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Relations of the spaces $A^p(\Omega)$ and $C^p(\partial\Omega)$

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Abstract Let Ω be a Jordan domain in $\mathbb{C}$, J an open arc of $\partial\Omega$ and $\phi : D \to \Omega$ a Riemann map from the open unit disk D onto Ω. Under certain assumptions on ϕ we prove that if a holomorphic function $f \in H(\Omega)$ extends continuously on $\Omega \cup J$ and $p \in \{1, 2, \dots\} \cup \{\infty\}$, then the following equivalence holds: the derivatives $f^{(l)}, 1 \leq l \leq p, l \in \mathbb{N}$, extend continuously on $\Omega \cup J$ if and only if the function $f|_J$ has continuous derivatives on J with respect to the position of orders $l, 1 \leq l \leq p, l \in \mathbb{N}$. Moreover, we show that for the relevant function spaces, the topology induced by the $l-$derivatives on $\Omega, 0 \leq l \leq p, l \in \mathbb{N}$, coincides with the topology induced by the same derivatives taken with respect to the position on J.

Keywords Riemann map · Poisson Kernel · Jordan curve · Smoothness on the boundary

1 Introduction

In this paper we investigate the relationship between the continuous extendability of the derivatives of a function $f \in A(\Omega)$, for some Jordan domain Ω, and the differentiability of the map $t \mapsto f(\gamma(t))$ for some parametrization γ of

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$\partial\Omega$. Here, $A(\Omega)$ is the collection of all complex functions holomorphic on Ω and continuous on $\overline{\Omega}$. Specifically, it is well known that the first p derivatives of a function $f \in A(D)$, D being the unit disk in $\mathbb{C}$, continuously extend over $\overline{D}$ if and only if the map $t \mapsto f(e^{it})$ is p times continuously differentiable ([6]). We generalize this for functions that are holomorphic on the unit disk but now continuously extend over an open arc of $\mathbb{T} = \partial D$ and prove an analogous equivalence for functions defined on Jordan domains that have sufficiently smooth Riemann maps.

The spaces $A^p(\Omega)$, $p \in \{0,1,2,...\} \cup \{\infty\}$, consist of all holomorphic functions f in Ω whose derivatives $f^{(l)}$, $l \in \{0,1,...\}, l \leq p$, extend continuously on $\overline{\Omega}$. It is well known that for any f in the disc algebra $A(D)$, $f \in A^p(D)$ if and only if the map $t \mapsto f(e^{it})$ is C^p smooth. In other words, $A^p(D) = A(D) \cap C^p(\mathbb{T})$ both as sets and as topological spaces. Additionally, if $f \in A(D)$ and $g(t) = f(e^{it}), t \in \mathbb{R}$, the equation that relates the continuous extension of f' on $\mathbb{T}$ and the derivative of g is as expected, i.e. $\frac{dg}{dt}(t) = ie^{it}f'(e^{it})$. To prove this, one can use the Poisson representation, to recover the values of f in the disk from its boundary values, i.e. $f(re^{it}) = (g * P_r)(t)$ where P_r denotes the Poisson kernel, differentiate both sides with respect to t and let $r \to 1^-$. A detailed proof can be found in [6].

In this paper we prove analogous results for functions $f \in A(D)$ whose derivatives continuously extend on an open arc of the unit circle but not necessarily the entire circle. Moreover, using Riemann's mapping theorem, we can drop our initial assumption $f \in A(D)$ and instead assume that it only extends continuously over the specific arc we are interested in. Precisely, if $f : D \to \mathbb{C}$ is holomorphic and continuously extends over an open arc $J \subseteq \mathbb{T}$, then its first p derivatives continuously extend over that arc if and only if the map $t \mapsto f(e^{it})$ is in $C^p(I)$, where $I = (a,b)$ is an interval in $\mathbb{R}$ with $J = \{e^{it} : t \in I\}$. This motivates a more general definition of the spaces A^p.

In section 3 we consider functions f holomorphic on a Jordan domain Ω and continuous on $\Omega \cup J$, for some open arc J of $\partial\Omega$. We prove that for any $p \in \{1,2,\dots\} \cup \{\infty\}$, the derivatives $f^{(l)}, 0 \leq l \leq p$, continuously extend over $\Omega \cup J$ if and only if the continuous extension of f on J is p times continuously differentiable on J with respect to the position ([2]). To do this, we place a smoothness assumption for the Riemann map $\phi : D \to \Omega$ from the open unit disk D onto Ω. The condition is that $(\phi^{-1})'$ has a continuous extension on $\Omega \cup J$ and that $(\phi^{-1})'(z) \neq 0$ on $\Omega \cup J$.

2 Extendability over an open arc of the unit circle

For $0 \leq p \leq +\infty$, $A^p(D)$ denotes the space of holomorphic functions on D whose derivatives of order $l \in \mathbb{N}, 0 \leq l \leq p$, extend continuously over $\overline{D}$. It is topologized via the semi-norms:

$$|f|_l = \sup_{z \in \overline{D}} |f^{(l)}(z)| = \sup_{z \in \mathbb{T}} |f^{(l)}(z)|, 0 \leq l \leq p, l \in \mathbb{N}.$$

The following theorem is well known. A detailed proof can be found in [6],

Theorem 1 *For all* $f \in A(D)$ *the following equivalence holds:* $f \in A^p(D)$ *if and only if the map* $g(t) = f(e^{it})$, $t \in \mathbb{R}$, *is* p *times continuously differentiable. In that case:*

$$\frac{dg}{dt}(t) = ie^{it} f'(e^{it}). \tag{1}$$

We now generalize this on functions that are holomorphic on D and continuously extend over an open arc J of the unit circle. We prove that for any such function f and $p \in \{1, 2, ...\} \cup \{\infty\}$ the first p derivatives of f continuously extend over $D \cup J$ if and only if the map $t \mapsto f(e^{it})$ is p times continuously differentiable in $I = \{t \in [a, a + 2\pi] : e^{it} \in J\}$ for a suitable $a \in \mathbb{R}$. Denote by $A^p(D, J)$ the space of holomorphic functions whose first p derivatives continuously extend over $D \cup J$ and let $C^p(J)$ be the class of functions $f : J \to \mathbb{C}$, such that the map $t \mapsto f(e^{it})$, $t \in I$, is p times continuously differentiable. The aim is to show the equality $A^p(D, J) = A(D, J) \cap C^p(J)$. For simplicity, we take $J = \{e^{it} : 0 < t < 1\}$ throughout this section. For any $z = re^{i\theta} \in \mathbb{C}$ we denote by $P_z(t)$ or $P_r(t)$ the Poisson kernel [1].

Proposition 1 *If* $u : [0, 1] \to \mathbb{R}$ *is a continuous function then:*

$$A(z) = \frac{1}{2\pi} \int_0^1 u(t) P_z(t) dt \tag{2}$$

is well defined in $\mathbb{C} \setminus \bar{J}$ *and* C^∞ *harmonic.*

Proof To see that $A(z)$ is well defined in $\mathbb{C} \setminus \bar{J}$ observe that:

$$A(z) = \frac{1}{2\pi} \int_0^1 u(t) P_z(t) dt = Re \left(\frac{1}{2\pi} \int_0^1 u(t) \frac{1 + e^{-it} z}{1 - e^{-it} z} dt \right) \tag{3}$$

and $1 - e^{-it} z = 0 \Leftrightarrow z = e^{i(t+2k\pi)}, k \in \mathbb{Z}$. For a fixed $z \in \mathbb{C} \setminus \{e^{it} : 0 \le t \le 1\}$ we have $\delta_z = dist(1, \{ze^{-it} : 0 \le t \le 1\}) > 0$ and:

$$\left| u(t) \frac{1 + e^{-it} z}{1 - e^{-it} z} \right| \le \sup_{t \in [0,1]} |u(t)| \frac{1 + |z|}{\delta_z} < +\infty \tag{4}$$

for all $t \in [0, 2\pi]$. Since the quantity on the right hand side is integrable we deduce that A(z) is indeed well defined in $\mathbb{C} \setminus \bar{J}$.

In order to prove that A is C^∞ harmonic it suffices to show that $g(z) = \frac{1}{2\pi} \int_0^1 u(t) \frac{1+e^{-it} z}{1-e^{-it} z} dt$ is holomorphic in $\mathbb{C} \setminus \bar{J}$, since A is the real part of g according to (3). Note that for $z \neq z_0$:

$$\frac{g(z) - g(z_0)}{z - z_0} = \frac{1}{2\pi} \int_0^1 u(t) \frac{2e^{-it}}{(1 - e^{-it} z)(1 - e^{-it} z_0)} dt \tag{5}$$

and hence for z sufficiently close to z_0:

$$\left|\frac{g(z)-g(z_0)}{z-z_0}-\frac{1}{2\pi}\int_0^1 u(t)\frac{2e^{-it}}{(1-e^{-it}z_0)^2}dt\right| = \tag{6}$$

$$\left|\frac{1}{2\pi}\int_0^1 u(t)\frac{2e^{-2it}(z-z_0)}{(1-e^{-it}z)(1-e^{-it}z_0)^2}dt\right| \tag{7}$$

$$\leq \frac{1}{2\pi}\sup_{t\in[0,1]}|u(t)|2\left(\frac{2}{\delta_{z_0}}\right)^3|z-z_0| \xrightarrow{z\to z_0} 0. \tag{8}$$

Therefore g is holomorphic and the proof is complete.

□

Lemma 1 *Let $u:[0,1]\to\mathbb{R}$ be a continuous function and define $A(z)$ as in Proposition 1. For all $z=re^{i\theta}\in\mathbb{C}\setminus\bar{J}$:*

$$\frac{dA}{d\theta}(re^{i\theta}) = \frac{1}{2\pi}\int_0^1 u(t)\frac{-2r(1-r^2)\sin(\theta-t)}{(1+r^2-2r\cos(\theta-t))^2}dt. \tag{9}$$

Thus, for $\theta\in(1,2\pi)$ and $r=1$:

$$\frac{dA}{d\theta}(e^{i\theta}) = 0. \tag{10}$$

Proof Since $A(re^{i\theta}) = \frac{1}{2\pi}\int_0^1 u(t)P_r(\theta-t)dt = \frac{1}{2\pi}(f*P_r)$ and P_r is differentiable in respect to θ we have that $A(re^{i\theta})$ is differentiable in respect to θ,

$$\frac{dA}{d\theta}(re^{i\theta}) = \frac{1}{2\pi}\frac{d(f*P_r)}{d\theta}(re^{i\theta}) = \frac{1}{2\pi}(f*\frac{dP_r}{d\theta})(re^{i\theta}) \Longrightarrow \tag{11}$$

$$\frac{dA}{d\theta}(re^{i\theta}) = \frac{1}{2\pi}\int_0^1 u(t)\frac{-2r(1-r^2)\sin(\theta-t)}{(1+r^2-2r\cos(\theta-t))^2}dt. \tag{12}$$

(10) is derived from (9) substituting $r=1$ and $\theta\in(1,2\pi)$.

□

Because every continuous function $u:[0,1]\to\mathbb{C}$ can be considered a 2π-periodic function $u:\mathbb{R}\to\mathbb{C}$ such that $v(x)=u(x)$ for $x\in[0,1]+2\pi\mathbb{Z}$ and $v(x)=0$ otherwise, one can expect that such a function when convolved with the Poisson kernel would retain the nice properties. More specifically, it will uniformly converge to 0 and to $u(x)$ on the compact subsets of the respective open arcs. We prove this in Propositions 2 and 3.

Proposition 2 *If $A(z)$ is as before, then for all $\theta\in(1,2\pi)$ and $l\in\mathbb{N}$:*

$$\lim_{r\to 1^-}\frac{d^lA}{d\theta^l}(re^{i\theta}) = 0. \tag{13}$$

The convergence is uniform in the compact subsets of $(1,2\pi)$.

Proof We start with $l = 1$. It suffices to prove that for all $[\theta_1, \theta_2] \subset (1, 2\pi)$ the convergence is uniform. Observe that $0 < \theta_1 - t < \theta_2 - t < 2\pi$ for all $t \in [0, 1]$ and therefore $\cos(\theta - t) \leq \max\{\cos(\theta_1 - 1), \cos(\theta_2)\} = M < 1, \forall t \in [0, 1]$ and $\forall\theta \in [\theta_1, \theta_2]$. This implies that $1 + r^2 - 2r\cos(\theta - t) \geq 1 + r^2 - 2rM = (1 - r)^2 + 2r(1 - M)$ for all $t \in [0, 1]$ and $\theta \in [\theta_1, \theta_2]$. Thus, for all $\theta \in [\theta_1, \theta_2]$ and $0 < r < 1$:

$$\left|\frac{dA}{d\theta}(re^{i\theta})\right| = \left|\frac{1}{2\pi}\int_0^1 u(t)\frac{-2r(1-r^2)\sin(\theta - t)}{(1 + r^2 - 2r\cos(\theta - t))^2}dt\right| \tag{14}$$

$$\leq \frac{1}{2\pi}\|u\|_\infty \frac{2r(1-r^2)}{((1-r)^2 + 2r(1-M))^2} \tag{15}$$

and hence $\sup\limits_{\theta\in[\theta_1,\theta_2]} \left|\frac{dA}{d\theta}(re^{i\theta})\right| \xrightarrow{r\to 1^-} 0$ since the right hand side of (15) converges to 0 as $r \to 1^-$.
Note that no matter how many times we differentiate P_r in respect to θ we will have a finite sum of fractions with numerator $c(1-r^2)^k\cos(\theta)^l\sin(\theta)^m$ for $c \neq 0, k, l, m \in \mathbb{N}$ and denominator a power of $(1 + r^2 - 2r\cos(\theta - t))$. So for any $l \geq 2$ the same arguments apply.

□

Denote by $C^p([0, 1])$ the class of functions $u : [0, 1] \to \mathbb{C}$ that are p times continuously differentiable in $(0, 1)$ and $u^{(l)}$ continuously extend on $[0, 1]$ for all $0 \leq l \leq p, l \in \mathbb{N}$.

Proposition 3 *Let* $p \in \{0, 1, ...\} \cup \{\infty\}$*,* $u : [0, 1] \to \mathbb{R}$ *of class* $C^p([0, 1])$ *and define* $A(z)$ *as in Proposition 1. For all* $\theta_0 \in (0, 1)$ *and* $0 \leq l \leq p, l \in \mathbb{N}$*,*

$$\lim_{\substack{z\to e^{i\theta_0}\\|z|<1}} \frac{d^l A}{d\theta^l}(z) = u^{(l)}(\theta_0). \tag{16}$$

The convergence is uniform in the compact subsets of $(0, 1)$*.*

Proof By a theorem of Borel (see [5]), we can find a function $q : \mathbb{R} \to \mathbb{R}$ of class $C^\infty(\mathbb{R})$ such that $q^l(0) = u^l(0)$ and $q^l(1) = u^l(1)$ for $0 \leq l \leq p, l \in \mathbb{N}$. Thus, the function:

$$g(t) = \begin{cases} u(t), t \in [0, 1] \\ q(t), t \in (1, 2\pi) \end{cases}$$

is of class $C^p(\mathbb{T})$. Define:

$$G(z) = \frac{1}{2\pi}\int_0^{2\pi} g(t)P_z(t)dt. \tag{17}$$

It is well known that uniformly for all $\theta \in \mathbb{R}, 0 \leq l \leq p, l \in \mathbb{N}$:

$$\lim_{\substack{z\to e^{i\theta}\\|z|<1}} \frac{d^l G}{d\theta^l}(z) = g^{(l)}(\theta) \tag{18}$$

Let

$$A(z) = \frac{1}{2\pi}\int_0^1 u(t)P_z(t)dt \text{ and } B(z) = \frac{1}{2\pi}\int_1^{2\pi} q(t)P_z(t)dt. \tag{19}$$

Note that $G(z) = A(z) + B(z)$ for all $|z| < 1$ and hence for $0 \leq l \leq p, l \in \mathbb{N}$:

$$\frac{d^l G}{d\theta^l}(z) = \frac{d^l A}{d\theta^l}(z) + \frac{d^l B}{d\theta^l}(z). \tag{20}$$

From (18) Proposition 2 we have that $\frac{d^l G}{d\theta^l}(re^{i\theta}) \to u^{(l)}(\theta)$ and $\frac{d^l B}{d\theta^l}(re^{i\theta}) \to 0$ as $r \to 1^-$, uniformly in the compact subsets of $(0, 1)$. As a result,

$$\lim_{\substack{z \to e^{i\theta_0} \\ |z|<1}} \frac{d^l A}{d\theta^l}(z) = \lim_{\substack{z \to e^{i\theta_0} \\ |z|<1}} \left(\frac{d^l G}{d\theta^l}(z) - \frac{d^l B}{d\theta^l}(z)\right) = u^{(l)}(e^{i\theta_0}) \tag{21}$$

while the convergence is uniform in the compact subsets of $(0, 1)$.

□

Remark 1 By linearity, Propositions 1, 2, 3 and Lemma 1 hold for complex functions $f = u + iv$ where $u = Ref$ and $v = Imf$, since we can apply them to the real and imaginary part separately.

We now adapt the proof of Theorem 1 for functions whose derivatives only extend over an open arc of the unit circle.

Theorem 2 *Let* $p \in \{1, 2, ...\} \cup \{\infty\}$, $f \in A(D)$ *and* $g(t) = f(e^{it}), t \in (0, 1)$. *The following are equivalent:* $f^{(l)}$ *continuously extends on* $D \cup J$ *for all* $0 \leq l \leq p, l \in \mathbb{N}$ *if and only if* g *is* p *times continuously differentiable in* $(0, 1)$. *In that case, for all* $t \in (0, 1)$

$$\frac{dg}{dt}(t) = ie^{it}f'(e^{it}). \tag{22}$$

Proof We prove it by induction on p. For $p = 1$, let $t_0 \in (0, 1)$ and $t_1 < t_0 < t_2$ such that $[t_1, t_2] \subset (0, 1)$. Assuming that $f^{(l)}$ continuously extends over $D \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$, let $f_r(t) = f(re^{it})$, $h(t) = ie^{it}f'(e^{it})$ and:

$$h_r(t) = \frac{df_r}{dt}(t) = ire^{it}f'(re^{it}) \tag{23}$$

for all $t \in (0, 1)$ and $0 < r < 1$. We have $h_r \to h$ as $r \to 1^-$ uniformly in $[t_1, t_2]$, since f' is continuous in $D \cup J$. Note that $f_r(t_0) \to f(e^{it_0})$ and hence from a well-known theorem $f_r \to \int hdt + c$, for a some $c \in \mathbb{C}$, as $r \to 1$, while the convergence is uniform in $[t_1, t_2]$. Additionally, $f_r \to g$, as $r \to 1$, uniformly in $[t_1, t_2]$ and therefore $g'(t) = h(t) = ie^{it}f'(e^{it}), t \in (t_1, t_2)$. However, t_0 was arbitrary thus, $g \in C^1((0, 1))$ and (22) holds.

For the converse let $g \in C^1((0, 1))$. We now use the Poisson representation:

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})P_z(t)dt. \tag{24}$$

for $|z|<1$. Define:

$$A(z)=\frac{1}{2\pi}\int_{t_1}^{t_2} g(t)P_z(t)dt \text{ and } B(z)=\frac{1}{2\pi}\int_{t_2}^{2\pi+t_1} f(e^{it})P_z(t)dt. \tag{25}$$

Note that $f(z)=A(z)+B(z)$ for all $|z|<1$. Consequently,

$$\frac{df}{dt}(re^{it})=\frac{dA}{dt}(re^{it})+\frac{dB}{dt}(re^{it}) \tag{26}$$

for all $0<r<1$ and $t\in\mathbb{R}$. Since $[t_1,t_2]\subset(0,1)$, Propositions 2 and 3 imply that:

$$\lim_{r\to1^-}\frac{dA}{dt}(re^{it})=\frac{dg}{dt}(t) \tag{27}$$

and

$$\lim_{r\to1^-}\frac{dB}{dt}(re^{it})=0 \tag{28}$$

uniformly for $t\in[t_1,t_2]$. Combining (26), (27) and (28) we get:

$$\lim_{r\to1^-} f'(re^{it})=\lim_{r\to1^-}\frac{1}{ire^{it}}\frac{df}{dt}(re^{it})=\frac{1}{ie^{it}}\frac{dg}{dt}(t) \tag{29}$$

uniformly for $t\in[t_1,t_2]$. Since t_0 was arbitrarily chosen in $(0,1)$ we deduce that f' extends continuously on $D\cup J$. To complete the induction, let us assume that the theorem holds for some $p\geq1$. If $f^{(l)}$ continuously extends on $D\cup J$ for all $0\leq l\leq p+1, l\in\mathbb{N}$ it follows that $(f')^{(l)}$ continuously extends on $D\cup J$ for all $0\leq l\leq p, l\in\mathbb{N}$. By the induction hypothesis the map $t\mapsto f'(e^{it})$ belongs in the class $C^p((0,1))$ and since, by the case of $p=1$, $g'(t)=ie^{it}f'(e^{it})$ we have $g'\in C^p((0,1))$ and hence $g\in C^{p+1}((0,1))$. For the converse, if $g\in C^{p+1}((0,1))$ it follows from (22) that $g'(t)=ie^{it}f'(e^{it})\in C^p((0,1))$ and therefore, the map $t\mapsto f'(e^{it})$ is of class $C^p((0,1))$. By the induction hypothesis, $(f')^{(l)}$ continuously extends on $D\cup J$, for all $0\leq l\leq p, l\in\mathbb{N}$ and hence $f^{(l)}$ continuously extends on $D\cup J$, for all $0\leq l\leq p+1, l\in\mathbb{N}$. The case of $p=\infty$ follows easily.

□

Using Riemann's mapping theorem [1] we can drop the assumption of continuity over $\overline{D}$. Indeed, we can just have f continuously extend over the open arc we are dealing with, i.e. $f\in A(D,J)$.

Theorem 3 *Let $p\in\{1,2,...\}\cup\{\infty\}$, $f:D\cup J\to\mathbb{C}$ continuous on $D\cup J$ and holomorphic in D and $g(t)=f(e^{it}), t\in(0,1)$. The following are equivalent: $f^{(l)}$ continuously extends over $D\cup J$, for all $0\leq l\leq p, l\in\mathbb{N}$, if and only if g is p times continuously differentiable in $(0,1)$. In that case:*

$$\frac{dg}{dt}(t)=ie^{it}f'(e^{it}) \tag{30}$$

for all $t\in(0,1)$.

Proof We prove it by induction on p. For $p = 1$, the only if part is proven like Theorem 2; we also obtain (30). For the converse, let $t_0 \in (0,1)$, $t_1 < t_0 < t_2$ such that $[t_1, t_2] \subseteq (0,1)$ and $J' = \{e^{it} : t_1 < t < t_2\}$. Set $V = \{|z| < 1, z \neq 0 : \frac{z}{|z|} \in J'\}$. It is easily verified that V is simply connected and hence there is a conformal map $\phi : D \to V$ which extends to homeomorphism over the closures $\phi : \overline{D} \to \overline{V}$, by the Osgood-Carathéodory theorem ([4]). Since $\phi(\mathbb{T}) = \{e^{it} : t_1 \leq t \leq t_2\} \cup \{re^{it_1} : 0 \leq r \leq 1\} \cup \{re^{it_2} : 0 \leq r \leq 1\}$ and $\{e^{it} : t_1 \leq t \leq t_2\}$ is connected we deduce that $\phi^{-1}(\{e^{it} : t_1 \leq t \leq t_2\})$ is closed and connected in $\mathbb{T}$. Without loss of generality, assume that $\phi^{-1}(\{e^{it} : t_1 \leq t \leq t_2\}) = \{e^{it} : 0 \leq t \leq 1\}$. Consequently, $f \circ \phi : \overline{D} \to \mathbb{C}$ is continuous on $\overline{D}$ and holomorphic in D that is, of class $A(D)$. Using the Reflection Principle, we deduce that ϕ conformally extends on an open $G \supset V \cup \{e^{it} : t_1 \leq t \leq t_2\}$ thus, $t \mapsto \phi(e^{it}), t \in (t_1, t_2)$ is of class $C^\infty((t_1, t_2))$ with non-vanishing derivative (see [1] p. 233-235). Since $g \in C^1((t_1, t_2))$ and $t \mapsto \phi(e^{it})$ is in $C^\infty((0,1))$ we have that the map $t \mapsto f(\phi(e^{it}))$ is in $C^1((0,1))$. By Theorem 2 the derivative $(f \circ \phi)'$ continuously extends on $D \cup J$ and hence f' continuously extends in $D \cup J'$, because $f = (f \circ \phi) \circ \phi^{-1}$ and $f' = (f \circ \phi)' \circ \phi^{-1} \cdot (\phi^{-1})'$. Our choice of t_0 was arbitrary and therefore f' continuously extends on $D \cup J$. To complete the induction, we follow the proof of Theorem 2. The case of $p = \infty$ follows easily.

□

In other words, Theorem 3 states that for any open arc J of $\mathbb{T}$ and $p \in \{0, 1, 2, ...\} \cup \{\infty\}$, $A^p(D, J) = A(D, J) \cap C^p(J)$ holds.

Remark 2 In all Propositions and Lemmas of this section, $[0,1]$ can be replaced by any interval $[a, b]$ with $0 \leq a < b < a + 2\pi$.

3 Jordan Domains

Motivated by Theorem 3 we now give a more general definition of the spaces A^p. Let Ω be a Jordan domain and J an open arc of $\partial\Omega$. Denote by $A^p(\Omega, J)$ the collection of all functions f holomorphic on Ω such that $f^{(l)}$ continuously extends on $\Omega \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$. Note that $A^p(\Omega, \partial\Omega) = A^p(\Omega)$. If $\gamma : I \to J$ is a parametrization of J, denote by $C^p_\gamma(J)$ the class of all functions $f : G \to \mathbb{C}$ defined on a varying set G, $J \subset G$, such that $f \circ \gamma : I \to \mathbb{C}$ is p times continuously differentiable. Moreover, if $\phi : \overline{D} \to \overline{\Omega}$ is a Riemann map, denote by $C^p_\phi(J)$ the class $C^p_\gamma(J)$ where $\gamma(t) = \phi(e^{it})$. Since any two Riemann maps differ by an automorphism of the unit disk it is easily verified that the spaces $C^p_\phi(J)$ do not depend on the chosen Riemann map. Next, we consider differentiability on J with respect to the position [2].

Definition 1 Let J be a Jordan arc and $f : J \to \mathbb{C}$. We define the derivative of f on $z_0 \in J$ by:

$$\frac{df}{dz}(z_0) = \lim_{z \to z_0, z \in J} \frac{f(z) - f(z_0)}{z - z_0} \tag{31}$$

if this limit exists and is a complex number.

In order to go one step further , we consider the derivative $\frac{df}{dz}$ on J of Definition 1 and we take its derivative on J with respect to the position.

Definition 2 A function $f : J \to \mathbb{C}$ belongs to the class $C^1(J)$ if $\frac{df}{dz}(z)$ exists and is continuous for $z \in J$. Inductively, suppose that $\frac{d^{p-1}f}{dz^{p-1}}$ is well defined on J for some $p = 2, \dots, +\infty$, we say that f is of class $C^p(J)$ if

$$\frac{d^p f}{dz^p}(z) = \frac{d(\frac{d^{p-1}f}{dz^{p-1}})}{dz}(z) \tag{32}$$

exists and is continuous on J.

Remark 3 In [2] the following fact is proven. If $\gamma : I \to J$ is a C^n regular (with non vanishing derivative) parametrization of a Jordan arc $J, n \in \{1, 2, \dots\} \cup \{\infty\}$, a function f is of class $C^p(J)$ if and only if $g(t) = (f \circ \gamma)(t), t \in I$, is of class $C^p(I), p \in \{1, 2, ...\} \cup \{\infty\}, p \leq n$. Additionally,

$$\frac{dg}{dt}(t) = \frac{df}{dz}(\gamma(t)) \cdot \gamma'(t). \tag{33}$$

In this section we prove that given a Jordan domain Ω, $\phi : \overline{D} \to \overline{\Omega}$ a Riemann map such that $\phi^{-1} \in A^1(\Omega, J)$ and $(\phi^{-1})'(z) \neq 0, z \in \Omega \cup J$, $f \in A(\Omega, J)$ and $p \in \{1, 2, \dots\} \cup \{\infty\}$ the following are equivalent: $f^{(l)}$ continuously extend over $\Omega \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$ if and only if $f|_J$ is of class $C^p(J)$. That is $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J)$.

Let us first prove a straightforward fact concerning the parametrization of an arc J of $\partial\Omega$ induced by a Riemann map ϕ.

Theorem 4 *Let Ω be a Jordan domain, $\phi : \overline{D} \to \overline{\Omega}$ a Riemann map and J an open arc of $\partial\Omega$ such that ϕ^{-1} is of class $A^n(\Omega, J)$ for some $n \in \{1, 2, \dots\} \cup \{\infty\}$ and $(\phi^{-1})'(z) \neq 0, z \in \Omega \cup J$. The following holds for any $p \in \{1, 2, \dots\} \cup \{\infty\}, p \leq n$: $A^p(\Omega, J) = A(\Omega, J) \cap C^p_\phi(J)$. In that case, if $f \in A^p(\Omega, J)$ and $g(t) = (f \circ \phi)(e^{it})$:*

$$\frac{dg}{dt}(t) = ie^{it} f'(\phi(e^{it}))\phi'(e^{it}) \tag{34}$$

for all $t \in \{s \in \mathbb{R} : \phi(e^{is}) \in J\}$.

On the right hand side of (34) f' denotes the continuous extension of $f'(z)$ from Ω to $\Omega \cup J$.

Before proceeding to the proof, let us note that the choice of the Riemann map is irrelevant. Suppose that Φ, Ψ are two Riemann maps and J an open arc of $\partial\Omega$ such that Φ^{-1} is of class $A^n(\Omega, J), n \in \{1, 2, ...\} \cup \{\infty\}$, and $(\Phi^{-1})'(z) \neq 0$ for all $z \in \Omega \cup J$. Then, Ψ^{-1} is also of class $A^n(\Omega, J)$ and $(\Psi^{-1})'$ is non zero in $\Omega \cup J$. To see this, observe that $\Phi^{-1} \circ \Psi$ is an automorphism of the unit disk and hence there are $a \in D$ and $c \in \mathbb{T}$ such that $\Psi = \Phi \circ \phi_a$, where $\phi_a(z) = c\frac{z-a}{1-\bar{a}z}, z \in D$. Note that ϕ_a is holomorphic in $D(0, \frac{1}{a}) \supset D$ with non vanishing derivative thus, $\Psi^{(l)}$ can be extended over $\overline{D}$ for all $0 \leq$

$l \leq n, l \in \mathbb{N}$. Additionally, $(\Psi^{-1})'(z) = \frac{1-|a|^2}{(1+\bar{a}\Phi^{-1}(z))^2} \cdot (\Phi^{-1})'(z) \neq 0$ for all $z \in \Omega \cup J$. Moreover, Φ^{-1} being of class $A^n(\Omega, J)$ is equivalent to Φ being of class $A^n(D, I)$, $I = \phi^{-1}(J)$. This is easy to see given that $(\Phi^{-1})'(z) \neq 0$ for $z \in \Omega \cup J$. In other words, if a Riemann map induces a regular n times continuously differentiable parametrization of J then, the same holds for any other Riemann map.

Proof (Theorem 4) Set $I = \phi^{-1}(J)$ and $\tilde{I} = \{s \in \mathbb{R} : \phi(e^{is}) \in J\}$. Given an $f \in A(\Omega, J)$ one can easily verify that f is of class $A^p(\Omega, J)$ if and only if $f \circ \phi$ is of class $A^p(D, I)$, since ϕ' is non zero in $D \cup I$ and $p \leq n$. By Theorem 3, $f \circ \phi$ is of class $A^p(D, I)$ if and only if the map $t \mapsto (f \circ \phi)(e^{it}), t \in \tilde{I}$, is p times continuously differentiable which by definition is equivalent to f being of class $C^p_\phi(J)$. To see that (34) holds, set $g_r(t) = f(\phi(re^{it})), t \in \mathbb{R}, 0 < r < 1$. Note that $g_r \to g$ as $r \to 1^-$ uniformly in the compact subsets of $\tilde{I}$. Additionally, $\frac{dg_r}{dt}(t) = ire^{it} f'(\phi(re^{it}))\phi'(e^{it})$ which uniformly converges to $ie^{it} f'(\phi(e^{it}))\phi'(e^{it})$ as $r \to 1^-$ in the compact subsets of $\tilde{I}$. From a well known theorem of calculus (34) follows.

□

In the next theorem we prove that for any function $f \in A(\Omega, J)$ the first p derivatives of f continuously extend over $\Omega \cup J$ if and only if $f|_J$ is of class $C^p(J)$, given that J is a smooth enough open arc of the Jordan domain Ω. That is, if ϕ is a Riemann map and ϕ^{-1} is of class $A^1(\Omega, J)$ with $(\phi^{-1})'(z) \neq 0, z \in \Omega \cup J$ then $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J)$ for all $p \in \{1, 2, ...\} \cup \{\infty\}$.

Theorem 5 *Let Ω be a Jordan domain, $\phi : D \to \Omega$ a Riemann map and J an open arc of $\partial\Omega$ such that ϕ^{-1} is of class $A^1(\Omega, J)$ with $(\phi^{-1})'(z) \neq 0, z \in \Omega \cup J$. Given an $f \in A(\Omega, J)$ and $p \in \{1, 2, ...\} \cup \{\infty\}$ we have that: $f^{(l)}$ continuously extend over $\Omega \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$ if and only if $f|_J$ is of class $C^p(J)$. That is, $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J)$. In that case:*

$$f^{(l)}(z) = \frac{d^l\, f|_J}{dz^l}(z) \tag{35}$$

for all $z \in J$ and $0 \leq l \leq p, l \in \mathbb{N}$.

On the left hand side of (35), $f^{(l)}(z)$ denotes the continuous extension of $f^{(l)}$ on $\Omega \cup J$, while on the right hand side the differentiation is with respect to the position.

Proof Let $p = 1$ and $f \in A(\Omega, J)$. Denote by $\tilde{I} = \{s \in \mathbb{R} : \phi(e^{is}) \in J\}$. By Theorem 4 we have that $f \in A^1(\Omega, J)$ if and only if the map $g(t) = f(\phi(e^{it})), t \in \tilde{I}$ is continuously differentiable. Additionally, $\frac{dg}{dt}(t) = ie^{it}\phi'(e^{it})f'(\phi(e^{it}))$ and:

$$\frac{dg}{dt}(e^{it}) = \frac{d\, f|_J}{dz}(\phi(e^{it})) \cdot \frac{d\, \phi|_{\mathbb{T}}}{dz}(e^{it}) \cdot ie^{it} = ie^{it}\phi'(e^{it})\frac{d\, f|_J}{dz}(\phi(e^{it})) \tag{36}$$

Since $ie^{it}\phi'(e^{it})$ is non zero we have that g is continuously differentiable if and only if $f|_J$ is of class $C^1(J)$. Consequently, f is of class $A^1(\Omega, J)$ if and only if g is of class $C^1(\tilde{I})$ if and only if $f|_J$ is of class $C^1(J)$. Equation (35) follows from equations (34) and (36). For the induction step, assume that $p > 1$. We have that $f \in A^p(\Omega, J)$ if and only if $f' \in A^{p-1}(\Omega, J)$ which by the induction hypothesis is equivalent to $f'|_J \in C^{p-1}(J)$ and that is equivalent to $f|_J \in C^p(J)$, by (35) for $l = 1$. Moreover,

$$\frac{d^l\, f|_J}{dz^l}(z) = \frac{d^{(l-1)}}{dz^{(l-1)}}\left(\frac{d\, f|_J}{dz}(z)\right) = \frac{d^{(l-1)}}{dz^{(l-1)}}\,(f'(z)) = (f')^{(l-1)}(z) = f^{(l)}(z) \tag{37}$$

for all $z \in J$ and $1 \leq l \leq p, l \in \mathbb{N}$.

$\square$

Remark 4 Note that if Ω is a Jordan domain, J an analytic arc of $\partial\Omega$ and $\phi : D \to \Omega$ a Riemann map we know from [1] (p. 235) that ϕ^{-1} has a conformal extension over an open set $G \supset \Omega \cup J$. Consequently, if J is an analytic arc we immediately have $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J)$ for all $p \in \{1, 2, ...\} \cup \{\infty\}$.

In addition, if we have γ a C^n regular parametrization of $J, n \in \{1, 2, ...\} \cup \{\infty\}$ we have a triple equivalence as derived from Remark 3 and Theorem 5.

Theorem 6 *Let Ω be a Jordan domain, ϕ a Riemann map and J an open arc of $\partial\Omega$ such that ϕ^{-1} is of class $A^1(\Omega, J)$ and $(\phi^{-1})'(z) \neq 0, z \in \Omega \cup J$, γ a C^n regular parametrization of $J, n \in \{1, 2, ...\} \cup \{\infty\}$, $f \in A(\Omega, J)$ and $p\{1, 2, ...\} \cup \{\infty\}, p \leq n$. The following are equivalent:*

1. *$f^{(l)}$ continuously extend over $\Omega \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$. That is, f is of class $A^p(\Omega, J)$.*
2. *$f|_J$ is of class $C^p(J)$.*
3. *$g(t) = (f \circ \gamma)(t), t \in I$ is p times continuously differentiable. That is f is of class $C^p_\gamma(J)$.*

In other words, $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J) = A(\Omega, J) \cap C^p_\gamma(\Omega, J)$. In that case:

$$\frac{dg}{dt}(t) = \frac{d\, f|_J}{dz}(\gamma(t))\gamma'(t) = f'(\gamma(t))\gamma'(t) \tag{38}$$

for all $t \in I$.

Remark 5 We notice that the assumption on Theorem 5 that the Riemann map $\phi : \overline{D} \to \overline{\Omega}$ is such that $\phi^{-1} \in A(\Omega, J)$ with $(\phi^{-1})'(z) \neq 0$ for $z \in \Omega \cup J$ is in fact equivalent to say that the parametrization of J induced by ϕ is C^1 regular.

We topologize $A^p(\Omega)$ by the semi-norms

$$|f|_l = \sup_{z \in \overline{\Omega}} |f^{(l)}(z)| = \sup_{z \in \partial\Omega} |f^{(l)}(z)|, 0 \leq l \leq p, l \in \mathbb{N} \tag{39}$$

and $C^p(\partial\Omega)$ by the semi-norms

$$|g|_l = \sup_{z \in \partial\Omega} |\frac{d^l g}{dz^l}(z)|, 0 \leq l \leq p, l \in \mathbb{N} \tag{40}$$

see also [2]. By equation (35) it easily seen that the topology on $A^p(\Omega)$ and $A(\Omega) \cap C^p(\partial\Omega)$ induced by $C^p(\partial\Omega)$ coincide. Summarizing:

Corollary 1 *Let Ω be a Jordan domain and $\phi : D \to \Omega$ a Riemann map of class $A^1(D)$ such that $\phi'(z) \neq 0$ for all $z \in \overline{D}$. The following equivalence holds for all $f \in A(\Omega)$ and $p \in \{1, 2, ...\} \cup \{\infty\}$: $f \in A^p(\Omega)$ if and only if $f|_{\partial\Omega}$ is p times continuously differentiable with respect to the position. In this case we have:*

$$\frac{d^l f}{dz^l}(z) = f^{(l)}(z) \tag{41}$$

For all $z \in \partial\Omega$ and $0 \leq l \leq p, l \in \mathbb{N}$. On the right hand side of (41) $f^{(l)}(z)$ denotes the continuous extension of $f^{(l)}$ from Ω to $\Omega \cup J$. We also have, $A^p(\Omega) = A(\Omega) \cap C^p(\partial\Omega)$ as topological spaces.

Remark 6 There is a natural way to topologize the spaces $A^p(\Omega, J)$ for any Jordan domain Ω, open arc $J \subset \partial\Omega$ and $p \in \{1, 2, \dots\} \cup \{\infty\}$. Then, the equality $A^p(\Omega, J) = A(\Omega, J) \cap C^p(J)$ holds taking into account the topologies of the spaces as well. However, we will not deal with the topological properties of the spaces $A^p(\Omega, J)$ in this paper.

One can easily extend the results of this section for the complement of a Jordan domain Ω given that the functions considered vanish at infinity. That is, if $A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega})$ is the class of all functions f with $\lim_{z\to\infty} f(z) = 0$ and holomorphic in $\mathbb{C} \setminus \overline{\Omega}$ such that their first p derivatives extend over $\mathbb{C} \setminus \Omega$ for all $0 \leq l \leq p, l \in \mathbb{N}$, then $A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega}) = A_0(\hat{\mathbb{C}} \setminus \overline{\Omega}) \cap C^p(\partial\Omega)$ under similar assumptions to that of Theorem 5. For J an open arc of $\partial\Omega$ we define $A_0^p(\hat{\mathbb{C}} \setminus \overline{\Omega}, J)$ similarly.

Remark 7 The previous results can be generalized to the case of finitely connected domains bounded by a finite set of disjoint Jordan curves. For this we use the Laurent Decomposition [3].

We will take a moment to sketch the proof. Let Ω be a bounded domain whose boundary consists of a finite number of disjoint Jordan curves. If $V_0, V_1, ..., V_{n-1}$ are the connected components of $\hat{\mathbb{C}} \setminus \Omega$, $\infty \in V_0$ and $\Omega_0 = \hat{\mathbb{C}} \setminus V_0$, ..., $\Omega_{n-1} = \hat{\mathbb{C}} \setminus V_{n-1}$, then for every f which is holomorphic in Ω we know that there exist functions $f_0, ..., f_{n-1}$ which are holomorphic in $\Omega_0, ..., \Omega_{n-1}$ respectively such that $f = f_0 + f_1 + ... + f_{n-1}$ and $\lim_{z\to\infty} f_j(z) = 0, j \in \{1, \dots, n-1\}$ [3]. Let $\phi_j : \overline{D} \to \overline{\Omega_j}, j \in \{0, 1, \dots, n-1\}$, be the respective conformal maps. Without loss of generality, take J an open arc of $\partial\Omega_0$ such that ϕ_0^{-1} is of class $A^1(\Omega_0, J)$ with non vanishing derivative. Let also f be a holomorphic function in Ω and continuous in $\Omega \cup J$ such that $f = f_0 + f_1 + \cdots + f_{n-1}$, f_j holomorphic in Ω_j, is the extended Laurent decomposition of f. Since f_j is holomorphic in a neighborhood of $\partial\Omega_0$ for all $j \neq 0$, we have that f_0 continuously extends over $\Omega_0 \cup J$ since f continuously extends over $\Omega \cup J$. Similarly, $f^{(l)}$ continuously extends over $\Omega \cup J$ for all $0 \leq l \leq p, l \in \mathbb{N}$, if and only if f_0 is of class $A^p(\Omega_0, J)$. By Theorem 5, f is of class $A^p(\Omega_0, J)$ if and only if $f_0|_J$ is of class $C^p(J)$, which happens if and only if $f|_J$ is of class $C^p(J)$.

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