SYMMETRIC POWER FUNCTORIALITY FOR HOLOMORPHIC MODULAR FORMS

JAMES NEWTON AND JACK A. THORNE

Abstract. Let f be a cuspidal Hecke eigenform of level 1. We prove the automorphy of the symmetric power lifting $\mathrm{Sym}^n f$ for every $n \geq 1$.

We establish the same result for a more general class of cuspidal Hecke eigenforms, including all those associated to semistable elliptic curves over $\mathbf{Q}$.

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Introduction

Context. Let F be a number field, and let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Langlands's functoriality principle [Lan70, Question 5] predicts the existence, for any algebraic representation $R : \mathrm{GL}_n \to \mathrm{GL}_N$, of a functorial lift of π along R; more precisely, an automorphic representation $R(\pi)$ of $\mathrm{GL}_N(\mathbf{A}_F)$ which may be characterized by the following property: for any place v of F, the Langlands parameter of $R(\pi)_v$ is the image, under R, of the Langlands parameter of π_v. The Langlands parameter is defined for each place v of F using the local Langlands correspondence for $\mathrm{GL}_n(F_v)$ (see [Lan89, HT01, Hen00]).

The simplest interesting case is when $n = 2$ and $R = \mathrm{Sym}^m$ is the m^{th} symmetric power of the standard representation of GL_2. In this case the automorphy of $\mathrm{Sym}^m \pi$ was proved for $m = 2$ by Gelbart and Jacquet [GJ78] and for $m = 3, 4$ by Kim and Shahidi [KS02, Kim03].

More recently, Clozel and the second author have proved the automorphy of $\mathrm{Sym}^m \pi$ for $m \leq 8$ under the assumption that π can be realised in a space of Hilbert modular forms of regular weight [CT14, CT15, CT17]; equivalently, that the number field F is totally real and the automorphic representation π is regular algebraic, in the sense of [Clo90b]. This includes the most classical case of automorphic representations arising from holomorphic modular forms of weight $k \geq 2$. We also mention the work of Dieulefait [Die15], which shows automorphy of the 5^{th} symmetric power for cuspidal Hecke eigenforms of level 1 and weight $k \geq 2$.

On the other hand, the *potential* automorphy (i.e. the existence of the symmetric power lifting after making some unspecified Galois base change) of *all* symmetric powers for automorphic representations π associated to Hilbert modular forms was obtained by Barnet-Lamb, Gee and Geraghty [BLGG11] (the case of elliptic modular forms is due to Barnet-Lamb, Geraghty, Harris and Taylor [BLGHT11]).

Results of this paper. In this paper, we prove the automorphy of all symmetric powers for cuspidal Hecke eigenforms of level 1 and weight $k \geq 2$. More precisely:

Theorem A. *Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of level 1 (i.e. which is everywhere unramified). Then for each integer $n \geq 2$, the symmetric power lifting $\mathrm{Sym}^{n-1}\pi$ exists, as a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_\mathbf{Q})$.*

In fact, we establish a more general result in which ramification is allowed:

Theorem B. *Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of conductor $N \geq 1$, which does not have CM.*[1] *Suppose that for each prime $l|N$, the Jacquet module of π_l is non-trivial; equivalently, that π_l is not supercuspidal. Then for each integer $n \geq 2$, the symmetric power lifting $\mathrm{Sym}^{n-1}\pi$ exists, as a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_\mathbf{Q})$.*

The class of automorphic representations described by Theorem B includes all those associated to holomorphic newforms of level $\Gamma_0(N)$, for some squarefree integer $N \geq 1$; in particular those associated to semistable elliptic curves over $\mathbf{Q}$. We can therefore offer the following corollary in more classical language:

Corollary C. *Let E be a semistable elliptic curve over $\mathbf{Q}$. Then, for each integer $n \geq 2$, the completed symmetric power L-function $\Lambda(\mathrm{Sym}^n E, s)$ as defined in e.g.* [DMW09]*, admits an analytic continuation to the entire complex plane.*

We remark that the meromorphic, as opposed to analytic, continuation of the completed L-function $\Lambda(\mathrm{Sym}^n E, s)$ was already known, as a consequence of the potential automorphy results mentioned above. Potential automorphy results were sufficient to prove the Sato–Tate conjecture, but our automorphy results make it possible to establish *effective* versions of Sato–Tate (we thank Ana Caraiani and Peter Sarnak for pointing this out to us). See, for example, [Tho14b] for an unconditional result and [Mur85, BK16, RT17] for results conditional on the Riemann Hypothesis for the symmetric power L-functions.

Strategy. Algebraic automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$ are conjectured to admit associated Galois representations [Clo90b]. When F is totally real and π is a self-dual regular algebraic automorphic representation, these Galois representations are known to exist; their Galois deformation theory is particularly well-developed; and they admit p-adic avatars, which fit into p-adic families of overconvergent automorphic forms. We make use of all of these tools. We begin by proving the following theorem:

[1] In other words, there is no quadratic Hecke character χ such that $\pi \cong \pi \otimes \chi$.

Theorem D. *Let $n \geq 2$ be an integer and suppose that the n^{th} symmetric power lifting exists for one regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of level 1. Then the n^{th} symmetric power lifting exists for every regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of level 1.*

We sketch the proof of Theorem D, which is based on the properties of the Coleman–Mazur eigencurve $\mathcal{E}_p$. We recall that if p is a prime, the eigencurve $\mathcal{E}_p$ is a p-adic rigid analytic space that admits a Zariski dense set of classical points corresponding to pairs (f, α) where f is a cuspidal eigenform of level 1 and some weight $k \geq 2$ and α is a root of the Hecke polynomial $X^2 - a_p(f)X + p^{k-1}$. The eigencurve admits a map $\kappa : \mathcal{E}_p \to \mathcal{W}_p = \mathrm{Hom}(\mathbf{Z}_p^\times, \mathbf{G}_m)$ to weight space with discrete fibres; the image of (f, α) is the character $x \mapsto x^{k-2}$.

We first show that for a fixed $n \geq 1$, the automorphy of $\mathrm{Sym}^n f$ is a property which is "constant on irreducible components of $\mathcal{E}_p$". (Here we confuse f and the automorphic representation π that it generates in order to simplify notation.) More precisely, if (f, α) and (f', α') determine points on the same irreducible component of $\mathcal{E}_p$, then the automorphy of $\mathrm{Sym}^n f$ is equivalent to that of $\mathrm{Sym}^n f'$. This part of the argument, which occupies §2 of this paper, does not require a restriction to cusp forms of level 1 – see Theorem 2.33. It is based on an infinitesimal $R = \mathbf{T}$ theorem on the eigenvariety associated to a definite unitary group in n variables. Kisin (for GL_2) [Kis03] and Bellaïche–Chenevier (for higher rank) [BC09] have observed that such theorems are often implied by the vanishing of adjoint Bloch–Kato Selmer groups. We are able to argue in this fashion here because we have proved the necessary vanishing results in [NT20].

To exploit this geometric property, we need to understand the irreducible components of $\mathcal{E}_p$. This is a notorious problem. However, conjectures predict that $\mathcal{E}_p$ has a simple structure over a suitably thin boundary annulus of a connected component of weight space $\mathcal{W}_p$ (see e.g. [LWX17, Conjecture 1.2]). We specialise to the case $p = 2$, in which case Buzzard–Kilford give a beautifully simple and explicit description of the geometry of $\mathcal{E}_p$ "close to the boundary of weight space" [BK05].

More precisely, $\mathcal{E}_2$ is supported above a single connected component $\mathcal{W}_2^+ \subset \mathcal{W}_2$, which we may identify with the rigid unit disc $\{|w| < 1\}$. The main theorem of [BK05] is that the pre-image $\kappa^{-1}(\{|8| < |w| < 1\})$ decomposes as a disjoint union $\sqcup_{i=1}^\infty X_i$ of rigid annuli, each of which maps isomorphically onto $\{|8| < |w| < 1\}$. Moreover, X_i has the following remarkable property: if $(f, \alpha) \in X_i$ is a point corresponding to a classical modular form, then the p-adic valuation $v_p(\alpha)$ (otherwise known as the slope of the pair (f, α)) equals $iv_p(w(\kappa(f, \alpha)))$.

We can now explain the second part of the proof of Theorem D, which occupies §3 of the paper. Since each irreducible component of $\mathcal{E}_2$ meets $\kappa^{-1}(\{|8| < |w| < 1\})$, it is enough to show that each X_i contains a point (f, α) such that $\mathrm{Sym}^n f$ is automorphic. This property only depends on f and not on the pair (f, α)! Moreover, the level 1 form f determines two points (f, α), (f, β) of $\kappa^{-1}(\{|8| < |w| < 1\})$, which lie on components X_i and $X_{i'}$ satisfying $i + i' = (k-1)/v_p(w(\kappa(f, \alpha)))$. Starting with a well-chosen initial point on a given annulus X_i, we can jump to any other $X_{i'}$ in a finite series of swaps between pairs (f', α'), (f', β') and moves within an annulus. We call this procedure playing ping pong, and it leads to a complete proof of Theorem D.

We remark that for this second step of the proof it is essential that we work with level 1 forms, since it is only in the level 1, $p = 2$ case that the eigencurve $\mathcal{E}_p$ admits such a simple structure (in particular, the eigencurve is supported above a single connected component of weight space and every Galois representation appearing in $\mathcal{E}_2$ admits the same residual representation, namely the trivial 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ over $\mathbf{F}_2$). We note as well that it is necessary to work with classical forms which may be ramified at the prime 2 in order for their weight characters to lie in the boundary annulus of $\mathcal{W}_2^+$. We have suppressed this minor detail here.

Theorem D implies that to prove Theorem A, it is enough to prove the following result:

Theorem E. *For each integer $n \geq 2$, there is a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of level 1 such that $\mathrm{Sym}^{n-1}\pi$ exists.*

As in the previous works of Clozel and the second author [CT14, CT15, CT17], we achieve this by combining an automorphy lifting theorem with the construction of level-raising congruences. We aim to find f and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ such that (writing $r_{f,\iota} : G_\mathbf{Q} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ for the p-adic Galois representation associated to f) the residual representation

$$\mathrm{Sym}^{n-1}\,\overline{r}_{f,\iota} : G_\mathbf{Q} \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$$

is automorphic; then we hope to use an automorphy lifting theorem to verify that $\mathrm{Sym}^{n-1}\,r_{f,\iota}$ is automorphic, and hence that $\mathrm{Sym}^{n-1} f$ is automorphic. In contrast to the papers just cited, where we chose $\overline{r}_{f,\iota}$ to have large image but p to be small, in order to exploit the reducibility of the symmetric power representations of GL_2 in small characteristic, here we choose $\overline{r}_{f,\iota}$ to have small image, and p to be large.

More precisely, we choose f to be congruent modulo p to a theta series, so that $\overline{r}_{f,\iota} \cong \mathrm{Ind}_{G_K}^{G_\mathbf{Q}} \overline{\psi}$ is induced. In this case $\mathrm{Sym}^{n-1}\,\overline{r}_{f,\iota}|_{G_K}$ is a sum of characters, so its residual automorphy can be verified using the endoscopic classification for unitary groups in n variables. The wrinkle is that the automorphy lifting theorems proved in [ANT20] (generalizing those of [Tho15]) require the automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_K)$ (say) verifying residual automorphy to have a local component which is a twist of the Steinberg representation. To find such a π we need to combine the endoscopic classification with the existence of level-raising congruences.

In fact, we combine two different level-raising results in order to construct the desired congruences. The first of these, original to this paper, suffices to prove Theorem E in the case that n is odd. The argument is based on a generalization of the following simple observation, which suffices to prove Theorem E in the case $n = 3$: let q be an odd prime power, and let $U_3(q)$ denote the finite group of Lie type associated to the outer form of GL_3 over $\mathbf{F}_q$. Let p be a prime such that $q \pmod p$ is a primitive 6^{th} root of unity. Then the unique cuspidal unipotent representation of $U_3(q)$ remains irreducible on reduction modulo p, and this reduction occurs as a constituent of the reduction modulo p of a generic cuspidal representation of $U_3(q)$ (see Proposition 1.15). Using the theory of depth zero types, this observation has direct consequences for the existence of congruences between automorphic representations of U_3. Similar arguments work for general odd n, for carefully chosen global data. We leave a discussion of the (quite intricate) details to §4.

The second level-raising result, proved by Anastassiades in his thesis, allows us to pass from the existence of $\mathrm{Sym}^{n-1} f$ to the existence of $\mathrm{Sym}^{2n-1} f$. We refer to the paper [AT21] for a more detailed discussion.

It remains to extend Theorem A to the ramified case, and prove Theorem B. For this we induct on the number of primes dividing the conductor, and use an argument of 'killing ramification' as in the proof of Serre's conjecture [KW09]. Thus to remove a prime l from the level we need to be able to move within a family of l-adic overconvergent modular forms to a classical form of the same tame level, but now unramified at l. This explains our assumption in Theorem B that the Jacquet module of π_l is non-trivial for every prime l: it implies the existence of a point associated to (a twist of) π on an l-adic eigencurve for every prime l.

In a sequel to this paper [NT], we prove a new kind of automorphy lifting theorem for symmetric power Galois representations. This allows us to finally prove a version of Theorem B where the hypothesis that no local component π_l is supercuspidal is removed. The arguments of [NT] use only fixed weight classical automorphic forms (as opposed to overconvergent automorphic forms) but do require the results of this paper (in particular, Theorem B) as a starting point.

Organization of this paper. We begin in §1 by recalling known results on the classification of automorphic representations of definite unitary groups. We make particular use of the construction of L-packets of discrete series representations of p-adic unitary groups given by Mœglin [Mœg07, Mœg14], the application of Arthur's simple trace formula for definite unitary groups as explicated in [Lab11], and Kaletha's results on the normalisation of transfer factors (in the simplest case of pure inner forms) [Kal16].

In §2 we study the interaction between the existence of symmetric power liftings of degree n with the geometry of the eigenvariety associated to a definite unitary group in n variables. The basic geometric idea is described in §2.1. In §3 we combine these results with the explicit description of the tame level 1, $p = 2$ Coleman–Mazur eigencurve to complete the proof of Theorem D.

We then turn to the proof of Theorem E, which rests upon two level-raising results, only the first of which is proved here. The proof of this result is in turn split into two halves; first we give in §4 an automorphic construction of level-raising congruences using types, in the manner sketched above. Then in §6 we establish level-raising congruences of a different kind using deformation theory for residually reducible representations, as developed in [Tho15, ANT20]. These two results are applied in turn to construct our desired level-raising congruences for odd n (Proposition 7.4). A key intermediate result is a finiteness result for certain Galois deformation rings, established in §5, and which may be of independent interest. We use this to control the dimension of the locus of reducible deformations.

Finally, we are in a position to prove our main theorems. In §7 we combine the preceding constructions with the main theorem of [AT21] in order to prove Theorem E and therefore Theorem A. In §8, we carry out the argument of 'killing ramification' in order to obtain Theorem B. The main technical challenge is to manage the hypothesis of 'n-regularity' which appears in our analytic continuation results (see especially Theorem 2.33). To do this we prove a result (Proposition 8.3) which takes a given automorphic representation π and constructs a congruence to an n-regular one π'. This may also be of independent interest.

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Notation. If F is a perfect field, we generally fix an algebraic closure $\overline{F}/F$ and write G_F for the absolute Galois group of F with respect to this choice. We make the convention that a soluble extension F'/F is a (finite) *Galois* extension with soluble Galois group $\mathrm{Gal}(F'/F)$.

When the characteristic of F is not equal to p, we write $\epsilon : G_F \to \mathbf{Z}_p^\times$ for the p-adic cyclotomic character. We write $\zeta_n \in \overline{F}$ for a fixed choice of primitive n^{th} root of unity (when this exists). If F is a number field, then we will also fix embeddings $\overline{F} \to \overline{F}_v$ extending the map $F \to F_v$ for each place v of F; this choice determines a homomorphism $G_{F_v} \to G_F$. When v is a finite place, we will write $\mathcal{O}_{F_v} \subset F_v$ for the valuation ring, $\varpi_v \in \mathcal{O}_{F_v}$ for a fixed choice of uniformizer, $\mathrm{Frob}_v \in G_{F_v}$ for a fixed choice of (geometric) Frobenius lift, $k(v) = \mathcal{O}_{F_v}/(\varpi_v)$ for the residue field, and $q_v = \#k(v)$ for the cardinality of the residue field. When v is a real place, we write $c_v \in G_{F_v}$ for complex conjugation. If S is a finite set of finite places of F then we write F_S/F for the maximal subextension of $\overline{F}$ unramified outside S and $G_{F,S} = \mathrm{Gal}(F_S/F)$.

If p is a prime, then we call a coefficient field a finite extension $E/\mathbf{Q}_p$ contained inside our fixed algebraic closure $\overline{\mathbf{Q}}_p$, and write $\mathcal{O}$ for the valuation ring of E, $\varpi \in \mathcal{O}$ for a fixed choice of uniformizer, and $k = \mathcal{O}/(\varpi)$ for the residue field. If A is a local ring, we write $\mathcal{C}_A$ for the category of complete Noetherian local A-algebras with residue field $A/\mathfrak{m}_A$. We will use this category mostly with $A = E$ or $A = \mathcal{O}$. If G is a profinite group and $\rho : G \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is a continuous representation, then we write $\overline{\rho} : G \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ for the associated semisimple residual representation (which is well-defined up to conjugacy).

If F is a CM number field (i.e. a totally imaginary quadratic extension of a totally real number field), then we write F^+ for its maximal totally real subfield, $c \in \mathrm{Gal}(F/F^+)$ for the unique non-trivial element, and $\delta_{F/F^+} : \mathrm{Gal}(F/F^+) \to \{\pm 1\}$ for the unique non-trivial character. If S is a finite set of finite places of F^+, containing the places at which F/F^+ is ramified, we set $F_S = F_S^+$ and $G_{F,S} = \mathrm{Gal}(F_S/F)$.

We write $T_n \subset B_n \subset \mathrm{GL}_n$ for the standard diagonal maximal torus and upper-triangular Borel subgroup. Let K be a non-archimedean characteristic 0 local field, and let Ω be an algebraically closed field of characteristic 0. If $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is a continuous representation (which is de Rham if p equals the residue characteristic of K), then we write $\mathrm{WD}(\rho) = (r, N)$ for the associated Weil–Deligne representation of ρ, and $\mathrm{WD}(\rho)^{F-ss}$ for its Frobenius semisimplification. We use the cohomological normalisation of class field theory: it is the isomorphism $\mathrm{Art}_K : K^\times \to W_K^{ab}$ which sends uniformizers to geometric Frobenius elements. When $\Omega = \mathbf{C}$, we have the local Langlands correspondence rec_K for $\mathrm{GL}_n(K)$: a bijection between the sets of isomorphism classes of irreducible, admissible $\mathbf{C}[\mathrm{GL}_n(K)]$-modules and Frobenius-semisimple Weil–Deligne representations over $\mathbf{C}$ of rank n. In general, we have the Tate normalisation rec_K^T of the local Langlands correspondence for GL_n as described in [CT14, §2.1]. When $\Omega = \mathbf{C}$, we have $\mathrm{rec}_K^T(\pi) = \mathrm{rec}_K(\pi \otimes |\cdot|^{(1-n)/2})$.

If G is a reductive group over K and $P \subset G$ is a parabolic subgroup and π is an admissible $\Omega[P(K)]$-module, then we write $\mathrm{Ind}_{P(K)}^{G(K)} \pi$ for the usual smooth induction. If $\Omega = \mathbf{C}$ then we write $i_P^G \pi$ for the normalised induction, defined as $i_P^G \pi = \mathrm{Ind}_{P(K)}^{G(K)} \pi \otimes \delta_P^{1/2}$, where $\delta_P : P(K) \to \mathbf{R}_{>0}$ is the character $\delta_P(x) = |\det(\mathrm{Ad}(x)|_{\mathrm{Lie}\, N_P})|_K$ (and N_P is the unipotent radical of P).

If $\psi : K^\times \to \mathbf{C}^\times$ is a smooth character, then we write $\mathrm{Sp}_n(\psi) = (r, N)$ for the Weil–Deligne representation on $\mathbf{C}^n = \oplus_{i=1}^n \mathbf{C} \cdot e_i$ given by $r = (\psi \circ \mathrm{Art}_K^{-1}) \oplus (\psi |\cdot|^{-1} \circ \mathrm{Art}_K^{-1}) \oplus \cdots \oplus (\psi |\cdot|^{1-n} \circ \mathrm{Art}_K^{-1})$ and $N e_1 = 0$, $N e_{i+1} = e_i$ $(1 \leq i \leq n-1)$. We write $\mathrm{St}_n(\psi)$ for the unique irreducible quotient of $i_{B_n}^{\mathrm{GL}_n}(\psi \circ \det)\delta_{B_n}^{-1/2} = \mathrm{Ind}_{B_n(K)}^{\mathrm{GL}_n(K)} \psi \circ \det$. We have $\mathrm{rec}_K^T(\mathrm{St}_n(\psi)) = \mathrm{Sp}_n(\psi)$.

If F is a number field and $\chi : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ is a Hecke character of type A_0 (equivalently: algebraic), then for any isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ there is a continuous character $r_{\chi,\iota} : G_F \to \overline{\mathbf{Q}}_p^\times$ which is de Rham at the places $v|p$ of F and such that for each finite place v of F, $\mathrm{WD}(r_{\chi,\iota}) \circ \mathrm{Art}_{F_v} = \iota^{-1}\chi|_{F_v^\times}$. Conversely, if $\chi' : G_F \to \overline{\mathbf{Q}}_p^\times$ is a continuous character which is de Rham and unramified at all but finitely many places, then there exists a Hecke character $\chi : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ of type A_0 such that $r_{\chi,\iota} = \chi'$. In this situation we abuse notation slightly by writing $\chi = \iota\chi'$.

If F is a CM or totally real number field and π is an automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, we say that π is regular algebraic if π_∞ has the same infinitesimal character as an irreducible algebraic representation W of $(\mathrm{Res}_{F/\mathbf{Q}} \mathrm{GL}_n)_{\mathbf{C}}$. We identify $X^*(T_n)$ with $\mathbf{Z}^n$ in the usual way, and write $\mathbf{Z}_+^n \subset \mathbf{Z}^n$ for the subset of weights which are B_n-dominant. If $W^\vee$ has highest weight $\lambda = (\lambda_\tau)_{\tau \in \mathrm{Hom}(F,\mathbf{C})} \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F,\mathbf{C})}$, then we say that π has weight λ.

When F is CM, the automorphic representation π is said to be conjugate self-dual if $\pi^c \cong \pi^\vee$. We refer to [BLGGT14, §2.1] for the more general notion of a polarizable automorphic representation. Note that if π is conjugate self-dual, then $(\pi, \delta_{F/F^+}^n)$ is polarized and therefore π is polarizable.

If π is cuspidal, regular algebraic, and polarizable, then for any isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ there exists a continuous, semisimple representation $r_{\pi,\iota} : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ such that for each finite place v of F, $\mathrm{WD}(r_{\pi,\iota}|_{G_{F_v}})^{F-ss} \cong \mathrm{rec}_{F_v}^T(\iota^{-1}\pi_v)$ (see e.g. [Car14]). (When $n = 1$, this is compatible with our existing notation.) We use the convention that the Hodge–Tate weight of the cyclotomic character is -1. Thus if π is of weight λ, then for any embedding $\tau : F \to \overline{\mathbf{Q}}_p$ the τ-Hodge–Tate weights of $r_{\pi,\iota}$ are given by

$$\mathrm{HT}_\tau(r_{\pi,\iota}) = \{\lambda_{\iota\tau,1} + (n-1), \lambda_{\iota\tau,2} + (n-2), \ldots, \lambda_{\iota\tau,n}\}.$$

For $n \geq 1$, we define a matrix

$$\Phi_n = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & \iddots & & \\ (-1)^{n-1} & & & \end{pmatrix}.$$

If E/F is a quadratic extension of fields of characteristic 0 then we write $\theta = \theta_n : \mathrm{Res}_{E/F} \mathrm{GL}_n \to \mathrm{Res}_{E/F} \mathrm{GL}_n$ for the involution given by the formula $\theta(g) = \Phi_n c(g)^{-t} \Phi_n^{-1}$. We write $U_n \subset \mathrm{Res}_{E/F} \mathrm{GL}_n$ for the fixed subgroup of θ_n. Then

U_n is a quasi-split unitary group. The standard pinning of GL_n (consisting of the maximal torus of diagonal matrices, Borel subgroup of upper-triangular matrices, and set $\{E_{i,i+1} \mid i = 1, \dots, n-1\}$ of root vectors) is invariant under the action of θ and defines an F-pinning of U_n, that we call its standard pinning. If F is a number field or a non-archimedean local field, then we also write U_n for the extension of U_n to a group scheme over $\mathcal{O}_F$ with functor of points

$$U_n(R) = \{g \in \mathrm{GL}_n(R \otimes_{\mathcal{O}_F} \mathcal{O}_E) \mid g = \Phi_n(1 \otimes c)(g)^{-t}\Phi_n^{-1}\}.$$

When F is a number field or a local field, we identify the dual group ${}^L U_n = \mathrm{GL}_n(\mathbf{C}) \rtimes W_F$, where W_E acts trivially on $\mathrm{GL}_n(\mathbf{C})$ and an element $w_c \in W_F - W_E$ acts by the formula $w_c \cdot g = \Phi_n g^{-t} \Phi_n^{-1}$ (therefore preserving the standard pinning of $\mathrm{GL}_n(\mathbf{C})$).

Given a partition of n (i.e. a tuple $(n_1, n_2, \dots, n_k)$ of natural numbers such that $n_1 + n_2 + \cdots + n_k = n$), we write $L_{(n_1,\dots,n_k)}$ for the corresponding standard Levi subgroup of GL_n (i.e. the block diagonal subgroup $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k} \subset \mathrm{GL}_n$), and $P_{(n_1,\dots,n_k)}$ for the corresponding standard parabolic subgroup (i.e. block upper-triangular matrices with blocks of sizes $n_1, \dots, n_k$). If E is a non-archimedean characteristic 0 local field and $\pi_1, \dots, \pi_k$ are admissible representations of $\mathrm{GL}_{n_1}(E), \dots, \mathrm{GL}_{n_k}(E)$, respectively, then we write $\pi_1 \times \pi_2 \times \cdots \times \pi_k = i^{\mathrm{GL}_n}_{P_{(n_1,\dots,n_k)}} \pi_1 \otimes \cdots \otimes \pi_k$. We write $\pi_1 \boxplus \cdots \boxplus \pi_k$ for the irreducible admissible representation of $\mathrm{GL}_n(E)$ defined by $\mathrm{rec}_E(\boxplus_{i=1}^k \pi_i) \cong \oplus_{i=1}^k \mathrm{rec}_E(\pi_i)$; it is a subquotient of $\pi_1 \times \cdots \times \pi_k$.

Given a tuple $(n_1, n_2, \dots, n_k)$ of natural numbers such that $2(n_1 + \cdots + n_{k-1}) + n_k = n$, we write $M_{(n_1,\dots,n_k)}$ for the Levi subgroup of U_n given by block diagonal matrices with blocks of size $n_1, n_2, \dots, n_{k-1}, n_k, n_{k-1}, \dots, n_1$. Then $M_{(n_1,\dots,n_k)}$ is a standard Levi subgroup (with respect to the diagonal maximal torus of U_n), and projection to the first k blocks gives an isomorphism $M_{(n_1,\dots,n_k)} \cong (\mathrm{Res}_{E/F} \mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{E/F} \mathrm{GL}_{n_{k-1}}) \times U_{n_k}$. We write $Q_{(n_1,\dots,n_k)}$ for the parabolic subgroup given by block upper triangular matrices (with blocks of the same sizes). If F is a non-archimedean characteristic 0 local field and $\pi_1, \dots, \pi_{k-1}, \pi_k$ are admissible representations of $\mathrm{GL}_{n_1}(E), \dots, \mathrm{GL}_{n_{k-1}}(E), U_{n_k}(F)$, respectively, then we write $\pi_1 \times \pi_2 \times \cdots \times \pi_k = i^{U_n}_{Q_{(n_1,\dots,n_k)}} \pi_1 \otimes \cdots \otimes \pi_k$.

1. Definite unitary groups

In this paper we will often use the following assumptions and notation, which we call the "standard assumptions":

- F is a CM number field such that F/F^+ is everywhere unramified. We note this implies that $[F^+ : \mathbf{Q}]$ is even (the quadratic character of $(F^+)^\times \backslash \mathbf{A}_{F^+}^\times / \widehat{\mathcal{O}}_{F^+}^\times$ cutting out F has non-trivial restriction to F_v^+ for each $v|\infty$ but is trivial on $(-1)_{v|\infty} \in (F_\infty^+)^\times$).
- p is a prime. We write S_p for the set of p-adic places of F^+.
- S is a finite set of finite places of F^+, all of which split in F. S contains S_p.
- For each $v \in S$, we suppose fixed a factorization $v = \widetilde{v}\widetilde{v}^c$ in F, and write $\widetilde{S} = \{\widetilde{v} \mid v \in S\}$.

Let $n \geq 1$ be an integer. Under the above assumptions we can fix the following data:

- The unitary group $G_n = G$ over F^+ with R-points given by the formula

$$G(R) = \{g \in \mathrm{GL}_n(R \otimes_{F^+} F) \mid g = (1 \otimes c)(g)^{-t}\}. \tag{1.0.1}$$

We observe that for each finite place v of F^+, $G_{F_v^+}$ is quasi-split, while for each place $v|\infty$ of F^+, $G(F_v^+)$ is compact. We use the same formula to extend G to a reductive group scheme over $\mathcal{O}_{F^+}$ (this uses that F/F^+ is everywhere unramified).[2]

- The inner twist $\xi : U_{n,F} \to G_F$, given by the formula
$$\xi(g_1, g_2) = (g_1, \Phi_n^{-1} g_2 \Phi_n)$$
with respect to the identifications
$$U_{n,F} = \{(g_1, g_2) \in \mathrm{GL}_n \times \mathrm{GL}_n \mid g_2 = \Phi_n g_1^{-t} \Phi_n^{-1}\}$$
and
$$G_F = \{(g_1, g_2) \in \mathrm{GL}_n \times \mathrm{GL}_n \mid g_2 = g_1^{-t}\}.$$
- A lift of ξ to a pure inner twist $(\xi, u) : U_{n,F} \to G_F$. We recall (see e.g. [Kal11]) that by definition, this means that $u \in Z^1(F^+, U_n)$ is a cocycle such that for all $\sigma \in G_{F^+}$, we have $\xi^{-1\sigma}\xi = \mathrm{Ad}(u_\sigma)$. When n is odd, we define u to be the cocycle inflated from $Z^1(\mathrm{Gal}(F/F^+), U_n(F))$, defined by the formula $u_1 = 1$, $u_c = (\Phi_n, \Phi_n)$. When n is even, we choose an element $\zeta \in F^\times$ with $\mathrm{tr}_{F/F^+}(\zeta) = 0$ and define u to be the cocycle inflated from $Z^1(\mathrm{Gal}(F/F^+), U_n(F))$, defined by the formula $u_1 = 1$, $u_c = (\zeta\Phi_n, \zeta^{-1}\Phi_n)$. (In fact, we will make essential use of this structure only when n is odd.)
- We also fix a choice of continuous character $\mu_F = \mu : F^\times \backslash \mathbf{A}_F^\times \to \mathbf{C}^\times$ such that $\mu|_{\mathbf{A}_{F^+}^\times} = \delta_{F/F^+} \circ \mathrm{Art}_{F^+}$ and such that if v is any place of F which is inert over F^+, then $\mu|_{F_v^\times}$ is unramified.

If v is a finite place of F^+, then the image of the cocycle u in $H^1(F_v^+, U_n)$ is trivial (this is true by Hilbert 90 if v splits in F, and true because $\det u_c \in \mathbf{N}_{F_{\widetilde{v}}/F_v^+} F_{\widetilde{v}}^\times$ if v is inert in F, cf. [Rog90, §1.9]). Our choice of pure inner twist (ξ, u) therefore determines a $U_n(F_v^+)$-conjugacy class of isomorphisms $\iota_v : G(F_v^+) \to U_n(F_v^+)$ (choose $g \in U_n(\overline{F}_v^+)$ such that $g^{-1c}g = u_c$; then ι_v is the map induced on F_v^+-points by the map $\mathrm{Ad}(g) \circ \xi^{-1} : G_{\overline{F}_v^+} \to U_{n,\overline{F}_v^+}$, which descends to F_v^+). If v splits $v = ww^c$ in F, then we have an isomorphism $\iota_w : G(F_v^+) \to \mathrm{GL}_n(F_w)$ (composite of inclusion $G(F_v^+) \subset (\mathrm{Res}_{F/F^+} \mathrm{GL}_n)(F_v^+)$ and canonical projection $(\mathrm{Res}_{F/F^+} \mathrm{GL}_n)(F_v^+) \to \mathrm{GL}_n(F_w)$).

If L^+/F^+ is a finite totally real extension, then we will use the following standard notation:

- We set $L = L^+F$.
- If T is a set of places of F^+ then we write T_L for the set of places of L^+ lying above T. If $w \in T_L$ lies above $v \in T$ and v splits $v = \widetilde{v}\widetilde{v}^c$ in F (in particular, we suppose that we have made a choice of $\widetilde{v}|v$), then we will write $\widetilde{w}$ for the unique place of L which lies above both w and $\widetilde{v}$ (in which case w splits $w = \widetilde{w}\widetilde{w}^c$ in L). We write e.g. $\widetilde{S}_L$ for the set of places of the form $\widetilde{w}$ ($w \in S_L$).

We note that formation of G is compatible with base change, in the sense that the group G_{L^+} is the same as the one given by formula (1.0.1) relative to the quadratic extension L/L^+. The same remark applies to the pure inner twist (ξ, u). When we

[2] The authors apologize for using the same notation G_L to denote both an extension of scalars of the algebraic group G and an absolute Galois group. We hope no confusion will arise.

need to compare trace formulae over F^+ and its extension L^+/F^+ (a situation that arises in §4), we will use the character $\mu_L = \mu_F \circ \mathbf{N}_{L/F}$.

1.1. **Base change and descent – first cases.** In the next few sections we summarise some results from the literature concerning automorphic representations of the group $G(\mathbf{A}_{F^+})$. We first give some results which do not rely on an understanding of the finer properties of L-packets for p-adic unitary groups at inert places of the extension F/F^+.

Theorem 1.2. *Let σ be an automorphic representation of $G(\mathbf{A}_{F^+})$. Then there exist a partition $n = n_1 + \cdots + n_k$ and discrete, conjugate self-dual automorphic representations*

$$\pi_1, \ldots, \pi_k$$

of

$$\mathrm{GL}_{n_1}(\mathbf{A}_F), \ldots, \mathrm{GL}_{n_k}(\mathbf{A}_F),$$

respectively, with the following properties:

(1) *Let $\pi = \pi_1 \boxplus \cdots \boxplus \pi_k$. Then for each finite place w of F below which σ is unramified, π_w is unramified and is the unramified base change of $\sigma_{w|_{F^+}}$.*
(2) *For each place $v = ww^c$ of F^+ which splits in F, $\pi_w \cong \sigma_v \circ \iota_w^{-1}$.*
(3) *For each place $v|\infty$ of F, π_v has the same infinitesimal character as $\otimes_{\tau : F_v \to \mathbf{C}} W_\tau$, where W_τ is the algebraic representation of $\mathrm{GL}_n(F_v) \cong \mathrm{GL}_n(\mathbf{C})$ such that $\sigma_v \cong W_\tau|_{G(F_v^+)}$.*

Proof. This follows from [Lab11, Corollaire 5.3]. □

We call π the base change of σ. If $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ is an isomorphism, we say that σ is ι-ordinary if π is ι-ordinary at all places $w|p$ in the sense of [Ger19, Definition 5.3]. We note that this depends only on π_p and the weight of π (equivalently, on σ_p and σ_∞).

Corollary 1.3. *Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism. Then there exists a unique continuous semisimple representation $r_{\sigma,\iota} : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ with the following properties:*

(1) *For each prime-to-p place w of F below which σ is unramified, $r_{\sigma,\iota}|_{G_{F_w}}$ is unramified.*
(2) *For each place $v \in S_p$, $r_{\sigma,\iota}|_{G_{F_{\widetilde{v}}}}$ is de Rham.*
(3) *For each place $v = ww^c$ of F^+ which splits in F, $\mathrm{WD}(r_{\sigma,\iota}|_{G_{F_w}})^{F-ss} \cong \mathrm{rec}^T_{F_w}(\sigma_v \circ \iota_w^{-1})$.*

Proof. This follows from the classification of discrete automorphic representations of $\mathrm{GL}_{n_i}(\mathbf{A}_F)$ [MW89], together with the known existence of Galois representations attached to RACSDC (regular algebraic, conjugate self-dual, cuspidal) automorphic representations of $\mathrm{GL}_{n_i}(\mathbf{A}_F)$ (cf. [AT21, Corollary 3.4]). □

We remark that if $\overline{r}_{\sigma,\iota}|_{G_{F(\zeta_p)}}$ is multiplicity free, then the base change of σ is $\pi = \pi_1 \boxplus \cdots \boxplus \pi_k$, where each π_i is a *cuspidal* automorphic representation of $\mathrm{GL}_{n_i}(\mathbf{A}_F)$. Indeed, [MW89] shows that a non-cuspidal π_i would contribute a direct sum of copies of a single Galois representation twisted by powers of the cyclotomic character to $r_{\sigma,\iota}$, which gives a factor with multiplicity > 1 in $\overline{r}_{\sigma,\iota}|_{G_{F(\zeta_p)}}$. In particular, if $\overline{r}_{\sigma,\iota}|_{G_{F(\zeta_p)}}$ is multiplicity free then π is tempered (as each π_i is, by the results of [Shi11, Clo13, Car12]).

Theorem 1.4. *Let π be a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Suppose that π is unramified outside S. Then there exists an automorphic representation σ of $G(\mathbf{A}_{F^+})$ with the following properties:*

(1) *For each finite place $v \notin S$ of F, $\sigma_v^{\iota_v^{-1}(U_n(\mathcal{O}_{F_v^+}))} \neq 0$.*
(2) *π is the base change of σ.*

Proof. This follows from [Lab11, Théorème 5.4]. □

1.5. **Endoscopic data and normalisation of transfer factors.** To go further we need to use some ideas from the theory of endoscopy, both for the unitary group G and for the twisted group $\mathrm{Res}_{F/F^+} \mathrm{GL}_n \rtimes \theta$. We begin by describing endoscopic data for G (cf. [Lab11, §4.2], [Rog90, §4.6]). The equivalence classes of endoscopic data for G are in bijection with pairs (p, q) of integers such that $p + q = n$ and $p \geq q \geq 0$. Define $\mu_+ = 1$, $\mu_- = \mu$. We identify $\mu_\pm$ with characters of the global Weil group W_F using Art_F. Then we can write down an extended endoscopic triple $\mathcal{E} = (H, s, \eta)$ giving rise to each equivalence class as follows:

- The group H is $U_p \times U_q$.
- $s = \mathrm{diag}(1, \dots, 1, -1, \dots, -1)$ (with p occurrences of 1 and q occurences of -1).
- $\eta : {}^L H \to {}^L G$ is given by the formulae:

$$\eta : (g_1, g_2) \rtimes 1 \mapsto \mathrm{diag}(g_1, g_2) \rtimes 1 \in \mathrm{GL}_n(\mathbf{C}) = \widehat{G},\ (g_1, g_2) \in \mathrm{GL}_p(\mathbf{C}) \times \mathrm{GL}_q(\mathbf{C}) = \widehat{H};$$
$$(1_p, 1_q) \rtimes w \mapsto \mathrm{diag}(\mu_{(-1)^q}(w) 1_p, \mu_{(-1)^p}(w) 1_q) \rtimes w \ (w \in W_F)$$
$$(1_p, 1_q) \rtimes w_c \mapsto \mathrm{diag}(\Phi_p, \Phi_q) \Phi_n^{-1} \rtimes w_c,$$

where $w_c \in W_{F^+} - W_F$ is any fixed element.

As described in [Lab11, §4.5], a choice of extended endoscopic triple $\mathcal{E}$ determines a normalisation of the local transfer factor $\Delta_v^{\mathcal{E}}$ (v a place of F^+) up to non-zero scalar. We will fix a normalisation of local transfer factors only when n is odd, using the following observations:

- The quasi-split group U_n, with its standard pinning, has a canonical normalisation of transfer factors. Indeed, in this case the Whittaker normalisation of transfer factors defined in [KS99, §5] is independent of the choice of additive character and coincides with the transfer factor denoted Δ_0 in [LS87].
- Our choice of pure inner twist $(\xi, u) : U_n \to G$ defines a normalisation of the local transfer factors for G. This normalisation of local transfer factors satisfies the adelic product formula (a very special case of [Kal18, Proposition 4.4.1]).

A local transfer factor having been fixed, one can define what it means for a function $f^H \in C_c^\infty(H(F_v^+))$ (resp. $f^H \in C_c^\infty(H(\mathbf{A}_{F^+}))$) to be an endoscopic transfer of a function $f \in C_c^\infty(G(F_v^+))$ (resp. $f \in C_c^\infty(G(\mathbf{A}_{F^+}))$). After the work of Waldspurger, Laumon, and Ngô, any function $f \in C_c^\infty(G(F_v^+))$ (resp. $C_c^\infty(G(\mathbf{A}_{F^+}))$) admits an endoscopic transfer (see [Lab11, Théorème 4.3] for detailed references).

We next discuss base change, or in other words, endoscopy for the twisted group $\mathrm{Res}_{F/F^+} \mathrm{GL}_n \rtimes \theta_n$. We will only require the principal extended endoscopic triple $(U_n, 1_n, \eta)$, where $\eta : {}^L U_n \to {}^L \mathrm{Res}_{F/F^+} \mathrm{GL}_n$ is defined as follows: first, identify ${}^L \mathrm{Res}_{F/F^+} \mathrm{GL}_n = (\mathrm{GL}_n(\mathbf{C}) \times \mathrm{GL}_n(\mathbf{C})) \rtimes W_{F^+}$, where W_{F^+} acts through its quotient $\mathrm{Gal}(F/F^+)$ and an element $w_c \in W_{F^+} - W_F$ acts by the automorphism $(g_1, g_2) \mapsto (\Phi_n g_2^{-t} \Phi_n^{-1}, \Phi_n g_1^{-t} \Phi_n^{-1})$. Then $\eta : {}^L U_n \to {}^L \operatorname{Res}_{F/F^+} \mathrm{GL}_n$ is given by the formulae:

$$\begin{aligned} \eta : (g) \rtimes 1 &\mapsto \mathrm{diag}(g, {}^t g^{-1}) \rtimes 1 \in \mathrm{GL}_n(\mathbf{C}) \times \mathrm{GL}_n(\mathbf{C}); \\ (1_n) \rtimes w &\mapsto \mathrm{diag}(1_n, 1_n) \rtimes w, \ (w \in W_F); \\ (1_n) \rtimes w_c &\mapsto \mathrm{diag}(\Phi_n, \Phi_n^{-1}) \rtimes w_c. \end{aligned}$$

Following [Lab11, §4.5], we fix the trivial transfer factors in this case. By [Lab11, Lemme 4.1], each function $\phi \in C_c^\infty(\operatorname{Res}_{F/F^+} \mathrm{GL}_n(F_v^+) \rtimes \theta_n)$ admits an endoscopic transfer $\phi^{U_n} \in C_c^\infty(U_n(F_v^+))$, and every function in $C_c^\infty(U_n(F_v^+))$ arises this way. We will follow *op. cit.* in using the following notation: if $f \in C_c^\infty(U_n(F_v^+))$ (or more generally, if U_n is replaced by a product of unitary groups) then we write $\widetilde{f} \in C_c^\infty(\operatorname{Res}_{F/F^+} \mathrm{GL}_n(F_v^+) \rtimes \theta_n)$ for any function that admits f as endoscopic transfer (with respect to the principal extended endoscopic triple defined above).

If $\mathcal{E} = (H, s, \eta)$ is one of the extended endoscopic triples for G as above then, following [Lab11, §4.7], we set $M^H = \operatorname{Res}_{F/F^+} H_F$, and write $\widetilde{M}^H$ for the twisted space on M^H associated to the non-trivial element of $\mathrm{Gal}(F/F^+)$. Then we may canonically identify $M^H = \operatorname{Res}_{F/F^+} \mathrm{GL}_p \times \mathrm{GL}_q$ and $\widetilde{M}^H = (\operatorname{Res}_{F/F^+} \mathrm{GL}_p \times \mathrm{GL}_q) \rtimes (\theta_p \times \theta_q)$. We will use the same notation to describe stable base change for M^H. In particular, if $f \in C_c^\infty(H(\mathbf{A}_{F^+}))$, then we will use $\widetilde{f} \in C_c^\infty(\widetilde{M}^H(\mathbf{A}_{F^+}))$ to denote a function whose endoscopic transfer (with respect to the principal extended endoscopic triple for M^H, defined as above) with respect to the trivial transfer factors is f (cf. [Lab11, Proposition 4.9]).

Having fixed the above normalisations, we can now formulate some simple propositions.

Proposition 1.6. *Let $n \geq 1$ be odd, and let v be an infinite place of F^+. Suppose given an extended endoscopic triple $\mathcal{E} = (H, s, \eta)$ as above and a Langlands parameter $\varphi_H : W_{F_v^+} \to {}^L H$ such that $\eta \circ \varphi_H$ is the Langlands parameter of an irreducible representation σ_v of $G(F_v^+)$. Let π be the (necessarily tempered, θ-invariant) irreducible admissible representation of $H(F_{\widetilde{v}})$ associated to the Langlands parameter $\varphi_H|_{W_{F_v}}$, and let $f_v \in C^\infty(G(F_v^+))$ be a coefficient for σ_v. Then there is a sign $\epsilon(v, \mathcal{E}, \varphi_H) \in \{\pm 1\}$ such that the identity $\widetilde{\pi}(\widetilde{f}_v^H) = \epsilon(v, \mathcal{E}, \varphi_H)\sigma_v(f_v) = \epsilon(v, \mathcal{E}, \varphi_H)$ holds, where the twisted trace is Whittaker normalised (cf.* [Lab11, §3.6]*).*

Proof. Let $\Pi(\varphi_H)$ be the L-packet of discrete series representations of $H(F_v^+)$ associated to φ_H. According to the main result of [Clo82], there is a sign $\epsilon_1 \in \{\pm 1\}$ such that $\widetilde{\pi}(\widetilde{f}_v^H) = \epsilon_1 \sum_{\sigma_{v,H} \in \Pi(\varphi_H)} \sigma_{v,H}(f_v^H)$. According to [Kal16, Proposition 5.10], there is a sign $\epsilon_2 \in \{\pm 1\}$ such that $\epsilon_2 \sum_{\sigma_{v,H} \in \Pi(\varphi_H)} \sigma_{v,H}(f_v^H) = \sigma_v(f_v)$. We may take $\epsilon(v, \mathcal{E}, \varphi_H) = \epsilon_1 \epsilon_2$. □

The sign in Proposition 1.6 depends on our fixed choice of pure inner twist (because it depends on the normalisation of transfer factors). We make the following basic but important remark, which is used in the proof of Proposition 4.6: let L^+/F^+ be a finite totally real extension, and let $L = L^+F$. Then G_{L^+} satisfies our standard assumptions, and comes equipped with a pure inner twist by base extension. If v is an infinite place of L^+, then we have the identity $\epsilon(v, \mathcal{E}_{L^+}, \varphi_H|_{W_{L_v^+}}) = \epsilon(v|_{F^+}, \mathcal{E}, \varphi_H)$.

Proposition 1.7. *Let $n \geq 1$ be odd, let v be a finite place of F^+, and let $f_v \in C_c^\infty(G(F_v^+))$. Suppose given an extended endoscopic triple $\mathcal{E} = (H, s, \eta)$.*

(1) *Suppose that v is inert in F and that f_v is unramified (i.e. $G(\mathcal{O}_{F_v^+})$-biinvariant). Suppose given an unramified Langlands parameter $\varphi_H : W_{F_v^+} \to {}^L H$ and let $\sigma_{v,H}$, σ_v be the unramified irreducible representations of $H(F_v^+)$, $G(F_v^+)$ associated to the parameters $\varphi_H, \eta \circ \varphi_H$, respectively. Let π be the unramified irreducible representation of $M^H(F_v^+)$ associated to $\varphi_H|_{W_{F_{\widetilde{v}}}}$. Then there are identities $\widetilde{\pi}(\widetilde{f}_v^H) = \sigma_{v,H}(f_v^H) = \sigma_v(f_v)$, where the twisted trace is normalised so that θ fixes the unramified vector of π. (If π is generic, this agrees with the Whittaker normalisation of the twisted trace.)*

(2) *Suppose that $v = \widetilde{v}\widetilde{v}^c$ splits in F. Suppose given a bounded Langlands parameter $\varphi_H : W_{F_v^+} \to {}^L H$ and let $\sigma_{v,H}$, σ_v be the representations of $H(F_v^+)$, $G(F_v^+)$ associated to the parameters φ_H, $\eta \circ \varphi_H$, respectively (by the local Langlands correspondence $\mathrm{rec}_{F_{\widetilde{v}}}$ for general linear groups). Let π_v be the irreducible representation of $M^H(F_v^+)$ associated to $\varphi_H|_{W_{F_{\widetilde{v}}}}$. Then there is an identity $\widetilde{\pi}_v(\widetilde{f}_v^H) = \sigma_{v,H}(f_v^H) = \sigma_v(f_v)$, where the twisted trace is Whittaker normalised.*

Proof. It is well-known that these identities hold up to non-zero scalar, which depends on the choice of transfer factor; the point here is that, with our choices, the scalar disappears. In the first part, the identity $\widetilde{\pi}(\widetilde{f}_v^H) = \sigma_{v,H}(f_v^H)$ is the fundamental lemma for stable base change [Clo90a]. The identity $\sigma_{v,H}(f_v^H) = \sigma_v(f_v)$ is the fundamental lemma for standard endoscopy [LN08], which holds on the nose because our transfer factors are identified, by the isomorphism $\iota_v : G(F_v^+) \to U_n(F_v^+)$, with those defined in [LS87] with respect to our fixed pinning of U_{n,F_v^+}; this is the 'canonical normalisation' of [Hal93]. If π is generic then a Whittaker functional is non-zero on the unramified vector, which gives the final assertion of the first part of the proposition.

In the second part, the equality $\widetilde{\pi}_v(\widetilde{f}_v^H) = \sigma_{v,H}(f_v^H)$ is the fundamental lemma for stable base change in the split case, cf. [Rog90, Proposition 4.13.2] (where the result is stated for $U(3)$ but the proof is valid in general). The equality $\sigma_{v,H}(f_v^H) = \sigma_v(f_v)$ holds because σ_v can be expressed as the normalised induction of a character twist of $\sigma_{v,H}$ (after choosing an appropriate embedding $H_{F_v^+} \to G_{F_v^+}$ and a parabolic subgroup of $G_{F_v^+}$ containing $H_{F_v^+}$) and because the correspondence $f_v \mapsto f_v^H$ can in this case be taken to be the corresponding character twist of the constant term along $H_{F_v^+}$ (cf. [Rog90, Lemma 4.13.1] and [Shi11, §§3.3–3.4], noting that our normalisation of transfer factors at the place v in this case agrees on the nose with the analogue of the factor written as Δ_v^0 in *loc. cit.*, as follows from the definition in [LS87]). □

1.8. ***L*-packets and types for *p*-adic unitary groups.** Let v be a place of F^+ inert in F. In this section we follow Mœglin [Mœg07, Mœg14] in defining L-packets of tempered representations for the group $G(F_v^+)$ (equivalently, given our choice of pure inner twist, $U_n(F_v^+)$).

We write $\mathcal{A}(\mathrm{GL}_n(F_{\widetilde{v}}))$ for the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(F_{\widetilde{v}})$ over $\mathbf{C}$, and $\mathcal{A}_t(\mathrm{GL}_n(F_{\widetilde{v}}))$ for its subset of tempered representations. We define $\mathcal{A}(U_n(F_v^+))$ and $\mathcal{A}_t(U_n(F_v^+))$ similarly. We write

$\mathcal{A}^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))$ and $\mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))$ for the respective subsets of θ-invariant representations (so e.g. $\mathcal{A}^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))$ is the set of irreducible representations of $\mathrm{GL}_n(F_{\widetilde{v}})$ such that $\pi^\theta := \pi \circ \theta \cong \pi$). Using the local Langlands correspondence $\mathrm{rec}_{F_{\widetilde{v}}}$ for $\mathrm{GL}_n(F_{\widetilde{v}})$ (and the Jacobson–Morozov theorem), we can identify $\mathcal{A}(\mathrm{GL}_n(F_{\widetilde{v}}))$ with the set of $\mathrm{GL}_n(\mathbf{C})$-conjugacy classes of Langlands parameters, i.e. the set of $\mathrm{GL}_n(\mathbf{C})$-conjugacy classes of continuous homomorphisms $\varphi : W_{F_{\widetilde{v}}} \times \mathrm{SL}_2(\mathbf{C}) \to \mathrm{GL}_n(\mathbf{C})$ satisfying the following conditions:

- $\varphi|_{W_{F_{\widetilde{v}}}}$ is semisimple;
- $\varphi|_{\mathrm{SL}_2(\mathbf{C})}$ is algebraic.

Then $\mathcal{A}_t(\mathrm{GL}_n(F_{\widetilde{v}}))$ is identified with the set of parameters φ such that the $\varphi(W_{F_{\widetilde{v}}})$ is relatively compact, and $\mathcal{A}^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))$ is identified with the set of conjugate self-dual parameters. We write $\mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))_+ \subset \mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))$ for the subset of parameters φ which extend to a homomorphism $\varphi_{F_v^+} : W_{F_v^+} \times \mathrm{SL}_2(\mathbf{C}) \to {}^L U_n$. Such an extension, if it exists, is unique up to $\mathrm{GL}_n(\mathbf{C})$-conjugacy (see e.g. [GGP12, Theorem 8.1]). The existence of such an extension $\varphi_{F_v^+}$ can be equivalently phrased as follows: fix a decomposition $\varphi = \oplus_{i\in I}\rho_i^{l_i} \oplus_{j\in J} \sigma_j^{m_j} \oplus_{k\in K} (\tau_k \oplus \tau_k^{w_c\vee})^{n_k}$, where:

- The integers l_i, m_j, n_k are all non-zero.
- Each representation ρ_i, σ_j, τ_k is irreducible and no two are isomorphic.
- For each i we have $\rho_i \cong \rho_i^{w_c,\vee}$ and for each j we have $\sigma_j \cong \sigma_j^{w_c,\vee}$. For each k we have $\tau_k \ncong \tau_k^{w_c,\vee}$.
- For each i, ρ_i is conjugate self-dual of sign $(-1)^{n-1}$ and for each j, σ_j is conjugate self-dual of sign $(-1)^n$, in the sense of [GGP12, p. 10].

Then an extension $\varphi_{F_v^+}$ exists if and only if each integer m_j is even. If the extension $\varphi_{F_v^+}$ is discrete, in the sense that $\mathrm{Cent}(\mathrm{GL}_n(\mathbf{C}), \mathrm{im}\,\varphi_{F_v^+})$ is finite, then $l_i = 1$ for each $i \in I$ and the sets J, K are empty. If the parameter $\varphi_{F_v^+}$ corresponding to a representation $\pi \in \mathcal{A}^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))_+$ is discrete, then we say that π is θ-discrete.

Let $\mathcal{S}_n$ denote the set of equivalence classes of pairs $((n_1, \dots, n_k), (\pi_1, \dots, \pi_k))$, where $(n_1, \dots, n_k)$ is a partition of n and $\pi_1, \dots, \pi_k$ are supercuspidal representations of $\mathrm{GL}_{n_1}(F_{\widetilde{v}}), \dots, \mathrm{GL}_{n_k}(F_{\widetilde{v}})$, respectively. Two such pairs are said to be equivalent if they are isomorphic after permutation of the indices $\{1, \dots, k\}$. Thus we may think of an element of $\mathcal{S}_n$ as a formal sum of supercuspidal representations. We recall (see e.g. [BZ77]) that to any $\pi \in \mathcal{A}(\mathrm{GL}_n(F_{\widetilde{v}}))$ we may associate the supercuspidal support $sc(\pi) \in \mathcal{S}_n$, defined by the condition that π occurs as an irreducible subquotient of the representation $\pi_1 \times \pi_2 \times \cdots \times \pi_k$ (notation for induction as defined at the beginning of this paper).

Mœglin associates to any element $\tau \in \mathcal{A}_t(U_n(F_v^+))$ its *extended cuspidal support* $esc(\tau) \in \mathcal{S}_n$. We do not recall the definition here but note that its definition can be reduced to the case where τ is supercuspidal, in the following sense: suppose that τ is a subquotient of a representation

$$\pi_1 \times \cdots \times \pi_{k-1} \times \tau_0 = i_{Q_{(n_1,\dots,n_k)}}^{U_n} \pi_1 \otimes \cdots \otimes \pi_{k-1} \otimes \tau_0,$$

where τ_0 is a supercuspidal representation of $U_{n_k}(F_v^+)$. Then $esc(\tau) = sc(\pi_1) + \cdots + sc(\pi_{k-1}) + esc(\tau_0) + sc(\pi_{k-1}^\theta) + \cdots + sc(\pi_1^\theta)$.

Proposition 1.9. *If $\tau \in \mathcal{A}_t(U_n(F_v^+))$ then there is a unique element $\pi_\tau \in \mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))_+$ such that $esc(\tau) = sc(\pi_\tau)$.*

Proof. [Mœg07, Lemme 5.4] states that there is a unique element $\pi = \pi_\tau \in \mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))$ such that $esc(\tau) = sc(\pi)$. We need to explain why in fact $\pi \in \mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))_+$. [Mœg07, Théorème 5.7] states that this is true when τ is square-integrable. In general, we can find a Levi subgroup $M_{(n_1,\ldots,n_k)} \subset U_n$ and an irreducible square-integrable representation $\pi_1 \otimes \cdots \otimes \pi_{k-1} \otimes \tau_0$ of $M_{(n_1,\ldots,n_k)}(F_v^+)$ such that τ is a subquotient of $i_{Q_{(n_1,\ldots,n_k)}}^{U_n} \pi_1 \otimes \cdots \otimes \pi_{k-1} \otimes \tau_0$ (see [Wal03, Proposition III.4.1]). Then $\pi_\tau = (\pi_1 \times \pi_1^\theta) \times (\pi_2 \times \pi_2^\theta) \times \cdots \times (\pi_{k-1} \times \pi_{k-1}^\theta) \times \pi_{\tau_0}$, so the result follows from the square-integrable case. □

According to the proposition, there is a well-defined map

$$BC : \mathcal{A}_t(U_n(F_v^+)) \to \mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))_+$$

defined by $BC(\tau) = \pi_\tau$ (which might be called stable base change).

Proposition 1.10. *The map BC is surjective, and it has finite fibres.*

Proof. The image of BC contains the θ-discrete representations, and the fibres of BC above such representations are finite, by [Mœg07, Théorème 5.7]. The general case can again be reduced to this one. □

If $\pi \in \mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))_+$, then we define $\Pi(\pi) = BC^{-1}(\pi)$. By definition, the sets $\Pi(\pi)$ partition $\mathcal{A}_t(U_n(F_v^+))$ and therefore deserve to be called L-packets. The following proposition is further justification for this.

Proposition 1.11. *Let $\pi \in \mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))_+$, and fix an extension $\widetilde{\pi}$ to the twisted group $\mathrm{GL}_n(F_{\widetilde{v}}) \rtimes \theta$. Then there are constants $c_\tau \in \mathbf{C}^\times$ such that for any $f \in C_c^\infty(\mathrm{GL}_n(F_{\widetilde{v}}) \rtimes \theta)$:*

$$\widetilde{\pi}(f) = \sum_{\tau \in \Pi(\pi)} c_\tau \tau(f^{U_n}).$$

Proof. When π is θ-discrete, this is the content of [Mœg07, Proposition 5.5]. In general, $\Pi(\pi)$ admits the following explicit description: decompose $\pi = \pi_1 \times \pi_2 \times \pi_1^\theta$, where $\pi_1 \in \mathcal{A}_t(\mathrm{GL}_{n_1}(F_{\widetilde{v}}))$ and $\pi_2 \in \mathcal{A}_t^\theta(\mathrm{GL}_{n-2n_1}(F_{\widetilde{v}}))_+$ is θ-discrete. Then $\Pi(\pi)$ is the set of Jordan–Hölder factors of the induced representations $\pi_1 \times \tau_2$ as τ_2 varies over the set of elements of $\Pi(\pi_2)$. Using the compatibility of transfer with normalised constant terms along a parabolic (see [Mor10, Lemma 6.3.4]) we thus have an identity

$$\widetilde{\pi}(f) = (\pi_1 \times \pi_2 \times \pi_1^\theta)^\sim(f) = \sum_{\tau_2 \in \Pi(\pi_2)} c_{\tau_2}(\pi_1 \times \tau_2)(f^{U_n})$$

for some constants $c_{\tau_2} \in \mathbf{C}^\times$. To prove the proposition, it is enough to show that if $\tau_2, \tau_2' \in \Pi(\pi_2)$ are non-isomorphic then the induced representations $\pi_1 \times \tau_2$, $\pi_1 \times \tau_2'$ have no Jordan–Hölder factors in common. This follows from [Wal03, Proposition III.4.1]. □

We now introduce some particular representations of U_n. These are built out of depth zero supercuspidal representations of U_3. Accordingly we first introduce some cuspidal representations of the finite group of Lie type $U_3(k(v))$:

- We write $\tau(v)$ for the unique cuspidal unipotent representation of $U_3(k(v))$ (see [Lus77, §9]).

- Let $k_3/k(v)$ be a degree 3 extension, and define
$$C = \ker(\mathbf{N}_{k_3 k(\widetilde{v})/k_3} : \mathrm{Res}_{k_3 k(\widetilde{v})/k(v)} \mathbf{G}_m \to \mathrm{Res}_{k_3/k(v)} \mathbf{G}_m).$$
Then there is a unique $U_3(k(v))$-conjugacy class of embeddings $C \to U_{3,k(v)}$ (as can be proved using e.g. [DL76, Corollary 1.14]).
Let p be a prime such that q_v is a primitive 6^{th} root of unity modulo p, and let $\theta : C(k(v)) \to \mathbf{C}^\times$ be a character of order p. Then we write $\lambda(v, \theta)$ for the (negative of the) Deligne–Lusztig induction $-R_C^{U_{3,k(v)}} \theta$. Then $\lambda(v, \theta)$ is a cuspidal irreducible representation of $U_3(k(v))$ (note that C is not contained in any proper $k(v)$-rational parabolic of $U_{3,k(v)}$).

We define $\widetilde{C} = \mathrm{Res}_{k_3 k(\widetilde{v})/k(\widetilde{v})} \mathbf{G}_m$. Then the homomorphism $\widetilde{C}(k(\widetilde{v})) \to C(k(v))$, $z \mapsto z/z^c$, is surjective, and we define a character $\widetilde{\theta} : \widetilde{C}(k(\widetilde{v})) \to \mathbf{C}^\times$ of order p by $\widetilde{\theta}(z) = \theta(z/z^c)$. There is a unique $\mathrm{GL}_3(k(\widetilde{v}))$-conjugacy class of embeddings $\widetilde{C} \to \mathrm{GL}_{3,k(\widetilde{v})}$, and we write $\widetilde{\lambda}(\widetilde{v}, \widetilde{\theta})$ for the Deligne–Lusztig induction $R_{\widetilde{C}}^{\mathrm{GL}_{3,k(\widetilde{v})}} \widetilde{\theta}$. Then $\widetilde{\lambda}(\widetilde{v}, \widetilde{\theta})$ is a cuspidal irreducible representation of $\mathrm{GL}_3(k(\widetilde{v}))$.

We now assume that the residue characteristic of $k(v)$ is odd.

Proposition 1.12. *(1) Let $\tau_v = \text{c-Ind}_{U_3(\mathcal{O}_{F_v^+})}^{U_3(F_v^+)} \tau(v)$ (compact induction). Then τ_v is a supercuspidal irreducible admissible representation of $U_3(F_v^+)$ and $BC(\tau_v) = \mathrm{St}_2(\chi) \boxplus \mathbf{1}$, where $\chi : F_{\widetilde{v}}^\times \to \mathbf{C}^\times$ is the unique non-trivial quadratic unramified character.*

(2) Let $\lambda_v(\theta) = \text{c-Ind}_{U_3(\mathcal{O}_{F_v^+})}^{U_3(F_v^+)} \lambda(v, \theta)$. Then $\lambda_v(\theta)$ is a supercuspidal irreducible admissible representation of $U_3(F_v^+)$.

(3) Extend $\widetilde{\lambda}(\widetilde{v}, \widetilde{\theta})$ to a representation of $F_{\widetilde{v}}^\times \mathrm{GL}_3(\mathcal{O}_{F_{\widetilde{v}}})$ by making $F_{\widetilde{v}}^\times$ act trivially, and let $\widetilde{\lambda}_{\widetilde{v}}(\widetilde{\theta}) = \text{c-Ind}_{F_{\widetilde{v}}^\times \mathrm{GL}_3(\mathcal{O}_{F_{\widetilde{v}}})}^{\mathrm{GL}_3(F_{\widetilde{v}})} \widetilde{\lambda}(\widetilde{v}, \widetilde{\theta})$. Then $\widetilde{\lambda}_{\widetilde{v}}(\widetilde{\theta})$ is a supercuspidal irreducible admissible representation of $\mathrm{GL}_3(F_{\widetilde{v}})$, and $BC(\lambda_v(\theta)) = \widetilde{\lambda}_{\widetilde{v}}(\widetilde{\theta})$.

Proof. If μ_0 is a cuspidal irreducible representation of $U_3(k(v))$, then $\text{c-Ind}_{U_3(\mathcal{O}_{F_v^+})}^{U_3(F_v^+)} \mu_0$ is a supercuspidal, irreducible admissible representation of $U_3(F_v^+)$ (see [MP96, Proposition 6.6] – we will return to this theme shortly). The essential point therefore is to calculate the extended cuspidal support in each case, which can be done using the results of [LS20] (which require the assumption that $k(v)$ has odd characteristic). Indeed §8 in *op. cit.* explains how to compute the reducibility points $\mathrm{Red}(\pi)$ (defined in [Mœg07, §4]) of a depth 0 supercuspidal representation, at least up to unramified twist. We compute that $\mathrm{Red}(\tau_v) = \{(\mathbf{1}, 3/2), (\chi, 1)\}$ or $\{(\mathbf{1}, 1), (\chi, 3/2)\}$ which corresponds to $BC(\tau_v) = \mathrm{St}_2(\mathbf{1}) \boxplus \chi$ or $BC(\tau_v) = \mathrm{St}_2(\chi) \boxplus \mathbf{1}$. Since $BC(\tau_v) \in \mathcal{A}_t^\theta(\mathrm{GL}_3(F_{\widetilde{v}}))_+$, the second alternative holds. For $\lambda_v(\theta)$, we deduce that $\mathrm{Red}(\lambda_v(\theta)) = \{(\rho, 1)\}$, where ρ is a conjugate self-dual unramified twist of $\widetilde{\lambda}_{\widetilde{v}}(\widetilde{\theta})$. We again conclude by sign considerations. □

Corollary 1.13. *Let $n = 2k+1$ be an odd integer, and consider a representation*
$$\pi = \mathrm{St}_2(\chi) \boxplus \mathbf{1} \boxplus (\boxplus_{i=1}^{2k-2} \chi_i) \in \mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))_+,$$
where $\chi : F_{\widetilde{v}}^\times \to \mathbf{C}^\times$ is the unique non-trivial quadratic unramified character and for each $i = 1, \dots, 2k-2$, $\chi_i : F_{\widetilde{v}}^\times \to \mathbf{C}^\times$ is a character such that $\chi_i|_{\mathcal{O}_{F_{\widetilde{v}}}^\times}$ has order 2. We can assume, after relabelling, that $\chi_i = \chi_{2k-1-i}^{w_c,\vee}$ $(i = 1, \dots, k-1)$,

and then $\Pi(\pi)$ *contains each irreducible subquotient of the induced representation* $\chi_1 \times \chi_2 \times \cdots \times \chi_{k-1} \times \tau_v$.

Proof. First we explain why we can relabel the characters so that $\chi_i = \chi_{2k-1-i}^{w_c,\vee}$. Considering the explicit description of $\mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))_+$, we need to explain why conjugate self-dual characters must appear with even multiplicity amongst the χ_i. Suppose χ_1 is conjugate self-dual. We know that $\chi_1|_{\mathcal{O}_{F_{\widetilde{v}}}^\times}$ is the non-trivial quadratic character, so there are two possibilities for χ_1 determined by $\chi_1(\varpi_v) = -1$ or 1 (this value is also the sign of χ_1). If the sign is -1, the multiplicity of χ_1 is one of the even exponents m_j. If the sign is $+1$, dimension reasons force its multiplicity to be even. The rest of the Corollary follows from the definition of $\Pi(\pi)$ in terms of extended supercuspidal supports. Note that we do not claim that $\Pi(\pi)$ contains *only* the subquotients of this induced representation – this is not true even when $k = 1$. □

To exploit Corollary 1.13 we need to introduce some results from the theory of types. We state only the results we need, continuing to assume that $n = 2k+1$ is odd. Let $\mathfrak{p}_v$ denote the standard parahoric subgroup of $U_n(\mathcal{O}_{F_v^+})$ associated to the partition $(1,1,\ldots,1,3)$; in other words, the pre-image under the reduction modulo ϖ_v map $U_n(\mathcal{O}_{F_v^+}) \to U_n(k(v))$ of $Q_{(1,1,\ldots,1,3)}(k(v))$. Projection to the Levi factor gives a surjective homomorphism $\mathfrak{p}_v \to M_{(1,1,\ldots,1,3)}(k(v)) \cong (k(\widetilde{v})^\times)^{k-1} \times U_3(k(v))$.

Given a cuspidal representation $\sigma(v)$ of $M_{(1,1,\ldots,1,3)}(k(v)) \cong (k(\widetilde{v})^\times)^{k-1} \times U_3(k(v))$, the pair $(\mathfrak{p}_v, \sigma(v))$ defines a depth zero unrefined minimal K-type in the sense of [MP96]. In this case we write $\mathcal{E}(\sigma_v)$ for the set of irreducible representations of $(F_{\widetilde{v}}^\times)^{k-1} \times U_3(\mathcal{O}_{F_v^+}) \subset M_{(1,1,\ldots,1,3)}(F_v^+)$ whose restriction to $M_{(1,1,\ldots,1,3)}(\mathcal{O}_{F_v^+}) = (\mathcal{O}_{F_{\widetilde{v}}}^\times)^{k-1} \times U_3(\mathcal{O}_{F_v^+})$ is isomorphic to (the inflation of) $\sigma(v)$. We have the following result.

Proposition 1.14. *Let* $\sigma(v)$ *be a cuspidal irreducible representation of* $M_{(1,1,\ldots,1,3)}(k(v))$. *Then:*

1. *For any* $\sigma' \in \mathcal{E}(\sigma(v))$, *the compact induction* $\text{c-Ind}_{(F_{\widetilde{v}}^\times)^{k-1} \times U_3(\mathcal{O}_{F_v^+})}^{M_{(1,1,\ldots,1,3)}(F_v^+)} \sigma'(v)$ *is irreducible and supercuspidal.*
2. *Let* π *be an irreducible admissible representation of* $U_n(F_v^+)$. *Then* $\pi|_{\mathfrak{p}_v}$ *contains* $\sigma(v)$ *if and only if* π *is a subquotient of an induced representation*
$$i_{Q_{(1,1,\ldots,1,3)}}^{U_n} \text{c-Ind}_{(F_{\widetilde{v}}^\times)^{k-1} \times U_3(\mathcal{O}_{F_v^+})}^{M_{(1,1,\ldots,1,3)}(F_v^+)} \sigma'(v)$$
for some $\sigma' \in \mathcal{E}(\sigma(v))$.

Proof. See [MP96, Proposition 6.6] and [MP96, Theorem 6.11]. □

We now describe explicitly the two types that we need. Recall that we are assuming that the characteristic of $k(v)$ is odd. Let $\omega(\widetilde{v}) : k(\widetilde{v})^\times \to \{\pm 1\}$ denote the unique non-trivial quadratic character of $k(\widetilde{v})^\times$.

- The representation $\tau(v,n)$ of $\mathfrak{p}_v$ inflated from the representation
$$\omega(\widetilde{v}) \otimes \cdots \otimes \omega(\widetilde{v}) \otimes \tau(v)$$
of $M_{(1,1,\ldots,1,3)}(k(v))$.

- The representation $\lambda(v,\theta,n)$ of $\mathfrak{p}_v$ inflated from the representation
$$\omega(\widetilde{v})\otimes\cdots\otimes\omega(\widetilde{v})\otimes\lambda(v,\theta)$$
of $M_{(1,1,\ldots,1,3)}(k(v))$ (where θ as above is a character of $C(k(v))$ of order p, and we assume q_v mod p is a primitive 6^{th} root of unity).

These types are introduced because they are related by a congruence modulo p, because of our assumption that q_v mod p is a primitive 6^{th} root of unity:

Proposition 1.15. *Fix an isomorphism $\overline{\mathbf{Q}}_p \to \mathbf{C}$ and use this to view $\tau(v,n)$ and $\lambda(v,\theta,n)$ as representations with coefficients in $\overline{\mathbf{Q}}_p$. Then:*

(1) $\overline{\tau}(v,n)$ is irreducible.
(2) $\overline{\lambda}(v,\theta,n)$ contains $\overline{\tau}(v,n)$ as a Jordan–Hölder factor with multiplicity 1.

(As usual, overline denotes semi-simplified residual representation over $\overline{\mathbf{F}}_p$.)

Proof. The modular irreducibility of cuspidal unipotent representations is a general phenomenon (see [DM18]). The proposition is a statement about representations of $U_3(k(v))$, which can be proved by explicit computation with Brauer characters; see [Gec90, Theorem 4.2] (although note that there is a typo in the proof: the right-hand side of the first displayed equation should have $\widehat{\chi}_1$ in place of $\widehat{\chi}_{q^2-q}$). $\square$

The following proposition will be a useful tool for exploiting the type $(\mathfrak{p}_v, \lambda(v,\theta,n))$. We introduce an associated test function $\phi(v,\theta,n) \in C_c^\infty(U_n(F_v^+))$: it is the function supported on $\mathfrak{p}_v$ and inflated from the character of $\lambda(v,\theta,n)^\vee$. If π is an admissible representation of $U_n(F_v^+)$, then $\pi(\phi(v,\theta,n))$ is (up to a positive real scalar depending on normalisation of measures) the dimension of the space $\mathrm{Hom}_{\mathfrak{p}_v}(\lambda(v,\theta,n),\pi|_{\mathfrak{p}_v})$.

Proposition 1.16. *Assume that the characteristic of $k(v)$ is greater than n. Let $\phi = \phi(v,\theta,n)$, and let $\mathcal{E} = (H,s,\eta)$ be one of our fixed endoscopic triples for U_n, with $H = U_p \times U_q$. Suppose given representations π_p, π_q in $\mathcal{A}_t^\theta(\mathrm{GL}_p(F_{\widetilde{v}}))_+$, $\mathcal{A}_t^\theta(\mathrm{GL}_q(F_{\widetilde{v}}))_+$, respectively, such that $(\pi_p\otimes\pi_q)^\sim(\widetilde{\phi}^H)\neq 0$. Then $sc(\pi_p)+sc(\pi_q) = \lambda_{\widetilde{v}}(\widetilde{\theta})+\chi_1+\cdots+\chi_{2k-2}$, where $\chi_1,\ldots,\chi_{2k-2} : F_{\widetilde{v}}^\times \to \mathbf{C}^\times$ are characters such that for each $i=1,\ldots,2k-2$, $\chi_i|_{\mathcal{O}_{F_{\widetilde{v}}}^\times} = \omega(\widetilde{v})$.*

Proof. By Proposition 1.11, there is an identity
$$(\pi_p\otimes\pi_q)^\sim(\widetilde{\phi}^H) = \sum_{\substack{\tau_p\in\Pi(\pi_p)\\ \tau_q\in\Pi(\pi_q)}} c_{\tau_p}c_{\tau_q}(\tau_p\otimes\tau_q)(\phi^H)$$
for some constants $c_{\tau_p}, c_{\tau_q}\in\mathbf{C}^\times$. Now, [KV12, Theorem 2.2.6] shows that ϕ^H can be taken to be a weighted sum of inflations to $H(\mathcal{O}_{F_v^+})$ of characters $R_{C_i}^{H_{k(v)}}(\theta^{-1}\otimes\omega(\widetilde{v})^{\otimes(k-1)})$ associated to conjugacy classes of embeddings $C_i : C\times\mathrm{Res}_{k(\widetilde{v})/k(v)}\mathbf{G}_m^{k-1}\to H_{k(v)}$. (Our appeal to this reference is the reason for the additional assumption on the characteristic of $k(v)$ in the statement of the theorem.) If $(\pi_p\otimes\pi_q)^\sim(\widetilde{\phi}^H)\neq 0$, then there exists a summand on the right-hand side such that $\tau_p\otimes\tau_q$ contains the inflation to $H(\mathcal{O}_{F_v^+})$ of the (irreducible) representation with character $-R_{C_i}^{H_{k(v)}}(\theta\otimes\omega(\widetilde{v})^{\otimes(k-1)})$. Taking into account the compatibility between parabolic induction and Deligne–Lusztig induction, the transitivity of Deligne–Lusztig induction [Lus76], and Proposition 1.14, we see that for one of the representations τ_p,τ_q (the one for the factor of even rank), the extended cuspidal support is a sum of

characters of $F_{\widetilde{v}}^\times$, each of which is the twist of an unramified character by a ramified quadratic character; and for the other of the representations τ_p, τ_q, the extended cuspidal support is a sum of such characters, together with $\lambda_{\widetilde{v}}(\widetilde{\theta})$. This completes the proof. $\square$

1.17. **Types for the general linear group.** In this section we record some analogues of the results of the previous section for general linear groups. Let $2 \leq n_1 \leq n$ be an integer. Let $\widetilde{v}$ be a finite place of F. We assume that the characteristic of $k(\widetilde{v})$ is odd. We have already introduced the notation $\omega(\widetilde{v})$ for the unique non-trivial quadratic character of $k(\widetilde{v})^\times$. We introduce a further representation of the finite group $\mathrm{GL}_{n_1}(k(\widetilde{v}))$ of Lie type:

- Let $k_{n_1}/k(\widetilde{v})$ be an extension of degree n_1, and suppose that $q_{\widetilde{v}} \bmod p$ is a primitive n_1^{th} root of unity modulo p. Let $\Theta : k_{n_1}^\times \to \mathbf{C}^\times$ be a character of order p. Then Θ is distinct from its conjugates by $\mathrm{Gal}(k_{n_1}/k_{\widetilde{v}})$, and we write $\widetilde{\lambda}(\widetilde{v}, \Theta) = (-1)^{n_1-1} R^{\mathrm{GL}_{n_1}}_{\mathrm{Res}_{k_{n_1}/k(\widetilde{v})} \mathbf{G}_m} \Theta$ for the Deligne–Lusztig induction. Then $\widetilde{\lambda}(\widetilde{v}, \Theta)$ is an irreducible representation of $\mathrm{GL}_{n_1}(k(\widetilde{v}))$.

The notation $\widetilde{\lambda}(\widetilde{v}, \Theta)$ thus generalises that introduced in the previous section (where $n_1 = 3$ and $\Theta = \widetilde{\theta}$).

Proposition 1.18. *Let π be an irreducible admissible representation of $\mathrm{GL}_{n_1}(F_{\widetilde{v}})$, and let $F_{\widetilde{v},n_1}/F_{\widetilde{v}}$ denote an unramified extension of degree n_1. Then the following are equivalent:*

(1) The restriction of π to $\mathrm{GL}_{n_1}(\mathcal{O}_{F_{\widetilde{v}}})$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta)$.
(2) There exists a continuous character $\chi : F_{\widetilde{v},n_1}^\times \to \mathbf{C}^\times$ such that $\chi|_{\mathcal{O}^\times_{F_{\widetilde{v},n_1}}} = \Theta$ and $\mathrm{rec}_{F_{\widetilde{v}}} \pi \cong \mathrm{Ind}_{W_{F_{\widetilde{v},n_1}}}^{W_{F_{\widetilde{v}}}} (\chi \circ \mathrm{Art}^{-1}_{F_{\widetilde{v},n_1}})$. In particular, π is supercuspidal.

Proof. This follows from the results of [Hen92] (see especially §3.4 of that paper) and [MP96]. $\square$

Let $n_2 = n - n_1$. We write $\mathfrak{q}_{\widetilde{v}} \subset \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$ for the standard parahoric subgroup associated to the partition (n_1, n_2), i.e. the pre-image under the reduction modulo $\varpi_{\widetilde{v}}$ map $\mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}}) \to \mathrm{GL}_n(k(\widetilde{v}))$ of $P_{(n_1,n_2)}(k(\widetilde{v}))$. We write $\widetilde{\lambda}(\widetilde{v}, \Theta, n)$ for the irreducible representation of $\mathfrak{q}_{\widetilde{v}}$ inflated from the representation $\widetilde{\lambda}(\widetilde{v}, \Theta) \otimes (\omega(\widetilde{v}) \circ \det)$ of $L_{(n_1,n_2)}(k(\widetilde{v}))$. We write $\mathfrak{r}_{\widetilde{v}} \subset \mathfrak{q}_{\widetilde{v}}$ for the standard parahoric subgroup associated to the partition $(n_1, 1, 1, \dots, 1)$. Then we have the following analogue of Proposition 1.14:

Proposition 1.19. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(F_{\widetilde{v}})$. Then the following are equivalent:*

- *$\pi|_{\mathfrak{r}_{\widetilde{v}}}$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta, n)|_{\mathfrak{r}_{\widetilde{v}}}$.*
- *$sc(\pi) = \pi_1 + \chi_1 + \cdots + \chi_{n_2}$, where π_1 satisfies the equivalent conditions of Proposition 1.18 and $\chi_1, \dots, \chi_{n_2} : F_{\widetilde{v}}^\times \to \mathbf{C}^\times$ are characters such that for each $i = 1, \dots, n_2$, $\chi_i|_{\mathcal{O}^\times_{F_{\widetilde{v}}}} = \omega(\widetilde{v})$.*

Proof. This once again follows from the results of [MP96]. $\square$

The pair $(\mathfrak{q}_{\widetilde{v}}, \widetilde{\lambda}(\widetilde{v}, \Theta, n))$ is not in general a type (because $\widetilde{\lambda}(\widetilde{v}, \Theta, n)$ is not a cuspidal representation of $L_{(n_1,n_2)}(k(\widetilde{v}))$ unless $n_2 = 1$). Nevertheless, we have the following proposition:

Proposition 1.20. *Let π be an irreducible admissible representation of $\mathrm{GL}_n(F_{\widetilde{v}})$. Then the following are equivalent:*

(1) *The restriction of π to $\mathfrak{q}_{\widetilde{v}}$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta, n)$.*
(2) *There exist irreducible admissible representations π_i of $\mathrm{GL}_{n_i}(F_{\widetilde{v}})$ ($i = 1, 2$) such that $\pi = \pi_1 \boxplus \pi_2$, the restriction of π_1 to $\mathrm{GL}_{n_1}(\mathcal{O}_{F_{\widetilde{v}}})$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta)$, and the restriction of π_2 to $\mathrm{GL}_{n_2}(\mathcal{O}_{F_{\widetilde{v}}})$ contains $\omega(\widetilde{v}) \circ \det$.*

We note that in the situation of the proposition, π_2 is the twist of an unramified representation by a quadratic ramified character.

Proof. Let $P = P_{(n_1,n_2)}$, $L = L_{(n_1,n_2)}$, and let N_P denote the unipotent radical of P. Abbreviate $\widetilde{\lambda} = \widetilde{\lambda}(\widetilde{v}, \Theta, n)$ and $\widetilde{\lambda}_{N_P} = \widetilde{\lambda}|_{L(\mathcal{O}_{F_{\widetilde{v}}})}$. If π is an irreducible admissible representation of $\mathrm{GL}_n(F_{\widetilde{v}})$ then we define $\pi^{\widetilde{\lambda}} = \mathrm{Hom}_{\mathfrak{q}_{\widetilde{v}}}(\widetilde{\lambda}, \pi|_{\mathfrak{q}_{\widetilde{v}}})$. We first show that for any admissible representation π of $\mathrm{GL}_n(F_{\widetilde{v}})$, the natural projection $\pi^{\widetilde{\lambda}} \to \pi_{N_P}^{\widetilde{\lambda}_{N_P}}$ (restriction of projection to unnormalised Jacquet module) is an isomorphism. Indeed, it is surjective by [Vig98, II.10.1, 1)]. To show that it is injective, let $\widetilde{\mu} = \widetilde{\lambda}|_{\mathfrak{r}_{\widetilde{v}}}$ and let $R = P_{(n_1,1,1,\ldots,1)}$, N_R the unipotent radical of R. Then the pair $(\mathfrak{r}_{\widetilde{v}}, \widetilde{\mu})$ is a depth zero unrefined minimal K-type in the sense of [MP96]. We now have a commutative diagram

$$\begin{tikzcd} \pi^{\widetilde{\lambda}} \arrow[r] \arrow[d] & \pi_{N_P}^{\widetilde{\lambda}_{N_P}} \arrow[d] \\ \pi^{\widetilde{\mu}} \arrow[r] & \pi_{N_R}^{\widetilde{\mu}_{N_R}}, \end{tikzcd}$$

where the left vertical arrow is the natural inclusion and the right vertical arrow is the natural projection to co-invariants. The lower horizontal arrow is an isomorphism, by [Mor99, Lemma 3.6]. We conclude that the top horizontal arrow is injective, and therefore an isomorphism.

Suppose now that π is an irreducible admissible representation of $\mathrm{GL}_n(F_{\widetilde{v}})$ and that $\pi^{\widetilde{\lambda}} \neq 0$. Then $\pi^{\widetilde{\mu}} \neq 0$, so by Proposition 1.19, π is an irreducible subquotient of an induced representation $\pi' = \pi_1 \times \chi_1 \times \cdots \times \chi_{n_2}$, where the inducing data is as in the statement of that proposition. Computation of the Jacquet module (using the geometric lemma [BZ77, Lemma 2.12]) shows that $(\pi')^{\widetilde{\lambda}}$ has dimension 1; therefore π must be isomorphic to the unique irreducible subquotient of π' which contains $\widetilde{\lambda}$. This is $\pi_1 \times \pi_2$, where π_2 is the unique irreducible subquotient of $\chi_1 \times \cdots \times \chi_{n_2}$ such that $\pi_2|_{\mathrm{GL}_{n_2}(\mathcal{O}_{F_{\widetilde{v}}})}$ contains $\omega(\widetilde{v}) \circ \det$ (note that $\pi_1 \times \pi_2$ is irreducible, by [Zel80, Proposition 8.5]).

Suppose instead that $\pi = \pi_1 \boxplus \pi_2 = \pi_1 \times \pi_2$, with π_1, π_2 as in the statement of the proposition. Then the geometric lemma shows that $\pi_{N_P}^{\widetilde{\lambda}_{N_P}} \neq 0$, hence $\pi^{\widetilde{\lambda}} \neq 0$. □

We now introduce the local lifting ring associated to the inertial type which is the analogue, on the Galois side, of the pair $(\mathfrak{q}_{\widetilde{v}}, \widetilde{\lambda}(\widetilde{v}, \Theta, n))$ introduced above. We recall that $k_{n_1}/k(\widetilde{v})$ is an extension of degree n_1, $q_{\widetilde{v}} \bmod p$ is a primitive n_1^{th} root of unity modulo p, and $\Theta : k_{n_1}^\times \to \mathbf{C}^\times$ is a character of order p. Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism, so that $\iota^{-1}\Theta : k_{n_1}^\times \to \overline{\mathbf{Q}}_p^\times$ is a character with trivial reduction modulo

p. Fix a coefficient field E and suppose given a representation $\overline{\rho}_{\widetilde{v}} : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(k)$ of the form $\overline{\rho}_v = \overline{\sigma}_{\widetilde{v},1} \oplus \overline{\sigma}_{\widetilde{v},2}$, where:

- Let $F_{\widetilde{v},n_1}/F_{\widetilde{v}}$ be the unramified extension of degree n_1 and residue field k_{n_1}. Then there is an unramified character $\overline{\psi}_{\widetilde{v}} : G_{F_{\widetilde{v},n_1}} \to k^\times$ and an isomorphism $\overline{\sigma}_{\widetilde{v},1} \cong \mathrm{Ind}_{G_{F_{\widetilde{v},n_1}}}^{G_{F_{\widetilde{v}}}} \overline{\psi}_{\widetilde{v}}$.
- $\overline{\sigma}_{\widetilde{v},2}|_{I_{F_{\widetilde{v}}}} \otimes \omega(\widetilde{v}) \circ \mathrm{Art}_{F_{\widetilde{v}}}^{-1}$ is trivial. (In other words, $\overline{\sigma}_{\widetilde{v},2}$ is the twist of an unramified representation by a ramified quadratic character.)

We recall that $\mathcal{C}_{\mathcal{O}}$ denotes the category of complete Noetherian local $\mathcal{O}$-algebras with residue field $\mathcal{O}/\varpi = k$.

Lemma 1.21. *Let $R \in \mathcal{C}_{\mathcal{O}}$ and let $\rho_{\widetilde{v}} : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(R)$ be a continuous lift of $\overline{\rho}_{\widetilde{v}}$ (i.e. a continuous homomorphism such that $\rho_{\widetilde{v}}$ mod $\mathfrak{m}_R = \overline{\rho}_{\widetilde{v}}$). Then there are continuous lifts $\sigma_{\widetilde{v},i} : G_{F_{\widetilde{v}}} \to \mathrm{GL}_{n_i}(R)$ of $\overline{\sigma}_{\widetilde{v},i}$ $(i = 1, 2)$ with the property that $\sigma_{\widetilde{v},1} \oplus \sigma_{\widetilde{v},2}$ is $1 + M_n(\mathfrak{m}_R)$-conjugate to $\rho_{\widetilde{v}}$. Moreover, each $\sigma_{\widetilde{v},i}$ is itself unique up to $1 + M_{n_i}(\mathfrak{m}_R)$-conjugacy.*

Proof. The splitting exists and is unique because the groups $H^i(F_{\widetilde{v}}, \mathrm{Hom}(\overline{\sigma}_{\widetilde{v},1}, \overline{\sigma}_{\widetilde{v},2}))$ and $H^i(F_{\widetilde{v}}, \mathrm{Hom}(\overline{\sigma}_{\widetilde{v},2}, \overline{\sigma}_{\widetilde{v},1}))$ vanish for $i = 0, 1$. Compare [Sho18, Lemma 2.3]. □

Let $R_{\widetilde{v}}^{\square} \in \mathcal{C}_{\mathcal{O}}$ denote the universal lifting ring, i.e. the representing object of the functor of all continuous lifts of $\overline{\rho}_{\widetilde{v}}$. We write $R(\widetilde{v}, \Theta, \overline{\rho}_{\widetilde{v}})$ for the quotient of $R_{\widetilde{v}}^{\square}$ associated by [Sho18, Definition 3.5] to the inertial type $\tau_{\widetilde{v}} : I_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$, $\tau_{\widetilde{v}} = \oplus_{i=1}^{n_1} (\iota^{-1}\Theta^{q_{\widetilde{v}}^{i-1}} \circ \mathrm{Art}_{F_{\widetilde{v}}}^{-1}) \oplus (\omega(\widetilde{v}) \circ \mathrm{Art}_{F_{\widetilde{v}}}^{-1})^{\oplus n_2}$. We record the following properties of $R(\widetilde{v}, \Theta, \overline{\rho}_{\widetilde{v}})$.

Proposition 1.22. *(1) The ring $R(\widetilde{v}, \Theta, \overline{\rho}_{\widetilde{v}})$ is reduced, p-torsion-free, and is supported on a union of irreducible components of $R_{\widetilde{v}}^{\square}$. In particular, $\mathrm{Spec}\, R(\widetilde{v}, \Theta, \overline{\rho}_{\widetilde{v}})$ is $\mathcal{O}$-flat and equidimensional of dimension $1 + n^2$.*

(2) Let $x : R_{\widetilde{v}}^{\square} \to \overline{\mathbf{Q}}_p$ be a homomorphism, and let $\rho_x : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be the pushforward of the universal lifting, with its associated direct sum decomposition $\rho_x \cong \sigma_{x,1} \oplus \sigma_{x,2}$. Then x factors through $R(\widetilde{v}, \Theta, \overline{\rho}_{\widetilde{v}})$ if and only if there is an isomorphism $\sigma_{x,1} \cong \mathrm{Ind}_{G_{F_{\widetilde{v},n_1}}}^{G_{F_{\widetilde{v}}}} \psi_x$ for a character $\psi_x : G_{F_{\widetilde{v},n_1}} \to \overline{\mathbf{Q}}_p^\times$ such that $\psi_x|_{I_{F_{\widetilde{v},n_1}}} \circ \mathrm{Art}_{F_{\widetilde{v},n_1}} = \iota^{-1}\Theta$ and $\sigma_{x,2}|_{I_{F_{\widetilde{v}}}} \otimes \omega(\widetilde{v}) \circ \mathrm{Art}_{F_{\widetilde{v}}}^{-1}$ is trivial.

(3) Let $\sigma_{\widetilde{v},1} : G_{F_{\widetilde{v}}} \to \mathrm{GL}_{n_1}(R(\widetilde{v}, \Theta, \overline{\rho}_{\widetilde{v}}))$ be the representation associated to the universal lifting by Lemma 1.21. There exists $\alpha_{\widetilde{v}} \in R(\widetilde{v}, \Theta, \overline{\rho}_{\widetilde{v}})/(\varpi)$ such that for any Frobenius lift $\phi_{\widetilde{v}} \in G_{F_{\widetilde{v}}}$, $\det(X - \sigma_{\widetilde{v},1}(\phi_{\widetilde{v}}^{n_1})) \equiv (X - \alpha_{\widetilde{v}})^{n_1}$ mod ϖ.

(4) Let $L_{\widetilde{v}}/F_{\widetilde{v}}$ be a finite extension such that $\tau_{\widetilde{v}}|_{I_{L_{\widetilde{v}}}}$ is trivial, and let $R_{L_{\widetilde{v}}}^{\square}$ denote the universal lifting ring of $\overline{\rho}|_{G_{L_{\widetilde{v}}}}$. Then the natural morphism $R_{L_{\widetilde{v}}}^{\square} \to R(\widetilde{v}, \Theta, \overline{\rho}_{\widetilde{v}})$ (classifying restriction of the universal lifting to $G_{L_{\widetilde{v}}}$) factors over the quotient $R_{L_{\widetilde{v}}}^{\square} \to R_{L_{\widetilde{v}}}^{ur}$ that classifies unramified liftings of $\overline{\rho}_{\widetilde{v}}|_{G_{L_{\widetilde{v}}}}$.

Proof. The first two properties follow from [Sho18, Proposition 3.6]. For the third, let $\phi_{\widetilde{v}}$ be a Frobenius lift. We note that $\det(X - \sigma_{\widetilde{v},1}(\phi_{\widetilde{v}})) = X^{n_1} + (-1)^{n_1} \det \sigma_{\widetilde{v},1}(\phi_{\widetilde{v}})$. Indeed, this can be checked at $\overline{\mathbf{Q}}_p$-points, at which $\sigma_{\widetilde{v},1}$ is irreducible, induced from a character of $G_{F_{\widetilde{v},n_1}}$ which extends $\iota^{-1}\Theta \circ \mathrm{Art}_{F_{\widetilde{v},n_1}}^{-1}$. Reducing modulo ϖ and applying Hensel's lemma, we find that there is an element $\alpha'_{\widetilde{v}} \in R(\widetilde{v}, \Theta, \overline{\rho}_{\widetilde{v}})/(\varpi)$ such that $\det(X - \sigma_{\widetilde{v},1}(\phi_{\widetilde{v}})) \equiv \prod_{i=1}^{n_1}(X - q_{\widetilde{v}}^{i-1}\alpha'_{\widetilde{v}}) \bmod \varpi$. If $\alpha_{\widetilde{v}} = (\alpha'_{\widetilde{v}})^{n_1}$ then $\det(X - \sigma_{\widetilde{v},1}(\phi_{\widetilde{v}}^{n_1})) = (X - \alpha_{\widetilde{v}})^{n_1}$. For the fourth part of the lemma, we need to show that the universal lifting is unramified on restriction to $G_{L_{\widetilde{v}}}$. Since $R(\widetilde{v}, \Theta, \overline{\rho}_{\widetilde{v}})$ is reduced, it suffices to check this at each geometric generic point. At such a point $\sigma_{\widetilde{v},1}$ is irreducible, induced from a character of $G_{F_{\widetilde{v},n_1}}$, while $\sigma_{\widetilde{v},2}$ is a quadratic ramified twist of an unramified representation. The result follows. □

1.23. **Algebraic modular forms.** Finally, we define notation for algebraic modular forms on the group G. Retaining our standard assumptions, fix a coefficient field $E \subset \overline{\mathbf{Q}}_p$ containing the image of each embedding $F \to \overline{\mathbf{Q}}_p$, with ring of integers $\mathcal{O}$, and let $\widetilde{I}_p$ denote the set of embeddings $\tau : F \to E$ inducing a place of $\widetilde{S}_p$. Given $\lambda = (\lambda_\tau)_\tau \in (\mathbf{Z}^n_+)^{\widetilde{I}_p}$, we write V_λ for the $E[\prod_{v \in S_p} \mathrm{GL}_n(F_{\widetilde{v}})]$-module denoted W_λ in [Ger19, Definition 2.3]; it is the restriction to $\mathrm{GL}_n(F_{\widetilde{v}})$ of a tensor product of highest weight representations of $\mathrm{GL}_n(E)$. We write $\mathcal{V}_\lambda \subset V_\lambda$ for the $\mathcal{O}[\prod_{v \in S_p} \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})]$-submodule denoted M_λ in *loc. cit.*; it is an $\mathcal{O}$-lattice.

In this paper we will only consider algebraic modular forms with respect to open compact subgroups $U \subset G(\mathbf{A}^\infty_{F^+})$ which decompose as a product $U = \prod_v U_v$, and such that for each $v \in S_p$, $U_v \subset \iota_{\widetilde{v}}^{-1}\mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$. Given such a subgroup, together with a finite set Σ of finite places of F^+ and a smooth $\mathcal{O}[U_\Sigma]$-module M, finite as $\mathcal{O}$-module, we define $S_\lambda(U, M)$ to be the set of functions $f : G(F^+)\backslash G(\mathbf{A}^\infty_{F^+}) \to \mathcal{V}_\lambda \otimes_\mathcal{O} M$ such that for each $u \in U$ and $g \in G(\mathbf{A}^\infty_{F^+})$, $u \cdot f(gu) = f(g)$. (Here U acts on $\mathcal{V}_\lambda \otimes_\mathcal{O} M$ via projection to $U_p \times U_\Sigma$.) If $\lambda = 0$, we drop it from the notation and simply write $S(U, M)$.

We recall the definition of some useful open compact subgroups and Hecke operators (see [Ger19, §2.3] for more details):

- For any place v of F^+ which splits $v = ww^c$ in F, the maximal compact subgroup $\mathrm{GL}_n(\mathcal{O}_{F_w})$. If $v \notin \Sigma \cup S_p$, $U_v = \iota_w^{-1}\mathrm{GL}_n(\mathcal{O}_{F_w})$, and $1 \leq j \leq n$, then the unramified Hecke operator T^j_w given by the double coset operator
$$T^j_w = \left[\iota_w^{-1}\left(\mathrm{GL}_n(\mathcal{O}_{F_w})\begin{pmatrix}\varpi_w \mathrm{Id}_j & 0 \\ 0 & \mathrm{Id}_{n-j}\end{pmatrix}\mathrm{GL}_n(\mathcal{O}_{F_w})\right) \times U^v\right]$$
acts on $S_\lambda(U, M)$.
- For any place v of F^+ which splits $v = ww^c$ in F, the Iwahori subgroup $\mathrm{Iw}_w \subset \mathrm{GL}_n(\mathcal{O}_{F_w})$ of matrices which are upper-triangular modulo ϖ_w.
- For any place $v \in S_p$ and $c \geq b \geq 0$ with $c \geq 1$, the subgroup $\mathrm{Iw}_{\widetilde{v}}(b, c) \subset \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$ of matrices which are upper-triangular $\varpi^c_{\widetilde{v}}$ and unipotent upper-triangular modulo $\varpi^b_{\widetilde{v}}$. If $U_v = \iota_{\widetilde{v}}^{-1}\mathrm{Iw}_{\widetilde{v}}(b, c)$ for each $v \in S_p$ and $1 \leq j \leq n$, then the re-normalised Hecke operator $U^j_{\widetilde{v},\lambda}$ of [Ger19, Definition 2.8] acts on $S_\lambda(U, M)$. (This Hecke operator depends on our choice of uniformizer $\varpi_{\widetilde{v}}$. However, the ordinary part of $S_\lambda(U, M)$, defined below using these operators, is independent of choices.)
- For any place v of F^+ which splits $v = ww^c$ in F, the principal congruence subgroup $K_{\widetilde{v}}(1) = \ker(\mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}}) \to \mathrm{GL}_n(k(\widetilde{v})))$.

When $U_v = \iota_{\widetilde{v}}^{-1}\mathrm{Iw}_{\widetilde{v}}(b, c)$ for each $v \in S_p$, there is a canonical direct sum decomposition $S_\lambda(U, M) = S^{ord}_\lambda(U, M) \oplus S^{n-ord}_\lambda(U, M)$ with the property that $S^{ord}_\lambda(U, M)$ is the largest submodule of $S_\lambda(U, M)$ where each operator $U^j_{\tilde{v},\lambda}$ ($v \in S_p$, $j = 1, \ldots, n$) acts invertibly ([Ger19, Definition 2.13]).

We recall some basic results about the spaces $S_\lambda(U, M)$. We say that U is sufficiently small if for $g \in G(\mathbf{A}^\infty_{F^+})$, the group $G(F^+) \cap gUg^{-1}$ is trivial. We have the following simple lemma (cf. [Ger19, p. 1351]):

Lemma 1.24. *Suppose that U is sufficiently small and that M is $\mathcal{O}$-flat. Then for any $c \geq 1$, the natural map $S_\lambda(U, M) \otimes_{\mathcal{O}} \mathcal{O}/\varpi^c \to S_\lambda(U, M/(\varpi^c))$ is an isomorphism.*

After fixing an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, we can describe the spaces $S_\lambda(U, M)$ in classical terms ([Ger19, Lemma 2.5]):

Lemma 1.25. *Let $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ be an isomorphism. Then there is an isomorphism*

$$S_\lambda(U, M) \otimes_{\mathcal{O},\iota} \mathbf{C} \cong \oplus_\sigma m(\sigma) \operatorname{Hom}_U((M \otimes_{\mathcal{O},\iota} \mathbf{C})^\vee, \sigma^\infty)$$

respecting the action of Hecke operators at finite places away from $\Sigma \cup S_p$, where the sum runs over automorphic representations σ of $G(\mathbf{A}_{F^+})$ such that for each embedding $\tau : F \to \mathbf{C}$ inducing a place v of F^+, σ_v is the restriction to $G(F^+_v)$ of the dual of the irreducible algebraic representation of $\mathrm{GL}_n(\mathbf{C})$ of highest weight $\lambda_{\iota^{-1}\tau}$.

Part I: Analytic continuation of functorial liftings

The first part of this paper (§§2 – 3) is devoted to the proof of Theorem D from the introduction, which shows that automorphy of the n^{th} symmetric power for one level 1 cuspidal Hecke eigenform implies automorphy of the n^{th} symmetric power for all level 1 cuspidal Hecke eigenforms.

As described in the introduction, the proof has two main ingredients. The first, which is the main result of §2, is that automorphy of symmetric powers can be propagated along irreducible components of the Coleman–Mazur eigencurve. The second ingredient, which is explained in §3, uses the main result of [BK05] and has already been sketched in the introduction.

Here we make some further introductory remarks on §2. By making a suitable (in particular, soluble) base change to a CM field, we translate ourselves to the setting of definite unitary groups. We start from a classical point z_0 of an eigenvariety for a rank 2 unitary group, $\mathcal{E}_2$, such that the n^{th} symmetric power of the associated Galois representation is known to be automorphic. We use Emerton's construction of eigenvarieties (involving his locally analytic Jacquet functor), and our point of view on eigenvarieties and Galois representations is particularly influenced by those of [BC09] and [BHS17]. Like the authors of [BHS17], we rely in an essential way on the results of [KPX14], which make it possible to spread out pointwise triangulations to global triangulations. We consider the diagram:

$$\begin{tikzcd} \mathcal{E}_2 \arrow[r, "i_2"] & \mathcal{X}_{ps,2} \times \mathcal{T}_2 \arrow[d, "\mathrm{Sym}^n"] \\ \mathcal{E}_{n+1} \arrow[r, "i_{n+1}"] & \mathcal{X}_{ps,n+1} \times \mathcal{T}_{n+1} \end{tikzcd}$$

Here, $\mathcal{E}_n$ is an eigenvariety for a rank $n + 1$ unitary group, $\mathcal{X}_{ps,d}$ is a certain rigid space of d-dimensional p-adic Galois pseudocharacters and $\mathcal{T}_d$ is a rigid space parameterising characters of a p-adic torus. Our eigenvarieties come equipped with maps to these character varieties as part of their construction; combining this with the existence of a family of Galois pseudocharacters over the eigenvariety interpolating the global Langlands correspondence at classical points gives the closed immersions i_d appearing in the diagram. The map Sym^n corresponds to taking the n^{th} symmetric power of the 2-dimensional pseudocharacter.

Our task is to show that if $\mathcal{C}$ is an irreducible component of $\mathcal{E}_2$ containing z_0, then $\mathrm{Sym}^n(i_2(C))$ is contained in the image of i_n. A classicality result (Lemma 2.30) will then be used to show that for another classical point z_1 of $\mathcal{C}$, its symmetric power $\mathrm{Sym}^n(i_2(z_1))$ is actually the image of a *classical* point of $\mathcal{E}_{n+1}$.

To show that $\mathrm{Sym}^n(i_2(C))$ is indeed contained in the image of i_n, we combine a simple lemma in rigid geometry (Lemma 2.2) with information coming from the local geometry of a certain natural locally closed neighbourhood of $\mathrm{Sym}^n(i_2(z_0))$ in $\mathcal{X}_{ps,n+1} \times \mathcal{T}_{n+1}$ which contains open subspaces of both $\mathcal{E}_{n+1}$ and $\mathrm{Sym}^n(i_2(C))$. This subspace is essentially the trianguline variety, but since we work with spaces of pseudocharacters instead of representations we restrict to open neighbourhoods in which our pseudocharacters are absolutely irreducible and hence naturally lift to representations. Our results on the vanishing of adjoint Selmer groups [NT20] are used to compare $\mathcal{E}_{n+1}$ and the trianguline variety. We proceed in a similar way to the proof of [BC09, Corollary 7.6.11], which shows that vanishing of an adjoint Selmer group implies that i_{n+1} induces an isomorphism between completed local rings of the eigenvariety and the trianguline variety.

2. Trianguline representations and eigenvarieties

Throughout this section, we let p be a prime and let $E \subset \overline{\mathbf{Q}}_p$ be a coefficient field. We write $\mathbf{C}_p$ for the completion of $\overline{\mathbf{Q}}_p$. If $\mathcal{X}$ is a quasi-separated E-rigid space we let $\mathcal{X}(\overline{\mathbf{Q}}_p) = \bigcup_{E' \subset \overline{\mathbf{Q}}_p} \mathcal{X}(E')$, where the union is over finite extensions of E. We can naturally view $\mathcal{X}(\overline{\mathbf{Q}}_p)$ as a subset of the set of closed points of the rigid space $\mathcal{X}_{\mathbf{C}_p}$ (where base extension of a quasi-separated rigid space is as defined in [BGR84, §9.3.6], see also [Con99, §3.1]).

2.1. **An 'analytic continuation' lemma.** Suppose given a diagram of E-rigid spaces

$$\mathcal{Y} \xrightarrow[\beta]{} \mathcal{G} \xleftarrow[\alpha]{} \mathcal{X},$$

where α is a closed immersion. We identify $\mathcal{X}$ with a subspace of $\mathcal{G}$. Let $x \in \mathcal{Y}$ be a point such that $\beta(x) \in \mathcal{X}$.

Lemma 2.2. *Suppose that $\beta^{-1}(\mathcal{X})$ contains an affinoid open neighbourhood of x. Then for each irreducible component $\mathcal{C}$ of $\mathcal{Y}$ containing x, we have $\beta(\mathcal{C}) \subset \mathcal{X}$.*

Proof. We observe that $\beta^{-1}(\mathcal{X}) \cap \mathcal{C}$ is a Zariski closed subset of $\mathcal{C}$ which contains a non-empty affinoid open subset. This forces $\beta^{-1}(\mathcal{X}) \cap \mathcal{C} = \mathcal{C}$ (apply [Con99, Lemma 2.2.3]), hence $\beta(\mathcal{C}) \subset \mathcal{X}$. □

2.3. **A Galois deformation space.** Let F, S, p be as in our standard assumptions (§1). We assume that E contains the image of every embedding $\tau : F \to \overline{\mathbf{Q}}_p$.

2.3.1. *Trianguline deformations – infinitesimal geometry.* This section has been greatly influenced by works of Bellaïche and Chenevier [BC09, Che11]. We use the formalism of families of $(\varphi, \Gamma_{F_{\widetilde{v}}})$-modules, as in [KPX14]. Thus if $v \in S_p$ and X is an E-rigid space, one can define the Robba ring $\mathcal{R}_{X,F_{\widetilde{v}}}$; if $\mathcal{V}$ is a family of representations of $G_{F_{\widetilde{v}}}$ over X, then the functor $D^{\dagger}_{rig}$ of [KPX14, Theorem 2.2.17] associates to $\mathcal{V}$ the family $D^{\dagger}_{rig}(\mathcal{V})$ of $(\varphi, \Gamma_{F_{\widetilde{v}}})$-modules over X which is, locally on X, finite free over $\mathcal{R}_{X,F_{\widetilde{v}}}$. We refer to [HS16, §2] for the definitions of these objects, as well as more detailed references. If $X = \operatorname{Sp} A$, where A is an E-affinoid algebra, we write $\mathcal{R}_{X,F_{\widetilde{v}}} = \mathcal{R}_{A,F_{\widetilde{v}}}$. If $\delta : F_{\widetilde{v}}^{\times} \to A^{\times}$ is a continuous character, we have a rank one $(\varphi, \Gamma_{F_{\widetilde{v}}})$-module $\mathcal{R}_{A,F_{\widetilde{v}}}(\delta_v)$ defined by [KPX14, Construction 6.2.4]. We will also have cause to mention the (φ, Γ)-cohomology groups $H^{*}_{\varphi,\gamma_{F_{\widetilde{v}}}}(-)$ which are defined in [KPX14, §2.3].

Let $v \in S_p$, and let $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(E)$ be a continuous representation. If $\delta_v = (\delta_{v,1}, \dots, \delta_{v,n}) : (F_{\widetilde{v}}^{\times})^n \to E^{\times}$ is a continuous character, we call a triangulation of ρ_v of parameter δ_v an increasing filtration of $D^{\dagger}_{\mathrm{rig}}(\rho_v)$ by direct summand $(\varphi, \Gamma_{F_{\widetilde{v}}})$-stable $\mathcal{R}_{E,F_{\widetilde{v}}}$-submodules such that the successive graded pieces are isomorphic to $\mathcal{R}_{E,F_{\widetilde{v}}}(\delta_{v,1}), \dots, \mathcal{R}_{E,F_{\widetilde{v}}}(\delta_{v,n})$. We say that ρ_v is trianguline of parameter δ_v if it admits a triangulation of parameter δ_v. If δ_v satisfies $\delta_{v,i}(\varpi_{\widetilde{v}}) \in \mathcal{O}^{\times}$ for each i, then we say that δ_v is an ordinary parameter. Equivalently, δ_v is ordinary if $\delta_{v,i} \circ \mathrm{Art}^{-1}_{F_{\widetilde{v}}}$ extends to a continuous character of $G_{F_{\widetilde{v}}}$ for each i. For an ordinary parameter δ_v, ρ_v is trianguline of parameter δ_v if and only if ρ_v has a filtration with successive graded pieces isomorphic to $\delta_{v,1} \circ \mathrm{Art}^{-1}_{F_{\widetilde{v}}}, \dots, \delta_{v,n} \circ \mathrm{Art}^{-1}_{F_{\widetilde{v}}}$.

We say that the character δ_v is regular if for all $1 \leq i < j \leq n$, we have $\delta_{v,i}/\delta_{v,j} \neq x^{a_v}$ for any $a_v = (a_{v,\tau})_{\tau} \in \mathbf{Z}_{\geq 0}^{\mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}},E)}$, where by definition $x^{a_v}(y) = \prod_{\tau} \tau(y)^{a_{v,\tau}}$. Note that the characters x^{a_v} satisfy $|x^{a_v}(p)|_p = p^{-\sum_{\tau} a_{v,\tau}}$, so there is an affinoid cover of the rigid space $\mathrm{Hom}(F_{\widetilde{v}}^{\times}, \mathbf{G}_m)$ with each open containing only finitely many x^{a_v}.

We define $\mathcal{T}_v = \mathrm{Hom}((F_{\widetilde{v}}^{\times})^n, \mathbf{G}_m)$, a smooth rigid space over E, and write $\mathcal{T}_v^{reg} \subset \mathcal{T}_v$ for the Zariski open subspace of regular characters (Zariski open by the finiteness observation in the preceding paragraph). We define $\mathcal{W}_v = \mathrm{Hom}((\mathcal{O}_{F_{\widetilde{v}}}^{\times})^n, \mathbf{G}_m)$ and write $r_v : \mathcal{T}_v \to \mathcal{W}_v$ for the natural restriction map.

Lemma 2.4. *Let $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(E)$ be a continuous representation. Then for any $\delta_v \in \mathcal{T}_v^{reg}(E)$, ρ_v admits at most one triangulation of parameter δ_v. If such a triangulation exists, then ρ_v is strictly trianguline of parameter δ_v in the sense of* [KPX14, Definition 6.3.1].

Proof. Suppose ρ_v admits a triangulation of parameter δ_v, so $D^{\dagger}_{\mathrm{rig}}(\rho_v)$ is equipped with an increasing filtration $\mathrm{Fil}_{\bullet}$. Following [KPX14, Definition 6.3.1], we need to show that for each $0 \leq i \leq n$ the cohomology group $H^0_{\varphi,\gamma_{F_{\widetilde{v}}}}\left(\left(D^{\dagger}_{\mathrm{rig}}(\rho_v)/\mathrm{Fil}_i\right)(\delta_{v,i+1})^{-1}\right)$ is one-dimensional. It follows from [KPX14, Proposition 6.2.8] that $H^0_{\varphi,\gamma_{F_{\widetilde{v}}}}\left(\mathrm{gr}_j\left(D^{\dagger}_{\mathrm{rig}}(\rho_v)\right)(\delta_{v,i})^{-1}\right)$ vanishes when $i < j$ and is one-dimensional when $i = j$. The vanishing holds precisely because δ_v is regular. A dévissage completes the proof. □

Definition 2.5. *If $\delta : F_{\widetilde{v}}^{\times} \to E^{\times}$ is a continuous character (hence locally $\mathbf{Q}_p$-analytic) we let the tuple $(wt_{\tau}(\delta))_{\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}},E)}$ be such that the derivative of δ is* *the map*

$$F_{\widetilde{v}} \to E$$
$$x \mapsto \sum_{\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}}, E)} -wt_\tau(\delta)\tau(x).$$

We can extend this discussion to Artinian local rings. Let $\mathcal{C}'_E$ denote the category of Artinian local E-algebras with residue field E. If $A \in \mathcal{C}'_E$, then $\mathcal{R}_{A,F_{\widetilde{v}}} = \mathcal{R}_{E,F_{\widetilde{v}}} \otimes_E A$. If $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(A)$ is a continuous representation, then $D^\dagger_{\mathrm{rig}}(\rho_v)$ is a free $\mathcal{R}_{A,F_{\widetilde{v}}}$-module. If $\delta_v \in \mathcal{T}_v(A)$, we call a triangulation of ρ_v of parameter δ_v an increasing filtration of $D^\dagger_{\mathrm{rig}}(\rho_v)$ by direct summand $(\varphi, \Gamma_{F_{\widetilde{v}}})$-stable $\mathcal{R}_{A,F_{\widetilde{v}}}$-submodules such that the successive graded pieces are isomorphic to $\mathcal{R}_{A,F_{\widetilde{v}}}(\delta_{v,1}), \ldots, \mathcal{R}_{A,F_{\widetilde{v}}}(\delta_{v,n})$.

If $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(E)$ is a continuous representation and $\mathcal{F}_v$ is a triangulation of parameter $\delta_v \in \mathcal{T}_v(E)$, then we write $\mathcal{D}_{\rho_v,\mathcal{F}_v,\delta_v} : \mathcal{C}'_E \to \mathrm{Sets}$ for the functor which associates to any A the set of equivalence classes of triples $(\rho'_v, \mathcal{F}'_v, \delta'_v)$, where:

- $\rho'_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(A)$ is a lifting of ρ_v, continuous with respect to the p-adic topology on A.
- $\delta'_v \in \mathcal{T}_v(A)$ is a lifting of δ_v.
- $\mathcal{F}'_v$ is a triangulation of ρ'_v of parameter δ'_v which lifts $\mathcal{F}_v$ (note that there is a canonical isomorphism $D^\dagger_{rig}(\rho'_v) \otimes_A E \cong D^\dagger_{rig}(\rho_v)$, as $D^\dagger_{rig}$ commutes with base change).

Triples $(\rho'_v, \mathcal{F}'_v, \delta'_v)$, $(\rho''_v, \mathcal{F}''_v, \delta''_v)$ are said to be equivalent if there exists $g \in 1 + M_n(\mathfrak{m}_A)$ which conjugates ρ'_v to ρ''_v and takes $\mathcal{F}'_v$ to $\mathcal{F}''_v$.

We write $\mathcal{D}_{\rho_v}$ for the functor of equivalence classes of lifts $\rho'_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(A)$. Thus forgetting the triangulation determines a natural transformation $\mathcal{D}_{\rho_v,\mathcal{F}_v,\delta_v} \to \mathcal{D}_{\rho_v}$.

Proposition 2.6. *Suppose that $\delta_v \in \mathcal{T}_v^{reg}(E)$. Then the natural transformation $\mathcal{D}_{\rho_v,\mathcal{F}_v,\delta_v} \to \mathcal{D}_{\rho_v}$ is relatively representable, and injective on A-points for every $A \in \mathcal{C}'_E$. If ρ_v is absolutely irreducible, then both functors are pro-representable, in which case there is a surjective morphism $R_{\rho_v} \to R_{\rho_v,\mathcal{F}_v,\delta_v}$ of (pro-)representing objects.*

Proof. If $F_{\widetilde{v}} = \mathbf{Q}_p$, this is contained in [BC09, Proposition 2.3.6] and [BC09, Proposition 2.3.9]. The general case is given by [Nak13, Lemma 2.35, Proposition 2.37, Corollary 2.38] (noting that Nakamura works with Berger's category of B-pairs, which is equivalent to the category of (φ, Γ)-modules over the Robba ring). □

A consequence of Proposition 2.6 is that when δ_v is regular, $\mathcal{D}_{\rho_v,\mathcal{F}_v,\delta_v}(E[\epsilon])$ can be identified with a subspace of $\mathcal{D}_{\rho_v}(E[\epsilon]) = H^1(F_{\widetilde{v}}, \mathrm{ad}\,\rho_v)$. Since $\mathcal{F}_v$ is moreover uniquely determined by δ_v (Lemma 2.4), this subspace depends only on δ_v, when it is defined. We write $H^1_{tri,\delta_v}(F_{\widetilde{v}}, \mathrm{ad}\,\rho_v)$ for this subspace. We observe that there is a natural transformation $\mathcal{D}_{\rho_v,\mathcal{F}_v,\delta_v} \to \mathrm{Spf}\,\widehat{\mathcal{O}}_{\mathcal{W}_v, r_v(\delta_v)}$, which sends a triple $(\rho'_v, \mathcal{F}'_v, \delta'_v)$ to the character $r_v(\delta'_v)$. Evaluating on $E[\epsilon]$-points, we obtain an E-linear map

$$H^1_{tri,\delta_v}(F_{\widetilde{v}}, \mathrm{ad}\,\rho_v) \to T_{r_v(\delta_v)}\mathcal{W}_v,$$

where $T_{r_v(\delta_v)}\mathcal{W}_v$ denotes the Zariski tangent space of $\mathcal{W}_v$ at the point $r_v(\delta_v)$. This map appears in the statement of the following lemma:

Lemma 2.7. *Let $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(E)$ be a continuous representation. Suppose that:*

(1) ρ_v is de Rham.
(2) ρ_v is trianguline of parameter $\delta_v \in \mathcal{T}_v^{reg}(E)$.
(3) For each $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}}, E)$, we have

$$wt_\tau(\delta_{v,1}) < wt_\tau(\delta_{v,2}) < \cdots < wt_\tau(\delta_{v,n}).$$

We note that the labelled weights wt_τ coincide with the labelled Hodge–Tate weights of ρ_v (cf. [KPX14, Lemma 6.2.12]*).*

Then the natural map

$$\ker\left(H^1_{tri,\delta_v}(F_{\widetilde{v}}, \mathrm{ad}\,\rho) \to T_{r_v(\delta_v)}\mathcal{W}_v\right) \to H^1(F_{\widetilde{v}}, \mathrm{ad}\,\rho)$$

has image contained in

$$H^1_g(F_{\widetilde{v}}, \mathrm{ad}\,\rho) = \ker(H^1(F_{\widetilde{v}}, \mathrm{ad}\,\rho) \to H^1(F_{\widetilde{v}}, \mathrm{ad}\,\rho \otimes_{\mathbf{Q}_p} B_{dR})).$$

Proof. We must show that if $(\rho'_v, \mathcal{F}'_v, \delta'_v) \in \mathcal{D}_{\rho_v, \mathcal{F}_v, \delta_v}(E[\epsilon])$ is an element in the kernel of the map to $T_{r_v(\delta_v)}\mathcal{W}_v$, then ρ'_v is de Rham. When $F_{\widetilde{v}} = \mathbf{Q}_p$, this follows from [BC09, Proposition 2.3.4]; in general it follows from modifying their argument as in [HS16, Proposition 2.6] (the coefficients in this latter result are assumed to be a field, whilst we need coefficients $E[\epsilon]$, but the same proof works with any Artin local E-algebra as coefficient ring). □

We say that a triangulation of a representation $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ of parameter δ_v is non-critical if for each $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}}, E)$, the labelled weights are an increasing sequence of integers:

$$wt_\tau(\delta_{v,1}) < wt_\tau(\delta_{v,2}) < \cdots < wt_\tau(\delta_{v,n}).$$

In other words, if δ_v satisfies condition (3) of Lemma 2.7.

We now give a criterion for a de Rham representation to have a triangulation satisfying this condition. This generalizes [HS16, Lemma 2.9], which treats the crystalline case.

Lemma 2.8. *Let $v \in S_p$, and let $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a de Rham representation satisfying the following conditions:*

(1) There exists an increasing filtration of the associated Weil–Deligne representation $\mathrm{WD}(\rho_v)$ (by sub-Weil–Deligne representations) with associated gradeds given by characters $\chi_{v,1}, \ldots, \chi_{v,n} : W_{F_{\widetilde{v}}} \to \overline{\mathbf{Q}}_p^\times$.
(2) For each embedding $\tau : F_{\widetilde{v}} \to E$, the τ-Hodge–Tate weights of ρ_v are distinct.
(3) For each embedding $\tau : F_{\widetilde{v}} \to E$, let $k_{\tau,1} < \cdots < k_{\tau,n}$ denote the strictly increasing sequence of τ-Hodge–Tate weights of ρ_v. Then we have for all $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}}, E)$:

$$v_p(\chi_{v,1}(p)) < k_{\tau,2} + \sum_{\tau' \neq \tau} k_{\tau',1}$$

and for all $i = 2, \ldots, n-1$:

$$v_p((\chi_{v,1} \ldots \chi_{v,i})(p)) < k_{\tau,i+1} + \sum_{\tau' \neq \tau} k_{\tau',i} + \sum_{\tau'} \sum_{j=1}^{i-1} k_{\tau',j}.$$

Then ρ_v is trianguline of parameter δ_v, where for each $i = 1, \ldots, n$, $\delta_{v,i} : F_{\widetilde{v}}^\times \to \overline{\mathbf{Q}}_p^\times$ is defined by the formula $\delta_{v,i}(x) = (\chi_{v,i} \circ \mathrm{Art}_{F_{\widetilde{v}}}(x)) \prod_\tau \tau(x)^{-k_{\tau,i}}$. In particular, the pair (ρ_v, δ_v) satisfies condition (3) of Lemma 2.7.

Proof. The filtration of $\mathrm{WD}(\rho_v)$ determines an increasing filtration $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = D_{pst}(\rho_v)$ of $D_{pst}(\rho_v)$ by sub-$(\varphi, N, G_{F_{\widetilde{v}}})$-modules (via the equivalence of categories of [BS07, Proposition 4.1]). The main result of [Ber08] states that there is an equivalence of tensor categories between the category of filtered $(\varphi, N, G_{F_{\widetilde{v}}})$-modules and a certain category of $(\varphi, \Gamma_{F_{\widetilde{v}}})$-modules (restricting to the usual equivalence between weakly admissible filtered $(\varphi, N, G_{F_{\widetilde{v}}})$-modules and $(\varphi, \Gamma_{F_{\widetilde{v}}})$-modules associated to de Rham representations). We thus obtain a triangulation of the associated $(\varphi, \Gamma_{F_{\widetilde{v}}})$-module of ρ_v.

A filtered $(\varphi, N, G_{F_{\widetilde{v}}})$-module M of rank 1 is determined up to isomorphism by its associated character $\chi : W_{F_{\widetilde{v}}} \to \overline{\mathbf{Q}}_p^\times$ and the (unique) integers a_τ such that $\mathrm{gr}^{a_\tau}(M_{F_{\widetilde{v}}} \otimes_{F_{\widetilde{v}} \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p, \tau \otimes \mathrm{id}} \overline{\mathbf{Q}}_p) \neq 0$. The corresponding rank-1 $(\varphi, \Gamma_{F_{\widetilde{v}}})$-module is the one associated to the character $\delta : F_{\widetilde{v}}^\times \to \overline{\mathbf{Q}}_p^\times$ given by the formula $\delta = x^{-a_v}(\chi \circ \mathrm{Art}_{F_{\widetilde{v}}})$ (cf. [KPX14, Example 6.2.6(3)]). What we therefore need to verify is that if $a_{\tau,i} \in \mathbf{Z}$ are the integers for which $\mathrm{gr}^{a_{\tau,i}}(M_i/M_{i-1} \otimes_{F_{\widetilde{v},0}, \tau} \overline{\mathbf{Q}}_p) \neq 0$, then $a_{\tau,i} = k_{\tau,i}$.

This follows from hypothesis (3) of the lemma, together with the fact that $D_{pst}(\rho_v)$ is a weakly admissible filtered $(\varphi, N, G_{F_{\widetilde{v}}})$-module, as we now explain. We show by induction on i that the jumps in the induced Hodge–de Rham filtration of M_i are as claimed. For M_1, if these jumps are k_{τ,j_τ} then we have for each τ

$$\frac{1}{[F_{\widetilde{v}} : \mathbf{Q}_p]} \left(k_{\tau,2} + \sum_{\tau' \neq \tau} k_{\tau',1} \right) > \frac{1}{[F_{\widetilde{v},0} : \mathbf{Q}_p]} v_p(\chi_{v,1}(\varpi_v))$$
$$= t_N(M_1) \geq t_H(M_1) = \frac{1}{[F_{\widetilde{v}} : \mathbf{Q}_p]} \sum_{\tau'} k_{\tau',j_{\tau'}}.$$

Since the sequences $k_{\tau,i}$ are strictly increasing, this is possible only if $j_\tau = 1$ for each τ. In general, if the jumps of M_{i-1} are as expected and M_i/M_{i-1} has jumps k_{τ,j_τ} then we have for each τ

$$\frac{1}{[F_{\widetilde{v}} : \mathbf{Q}_p]} \left(k_{\tau,i+1} + \sum_{\tau' \neq \tau} k_{\tau',i} + \sum_{\tau'} \sum_{j=1}^{i-1} k_{\tau',j} \right) > t_N(M_i)$$
$$\geq t_H(\mathrm{Fil}_i) = \frac{1}{[F_{\widetilde{v}} : \mathbf{Q}_p]} \left(\sum_{\tau'} k_{\tau',j_{\tau'}} + \sum_{\tau'} \sum_{j=1}^{i-1} k_{\tau',j} \right).$$

Once again this is possible only if $j_\tau = i$ for each τ. □

Definition 2.9. *We say that a character $\delta_v \in \mathcal{T}_v(\overline{\mathbf{Q}}_p)$ is numerically non-critical if it satisfies the following conditions:*

1. *For each $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}}, E)$, the labelled weights $wt_\tau(\delta_{v,1}), \ldots, wt_\tau(\delta_{v,n})$ are an increasing sequence of integers.*
2. *For each $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}}, E)$, and for each $i = 1, \ldots, n-1$, we have*
$$v_p((\delta_{v,1} \ldots \delta_{v,i})(p)) < wt_\tau(\delta_{v,i+1}) - wt_\tau(\delta_{v,i}).$$

Following [BC09, Remark 2.4.6], we may reformulate Lemma 2.8 as follows: let $\rho_v : G_{F_{\tilde{v}}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be a Hodge–Tate regular de Rham representation, and suppose that $\mathrm{WD}(\rho_v)$ is equipped with an increasing filtration such that the associated gradeds are given by characters $\chi_{v,1}, \dots, \chi_{v,n} : W_{F_{\tilde{v}}} \to \overline{\mathbf{Q}}_p^\times$. Let $k_{\tau,1} < \cdots < k_{\tau,n}$ be the strictly increasing sequences of τ-Hodge–Tate weights, and let $\delta_v \in \mathcal{T}(\overline{\mathbf{Q}}_p)$ be the character defined by the formula $\delta_{v,i}(x) = (\chi_{v,i} \circ \mathrm{Art}_{F_{\tilde{v}}}(x)) \prod_\tau \tau(x)^{-k_{\tau,i}}$. Then if δ_v is numerically non-critical, the representation ρ_v admits a non-critical triangulation with parameter δ_v.

The most important case for us is that of 2-dimensional de Rham representations of $G_{\mathbf{Q}_p}$, and their symmetric powers. In this case the possible triangulations admit a particularly explicit description (cf. [Col08]; this description can be easily justified, including the case $p = 2$, using the results of [Ber08]):

Example 2.10. *Let $\rho : G_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be a de Rham representation with Hodge–Tate weights $k_1 < k_2$ such that $\mathrm{WD}(\rho)^{ss} = \chi_1 \oplus \chi_2$ with $\chi_i : W_{\mathbf{Q}_p} \to \overline{\mathbf{Q}}_p^\times$. Assume for simplicity that $\chi_1 \neq \chi_2$. Then we have the following possibilities:*

(1) If ρ is not potentially crystalline, then we can choose χ_1, χ_2 so that $\chi_1 = \chi_2(|\cdot|_p \circ \mathrm{Art}_{\mathbf{Q}_p}^{-1})$. In this case ρ has a unique triangulation. It is non-critical, of parameter

$$\delta = (x^{-k_1}\chi_1 \circ \mathrm{Art}_{\mathbf{Q}_p}, x^{-k_2}\chi_2 \circ \mathrm{Art}_{\mathbf{Q}_p}).$$

(2) If ρ is potentially crystalline and irreducible, or reducible and indecomposable, ρ has two triangulations. Both of these are non-critical, and their respective parameters are

$$\delta = (x^{-k_1}\chi_1 \circ \mathrm{Art}_{\mathbf{Q}_p}, x^{-k_2}\chi_2 \circ \mathrm{Art}_{\mathbf{Q}_p})$$

and

$$\delta = (x^{-k_1}\chi_2 \circ \mathrm{Art}_{\mathbf{Q}_p}, x^{-k_2}\chi_1 \circ \mathrm{Art}_{\mathbf{Q}_p}).$$

(3) If ρ is decomposable, we can assume $\rho \cong \psi_1 \oplus \psi_2$ where ψ_i has Hodge–Tate weight k_i and $\mathrm{WD}(\psi_i) = \chi_i$. In this case ρ admits two triangulations. The non-critical triangulation has parameter

$$\delta = (x^{-k_1}\chi_1 \circ \mathrm{Art}_{\mathbf{Q}_p}, x^{-k_2}\chi_2 \circ \mathrm{Art}_{\mathbf{Q}_p})$$

and the critical triangulation has parameter

$$\delta = (x^{-k_2}\chi_2 \circ \mathrm{Art}_{\mathbf{Q}_p}, x^{-k_1}\chi_1 \circ \mathrm{Art}_{\mathbf{Q}_p})$$

(see for example [Ber17, Example 3.7] *for the crystalline case).*

We now consider the global situation. We define $\mathcal{T} = \prod_{v \in S_p} \mathcal{T}_v$, $\mathcal{T}^{reg} = \prod_{v \in S_p} \mathcal{T}_v^{reg}$, and $\mathcal{W} = \prod_{v \in S_p} \mathcal{W}_v$. We write $r = \prod_{v \in S_p} r_v : \mathcal{T} \to \mathcal{W}$ for the product of restriction maps. Let $\mathcal{G}_n = (\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \{\pm 1\}$ denote the group scheme defined in [CHT08, §2.1], $\nu_{\mathcal{G}_n} : \mathcal{G}_n \to \mathrm{GL}_1$ its character, and suppose given a continuous homomorphism $\rho : G_{F^+,S} \to \mathcal{G}_n(E)$ such that $\nu_{\mathcal{G}_n} \circ \rho = \epsilon^{1-n}\delta_{F/F^+}^n$. We write $\mathrm{ad}\,\rho$ for the $E[G_{F^+,S}]$-module given by adjoint action of $\mathcal{G}_n$ on the Lie algebra of GL_n. We write $\mathcal{D}_\rho : \mathcal{C}'_E \to \mathrm{Sets}$ for the functor which associates to each $A \in \mathcal{C}'_E$ the set of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(E))$-conjugacy classes of lifts $\rho' : G_{F^+,S} \to \mathcal{G}_n(A)$ of ρ such that $\nu_{\mathcal{G}_n} \circ \rho' = \epsilon^{1-n}\delta_{F/F^+}^n$.

If $\Delta \subset G_{F,S}$ is a subgroup, then $\rho(\Delta) \subset \mathcal{G}_n^\circ(E) = \mathrm{GL}_n(E) \times \mathrm{GL}_1(E)$, and we follow [CHT08] in writing $\rho|_\Delta : \Delta \to \mathrm{GL}_n(E)$ for the projection to the first factor. If $v \in S$, then there is a natural functor $\mathcal{D}_\rho \to \mathcal{D}_{\rho|_{G_{F_{\widetilde{v}}}}}$, given by restriction $\rho' \mapsto \rho'|_{G_{F_{\widetilde{v}}}}$.

Let $\delta = (\delta_v)_{v \in S_p} \in \mathcal{T}^{reg}(E)$ be such that for each $v \in S_p$, $\rho_v = \rho|_{G_{F_{\widetilde{v}}}}$ is trianguline of parameter δ_v. We define a functor

$$\mathcal{D}_{\rho,\mathcal{F},\delta} = \mathcal{D}_\rho \times_{\prod_{v \in S_p} \mathcal{D}_{\rho_v}} \prod_{v \in S_p} \mathcal{D}_{\rho_v,\mathcal{F}_v,\delta_v}$$

and $H^1_{tri,\delta}(F_S/F^+, \mathrm{ad}\,\rho) = \mathcal{D}_{\rho,\mathcal{F},\delta}(E[\epsilon]) \subset H^1(F_S/F^+, \mathrm{ad}\,\rho)$.

Proposition 2.11. *Keeping assumptions as above, suppose further that the following conditions are satisfied:*

(1) For each $v \in S_p$, ρ_v is de Rham.
(2) For each $v \in S$, $\mathrm{WD}(\rho|_{G_{F_{\widetilde{v}}}})$ is generic, in the sense of [All16, Definition 1.1.2].
(3) For each $v \in S_p$ and for each $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}}, E)$, we have

$$wt_\tau(\delta_{v,1}) < wt_\tau(\delta_{v,2}) < \cdots < wt_\tau(\delta_{v,n}).$$

In other words, the triangulation of ρ_v with parameter δ_v is non-critical.
(4) $H^1_f(F^+, \mathrm{ad}\,\rho) = 0$.

Then $\dim_E H^1_{tri,\delta}(F_S/F^+, \mathrm{ad}\,\rho) \leq \dim_E \mathcal{W}$.

Proof. Our assumption that $\mathrm{WD}(\rho|_{G_{F_{\widetilde{v}}}})$ is generic means that for each $v \in S_p$, $H^1_f(F_{\widetilde{v}}, \mathrm{ad}\,\rho) = H^1_g(F_{\widetilde{v}}, \mathrm{ad}\,\rho)$ and for each $v \in S - S_p$, $H^1_f(F_{\widetilde{v}}, \mathrm{ad}\,\rho) = H^1(F_{\widetilde{v}}, \mathrm{ad}\,\rho)$ (see [All16, Remark 1.2.9]). Lemma 2.7 then implies that the map $H^1_{tri,\delta}(F_S/F^+, \mathrm{ad}\,\rho) \to T_{r(\delta)}\mathcal{W}$ is injective (its kernel being contained in $H^1_f(F^+, \mathrm{ad}\,\rho) = 0$). $\square$

2.11.1. *Trianguline representations – global geometry.* We fix a continuous pseudocharacter $\overline{\tau} : G_{F,S} \to k$ of dimension $n \geq 1$ which is conjugate self-dual, in the sense that $\overline{\tau} \circ c = \overline{\tau}^\vee \otimes \epsilon^{1-n}$. (We define pseudocharacters following Chenevier [Che14], where they are called determinants. For a summary of this theory, including what it means for a pseudocharacter of $G_{F,S}$ to be conjugate self-dual, see [NT20, §2].) Let R_{ps} denote the universal pseudodeformation ring representing the functor of lifts of $\overline{\tau}$ to conjugate self-dual pseudocharacters over objects of $\mathcal{C}_\mathcal{O}$ (cf. [NT20, §2.19]). If $v \in S_p$, let $R_{ps,v}$ denote the pseudodeformation ring of $\overline{\tau}|_{G_{F_{\widetilde{v}}}}$. We write $\mathcal{X}_{ps}$ for the rigid generic fibre of R_{ps}, and $\mathcal{X}_{ps,v}$ for the rigid generic fibre of $R_{ps,v}$. Then there is a natural morphism $\mathcal{X}_{ps} \to \mathcal{X}_{ps,p} = \prod_{v \in S_p} \mathcal{X}_{ps,v}$ of rigid spaces over E. We recall that to any representation $\rho : G_{F,S} \to \mathrm{GL}_n(E)$ such that $\mathrm{tr}\,\overline{\rho} = \overline{\tau}$, and which is conjugate self-dual in the sense that $\rho^c \cong \rho^\vee \otimes \epsilon^{1-n}$, is associated a closed point $\mathrm{tr}\,\rho \in \mathcal{X}_{ps}(E)$. Conversely, if $t \in \mathcal{X}_{ps}(E)$, then there exists a semi-simple conjugate self-dual representation $\rho : G_{F,S} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ such that $\mathrm{tr}\,\rho = t$, and this representation is unique up to isomorphism.

If $\rho : G_{F,S} \to \mathrm{GL}_n(E)$ is an absolutely irreducible representation such that $\rho^c \cong \rho^\vee \otimes \epsilon^{1-n}$, then there is a homomorphism $\rho_1 : G_{F^+,S} \to \mathcal{G}_n(E)$ such that $\rho_1|_{G_{F,S}} = \rho$ and $\nu_{\mathcal{G}_n} \circ \rho_1 = \epsilon^{1-n}\delta^a_{F/F^+}$. The integer $a \in \{0,1\}$ is uniquely determined by ρ, and any two such extensions are conjugate by an element of $\mathrm{GL}_n(\overline{\mathbf{Q}}_p)$. (See [CHT08, Lemma 2.1.4].) The following lemma extends extends this to objects of $\mathcal{C}_E$:

Lemma 2.12. *Let $\rho : G_{F,S} \to \mathrm{GL}_n(E)$ be an absolutely irreducible representation such that $\rho^c \cong \rho^\vee \otimes \epsilon^{1-n}$, and fix an extension ρ_1 as in the previous paragraph. Let $A \in \mathcal{C}_E$. Then the following sets are in canonical bijection:*

1. *The set of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(E))$-conjugacy classes of liftings $\rho' : G_{F,S} \to \mathrm{GL}_n(A)$ of ρ such that $\mathrm{tr}\,\rho' \circ c = \mathrm{tr}(\rho')^\vee \otimes \epsilon^{1-n}$.*
2. *The set of $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(E))$-conjugacy classes of liftings $\rho_1' : G_{F^+,S} \to \mathcal{G}_n(A)$ of ρ_1 such that $\nu \circ \rho_1' = \epsilon^{1-n}\delta^a_{F/F^+}$.*

Proof. There is an obvious map sending ρ_1' to $\mathrm{tr}\,\rho_1'|_{G_{F,S}}$. We need to check that this is bijective (at the level of conjugacy classes). To check injectivity, let $\rho_1', \rho_1'' : G_{F^+,S} \to \mathcal{G}_n(A)$ be two such homomorphisms and suppose that $\rho_1'|_{G_{F,S}}$, $\rho_1''|_{G_{F,S}}$ are $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(E))$-conjugate. We must show that ρ_1', ρ_1'' are themselves conjugate. We may suppose that in fact $\rho_1'|_{G_{F,S}} = \rho_1''|_{G_{F,S}}$. In this case Schur's lemma (cf. [CHT08, Lemma 2.1.8]), applied to $\rho_1'(c)^{-1}\rho_1''(c)$, shows that ρ_1', ρ_1'' are equal.

Now suppose given $\rho' : G_{F,S} \to \mathrm{GL}_n(A)$ lifting ρ, and such that $\mathrm{tr}\,\rho' \circ c = \mathrm{tr}(\rho')^\vee \otimes \epsilon^{1-n}$. Let $J \in \mathrm{GL}_n(E)$ be defined by $\rho_1(c) = (J, -\nu_{\mathcal{G}_n} \circ \rho_1(c))\jmath$ (cf. [CHT08, Lemma 2.1.1]), so that $\rho(\sigma^c) = J\rho(\sigma)^{-t}J^{-1} \otimes \epsilon^{1-n}$ for all $\sigma \in G_{F,S}$. Then [Che14, Example 3.4] implies the existence of a matrix $J' \in \mathrm{GL}_n(A)$ lifting J such that $\rho'(\sigma^c) = J'(\rho'(\sigma))^{-t}(J')^{-1}$ for all $\sigma \in G_{F,S}$. By [CHT08, Lemma 2.1.1], this implies the existence of a homomorphism $\rho_1' : G_{F^+,S} \to \mathcal{G}_n(A)$ lifting ρ_1 and such that $\rho_1'|_{G_{F,S}} = \rho'$. This completes the proof. □

There is a Zariski open subspace $\mathcal{X}^{v-irr}_{ps,v} \subset \mathcal{X}_{ps,v}$ consisting of those points at which the universal pseudocharacter is absolutely irreducible. We write $\mathcal{X}^{p-irr}_{ps,p} = \prod_{v \in S_p} \mathcal{X}^{v-irr}_{ps,v}$ and $\mathcal{X}^{p-irr}_{ps}$ for the pre-image of $\mathcal{X}^{p-irr}_{ps,p}$. Thus again there is a canonical morphism $\mathcal{X}^{p-irr}_{ps} \to \mathcal{X}^{p-irr}_{ps,p}$. According to [Che14, §4.2], there exists an Azumaya algebra $\mathcal{A}$ over $\mathcal{X}^{v-irr}_{ps,v}$ and a homomorphism $\rho_v^u : G_{F_{\widetilde{v}}} \to \mathcal{A}^\times$ such that $\mathrm{tr}\,\rho_v^u$ is the universal pseudocharacter.

Lemma 2.13. *Let $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(E)$ be an absolutely irreducible representation, corresponding to a point $z = \mathrm{tr}\,\rho_v \in \mathcal{X}^{v-irr}_{ps,v}(E)$. Then there exists an affinoid open neighbourhood $z \in \mathcal{U} \subset \mathcal{X}^{v-irr}_{ps,v}(E)$ and an isomorphism $\mathcal{A}|_{\mathcal{U}} \cong M_n(\mathcal{O}_{\mathcal{U}})$.*

Proof. Let $\mathcal{U}$ be an open affinoid neighbourhood of z. The stalk $\mathcal{O}_{\mathcal{U},z}$ is a Henselian local ring ([FvdP04, Proposition 7.1.8]). Thus the stalk $\mathcal{A}_z$ is an Azumaya algebra over a Henselian local ring which is split over the closed point; it is therefore split, i.e. there exists an isomorphism $\mathcal{A}_z \cong \mathcal{M}_n(\mathcal{O}_{\mathcal{U},z})$. After shrinking $\mathcal{U}$, this extends to an isomorphism $\mathcal{A}(\mathcal{U}) \cong M_n(\mathcal{O}(\mathcal{U}))$, as desired. □

Lemma 2.14.
1. *Let $z \in \mathcal{X}^{p-irr}_{ps,p}(E)$ be the closed point corresponding to the isomorphism class of a tuple $(\rho_v)_{v \in S_p}$ of absolutely irreducible representations $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(E)$. Then there is a canonical isomorphism*
$$T_z\mathcal{X}^{p-irr}_{ps,p} \cong \oplus_{v \in S_p} H^1(F_{\widetilde{v}}, \mathrm{ad}\,\rho_v).$$
2. *Let $z \in \mathcal{X}^{p-irr}_{ps}(E)$ be the closed point determined by a representation $\rho : G_{F^+,S} \to \mathcal{G}_n(E)$ such that for each $v \in S_p$, $\rho|_{G_{F_{\widetilde{v}}}}$ is absolutely irreducible. Then there is a canonical isomorphism*
$$T_z\mathcal{X}^{p-irr}_{ps} \cong H^1(G_{F^+,S}, \mathrm{ad}\,\rho),$$

which has the property that the diagram

$$\begin{tikzcd} T_z \mathcal{X}_{ps}^{p-irr} \arrow[r] \arrow[d] & H^1(G_{F^+,S}, \operatorname{ad}\rho) \arrow[d] \\ T_z \mathcal{X}_{ps,p}^{p-irr} \arrow[r] & \oplus_{v \in S_p} H^1(F_{\widetilde{v}}, \operatorname{ad}\rho_v) \end{tikzcd}$$

commutes.

Proof. The first part follows from [Che14, §4.1], which states that the completed local ring of $\mathcal{X}_{ps,v}^{v-irr}$ at the E-point corresponding to an absolutely irreducible representation $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(E)$ pro-represents the functor D_{ρ_v}. The second follows from this and Lemma 2.12. $\square$

Proposition 2.15. *Let $v \in S_p$, and let $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(E)$ be an absolutely irreducible representation which is trianguline of parameter $\delta_v \in \mathcal{T}_v^{reg}(E)$. Let $z \in \mathcal{X}_{ps,v}^{v-irr} \times \mathcal{T}_v^{reg}$ be the closed point corresponding to the pair (ρ_v, δ_v). Then:*

1. *There exists an affinoid open neighbourhood $\mathcal{U}_v \subset \mathcal{X}_{ps,v}^{v-irr} \times \mathcal{T}_v^{reg}$ of z over which there exists a universal representation $\rho_v^u : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\mathcal{O}(\mathcal{U}_v))$. Let $\mathcal{V}_v \subset \mathcal{U}_v$ denote the set of points (ρ_v', δ_v') such that ρ_v' is trianguline of parameter δ_v', and let $\mathcal{Z}_v \subset \mathcal{U}_v$ denote the Zariski closure of $\mathcal{V}_v$. Then $\mathcal{V}_v$ is the set of points of a Zariski open subspace of $\mathcal{Z}_v$.*
2. *The Zariski tangent space of $\mathcal{Z}_v$ at z is contained in the subspace $H^1_{tri,\delta_v}(F_{\widetilde{v}}, \operatorname{ad}\rho_v)$ of the Zariski tangent space of $\mathcal{X}_{ps,v}^{v-irr} \times \mathcal{T}_v^{reg}$ at z.*

Proof. By Lemma 2.13, there is an affinoid neighbourhood $\mathcal{U}_v \subset \mathcal{X}_{ps,v}^{v-irr} \times \mathcal{T}_v^{reg}$ of z over which there exists a universal representation $\rho_v^u : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\mathcal{O}(\mathcal{U}_v))$. We can assume without loss of generality that $\mathcal{U}_v$ is connected. By [KPX14, Corollary 6.3.10], there is a reduced rigid space $\mathcal{Z}'$ over E and a proper birational morphism $f : \mathcal{Z}' \to \mathcal{Z}_v$ having the following properties:

- For every point $z' \in \mathcal{Z}'$, the absolutely irreducible representation $\rho_{f(z')}$ is trianguline.
- There is an increasing filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = D^\dagger_{rig}(f^*(\rho^u)))$ by coherent $(\varphi, \Gamma_{F_{\widetilde{v}}})$-stable $\mathcal{R}_{\mathcal{Z}', F_{\widetilde{v}}}$-submodules.
- There exists a Zariski closed subset $\mathcal{Z}_b' \subset \mathcal{Z}'$ such that $\mathcal{Z}_b' \cap f^{-1}(\mathcal{V}_v) = \emptyset$ and for each $z' \in \mathcal{Z}' - \mathcal{Z}_b'$, the pullback of $\mathcal{F}_\bullet$ to $D^\dagger_{rig}(\rho_{f(z')})$ is a triangulation of parameter $\delta_{f(z')}$.
- Over $\mathcal{Z}' - \mathcal{Z}_b'$, $\mathcal{F}_\bullet$ is in fact a filtration by local direct summand $\mathcal{R}_{\mathcal{Z}', F_{\widetilde{v}}}$-submodules.
- The map f factors through a proper birational morphism $\tilde{f} : \mathcal{Z}' \to \widetilde{\mathcal{Z}}_v$, where $\widetilde{\mathcal{Z}}_v$ is the normalisation of $\mathcal{Z}_v$. Moreover, $\tilde{f}$ factors as the composition of a sequence of proper birational morphisms between normal rigid spaces
$$\mathcal{Z}' = \mathcal{Z}_m \to \mathcal{Z}_{m-1} \to \cdots \mathcal{Z}_1 = \widetilde{\mathcal{Z}}_v$$
where each morphism $\mathcal{Z}_i \to \mathcal{Z}_{i-1}$ is glued, locally on the target, from analytifications[3] of birational projective schemes over $\mathrm{Spec}(A)$, with $\mathrm{Sp}(A) \subset \mathcal{Z}_{i-1}$ an affinoid open.

[3]Here we mean the relative analytification defined by Köpf [Köp74], see also [Con06, Example 2.2.11].

Note that the final point is a consequence of the construction in the proof of [KPX14, Theorem 6.3.9]. The third point actually implies that $\mathcal{Z}' = \mathcal{Z}'_b \sqcup f^{-1}(\mathcal{V}_v)$, hence $\mathcal{Z}_v = f(\mathcal{Z}'_b) \sqcup \mathcal{V}_v$. Since f is proper this shows that $\mathcal{V}_v$ is Zariski open in $\mathcal{Z}_v$.

Let $\tilde{z}_1, \dots, \tilde{z}_m \in \widetilde{\mathcal{Z}}_v$ be the closed points of the normalisation with image in $\mathcal{Z}_v$ equal to z. For each $1 \leq j \leq m$, let z'_j be a closed point of the preimage of $\tilde{z}_j$ in $\mathcal{Z}'$. We denote by z_j the image of z'_j in any of the $\mathcal{Z}_i$, for $1 \leq i \leq m$. We claim that the map $\widehat{\mathcal{O}}_{\widetilde{\mathcal{Z}}_v, \tilde{z}_j} \to \widehat{\mathcal{O}}_{\mathcal{Z}', z'_j}$ on completed local rings is injective; indeed, it follows from the final point in the itemized list above that we need to show injectivity for each map $\widehat{\mathcal{O}}_{\mathcal{Z}_i, z_j} \to \widehat{\mathcal{O}}_{\mathcal{Z}_{i+1}, z_j}$. Each of these maps coincides with the map on complete local rings $\widehat{A}_x \to \widehat{\mathcal{O}}_{X,x}$ associated with a (projective, birational) morphism of schemes $X \to \operatorname{Spec}(A)$, where A is the ring of functions on an open affinoid neighbourhood of $z_j \in \mathcal{Z}_i$, $x \in \operatorname{Spec}(A)$ is the maximal ideal given by z_j and $x' \in X$ is a closed point mapping to x. The complete local ring $\widehat{A}_x$ is a domain (by normality and excellence of A_x) and the map $A_x \to \mathcal{O}_{X,x}$ is injective (by dominance of $X \to \operatorname{Spec}(A)$), so [GD71, Corollaire 3.9.8] gives the desired injectivity. The map $\widehat{\mathcal{O}}_{\mathcal{Z}_v, z} \to \prod_{i=1}^m \widehat{\mathcal{O}}_{\widetilde{\mathcal{Z}}_v, \tilde{z}_i}$ is the normalisation of $\widehat{\mathcal{O}}_{\mathcal{Z}_v, z}$, so it is also injective. Putting everything together, we have shown that the map $\widehat{\mathcal{O}}_{\mathcal{Z}_v, z} \to \prod_{i=1}^m \widehat{\mathcal{O}}_{\mathcal{Z}'_v, z'_i}$ is injective.

After possibly extending E, we can assume that all of the points z'_i in $\mathcal{Z}'$ have residue field E. The existence of a global triangulation over $\mathcal{Z}' - \mathcal{Z}'_b$ implies that for each $i = 1, \dots, m$, there is a classifying map $R_{\rho_v, \mathcal{F}_v, \delta_v} \to \widehat{\mathcal{O}}_{\mathcal{Z}', z'_i}$, where $\mathcal{F}_v$ is the unique triangulation of ρ_v of parameter δ_v. This implies the existence of a commutative diagram

$$\begin{tikzcd} \widehat{\mathcal{O}}_{\mathcal{U}_v, z} \arrow[r] \arrow[d] & \widehat{\mathcal{O}}_{\mathcal{Z}_v, z} \arrow[d] \\ R_{\rho_v, \mathcal{F}_v, \delta_v} \arrow[r] & \prod_{i=1}^m \widehat{\mathcal{O}}_{\mathcal{Z}', z'_i}, \end{tikzcd}$$

where the left vertical arrow is surjective (Proposition 2.6) and the top horizontal arrow is surjective. We have already noted that the right vertical arrow is injective. These facts together imply that the top horizontal arrow factors through a surjective map $R_{\rho_v, \mathcal{F}_v, \delta_v} \to \widehat{\mathcal{O}}_{\mathcal{Z}_v, z}$. This implies the desired result at the level of Zariski tangent spaces. □

Remark 2.16. Prompted by a referee, we note that the definition of 'Zariski dense' given in [KPX14, Definition 6.3.2] is somewhat non-standard. In this paper (and in other references we cite such as [BHS17]), a subset Z of a rigid space X is called Zariski dense if the smallest closed analytic subset of X which contains Z is X. In [KPX14, Definition 6.3.2] the stronger condition is imposed that Z is Zariski dense (in the usual sense) in each member of some admissible affinoid cover of X.

When we apply [KPX14, Corollary 6.3.10] in the above proof, we have a Zariski dense subset of an affinoid, so there is no discrepancy between the definitions in this case. Eugen Hellmann has explained to us that the crucial result [KPX14, Theorem 6.3.9] does in fact hold with the weaker, standard definition of Zariski dense. Since it may be of interest, we sketch their argument.

We start with X, δ, M as in [KPX14, Theorem 6.3.9] and suppose we have a Zariski dense (in the usual sense) subset $X_{\mathrm{alg}} \subset X$ satisfying the assumptions of *loc. cit.* We may assume that X is normal and connected, and will show that X_{alg} can be enlarged to a subset which is Zariski dense in the stronger sense of [KPX14].

There are coherent sheaves $H^i_{\varphi,\gamma_K}(M^\vee(\delta))$, $H^i_{\varphi,\gamma_K}(M^\vee(\delta)/t_\sigma)$ on X, which are locally free over a non-empty (hence dense) Zariski open subset $U \subset X$. At points z in the Zariski dense subset $X_{\mathrm{alg}} \cap U$, the fibre $H^0_{\varphi,\gamma_K}(M^\vee(\delta)) \otimes_{\mathcal{O}_X} k(z) \cong H^0_{\varphi,\gamma_K}(M^\vee_z(\delta_z))$ has dimension one and the map $M_z \to \mathcal{R}_{k(z)}(\pi_K)(\delta_z)$ dual to a non-zero element of this fibre is surjective. The latter condition is equivalent to non-vanishing of the map $H^0_{\varphi,\gamma_K}(M^\vee(\delta)) \otimes_{\mathcal{O}_X} k(z) \to H^0_{\varphi,\gamma_K}(M^\vee(\delta)/t_\sigma) \otimes_{\mathcal{O}_X} k(z)$ for every p-adic embedding σ. These conditions hold over a Zariski open subset $U' \subset U$. Since U' contains $X_{\mathrm{alg}} \cap U$, it is also Zariski dense in X. Moreover, U' contains a Zariski dense subset of every affinoid open $V \subset X$. Indeed, the intersection $V \cap U$ with the Zariski open and dense subset U contains an affinoid open subset of V. Repeating this step, the intersection $V \cap U'$ also contains an affinoid open subset of V. We have shown that we obtain the desired enlargement of X_{alg} by adjoining U'.

2.17. **The unitary group eigenvariety.** Now let $F, S, p, G = G_n$ be as in our standard assumptions (§1). We continue to assume that E contains the image of every embedding $\tau : F \to \overline{\mathbf{Q}}_p$. In particular, the reductive group $\mathrm{Res}_{F^+/\mathbf{Q}} G_n$ splits over E.

Let $U_n = \prod_v U_{n,v} \subset G_n(\mathbf{A}^\infty_{F^+})$ be an open compact subgroup such that for every finite place $v \notin S$ of F^+, $U_{n,v}$ is hyperspecial maximal compact subgroup of $G_n(F^+_v)$. We define $\mathbf{T}^S_n = \mathcal{O}[T^1_w, \ldots, T^n_w, (T^n_w)^{-1}] \subset \mathcal{O}[U^S_n \backslash G_n(\mathbf{A}^{\infty,S}_{F^+})/U^S_n]$ to be the algebra generated by the unramified Hecke operators at split places $v = ww^c$ of F^+ not lying in S. These operators were defined in §1.23.

We write $T_n \subset B_n = T_n N_n \subset \mathrm{GL}_n$ for the usual maximal torus and upper triangular Borel subgroup, and define E-rigid spaces

$$\mathcal{W}_n = \mathrm{Hom}(\prod_{v \in S_p} T_n(\mathcal{O}_{F_{\widetilde{v}}}), \mathbf{G}_m)$$

and

$$\mathcal{T}_n = \mathrm{Hom}(\prod_{v \in S_p} T_n(F_{\widetilde{v}}), \mathbf{G}_m).$$

Restriction of characters determines a morphism $r : \mathcal{T}_n \to \mathcal{W}_n$ of rigid spaces. Note that the spaces $\mathcal{T}_n, \mathcal{W}_n$ may be canonically identified with the spaces $\mathcal{T}, \mathcal{W}$ of the previous section.

We fix a choice of isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. If π is an automorphic representation of $G_n(\mathbf{A}_{F^+})$ with $\pi^{U_n} \neq 0$, there is a corresponding semisimple Galois representation $r_{\pi,\iota} : G_{F,S} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ (cf. Corollary 1.3), which satisfies local-global compatibility at each place of F. The space $\iota^{-1}(\pi^\infty)^{U_n}$ is naturally an isotypic $\mathbf{T}^S_n$-module, which therefore determines a homomorphism $\psi_\pi : \mathbf{T}^S_n \to \overline{\mathbf{Q}}_p$. We call an *accessible refinement* of π a choice $\chi = (\chi_v)_{v \in S_p}$ for each $v \in S_p$ of a (necessarily smooth) character $\chi_v : T_n(F_{\widetilde{v}}) \to \overline{\mathbf{Q}}_p^\times$ which appears as a subquotient of the normalised Jacquet module $\iota^{-1} r_{N_n}(\pi_v) = \iota^{-1}(\pi_{v,N_n(F_{\widetilde{v}})} \delta_{B_n}^{-1/2})$; equivalently, for which there is an embedding of π_v into the normalised induction $i_{B_n}^{\mathrm{GL}_n} \iota\chi_v$. Note that $\chi \in \mathcal{T}_n(\overline{\mathbf{Q}}_p)$.

Lemma 2.18. *Let π be an automorphic representation of $G_n(\mathbf{A}^+_F)$, and let $\chi = (\chi_v)_{v \in S_p}$ be an accessible refinement of π. Then for each $v \in S_p$, there is an* *increasing filtration of* $\mathrm{rec}^T_{F_{\widetilde{v}}}(\iota^{-1}\pi_v)$ *by sub-Weil–Deligne representations with graded pieces*

$$\chi_{v,1}|\cdot|^{(1-n)/2} \circ \mathrm{Art}_{F_{\widetilde{v}}}, \dots, \chi_{v,n}|\cdot|^{(1-n)/2} \circ \mathrm{Art}_{F_{\widetilde{v}}}.$$

Proof. Since π_v admits an accessible refinement, it is a subquotient of a principal series representation. Suppose that $\mathrm{rec}_{F_{\widetilde{v}}}(\pi_v) = \oplus_{i=1}^k \mathrm{Sp}_{n_i}(\psi_i|\cdot|^{(n_i-1)/2})$ for some characters $\psi_i : F_{\widetilde{v}}^\times \to \mathbf{C}^\times$. By the Langlands classification, π_v is isomorphic to a subquotient of the normalised induction

$$\Pi = i_P^{\mathrm{GL}_n} \mathrm{St}_{n_1}(\psi_1) \otimes \cdots \otimes \mathrm{St}_{n_k}(\psi_k),$$

where $P \subset \mathrm{GL}_n$ is the standard parabolic subgroup corresponding to the partition $n = n_1 + n_2 + \cdots + n_k$. It will therefore suffice to show the stronger statement that if $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_n$ is a subquotient of the normalised Jacquet module of Π, then there is an increasing filtration of $\oplus_{i=1}^k \mathrm{Sp}_{n_i}(\psi_i|\cdot|^{(n_i-1)/2})$ by sub-Weil–Deligne representations with graded pieces given by $\alpha_1 \circ \mathrm{Art}_{F_{\widetilde{v}}}, \dots, \alpha_n \circ \mathrm{Art}_{F_{\widetilde{v}}}$. We recall that each $\mathrm{Sp}_{n_i}(\psi_i|\cdot|^{(n_i-1)/2})$ comes with a standard basis $e_1, e_2, \dots, e_{n_i}$. We concatenate and relabel these bases so that $e_1, e_2, \dots, e_n$ is a basis for $\oplus_{i=1}^k \mathrm{Sp}_{n_i}(\psi_i|\cdot|^{(n_i-1)/2})$ with $e_{1+\sum_{i=1}^{j-1} n_i}, \dots e_{\sum_{i=1}^{j} n_i}$ the standard basis for $\mathrm{Sp}_{n_j}(\psi_j|\cdot|^{(n_j-1)/2})$.

We first treat the case $k = 1$, $n_1 = n$. After twisting we can assume that $\psi = 1$. Then the normalised Jacquet module of St_n equals $|\cdot|^{(n-1)/2} \otimes \cdots \otimes |\cdot|^{(1-n)/2}$, while there is a unique invariant flag of $\mathrm{Sp}_n(|\cdot|^{(n-1)/2})$ given by $\mathrm{Fil}_i = \mathrm{span}(e_1, \dots, e_i)$ $(i = 1, \dots, n)$ which has the desired graded pieces.

Now we return to the general case. Using [Zel80, Theorem 1.2], we see that the irreducible subquotients of the normalised Jacquet module of Π are precisely the characters $\beta_{w^{-1}(1)} \otimes \cdots \otimes \beta_{w^{-1}(n)}$, where $w \in S_n$ is any permutation which is increasing on each of the sets $\{1, \dots, n_1\}, \{n_1 + 1, \dots, n_1 + n_2\}, \dots, \{n_1 + \cdots + n_{k-1} + 1, \dots, n_1 + \cdots + n_k\}$, and $(\beta_1, \dots, \beta_n)$ is the concentation of the tuples $(\psi_i|\cdot|^{(n_i-1)/2}, \dots, \psi_i|\cdot|^{(1-n_i)/2})$ for $i = 1, \dots, k$.

We see that the increasing filtration of

$$\oplus_{i=1}^k \mathrm{Sp}_{n_i}(\psi_i|\cdot|^{(n_i-1)/2})$$

given by $\mathrm{Fil}_j = \mathrm{span}(e_{w^{-1}(1)}, \dots, e_{w^{-1}(j)})$ is a filtration by sub-Weil–Deligne representations which has the desired property. This completes the proof. $\square$

If χ is an accessible refinement of π, then we write $\nu(\pi, \chi) \in \mathcal{T}_n(\overline{\mathbf{Q}}_p)$ for the character

$$\nu(\pi, \chi) = \kappa(\pi) \cdot (\chi_v \iota^{-1} \delta_{B_n}^{-1/2})_{v \in S_p}, \tag{2.18.1}$$

where $\kappa(\pi) \in \mathcal{T}_n(E)$ is the (B_n-dominant) $\mathbf{Q}_p$-algebraic character which is the highest weight of $\iota^{-1}\pi_\infty$. If $\kappa(\pi)_v = (\kappa_{\tau,1} \geq \kappa_{\tau,2} \geq \cdots \geq \kappa_{\tau,n})_{\tau : F_{\widetilde{v}} \to \overline{\mathbf{Q}}_p}$ then the labelled Hodge–Tate weights of $r_{\pi,\iota}|_{G_{F_{\widetilde{v}}}}$, in increasing order, are $(-\kappa_{\tau,1} < -\kappa_{\tau,2} + 1 < \cdots < -\kappa_{\tau,n} + n - 1)_{\tau : F_{\widetilde{v}} \to \overline{\mathbf{Q}}_p}$.

We write $\jmath_n : \mathcal{T}_n \to \mathcal{T}_n$ for the map defined by the formula

$$\jmath_n(\nu)_v = \nu_v \cdot (1, \epsilon^{-1} \circ \mathrm{Art}_{F_{\widetilde{v}}}, \dots, \epsilon^{1-n} \circ \mathrm{Art}_{F_{\widetilde{v}}}).$$

The reason for introducing this map is that if π is an automorphic representation of $G_n(\mathbf{A}_{F^+})$ and χ is an accessible refinement, then the parameter δ associated to $\chi|\cdot|^{(1-n)/2}$ by the formula of Lemma 2.8 satisfies $\jmath_n(\nu(\pi, \chi)) = \delta$. We call the accessible refinement χ numerically non-critical or ordinary if δ is. Note that this property depends on the pair (π, χ) and not just on χ.

2.18.1. *Emerton's eigenvariety construction.* We now describe the construction, following Emerton [Eme06b], of the (tame level U_n) eigenvariety for G_n. We use Emerton's construction because we do not want to restrict to considering π with Iwahori-fixed vectors at places in S_p (as is done, for example, in [BC09]) and it seems to us that Emerton's representation-theoretic viewpoint is the most transparent way to handle this level of generality.

We recall the set-up of §1.23, so for each dominant weight λ we have a module $S_\lambda(U_n, \mathcal{O}/\varpi^n)$ of algebraic modular forms, which has a natural action of $\mathbf{T}_n^S$. When λ is trivial we omit it from the notation.

We define

$$\widetilde{S}(U_n^p, \mathcal{O}) := \varprojlim_s \left(\varinjlim_{U_p} S(U_n^p U_p, \mathcal{O}/\varpi^s) \right)$$

and

$$\widetilde{S}(U_n^p, E) := \widetilde{S}(U_n^p, \mathcal{O}) \otimes_{\mathcal{O}} E,$$

so $\widetilde{S}(U_n^p, E)$ is an E-Banach space (with unit ball $\widetilde{S}(U_n^p, \mathcal{O})$), equipped with an admissible continuous representation of $G_n(F_p^+)$. For dominant weights λ, we can consider the space of locally $V_\lambda^\vee$-algebraic vectors $\widetilde{S}(U_n^p, E)^{V_\lambda^\vee-alg}$. We have a $(G_n(F_p^+) \times \mathbf{T}_n^S)$-equivariant isomorphism

$$\varinjlim_{U_p} S_\lambda(U_n^p U_p, \mathcal{O}) \otimes_{\mathcal{O}} V_\lambda^\vee \cong \widetilde{S}(U_n^p, E)^{V_\lambda^\vee-alg}$$

(see [Eme06b, Corollary 2.2.25]). We can also consider the space of locally $\mathbf{Q}_p$-analytic vectors $\widetilde{S}(U_n^p, E)^{an}$, and apply Emerton's locally analytic Jacquet functor J_{B_n} to this locally analytic representation of $G_n(F_p^+)$. We thereby obtain an essentially admissible locally analytic representation $J_{B_n}\widetilde{S}(U_n^p, E)^{an}$ of $\prod_{v \in S_p} T_n(F_{\widetilde{v}})$, and by duality a coherent sheaf $\mathcal{M}_n$ on $\mathcal{T}_n$, equipped with an action of $\mathbf{T}_n^S$. We denote by $\mathcal{A}_n \subset \mathrm{End}(\mathcal{M}_n)$ the coherent $\mathcal{O}_{\mathcal{T}_n}$-algebra subsheaf generated by $\mathbf{T}_n^S$. Now we can define the eigenvariety, an E-rigid space, as a relative rigid analytic spectrum

$$\mathcal{E}_n := \mathrm{Sp}_{\mathcal{T}_n} \mathcal{A}_n \xrightarrow{\nu'} \mathcal{T}_n$$

equipped with the canonical finite morphism ν'.

We define another finite morphism $\nu : \mathcal{E}_n \to \mathcal{T}_n$ by twisting ν' by $\delta_{B_n}^{-1}$ (see Remark 2.21). By construction, we also have a ring homomorphism $\psi : \mathbf{T}_n^S \to \mathcal{O}(\mathcal{E}_n)$, so we obtain a map on points:

$$\psi^* \times \nu : \mathcal{E}_n(\overline{\mathbf{Q}}_p) \to \mathrm{Hom}(\mathbf{T}_n^S, \overline{\mathbf{Q}}_p) \times \mathcal{T}_n(\overline{\mathbf{Q}}_p).$$

For E'/E finite (with $E' \subset \overline{\mathbf{Q}}_p$), a point $(\psi_0, \nu_0) \in \mathrm{Hom}(\mathbf{T}_n^S, E') \times \mathcal{T}_n(E')$ is in the image of $\psi^* \times \nu$ if and only if the eigenspace

$$J_{B_n}\left(\widetilde{S}(U_n^p, E')^{an}\right)[\psi_0, \nu_0 \delta_{B_n}]$$

is non-zero, or in other words if there is a non-zero $\prod_{v \in S_p} T(F_{\widetilde{v}})$-equivariant map

$$\nu_0 \delta_{B_n} \to J_{B_n}\left(\widetilde{S}(U_n^p, E')^{an}\right)[\psi_0].$$

We define the subset $Z_n \subset \mathcal{E}_n(\overline{\mathbf{Q}}_p)$ of classical points to be those for which there is moreover a non-zero map to the Jacquet module of the locally algebraic vectors:

$$\nu_0 \delta_{B_n} \to J_{B_n}\left(\widetilde{S}(U_n^p, E')^{alg}\right)[\psi_0].$$

Lemma 2.19. *For any characters $\psi : \mathbf{T}_n^S \to E$ and $\chi : \prod_{v \in S_p} T(F_{\widetilde{v}}) \to E^\times$, we have*

$$\mathrm{Hom}_{\prod_{v\in S_p} T(F_{\widetilde{v}})}\left(\chi, J_{B_n}\left(\widetilde{S}(U_n^p, E)^{an}\right)[\psi]\right) = \mathrm{Hom}_{\prod_{v\in S_p} T(F_{\widetilde{v}})}\left(\chi, J_{B_n}\left(\widetilde{S}(U_n^p, E)^{an}[\psi]\right)\right).$$

Proof. This can be seen using Emerton's canonical lift [Eme06a, Proposition 3.4.9], which identifies both sides of the equality with the same eigenspace in $\widetilde{S}(U_n^p, E)^{an}$. Alternatively, we can use the left exactness of the Jacquet functor. In the latter argument we need to use the fact that $\mathbf{T}_n^S$ acts on $\widetilde{S}(U_n^p, E)^{an}$ via a Noetherian ring and we then deduce that passing to an eigenspace for a (finitely generated) ideal in this ring commutes with the Jacquet functor. $\square$

We now relate the classical points Z_n to refined automorphic representations. Let $\mathcal{A}_n$ denote the set of automorphic representations π of $G_n(\mathbf{A}_{F^+})$ such that $(\pi^\infty)^{U_n^{S_p}} \neq 0$, let $\mathcal{RA}_n$ denote the set of pairs (π, χ) where $\pi \in \mathcal{A}_n$ and χ is an accessible refinement of π, and let $\mathcal{Z}_n \subset \mathrm{Hom}(\mathbf{T}_n^S, \overline{\mathbf{Q}}_p) \times \mathcal{T}_n(\overline{\mathbf{Q}}_p)$ denote the set of points of the form $(\psi_\pi, \nu(\pi, \chi))$, where $(\pi, \chi) \in \mathcal{RA}_n$. We note in particular the existence of the surjective map $\gamma_n : \mathcal{RA}_n \to \mathcal{Z}_n$.

Lemma 2.20. *The map $\psi^* \times \nu$ restricts to a bijection $Z_n \to \mathcal{Z}_n$.*

Proof. If $z \in Z_n \cap \mathcal{E}_n(E')$ is a classical point defined over E', it follows from the above discussion on locally algebraic vectors that z arises arises from a non-zero map

$$\nu(z)\delta_{B_n} \to J_{B_n}\left(\varinjlim_{U_p} S_\lambda(U_n^p U_p, E')[\psi^*(z)] \otimes_{E'} V_\lambda^\vee\right)$$

for some dominant weight λ. It follows from [Eme06a, Prop. 4.3.6] that such maps correspond bijectively with non-zero maps

$$\nu(z)\delta_{B_n}(\lambda^\vee)^{-1} \to J_{B_n}\left(\varinjlim_{U_p} S_\lambda(U_n^p U_p, E')[\psi^*(z)]\right),$$

where $\lambda^\vee$ is the highest weight of $V_\lambda^\vee$.

By Lemma 1.25, we have $(\pi, \chi) \in \mathcal{RA}_n$ where π_∞ has highest weight $\iota\lambda^\vee$, $\psi_\pi = \psi^*(z)$ and $\nu(z) = \lambda^\vee \chi \delta_{B_n}^{-1/2} = \nu(\pi, \chi)$. This shows that we do indeed have an induced map $Z_n \to \mathcal{Z}_n$, and it is easy to see that this is a bijection. $\square$

Remark 2.21. An accessible refinement χ is numerically non-critical if and only if for every $v \in S_p$ the character $\nu(\pi,\chi)_v \delta_{B_n} = \kappa(\pi)_v \chi_v \delta_{B_n}^{1/2}$ has non-critical slope, in the sense of [Eme06a, Defn. 4.4.3]. The renormalisation (replacing $\chi_v \delta_{B_n}^{-1/2}$ with $\chi_v \delta_{B_n}^{1/2}$) appears in Emerton's eigenvariety construction because $\chi_v \delta_{B_n}^{1/2}$ is a smooth character appearing in the (non-normalised) Jacquet module $\iota^{-1}\pi_{v, N_n(F_{\widetilde{v}})}$, whilst Bellaïche–Chenevier normalise things to be compatible with the Hecke action on Iwahori-fixed vectors (see [BC09, Prop. 6.4.3]).

Our next task is to recall some well known properties of the eigenvariety $\mathcal{E}_n$ (cf. [Bre15, §7]), variants of which are established by numerous authors in slightly different contexts (e.g. [Che04, Buz07, Eme06b, Loe11]). We follow the exposition of [BHS17] which establishes these properties for the patched eigenvariety. In order to at least sketch the proofs of these properties in our context, we first introduce a 'spectral variety' which will turn out to be a Fredholm hypersurface over $\mathcal{W}_n$.

We fix the element $z = (z_v)_{v \in S_p} \in \prod_{v \in S_p} T_n(F_{\widetilde{v}})$ with $z_v = \mathrm{diag}(\varpi_{\widetilde{v}}^{n-1}, \ldots, \varpi_{\widetilde{v}}, 1)$, and let Y be the closed subgroup of $\prod_{v \in S_p} T_n(F_{\widetilde{v}})$ generated by $\prod_{v \in S_p} T_n(\mathcal{O}_{F_{\widetilde{v}}})$ and z. The rigid space $\widehat{Y} = \mathrm{Hom}(Y, \mathbf{G}_m)$ is then identified with $\mathcal{W}_n \times \mathbf{G}_m$. As in [BHS17, §3.3], it follows from [Eme06a, Proposition 3.2.27] that $J_{B_n} \widetilde{S}(U_n^p, E)^{an}$ has dual equal to the space of global sections of a coherent sheaf $\mathcal{N}_n$ on $\mathcal{W}_n \times \mathbf{G}_m$. We define $\mathcal{Y}_z$ to be the schematic support (cf. above Définition 3.6 in [BHS17]) of $\mathcal{N}_n(\delta_{B_n}^{-1})$. This rigid space comes equipped with a closed immersion $\mathcal{Y}_z \hookrightarrow \mathcal{W}_n \times \mathbf{G}_m$. The twist in the definition of $\mathcal{Y}_z$ is there to ensure that this closed immersion is compatible with the map ν. Indeed, the map from $\mathcal{E}_n$ given by composing ν with the restriction map to $\mathcal{W}_n \times \mathbf{G}_m$ factors through a finite map $f : \mathcal{E}_n \to \mathcal{Y}_z$, giving us a commutative diagram:

$$\begin{tikzcd} \mathcal{E}_n \arrow[r, "\nu"] \arrow[d, "f"] & \mathcal{T}_n \arrow[d] \\ \mathcal{Y}_z \arrow[r, hook] & \mathcal{W}_n \times \mathbf{G}_m \end{tikzcd}$$

Now we state our proposition summarising the key properties of the eigenvariety $\mathcal{E}_n$.

Proposition 2.22. *The tuple $(\mathcal{E}_n, \psi, \nu, Z_n)$ has the following properties:*

(1) *$\mathcal{E}_n$ is a reduced E-rigid space, equipped with a finite morphism $\nu : \mathcal{E}_n \to \mathcal{T}_n$. We write κ for the induced map $\kappa : \mathcal{E}_n \to \mathcal{W}_n$.*
(2) *$Z_n \subset \mathcal{E}_n(\overline{\mathbf{Q}}_p)$ is a Zariski dense subset which accumulates at every point of Z_n (in other words, each point of Z_n admits a basis of affinoid neighbourhoods V such that $V \cap Z_n$ is Zariski dense in V), and the map $\psi^* \times \nu : \mathcal{E}_n(\overline{\mathbf{Q}}_p) \to \mathrm{Hom}(\mathbf{T}_n^S, \overline{\mathbf{Q}}_p) \times \mathcal{T}_n(\overline{\mathbf{Q}}_p)$ restricts to a bijection $Z_n \to \mathcal{Z}_n$.*
(3) *For any affinoid open $V \subset \mathcal{T}_n$, the map $\mathbf{T}_n^S \otimes \mathcal{O}(V) \to \mathcal{O}(\nu^{-1}V)$ is surjective.*
(4) *$\mathcal{E}_n$ is equidimensional of dimension equal to $\dim \mathcal{W}_n$. For any irreducible component $\mathcal{C} \subset \mathcal{E}_n$, $\kappa(\mathcal{C})$ is a Zariski open subset of $\mathcal{W}_n$.*
(5) *Let $z \in \mathcal{E}_n(\overline{\mathbf{Q}}_p)$ be a point, and suppose that $\delta = \jmath_n(\nu(z))$ factors as $\delta = \delta_{alg}\delta_{sm}$, where δ_{alg} is a strictly dominant algebraic character and δ_{sm} is smooth, and that δ is numerically non-critical. Then $z \in Z_n$.*
(6) *ψ takes values in the subring $\mathcal{O}(\mathcal{E}_n)^{\leq 1}$ of bounded elements.*

Proof. First we note that a tuple satisfying the first three properties is unique - we will not actually use this fact, but it can be proved in the same way as [BC09, Proposition 7.2.8] (our context is slightly different, as we equip our eigenvarieties with a map to $\mathcal{T}_n$ instead of $\mathcal{W}_n \times \mathbf{G}_m$). We also note that it is not essential for our purposes to show that $\mathcal{E}_n$ is reduced (this is the most delicate of the listed properties); we could instead replace $\mathcal{E}_n$ with its underlying reduced subspace.

Now we summarise how to verify these properties. Property (5) follows from Emerton's 'classicality criterion' for his Jacquet functor [Eme06a, Theorem 4.4.5] (cf. Remark 2.21). Property (3) holds by construction.

Property (4) can be established as in [BHS17, §3.3] using the spectral variety $\mathcal{Y}_z$. More precisely, (the proof of) Lemma 3.10 in this reference shows that the closed analytic subset of $\mathcal{W}_n \times \mathbf{G}_m$ underlying $\mathcal{Y}_z$ is a Fredholm hypersurface, and $\mathcal{Y}_z$ has an admisible cover by affinoids $(U'_i)_{i\in I}$ on which the map to $\mathcal{W}_n$ is finite and surjective with image an open affinoid $W_i \subset \mathcal{W}_n$. Moreover, each U'_i is disconnected from its complement in the inverse image of W_i and $\Gamma(U'_i, \mathcal{N}_n)$ is a finite projective $\mathcal{O}_{\mathcal{W}_n}(W_i)$-module.

Having established the existence of a good affinoid cover of the spectral variety, we set $U_i = f^{-1}(U'_i)$. Since f is a finite map, $(U_i)_{i\in I}$ is an admissible affinoid cover of $\mathcal{E}_n$. It can then be shown, as in [BHS17, Proposition 3.11], that each affinoid $\mathcal{O}_{\mathcal{E}_n}(U_i)$ is isomorphic to a $\mathcal{O}_{\mathcal{W}_n}(W_i)$-algebra of endomorphisms of the finite projective $\mathcal{O}_{\mathcal{W}_n}(W_i)$-module $\Gamma(U_i, \mathcal{M}_n)$. We can now prove Property (4) as in [BHS17, Corollaire 3.12]: this shows that $\mathcal{E}_n$ is equidimensional of dimension equal to $\mathcal{W}_n$, without embedded components, and each irreducible component maps surjectively to an irreducible component of $\mathcal{Y}_z$. Since irreducible components of Fredholm hypersurfaces are again Fredholm hypersurfaces, the image of such an irreducible component is Zariski open in $\mathcal{W}_n$ (cf. [BHS17, Corollaire 3.13]).

Now to establish property (2), using property (5), it suffices to show that points $z \in \mathcal{E}_n(\overline{\mathbf{Q}}_p)$ with numerically non-critical $\delta = \jmath_n(\nu(z))$ accumulate at any point z_0 with $\kappa(z_0)$ locally algebraic (cf. [BHS17, Théorème 3.19]). Using the good affinoid cover described in the previous paragraph, we have an affinoid neighbourhood U of z_0 which is a finite cover of an affinoid $W \subset \mathcal{W}_n$. In fact, such U form a neighbourhood basis at z_0 (cf. [Taï16, Theorem 2.1.1, Lemma 2.1.2]). The valuations $v_p(\delta_{v,i}(p))$ are bounded as z varies in U (with $\delta = \jmath_n(\nu(z))$). It follows from the description in Definition 2.9 that there is a subset Σ of $\mathcal{W}_n$ accumulating at $\kappa(z_0)$ such that $\kappa^{-1}(\Sigma) \cap U$ consists entirely of points with numerically non-critical δ. The subset $\kappa^{-1}(\Sigma) \cap U$ is Zariski dense in U.

Finally, to establish property (1) it remains to prove that $\mathcal{E}_n$ is reduced. Since we showed that $\mathcal{E}_n$ is without embedded components, it suffices to prove that every irreducible component of $\mathcal{E}_n$ contains a reduced point. Using (4) and the Zariski density of algebraic characters in $\mathcal{W}_n$, it suffices to show that $\mathcal{E}_n$ is reduced at every point z_0 with $\kappa(z_0)$ algebraic. We use a good affinoid neighbourhood $U = \mathrm{Sp}(B)$ of z_0 as in the previous paragraph, with $W = \kappa(U) = \mathrm{Sp}(A)$. The finite A-algebra B is identified with a sub-A-algebra of $\mathrm{End}_A(M)$, where $M = \Gamma(U, \mathcal{M}_n)$ is a finite projective A-module. As in the proof of [Che05, Proposition 3.9], it now suffices to show that for w in a Zariski dense subset of W, the Hecke algebra $\mathbf{T}_n^S$ and $\prod_{v\in S_p} T_n(F_{\widetilde{v}})$ act semisimply on the fibre $M \otimes_A k(w)$ — we use the fact that an endomorphism of a projective A-module which vanishes in the fibres at a Zariski dense subset of points in W necessarily vanishes. The proof of [BHS17, Corollaire 3.20] shows that we can achieve this by choosing w so that their pre-images in U have *très classique* associated characters δ [BHS17, Définition 3.17] (this is a condition on characters with algebraic image in $\mathcal{W}_n$ which can be guaranteed by a 'numerical' condition as in the proof of [BHS17, Théorème 3.19], in particular it gives a Zariski-dense and self-accumulating subset of $\mathcal{E}_n$). We can replace the reference to [CEG+16] in the proof with the well-known assertion that the Hecke algebra $\mathbf{T}_n^S$

acts semisimply on $\varinjlim_{U_p} S_\lambda(U_n^p U_p, E)$ for dominant λ. Finally, property (6) follows from the fact that the $\mathbf{T}_n^S$-action stabilizes the unit ball $\widetilde{S}(U_n^p, \mathcal{O}) \subset \widetilde{S}(U_n^p, E)$. $\square$

The properties established in Proposition 2.22 imply the existence of a conjugate self-dual Galois pseudocharacter $T_n : G_{F,S} \to \mathcal{O}(\mathcal{E}_n)$ with the property that for any point $z \in Z_n$ corresponding to a pair (π, χ), $T_{n,z} = \operatorname{tr} r_{\pi,\iota}$. This is proved as in [BC09, Proposition 7.5.4] and [Che04, Proposition 7.1.1]. The key points are that $\mathcal{O}(\mathcal{E}_n)^{\leq 1}$ is compact [BC09, Lemma 7.2.11] and the map $\mathcal{O}(\mathcal{E}_n)^{\leq 1} \to \prod_{z \in Z_n} \mathbf{C}_p$ given by the evaluation maps at each $z \in Z_n$ is a continuous injection (by Zariski density of Z_n and reducedness of $\mathcal{E}_n$). Then [Che14, Example 2.32] is used to glue together the pseudocharacters $\operatorname{tr} r_{\pi,\iota}$ to form the continuous pseudocharacter T_n.

The pseudocharacter T_n determines an admissible cover $\mathcal{E}_n = \sqcup_{\overline{\tau}_n} \mathcal{E}_n(\overline{\tau}_n)$ as a disjoint union of finitely many open subspaces indexed by G_k-orbits of pseudocharacters $\overline{\tau}_n : G_{F,S} \to \overline{\mathbf{F}}_p$, over which which the residual pseudocharacter satisfies the condition $\overline{T}_{n,z} = \overline{\tau}_n$ (cf. [Che14, Theorem 3.17]).

Fix a pseudocharacter $\overline{\tau}_n : G_{F,S} \to \overline{\mathbf{F}}_p$. Extending E if necessary, we may assume that $\overline{\tau}_n$ takes values in k. We recall some of the E-rigid spaces of Galois representations defined in §2.11.1, now decorated with n subscripts. Thus $\mathcal{X}_{ps,n}$ is the space of conjugate self-dual deformations of $\overline{\tau}_n$, $\mathcal{X}_{ps,n}^{p-irr}$ is its Zariski open subspace of pseudocharacters which are irreducible at the p-adic places. We also have the subspace $\mathcal{X}_{ps,n}^{irr}$ of pseudocharacters which are (globally) irreducible. The existence of T_n determines a morphism $\lambda : \mathcal{E}(\overline{\tau}_n) \to \mathcal{X}_{ps,n}$, and the morphism $i_n = \lambda \times (\jmath \circ \nu) : \mathcal{E}_n(\overline{\tau}_n) \to \mathcal{X}_{ps,n} \times \mathcal{T}_n$ is a closed immersion, by point (3) in the list of defining properties of $\mathcal{E}_n$.

Now assume that $n \geq 3$, and let $\overline{\tau}_2 : G_{F,S} \to \overline{\mathbf{F}}_p$ be a conjugate self-dual pseudocharacter of dimension 2. Let $\overline{\tau}_n = \operatorname{Sym}^{n-1} \overline{\tau}_2$; then $\overline{\tau}_n$ is a conjugate self-dual pseudocharacter of $G_{F,S}$ of dimension n. Taking symmetric powers of pseudocharacters determines a morphism $\sigma_{n,g} : \mathcal{X}_{ps,2} \to \mathcal{X}_{ps,n}$. On the other hand, we can define a map $\sigma_{n,p} : \mathcal{T}_2 \to \mathcal{T}_n$ by the formula

$$((\delta_{v,1}, \delta_{v,2}))_{v \in S_p} \mapsto ((\delta_{v,1}^{n-1}, \delta_{v,1}^{n-2}\delta_{v,2}, \ldots, \delta_{v,2}^{n-1}))_{v \in S_p}.$$

We write $\sigma_n = \sigma_{n,g} \times \sigma_{n,p} : \mathcal{X}_{ps,2} \times \mathcal{T}_2 \to \mathcal{X}_{ps,n} \times \mathcal{T}_n$ for the product of these two morphisms. We have constructed a diagram

$$\mathcal{E}_2(\overline{\tau}_2) \xrightarrow{\sigma_n \circ i_2} \mathcal{X}_{ps,n} \times \mathcal{T}_n \xleftarrow{i_n} \mathcal{E}_n(\overline{\tau}_n).$$

Compare Lemma 2.2.

Definition 2.23. *Let π be an automorphic representation of $G_2(\mathbf{A}_{F^+})$, let $\chi = (\chi_v)_{v \in S_p}$ be an accessible refinement of π, and let $n \geq 2$. We say that χ is n-regular if for each $v \in S_p$ the character $\chi_v = \chi_{v,1} \otimes \chi_{v,2}$ satisfies $(\chi_{v,1}/\chi_{v,2})^i \neq 1$ for $1 \leq i \leq n-1$.*

Theorem 2.24. *Let $(\pi_2, \chi_2) \in \mathcal{RA}_2$ satisfy $\operatorname{tr} \overline{r}_{\pi_2,\iota} = \overline{\tau}_2$, and let $z_2 = \gamma_2(\pi_2, \chi_2) \in \mathcal{E}_2(\overline{\tau}_2)(\overline{\mathbf{Q}}_p)$. Suppose that:*

(1) The refinement χ_2 is numerically non-critical and n-regular.
(2) There exists $(\pi_n, \chi_n) \in \mathcal{RA}_n$ such that $(\sigma_n \circ i_2)(z_2) = i_n(z_n)$, where $z_n = \gamma_n(\pi_n, \chi_n)$.
(3) For each $v \in S_p$, the Zariski closure of $r_{\pi_2,\iota}(G_{F_{\widetilde{v}}})$ (in $\mathrm{GL}_2/\overline{\mathbf{Q}}_p$) contains SL_2.

Then each irreducible component $\mathcal{C}$ of $\mathcal{E}_2(\overline{\tau}_2)_{\mathbf{C}_p}$ containing z_2 satisfies $(\sigma_n \circ i_2)(\mathcal{C}) \subset i_n(\mathcal{E}_n(\overline{\tau}_n)_{\mathbf{C}_p})$.

Proof. Extending E (the field over which $\mathcal{E}_2$ is defined) if necessary, we may assume that $z_2 \in \mathcal{E}_2(\overline{\tau}_2)(E)$ and $r_{\pi_2,\iota}$ takes values in $\mathrm{GL}_2(E)$. By [Con99, Theorem 3.4.2] (which says that an irreducible component of $\mathcal{E}_2(\overline{\tau}_2)_{\mathbf{C}_p}$ is contained in the base change of an irreducible component of $\mathcal{E}_2(\overline{\tau}_2)$), it suffices to show that each irreducible component $\mathcal{C}$ of $\mathcal{E}_2(\overline{\tau}_2)$ containing z_2 satisfies $(\sigma_n \circ i_2)(\mathcal{C}) \subset i_n(\mathcal{E}_n(\overline{\tau}_n))$. By Lemma 2.2, it is enough to show that $(\sigma_n \circ i_2)^{-1}(i_n(\mathcal{E}_n(\overline{\tau}_n)))$ contains an affinoid open neighbourhood of z_2. To prove this, we will use a number of the results established so far.

Since χ_2 is a numerically non-critical refinement, the parameter δ_2 of the associated triangulation is non-critical, in the sense that for each $\tau \in \mathrm{Hom}(F, E)$, the sequence of τ-weights of δ_2 is strictly increasing (Lemma 2.8). Passing to symmetric powers, we see that $r_{\pi_n,\iota}$ is trianguline of parameter $\delta_n = \sigma_{n,p}(\delta_2)$, and that δ_n is non-critical (although it is not necessarily numerically non-critical).

The n-regularity of χ_2 implies that $\delta_n \in \mathcal{T}_n^{reg}(E)$. We are going to apply Proposition 2.11 to conclude that $\dim_E H^1_{tri,\delta_n}(F_S/F^+, \operatorname{ad} r_{\pi_n,\iota}) \leq \dim \mathcal{W}_n = \dim \mathcal{E}_n(\overline{\tau}_n)$. Note that if $v \in S$ then $\mathrm{WD}(r_{\pi_n,\iota}|_{G_{F_{\widetilde{v}}}})$ is generic, because it is pure: the base change of π_n to $\mathrm{GL}_n(\mathbf{A}_F)$ exists (for example, by [Lab11, Corollaire 5.3]) and is cuspidal (because $r_{\pi_n,\iota}$ is irreducible), so we can appeal to the main theorem of [Car12] (which establishes the general case; under various additional hypotheses, purity was established in [HT01, TY07, Shi11, Clo13]). If $v \in S_p$, let us write $f_v : \mathcal{X}_{ps,n} \times \mathcal{T}_n \to \mathcal{X}_{ps,n,v} \times \mathcal{T}_v$ for the natural restriction map. By Proposition 2.15 and Lemma 2.14, we can find for each $v \in S_p$ an affinoid open neighbourhood $\mathcal{U}_v \subset \mathcal{X}_{ps,n,v} \times \mathcal{T}_v$ of the point $f_v(z_n)$ such that the following properties hold:

- In fact, $\mathcal{U}_v \subset \mathcal{X}_{ps,n,v}^{v-irr} \times \mathcal{T}_v^{reg}$ and there exists a universal representation $\rho_v^u : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\mathcal{O}(\mathcal{U}_v))$ over $\mathcal{U}_v$.
- Let $\mathcal{Z}_v \subset \mathcal{U}_v$ denote the Zariski closure of the set $\mathcal{V}_v \subset \mathcal{U}_v$ of points corresponding to pairs (ρ_v, δ_v) such that ρ_v is trianguline of parameter δ_v. Then the Zariski tangent space of $\mathcal{Z}_v$ at $f_v(z_n)$ is contained in $H^1_{tri,\delta_v}(F_{\widetilde{v}}, \operatorname{ad} r_{\pi_n,\iota}|_{G_{F_{\widetilde{v}}}})$.

We can then find an affinoid open neighbourhood $\mathcal{U} \subset \mathcal{X}_{ps,n} \times \mathcal{T}_n$ of the point z_n such that the following properties hold:

- $\mathcal{U} \subset \cap_{v \in S_p} f_v^{-1}(\mathcal{U}_v)$ and there exists a universal representation $\rho^u : G_{F,S} \to \mathrm{GL}_n(\mathcal{O}(\mathcal{U}))$ over $\mathcal{U}$.
- Let $\mathcal{Z} = \mathcal{U} \cap (\cap_{v \in S_p} f_v^{-1}(\mathcal{Z}_v))$. Then $\mathcal{Z}$ is a closed analytic subset of $\mathcal{U}$ and the Zariski tangent space of $\mathcal{Z}$ at the point z_n is contained in $H^1_{tri,\delta}(F_S/F^+, \operatorname{ad} r_{\pi_n,\iota})$. By the main theorem of [NT20], we have $H^1_f(F^+, \operatorname{ad} r_{\pi_n,\iota}) = 0$, so Proposition 2.11 implies that the Zariski tangent space of $\mathcal{Z}$ at point z_n has dimension at most $\dim \mathcal{W}_n$.

Let $\mathcal{U}' = \mathcal{E}_n(\overline{\tau}_n) \cap \mathcal{U}$. Then $\mathcal{U}'$ is an affinoid open neighbourhood of z_n in $\mathcal{E}_n(\overline{\tau}_n)$. We note that if $z_n' = \gamma_n(\pi_n', \chi_n') \in \mathcal{U}'$, where χ_n' is a numerically non-critical refinement, then $z_n' \in \mathcal{Z}$ (by Lemma 2.8, and the definition of $\mathcal{Z}$). Such points accumulate at z_n, implying that every irreducible component of $\mathcal{U}'$ containing the point z_n is contained in $\mathcal{Z}$. In particular, $\mathcal{Z}$ contains an affinoid open neighbourhood of z_n in $\mathcal{U}'$, so we have $\dim \mathcal{Z} \geq \dim \mathcal{E}_n(\overline{\tau}_n) = \dim \mathcal{W}_n$. It follows that $\dim \mathcal{Z} = \dim \mathcal{W}_n$, that $\widehat{\mathcal{O}}_{\mathcal{Z},z_n}$ is a regular local ring, and that $\mathcal{Z}$ is smooth at the point z_n. Consequently, $\mathcal{U}'$ and $\mathcal{Z}$ are locally isomorphic at z_n, $\mathcal{E}_n(\overline{\tau}_n)$ is smooth at the point z_n, and $\mathcal{E}_n(\overline{\tau}_n)$

has a unique irreducible component passing through z_n. Applying Lemma 2.2, we can also deduce that the unique irreducible component $\mathcal{Z}'$ of $\mathcal{Z}$ containing z_n is contained in $\mathcal{U}'$.

Now let $\mathcal{U}'' = (\sigma_n \circ i_2)^{-1}(\mathcal{U})$, and let $g = (\sigma_n \circ i_2)|_{\mathcal{U}''} : \mathcal{U}'' \to \mathcal{U}$. Then $\mathcal{U}''$ is an admissible open of $\mathcal{E}_2(\overline{\tau}_2)$, and $g^{-1}(\mathcal{Z}) \subset \mathcal{U}''$ is a non-empty closed analytic subset. Let $(\pi_2', \chi_2') \in \mathcal{RA}_2$ be a pair such that χ_2' satisfies the analogue of property (1) in the theorem. Arguing again as in the second paragraph of the proof, we see that if the point $(\sigma_n \circ i_2)(\gamma_2(\pi_2', \chi_2'))$ lies in $\mathcal{U}$, then it in fact lies in $\mathcal{Z}$. Since such points accumulate at z_2, we see that $g^{-1}(\mathcal{Z}')$ contains each irreducible component of $\mathcal{U}''$ which passes through z_2 (and hence contains an affinoid open neighbourhood of z_2). Since $\mathcal{Z}' \subset \mathcal{U}'$ we deduce that $(\sigma_n \circ i_2)^{-1}(\mathcal{E}_n(\overline{\tau}_n))$ contains an affinoid open neighbourhood of z_2. This completes the proof. □

Remark 2.25. Note that assumption (3) on the image of the local Galois representation ensures that all symmetric powers remain locally irreducible. We need this to apply the results of §2.11.1. The authors expect that, with some effort, this material could be adjusted to allow locally reducible (but globally irreducible) families of Galois representations.

We also prove a version of this result in the ordinary case. We first note a well-known consequence of Hida theory:

Lemma 2.26. *The Zariski closure of the classical points with ordinary refinements $\mathcal{E}_n(\overline{\tau}_n)^{ord} \subset \mathcal{E}_n(\overline{\tau}_n)$ is a union of connected components of $\mathcal{E}_n(\overline{\tau}_n)$ which are finite over $\mathcal{W}_n$, and every classical point of $\mathcal{E}_n(\overline{\tau}_n)^{ord}$ has an ordinary refinement. All points of $\mathcal{E}_n(\overline{\tau}_n)^{ord}$ with dominant locally algebraic image in $\mathcal{W}_n$ are classical.*

Proof. We can identify $\mathcal{E}_n(\overline{\tau}_n)^{ord}$ with the generic fibre of the formal spectrum of Hida–Hecke algebra (a localization of the ring denoted by $\widetilde{\mathbf{T}}^{S,\mathrm{ord}}(U_n(\mathfrak{p}^\infty), \mathcal{O})$ in [Ger19, §2]), since this is naturally a Zariski closed subspace of $\mathcal{X}_{ps,n} \times \mathcal{T}_n$ in which the classical points with ordinary refinements are Zariski dense. We deduce from Hida theory that $\mathcal{E}_n(\overline{\tau}_n)^{ord}$ is finite over $\mathcal{W}_n$ and equidimensional of dimension $\dim \mathcal{W}_n$. Moreover, the map $\nu : \mathcal{E}_n(\overline{\tau}_n)^{ord} \to \mathcal{T}_n$ factors through the open subspace $\mathcal{T}_n^\circ \subset \mathcal{T}_n$ classifying unitary characters of $\prod_{v \in S_p} T_n(F_{\widetilde{v}})$.

On the other hand, we claim that every point of $\mathcal{E}_n(\overline{\tau}_n) \times_{\nu, \mathcal{T}_n} \mathcal{T}_n^\circ$ is contained in $\mathcal{E}_n(\overline{\tau}_n)^{ord}$. Assuming this, these (reduced) subspaces of $\mathcal{E}_n(\overline{\tau}_n)$ are equal and $\mathcal{E}_n(\overline{\tau}_n)^{ord}$ is an open and closed subspace of $\mathcal{E}_n(\overline{\tau}_n)$. The final part of the lemma follows from the classicality theorem in Hida theory [Ger19, Lemma 2.25].

It remains to show the claimed inclusion of $\mathcal{E}_n(\overline{\tau}_n) \times_{\nu, \mathcal{T}_n} \mathcal{T}_n^\circ$ in $\mathcal{E}_n(\overline{\tau}_n)^{ord}$. Suppose z is an E-point of $\mathcal{E}_n(\overline{\tau}_n) \times_{\nu, \mathcal{T}_n} \mathcal{T}_n^\circ$ (extending scalars deals with the general case). The character $\nu(z)\delta_{B_n}$ then appears in the eigenspace $\left(J_{B_n} \widetilde{S}(U_n^p, E)^{an}\right)[\psi^*(z)]$. This character therefore also appears in $J_{B_n}\left(\widetilde{S}(U_n^p, E)^{an}[\psi^*(z)]\right)$, by Lemma 2.19. Applying [Sor17, Corollary 6.4] to $\widetilde{S}(U_n^p, \mathcal{O})[\psi^*(z)]$ (note that Sorensen's Jacquet modules are twisted by $\delta_{B_n}^{-1}$ compared to ours), we deduce that the unitary character $\nu(z)$ appears in the ordinary part $\mathrm{Ord}_{B_n} \widetilde{S}(U_n^p, \mathcal{O})[\psi^*(z)]$. This shows that z is a point of $\mathcal{E}_n(\overline{\tau}_n)^{ord}$. □

Theorem 2.27. *Let $(\pi_2, \chi_2) \in \mathcal{RA}_2$ satisfy $\mathrm{tr}\, \overline{r}_{\pi_2, \iota} = \overline{\tau}_2$, and let $z_2 = \gamma_2(\pi_2, \chi_2) \in \mathcal{E}_2(\overline{\tau}_2)(\overline{\mathbf{Q}}_p)$. Suppose that:*

(1) *The refinement* χ_2 *is ordinary.*
(2) *There exists* $(\pi_n, \chi_n) \in \mathcal{RA}_n$ *such that* $(\sigma_n \circ i_2)(z_2) = i_n(z_n)$*, where* $z_n = \gamma_n(\pi_n, \chi_n)$.
(3) *The Zariski closure of* $r_{\pi_2,\iota}(G_F)$ *contains* SL_2.

Then each irreducible component $\mathcal{C}$ *of* $\mathcal{E}_2(\overline{\tau}_2)_{\mathbf{C}_p}$ *containing* z_2 *satisfies* $(\sigma_n \circ i_2)(\mathcal{C}) \subset i_n(\mathcal{E}_n(\overline{\tau}_n)_{\mathbf{C}_p})$.

Proof. Extending E if necessary, we may assume that $z_2 \in \mathcal{E}_2(\overline{\tau}_2)(E)$ and $r_{\pi_2,\iota}$ takes values in $\mathrm{GL}_2(E)$. We denote by $\mathcal{T}_n^{HT-reg} \subset \mathcal{T}_n$ the Zariski open subset where for each $v \in S_p$ and $\tau \in \mathrm{Hom}_{\mathbf{Q}_p}(F_{\widetilde{v}}, E)$ the labelled weights $\mathrm{wt}_\tau(\delta_{n,v,i})$ are distinct for $i = 1, \dots, n$.

By Lemma 2.12 and (a global variant of) Lemma 2.13, there is an open affinoid neighbourhood

$$z_n = (\mathrm{tr}\, r_{\pi_n,\iota}, \delta_n) \in \mathcal{U} \subset \mathcal{X}_{ps,n}^{irr} \times \mathcal{T}_n^{reg}$$

and a universal representation $\rho^u : G_{F,S} \to \mathrm{GL}_n(\mathcal{O}(\mathcal{U}))$ such that the induced representation $(\rho^u)^\wedge_{z_n} : G_F \to \mathrm{GL}_n(\mathcal{O}(\mathcal{U})^\wedge_{z_n})$ with coefficients in the completed local ring at z_n extends to a homomorphism $(\rho^u)^\wedge_{z_n} : G_{F^+,S} \to \mathcal{G}_n(\mathcal{O}(\mathcal{U})^\wedge_{z_n})$ with $\nu_{\mathcal{G}_n} \circ (\rho^u)^\wedge_{z_n} = \epsilon^{1-n}\delta_{F/F^+}^n$.

Since χ_2 is ordinary, the parameter $\delta_n = \sigma_{n,p}(\delta_2)$ is ordinary. We denote by $\mathcal{FL}_p(\mathcal{O}_{\mathcal{U}}^n) \xrightarrow{\alpha} \mathcal{U}$ the rigid space (equipped with a proper map to $\mathcal{U}$) classifying S_p-tuples $(\mathcal{F}_v)_{v \in S_p}$ of full flags in $\mathcal{O}_{\mathcal{U}}^n$. We consider the closed subspace

$$\mathcal{Z}^{ord} \subset \mathcal{FL}_p(\mathcal{O}_{\mathcal{U}}^n)$$

whose points z correspond to flags $\mathcal{F}_v$ which are $G_{F_{\widetilde{v}}}$-stable (under the ρ^u-action) for each $v \in S_p$ and the action of $G_{F_{\widetilde{v}}}$ on $\mathrm{gr}^i(\mathcal{F}_v)$ is given by $\delta_{z,v,i} \circ \mathrm{Art}_{F_{\widetilde{v}}}^{-1}$ where δ_z is the parameter of $\alpha(z)$. Since our parameters lie in $\mathcal{T}_n^{HT-reg}$, $\mathcal{Z}^{ord} \to \mathcal{U}$ is a closed immersion (it is a proper monomorphism).

Using the existence of $(\rho^u)^\wedge_{z_n}$, we can view the tangent space $T_{z_n}\mathcal{Z}^{ord}$ as a subspace of $H^1(G_{F^+,S}, \mathrm{ad}\, r_{\pi_n,\iota})$. By a similar argument to Proposition 2.11, it follows from e.g. [Ger19, Lemma 3.9] (which gives the analogue of Lemma 2.7 in the ordinary case) and the main theorem of [NT20] that the map $T_{z_n}\mathcal{Z}^{ord} \to T_{r(\delta_n)}\mathcal{W}_n$ is injective. On the other hand, $\mathcal{E}_n(\overline{\tau}_n)^{ord} \cap \mathcal{U}$, a subspace of $\mathcal{Z}^{ord}$ containing z_n, is equidimensional of dimension $\dim \mathcal{W}_n$. We deduce that $\mathcal{Z}^{ord}$ is smooth at z_n, and that $\mathcal{E}_n(\overline{\tau}_n)$ is locally isomorphic to $\mathcal{Z}^{ord}$ at z_n. We complete the proof in the same way as Theorem 2.24. ∎

We restate Theorem 2.24 and Theorem 2.27 in a way that does not make explicit reference to $\mathcal{E}_n$.

Corollary 2.28. *Let* $(\pi_2, \chi_2), (\pi_2', \chi_2') \in \mathcal{RA}_2$*, and let* $z_2, z_2' \in \mathcal{E}_2(\overline{\mathbf{Q}}_p)$ *be the corresponding points of the eigenvariety. Suppose that one of the following two sets of conditions are satisfied:*

(1) *The refinement* χ_2 *is numerically non-critical and* n*-regular.*
(2) *For each* $v \in S_p$*, every triangulation of* $r_{\pi_2',\iota}|_{G_{F_{\widetilde{v}}}}$ *is non-critical. The refinement* χ_2' *is* n*-regular.*
(3) *For each* $v \in S_p$*, the Zariski closures of the images of* $r_{\pi_2,\iota}|_{G_{F_{\widetilde{v}}}}$ *and* $r_{\pi_2',\iota}|_{G_{F_{\widetilde{v}}}}$ *contain* SL_2.
(4) *There exists an automorphic representation* π_n *of* $G_n(\mathbf{A}_{F^+})$ *such that*

$$\mathrm{Sym}^{n-1} r_{\pi_2,\iota} \cong r_{\pi_n,\iota}.$$

(5) The points z_2, z_2' lie on a common irreducible component of $\mathcal{E}_{2,\mathbf{C}_p}$;[4]

or

(1^{ord}) The refinement χ_2 is ordinary.

(2^{ord}) The Zariski closure of the image of $r_{\pi_2,\iota}|_{G_F}$ contains SL_2.

(3^{ord}) There exists an automorphic representation π_n of $G_n(\mathbf{A}_{F^+})$ such that

$$\mathrm{Sym}^{n-1} r_{\pi_2,\iota} \cong r_{\pi_n,\iota}.$$

(4^{ord}) The points z_2, z_2' lie on a common irreducible component of $\mathcal{E}_{2,\mathbf{C}_p}$ (this implies that the refinement χ_2' is also ordinary, by Lemma 2.26).

Then there exists an automorphic representation π_n' of $G_n(\mathbf{A}_{F^+})$ such that

$$\mathrm{Sym}^{n-1} r_{\pi_2',\iota} \cong r_{\pi_n',\iota}.$$

Proof. Choose $U_n \subset G_n(\mathbf{A}_{F^+}^\infty)$ so that $(\pi_n^\infty)^{U_n^{S_p}} \neq 0$ and take $\overline{\tau}_2 = \mathrm{tr}\,\overline{r}_{\pi_2,\iota}$. Then $(\sigma_n \circ i_2)(z_2) \in i_n(\mathcal{E}_n(\overline{\tau}_n)(\overline{\mathbf{Q}}_p))$. We claim that setting $\chi_{n,v} = \chi_{2,v,1}^{n-1} \otimes \chi_{2,v,1}^{n-2}\chi_{2,v,2} \otimes \cdots \otimes \chi_{2,v,2}^{n-1}$ for $v \in S_p$ defines an accessible refinement χ_n of π_n. Fix $v \in S_p$. To temporarily simplify notation, we write $\chi = \chi_1 \otimes \chi_2$ for $\chi_{2,v}$. The representation $\pi_{2,v}$ is isomorphic to either $\mathrm{St}_2(\iota\chi_1|\cdot|^{-1/2})$ or to an irreducible parabolic induction $i_{B_2}^{\mathrm{GL}_2}\iota\chi$. In the first case,

$$\mathrm{rec}^T_{F_v^+}(\iota^{-1}\pi_{n,v}) \cong \mathrm{Sym}^{n-1}\mathrm{rec}^T_{F_v^+}(\iota^{-1}\pi_{2,v}) \cong \mathrm{Sp}_n(\chi_1^{n-1}|\cdot|^{(1-n)/2})$$

and $\chi_{n,v}$ is the unique accessible refinement of $\pi_{n,v}$. In the second case,

$$\mathrm{rec}^T_{F_v^+}(\iota^{-1}\pi_{n,v}) \cong \mathrm{Sym}^{n-1}\mathrm{rec}^T_{F_v^+}(\iota^{-1}\pi_{2,v}) \cong \bigoplus_{i=0}^{n-1}\chi_1^{n-1-i}\chi_2^i|\cdot|^{(1-n)/2} \circ \mathrm{Art}^{-1}_{F_v^+}.$$

Note that $\pi_{n,v}$ is generic (the base change of π_n to $\mathrm{GL}_n(\mathbf{A}_F)$ is cuspidal, since $r_{\pi_n,\iota}$ is irreducible). We can now use the characterisation of generic representations in the Bernstein–Zelevinsky classification [Zel80, Theorem 9.7] and the compatibility with local Langlands [Rod82, §4.4]. It follows that no pair of characters in the above direct sum decomposition have ratio equal to the norm character, so the parabolic induction $i_{B_n}^{\mathrm{GL}_n}\iota\chi_{n,v}$ is irreducible and isomorphic to $\pi_{n,v}$. In particular, $\chi_{n,v}$ is an accessible refinement of $\pi_{n,v}$.

Taking the above discussion into account, it is straightforward to see that $(\sigma_n \circ i_2)(z_2)$ is associated to the pair $(\pi_n, \chi_n) \in \mathcal{RA}_n$. Thus the hypotheses of Theorem 2.24 or 2.27 are satisfied, and for any (π_2', χ_2') as in the statement of the corollary there exists a point $z_n' \in \mathcal{E}_n(\overline{\tau}_n)(\overline{\mathbf{Q}}_p)$ such that $T_{n,z'} = \mathrm{tr}\,\mathrm{Sym}^{n-1} r_{\pi_2',\iota}$. It remains to show that $z_n' \in Z_n$, or in other words that z_n' is associated to a classical automorphic representation.

In the ordinary case, this follows from Lemma 2.26. In the remaining case, Lemma 2.29 shows that for each $v \in S_p$, every triangulation of $\mathrm{Sym}^{n-1} r_{\pi_2',\iota}|_{G_{F_{\widetilde{v}}}}$ is non-critical. It follows from Lemma 2.30 that $z_n' \in Z_n$. □

Lemma 2.29. *Let $v \in S_p$, and let $\rho_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be a continuous, regular de Rham representation such that $\mathrm{WD}(\rho_v)$ has two distinct characters χ_1, χ_2 as Jordan–Hölder factors, which satisfy $(\chi_1/\chi_2)^i \neq 1$ for each $i = 1, \ldots, n-1$. Suppose*

[4] By [Con99, Theorem 3.4.2], this assumption is equivalent to requiring that there is a finite extension of coefficient fields E'/E such that z_2, z_2' lie on a common geometrically irreducible component of $\mathcal{E}_{2,E'}$.

moreover that every triangulation of ρ_v is non-critical. Then every triangulation of $\mathrm{Sym}^{n-1}\rho_v$ is non-critical.

Proof. We begin by describing the data of the triangulation of ρ_v in a bit more detail. Let $K = F_{\widetilde{v}}$ and let L/K be a Galois extension over which ρ_v becomes semi-stable. Let L_0 be the maximal unramified extension of $L/\mathbf{Q}_p$. After enlarging E, we can assume that every embedding of L in $\overline{\mathbf{Q}}_p$ lands in E, and that ρ_v is defined over E. The filtered $(\varphi, N, \mathrm{Gal}(L/K))$-module D associated to ρ_v consists of the following data:

(1) A free $L_0 \otimes_{\mathbf{Q}_p} E$-module D of rank 2, equipped with a $\sigma \otimes 1$-semilinear endomorphism φ.
(2) An $L_0 \otimes_{\mathbf{Q}_p} E$-linear endomorphism N of D satisfying the relation $N\varphi = p\varphi N$.
(3) An L_0-semilinear, E-linear action of the group $\mathrm{Gal}(L/K)$ on D that commutes with the action of both φ and N.
(4) A decreasing, $\mathrm{Gal}(L/K)$-stable, filtration $\mathrm{Fil}_\bullet D_L$ of $D_L = D \otimes_{L_0} L$.

For each embedding $\tau : L \to E$, we write $l_\tau \subset D_\tau = D_L \otimes_{L \otimes E, \tau} E$ for the image of the rank 1 step of the filtration $\mathrm{Fil}_\bullet$. We can define an action of the group W_K on D by the formula $g \cdot v = (g \bmod W_L) \circ \varphi^{-\alpha(g)}$, where $\alpha(g)$ is the power of the absolute arithmetic Frobenius induced by g on the residue field of $\overline{K}$.

This action preserves the factors of the product decomposition $D = \prod_t D_t$, where t ranges over embeddings $t : L_0 \to E$ and $D_t = D \otimes_{L_0 \otimes E, t} E$. Moreover, the isomorphism class of the Weil–Deligne representation D_t is independent of t. The data of a triangulation of ρ_v is equivalent to the data of a choice of character appearing in some (hence every) D_t. If N is non-zero on D_t, then there is a unique N-stable line in $\mathrm{Sym}^{n-1} D_t$. Hence there is a unique triangulation of $\mathrm{Sym}^{n-1}\rho_v$, induced by the unique (non-critical) triangulation of ρ_v, and it is also non-critical. From now on we assume that $N = 0$, and we proceed as indicated in [Che11, Example 3.26].

We can choose a basis e_1, e_2 for D as $L_0 \otimes_{\mathbf{Q}_p} E$-module such that the projection of the vectors e_1, e_2 to each D_t is a basis of eigenvectors for the group W_K.

Having made this choice of basis, each line l_τ is spanned by a linear combination of e_1, e_2. Our assumption that every triangulation of ρ_v is non-critical is equivalent to the requirement that l_τ may be spanned by a vector $e_1 + a_\tau e_2$, where $a_\tau \in E^\times$ for all τ. Indeed, if l_τ is spanned by e_i for some i, then the triangulation corresponding to the submodule of D spanned by e_i will fail the condition required for non-criticality with respect to the embedding τ.

Having made these normalisations, the condition that every triangulation of $\mathrm{Sym}^{n-1}\rho_v$ be non-critical is equivalent to the following statement: let $I \subset \{0, \dots, n-1\}$ be a subset, and let $\sum_{i \in I} a_i x^i \in E[x]$ be a polynomial, which is equal to $(1 + a_\tau x)^{|I|} Q(x)$ for some polynomial $Q(x) \in E[x]$ of degree at most $n - 1 - |I|$, then $Q(x) = 0$. Polynomials of the latter form correspond to elements of the $|I|$th step of the Hodge filtration on $\mathrm{Sym}^{n-1} D_\tau$ and the statement implies that this Hodge filtration is in general position compared to the filtration induced by every triangulation. Replacing the variable x with $-a_\tau x$, we can assume that $a_\tau = -1$. As in [Che11, Example 3.26], the vanishing of the $|I|$ successive derivatives at 1 of $\sum_{i \in I} a_i x^i$ gives a non-degenerate linear system of $|I|$ equations satisfied by the a_i, and therefore the a_i are all zero. Non-degeneracy is checked by noticing that the determinant of the linear system is the Vandermonde determinant $\prod_{i<j\in I}(i-j)$. $\square$

Lemma 2.30. *Let $z \in \mathcal{E}_n(\overline{\tau}_n)(\overline{\mathbf{Q}}_p)$ be a point with $i_n(z) = (\operatorname{tr} r_z, \delta) \in \mathcal{X}^{p-irr}_{ps,n}(\overline{\mathbf{Q}}_p) \times \mathcal{T}^{reg}_n(\overline{\mathbf{Q}}_p)$. Suppose that $\delta = \jmath_n(\nu(z)) = \delta_{alg}\delta_{sm}$ with δ_{alg} algebraic and δ_{sm} smooth. Suppose moreover that, for each $v \in S_p$, every triangulation of $r_z|_{G_{F_{\widetilde{v}}}}$ is non-critical. Then $z \in Z_n$ (in particular, δ_{alg} is strictly dominant).*

Proof. After extending E, we may assume that $z \in \mathcal{E}_n(\overline{\tau}_n)(E)$ and r_z takes values in $\mathrm{GL}_n(E)$. We note that, since δ is locally algebraic, it follows from property (5) of the eigenvariety that the subset of numerically non-critical classical points in $i_n^{-1}(\mathcal{X}^{p-irr}_{ps,n} \times \mathcal{T}^{reg}_n)$ accumulates at z. It follows from [KPX14, Corollary 6.3.10], applied as in Proposition 2.15, that there is a connected affinoid neighbourhood $\mathcal{U}$ of z in $i_n^{-1}(\mathcal{X}^{p-irr}_{ps,n} \times \mathcal{T}^{reg}_n)$, over which there exist representations $\rho^u_v : G_{F_{\widetilde{v}}} \to \mathrm{GL}_n(\mathcal{O}(\mathcal{U}))$ for each $v \in S_p$ with trace equal to the restriction to $G_{F_{\widetilde{v}}}$ of the universal pseudocharacter and a non-empty Zariski open and dense subspace $\mathcal{V} \subset \mathcal{U}$ such that for every $z' \in \mathcal{V}$ with $i_n(z') = (\operatorname{tr} r', \delta')$, r' is trianguline of parameter δ'. Now we can apply [BHS17, Lemme 2.11][5] to deduce that δ_{alg} is strictly dominant and r_z is trianguline of parameter δ.

We now argue as in [BHS17, Prop. 3.28] (which is itself similar to the argument of [Che11, Prop. 4.2]). The idea of the argument is to show that failure of classicality would entail the existence of a 'companion point' to z, with the same associated Galois representation and a locally algebraic weight which is *not* strictly dominant. This would contradict [BHS17, Lemma 2.11].

Let $\eta = \nu(z)\delta_{B_n} = \eta_{alg}\eta_{sm}$, with η_{alg} dominant algebraic (since δ_{alg} is strictly dominant) and η_{sm} smooth. By the construction of $\mathcal{E}_n$ and Lemma 2.19 we have a non-zero space of morphisms

$$0 \neq \operatorname{Hom}_{\prod_{v\in S_p} T_n(F_{\widetilde{v}})}\left(\eta, J_{B_n}\left(\widetilde{S}(U^p_n, E)^{an}[\psi^*(z)]\right)\right).$$

Now we use some of the work of Orlik–Strauch [OS15], with notation as in [Bre15, §2]. We denote by $\mathfrak{g}_n$ the $\mathbf{Q}_p$-Lie algebra of $\prod_{v\in S_p} G_n(F^+_v) \cong \prod_{v\in S_p} \mathrm{GL}_n(F_{\widetilde{v}})$ and denote by $\overline{\mathfrak{b}}_n \subset \mathfrak{g}_n$ the lower triangular Borel. We define a locally analytic representation of $\prod_{v\in S_p} G_n(F^+_v)$ (see [Bre15, Thm. 2.2] for the definition of the functor $\mathcal{F}^{G_n}_{\overline{B}_n}$):

$$\mathcal{F}^{G_n}_{\overline{B}_n}(\eta\delta^{-1}_{B_n}) := \mathcal{F}^{G_n}_{\overline{B}_n}\left(\left(U(\mathfrak{g}_{n,E}) \otimes_{U(\overline{\mathfrak{b}}_{n,E})} \eta^{-1}_{alg}\right)^{\vee}, \eta_{sm}\delta^{-1}_{B_n}\right).$$

Note that $\left(U(\mathfrak{g}_{n,E}) \otimes_{U(\overline{\mathfrak{b}}_{n,E})} \eta^{-1}_{alg}\right)^{\vee}$ has a unique simple submodule (isomorphic to the unique simple quotient of $U(\mathfrak{g}_{n,E}) \otimes_{U(\overline{\mathfrak{b}}_{n,E})} \eta^{-1}_{alg}$), the algebraic representation $V(\eta_{alg})^{\vee}$ with lowest (with respect to B_n) weight η^{-1}_{alg}. It follows from [Bre15, Thm. 2.2] that $\mathcal{F}^{G_n}_{\overline{B}_n}(\eta\delta^{-1}_{B_n})$ has a locally algebraic quotient isomorphic to $V(\eta_{alg}) \otimes_E \operatorname{Ind}^{G_n}_{\overline{B}_n} \eta_{sm}\delta^{-1}_{B_n}$.

[5]We caution the reader that the version of this paper currently available on the arXiv contains a less general result than the published version, to which we appeal here. In particular, it restricts to Galois representations which are known in advance to be crystalline.

By [Bre15, Thm. 4.3], there is a non-zero space of morphisms

$$0 \neq \mathrm{Hom}_{\prod_{v\in S_p} G_n(F_v^+)} \left(\mathcal{F}_{\overline{B}_n}^{G_n}(\eta\delta_{B_n}^{-1}), \widetilde{S}(U_n^p, E)^{an}[\psi^*(z)] \right).$$

The Jordan–Hölder factors of $\mathcal{F}_{\overline{B}_n}^{G_n}(\eta\delta_{B_n}^{-1})$ can be described using [Bre15, Thm. 2.2] and standard results on the Jordan–Hölder factors of Verma modules (see [Bre15, Cor. 4.6]). Suppose $\lambda \in \mathcal{T}_n(E)$ is an algebraic character. Denote by M_λ the unique simple submodule of the dual Verma module $\left(U(\mathfrak{g}_{n,E}) \otimes_{U(\overline{\mathfrak{b}}_{n,E})} \lambda^{-1}\right)^\vee$. Then the Jordan–Hölder factors of $\mathcal{F}_{\overline{B}_n}^{G_n}(\eta\delta_{B_n}^{-1})$ are all of the form

$$JH(w,\pi) = \mathcal{F}_{\overline{P}_n}^{G_n}\left(M_{w\cdot\eta_{alg}}, \pi\right)$$

with $\overline{P}_n$ a parabolic subgroup of $\prod_{v\in S_p} G_n(F_v^+)$ containing $\overline{B}_n$, π a Jordan–Hölder factor of the parabolic induction of $\eta_{sm}\delta_{B_n}^{-1}$ from $\overline{B}_n$ to the Levi of $\overline{P}_n$, and w an element of the Weyl group of $(\mathrm{Res}_{F^+/\mathbf{Q}} G_n) \times_{\mathbf{Q}} E$, acting by the 'dot action' on η_{alg}. Here $\overline{P}_n$ is maximal for $M_{w\cdot\eta_{alg}}$, in the sense of [Bre15, §2].

We claim that there cannot be a non-zero morphism $JH(w,\pi) \to \widetilde{S}(U_n^p, E)^{an}[\psi^*(z)]$ for $w \neq 1$. Suppose, for a contradiction, that there is such a map. It follows from [Bre16, Cor. 3.4] that we have $\psi^*(z) = \psi^*(z')$ (and hence an isomorphism of Galois representations $r_z \cong r_{z'}$) for a point $z' \in \mathcal{E}_n(\overline{\tau}_n)(E)$ with $\jmath_n(\nu(z'))$ locally algebraic but *not* strictly dominant (its algebraic part matches the algebraic part of $\jmath_n(w\cdot\eta_{alg})$). The argument in the first paragraph of this proof, using [BHS17, Lemma 2.11], then gives a contradiction.

We deduce from this that any map

$$\mathcal{F}_{\overline{B}_n}^{G_n}(\eta\delta_{B_n}^{-1}) \to \widetilde{S}(U_n^p, E)^{an}[\psi^*(z)]$$

factors through the locally algebraic quotient $V(\eta_{alg}) \otimes_E \mathrm{Ind}_{\overline{B}_n}^{G_n} \eta_{sm}\delta_{B_n}^{-1}$. Applying [Bre15, Thm. 4.3] again, we deduce that we have equalities

$$\begin{aligned} &\mathrm{Hom}_{\prod_{v\in S_p} T_n(F_{\tilde{v}})}\left(\eta, J_{B_n}\left(\widetilde{S}(U_n^p, E)^{an}[\psi^*(z)]\right)\right) \\ &\qquad = \mathrm{Hom}_{\prod_{v\in S_p} G_n(F_v^+)}\left(\mathcal{F}_{\overline{B}_n}^{G_n}(\eta\delta_{B_n}^{-1}), \widetilde{S}(U_n^p, E)^{an}[\psi^*(z)]\right) \\ &\qquad = \mathrm{Hom}_{\prod_{v\in S_p} G_n(F_v^+)}\left(\mathcal{F}_{\overline{B}_n}^{G_n}(\eta\delta_{B_n}^{-1}), \widetilde{S}(U_n^p, E)^{alg}[\psi^*(z)]\right) \\ &\qquad = \mathrm{Hom}_{\prod_{v\in S_p} T_n(F_{\tilde{v}})}\left(\eta, J_{B_n}\left(\widetilde{S}(U_n^p, E)^{alg}[\psi^*(z)]\right)\right). \end{aligned}$$

In particular, our point z arises from a non-zero map

$$\eta \to J_{B_n}\left(\widetilde{S}(U_n^p, E)^{alg}[\psi^*(z)]\right).$$

Applying [Eme06a, Prop. 4.3.6] and computing locally algebraic vectors as in §2.18.1 we see that such a map corresponds to a non-zero map of smooth representations $\eta_{sm} \to J_{B_n}\left(\varinjlim_{U_p} S_{\eta_{alg}^\vee}(U_n^p U_p, E)[\psi^*(z)]\right)$ and hence a pair $(\pi_n, \iota\circ\eta_{sm}\delta_{B_n}^{-1/2}) \in \mathcal{RA}_n$ with corresponding classical point equal to z. We therefore have $z \in Z_n$. $\square$

2.31. **Application to the eigencurve.** Thus far in this section we have found it convenient to phrase our arguments in terms of automorphic forms on unitary groups. Since our intended application will rely on particular properties of the Coleman–Mazur eigencurve for GL_2, we now show how to deduce what we need for the eigencurve from what we have done so far.

We first introduce the version of the eigencurve that we use. Fix an integer $N \geq 1$, prime to p. Let $\mathcal{T}_0 = \mathrm{Hom}(\mathbf{Q}_p^\times/\mathbf{Z}_p^\times \times \mathbf{Q}_p^\times, \mathbf{G}_m)$; it is the E-rigid space parameterising characters $\chi_0 = \chi_{0,1} \otimes \chi_{0,2}$ of $(\mathbf{Q}_p^\times)^2$ such that $\chi_{0,1}$ is unramified. Let $\mathcal{W}_0 = \mathrm{Hom}(\mathbf{Z}_p^\times, \mathbf{G}_m)$, and write $r_0 : \mathcal{T}_0 \to \mathcal{W}_0$ for the morphism given by $r_0(\chi_{0,1} \otimes \chi_{0,2}) = \chi_{0,1}/\chi_{0,2}|_{\mathbf{Z}_p^\times} = \chi_{0,2}^{-1}|_{\mathbf{Z}_p^\times}$. We denote the map $r_0 \circ \nu_0 : \mathcal{E}_0 \to \mathcal{W}_0$ by κ. Let $\mathbf{T}_0^{pN} = \mathcal{O}[\{T_l, S_l\}_{l \nmid pN}]$ denote the polynomial ring in unramified Hecke operators at primes not dividing Np. Here T_l and S_l are the double coset operators for the matrices $\left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} l & 0 \\ 0 & l \end{smallmatrix}\right)$. Let $U_1(N) = \prod_l U_1(N)_l \subset \mathrm{GL}_2(\widehat{\mathbf{Z}}) = \prod_l \mathrm{GL}_2(\mathbf{Z}_l)$ be defined by

$$U_1(N)_l = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_l) : c, d-1 \in N\mathbf{Z}_l \right\}.$$

The eigencurve is a tuple $(\mathcal{E}_0, \psi_0, \nu_0, Z_0)$, where:

(1) $\mathcal{E}_0$ is a reduced E-rigid space, equipped with a finite morphism $\nu_0 : \mathcal{E}_0 \to \mathcal{T}_0$.
(2) $\psi_0 : \mathbf{T}_0^{pN} \to \mathcal{O}(\mathcal{E}_0)$ is a ring homomorphism, which takes values in the subring $\mathcal{O}(\mathcal{E}_0)^{\leq 1}$ of bounded elements.
(3) $Z_0 \subset \mathcal{E}_0(\overline{\mathbf{Q}}_p)$ is a Zariski dense subset which accumulates at itself.

The following properties are satisfied:

(1) $\mathcal{E}_0$ is equidimensional of dimension $\dim \mathcal{W}_0 = 1$. For any irreducible component $\mathcal{C} \subset \mathcal{E}_0$, $\kappa(\mathcal{C})$ is a Zariski open subset of $\mathcal{W}_0$.
(2) Let $\mathcal{A}_0$ denote the set of cuspidal automorphic representations π_0 of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ such that $(\pi_0^\infty)^{U_1(N)^p} \neq 0$ and $\pi_{0,\infty}$ has the same infinitesimal character as $(\mathrm{Sym}^{k-2}\,\mathbf{C}^2)^\vee$ for some $k \geq 2$ (in which case we say π_0 has weight k), and let $\mathcal{RA}_0$ denote the set of pairs (π_0, χ_0), where $\pi_0 \in \mathcal{A}_0$ and $\chi_0 = \chi_{0,1} \otimes \chi_{0,2}$ is an accessible refinement of $\pi_{0,p}$ such that $\chi_{0,1}$ is *unramified*. As in the unitary case we considered above, for $(\pi_0, \chi_0) \in \mathcal{RA}_0$ we have a homomorphism $\psi_{\pi_0} : \mathbf{T}_0^{pN} \to \overline{\mathbf{Q}}_p^\times$ determined by the action of the Hecke operators on $\iota^{-1}(\pi_0^\infty)^{U_1(N)^p}$. There is also a character $\nu_0(\pi_0, \chi_0) \in \mathcal{T}_0(\overline{\mathbf{Q}}_p)$ defined in exactly the same way as in the unitary case (2.18.1). An explicit formula appears below (2.31.1). Our assumption that $\chi_{0,1}$ is unramified implies that this character does indeed give a point of $\mathcal{T}_0$. Now we can let $\mathcal{Z}_0 \subset \mathrm{Hom}(\mathbf{T}_0^{pN}, \overline{\mathbf{Q}}_p) \times \mathcal{T}(\overline{\mathbf{Q}}_p)$ denote the set of points of the form $(\psi_{\pi_0}, \nu_0(\pi_0, \chi_0))$, where $(\pi_0, \chi_0) \in \mathcal{RA}_0$. Then the map $\psi_0^* \times \nu_0 : \mathcal{E}_0(\overline{\mathbf{Q}}_p) \to \mathrm{Hom}(\mathbf{T}_0^{pN}, \overline{\mathbf{Q}}_p) \times \mathcal{T}_0(\overline{\mathbf{Q}}_p)$ restricts to a bijection $Z_0 \to \mathcal{Z}_0$.
(3) For any affinoid open $\mathcal{V}_0 \subset \mathcal{T}_0$, the map $\mathbf{T}_0^{pN} \otimes \mathcal{O}(\mathcal{V}_0) \to \mathcal{O}(\nu_0^{-1}\mathcal{V}_0)$ is surjective.

The uniqueness of the tuple $(\mathcal{E}_0, \psi_0, \nu_0, Z_0)$ follows from [BC09, Proposition 7.2.8]. Its existence can be proved in various ways. A construction using overconvergent modular forms is given in [Buz07]. We note that in this case, in contrast to the unitary group case, the map $\mathcal{RA}_0 \to \mathcal{Z}_0$ is bijective – a consequence of the strong multiplicity one theorem. We will therefore feel free to speak of the cuspidal automorphic representation $\pi_0 \in \mathcal{A}_0$ associated to a point lying in Z_0. As in the unitary group case, there is a Galois pseudocharacter $t : G_{\mathbf{Q},Np} \to \mathcal{O}(\mathcal{E}_0)$ with the property that for $z \in Z_0$ associated to (π_0, χ_0), $t_z = \operatorname{tr} r_{\pi_0,\iota}$.

Let us describe explicitly the link with more classical language. We are using the normalisations of [DI95, §11]. If $(\pi_0, \chi_0) \in \mathcal{A}_0$, then there is a cuspidal holomorphic modular form $f = q + \sum_{n\geq 2} a_n(f)q^n$ of level $\Gamma_1(Np^r)$ (for some $r \geq 1$) which is an eigenform for all the Hecke operators T_l ($l \nmid Np$) and U_p, in their classical normalisations, and we have the formulae

$$a_l(f) = \text{eigenvalue of } T_l \text{ on } \pi_{0,l}^{\mathrm{GL}_2(\mathbf{Z}_l)},\ p^{-1/2}a_p(f) = \iota\chi_{0,1}(p).$$

Note that the central character of π_0 is a Hecke character ψ_{π_0} with $\psi_{\pi_0}|_{\mathbf{R}_{>0}}(z) = z^{2-k}$. So $\psi_{\pi_0}|_{\mathbf{Q}_p^\times} = \iota(\chi_{0,1}\chi_{0,2})$ is a finite order twist of the character $z \mapsto |z|^{2-k}$. To convince the reader that these formulae are correct, we observe that if $\pi_{0,l}$ is a normalised induction $i_{B_2}^{\mathrm{GL}_2}\mu_1 \otimes \mu_2$, then the eigenvalue of T_l on $\pi_{0,l}^{\mathrm{GL}_2(\mathbf{Z}_l)}$ is $l^{1/2}(\mu_1(l) + \mu_2(l))$ [DI95, (11.2.4)], whilst considering the central character shows that $\mu_1(l)\mu_2(l)$ has (complex) absolute value l^{k-2}. This is compatible with the fact that $a_l(f)$ is a sum of numbers with absolute values $l^{(k-1)/2}$.

We can define a map $s : \mathcal{E}_0(\overline{\mathbf{Q}}_p) \to \mathbf{R}$, called the slope, by composing the projection to $\mathcal{T}_0(\overline{\mathbf{Q}}_p)$ with the map $\chi \mapsto v_p\left(\chi\left(\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right)\right)$. Note that if $(\pi_0, \chi_{0,1} \otimes \chi_{0,2}) \in \mathcal{RA}_0$ we have

$$\nu_0(\pi_0, \chi_0)\left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2\end{smallmatrix}\right) = t_2^{2-k}\chi_{0,1}(t_1)\chi_{0,2}(t_2)\left|\frac{t_1}{t_2}\right|_p^{-1/2} \tag{2.31.1}$$

so the slope map sends $(\pi_0, \chi_{0,1} \otimes \chi_{0,2}) \in \mathcal{RA}_0$ to $1/2 + v_p(\chi_{0,1}(p))$.

In particular, at a point $z_0 \in Z_0$ corresponding to a classical holomorphic modular form f, $s(z_0)$ equals the p-adic valuation of $\iota^{-1}a_p(f)$. Note that the corresponding pair (π_0, χ_0) is numerically non-critical exactly when $s(z_0) < k - 1$ and ordinary exactly when $s(z_0) = 0$. The classicality criterion of Coleman [Col96, Col97] shows that a point $z \in \mathcal{E}_0(\overline{\mathbf{Q}}_p)$ with $\kappa(z)$ restricting to $t \mapsto t^{k-2}$ on a finite index subgroup of $\mathbf{Z}_p^\times$ and $s(z) < k - 1$ is necessarily in Z_0.

Let $Z_0^{pc} \subset Z_0$ denote the subset of points corresponding to pairs (π, χ) where π_p is not a twist of the Steinberg representation (pc stands for *potentially crystalline*). We now define a 'twin' map $\tau : Z_0^{pc} \to Z_0^{pc}$. Let (π_0, χ_0) be the pair corresponding to a point $z \in Z_0^{pc}$. Write $\chi_0 = \chi_{0,1} \otimes \chi_{0,2}$. Since $\pi_{0,p}$ is not a twist of the Steinberg representation, $\pi_{0,p}$ equals the full normalised induction $i_{B_2}^{\mathrm{GL}_2}\iota\chi_0$, which is irreducible. Let $\psi : \mathbf{Q}^\times\backslash\mathbf{A}_{\mathbf{Q}}^\times \to \overline{\mathbf{Z}}_p^\times$ be the unique finite order character which is unramified outside p and such that $\psi|_{\mathbf{Z}_p^\times} = \chi_{0,2}|_{\mathbf{Z}_p^\times}^{-1}$. Then the character $\chi_{0,2}\psi|_{\mathbf{Z}_p^\times}$ is unramified and $\chi_0' = \chi_{0,2}\psi|_{\mathbf{Z}_p^\times} \otimes \chi_{0,1}\psi|_{\mathbf{Z}_p^\times}$ is an accessible refinement of the twist $\pi_0 \otimes \iota\psi$. We therefore have a point $\tau(z) \in Z_0^{pc}$ corresponding to the pair $\tau(\pi_0, \chi_0) = (\pi_0 \otimes \iota\psi, \chi_0')$, that we call the twin of z. Note that $\tau^2 = 1$ and if $\pi_{0,p}$ is unramified then $\tau(z)$ is the usual companion point appearing in the Gouvea–Mazur construction of the infinite fern [Maz97, §18]. The following lemma is an easy computation.

Lemma 2.32. *Let $z \in Z_0^{pc}$, and let $z' = \tau(z)$. Let s, s' denote the slopes of these two points, and $\kappa(z), \kappa(z') \in \mathcal{W}_0(\overline{\mathbf{Q}}_p)$ their images in weight space. Then $s + s' = k - 1$ and $v_p(\kappa(z)(1 + q) - 1) = v_p(\kappa(z')(1 + q) - 1)$, where $q = p$ if p is odd and $q = 4$ if p is even.*

Here is the main result of §2.

Theorem 2.33. *Let $(\pi_0, \chi_0), (\pi_0', \chi_0') \in \mathcal{RA}_0$ and let $n \geq 2$. Let $z_0, z_0' \in Z_0$ be the corresponding points. Suppose that one of following two sets of conditions are satisfied:*

(1) The refinement χ_0 is numerically non-critical and n-regular.
(2) The refinement χ_0' is n-regular.
(3) The Zariski closures of $r_{\pi_0,\iota}(G_{\mathbf{Q}_p})$ and $r_{\pi_0',\iota}(G_{\mathbf{Q}_p})$ contain SL_2.
(4) $\mathrm{Sym}^{n-1} r_{\pi_0,\iota}$ is automorphic;

or

(1^{ord}) The refinement χ_0 is ordinary.
(2^{ord}) π_0 and π_0' are not CM (so the Zariski closures of $r_{\pi_0,\iota}(G_{\mathbf{Q}})$ and $r_{\pi_0',\iota}(G_{\mathbf{Q}})$ contain SL_2).
(3^{ord}) $\mathrm{Sym}^{n-1} r_{\pi_0,\iota}$ is automorphic.

If the points z_0, z_0' lie on a common irreducible component of $\mathcal{E}_{0,\mathbf{C}_p}$, then $\mathrm{Sym}^{n-1} r_{\pi_0',\iota}$ is also automorphic.

Proof. We want to apply Corollary 2.28. We first need to specify suitable data F, S, G_2, U_2. Let $F'/\mathbf{Q}$ be an abelian CM extension satisfying the following conditions:

- Each prime dividing Np splits in F'.
- $[(F')^+ : \mathbf{Q}]$ is even.
- The extension $F'/(F')^+$ is everywhere unramified.

After extending E, we may assume that $z_0, z_0' \in \mathcal{E}_0(E)$ and that there is an irreducible component $\mathcal{C} \subset \mathcal{E}_0$ containing the points z_0, z_0'. Moreover, by the first part of [Con99, Theorem 3.4.2], we may assume that $\mathcal{C}$ is geometrically irreducible. Let W denote the unique connected component of $\mathcal{W}_0$ containing $\kappa(\mathcal{C})$. We can find a character $\chi : G_{\mathbf{Q}} \to \mathcal{O}(W)^\times$ such that the determinant of the universal pseudocharacter over $\mathcal{C}$ equals $\epsilon^{-1}\chi$ (χ is the product of a finite order p-unramified character and the composition of ϵ with the universal character $\mathbf{Z}_p^\times \to \mathcal{O}(\mathcal{W}_0)$). By Lemma 2.34, we can find a finite étale morphism $\eta : \widetilde{W} \to W$ and a character $\psi : G_{F'} \to \mathcal{O}(\widetilde{W})^\times$, unramified almost everywhere, such that $\psi\psi^c = \chi|_{G_{F'}}$, and such that for each place $v|p$ of $(F')^+$, there is a place $\widetilde{v}|v$ of F' such that $\psi|_{G_{F'_{\widetilde{v}}}}$ is unramified. We now let $F/\mathbf{Q}$ be a soluble, Galois, CM extension, containing F', such that:

- Each prime dividing Np splits in F.
- The extension F/F^+ is everywhere unramified.
- The character $\psi|_{G_F}$ is unramified away from p.

Let S denote the set of places of F^+ dividing Np. Fix as usual a set of factorisations $v = \widetilde{v}\widetilde{v}^c$ for $v \in S$. Fix the unitary group G_2 as in our standard assumptions (§1). Then for each $v \in S$, there is an isomorphism $\iota_{\widetilde{v}} : G_2(F_v^+) \to \mathrm{GL}_2(F_{\widetilde{v}})$. We let $U_2 = \prod_v U_{2,v} \subset G(\mathbf{A}_{F^+}^\infty)$ be an open subgroup with the property that $U_{2,v}$ is hyperspecial maximal compact if $v \notin S$, and $U_{2,v}$ is the pre-image under $\iota_{\widetilde{v}}$ of the subgroup $U_1(N)_l$ of $\mathrm{GL}_2(\mathbf{Q}_l)$ if $v \in S$ has residue characteristic l (in which case $F_{\widetilde{v}} = \mathbf{Q}_l$).

We recall that $\mathcal{T}_2 = \prod_{v \in S_p} \mathrm{Hom}((F_{\widetilde{v}}^\times)^2, \mathbf{G}_m)$. Let $\widetilde{\mathcal{T}} = \mathcal{T}_0 \times_{\mathcal{W}_0} \widetilde{W}$, with $\tilde{\kappa} : \widetilde{\mathcal{T}} \to \widetilde{W}$ the projection map. If $\chi^u = \chi_1^u \otimes \chi_2^u \in \mathcal{T}_0(\mathcal{T}_0)$ denotes the universal character, then the tuple of characters

$$((\chi_1^u \circ \mathbf{N}_{F_{\widetilde{v}}/\mathbf{Q}_p} \cdot \psi^{-1}|_{G_{F_{\widetilde{v}}}} \circ \mathrm{Art}_{F_{\widetilde{v}}}) \otimes (\chi_2^u \circ \mathbf{N}_{F_{\widetilde{v}}/\mathbf{Q}_p} \cdot \psi^{-1}|_{G_{F_{\widetilde{v}}}} \circ \mathrm{Art}_{F_{\widetilde{v}}}))_{v \in S_p}$$

in $\mathcal{T}_2(\widetilde{\mathcal{T}})$ determines a morphism $b_p : \widetilde{\mathcal{T}} \to \mathcal{T}_2$. Writing $\mathcal{X}_{0,ps}$ for the rigid space of 2-dimensional pseudocharacters of $G_{\mathbf{Q}}$, unramified outside Np, there is a base change morphism $b : \mathcal{X}_{0,ps} \times \widetilde{\mathcal{T}} \to \mathcal{X}_{2,ps} \times \mathcal{T}_2$ covering b_p and sending a pair (τ, μ) to $(\tau|_{G_F} \otimes \psi^{-1}_{\widetilde{\kappa}(\mu)}, b_p(\mu))$. This leads to a diagram of rigid spaces

$$\begin{tikzcd} \mathcal{X}_{0,ps} \times \widetilde{\mathcal{T}} \arrow[r, "b"] & \mathcal{X}_{2,ps} \times \mathcal{T}_2 . \\ \mathcal{E}_0 \times_{\mathcal{T}_0} \widetilde{\mathcal{T}} \arrow[u, "\widetilde{i}"] & \mathcal{E}_2 \arrow[u, "i_2"'] \end{tikzcd}$$

Let $\widetilde{\mathcal{C}} = \mathcal{C} \times_W \widetilde{W} = \mathcal{C} \times_{\mathcal{T}_0} \widetilde{\mathcal{T}}$. Then the morphism $\widetilde{\mathcal{C}} \to \mathcal{C}$ is finite étale. In particular, each irreducible component of $\widetilde{\mathcal{C}}$ maps surjectively to $\mathcal{C}$. Choose E'/E so that the irreducible components of $\widetilde{\mathcal{C}}_{E'}$ are geometrically irreducible (we apply [Con99, Theorem 3.4.2] again). Since $\mathcal{C}$ is geometrically irreducible, we still know that each irreducible component of $\widetilde{\mathcal{C}}_{E'}$ maps surjectively to $\mathcal{C}_{E'}$. Consequently, we can find points z_1, z_1' of $\widetilde{\mathcal{C}}_{E'}$ lifting z_0, z_0' and lying on a common geometrically irreducible component $\widetilde{\mathcal{C}}'$ of $\widetilde{\mathcal{C}}_{E'}$. We next wish to show that $b \circ \widetilde{i}(\widetilde{\mathcal{C}}') \subset i_2(\widetilde{\mathcal{E}}_{2,E'})$, or equivalently that $(b \circ \widetilde{i})^{-1}(i_2(\mathcal{E}_{2,E'}))$ contains $\widetilde{\mathcal{C}}'$. Since i_2 is a closed immersion, it suffices to show that $\widetilde{Z}_0'$, the pre-image of Z_0 in $\widetilde{\mathcal{C}}'$, satisfies $b \circ \widetilde{i}(\widetilde{Z}_0') \subset i_2(\mathcal{E}_2(\overline{\mathbf{Q}}_p))$ (the accumulation property of $\widetilde{Z}_0'$ in $\widetilde{\mathcal{C}}'$ is inherited from the corresponding property of the subset $Z_0 \cap \mathcal{C} \subset \mathcal{C}$).

To see this, we note that for any $(\pi, \chi) \in \mathcal{Z}_0$, with lift $\widetilde{z}' \in \widetilde{Z}_0'$, the base change π_F (which exists since $F/\mathbf{Q}$ is soluble) is still cuspidal. Indeed, if not then $r_{\pi,\iota}|_{G_F}$ would be reducible, implying that π was automorphically induced from a quadratic imaginary subfield $K/\mathbf{Q}$ of $F/\mathbf{Q}$. This is a contradiction, since we chose F so that all primes dividing Np split in F, yet K must be ramified at at least one such prime. The descent of $\pi_F \otimes \iota\psi^{-1}_{\widetilde{z}'}$ to G_2 (which exists, by [Lab11, Théorème 5.4]) gives (together with $b_p(\nu_0(\pi, \chi))$) a point of $\mathcal{E}_2$ which equals the image of $\widetilde{z}'$ under the map $b \circ \widetilde{i}$.

We can now complete the proof. Indeed, the points $b \circ \widetilde{i}(z_1)$, $b \circ \widetilde{i}(z_1')$ lie on a common geometrically irreducible component of $\mathcal{E}_{2,E'}$, by construction. They satisfy the conditions of Corollary 2.28 (in particular, Example 2.10 shows that our assumption on $r_{\pi_0',\iota}(G_{\mathbf{Q}_p})$ in the non-ordinary case implies that all of its triangulations are non-critical). We therefore conclude the existence of an automorphic representation π_n' of $G_n(\mathbf{A}_{F^+})$ such that $\mathrm{Sym}^{n-1} r_{\pi_0',\iota}|_{G_F} \cong r_{\pi_n',\iota}$. Our assumptions (cf. Lemma 3.5(2)) imply that $\mathrm{Sym}^{n-1} r_{\pi_0',\iota}|_{G_F}$ is irreducible, and therefore that the base change of π_n' is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$. Soluble descent for GL_n now implies that $\mathrm{Sym}^{n-1} r_{\pi_0',\iota}$ is itself automorphic. $\square$

Lemma 2.34. *Let F be a CM number field. Suppose that each p-adic place of F^+ splits in F, and let $\widetilde{S}_p$ be a set of p-adic places of F such that $\widetilde{S}_p \sqcup \widetilde{S}_p^c$ is the set of all p-adic places of F. Let W be a connected E-rigid space, and let $\chi : G_{\mathbf{Q}} \to \mathcal{O}(W)^\times$ be a continuous character (continuity defined by demanding that the induced characters with values in $\mathcal{O}(U)^\times$ are continuous for all affinoid admissible opens $U \subset W$, as* *in* [Buz04, §2]*), unramified almost everywhere. Then we can find a finite étale morphism* $\eta : \widetilde{W} \to W$ *and a continuous character* $\psi : G_F \to \mathcal{O}(\widetilde{W})^\times$ *such that the following properties hold:*

1. *(1)* ψ *is unramified almost everywhere.*
2. *(2) For each* $\widetilde{v} \in \widetilde{S}_p$, $\psi|_{G_{F_{\widetilde{v}}}}$ *is unramified.*
3. *(3)* $\psi\psi^c = \eta^*(\chi)|_{G_F}$.

Proof. We first claim that we can find a finite étale morphism $W' \to W$ and a continuous character $\lambda : G_F \to \mathcal{O}(W')^\times$ with the following properties:

- λ is unramified almost everywhere.
- $\chi|_{G_F}\lambda\lambda^c$ has finite order.

Indeed, let $L : \prod_{w|p} \mathcal{O}_{F_w}^\times \to \mathcal{O}(W)^\times$ be defined by the formula

$$L((u_w)_w) = \prod_{\widetilde{v} \in \widetilde{S}_p^c} \chi|_{G_F}^{-1} \circ \mathrm{Art}_{F_{\widetilde{v}}}(u_{\widetilde{v}}).$$

Then L is continuous, and trivial on a finite index subgroup of $\mathcal{O}_F^\times$ (it is trivial on the norm 1 units in $\mathcal{O}_{F+}^\times$). It follows from Chevalley's theorem [Che51, Théorème 1] that there is a compact open subgroup U^p of $\prod_{w\nmid p} \mathcal{O}_{F_w}^\times$ such that L is trivial on $\Gamma(U^p) := \left(U^p \times \prod_{w|p} \mathcal{O}_{F_w}^\times\right) \cap \mathcal{O}_F^\times$.

Note that if H is a product of a finite abelian group and a finite $\mathbf{Z}_p$-module, and $H' \subset H$ is a finite index subgroup, then the natural map $\mathrm{Hom}(H, \mathbf{G}_m) \to \mathrm{Hom}(H', \mathbf{G}_m)$ of rigid spaces is finite étale. Maps of rigid spaces $W \to \mathrm{Hom}(H, \mathbf{G}_m)$ biject with continuous characters $H \to \mathcal{O}(W)^\times$.

It follows that we may extend L to a continuous character $L' : F^\times\backslash\mathbf{A}_F^{\infty,\times} \to \mathcal{O}(W')^\times$, for some finite étale morphism $W' \to W$. Indeed, we apply the preceding remark with H' the quotient of $\prod_{w|p} \mathcal{O}_{F_w}^\times$ by the closure of $\Gamma(U^p)$ and H the quotient of $F^\times\backslash\mathbf{A}_F^{\infty,\times}$ by the closure of the image of U^p (cf. the discussion in [Buz04, §2]).

We define λ by $\lambda \circ \mathrm{Art}_F = L'$. The character $\chi|_{G_F}\lambda\lambda^c$ has finite order because it factors through the Galois group of an abelian extension of F which is unramified at all but finitely many places and unramified at the primes above p.

Replacing W' by a connected component, we may suppose that W' is connected, in which case the character $\chi|_{G_F}\lambda\lambda^c$ is constant (i.e. pulled back from a morphism $W' \to \mathrm{Sp}\, E'$, for a finite extension E'/E). Applying [BLGGT14, Lemma A.2.5], we may find a finite extension E''/E' and a continuous character $\varphi : G_F \to (E'')^\times$ of finite order such that $\chi|_{G_F}\lambda\lambda^c = \varphi\varphi^c$. The proof is complete on taking $\widetilde{W} = W'_{E''}$ and $\psi = \varphi\lambda^{-1}$. □

We conclude this section with a lemma that will be used in §8. It uses the existence of the universal pseudocharacter t over $\mathcal{E}_0$.

Lemma 2.35. *Fix* $n \geq 1$*, and let* $\mathcal{Z} \subset \mathcal{E}_0$ *denote the set of points* x *satisfying one of the following conditions:*

1. *(1)* t_x *is absolutely reducible;*
2. *(2)* $t_x = \mathrm{tr}\,\rho_x$ *for an absolutely irreducible representation* $\rho_x : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$*, and the Zariski closure of the image of* ρ_x *does not contain* SL_2.
3. *(3) There exists a prime* $l|N$ *such that* $t_x|_{G_{\mathbf{Q}_l}} = \chi_1 + \chi_2$ *for characters* $\chi_i : G_{\mathbf{Q}_l} \to \overline{\mathbf{Q}}_p^\times$ *such that* $(\chi_1/\chi_2)^i = 1$ *for some* $i = 1, \ldots, n-1$.

Then $\mathcal{Z}$ is Zariski closed.

Proof. The discussion in [Che14, §4.2] shows that the locus where t_x is absolutely reducible is Zariski closed. If $\rho_x : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is irreducible, then the Zariski closure of the image of ρ_x contains SL_2 if and only if $\mathrm{Sym}^6 \rho_x$ is irreducible. Indeed, the Zariski closure of the image of ρ_x contains SL_2 if and only if the Zariski closure G_x of the image of the associated projective representation $\mathrm{Proj}\, \rho_x : G_{\mathbf{Q}} \to \mathrm{PGL}_2(\overline{\mathbf{Q}}_p)$ is PGL_2. There are two possibilities for the group G_x, which is a (possibly disconnected) reductive group: the first is that it is finite, hence either dihedral or conjugate to one of A_4, S_4, or A_5. In any of these cases $\mathrm{Sym}^6 \rho_x$ is reducible. The next is that G_x has a non-trivial identity component, which therefore contains a maximal torus of PGL_2. The only possibilities are therefore that either G_x equals the normaliser of this maximal torus (in which case $\mathrm{Sym}^6 \rho_x$ is again reducible) or that $G_x = \mathrm{PGL}_2$ (in which case $\mathrm{Sym}^6 \rho_x$ is irreducible).

This shows that the set $\mathcal{Z}_{12}$ of points satisfying conditions (1) or (2) of the Lemma is Zariski closed. Finally, if $l|N$ and $i = 1, \dots, n-1$, let $\mathcal{Z}_{3,l,i}$ denote the set of points x such that $t_x|_{G_{\mathbf{Q}_l}} = \chi_1 + \chi_2$ for some characters χ_1, χ_2 such that $(\chi_1/\chi_2)^i = 1$. It remains to show that $\mathcal{Z}_{3,l,i}$ is Zariski closed. Its complement is the set of points such that either $t_x|_{G_{\mathbf{Q}_l}}$ is absolutely irreducible, or $t_x|_{G_{\mathbf{Q}_l}}$ is absolutely reducible and there exists $g \in G_{\mathbf{Q}_l}$ such that the discriminant of the characteristic polynomial of g^i under the pseudocharacter t_x is non-zero. This is a union of Zariski open sets. □

3. Ping pong

In this section we use the rigid analytic results of §2 to prove the following theorem. We recall that we say that an automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ has "weight k" for an integer $k \geq 2$ if π_∞ has the same infinitesimal character as the dual of the algebraic representation $\mathrm{Sym}^{k-2} \mathbf{C}^2$.

Theorem 3.1. *Fix an integer $n \geq 2$. Let π_0 be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ which is everywhere unramified and of weight k, for some $k \geq 2$. Suppose that $\mathrm{Sym}^{n-1} r_{\pi_0,\iota}$ is automorphic for some (equivalently, any) prime p and isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. Then for any everywhere unramified cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ of weight $l \geq 2$, $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ is automorphic.*

To prove Theorem 3.1, we will use the properties of the eigencurve $\mathcal{E}_0$, as defined in §2.31. More precisely, we henceforth let $p = 2$, $N = 1$, and let $\mathcal{E}_0$ denote the eigencurve defined with respect to this particular choice of parameters. We fix an isomorphism $\iota : \overline{\mathbf{Q}}_2 \to \mathbf{C}$. $\mathcal{E}_0$ is supported on the connected component $\mathcal{W}_0^+ \subset \mathcal{W}_0$ defined by $\chi(-1) = 1$. We write $\chi^u : \mathbf{Z}_2^\times \to \mathcal{O}(\mathcal{W}_0)$ for the universal character. We have the following explicit result of Buzzard and Kilford on the geometry of the morphism $\kappa : \mathcal{E}_0 \to \mathcal{W}_0^+$ and the slope map $s : \mathcal{E}_0(\overline{\mathbf{Q}}_p) \to \mathbf{R}$:

Theorem 3.2. *Let $w \in \mathcal{O}(\mathcal{W}_0)$ denote the function $\chi^u(5) - 1$. Then:*

(1) *w restricts to an isomorphism between $\mathcal{W}_0^+$ and the open unit disc $\{|w| < 1\}$.*
(2) *Let $\mathcal{W}_0(b) \subset \mathcal{W}_0^+$ denote the open subset where $|8| < |w| < 1$, and let $\mathcal{E}_0(b) = \kappa^{-1}(\mathcal{W}_0(b))$. Then there is a decomposition $\mathcal{E}_0(b) = \sqcup_{i=1}^\infty X_i$ of $\mathcal{E}_0(b)$ as a countable disjoint union of admissible open subspaces such that for each $i \geq 1$, $\kappa|_{X_i} : X_i \to \mathcal{W}_0(b)$ is an isomorphism.*

(3) *For each $i = 1, 2, \ldots$, the map $s \circ \kappa|_{X_i}^{-1} : \mathcal{W}_0(b)(\overline{\mathbf{Q}}_p) \to X_i(\overline{\mathbf{Q}}_p) \to \mathbf{R}$ equals the map $iv_p \circ w$.*

Proof. This is almost the main theorem of [BK05], except that here we are using the cuspidal version of the eigencurve. However, if $\mathcal{E}_1$ denotes the full eigencurve used in [BK05], then there is a decomposition $\mathcal{E}_1 = \mathcal{E}_1^{ord} \sqcup \mathcal{E}_1^{non-ord}$ as a union of open and closed subspaces. This follows from the fact that the ordinary locus $\mathcal{E}_1^{ord}$ in the eigencurve can also be constructed using Hida theory (see [Pil13, §6]), so is finite over $\mathcal{W}_0$. Since $\mathcal{E}_1$ is separated the open immersion $\mathcal{E}_1^{ord} \hookrightarrow \mathcal{E}_1$ is therefore also finite, hence a closed immersion. In our particular case ($p = 2, N = 1$), we have $\mathcal{E}_1^{ord} \cong \mathcal{W}_0^+$ (the unique ordinary family is the family of Eisenstein series) and therefore $\mathcal{E}_1^{non-ord} = \mathcal{E}_0$, giving the statement we have here. See Lemma 7.4 of the (longer) arXiv version of [BC05] for an alternative argument.

We note as well that in our normalisation, the trivial character in $\mathcal{W}_0$ corresponds to forms of weight 2, whereas in the notation of [BK05], the character x^2 corresponds to forms of weight 2. However, this renormalisation does not change the region $\mathcal{W}_0(b)$. □

Before giving the proof of Theorem 3.1, we record some useful lemmas.

Lemma 3.3. *Let $z \in Z_0^{pc} \cap \mathcal{E}_0(b)$, and suppose $z \in X_i$. Let $z' = \tau(z)$ be the twin of z. Then $z \in X_{i'}$, where i' satisfies the relation $i + i' = (k-1)/v_p(w(z))$.*

Proof. By Lemma 2.32, z' lies in $\mathcal{E}_0(b)$, so in $X_{i'}$ for a unique integer $i' \geq 1$. Writing s, s' for the slopes of these two points and k for their weights, we have $k - 1 = s + s' = iv_p(w(z)) + i'v_p(w(z))$, hence $i + i' = (k-1)/v_p(w(z))$. □

As a sanity check, we observe that in the context of the proof of Lemma 3.3, $(k-1)/v_p(w(z))$ is always an integer. Indeed, $\kappa(z)$ satisfies $\kappa(z)(5) = 5^{k-2}\zeta_{2^m}$ for some $k \geq 2$ and $m \geq 0$. This weight lies in $\mathcal{W}_0(b)$ if and only if either k is odd, or k is even and $m \geq 1$. If $m \geq 1$, then $v_p(w(z)) = 2^{1-m}$. If $m = 0$ and k is odd, then $v_p(w(z)) = 2$. In either case we see that $(k-1)/v_p(w(z))$ is an integer.

Lemma 3.4. *Let π be an everywhere unramified cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of weight $k \geq 2$. Every accessible refinement of π is numerically non-critical and n-regular for every $n \geq 2$ (recall that we have fixed $p = 2$ and these notions refer to the local factor at 2, π_2).*

Proof. Numerical non-criticality of every refinement is immediate from the fact that there are no cusp forms of level 1 that are ordinary at 2. For regularity, if we fix a refinement $\chi = \chi_1 \otimes \chi_2$ then $\alpha = p^{1/2}\iota\chi_1(p)$ and $\beta = p^{1/2}\iota\chi_2(p)$ are the roots of the polynomial $X^2 - a_2X + 2^{k-1}$, with a_2 the T_2-eigenvalue of the level 1 weight k normalised eigenform f associated to π. We need to show that α/β is not a root of unity.

Suppose $\alpha/\beta = \zeta$ is a root of unity. If we fix $\iota_5 : \overline{\mathbf{Q}}_5 \cong \mathbf{C}$, the semisimplified mod 5 Galois representation $\overline{r}_{f,\iota_5}$ arises up to twist from a level 1 eigenform of weight ≤ 6 (i.e. the level 1 Eisenstein series of weight 4 or 6). This shows that $\iota_5^{-1}(\zeta) \equiv 2^3$ or 2^5 mod $\mathfrak{m}_{\overline{\mathbf{Z}}_5}$, and therefore ζ is the product of a 5-power root of unity and $\pm i$ (since 2 has order 4 in $\mathbf{F}_5^\times$). Applying a similar argument at the prime 7 with $\iota_7 : \overline{\mathbf{Q}}_7 \cong \mathbf{C}$, we see that $\iota_7^{-1}(\zeta) \equiv 2^3, 2^5$ or 2^7 mod $\mathfrak{m}_{\overline{\mathbf{Z}}_7}$ and therefore ζ is the product of a 7-power root of unity and a cube root of unity. This gives the desired contradiction. (We thank Fred Diamond for pointing out this argument to us, and thank an anonymous referee for explaining how to avoid using Hatada's congruence which appeared in the first version of this argument.) □

Lemma 3.5. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of weight $k \geq 2$. We temporarily let p be an arbitrary prime. Then:*

(1) $r_{\pi,\iota}|_{G_{\mathbf{Q}_p}}$ is reducible if and only if π is ι-ordinary.
(2) Suppose either that $r_{\pi,\iota}|_{G_{\mathbf{Q}_p}}$ is irreducible and π_p admits a 3-regular refinement, or that $k > 2$ and $r_{\pi,\iota}$ is not potentially crystalline. Then the Zariski closure of $r_{\pi,\iota}(G_{\mathbf{Q}_p})$ (in $\mathrm{GL}_2/\overline{\mathbf{Q}}_p$) contains SL_2.
(3) Suppose again that $p = 2$, and that π is everywhere unramified. Then the Zariski closure of $r_{\pi,\iota}(G_{\mathbf{Q}_2})$ contains SL_2.

Proof. For the first part, ι-ordinarity implies reducibility by local-global compatibility at p, as in [Tho15, Theorem 2.4]. For the converse, if $r_{\pi,\iota}|_{G_{\mathbf{Q}_p}}$ is reducible, then its Jordan–Hölder factors are de Rham characters of $G_{\mathbf{Q}_p}$ and therefore have the form $\psi_i \epsilon^{-k_i}$, where the ψ_i are $\overline{\mathbf{Z}}_p^\times$-valued characters with finite order restriction to inertia and the k_i are the Hodge–Tate weights (we can assume $k_1 = 0$ and $k_2 = k-1$ in our situation). The Weil representation part of $\mathrm{WD}(r_{\pi,\iota}|_{G_{\mathbf{Q}_p}})$ is therefore equal to $\psi_1 \oplus \psi_2 |\cdot|^{1-k} \circ \mathrm{Art}_{\mathbf{Q}_p}^{-1}$. Since $\mathrm{WD}(r_{\pi,\iota}|_{G_{\mathbf{Q}_p}}) = \mathrm{rec}^T_{\mathbf{Q}_p}(\iota^{-1}\pi_p)$, π_p is a subquotient of the normalised induction $i_{B_2}^{\mathrm{GL}_2}(\iota\psi_1 \circ \mathrm{Art}_{\mathbf{Q}_p}|\cdot|^{1/2} \otimes \iota\psi_2 \circ \mathrm{Art}_{\mathbf{Q}_p}|\cdot|^{3/2-k})$. It follows from [Tho15, Lemma 2.3] that π is ι-ordinary.

For the second part, we note that $r_{\pi,\iota}|_{G_{\mathbf{Q}_p}}$ is irreducible. Indeed, if $r_{\pi,\iota}|_{G_{\mathbf{Q}_p}}$ is reducible then the first part of the lemma shows that π is ι-ordinary, and [Tho15, Lemma 2.3] implies that π_p is a subquotient of $i_{B_2}^{\mathrm{GL}_2}\chi_1 \otimes \chi_2$ with $v_p\left(\iota^{-1}(\chi_1/\chi_2(p))\right) = 1-k$. If $r_{\pi,\iota}$ is not potentially crystalline then $\mathrm{WD}(r_{\pi,\iota}|_{G_{\mathbf{Q}_p}})$ has $N \neq 0$ and local-global compatibility implies that if π_p is a subquotient of $i_{B_2}^{\mathrm{GL}_2}\chi_1 \otimes \chi_2$, then $\chi_1/\chi_2 = |\cdot|^{\pm 1}$. This is a contradiction if $k > 2$.

Thus $r_{\pi,\iota}|_{G_{\mathbf{Q}_p}}$ is irreducible and the Zariski closure H of its image is a reductive subgroup of GL_2. Let T be a maximal torus of H. Since $r_{\pi,\iota}$ is Hodge–Tate regular, T is regular in GL_2 (i.e. its centralizer is a maximal torus of GL_2) by [Sen73, Theorem 1]. If H does not contain SL_2, then it is contained in the normaliser of a maximal torus of GL_2 and $r_{\pi,\iota}|_{G_{\mathbf{Q}_p}}$ is induced from a character of an index two subgroup. This forces $\mathrm{WD}(r_{\pi,\iota}|_{G_{\mathbf{Q}_p}})$ to likewise be induced, so any refinement $\chi = \chi_1 \otimes \chi_2$ of π_p satisfies $\chi_1^2 = \chi_2^2$, and this Weil–Deligne representation has $N = 0$. This is a contradiction, since we are assuming either that there exists a 3-regular refinement or that N is non-zero.

For the third part, we have already observed (see the proof of Theorem 3.2) that there are no cusp forms of level 1 that are ordinary at 2, so $r_{\pi,\iota}|_{G_{\mathbf{Q}_2}}$ is irreducible. Suppose that $r_{\pi,\iota}|_{G_{\mathbf{Q}_2}}$ is induced. Then $\mathrm{WD}(r_{\pi,\iota}|_{G_{\mathbf{Q}_2}}) = \mathrm{Ind}_{W_K}^{W_{\mathbf{Q}_2}} \psi$ for some quadratic extension $K/\mathbf{Q}_2$ and character $\psi : W_K \to \overline{\mathbf{Q}}_2^\times$. Since this Weil–Deligne representation must be unramified, we see that K and ψ are both unramified, and therefore that ψ extends to a character $\psi : W_{\mathbf{Q}_2} \to \overline{\mathbf{Q}}_2^\times$ such that $\mathrm{WD}(r_{\pi,\iota}|_{G_{\mathbf{Q}_2}}) = \psi \oplus (\psi \otimes \delta_{K/\mathbf{Q}_2})$. In particular, the T_2-eigenvalue (which equals the trace of Frobenius in this representation) is 0, but, as shown in the proof of Lemma 3.4, this is impossible. □

Lemma 3.6. *Let $i \geq 1$ be an integer, and let $z \in Z_0 \cap X_i(\overline{\mathbf{Q}}_p)$ be the point corresponding to a pair (π, χ). Suppose that χ is n-regular. Let $z' \in Z_0 \cap X_i(\overline{\mathbf{Q}}_p)$ be any other point, corresponding to a pair (π', χ'), with χ' n-regular. If $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ is automorphic, then $\mathrm{Sym}^{n-1} r_{\pi',\iota}$ is also.*

Proof. We can assume $n \geq 3$. We apply Theorem 2.33 (note that $X_{i,\mathbf{C}_p} \cong \mathcal{W}_0(b)_{\mathbf{C}_p}$ is irreducible): since there are no ordinary points in $\mathcal{E}_0$, χ is numerically non-critical and Lemma 3.5 implies that the Zariski closures of $r_{\pi,\iota}(G_{\mathbf{Q}_p})$ and $r_{\pi',\iota}(G_{\mathbf{Q}_p})$ contain SL_2 (the first part of the lemma shows that we are in the locally irreducible and 3-regular case of the second part of the lemma). $\square$

Lemma 3.7. *Let π be a cuspidal, everywhere unramified automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of weight $k \geq 2$, and let χ be a choice of accessible refinement. Then there exists an integer $m_\pi \geq 1$ such that for any integer $m \geq m_\pi$, we can find a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ satisfying the following conditions:*

(1) π' is unramified outside 2.
(2) π' admits two accessible refinements of distinct slopes (in particular these refinements are n-regular for every $n \geq 2$).
(3) There is an accessible refinement χ' of π' such that (π, χ) and (π', χ') define points z, z' on the same irreducible component of $\mathcal{E}_{0,\mathbf{C}_p}$.
(4) $\kappa(z') \in \mathcal{W}_0(b)$. In particular, $z' \in X_i$ for some $i \geq 1$ (notation as in the statement of Theorem 3.2).
(5) Set $z'' = \tau(z')$, and let $(\pi'', \chi'') \in Z_0^{pc}$ be the associated pair.[6] Then $z'' \in X_{2^m-1}$.
(6) The automorphy of any one of the three representations

$$\mathrm{Sym}^{n-1} r_{\pi,\iota}, \mathrm{Sym}^{n-1} r_{\pi',\iota}, \mathrm{Sym}^{n-1} r_{\pi'',\iota}$$

implies automorphy of all three.

Proof. We use Theorem 3.2. Extending E if necessary, we may assume that $z \in \mathcal{E}_0(E)$ and every irreducible component of $\mathcal{E}_0$ containing z is geometrically irreducible. Fix one of these irreducible components and fix i such that this irreducible component contains X_i (such an i exists, because every irreducible component of $\mathcal{E}_0$ has Zariski open image in $\mathcal{W}_0^+$, hence intersects $\mathcal{E}_0(b)$ and therefore contains a non-empty union of irreducible components of $\mathcal{E}_0(b)$). We define m_π to be least integer $m_\pi \geq 1$ satisfying the inequality

$$(2i + 2^{m_\pi+1} - 3)/2 > 2i.$$

Given $m \geq m_\pi$, we choose $z' \in X_i(\overline{\mathbf{Q}}_p)$ to be the point such that $\kappa(z')(5) = 5^{k'-2}$, where $k' = 2i + 2^{m+1} - 1$. Then $(k'-2)/2 > 2i = s(z')$. By Coleman's classicality criterion, $z' \in Z_0$. If z' was not in Z_0^{pc}, its slope would be $(k'-2)/2$. So $z' \in Z_0^{pc}$ and if (π', χ') denotes the corresponding pair, then the two accessible refinements of π' have distinct slopes ($2i$ and $k' - 1 - 2i$).

Let $z'' = \tau(z')$ denote the twin point, and (π'', χ'') the corresponding pair. Then z'' lies on $X_{i'}$, where $i' = (k'-1)/2 - i = 2^m - 1$, by Lemma 3.3. We're done: the first 5 properties of (π', χ') follow by construction, whilst the 6th follows from Theorem 2.33, Lemma 3.4 and Lemma 3.5. $\square$

[6] Bouncing from z' to its twin point z'' reminded the authors of a game of ping pong, whence the section title. Earlier versions of the argument involved longer rallies!

We can now complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let π_0, π_0' be everywhere unramified cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A_Q})$ of weights $k_0, k_0' \geq 2$, respectively. Define integers $m_{\pi_0}, m_{\pi_0'}$ as in Lemma 3.7 and fix an integer $m \geq \max(m_{\pi_0}, m_{\pi_0'})$. Combining Lemma 3.7 and Lemma 3.6 (applied with $i = 2^m - 1$) we see that the automorphy of $\mathrm{Sym}^{n-1} r_{\pi_0,\iota}$ implies that of $\mathrm{Sym}^{n-1} r_{\pi_0',\iota}$. Since π_0' was arbitrary, this completes the proof. □

Part II: Raising the level

Most of the remainder of this paper (§§4 – 7) is devoted to the proof of Theorem E from the introduction, namely the existence for each $n \geq 2$ of a single regular algebraic, cuspidal, everywhere unramified automorphic representation π of $\mathrm{GL}_2(\mathbf{A_Q})$ such that $\mathrm{Sym}^{n-1} \pi$ exists. As a guide to what follows, we now give an expanded sketch of the proof of this theorem.

Fix, for the sake of argument, a regular algebraic, cuspidal, everywhere unramified automorphic representation π of $\mathrm{GL}_2(\mathbf{A_Q})$. We will try to establish the existence of $\mathrm{Sym}^{n-1} \pi$ by proving the automorphy of one of the Galois representations $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ associated to a choice of prime p and isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, using an automorphy lifting theorem. First, if $K/\mathbf{Q}$ is an imaginary quadratic extension then we can find (using e.g. [BLGGT14, Lemma A.2.5]) a (de Rham) character $\omega : G_K \to \overline{\mathbf{Q}}_p^\times$ such that $\omega\omega^c = (\det r_{\pi,\iota} \epsilon^{-1})^{n-1}$. Then the representation $\rho = \omega \otimes \mathrm{Sym}^{n-1} r|_{G_K}$ satisfies $\rho^c \cong \rho^\vee \otimes \epsilon^{1-n}$, so has the potential to be associated to a RACSDC automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$. This means we can use an automorphy lifting theorem adapted to such automorphic representations. (The automorphy of ρ will imply that of $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ by quadratic descent.)

We need to select π and ι so that the residual representation $\overline{\rho}$ is automorphic. For "most" ι (say, for all but finitely many primes p) the image of $\overline{r}_{\pi,\iota}$ will contain a conjugate of $\mathrm{SL}_2(\mathbf{F}_p)$ and $\mathrm{Sym}^{n-1} \overline{\rho}$ will be irreducible, and it is not clear how to proceed. We therefore want to avoid this generic case. Here we choose π and ι so that there is an isomorphism $\overline{r}_{\pi,\iota} \cong \mathrm{Ind}_{G_K}^{G_\mathbf{Q}} \overline{\chi}$ for some imaginary quadratic extension $K/\mathbf{Q}$ and character $\overline{\chi} : G_K \to \overline{\mathbf{F}}_p^\times$. Then there is an isomorphism

$$\overline{\rho} \cong \oplus_{i=1}^n \overline{\omega \chi}^{n-i} (\overline{\chi}^c)^{i-1}.$$

In particular, this residual representation is highly reducible, being a sum of n characters. Most automorphy lifting theorems in the literature require the residual representation to be irreducible; we will apply [ANT20, Theorem 1.1], an automorphy lifting theorem that does not have this requirement, but that does have some other stringent conditions. These conditions include the requirement that there exist a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_K)$ such that $\overline{r}_{\Pi,\iota} \cong \overline{\rho}$, and satisfying the following:

- Π is ι-ordinary (and so is π).
- There exists a prime $l \neq p$ and a place $v|l$ of K such that both π_l and Π_v are twists of the Steinberg representation (of $\mathrm{GL}_2(\mathbf{Q}_l)$ and $\mathrm{GL}_n(K_v)$, respectively).

It is easy to arrange that the first requirement be satisfied, by choosing p to be a prime which splits in K. The second is more difficult. First it requires that π is ramified at l, whereas we have to this point asked for π to be everywhere unramified. We will thus first find a ramified π for which $\mathrm{Sym}^{n-1}\pi$ exists, and eventually remove the primes of ramification using the l-adic analytic continuation of functoriality results proved in the first part of the paper. The main problem is then to find a Π verifying the residual automorphy of $\overline{\rho}$ such that Π_v is a twist of the Steinberg representation. This is what will occupy us in §§4 – 6 below. The above argument is then laid out carefully in §7 in order to finally prove Theorem E.

Here is how we get our hands on Π. By choosing an appropriate lift of the character $\overline{\chi}$, we can choose characters $X_1, \dots, X_n : K^\times \backslash \mathbf{A}_K^\times \to \mathbf{C}^\times$ such that $\Pi_0 = X_1 \boxplus \cdots \boxplus X_n$ is a regular algebraic and conjugate self-dual (although not cuspidal!) automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$ whose associated residual representation is $\overline{\rho}$. If G is a definite unitary group in n variables associated to the extension $K/\mathbf{Q}$, quasi-split at finite places, then we can hope that Π_0 transfers to an automorphic representation of $G(\mathbf{A}_\mathbf{Q})$. There is a slight wrinkle here in that such a group G does not exist if n is even, and even in the case that n is odd there is a potential obstruction to the existence of this transfer given, at least conjecturally, by Arthur's multiplicity formula. Both of these obstacles can be avoided by replacing $\mathbf{Q}$ with a suitable soluble totally real extension $F/\mathbf{Q}$. In order to avoid introducing additional notation in this sketch, we pretend they can be dealt with already in the case $F = \mathbf{Q}$. (Actually, we will find it convenient to take Π_0 to be the box sum of two cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_K)$ and $\mathrm{GL}_{n-2}(\mathbf{A}_K)$, respectively. This means that the final form of the proof of Theorem E will be a kind of induction on n.)

We thus find ourselves with an automorphic representation Σ_0 of $G(\mathbf{A}_\mathbf{Q})$, whose base change (in the sense of Theorem 1.2) is Π_0. Say for the sake of argument that l splits in K, so that we can identify $G(\mathbf{Q}_l)$ with $\mathrm{GL}_n(K_v)$. If we can find another automorphic representation Σ_1 of $G(\mathbf{A}_\mathbf{Q})$, congruent modulo p to Σ_0 and such that $\Sigma_{1,v}$ is a twist of the Steinberg representation, then we will have solved our problem. We can therefore now focus on this problem of level-raising for the definite unitary group G.

There are differing approaches to this problem in the literature. First there is the purely automorphic approach, pioneered by Ribet for $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ [Rib84]. Some generalisations to higher rank groups of this statement do exist (see for example [Tho14a]), but nothing that is applicable in the level of generality considered here. Then there is the purely Galois theoretic approach, based on the powerful automorphy lifting theorems which are now available for Galois representations in arbitrary rank (see for example [Gee11]). We can not directly apply such results here because the only automorphy lifting theorems applicable in the residually reducible case (namely those of [ANT20]) require the existence of at least one place at which the starting automorphic representation is sufficiently non-degenerate.

We solve the problem here by combining aspects of these approaches. A similar combination of techniques is used in the paper [CT17]: the idea is to first use an automorphic technique to replace Σ_0 with a representation Σ_0' such that $\Sigma_{0,v}'$ is so ramified that, in conjunction with other conditions in place, Σ_0' is forced to be stable (i.e. its base change is cuspidal). (The possibility of doing this is the reason for choosing Π_0 to in fact be a box sum of two cuspidal factors, as mentioned above.) This situation is reflected in the deformation theory, where one finds that (in the big ordinary Galois deformation ring) the locus of reducible deformations is small enough that something like the techniques of [Gee11] can be applied to construct an automorphic lift of $\overline{\rho}$ with the required local properties. These Galois theoretic arguments are carried out in §§5, 6.

What remains to be explained then is the automorphic level-raising technique developed in §4. The approach to creating congruences here is based on types. We recall (using the language of [BK98]) that if $\mathfrak{s}$ is an inertial equivalence class of $G(\mathbf{Q}_l)$ (i.e. a supercuspidal representation of a Levi factor of $G(\mathbf{Q}_l)$, up to unramified twist), then an $\mathfrak{s}$-type is a pair (U, τ), where U is an open compact subgroup of $G(\mathbf{Q}_l)$ and τ is an irreducible finite-dimensional representation of U such that for each irreducible admissible representation σ_v of $G(\mathbf{Q}_l)$, the supercuspidal support of σ_v is in class $\mathfrak{s}$ if and only if $\sigma_v|_U$ contains τ.

It can sometimes happen that two inertial equivalence classes $\mathfrak{s}, \mathfrak{s}'$ admit types (U, τ) and (U', τ') with the property that $U = U'$, the reduction modulo p $\overline{\tau}$ of $\iota^{-1}\tau$ is irreducible, and the reduction modulo p $\overline{\tau}'$ of $\iota^{-1}\tau'$ contains $\overline{\tau}$ as a Jordan–Hölder factor. This situation might be called a congruence of types. If this is the case then the theory of algebraic modular forms implies that any automorphic representation Σ of $G(\mathbf{A}_{\mathbf{Q}})$ such that Σ_l is of type $\mathfrak{s}$ is congruent to another Σ' such that Σ'_l is of type $\mathfrak{s}'$. The existence of such global congruences is explained in [Vig01, §3]. It gives an efficient way to construct congruences between automorphic representations Σ, Σ' such that Σ_l, Σ'_l are in different inertial equivalence classes, although it is not usually possible to change the Levi subgroup underlying the inertial equivalence class. Since $G(\mathbf{Q}_l) \cong \mathrm{GL}_n(K_v)$ and the initial representation Σ_0 is certainly not supercuspidal at l, it is not immediately clear how to use this.

We therefore instead introduce an auxiliary imaginary quadratic extension $E/\mathbf{Q}$ in which l is inert, as well as an associated definite unitary group G', and carry out the first step of the automorphic part of the level-raising argument using algebraic modular forms on G'. The importance of the group G' is that there are conjugate self-dual irreducible admissible representations of $\mathrm{GL}_3(E_l)$ which are not supercuspidal, but for which the associated L-packets of representations of $U_3(\mathbf{Q}_l)$ contain supercuspidal elements. For carefully chosen local data, we can find use the method of types to find congruences to supercuspidal representations of $U_3(\mathbf{Q}_l)$ whose base change to $\mathrm{GL}_3(E_l)$ is supercuspidal. We have already constructed such congruences of types in §1.8. In terms of automorphic representations of $\mathrm{GL}_n(\mathbf{A}_E)$, this will allow us to change the Levi subgroup underlying the inertial equivalence class at l from the maximal torus of $\mathrm{GL}_n(E_l)$ to the group $\mathrm{GL}_3 \times \mathrm{GL}_1^{n-3}$. This will be enough for our intended application.

4. Raising the level – automorphic forms

Let $n = 2k+1 \geq 3$ be an odd integer, and let $F, p, S, G = G_n$ etc. be as in our standard assumptions (see §1). Suppose given cuspidal, conjugate self-dual automorphic representations π_2 of $\mathrm{GL}_2(\mathbf{A}_F)$ and π_{n-2} of $\mathrm{GL}_{n-2}(\mathbf{A}_F)$ with the following properties:

(1) $\pi = \pi_2 \boxplus \pi_{n-2}$ is regular algebraic.

Consequently, $\pi_2|\cdot|^{(2-n)/2}$ and π_{n-2} are regular algebraic and the representations $\overline{\rho}_2 = \overline{r}_{\pi_2|\cdot|^{(2-n)/2},\iota}$, $\overline{\rho}_{n-2} = \overline{r}_{\pi_{n-2}|\cdot|^{-1},\iota}$ are defined. We set $\overline{\rho} = \overline{r}_{\pi,\iota} = \overline{\rho}_2 \oplus \overline{\rho}_{n-2}$. Moreover, π_2 and π_{n-2} are both tempered, cf. the remark after Corollary 1.3.

(2) π is ι-ordinary.

(3) We are given disjoint, non-empty sets T_1, T_2, T_3 of places of F^+ with the following properties:
   (a) For all $v \in T = T_1 \cup T_2 \cup T_3$, $v \notin S$ and q_v is odd. The representation π is unramified outside $S \cup T$. If $\widetilde{v}$ is a place of F lying above a place in T then, as in §§1.8, 1.17, we write $\omega(\widetilde{v}) : k(\widetilde{v})^\times \to \mathbf{C}^\times$ for the unique quadratic character.
   (b) For each $v \in T_1$, v is inert in F, $q_v \bmod p$ is a primitive 6^{th} root of unity, and the characteristic of $k(v)$ is greater than n. There are characters $\chi_{\widetilde{v}}, \chi_{\widetilde{v},0}, \chi_{\widetilde{v},1}, \ldots, \chi_{\widetilde{v},2k-2} : F_{\widetilde{v}}^\times \to \mathbf{C}^\times$ such that $\chi_{\widetilde{v}}, \chi_{\widetilde{v},0}$ are unramified and for each $i = 1, \ldots, 2k-2$, $\chi_{\widetilde{v},i}|_{\mathcal{O}_{F_{\widetilde{v}}}^\times} = \omega(\widetilde{v})$. We have $\pi_{2,\widetilde{v}} \cong \mathrm{St}_2(\chi_{\widetilde{v}})$ and $\pi_{n-2,\widetilde{v}} \cong \boxplus_{i=0}^{2k-2} \chi_{\widetilde{v},i}$.
   (c) For each $v \in T_2$, v splits $v = \widetilde{v}\widetilde{v}^c$ in F, $q_v \bmod p$ is a primitive 2^{nd} root of unity, $\pi_{2,\widetilde{v}}|_{\mathrm{GL}_2(\mathcal{O}_{F_{\widetilde{v}}})}$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}})$ (for some order p character $\Theta_{\widetilde{v}}$ as in §1.17, with $n_1 = 2$), and $\pi_{n-2,\widetilde{v}}|_{\mathrm{GL}_{n-2}(\mathcal{O}_{F_{\widetilde{v}}})}$ contains $\omega(\widetilde{v}) \circ \det$. Thus $\pi_{\widetilde{v}}$ satisfies the equivalent conditions of Proposition 1.20, and $\pi_{\widetilde{v}}|_{\mathfrak{q}_{\widetilde{v}}}$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}}, n)$.
   (d) For each $v \in T_3$, v splits $v = \widetilde{v}\widetilde{v}^c$ in F, $q_v \bmod p$ is a primitive $(n-2)^{\text{th}}$ root of unity, $\pi_{n-2,\widetilde{v}}|_{\mathrm{GL}_{n-2}(\mathcal{O}_{F_{\widetilde{v}}})}$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}})$ (for some order p character $\Theta_{\widetilde{v}}$ as in §1.17, with $n_1 = n-2$), and $\pi_{2,\widetilde{v}}|_{\mathrm{GL}_2(\mathcal{O}_{F_{\widetilde{v}}})}$ contains $\omega(\widetilde{v}) \circ \det$. Thus $\pi_{\widetilde{v}}$ satisfies the equivalent conditions of Proposition 1.20, and $\pi_{\widetilde{v}}|_{\mathfrak{q}_{\widetilde{v}}}$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}}, n)$.

Let $\widetilde{T} = \{\widetilde{v} \mid v \in T\}$. We fix for each $v \in T_1$ a character $\theta_v : C(k(v)) \to \mathbf{C}^\times$ of order p (notation as in Proposition 1.12). In the rest of this section, we will prove the following theorem.

Theorem 4.1. *With hypotheses as above, let L^+/F^+ be a totally real $S \cup T$-split quadratic extension, and let $L = L^+F$. Then there exists a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_L)$ with the following properties:*

(1) Π is ι-ordinary, and unramified at any place not dividing $S \cup T$.
(2) $\overline{r}_{\Pi,\iota} \cong \overline{r}_{\pi,\iota}|_{G_L}$.
(3) For each place $v \in T_{1,L}$, $\Pi_{\widetilde{v}}|_{\mathfrak{r}_{\widetilde{v}}}$ contains the representation $\widetilde{\lambda}(\widetilde{v}, \widetilde{\theta}_{v|_{F^+}}, n)|_{\mathfrak{r}_{\widetilde{v}}}$ (thus satisfying the equivalent conditions of Proposition 1.19).
(4) For each place $v \in T_{2,L} \cup T_{3,L}$, $\Pi_{\widetilde{v}}|_{\mathfrak{q}_{\widetilde{v}}}$ contains the representation $\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}|_F}, n)$ (thus satisfying the equivalent conditions of Proposition 1.20 with $n_2 = 2$ if $v \in T_{2,L}$ and $n_2 = n-2$ if $v \in T_{3,L}$).

Remark 4.2. The places $v \in T_2 \cup T_3$ play a role in ensuring that Π is cuspidal (using Lemma 4.5 below). Our set-up is adapted to the proof of Proposition 7.4, which uses an induction on the dimension to construct automorphic representations with an unramified twist of Steinberg local factor which are congruent to some very special odd-dimensional symmetric powers.

We begin with two important observations.

Lemma 4.3. *Let v be a finite place of F^+ which is inert in F. Then $\pi_{\widetilde{v}} \in \mathcal{A}_t^\theta(\mathrm{GL}_n(F_{\widetilde{v}}))_+$.*

Proof. The representation $\pi_{\widetilde{v}}$ is tempered because both $\pi_{2,\widetilde{v}}$ and $\pi_{n-2,\widetilde{v}}$ are tempered. By the main theorem of [BC11], $r_{\pi,\iota}$ extends to a homomorphism $r : G_{F^+} \to \mathcal{G}_n(\overline{\mathbf{Q}}_p)$

such that $\nu \circ r = \epsilon^{1-n}\delta^n_{F/F^+}$. Restricting to $W_{F_v^+}$ and twisting by an appropriate character, we see that the Langlands parameter of $\pi_{\widetilde{v}}$ extends to a parameter $W_{F_v^+} \times \mathrm{SL}_2 \to {}^L G$. $\square$

Remark 4.4. A consequence of this lemma is that for $v \in T_1$ the character $\chi_{\widetilde{v}}$ is non-trivial quadratic and the character $\chi_{\widetilde{v},0}$ is trivial.

Lemma 4.5. *Suppose given a partition $n = n_1 + \cdots + n_r$ and cuspidal, conjugate self-dual automorphic representations $\pi'_1, \ldots, \pi'_r$ of $\mathrm{GL}_{n_i}(\mathbf{A}_F)$ such that $\pi' = \pi'_1 \boxplus \cdots \boxplus \pi'_r$ is regular algebraic. Suppose moreover that the following conditions are satisfied:*

(1) *There is an isomorphism $\overline{r}_{\pi',\iota} \cong \overline{r}_{\pi,\iota}$.*
(2) *If $v \in T_2 \cup T_3$ then $\pi'_{\widetilde{v}}|_{\mathfrak{q}_{\widetilde{v}}}$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}}, n)$.*

Then one of the following two statements holds:

(1) *We have $r = 1$, $n_1 = n$, and so π' is cuspidal.*
(2) *After re-ordering we have $r = 2$, $n_1 = n - 2$, $n_2 = 2$. If $v \in T_2$ then $\pi'_{1,\widetilde{v}}|_{\mathrm{GL}_{n-2}(\mathcal{O}_{F_{\widetilde{v}}})}$ contains $\omega(\widetilde{v}) \circ \det$ and $\pi'_{2,\widetilde{v}}|_{\mathrm{GL}_2(\mathcal{O}_{F_{\widetilde{v}}})}$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}})$, while if $v \in T_3$ then $\pi'_{1,\widetilde{v}}|_{\mathrm{GL}_{n-2}(\mathcal{O}_{F_{\widetilde{v}}})}$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}})$ and $\pi'_{2,\widetilde{v}}|_{\mathrm{GL}_2(\mathcal{O}_{F_{\widetilde{v}}})}$ contains $\omega(\widetilde{v}) \circ \det$. We have isomorphisms of semisimplified residual representations $\overline{r}_{\pi'_1|\cdot|^{-1},\iota} \cong \overline{\rho}_{n-2}$ and $\overline{r}_{\pi'_2|\cdot|^{(2-n)/2},\iota} \cong \overline{\rho}_2$.*

Proof. Before beginning the proof, we observe that the representations $\overline{\rho}_2$, $\overline{\rho}_{n-2}$ have the following properties:

- If $v \in T_2$, then $\overline{\rho}_2|^{ss}_{G_{F_{\widetilde{v}}}}$ is unramified and $\overline{\rho}_{n-2}|^{ss}_{G_{F_{\widetilde{v}}}}$ is unramified after twisting by a ramified quadratic character. (The character $\Theta_{\widetilde{v}}$ has order p.)
- If $v \in T_3$, then $\overline{\rho}_2|^{ss}_{G_{F_{\widetilde{v}}}}$ is unramified after twisting by a ramified quadratic character and $\overline{\rho}_{n-2}|^{ss}_{G_{F_{\widetilde{v}}}}$ is unramified.

We can suppose without loss of generality that $r \geq 2$. Fix places $v_2 \in T_2, v_3 \in T_3$. By Proposition 1.20, we can assume after relabelling that $\pi'_{1,\widetilde{v}_3}|_{\mathrm{GL}_{n_1}(\mathcal{O}_{F_{\widetilde{v}_3}})}$ contains $\widetilde{\lambda}(\widetilde{v}_3, \Theta_{\widetilde{v}_3}, n_1)$ (and in particular, $n_1 \geq n-2$), in which case $(\pi'_{2,\widetilde{v}_3} \boxplus \cdots \boxplus \pi'_{r,\widetilde{v}_3})|_{\mathrm{GL}_{n-n_1}(\mathcal{O}_{F_{\widetilde{v}_3}})}$ contains $\omega(\widetilde{v}_3) \circ \det$. There is an isomorphism

$$\overline{r}_{\pi',\iota} = \oplus_{i=1}^r \overline{r}_{\pi'_i|\cdot|^{(n_i-n)/2},\iota} \cong \overline{\rho}_2 \oplus \overline{\rho}_{n-2},$$

hence

$$\oplus_{i=1}^r \overline{r}_{\pi'_i|\cdot|^{(n_i-n)/2},\iota}|^{ss}_{G_{F_{\widetilde{v}_3}}} \cong \overline{\rho}_2|^{ss}_{G_{F_{\widetilde{v}_3}}} \oplus \overline{\rho}_{n-2}|^{ss}_{G_{F_{\widetilde{v}_3}}}.$$

Since $\oplus_{i=2}^r \overline{r}_{\pi'_i|\cdot|^{(n_i-n)/2},\iota}|^{ss}_{G_{F_{\widetilde{v}_3}}}$ contains no unramified subrepresentation, we conclude that $\overline{r}_{\pi'_1|\cdot|^{(n_1-n)/2},\iota}$ contains $\overline{\rho}_{n-2}$ as a subrepresentation.

We now look at the place v_2. There are two possibilities for the representation $\pi'_{1,\widetilde{v}_2}$: either $\pi'_{1,\widetilde{v}_2}|_{\mathrm{GL}_{n_1}(\mathcal{O}_{F_{\widetilde{v}_2}})}$ contains $\widetilde{\lambda}(\widetilde{v}_2, \Theta_{\widetilde{v}_2}, n_1)$, or it contains $\omega(\widetilde{v}_2) \circ \det$. We claim that the first possibility does not occur. Indeed, in this case arguing as above shows that $\oplus_{i=2}^r \overline{r}_{\pi'_i|\cdot|^{(n_i-n)/2},\iota}|^{ss}_{G_{F_{\widetilde{v}_2}}}$ contains no unramified subrepresentation, and therefore that $\overline{\rho}_2$ is a subrepresentation of $\overline{r}_{\pi'_1|\cdot|^{(n_1-n)/2},\iota}$. This forces $n_1 = n$ and $r = 1$, a contradiction. Therefore we must have $r = 2$, $n_2 = 2$, and $\pi'_{2,\widetilde{v}_2}|_{\mathrm{GL}_2(\mathcal{O}_{F_{\widetilde{v}_2}})}$ contains $\widetilde{\lambda}(\widetilde{v}_2, \Theta_{\widetilde{v}_2})$. Since v_2, v_3 were arbitrary, this completes the proof. $\square$

We now commence the proof of the theorem. Let $U = \prod_v U_v \subset G(\mathbf{A}^\infty_{F^+})$ be an open compact subgroup with the following properties:

- For each $v \in S$, $\pi_{\widetilde{v}}^{\iota_{\widetilde{v}}(U_v)} \neq 0$.
- If $v \notin S \cup T$, then $U_v = G(\mathcal{O}_{F_v^+})$.
- If $v \in S_p$, then $U_v = \iota_{\widetilde{v}}^{-1}\mathrm{Iw}_{\widetilde{v}}(c,c)$ for some $c \geq 1$ such that $\pi_{\widetilde{v}}^{\mathrm{Iw}_{\widetilde{v}}(c,c),ord} \neq 0$ (notation as in [Ger19, §5.1]) and U_v contains no non-trivial torsion elements (note this implies that U is sufficiently small).
- If $v \in T_1$ then $U_v = \iota_v^{-1}\mathfrak{p}_v$ (notation as in §1.8).
- If $v \in T_2 \cup T_3$ then $U_v = \iota_{\widetilde{v}}^{-1}\mathfrak{q}_{\widetilde{v}}$ (notation as in §1.17, defined with $n_1 = 2$ if $v \in T_2$ and $n_1 = n-2$ if $v \in T_3$).

We define $\tau_g = \otimes_{v \in T_1}\tau(v,n) \otimes_{v \in T_2 \cup T_3} \widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}}, n)$, where $\tau(v,n)$, $\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}}, n)$ are the representations of $\mathfrak{p}_v$, $\mathfrak{q}_{\widetilde{v}}$ defined in §1.8, §1.17 respectively. Thus τ_g is an irreducible $\mathbf{C}[U_T]$-module, which we view as a $\mathbf{C}[U]$-module by projection to the T-component. Similarly we define $\lambda_g = \otimes_{v \in T_1}\lambda(v, \theta_v, n) \otimes_{v \in T_2 \cup T_3} \widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}}, n)$. Fixing a sufficiently large coefficient field, we can choose $\mathcal{O}$-lattices $\mathring{\tau}_g$ and $\mathring{\lambda}_g$ in $\iota^{-1}\tau_g$ and $\iota^{-1}\lambda_g$, respectively.

If L^+/F^+ is an $S \cup T$-split totally real quadratic extension, then we define an open compact subgroup $U_L = \prod_v U_{L,v} \subset G(\mathbf{A}_{L^+}^\infty)$ and representations $\tau_{g,L}$, $\lambda_{g,L}$ by the same recipe (where we now replace the sets S, T_i by their lifts S_L, $T_{i,L}$ to L^+).

Proposition 4.6. *Let L^+/F^+ be an $S \cup T$-split totally real quadratic extension and let $L = L^+F$. Then either there exists an automorphic representation σ of $G(\mathbf{A}_{F^+})$ with the following properties:*

1. *π is the base change of σ (cf. Theorem 1.2);*
2. *For each place $v \notin T$, $\sigma_v^{U_v} \neq 0$;*
3. *$\sigma_T|_{U_T}$ contains τ_g.*

or there exists an automorphic representation σ of $G(\mathbf{A}_{L^+})$ with the following properties:

1. *Let π_L denote the base change of π with respect to the quadratic extension L/F. Then π_L is the base change of σ;*
2. *For each place $v \notin T_L$, $\sigma_v^{U_{L,v}} \neq 0$;*
3. *$\sigma_{T_L}|_{U_{L,T_L}}$ contains $\tau_{g,L}$.*

Proof. By [Lab11, Théorème 5.1], there is an identity

$$T_{disc}^G(f) = \sum_H \iota(G,H) T_{disc}^{\widetilde{M}^H}(\widetilde{f}^H), \tag{4.6.1}$$

for any $f = f^\infty \otimes f_\infty \in C_c^\infty(G(\mathbf{A}_{F^+}))$ such that f_∞ is a pseudocoefficient of discrete series. Here the sum on the right-hand side is over representatives for equivalence classes of endoscopic data for G, represented here by the associated endoscopic group H (recall that we have fixed representative endoscopic triples in §1). The coefficients $\iota(G,H)$ are given in [Lab11, Proposition 4.11], while the expression $T_{disc}^{\widetilde{M}^H}(\widetilde{f}^H)$ is given in [Lab11, Proposition 3.4] as a formula

(4.6.2)
$$\sum_{L \in \mathcal{L}^0/W^{M^H}} \sum_{s \in W^{\widetilde{M}^H}(L)_{reg}} \sum_{\widetilde{\pi}^L \in \Pi_{disc}(\widetilde{L}_s)} (|\det(s-1 \mid \mathfrak{a}_L/\mathfrak{a}_{M^H})||W^{M^H}(L)|)^{-1} \operatorname{tr} I_Q(\widetilde{\pi}^L)(\widetilde{f}^H),$$

where (summarizing the notation of *op. cit.*):

- $\widetilde{M}^H$ is a twisted space on a Levi of $\mathrm{Res}_{F/F^+}\mathrm{GL}_n$, as in §1.5;
- $\mathcal{L}^0$ is the set of standard Levi subgroups of M^H;

- $W^{\widetilde{M}^H}(L)_{reg}$ is the quotient by the Weyl group W^L of the set of elements s in the twisted Weyl group $W^{\widetilde{M}^H} = W^{M^H} \rtimes \theta_{M^H}$ which normalise L and such that $\det(s-1 \mid \mathfrak{a}_L/\mathfrak{a}_{M^H}) \neq 0$, where $\mathfrak{a}_?$ denotes the Lie algebra of the maximal $\mathbf{Q}$-split subtorus of the centre of a reductive group.
- $\Pi_{disc}(\widetilde{L}_s)$ is the set of isomorphism classes of irreducible representations of the twisted space $\widetilde{L}_s(\mathbf{A}_{F^+})$ which appear as subrepresentations of the discrete spectrum of L.
- $I_Q(\widetilde{\pi}^L)(\widetilde{f}^H)$ is a certain intertwining operator, with Q a parabolic subgroup with Levi L.

We fix our choice of f_∞ so that it only has non-zero traces on representations of $G(F_\infty^+)$ whose infinitesimal character is related, by twisted base change, to that of π. The argument of [Shi11, Proposition 4.8] then shows that for each $L \in \mathcal{L}^0/W^{M^H}$, there is at most one element $s \in W^{\widetilde{M}^H}(L)_{reg}$ for which the corresponding summand in (4.6.2) can be non-zero (and a representative for s can be chosen which acts as conjugate inverse transpose on each simple factor of L). Using Proposition 1.7, linear independence of characters, the description of the discrete spectrum of general linear groups [MW89], and the Jacquet–Shalika theorem [JS81], we can combine (4.6.1) and (4.6.2) to obtain a refined identity

$$\sum_i m(\sigma_i)\sigma_i(f) = \frac{1}{2}\left(\widetilde{\pi}(\widetilde{f}) + (\pi_{n-2} \otimes (\pi_2 \otimes \mu^{-1} \circ \det))^{\sim}(\widetilde{f}^{U_{n-2}\times U_2})\right), \tag{4.6.3}$$

where:

- The sum on the left-hand side is over the finitely many automorphic representations σ_i of $G(\mathbf{A}_{F^+})$ which are unramified at all places below which π is unramified, have infinitesimal character related to that of π_∞ by twisted base change, and which are related to π by (either unramified or split) base change at places $v \notin T_1$ of F^+, each occurring with its multiplicity $m(\sigma_i)$.
- The twisted traces on the right-hand side are Whittaker-normalised. (These two terms arise from $H = U_n, L = \mathrm{Res}_{F/F^+} \mathrm{GL}_{n-2} \times \mathrm{GL}_2$ and $H = U_{n-2} \times U_2$, $L = M^H$, respectively. The same argument as in [Lab11, Proposition 3.7] shows that Arthur's normalisation of the twisted trace, implicit in the term $I_Q(?)$ of (4.6.2), agrees with the Whittaker normalisation on the corresponding terms.)

We remark that the representations π_2, π_{n-2} are tempered and that the representations $\sigma_i^{T_1}$ (i.e. prime to T_1-part) are isomorphic. If $v \in T_1$, then we can find (combining Proposition 1.11 for $U_{n-2} \times U_2$ and e.g. [Hir04, Proposition 4.6]) a finite set $\{\lambda_{v,i}\}$ of irreducible admissible representations of $G(F_v^+)$ and scalars $d_{v,i} \in \mathbf{C}$ such that $(\pi_{n-2,v} \otimes (\pi_{2,v} \otimes \mu_v^{-1} \circ \det))^{\sim}(\widetilde{f}_v^{U_{n-2}\times U_2}) = \sum_i d_{v,i}\lambda_{v,i}(f_v)$. By Proposition 1.6, Proposition 1.7 and Proposition 1.11, we therefore have an identity:

$$\sum_i m(\sigma_i)\sigma_{i,T_1}(f_{T_1}) = \frac{1}{2}\left(\prod_{v|\infty} \epsilon(v, U_n, \varphi_{U_n}) \prod_{v\in T_1} \sum_{\tau\in\Pi(\pi_v)} c_\tau \tau(f_v)\right.$$
$$\left. + \prod_{v|\infty} \epsilon(v, U_{n-2} \times U_2, \varphi_{U_{n-2}\times U_2}) \prod_{v\in T_1} \sum_i d_{v,i}\lambda_{v,i}(f_v)\right),$$

Choose for each $v \in T_1$ a representation $\tau_v \in \Pi(\pi_v)$ such that $\tau_v|_{U_v}$ contains $\tau(v,n)$ (this is possible by Corollary 1.13 and Proposition 1.14). We can assume that for each $v \in T_1$, $\lambda_{v,1} = \tau_v$ (possibly with $d_{v,1} = 0$). We conclude that there is at most one automorphic representation σ of $G(\mathbf{A}_{F^+})$ with the following properties:

- σ is unramified outside $S \cup T$, and is related to π by split or unramified base change at all places $v \notin T_1$;
- If $v \in T_1$, then $\sigma_v \cong \tau_v$.

The representation σ occurs with multiplicity

$$m(\sigma) = \frac{1}{2}\left(\prod_{v|\infty} \epsilon(v, U_n, \varphi_{U_n}) \prod_{v \in T_1} c_{\tau_v} + \prod_{v|\infty} \epsilon(v, U_{n-2} \times U_2, \varphi_{U_{n-2}\times U_2}) \prod_{v \in T_1} d_{v,1}\right).$$

We note that the numbers c_{τ_v} are all non-zero, by Proposition 1.11. If $m(\sigma)$ is non-zero, then we're done (we are in the first case in the statement of the proposition). Otherwise, $\prod_{v\in T_1} c^2_{\tau_v} = \prod_{v \in T_1} d^2_{v,1}$, which we now assume.

In this case, let L^+/F^+ be a totally real quadratic $S \cup T$-split extension, let $L = L^+F$, and let π_L denote the base change of π with respect to the quadratic extension L/F. If $v \in T_{1,L}$, let $\tau_v = \tau_{v|_{F^+}}$. Then repeating the same argument shows that there is at most one automorphic representation σ of $G(\mathbf{A}_{L^+})$ with the following properties:

- σ is unramified outside $S_L \cup T_L$, and is related to π_L by split or unramified base change at all places $v \notin T_{1,L}$;
- If $v \in T_{1,L}$, then $\sigma_v \cong \tau_v$.

Using the remark after Proposition 1.6, we see that the representation σ occurs with multiplicity

$$m(\sigma) = \frac{1}{2}\left(\prod_{v\in T_{1,L}} c_{\tau_v} + \prod_{v \in T_{1,L}} d_{v,1}\right) = \prod_{v\in T_1} c^2_{\tau_v}.$$

This is non-zero, so we're done in this case also (and we are in the second case of the proposition). $\square$

We now show how to complete the proof of Theorem 4.1, assuming first that we are in the first case of Proposition 4.6. We let σ be the automorphic representation of $G(\mathbf{A}_{F^+})$ whose existence is asserted by Proposition 4.6. Let $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,\overline{\mathbf{Q}}_p)}$ be such that σ contributes to $S^{ord}_\lambda(U, \iota^{-1}\tau_g)$ under the isomorphism of Lemma 1.25. Let $\mathbf{T} \subset \mathrm{End}_{\mathcal{O}}(S^{ord}_\lambda(U, \iota^{-1}\tau_g))$ be the commutative $\mathcal{O}$-subalgebra generated by unramified Hecke operators T^j_w at split places $v = ww^c \notin S$ of F^+, and let $\mathfrak{m} \subset \mathbf{T}$ be the maximal ideal determined by σ.

Then $S^{ord}_\lambda(U, \mathring{\tau}_g \otimes_{\mathcal{O}} k)_{\mathfrak{m}}$ is non-zero, by Lemma 1.24, hence (using the exactness of $S^{ord}_\lambda(U, -)$ as a functor on $k[U]$-modules, together with Proposition 1.15) $S^{ord}_\lambda(U, \mathring{\lambda}_g \otimes_{\mathcal{O}} k)_{\mathfrak{m}} \neq 0$, hence $S^{ord}_\lambda(U, \mathring{\lambda}_g)_{\mathfrak{m}} \neq 0$. Applying Lemma 1.25 once again, we conclude the existence of an automorphic representation Σ of $G_n(\mathbf{A}_{F^+})$ with the following properties:

- $\overline{r}_{\Sigma,\iota} \cong \overline{r}_{\pi,\iota}$.
- $\Sigma_T|_{U_T}$ contains λ_g.
- Σ is ι-ordinary and is unramified outside $S \cup T$.

Let Π denote the base change of Σ, let L^+/F^+ be a quadratic totally real extension as in the statement of Theorem 4.1, and let Π_L denote base change of Π with respect to the extension L/F. We claim that Π_L satisfies the requirements of Theorem 4.1. The only points left to check are that Π_L is cuspidal and that if $v \in T_{1,L}$ then $\Pi_{L,v}$ satisfies condition (3) in the statement of Theorem 4.1. In fact, it is enough to show that Π is cuspidal and that if $v \in T_1$ then $\Pi_{\widetilde{v}}|_{\mathfrak{r}_{\widetilde{v}}}$ contains $\widetilde{\lambda}(\widetilde{v}, \widetilde{\theta}_v, n)|_{\mathfrak{r}_{\widetilde{v}}}$. We first show that Π is cuspidal. If Π is not cuspidal, then Lemma 4.5 shows that $\Pi = \Pi_{n-2} \boxplus \Pi_2$ where Π_{n-2}, Π_2 are cuspidal, conjugate self-dual automorphic representations of $\mathrm{GL}_{n-2}(\mathbf{A}_F)$, $\mathrm{GL}_2(\mathbf{A}_F)$, respectively. Arguing as in proof of Proposition 4.6, we obtain an identity

$$\sum_i m(\Sigma_i)\Sigma_i(f) = \frac{1}{2}\left(\widetilde{\Pi}(\widetilde{f}) + (\Pi_{n-2} \otimes (\Pi_2 \otimes \mu^{-1} \circ \det))^\sim(\widetilde{f}^{U_{n-2}\times U_2})\right), \tag{4.6.4}$$

where the sum on the left-hand side is over the finitely many automorphic representations Σ_i of $G(\mathbf{A}_{F^+})$ which are unramified at all places below which Π is unramified, have infinitesimal character related to that of Π_∞ by twisted base change, and which are related to Π by (either unramified or split) base change at places $v \notin T_1$ of F^+.

Fix $v \in T_1$, and consider a test function of the form $f = f_v \otimes f_\infty \otimes f^{v,\infty}$, where:

- f_∞ is a coefficient for Σ_∞.
- f_v is the test function denoted ϕ in the statement of Proposition 1.16.
- $f^{v,\infty}$ is the characteristic function of an open compact subgroup of $G(\mathbf{A}^\infty_{F^+})$.
- $\Sigma(f) \neq 0$.

Then $\Sigma_i(f)$ is non-negative for any i, and the left-hand side of (4.6.4) is non-zero. We conclude that at least one of the terms $\widetilde{\Pi}(\widetilde{f})$ and $(\Pi_{n-2}\otimes(\Pi_2\otimes\mu^{-1}\circ\det))^\sim(\widetilde{f}^{U_{n-2}\times U_2})$ is non-zero. In either case Proposition 1.16 implies that the cuspidal support of $\Pi_{\widetilde{v}}$, and therefore $\Pi_{n-2,\widetilde{v}}$, contains a supercuspidal representation Ψ of $\mathrm{GL}_3(F_{\widetilde{v}})$ such that the semisimple residual representation attached to $\mathrm{rec}^T_{F_{\widetilde{v}}}(\iota^{-1}\Psi)$ is unramified. This contradicts Lemma 4.5, which implies that $\overline{r}_{\Pi_{n-2}|\cdot|^{-1},\iota}|^{ss}_{G_{F_{\widetilde{v}}}}$ is the sum of an unramified character and the twist of an unramified representation by a quadratic ramified character.

Therefore Π is cuspidal, and a similar argument now gives an identity

$$\sum_i m(\Sigma_i)\Sigma_i(f) = \widetilde{\Pi}(\widetilde{f}). \tag{4.6.5}$$

With the same choice of test function we have $\widetilde{\Pi}(\widetilde{f}) \neq 0$, so another application of Proposition 1.16 shows that $\Pi_{\widetilde{v}}$ has the required property. This completes the proof of Theorem 4.1, assuming that the first case of Proposition 4.6 holds. If the second case holds, the argument is very similar, except that there is no need to replace Π by its base change with respect to a quadratic extension L/F. In either case, this completes the proof.

5. A finiteness result for Galois deformation rings

In this section we prove that certain Galois deformation rings are finite over the Iwasawa algebra (Theorem 5.2), and use this to give a criterion for a given deformation to have an irreducible specialization with useful properties (Theorem 5.7). These technical results form the basis for the arguments in §6, where we will apply our criterion to the Galois representation valued over a big ordinary Hecke algebra.

The novelty of the results proved in this section is that we assume that the residual representation is reducible (in fact, to simplify the exposition we assume that this representation is a sum of characters). The main tools are the automorphy lifting theorems proved in [ANT20] and the idea of potential automorphy, for which we use [BLGGT14] as a reference. The notation and definitions we use for Galois deformation theory in the ordinary case are summarized in [ANT20, §3], and we refer to that paper in particular for the notion of local and global deformation problem, and the definitions of the particular local deformation problems used below.

Before getting stuck into the details, we record a useful lemma. If Γ is a profinite group, k is a field with the discrete topology, and $\overline{\rho} : \Gamma \to \mathrm{GL}_n(k)$ is a continuous representation, we say that $\overline{\rho}$ is primitive if it is not isomorphic to a representation of the form $\mathrm{Ind}_{\Gamma'}^{\Gamma} \overline{\sigma}$ for some finite index proper closed subgroup $\Gamma' \subset \Gamma$ and representation $\overline{\sigma} : \Gamma' \to \mathrm{GL}_{n/[\Gamma:\Gamma']}(k)$. This condition appears as a hypothesis in the automorphy lifting theorem proved in [ANT20].

Lemma 5.1. *Suppose that $\overline{\rho} = \overline{\chi}_1 \oplus \cdots \oplus \overline{\chi}_n$, for some continuous characters $\overline{\chi}_i : \Gamma \to k^\times$ such that for each $i \neq j$, $\overline{\chi}_i/\overline{\chi}_j$ has order greater than n. Then $\overline{\rho}$ is primitive.*

Proof. Suppose that there is an isomorphism $\overline{\rho} \cong \mathrm{Ind}_{\Gamma'}^{\Gamma} \overline{\sigma}$. Then Frobenius reciprocity implies that $\overline{\sigma}$ contains each character $\overline{\chi}_i|_{\Gamma'}$. These n characters are distinct: if $\overline{\chi}_i|_{\Gamma'} = \overline{\chi}_j|_{\Gamma'}$, then $(\overline{\chi}_i/\overline{\chi}_j)^{[\Gamma:\Gamma']} = 1$, which would contradict our assumption that $\overline{\chi}_i/\overline{\chi}_j$ has order greater than n if $i \neq j$. Thus $\overline{\sigma}$ must have dimension at least n, implying that $\Gamma = \Gamma'$. It follows that $\overline{\rho}$ is primitive. $\square$

Now let $n \geq 2$ and let F, S, p be as in our standard assumptions (see §1), and let $E \subset \overline{\mathbf{Q}}_p$ be a coefficient field. We recall the definition of the Iwasawa algebra Λ. If $v \in S_p$, then we write $\Lambda_v = \mathcal{O}[\![(I_{F_{\widetilde{v}}}^{\mathrm{ab}}(p))^n]\!]$, where $I_{F_{\widetilde{v}}}^{\mathrm{ab}}(p)$ denotes the inertia subgroup of the Galois group of the maximal abelian pro-p extension of $F_{\widetilde{v}}$. We set $\Lambda = \widehat{\otimes}_{v \in S_p} \Lambda_v$, the completed tensor product being over $\mathcal{O}$. For each $v \in S_p$ and $i = 1, \ldots, n$ there is a universal character $\psi_v^i : I_{F_{\widetilde{v}}}^{\mathrm{ab}}(p) \to \Lambda_v^\times$. At times we will need to introduce Iwasawa algebras also for extension fields F'/F and for representations of degree $n' \neq n$, in which case we will write e.g. $\Lambda_{F',n'}$ for the corresponding Iwasawa algebra, dropping a subscript when either $F' = F$ or $n' = n$.

Let $\mu : G_{F^+,S} \to \mathcal{O}^\times$ be a continuous character which is de Rham and such that $\mu(c_v) = -1$ for each place $v|\infty$ of F^+. Fix an integer $n \geq 2$, and suppose given characters $\overline{\chi}_1, \ldots, \overline{\chi}_n : G_{F,S} \to k^\times$ such that for each $i = 1, \ldots, n$, $\overline{\chi}_i \overline{\chi}_i^c = \overline{\mu}|_{G_{F,S}}$. We set $\overline{\rho} = \oplus_{i=1}^n \overline{\chi}_i$; then $\overline{\rho}$ extends to a homomorphism $\overline{r} : G_{F^+,S} \to \mathcal{G}_n(k)$ such that $\nu_{\mathcal{G}_n} \circ \overline{r} = \overline{\mu}$, by setting $\overline{r}(c) = (1_n, 1)\jmath \in \mathcal{G}_n(k)$. We suppose that for each $v \in S_p$, $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial.

Let Σ be a set of finite places of F^+ split in F and disjoint from S, and $\widetilde{\Sigma}$ a lift of Σ to F. If for each $v \in \Sigma$, $q_v \equiv 1 \bmod p$ and $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial, then we can define the global deformation problem

$$\mathcal{S}_\Sigma = (F/F^+, S \cup \Sigma, \widetilde{S} \cup \widetilde{\Sigma}, \Lambda, \overline{r}, \mu, \{R_v^\triangle\}_{v \in S_p} \cup \{R_v^\square\}_{v \in S - S_p} \cup \{R_v^{St}\}_{v \in \Sigma}).$$

(For the convenience of the reader, we summarize the notation from [ANT20, §3]. Thus the local lifting ring $R_v^\triangle$ represents the functor of ordinary, variable weight liftings; $R_v^\square$ the functor of all liftings; and R_v^{St} the functor of Steinberg liftings.) If $\overline{r}$ is Schur, in the sense of [CHT08, Definition 2.1.6], then the corresponding global deformation functor is represented by an object $R_{\mathcal{S}_\Sigma} \in \mathcal{C}_\Lambda$. If Σ is empty, then we write simply $\mathcal{S} = \mathcal{S}_\emptyset$.

Theorem 5.2. *Suppose that the following conditions are satisfied:*

(1) $p > 2n$.
(2) For each $1 \leq i < j \leq n$, $\overline{\chi}_i/\overline{\chi}_j|_{G_{F(\zeta_p)}}$ *has order greater than* $2n$. *(In particular,* $\overline{r}$ *is Schur.)*
(3) $[F(\zeta_p) : F] = p - 1$.
(4) Σ *is non-empty.*

Then $R_{\mathcal{S}_\Sigma}$ *is a finite* Λ*-algebra.*

Proof. We will compare $R_{\mathcal{S}_\Sigma}$ with a deformation ring for Galois representations to $\mathcal{G}_{2n}$. First, fix a place v_q of F prime to $S \cup \Sigma$, lying above a rational prime $q > 2n$ which splits in F. After possibly enlarging k, we can find a character $\overline{\psi} : G_F \to k^\times$ satisfying the following conditions:

- $\overline{\psi\psi}^c = \epsilon^{1-2n}\overline{\mu}|_{G_F}^{-1}$.
- For each $v \in S \cup \Sigma$, $\overline{\psi}|_{G_{F_{\tilde{v}}}}$ is unramified.
- q divides the order of $\overline{\psi}/\overline{\psi}^c(I_{F_{v_q}})$.

Using the formulae in [BLGGT14, §1.1], we can write down a character

$$(\overline{\psi}, \epsilon^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}) : G_{F^+} \to \mathcal{G}_1(k),$$

the tensor product representation

$$\overline{r} \otimes (\overline{\psi}, \epsilon^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}) : G_{F^+} \to \mathcal{G}_n(k),$$

which has multiplier ϵ^{1-2n}, and the representations

$$\overline{r}_1 = I(\overline{r} \otimes (\overline{\psi}, \epsilon^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+})) : G_{F^+} \to \mathrm{GSp}_{2n}(k)$$

and

$$\overline{r}_2 = \widehat{(\overline{r}_1)}_{G_F} : G_{F^+} \to \mathcal{G}_{2n}(k).$$

These representations have the following properties:

- The multiplier character of $\overline{r}_1$ equals ϵ^{1-2n}.
- The multiplier character $\nu_{\mathcal{G}_{2n}} \circ \overline{r}_2$ equals ϵ^{1-2n}.
- The representations $\overline{r}_1|_{G_F}$ and $\overline{r}_2|_{G_F}$ are both conjugate in $\mathrm{GL}_{2n}(k)$ to $\overline{\rho} \otimes \overline{\psi} \oplus \overline{\rho}^c \otimes \overline{\psi}^c$.

Let $\overline{\rho}_2 = \overline{r}_2|_{G_F}$. We observe that the following conditions are satisfied:

- $\zeta_p \notin \overline{F}^{\ker \mathrm{ad}\, \overline{\rho}_2}$ and $F \not\subset F^+(\zeta_p)$.
- $\overline{\rho}_2$ is primitive.
- The irreducible constituents of $\overline{\rho}_2|_{G_{F(\zeta_p)}}$ occur with multiplicity 1.

Indeed, the condition $F \not\subset F^+(\zeta_p)$ holds because $[F(\zeta_p) : F] = p - 1$. We have $\overline{F}^{\ker \mathrm{ad}\, \overline{\rho}_2} \subset F(\{\overline{\chi}_i/\overline{\chi}_j\}_{i \neq j}, \overline{\psi}/\overline{\psi}^c, \{\overline{\chi}_i\overline{\chi}_j\overline{\mu}^{-1}\}_{i,j}) = M$, say, and c acts on $\mathrm{Gal}(M/F)$ as -1. It follows that $F(\zeta_p) \cap M$ has degree at most 2 over F, showing that $\zeta_p \notin M$. To see that $\overline{\rho}_2$ is primitive, it is enough (by Lemma 5.1) to show that $\overline{\chi}_i/\overline{\chi}_j$ has order greater than $2n$ if $i \neq j$ and that $(\overline{\chi}_i\overline{\psi})/(\overline{\chi}_j^c\overline{\psi}^c)$ has order greater than $2n$ for any i, j. These properties hold by hypothesis in the first case and since $q > 2n$ in the second. Finally, the constituents of $\overline{\rho}_2$ are, with multiplicity, $\overline{\chi}_1 \otimes \overline{\psi}, \ldots, \overline{\chi}_n \otimes \overline{\psi}, \overline{\chi}_1^c \otimes \overline{\psi}^c, \ldots, \overline{\chi}_n^c \otimes \overline{\psi}^c$. Our hypotheses include the condition that $\overline{\chi}_i \otimes \overline{\psi}|_{G_{F(\zeta_p)}} \neq \overline{\chi}_j \otimes \overline{\psi}|_{G_{F(\zeta_p)}}$ if $i \neq j$. If $\overline{\chi}_i \otimes \overline{\psi}|_{G_{F(\zeta_p)}} = \overline{\chi}_j^c \otimes \overline{\psi}^c|_{G_{F(\zeta_p)}}$ then $\overline{\psi}/\overline{\psi}^c|_{I_{F_{v_q}}}$ is trivial, a contradiction.

Fix an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. By [BLGGT14, Theorem 3.1.2], we can find a Galois totally real extension L^+/F^+ and a regular algebraic, self-dual, cuspidal, automorphic representation π of $\mathrm{GL}_{2n}(\mathbf{A}_{L^+})$, with the following properties:

- Let $L = FL^+$. Then L/F is linearly disjoint from the extension of $F(\zeta_p)$ cut out by $\overline{\rho}_2|_{G_{F(\zeta_p)}}$. In particular, $[L(\zeta_p) : L] = p - 1$, $\zeta_p \notin \overline{L}^{\ker \mathrm{ad}\, \overline{\rho}_2|_{G_L}}$, and $L \not\subset L^+(\zeta_p)$.
- There is an isomorphism $\overline{r}_{\pi,\iota} \cong \overline{r}_1|_{G_{L^+}}$.
- π is ι-ordinary. More precisely, π is of weight 0 and for each place $v|p$ of L^+, π_v is an unramified twist of the Steinberg representation.
- $\overline{\rho}_2|_{G_L}$ is primitive.
- The irreducible constituents of $\overline{\rho}_2|_{G_{L(\zeta_p)}}$ occur with multiplicity 1.
- For each place v of L^+ lying above a place of Σ, π_v is an unramified twist of the Steinberg representation.

More precisely, [BLGGT14, Theorem 3.1.2] guarantees the existence of L^+ satisfying the first condition and an essentially self-dual π satisfying all the remaining conditions (except possibly the last one). The last paragraph of the proof notes that the π constructed is in fact self-dual and π_v is an unramified twist of the Steinberg representation for each place $v|p$ of L^+. We can moreover ensure that π is Steinberg at the places of L^+ lying above Σ by inserting the condition "$t(P) < 0$ for all places $v|\Sigma$ of L^+" in the first list of conditions on [BLGGT14, p. 549].

After possibly adjoining another soluble totally real extension of F^+ to L^+, we can assume that the following further conditions are satisfied:

- $\overline{\rho}_2|_{G_L}$ is unramified at those finite places not dividing $S_L \cup \Sigma_L$.
- Each place of L at which π_L is ramified is split over L^+.
- For each place $v \in S_L \cup \Sigma_L$, $\overline{\psi}|_{G_{L_{\widetilde{v}}}}$ is trivial.
- For each place $v \in S_{p,L}$, $[L_{\widetilde{v}} : \mathbf{Q}_p] > 2n(2n-1)/2 + 1$ and $\overline{\rho}_2|_{G_{L_{\widetilde{v}}}}$ is trivial.

Here π_L denotes the base change of π. It is a RACSDC automorphic representation of $\mathrm{GL}_{2n}(\mathbf{A}_L)$. By construction, then, π_L satisfies the hypotheses of [ANT20, Theorem 6.2]. Therefore, if we define the global deformation problem

$$\begin{aligned} \mathcal{S}' = (L/L^+, S_L \cup \Sigma_L, \widetilde{S}_L \cup \widetilde{\Sigma}_L, \Lambda_{L,2n}, \overline{r}_2|_{G_{L^+}}, \epsilon^{1-2n}, \\ \{R_v^{\triangle}\}_{v \in S_{p,L}} \cup \{R_v^{\square}\}_{v \in S_L - S_{p,L}} \cup \{R_v^{St}\}_{v \in \Sigma_L}), \end{aligned}$$

then $R_{\mathcal{S}'}$ is a finite $\Lambda_{L,2n}$-algebra. (Here we have written $\Lambda_{L,2n}$ to distinguish from $\Lambda = \Lambda_{F,n}$ used above.)

We now need to relate the rings $R_{\mathcal{S}'}$ and $R_{\mathcal{S}_\Sigma}$. In fact, it will be enough to construct a commutative diagram

$$\begin{tikzcd} R_{\mathcal{S}'}/(\varpi) \arrow[r] & R_{\mathcal{S}_\Sigma}/(\varpi) \\ \Lambda_{L,2n}/(\varpi) \arrow[r] \arrow[u] & \Lambda_{F,n}/(\varpi) \arrow[u] \end{tikzcd}$$

where the top horizontal morphism is finite. We first specify the map $\Lambda_{L,2n}/(\varpi) \to \Lambda_{F,n}/(\varpi)$. It is the map that for each place $w \in \widetilde{S}_{p,L}$ lying above a place $\widetilde{v}$ of F

classifies the tuple of characters

$$(\psi_1^v|_{I_{L_w}}, \dots, \psi_n^v|_{I_{L_w}}, \psi_n^v|_{I_{L_w}}^{-1}, \dots, \psi_1^v|_{I_{L_w}}^{-1}).$$

This endows the ring $R_{\mathcal{S}_\Sigma}/(\varpi)$ with the structure of $\Lambda_{L,2n}$-algebra. To give a map $R_{\mathcal{S}'}/(\varpi) \to R_{\mathcal{S}_\Sigma}/(\varpi)$, we must give a lifting of $\overline{r}_2|_{G_{L^+}}$ over $R_{\mathcal{S}_\Sigma}/(\varpi)$ which is of type $\mathcal{S}'$. To this end, let r denote a representative of the universal deformation (of $\overline{r}$) to $R_{\mathcal{S}_\Sigma}/(\varpi)$, and let $r' = I(r \otimes (\overline{\psi}, \epsilon^{1-2n}\overline{\mu}^{-1}\delta_{F/F^+}))^{\wedge}_{G_F}|_{G_{L^+}}$ (notation as in [BLGGT14, §1.1]). Then r' is a lift of $\overline{r}_2$ and $r'|_{G_L}$ is the restriction of $r|_{G_F} \otimes \overline{\psi} \oplus r^c|_{G_F} \otimes \overline{\psi}^c$ to G_L. We need to check that for each $v \in S_{p,L}$, $r'|_{G_{L_{\widetilde{v}}}}$ is of type $R_v^{\triangle}$; and that for each $v \in \Sigma_L$, $r'|_{G_{L_{\widetilde{v}}}}$ is of type R_v^{St}. These statements can be reduced to a universal local computation.

It follows that r' is of type $\mathcal{S}'$, and so determines a morphism $R_{\mathcal{S}'}/(\varpi) \to R_{\mathcal{S}_\Sigma}/(\varpi)$. To complete the proof, it will be enough to show that this is a finite ring map. We can enlarge the above commutative diagram to a diagram

$$\begin{tikzcd} R_{\mathcal{S}'}/(\varpi) \arrow[r] & R_{\mathcal{S}_\Sigma}/(\varpi) \\ Q_{\overline{t}_2|_{G_L}}/(\varpi)\widehat{\otimes}_{\mathcal{O}}\Lambda_{L,2n}/(\varpi) \arrow[r] \arrow[u] & Q_{\overline{t}}/(\varpi)\widehat{\otimes}_{\mathcal{O}}\Lambda_{F,n}/(\varpi), \arrow[u] \end{tikzcd}$$

where $Q_{\overline{t}_2|_{G_L}}$ is the complete Noetherian local $\mathcal{O}$-algebra classifying pseudocharacters of $G_{L,S_L\cup\Sigma_L}$ lifting the restriction of $\overline{t}_2 = \operatorname{tr}\overline{r}_2|_{G_F}$ to G_L, $Q_{\overline{t}}$ is defined similarly with respect to the pseudocharacter $\overline{t} = \operatorname{tr}\overline{r}|_{G_F}$ of $G_{F,S\cup\Sigma}$, and the map $Q_{\overline{t}_2|_{G_L}}/(\varpi) \to Q_{\overline{t}}/(\varpi)$ is the one classifying the natural transformation sending a pseudocharacter t lifting $\overline{t}$ to the pseudocharacter $(t|_{G_L} \otimes \overline{\psi}) + (t^c|_{G_L} \otimes \overline{\psi}^c)$. We deduce from [Tho15, Proposition 3.29] that the vertical arrows are finite ring maps. The map $\Lambda_{L,2n} \to \Lambda_{F,n}$ is also finite, so it's enough finally to show that the map $Q_{\overline{t}_2|_{G_L}}/(\varpi) \to Q_{\overline{t}}/(\varpi)$ is finite. This map can in turn be written as a composite

$$Q_{\overline{t}_2|_{G_L}}/(\varpi) \to Q_{\overline{t}_2|_{G_F}}/(\varpi) \to Q_{\overline{t}}/(\varpi),$$

where the first map classifies restriction of pseudocharacters from G_F to G_L. Since $\overline{r}_2|_{G_F}$ is multiplicity free, [ANT20, Proposition 2.5] (specifically, the uniqueness of the expression as a sum of pseudocharacters) implies that the second map is in fact surjective. We finally just need to show that the first map is finite, and this follows from the following general lemma. $\square$

Lemma 5.3. *Let Γ be a topologically finitely generated profinite group, let Σ be a closed subgroup of finite index, and let $\overline{t}$ be a pseudocharacter of Γ with coefficients in k of some dimension n. Let $Q_{\overline{t}}$ be the complete Noetherian local $\mathcal{O}$-algebra classifying lifts of $\overline{t}$. Then the map $Q_{\overline{t}|_\Sigma} \to Q_{\overline{t}}$ classifying restriction to Σ is a finite ring map.*

Proof. It suffices to show that $Q_{\overline{t}}/(\mathfrak{m}_{Q_{\overline{t}|_\Sigma}})$ is Artinian. If not, we can find a prime ideal $\mathfrak{p}$ of this ring of dimension 1; let A be its residue ring (which is a k-algebra), and let t_A be the induced pseudocharacter of Γ with coefficients in A. Let $N = [\Gamma : \Sigma]$. If $\gamma \in \Gamma$ then $\gamma^{N!} \in \Sigma$. If we factor the characteristic polynomial of $X - \gamma$ under t as $\prod_{i=1}^n (X - \alpha_i)$ for some elements α_i in the algebraic closure of $\operatorname{Frac} A$, then the characteristic polynomial of $\gamma^{N!}$ under t, namely $\prod_{i=1}^n (X - \alpha_i^{N!})$, lies in $k[X]$ and equals the characteristic polynomial of $\gamma^{N!}$ under $\overline{t}$. This shows that the elements

α_i are in fact algebraic over k, and thus (using [Che14, Corollary 1.14]) that t_A can be defined over k, and must in fact equal $\bar{t}$. This is a contradiction. $\square$

Corollary 5.4. *With hypotheses as in Theorem 5.2, fix $\lambda \in (\mathbf{Z}^n_+)^{\mathrm{Hom}(F,\overline{\mathbf{Q}}_p)}$ such that for each $i = 1, \ldots, n$ and $\tau \in \mathrm{Hom}(F, \overline{\mathbf{Q}}_p)$, we have $\lambda_{\tau c,i} = -\lambda_{\tau,n+1-i}$. Suppose further that for each $v \in S_p$, $[F_{\widetilde{v}} : \mathbf{Q}_p] > n(n-1)/2 + 1$. Then there exists a homomorphism $r : G_{F^+,S} \to \mathcal{G}_n(\overline{\mathbf{Z}}_p)$ lifting $\overline{r}$ such that $r|_{G_{F,S}}$ is ordinary of weight λ, in the sense of* [Tho15, Definition 2.5].

Proof. We observe that [Tho15, Proposition 3.9, Proposition 3.14] show that for each minimal prime $Q \subset R_{\mathcal{S}_\Sigma}$, $\dim R_{\mathcal{S}_\Sigma}/Q = \dim \Lambda$; consequently, there is a minimal prime Q_Λ of Λ and a finite injective algebra morphism $\Lambda/Q_\Lambda \to R_{\mathcal{S}_\Sigma}/Q$. The corollary follows on choosing any prime of $R_{\mathcal{S}_\Sigma}/Q[1/p]$ lying above a maximal ideal of $\Lambda/Q_\Lambda[1/p]$ associated to the weight λ as in [Ger19, Definition 2.24]. $\square$

Corollary 5.5. *With hypotheses (1) – (3) of Theorem 5.2, choose a place $v_0 \notin S$ of F^+ split in F and a lift $\widetilde{v}_0$ to F such that $q_{v_0} \equiv 1 \bmod p$ and $\overline{r}|_{G_{F_{\widetilde{v}_0}}}$ is trivial. Consider the quotient*

$$A = R_{\mathcal{S}_\emptyset}/(\varpi, \{\mathrm{tr}\, r_{\mathcal{S}_\emptyset}(\mathrm{Frob}^i_{\widetilde{v}_0}) - n\}_{i=1,\ldots,n}).$$

Then A is a finite Λ-algebra. Consequently, $\dim R_{\mathcal{S}_\emptyset}/(\varpi) \leq n[F^+ : \mathbf{Q}] + n$.

Proof. It suffices to verify that the quotient of $R_{\mathcal{S}_\emptyset}$ where the characteristic polynomial of $\mathrm{Frob}_{\widetilde{v}_0}$ equals $\prod_{i=1}^n (X - q_{v_0}^{1-i})$ is a quotient of $R_{\mathcal{S}_{\{v_0\}}}$. This in turn means checking that the quotient A_{v_0} of the local unramified lifting ring $R^{ur}_{v_0}$ where the characteristic polynomial of Frobenius equals $\prod_{i=1}^n (X - q_{v_0}^{1-i})$ is a quotient of $R^{St}_{v_0}$. Since A_{v_0} is flat over $\mathcal{O}$, this follows from the definition of $R^{St}_{v_0}$ (see [Tay08, §3]). $\square$

For the statement of the next proposition, suppose given a surjection $R_{\mathcal{S}}/(\varpi) \to A$ in $\mathcal{C}_\Lambda$, where A is a domain, and let $r : G_{F^+,S} \to \mathcal{G}_n(A)$ denote the pushforward of (a representative of) the universal deformation. Suppose given the following data:

- A decomposition $r = r_1 \oplus r_2$, where the $r_i : G_{F^+,S} \to \mathcal{G}_{n_i}(A)$ satisfy $\nu_{\mathcal{G}_{n_i}} \circ r_i = \nu_{\mathcal{G}_n} \circ r$. (In other words, $r|_{G_F} = r_1|_{G_F} \oplus r_2|_{G_F}$ and if $r_i(c) = (A_i, 1)\jmath$ then $r(c) = \mathrm{diag}(A_1, A_2)\jmath$.)
- A subset $R \subset S - S_p$ (consisting of places of odd residue characteristic) with the following property: for each $v \in R$ we are given an integer $1 \leq n_{\widetilde{v}} \leq n$ such that $q_{\widetilde{v}} \bmod p$ is a primitive $n_{\widetilde{v}}^{\mathrm{th}}$ root of unity and there is a decomposition $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{\sigma}_{\widetilde{v},1} \oplus \overline{\sigma}_{\widetilde{v},2}$, where $\overline{\sigma}_{\widetilde{v},1} = \mathrm{Ind}_{G_{F_{\widetilde{v},n_{\widetilde{v}}}}}^{G_{F_{\widetilde{v}}}} \overline{\psi}_{\widetilde{v}}$ with $F_{\widetilde{v},n_{\widetilde{v}}}/F_{\widetilde{v}}$ the unramified extension of degree $n_{\widetilde{v}}$ and $\overline{\psi}_{\widetilde{v}}$ an unramified character of $G_{F_{\widetilde{v},n_{\widetilde{v}}}}$, and $\overline{\sigma}_{\widetilde{v},2}$ is the twist of an unramified representation of $G_{F_{\widetilde{v}}}$ of dimension $n - n_{\widetilde{v}}$ by a ramified quadratic character.
- An isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ and for each $v \in R$, a character $\Theta_{\widetilde{v}} : \mathcal{O}^\times_{F_{\widetilde{v},n_{\widetilde{v}}}} \to \mathbf{C}^\times$ of order p. Thus the lifting ring $R(\widetilde{v}, \Theta_{\widetilde{v}}, n)$ is defined (notation as in §1.17).

Proposition 5.6. *With the above assumptions on $R_{\mathcal{S}}/(\varpi) \to A$, suppose that the following additional conditions are satisfied:*

(1) $p > 2n$.
(2) For each $1 \leq i < j \leq n$, $\overline{\chi}_i/\overline{\chi}_j|_{G_{F(\zeta_p)}}$ has order greater than $2n$. (In particular, this character is non-trivial and $\overline{r}$ is Schur.)

(3) For each $v \in S_p$, $[F_{\widetilde{v}} : \mathbf{Q}_p] > n(n-1)/2 + 1$.
(4) $[F(\zeta_p) : F] = p - 1$.
(5) For each $v \in R$, both $\overline{r}_1|_{G_{F_{\widetilde{v}}}}$ and $\overline{r}_2|_{G_{F_{\widetilde{v}}}}$ admit a non-trivial unramified subquotient and the composite map $R_v^\square \to R_{\mathcal{S}} \to A$ factors over $R(\widetilde{v}, \Theta_{\widetilde{v}}, n)$.

Let L_{S_p} denote the maximal abelian pro-p extension of F unramified outside S_p, and let $\Delta = \mathrm{Gal}(L_{S_p}/F)/(c+1)$. Let d_R denote the $\mathbf{Z}_p$-rank of the subgroup of Δ generated by the elements $\mathrm{Frob}_{\widetilde{v}}$, $v \in R$. Then $\dim A \leq n[F^+ : \mathbf{Q}] + n - d_R$.

Proof. Fix a place $\widetilde{v}_0$ of F split over F^+, prime to S, and such that $q_{\widetilde{v}_0} \equiv 1 \bmod p$ and $\overline{r}|_{G_{F_{\widetilde{v}_0}}}$ is trivial. Let I denote the ideal of A generated by the coefficients of the polynomial $\det(X - r|_{G_{F,S}}(\mathrm{Frob}_{\widetilde{v}_0})) - (X-1)^n$. Then $\dim A/I \geq \dim A - n$. After replacing A by $A/\mathfrak{p}$, where $\mathfrak{p} \subset A$ is a prime ideal minimal among those containing I, we can assume that $r|_{G_{F,S}}(\mathrm{Frob}_{\widetilde{v}_0})$ is unipotent, and must show $\dim A \leq n[F^+ : \mathbf{Q}] - d_R$. Consider the deformation problems ($i = 1, 2$):

$$\mathcal{S}_i = (F/F^+, S \cup \{v_0\}, \widetilde{S} \cup \{\widetilde{v}_0\}, \Lambda_{n_i}, \overline{r}_i, \mu, \{R_v^\Delta\}_{v \in S_p} \cup \{R_v^\square\}_{v \in S - S_p} \cup \{R_{v_0}^{St}\}).$$

Let $K = \mathrm{Frac}\, A$. We now repeat the argument of [ANT20, Lemma 3.6]: if $v \in S_p$, then (since we assume $[F_{\widetilde{v}} : \mathbf{Q}_p] > n(n-1)/2+1$) we can appeal to [Tho15, Corollary 3.12], which implies the existence of an increasing filtration

$$0 \subset \mathrm{Fil}_v^1 \subset \mathrm{Fil}_v^2 \subset \cdots \subset \mathrm{Fil}_v^n = \overline{K}^n$$

of $r|_{G_{F_{\widetilde{v}}}} \otimes_A \overline{K}$ by $G_{F_{\widetilde{v}}}$-invariant subspaces, such that each $\mathrm{gr}^i \mathrm{Fil}_v^\bullet = \mathrm{Fil}_v^i / \mathrm{Fil}_v^{i-1}$ ($i = 1, \ldots, n$) is 1-dimensional, and such that the character $I_{F_{\widetilde{v}}}(p) \to \overline{K}^\times$ afforded by $\mathrm{gr}^i \mathrm{Fil}_v^\bullet$ agrees with the pushforward of the universal character $\psi_v^i : I_{F_{\widetilde{v}}} \to \Lambda_v^\times$. Using the decomposition $r = r_1 \oplus r_2$, we obtain induced filtrations $\mathrm{Fil}_v^\bullet \cap (\overline{K}^{n_1} \oplus 0^{n_2})$ of $r_1|_{G_{F_{\widetilde{v}}}} \otimes_A \overline{K}$ and $\mathrm{Fil}_v^\bullet \cap (0^{n_1} \oplus \overline{K}^{n_2})$ of $r_2 \otimes_A \overline{K}$ and, applying [Tho15, Corollary 3.12] once more, we see that we can find an isomorphism $\Lambda_{n_1} \widehat{\otimes} \Lambda_{n_2} \cong \Lambda = \Lambda_n$ such that, endowing A with the induced Λ_i-algebra structure, r_i is a lifting of $\overline{r}_i$ of type $\mathcal{S}_i$ for each $i = 1, 2$. We deduce the existence of a surjective Λ-algebra homomorphism $R_{\mathcal{S}_1} \widehat{\otimes}_{\mathcal{O}} R_{\mathcal{S}_2} \to A$. We observe that Theorem 5.2 applies to the deformation problems $\mathcal{S}_1$ and $\mathcal{S}_2$, showing that $\dim R_{\mathcal{S}_i}/(\varpi) \leq n_i[F^+ : \mathbf{Q}]$.

Let $\psi_i : G_{F,S} \to \mathcal{O}^\times$ denote the Teichmüller lift of $\overline{\psi}_i = \det \overline{r}_i|_{G_{F,S}}$, and let $R_{\mathcal{S}_i}^{\psi_i}$ denote the quotient of $R_{\mathcal{S}_i}$ over which the determinant of the universal deformation equals ψ_i. Then [Tho15, Lemma 3.36] states that there is an isomorphism $R_{\mathcal{S}_i} \cong R_{\mathcal{S}_i}^{\psi_i} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[\![\Delta]\!]$. In particular, $\dim R_{\mathcal{S}_i}^{\psi_i}/(\varpi) \leq (n_i - 1)[F^+ : \mathbf{Q}]$. To complete the proof, it is enough to show that if $A' = A/(\mathfrak{m}_{R_{\mathcal{S}_1}^{\psi_1}}, \mathfrak{m}_{R_{\mathcal{S}_2}^{\psi_2}})$, then $\dim A' \leq 2[F^+ : \mathbf{Q}] - d_R = \dim k[\![\Delta \times \Delta]\!] - d_R$.

To this end, we observe that by construction there is a surjection $k[\![\Delta \times \Delta]\!] \to A'$. If $\Psi_1, \Psi_2 : G_{F,S} \to k[\![\Delta \times \Delta]\!]^\times$ are the two universal characters, then the third part of Proposition 1.22 (together with our assumption that both $\overline{r}_1|_{G_{F_{\widetilde{v}}}}$ and $\overline{r}_2|_{G_{F_{\widetilde{v}}}}$ admit an unramified subquotient) implies that the relation $\Psi_1(\mathrm{Frob}_{\widetilde{v}})^{n_{\widetilde{v}}} = \Psi_2(\mathrm{Frob}_{\widetilde{v}})^{n_{\widetilde{v}}}$ holds in A' for each $v \in R$. Since Δ is a pro-p group, this implies that $\Psi_1(\mathrm{Frob}_{\widetilde{v}}) = \Psi_2(\mathrm{Frob}_{\widetilde{v}})$ in A', and hence that the map $k[\![\Delta \times \Delta]\!] \to A'$ factors over the completed group algebra of the quotient of $\Delta \times \Delta$ by the subgroup topologically generated by the elements $(\mathrm{Frob}_{\widetilde{v}}, -\mathrm{Frob}_{\widetilde{v}})_{v \in R}$. This completes the proof. $\square$

We are now in a position to prove the main theorem of this section, which guarantees the existence of generic primes in sufficiently large quotients of a certain deformation ring. For the convenience of the reader, we state our assumptions from scratch.

Thus we take F, S, p as in our standard assumptions (see §1). We assume that $[F(\zeta_p) : F] = (p-1)$. We let E be a coefficient field, and suppose given an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ and a continuous character $\mu : G_{F^+,S} \to \mathcal{O}^\times$ which is de Rham and such that $\mu(c_v) = -1$ for each place $v|\infty$ of F^+. We fix an integer $2 \leq n < p/2$ and characters $\overline{\chi}_1, \ldots, \overline{\chi}_n : G_{F,S} \to k^\times$ such that for each $i = 1, \ldots, n$, $\overline{\chi}_i \overline{\chi}_i^c = \overline{\mu}|_{G_{F,S}}$. We set $\overline{\rho} = \oplus_{i=1}^n \overline{\chi}_i$; then $\overline{\rho}$ naturally extends to a homomorphism $\overline{r} : G_{F,S} \to \mathcal{G}_n(k)$ such that $\nu_{\mathcal{G}_n} \circ \overline{r} = \overline{\mu}$. We suppose for each $1 \leq i < j \leq n$, $\overline{\chi}_i/\overline{\chi}_j|_{G_{F(\zeta_p)}}$ has order greater than $2n$. This implies that $\overline{r}$ is Schur. We suppose that for each $v \in S_p$, $\overline{r}|_{G_{F_{\widetilde{v}}}}$ is trivial and $[F_{\widetilde{v}} : \mathbf{Q}_p] > n(n-1)/2 + 1$.

We suppose given a subset $R = R_1 \sqcup R_2 \subset S - S_p$ (consisting of places of odd residue characteristic) and integers $1 \leq n_{\widetilde{v}} \leq n$ ($v \in R$) such that for each $v \in R$, $q_{\widetilde{v}} \bmod p$ is a primitive $n_{\widetilde{v}}^{\text{th}}$ root of unity, and there is a decomposition $\overline{r}|_{G_{F_{\widetilde{v}}}} = \overline{\sigma}_{\widetilde{v},1} \oplus \overline{\sigma}_{\widetilde{v},2}$, where $\overline{\sigma}_{\widetilde{v},1} = \mathrm{Ind}_{G_{F_{\widetilde{v},n_{\widetilde{v}}}}}^{G_{F_{\widetilde{v}}}} \overline{\psi}_{\widetilde{v}}$ is induced from an unramified character of the unramified degree $n_{\widetilde{v}}$ extension of $F_{\widetilde{v}}$, and $\overline{\sigma}_{\widetilde{v},2}$ is the twist of an unramified representation of dimension $n - n_{\widetilde{v}}$ by a ramified quadratic character. We fix for each $v \in R$ a character $\Theta_{\widetilde{v}} : \mathcal{O}_{F_{\widetilde{v},n_{\widetilde{v}}}}^\times \to \mathbf{C}^\times$ of order p.

Assuming (as we may) that E is large enough, we may then (re-)define the global deformation problem

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \Lambda, \overline{r}, \mu, \{R_v^\triangle\}_{v \in S_p} \cup \{R(\widetilde{v}, \Theta_{\widetilde{v}}, n)\}_{v \in R} \cup \{R_v^\square\}_{v \in S - (S_p \cup R)}).$$

Following [ANT20, Definition 3.7], we say that a prime $\mathfrak{p} \subset R_{\mathcal{S}}$ of dimension 1 and characteristic p is generic at p if it satisfies the following conditions:

- Let $A = R_{\mathcal{S}}/\mathfrak{p}$, and let $r_{\mathfrak{p}} : G_{F^+,S} \to \mathcal{G}_n(A)$ be the pushforward of (a representative of) the universal deformation. Then for each $v \in S_p$, the (pushforwards from Λ of the) universal characters $\psi_1^v, \ldots, \psi_n^v : I_{F_{\widetilde{v}}}^{ab}(p) \to A^\times$ are distinct.
- There exists $v \in S_p$ and $\sigma \in I_{F_{\widetilde{v}}}^{ab}(p)$ such that the elements $\psi_1^v(\sigma), \ldots, \psi_n^v(\sigma) \in A^\times$ satisfy no non-trivial $\mathbf{Z}$-linear relation.

We say that $\mathfrak{p}$ is generic if it is generic at p and if $r_{\mathfrak{p}}|_{G_{F,S}} \otimes_A \operatorname{Frac} A$ is absolutely irreducible.

Theorem 5.7. *With assumptions as above, let $R_{\mathcal{S}} \to B$ be a surjection in $\mathcal{C}_\Lambda$, where B is a finite $\Lambda/(\varpi)$-algebra. Let L_{S_p} denote the maximal abelian pro-p extension of F unramified outside S_p, and let $\Delta = \mathrm{Gal}(L_{S_p}/F)/(c+1)$. Let d_{R_i} denote the $\mathbf{Z}_p$-rank of the subgroup of Δ topologically generated by the elements* $\mathrm{Frob}_{\widetilde{v}}$*, $v \in R_i$.*

Suppose that the following conditions are satisfied:

1. *Each irreducible component of* $\operatorname{Spec} B$ *has dimension strictly greater than* $\sup(\{n[F^+ : \mathbf{Q}] + n - d_{R_i}\}_{i=1,2}, \{n[F^+ : \mathbf{Q}] - [F_{\widetilde{v}} : \mathbf{Q}_p]\}_{v \in S_p})$.
2. *For each direct sum decomposition $\overline{r} = \overline{r}_1 \oplus \overline{r}_2$ with $\overline{r}_j : G_{F^+,S} \to \mathcal{G}_{n_j}(k)$ ($j = 1, 2$) and $n_1 n_2 \neq 0$, there exists $i \in \{1, 2\}$ such that for each $v \in R_i$, both $\overline{r}_1|_{G_{F_{\widetilde{v}}}}$ and $\overline{r}_2|_{G_{F_{\widetilde{v}}}}$ admit a non-trivial unramified subquotient.*

Then there exists a prime $\mathfrak{p} \subset R_{\mathcal{S}}$ of dimension 1 and characteristic p, containing the kernel of $R_{\mathcal{S}} \to B$, which is generic.

Proof. After passage to a quotient by a minimal prime, we can assume that B is a domain. The argument is now very similar to that of [ANT20, Lemma 3.9]. Indeed, by [ANT20, Lemma 3.8], we can find countable collection $(I_i)_{i\geq 1}$ of ideals $I_i \subset \Lambda/(\varpi)$ such that for all $i \geq 1$, $\dim \Lambda/(\varpi, I_i) \leq \sup\{n[F^+ : \mathbf{Q}] - [F_{\widetilde{v}} : \mathbf{Q}_p]\}_{v \in S_p}$ and if $\mathfrak{p} \subset R_{\mathcal{S}}$ is a prime of dimension 1 and characteristic p which is not generic at p, then $I_i R_{\mathcal{S}} \subset \mathfrak{p}$ for some $i \geq 1$. Let $I_{\mathcal{S}}^{red} \subset R_{\mathcal{S}}$ be the reducibility ideal defined just before [ANT20, Lemma 3.4], and let $I_0 = (I_{\mathcal{S}}^{red}, \varpi)R_{\mathcal{S}}$. Proposition 5.6 shows that $\dim R_{\mathcal{S}}/I_0 \leq \sup\{n[F^+ : \mathbf{Q}] + n - d_{R_i}\}_{i=1,2}$.

Since B is a finite $\Lambda/(\varpi)$-algebra, we have $\dim B/I_i \leq \sup\{n[F^+ : \mathbf{Q}] - [F_{\widetilde{v}} : \mathbf{Q}_p]\}_{v \in S_p}$. We also have $\dim B/I_0 \leq \sup\{n[F^+ : \mathbf{Q}] + n - d_{R_i}\}_{i=1,2}$. The existence of a generic prime $\mathfrak{p}$ containing the kernel of the map $R_{\mathcal{S}} \to B$ thus follows from [Tho15, Lemma 1.9]. □

We conclude this section with a result concerning the existence of automorphic lifts of prescribed types, under the hypothesis of residual automorphy over a soluble extension. It only uses the results of [ANT20] and not the results proved earlier in this section, and is very similar in statement and proof to [BG19, Theorem 5.2.1].

We begin by re-establishing notation. We therefore let F_0 be an imaginary CM field such that F_0/F_0^+ is everywhere unramified. We fix a prime p and write $S_{0,p}$ for the set of of p-adic places of F_0^+. We fix a finite set S_0 of finite places of F_0^+ containing $S_{0,p}$. We assume that each place of $S_{0,p}$ splits in F_0, but not necessarily that each place of $S_0 - S_{0,p}$ splits in F_0. We choose for each $v \in S_0$ a place $\widetilde{v}$ of F_0 lying above v, and write $\widetilde{S}_0 = \{\widetilde{v} \mid v \in S_0\}$. We fix a coefficient field $E \subset \overline{\mathbf{Q}}_p$. Fix an integer $n \geq 2$, and suppose given a continuous representation $\overline{\rho} : G_{F_0, S_0} \to \mathrm{GL}_n(k)$ satisfying the following conditions:

- There is an isomorphism $\overline{\rho} \cong \oplus_{i=1}^r \overline{\rho}_i$, where each representation $\overline{\rho}_i$ is absolutely irreducible and satisfies $\overline{\rho}_i^c \cong \overline{\rho}_i^\vee \otimes \epsilon^{1-n}$. Moreover, for each $1 \leq i < j \leq r$, we have $\overline{\rho}_i \not\cong \overline{\rho}_j$.

Proposition 5.8. *Fix disjoint subsets $T_0, \Sigma_0 \subset S_0$ consisting of prime-to-p places which split in F_0. We assume that for each $v \in \Sigma_0$, we have $q_{\widetilde{v}} \equiv 1 \bmod p$ and $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ is trivial. We fix for each $v \in T_0$ a quotient $R_v^\square \to \overline{R}_v$ of the universal lifting ring of $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ corresponding to a non-empty union of irreducible components of $\operatorname{Spec} R_v^\square[1/p]$. We suppose that if $v \in S_0$ and v is inert in F_0, then $\overline{\rho}(I_{F_{0,\widetilde{v}}})$ is of order prime to p. Fix a weight $\lambda \in (\mathbf{Z}_+^n)^{\mathrm{Hom}(F_0, \overline{\mathbf{Q}}_p)}$ such that for each $i = 1, \ldots, n$ and $\tau \in \mathrm{Hom}(F_0, \overline{\mathbf{Q}}_p)$, $\lambda_{\tau c, i} = -\lambda_{\tau, n+1-i}$, and suppose that for each $v \in S_{0,p}$, $\overline{\rho}|_{G_{F_{0,\widetilde{v}}}}$ admits a lift to $\overline{\mathbf{Z}}_p$ which is ordinary of weight $\lambda_{\widetilde{v}}$, in the sense of* [Ger19, Definition 3.8]. *Suppose that there exists a soluble CM extension F/F_0 such that the following conditions are satisfied:*

(1) $p > \max(n, 3)$. For each place $v|p$ of F, we have $[F_v : \mathbf{Q}_p] > n(n-1)/2 + 1$ and $\overline{\rho}|_{G_{F_v}}$ is trivial.

(2) $F(\zeta_p)$ is not contained in $\overline{F}^{\ker \operatorname{ad} \overline{\rho}}$ and F is not contained in $F^+(\zeta_p)$. For each $1 \leq i < j \leq r$, $\overline{\rho}_i|_{G_{F(\zeta_p)}}$ is absolutely irreducible and $\overline{\rho}_i|_{G_{F(\zeta_p)}} \not\cong \overline{\rho}_j|_{G_{F(\zeta_p)}}$. Moreover, $\overline{\rho}|_{G_F}$ is primitive and $\overline{\rho}(G_F)$ has no quotient of order p.

(3) There exists a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$ such that $\overline{r}_{\pi,\iota} \cong \overline{\rho}|_{G_F}$. Moreover, π is ι-ordinary

and there exists a place v of F lying above Σ_0 such that π_v is an unramified twist of the Steinberg representation.

(4) If S denotes the set of places of F^+ lying above S_0, then each place of S splits in F.

Then there exists a RACSDC automorphic representation π_0 of $\mathrm{GL}_n(\mathbf{A}_{F_0})$ satisfying the following conditions:

(1) π_0 is unramified outside S_0 and there is an isomorphism $\overline{r}_{\pi_0,\iota} \cong \overline{\rho}$.
(2) π_0 is ι-ordinary of weight $\iota\lambda$.
(3) For each place $v \in T_0$, $r_{\pi_0,\iota}|_{G_{F_{0,\widetilde{v}}}}$ defines a point of $\overline{R}_v$.
(4) For each place $v \in \Sigma_0$, $\pi_{0,\widetilde{v}}$ is an unramified twist of the Steinberg representation.
(5) For each place $v \in S_0$ which is inert in F_0, reduction modulo p induces an isomorphism $r_{\pi_0,\iota}(I_{F_0,\widetilde{v}}) \to \overline{r}_{\pi_0,\iota}(I_{F_0,\widetilde{v}})$.

Proof. Let $v \in S_{0,p}$, and let $\rho_v : G_{F_{0,\widetilde{v}}} \to \mathrm{GL}_n(\overline{\mathbf{Z}}_p)$ be the lift of $\overline{\rho}|_{G_{F_{\widetilde{v}}}}$ which is ordinary of weight $\lambda_{\widetilde{v}}$ and which exists by assumption. Thus ρ_v is conjugate over $\overline{\mathbf{Q}}_p$ to an upper-triangular representation with the property that if $\alpha_{v,1}, \dots, \alpha_{v,n} : G_{F_{0,\widetilde{v}}} \to \overline{\mathbf{Q}}_p^\times$ are the characters appearing on the diagonal, then for each $i = 1, \dots, n$ the character $\alpha_{v,i}$ is equal, on restriction to some open subgroup of $I_{F_{0,\widetilde{v}}}$, to the character

$$\chi_{\lambda_{\widetilde{v}},i} : \sigma \in I_{F_{0,\widetilde{v}}} \mapsto \epsilon(\sigma)^{1-i} \prod_{\tau : F_{0,\widetilde{v}} \to \overline{\mathbf{Q}}_p} \tau(\mathrm{Art}_{F_{0,\widetilde{v}}}^{-1}(\sigma))^{-\lambda_{\tau,n-i+1}}.$$

After enlarging E, we can assume that each character $\alpha_{v,i}$ takes values in $\mathcal{O}$. We use the restricted characters $\alpha_{v,i}|_{I_{F_{0,\widetilde{v}}}(p)} : I_{F_{0,\widetilde{v}}}(p) \to \mathcal{O}^\times$ ($v \in S_{0,p}, i = 1, \dots, n$) to define a homomorphism $\Lambda_{F_0} \to \mathcal{O}$.

Let $\beta_{\widetilde{v},i} = \alpha_{v,i}\chi_{\lambda_{\widetilde{v}},i}^{-1}$. Then $\tau_{\widetilde{v}} = \oplus_{i=1}^n \beta_{\widetilde{v},i}$ is an inertial type and the type $\tau_{\widetilde{v}}$, Hodge type $\lambda_{\widetilde{v}}$ lifting ring $R_{\widetilde{v}}^{\lambda_{\widetilde{v}},\tau_{\widetilde{v}}}$ is defined and equidimensional of dimension $1 + n^2 + n(n-1)[F_{0,\widetilde{v}} : \mathbf{Q}_p]/2$ (see [Kis08, Theorem 3.3.4]). When $\tau_{\widetilde{v}}$ is trivial, [Ger19, Lemma 3.10] shows that there is a minimal prime ideal of $R_{\widetilde{v}}^{\lambda_{\widetilde{v}},\tau_{\widetilde{v}}}$ such that, writing R_v for the corresponding quotient, the following properties are satisfied:

- R_v is $\mathcal{O}$-flat of dimension $1 + n^2 + n(n-1)[F_{0,\widetilde{v}} : \mathbf{Q}_p]/2$.
- The map $R_{\widetilde{v}}^{\lambda_{\widetilde{v}},\tau_{\widetilde{v}}} \to \overline{\mathbf{Q}}_p$ determined by ρ_v factors through R_v.
- For every homomorphism $R_v \to \overline{\mathbf{Q}}_p$, the corresponding Galois representation $G_{F_{0,\widetilde{v}}} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is ordinary of weight $\lambda_{\widetilde{v}}$, in the sense of [Ger19, Definition 3.8].
- The homomorphism $R_v^\square \widehat{\otimes}_{\mathcal{O}} \Lambda_v \to R_v$ (completed tensor product of the tautological quotient map $R_v^\square \to R_v$ and the composite $\Lambda_v \to \mathcal{O} \to R_v$) factors over the quotient $R_v^\square \widehat{\otimes}_{\mathcal{O}} \Lambda_v \to R_v^\triangle$ (defined in e.g. [Tho15, §3.3.2]).

In fact, the same proof shows that these properties hold also in the case that $\tau_{\widetilde{v}}$ is non-trivial.

Our hypotheses imply that we can extend $\overline{\rho}$ to a continuous homomorphism $\overline{r} : G_{F_0^+,S_0} \to \mathcal{G}_n(k)$ with the property that $\nu \circ \overline{r} = \epsilon^{1-n}\delta_{F_0/F_0^+}^n$. Let R^{univ} denote the deformation ring, defined as in [BG19, Corollary 5.1.1], of $\epsilon^{1-n}\delta_{F_0/F_0^+}^n$-polarised deformations of $\overline{r}$, where the quotients of the local lifting rings for $v \in S_0$ are specified as follows:

- If $v \in S_{0,p}$, we take the quotient R_v defined above.
- If $v \in T_0$, take $\overline{R}_v$.
- If $v \in \Sigma_0$, take the Steinberg lifting ring R_v^{St}.
- If $v \in S_0$ and v is inert in F_0, take the component corresponding to the functor of lifts r of $\overline{r}|_{G_{F_{0,v}^+}}$ such that the reduction map induces an isomorphism $r(I_{F_{0,v}^+}) \to \overline{r}(I_{F_{0,v}^+})$.

We can invoke [BG19, Corollary 5.1.1] to conclude that R^{univ} has Krull dimension at least 1. We remark that this result includes the hypothesis that $\overline{r}|_{G_{F_0(\zeta_p)}}$ is irreducible, but this is used only to know that the groups $H^0(F_0^+, \operatorname{ad}\overline{r})$ and $H^0(F_0^+, \operatorname{ad}\overline{r}(1))$ vanish, which is true under the weaker condition that $\overline{r}|_{G_{F_0^+(\zeta_p)}}$ is Schur, which follows from our hypotheses. (The vanishing of these groups implies that the deformation functor is representable and that the Euler characteristic formula gives the correct lower bound for its dimension.)

We consider as well the deformation problem

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \Lambda_F, \overline{r}|_{G_F}, \epsilon^{1-n}\delta_{F/F^+}^n, \{R_v^{\triangle}\}_{v \in S_p} \cup \{R_v^{\square}\}_{v \in S-(S_p \cup \Sigma)} \cup \{R_v^{St}\}_{v \in \Sigma}),$$

where we define S, T, Σ to be the sets of places of F^+ above S_0, T_0, Σ_0, respectively. Then there is a natural morphism $R_{\mathcal{S}} \to R^{\mathrm{univ}}$ of Λ_F-algebras, which is finite (apply Lemma 5.3 and [Tho15, Proposition 3.29(2)]). By [ANT20, Theorem 6.2], $R_{\mathcal{S}}$ is a finite Λ_F-algebra. The map $\Lambda_F \to R^{\mathrm{univ}}$ factors through a homomorphism $\Lambda_F \to \mathcal{O}$ (by construction), so R^{univ} is a finite $\mathcal{O}$-algebra (of Krull dimension at least 1, as we have already remarked).

We deduce the existence of a lift $r : G_{F_0^+, S_0} \to \mathcal{G}_n(\overline{\mathbf{Z}}_p)$ of $\overline{r}$ arising from a homomorphism $R^{univ} \to \overline{\mathbf{Z}}_p$. We can now apply [ANT20, Theorem 6.1] and soluble descent to conclude that $r|_{G_{F_0}}$ is automorphic, associated to an automorphic representation π_0 of $\mathrm{GL}_n(\mathbf{A}_{F_0})$ with the desired properties. $\square$

6. Raising the level – Galois theory

This section is devoted to the proof of a single theorem that will bridge the gap between Theorem 4.1 and our intended applications. Let F, S, p, G be as in our standard assumptions (see §1), and let $n \geq 3$ be an odd integer such that $p > 2n$. Fix an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. We suppose given a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$, and that the following conditions are satisfied:

(1) π is ι-ordinary.
(2) $[F(\zeta_p) : F] = p - 1$.
(3) There exist characters $\overline{\chi}_1, \ldots, \overline{\chi}_n : G_F \to \overline{\mathbf{F}}_p^\times$ and an isomorphism $\overline{r}_{\pi,\iota} \cong \overline{\chi}_1 \oplus \cdots \oplus \overline{\chi}_n$, where for each $i = 1, \ldots, n$, we have $\overline{\chi}_i^c = \overline{\chi}_i^\vee \epsilon^{1-n}$ and for each $1 \leq i < j \leq n$, $\overline{\chi}_i/\overline{\chi}_j|_{G_{F(\zeta_p)}}$ has order strictly greater than $2n$.
(4) For each $v \in S_p$, $\overline{r}_{\pi,\iota}|_{G_{F_{\widetilde{v}}}}$ is trivial and $[F_{\widetilde{v}} : \mathbf{Q}_p] > n(n-1)/2 + 1$.
(5) There is a set $R = R_1 \sqcup R_2 \subset S - S_p$ with the following properties:
    (a) The sets R_1 and R_2 are both non-empty and for each $v \in R$, the characteristic of $k(v)$ is odd. As in §1.17, we write $\omega(\widetilde{v}) : k(\widetilde{v})^\times \to \{\pm 1\}$ for the unique quadratic character.
    (b) If $v \in R_1$, then q_v mod p is a primitive 3rd root of unity, and there exists a character $\Theta_{\widetilde{v}} : k_3^\times \to \mathbf{C}^\times$ of order p such that $\pi_{\widetilde{v}}|_{\mathfrak{q}_{\widetilde{v}}}$ contains

$\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}}, n)$ (notation as in §1.17, this representation of $\mathfrak{q}_{\widetilde{v}}$ defined with respect to $n_1 = 3$).

(c) If $v \in R_2$, then $q_v \bmod p$ is a primitive $(n-2)^{\text{th}}$ root of unity, and there exists a character $\Theta_{\widetilde{v}} : k_{n-2}^\times \to \mathbf{C}^\times$ of order p such that $\pi_{\widetilde{v}}|_{\mathfrak{q}_{\widetilde{v}}}$ contains $\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}}, n)$ (notation as in §1.17, this representation of $\mathfrak{q}_{\widetilde{v}}$ defined with respect to $n_1 = n-2$).

(d) For each non-trivial direct sum decomposition $\overline{r}_{\pi,\iota} = \overline{\rho}_1 \oplus \overline{\rho}_2$, there exists $i \in \{1,2\}$ such that for each $v \in R_i$, $\overline{\rho}_1|_{G_{F_{\widetilde{v}}}}$ and $\overline{\rho}_2|_{G_{F_{\widetilde{v}}}}$ both admit a non-trivial unramified subquotient.

(This is the situation we will find ourselves in after applying Theorem 4.1. The sets of places R_1 and R_2 here will correspond to the sets T_1 and T_3 respectively from §4, and it will be possible to label the characters $\overline{\chi}_i$ so that we have the following ramification properties:

- If $v \in R_1$, then $\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3$ are unramified at $\widetilde{v}$ and $\overline{\chi}_4, \dots, \overline{\chi}_n$ are ramified at $\widetilde{v}$ (and the image of inertia under each of these characters has order 2).
- If $v \in R_2$, then $\overline{\chi}_1, \overline{\chi}_2$ are ramified at $\widetilde{v}$ (and the image of inertia under each of these characters has order 2) and $\overline{\chi}_3, \dots, \overline{\chi}_n$ are unramified at $\widetilde{v}$.

These properties imply condition 5(d) above, which is what we actually need for the proofs in this section.)

The theorem we prove in this section is the following one:

Theorem 6.1. *With assumptions as above, fix a place v_{St} of F lying above $S - (S_p \cup R)$ such that $q_{v_{St}} \equiv 1 \pmod p$ and $\overline{r}_{\pi,\iota}|_{G_{F_{v_{St}}}}$ is trivial. Then we can find a RACSDC ι-ordinary automorphic representation π' of $\mathrm{GL}_n(\mathbf{A}_F)$ satisfying the following conditions:*

(1) There is an isomorphism $\overline{r}_{\pi',\iota} \cong \overline{r}_{\pi,\iota}$.

(2) For each embedding $\tau : F \to \overline{\mathbf{Q}}_p$, we have
$$\mathrm{HT}_\tau(r_{\pi',\iota}) = \mathrm{HT}_\tau(r_{\pi,\iota}).$$

(3) $\pi'_{v_{St}}$ is an unramified twist of the Steinberg representation.

Our proof of this theorem follows a similar template to the proof of [CT17, Theorem 5.1]. Briefly, we use our local conditions at the places of R, together with Theorem 5.7 to show that (after a suitable base change) we can find a generic prime in the spectrum of the big ordinary Hecke algebra. This puts us in a position to use the "$R_{\mathfrak{p}} = \mathbf{T}_{\mathfrak{p}}$" theorem proved in [ANT20], which is enough to construct automorphic lifts of $\overline{r}_{\pi,\iota}$ (or its base change) with the desired properties.

We now begin the proof. Let F^a/F denote the extension of $F(\zeta_p)$ cut out by $\overline{r}_{\pi,\iota}|_{G_{F(\zeta_p)}}$, and let Y^a be a finite set of finite places of F with the following properties:

- For each place $v \in Y^a$, v is split over F^+, prime to S, and π_v is unramified.
- For each intermediate Galois extension $F^a/M/F$ such that $\mathrm{Gal}(M/F)$ is simple, there exists $v \in Y^a$ which does not split in M.

Then any Y^a-split finite extension L/F is linearly disjoint from F^a/F. After conjugation, we can find a coefficient field E such that $r_{\pi,\iota}$ is valued in $\mathrm{GL}_n(\mathcal{O})$, and extend it to a homomorphism $r : G_{F^+,S} \to \mathcal{G}_n(\mathcal{O})$ such that $\nu \circ r = \epsilon^{1-n}\delta^n_{F/F^+}$. We write $\overline{r} : G_{F^+,S} \to \mathcal{G}_n(k)$ for the reduction modulo ϖ of r.

Lemma 6.2. *Let L/F be an Y^a-split finite CM extension. Then:*

(1) $\overline{r}|_{G_{L^+(\zeta_p)}}$ *is Schur.*
(2) $\overline{r}|_{G_L}$ *is primitive.*
(3) Suppose moreover that L/F *is soluble. Then the base change of* π *with respect to the extension* L/F *is cuspidal.*
(4) More generally, suppose that $L/F_0/F$ *is an intermediate field with* L/F_0 *soluble, and let* Π *be a RACSDC automorphic representation of* $\mathrm{GL}_n(\mathbf{A}_{F_0})$ *such that* $\overline{r}_{\Pi,\iota} \cong \overline{r}|_{G_{F_0}}$*. Then the base change of* Π *with respect to the extension* L/F_0 *is cuspidal.*

Proof. For the first part, it is enough to check that $L \not\subset L^+(\zeta_p)$ and $\overline{\chi}_i/\overline{\chi}_j|_{G_{L(\zeta_p)}}$ is non-trivial for each $1 \leq i < j \leq n$. We have $[L(\zeta_p) : L] = p - 1$, which implies $L \not\subset L^+(\zeta_p)$, while $\overline{\chi}_i/\overline{\chi}_j(G_{L(\zeta_p)}) = \overline{\chi}_i/\overline{\chi}_j(G_{F(\zeta_p)})$, so this ratio of characters is non-trivial. The second part follows from Lemma 5.1 and the fact that $\overline{\chi}_i/\overline{\chi}_j|_{G_L}$ has order greater than $2n$ for any $i \neq j$, because L/F is linearly disjoint from the extension F^a/F.

The third part is a special case of the fourth part, so we just prove this. Suppose for contradiction that the base change of Π with respect to the extension L/F_0 is not cuspidal. Then we can find intermediate extensions $L/F_2/F_1/F_0$ such that there is a tower $F_1 = M_m/M_{m-1}/\dots/M_0 = F_0$, where each extension M_{i+1}/M_i is cyclic of prime degree; F_2/F_1 is cyclic of prime degree l; the base change Π_{F_1} of Π to F_1 (constructed as the iterated cyclic base change with respect to the tower $M_m/M_{m-1}/\dots/M_0$) is cuspidal; but the base change of Π_{F_1} to F_2 is not cuspidal. We will derive a contradiction. By the second part of the lemma, $\overline{r}|_{G_{F_1}}$ is primitive. Let $\sigma \in \mathrm{Gal}(F_2/F_1)$ be a generator. Since the base change of Π_{F_1} with respect to the extension F_2/F_1 is not cuspidal, there exists a cuspidal automorphic representation Ξ of $\mathrm{GL}_{n/l}(\mathbf{A}_{F_2})$ such that the base change of Π_{F_1} is $\Xi \boxplus \Xi^\sigma \boxplus \cdots \boxplus \Xi^{\sigma^{l-1}}$ (see [AC89, Theorem 4.2]). We claim that Ξ is in fact conjugate self-dual. The representation $\Xi|\cdot|^{(n/l-n)/2}$ is regular algebraic (by [AC89, Theorem 5.1]). Since Π_{F_1} is conjugate self-dual, [AC89, Proposition 4.4]) shows that $\Xi \boxplus \Xi^\sigma \boxplus \cdots \boxplus \Xi^{\sigma^{l-1}}$ is also conjugate self-dual. The classification of automorphic representations of GL_n then implies that there is an isomorphism $\Xi^{c,\vee} \cong \Xi^{\sigma^i}$ for some $0 \leq i < l$. If w is an infinite place of F_2, then the purity lemma ([Clo90b, Lemma 4.9]) implies that $\Xi_w \cong \Xi_w^{c,\vee}$, hence $\Xi_w \cong \Xi_w^{\sigma^i}$. Since $\Xi \boxplus \Xi^\sigma \boxplus \cdots \boxplus \Xi^{\sigma^{l-1}}$ is regular algebraic, this is possible only if $i = 0$ and Ξ is indeed conjugate self-dual.

Therefore $r_{\Xi|\cdot|^{(n/l-n)/2},\iota}$ is defined and there is an isomorphism

$$r_{\Pi_{F_1},\iota} \cong \mathrm{Ind}_{G_{F_2}}^{G_{F_1}} r_{\Xi|\cdot|^{(n/l-n)/2},\iota}.$$

This contradicts the second part of the lemma, which implies that $\overline{r}_{\Pi_{F_1},\iota}$ is primitive. This contradiction completes the proof. □

Combining Lemma 6.2 and Proposition 5.8, we see that Theorem 6.1 will follow provided we can find a soluble CM extension L/F with the following properties:

- L/F is Y^a-split.
- There exists a RACSDC ι-ordinary automorphic representation π'' of $\mathrm{GL}_n(\mathbf{A}_L)$ and a place v'' of L lying above v_{St} such that $\overline{r}_{\pi'',\iota} \cong \overline{r}_{\pi,\iota}|_{G_L}$ and $\pi''_{v''}$ is an unramified twist of the Steinberg representation.

After first replacing F by a suitable $Y^a \cup R$-split soluble extension, we can assume in addition that $S = S_p \cup R \cup \{v_{St}|_{F^+}\}$ and that π is unramified outside $S_p \cup R$ (use the Skinner–Wiles base change trick as in [CHT08, Lemma 4.4.1]).

Lemma 6.3. *There exist infinitely many prime-to-S places $\widetilde{v}_a$ of F with the following property: $\widetilde{v}_a$ does not split in $F(\zeta_p)$ and $\overline{r}(\mathrm{Frob}_{\widetilde{v}_a})$ is scalar.*

Proof. By the Chebotarev density theorem, it is enough to find $\tau \in G_F$ such that $\overline{r}(\tau)$ is scalar and $\overline{\epsilon}(\tau) \neq 1$. We can choose any $\tau_0 \in G_F$ such that $\overline{\epsilon}(\tau_0)^2 \neq 1$, and set $\tau = \tau_0 \tau_0^c$. □

Choose a place $\widetilde{v}_a$ of F as in Lemma 6.3 which is absolutely unramified and of odd residue characteristic. We set $S_a = \{\widetilde{v}_a|_{F^+}\}$ and $\widetilde{S}_a = \{\widetilde{v}_a\}$.

We will need to consider several field extensions L/F and global deformation problems. We therefore introduce some new notation. We define a deformation datum to be a pair $\mathcal{D} = (L, \{R_v\}_{v\in X})$ consisting of the following data:

- A $Y^a \cup \widetilde{S}_a$-split, soluble CM extension L/F.
- A subset $X \subset S - S_p$, which may be empty. We write $\widetilde{X}$ for the pre-image of X in $\widetilde{S}$.
- For each $v \in X$, one of the following complete Noetherian local rings R_v, representing a local deformation problem:
  - For any $v \in R_L$ such that $\widetilde{v}$ is split over F, the ring $R_v = R(\widetilde{v}, \Theta_{\widetilde{v}}, \overline{r}|_{G_{L_{\widetilde{v}}}})$ (notation as in Proposition 1.22 – we define $\Theta_{\widetilde{v}} = \Theta_{\widetilde{v}|_F}$).
  - For any $v \in S_L$ such that $q_v \equiv 1 \bmod p$ and $\overline{r}|_{G_{L_{\widetilde{v}}}}$ is trivial, the unipotently ramified local lifting ring $R_v = R_v^1$ considered in [Tho15, §3.3.3].
  - For any $v \in S_L$ such that $q_v \equiv 1 \bmod p$ and $\overline{r}|_{G_{L_{\widetilde{v}}}}$ is trivial, the Steinberg local lifting ring $R_v = R_v^{St}$ considered in [Tho15, §3.3.4].

If $\mathcal{D} = (L, \{R_v\}_{v\in X})$ is a deformation datum, then we can define the global deformation problem

$$\mathcal{S}_{\mathcal{D}} = (L/L^+, S_{p,L} \cup X, \widetilde{S}_{p,L} \cup \widetilde{X}, \Lambda_L, \overline{r}|_{G_{L^+}}, \epsilon^{1-n}\delta^n_{L/L^+}, \{R_v^{\triangle}\}_{v\in S_{p,L}} \cup \{R_v\}_{v\in X}).$$

We write $R_{\mathcal{D}} = R_{\mathcal{S}_{\mathcal{D}}} \in \mathcal{C}_{\Lambda_L}$ for the representing object of the corresponding deformation functor.

Lemma 6.4. *If $\mathcal{D} = (L, \{R_v\}_{v\in X})$ is a deformation datum, then each irreducible component of $R_{\mathcal{D}}$ has dimension at least $1 + n[L^+ : \mathbf{Q}]$.*

Proof. This follows from [Tho15, Proposition 3.9], noting that the term $H^0(L^+, \mathrm{ad}\, \overline{r}(1))$ vanishes because $\overline{r}|_{G_{L^+(\zeta_p)}}$ is Schur (cf. [CHT08, Lemma 2.1.7]). □

Given a deformation datum $\mathcal{D}$, we define an open compact subgroup $U_{\mathcal{D}} = \prod_{v\in S_L - S_{p,L}} U_{\mathcal{D},v} \subset \prod_{v\in S_L - S_{p,L}} \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$ and a smooth $\mathcal{O}[U_{\mathcal{D}}]$-module $M_{\mathcal{D}} = \otimes_{v\in S_L - S_{p,L}} M_{\mathcal{D},v}$ as follows:

- If $v \notin X$, then $U_{\mathcal{D},v} = \mathrm{GL}_n(\mathcal{O}_{L_{\widetilde{v}}})$ and $M_{\mathcal{D},v} = \mathcal{O}$.
- If $v \in X \cap R_L$ and $R_v = R(\widetilde{v}, \Theta_{\widetilde{v}}, \overline{r}|_{G_{L_{\widetilde{v}}}})$, then $U_{\mathcal{D},v} = \mathfrak{q}_{\widetilde{v}}$ and $M_{\mathcal{D},v}$ is an $\mathcal{O}$-lattice in $\iota^{-1}\widetilde{\lambda}(\widetilde{v}, \Theta_{\widetilde{v}}, n)^\vee$ (notation as in §1.17).
- If $v \in X$ and $R_v = R_v^1$ or R_v^{St}, then $U_{\mathcal{D},v} = \mathrm{Iw}_{\widetilde{v}}$ and $M_{\mathcal{D},v} = \mathcal{O}$.

We now define a Hecke algebra $\mathbf{T}_{\mathcal{D}}$ associated to any deformation datum $\mathcal{D} = (L, \{R_v\}_{v \in X})$. It is to be a finite Λ_L-algebra (or zero). If $c \geq 1$, let $U(\mathcal{D}, c) \subset G(\mathbf{A}_{L^+}^\infty)$ be the open compact subgroup defined as follows:

$$U(\mathcal{D}, c) = \prod_{v \in S_{p,L}} \iota_{\widetilde{v}}^{-1} \mathrm{Iw}_{\widetilde{v}}(c, c) \times (\prod_{v \in S_L - S_{p,L}} \iota_{\widetilde{v}}^{-1}) U_{\mathcal{D}} \times \prod_{v \in S_{a,L}} \iota_{\widetilde{v}}^{-1} K_{\widetilde{v}}(1) \times G(\widehat{\mathcal{O}}_{L^+}^{S_L \cup S_{a,L}})$$

(here we are using the notation for open compact subgroups established in §1.23). Note that $U(\mathcal{D}, 1)$ is sufficiently small, because of our choice of v_a. We write

$$\mathbf{T}^{ord}(\mathcal{D}, c) \subset \mathrm{End}_{\mathcal{O}}(S^{ord}(U(\mathcal{D}, c), M_{\mathcal{D}}))$$

for the $\mathcal{O}$-subalgebra generated by the unramified Hecke operators T_w^j at split places $v = ww^c \notin S_L \cup S_{a,L}$ and the diamond operators $\langle u \rangle$ for $u \in \mathrm{Iw}_{\widetilde{v}}(1, c)$ $(v \in S_{p,L})$.

Following [Ger19, §2.4], we define $\mathbf{T}^{ord}(\mathcal{D}) = \varprojlim_c \mathbf{T}^{ord}(\mathcal{D}, c)$ and $H^{ord}(\mathcal{D}) = \varprojlim_c \mathrm{Hom}(S^{ord}(U(\mathcal{D}, c), M_{\mathcal{D}}), \mathcal{O})$. We endow $\mathbf{T}^{ord}(\mathcal{D})$ with a Λ_L-algebra structure using the same formula as in [Ger19, Definition 2.6.2]. We then have the following result.

Proposition 6.5. *$H^{ord}(\mathcal{D})$ is a finite free Λ_L-module and $\mathbf{T}^{ord}(\mathcal{D})$ is a finite faithful Λ_L-algebra, if it is non-zero.*

Proof. This can be proved in the same way as [Ger19, Proposition 2.20] and [Ger19, Corollary 2.21]. The proof uses that $U(\mathcal{D}, 1)$ is sufficiently small. $\square$

We write $\mathfrak{m}_{\mathcal{D}} \subset \mathbf{T}^{ord}(\mathcal{D})$ for the ideal generated by $\mathfrak{m}_{\Lambda_L}$ and the elements $T_w^j - q_w^{j(j-1)/2} \operatorname{tr} \wedge^j \overline{r}(\mathrm{Frob}_w)$ (w a split place of L/L^+ not lying above a place of $S_L \cup S_{a,L}$). It is either a maximal ideal with residue field k, or the unit ideal. In either case we set $\mathbf{T}_{\mathcal{D}} = \mathbf{T}^{ord}(\mathcal{D})_{\mathfrak{m}_D}$, which is either a finite local Λ_L-algebra or the zero ring. (In the cases we consider, it will be non-zero, but this will require proof.)

For any deformation datum $\mathcal{D}$, we write $P_{\mathcal{D}}$ for the Λ_L-subalgebra of $R_{\mathcal{D}}$ topologically generated by the coefficients of the characteristic polynomials of elements of G_L in (a representative of) the universal deformation $r_{\mathcal{S}_{\mathcal{D}}}$. By [Tho15, Proposition 3.26], the group determinant $\det r_{\mathcal{S}_{\mathcal{D}}}|_{G_F}$ is valued in $P_{\mathcal{D}}$, and $P_{\mathcal{D}}$ is a complete Noetherian local Λ_L-algebra. By [Tho15, Proposition 3.29], $R_{\mathcal{D}}$ is a finite $P_{\mathcal{D}}$-algebra.

Lemma 6.6. *Let $\mathcal{D} = (L, \{R_v\}_{v \in X})$ be a deformation datum, and suppose that $R_v \neq R_v^{St}$ for all $v \in X$. Then there is a natural surjective morphism $P_{\mathcal{D}} \to \mathbf{T}_{\mathcal{D}}$ of Λ_L-algebras.*

Proof. The proof is the essentially the same as the proof of [Tho15, Proposition 4.12], but we give the details for completeness. It is enough to construct maps $P_{\mathcal{D}} \to \mathbf{T}^{ord}(\mathcal{D}, c)_{\mathfrak{m}_D}$ which are compatible as $c \geq 1$ varies. Let $\Pi(\mathcal{D}, c)$ denote the set of automorphic representations σ of $G(\mathbf{A}_{L^+})$ with the following properties:

- $\overline{r}_{\sigma,\iota} \cong \overline{r}|_{G_L}$.
- σ_∞ is the trivial representation.
- The subspace

  $$\mathrm{Hom}_{U(\mathcal{D},c)}(M_{\mathcal{D}}^\vee, \iota^{-1}\sigma^\infty)^{ord} \subset \mathrm{Hom}_{U(\mathcal{D},c)}(M_{\mathcal{D}}^\vee, \iota^{-1}\sigma^\infty)$$

  where all the Hecke operators $U_{\widetilde{v},0}^j$ ($v \in S_p$, $j = 1, \ldots, n$) act with eigenvalues which are p-adic units is non-zero.

Then there is an injection

$$\mathbf{T}^{ord}(\mathcal{D}, c)_{\mathfrak{m}_D} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p \to \oplus_{\sigma \in \Pi(\mathcal{D}, c)} \overline{\mathbf{Q}}_p \tag{6.6.1}$$

which sends any Hecke operator to the tuple of its eigenvalues on each $(\iota^{-1}\sigma^\infty)^{U(\mathcal{D},c)}$. We can find a coefficient field E_c/E with ring of integers $\mathcal{O}_c$ and for each $\sigma \in \Pi(\mathcal{D}, c)$, a homomorphism $r_\sigma : G_{L^+,S} \to \mathcal{G}_n(\mathcal{O}_c)$ lifting $\overline{r}$ and such that $r_\sigma|_{G_L} \cong r_{\sigma,\iota}$ (apply [CHT08, Lemmas 2.1.5, 2.1.7]).

Let $A_c \subset k \oplus \bigoplus_{\sigma \in \Pi(\mathcal{D},c)} \mathcal{O}_c$ be the subring consisting of elements $(a, (a_\sigma)_\sigma)$ such that for each σ, $a_\sigma \bmod \varpi_c = a$. Then A_c is a local ring containing the image of the map (6.6.1), and the representation $\overline{r} \times (\times_{\sigma \in \Pi(\mathcal{D},c)} r_\sigma)$ is valued in $\mathcal{G}_n(A_c)$ and is of type $\mathcal{S}_{\mathcal{D}}$ (by our choice of deformation problems and level structures). Writing $Q_{S_L} \in \mathcal{C}_{\mathcal{O}}$ for the ring classifying pseudocharacters which lift $\operatorname{tr} \overline{r}|_{G_{L,S_L}}$, we see that there is a commutative diagram

$$\begin{array}{ccc} R_{\mathcal{D}} & \longrightarrow & A_c \\ \uparrow & & \uparrow \\ Q_{S_L} \widehat{\otimes}_{\mathcal{O}} \Lambda_L & \longrightarrow & \mathbf{T}^{ord}(\mathcal{D}, c)_{\mathfrak{m}_D} \end{array}$$

The ring $P_{\mathcal{D}}$ is equal to the image of the map $Q_{S_L} \widehat{\otimes}_{\mathcal{O}} \Lambda_L \to R_{\mathcal{D}}$. The proof is thus complete on noting that the right vertical arrow is injective and the bottom horizontal arrow is surjective. □

We define $J_{\mathcal{D}} = \ker(P_{\mathcal{D}} \to \mathbf{T}_{\mathcal{D}})$; this is a proper ideal if and only if $\mathbf{T}_{\mathcal{D}} \neq 0$.

We now fix a place $\overline{v}_{St}$ of $\overline{F}$ above v_{St}. If L/F is a CM extension, we write $v_{St,L} = \overline{v}_{St}|_L$.

Lemma 6.7. *We can find a deformation datum $\mathcal{D}_1 = (L_1, \emptyset)$ with the following properties:*

(1) $\mathbf{T}_{\mathcal{D}_1} \neq 0$.

(2) There exists a prime ideal $\mathfrak{p}_1 \subset R_{\mathcal{D}_1}$ of dimension 1 and characteristic p which is generic, such that $J_{\mathcal{D}_1} \subset \mathfrak{p}_1$, and such that $r_{\mathfrak{p}_1}|_{G_{L_1,v_{St,L_1}}}$ is the trivial representation.

Proof. We first claim that we can find an $\widetilde{R} \cup Y^a \cup \{\widetilde{v}_a\}$-split soluble CM extension L_0/F with the following properties:

- For each $i \in \{1, 2\}$, let L_{0,S_p} denote the maximal abelian pro-p extension of L_0 unramified outside S_{p,L_0}, and let $\Delta_{L_0} = \mathrm{Gal}(L_{0,S_p}/L_0)/(c+1)$. Let $d_{R_{i,L_0}}$ denote the $\mathbf{Z}_p$-rank of the subgroup of Δ_{L_0} topologically generated by the elements $\mathrm{Frob}_{\widetilde{v}}$, $v \in R_{i,L_0}$. Then $d_{R_{i,L_0}} > n + n^2$.
- For each $v \in S_{p,L_0}$, $[L_{0,\widetilde{v}} : \mathbf{Q}_p] > n^2$.
- For each $v \in S_{L_0} - (S_{p,L_0} \cup R_{L_0})$, $\overline{r}|_{G_{L_{0,\widetilde{v}}}}$ is trivial, and $q_v \equiv 1 \bmod p$.

The third property is automatic, since $S_{L_0} - (S_{p,L_0} \cup R_{L_0})$ consists of primes above $v_{St}|_{F^+}$. We can construct an extension satisfying the first two properties using a similar idea to the proof of [ANT20, Theorem 7.1]. Indeed, we can find, for any odd integer $d \geq 1$, a cyclic totally real extension M_d/F^+ which is $R \cup \{v|_{F^+} \mid v \in Y^a\} \cup S_a$-split and in which each place $v \in S_p$ is totally inert. If $d > n^2$ and $L_0 = M_d \cdot F$ then L_0/F will be a $\widetilde{R} \cup Y^a \cup \{\widetilde{v}_a\}$-split soluble CM extension which also satisfies the second point above. We need to explain how to arrange that the first point is also satisfied. By class field theory, $d_{R_{i,L_0}}$ is equal to the $\mathbf{Z}_p$-rank of the subgroup of $(\mathcal{O}_{L_0} \otimes_{\mathbf{Z}} \mathbf{Z}_p)^\times$ topologically generated by $(\mathcal{O}^\times_{L_0,R_{i,L_0}})^{c=-1}$. Since $\mathcal{O}^{c=-1}_{L_0,R_{i,L_0}} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}}_p$ decomposes as a $\overline{\mathbf{Q}}_p[\mathrm{Gal}(L_0/F^+)]$-module with multiplicity 1, [Mai02, Proposition 19] shows that this rank equals the $\mathbf{Z}$-rank of $(\mathcal{O}^\times_{L_0,R_{i,L_0}})^{c=-1}$, which is $d|R_i|$. Choosing any $d > n + n^2$ therefore gives an extension with the desired properties.

Let π_0 be the base change of π with respect to the extension L_0/F. It is cuspidal by Lemma 6.2. Let $\mathcal{D}_0 = (L_0, \{R(\widetilde{v}, \Theta_{\widetilde{v}}, \overline{r}|_{G_{L_{0,\widetilde{v}}}})\}_{v \in R_{L_0}})$. Then $\mathcal{D}_0$ is a deformation datum and the existence of π_0, together with Theorem 1.4, shows that $\mathbf{T}_{\mathcal{D}_0} \neq 0$. Let $B = R_{\mathcal{D}_0}/(J_{\mathcal{D}_0}, \mathfrak{m}_{R^{ur}_{v_{St,L_0}}})$. Then $\dim B \geq n[L_0^+ : \mathbf{Q}] - n^2$, and we may apply Theorem 5.7 to conclude the existence of a generic prime $\mathfrak{p}_0 \subset R_{\mathcal{D}_0}$ of dimension 1 and characteristic p which contains $(J_{\mathcal{D}_0}, \mathfrak{m}_{R^{ur}_{v_{St,L_0}}})$.

We now make another base change. Let L_1/L_0 be a CM extension with the following properties:

- L_1/F is soluble and $Y^a \cup \{\widetilde{v}_a\}$-split.
- For each $v \in R_{L_1}$, the natural morphism $R^\square_{\widetilde{v}} \to R(\widetilde{v}, \Theta_{\widetilde{v}}, \overline{r}|_{G_{L_{0,\widetilde{v}}}})$ factors over the unramified quotient $R^\square_{\widetilde{v}} \to R^{ur}_{\widetilde{v}}$ (cf. Proposition 1.22).
- For each $v \in S_{L_1} - S_{p,L_1}$, $q_v \equiv 1 \bmod p$ and $\overline{r}|_{G_{L_{1,\widetilde{v}}}}$ is trivial.

Let π_1 denote the base change of π_0 to L_1. Then π_1 is a RACSDC, ι-ordinary automorphic representation of $\mathrm{GL}_n(\mathbf{A}_{L_1})$ which is unramified outside $S_{p,L}$. Thus $\mathcal{D}_1 = (L_1, \emptyset)$ is a deformation datum and $\mathbf{T}_{\mathcal{D}_1} \neq 0$.

To complete the proof, it is enough to produce a commutative diagram

$$\begin{array}{ccccc} R_{\mathcal{D}_1} & \longleftarrow & P_{\mathcal{D}_1} & \longrightarrow & \mathbf{T}_{\mathcal{D}_1} \\ \downarrow & & \downarrow & & \downarrow \\ R_{\mathcal{D}_0} & \longleftarrow & P_{\mathcal{D}_0} & \longrightarrow & \mathbf{T}_{\mathcal{D}_0}. \end{array}$$

Indeed, then we can take $\mathfrak{p}_1$ to be the pullback of $\mathfrak{p}_0$ along the map $R_{\mathcal{D}_1} \to R_{\mathcal{D}_0}$. The only arrow which has not already been constructed is the arrow $\mathbf{T}_{\mathcal{D}_1} \to \mathbf{T}_{\mathcal{D}_0}$. This may be constructed in exactly the same way as [Tho15, Proposition 4.18], using the construction of the Hecke algebra as an inverse limit as in the proof of Lemma 6.6, provided we can prove the following statement: for any automorphic representation σ_0 of $G(\mathbf{A}_{L_0^+})$ satisfying the following conditions:

- There exists $c \geq 1$ such that $\mathrm{Hom}_{U(\mathcal{D}_0,c)}(M^\vee_{\mathcal{D}_0}, \iota^{-1}\sigma_0^\infty)^{ord} \neq 0$;
- $\sigma_{0,\infty}$ is trivial;
- There is an isomorphism $\overline{r}_{\sigma_0,\iota} \cong \overline{r}_{\pi,\iota}|_{G_{L_0}}$;

(in other words, such that σ_0 contributes to $S^{ord}(U(\mathcal{D}, c), M_{\mathcal{D}})_{\mathfrak{m}_{\mathcal{D}_0}}$ for some $c \geq 1$), there exists an automorphic representation σ_1 of $G(\mathbf{A}_{L_1^+})$ satisfying the following conditions:

- There exists $c \geq 1$ such that $\mathrm{Hom}_{U(\mathcal{D}_1,c)}(M^\vee_{\mathcal{D}_1}, \iota^{-1}\sigma_1^\infty)^{ord} \neq 0$;
- $\sigma_{1,\infty}$ is trivial;
- There is an isomorphism $r_{\sigma_1,\iota} \cong r_{\sigma_0,\iota}|_{G_{L_1}}$.

To see this, we first show that the base change of any such σ_0 to L_0 (in the sense of Theorem 1.2) must be cuspidal. We will show that in fact $r_{\sigma_0,\iota}$ is irreducible. Suppose that there is a decomposition $r_{\sigma_0,\iota} \cong \rho_1 \oplus \rho_2$. By assumption, there exists $v \in R_L$ such that both $\overline{\rho}_1|_{G_{L_{0,\widetilde{v}}}}$ and $\overline{\rho}_2|_{G_{L_{0,\widetilde{v}}}}$ admit an unramified subquotient.

However, local-global compatibility (together with Proposition 1.20) shows that $r_{\sigma_0,\iota}|_{G_{L_{0,\tilde{v}}}} \cong \rho'_1 \oplus \rho'_2$, where ρ'_1 is an irreducible representation of $G_{L_{0,\tilde{v}}}$ with unramified residual representation and ρ'_2 is a representation of $G_{L_{0,\tilde{v}}}$ such that $\overline{\rho}'_2$ is a sum of ramified characters. This is a contradiction unless one of ρ_1 and ρ_2 is the zero representation. If μ_0 denotes the base change of σ_0, a RACSDC ι-ordinary automorphic representation of $\mathrm{GL}_n(\mathbf{A}_{L_0})$, then the base change of μ_0 with respect to the soluble extension L_1/L_0 is also cuspidal, by Lemma 6.2, and the existence of σ_1 follows from Theorem 1.4. This completes the proof. □

We can now complete the proof of Theorem 6.1. We recall that it is enough to construct a RACSDC, ι-ordinary automorphic representation π'' of $\mathrm{GL}_n(\mathbf{A}_{L_1})$ such that $\overline{r}_{\pi'',\iota} \cong \overline{r}|_{G_{L_1}}$ and $\pi''_{v_{St,L_1}}$ is an unramified twist of the Steinberg representation. Let $\mathcal{D}_1$ and $\mathfrak{p}_1$ be as in the statement of Lemma 6.7. Consider the deformation data $\mathcal{D}_{1,a} = (L_1, \{R^{St}_{v_{St,L_1}}\})$ and $\mathcal{D}_{1,b} = (L_1, \{R^{1}_{v_{St,L_1}}\})$. Then there are surjections $R_{\mathcal{D}_{1,b}} \to R_{\mathcal{D}_{1,a}}$ and $R_{\mathcal{D}_{1,b}} \to R_{\mathcal{D}_1}$ and the prime $\mathfrak{p}_1$ lies in intersection of $\operatorname{Spec} R_{\mathcal{D}_1}$ and $\operatorname{Spec} R_{\mathcal{D}_{1,a}}$ in $\operatorname{Spec} R_{\mathcal{D}_{1,b}}$. We see that the hypotheses of [ANT20, Theorem 4.1] are satisfied for $R_{\mathcal{D}_{1,b}}$ (in the notation of loc. cit., we set $R = \{v_{St,L_1}\}, \chi_{v_{St,L_1}} = 1, S(B) = \emptyset$), and conclude that for any minimal prime $Q \subset R_{\mathcal{D}_{1,b}}$ contained in $\mathfrak{p}_1$, we have $J_{\mathcal{D}_{1,b}} \subset Q$. In particular, $\dim R_{\mathcal{D}_{1,b}}/Q = \dim \Lambda_{L_1}$. (We remark that the essential condition for us in applying [ANT20, Theorem 4.1] is that there is no ramification outside p; this is the reason for proving Lemma 6.7.)

Lemma 6.4 and Theorem 5.2 show together that each minimal prime of $R_{\mathcal{D}_{1,a}}$ has dimension equal to $\dim \Lambda_{L_1}$. Let $Q_a \subset R_{\mathcal{D}_{1,a}}$ be a minimal prime contained in $\mathfrak{p}_1$. Then Q_a is also a minimal prime of $R_{\mathcal{D}_{1,b}}$, $J_{\mathcal{D}_{1,b}} \subset Q_a$, and there exists a minimal prime Q_0 of Λ_{L_1} such that $R_{\mathcal{D}_{1,a}}/Q_a$ is a finite faithful Λ_{L_1}/Q_0-algebra. If $Q_1 = Q_a \cap P_{\mathcal{D}_{1,a}}$, then there are finite injective algebra maps

$$\Lambda_{L_1}/Q_0 \to P_{\mathcal{D}_{1,a}}/Q_1 \cong \mathbf{T}_{\mathcal{D}_{1,b}}/Q_1 \to R_{\mathcal{D}_{1,a}}/Q_a.$$

Using [Ger19, Lemma 2.25] and Theorem 1.2, we conclude the existence of an automorphic representation π'' of $\mathrm{GL}_n(\mathbf{A}_{L_1})$ with the required properties.

7. Level 1 case

The goal of this section is to prove Theorem E, using the level-raising results established in the last few sections. Combining this with the results of §§2 − 3, we will then be able to deduce Theorem A.

Our starting point is σ_0, the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ of weight 5 associated to the unique normalised newform

$$f_0(q) = q - 4q^2 + 16q^4 - 14q^5 - 64q^8 + \dots$$

of level $\Gamma_1(4)$ and weight 5; it is the automorphic induction from the quadratic extension $K = \mathbf{Q}(i)$ of the unique unramified Hecke character with ∞-type $(4, 0)$. For any prime $p \equiv 1 \bmod 4$ and isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, σ_0 is ι-ordinary. We observe that $r_{\sigma_0,\iota} \cong \mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \psi$ for a character $\psi : G_K \to \overline{\mathbf{Q}}_p^\times$ (which depends on p) and $\det r_{\sigma_0,\iota} = \delta_{K/\mathbf{Q}}\epsilon^{-4}$, where $\delta_{K/\mathbf{Q}} : G_{\mathbf{Q}} \to \{\pm 1\}$ is the quadratic character with kernel G_K.

The main technical result of this section is the following theorem:

Theorem 7.1. *Let $n \geq 3$ be an integer. Suppose given the following data:*

(1) A prime $p \equiv 1 \pmod{48n!}$ and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$.

(2) A prime $q \neq p$.
(3) A finite set X_0 of places of K, each prime to $2pq$.
(4) A de Rham character $\omega : G_K \to \overline{\mathbf{Q}}_p^\times$ such that $\omega\omega^c = \epsilon^3$ and $\omega|_{G_{K_v}}$ is unramified if $v \in X_0$.

Then there exists a soluble CM extension F/K with the following properties:
(1) F/K is X_0-split.
(2) There is a RACSDC, ι-ordinary automorphic representation Π of $\mathrm{GL}_n(\mathbf{A}_F)$ with the following properties:
(a) There is an isomorphism

$$\overline{r}_{\Pi,\iota} \cong \overline{\omega}^{n-1}|_{G_F} \otimes \mathrm{Sym}^{n-1}\,\overline{r}_{\sigma_0,\iota}|_{G_F}.$$

(b) For each embedding $\tau : F \to \overline{\mathbf{Q}}_p$, we have

$$\mathrm{HT}_\tau(r_{\Pi,\iota}) = \mathrm{HT}_\tau(\omega^{n-1}|_{G_F} \otimes \mathrm{Sym}^{n-1}\, r_{\sigma_0,\iota}|_{G_F}).$$

(c) There exists a place $v|q$ of F such that Π_v is an unramified twist of the Steinberg representation.

The following lemma will be used repeatedly.

Lemma 7.2. *Let $n \geq 3$ be an integer, and let $p \equiv 1 \pmod{48n!}$. Let $\omega : G_K \to \overline{\mathbf{Q}}_p^\times$ be a de Rham character such that $\omega\omega^c = \epsilon^3$.*

Let F/K be a finite CM extension which is linearly disjoint from the extension of $K(\zeta_p)$ cut out by $\overline{r}_{\sigma_0,\iota}|_{G_{K(\zeta_p)}}$, and set $\overline{\chi}_i = \overline{\omega}^{n-1} \otimes \overline{\psi}^{n-i}(\overline{\psi}^c)^{i-1}$, $\overline{\rho} = \overline{\chi}_1 \oplus \cdots \oplus \overline{\chi}_n$, so that there is an isomorphism

$$\overline{\rho} \cong \overline{\omega}^{n-1} \otimes \mathrm{Sym}^{n-1}\,\overline{r}_{\sigma_0,\iota}|_{G_K}.$$

Then:
(1) $[F(\zeta_p) : F] = p - 1$.
(2) $\overline{\psi}/\overline{\psi}^c|_{G_{F(\zeta_p)}}$ has order greater than $2n(n-1)$ and for each $1 \leq i < j \leq n$, $\overline{\chi}_i/\overline{\chi}_j|_{G_{F(\zeta_p)}}$ has order greater than $2n$.
(3) For each $1 \leq i \leq n$, $\overline{\chi}_i\overline{\chi}_i^c = \epsilon^{1-n}$.
(4) $\zeta_p \notin F^{\ker \mathrm{ad}\,\overline{\rho}}$ and $F \not\subset F^+(\zeta_p)$.
(5) $\overline{\rho}|_{G_F}$ is primitive.

Proof. We have $[F(\zeta_p) : F] = p - 1$ because F/K is disjoint from $K(\zeta_p)/K$. To justify the second point, let L/K denote the extension cut out by $\overline{\psi}/\overline{\psi}^c$. We must show that $[L \cdot F(\zeta_p) : F(\zeta_p)] > 2n(n-1)$. We note that $[L : K] \geq (p-1)/4$, because the restriction of $\overline{\psi}/\overline{\psi}^c$ to an inertia group at p has order $(p-1)/4$. Moreover, $L \cap K(\zeta_p)$ has degree at most 2 (since c acts as 1 on $\mathrm{Gal}(K(\zeta_p)/K)$ and as -1 on $\mathrm{Gal}(L/K)$), so $[L(\zeta_p) : K] \geq (p-1)^2/8$.

Since F/K is supposed disjoint from $L(\zeta_p)/K$, we have $[F \cdot L(\zeta_p) : K] \geq (p-1)^2[F : K]/8$. Since $[F(\zeta_p) : F] = p - 1$, we have $[F(\zeta_p) : K] = (p-1)[F : K]$. Putting these together we find

$$[F \cdot L(\zeta_p) : F(\zeta_p)] = \frac{[F \cdot L(\zeta_p) : K]}{[F(\zeta_p) : K]} \geq (p-1)/8.$$

Since we assume $p \equiv 1 \pmod{48n!}$, we in particular have $p - 1 \geq 48n!$, hence $(p-1)/8 > 2n(n-1)$.

If $1 \leq i < j \leq n$ then $\overline{\chi}_i/\overline{\chi}_j = (\overline{\psi}^c/\overline{\psi})^{i-j}$, so this shows the second point of the lemma. For the third point we compute

$$\overline{\chi}_i\overline{\chi}_i^c = (\overline{\omega\omega}^c|_{G_F})^{n-1}(\overline{\psi\psi}^c|_{G_F})^{n-1} = \epsilon^{1-n}.$$

We now come to the fourth point. To show that $\zeta_p \notin F^{\ker \operatorname{ad} \overline{\rho}}$, we must find $\tau \in G_F$ such that $\overline{\rho}(\tau)$ is scalar but $\overline{\epsilon}(\tau) \neq 1$. We can choose $\tau = \tau_0\tau_0^c$ for any $\tau_0 \in G_F$ such that $\overline{\epsilon}(\tau_0)^2 \neq 1$. Such a τ_0 exists because $[F(\zeta_p) : F] = p - 1$, and $\overline{\rho}(\tau)$ is scalar by the third part of the lemma. If $F \subset F^+(\zeta_p)$ then $F(\zeta_p) = F^+(\zeta_p)$ and $[F(\zeta_p) : F] = [F^+(\zeta_p) : F^+]/[F : F^+] = (p-1)/2$, contradicting the first part of the lemma.

For the fifth point it is enough, by Lemma 5.1, to show that for each $1 \leq i < j \leq n$, $\overline{\chi}_i/\overline{\chi}_j$ has order greater than n. This follows from the second point. □

Before giving the proof of Theorem 7.1, we give a corollary which establishes the existence of the automorphic representations necessary for the proof of Theorem 7.6.

Corollary 7.3. *Let $n \geq 3$ be an integer. Then there exists a cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ of weight 5 with the following properties:*

1. *σ is unramified away from 2 and a prime $q \equiv 3 \bmod 4$.*
2. *σ_2 is isomorphic to a principal series representation $i_{B_2}^{\mathrm{GL}_2}\chi_1 \otimes \chi_2$, where χ_1 is unramified and χ_2 has conductor 4.*
3. *σ_q is an unramified twist of the Steinberg representation.*
4. *For any prime p and any isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, $\mathrm{Sym}^{n-1} r_{\sigma,\iota}$ is automorphic.*

Proof. Choose a prime $p \equiv 1 \pmod{48n!}$ and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. It suffices to construct σ as in the statement of the corollary such that $\mathrm{Sym}^{n-1} r_{\sigma,\iota}$ is automorphic for our fixed choice of ι.

Let $F^{avoid}/\mathbf{Q}$ denote the extension cut out by $\overline{r}_{\sigma_0,\iota} \oplus \overline{\epsilon}$, and choose a prime q satisfying $q \equiv 3 \bmod 4$ (so $a_q(f_0) = 0$) and $q \equiv -1 \bmod p$. This implies that σ_0 satisfies the level-raising congruence at q. By a level-raising result for $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ (e.g. [Dia89, Corollary 6.9]), we can find an ι-ordinary cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ satisfying the following conditions:

- σ has weight 5, and $\overline{r}_{\sigma,\iota} \cong \overline{r}_{\sigma_0,\iota}$.
- σ is unramified at primes not dividing $2q$; σ_2 is isomorphic to a principal series representation $i_{B_2}^{\mathrm{GL}_2}\chi_1 \otimes \chi_2$, where χ_1 is unramified and χ_2 has conductor 4; and σ_q is an unramified twist of the Steinberg representation.

Let $\omega : G_K \to \overline{\mathbf{Q}}_p^\times$ be a character crystalline at p and unramified at q and such that $\omega\omega^c = \epsilon^3$. Then $(\psi\omega)(\psi\omega)^c = \epsilon^{-1}$. We take X_0 be a set of prime-to-$2pq$ places of K at which ω is unramified, and with the property that any X_0-split extension of K is linearly disjoint from F^{avoid}/K.

Let F/K and π be the soluble CM extension and RACSDC automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$ whose existence is asserted by Theorem 7.1. Thus in particular π is ι-ordinary, is an unramified twist of Steinberg at some place $v|q$ of F, and there are isomorphisms

$$\overline{r}_{\pi,\iota} \cong \overline{\omega}|_{G_F}^{n-1} \otimes (\oplus_{i=1}^n \overline{\psi}^{n-i}(\overline{\psi}^c)^{i-1}|_{G_F}) \cong \overline{\omega}^{n-1}|_{G_F} \otimes \mathrm{Sym}^{n-1}\,\overline{r}_{\sigma,\iota}|_{G_F}.$$

We now want to apply [ANT20, Theorem 6.1] (an automorphy lifting theorem) to conclude that the representation $\omega^{n-1} \otimes \mathrm{Sym}^{n-1} r_{\sigma,\iota}|_{G_F}$ is automorphic. This will in turn imply, by soluble descent, that $\mathrm{Sym}^{n-1} r_{\sigma,\iota}$ is automorphic. The hypotheses of [ANT20, Theorem 6.1] may be checked using Lemma 7.2. This concludes the proof. □

We first prove Theorem 7.1 in the case where $n = 2k+1$ is an odd integer, using the results of §§4 – 6.

Proposition 7.4. *Theorem 7.1 holds when $n = 2k+1$ is odd.*

Proof. We prove the proposition by induction on odd integers $n = 2k+1$. Let p, q, X_0, ω be as in the statement of Theorem 7.1. Let Z denote the set of rational primes below which ω is ramified, together with $2, p, q$. Let F^{avoid}/K denote the extension of K cut out by $\overline{r}_{\sigma_0,\iota} \oplus \overline{\epsilon}$. We fix a finite set X of finite places of K with the following properties:

- X contains X_0.
- If $v \in X$ then v is prime to Z. In particular, $\omega|_{G_{K_v}}$ is unramified.
- For each subextension M/K of F^{avoid}/K with $\mathrm{Gal}(M/K)$ simple and non-trivial, there exists $v \in X$ which does not split in M.

Let q_0 be a prime not in Z and which does not split in K, and let Y denote the set of rational primes dividing q_0 or an element of X. We make the following observations:

- If F/K is a finite X-split extension, then F/K is linearly disjoint from F^{avoid}/K.
- If $F_0/\mathbf{Q}$ is a finite Y-split extension, then $F_0/\mathbf{Q}$ is linearly disjoint from $F^{avoid}/\mathbf{Q}$ and F_0K/K is linearly disjoint from F^{avoid}/K.

Note in particular that Y-split extensions are linearly disjoint from $K/\mathbf{Q}$. We can find distinct rational primes q_1, q_2, q_3 satisfying the following conditions:

- For each $i = 1, 2, 3$, we have $q_i \notin Y \cup Z$ and q_i splits in K. In particular, q_i is odd.
- We have $q_1 > n$ and $q_1 \bmod p$ is a primitive 6^{th} root of unity. The eigenvalues of Frob_{q_1} on $\mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi}^{2k}$ have ratio $q_1^{\pm 1} \bmod p$, while the eigenvalues of Frob_{q_1} on $\mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi}$ have ratio which is a primitive $12k^{\text{th}}$ root of unity in $\mathbf{F}_p^\times$.
- We have $q_2 \equiv -1 \bmod p$ and the eigenvalues of Frob_{q_2} on $\mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi}^{2k}$ have ratio -1.
- The number $q_3 \bmod p$ is a primitive $(n-2)^{\text{th}}$ root of unity and the eigenvalues of Frob_{q_3} on $\mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi}$ have ratio $q_3^{\pm 1} \bmod p$.

To construct q_1, q_2, q_3 we use the Chebotarev density theorem. After conjugation, we can assume that $\overline{r}_{\sigma_0,\iota}|_{G_K} = \overline{\psi} \oplus \overline{\psi}^c$ is diagonal. Consideration of the restriction of $\overline{r}_{\sigma_0,\iota}|_{G_K}$ to the inertia groups at p shows that $(\overline{r}_{\sigma_0,\iota} \oplus \overline{\epsilon})(G_K)$ contains the subgroup

$$\{(\mathrm{diag}(a^4, (c/a)^4), c^{-1}) \mid a, c \in \mathbf{F}_p^\times\} \subset \mathrm{GL}_2(\overline{\mathbf{F}}_p) \times \mathbf{F}_p^\times.$$

We assume $p \equiv 1 \pmod{48n!}$, hence in particular $p \equiv 1 \pmod{96k}$. Let $z \in \mathbf{F}_p^\times$ be an element of order $96k$, and let $x_1 = z^{1+8k}$, $y_1 = z^{16k}$. Then y_1 is a primitive 6^{th} root of unity and $x_1^{16k} = y_1^{1+8k}$. If the prime q_1 is chosen so that $q_1 > n$ and $(\overline{r}_{\sigma_0,\iota} \oplus \overline{\epsilon})(\mathrm{Frob}_{q_1}) = (\mathrm{diag}(x_1^4, (y_1/x_1)^4), y_1^{-1})$, then the eigenvalues of Frob_{q_1} in $\mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi}$ have ratio $x_1^8/y_1^4 = z^{8+64k-64k} = z^8$, a primitive $12k^{\text{th}}$ root of unity, while the eigenvalues of Frob_{q_1} in $\mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi}^{2k}$ have ratio

$$z^{16k} = y_1 = \epsilon^{-1}(\mathrm{Frob}_{q_1}) \equiv q_1 \pmod{p}.$$

We can choose the prime q_2 so that $(\overline{r}_{\sigma_0,\iota} \oplus \overline{\epsilon})(\mathrm{Frob}_{q_2}) = (\mathrm{diag}(x_2^4, x_2^{-4}), -1)$, where $x_2 \in \mathbf{F}_p^\times$ satisfies $x_2^{16k} = -1$; and we can choose q_3 so that $(\overline{r}_{\sigma_0,\iota} \oplus \overline{\epsilon})(\mathrm{Frob}_{q_3}) = (\mathrm{diag}(x_3^4, (y_3/x_3)^4), y_3^{-1})$, where $x_3, y_3 \in \mathbf{F}_p^\times$, y_3 is a primitive $(n-2)^{\text{th}}$ root of unity, and x_3 is chosen so that $x_3^8 = y_3^3$. These choices of x_i, y_i are again possible because of the congruence $p \equiv 1 \bmod 48n!$.

We fix real quadratic extensions $M_i/\mathbf{Q}$ $(i = 1, 2, 3)$ with the following properties:

- M_1 is $Y \cup \{p, q, q_1, q_2\}$-split, and q_3 is ramified in M_1.
- M_2 is $Y \cup \{p, q, q_3\}$-split, and q_1, q_2 are ramified in M_2.
- M_3 is $Y \cup \{p, q, q_1, q_3\}$-split, and q_2 is ramified in M_3.

We write $\omega_i : G_{\mathbf{Q}} \to \{\pm 1\}$ for the quadratic character of kernel G_{M_i}.

By a level-raising result for $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ (e.g. [DT94, Theorem A], we can find a cuspidal, regular algebraic automorphic representation τ of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ with the following properties:

- τ is unramified outside $2, q_1, q_2$.
- There is an isomorphism $\overline{r}_{\tau,\iota} \cong \mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi}^{2k}$ and $\det r_{\tau,\iota} = \det \mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \psi^{2k} = \epsilon^{-8k} \delta_{K/\mathbf{Q}}$.
- τ_{q_1} is an unramified twist of the Steinberg representation, and there is an isomorphism $\mathrm{rec}_{\mathbf{Q}_{q_2}} \tau_{q_2} \cong \mathrm{Ind}_{W_{\mathbf{Q}_{q_2^2}}}^{W_{\mathbf{Q}_{q_2}}} \chi_{q_2}$, where $\chi_{q_2}|_{I_{\mathbf{Q}_q}}$ is a character of order p. In particular, τ_{q_2} is supercuspidal.
- τ is ι-ordinary and $r_{\tau,\iota}$ has the same Hodge–Tate weights as $\mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \psi^{2k}$.

For the ordinary condition in the last point, note that since $p > 8k$ and $r_{\tau,\iota}$ has Hodge–Tate weights $(0, 8k)$ and reducible local residual representation $\overline{r}_{\tau,\iota}|_{G_{\mathbf{Q}_p}}$ at p, τ is necessarily ι-ordinary [Edi92, Theorem 2.6].

By induction, there exists a soluble CM extension F_{-1}/K with the following properties:

- F_{-1}/K is X-split.
- There exists a RACSDC, ι-ordinary automorphic representation π of $\mathrm{GL}_{n-2}(\mathbf{A}_{F_{-1}})$ with the following properties:
  (1) There is an isomorphism
  $$\overline{r}_{\pi,\iota} \cong \overline{\omega}^{n-3}|_{G_{F_{-1}}} \otimes \mathrm{Sym}^{n-3}\, \overline{r}_{\sigma_0,\iota}|_{G_{F_{-1}}}.$$
  (2) For each embedding $\tau : F_{-1} \to \overline{\mathbf{Q}}_p$, we have
  $$\mathrm{HT}_\tau(r_{\pi,\iota}) = \mathrm{HT}_\tau(\omega^{n-3}|_{G_E} \otimes \mathrm{Sym}^{n-3}\, r_{\sigma_0,\iota}|_{G_E}).$$
  (3) There exists a place $v_{-1}|q$ of F_{-1} such that $\pi_{v_{-1}}$ is an unramified twist of the Steinberg representation.

We can find a soluble CM extension $F_0/\mathbf{Q}$ with the following properties:

- F_0 is Y-split.
- The prime q_1 is split in F_0^+, and each place of F_0^+ above q_1 is inert in F_0. The primes q_2, q_3 split in F_0.
- F_0/F_0^+ is everywhere unramified.
- For each place $v|p$ of F_0, v is split over F_0^+ and $[F_{0,v} : \mathbf{Q}_p] > n(n-1)/2 + 1$.
- For each place $v|q$ of F_0, v is split over F_0^+ and $q_v \equiv 1 \bmod p$.
- There exists a crystalline character $\omega_0 : G_{F_0} \to \overline{\mathbf{Q}}_p^\times$, unramified outside p, such that $\omega_0 \omega_0^c = \epsilon^3 \delta_{K/\mathbf{Q}}|_{G_{F_0}}$. (Use [BLGGT14, Lemma A.2.5].)

- For each place $v|pq$ of F_0, the representations $\overline{r}_{\sigma_0,\iota}|_{G_{F_{0,v}}}$ and $\overline{\omega}_0|_{G_{F_{0,v}}}$ are trivial.

Define

$$\overline{\rho}_0 = \overline{\omega}_0^{n-3} \otimes \left(\overline{\omega}_2 \otimes (\oplus_{i=1}^{k-1}(\mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi}^{2k-i-1}\overline{\psi}^{c,i-1})) \oplus \overline{\omega}_3 \epsilon^{-4(k-1)}\right)|_{G_{F_0}}.$$

Then for each place $v|pq$ of F_0, $\overline{\rho}_0|_{G_{F_{0,v}}}$ is trivial.

We now apply Proposition 5.8 with the following choices:

- $F_1 = F_{-1} \cdot F_0 \cdot M_1 \cdot M_2 \cdot M_3$.
- $\overline{\rho}_0$ is the residual representation defined above.
- Σ_0 is the set of places of F_0^+ lying above q; T_0 is the set of places of F_0^+ lying above q_2 or q_3; and S_0 is the set of places of F_0^+ lying above p, q, q_1, q_2, or q_3.
- If $v|q_2$, then $\overline{R}_v$ is the fixed type deformation ring (defined as in [Sho18, Definition 3.5]) associated to the inertial type $\oplus_{i=0}^{2k-2}\omega(v) \circ \mathrm{Art}_{F_{0,v}}|_{\mathcal{O}_{F_{0,v}}^\times}^{-1}$ (where as usual, $\omega(v)$ denotes the unique quadratic character of $k(v)^\times$ provided that $k(v)$ has odd characteristic). If $v|q_3$, then there is a character $\Theta_v : \mathcal{O}_{F_{0,v,n-2}}^\times \to \mathbf{C}^\times$ of order p such that $\overline{R}_v$ is the fixed type deformation ring associated to the inertial type $\oplus_{i=0}^{2k-2}\iota^{-1}\Theta_v^{q_v^{i-1}} \circ \mathrm{Art}_{F_{0,v,n-2}}|_{\mathcal{O}_{F_{0,v,n-2}}^\times}^{-1}$ (where $F_{0,v,n-2}/F_{0,v}$ is an unramified extension of degree $n-2$).
- π_1 is the twist of the base change of π with respect to the soluble CM extension F_1/F_{-1} by the character $\iota\omega_0^{n-3}|_{G_{F_1}}/\omega^{n-3}|_{G_{F_1}}$.

(Note that F_1/K is X-split, so Lemma 7.2 may be applied to $\overline{r}_{\pi_1,\iota}$.) We conclude the existence of a RACSDC, ι-ordinary automorphic representation π_0 of $\mathrm{GL}_{n-2}(\mathbf{A}_{F_0})$ satisfying the following conditions:

- There is an isomorphism $\overline{r}_{\pi_0,\iota} \cong \overline{\rho}_0$.
- π_0 is unramified outside S_0.
- For each place $v|q$ of F_0, $\pi_{0,v}$ is an unramified twist of the Steinberg representation.
- For each place $v|q_1$ of F_0, there are characters $\chi_{v,0}, \chi_{v,1} \ldots, \chi_{v,2k-2} : F_{0,v}^\times \to \mathbf{C}$ such that $\pi_{0,v} \cong \chi_{v,0} \boxplus \chi_{v,1} \boxplus \cdots \boxplus \chi_{v,2k-2}$, $\chi_{v,0}$ is unramified,and for each $i = 1, \ldots, 2k-2$, $\chi_{v,i}|_{\mathcal{O}_{F_{0,v}}^\times} = \omega(v)$.
- For each place $v|q_2$ of F_0, $\pi_{0,v}|_{\mathrm{GL}_{n-2}(\mathcal{O}_{F_{0,v}})}$ contains $\omega(v) \circ \det$.
- For each place $v|q_3$ of F_0, $\pi_{0,v}|_{\mathrm{GL}_{n-2}(\mathcal{O}_{F_{0,v}})}$ contains the representation $\widetilde{\lambda}(v, \Theta_v)$ (notation as in Proposition 1.18).

Let T_i denote the set of places of F_0^+ lying above q_i, and let $T = T_1 \cup T_2 \cup T_3$. Let τ_0 denote the base change of τ to F_0, and let $\pi_2 = \tau_0 \otimes |\cdot|^{(n-2)/2}\iota\omega_1\omega_0^{n-1}$. Let $\pi_{n-2} = \pi_0 \otimes |\cdot|^{-3}\iota\omega_0^2$. We see that the hypotheses of Theorem 4.1 are now satisfied, and we conclude the existence of a T-split quadratic totally real extension L_0^+/F_0^+ and a RACSDC ι-ordinary automorphic representation Π_0 of $\mathrm{GL}_n(\mathbf{A}_{L_0})$ satisfying the following conditions:

- The extension $L_0/\mathbf{Q}$ is soluble and Y-split.
- There is an isomorphism

$$\overline{r}_{\Pi_0,\iota} \cong \overline{\omega}_0^{n-1}|_{G_{L_0}} \otimes \Big(\overline{\omega}_1 \otimes \mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi}^{2k} \oplus \overline{\omega}_2 \otimes (\oplus_{i=1}^{k-1}(\mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi}^{2k-i}\overline{\psi}^{c,i})) \oplus \overline{\omega}_3\epsilon^{-4k}\Big)|_{G_{L_0}}.$$

- For each place $v|q_1$ of L_0, there exists a character $\Theta_v : \mathcal{O}_{L_{0,v,3}}^\times \to \mathbf{C}^\times$ of order p such that $\Pi_{0,v}|_{\mathfrak{r}_v}$ contains $\widetilde{\lambda}(v, \Theta_v, n)|_{\mathfrak{r}_v}$ (notation as in §1.17).
- For each place $v|q_3$ of L_0, $\Pi_{0,v}|_{\mathfrak{q}_v}$ contains the representation $\widetilde{\lambda}(v, \Theta_v, n)$ (where we define $\Theta_v = \Theta_{v|_{F_0}}$).

In fact, Π_0 has the following stronger property:

- For each place $v|q_1$ of L_0, $\Pi_{0,v}|_{\mathfrak{q}_v}$ contains $\widetilde{\lambda}(v, \Theta_v, n)$.

To see this, it is enough to check that no two eigenvalues $\alpha, \beta \in \overline{\mathbf{F}}_p^\times$ of the representation $(\mathrm{Sym}^{n-3} \mathrm{Ind}_{G_K}^{G_{\mathbf{Q}}} \overline{\psi})(\mathrm{Frob}_{q_1})$ satisfy $(\alpha/\beta)^2 = q_1^2$ (recall that if $v|q_1$ is a place of L_0, then $L_{0,v}/\mathbf{Q}_{q_1}$ is an unramified quadratic extension). Recalling the numbers $x_1, y_1 \in \mathbf{F}_p^\times$, we see that we must check that $(y_1^8/x_1^{16})^i \neq y_1^{\pm 2}$ for $i = 1, \dots, 2k-2$. However, by construction y_1^2 is a primitive 3^{rd} root of unity and y_1^8/x_1^{16} is a primitive $6k^{\text{th}}$ root of unity, so we cannot have $(y_1^8/x_1^{16})^{3i} = 1$ if $1 \leq i < 2k$.

Let $L_1 = L_0 K$. Then the following conditions are satisfied:

- The extension L_1/K is soluble and X-split.
- Let Π_1 denote the base change of Π_0 with respect to the quadratic extension L_1/L_0. Then Π_1 is RACSDC and ι-ordinary. (It is cuspidal because L_1/L_0 is quadratic and n is odd, cf. [AC89, Theorem 4.2].)
- For each place v of L_1 of residue characteristic q_1, q_3, v is split over L_0 and over L_1^+. (The prime q_i splits in K.)

Thus the hypotheses of Theorem 6.1 are satisfied with R_1 (resp. R_2) the set of places of L_1^+ of residue characteristic q_1 (resp. q_3), and we conclude the existence of a RACSDC ι-ordinary automorphic representation Π_1' of $\mathrm{GL}_n(\mathbf{A}_{L_1})$ satisfying the following conditions:

- There is an isomorphism $\overline{r}_{\Pi_1',\iota} \cong \overline{r}_{\Pi_0,\iota}|_{G_{L_1}}$.
- There exists a place $v|q$ of L_1 such that $\Pi_{1,v}'$ is an unramified twist of the Steinberg representation.

Finally, let $F = L_1 M_1 M_2 M_3$, and let Π' be the base change of Π_1' with respect to the extension F/L_1. We see that the conclusion of Theorem 7.1 holds with $\Pi = \Pi' \otimes \iota(\omega|_{G_F}/\omega_0|_{G_F})^{n-1}$. □

Proof of Theorem 7.1. If n is odd then the statement reduces to Proposition 7.4. Let $m \geq 1$ be an odd integer. We will prove by induction on $r \geq 0$ that the conclusion of Theorem 7.1 holds for all integers of the form $n = 2^r m$.

The case $r = 0$ is already known. Supposing the theorem known for a fixed $r \geq 0$ (hence $n = 2^r m$), we will now establish it for $r+1$ (hence $n' = 2^{r+1} m = 2n$). Fix data p, q, X_0, ω as in the statement of Theorem 7.1. In particular $p \equiv 1 \pmod{48 n'!}$. Once again we enlarge X_0 so that any X_0-split extension F/K is forced to be linearly disjoint from the fixed field of $\ker(\overline{r}_{\sigma_0,\iota} \oplus \overline{\epsilon})$.

By induction, we can find a soluble CM extension F/K and a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ such that the following conditions are satisfied:

- π is ι-ordinary. There is an isomorphism $\overline{r}_{\pi,\iota} \cong \overline{\omega}^{n-1}|_{G_F} \otimes \mathrm{Sym}^{n-1} \overline{r}_{\sigma_0,\iota}|_{G_F}$. The representations $r_{\pi,\iota}$ and $\omega^{n-1}|_{G_F} \otimes \mathrm{Sym}^{n-1} r_{\sigma_0,\iota}$ have the same Hodge–Tate weights (with respect to any embedding $\tau : F \to \overline{\mathbf{Q}}_p$).
- There exists a place $v|q$ such that π_v is an unramified twist of the Steinberg representation.

- F is X_0-split.

After possibly enlarging F, we can assume that the following additional conditions are satisfied:

- $q_v \equiv 1 \bmod p$ and $\overline{r}_{\sigma_0,\iota}(\mathrm{Frob}_v)$ is trivial.
- Each place of F which is either p-adic or at which π is ramified is split over F^+.

Let $\Omega, \Psi : K^\times \backslash \mathbf{A}_K^\times \to \mathbf{C}^\times$ be the Hecke characters of type A_0 with $r_{\Omega,\iota} = \omega$ and $r_{\Psi,\iota} = \psi$. Define $\pi_1 = \pi \otimes (|\cdot|^{n/2}(\Omega\Psi \circ \mathbf{N}_{F/K})^n)$, $\pi_2 = \pi \otimes (|\cdot|^{n/2}(\Omega\Psi^c \circ \mathbf{N}_{F/K})^n)$. We make the following observations:

- π_1 and π_2 are cuspidal, conjugate self-dual automorphic representations of $\mathrm{GL}_n(\mathbf{A}_F)$.
- Let $\pi_0 = \pi_1 \boxplus \pi_2$ and define $r_{\pi_0,\iota} = r_{\pi_1|\cdot|^{-n/2},\iota} \oplus r_{\pi_2|\cdot|^{-n/2},\iota}$. Then π_0 is regular algebraic and ι-ordinary. Moreover, for each finite place v of F there is an isomorphism $\mathrm{WD}(r_{\pi_0,\iota}|_{G_{F_v}})^{F-ss} \cong \mathrm{rec}^T_{F_v}(\pi_{0,v})$, and there is an isomorphism
$$\overline{r}_{\pi_0,\iota} \cong \overline{\omega}^{2n-1}|_{G_F} \otimes \mathrm{Sym}^{2n-1}\, \overline{r}_{\sigma_0,\iota}|_{G_F}.$$
- There are unramified characters $\xi_i : F_v^\times \to \mathbf{C}^\times$ such that $\pi_i \cong \mathrm{St}_n(\xi_i)$ and $\iota^{-1}\xi_1/\xi_2(\varpi_v) \equiv q_v^n \bmod \mathfrak{m}_{\overline{\mathbf{Z}}_p}$.
- $\overline{r}_{\pi_0,\iota}$ is not isomorphic to a twist of $1 \oplus \epsilon^{-1} \oplus \cdots \oplus \epsilon^{1-2^{r+1}m}$.

We justify each of these points in turn. Since π is conjugate self-dual, the first point follows from the fact that $(\Omega\Psi)(\Omega\Psi)^c = |\cdot|^{-1}$ (in turn a consequence of the identity $(\omega\psi)(\omega\psi)^c = \epsilon^{-1}$). The second follows from the identity
$$\mathrm{Sym}^{2n-1}\, r_{\sigma_0,\iota}|_{G_K} \cong \psi^{2n-1} \oplus \psi^{2n-2}\psi^c \oplus \cdots \oplus (\psi^c)^{2n-1} \cong (\psi^n \oplus (\psi^c)^n) \otimes \mathrm{Sym}^{n-1}\, r_{\sigma_0,\iota}|_{G_K}.$$
The third point holds by construction ($q_v \equiv 1 \pmod p$ and $\overline{r}_{\pi_0,\iota}(\mathrm{Frob}_v)$ is scalar). The fourth holds since otherwise $\overline{r}_{\pi_0,\iota}|_{G_{F(\zeta_p)}}$ would be a twist of the trivial representation, contradicting part 2 of Lemma 7.2.

We see that the hypotheses of [AT21, Theorem 5.1] are satisfied. This theorem implies that we can find a quadratic CM extension F'/F such that F'/K is soluble X_0-split, as well as a RACSDC automorphic representation π' of $\mathrm{GL}_{n'}(\mathbf{A}_{F'})$ satisfying the following conditions:

- π' is ι-ordinary, and there is an isomorphism
$$\overline{r}_{\pi',\iota} \cong \overline{\omega}^{n'-1}|_{G_{F'}} \otimes \mathrm{Sym}^{n'-1}\, \overline{r}_{\sigma_0,\iota}|_{G_{F'}}.$$
- The weight of π' is the same as that of π_0.
- There exists a place $v'|v$ of F' such that $\pi'_{v'}$ is an unramified twist of the Steinberg representation.

This existence of F' and π' completes the induction step, and therefore the proof of the theorem. □

Remark 7.5. We observe that the results of [AT21] already suffice to prove Theorem 7.1 (and hence Theorem 7.7) when n is a power of two, without using the level raising results of Sections 4–6.

We can now put everything together to deduce our main results on automorphy of symmetric powers.

Theorem 7.6. *Let $n \geq 3$. Then there exists a cuspidal, everywhere unramified automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of weight $k \geq 2$ such that, for any isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ is automorphic.*

Proof. By Corollary 7.3, we can find odd primes $p \neq q$, with $q \equiv 3 \bmod 4$, and a cuspidal automorphic representation σ of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of weight 5 satisfying the following conditions:

- σ is unramified at primes not dividing $2q$; σ_2 is isomorphic to a principal series representation $i_{B_2}^{\mathrm{GL}_2} \chi_1 \otimes \chi_2$, where χ_1 is unramified and χ_2 has conductor 4; and σ_q is an unramified twist of the Steinberg representation.
- For any isomorphism $\iota_p : \overline{\mathbf{Q}}_p \to \mathbf{C}$, $\mathrm{Sym}^{n-1} r_{\pi,\iota_p}$ is automorphic.

Now we choose an isomorphism $\iota_q : \overline{\mathbf{Q}}_q \to \mathbf{C}$. By the second part of Lemma 3.5, the Zariski closure of $r_{\sigma,\iota_q}(G_{\mathbf{Q}_q})$ contains SL_2. Since σ_q is an unramified twist of Steinberg it has a unique (q-adic) accessible refinement, which is numerically non-critical and n-regular. We can therefore apply Theorem 2.33 to the point of the q-adic, tame level 4 eigencurve associated to σ with its unique accessible refinement. Using the accumulation property of the eigencurve to find a suitable classical point in the same (geometric) irreducible component as this point, we deduce the existence of a cuspidal automorphic representation σ' of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of weight $k > 2$ satisfying the following conditions:

(1) σ' is unramified outside 2, and σ'_2 is isomorphic to a principal series representation $i_{B_2}^{\mathrm{GL}_2} \chi_1 \otimes \chi_2$, where χ_1 is unramified and χ_2 has conductor 4.
(2) The weight of σ' satisfies $k \equiv 3 \bmod 4$ (this is possible because $q \equiv 3 \bmod 4$, and we can choose any $k \equiv 5 \bmod (q-1)q^\alpha$ for sufficiently large α).
(3) $\mathrm{Sym}^{n-1} r_{\sigma',\iota_q}$ is automorphic.

Let $\iota : \overline{\mathbf{Q}}_2 \to \mathbf{C}$ be an isomorphism. These conditions imply that the Zariski closure of $r_{\sigma',\iota}(G_{\mathbf{Q}_2})$ must contain SL_2. Indeed, we have already observed in §3 that there are no 2-ordinary cusp forms of tame level 1, so (invoking Lemma 3.5) if this Zariski closure does not contain SL_2 then $r_{\sigma',\iota}|_{G_{\mathbf{Q}_2}}$ must be irreducible and induced from a quadratic extension of $\mathbf{Q}_2$, implying that both refinements of σ' at the prime 2 have slope $(k-1)/2$, an odd integer. However, Theorem 3.2 implies that there are no newforms of level 4 and odd slope (see [BK05, Corollary of Theorem B]); a contradiction. The same argument shows that the refinement $\chi_1 \otimes \chi_2$ is n-regular, since the two refinements of σ' have distinct slopes.

We see that $(\sigma', \chi_1 \otimes \chi_2)$ satisfies the hypotheses of Theorem 2.33. Using the accumulation property of the (tame level 1, 2-adic) eigencurve, we deduce the existence of a cuspidal, everywhere unramified automorphic representation π of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ such that $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ is automorphic. This completes the proof. □

Combining Theorem 7.6 with Theorem 3.1, we deduce:

Theorem 7.7. *Let $n \geq 3$, and let π be a cuspidal, everywhere unramified automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of weight $k \geq 2$. Then for any isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ is automorphic.*

8. Higher levels

In this section we extend our main theorem to higher levels as follows:

Theorem 8.1. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of weight $k \geq 2$ satisfying the following two conditions:*

(1) For each prime l, π_l has non-trivial Jacquet module (equivalently, π_l admits an accessible refinement).
(2) π is not a CM form.

Then for any $n \geq 3$ and any isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ is automorphic.

For example, these conditions are satisfied if π is associated to a non-CM cuspidal eigenform f of level $\Gamma_1(N)$ for some squarefree integer $N \geq 1$; in particular, if $k = 2$ and π is associated to a semistable elliptic curve over $\mathbf{Q}$.

Fix $n \geq 3$ for the remainder of this section. We first prove the following special case of Theorem 8.1:

Proposition 8.2. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of weight $k \geq 2$ satisfying the following conditions:*

(1) For each prime l such that π_l is ramified, π_l has an accessible refinement which is n-regular, in the sense of Definition 2.23.
(2) π is not a CM form.

Then for any isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ is automorphic.

Proof. We prove the proposition by induction on the number of primes r dividing the conductor N of π. The case $r = 0$ (equivalently, $N = 1$) is Theorem 7.7.

Suppose therefore that $r > 0$ and that the theorem is known for automorphic representations of conductor divisible by strictly fewer than r primes. Let π be a cuspidal automorphic representation as in the statement of the proposition. Fix a prime p at which π is ramified, and an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$. Factor $N = Mp^s$, where $(M, p) = 1$.

Suppose first that $r_{\pi,\iota}|_{G_{\mathbf{Q}_p}}$ is reducible. Then π is ι-ordinary and π admits an ordinary refinement χ. After twisting by a finite order character, we can assume that $(\pi, \chi) \in \mathcal{RA}_0$ (here we use the notation established for the Coleman–Mazur eigencurve in §2.31). Let $\mathcal{C}$ be an irreducible component of the (tame level M, p-adic) eigencurve $\mathcal{E}_{0,\mathbf{C}_p}$ containing the point x corresponding to (π, χ), and let $\mathcal{Z} \subset \mathcal{E}_0$ denote the Zariski closed set defined in Lemma 2.35. Our hypotheses imply that $x \notin \mathcal{Z}_{\mathbf{C}_p}$.

We can therefore find a point $x'' \in \mathcal{C} - \mathcal{Z}_{\mathbf{C}_p}$ such that the image of x'' in $\mathcal{W}_{0,\mathbf{C}_p}$ is a character of the form $y \mapsto y^{k''-2}$ for some integer $k'' \geq 2$. Indeed, since the image of $\mathcal{C}$ in $\mathcal{W}_{0,\mathbf{C}_p}$ is Zariski open, we can find such a point in $\mathcal{C}$. There is an affinoid neighbourhood U'' of this point which maps in a finite and surjective fashion onto an affinoid open in $\mathcal{W}_{0,\mathbf{C}_p}$. The image of $\mathcal{Z}_{\mathbf{C}_p} \cap U''$ in this affinoid open is Zariski closed, and we can therefore find another such point $x'' \in \mathcal{C} - \mathcal{Z}_{\mathbf{C}_p}$. (In fact, the ordinary component $\mathcal{C}$ surjects onto a connected component of $\mathcal{W}_{0,\mathbf{C}_p}$, but we will apply the same argument for a non-ordinary component.)

Choosing another point in a sufficiently small affinoid neighbourhood of x'' in $\mathcal{C} - \mathcal{Z}_{\mathbf{C}_p}$ and applying the classicality criterion, we can find a point $x' \in \mathcal{C} - \mathcal{Z}_{\mathbf{C}_p}$ corresponding to an ι-ordinary cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$ of weight $k' \geq 2$ with the following properties:

(1) Let χ' denote the ordinary refinement of π'. Then (π', χ') determines a point on the same irreducible component of the (tame level M, p-adic) eigencurve $\mathcal{E}_{0,\mathbf{C}_p}$ as (π, χ).

(2) The level of π' is prime to p.
(3) For each prime $l|M$, each accessible refinement of π'_l is n-regular.
(4) The Zariski closure of $r_{\pi',\iota}(G_{\mathbf{Q}})$ (in $\mathrm{GL}_2/\overline{\mathbf{Q}}_p$) contains SL_2.

(The latter two properties follow from the definition of the set $\mathcal{Z}$ in Lemma 2.35. In fact we can take $x' = x''$, since ordinary points of classical weights are classical; however, we will repeat the same argument in the next paragraph also for a non-ordinary component of the eigenvariety, in which case two steps are required.) By induction, $\mathrm{Sym}^{n-1} r_{\pi',\iota}$ is automorphic. We may then apply the ordinary case of Theorem 2.33 to conclude that $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ is automorphic.

Suppose instead that $r_{\pi,\iota}|_{G_{\mathbf{Q}_p}}$ is irreducible, and let χ be an accessible, n-regular refinement. The existence of χ implies that the Zariski closure of $r_{\pi,\iota}(G_{\mathbf{Q}_p})$ in $\mathrm{GL}_2/\overline{\mathbf{Q}}_p$ contains SL_2, by Lemma 3.5. Again, after twisting by a finite order character, we can assume that $(\pi, \chi) \in \mathcal{RA}_0$. Repeating the same argument as in the ordinary case, we can find a cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ of weight $k' \geq 2$ with the following properties:

(1) π' admits a non-ordinary refinement χ' which is numerically non-critical and n-regular. (This again implies that the Zariski closure of $r_{\pi',\iota}(G_{\mathbf{Q}_p})$ contains SL_2.)
(2) The pair (π', χ') determines a point on the same irreducible component of the (tame level M, p-adic) eigencurve $\mathcal{E}_{0,\mathbf{C}_p}$ as (π, χ).
(3) For each prime $l|M$, each accessible refinement of π'_l is n-regular.
(4) The level of π' is prime to p.

By induction, $\mathrm{Sym}^{n-1} r_{\pi',\iota}$ is automorphic. We can then appeal to Theorem 2.33 to conclude that $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ is automorphic.

In either case we are done, by induction. $\square$

To reduce the general case of Theorem 8.1 to Proposition 8.2, we establish the following intermediate result.

Proposition 8.3. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ of weight $k \geq 2$, without CM. Suppose that for each prime l, π_l has non-trivial Jacquet module. Then we can find a prime p, an isomorphism $\iota : \overline{\mathbf{Q}}_p \to \mathbf{C}$, and another cuspidal automorphic representation π' of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ of weight k with the following properties:*

(1) $p > \max(2(n+1), (n-1)k)$.
(2) The image of $\overline{r}_{\pi,\iota}$ contains a conjugate of $\mathrm{SL}_2(\mathbf{F}_p)$.
(3) Both π_p and π'_p are unramified.
(4) There is an isomorphism $\overline{r}_{\pi,\iota} \cong \overline{r}_{\pi',\iota}$.
(5) For each prime l, π'_l has non-trivial Jacquet module. If π'_l is ramified, then each accessible refinement of π'_l is n-regular.

Proof. We use Taylor–Wiles–Kisin patching. The idea is that if all the automorphic representations congruent to π mod p fail to have n-regular refinements at l then the patched module will be supported on a codimension one quotient of the local deformation ring at l, which contradicts the numerology of the Taylor–Wiles–Kisin method.

Let M denote the conductor of π. We can choose a prime p satisfying (1) and (2), $p > M$, such that π_p is unramified, and satisfying the following additional condition:

- For each prime $l \neq p$ such that π_l is ramified, the universal lifting ring classifying lifts of $\overline{r}_{\pi,\iota}|_{G_{\mathbf{Q}_l}}$ of determinant equal to $\det r_{\pi,\iota}$ is formally smooth.

Indeed, it is sufficient that for each such prime l, the group $H^0(\mathbf{Q}_l, \mathrm{ad}^0\, \overline{r}_{\pi,\iota}(1))$ vanishes. Such a prime exists thanks to [Wes04, Proposition 3.2, Proposition 5.3].

Fix an additional prime $q_a > p$ such that π_{q_a} is unramified and such that the universal lifting ring classifying lifts of $\overline{r}_{\pi,\iota}|_{G_{\mathbf{Q}_{q_a}}}$ of determinant equal to $\det r_{\pi,\iota}$ is formally smooth. This is possible by e.g. [DT94, Lemma 11].

Fix a coefficient field $E/\mathbf{Q}_p$, large enough that there is a conjugate $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ of $\overline{r}_{\pi,\iota}$ and such that $\chi = \det r_{\pi,\iota}$ takes values in $\mathcal{O}$. We assume moreover that for each $\sigma \in G_{\mathbf{Q}}$, the roots of the characteristic polynomial of $\overline{\rho}(\sigma)$ lie in k. Let S denote the set of primes at which $r_{\pi,\iota}$ is ramified (equivalently, at which $\overline{\rho}$ is ramified), together with q_a. We consider the global deformation problem (in the sense of [Tho16, Definition 5.6])

$$\mathcal{S} = (\overline{\rho}, \chi, S, \{\mathcal{O}\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S}),$$

where $\mathcal{D}_p$ is the functor of lifts of $\overline{\rho}|_{G_{\mathbf{Q}_p}}$ of determinant χ which are Fontaine–Laffaille with the same Hodge–Tate weights as $r_{\pi,\iota}$, and if $l \in S - \{p\}$ then $\mathcal{D}_l$ is the functor of all lifts of $\overline{\rho}|_{G_{\mathbf{Q}_l}}$ of determinant χ. Since $\overline{\rho}$ is absolutely irreducible, the functor of deformations of type $\mathcal{S}$ is represented by an object $R_{\mathcal{S}} \in \mathcal{C}_{\mathcal{O}}$ (cf. [Tho16, Theorem 5.9]). We may choose a representative $\rho_{\mathcal{S}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(R_{\mathcal{S}})$ of the universal deformation. We set $H = H^1(Y_{U_1(Mq_a)}, \mathrm{Sym}^{k-2}\, \mathcal{O}^2)$, where $U_1(Mq_a)$ is the open compact subgroup of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}}^\infty)$ defined in §2.31 and Y_U is the modular curve of level U (denoted $\widetilde{Y}(U)$ in [Eme06b, §4.1]). We write $\mathbf{T}^S \subset \mathrm{End}_{\mathcal{O}}(H)$ for the commutative $\mathcal{O}$-subalgebra generated by the unramified Hecke operators T_l, S_l for $l \notin S$. Then there is a unique maximal ideal $\mathfrak{m} \subset \mathbf{T}^S$ with residue field k such that for each prime $l \notin S$, the characteristic polynomial of $\overline{\rho}(\mathrm{Frob}_l)$ equals $X^2 - T_l X + l^{k-1} S_l \bmod \mathfrak{m}$. The localization $H_{\mathfrak{m}}$ is a finite free $\mathcal{O}$-module, and there is a unique strict equivalence class of liftings $\rho_{\mathfrak{m}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}^S_{\mathfrak{m}})$ of type $\mathcal{S}$ such that for each prime $l \notin S$, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_l)$ equals the image of $X^2 - T_l X + l^{k-1} S_l$ in $\mathbf{T}^S_{\mathfrak{m}}[X]$. (See [Tho16, Proposition 6.5] for justification of a very similar statement in the context of Shimura curves.) In particular, there is an $\mathcal{O}$-algebra morphism $R_{\mathcal{S}} \to \mathbf{T}^S_{\mathfrak{m}}$ classifying $\rho_{\mathfrak{m}}$, which is surjective.

Suppose given a finite set Q of primes satisfying the following conditions:

(a) $Q \cap S = \emptyset$.
(b) For each $q \in Q$, $q \equiv 1 \bmod p$ and $\overline{\rho}(\mathrm{Frob}_q)$ has distinct eigenvalues $\alpha_q, \beta_q \in k$.

In this case we can define the following additional data:

- The group $\Delta_Q = \prod_{q \in Q} (\mathbf{Z}/q\mathbf{Z})^\times(p)$ (i.e. the maximal p-power quotient of the product of the units in each residue field).
- The augmented global deformation problem

$$\mathcal{S}_Q = (\overline{\rho}, \chi, S \cup Q, \{\mathcal{O}\}_{v \in S \cup Q}, \{\mathcal{D}_v\}_{v \in S \cup Q}),$$

  where for each $q \in Q$, $\mathcal{D}_q$ is the functor all lifts of $\overline{\rho}|_{G_{\mathbf{Q}_q}}$ of determinant χ. The labelling of α_q, β_q for each $q \in Q$ determines an algebra homomorphism $\mathcal{O}[\Delta_Q] \to R_{\mathcal{S}_Q}$ in the following way: if $\rho_{\mathcal{S}_Q}$ is a representative of the universal deformation, then $\rho_{\mathcal{S}_Q}|_{G_{\mathbf{Q}_q}}$ is conjugate to a representation of the form $A_q \oplus B_q$, where $A_q : G_{\mathbf{Q}_q} \to R_{\mathcal{S}_Q}^\times$ is a character such that $A_q \bmod \mathfrak{m}_{R_{\mathcal{S}_Q}}$

is unramified and $A_q \bmod \mathfrak{m}_{R_{\mathcal{S}_Q}}(\mathrm{Frob}_q) = \alpha_q$ (and similarly for B_q). Then $A_q \circ \mathrm{Art}_{\mathbf{Q}_q}|_{\mathbf{Z}_q^\times}$ factors through a homomorphism $(\mathbf{Z}/q\mathbf{Z})^\times(p) \to R_{\mathcal{S}_Q}^\times$. These homomorphisms for $q \in Q$ collectively determine the algebra homomorphism $\mathcal{O}[\Delta_Q] \to R_{\mathcal{S}_Q}$.

- The cohomology module $H_Q = H^1(Y_{U_1(Mq_a)\cap U_2(Q)}, \mathrm{Sym}^{k-2}\,\mathcal{O}^2)$, where we define
$$U_2(Q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbf{Z}}) : c \equiv 0 \bmod (\prod_{q\in Q} q), ad^{-1} \mapsto 1 \in \Delta_Q \right\},$$
and commutative $\mathcal{O}$-subalgebras $\mathbf{T}^{S\cup Q} \subset \mathbf{T}_Q^{S\cup Q} \subset \mathrm{End}_{\mathcal{O}}(H_Q)$. By definition, $\mathbf{T}^{S\cup Q}$ is generated by the unramified Hecke operators T_l, S_l for $l \notin S \cup Q$ and $\mathbf{T}_Q^{S\cup Q}$ is generated by $\mathbf{T}_Q^{S\cup Q}$ and the operators U_q for $q \in Q$. There are maximal ideals $\mathfrak{m}_Q \subset \mathbf{T}^{S\cup Q}$ and $\mathfrak{m}_{Q,1} \subset \mathbf{T}_Q^{S\cup Q}$ with residue field k defined as follows: $\mathfrak{m}_Q$ is the unique maximal ideal such that for each prime $l \notin S \cup Q$, the characteristic polynomial of $\overline{\rho}(\mathrm{Frob}_l)$ equals $X^2 - T_l X + l^{k-1}S_l \bmod \mathfrak{m}_Q$. The ideal $\mathfrak{m}_{Q,1}$ is generated by $\mathfrak{m}_Q$ and the elements $U_q - \alpha_q$ for $q \in Q$. There is a unique strict equivalence class of liftings $\rho_{\mathfrak{m}_Q} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{T}^{S\cup Q}_{\mathfrak{m}_Q})$ of type $\mathcal{S}_Q$ such that for each $l \notin S \cup Q$, the characteristic polynomial of $\rho_{\mathfrak{m}_Q}(\mathrm{Frob}_l)$ equals the image of $X^2 - T_l X + l^{k-1} S_l$ in $\mathbf{T}^{S\cup Q}_{\mathfrak{m}_Q}[X]$. There is an $\mathcal{O}$-algebra morphism $R_{\mathcal{S}_Q} \to \mathbf{T}^{S\cup Q}_{\mathfrak{m}_Q}$ classifying $\rho_{\mathfrak{m}_Q}$, which is surjective. Moreover, if we view $H_{Q,\mathfrak{m}_{Q,1}}$ as an $R_{\mathcal{S}_Q}$-module via this map, then the two $\mathcal{O}[\Delta_Q]$-module structures on $H_{Q,\mathfrak{m}_{Q,1}}$, one arising from $R_{\mathcal{S}_Q}$, the other arising from the action of Δ_Q via Hecke operators, coincide. (These statements in turn may be justified as in the proof of [Tho16, Lemma 6.8].) Finally, $H_{Q,\mathfrak{m}_{Q,1}}$ is a free $\mathcal{O}[\Delta_Q]$-module and there is an isomorphism $H_{Q,\mathfrak{m}_{Q,1}} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong H_{\mathfrak{m}}$ of $R_{\mathcal{S}_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \cong R_{\mathcal{S}}$-modules. (This is again proved in a similar way to [Tho16, Lemma 6.8], using the fact that $H^i(Y_{U_1(Mq_a)\cap U_0(Q)}, \mathrm{Sym}^{k-2}(\mathcal{O}/\varpi)^2)$ is Eisenstein for $i \neq 1$, together with [KT17, Corollary 2.7], to justify the freeness.)

If $l \in S$, let $R_l \in \mathcal{C}_{\mathcal{O}}$ denote the universal lifting ring representing the local deformation problem $\mathcal{D}_l$. By construction (if $l \neq p$) or arguing as in [CHT08, §2.4.1] (if $l = p$) R_l is a formally smooth $\mathcal{O}$-algebra; if $l \neq p$, then R_l is formally smooth over $\mathcal{O}$ of relative dimension 3, while R_p has relative dimension 4. We set $T = S - \{p, q_a\}$ and $A_{\mathcal{S}}^T = \widehat{\otimes}_{l\in T} R_l$ (the completed tensor product being over $\mathcal{O}$). The T-framed deformation rings $R_{\mathcal{S}}^T$ and $R_{\mathcal{S}_Q}^T$ are defined (see [Tho16, §5.2]) and there are canonical homomorphisms $A_{\mathcal{S}}^T \to R_{\mathcal{S}}^T$ and $A_{\mathcal{S}}^T \to R_{\mathcal{S}_Q}^T$.

By the argument of [Kis09, Proposition 3.2.5] and [Tho16, Proposition 5.10], we can find an integer $q_0 \geq 0$ with the following property: for each $N \geq 1$, there exists a set $Q = Q_N$ of primes satisfying conditions (a), (b) above and also:

(c) $|Q_N| = q_0$.
(d) For each $q \in Q_N$, $q \equiv 1 \bmod p^N$.
(e) The algebra map $A_{\mathcal{S}}^T \to R_{\mathcal{S}_{Q_N}}^T$ extends to a surjective algebra homomorphism $A_{\mathcal{S}}^T[\![X_1, \dots, X_g]\!] \to R_{\mathcal{S}^T_{Q_N}}$, where $g = q + |T| - 1$.

We choose for each $N \geq 1$ a representative $\rho_{\mathcal{S}_{Q_N}}$ of the universal deformation over $R_{\mathcal{S}_{Q_N}}$ which lifts $\rho_{\mathcal{S}}$. This choice determines an isomorphism $R_{\mathcal{S}_{Q_N}}^T \cong$

$R_{\mathcal{S}_{Q_N}} \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$, where $\mathcal{T}$ is a power series ring over $\mathcal{O}$ in $4|T|-1$ variables. We set $H^T_{Q_N} = H_{Q_N, \mathfrak{m}_{Q_N,1}} \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$. It is a free $\mathcal{T}[\Delta_{Q_N}]$-module, and there is an isomorphism $H^T_{Q_N} \otimes_{\mathcal{T}[\Delta_{Q_N}]} \mathcal{O} \cong H_{\mathfrak{m}}$ of $R_{\mathcal{S}^T_{Q_N}} \otimes_{\mathcal{T}[\Delta_{Q_N}]} \mathcal{O} \cong R_{\mathcal{S}}$-modules.

We now come to the essential point of the proof. Let $l \in S - \{p\}$, and fix a Frobenius lift $\phi_l \in G_{\mathbf{Q}_l}$.

Lemma 8.4. *With our current assumptions, there is a principal ideal $I_l \subset R_l$ with the following property: for any homomorphism $f : R_l \to \overline{\mathbf{Q}}_p$, the resulting homomorphism $\rho_f : G_{\mathbf{Q}_l} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ has the property that the eigenvalues α_l, β_l of $\rho_f(\phi_l)$ satisfy $(\alpha_l/\beta_l)^i = 1$ for some $i = 1, \ldots, n-1$ if and only if $f(I_l) = 0$. Moreover, the quotient R_l/I_l has dimension strictly smaller than the dimension of R_l.*

Proof. Let $(r, N) = \mathrm{rec}^T_{\mathbf{Q}_l}(\iota^{-1}\pi_l)$, a Weil–Deligne representation that we may assume is defined over E. The proof will use the fact that the Jacquet module of π_l is non-trivial (equivalently, that the Weil–Deligne representation (r, N) is reducible).

We recall that the ring R_l is a formally smooth $\mathcal{O}$-algebra of relative dimension 3. Let $r_l^{\mathrm{univ}} : G_{\mathbf{Q}_l} \to \mathrm{GL}_2(R_l)$ be the universal lifting. We can take I_l to be the ideal generated by the discriminant of the characteristic polynomial of $\mathrm{Sym}^{n-1} r_l^{\mathrm{univ}}(\phi_l)$. To complete the proof of the lemma, we need to show that $\dim R_l/I_l < \dim R_l$. Since R_l is an integral domain, it is equivalent to show that I_l is not the zero ideal.

To show this, we split into cases. If π_l is a twist of the Steinberg representation then the discriminant of the characteristic polynomial of $\mathrm{Sym}^{n-1} r(\phi_l)$ is non-zero (as the eigenvalues of $r(\phi_l)$ have eigenvalues whose ratio is a non-zero power of l), so we see that I_l is not the zero ideal in this case. Otherwise, $N = 0$ and $r = \chi_1 \oplus \chi_2$ is a direct sum of two characters of $W_{\mathbf{Q}_l}$. Let $\psi : W_{\mathbf{Q}_l} \to E[\![T]\!]$ be the unramified character which sends ψ to $1 + T$; then $r' = \chi_1\psi \oplus \chi_2\psi^{-1}$ is a deformation of r to $E[\![T]\!]$ of determinant χ with the property that the discriminant of the characteristic polynomial of $\mathrm{Sym}^{n-1} r'(\phi_l)$ is non-zero in $E[\![T]\!]$. The existence of this deformation, together with [Gee11, Proposition 2.1.5], implies that I_l cannot be the zero ideal in this case either. □

We set $I = \prod_{l \in S - \{p\}} I_l A^T_{\mathcal{S}} \subset A^T_{\mathcal{S}}$. Then $\dim A^T_{\mathcal{S}}/I = \dim A^T_{\mathcal{S}} - 1$.

Suppose for contradiction that for each automorphic representation π' contributing to H_{Q_N} for some $N \geq 1$, there is a prime $l \in S$ such that π'_l is ramified and there is an accessible refinement of π'_l which is not n-regular. Then $IH^T_{Q_N} = 0$. On the other hand, a standard patching argument (cf. [Tho12, Lemma 6.10]) implies the existence of the following objects:

- A ring $S_\infty = \mathcal{T}[\![S_1, \ldots, S_{q_0}]\!]$ and an algebra homomorphism $S_\infty \to R_\infty = (A^T_{\mathcal{S}}/I)[\![X_1, \ldots, X_g]\!]$.
- A finite R_∞-module H_∞, which is finite free as S_∞-module.

This is a contradiction. Indeed, [KT17, Lemma 2.8] shows that the dimension of H_∞ is the same, whether considered as R_∞- or S_∞-module. By freeness, its dimension as S_∞-module is $\dim S_\infty = 4|T| + q_0$. On the other hand, its dimension as R_∞-module is bounded above by $\dim R_\infty = \dim A^T_{\mathcal{S}} - 1 + g = 4|T| + q_0 - 1$.

We conclude that there exists an automorphic representation π' contributing to H_{Q_N} for some $N \geq 1$ such that for each prime $l \in S$ such that π'_l is ramified, each accessible refinement of π'_l is n-regular. To complete the proof, we just need to explain why π'_q is n-regular for each prime $q \in Q_N$ such that π_q is ramified. However, our construction shows that $r_{\pi',\iota}|_{I_{\mathbf{Q}_q}}$ has the form $C_q \oplus C_q^{-1}$, where $C_q : I_{\mathbf{Q}_q} \to \overline{\mathbf{Q}}_p^\times$ has order a power of p. Since $p > 2n$, by hypothesis, this is a fortiori n-regular. This completes the proof. □

We can now finish the proof of Theorem 8.1.

Proof of Theorem 8.1. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_{\mathbf{Q}})$ of weight $k \geq 2$, without CM, and such that each local component π_l admits an accessible refinement. Let p, ι, and π' be as in the statement of Proposition 8.3. Then $\mathrm{Sym}^{n-1} r_{\pi',\iota}$ is automorphic, by Proposition 8.2.

On the other hand, our assumptions imply that the residual representation $\mathrm{Sym}^{n-1} \overline{r}_{\pi,\iota} \cong \mathrm{Sym}^{n-1} \overline{r}_{\pi',\iota}$ is irreducible. We can therefore apply [BLGGT14, Theorem 4.2.1] to conclude that $\mathrm{Sym}^{n-1} r_{\pi,\iota}$ is automorphic. This completes the proof. □

References

[AC89] James Arthur and Laurent Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton University Press, Princeton, NJ, 1989.

[All16] Patrick B. Allen, *Deformations of polarized automorphic Galois representations and adjoint Selmer groups*, Duke Math. J. **165** (2016), no. 13, 2407–2460.

[ANT20] Patrick B. Allen, James Newton, and Jack A. Thorne, *Automorphy lifting for residually reducible l-adic Galois representations, II*, Compos. Math. **156** (2020), no. 11, 2399–2422.

[AT21] Christos Anastassiades and Jack A. Thorne, *Raising the level of automorphic representations of* GL_{2n} *of unitary type*, To appear in Journal of the Institute of Mathematics of Jussieu (2021).

[BC05] Kevin Buzzard and Frank Calegari, *The 2-adic eigencurve is proper*, arXiv Mathematics e-prints (2005), math/0503362.

[BC09] Joël Bellaïche and Gaëtan Chenevier, *Families of Galois representations and Selmer groups*, Astérisque (2009), no. 324, xii+314.

[BC11] ———, *The sign of Galois representations attached to automorphic forms for unitary groups*, Compos. Math. **147** (2011), no. 5, 1337–1352.

[Ber08] Laurent Berger, *Équations différentielles p-adiques et* (ϕ, N)*-modules filtrés*, Astérisque (2008), no. 319, 13–38, Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

[Ber17] John Bergdall, *Paraboline variation over p-adic families of* (ϕ, Γ)*-modules*, Compos. Math. **153** (2017), no. 1, 132–174.

[BG19] Rebecca Bellovin and Toby Gee, *G-valued local deformation rings and global lifts*, Algebra Number Theory **13** (2019), no. 2, 333–378.

[BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 261, Springer-Verlag, Berlin, 1984, A systematic approach to rigid analytic geometry.

[BHS17] Christophe Breuil, Eugen Hellmann, and Benjamin Schraen, *Une interprétation modulaire de la variété trianguline*, Math. Ann. **367** (2017), no. 3-4, 1587–1645.

[BK98] Colin J. Bushnell and Philip C. Kutzko, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. (3) **77** (1998), no. 3, 582–634.

[BK05] Kevin Buzzard and L. J. P. Kilford, *The 2-adic eigencurve at the boundary of weight space*, Compos. Math. **141** (2005), no. 3, 605–619.

[BK16] Alina Bucur and Kiran S. Kedlaya, *An application of the effective Sato-Tate conjecture*, Frobenius distributions: Lang-Trotter and Sato-Tate conjectures, Contemp. Math., vol. 663, Amer. Math. Soc., Providence, RI, 2016, pp. 45–56.

[BLGG11] Tom Barnet-Lamb, Toby Gee, and David Geraghty, *The Sato-Tate Conjecture for Hilbert Modular Forms*, J. Amer. Math. Soc. **24** (2011), no. 2, 411–469.

[BLGGT14] Thomas Barnet-Lamb, Toby Gee, David Geraghty, and Richard Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609.

[BLGHT11] Tom Barnet-Lamb, David Geraghty, Michael Harris, and Richard Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 1, 29–98.

[Bre15] Christophe Breuil, *Vers le socle localement analytique pour* GL_n *II*, Math. Ann. **361** (2015), no. 3-4, 741–785.

[Bre16] ———, *Socle localement analytique I*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 2, 633–685.

[BS07] Christophe Breuil and Peter Schneider, *First steps towards p-adic Langlands functoriality*, J. Reine Angew. Math. **610** (2007), 149–180.

[Buz04] Kevin Buzzard, *On p-adic families of automorphic forms*, Modular curves and abelian varieties, Progr. Math., vol. 224, Birkhäuser, Basel, 2004, pp. 23–44.

[Buz07] ———, *Eigenvarieties*, L-functions and Galois representations, London Math. Soc. Lecture Note Ser., vol. 320, Cambridge Univ. Press, Cambridge, 2007, pp. 59–120.

[BZ77] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive $\mathfrak{p}$-adic groups. I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 441–472.

[Car12] Ana Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), no. 12, 2311–2413.

[Car14] ———, *Monodromy and local-global compatibility for $l = p$*, Algebra Number Theory **8** (2014), no. 7, 1597–1646.

[CEG+16] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin, *Patching and the p-adic local Langlands correspondence*, Cambridge Journal of Mathematics **4** (2016), no. 2, 197–287.

[Che51] Claude Chevalley, *Deux théorèmes d'arithmétique*, J. Math. Soc. Japan **3** (1951), 36–44.

[Che04] Gaëtan Chenevier, *Familles p-adiques de formes automorphes pour* GL_n, J. Reine Angew. Math. **570** (2004), 143–217.

[Che05] ———, *Une correspondance de Jacquet-Langlands p-adique*, Duke Math. J. **126** (2005), no. 1, 161–194.

[Che11] ———, *On the infinite fern of Galois representations of unitary type*, Ann. Sci. Éc. Norm. Supér. (4) **44** (2011), no. 6, 963–1019.

[Che14] Gaëtan Chenevier, *The p-adic analytic space of pseudocharacters of a profinite group and pseudorepresentations over arbitrary rings*, Automorphic forms and Galois representations. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 414, Cambridge Univ. Press, Cambridge, 2014, pp. 221–285.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.

[Clo82] Laurent Clozel, *Changement de base pour les représentations tempérées des groupes réductifs réels*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 1, 45–115.

[Clo90a] ———, *The fundamental lemma for stable base change*, Duke Math. J. **61** (1990), no. 1, 255–302.

[Clo90b] ———, *Motifs et formes automorphes: applications du principe de fonctorialité*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 77–159.

[Clo13] ———, *Purity reigns supreme*, Int. Math. Res. Not. IMRN (2013), no. 2, 328–346.

[Col96] Robert F. Coleman, *Classical and overconvergent modular forms*, Invent. Math. **124** (1996), no. 1-3, 215–241.

[Col97] ———, *Classical and overconvergent modular forms of higher level*, J. Théor. Nombres Bordeaux **9** (1997), no. 2, 395–403.

[Col08] Pierre Colmez, *Représentations triangulines de dimension 2*, Astérisque (2008), no. 319, 213–258, Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

[Con99] Brian Conrad, *Irreducible components of rigid spaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 2, 473–541.

[Con06] ———, *Relative ampleness in rigid geometry*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 4, 1049–1126.

[CT14] Laurent Clozel and Jack A. Thorne, *Level-raising and symmetric power functoriality, I*, Compos. Math. **150** (2014), no. 5, 729–748.

[CT15] ———, *Level raising and symmetric power functoriality, II*, Ann. of Math. (2) **181** (2015), no. 1, 303–359.

[CT17] ———, *Level-raising and symmetric power functoriality, III*, Duke Math. J. **166** (2017), no. 2, 325–402.

[DI95] Fred Diamond and John Im, *Modular forms and modular curves*, Seminar on Fermat's Last Theorem (Toronto, ON, 1993–1994), CMS Conf. Proc., vol. 17, Amer. Math. Soc., Providence, RI, 1995, pp. 39–133.

[Dia89] Fred Diamond, *On congruence modules associated to* Λ*-adic forms*, Compositio Math. **71** (1989), no. 1, 49–83.

[Die15] Luis Dieulefait, *Automorphy of* $\mathrm{Symm}^5(\mathrm{GL}(2))$ *and base change*, J. Math. Pures Appl. (9) **104** (2015), no. 4, 619–656.

[DL76] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161.

[DM18] Olivier Dudas and Gunter Malle, *Modular irreducibility of cuspidal unipotent characters*, Invent. Math. **211** (2018), no. 2, 579–589.

[DMW09] Neil Dummigan, Phil Martin, and Mark Watkins, *Euler factors and local root numbers for symmetric powers of elliptic curves*, Pure Appl. Math. Q. **5** (2009), no. 4, Special Issue: In honor of John Tate. Part 1, 1311–1341.

[DT94] Fred Diamond and Richard Taylor, *Nonoptimal levels of mod* l *modular representations*, Invent. Math. **115** (1994), no. 3, 435–462.

[Edi92] Bas Edixhoven, *The weight in Serre's conjectures on modular forms*, Invent. Math. **109** (1992), no. 3, 563–594.

[Eme06a] Matthew Emerton, *Jacquet modules of locally analytic representations of* p*-adic reductive groups. I. Construction and first properties*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 5, 775–839.

[Eme06b] ———, *On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms*, Invent. Math. **164** (2006), no. 1, 1–84.

[FvdP04] Jean Fresnel and Marius van der Put, *Rigid analytic geometry and its applications*, Progress in Mathematics, vol. 218, Birkhäuser Boston, Inc., Boston, MA, 2004.

[GD71] A. Grothendieck and J. A. Dieudonné, *Éléments de géométrie algébrique. I*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 166, Springer-Verlag, Berlin, 1971.

[Gec90] Meinolf Geck, *Irreducible Brauer characters of the* 3*-dimensional special unitary groups in nondefining characteristic*, Comm. Algebra **18** (1990), no. 2, 563–584.

[Gee11] Toby Gee, *Automorphic lifts of prescribed types*, Math. Ann. **350** (2011), no. 1, 107–144.

[Ger19] David Geraghty, *Modularity lifting theorems for ordinary Galois representations*, Math. Ann. **373** (2019), no. 3-4, 1341–1427.

[GGP12] Wee Teck Gan, Benedict H. Gross, and Dipendra Prasad, *Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups*, Astérisque (2012), no. 346, 1–109, Sur les conjectures de Gross et Prasad. I.

[GJ78] Stephen Gelbart and Hervé Jacquet, *A relation between automorphic representations of* $\mathrm{GL}(2)$ *and* $\mathrm{GL}(3)$, Ann. Sci. École Norm. Sup. (4) **11** (1978), no. 4, 471–542.

[Hal93] Thomas C. Hales, *A simple definition of transfer factors for unramified groups*, Representation theory of groups and algebras, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 109–134.

[Hen92] Guy Henniart, *Correspondance de Langlands-Kazhdan explicite dans le cas non ramifié*, Math. Nachr. **158** (1992), 7–26.

[Hen00] ———, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps* p*-adique*, Invent. Math. **139** (2000), no. 2, 439–455.

[Hir04] Kaoru Hiraga, *On functoriality of Zelevinski involutions*, Compos. Math. **140** (2004), no. 6, 1625–1656.

[HS16] Eugen Hellmann and Benjamin Schraen, *Density of potentially crystalline representations of fixed weight*, Compos. Math. **152** (2016), no. 8, 1609–1647.

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[JS81] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic forms. II*, Amer. J. Math. **103** (1981), no. 4, 777–815.

[Kal11] Tasho Kaletha, *Endoscopic character identities for depth-zero supercuspidal L-packets*, Duke Math. J. **158** (2011), no. 2, 161–224.

[Kal16] ———, *Rigid inner forms of real and p-adic groups*, Ann. of Math. (2) **184** (2016), no. 2, 559–632.

[Kal18] ———, *Global rigid inner forms and multiplicities of discrete automorphic representations*, Invent. Math. **213** (2018), no. 1, 271–369.

[Kim03] Henry H. Kim, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2, J. Amer. Math. Soc. **16** (2003), no. 1, 139–183, With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

[Kis03] Mark Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math. **153** (2003), no. 2, 373–454.

[Kis08] ———, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546.

[Kis09] ———, *Moduli of finite flat group schemes, and modularity*, Ann. of Math. (2) **170** (2009), no. 3, 1085–1180.

[Köp74] Ursula Köpf, *Über eigentliche Familien algebraischer Varietäten über affinoiden Räumen*, Schr. Math. Inst. Univ. Münster (2) (1974), no. Heft 7, iv+72.

[KPX14] Kiran S. Kedlaya, Jonathan Pottharst, and Liang Xiao, *Cohomology of arithmetic families of* (φ, Γ)*-modules*, J. Amer. Math. Soc. **27** (2014), no. 4, 1043–1115.

[KS99] Robert E. Kottwitz and Diana Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190.

[KS02] Henry H. Kim and Freydoon Shahidi, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and the symmetric cube for* GL_2, Ann. of Math. (2) **155** (2002), no. 3, 837–893, With an appendix by Colin J. Bushnell and Guy Henniart.

[KT17] Chandrashekhar B. Khare and Jack A. Thorne, *Potential automorphy and the Leopoldt conjecture*, Amer. J. Math. **139** (2017), no. 5, 1205–1273.

[KV12] David Kazhdan and Yakov Varshavsky, *On endoscopic transfer of Deligne-Lusztig functions*, Duke Math. J. **161** (2012), no. 4, 675–732.

[KW09] Chandrashekhar Khare and Jean-Pierre Wintenberger, *Serre's modularity conjecture. I*, Invent. Math. **178** (2009), no. 3, 485–504.

[Lab11] J.-P. Labesse, *Changement de base CM et séries discrètes*, On the stabilization of the trace formula, Stab. Trace Formula Shimura Var. Arith. Appl., vol. 1, Int. Press, Somerville, MA, 2011, pp. 429–470.

[Lan70] R. P. Langlands, *Problems in the theory of automorphic forms*, Lectures in modern analysis and applications, III, 1970, pp. 18–61. Lecture Notes in Math., Vol. 170.

[Lan89] R. Langlands, *On the classification of irreducible representations of real algebraic groups*, Representation theory and harmonic analysis on semisimple Lie groups, Math. Surveys Monogr., 31, AMS, 1989, pp. 101–170.

[LN08] Gérard Laumon and Bao Châu Ngô, *Le lemme fondamental pour les groupes unitaires*, Ann. of Math. (2) **168** (2008), no. 2, 477–573.

[Loe11] David Loeffler, *Overconvergent algebraic automorphic forms*, Proc. Lond. Math. Soc. (3) **102** (2011), no. 2, 193–228.

[LS87] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), no. 1-4, 219–271.

[LS20] Jaime Lust and Shaun Stevens, *On depth zero L-packets for classical groups*, Proc. Lond. Math. Soc. (3) **121** (2020), no. 5, 1083–1120.

[Lus76] G. Lusztig, *On the finiteness of the number of unipotent classes*, Invent. Math. **34** (1976), no. 3, 201–213.

[Lus77] ———, *Irreducible representations of finite classical groups*, Invent. Math. **43** (1977), no. 2, 125–175.

[LWX17] Ruochuan Liu, Daqing Wan, and Liang Xiao, *The eigencurve over the boundary of weight space*, Duke Math. J. **166** (2017), no. 9, 1739–1787.

[Mai02] Christian Maire, *On the $\mathbf{Z}_l$-rank of abelian extensions with restricted ramification*, J. Number Theory **92** (2002), no. 2, 376–404.

[Maz97] B. Mazur, *An "infinite fern" in the universal deformation space of Galois representations*, vol. 48, 1997, Journées Arithmétiques (Barcelona, 1995), pp. 155–193.

[Mœg07] Colette Mœglin, *Classification et changement de base pour les séries discrètes des groupes unitaires p-adiques*, Pacific J. Math. **233** (2007), no. 1, 159–204.

[Mœg14] ———, *Paquets stables des séries discrètes accessibles par endoscopie tordue; leur paramètre de Langlands*, Automorphic forms and related geometry: assessing the legacy of I. I. Piatetski-Shapiro, Contemp. Math., vol. 614, Amer. Math. Soc., Providence, RI, 2014, pp. 295–336.

[Mor99] Lawrence Morris, *Level zero* **G***-types*, Compositio Math. **118** (1999), no. 2, 135–157.

[Mor10] Sophie Morel, *On the cohomology of certain noncompact Shimura varieties. With an appendix by Robert Kottwitz.*, vol. 173, Princeton, NJ: Princeton University Press, 2010 (English).

[MP96] Allen Moy and Gopal Prasad, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), no. 1, 98–121.

[Mur85] V. Kumar Murty, *Explicit formulae and the Lang-Trotter conjecture*, Rocky Mountain J. Math. **15** (1985), no. 2, 535–551, Number theory (Winnipeg, Man., 1983).

[MW89] C. Mœglin and J.-L. Waldspurger, *Le spectre résiduel de* $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) **22** (1989), no. 4, 605–674.

[Nak13] Kentaro Nakamura, *Deformations of trianguline B-pairs and Zariski density of two dimensional crystalline representations*, J. Math. Sci. Univ. Tokyo **20** (2013), no. 4, 461–568.

[NT] James Newton and Jack A. Thorne, *Symmetric power functoriality for holomorphic modular forms, II*, Preprint, arXiv:2009.07180.

[NT20] James Newton and Jack A. Thorne, *Adjoint Selmer groups of automorphic Galois representations of unitary type*, 2020, arXiv:1912.11265, to appear in J. Eur. Math. Soc.

[OS15] Sascha Orlik and Matthias Strauch, *On Jordan-Hölder series of some locally analytic representations*, J. Amer. Math. Soc. **28** (2015), no. 1, 99–157.

[Pil13] Vincent Pilloni, *Overconvergent modular forms*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 1, 219–239.

[Rib84] Kenneth A. Ribet, *Congruence relations between modular forms*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), PWN, Warsaw, 1984, pp. 503–514.

[Rod82] François Rodier, *Représentations de* $\mathrm{GL}(n,\, k)$ *où k est un corps p-adique*, Bourbaki Seminar, Vol. 1981/1982, Astérisque, vol. 92, Soc. Math. France, Paris, 1982, pp. 201–218.

[Rog90] Jonathan D. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, vol. 123, Princeton University Press, Princeton, NJ, 1990.

[RT17] Jeremy Rouse and Jesse Thorner, *The explicit Sato-Tate conjecture and densities pertaining to Lehmer-type questions*, Trans. Amer. Math. Soc. **369** (2017), no. 5, 3575–3604.

[Sen73] Shankar Sen, *Lie algebras of Galois groups arising from Hodge-Tate modules*, Ann. of Math. (2) **97** (1973), 160–170.

[Shi11] Sug Woo Shin, *Galois representations arising from some compact Shimura varieties*, Ann. of Math. (2) **173** (2011), no. 3, 1645–1741.

[Sho18] Jack Shotton, *The Breuil-Mézard conjecture when $l \neq p$*, Duke Math. J. **167** (2018), no. 4, 603–678.

[Sor17] Claus Sorensen, *A note on Jacquet functors and ordinary parts*, Math. Scand. **121** (2017), no. 2, 311–319.

[Taï16] Olivier Taïbi, *Eigenvarieties for classical groups and complex conjugations in Galois representations*, Math. Res. Lett. **23** (2016), no. 4, 1167–1220.

[Tay08] Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations. II*, Pub. Math. IHES **108** (2008), 183–239.

[Tho12] Jack Thorne, *On the automorphy of l-adic Galois representations with small residual image*, J. Inst. Math. Jussieu **11** (2012), no. 4, 855–920, With an appendix by Robert Guralnick, Florian Herzig, Richard Taylor and Thorne.

[Tho14a] Jack A. Thorne, *Raising the level for* GL_n, Forum Math. Sigma **2** (2014), Paper No. e16, 35.

[Tho14b] Jesse Thorner, *The error term in the Sato-Tate conjecture*, Arch. Math. (Basel) **103** (2014), no. 2, 147–156.

[Tho15] Jack A. Thorne, *Automorphy lifting for residually reducible l-adic Galois representations*, J. Amer. Math. Soc. **28** (2015), no. 3, 785–870.

[Tho16] ———, *Automorphy of some residually dihedral Galois representations*, Math. Ann. **364** (2016), no. 1-2, 589–648.

[TY07] Richard Taylor and Teruyoshi Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493 (electronic).

[Vig98] Marie-France Vignéras, *Induced R-representations of p-adic reductive groups*, Selecta Math. (N.S.) **4** (1998), no. 4, 549–623.

[Vig01] ———, *Correspondance de Langlands semi-simple pour* $\mathrm{GL}(n, F)$ *modulo* $l \neq p$, Invent. Math. **144** (2001), no. 1, 177–223.

[Wal03] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra)*, J. Inst. Math. Jussieu **2** (2003), no. 2, 235–333.

[Wes04] Tom Weston, *Unobstructed modular deformation problems*, Amer. J. Math. **126** (2004), no. 6, 1237–1252.

[Zel80] A. V. Zelevinsky, *Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of* $\mathrm{GL}(n)$, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 2, 165–210.