) \\ &= \mathbb{E}_{\nu_t}\left(\Phi'(F_t)\boldsymbol{L}_t F_t + \Phi''(F_t)\frac{1}{2}(\nabla F_t)^2_{\dot{\boldsymbol{C}}_t} - \Phi'(F_t)\dot{F}_t\right) \\ &= \frac{1}{2}\mathbb{E}_{\nu_t}\big(\Phi''(F_t)(\nabla F_t)^2_{\dot{\boldsymbol{C}}_t}\big).\end{aligned} \tag{2.16}$$

Integrating this relation using (2.10), with $\Phi''(x) = 1/x$, it follows that

$$\operatorname{Ent}_{\nu_0}(F) = \frac{1}{2}\int_0^\infty \mathbb{E}_{\nu_t}\frac{(\nabla \boldsymbol{P}_{0,t}F)^2_{\dot{\boldsymbol{C}}_t}}{\boldsymbol{P}_{0,t}F}\,dt = 2\int_0^\infty \mathbb{E}_{\nu_t}(\nabla\sqrt{\boldsymbol{P}_{0,t}F})^2_{\dot{\boldsymbol{C}}_t}\,dt. \tag{2.17}$$

To be precise, recall that $\boldsymbol{C}_t$ is differentiable except for at most countably many t. For all t such that $\boldsymbol{C}_t$ is differentiable, the identity (2.16) holds and implies that the continuous function $t \mapsto \mathbb{E}_{\nu_t}\Phi(F_t)$ is differentiable at t with nonpositive derivative. In particular, this implies that $\mathbb{E}_{\nu_t}\Phi(F_t)$ is decreasing, which justifies the use of the fundamental theorem of calculus and together with (2.5) with $t = +\infty$ for the limit gives (2.17).

In order to obtain a log-Sobolev inequality, the right-hand side of (2.17) must be bounded by the Dirichlet form with respect to the measure ν_0. The same argument with $\Phi(x) = x^2$ would give a bound on the variance rather than the entropy and correspondingly a spectral gap inequality; the required bound is easier to obtain in this case.

For measures that are log-concave (or, more generally, ones that satisfy a curvature dimension condition; see [4]), sharp estimates have been obtained by celebrated arguments of Lichnerowicz (for the spectral gap) and of Bakry-Émery. We review the latter briefly now.

Example 2.4 (Bakry-Émery [2, 3]). Assume the measure $\nu = \nu_0$ has expectation given by (1.12). Let $\nu_t = \nu_0$ for all $t \geqslant 0$, and define the semigroup $\boldsymbol{T}_{s,t} = \boldsymbol{T}_{t-s}$ with generator

$$\boldsymbol{L}F = \Delta F - (\nabla H, \nabla F). \tag{2.18}$$

This semigroup leaves ν_0 invariant. Bakry-Émery showed, for all $F \geqslant 0$,

$$\begin{aligned} &\frac{\partial}{\partial t}\mathbb{E}_{\nu_0}(\nabla\sqrt{\boldsymbol{T}_t F})^2 \\ &\quad = -\frac{1}{2}\mathbb{E}_{\nu_0}\big(\boldsymbol{T}_t F\big(|\mathrm{Hess}\log \boldsymbol{T}_t F|_2^2 + (\nabla\log \boldsymbol{T}_t F, (\mathrm{Hess}\, H)\nabla \log \boldsymbol{T}_t F)\big)\big) \\ &\quad \leqslant -\frac{1}{2}\mathbb{E}_{\nu_0}\big(\boldsymbol{T}_t F(\nabla\log \boldsymbol{T}_t F, (\mathrm{Hess}\, H)\nabla \log \boldsymbol{T}_t F)\big). \end{aligned} \tag{2.19}$$

If Hess $H(\varphi) \geqslant \lambda\,\mathrm{id} > 0$ as quadratic forms, uniformly in $\varphi \in \mathbb{R}^N$, it follows that

$$\begin{aligned} \frac{\partial}{\partial t}\mathbb{E}_{\nu_0}(\nabla\sqrt{\boldsymbol{T}_t F})^2 &\leqslant -2\lambda\mathbb{E}_{\nu_0}(\nabla\sqrt{\boldsymbol{T}_t F})^2, \\ \mathbb{E}_{\nu_0}(\nabla\sqrt{\boldsymbol{T}_t F})^2 &\leqslant e^{-2\lambda t}\mathbb{E}_{\nu_0}(\nabla\sqrt{F})^2. \end{aligned} \tag{2.20}$$

Substituting this into (2.17) yields the log-Sobolev inequality

$$\mathrm{Ent}_{\nu_0}(F) = 4\int_0^\infty \mathbb{E}_{\nu_0}\big(\nabla\sqrt{\boldsymbol{T}_t F}\big)^2\,dt \leqslant \frac{2}{\lambda}\mathbb{E}_{\nu_0}\big(\nabla\sqrt{F}\big)^2. \tag{2.21}$$

In fact, (2.19) follows as in Lemma 2.8 below.

2.3 Variations of Theorem 1.2

The following theorem generalises Theorem 1.2 by not assuming that $\dot{C}_t$ is given by the heat kernel.

THEOREM 2.5. *Let $\dot{C}_t$ and V_t be as in Section* 2.1, *assume that $\dot{C}_t$ is differentiable for all t, and that* (2.5) *holds. Suppose there are $\dot{\lambda}_t$* (*allowed to be negative*) *such that*

$$\dot{C}_t\,\mathrm{Hess}\,V_t(\varphi)\dot{C}_t - \frac{1}{2}\ddot{C}_t \geqslant \dot{\lambda}_t\dot{C}_t \quad \text{for all } t \geqslant 0 \text{ and all } \varphi \in X, \tag{2.22}$$

and define

$$\lambda_t = \int_0^t \dot{\lambda}_s\,ds, \qquad \frac{1}{\gamma} = |\dot{C}_0|\int_0^\infty e^{-2\lambda_s}\,ds \tag{2.23}$$

where $|\dot{C}_0|$ is the largest eigenvalue of $\dot{C}_0$. Then ν_0 satisfies the log-Sobolev inequality

$$\mathrm{Ent}_{\nu_0}(F) \leqslant \frac{2}{\gamma}\mathbb{E}_{\nu_0}(\nabla\sqrt{F})^2. \tag{2.24}$$

The proof of the theorem is given in Section 2.5. When $\dot{C}_t$ is given by the heat kernel, as in the context of Theorem 1.2, the term $\ddot{C}_t$ in (2.22) can be eliminated explicitly and we can thus deduce Theorem 1.2 as follows.

PROOF OF THEOREM 1.2. Let $Q_t = e^{-tA/2}$ and $\dot{C}_t = e^{-tA} = Q_t^2$. Then $\ddot{C}_t = -A\dot{C}_t = -Q_tAQ_t$, and the left-hand side of (2.22) is equal to

$$Q_t\left[Q_t \operatorname{Hess} V_t(\varphi)Q_t + \frac{1}{2}A\right]Q_t. \tag{2.25}$$

Since by assumption $A \geqslant \lambda$ and $Q_t \operatorname{Hess} V_t Q_t \geqslant \dot{\mu}_t$, we can choose $\dot{\lambda}_t = \frac{1}{2}\lambda + \dot{\mu}_t$ to get

$$\frac{1}{2}A + Q_t \operatorname{Hess} V_t(\varphi)Q_t \geqslant \dot{\lambda}_t \,\mathrm{id}, \tag{2.26}$$

which with $Q_t^2 = \dot{C}_t$ implies (2.22). The claim (1.9) is thus implied by Theorem 2.5. □

The next theorem provides a variation of Theorem 2.5 that does not rely on differentiability or even continuity of $\dot{C}_t$ in t, and can therefore be applied with more general covariance decompositions. The price is the less symmetric condition (2.27). However, this condition can for example be applied to discrete decompositions $C_\infty = C_0 + C_1 + \cdots$ by setting $\dot{C}_s = \sum_j 1_{(j,j+1]}(s)C_j$. In particular, this applies to the hierarchical spin models that we studied in [6]; see Example 2.7.

THEOREM 2.6. *Let $\dot{C}_t$ and V_t be as in Section* 2.1, *and let $X_t \subseteq X$ be the image of the matrix $C_\infty - C_t$. Assume that* (2.5) *holds and that there are $\dot{\lambda}_t$ (allowed to be negative) such that*

$$\frac{1}{2}\left[\dot{C}_t \operatorname{Hess} V_t(\varphi) + \operatorname{Hess} V_t(\varphi)\dot{C}_t\right] \geqslant \dot{\lambda}_t \,\mathrm{id} \quad \textit{for all } t \geqslant 0 \textit{ and all } \varphi \in X_t, \tag{2.27}$$

and define

$$\lambda_t = \int_0^t \dot{\lambda}_s \, dt, \qquad \frac{1}{\gamma} = \int_0^\infty e^{-2\lambda_s}|\dot{C}_s| ds \tag{2.28}$$

where $|\dot{C}_t|$ is the largest eigenvalue of $\dot{C}_t$. Then ν_0 satisfies the log-Sobolev inequality (2.24).

Again the proof is given in Section 2.5.

Example 2.7 (Hierarchical models). Let $C_j = \mu_j Q_j$ be the decomposition of the hierarchical Green function as in [6, sec. 2.1] (where we here write μ_j instead of λ_j) and set $\dot{C}_t = \sum_j 1_{(j,j+1]}(t)C_j$ and $\dot{Q}_t = \sum_j 1_{(j,j+1]}(t)Q_j$. Using the structure of the hierarchical decomposition, for $\varphi \in X_t$, the matrix Hess $V_t(\varphi)$ is block diagonal with respect to scale-j blocks (see [6, sec. 1.3]) where $t \in (j, j+1]$ and constant on each such block. This means that Hess $V_t(\varphi)$ commutes with Q_t and by the hierarchical structure thus with $\dot{C}_t$. In particular, for $\varphi \in X_t$,

$$\dot{C}_t^{1/2} \operatorname{Hess} V_t(\varphi) \dot{C}_t^{1/2} \geqslant \dot{\lambda}_t \operatorname{id} \tag{2.29}$$

implies (2.27). For hierarchical versions of the four-dimensional lattice $|\varphi|^4$ model in the approach of the critical point, and for the two-dimensional lattice sine-Gordon model in the rough (Kosterlitz–Thouless) phase, we established the estimate (2.29) for integer t (and appropriate $\dot{\lambda}_t$) in [6]. By the same methods, one can extend those estimates to noninteger t with $-\dot{\lambda}_t = O(-\dot{\lambda}_j)$ for $t \in (j, j+1]$. Using Theorem 2.6 instead of [6, theorem 2.1], the theorems for the spectral gap in [6] can thus be extended to analogous ones for the log-Sobolev constant.

Further variations of the conditions (2.22) and (2.27) for the log-Sobolev inequality are possible and might be useful in other applications, but we do not investigate these here.

2.4 Proof of Proposition 2.1

We start with the proof of Proposition 2.1. This is a straightforward computation from the definitions.

PROOF OF PROPOSITION 2.1. Let $Z_t(\varphi) = \boldsymbol{E}_{C_t} e^{-V_0(\varphi+\zeta)}$. By a well-known computation (see, e.g., [7, sec. 2]), it follows that the Gaussian convolution acts as the heat semigroup with time-dependent generator $\frac{1}{2}\Delta_{\dot{C}_t}$, i.e., if Z_0 is C^2 in φ, so is Z_t for any $t > 0$, that $Z_t(\varphi) > 0$ for any t and φ, and that for any $t > 0$ such that C_t is differentiable,

$$\frac{\partial}{\partial t} Z_t = \frac{1}{2}\Delta_{\dot{C}_t} Z_t, \quad Z_0 = e^{-V_0}. \tag{2.30}$$

Therefore $V_t = -\log Z_t$ satisfies the Polchinski equation

$$\begin{aligned} \frac{\partial}{\partial t} V_t = -\frac{\frac{\partial}{\partial t} Z_t}{Z_t} = -\frac{\Delta_{\dot{C}_t} Z_t}{2Z_t} &= -\frac{1}{2} e^{V_t} \Delta_{\dot{C}_t} e^{-V_t} \\ &= \frac{1}{2}\Delta_{\dot{C}_t} V_t - \frac{1}{2}(\nabla V_t)^2_{\dot{C}_t}. \end{aligned} \tag{2.31}$$

That $(\boldsymbol{P}_{s,t})$ is a semigroup, i.e., that $\boldsymbol{P}_{r,t}\boldsymbol{P}_{s,r} = \boldsymbol{P}_{s,t}$ and $\boldsymbol{P}_{t,t} =$ id for any $s \leqslant r \leqslant t$, follows immediately from the definition (1.4) and the convolution property of Gaussian measures, i.e., that the sum of two independent Gaussian vectors is Gaussian with covariance given by the sum of the covariances (again see, e.g., [7, sec. 2]). The Markov property is obvious. To verify that its generator $\boldsymbol{L}_t$ is given by (2.8), set $F_t(\varphi) = \boldsymbol{P}_{0,t}F(\varphi) = e^{V_t(\varphi)}\boldsymbol{E}_{C_t}(e^{-V_0(\varphi+\zeta)}F(\varphi+\zeta))$.

Then

$$\begin{aligned}
\frac{\partial}{\partial t}F_t &= \left(\frac{\partial}{\partial t}V_t\right)F_t + e^{V_t}\frac{1}{2}\Delta_{\dot C_t}\mathbb{E}_{C_t}(e^{-V_0(\cdot+\zeta)}F(\cdot+\zeta)) \\
&= \left(\frac{\partial}{\partial t}V_t\right)F_t + e^{V_t}\frac{1}{2}\Delta_{\dot C_t}(e^{-V_t}F_t) \\
&= \left(\frac{\partial}{\partial t}V_t\right)F_t - \left(\frac{1}{2}\Delta_{\dot C_t}V_t\right)F_t + \frac{1}{2}(\nabla V_t)^2_{\dot C_t}F_t \\
&\quad + \frac{1}{2}\Delta_{\dot C_t}F_t - (\nabla V_t, \nabla F_t)_{\dot C_t} \\
&= \frac{1}{2}\Delta_{\dot C_t}F_t - (\nabla V_t, \nabla F_t)_{\dot C_t} \\
&= \boldsymbol{L}_t F_t, \qquad (2.32)
\end{aligned}$$

which is the second equality in (2.7). The third inequality in (2.7) follows analogously, and the first inequality is clear from the fact that the Gaussian measure with covariance 0 is the Dirac measure at 0.

The first equality in (2.9) holds by definition, and the second one is a direct computation from the definition (1.3) and the fact that V satisfies (1.10):

$$\begin{aligned}
-\frac{\partial}{\partial t}\mathbb{E}_{\nu_t}F &= \mathbb{E}_{\nu_t}\Bigg(\left(\frac{\partial}{\partial t}V_t\right)F - \frac{1}{2}(\Delta_{\dot C_t}V_t)F + \frac{1}{2}(\nabla V_t)^2_{\dot C_t}F \\
&\qquad + \frac{1}{2}\Delta_{\dot C_t}F - (\nabla V_t, \nabla F)_{\dot C_t}\Bigg) \\
&= \mathbb{E}_{\nu_t}\left(\frac{1}{2}\Delta_{\dot C_t}F - (\nabla V_t, \nabla F)_{\dot C_t}\right) = \mathbb{E}_{\nu_t}\boldsymbol{L}_t F. \qquad (2.33)
\end{aligned}$$

Finally, (2.10) follows from (2.5). Indeed, if F takes values in a compact interval $I \subset (0,\infty)$, then $\boldsymbol{P}_{0,t}F$ also takes values in I. The function Φ is smooth on I and can be extended to a bounded smooth function g on $\mathbb{R}$ such that $g|_I = \Phi|_I$. The claim now follows from (2.5). □

2.5 Proofs of Theorems 2.5–2.6

Theorems 2.5–2.6 can be proved in the same way as the Bakry-Émery criterion with the crucial difference that the original semigroup is replaced by the Polchinski semigroup, that the corresponding potentials depend on time, and that gradients are taken in terms of a time-dependent quadratic form. We present the primary proofs along the lines of [4]; see Remark 2.9 for alternative proofs using synchronous coupling as in [15].

LEMMA 2.8. *Let $\boldsymbol{L}_t$, $\boldsymbol{P}_{0,t}$, $\dot{C}_t$, V_t be as in Section* 2.1. *Then the following identity holds for any t-independent positive definite matrix Q:*

$$(\boldsymbol{L}_t - \partial_t)(\nabla\sqrt{\boldsymbol{P}_{0,t}F})_Q^2 = 2(\nabla\sqrt{\boldsymbol{P}_{0,t}F}, \operatorname{Hess} V_t \dot{C}_t \nabla\sqrt{\boldsymbol{P}_{0,t}F})_Q + \frac{1}{4}(\boldsymbol{P}_{0,t}F)\big|\dot{C}_t^{1/2}(\operatorname{Hess}\log \boldsymbol{P}_{0,t}F)Q^{1/2}\big|_2^2, \tag{2.34}$$

where $|M|_2^2 = \sum_{p,q}|M_{pq}|^2$ denotes the squared Frobenius norm of a matrix $M = (M_{pq})$.

PROOF. Throughout the proof, we drop the fixed index t, i.e., write F instead of $\boldsymbol{P}_{0,t}F$, and $\boldsymbol{L}$ for $\boldsymbol{L}_t$, and similarly for $\dot{C}_t$ and V_t. Then the left-hand side of (2.34) can be written as

$$\frac{1}{2}\left[\boldsymbol{L}\frac{(\nabla F)_Q^2}{2F} - \frac{(\nabla \boldsymbol{L}F, \nabla F)_Q}{F} + \frac{(\nabla F)_Q^2}{2F^2}\boldsymbol{L}F\right]. \tag{2.35}$$

To compute the three terms, we denote derivatives by subscripts i, j, k, and l, and use the summation convention for these subscripts. The first term then is

$$\begin{aligned}\boldsymbol{L}\frac{(\nabla F)_Q^2}{2F} &= \frac{1}{2}\dot{C}_{ij}Q_{kl}\left[\left(\frac{F_kF_l}{2F}\right)_{ij} - 2V_i\left(\frac{F_kF_l}{2F}\right)_j\right] \\ &= \frac{1}{2}\dot{C}_{ij}Q_{kl}\left[\left(\frac{F_{ik}F_l}{F} - \frac{F_kF_lF_i}{2F^2}\right)_j - 2V_i\left(\frac{F_kF_l}{2F}\right)_j\right]\end{aligned} \tag{2.36}$$

where the last bracket can be expanded as

$$\begin{aligned}&\left[\frac{F_{ijk}F_l + F_{ik}F_{jl}}{F} - \frac{F_{ik}F_lF_j}{F^2} - \frac{2F_{kj}F_lF_i + F_kF_lF_{ij}}{2F^2}\right. \\ &\qquad \left. + \frac{F_kF_lF_iF_j}{F^3} - 2V_i\left(\frac{F_{jk}F_l}{F} - \frac{F_kF_lF_j}{2F^2}\right)\right].\end{aligned} \tag{2.37}$$

The sum of the second and third terms in (2.35) is

$$\begin{aligned}&-\frac{(\nabla \boldsymbol{L}F, \nabla F)_Q}{F} + \frac{(\nabla F)_Q^2}{2F^2}\boldsymbol{L}F \\ &= \frac{1}{2}\dot{C}_{ij}Q_{kl}\left[\frac{-(F_{kij} - 2V_iF_{kj} - 2V_{ik}F_j)F_l}{F} + \frac{(F_{ij} - 2V_iF_j)F_kF_l}{2F^2}\right] \\ &= \frac{1}{2}\dot{C}_{ij}Q_{kl}\left[2V_{ik}\frac{F_jF_l}{F} - \frac{F_{kij}F_l}{F} + \frac{F_{ij}F_kF_l}{2F^2}\right. \\ &\qquad \left. + 2V_i\left(\frac{F_{kj}F_l}{F} - \frac{F_jF_kF_l}{2F^2}\right)\right].\end{aligned}$$

By adding all three terms, we obtain that (2.35) equals

$$\begin{aligned}&\frac{1}{2}\dot{C}_{ij}Q_{kl}\frac{V_{ik}F_jF_l}{F}\\ &\quad+\frac{1}{4}\dot{C}_{ij}Q_{kl}\left[\frac{F_{ik}F_{jl}}{F}-\frac{F_{ik}F_lF_j+F_{jl}F_iF_k}{F^2}+\frac{F_kF_lF_iF_j}{F^3}\right].\end{aligned}\tag{2.38}$$

Using that for any given indices i, j, k, l,

$$\begin{aligned}(\log F)_{ik}&=\left(\frac{F_i}{F}\right)_k=\frac{F_{ik}}{F}-\frac{F_iF_k}{F^2},\\ (\log F)_{jk}&=\left(\frac{F_j}{F}\right)_l=\frac{F_{jl}}{F}-\frac{F_jF_l}{F^2},\end{aligned}\tag{2.39}$$

equation (2.38) can be written as

$$\frac{1}{2}\dot{C}_{ij}Q_{kl}\frac{V_{ki}F_jF_l}{F}+\frac{1}{4}F\dot{C}_{ij}Q_{kl}(\log F)_{ik}(\log F)_{jl}.\tag{2.40}$$

Using that $2(\sqrt{F})_j = F_j/\sqrt{F}$ for the first term, and that, for any symmetric matrix M,

$$\begin{aligned}\dot{C}_{ij}Q_{kl}M_{ik}M_{jl}&=\dot{C}_{ip}^{1/2}\dot{C}_{jp}^{1/2}Q_{kq}^{1/2}Q_{lq}^{1/2}M_{ik}M_{jl}\\ &=\dot{C}_{ip}^{1/2}\dot{C}_{jp}^{1/2}(MQ^{1/2})_{iq}(MQ^{1/2})_{jq}\\ &=(\dot{C}^{1/2}MQ^{1/2})_{pq}(\dot{C}^{1/2}MQ^{1/2})_{pq}\end{aligned}\tag{2.41}$$

for the second term, (2.40) can therefore be written as

$$2(\nabla\sqrt{F},\operatorname{Hess}V\dot{C}\nabla\sqrt{F})_Q+\frac{1}{4}F|\dot{C}^{1/2}(\operatorname{Hess}\log F)Q^{1/2}|_2^2.\tag{2.42}$$

□

PROOF OF THEOREM 2.5. Lemma 2.8 with $Q=\dot{C}_s$ implies

$$\begin{aligned}&(\boldsymbol{L}_s-\partial_s)(\nabla\sqrt{\boldsymbol{P}_{0,s}F})^2_{\dot{C}_s}\\ &\quad=2(\nabla\sqrt{\boldsymbol{P}_{0,s}F},\operatorname{Hess}V_s\dot{C}_s\nabla\sqrt{\boldsymbol{P}_{0,s}F})_{\dot{C}_s}-(\nabla\sqrt{\boldsymbol{P}_{0,s}F})^2_{\ddot{C}_s}\\ &\qquad+\frac{1}{4}(\boldsymbol{P}_{0,s}F)|\dot{C}_s^{1/2}(\operatorname{Hess}\log\boldsymbol{P}_{0,s}F)\dot{C}_s^{1/2}|_2^2.\end{aligned}\tag{2.43}$$

By the assumption (2.22) and since the last term is positive, it follows that

$$(\boldsymbol{L}_s-\partial_s)(\nabla\sqrt{\boldsymbol{P}_{0,s}F})^2_{\dot{C}_s}\geqslant 2\dot{\lambda}_s(\nabla\sqrt{\boldsymbol{P}_{0,s}F})^2_{\dot{C}_s}.\tag{2.44}$$

Equivalently,

$$\psi(s):=e^{-2\lambda_t+2\lambda_s}\boldsymbol{P}_{s,t}\big[(\nabla\sqrt{\boldsymbol{P}_{0,s}F})^2_{\dot{C}_s}\big]\tag{2.45}$$

satisfies $\psi'(s)\leqslant 0$ for $s<t$. This implies

$$\begin{aligned}(\nabla\sqrt{\boldsymbol{P}_{0,t}F})^2_{\dot{C}_t}=\psi(t)\leqslant\psi(0)&=e^{-2\lambda_t}\boldsymbol{P}_{0,t}\big[(\nabla\sqrt{F})^2_{\dot{C}_0}\big]\\ &\leqslant|\dot{C}_0|\,e^{-2\lambda_t}\boldsymbol{P}_{0,t}[(\nabla\sqrt{F})^2].\end{aligned}\tag{2.46}$$

By (2.17), then (2.24) follows. □

PROOF OF THEOREM 2.6. Lemma 2.8 with $Q = \mathrm{id}$ implies

$$(\boldsymbol{L}_s - \partial_s)(\nabla\sqrt{\boldsymbol{P}_{0,s}F})^2 = 2(\nabla\sqrt{\boldsymbol{P}_{0,s}F}, \mathrm{Hess}\, V_s \dot{C}_s \nabla\sqrt{\boldsymbol{P}_{0,s}F}) + \frac{1}{4}(\boldsymbol{P}_{0,s}F)|\dot{C}_s^{1/2}(\mathrm{Hess}\log \boldsymbol{P}_{0,s}F)|_2^2. \tag{2.47}$$

By the assumption (2.27) and since the last term is positive, it follows that, on X_s,

$$(\boldsymbol{L}_s - \partial_s)(\nabla\sqrt{\boldsymbol{P}_{0,s}F})^2 \geqslant 2\dot{\lambda}_s(\nabla\sqrt{\boldsymbol{P}_{0,s}F})^2. \tag{2.48}$$

Equivalently, pointwise on X_t, $\psi(s) := e^{-2\lambda_t+2\lambda_s}\boldsymbol{P}_{s,t}[(\nabla\sqrt{\boldsymbol{P}_{0,s}F})^2]$ satisfies $\psi'(s) \leqslant 0$ for $s < t$. This implies, on X_t,

$$(\nabla\sqrt{\boldsymbol{P}_{0,t}F})^2_{\dot{C}_t} \leqslant |\dot{C}_t|(\nabla\sqrt{\boldsymbol{P}_{0,t}F})^2 = |\dot{C}_t|\psi(t) \leqslant |\dot{C}_t|\psi(0) = |\dot{C}_t|e^{-2\lambda_t}\boldsymbol{P}_{0,t}[(\nabla\sqrt{F})^2]. \tag{2.49}$$

Again by (2.17), using that ν_t is supported on X_t, (2.24) follows. □

Remark 2.9. Using the representation (2.12)–(2.13) of the semigroup $\boldsymbol{P}_{s,t}$ in terms of a stochastic process (that evolves backwards in time from t to s), one can alternatively prove the theorems using synchronous coupling as in [15].

3 Application to the Continuum Sine-Gordon Model

In this section, we prove Theorems 1.6 and 1.7 by applying Theorem 1.2. While it is not necessary, we find it clearest to rescale the continuum sine-Gordon model at scale ε to a unit lattice problem.

3.1 Rescaling and Heat Kernel Decomposition

Identifying $\Omega_{\varepsilon,L}$ with the unit lattice $\Lambda = \frac{1}{\varepsilon}\Omega_{\varepsilon,L}$, the continuum sine-Gordon model $\nu_{\varepsilon,L}$ is equivalent to a spin system whose coupling matrix is given by the nearest-neighbour Laplacian on $\mathbb{Z}^d$. We will thus drop the subscripts ε and L now, and write ν_0 for the measure of the form (1.1) with $X = \mathbb{R}^\Lambda$ and

$$A = -\Delta_\Lambda + \varepsilon^2 m^2, \quad V_0(\varphi) = \sum_{x\in\Lambda} z\varepsilon^{2-\beta/4\pi}\cos\big(\sqrt{\beta}\varphi_x\big), \tag{3.1}$$

where Δ_Λ is the standard unit lattice Laplacian acting on the discrete torus of side length L/ε. We emphasise that throughout this section Δ_Λ denotes the lattice Laplacian on Λ and not the Laplacian on $\mathbb{R}^\Lambda$, which we denoted $\Delta_{\dot{C}_t}$ in the previous section. Note that φ is not rescaled. As is natural in this normalisation, we normalise the Glauber Dirichlet form, for $F : \mathbb{R}^\Lambda \to \mathbb{R}$, by

$$\sum_{x\in\Lambda} \mathbb{E}_{\nu_0}\bigg[\bigg(\frac{\partial F}{\partial\varphi_x}\bigg)^2\bigg]. \tag{3.2}$$

We note that in this normalisation the log-Sobolev constant of the noninteracting (Gaussian) model with $z = 0$ scales as $\varepsilon^2 m^2$ (corresponding to the unit order log-Sobolev constant $m^2 > 0$ in the continuum scaling). Also note that the correlation length of the noninteracting model scales as $1/(m\varepsilon)$, making it natural to assume $L \geqslant 1/m$ as in the statements of the theorems.

In the following, we will use Theorem 1.2 to prove the same scaling in ε for the log-Sobolev constant of the interacting model. To verify the assumptions of Theorem 1.2, we will prove the following estimates on V_t as defined in (1.3). We recall that $Q_t = e^{-tA/2}$ denotes the heat kernel on the index space Λ.

PROPOSITION 3.1. *Let $\beta < 6\pi$, and $L > 0$, $m > 0$, and $z \in \mathbb{R}$. Then* (1.6) *holds, and for all $t \geqslant 0$,*

$$Q_t \operatorname{Hess} V_t(\varphi) Q_t \geqslant \dot{\mu}_t \operatorname{id}, \tag{3.3}$$

where $\mu_t = \int_0^t \dot{\mu}_s \, ds$ satisfies

$$|\mu_t| \leqslant \mu^* \tag{3.4}$$

with $\mu^* = \mu^*(\beta, z, m, L)$ independent of $\varepsilon > 0$. Moreover, there is $\delta_\beta > 0$ such that if

$$Lm \geqslant 1 \quad \text{and} \quad |z| m^{-2+\beta/4\pi} \leqslant \delta_\beta, \tag{3.5}$$

then the optimal bound satisfies $\mu^* = O_\beta(|z| m^{-2+\beta/4\pi})$ uniformly in L.

Indeed, Theorem 1.6 is an immediate consequence of these estimates and Theorem 1.2.

PROOF OF THEOREM 1.6. The smallest eigenvalue of A is $\lambda = \varepsilon^2 m^2$. By (1.9) and (3.4), therefore

$$\frac{1}{\gamma} = \int_0^\infty e^{-\lambda t - 2\mu_t} \, dt \leqslant e^{2\mu^*} \int_0^\infty e^{-\lambda t} \, dt = \frac{e^{2\mu^*}}{\lambda} = \frac{e^{2\mu^*}}{\varepsilon^2 m^2}, \tag{3.6}$$

and Theorem 1.2 implies that ν_0 satisfies a log-Sobolev inequality with constant γ. In the continuum normalisation of the Dirichlet form (1.17), the sine-Gordon measure thus satisfies a log-Sobolev inequality with constant given by $m^2 e^{-2\mu^*}$. Moreover, if (3.5) holds, then $m^2 e^{-2\mu^*} = m^2 + O_\beta(m^{\beta/4\pi}|z|)$. $\square$

The proof of Theorem 1.7 for Kawasaki dynamics is almost the same as that of Theorem 1.6. The constrained measure ν_0^0 can be written as in (2.1), with the degenerate covariance matrix C_∞^0 supported on the subspace $X = \mathbb{R}_0^\Lambda = \{\varphi \in \mathbb{R}^\Lambda : \sum_x \varphi_x = 0\}$ given by

$$C_\infty^0 = PA^{-1}P \quad \text{where } P\varphi_x = \varphi_x - \frac{1}{|\Lambda|} \sum_{y \in \Lambda} \varphi_y. \tag{3.7}$$

In unit lattice scaling, the Dirichlet form for Kawasaki dynamics is given, for $F : \mathbb{R}_0^\Lambda \to \mathbb{R}$, by

$$\sum_{x \sim y \in \Lambda} \mathbb{E}_{\nu_0^0}\left[\left(\frac{\partial F}{\partial \varphi_x} - \frac{\partial F}{\partial \varphi_y}\right)^2\right]. \tag{3.8}$$

We decompose the covariance matrix C_∞^0 in terms of

$$\dot{C}_t^0 = e^{-tA} P, \qquad Q_t^0 = e^{-tA/2} P, \tag{3.9}$$

and define V_t^0 as in (1.3) with respect to $\dot{C}_t^0$. From now on, we will refer to the case that V_t is replaced by V_t^0 and $\dot{C}_t$ by $\dot{C}_t^0$ as the *conservative case*. Then the statement of Proposition 3.1 remains true in the conservative case.

PROPOSITION 3.2. *Let $\beta < 6\pi$ and $L > 0$, $m > 0$, and $z \in \mathbb{R}$. Then* (1.6) *holds, and for all $t \geqslant 0$,*

$$Q_t^0 \operatorname{Hess} V_t^0(\varphi) Q_t^0 \geqslant \dot{\mu}_t P, \tag{3.10}$$

where μ_t satisfies (3.4) *with the same bound on μ^* if* (3.5) *holds.*

Analogously as in the proof of Theorem 1.6, we deduce Theorem 1.7 from Proposition 3.2.

PROOF OF THEOREM 1.7. Since Λ is a discrete torus of side length L/ε, the smallest nonzero eigenvalue of the lattice Laplacian $-\Delta_\Lambda$ on Λ is of order $(\varepsilon/L)^2$. We thus denote the smallest nonzero eigenvalue of $-\Delta_\Lambda$ on Λ by $\zeta^2\varepsilon^2$. Explicitly, as $\varepsilon \to 0$,

$$\zeta^2 \to \left(\frac{2\pi}{L}\right)^2. \tag{3.11}$$

As in the proof of Theorem 1.6, with λ the smallest eigenvalue on X of $A = -\Delta_\Lambda + \varepsilon^2 m^2$,

$$\frac{1}{\gamma^0} \leqslant \frac{e^{2\mu^*}}{\lambda} = \frac{e^{2\mu^*}}{\varepsilon^2(\zeta^2 + m^2)}, \tag{3.12}$$

and Theorem 1.2 implies that ν_0^0 satisfies a log-Sobolev inequality with constant γ^0:

$$\begin{aligned} \operatorname{Ent}_{\nu_0^0}(F) &\leqslant \frac{e^{2\mu^*}}{\varepsilon^2(m^2 + \zeta^2)} \mathbb{E}_{\nu_0^0}(\nabla F, P \nabla F) \\ &\leqslant \frac{e^{2\mu^*}}{\varepsilon^4 \zeta^2 (m^2 + \zeta^2)} \mathbb{E}_{\nu_0^0}(\nabla F, -\Delta_\Lambda P \nabla F) \end{aligned} \tag{3.13}$$

where the last inequality again uses that the smallest nonzero eigenvalue of the lattice Laplacian $-\Delta$ is $\varepsilon^2\zeta^2$. We emphasise that ∇ denotes the continuous gradient on $\mathbb{R}^\Lambda$ while Δ_Λ is the lattice Laplacian on Λ. Recalling the continuum normalisation of the Dirichlet form given by (1.21) and (3.4), this is the claim of Theorem 1.7. □

3.2 Outline, Scaling Conventions, and Heat Kernel

To prove Propositions 3.1 and 3.2, we proceed in the following steps. We first consider the main case (3.5). The proofs are simpler for $\beta < 4\pi$, and we begin with this case in Section 3.4. In Sections 3.5-3.7, we extend this analysis to the case $\beta < 6\pi$. Finally, in Section 3.8, we show that a crude argument suffices to remove the assumption (3.5) at the cost of constants that are uniform in ε but not in L.

To prove Propositions 3.1–3.2, we will require estimates on the heat kernel decomposition

$$C_t = \int_0^t \dot{C}_s \, ds, \quad \dot{C}_s = Q_s^2 = e^{-sA}. \tag{3.14}$$

In this section, we set up a convenient normalisation and also collect some elementary estimates. We have chosen the heat kernel decomposition (and not a finite range decomposition, for example) to be able to directly apply Theorem 1.2. The *characteristic length scale* of the heat kernel is defined by

$$\ell_t = (1 \vee \sqrt{t}) \wedge \frac{1}{\varepsilon m}, \tag{3.15}$$

and we set

$$\mathsf{Q}_t = \ell_t Q_t, \qquad \dot{\mathsf{C}}_t = \ell_t^2 \dot{C}_t, \qquad \vartheta_t = e^{-\frac{1}{2}m^2\varepsilon^2 t}. \tag{3.16}$$

Standard estimates on the heat kernel imply that $\dot{C}_t(x,y)$ is essentially supported on $|x-y| \lesssim \ell_t$, and the above normalisation is such that $\dot{\mathsf{C}}_{\lambda^2 t}(\lambda x, \lambda y) \approx \dot{\mathsf{C}}_t(x,y)$ and $\mathsf{Q}_t^2 = \dot{\mathsf{C}}_t$. We will often express estimates in terms of these quantities and in terms of ℓ_t (instead of t), and write integrals over the scale in terms of the approximately scale-invariant measure $dt/\ell_t^2 \approx dt/t$ (instead of dt). For estimates involving the heat kernels Q_t, $\dot{C}_t$, C_t, and its scaled versions, we will always impose the following assumption:

$$Lm \geqslant 1 \quad \text{or} \quad t \leqslant \frac{1}{\varepsilon^2}\left(\frac{1}{m^2} \wedge L^2\right). \tag{3.17}$$

The next lemma provides some elementary estimates on the heat kernel. These are sufficient for the case $\beta < 4\pi$; for $\beta > 4\pi$ more precise estimates are required (and will be stated in the section in which they are used). All of these estimates on the heat kernel are collected in Appendix 3.8.

LEMMA 3.3. *Assume* (3.17). *For any $x \in \Lambda$,*

$$C_t(x,x) = \frac{1}{2\pi} \log \ell_t + O(1), \qquad \sup_x \sum_y |\dot{\mathsf{C}}_t(x,y)| = O(\ell_t^2 \vartheta_t^2), \tag{3.18}$$

and the same estimates hold in the conservative case.

PROOF. This follows from standard estimates on the heat kernel on $\mathbb{Z}^2$; see Appendix 3.8. □

Further, we define the *scale-dependent coupling constant* z_t and its microscopic version z_t by

$$\mathsf{z}_t = \ell_t^2 z_t, \quad z_t = e^{-\frac{\beta}{2}C_t(0,0)} z_0, \quad \text{where } z_0 = \varepsilon^{2-\beta/4\pi} z. \tag{3.19}$$

For later purposes, we will now collect some basic properties of this definition. By (3.18) and the definitions of z_t and ℓ_t, uniformly in $t > 0$,

$$\mathsf{z}_t = O_\beta(|z|(\varepsilon\ell_t)^{2-\beta/4\pi}) = O_\beta(|z|m^{-2+\beta/4\pi}). \tag{3.20}$$

In the following, we write $x \lesssim y$ or $x = O_\beta(y)$ if $|x| \leqslant C_\beta |y|$ for a β-dependent constant C_β. For any $\beta < 8\pi$, by (3.20) then

$$\int_0^t |\mathsf{z}_s| \vartheta_s^2 \frac{ds}{\ell_s^2} \lesssim |\mathsf{z}_t|, \tag{3.21}$$

as is straightforward to check from the definitions. For use in the proof for $\beta > 4\pi$, we also record the following estimates (again straightforward from the definitions): for all positive integers n,

$$\int_0^t |z_s|^n \ell_s^{2(n-1)} \vartheta_s^2 \frac{ds}{\ell_s^2} \lesssim \frac{1}{n} |z_t|^n (C_\beta \ell_t^2)^{n-1} \quad \text{for } \beta < 8\pi(1 - 1/n), \tag{3.22}$$

$$\int_0^t |z_s|^n \ell_s^{2(n-1)} \ell_s^{\beta/4\pi} \vartheta_s^2 \frac{ds}{\ell_s^2} \lesssim \frac{1}{n} |z_t|^n (C_\beta \ell_t^2)^{n-1} \ell_t^{\beta/4\pi} \quad \text{for } \beta < 8\pi. \tag{3.23}$$

3.3 Fourier Representation

To estimate the Hessian of the renormalised potential V_t, we use the Brydges–Kennedy approach [14]. Namely, for any function $V : \mathbb{R}^\Lambda \to \mathbb{R}$ that is $\frac{2\pi}{\sqrt{\beta}}$-periodic in each variable, we will write its Fourier series (assuming it converges absolutely) as

$$\begin{aligned} V(\varphi) &= \sum_{n=0}^\infty V^{(n)}(\varphi), \\ V^{(n)}(\varphi) &= \frac{1}{n!} \sum_{\xi_1,\dots,\xi_n} \tilde{V}^{(n)}(\xi_1,\dots,\xi_n) e^{i\sqrt{\beta}\sum_{k=1}^n \varphi_{x_k}\sigma_k}, \end{aligned} \tag{3.24}$$

where $\tilde{V}^{(n)} : (\Lambda \times \{\pm 1\})^n \to \mathbb{R}$ and

$$\xi_i = (x_i, \sigma_i) \in \Lambda \times \{\pm 1\}. \tag{3.25}$$

We think of ξ_i as a particle with position x_i and charge σ_i. Since the index n is determined from the number of arguments of $\tilde{V}^{(n)}$, we will often omit it and write $\tilde{V}(\xi_1,\dots,\xi_n) = \tilde{V}^{(n)}(\xi_1,\dots,\xi_n)$. The representation (3.24) is *not* manifestly unique without further conditions, but in the relevant cases we will in fact construct coefficients $\tilde{V}(\xi_1,\dots,\xi_n)$ such that (3.24) holds.

The initial potential V_0 of the sine-Gordon model corresponds to

$$\tilde{V}_0(\varnothing) = 0, \quad \tilde{V}_0(\xi_1) = z_0, \quad \tilde{V}_0(\xi_1,\dots,\xi_n) = 0 \ \ (n > 1). \tag{3.26}$$

Set

$$\dot{u}_s(\xi_i, \xi_j) = \beta \dot{C}_s(x_i, x_j)\sigma_i\sigma_j,$$
$$\dot{\mathsf{u}}_s(\xi_i, \xi_j) = \ell_s^2 \dot{u}_s(\xi_i, \xi_j) = \beta \dot{\mathsf{C}}_s(x_i, x_j)\sigma_i\sigma_j, \tag{3.27}$$

and

$$\dot{W}_s(\xi_1, \dots, \xi_n) = \frac{1}{2}\sum_{k,l\in[n]} \dot{u}_s(\xi_k, \xi_l), \tag{3.28}$$

where $[n] = \{1, \dots, n\}$. We define u_s and W_s analogously by replacing $\dot{C}_s$ by C_s. For later use, we note that $W_t - W_s \geqslant 0$ holds for all arguments by positive definiteness of $\dot{C}_s$.

Then in terms of the Fourier representation (3.24), the two terms on the right-hand side of the Polchinski equation (1.10) are represented by

$$\begin{aligned}\frac{1}{2}\widetilde{(\Delta_{\dot{C}_s} V)}(\xi_1, \dots, \xi_n) &= -\frac{1}{2}\sum_{i,j\in[n]} \dot{u}_s(\xi_i, \xi_j)\tilde{V}(\xi_1, \dots, \xi_n)\\ &= -\dot{W}_s(\xi_1, \dots, \xi_n)\tilde{V}(\xi_1, \dots, \xi_n)\end{aligned} \tag{3.29}$$

and

$$\begin{aligned}&\frac{1}{2}\widetilde{(\nabla V, \nabla V)}_{\dot{C}_s}(\xi_1, \dots, \xi_n)\\ &\quad= -\frac{1}{2}\sum_{I_1\dot{\cup}I_2=[n]} \tilde{V}(\xi_{I_1})\tilde{V}(\xi_{I_2})\sum_{i\in I_1, j\in I_2} \dot{u}_s(\xi_i, \xi_j).\end{aligned} \tag{3.30}$$

The sum over $I_1\dot{\cup}I_2 = [n]$ is over all nonempty disjoint subsets I_1 and I_2 of $[n]$ with $I_1 \cup I_2 = [n]$. Moreover, given $\xi_1, \dots, \xi_n$ and $I = \{i_1, \dots, i_k\} \subset [n]$, we denote by ξ_I the vector $(\xi_{i_1}, \dots, \xi_{i_k})$.

Indeed, (3.29) is straightforward to verify in the sense that if V is given by (3.24) and $\widetilde{\Delta_{\dot{C}_s} V}$ by (3.29), then

$$\Delta_{\dot{C}_s} V(\varphi) = \sum_n \frac{1}{n!}\sum_{\xi_1,\dots,\xi_n} \widetilde{(\Delta_{\dot{C}_s} V)}(\xi_1, \dots, \xi_n)e^{i\sqrt{\beta}\sum_{k=1}^n \varphi_{x_k}\sigma_k}. \tag{3.31}$$

To see (3.30), note that differentiating (3.24) gives

$$\begin{aligned}&\frac{\partial}{\partial\varphi_x} V^{(p)}(\varphi)\\ &\quad= \frac{1}{p!}\sum_{\xi_1,\dots,\xi_p} \tilde{V}(\xi_1, \dots, \xi_p)\sum_{k=1}^p i\sqrt{\beta}\sigma_k 1_{x=x_k} e^{i\sqrt{\beta}\sum_{k=1}^p \varphi_{x_k}\sigma_k}\end{aligned} \tag{3.32}$$

and thus

$$(3.33)\quad \begin{aligned}&(\nabla V^{(p)}, \nabla V^{(q)})_{\dot{C}_s}(\varphi)\\ &\quad= \frac{-1}{p!q!} \sum_{\xi_1,\dots,\xi_{p+q}} \widetilde{V}(\xi_1,\dots,\xi_p)\widetilde{V}(\xi_{p+1},\dots,\xi_{p+q})\\ &\qquad \sum_{i=1}^{p}\sum_{j=p+1}^{p+q} \dot{u}_s(\xi_i,\xi_j) e^{i\sqrt{\beta}\sum_{k=1}^{p+q}\varphi_{x_k}\sigma_k}.\end{aligned}$$

Therefore taking the sum over p and q, using that the number partitions of $[n]$ into two subsets with p and $q = n - p$ elements is $n!/(p!q!)$ and that $\widetilde{V}$ is symmetric in its arguments, we find

$$(3.34)\quad \begin{aligned}&(\nabla V, \nabla V)_{\dot{C}_s}(\varphi)\\ &\quad= \sum_n \frac{1}{n!} \sum_{\xi_1,\dots,\xi_n} \widetilde{(\nabla V, \nabla V)}_{\dot{C}_s}(\xi_1,\dots,\xi_n) e^{i\sqrt{\beta}\sum_{k=1}^{n}\varphi_{x_k}\sigma_k}\end{aligned}$$

if $\widetilde{(\nabla V, \nabla V)}_{\dot{C}_s}$ is given by (3.30).

By (3.29)-(3.30) and the Duhamel principle, the Polchinski equation has the following formulation as an integral equation:

$$(3.35)\quad \begin{aligned}&\widetilde{V}_t(\xi_1,\dots,\xi_n)\\ &\quad= e^{-W_t(\xi_1,\dots,\xi_n)}\widetilde{V}_0(\xi_1,\dots,\xi_n)\\ &\qquad+ \frac{1}{2}\int_0^t ds \sum_{I_1\dot{\cup} I_2=[n]} \sum_{i\in I_1, j\in I_2} \dot{u}_s(\xi_i,\xi_j)\widetilde{V}_s(\xi_{I_1})\widetilde{V}_s(\xi_{I_2})\\ &\qquad\qquad e^{-(W_t(\xi_1,\dots,\xi_n)-W_s(\xi_1,\dots,\xi_n))}.\end{aligned}$$

For $n \leqslant 1$, the unique solution to (3.35) is simply

$$(3.36)\quad \widetilde{V}_t(\varnothing) = \widetilde{V}_0(\varnothing) = 0, \quad \widetilde{V}_t(\xi_1) = e^{-\frac{1}{2}u_t(\xi_1,\xi_1)}\widetilde{V}_0(\xi_1) = z_t,$$

with z_t defined in (3.19). For $n > 1$, $\widetilde{V}_t(\xi_1,\dots,\xi_n)$ is then determined explicitly by (3.35) in terms of $\widetilde{V}_s(\xi_1,\dots,\xi_k)$, $k < n$. Hence by induction, (3.35) has a unique solution for any n and t. This is summarised in the following lemma along with a uniqueness property.

LEMMA 3.4. *The integral equation* (3.35) *has a unique solution $\widetilde{V}$ for all n and t. Moreover, if V_t defined in terms of $\widetilde{V}_t$ by* (3.24) *converges absolutely, locally uniformly in $t > 0$, then V_t is equal to* (1.3)*, the convolution solution of the Polchinski equation.*

PROOF. We have already shown that (3.35) has a unique solution. For coefficients $\widetilde{V}_t$ such that (3.24) and its derivatives converge absolutely, the function V_t defined by (3.24) is smooth. Moreover, for smooth V_t, the integral equation

(3.35) implies the Polchinski equation (1.10). Uniqueness of bounded solutions to the Polchinski equation by Remark 1.4 then implies that V_t coincides with the convolution solution of the Polchinski equation. □

3.4 Up to the First Threshold: Proof of Propositions 3.1–3.2 for $\beta < 4\pi$ Assuming (3.5)

The following proposition, due to [14], gives good bounds when $\beta < 4\pi$. For completeness, we reproduce their argument here in our setup and notation. (See also [11, 30, 31, 38, 43] for related results.) We will then use the result to derive Proposition 3.1 in the case $\beta < 4\pi$. Let

$$\|\dot{u}_s\| = \sup_{\xi_1} \sum_{\xi_2} |\dot{u}_s(\xi_1, \xi_2)| \tag{3.37}$$

and

$$\begin{aligned} \|\tilde{V}^{(1)}\| &= \sup_{\xi_1} \big|\tilde{V}(\xi_1)\big|, \\ \|\tilde{V}^{(n)}\| &= \sup_{\xi_1} \sum_{\xi_2,\dots,\xi_n} |\tilde{V}(\xi_1,\dots,\xi_n)| \quad (n > 1). \end{aligned} \tag{3.38}$$

PROPOSITION 3.5. *For all $n \geqslant 1$, the solution to* (3.35) *satisfies*

$$\big\|\tilde{V}_t^{(n)}\big\| \leqslant n^{n-2} |z_t|^n M_t^{n-1} \quad \text{where } M_t = \int_0^t ds \|\dot{u}_s\| e^{\beta(C_t - C_s)(0,0)}, \tag{3.39}$$

with z_t defined in (3.19)*. In particular, if $z_t M_t < 1/e$, the Fourier series for V_t converges and V_t coincides with the convolution solution to the Polchinski equation. The analogous statements hold in the conservative case.*

PROOF. For $n = 1$, the bound (3.39) is obvious from (3.36). To prove the bounds (3.39) for $n > 1$, we use induction. Note that the first term on the right-hand side of (3.35) does not contribute for $n > 1$ since then $\tilde{V}_0^{(n)} = 0$ by (3.26). In the second term, we drop the exponential inside the integral (as $W_t - W_s \geqslant 0$) to obtain

$$|\tilde{V}_t(\xi_1,\dots,\xi_n)| \leqslant \frac{1}{2} \int_0^t ds \sum_{I_1 \dot\cup I_2 = [n]} \sum_{i \in I_1, j \in I_2} |\dot{u}_s(\xi_i, \xi_j) \tilde{V}_s(\xi_{I_1}) \tilde{V}_s(\xi_{I_2})|. \tag{3.40}$$

Note that if $|I_1| = n - k$ and $|I_2| = k$, then

$$\sup_{\xi_1} \sum_{\xi_2,\dots,\xi_n} |\dot{u}_s(\xi_i, \xi_j) \tilde{V}_s(\xi_{I_1}) \tilde{V}_s(\xi_{I_2})| \leqslant \|\dot{u}_s\| \big\|\tilde{V}_s^{(n-k)}\big\| \big\|\tilde{V}_s^{(k)}\big\|. \tag{3.41}$$

For example,

$$\begin{aligned}
&\sup_{\xi_1} \sum_{\xi_2,\xi_3,\xi_4} \left| \dot{u}_s(\xi_1,\xi_3) \widetilde{V}_s(\xi_1,\xi_2) \widetilde{V}_s(\xi_3,\xi_4) \right| \\
&\quad \leqslant \sup_{\xi_1} \sum_{\xi_3} |\dot{u}_s(\xi_1,\xi_3)| \sup_{\xi_1} \sum_{\xi_2} |\widetilde{V}_s(\xi_1,\xi_2)| \sup_{\xi_3} \sum_{\xi_4} |\widetilde{V}_s(\xi_3,\xi_4)| \\
&\quad \leqslant \|\dot{u}_s\| \|\widetilde{V}_s^{(2)}\|^2.
\end{aligned} \tag{3.42}$$

Assuming the bound (3.39) for integers less than n, therefore

$$\begin{aligned}
\|\widetilde{V}_t^{(n)}\| &\leqslant \frac{1}{2} \int_0^t ds\, \|\dot{u}_s\| \sum_{k=1}^{n-1} \binom{n}{k} k(n-k) \|\widetilde{V}_s^{(n-k)}\| \, \|\widetilde{V}_s^{(k)}\| \\
&\leqslant \frac{1}{2} \int_0^t ds\, \|\dot{u}_s\| \sum_{k=1}^{n-1} \binom{n}{k} |z_s|^n M_s^{n-2} (n-k)^{n-k-1} k^{k-1}.
\end{aligned} \tag{3.43}$$

Using that $\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2}$ and $n/2 \leqslant n-1$ for $n \geqslant 2$,

$$\begin{aligned}
\|\widetilde{V}_t^{(n)}\| &\leqslant n^{n-2} |z_t|^n (n-1) \int_0^t ds\, \|\dot{u}_s\| e^{\frac{n}{2}\beta(C_t - C_s)(0,0)} M_s^{n-2} \\
&\leqslant n^{n-2} |z_t|^n (n-1) \int_0^t ds \|\dot{u}_s\| e^{(n-1)\beta(C_t - C_s)(0,0)} M_s^{n-2} \\
&= n^{n-2} |z_t|^n M_t^{n-1}.
\end{aligned} \tag{3.44}$$

For $n > 2$, the last equality follows from the following change of variables,

$$(n-1) \int_0^t ds\, g(s) \left(\int_0^s ds'\, g(s') \right)^{n-2} = \left(\int_0^t ds\, g(s) \right)^{n-1}, \tag{3.45}$$

applied with $g(s) = \|\dot{u}_s\| e^{-\beta C_s(0,0)}$. Indeed,

$$\begin{aligned}
&(n-1) \int_0^t ds\, \|\dot{u}_s\| e^{\beta(n-1)(C_t - C_s)(0,0)} M_s^{n-2} \\
&\quad = (n-1) e^{\beta(n-1)C_t(0,0)} \\
&\qquad \cdot \int_0^t ds\, \|\dot{u}_s\| e^{-\beta C_s(0,0)} \left(\int_0^s ds'\, \|\dot{u}_{s'}\| e^{-\beta C_{s'}(0,0)} \right)^{n-2} \\
&\quad = M_t^{n-1}.
\end{aligned} \tag{3.46}$$

Finally, using the assumption $\sup_t z_t M_t < 1/e$ and the bounds (3.39) for $\widetilde{V}_t(\xi_1, \dots, \xi_n)$, the series (3.24) for $V_t(\varphi)$ converges absolutely since (by using $n^n/n! \leqslant e^n$),

$$\begin{aligned} \frac{|V_t(\varphi)|}{|\Lambda|} &\leqslant \sum_{n=1}^{\infty} \frac{1}{n!} n^{n-2} |z_t|^n M_t^{n-1} \\ &\leqslant \sum_{n=1}^{\infty} e^n |z_t|^n M_t^{n-1} = \frac{e|z_t|}{1 - e|z_t| M_t} \leqslant C < \infty, \end{aligned} \tag{3.47}$$

and analogously for derivatives. Hence V solves the Polchinski equation (1.10) by Lemma 3.4. □

Using the conclusion of the last proposition together with the basic estimates for $\dot{\mathsf{C}}_s$ given in Lemma 3.3, it is straightforward to complete the proof of Propositions 3.1–3.2 for $\beta < 4\pi$.

PROOF OF PROPOSITIONS 3.1–3.2 FOR $\beta < 4\pi$ ASSUMING (3.5). Since the proofs of the two propositions are identical, we only discuss Proposition 3.1. From (3.18),

$$\|\dot{u}_s\| \leqslant \beta \vartheta_s^2 \sup_x \sum_y |\dot{C}_s(x, y)| \leqslant O_\beta\big(\vartheta_s^2\big). \tag{3.48}$$

For $\beta < 4\pi$, the definition of M_t in (3.39), the definition of ℓ_t in (3.15), and (3.18) imply

$$M_t \leqslant C_\beta \ell_t^{\beta/(2\pi)} \int_0^t ds\, \vartheta_s^2\, \ell_s^{-\beta/(2\pi)} = O_\beta(\ell_t^2). \tag{3.49}$$

In this proof, the condition $\beta < 4\pi$ is only needed in order to achieve the scaling ℓ_t^2 in the previous upper bound. By (3.19)–(3.20) therefore, using in the last inequality that $|z| m^{-2+\beta/4\pi}$ is sufficiently small,

$$|z_t| M_t = O_\beta(|z_t|) = O_\beta(|z| m^{-2+\beta/4\pi}) \leqslant \frac{1}{2e}. \tag{3.50}$$

Let

$$\|\mathrm{Hess}\, V_t(\varphi)\| = \sup_x \sum_y \left| \frac{\partial^2}{\partial \varphi_x \partial \varphi_y} V_t(\varphi) \right|. \tag{3.51}$$

From (3.24) together with (3.39) and (3.49), and with $n^n/n! \leqslant e^n$, we obtain

$$\begin{aligned} \|\mathrm{Hess}\, V_t(\varphi)\| &\leqslant \beta \sum_{n=1}^{\infty} \frac{1}{n!} n^2 n^{n-2} |z_t|^n M_t^{n-1} \\ &\leqslant \beta \sum_{n=1}^{\infty} e^n |z_t|^n M_t^{n-1} = \frac{\beta e |z_t|}{1 - e|z_t| M_t} \leqslant 2\beta e |z_t|. \end{aligned} \tag{3.52}$$

Since $|(f, \operatorname{Hess} V_t(\varphi) f)| \leqslant \|\operatorname{Hess} V_t(\varphi)\| |f|_2^2$ and $|Q_t f|_2 \leqslant \vartheta_t |f|_2$, we obtain

$$|(Q_t f, \operatorname{Hess} V_t(\varphi) Q_t f)| \leqslant O_\beta\big(|z_t|\vartheta_t^2\big)|f|_2^2. \tag{3.53}$$

In the notation of Theorem 1.2 we thus have that $\dot{\mu}_t \geqslant -O_\beta(|z_t|\vartheta_t^2)$. Hence, using the bounds for z_t from (3.21) and (3.20), for all $t \geqslant 0$,

$$\begin{aligned}\mu_t &\geqslant -\int_0^t O_\beta\big(|\mathsf{z}_s|\vartheta_s^2\big)\frac{ds}{\ell_s^2} \geqslant -O_\beta(|\mathsf{z}_t|) \\ &\geqslant -O_\beta(|z|m^{-2+\beta/4\pi}) \equiv -\mu^*.\end{aligned} \tag{3.54}$$

Finally, the ergodicity assumption (1.6) follows from the weak-$*$ convergence $\nu_t \to \nu_\infty \equiv \delta_0$ and $\boldsymbol{P}_{0,t}F(\varphi) \to \boldsymbol{P}_{0,\infty}F(\varphi)$ uniformly in φ. Indeed, $\nu_t \to \nu_\infty$ holds since the Gaussian measure covariance $C_\infty - C_t$ converges to δ_0 and $V_t(\varphi)$ is bounded (uniformly in φ and t). The uniform convergence $\boldsymbol{P}_{0,t}F \to \boldsymbol{P}_{0,\infty}F$ holds since $V_t(\varphi) \to V_\infty(\varphi)$ and $\boldsymbol{E}_{C_s} e^{-V_0(\varphi+\zeta)}F(\varphi+\zeta) \to \boldsymbol{E}_{C_\infty} e^{-V_0(\varphi+\zeta)}F(\varphi+\zeta)$, both uniformly in φ, where the last claim holds since the integrand is a bounded Lipschitz function. □

3.5 Up to the Second Threshold: Proof of Propositions 3.1–3.2 for $\beta < 6\pi$ Assuming (3.5)

The remainder of Section 3 is devoted to extending the proof of Proposition 3.1 from $\beta < 4\pi$ to $\beta < 6\pi$. For this, we will estimate the $n = 2, 3, 4$ terms in (3.24) more carefully.

Indeed, for $n = 2$, a uniform bound on $\widetilde{V}_t(\xi_1, \xi_2)$ as used for $\beta < 4\pi$ is not true when $\beta \geqslant 4\pi$, and we rely crucially on the smoothing effect of the heat kernel Q_t in (1.8) to obtain the required bound stated in the following proposition. (Note that this estimate is best expressed in terms of Q_t and z_t rather than Q_t and z_t.)

PROPOSITION 3.6. *Let* $\beta < 8\pi$ *and assume* (3.17). *Then*

$$\big(\mathsf{Q}_t f, \operatorname{Hess} V_t^{(2)}(\varphi)\mathsf{Q}_t f\big) = O_\beta\big(|\mathsf{z}_t|^2\vartheta_t^2\big)|f|_2^2. \tag{3.55}$$

The analogous statement holds in the conservative case.

For the terms $n > 2$, the following proposition gives an analogue of Proposition 3.5 for $\beta < 6\pi$.

PROPOSITION 3.7. *Let* $\beta < 6\pi$ *and assume* (3.17). *Then there is* $C_\beta < \infty$ *such that for all* $n \geqslant 3$,

$$\big\|\widetilde{V}_t^{(n)}\big\| \leqslant n^{n-2}|z_t|^n(C_\beta \ell_t^2)^{n-1}. \tag{3.56}$$

The analogous statement holds in the conservative case.

These bounds together imply Propositions 3.1–3.2 when (3.5) holds.

PROOF OF PROPOSITIONS 3.1–3.2 ASSUMING (3.5). Since the proofs are once again the same, we only prove Proposition 3.1. The bound (3.56) (together with the qualitative fact that $V^{(1)}$ and $V^{(2)}$ are finite) implies that (3.24) converges, exactly as in (3.47). Moreover, exactly as in (3.52)–(3.53), for $|z|m^{-2+\beta/4\pi}$ sufficiently small, it follows that

$$(\mathsf{Q}_t f, (\text{Hess } V_t(\varphi) - \text{Hess } V_t^{(2)}(\varphi))\mathsf{Q}_t f) = O_\beta(|\mathsf{z}_t|\vartheta_t^2)|f|_2^2. \tag{3.57}$$

Combined with (3.55), this gives the required bound (3.3). The proof of the ergodicity assumption (1.6) is also identical to that in the proof of Proposition 3.1 for $\beta < 4\pi$. $\square$

To prove the above propositions, *neutral* configurations require more careful treatment compared to the case $\beta < 4\pi$, where neutral means the following. For a configuration $\xi = (\xi_1, \dots, \xi_k)$ we define the *charge* $\sigma(\xi) = \sum_{i=1}^k \sigma_i$ and call ξ *neutral* if $\sigma(\xi) = 0$ and call ξ *charged* otherwise. We will sometimes decompose

$$V^{(n)}(\varphi) = V^{(n,0)}(\varphi) + V^{(n,\pm)}(\varphi)$$
$$\widetilde{V}^{(0)}(\xi) = \widetilde{V}(\xi)1_{\sigma(\xi)=0}, \quad \widetilde{V}^{(\pm)}(\xi) = \widetilde{V}(\xi)1_{\sigma(\xi)\neq 0},$$

where $V^{(n,0)}$ is defined as in (3.24) with the sum over $\xi = (\xi_1, \dots, \xi_n)$ restricted to neutral ξ, and $V^{(n,\pm)}$ by restricting the sum to charged ξ. As in the proof for $\beta < 4\pi$, the starting point for the proofs is (3.35), but now without dropping the exponential inside the integral, i.e., for $n > 1$,

$$\begin{aligned} &\widetilde{V}_t(\xi_1, \dots, \xi_n) \\ &\quad = -\frac{1}{2} \sum_{I_1 \dot\cup I_2 = [n]} \int_0^t ds \left[\sum_{i \in I_1, j \in I_2} \dot{u}_s(\xi_i, \xi_j) \widetilde{V}_s(\xi_{I_1}) \widetilde{V}_s(\xi_{I_2}) \right] e^{-(W_t(\xi) - W_s(\xi))} \\ &\quad = -\frac{1}{2} \sum_{I_1 \dot\cup I_2 = [n]} \int_0^t \frac{ds}{\ell_s^2} \left[\sum_{i \in I_1, j \in I_2} \dot{\mathsf{u}}_s(\xi_i, \xi_j) \widetilde{V}_s(\xi_{I_1}) \widetilde{V}_s(\xi_{I_2}) \right] e^{-(W_t(\xi) - W_s(\xi))}. \end{aligned} \tag{3.58}$$

3.6 Proof of Proposition 3.6: The Term $n = 2$

The following two lemmas give the explicit form of $\widetilde{V}(\xi_1, \xi_2)$ and bounds on the heat kernel that imply the required bound.

LEMMA 3.8.

$$\widetilde{V}_t(\xi_1, \xi_2) = -z_t^2(1 - e^{-\beta\sigma_1\sigma_2 C_t(x_1, x_2)}). \tag{3.59}$$

PROOF. By (3.35) and using that $V_s(\xi) = z_s = z_0 e^{-\frac{\beta}{2}C_s(0,0)}$ by (3.36),

$$\begin{aligned} \widetilde{V}_t(\xi_1, \xi_2) &= -\int_0^t ds\, \dot{u}_s(\xi_1, \xi_2) \widetilde{V}_s(\xi_1) \widetilde{V}_s(\xi_2) e^{-(W_t(\xi_1, \xi_2) - W_s(\xi_1, \xi_2))} \\ &= -z_0^2 e^{-W_t(\xi_1, \xi_2)} \int_0^t ds\, \dot{u}_s(\xi_1, \xi_2) e^{-\beta C_s(0,0)} e^{W_s(\xi_1, \xi_2)}. \end{aligned} \tag{3.60}$$

Let $\sigma = \sigma_1\sigma_2$. By (3.28), $-\beta C_s(0,0) + W_s(\xi_1,\xi_2) = \sigma\beta C_s(x_1,x_2)$, so the integral can be evaluated as

$$\begin{aligned}(3.61)\qquad &\int_0^t ds\, \dot{u}_s(\xi_1,\xi_2) e^{-\beta C_s(0,0)} e^{W_s(\xi_1,\xi_2)} \\ &\quad = \int_0^t ds\, \beta\sigma \dot{C}_s(x_1,x_2) e^{\beta\sigma C_s(x_1,x_2)} = e^{\beta\sigma C_t(x_1,x_2)} - 1,\end{aligned}$$

which after rearranging gives

$$\begin{aligned}\tilde{V}_t(\xi_1,\xi_2) &= -z_0^2 e^{-\beta C_t(0,0) - \beta\sigma C_t(x_1,x_2)} (e^{\beta\sigma C_t(x_1,x_2)} - 1) \\ (3.62)\qquad &= -z_t^2 (1 - e^{-\beta\sigma C_t(x_1,x_2)}).\end{aligned}$$

□

LEMMA 3.9. *Let $U_t(x,y) = e^{\beta C_t(x,y)} - 1$. The following bounds hold for $t \geqslant 0$, $f : \Lambda \to \mathbb{R}$, $\beta < 8\pi$:*

$$(3.63)\qquad \sup_{x_1} \sum_{x_2} |1 - e^{-\beta C_t(x_1,x_2)}| = O_\beta(\ell_t^2),$$

$$(3.64)\qquad \sum_{x_1,x_2} |U_t(x_1,x_2)| (\mathsf{Q}_t f(x_1) - \mathsf{Q}_t f(x_2))^2 = O_\beta(\ell_t^4 \vartheta_t^2) |f|_2^2,$$

and again analogous estimates hold in the conservative case.

PROOF. The lemma again follows from estimates for the heat kernel and is given in Appendix 3.8. □

PROOF OF PROPOSITION 3.6. We first consider $V^{(2,\pm)}$. By (3.59) and (3.63),

$$\begin{aligned}(3.65)\qquad \sum_y |\tilde{V}_t((x,+1),(y,+1))| &= O(|z_t|^2) \sum_y |1 - e^{-\beta C_t(x,y)}| \\ &= O(|z_t|^2 \ell_t^2),\end{aligned}$$

which is analogous to the bound for $\beta < 4\pi$ and thus gives

$$\begin{aligned}(3.66)\qquad \left|\left(\mathsf{Q}_t f, \operatorname{Hess} V_t^{(2,\pm)}(\varphi) \mathsf{Q}_t f\right)\right| &= O_\beta\left(|z_t|^2 \ell_t^4 \vartheta_t^2\right) |f|_2^2 \\ &= O_\beta\left(|\mathsf{z}_t|^2 \vartheta_t^2\right) |f|_2^2,\end{aligned}$$

exactly as in (3.53). On the other hand, the neutral contribution to $V^{(2)}$ is given by

$$\begin{aligned}(3.67)\qquad &V_t^{(2,0)}(\varphi) = z_t^2 \sum_{x,y} U_t(x,y) \cos(\sqrt{\beta}\varphi_x - \sqrt{\beta}\varphi_y), \\ &U_t(x,y) = e^{\beta C_t(x,y)} - 1.\end{aligned}$$

Therefore

$$\begin{aligned}(3.68)\qquad &\left(\mathsf{Q}_t f, \operatorname{Hess} V_t^{(2,0)}(\varphi) \mathsf{Q}_t f\right) \\ &\quad = -z_t^2 \beta \sum_{x,y} U_t(x,y) \cos\left(\sqrt{\beta}\varphi_x - \sqrt{\beta}\varphi_y\right) (\mathsf{Q}_t f(x) - \mathsf{Q}_t f(y))^2.\end{aligned}$$

By (3.64), the right-hand side is bounded by

$$O_\beta\big(|z_t|^2\ell_t^4\vartheta_t^2\big)|f|_2^2 = O_\beta\big(|\mathsf{z}_t|^2\vartheta_t^2\big)|f|_2^2. \qquad \square \tag{3.69}$$

Remark 3.10. Similarly as in (3.64), for $t > 0$, $f : \Lambda \to \mathbb{R}$, $\beta < 6\pi$, assuming (3.17), we have

$$\sum_{x_1,x_2} |U_t(x_1,x_2)||Q_t f(x_1) - Q_t f(x_2)| = O_\beta(\ell_t^2\vartheta_t)|f|_1; \tag{3.70}$$

see Appendix 3.8. Therefore, as in (3.68),

$$\begin{aligned}
&\big(Q_t f, \nabla V_t^{(2,0)}\big)\\
&\quad = -z_t^2\sqrt{\beta}\sum_{x,y} U_t(x,y)\sin(\sqrt{\beta}\varphi_x - \sqrt{\beta}\varphi_y)(Q_t f(x) - Q_t f(y))\\
&\quad = O_\beta(|z_t|^2\ell_t^2\vartheta_t)|f|_1 = O_\beta(|\mathsf{z}_t z_t|\vartheta_t)|f|_1 = O_\beta(|z_t|\vartheta_t)|f|_1,
\end{aligned} \tag{3.71}$$

provided that $\mathsf{z}_t = O(1)$. Exactly as in (3.66), the same bound holds for $V^{(2,\pm)}$, and as in (3.57) for $V - V^{(2)}$. In summary, whenever $|\mathsf{z}_t|$ is sufficiently small and (3.17) holds,

$$\max_x |(Q_t\nabla V_t)_x| = O_\beta(|z_t|\vartheta_t). \tag{3.72}$$

3.7 Proof of Proposition 3.7: The Terms $n > 2$

To bound the contributions due to (3.59), we need the following bounds on the heat kernel. For the statement of the bounds, we set

$$\delta_{12}\dot{\mathsf{C}}_t(x_1,x_2,x_3) = \dot{\mathsf{C}}_t(x_1,x_3) - \dot{\mathsf{C}}_t(x_2,x_3), \tag{3.73}$$

$$\begin{aligned}
\delta_{34}\delta_{12}\dot{\mathsf{C}}_t(x_1,x_2,x_3,x_4) &= (\dot{\mathsf{C}}_t(x_1,x_3) - \dot{\mathsf{C}}_t(x_2,x_3))\\
&\quad - (\dot{\mathsf{C}}_t(x_1,x_4) - \dot{\mathsf{C}}_t(x_2,x_4)),
\end{aligned} \tag{3.74}$$

LEMMA 3.11. *Let $U_t(x,y) = e^{\beta C_t(x,y)} - 1$. The following bounds hold for $t \geqslant 0$, $\beta < 6\pi$:*

$$\sup_{x_1}\sum_{x_2,x_3} |U_t(x_1,x_2)\delta_{12}\dot{\mathsf{C}}_t(x_1,x_2,x_3)| = O_\beta\big(\ell_t^4\vartheta_t^2\big), \tag{3.75}$$

$$\begin{aligned}
&\sup_{x_1}\sum_{x_2,x_3,x_4} |U_t(x_1,x_2)U_t(x_3,x_4)\delta_{34}\delta_{12}\dot{\mathsf{C}}_t(x_1,x_2,x_3,x_4)|\\
&\quad = O_\beta\big(\ell_t^6\vartheta_t^2\big),
\end{aligned} \tag{3.76}$$

and the same bounds hold with the roles of the x_i exchanged. Also, for all $t > s > 0$, $x_i \in \Lambda$,

$$\begin{aligned}
(C_t - C_s)(0,0) &- (C_t - C_s)(x_1,x_2)\\
&+ (C_t - C_s)(x_1,x_3) - (C_t - C_s)(x_2,x_3) \geqslant -O(1).
\end{aligned} \tag{3.77}$$

Again, analogous estimates hold in the conservative case.

PROOF. The lemma again follows from estimates for the heat kernel and is given in Appendix 3.8. $\square$

LEMMA 3.12. *Let $\beta < 6\pi$. Then $\|\tilde{V}_t^{(3)}\| \lesssim |z_t|^3 \ell_t^4$. Analogous bounds hold in the conservative case.*

PROOF. We start from (3.58). We assume $I_1 = \{1, 2\}$ and $I_2 = \{3\}$ since the other cases are analogous. We first consider the case that ξ_{I_1} is neutral. Then

$$\begin{aligned} &-\int_0^t ds \sum_{i=1,2} \dot{u}_s(\xi_i, \xi_3) \tilde{V}_s(\xi_1, \xi_2) \tilde{V}_s(\xi_3) e^{-(W_t(\xi_1,\xi_2,\xi_3) - W_s(\xi_1,\xi_2,\xi_3))} \\ &= \pm\beta \int_0^t \frac{ds}{\ell_s^2} (\dot{\mathsf{C}}_s(x_1, x_3) - \dot{\mathsf{C}}_s(x_2, x_3)) U_s(x_1, x_2) z_s^3 e^{-(W_t(\xi_1,\xi_2,\xi_3) - W_s(\xi_1,\xi_2,\xi_3))}. \end{aligned} \tag{3.78}$$

By the definition of W in (3.28) and by (3.77),

$$\begin{aligned} W_t(\xi_1, \xi_2, \xi_3) - W_s(\xi_1, \xi_2, \xi_3) &\geqslant \frac{\beta}{2}(C_t - C_s)(0, 0) - O(1) \\ &= \frac{\beta}{4\pi} \log(\ell_t/\ell_s) - O(1). \end{aligned} \tag{3.79}$$

By (3.75),

$$\sup_{x_1} \sum_{x_2, x_3} |\delta_{12} \dot{\mathsf{C}}_s(x_1, x_2, x_3) U_s(x_1, x_2)| \lesssim \ell_s^4 \vartheta_s^2. \tag{3.80}$$

Substituting these bounds into (3.78) shows that the contribution to $\|\tilde{V}_t^{(3)}\|$ from neutral ξ_{I_1} is bounded by

$$\ell_t^{-\beta/4\pi} \int_0^t \frac{ds}{\ell_s^2} |z_s|^3 \ell_s^4 \ell_s^{\beta/4\pi} \vartheta_s^2 \lesssim |z_t|^3 \ell_t^4 \tag{3.81}$$

where we used (3.23).

We turn now to the charged case $\sigma_1 = \sigma_2$. Note that (3.79) follows as above if $\sigma_3 = -\sigma_1$ and in fact holds with the better lower bound $\frac{3\beta}{4\pi} \log(\ell_t/\ell_s) - O(1)$ by positive definiteness of $C_t - C_s$ if $\sigma_3 = \sigma_1$, i.e., if all charges are the same. From the explicit form (3.59) of $\tilde{V}_s(\xi_1, \xi_2)$, we thus get

$$\begin{aligned} &-\int_0^t ds \sum_{i=1,2} \dot{u}_s(\xi_i, \xi_3) \tilde{V}_s(\xi_1, \xi_2) \tilde{V}_s(\xi_3) e^{-(W_t(\xi_1,\xi_2,\xi_3) - W_s(\xi_1,\xi_2,\xi_3))} \\ &\lesssim \beta \int_0^t \frac{ds}{\ell_s^2} (\dot{\mathsf{C}}_s(x_1, x_3) + \dot{\mathsf{C}}_s(x_2, x_3)) |1 - e^{-\beta C_s(x_1,x_2)}| |z_s|^3 \left(\frac{\ell_s}{\ell_t}\right)^{\frac{\beta}{4\pi}}. \end{aligned}$$

As the sum over x_3 can be controlled uniformly in x_1, x_2 by $O(\ell_t^2 \vartheta_t^2)$ thanks to (3.18) and then the sum over x_2 can be estimated by $O(\ell_t^2)$ thanks to (3.63), we end up with the same upper bound as in (3.81). This completes the charged case. $\square$

LEMMA 3.13. *Let $\beta < 6\pi$ and assume* (3.17). *Then $\|\tilde{V}_t^{(4)}\| \lesssim |z_t|^4 \ell_t^6$. Analogous bounds hold in the conservative case.*

PROOF. We again start from (3.58). Up to permutation of the indices, there are terms with $|I_1| = 1$, $|I_2| = 3$, and $|I_1| = |I_2| = 2$. We begin with the case $|I_1| = 1$ and $|I_1| = 3$. Using that $|\dot{\mathsf{u}}_s| \lesssim \ell_s^2 \vartheta_s^2$ and that $\|\widetilde{V}_s^{(1)}\| \lesssim |z_s|$ and $\|\widetilde{V}_s^{(3)}\| \lesssim |z_s|^3 \ell_s^4$ (by (3.36) and Lemma 3.12),

$$
\begin{aligned}
\sup_{\xi_1} \sum_{\xi_2,\dots,\xi_n} \left|\dot{\mathsf{u}}_s(\xi_i,\xi_j)\widetilde{V}_s(\xi_{I_1})V_s(\xi_{I_2})\right| &\leqslant \|\dot{\mathsf{u}}_s\| \big\|\widetilde{V}_s^{(1)}\big\| \big\|\widetilde{V}_s^{(3)}\big\| \\
&\lesssim |z_s|^4 \ell_s^6 \vartheta_s^2,
\end{aligned} \tag{3.82}
$$

and we obtain the claimed bound exactly as in the proof for $\beta < 4\pi$.

In the remainder of the proof we bound the terms with $|I_1| = |I_2| = 2$. We begin with the case that ξ_{I_1} and ξ_{I_2} are both neutral. Up to permutation of the indices, we may then assume $\xi_{I_1} = ((x_1,+1),(x_2,-1))$ and $\xi_{I_2} = ((x_3,+1),(x_4,-1))$. By (3.59), using $\dot{\mathsf{u}}_t(\xi_1,\xi_j) + \dot{\mathsf{u}}_t(\xi_2,\xi_j) = \sigma_1\sigma_j(\dot{\mathsf{C}}_t(x_1,x_j) - \dot{\mathsf{C}}_t(x_2,x_j))$ and analogously for the sum over j,

$$
\begin{aligned}
&\sum_{i\in I_1, j\in I_2} \dot{\mathsf{u}}_t(\xi_i,\xi_j)\widetilde{V}_t(\xi_{I_1})\widetilde{V}_t(\xi_{I_2}) \\
&\quad = z_t^4 U_t(x_1,x_2)U_t(x_3,x_4)\delta_{34}\delta_{12}\dot{\mathsf{C}}_t(x_1,x_2,x_3,x_4).
\end{aligned} \tag{3.83}
$$

Hence, by (3.76) and (3.22) for $\beta < 6\pi$,

$$
\begin{aligned}
&\sup_{x_1} \sum_{x_2,x_3,x_4} \int_0^t \frac{ds}{\ell_s^2} \left| \sum_{i\in I_1, j\in I_2} \dot{\mathsf{u}}_s(\xi_i,\xi_j)\widetilde{V}_s(\xi_{I_1})\widetilde{V}_s(\xi_{I_2})\right| \\
&\quad \lesssim \int_0^t \frac{ds}{\ell_s^2} |z_s|^4 \ell_s^6 \vartheta_s^2 \lesssim |z_t|^4 \ell_t^6.
\end{aligned} \tag{3.84}
$$

In the case that I_1 is neutral and I_2 is charged, we similarly use

$$
\begin{aligned}
&\sup_{\xi_1} \sum_{\xi_2,\dots,\xi_n} \left| \frac{1}{2} \int_0^t \frac{ds}{\ell_s^2} \sum_{j\in I_2} \Big[\sum_{i\in I_1} \dot{\mathsf{u}}_s(\xi_i,\xi_j)\widetilde{V}_s(\xi_{I_1}) 1_{\sigma(\xi_{I_1})=0}\Big] \widetilde{V}_s(\xi_{I_2}) 1_{\sigma(\xi_{I_2})\neq 0}\right| \\
&\quad \leqslant \beta \int_0^t \frac{ds}{\ell_s^2} \left[\sup_{x_1} \sum_{x_2,x_3} \left|(\dot{\mathsf{C}}_s(x_1,x_3) - \dot{\mathsf{C}}_s(x_2,x_3))U_s(x_1,x_2)\right|\right] \\
&\qquad\qquad \cdot \left[\sup_{\xi_3} \sum_{\xi_4} |\widetilde{V}_s(\xi_{I_2})| 1_{\sigma(\xi_{I_2})\neq 0}\right].
\end{aligned}
$$

By (3.75), the first bracket is bounded by

$$
O_\beta\big(|z_t|^2 \ell_t^4 \vartheta_t^2\big). \tag{3.85}
$$

Since ξ_{I_2} is charged, the contribution from $V(\xi_{I_2})$ term is bounded using (3.63) by

$$
\sup_{\xi_3} \sum_{\xi_4} |\widetilde{V}_t(\xi_{I_2})| 1_{\sigma(\xi_{I_2})\neq 0} \lesssim |z_t|^2 \sup_{x_3} \sum_{x_4} |1 - e^{-\beta C_t(x_3,x_4)}| \lesssim |z_t|^2 \ell_t^2. \tag{3.86}
$$

So altogether these contributions to (3.85) are again bounded using (3.22) (and $\beta < 6\pi$) by

$$\int_0^t \frac{ds}{\ell_s^2} |z_s|^4 \ell_s^6 \vartheta_s^2 \lesssim |z_t|^4 \ell_t^6. \tag{3.87}$$

Again the case that ξ_{I_1} and ξ_{I_2} are both charged is easier and analogous to the proof for $\beta < 4\pi$ and so is omitted. $\square$

LEMMA 3.14. *Let $\beta < 6\pi$ and assume (3.17). Then*

$$\|\tilde{V}_t^{(n)}\| \leqslant n^{n-2}|z_t|^n (C_\beta \ell_t^2)^{n-1} \quad \text{for all } n \geqslant 5. \tag{3.88}$$

Analogous bounds hold in the conservative case.

PROOF. Similarly as in the proof of (3.39), we make the inductive assumption that, for some $n \geqslant 4$, the bound (3.56) holds for all $1 \leqslant k \leqslant n$, $k \neq 2$. By (3.36) and Lemmas 3.12–3.13, the inductive assumption is verified for $n = 4$. To advance the induction, we again start from

$$|\tilde{V}_t(\xi_1, \dots, \xi_n)| \leqslant \frac{1}{2} \sum_{I_1 \dot\cup I_2 = [n]} \int_0^t ds \left| \sum_{i \in I_1, j \in I_2} \dot{u}_s(\xi_i, \xi_j) \tilde{V}_s(\xi_{I_1}) \tilde{V}_s(\xi_{I_2}) \right|. \tag{3.89}$$

For $|I_1| = n - k \neq 2$ and $|I_2| = k \neq 2$, we use

$$\sup_{\xi_1} \sum_{\xi_2, \dots, \xi_n} |\dot{u}_s(\xi_i, \xi_j) \tilde{V}_s(\xi_{I_1}) \tilde{V}_s(\xi_{I_2})| \leqslant \|\dot{u}_s\| \|\tilde{V}_s^{(n-k)}\| \|\tilde{V}_s^{(k)}\|, \tag{3.90}$$

and bound the terms on the right-hand side using the inductive assumption. Then exactly as in the proof for $\beta < 4\pi$, i.e., of (3.39), the result is

$$\begin{aligned} &\sup_{\xi_1} \sum_{\xi_2, \dots, \xi_n} \sum_{\substack{I_1 \dot\cup I_2 = [n] \\ |I_1| \neq 2, |I_2| \neq 2}} \int_0^t ds \sum_{i \in I_1, j \in I_2} |\dot{u}_s(\xi_i, \xi_j) \tilde{V}_s(\xi_{I_1}) \tilde{V}_s(\xi_{I_2})| \\ &\quad \leqslant n^{n-2} |z_t|^n \big(C_\beta \ell_t^2\big)^{n-1}. \end{aligned} \tag{3.91}$$

The terms with $|I_1| = 2$ or $|I_2| = 2$ require special treatment. By symmetry we may assume that $|I_1| = 2$ and that $I_1 = \{1, 2\}$ and $I_2 = \{3, \dots, n\}$ with $n \geqslant 5$. If ξ_{I_1} is neutral, we use

$$\begin{aligned} &\sup_{\xi_1} \sum_{\xi_2, \dots, \xi_n} \left| \frac{1}{2} \int_0^t \frac{ds}{\ell_s^2} \sum_{j \in I_2} \left[\sum_{i \in I_1} \dot{\mathsf{u}}_s(\xi_i, \xi_j) \tilde{V}_s(\xi_{I_1}) 1_{\sigma(\xi_{I_1})=0} \right] \tilde{V}_s(\xi_{I_2}) \right| \\ &\quad \leqslant (n-2) \int_0^t \frac{ds}{\ell_s^2} \left[\sup_{x_1} \sum_{x_2, x_3} \big| (\dot{\mathsf{C}}_s(x_1, x_3) - \dot{\mathsf{C}}_s(x_2, x_3)) U_s(x_1, x_2) \big| \right] \\ &\qquad \left[\sup_{\xi_3} \sum_{\xi_4, \dots, \xi_n} |\tilde{V}_s(\xi_{I_2})| \right]. \end{aligned}$$

By (3.75), the first bracket is bounded by $O_\beta(z_t^2 \ell_t^4 \vartheta_t^2)$, while for the second term involving $V(\xi_{I_2})$, using inductive assumption for $\widetilde{V}(\xi_{I_2})$ (note that $n-2 \geqslant 3$), we get

$$\sup_{\xi_3} \sum_{\xi_4,\dots,\xi_n} \big|\widetilde{V}_t(\xi_{I_2})\big| \leqslant \big\|\widetilde{V}_t^{(n-2)}\big\| \leqslant (n-2)^{n-4}|z_t|^{n-2}\big(C_\beta \ell_t^2\big)^{n-3}. \tag{3.92}$$

So altogether these contributions to (3.92) are bounded by (using again (3.22) for $\beta < 6\pi$),

$$\begin{aligned} O_\beta(1)(n-2)^{n-3} C_\beta^{n-3} \int_0^t |z_s|^n \ell_s^{2(n-1)} \vartheta_s^2 \frac{ds}{\ell_s^2} \\ \lesssim C_\beta^{-2} n^{n-4} |z_t|^n \big(C_\beta \ell_t^2\big)^{n-1} \\ \leqslant n^{n-4} |z_t|^n \big(C_\beta \ell_t^2\big)^{n-1}, \end{aligned} \tag{3.93}$$

where in the last bound we have chosen C_β sufficiently large (independently of n). Summing over the $\binom{n}{2} \leqslant n^2$ choices for I_1 and I_2 with $|I_1| = 2$ leads to the expected upper bound. The charged case holds in the same way. □

PROOF OF PROPOSITION 3.7. The bounds (3.56) follow by combining the previous three lemmas. □

3.8 Proofs of Propositions 3.1–3.2 Without (3.5)

Finally, we remove the assumption (3.5) at the cost of constants that are uniform in ε but not uniform in L. For $t \leqslant t_0$, where t_0 is sufficiently small but of order $1/\varepsilon^2$, we can apply the same analysis as before. On the other hand, for $t \geqslant t_0$, a very crude argument is sufficient to show that the Hessian of the effective potential is bounded from below uniformly in ε. Our starting point for this is (2.15), i.e.,

$$\begin{aligned} (f, \operatorname{Hess} V_t f) = \boldsymbol{P}_{t_0,t}(f, \operatorname{Hess} V_{t_0} f) \\ - \big(\boldsymbol{P}_{t_0,t}\big((f, \nabla V_{t_0})^2\big) - \big(\boldsymbol{P}_{t_0,t}(f, \nabla V_{t_0})\big)^2\big). \end{aligned} \tag{3.94}$$

The input from the previous analysis is summarised in the following lemma.

LEMMA 3.15. *Let $\beta < 6\pi$. Then there is a constant $\alpha = \alpha(\beta) > 0$ such that for all $t \geqslant 0$ satisfying $|\mathsf{z}_t| \leqslant \alpha$ and (3.17), the following bounds hold uniformly in $\varphi \in X$, $f \in X$, and $x \in \Lambda$:*

$$|(Q_t f, \operatorname{Hess} V_t Q_t f)| \leqslant O_\beta(|z_t| \vartheta_t^2)|f|_2^2, \tag{3.95}$$

$$|(Q_t \nabla V_t)_x| \leqslant O_\beta(|z_t| \vartheta_t). \tag{3.96}$$

PROOF. For $\beta < 4\pi$, these bounds follow exactly as in (3.52)–(3.53). For $\beta < 6\pi$, the bound on the Hessian is as in (3.55) and (3.57), and for ∇V_t, see (3.72). □

PROOF OF THEOREMS 3.1–3.2 WITHOUT (3.5). Recall from (3.18) that

$$e^{-\frac{\beta}{2}C_t(0,0)} \asymp \ell_t^{-\beta/4\pi},$$

and hence $|z_t| \asymp \varepsilon^2(\varepsilon\ell_t)^{-\beta/4\pi}|z|$ and $|\mathsf{z}_t| \asymp (\varepsilon\ell_t)^{2-\beta/4\pi}|z|$. Here $a \asymp b$ denotes that $c_\beta \leqslant a/b \leqslant 1/c_\beta$ for some constant $c_\beta > 0$. Let $t_\alpha > 0$ be such that $|\mathsf{z}_{t_\alpha}| = \alpha$. Thus $\varepsilon\ell_{t_\alpha} \asymp (\alpha/|z|)^{1/(2-\beta/4\pi)}$ and hence

$$|z_{t_\alpha}| = O_\beta(\varepsilon^2(\varepsilon\ell_{t_\alpha})^{-\beta/4\pi}|z|) = O_\beta(\varepsilon^2|z|^{1/(1-\beta/8\pi)}). \tag{3.97}$$

Also, with $t_{m,L} = \varepsilon^{-2}(m^{-2} \wedge L^2)$ as in (3.17),

$$|z_{t_{m,L}}| = O_\beta(\varepsilon^2(m^{-1} \wedge L)^{-\beta/4\pi}|z|). \tag{3.98}$$

We choose $t_0 = t_\alpha \wedge t_{m,L}$ so that, since $|z_t|$ in decreasing in t (see (3.19)),

$$|z_{t_0}| = O_\beta(\varepsilon^2)\big((m^{-1} \wedge L)^{-\beta/4\pi}|z| + |z|^{1/(1-\beta/8\pi)}\big) = O_{\beta,z,m,L}(\varepsilon^2). \tag{3.99}$$

With this and since $|\Lambda| = \varepsilon^{-2}L^2$, it follows from (3.96) that, uniformly in φ,

$$|Q_{t_0}\nabla V_{t_0}|_2^2 = \sum_{x\in\Lambda}(Q_{t_0}\nabla V_{t_0})_x^2 \leqslant O_{\beta,z,m,L}\big(\varepsilon^2\vartheta_{t_0}^2\big). \tag{3.100}$$

For any $t \geqslant t_0$, by the Cauchy-Schwarz inequality and $|Q_{t-t_0}f|_2 \leqslant \vartheta_{t-t_0}|f|_2$, in particular,

$$(Q_t f, \nabla V_{t_0})^2 \leqslant O_{\beta,z,m,L}\big(\varepsilon^2\vartheta_{t_0}^2\big)|Q_{t-t_0}f|_2^2 \leqslant O_{\beta,z,m,L}\big(\varepsilon^2\vartheta_t^2\big)|f|_2^2. \tag{3.101}$$

Similarly, by (3.95),

$$|(Q_t f, \operatorname{Hess} V_{t_0} Q_t f)| \leqslant O_\beta(z_{t_0}\vartheta_{t_0}^2)|Q_{t-t_0}f|_2^2 = O_\beta\big(|z|\varepsilon^2\vartheta_t^2\big)|f|_2^2. \tag{3.102}$$

Substituting (3.101)-(3.102) into (3.94), using that $\boldsymbol{P}_{t_0,t}$ is a Markov operator, we conclude that, for all $t \geqslant t_0$,

$$(Q_t f, \operatorname{Hess} V_t Q_t f) \geqslant \dot{\mu}_t|f|_2^2 \quad \text{where } \dot{\mu}_t \geqslant -O_{\beta,z,m,L}\big(\varepsilon^2\vartheta_t^2\big). \tag{3.103}$$

For $t \leqslant t_0$, we have $\dot{\mu}_t = O_\beta(|z_t|\vartheta_t^2) = O_\beta(|z|)\varepsilon^2\vartheta_t^2$ exactly as in the proofs of the theorems in the case (3.5). In summary, for all $t \geqslant 0$,

$$\mu_t \geqslant -(O_\beta(|z|) + O_{\beta,z,m,L}(1))\int_0^\infty ds\,\varepsilon^2\vartheta_s^2 \geqslant -\mu^*(\beta,z,m,L), \tag{3.104}$$

with $\mu^*(\beta,z,m,L)$ independent of ε. From this bound, the remainder of the proof is the same as in the case (3.5). □

Appendix: Heat Kernel Estimates: Proof of Lemmas 3.3 and 3.9–3.11

In this appendix, we prove Lemmas 3.3 and 3.9–3.11. These follow from standard estimates for the lattice heat kernel $p_t(x) = e^{t\Delta}(0,x)$ on $\mathbb{Z}^d$ and its torus version $p_t^L(x) = \sum_{y\in\mathbb{Z}^d} p_t(x+Ly)$, where $L \in \mathbb{N}$. Throughout the appendix, Δ and ∇ denote the lattice Laplacian and derivative on $\mathbb{Z}^d$, not the Laplacian and gradient on $\mathbb{R}^\Lambda$.

A.1 Bounds on the Heat Kernel

We begin by collecting estimates on the heat kernel on $\mathbb{Z}^d$. To state these, let α be a sequence of $|\alpha| \equiv k$ unit vectors $\alpha_1, \dots, \alpha_k$ in $\mathbb{Z}^d$, i.e., $\alpha_i \in \{e_{1\pm}, \dots, e_{d\pm}\}$ is one of the $2d$ unit vectors $e_{i\pm}$ in $\mathbb{Z}^d$, and write $\nabla^\alpha = \prod_{i=1}^k \nabla_{\alpha_i}$ with $\nabla_e f(x) = f(x+e) - f(x)$ the lattice gradient. For $x \in \mathbb{Z}^d$, $|x|$ denotes any fixed norm unless stated.

LEMMA A.1. *The heat kernel p_t on $\mathbb{Z}^d$ satisfies the following upper bounds for $t \geqslant 1$, $x \in \mathbb{Z}^d$, and all sequences of unit vectors α:*

$$|\nabla^\alpha p_t(x)| = O_\alpha(t^{-d/2-|\alpha|/2} e^{-c|x|/\sqrt{t}}), \tag{A.1}$$

as well as the following asymptotics if $d = 2$, for $t \geqslant 1$ and $x \neq 0$,

$$\begin{aligned} p_t(0) &= \frac{1}{4\pi t} + O\left(\frac{1}{t^2}\right), \\ \int_0^t (p_s(0) - p_s(x))ds &= \frac{1}{2\pi}\log(|x| \wedge \sqrt{t}) + O(1). \end{aligned} \tag{A.2}$$

Moreover, the heat kernel p_t^L on a discrete torus of side length L satisfies, for $t \geqslant 1$, $|x|_\infty < L/2$,

$$\nabla^\alpha p_t^L(x) = \nabla^\alpha p_t(x) + O_\alpha(t^{-|\alpha|/2} L^{-d} e^{-cL/\sqrt{t}}), \tag{A.3}$$

and the mean 0 heat kernel on the torus is given by $p_t^{0,L}(x) = p_t^L(x) - 1/L^2$.

PROOF. Writing $\alpha_i = e_{j\sigma_j}$ with $j \in \{1, \dots, d\}$ and $\sigma_j \in \{\pm\}$ for each $i \in \{1, \dots, |\alpha|\}$, the bound (A.1) can be seen by writing $\nabla^\alpha p_t(x)$ in its Fourier representation:

$$\begin{aligned} & t^{d/2+|\alpha|/2} \nabla^\alpha p_t(x\sqrt{t}) \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \prod_{i=1}^{|\alpha|} \sqrt{t}(1 - e^{i\sigma_{\alpha_i} k_{\alpha_i}}) \\ &\qquad\qquad e^{t\sum_{j=1}^d (2\cos(k_j)-2)} e^{ikx\sqrt{t}} t^{d/2}\, dk, \end{aligned}$$

$$\begin{aligned} &= \frac{1}{(2\pi)^d} \int_{[-t\pi,t\pi]^d} \prod_{i=1}^{|\alpha|} \sqrt{t}(1 - e^{i\sigma_{\alpha_i} k_{\alpha_i}/\sqrt{t}}) \\ &\qquad\qquad e^{t\sum_{j=1}^d (2\cos(k_j/\sqrt{t})-2)} e^{ikx}\, dk. \end{aligned} \tag{A.4}$$

For $t \geqslant 1$, the integrand is analytic on a strip $k \in (\mathbb{R} + i[-c, c])^d$ with $c > 0$ independent of t, and hence (A.4) decays exponentially in $|x|$ (see, e.g., [41, chap. I.4, exer. 4]). The first estimate in (A.2) is standard and straightforward to verify by writing the left-hand side in terms of the Fourier transform; we thus omit its proof. The second estimate in (A.2) is similarly standard if $t = \infty$, in which case the left-hand side is the Green function of the discrete Laplacian:

$$\int_0^\infty (p_s(0) - p_s(x))ds = \frac{1}{2\pi}\log|x| + O(1). \tag{A.5}$$

This estimate can be found, for example, in [39, p. 198] or [46, theorem 4.4.4] (with normalisation there differing by a factor $2d = 4$). To prove the second estimate in (A.2) for $0 < |x| \leqslant \sqrt{t}$, we use that by (A.1) with $|\alpha| = 1$,

$$\int_t^\infty (p_s(0) - p_s(x))ds = O(|x|)\int_t^\infty s^{-3/2}\,ds = O(|x|/\sqrt{t}), \tag{A.6}$$

which using (A.5) implies

$$\begin{aligned}\int_0^t (p_s(0) - p_s(x))ds &= \int_0^\infty (p_s(0) - p_s(x))ds + O(|x|/\sqrt{t}) \\ &= \frac{1}{2\pi}\log|x| + O(1).\end{aligned} \tag{A.7}$$

For $|x| \geqslant \sqrt{t}$, we use that the first bound in (A.2) (and $p_t(0) \leqslant 1$ for $t < 1$) implies

$$\int_0^t p_s(0)ds = \frac{1}{2\pi}\log\sqrt{t} + O(1), \tag{A.8}$$

and hence with (A.1) to bound $p_s(x)$,

$$\int_0^t (p_s(0) - p_s(x))ds = \frac{1}{2\pi}\log\sqrt{t} + O(1) - \int_1^t O(s^{-1}e^{-c|x|/\sqrt{s}})ds, \tag{A.9}$$

where the integral is bounded by a multiple of

$$\int_1^t e^{-|x|/\sqrt{s}}\frac{ds}{s} = \int_{1/|x|^2}^{t/|x|^2} e^{-1/\sqrt{s}}\frac{ds}{s} \leqslant \int_0^1 e^{-1/\sqrt{s}}\frac{ds}{s} = O(1). \tag{A.10}$$

This completes the proof of (A.2).

For the torus of side length L, we use that $p_t^L(x) = \sum_{y\in\mathbb{Z}^d} p_t(x + Ly)$ and set $|x|_L = \inf_{y\in\mathbb{Z}^d} |x + Ly|$. Then

$$\sum_{y\in\mathbb{Z}^d} e^{-c|x+Ly|/\sqrt{t}} = e^{-c|x|_L/\sqrt{t}} + O((\sqrt{t}/L)^d e^{-\frac{1}{2}cL/\sqrt{t}}), \tag{A.11}$$

since the remainder between the left-hand side and the first term on the right-hand side of the last equation can be controlled by (approximating the sum by an integral and using polar coordinates)

$$\begin{aligned}\int_1^\infty e^{-crL/\sqrt{t}} r^{d-1}\,dr &\leqslant e^{-\frac{1}{2}cL/\sqrt{t}}\int_1^\infty e^{-\frac{1}{2}crL/\sqrt{t}} r^{d-1}\,dr \\ &\leqslant e^{-\frac{1}{2}cL/\sqrt{t}}(\sqrt{t}/L)^d\int_1^\infty e^{-\frac{1}{2}cr} r^{d-1}\,dr.\end{aligned} \tag{A.12}$$

This shows the estimates (A.3).

The expression for the mean 0 heat kernel follows from

$$p_t^{0,L}(x) = (\delta_0, Pe^{\Delta t}P\delta_x) = (\delta_0 - 1/L^2, e^{\Delta t}(\delta_x - 1/L^2))$$
$$= p_t^L(x) - 2/L^2 + 1/L^2 = p_t^L(x) - 1/L^2$$

with the projection P from (3.7). □

A.2 Proof of Lemma 3.3

We recall the definition $\dot{C}_t(x) = p_t^{L_\varepsilon}(x)e^{-\varepsilon^2 m^2 t} = p_t^{L_\varepsilon}(x)\vartheta_t^2$. Lemma 3.3 is an elementary combination of the estimates from Lemma A.1, whose details are given as follows.

PROOF OF LEMMA 3.3. Applying (A.1) and (A.3) with $x = 0$ to the torus of side length $L_\varepsilon = L/\varepsilon$ and, for $t \geqslant 1$, we have

$$|p_t(0) - p_t^{L_\varepsilon}(0)| \lesssim L_\varepsilon^{-d} e^{-cL_\varepsilon/\sqrt{t}}, \qquad p_t^{L_\varepsilon}(0) \lesssim t^{-d/2} \vee L_\varepsilon^{-d}. \tag{A.13}$$

By the assumption (3.17), either $t \leqslant 1/\varepsilon^2 m^2$ or $Lm \geqslant 1$ holds. By the above bound, if $Lm \geqslant 1$, the contribution to $C_t(0)$ from $t \geqslant 1/\varepsilon^2 m^2$ is negligible since

$$\begin{aligned}\int_{1/\varepsilon^2 m^2}^{\infty} p_t^{L_\varepsilon}(0)\, e^{-\varepsilon^2 m^2 t}\, dt &\lesssim \int_{1/\varepsilon^2 m^2}^{\infty} (t^{-1} \vee \varepsilon^2 L^{-2})\, e^{-\varepsilon^2 m^2 t}\, dt \\ &\lesssim \varepsilon^2 m^2 \int_{1/\varepsilon^2 m^2}^{\infty} e^{-\varepsilon^2 m^2 t}\, dt \lesssim 1.\end{aligned} \tag{A.14}$$

For $t \leqslant L^2/\varepsilon^2$ (and thus for $t \leqslant 1/m^2\varepsilon^2$ when $Lm \geqslant 1$), we may moreover replace $p_t^{L_\varepsilon}$ by p_t since

$$\int_0^t \big(p_s(0) - p_s^{L_\varepsilon}(0)\big) ds = O(L_\varepsilon^{-2} t) = O(1). \tag{A.15}$$

Finally, the contribution to $\dot{C}_t(0)$ from the infinite volume heat kernel $p_t(0)$ is

$$p_t(0)e^{-\varepsilon^2 m^2 t} = \left[\frac{1}{4\pi t} + O\left(\frac{1}{t^2}\right)\right]e^{-\varepsilon^2 m^2 t} = \frac{1}{4\pi t} + O\left(\frac{1}{t^2}\right) + O(\varepsilon^2 m^2), \tag{A.16}$$

which integrated up to $t \leqslant 1/\varepsilon^2 m^2$ gives the main contribution

$$\begin{aligned}C_t(0) &= \int_0^t p_s(0)e^{-\varepsilon^2 m^2 s}\, ds + O(1) \\ &= \frac{1}{4\pi}\log t + O(1) = \frac{1}{2\pi}\log \ell_t + O(1).\end{aligned} \tag{A.17}$$

This shows the first estimate in (3.18). The second estimate is straightforward since $\dot{\mathsf{C}}_s(x,y) = \dot{\mathsf{C}}_s(0, x-y) \geqslant 0$ and the fact that the heat kernel defines a probability density immediately imply

$$\sup_x \sum_y \dot{\mathsf{C}}_t(x,y) = \ell_t^2 \vartheta_t^2 \sum_{y \in \Lambda} p_t^L(y) = \ell_t^2 \vartheta_t^2 \sum_{y \in \mathbb{Z}^2} p_t(y) = \ell_t^2 \vartheta_t^2. \tag{A.18}$$

Finally, in the conservative case the estimates are unchanged since

$$
\begin{aligned}
C_t^0(0,0) = C_t(0,0) - \frac{1}{|\Lambda|}\int_0^t e^{-\varepsilon^2 m^2 s}\,ds &= C_t(0,0) - \frac{1-e^{-\varepsilon^2 m^2 t}}{L^2 m^2} \\
&= C_t(0,0) + O(1)
\end{aligned}
\tag{A.19}
$$

and

$$
\sum_x |\dot{\mathsf{C}}_t^0(0,x)| \leqslant \sum_x \biggl(\dot{\mathsf{C}}_t(0,x) + \frac{\ell_t^2\vartheta_t^2}{|\Lambda|}\biggr) = O(\ell_t^2\vartheta_t^2). \tag{A.20}
$$

□

A.3 Proof of Lemmas 3.9–3.11

To prepare for the proofs of the lemmas, we state the following consequences of Lemma A.1 in the notation used in the lemmas. In particular, recall (3.73)–(3.74). For $x \in \Lambda$, abusing notation slightly, we write $|x|$ for the torus distance $|x|_{L_\varepsilon} = \inf_{y\in\mathbb{Z}^d} |x + L_\varepsilon y|$. In particular, $|x| = O(L_\varepsilon)$ for all $x \in \Lambda$. Moreover, in all of the following lemmas, we impose the assumption (3.17) without stating it explicitly.

LEMMA A.2. *The following estimates hold for $\dot{\mathsf{C}}_t$, C_t for $t \geqslant 1$ and $|x-y| \geqslant 1$:*

$$
\begin{aligned}
C_t(x,y) &= -\frac{1}{2\pi}\log(|x-y|/\ell_t \wedge 1) + O(1), \\
|\dot{\mathsf{C}}_t(x,y)| &\lesssim \vartheta_t^2 e^{-c|x-y|/\ell_t}.
\end{aligned}
\tag{A.21}
$$

The first bounds also implies that

$$
C_t(x,y) = \int_1^t \frac{1}{4\pi s} e^{-|x-y|^2/2s} e^{-\varepsilon^2 m^2 s}\,ds + O(1). \tag{A.22}
$$

For any $c' > 0$ small enough,

$$
\begin{aligned}
&|\delta_{12}\dot{\mathsf{C}}_t(x,y,z)| e^{-c'|x-y|/\ell_t} \\
&\quad \lesssim \vartheta_t^2 (|x-y|/\ell_t) e^{-c'|x-z|/2\ell_t} e^{-c'|y-z|/2\ell_t},
\end{aligned}
\tag{A.23}
$$

$$
\begin{aligned}
&|\delta_{34}\delta_{12}\dot{\mathsf{C}}_t(x,y,w,z)| e^{-c'|x-y|/\ell_t} e^{-c'|w-z|/\ell_t} \\
&\quad \lesssim \vartheta_t^2 (|x-y|/\ell_t)(|w-z|/\ell_t) e^{-c'|x-w|/\ell_t}.
\end{aligned}
\tag{A.24}
$$

The same estimates hold with $\dot{\mathsf{C}}_t$ replaced by $\ell_t\vartheta_t\mathsf{Q}_t$, and if $\dot{\mathsf{C}}_t$ and Q_t are replaced by $\dot{\mathsf{C}}_t^0$ and Q_t^0.

PROOF. The estimates (A.21) follow easily from those for the heat kernel in (A.1)–(A.3). Indeed, the second bound in (A.21) is a special case of (A.1) and (A.3):

$$
\begin{aligned}
\dot{\mathsf{C}}_t(x,y) &= \ell_t^2\vartheta_t^2 p_t^{L_\varepsilon}(x,y) \\
&\lesssim \ell_t^2\vartheta_t^2 \biggl(\frac{1}{t} e^{-c|x-y|/\sqrt{t}} + \frac{1}{L_\varepsilon^2} e^{-cL_\varepsilon/\sqrt{t}}\biggr) \lesssim \vartheta_t^2 e^{-c|x-y|/\sqrt{t}},
\end{aligned}
\tag{A.25}
$$

where in the last inequality we used that $\ell_t/L_\varepsilon \leqslant 1$ follows from (3.17) and the definition of ℓ_t in (3.15). Indeed, by (3.17), either $t \leqslant L_\varepsilon^2$ which implies $\ell_t \leqslant L_\varepsilon$, or otherwise $Lm \geqslant 1$ and then also

$$\ell_t/L_\varepsilon = (\sqrt{t} \wedge 1/(\varepsilon m))/(L/\varepsilon) \leqslant \sqrt{\varepsilon^2 m^2 t} \wedge 1 \leqslant 1.$$

For the first bound in (A.21) we note that (A.2) implies

$$\begin{aligned} \int_0^t p_s(x)ds &= \frac{1}{2\pi}[\log\sqrt{t} - \log(|x| \wedge \sqrt{t})] + O(1) \\ &= -\frac{1}{2\pi}\log(|x|/\sqrt{t} \wedge 1) + O(1). \end{aligned} \tag{A.26}$$

The additional factor $e^{-\varepsilon^2 m^2 s}$ multiplying $p_s(x)$ leads to the replacement of $\sqrt{t}$ by ℓ_t exactly as in the proof of (3.18). By an analogous calculation, the same formula holds with the discrete heat kernel replaced by the continuous one, i.e.,

$$\int_1^t \frac{1}{4\pi s} e^{-|x|^2/2s}\, ds = -\frac{1}{2\pi}\log(|x|/\sqrt{t} \wedge 1) + O(1), \tag{A.27}$$

from which (A.22) follows after taking into account the additional factor $e^{-\varepsilon^2 m^2 s}$ as before.

To verify (A.23)–(A.24) for $x, y \in \mathbb{Z}^d$, let γ_{xy} be a path from x to y of length $|x-y|$ where $|x|$ denotes the 1-norm in this proof. Then (A.1) and (A.3) imply

$$\begin{aligned} \left|\delta_{12} p_t^{L_\varepsilon}(x,y,z)\right| &= \left|p_t^{L_\varepsilon}(x,z) - p_t^{L_\varepsilon}(y,z)\right| \\ &\leqslant \sum_{u\in\gamma_{xy}} \left|\nabla p_t^{L_\varepsilon}(u,z)\right| \lesssim \ell_t^{-3} \sum_{u\in\gamma_{xy}} e^{-c|u-z|/\ell_t}. \end{aligned} \tag{A.28}$$

For $u \in \gamma_{xy}$, we have $|x-z| \leqslant |x-u|+|u-z| \leqslant |x-y|+|u-z|$, and we deduce from the symmetric estimate in y that $-|u-z| \leqslant -|x-y|-|x-z|/2-|y-z|/2$. Choosing $c' < c$, we get

$$\left|\delta_{12} p_t^{L_\varepsilon}(x,y,z)\right| \lesssim \ell_t^{-2}(|x-y|/\ell_t) e^{-c'|x-z|/2\ell_t} e^{-c'|y-z|/2\ell_t} e^{+c'|x-y|/\ell_t}.$$

This completes (A.23). Analogously, again applying (A.1) and (A.3) and choosing $c' < c$, we get

$$\begin{aligned} &\left|\delta_{34}\delta_{12} p_t^{L_\varepsilon}(x,y,w,z)\right| \\ &\leqslant \sum_{u\in\gamma_{xy}} \sum_{v\in\gamma_{wz}} |\nabla^2 p_t^{L_\varepsilon}(u-v)| \\ &\lesssim \ell_t^{-4} \sum_{u\in\gamma_{xy}} \sum_{v\in\gamma_{wz}} e^{-c|u-v|/\ell_t} \\ &\lesssim \ell_t^{-2}(|x-y|/\ell_t)(|w-z|/\ell_t) e^{-c'|x-w|/\ell_t} e^{+c'|x-y|/\ell_t} e^{+c'|w-z|/\ell_t} \end{aligned} \tag{A.29}$$

using that $|x-w| \leqslant |x-u|+|u-v|+|v-w| \leqslant |x-y|+|u-v|+|w-z|$. □

LEMMA A.3. *For all $x, y, z \in \Lambda$, $0 \leqslant s \leqslant t$,*

$$(C_t - C_s)(0,0) - (C_t - C_s)(x,y) + (C_t - C_s)(x,z) - (C_t - C_s)(y,z) \geqslant -O(1). \tag{A.30}$$

PROOF. It suffices to assume that $s \geqslant 1$. Throughout this proof, $|x|$ denotes the Euclidean norm. Suppose first that $|x-y| \leqslant |x-z| \wedge |y-z|$. We will show that

$$|(C_t - C_s)(x,z) - (C_t - C_s)(y,z)| \leqslant \int_s^t |\dot{C}_u(x,z) - \dot{C}_u(y,z)|\,du \lesssim 1. \tag{A.31}$$

Indeed, this bound follows from the following two estimates: using (A.1) with $|\alpha| = 0$ for the first bound and with $|\alpha| = 1$ for the second bound, and also (A.3) for the error due to periodicity,

$$\begin{aligned} &\int_s^{|x-y|^2} (|\dot{C}_u(x,z)| + |\dot{C}_u(y,z)|)\,du \\ &\quad \lesssim 1 + \int_s^{|x-y|^2} u^{-1} e^{-c|x-y|/\sqrt{u}}\,du \lesssim 1, \end{aligned} \tag{A.32}$$

$$\begin{aligned} &\int_{|x-y|^2}^t |\dot{C}_u(x,z) - \dot{C}_u(y,z)|\,du \\ &\quad \lesssim 1 + |x-y| \int_{|x-y|^2}^t u^{-3/2}\,du \lesssim 1. \end{aligned} \tag{A.33}$$

Here we have used that the remainder in (A.3) due to the periodicity is bounded by

$$\begin{aligned} &\frac{|x-y|}{L_\varepsilon^2} \int_{|x-y|^2}^t u^{-1/2} e^{-cL_\varepsilon/\sqrt{u} - \varepsilon^2 m^2 u} \\ &\quad \lesssim 1 + \frac{|x-y|}{L_\varepsilon^2} \int_{|x-y|^2}^{\varepsilon^{-2}m^{-2}} u^{-1/2} e^{-cL_\varepsilon/\sqrt{u}} \lesssim 1 \end{aligned} \tag{A.34}$$

when $Lm \geqslant 1$, and that an analogous bound holds when $t \leqslant \varepsilon^{-2}(m^{-2} \wedge L^2)$. The bound (A.30) then follows from (A.31) and $(C_t - C_s)(0,0) - (C_t - C_s)(x,y) \geqslant 0$, which holds by the positive definiteness of $C_t - C_s$ and translation invariance.

The same argument as above also applies if $|y-z| \leqslant |x-z| \wedge |x-y|$. Therefore suppose that $|x-z| \leqslant |x-y| \wedge |y-z|$. From (A.22) recall that

$$C_t(x,z) = \int_1^t \frac{1}{4\pi u} e^{-|x-z|^2/2u} e^{-\varepsilon^2 m^2 u}\,du + O(1). \tag{A.35}$$

Since $e^{-|x-z|^2/2u} \geqslant e^{-|y-z|^2/2u}$,

$$(C_t - C_s)(x,z) - (C_t - C_s)(y,z) \geqslant -O(1). \tag{A.36}$$

The conclusion (A.30) now follows from $(C_t - C_s)(0,0) - (C_t - C_s)(x,y) \geqslant 0$. $\square$

LEMMA A.4. *Let* $U_t(x) = e^{\beta C_t(0,x)} - 1$. *Then for* $\beta < 2\pi(k+2)$ *and sufficiently small* $c' > 0$,

$$\sum_x |U_t(x)|(|x|/\ell_t)^k e^{c'|x|/\sqrt{t}} \lesssim \ell_t^2. \tag{A.37}$$

The analogous estimate holds in the conservative case.

PROOF. By (A.21), $C_s(0,x) = -\frac{1}{2\pi}\log(|x|/\ell_s \wedge 1) + O(1)$ and $|\dot{\mathsf{C}}_s(0,x)| \lesssim \vartheta_s^2 e^{-c|x|/\sqrt{s}}$. Therefore

$$\begin{aligned} |U_t(x)| = |e^{\beta C_t(0,x)} - 1| &\leqslant \int_0^t \beta |\dot{\mathsf{C}}_s(0,x)| e^{\beta C_s(0,x)} \frac{ds}{\ell_s^2} \\ &\lesssim \int_0^t \big(\ell_s^{\beta/2\pi} |x|^{-\beta/2\pi} e^{-c|x|/\sqrt{s}} e^{-\varepsilon^2 m^2 s}\big) \frac{ds}{\ell_s^2}. \end{aligned} \tag{A.38}$$

Choosing $c' < c/2$, we get $e^{c'|x|/\sqrt{t}} e^{-c|x|/\sqrt{s}} \leqslant e^{-\frac{1}{2}c|x|/\sqrt{s}}$ for $t \geqslant s$. Furthermore,

$$\sum_x |x|^{k-\beta/2\pi} e^{-\frac{1}{2}c|x|/\sqrt{s}} \lesssim \sqrt{s}^{2+k-\beta/2\pi} \tag{A.39}$$

holds if $2 + k > \beta/2\pi$ and $s \geqslant 1$. Therefore

$$\sum_x |U_t(x)|(|x|/\ell_t)^k e^{c'|x|/\sqrt{t}} \lesssim \ell_t^{-k} \int_0^t \big(\sqrt{s}^{2+k} e^{-\varepsilon^2 m^2 s}\big) \frac{ds}{\ell_s^2} \lesssim \ell_t^2. \tag{A.40}$$

The bounds are the same in the conservative case. $\square$

With the above preparation, we now prove Lemmas 3.9–3.11.

PROOF OF (3.63). For (3.63), we use $C_t(0,x) \geqslant 0$, which with $1 - e^{-x} \leqslant x$ for $x \geqslant 0$ gives the claim

$$\sum_x |1 - e^{-C_t(0,x)}| = \sum_x (1 - e^{-C_t(0,x)}) \leqslant \sum_x C_t(0,x) = O(\ell_t^2). \tag{A.41}$$

In the conservative case, $C_t^0(x) \geqslant -1/L^2$ and the claim follows similarly from $|1 - e^{-x}| \leqslant 2|x|$ for $x \geqslant -1$. $\square$

PROOF OF (3.64). For sufficiently small $c' > 0$, we write

$$\sum_{x,y} |U_t(x,y)|(\mathsf{Q}_t f(x) - \mathsf{Q}_t f(y))^2 = \sum_{x,y} A_{xy} B_{xy}^2, \tag{A.42}$$

where

$$A_{xy} = |U_t(x,y)|(|x-y|/\ell_t)^2 e^{2c'|x-y|/\ell_t}, \tag{A.43}$$

$$B_{xy} = \frac{|\mathsf{Q}_t f(x) - \mathsf{Q}_t f(y)|}{|x-y|/\ell_t} e^{-c'|x-y|/\ell_t} 1_{x \neq y}. \tag{A.44}$$

By (A.37), $\sup_x \sum_y A_{xy} \lesssim \ell_t^2$ for $c' > 0$ small enough. By (A.23) for $\ell_t \vartheta_t \mathsf{Q}_t$ instead of $\dot{\mathsf{C}}_t$ and the inequality $2ab \leqslant a^2 + b^2$, we have for $x \neq y$,

$$\begin{aligned} \frac{|\mathsf{Q}_t(x,z) - \mathsf{Q}_t(y,z)|}{|x-y|/\ell_t} e^{-c'|x-y|/\ell_t} &\lesssim \frac{\vartheta_t}{\ell_t} e^{-c'|x-z|/2\ell_t} e^{-c'|y-z|/2\ell_t} \\ &\leqslant \frac{\vartheta_t}{2\ell_t} (e^{-c'|x-z|/\ell_t} + e^{-c'|y-z|/\ell_t}). \end{aligned} \tag{A.45}$$

Thus there are positive

$$M_{xy} = M_{yx} = O(\vartheta_t \ell_t^{-1} e^{-c'|x-y|/\ell_t}),$$

i.e., $\sup_x \sum_y M_{xy} \lesssim \ell_t \vartheta_t$, such that

$$B_{xy} \leqslant \sum_z (M_{xz} + M_{yz})|f_z|. \tag{A.46}$$

Then (using $(a+b)^2 \leqslant 2a^2 + 2b^2$ and $A_{xy} = A_{yx}$),

$$\begin{aligned} &\sum_{x,y} A_{xy} B_{xy}^2 \\ &\quad \leqslant \sum_{x,y} A_{xy} \Big[\sum_z M_{xz}|f_z| + \sum_z M_{yz}|f_z| \Big]^2 \\ &\quad \leqslant 4 \sum_{x,y} A_{xy} \Big[\sum_z M_{xz}|f_z| \Big]^2 \leqslant 4 \Big[\sup_x \sum_y A_{xy} \Big] \sum_x \Big[\sum_z M_{xz}|f_z| \Big]^2. \end{aligned} \tag{A.47}$$

Similarly (with $2|ab| \leqslant a^2 + b^2$ and $M_{xy} = M_{yx}$)

$$\begin{aligned} &\sum_x \Big[\sum_z M_{xz}|f_z| \Big]^2 \\ &\quad = \sum_{x,z,w} M_{xz} M_{xw} |f_z f_w| \\ &\quad \leqslant \sum_{x,z,w} M_{xz} M_{xw} |f_z|^2 \leqslant \Big[\sup_z \sum_x M_{xz} \Big] \Big[\sup_x \sum_w M_{xw} \Big] \sum_z |f_z|^2. \end{aligned} \tag{A.48}$$

Therefore

$$\sum_{x,y} A_{xy} B_{xy}^2 \leqslant 4 \Big[\sup_x \sum_y A_{xy} \Big] \Big[\sup_z \sum_x M_{xz} \Big] \Big[\sup_x \sum_w M_{xw} \Big] |f|_2^2. \tag{A.49}$$

Since $\sup_x \sum_y A_{xy} \lesssim \ell_t^2$ and $\sup_x \sum_y M_{xy} \lesssim \vartheta_t \ell_t$, the desired bound $\lesssim \vartheta_t^2 \ell_t^4$ follows. The bounds are unchanged in the conservative case. □

PROOF OF (3.70). We proceed analogously to the proof of (3.64), i.e., for sufficiently small $c' > 0$, we write

$$\sum_{x,y} |U_t(x,y)||\mathsf{Q}_t f(x) - \mathsf{Q}_t f(y)| = \sum_{x,y} A_{xy} B_{xy}, \tag{A.50}$$

where

$$A_{xy} = |U_t(x,y)|(|x-y|/\ell_t) e^{c'|x-y|/\ell_t}, \tag{A.51}$$

$$B_{xy} = \frac{|\mathsf{Q}_t f(x) - \mathsf{Q}_t f(y)|}{|x-y|/\ell_t} e^{-c'|x-y|/\ell_t} 1_{x \neq y}. \tag{A.52}$$

By (A.37), again $\sup_x \sum_y A_{xy} \lesssim \ell_t^2$ for $c' > 0$ small enough, but now using that $\beta < 6\pi$ due to the different power in the definition of A_{xy}. The bound for B_{xy} is the same. From this, we conclude

$$\begin{aligned} \sum_{x,y} A_{xy} B_{xy} &\leqslant 2 \sum_{x,y} A_{xy} \Big[\sum_z M_{xz} |f_z| \Big] \\ &\leqslant 2 \Big[\sup_x \sum_y A_{xy} \Big] \Big[\sup_z \sum_x M_{xz} \Big] |f|_1 \lesssim \ell_t^3 \vartheta_t |f|_1. \end{aligned} \tag{A.53}$$

Since $\mathsf{Q}_t = \ell_t Q_t$, this is (3.70). The bounds are unchanged in the conservative case. ☐

PROOF OF (3.75). By (A.23) and (A.37) (with $\beta < 6\pi$), one can find $c' > 0$ small enough such that

$$\begin{aligned} &\sup_{x_1} \sum_{x_2,x_3} |U_t(x_1,x_2)||\delta_{12}\dot{\mathsf{C}}_t(x_1,x_2,x_3)| \\ &\lesssim \vartheta_t^2 \sup_{x_1} \sum_{x_2,x_3} |U_t(x_1,x_2)| e^{c'|x_1-x_2|/\ell_t} \frac{|x_1-x_2|}{\ell_t} \\ &\qquad e^{-c'|x_1-x_3|/2\ell_t - c'|x_2-x_3|/2\ell_t} \lesssim \ell_t^4 \vartheta_t^2, \end{aligned} \tag{A.54}$$

where a factor ℓ_t^2 comes first by summing over x_3 and another factor ℓ_t^2 from (A.37). The same applies when the roles of x_1, x_2, and x_3 in the sup and sum are exchanged. The bounds are unchanged in the conservative case. ☐

PROOF OF (3.76). By (A.24), there is $c' > 0$ small enough such that

$$\begin{aligned} &|\delta_{34}\delta_{12}\dot{\mathsf{C}}_t(x_1,x_2,x_3,x_4)| e^{-c.|x_1-x_2|/\ell_t - c'|x_3-x_4|/\ell_t} \\ &\quad \leq (|X_1 - x_2|/\ell_t)(|x_3-x_4|/\ell_t) e^{-c'|x_1-x_3|/\ell_t} \vartheta_t^2. \end{aligned} \tag{A.55}$$

Using (A.37) both for the sum over x_2 and x_4 (with $\beta < 6\pi$), as well as the elementary bound $\sup_{x_1} \sum_{x_3} e^{-c|x_1-x_3|/\ell_t} \lesssim \ell_t^2$, implies

$$\sup_{x_1} \sum_{x_2,x_3,x_4} |U_t(x_1,x_2)U_t(x_3,x_4)||\delta_{34}\delta_{12}\dot{\mathsf{C}}_t(x_1,x_2,x_3,x_4)| \lesssim \ell_t^6 \vartheta_t^2 \tag{A.56}$$

with one factor ℓ_t^2 from each of the sums. The bounds are unchanged in the conservative case. □

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ROLAND BAUERSCHMIDT
Centre of Mathematical Sciences
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM
E-mail: rb812@cam.ac.uk

THIERRY BODINEAU
École Polytechnique
Centre de Mathématiques Appliquées
Route de Saclay
91128 Palaiseau Cedex
FRANCE
E-mail: thierry.bodineau@polytechnique.edu

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