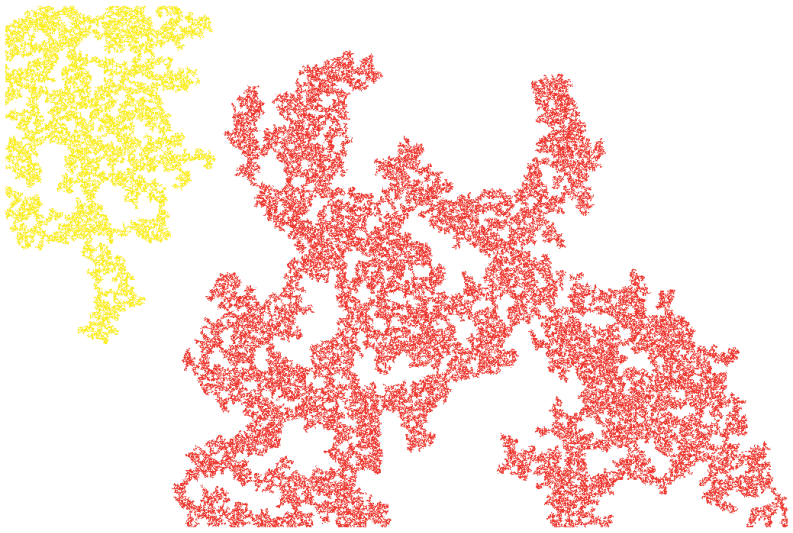
The geometry of non-Markovian interacting systems



Edward John Mottram

Churchill College

University of Cambridge

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Overview

The main goal of this thesis is to study two different self-interacting systems. The first of these is a model for frozen percolation, which we call *percolation with constant freezing*, and the second is a 1-dimensional Brownian motion conditioned to have its local time bounded by 1. A summary for each of the chapters is given below.

Chapter 1 starts by giving an introduction to percolation and reviews some of the existing tools and results. We then move on to looking at frozen percolation as a model of self-organised criticality, and conclude by discussing various methods used for constructing infinite volume limits for interacting systems.

The introductory material continues in Chapter 2. Here we begin by giving a brief overview of self-interacting 1-dimensional stochastic processes. The chapter then moves on to give an explanation of *entropic repulsion*, before presenting the Ray–Knight Theorems and the Donsker–Varadhan Theorem. These are the two key tools used in Chapter 4.

In Chapter 3 we introduce and study a model of *percolation with constant freezing* (*PCF*) where edges open at constant rate 1, and clusters freeze at rate α independently of their size. The main result is that the infinite volume process can be constructed on any amenable vertex transitive graph. This is in sharp contrast to models of percolation with freezing previously introduced, where the limit is known not to exist. Our interest is in the study of the percolative properties of the final configuration as a function of α. We also obtain more precise results in the case of trees. Surprisingly the algebraic exponent for the cluster size depends on the degree, suggesting that there is no lower critical dimension for the model. Moreover, even for $\alpha < \alpha_c$, it is shown that finite clusters have algebraic tail decay, which is a signature of self organised criticality. Partial results are obtained on $\mathbb{Z}^d$, and many open questions are discussed.

Chapter 4 is devoted to an investigation of the ballistic behaviour of a Brownian motion $(W_t)_{t\geq 0}$ conditioned to have bounded local time. In particular we condition on the event $\mathcal{E} = \{L_x(t) \leq 1 \text{ for all } x \text{ and } t\}$. Since $\mathcal{E}$ is an event with probability 0 it has to be realised as a limit of events with positive probability. The main result is to show that if we condition on the events $\mathcal{E}_T^\bullet = \{L_x(T) \leq 1 \text{ for all } x \in \mathbb{R}\}$ then $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ converges weakly to a measure $\mathbb{Q}^\bullet$ as $T \longrightarrow \infty$. Furthermore W_t has a $\mathbb{Q}^\bullet$-almost sure limiting speed $\gamma^\bullet$ as $t \longrightarrow \infty$. By comparing $\gamma^\bullet$ with γ^* – the limiting speed obtained in a paper of Benjamini and Berestycki where a different decomposition of $\mathcal{E}$ is used – we see that the speed of W_t is sensitive to the particular way in which $\mathcal{E}$ is decomposed. However, in both cases we have $\mathbb{P}(L_x(\infty) > 1 - \varepsilon) \asymp \varepsilon^3$, suggesting that there is a sense to which Brownian entropic repulsion is universal.

Statement of originality

I hereby declare that my dissertation entitled "The geometry of non-Markovian interacting systems" is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically acknowledged in the text.

I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any other degree, diploma or qualification at the University of Cambridge or any other university or similar institution.

Chapter 1 and Chapter 2 predominantly consist of literature review, whereas the results of Chapters 3 and 4 are original research, and appear in [Mot14b] and [Mot14a] respectively.

Acknowledgements

I am extremely grateful to my supervisor, Nathanaël Berestycki, who has guided and encouraged me throughout the last four years. During this time he has introduced me to many beautiful areas of Probability and enabled me to grow as a research student.

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Contents

Chapter 1

Frozen percolation

This chapter starts by giving an introduction to percolation, with a review of some of the existing tools and results. We then move on to looking at *frozen percolation* as a model of self-organised criticality, and conclude by discussing some of the various methods which can be used for constructing infinite volume limits of interacting systems. The ideas present in the proof of the uniqueness of the infinite percolation cluster, and the construction of an infinite volume measure for the random cluster model play a key role in Chapter 3. Therefore our treatment gives a particular emphasis to these.

1.1 Percolation

Statistical physics gives rise to many interesting random models which the probabilist can study in order to gain insight into various physical phenomena. From a mathematical point of view the simplest of these models is perhaps *bond percolation*. This was first analysed by Broadbent and Hammersley in 1957 as a way of investigating the passage of fluid through porous media, [BH57], and has since developed into a rich and beautiful mathematical theory.

1.1.1 Phase transition

For our purposes it will be helpful to view the bond percolation model as a cádlág process which evolves in time. Formally we let $\mathbf{G} = (\mathcal{V}, \mathcal{E})$ be a (possibly infinite) graph, and attach a uniform $[0,1]$ random variable U_e to each edge $e \in \mathcal{E}$. At time $p = 0$ each edge $e \in \mathcal{E}$ is *closed*, and remains so until $p = U_e$ at which point the edge e becomes *open*. Thus e is open at time p if and only if $U_e \leq p$.

If we use $\mathbb{P}_p$ to denote the probability measure of a configuration at time p then, because of the natural *coupling* in the model, it is clear that if $p' \geq p$ then a configuration with law $\mathbb{P}_{p'}$ will contain more open edges than a configuration with law $\mathbb{P}_p$. From this we get the notion of *stochastic ordering* which we formally define in the following way.

Given an edge configuration $\omega \in \{0,1\}^{\mathcal{E}}$ define ω^e by $\omega^e(e) = 1$ and $\omega(e') = \omega^e(e')$ for all $e' \neq e$. We now say that a measurable event $A \subseteq \{0,1\}^{\mathcal{E}}$ is *increasing* if $\omega \in A$ implies $\omega^e \in A$ for all $e \in \mathcal{E}$. By looking at these increasing events we can now say that a measure $\mathbb{P}$ is *stochastically dominated* by the measure $\mathbb{Q}$ (and write $\mathbb{P} \leq_{\text{st}} \mathbb{Q}$) if $\mathbb{P}(A) \leq \mathbb{Q}(A)$ for all increasing $A \subseteq \{0,1\}^{\mathcal{E}}$.

Now observe that the events $w \overset{\omega}{\leftrightarrow} v$ (there is an open path from v to w) and $v \overset{\omega}{\leftrightarrow} \infty$ (v is contained in an infinite open cluster) are both increasing. Therefore if we let $\mathbf{G} = \mathbb{Z}^d$, and define $\theta(p) = \theta_d(p) = \mathbb{P}_p(0 \overset{\omega}{\leftrightarrow} \infty)$ then we see that $\theta : [0,1] \longrightarrow [0,1]$ must be monotonically increasing as a function of p. By analysing the behaviour of $\theta(p)$ we see that bond percolation on $\mathbb{Z}^d$ exhibits a *phase transition*.

Theorem 1.1.1 (Broadbent and Hammersley, 1957)**.** *For each $d \geq 2$ there exists a critical probability $0 < p_c = p_c(d) < 1$ such that $\theta_d(p) = 0$ for all $p < p_c$, and $\theta_d(p) > 0$ for all $p > p_c$.*

In 1960 Harris showed that on $\mathbb{Z}^2$ we have $\theta\left(\frac{1}{2}\right) = 0$, [Har60]; then in 1980 Kesten also showed that $\theta(p) > 0$ for all $p > \frac{1}{2}$, [Kes80]. Therefore when $d = 2$ we have $p_c = \frac{1}{2}$. However, in higher dimension or on many other lattices an explicit value for the critical value p_c is not known. In many cases understanding the behaviour of $\theta(p)$ near p_c, and thus the nature of the phase transition, also remains an open problem. For $d \geq 19$ the lace expansion of Hara and Slade, [HS90], tells us that $\theta(p) \sim c(p - p_c)$ as $p \searrow p_c$; and for $d = 2$ it is conjectured that $\theta(p) = (p - p_c)^{-\frac{5}{36}+o(1)}$. However, in dimension 3 even the equality $\theta(p_c) = 0$ still remains to be proved.

1.1.2 Tools for percolation

The FKG inequality Suppose we condition on a percolation configuration having an open path between v and w. Intuitively one might expect that this would make it more likely for two other vertices x and y to be connected by an open path (since a path between x and y could use part of the path between v and w for free). This is indeed the case, and for a general increasing event A we have the following due to Fortuin, Kasteleyn and Ginibre, [FKG71].

Theorem 1.1.2 (FKG inequality)**.** *Suppose A and B are increasing events with $\mathbb{P}_p(A) > 0$, then*

$$\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A) \times \mathbb{P}_p(B). \tag{1.1.1}$$

From this we deduce that $\mathbb{P}_p \leq_{\text{st}} \mathbb{P}_p(\cdot \mid A)$ whenever A is increasing.

As well as being a useful tool for studying the percolation measure $\mathbb{P}_p$, it turns out that the *FKG inequality* also holds for any measure μ which satisfies the *FKG lattice condition*,

$$\mu(\omega \vee \omega') \times \mu(\omega \wedge \omega') \geq \mu(\omega) \times \mu(\omega') \tag{1.1.2}$$

for all $\omega, \omega' \in \{0,1\}^{\mathcal{E}}$. Therefore, since it is possible to check whether (1.1.2) holds by looking at what happens when ω and ω' differ by at most two edges, (1.1.1) provides a powerful tool for the study of many other lattice models.

Remark 1.1.3. It is known that (1.1.2) is a strictly stronger than (1.1.1), and we believe that the *percolation with constant freezing* model is an example of a system where (1.1.1) holds even though (1.1.2) does not. See Example 3.1.5 of Chapter 3 for further details.

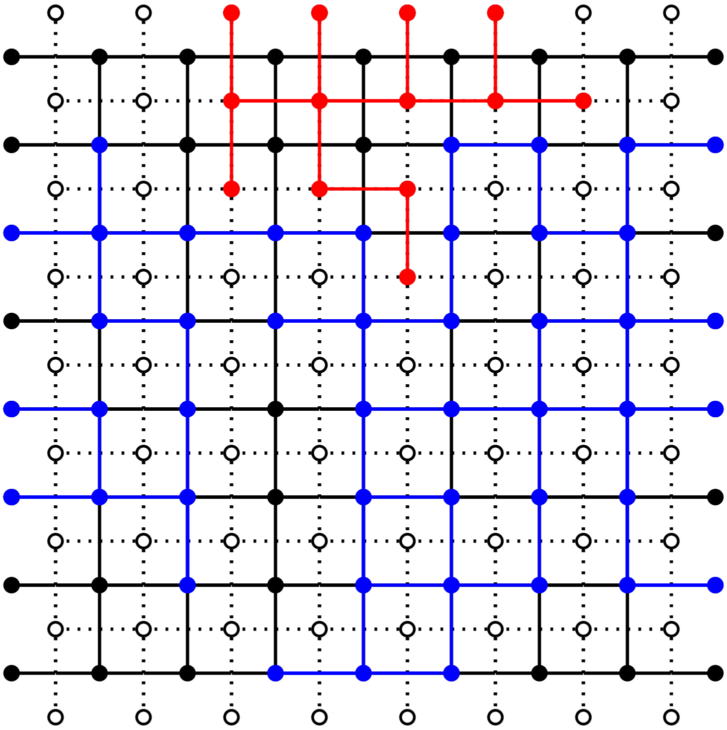


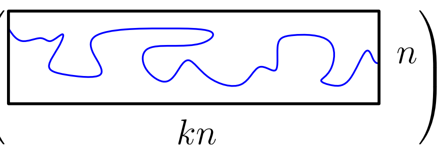
Figure 1.1.1: The dual to an n by $n+1$ box is an $n+1$ by n box. Observe that for any configuration there will either be a left–right crossing in the primal graph or a top–bottom crossing in the dual. In this case there is a primal crossing (in blue) blocking a dual crossing (in red).

The RSW Theorem Another powerful tool for studying percolation comes from the self-duality of $\mathbb{Z}^2$. This duality means that for all n we have that

$$\mathbb{P}_{\frac{1}{2}}(\text{there exists a left–right crossing of a } n \text{ by } n+1 \text{ box})$$
$$= \mathbb{P}_{\frac{1}{2}}(\text{there exists a top–bottom dual crossing of a } n \text{ by } n+1 \text{ box}) = \frac{1}{2}.$$

By revealing the states of edges in a clever order, and using the FKG inequality, Russo, [Rus78], and Seymour and Welsh, [SW78], were able to independently extend this to crossings of a n by kn box and show the following.

Theorem 1.1.4 (Russo, Seymour and Welsh, 1978)**.** *For each $k > 0$ there exists a constant $c(k) > 0$ such that*

$$c(k) < \mathbb{P}_{\frac{1}{2}}\Bigg($$


$$\Bigg) < 1 - c(k), \qquad (1.1.3)$$

for all $n \in \mathbb{N}$.

Ergodicity The event {there exists an infinite cluster} is tail measurable – meaning that the occurrence of the event does not depend on the configuration of any finite set of edges. Therefore, because $\mathbb{P}_p$ is a product measure, Kolmogorov's 0–1 law tells us that $\mathbb{P}_p(\{\text{there exists an infinite cluster}\}) \in \{0, 1\}$. However, since any two infinite clusters can be connected (or disconnected) via a finite bridge, then in order to ask how many infinite clusters there are we must look towards the property of *ergodicity*.

Definition 1.1.5 (Ergodicity)**.** Let $\mathbf{G}$ be a graph which is invariant under some translation $T : \mathbf{G} \longrightarrow \mathbf{G}$, and let μ be a lattice measure on $\mathbf{G}$. We say that μ is *ergodic* with respect to T if for every measurable event E with $T^{-1}(E) = E$ then either $\mu(E) = 0$ or $\mu(E) = 1$.

$\mathbb{P}_p$ is a product measure and thus when $\mathbf{G}$ is a translation invariant graph, Birkohoff's Ergodic Theorem tells us that $\mathbb{P}_p$ is ergodic. The ergodic property proves to be a powerful tool which applies for many other lattice measures. However, even in a situation where μ may not necessarily be ergodic we still have the following.

Theorem 1.1.6 (Ergodic Decomposition Theorem)**.** *For any translation invariant measure space Ω there exists a measurable map from Ω to the space of ergodic measures on Ω, $m : \Omega \longrightarrow \mathscr{E}(\Omega)$, such that*

$$\mu(E) = \int_{x \in \Omega} m_x(E) \, \mathrm{d}\mu(x), \tag{1.1.4}$$

for all translation invariant measures μ and all measurable E.

This theorem allows us to decompose translation invariant measures into ergodic parts, and is the result of some clever convex analysis. For details see [Kal02, Proposition 10.26] or [Var63].

1.1.3 The geometry of percolation

Number of infinite clusters Suppose we are given a supercritical percolation configuration ω. Since we know that $\mathbb{P}_p(0 \overset{\omega}{\leftrightarrow} \infty) > 0$, then by ergodicity we know that with probability 1 there must be an infinite cluster. But how many infinite clusters are there? In 1981 Newman and Schulman used a *finite energy* property (defined below) to show that the number of infinite clusters in a percolation configuration on a translation invariant graph must be either 0, 1 or ∞, [NS81].

Definition 1.1.7 (Finite energy)**.** A lattice measure μ is said to have the *finite energy property* if there exists a $c > 0$ such that for each $e \in \mathcal{E}$ and all configurations $\bar{\omega}$ on $\mathcal{E} \setminus \{e\}$ we have

$$c \leq \mu(e \text{ open} \,|\, \bar{\omega}) \leq 1 - c. \tag{1.1.5}$$

But can we do better? In 1989 Burton and Keane were able to use the amenability of $\mathbb{Z}^d$ (defined below) to show that any infinite cluster of $\mathbb{Z}^d$ is necessarily unique.

Definition 1.1.8 (Amenability)**.** Given a graph $\mathbf{G} = (\mathcal{V}, \mathcal{E})$, let $d : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{Z}_{\geq 0}$ denote the graph distance. Define $\mathbf{\Lambda}_n(v) = \{w \in \mathcal{V} : d(v, w) \leq n\}$ and set $\partial\mathbf{\Lambda}_n(v) = \{w \in \mathcal{V} : d(v, w) = n\}$. We say that $\mathbf{G}$ is *amenable* if

$$\limsup_{n\to\infty} \frac{|\partial\mathbf{\Lambda}_n(v)|}{|\mathbf{\Lambda}_n(v)|} = 0 \tag{1.1.6}$$

for all $v \in \mathcal{V}$.

By a quick calculation we can check that for $\mathbf{G} = \mathbb{Z}^d$ we have $|\mathbf{\Lambda}_n(v)| \sim C_d n^d$ and $|\mathbf{\Lambda}_n(v)| \sim c_d n^{d-1}$. Therefore by comparing with (1.1.6) we see that $\mathbb{Z}^d$ is amenable.

Theorem 1.1.9 (Burton and Keane, 1989)**.** *Suppose* $\mathbf{G}$ *is a translation invariant amenable graph and* μ *is an ergodic measure with the finite energy property, then the number of infinite clusters is either* 0 μ*-almost surely or* 1 μ*-almost surely.*

The theorem above uses many of the tools that were required for proving Proposition 3.2.13 – a key element in the construction of the PCF process – see Chapter 3. We therefore include a proof in order to motivate this later work.

Proof of Theorem 1.1.9. We start by using the argument of Newman and Schulman to show that the number of infinite clusters is in $\{0, 1, \infty\}$ μ-almost surely.

First observe that for each $k \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ the event $E_k = \{\omega \text{ has } k \text{ infinite clusters}\}$ is translation invariant. Therefore since μ is ergodic, for each $k \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ we have $\mu(E_k) \in \{0, 1\}$. Suppose for contradiction that $\mu(E_k) = 1$ for some $k \in \{2, 3, \ldots\}$. Since $k \geq 2$, we can take m sufficiently large to ensure that $\mu(E_k \cap \{\mathbf{\Lambda}_m = \mathbf{\Lambda}_m(0) \text{ intersects at least 2 infinite clusters}\}) \geq \frac{1}{2}$. Now fix such an m and observe that if $\bar{\omega}$ is a configuration on $\mathbb{Z}^d \setminus \mathbf{\Lambda}_m$ then by finite energy we have

$$\mu(\text{all edges of } \mathbf{\Lambda}_m \text{ are open} \,|\, \bar{\omega}) \geq c^{\#\mathbf{\Lambda}_m}, \tag{1.1.7}$$

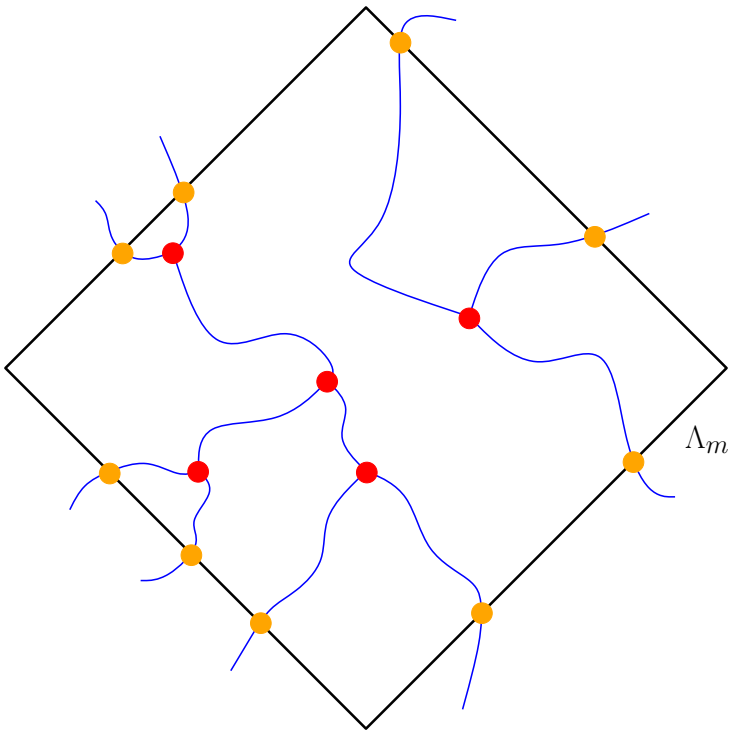


Figure 1.1.2: This diagram shows $\mathbf{\Lambda}_m$ containing five trifurcation points in two infinite clusters. In [BK89] an induction argument is used to show that any infinite cluster containing t trifurcation points must intersect the boundary of $\mathbf{\Lambda}_m$ in at least $t+2$ places.

where $\#\mathbf{\Lambda}_m$ denotes the number of edges in $\mathbf{\Lambda}_m$, and c is the constant from Definition 1.1.7. To get our contradiction we now observe that if ω is a configuration with k infinite clusters where at least two intersect $\mathbf{\Lambda}_m$, then by modifying ω so that all the edges of $\mathbf{\Lambda}_m$ are open we get a configuration ω' with strictly fewer than k infinite clusters – at least two of the infinite clusters will have been joined together. Therefore from (1.1.7) we get

$$\mu(\omega \text{ has fewer than } k \text{ infinite clusters}) \geq \frac{1}{2}\, c^{\#\mathbf{\Lambda}_m} > 0,$$

implying that $\mu(E_k)$ must be strictly less than 1, a contradiction. Hence we can conclude that the number of infinite clusters must be in $\{0, 1, \infty\}$ μ-almost surely.

We now use the argument of Burton and Keane to show that an amenable graph can not have infinitely many infinite open clusters. The first step is to define the notion of a *trifurcation point*. Given a configuration ω we say that the vertex v is a trifurction point of ω if v is contained in an infinite open cluster C, and removing v along with its incident edges would partition C into three disjoint infinite components. Now suppose $\mathbf{\Lambda}$ is any sub-graph of $\mathbf{G}$. A lemma of [BK89] tells us that $\partial\mathbf{\Lambda}$ must contain at least as many vertices as there are trifurcation points inside $\mathbf{\Lambda}$. See Figure 1.1.2. Therefore it is not possible for an amenable graph to have a positive density of trifurcation points.

To complete the proof we suppose for contradiction that ω contains infinitely many open clusters μ-almost surely, and let m be such that $\mu(\mathbf{\Lambda}_m$ intersects at least 3 infinite clusters$) \geq \frac{1}{2}$. We can now fix such a configuration on $\mathbb{Z}^2 \setminus \mathbf{\Lambda}_m$ and use the finite energy property to modify the edges of $\mathbf{\Lambda}_m$ so that there are three disjoint open paths from 0 to different infinite clusters on the boundary. This ensures that 0 is a trifurcation point. Therefore, from the finite energy property, it follows that

$$C = \mu(0 \text{ is a trifurcation point}) \geq \frac{1}{2}\, c^{\#\mathbf{\Lambda}_m} > 0.$$

From the Ergodic Theorem we must then have that

$$\mu\left(\mathbf{\Lambda}_n \text{ contains at least } \frac{1}{2}\, C\, |\mathbf{\Lambda}_n| \text{ trifurcation points}\right) \longrightarrow 1$$

as $n \longrightarrow \infty$. Thus we get a contradiction since $|\partial \mathbf{\Lambda}_n| < \frac{1}{2}\, C\, |\mathbf{\Lambda}_n|$ whenever n is sufficiently large, and so we must have $k \in \{0, 1\}$ μ-almost surely as claimed. □

Size of infinite clusters Another question we can ask about a percolation configuration is "how big is a typical cluster?" By answering this we find another way in which the percolation model exhibits a phase transition.

Theorem 1.1.10. *Consider bond percolation on $\mathbb{Z}^d$ for $d \geq 2$ and use C_0 to denote the cluster containing the origin. Suppose $0 < p < p_c$, then there exists a constant $0 < \zeta(p) < \infty$ such that*

$$\mathbb{P}_p(|C_0| = k) = \exp[-(\zeta(p) + o(1)\,)k]. \tag{1.1.8}$$

When we have $p_c < p < 1$ then there are constants $0 < \eta(p) \leq \gamma(p) < \infty$ such that

$$\exp\left[-\eta(p)\, k^{\frac{d-1}{d}}\right] \lesssim \mathbb{P}_p(|C_0| = k) \lesssim \exp\left[-\gamma(p)\, k^{\frac{d-1}{d}}\right], \tag{1.1.9}$$

and at criticality we get

$$\chi(p_c) = \sum_{k=1}^{\infty} k\, \mathbb{P}_{p_c}(|C_0| = k) = \infty, \tag{1.1.10}$$

suggesting that $\mathbb{P}_{p_c}(|C_0| = k)$ decays like a negative power of k.

This theorem is built from the results of many different authors, who are too numerous to mention here. However, for a good reference to these and other results in percolation

theory please see [Gri99].

The scaling limit To conclude our discussion of the geometry of percolation we recall a paper of Smirnov, [Smi01], in which it is shown that there exists a unique conformally invariant continuum scaling limit to critical site percolation on the triangular lattice. Using this Smirnov shows that the law of the exploration process must converge to SLE_6 as the mesh size tends to zero, [Smi01, Theorem 3]. Here SLE_κ denotes the Stochastic Loewner Evolution with parameter κ. This was introduced by Schramm in [Sch00]. From this scaling limit it is possible to deduce various facts about the geometry of the critical percolation process. For example

- $\mathbb{P}_{p_c}(|C_0| = n) = n^{-\frac{96}{91}+o(1)}$,
- $\mathbb{P}_{p_c}(0 \overset{\omega}{\leftrightarrow} v) = \|v\|^{-\frac{5}{24}+o(1)}$.

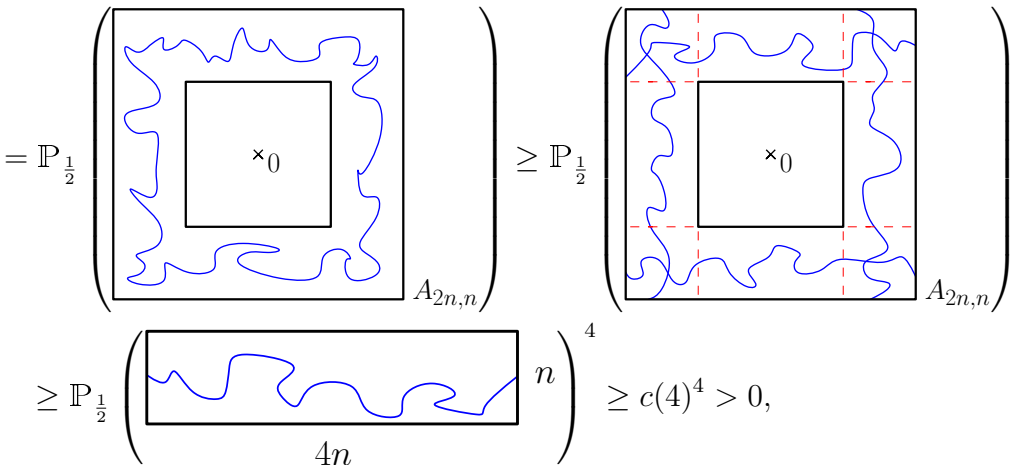
Since a SLE_6 curve has Hausdorff dimension $\frac{7}{4}$, then this must also be the Hausdorff dimension of the (limiting) percolation exploration process. Likewise, because we also know that SLE_6 has a boundary with Hausdorff dimension $\frac{4}{3}$, then so too does the outer boundary of a percolation cluster. See [Smi01, Corollary 6] for further details.

This scaling limit (and the related geometric properties) are conjectured to hold for critical percolation on a wide range of 2 dimensional lattices. However, Smirnov's proof requires three-fold rotational symmetry to show a conformal invariance property for the model, and so as yet the universality of the scaling limits remain unproven.

1.2 Frozen percolation and self organised criticality

Scale invariance For $n \in \mathbb{N}$ define $B_n = [-n, n]^2 \cap \mathbb{Z}^2$ and let $A_{2n,n} = B_{2n} \setminus B_{n-1}$. From the RSW Theorem we know that there exists a constant $c(4) > 0$ such that $\mathbb{P}_{\frac{1}{2}}$(there is an open crossing of a $4n$ by n box) $> c(4)$ for all $n \in \mathbb{N}$. Therefore by using the FKG inequality we get

$\mathbb{P}_{\frac{1}{2}}$(there is an open path around 0 contained in $A_{2n,n}$)



for all n. From this it is clear that in any box B_n there is a positive probability that we have a cluster whose diameter is of order n, and therefore we see that critical percolation has a kind of spacial *scale invariance* property.

Another indicator of scale invariance comes from looking at the radius of C_0 – the cluster containing the origin. In the case of site percolation on the triangular lattice a result of Smirnov and Werner, [SW01], tells us that at criticality

$$\mathbb{P}_{\frac{1}{2}}(\text{the radius of } C_0 \text{ is at least } r) = r^{-\frac{5}{48}+o(1)}. \tag{1.2.1}$$

This is proved by comparing the critical percolation process to SLE_6, and so is believed to also hold for critical (site or bond) percolation on any 2 dimensional translation invariant lattice. However, as Theorem 1.1.10 shows, when $p \neq p_c$ this behaviour no long occurs.

The power law decay demonstrated by (1.2.1) is present in many physical and social contexts. For example the size of forest fires, the luminosity of stars and the prevalence of rainfall; as well as the connections in a social network and the size of settlements (villages, towns, cities, etc.) all have distributions that decay like $k^{-\beta}$.

Self-organised criticality For the percolation model we know that this behaviour is only present for one special value of p, $p = p_c$. Therefore we ask if there are any

interacting systems where scale invariance is present without the need for any fine tuning of parameters. This question was first asked by Bak, Tang and Wiesenfeld in 1987, [BTW87]. By introducing the *sand-pile model*, they showed that is it possible to find dynamical systems where the critical point is an attractor. This behaviour was coined *self-organised criticality*.

1.2.1 The forest fire model

In 1992 Drossel and Schwable suggested that the forest fire model might also display self organised criticality, [DS92]. The dynamics of this model are similar to those of percolation – in that there are a collection of adjoining sites which open at a constant rate independently of each other. But there is also the added dynamic of *ignition*, and so each site is hit by lightning at a rate λ. A lightning strike at v causes the simultaneous destruction – i.e. closing – of all sites which are in the same cluster as v. These burnt sites will then open again at rate 1, forming new clusters as they do so.

In [Sta12] Stahl shows that for each fixed λ and each $d \geq 2$ there is a stationary measure for the forest fire model on $\mathbb{Z}^d$. However, in the mean-field case we know from a paper of Ráth and Tóth, [RT09], that for each fixed $\lambda > 0$ the lightning will regularly hit even small clusters, and thus the system will stay in a sub-critical state. An interesting question therefore is "what happens as $\lambda \longrightarrow 0$?" In this situation the dynamics are such that finite clusters will not be hit and so will never burn, but infinite clusters burn instantaneously.

A recent paper by Kiss, Manolescu and Sidoravicius, [KMS13], showed that no matter how thin an infinite cluster of $\mathbb{Z}^2$ is at the point it is burnt, there is a fixed lower bound on the amount of time it will take the system to recover. A consequence of this is that, as we take the limit $\lambda \longrightarrow 0$, the forest fire model becomes degenerate. However, Ahlberg, Duminil-Copin, Kozma and Sidoravicius have shown in [ADKS13] that when the dimension is sufficiently high then this behaviour no longer occurs, and so it is possible that there is a non-degenerate limiting measure as $\lambda \longrightarrow 0$.

1.2.2 Aldous's frozen percolation on the binary tree

A dynamic which is similar to ignition, and of particular interest to us, is the dynamic of *freezing*. When we say a cluster is *frozen* we mean that all the edges on the boundary are prevented from opening, and thus the shape of the cluster stays fixed forever.

In 1999 Aldous showed that it is possible to construct a bond percolation model on the binary tree where a cluster freezes as soon as it becomes infinite, [Ald99]. Of particular interest is the end state of the system where we have

- $\mathbb{P}_\infty(v \text{ is a singleton}) = \frac{1}{64}$, $\mathbb{P}_\infty(v \text{ is in a finite but non-singleton cluster}) = \frac{7}{64}$ and $\mathbb{P}_\infty(v \text{ is in an infinite cluster}) = \frac{7}{8}$. This means that there is a positive density of both finite and infinite clusters.

- If we condition on an edge e being in a finite cluster, C_e, then C_e has the same law as a critical percolation cluster – which in turn has the same law as a critical Galton–Watson tree. Therefore the size distribution of the finite clusters has a power law tail,

$$\mathbb{P}_\infty(|C_e| = k \,|\, C_e \text{ finite}) \sim ck^{-\frac{3}{2}}.$$

- The law of an infinite cluster is independent of the time at which it formed (and froze).

These properties combine to suggest that frozen percolation is a good model of self-organised criticality.

1.2.3 Frozen percolation on the lattice

In his paper Aldous asks if it is possible to construct this frozen percolation model on the lattice $\mathbb{Z}^d$. This was soon answered by a personal communication of Benjamini and Schramm, [BS99], details are given in [BT01, Section 3]. They showed that on $\mathbb{Z}^2$ no frozen percolation process can exist for the following reason.

Non-existence of frozen percolation on $\mathbb{Z}^2$ On $\mathbb{Z}^2$ we know that $\theta\left(\frac{1}{2}\right) = 0$, and therefore at $p = p_c = \frac{1}{2}$ there are no infinite clusters in either the primal or dual graphs. This means that there must be an infinite sequence of disjoint open clusters encircling 0,

$C_1, C_2, C_3, \ldots$ say. Now let $p_i > \frac{1}{2}$ denote the time at which C_i would join to C_{i+1} in the percolation model, and observe that $p_i \longrightarrow \frac{1}{2}$ as $i \longrightarrow \infty$ $\mathbb{P}$-almost surely.

Suppose for contradiction that the frozen percolation model does exist. Since the radii of the C_i must tend to infinity, then any frozen infinite cluster must contain a C_i for some i. Now let j be the smallest number larger than i such that $p_k < p_i$ for all $k \geq j$. At time p_{j-1}^- there must be an infinite cluster containing C_k for all $k \geq j$, but not C_i. Because this must freeze before connecting to C_i we get our contradiction.

N-parameter frozen percolation Since the infinite parameter frozen percolation model is known not to exist on $\mathbb{Z}^2$, various authors including van den Berg, Kiss, Lima and Nolin, [BdLN12], [BKN12] and [Kis13], have considered a percolation process where clusters freeze as soon as their diameter becomes larger than some N. Of particular interest is what happens as $N \longrightarrow \infty$, and if the non-existence of the ∞-parameter model is reflected in the asymptotic behaviour.

In [BdLN12] it is shown that if $C^{(N)}$ is the cluster at the origin in the final state of the N-parameter model, then for each $0 < a < b < 1$ we have

$$\liminf_{N\to\infty} \mathbb{P}^{(N)}(C^{(N)} \text{ has diameter } \in (aN, bN)) > 0.$$

From this we can deduce that $\limsup_{N\to\infty} \mathbb{P}^{(N)}(C^{(N)} \text{ frozen}) < 1$. In fact it has since been shown in [Kis13] that

$$\lim_{N\to\infty} \mathbb{P}^{(N)}(C^{(N)} \text{ frozen}) = 0.$$

Therefore, although it is also shown that there is a limiting process, we see that this is in some sense trivial.

However, on the tree things are as we would expect. In [BKN12] it is shown that if we consider frozen percolation on the binary tree where clusters freeze as soon as their volume is larger than N, then if v is a distinguished vertex and C_v is the cluster containing v we have

$$\lim_{N\to\infty} \mathbb{P}^{(N)}(C_v = C) = \mathbb{P}_\infty(C_v = C),$$

for each finite C. Therefore the process converges – in some sense – to Aldous's frozen percolation model as $N \longrightarrow \infty$.

1.3 Constructing models on the lattice

The method used by Aldous to construct frozen percolation on the binary tree relies on the freezing times of adjacent edges satisfying a certain distributional recurrence relation. Whilst very elegant, this construction is specific to regular trees, and so if we wish to show the existence of similar systems in different settings, then an alternative methodology is required. We now give a review of two possible approaches.

1.3.1 Finite range interacting particle systems

Definition 1.3.1 (Finite range interacting particle systems)**.** An *interacting particle system* is a Markov process consisting of countably many sites (e.g. edges and/or vertices) with pure jump processes that interact by modifying each other's (finite) transition rates. An interacting particle system is said to have *finite range* if there is a universal R such that the jump rate at one site only depends on the states of at most R other sites.

It is known that a finite range interacting particle system can be constructed by using the infinitesimal rates to write down a semi-group generator, and then by using the Hille–Yosida Theorem to construct the corresponding semi-group. Using general Markov process theory it is then possible to define a unique process from the semi-group. For details see [Pen08] or [Lig85].

The N-parameter frozen percolation models studied in [BdLN12], [BKN12] and [Kis13] have the property that clusters freeze as soon as they reach diameter N, and therefore their existence is assured by this general framework. However, since the event $v \overset{\omega}{\longleftrightarrow} \infty$ depends on infinitely many edges, then clearly any model in which infinite clusters freeze is not a finite range interacting system. Therefore if one wishes to show that the infinite volume limit of such a system exists then other methods are required. We shall now introduce the *random cluster model* in order to give an example of how *monotonicity* can be used to construct an infinite volume limit.

1.3.2 The infinite volume measure of the random cluster model

Definition 1.3.2 (Random cluster model)**.** Let $q > 0$, $0 \leq p \leq 1$ and suppose that $\mathbf{G} = (\mathcal{V}, \mathcal{E})$ is a finite graph. We now choose a subset of edges $F \subseteq \mathcal{E}$ according to the

following probability mass function

$$\varphi_{\mathbf{G},p,q}(F) = \frac{1}{Z_{\mathbf{G},p,q}} p^{|F|}(1-p)^{|\mathcal{E}\setminus F|} q^{k(F)}, \tag{1.3.1}$$

and define $\varphi_{\mathbf{G},p,q}$ to be the *random cluster measure* on $\mathbf{G}$. Here $k(F)$ denotes the number of connected components of the graph $(\mathcal{V}, F)$ and $Z_{\mathbf{G},p,q}$ is a normalising constant.

Observe that in the case where $q = 1$ then this becomes

$$\varphi_{\mathbf{G},p,1}(F) = p^{|F|}(1-p)^{|\mathcal{E}\setminus F|}, \tag{1.3.2}$$

and thus each of the edges are present with probability p independently of each other. Hence we have recovered the percolation model. What is more, if $q = 2$ then $\varphi_{\mathbf{G},p,2}$ gives us a representation of the Ising model of ferromagnetism, and for general $q \in \mathbb{N}$ there is a connection to the q-state Potts model. Therefore the random cluster model, as introduced by Fortuin and Kasteleyn [FK72], gives us a general framework for studying the percolation, Ising and Potts models. For a good general overview of the frozen cluster model please see [Gri06].

Suppose that $\mathbf{G} \subseteq \mathbb{Z}^d$ and use $\partial\mathbf{G}$ to denote the collection of vertices in $\mathbf{G}$ which have a neighbour in $\mathbb{Z}^d \setminus \mathbf{G}$. Let ξ be a configuration on the edges of $\mathbb{Z}^d \setminus \mathbf{G}$ and observe that if two connected of components of $(\mathcal{V}, F)$ both meet $\partial\mathbf{G}$ then it is possible for them to be connected by a path in ξ. Let $k^\xi(F)$ be equal to the number of connected components in $(\mathcal{V}, F)$ once these extra connections have been taken into account. I.e. $k^\xi(F)$ is equal to the number of connected components of $F \cup \xi$ which intersect $\mathbf{G}$. We can now define $\varphi^\xi_{\mathbf{G},p,q}$ – the *random cluster measure on* $\mathbf{G}$ *with boundary conditions* ξ – by using (1.3.1), replacing $k(F)$ by $k^\xi(F)$ and rescaling $Z_{\mathbf{G},p,q}$ accordingly.

Observe that by adding edges to $\mathbb{Z}^d \setminus \mathbf{G}$ we can only decrease the number of clusters which intersect $\mathbf{G}$. Therefore if ξ and ζ are two configurations on the edges of $\mathbb{Z}^d \setminus \mathbf{G}$ with $\xi \leq \zeta$ then $k^\xi(F) \geq k^\zeta(F)$ for each set of edges $F \subseteq \mathcal{E}$. If $q \geq 1$ then (1.3.1) gives greater mass to configurations with more clusters, and so from the previous observation it is possible to show that in fact $\varphi^\xi_{\mathbf{G},p,q} \leq_{\text{st}} \varphi^\zeta_{\mathbf{G},p,q}$ for all $\xi \leq \zeta$.

Boundary conditions There are now two extremal cases for ξ, the case $\xi = 1$ where all edges in $\mathbb{Z}^d \setminus \mathbf{G}$ are open – which gives us *wired boundary conditions*, and the case $\xi = 0$ where all edges in $\mathbb{Z}^d \setminus \mathbf{G}$ are closed – which gives us *free boundary conditions*.

Therefore for any configuration ξ we have

$$\varphi^0_{\mathbf{G},p,q} \leq_{\text{st}} \varphi^{\xi}_{\mathbf{G},p,q} \leq_{\text{st}} \varphi^1_{\mathbf{G},p,q}. \tag{1.3.3}$$

Now suppose that we have another finite sub-graph $\mathbf{G} \subseteq \mathbf{H} \subseteq \mathbb{Z}^d$. By conditioning on the edges of $\mathbf{H} \setminus \mathbf{G}$ we see that when we restrict the measures $\varphi^0_{\mathbf{H},p,q}$ and $\varphi^1_{\mathbf{H},p,q}$ to $\mathbf{G}$ we must get

$$\varphi^0_{\mathbf{G},p,q} \leq_{\text{st}} \varphi^0_{\mathbf{H},p,q}\big|_{\mathbf{G}} \leq_{\text{st}} \varphi^1_{\mathbf{H},p,q}\big|_{\mathbf{G}} \leq_{\text{st}} \varphi^1_{\mathbf{G},p,q}. \tag{1.3.4}$$

Therefore if we let $\mathbf{\Lambda}_1 \subseteq \mathbf{\Lambda}_2 \subseteq \ldots$ be any exhaustion of $\mathbb{Z}^d$ with each $\mathbf{\Lambda}_n$ finite, and E_S be the event that a finite set S of edges are all open, then the limits

$$\lim_{n\to\infty} \varphi^0_{\mathbf{\Lambda}_n,p,q}(E_S) = \varphi^0_{\mathbb{Z}^d,p,q}(E_S) \qquad \text{and} \qquad \lim_{n\to\infty} \varphi^0_{\mathbf{\Lambda}_n,p,q}(E_S) = \varphi^0_{\mathbb{Z}^d,p,q}(E_S)$$

both exist. The collection of events $\{E_S : S \subseteq \mathbb{Z}^d,\ S \text{ finite}\}$ are convergence determining, and so by applying Kolmogorov's Extension Theorem we get limiting measures $\varphi^0_{\mathbb{Z}^d,p,q}$ and $\varphi^1_{\mathbb{Z}^d,p,q}$. A quick check also reveals that these limiting measures are independent of the choice of exhaustion $\mathbf{\Lambda}_1 \subseteq \mathbf{\Lambda}_2 \subseteq \ldots$.

Of course $\varphi^0_{\mathbb{Z}^d,p,q}$ and $\varphi^1_{\mathbb{Z}^d,p,q}$ need not necessarily be equal, but it is known from a theorem of Grimmett, [Gri95], that for each $q \geq 1$ there is a countable, and possibly empty, subset $\mathcal{D}_q \subseteq [0,1]$ such that $\varphi^0_{\mathbb{Z}^d,p,q}$ and $\varphi^1_{\mathbb{Z}^d,p,q}$ are equal for each $p \in [0,1] \setminus \mathcal{D}_q$.

It turns out that a similar monotonicity condition (for a particular set of boundary conditions) holds for the PCF process of Chapter 3. Therefore showing that a particular sequence of measures is monotone, and using those to define an infinite volume limit, is one of the key ingredients in our proof of Theorem 3.1.1.

Chapter 2

Interacting walks in one dimension

This chapter starts by giving a brief overview to self-interacting 1-dimensional stochastic processes such as the *"true" self avoiding random walk* and *Domb–Joyce polymer model*. We then give an explanation of *entropic repulsion*, before presenting the Ray–Knight Theorems and the Donsker–Varadhan Theorem. These are the two key tools used in Chapter 4.

2.1 Non-Markovian random walks

The PCF process we shall encounter in Chapter 3 is non-Markovian in a spatial sense – to know what happens to the edges inside a box $\mathbf{\Lambda}$ we will need to know more than just the state of the edges on the boundary $\partial\mathbf{\Lambda}$. In fact to properly understand the configuration inside $\mathbf{\Lambda}$ we will need to know at what time each of the clusters on the boundary froze.

In this introductory chapter and Chapter 4 we look at look at 1-dimensional processes which are non-Markovian in a temporal sense. That is to say the future of the process depends not only on the present state, but also on the past. To begin we shall consider random walks in the discrete setting.

2.1.1 Self-interacting walks on $\mathbb{Z}$

Let $(X_n)_{n\geq 0}$ be a simple random walk on $\mathbb{Z}$. At a time N the amount of time X has spent at a site $i \in \mathbb{Z}$ is given by

$$L_i(N) = \sum_{n=0}^{N} \mathbb{1}_{\{X_n = i\}}. \tag{2.1.1}$$

We call $L_i(N)$ the *local time* of X at level i and time N. One way to construct a walk which interacts with its past is to modify $(X_n)_{n\geq 0}$ so that its transition probabilities at time n depend on $(L_i(n))_{i\in\mathbb{Z}}$.

The "true" self-avoiding random walk The "true" self-avoiding random walk was introduced by Amit, Parisi and Peliti in 1983, [APP83] . It gives a non-trivial model for a random walk with memory which behaves qualitatively differently to the usual diffusive behaviour of the simple random walk.

The process can be characterised by its transition probabilities

$$\mathbb{P}(X_{n+1} = i \pm 1 \,|\, X_n = 1, \mathcal{F}_n) = \frac{\exp\{-\beta L_{i\pm 1}(n)\}}{\exp\{-\beta L_{i-1}(n)\} + \exp\{-\beta L_{i+1}(n)\}},$$

where $\beta > 0$ is a parameter for the model. It is believed that for each positive β this process has a non-trivial limit when scaled by $n^{-\frac{2}{3}}$. However, as yet no rigorous results for this particular model are known. In [Tót95] Tóth considers a slightly different set-up – where the transition probabilities are determined by the number of visits to adjacent

edges rather than adjacent sites. In this setting it is shown (amongst other things) that $n^{-\frac{2}{3}}X_n$ converges to a non-trivial distribution as $n \longrightarrow \infty$.

Domb–Joyce polymer model The Domb–Joyce polymer model was first introduced in [DJ72] as a model which interpolates between the self avoiding walk and the simple random walk. Like the self avoiding walk it is not strictly a process, but is instead defined by a sequence of measures on nearest neighbour paths of length N.

$$\mu_N^\beta(\omega) = \frac{1}{Z_N^\beta} \exp\left(-\beta \sum_{0 \le i \le j \le N} \mathbb{1}_{\{\omega_i = \omega_j\}}\right) 2^{-N}. \tag{2.1.2}$$

Here Z_N^β is a normalising constant and $\beta > 0$ is a parameter for the model. Note that $\beta = 0$ would correspond to the simple random walk, and that μ_N^β converges to the SAW_N as $\beta \longrightarrow \infty$.

In [Kön93] König showed that the one dimensional Domb–Joyce model exhibits ballistic behaviour (with a speed depending on β). König strengthened this in 1996 to show that the end point also satisfies a central limit theorem, [Kön96].

In the discrete setting it is difficult to make exact computations of the speed (and variance) of the process. Indeed, showing that the speed monotonically increases with β remains an open problem. However, similar results have been established in the continuum – where different methods are used and so explicit calculation becomes easier.

2.1.2 Continuous processes with self interaction

It is also possible to define a notion of *local time* for a continuous semi-martingale $(W_t)_{t \ge 0}$ by

$$L_x(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{|W_s - x| < \varepsilon\}} \, \mathrm{d}s.$$

This limit exists almost surely for all continuous semi-martingales, every $x \in \mathbb{R}$ and all $t \ge 0$. Roughly speaking $L_x(t)$ tells us how long $(W_t)_{t \ge 0}$ has spent at a given point x before time t – see Figure 2.1.1 for the local time of a Brownian motion. We refer the reader to [RY99, Chapter VI] for an overview of the local time process.

Using this definition it is possible to construct one-dimensional continuum analogues to

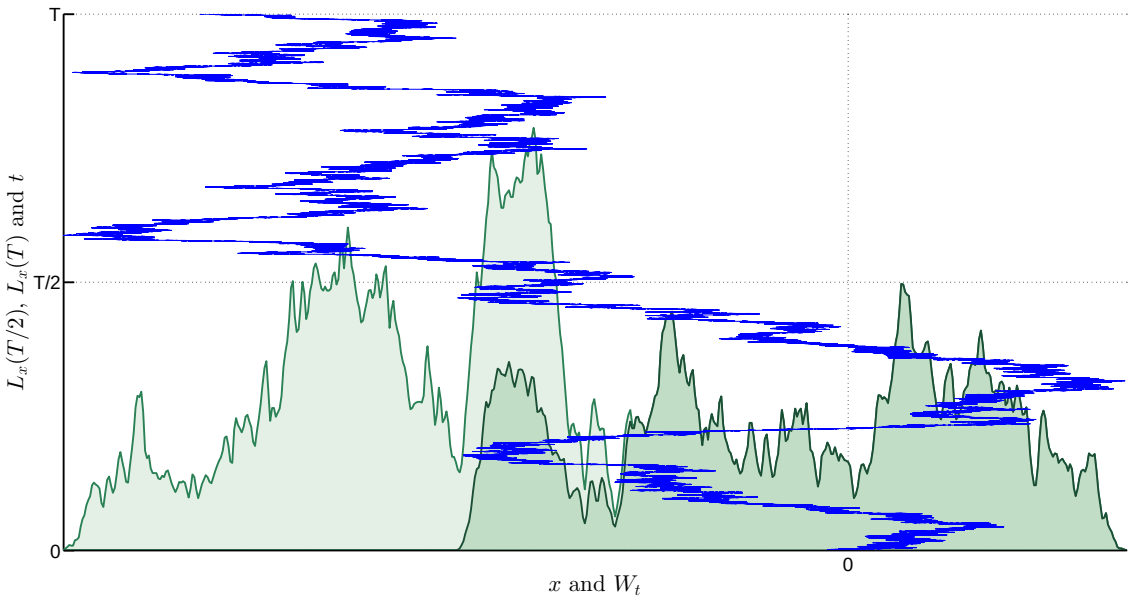


Figure 2.1.1: A Brownian motion $(W_t)_{0\le t\le T}$ (blue) together with its local time profiles $L_x(T/2)$ (darker green) and $L_x(T)$ (lighter green).

the self-interacting processes of Section 2.1.1. It is worth noting however, that in higher dimension a typical diffusion never revisits a point, and so the notion of local time no longer makes sense. Therefore, whilst both the "true" self-avoiding walk and Domb–Joyce models can be defined in higher dimension, their continuum counterparts can not.

Edwards' polymer model The Edwards polymer model can be thought of as a continuous version of the Domb–Joyce model, and is again defined by a sequence of measures on paths

$$\frac{\mathrm{d}\mu_T^\beta}{\mathrm{d}\mathbb{W}} = \exp\left(-\beta \int_{\mathbb{R}} L_x(T)^2 \,\mathrm{d}x\right). \tag{2.1.3}$$

Here $\mathbb{W}$ denotes Wiener measure. In 1984 Westwater established a weak law of large numbers for the model, [Wes84]. In fact it is shown that for every $\beta > 0$ there exists a $\theta^*(\beta) > 0$ such that

$$\lim_{T\to\infty} \mu_T^\beta\left(\left|\frac{W_T}{T} - \theta^*(\beta)\right| < \varepsilon \,|\, W_T > 0\right) = 1 \tag{2.1.4}$$

for every $\varepsilon > 0$. As an immediate consequence we see that the self-repulsive nature of Edwards' polymer model causes the process to have a type of ballistic behaviour (no matter how small the repulsive force may be).

It is interesting to note that Westwater proved this result by using the Ray–Knight Theorems to find the law of the local times at T, and then finding the end point by using tools from the theory of large deviations. This is essentially the approach we take in Chapter 4 in order to find the ballistic speed of a Brownian motion conditioned to have $L_x(t) \leq 1$ for all x and t.

In 1997 van der Hofstad, den Hollander and König showed that Edwards' model also satisfies a strong law of large numbers, [HHK97].

$$\lim_{T\to\infty} \mu_T^\beta \left(\frac{W_T - \theta^*(\beta)T}{\sigma^*(\beta)\sqrt{T}} \leq C \,|\, W_T \geq 0 \right) = \mathcal{N}((-\infty, C]) \quad \text{for all } C \in \mathbb{R}. \tag{2.1.5}$$

Here $\mathcal{N}$ denotes the measure of the normal distribution with mean 0 and variance 1. What is more, there are constants $b^* > 0$ and $c^* > 0$ such that

$$\theta^*(\beta) = b^* \beta^{\frac{1}{3}} \qquad \text{and} \qquad \sigma^*(\beta) = c^*.$$

Here b^* and c^* can both be calculated from the eigenvalues of a particular Strum–Liouville operator. Therefore there is again a similarity with the results of Chapter 4 – where the limiting speed of our conditioned process can be calculated by finding the principal eigenvalues of a particular class of differential operators.

Brownian motion with self intersection In [NRW87] Norris, Rogers and Williams asked if it is possible to construct a process in 3 dimensions which satisfies

$$X_t = W_t + \int_0^t \left(\int_0^s f(X_s - X_u)\, \mathrm{d}u \right) \mathrm{d}s.$$

A case they considered particularly interesting is when $f(x)$ is the electrostatic potential $\dfrac{x}{\|x\|^2}$. However, it is not clear that this stochastic differential equation even has a unique solution, nor what its properties might be. Thus, as a first step, they chose to work with the following SDE.

$$X_t = W_t - \int_0^t g(X_s, L_{X_s}(s))\, \mathrm{d}s. \tag{2.1.6}$$

They show that an analogue to the Ray–Knight Theorems holds for $(L_x(\infty))_{x\in\mathbb{R}}$, allowing one to prove in many cases of interest that $\lim\limits_{t\to\infty} \dfrac{X_t}{t}$ exists almost surely. In the special case where $X_t = W_t - \displaystyle\int_0^t L_{X_s}(s)\, \mathrm{d}s$, it is also shown that $\lim\limits_{t\to\infty} \dfrac{X_t}{t} = -\sqrt{\dfrac{\pi}{4}}$ almost surely.

2.1.3 Brownian motion with limited local time

The processes discussed thus far have been constructed by changing the measure of a Brownian motion (or simple ransom walk) by a function of the local time. It turns out that similar phenomena also occur when we condition on the local time of a Brownian motion satisfying certain conditions.

Local times bounded by 1 In [BB10] Benjamini and Berestycki consider the behaviour of a Brownian motion conditioned to have its local time bounded by 1. This means that the process can spend at most 1 unit of time in any unit interval – and so must exhibit self avoiding properties, and escape to infinity with a speed of at least 1. Because the local times of a Brownian motion fluctuate wildly an effect of *entropic repulsion* comes into play. Therefore the speed of the process ends up being strictly greater than 1. The convergence and precise speed of the limiting process depend upon the way in which we condition on the event $\{L_x(t) \leq 1 \text{ for all } x \in \mathbb{R} \text{ and } t \geq 0\}$. However, in several cases it is possible to show that the limiting process does indeed exist, and make explicit calculations of its velocity. For further details see Section 4.1.

[BB10] also includes a convergence result for a simple symmetric random walk conditioned to visit each site in $\mathbb{Z}$ at most N times (for $N \geq 2$). However, this is proved using different methods to the continuum case, and so whilst it can be shown that the speed of the process is strictly greater than N^{-1}, the exact speed remains unknown.

Local time at 0 bounded by a function $f(t)$ In a related work of Benjamini and Berestycki, [BB11], a Brownian motion $(W_s)_{s\geq 0}$ is conditioned to have its local time at 0 bounded by some positive function $f(t)$. $L_0(t) \leq f(t)$ for all $t \geq 0$ say. They show that if

$$\mathcal{I}(f) = \int_1^\infty \frac{f(t)}{t^{\frac{3}{2}}} \mathrm{d}t < \infty \tag{2.1.7}$$

then $(W_s)_{s\geq 0}$ is transient almost surely. Note that $\{L_0(t) \leq f(t) \text{ for all } t \geq 0\}$ is an event with probability 0 which is realised by conditioning on $\{L_0(t) \leq f(t) \text{ for all } 0 \leq t \leq T\}$ and taking the limit as $T \longrightarrow \infty$. Benjamini and Berestycki also conjecture that this result is sharp – meaning that if $\mathcal{I}(f) = \infty$ then $(W_s)_{s\geq 0}$ is almost surely recurrent. Recent work of Kolb and Savov makes significant progress towards this goal. In [KS13] they show that if $f(t)$ is monotonically increasing and $\dfrac{f(t)}{\sqrt{t}} \longrightarrow 0$ monotonically then $(W_s)_{s\geq 0}$ is almost surely recurrent if and only if $\mathcal{I}(f) = \infty$.

2.2 Entropic repulsion

The idea behind *entropic repulsion* is that in certain situations the easiest way for a process to satisfy a particular condition is to satisfy a seemingly stronger one. Perhaps the simplest example of this happening occurs when we consider 1-dimensional Brownian motion conditioned to stay positive.

Suppose $(W_t)_{t\geq 0}$ is a Brownian motion with Wiener measure $\mathbb{W}$ and $W_0 = x > 0$. If we condition on the event $\mathcal{E}_T^+ = \{W_t > 0 \text{ for all } 0 \leq t \leq T\}$ then as $T \longrightarrow \infty$ the measures $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^+)$ converge weakly to $\mathbb{Q}^+$, the measure of a process with generator

$$\mathcal{L}f(x) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}f(x). \tag{2.2.1}$$

See [Pin85b, Example 3] for a nice proof of this. We can recognise (2.2.1) as being the generator for the radial process of 3-dimensional Brownian motion. Therefore, since Brownian motion is transient in dimension 3 and above, it must follow that $(W_t)_{t\geq 0}$ is transient $\mathbb{Q}^+$-almost surely. This means that although we only conditioned on $\inf_{t\geq 0} W_t \geq 0$, what we have ended up with is $\liminf_{t\to\infty} W_t = \infty$. A seemingly stronger condition.

2.2.1 Brownian motion conditioned to stay in a bounded set

Another example – which is more relevant to Chapter 4 – comes from considering a d-dimensional Brownian motion conditioned to stay in a connected bounded set $0 \in U \subseteq \mathbb{R}^d$. Let $\mathcal{E}_T^U = \{W_t \in U \text{ for all } 0 \leq t \leq T\}$. From [Pin85b] we know that $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^U)$ converges weakly to $\mathbb{Q}^U$, the measure of a process with generator

$$\mathcal{L}f(x) = \frac{1}{2}\Delta f(x) + \frac{\varphi_0'(x)}{\varphi_0(x)}\nabla f(x). \tag{2.2.2}$$

Here φ_0 is the unique eigenfunction corresponding to the first eigenvalue of the Laplacian with zero boundary conditions on U. What is more, $\mathbb{Q}^U$ has a stationary distribution with a density given by φ_0^2 (where φ_0 is normalised so that $\|\varphi_0\|_2 = 1$).

Since φ_0^2 is smooth, positive and decaying to zero on the boundary of U then we see that the Brownian motion has a preference for spending time in the middle of U, rather than near the edges. Of course the proportion of time spent in the middle of U is very dependent upon the geometry of U. However, by looking at how likely it is for W_t to be

close to the boundary of U we find a sense in which the entropic repulsion of W_t from the boundary of U is universal.

Proposition 2.2.1. *Suppose $0 \in U \subseteq \mathbb{R}^d$ is connected bounded set, and let $d(x, U^{\mathrm{C}}) = \inf\{\|x - y\| : y \in \mathbb{R}^d \setminus U\}$, then*

$$\lim_{t\to\infty} \mathbb{Q}^U(d(W_t, U^{\mathrm{C}}) < \varepsilon) \sim C_U \varepsilon^3 \tag{2.2.3}$$

as $\varepsilon \longrightarrow 0$.

Remark 2.2.2. This proposition motivates Theorem 4.1.3 in Chapter 4.

This proposition could be proved by analysing φ_0 – the principle eigenfunction of the Laplacian on U. However, such an analytic proof would give us very little insight into why Brownian motion should behave in this way. Therefore we shall now assume that ∂U is smooth and give a simple probabilistic explanation of why the entropic repulsion of $(W_t)_{t\geq 0}$ from the boundary has this exponent.

Sketch proof. By using the Fundamental Theorem of Calculus it suffices to show that if we are given a fixed $t > 0$ then $\mathbb{Q}^U(d(W_t, U^{\mathrm{C}}) < \varepsilon)$ is differentiable as a function of ε and that

$$\lim_{\eta\to 0} \frac{1}{\eta} \mathbb{Q}^U(d(W_t, U^{\mathrm{C}}) \in [\varepsilon, \varepsilon + \eta)) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathbb{Q}^U(d(W_t, U^{\mathrm{C}}) < \varepsilon) \sim c_U \varepsilon^2 \tag{2.2.4}$$

as $\varepsilon \longrightarrow 0$. To do this we fix $\varepsilon > 0$, let $\eta \ll \varepsilon$ and suppose $W_t \in U$ with $d(W_t, U^{\mathrm{C}}) \in [\varepsilon, \varepsilon + \eta)$. Since ∂U is smooth then, provided ε is sufficiently small, U can be approximated by a half-space near W_t and so $d(W_s, U^{\mathrm{C}})$ can be approximated by a 1-dimensional Brownian motion. Because we are conditioning on $d(W_s, U^{\mathrm{C}}) \geq 0$ for all s, we therefore ask what is the probability that a 1-dimensional Brownian motion $(\tilde{W}_s)_{s\geq 0}$ stays positive near t given that $\tilde{W}_t \in (\varepsilon, \varepsilon + \eta)$. We now obtain

$$\begin{aligned}
\mathbb{Q}^U(d(W_t, U^{\mathrm{C}}) \in [\varepsilon, \varepsilon + \eta)) &= \lim_{T\to\infty} \frac{\mathbb{W}(d(W_t, U^{\mathrm{C}}) \in [\varepsilon, \varepsilon + \eta) \text{ and } W_s \in U \,\forall\, s \in [0, T])}{\mathbb{W}(W_s \in U \text{ for all } s \in [0, T])} \\
&\approx c_1 \,\tilde{\mathbb{W}} \left(\tilde{W}_t \in [\varepsilon, \varepsilon + \eta) \text{ and } \inf_{s\in[t-1,t+1]} \tilde{W}_s \geq 0 \right) \\
&\approx c_1 \,\tilde{\mathbb{W}}(\tilde{W}_t \in [\varepsilon, \varepsilon + \eta)) \times \tilde{\mathbb{W}} \left(\inf_{s\in[t-1,t+1]} \tilde{W}_s \geq 0 \,|\, \tilde{W}_t = \varepsilon \right).
\end{aligned}$$

Now observe that $\tilde{\mathbb{W}}(\tilde{W}_t \in [\varepsilon, \varepsilon + \eta)) \approx c_2 \eta$. Here the constant c_2 does not depend on $0 < \varepsilon \ll 1$. Notice also that because we have fixed $\tilde{W}_t = \varepsilon$, then $(\tilde{W}_s)_{t<s\leq t+1}$ and

$(\tilde{W}_s)_{t-1\le s<t}$ are independent. Therefore if we let $X \sim N(0,1)$, we can calculate

$$\tilde{\mathbb{W}}\left(\inf_{s\in[t-1,t+1]} \tilde{W}_s \ge 0 \,|\, \tilde{W}_t = \varepsilon\right) = \tilde{\mathbb{W}}\left(\inf_{s\in(t,t+1]} \tilde{W}_s \ge 0 \,|\, \tilde{W}_t = \varepsilon\right)^2$$
$$= (1 - 2\,\mathbb{P}(X > \varepsilon))^2 \sim c_3\varepsilon^2$$

as $\varepsilon \longrightarrow 0$. Thus it follows that $\mathbb{Q}^U(d(W_t, U^c) < \varepsilon)$ is differentiable near 0, and its derivative is asymptotically equal to $c_U\varepsilon^2$. □

2.2.2 Hard walls in the Gaussian free field

The phenomenon of entropic repulsion is also present in other contexts.

In [BDG01] and [BDZ95] the authors consider the Gaussian free field with a hard wall at 0, and show that if the field is conditioned to be positive on some open set $D \subseteq (\mathbb{Z}/N\mathbb{Z})^d$ then the value of the field is typically of order $\log N$ on D. Indeed, suppose $D \subset (0,1)^2$ is a smooth domain at a positive distance from the boundary of $(0,1)^2$ and that ϕ is a discrete Gaussian free field on $V_N = (\mathbb{Z}/N\mathbb{Z})^2$. If we let $D_N = \mathbb{Z}^2 \cap ND$ and set $\Omega^+_{D_N} = \{\phi_x \ge 0, \text{ for all } x \in D_N\}$, then in [BDG01, Theorem 4] it is proved that for each $\varepsilon > 0$

$$\lim_{N\to\infty} \sup_{x\in D_N} \mathbb{P}\left(\left|\phi_x - \sqrt{\frac{8}{\pi}}\log N\right| \ge \varepsilon \log N \,|\, \Omega^+_{D_N}\right) = 0. \tag{2.2.5}$$

In [BDG01] it is also shown that (in the unconditioned case) the expected maximum of the discrete Gaussian free field on V_N is $\sqrt{8\pi^{-1}}\log N + o(\log N)$. Thus the intuitive reason for (2.2.5) is that it is very unlikely for the maximum of the oscillations on D_N to be significantly less than $\sqrt{8\pi^{-1}}\log N$, and hence the easiest way for the free field to satisfy $\Omega^+_{D_N}$ is by globally shifting the free field on D_N by $\sqrt{8\pi^{-1}}\log N + o(\log N)$.

2.3 The Ray–Knight Theorems

The results of Chapter 4 rely on the Ray–Knight Theorems and the Donsker–Varadhan Theorem. Section 2.3 and Section 2.4 are designed to give an overview of these.

2.3.1 Local times via the square Bessel process

Recall that if $(W_t)_{t\geq 0}$ is a $\mathbb{R}$-valued Brownian motion with Wiener measure $\mathbb{W}$, then $\mathbb{W}$-almost surely the jointly continuous version of the *local time process* is given by

$$L_x(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{|W_s - x| < \varepsilon\}} \, \mathrm{d}s, \tag{2.3.1}$$

for all $x \in \mathbb{R}$ and all $t \geq 0$. Roughly speaking the local time tells us how long a Brownian motion has spent at a given point . We refer the reader to [RY99, Chapter VI] for an overview of the local time process of Brownian motion.

Recall also that for every $d \geq 0$ there is a unique strong solution of

$$\mathrm{d}Y_x = 2\sqrt{Y_x} \, \mathrm{d}B_x + d \, \mathrm{d}x \tag{2.3.2}$$

which we call the *square Bessel process of dimension d*. In the case where $Y_0 = c$, we shall write that $(Y_x)_{x\geq 0}$ is a BES$Q^d(c)$ process. If we suppose further that $(Y_x)_{0\leq x\leq a}$ is conditioned on the event $Y_a = b$, then $(Y_x)_{0\leq x\leq a}$ becomes a *square Bessel bridge of dimension d and length a*. Here we write that $(Y_x)_{0\leq x\leq a}$ is a BES$Q^d_a(c, b)$ bridge. Note that although the event $Y_a = b$ has zero probability, we can make sense of such a restriction by conditioning on the event $|Y_a - b| < \varepsilon$ and letting $\varepsilon \longrightarrow 0$. One can then show that the conditioned measures converge weakly to the law of the square Bessel bridge. See [RY99, Chapter XI, §3] for details.

In this thesis we are interested in the cases where $d = 2$ or $d = 0$ since the local times of a Brownian motion can be related to BESQ^2 and BESQ^0 processes via the Ray–Knight Theorems. These apply to a Brownian motion stopped at certain stopping times. For $a \in \mathbb{R}$ and $b \geq 0$ we define

$$\tau^a = \tau^a_0 = \inf\{t : W_t = a\} \quad \text{and} \quad \tau^a_b = \inf\{t : L_a(t) > b\}, \tag{2.3.3}$$

and then the Ray–Knight Theorems tell us the following:

Theorem 2.3.1 (First Ray–Knight Theorem)**.** *Let $(W_t)_{t\geq 0}$ be a Brownian motion, fix $a > 0$ and define $Y_x = L_{a-x}(\tau_0^a)$ for $0 \leq x \leq a$. The process $(Y_x)_{0\leq x\leq a}$ is then equal in law to a* BES$Q^2(0)$ *process.*

Theorem 2.3.2 (Second Ray–Knight Theorem)**.** *Let $(W_t)_{t\geq 0}$ be a Brownian motion, fix $b \geq 0$ and define $Y_x^+ = L_x(\tau_b^0)$ for $x \geq 0$ and $Y_x^- = L_{-x}(\tau_b^0)$ for $x \geq 0$. The processes $(Y_x^+)_{x\geq 0}$ and $(Y_x^-)_{x\geq 0}$ are then equal in law to two independent* BES$Q^0(b)$ *processes.*

For a reference to the Ray–Knight Theorems see [RY99, Chapter XI, §2]. Theorem 1.2 of [RY99, Chapter XI, §1] tells us that a BES$Q^{d_1}(b_1)$ process plus a BES$Q^{d_2}(b_2)$ process is equal in law to a BES$Q^{d_1+d_2}(b_1 + b_2)$ process. Therefore it is possible to combine Theorem 2.3.1 and Theorem 2.3.2 to describe $(L_x(\tau_b^a))_{x\in\mathbb{R}}$ for all $a \in \mathbb{R}$ and each $b \geq 0$. However, in Chapter 4 we will want to be able to describe $(L_x(T))_{x\in\mathbb{R}}$ at a fixed (rather than random) $T > 0$. To simplify notation we assume $W_T \geq 0$ and then define

$$S_T^- = \int_0^T \mathbb{1}_{\{W_s<0\}}\,\mathrm{d}s = \int_{-\infty}^0 L_x(T)\,\mathrm{d}x \quad \text{and} \quad S_T^+ = \int_0^T \mathbb{1}_{\{W_s>W_T\}}\,\mathrm{d}s = \int_{W_T}^\infty L_x(T)\,\mathrm{d}x.$$

Using $(Y_x)_{x\geq 0}$ to denote a square Bessel process, we let

- $q(a, c, \cdot)$ be the density of Y_a with respect to the law of a BES$Q^2(c)$ process.
- $f(c, \cdot)$ be the density of $\int_0^\infty Y_x\,\mathrm{d}x$ with respect to the law of a BES$Q^0(c)$ process.
- $g(a, c, b, \cdot)$ be the density of $\int_0^a Y_x\,\mathrm{d}x$ with respect to the law of a BES$Q_a^2(c, b)$ bridge.

In [Leu98] we are told that the density f has a relatively simple expression

$$f(c, s) = \frac{c}{\sqrt{8\pi}s^{\frac{3}{2}}} \exp\left(-\frac{c^2}{8s}\right), \tag{2.3.4}$$

whereas the expressions for q and g turn out to be more complicated. From a result of Leuridan, [Leu98, Theorem 1], we now have the following:

Theorem 2.3.3 (Leuridan, 1998)**.** *Let $T > 0$ be fixed and suppose $(W_t)_{t\geq 0}$ is a Brownian motion conditioned on the event $\{W_T \geq 0\}$. The joint distribution of W_T and $(L_x(T))_{x\in\mathbb{R}}$ is characterised by the following properties.*

- *The 5-tuple $(W_T, L_{W_T}(T), L_0(T), S_T^-, S_T^+)$ admits a probability density on $[0, \infty)^5$,*

$$(a, b, c, s^-, s^+) \longmapsto 2\,q(a, c, b)f(c, s^-)f(b, s^+)g(a, c, b, T - s^- - s^+).$$

- *Conditionally on* $(W_T, L_{W_T}(T), L_0(T), S_T^-, S_T^+) = (a, c, b, s^-, s^+)$
  - $(L_{-x}(T))_{x\geq 0}$, $(L_{a+x}(T))_{x\geq 0}$ *and* $(L_x(T))_{0\leq x\leq a}$ *are independent.*
  - $(L_{-x}(T))_{x\geq 0}$ *and* $(L_{a+x}(T))_{x\geq 0}$ *are equal in law to* $\mathrm{BESQ}^0(c)$ *and* $\mathrm{BESQ}^0(b)$ *processes conditioned on the events* $\int_0^\infty Y_x \,\mathrm{d}x = s^-$ *and* $\int_0^\infty Y_x \,\mathrm{d}x = s^+$.
  - $(L_x(T))_{0\leq x\leq a}$ *is equal in law to a* $\mathrm{BESQ}^2_a(c, b)$ *bridge conditioned on the event that* $\int_0^a Y_x \,\mathrm{d}x = T - s^- - s^+$.

Of course the events $\left\{\int_0^\infty Y_x^- \,\mathrm{d}x = s^-\right\}$, $\left\{\int_0^\infty Y_x^+ \,\mathrm{d}x = s^+\right\}$ and $\left\{\int_0^a Y_x \,\mathrm{d}x = T - s^- - s^+\right\}$ all have probability 0. However, as with the construction of the Brownian bridge, the conditioned processes can all be realised as a weak limit. In [Leu98, Theorem 1] we are given explicit generators for these conditioned processes.

- The joint processes $\left(L_{-x}(T), \int_{-\infty}^{-x} L_y(T)\,\mathrm{d}y\right)_{x\geq 0}$ and $\left(L_{a+x}(T), \int_{a+x}^{\infty} L_y(T)\,\mathrm{d}y\right)_{x\geq 0}$ are both Markovian with infinitesimal generator

$$2z_1\frac{\partial^2}{\partial z_1^2} + \left(4 - \frac{z_1^2}{z_2}\right)\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_2}. \tag{2.3.5}$$

- The joint process $\left(L_x(T), \int_x^a L_y(T)\,\mathrm{d}y, x\right)_{0\leq x\leq a}$ is Markovian with infinitesimal generator

$$2z_1\frac{\partial^2}{\partial z_1^2} + \left(2 + 4z_1\left(\frac{\partial_1 q_{z_3}}{q_{z_3}}(z_1, b) + \frac{\partial_1 g_{z_3}}{g_{z_3}}(z_1, b, z_2)\right)\right)\frac{\partial}{\partial z_1} - z_1\frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}. \tag{2.3.6}$$

In [Leu98] the generators (2.3.5) and (2.3.6) are both be deduced from *Doob's h-transform.* See [RW00b] for a detailed explanation of the h-transform. In 1999 Pitman also proved a similar result to Theorem 2.3.3 via a branching process approximation. See [Pit99].

2.4 Large deviations

The theory of *large deviations* deals with the convergence of probability measures when we condition on certain kinds of extreme events. For example if $(W_t)_{t\geq 0}$ is a Brownian motion then one might ask what happens when we condition on event $\{W_t \geq t\}$ and let $t \longrightarrow \infty$.

The central message of large deviations is that "any large deviation is achieved in the least unlikely of all the unlikely ways". In our example the least unlikely way for a Brownian motion to reach t by time t is for it to travel with a constant drift. Thus it turns out that $\mathbb{W}(\cdot \,|\, W_t \geq t)$ converges weakly to the law of a Brownian process with drift 1.

For a good introduction to the theory of large deviations please see [Hol08].

2.4.1 The invariant density of a diffusion process

Suppose M is a complete metric space and $(X_t)_{t\geq 0}$ is a diffusion process on M with infinitesimal generator $\mathcal{L}$, and starting point $X_0 = y$. For each $A \in \mathcal{B}(M)$ let

$$L((X_t), T, A) = \frac{1}{T}\int_0^T \mathbb{1}_{\{X_t \in A\}}\,\mathrm{d}t. \tag{2.4.1}$$

Then for a given $(X_t)_{t\geq 0}$ and T, $L((X_t), T, A)$ gives the proportion of time that $(X_t)_{0\leq t\leq T}$ spends in A, and $L((X_t), T, \cdot)$ defines a probability measure on M. We call $L((X_t), T, \cdot)$ the *occupation measure of* (X_t) *at time* T. Now for each probability measure $\mu \subseteq \mathcal{P}(\mathbb{R})$ define

$$I(\mu) = -\inf_{u\in\mathcal{D}, u>0} \int_M \frac{\mathcal{L}(u)}{u}\,\mathrm{d}\mu, \tag{2.4.2}$$

where $\mathcal{D}$ is the domain of $\mathcal{L}$. A theorem of Donsker and Varadhan, [DV75a, Theorem 1], now gives us the following.

Theorem 2.4.1 (Donsker–Varadhan)**.** *Let $\mathcal{P}(\mathbb{R})$ be the space of all probability measures on $\mathbb{R}$ equipped with the weak topology, and suppose $\mathbb{P}_y$ is the measure for a process generated by $\mathcal{L}$ and started at y. For all $C \subseteq \mathcal{P}(\mathbb{R})$ closed and all $O \subseteq \mathcal{P}(\mathbb{R})$ open, if we also*

assume that $y \in \text{support}(\mu)$ *for each* $\mu \in O$*, then we have*

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{P}_y(L((Y_x), T, \cdot) \in C) \leq -\inf_{\mu\in C} I(\mu) \tag{2.4.3}$$

$$\liminf_{T\to\infty} \frac{1}{T} \log \mathbb{P}_y(L((Y_x), T, \cdot) \in O) \geq -\inf_{\mu\in O} I(\mu). \tag{2.4.4}$$

In other words the occupation measure of the process generated by $\mathcal{L}$ *satisfies a* large deviations principle *with rate function* I*.*

Remark 2.4.2. By following the proof of the Donsker–Varadhan Theorem in [DV75a, Section 2] we see that (2.4.3) and (2.4.4) also hold in the case where y is a random variable on $\mathbb{R}$. In this situation we substitute $\mathbb{P}_y(L((Y_x), T, \cdot) \in \cdot)$ with $\mathbb{E}^y\big\{\mathbb{P}_y(L((Y_x), T, \cdot) \in \cdot)\big\}$, and replace the condition that $\big\{y \in \text{support}(\mu) \text{ for each } \mu \in O\big\}$ with the condition that $\big\{\text{support}(y) \subseteq \text{support}(\mu) \text{ for each } \mu \in O\big\}$.

In general the formula for $I(\mu)$ given by (2.4.2) is rather impenetrable. However, in [DV75a, Theorem 5] it is shown that when M is the real line equipped with the usual metric, and $\mathcal{L}$ is self-adjoint with respect to Lebesgue measure, then

$$I(\mu) = \begin{cases} \left\|\sqrt{-\mathcal{L}}g\right\|_2^2, & \text{where } g = \sqrt{\dfrac{\mathrm{d}\mu}{\mathrm{d}x}} \text{ exists and is in the domain of } \sqrt{-\mathcal{L}} \\ \infty, & \text{otherwise} \end{cases}. \tag{2.4.5}$$

In Chapter 4 we will only be interested in the case when $\mathcal{L}$ is a second order differential operator of the form

$$\mathcal{L}f(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) + b(x)\frac{\mathrm{d}}{\mathrm{d}x} f(x), \tag{2.4.6}$$

where a is continuous and b is continuously differentiable. In this setting we can use (2.4.5) to express (2.4.2) as an integral.

First suppose that the drift function b can be written as $b(x) = a(x)\dfrac{\mathrm{d}}{\mathrm{d}x}Q(x)$, for a continuously differentiable function $Q(x)$. In this case $\mathcal{L}$ can be realised as a self-adjoint operator with respect to a measure μ_{rev}, defined by $\mathrm{d}\mu_{\text{rev}} = \mathrm{e}^{2Q}\mathrm{d}x$. Therefore, provided $\dfrac{\mathrm{d}\mu}{\mathrm{d}\mu_{\text{rev}}}$ exists and $g = \sqrt{\dfrac{\mathrm{d}\mu}{\mathrm{d}\mu_{\text{rev}}}}$ belongs to the domain of $\sqrt{-\mathcal{L}}$, we get

$$I(\mu) = \left\|\sqrt{-\mathcal{L}}g\right\|^2_{2,\mu_{\text{rev}}} = \frac{1}{2}\int a(x)\left(\frac{\mathrm{d}}{\mathrm{d}x}g(x)\right)^2 \mathrm{e}^{2Q(x)}\,\mathrm{d}x. \tag{2.4.7}$$

See [Pin07] for more details. It is clear from (2.4.5) that if $\mathcal{L}$ is self-adjoint then I :

$\mathcal{P}(\mathbb{R}) \longrightarrow [0,\infty]$ is lower semi-continuous. Note however that I is not continuous since any measure μ with g in the domain of $\sqrt{-\mathcal{L}}$ can be approximated arbitrary closely by measures without a Radon-Nikodym derivative. We also remark that I is in fact lower semi-continuous for all infinitesimal generators $\mathcal{L}$. This follows from [Pin85a] where Pinsky shows that (2.4.2) can be replaced by

$$I(\mu) = -\inf_{u\in\mathcal{D},u>0}\int_M \frac{\mathcal{L}(u)}{u}\,\mathrm{d}\mu = -\inf_{u\in C^2(\mathbb{R}),u>0}\int_M \frac{\mathcal{L}(u)}{u}\,\mathrm{d}\mu. \tag{2.4.8}$$

2.4.2 A simple proof of the Donsker–Varadhan Theorem

The proof of the Donsker–Varadhan Theorem in the general case is somewhat complicated, and so below we present a version of the theorem for an irreducible reversible finite Markov chain. The proof is an adaptation of an idea of Berestycki, [Ber13].

Theorem 2.4.3 (Donsker–Varadhan for a finite Markov chain)**.** *Suppose $(X_t)_{t\geq 0}$ is an irreducible reversible continuous time Markov chain on the finite state space S. Denote its transition matrix by P, its invariant distribution by π, and its measure by $\mathbb{P}$. Now let $\phi : S \longrightarrow [0,1]$ be a function with $\sum_{x\in S}\phi(x) = 1$ and $\phi(x) > 0$ for all $x \in S$, then*

$$\lim_{\varepsilon\to 0}\lim_{T\to\infty}\frac{1}{T}\log\mathbb{P}\left((1-\varepsilon)\phi(x) \leq \frac{L_x(T)}{T} \leq (1+\varepsilon)\phi(x)\,\forall\, x\in S\right) = -\left\|\sqrt{-\nabla_P}\, g\right\|_{2,\pi}^2.$$

Here g is defined by $g(x) = \sqrt{\dfrac{\phi(x)}{\pi(x)}}$, and $\nabla_P g(x) = \sum_{y\in S} p(x,y)(g(y)-g(x))$ is the discrete Laplacian. Thus

$$\left\|\sqrt{-\nabla_P}\, g\right\|_{2,\pi}^2 = -\sum_{x\in S}\pi(x)g(g)\nabla_P g(x) = \sum_{x\in S}\pi(x)g(x)\sum_{y\in S}p(x,y)(g(x)-g(y)) \tag{2.4.9}$$

is the Dirichlet energy of g. Compare with (2.4.5).

Sketch of proof. The first step is to find another irreducible reversible Markov chain $(Y_t)_{t\geq 0}$ on the state space S such that ϕ is the invariant measure for Y. We shall write Q for the transition matrix of Y and $\mathbb{Q}$ for the associated measure.

Define $g : S \longrightarrow \mathbb{R}$ by $g(x) = \sqrt{\dfrac{\phi(x)}{\pi(x)}}$, and let Q be such that $q(x,y) = \dfrac{g(y)}{g(x)}p(x,y)$ for

each $x, y \in S$. We then have

$$\phi(x)q(x,y) = \frac{\sqrt{\phi(x)\phi(y)}}{\sqrt{\pi(x)\pi(y)}}\pi(x)p(x,y) = \frac{\sqrt{\phi(x)\phi(y)}}{\sqrt{\pi(x)\pi(y)}}\pi(y)p(y,x) = \phi(y)q(y,x).$$

Thus if $(Y_t)_{t\geq 0}$ is a Markov chain with transition matrix Q, then Y has ϕ as its invariant distribution. We now claim that $\mathbb{Q}$ is approximately equal to the conditional law of X given

$$\mathcal{D}_{T,\varepsilon} = \left\{(1-\varepsilon)\phi(x) \leq \frac{L_x(T)}{T} \leq (1+\varepsilon)\phi(x) \text{ for all } x \in S\right\}.$$

Assume $\omega \in S^{[0,\infty)}$ is a trajectory visiting sites $x_0, x_1, \ldots, x_n$ for times $\tau_0, \tau_1, \ldots, \tau_n$, where $\tau_0 + \tau_1 + \ldots + \tau_n = T$. Then for $\omega \in \mathcal{D}_{T,\varepsilon}$ we can compute

$$\begin{aligned}
\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(\omega) &= \frac{p(x_0)\mathrm{e}^{-p(x_0)\tau_0} \times \frac{p(x_0,x_1)}{p(x_0)}p(x_1)\mathrm{e}^{-p(x_1)\tau_1} \times \ldots \times \frac{p(x_{n-1},x_n)}{p(x_{n-1})}p(x_n)\mathrm{e}^{-p(x_n)\tau_n}}{q(x_0)\mathrm{e}^{-q(x_0)\tau_0} \times \frac{q(x_0,x_1)}{q(x_0)}q(x_1)\mathrm{e}^{-q(x_1)\tau_1} \times \ldots \times \frac{q(x_{n-1},x_n)}{q(x_{n-1})}q(x_n)\mathrm{e}^{-q(x_n)\tau_n}} \\
&= \frac{p(x_0,x_1)\ldots p(x_{n-1},x_n)p(x_n)\exp\left(-\sum_{i=0}^{n} p(x_i)\tau_i\right)}{q(x_0,x_1)\ldots q(x_{n-1},x_n)q(x_n)\exp\left(-\sum_{i=0}^{n} q(x_i)\tau_i\right)} \\
&= \frac{g(x_0)}{g(x_n)}\frac{p(x_n)}{q(x_n)}\exp\left(\sum_{x\in S}(q(x)-p(x))L_x(T)\right)
\end{aligned}$$

where $p(x) = \sum_{y\in S} p(x,y)$ and $q(x) = \sum_{y\in S} q(x,y)$ are the respective transition rates of X and Y at the point $x \in S$. Now

$$q(x) - p(x) = \sum_{y\in S} q(x,y) - p(x,y) = \sum_{y\in S} p(x,y)\left(\frac{g(y)-g(x)}{g(x)}\right) = \frac{1}{g(x)}\nabla_P g(x).$$

Thus since $\omega \in \mathcal{D}_{T,\varepsilon}$ we get

$$\begin{aligned}
\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(\omega) &= \frac{g(x_0)}{g(x_n)}\frac{p(x_n)}{q(x_n)}\exp\left(\sum_{x\in S}\frac{1}{g(x)}\nabla_P g(x)L_x(t)\right) \\
&= \frac{g(x_0)}{g(x_n)}\frac{p(x_n)}{q(x_n)}\exp\left((1+\mathcal{O}(\varepsilon))\,T\sum_{x\in S}\frac{1}{g(x)}\nabla_P g(x)\phi(x)\right) \\
&= \frac{g(x_0)}{g(x_n)}\frac{p(x_n)}{q(x_n)}\exp\left((1+\mathcal{O}(\varepsilon))\,T\sum_{x\in S}\pi(x)g(x)\nabla_P g(x)\right).
\end{aligned}$$

To complete the proof we note that the Ergodic Theorem for Markov chains tells us that for each fixed $\varepsilon > 0$ we have $\mathbb{Q}(\mathcal{D}_{T,\varepsilon}) \longrightarrow 1$ as $T \longrightarrow \infty$. Therefore

$$\begin{aligned}\mathbb{P}(\mathcal{D}_{T,\varepsilon}) &= \mathbb{E}_{\mathbb{Q}}\left(\mathbb{1}_{\mathcal{D}_{T,\varepsilon}} \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\right) \\ &= (1 - o(1)) \frac{g(x_0)}{g(x_n)} \frac{p(x_n)}{q(x_n)} \exp\left((1 + \mathcal{O}(\varepsilon))\, t \sum_{x \in S} \pi(x) g(x) \nabla_P g(x)\right),\end{aligned}$$

and so because $\dfrac{g(x_0)}{g(x_n)} \dfrac{p(x_n)}{q(x_n)}$ is bounded away from both 0 and ∞ we get

$$\lim_{\varepsilon \to 0} \lim_{T \to \infty} \frac{1}{T} \log \mathbb{P}(\mathcal{D}_{t,\varepsilon}) = - \left\| \sqrt{-\nabla_P}\, g \right\|_{2,\pi}^2,$$

as claimed. □

Chapter 3

Percolation with constant freezing

In this chapter we introduce and study a model of *percolation with constant freezing* (*PCF*) where edges open at constant rate 1, and clusters freeze at rate α independently of their size. The main result is that the infinite volume process can be constructed on any amenable vertex transitive graph. This is in sharp contrast to models of percolation with freezing previously introduced, where the limit is known not to exist. Our interest is in the study of the percolative properties of the final configuration as a function of α. We also obtain more precise results in the case of trees. Surprisingly the algebraic exponent for the cluster size depends on the degree, suggesting that there is no lower critical dimension for the model. Moreover, even for $\alpha < \alpha_c$, it is shown that finite clusters have algebraic tail decay, which is a signature of self organised criticality. Partial results are obtained on $\mathbb{Z}^d$, and many open questions are discussed.

This chapter is based on work appearing in [Mot14b].

3.1 Introduction

Let $\alpha > 0$, and let $\mathbf{G} = (\mathcal{V}, \mathcal{E})$ be a finite graph for now. We consider a modification of the percolation process defined as follows. Each edge $e \in \mathcal{E}$ opens independently at rate 1, and each open cluster freezes independently at a constant rate α (regardless of its size). Once a cluster has frozen all its neighbouring edges will remain closed forever. We are interested in the final configuration of the edges of $\mathbf{G}$, and its dependence on α. This is the *percolation with constant freezing* (*PCF*) model, and was introduced by Ben–Naim and Krapivski in the mean field case – where several interesting features were shown (through not entirely rigorous methods) [BNK05b]. In this chapter we prove that an infinite volume process can be defined on the finite dimensional lattice $\mathbb{Z}^d$ – or more generally on any amenable vertex transitive graph $\mathbf{G}$. The existence of this process on any countable tree $\mathbf{T}$ is also shown.

3.1.1 Existence of an infinite volume limit

It is straightforward to construct the PCF model on any finite graph $\mathbf{G} = (\mathcal{V}, \mathcal{E})$ – see Definition 3.2.1. By running the process until all the clusters are frozen, we induce the *PCF measure* $\mu_{\mathbf{G},\alpha}$ on $\{0,1\}^{\mathcal{E}}$. Our aim is to construct a PCF measure when $\mathbf{G}$ is an amenable vertex transitive graph.

The infinite volume version of PCF can be understood in terms of *local limits*. Given $\mathbf{G}$, we let $\mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \ldots$ be a fixed exhaustion of $\mathbf{G}$ with each $\mathbf{G}_n$ finite. Then for any finite $\mathbf{\Lambda} \subseteq \mathbf{G}$, there exists an N such that $\mathbf{\Lambda} \subseteq \mathbf{G}_n$ for all $n \geq N$. It is shown that the restriction of PCF on $\mathbf{G}_n$ to $\mathbf{\Lambda}$ tends in law to a unique limiting process as $n \longrightarrow \infty$. This limiting process is then equal in law to the infinite volume version of PCF restricted to $\mathbf{\Lambda}$.

Theorem 3.1.1. *For every amenable vertex transitive graph $\mathbf{G} = (\mathcal{V}, \mathcal{E})$, and every fixed rate of freezing $\alpha > 0$, there exists an infinite volume PCF process on $\mathbf{G}$ in the sense of local limits. This induces the PCF measure $\mu_{\mathbf{G},\alpha}$ on $\{0,1\}^{\mathcal{E}}$. Furthermore, the measure $\mu_{\mathbf{G},\alpha}$ is translation invariant.*

This result is in sharp contrast to Aldous's *Frozen percolation* model [Ald99], in which clusters freeze as soon as they become infinite. Although Aldous showed that this model can be rigorously defined on a binary tree – where it exhibits interesting behaviour – we

have seen in Section 1.2.3 that it can not be constructed on a general graph. Note that in Aldous's model edges open independently at times distributed uniformly on $[0, 1]$. This gives it a slightly different time parametrisation to the one used in PCF. However, by using the map $t \longmapsto 1 - \mathrm{e}^{-t}$ we see that both parametrisations are equivalent.

The construction of an infinite volume PCF process goes via a secondary process which we call *warm PCF*. Here the definition of the process is adapted so that clusters on the boundary of $\mathbf{G}_n \subseteq \mathbf{G}$ do not freeze. By modifying the process in this way we obtain a form of monotonicity which in turn leads to the infinite volume limit. We note that the technique of using a warm boundary can also be adapted to the construction of similar models. In particular the monotonicity properties can be used to construct an infinite volume version of the Drossel–Schwabl forest fire model, [DS92]. Indeed, the construction of a stationary measure for the forest fire model in [Sta12] uses similar ideas.

3.1.2 Properties of PCF

On a lattice Having seen that PCF can be constructed on any amenable vertex transitive graph it is natural to ask what the process looks like. Clearly the rate of freezing controls the behaviour of the model, and so we make the following conjecture.

Conjecture 3.1.2. *For every dimension $d \geq 2$ there is a critical value α_c such that when we run rate α PCF on $\mathbb{Z}^d$ then if $\alpha > \alpha_c$ all clusters are almost surely finite (the* sub-critical regime*), and if $\alpha < \alpha_c$ the final distribution contains infinite clusters (the* super-critical regime*).*

The following proposition is a first step in this direction.

Proposition 3.1.3. *Suppose we run PCF on the d dimensional lattice $\mathbb{Z}^d$, then provided $\alpha > 0$ is sufficiently large (depending on d) all clusters will remain finite almost surely.*

Figures 3.1.1, 3.1.2 and 3.1.3 show the largest clusters from simulations of PCF on a square grid for $\alpha = 0.60$, 0.55 and 0.50. These illustrate the transition from the sub-critical to super-critical regimes. Figure 3.1.4 then shows simulations of super-critical PCF on a cubic lattice. Here the freezing times of the largest clusters are such that we get 2 or 3 spanning clusters in the final configuration. Since the model is not too sensitive to the freezing times of the largest clusters, these images suggest that there is positive probability of 2 of more infinite clusters forming when we run super-critical PCF in dimension 3 or

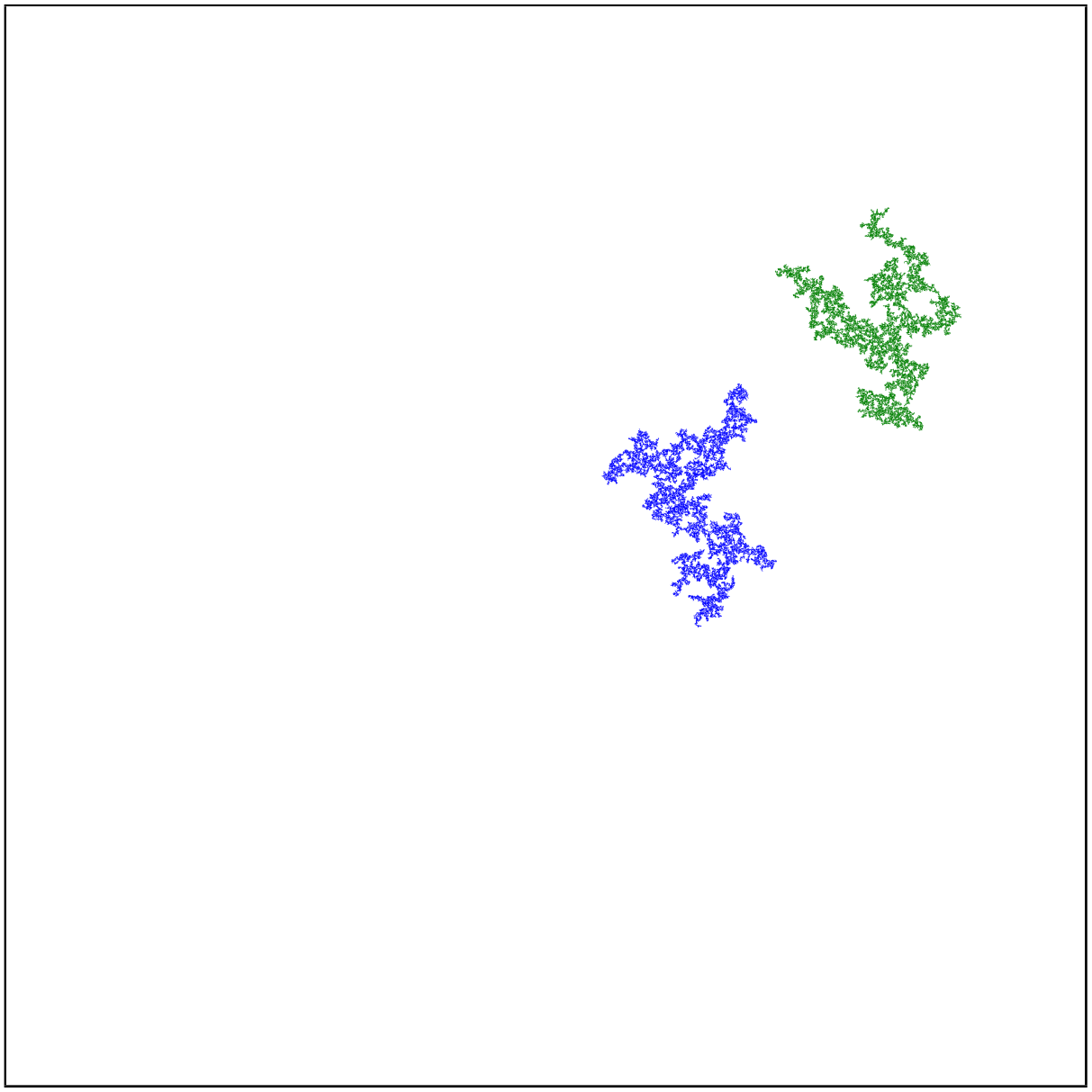


Figure 3.1.1: Sub-critical PCF
The largest two clusters generated in a simulation of sub-critical PCF on a 4096 by 4096 square grid. Here we have $\alpha = 0.60$ which our simulations suggest is sufficiently quick to prevent the formation of a spanning cluster.

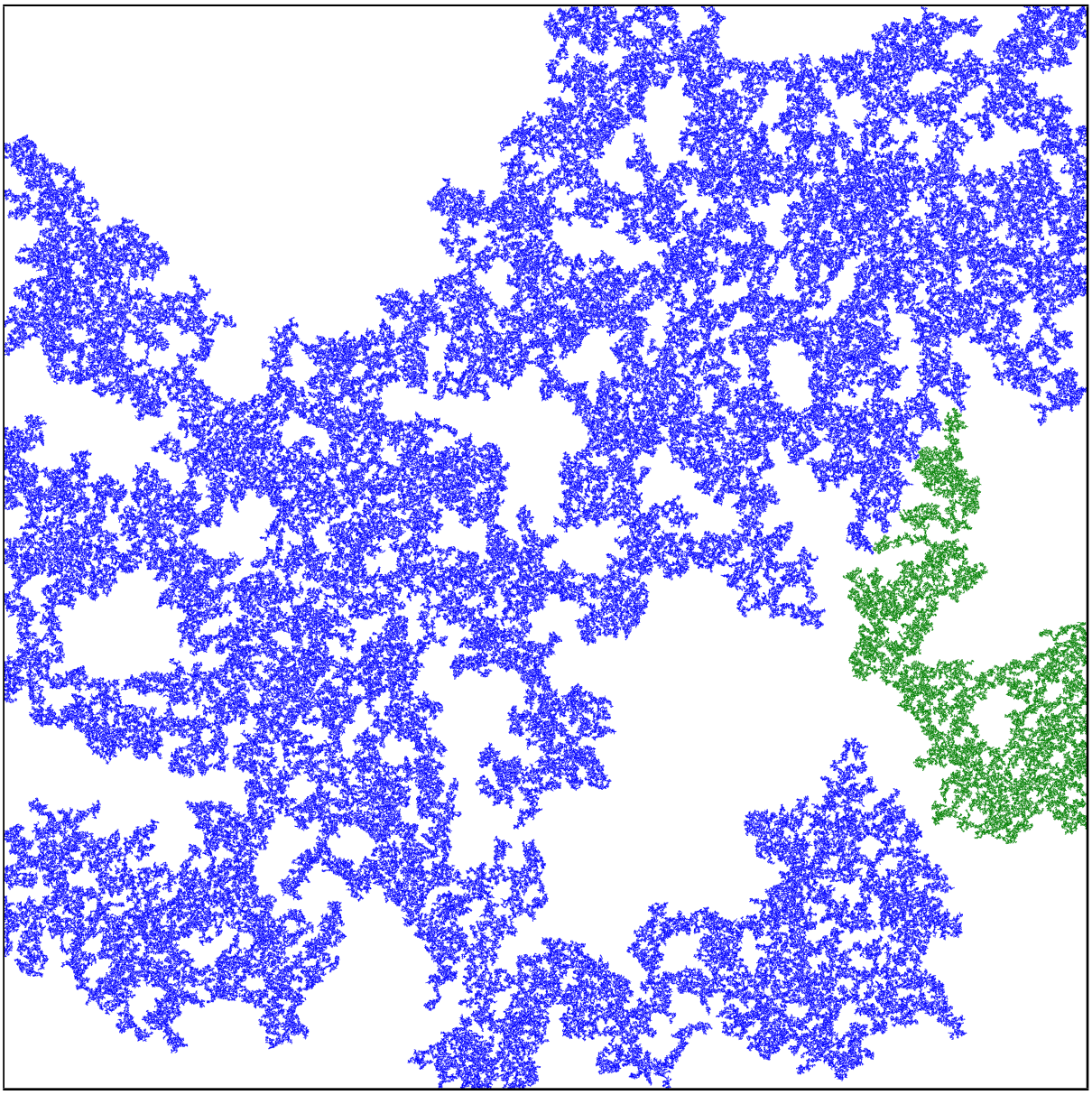


Figure 3.1.2: Near-critical PCF
Our simulations – presented in Section 3.5 – suggest that on $\mathbb{Z}^2$ we have $\alpha_c \approx 0.55$. Here are the two largest clusters generated in a simulation of PCF on a 4096 by 4096 square grid at this near critical value.

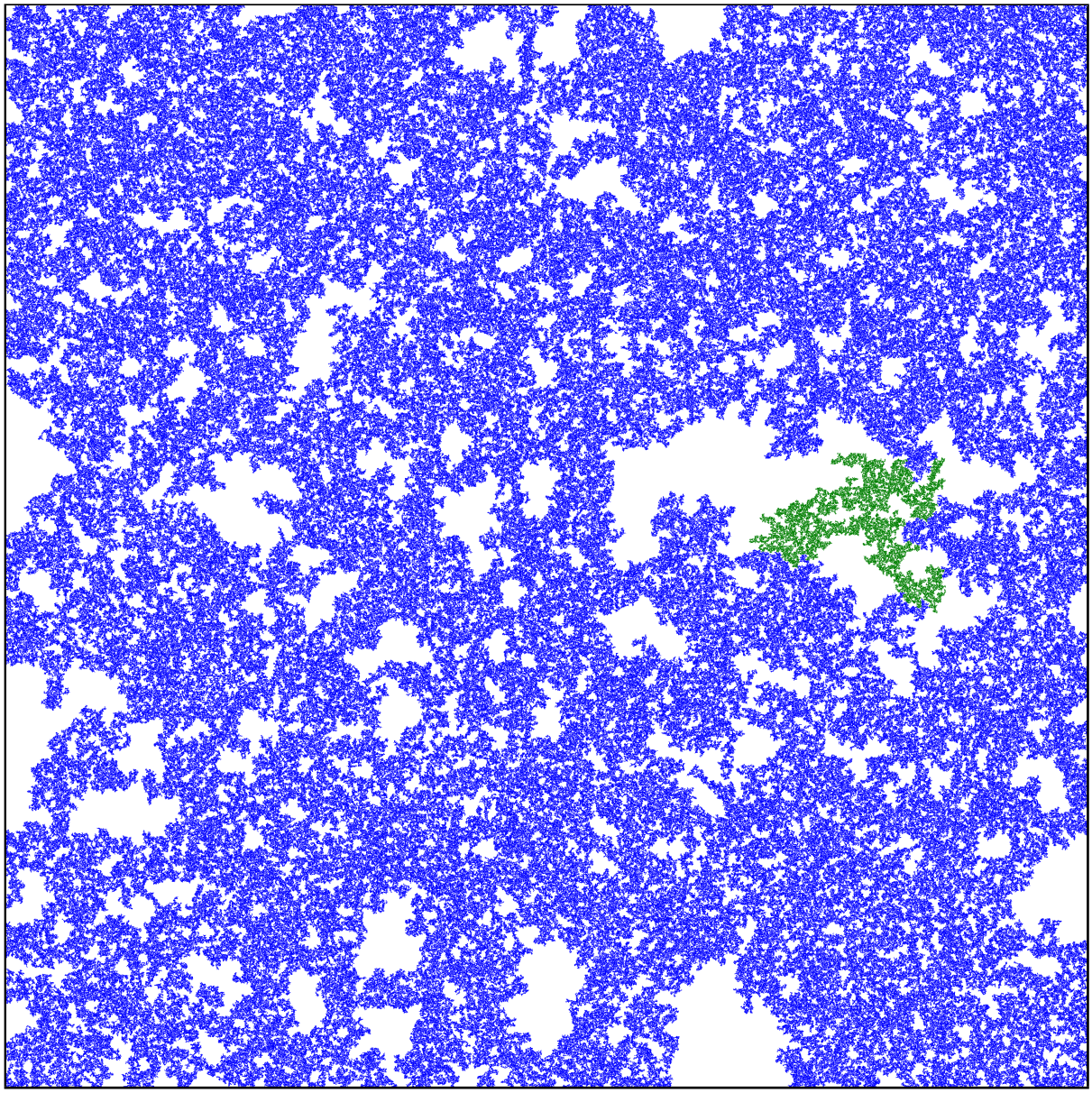


Figure 3.1.3: Super-critical PCF
The largest two clusters generated in a simulation of super-critical PCF on a 4096 by 4096 square grid. Here we have $\alpha = 0.50$, which appears to be too small to prevent the formation of a spanning cluster. In contrast to the percolation model, simulations also suggest that the size of the finite components (and thus the size of the holes) have a power law distribution.

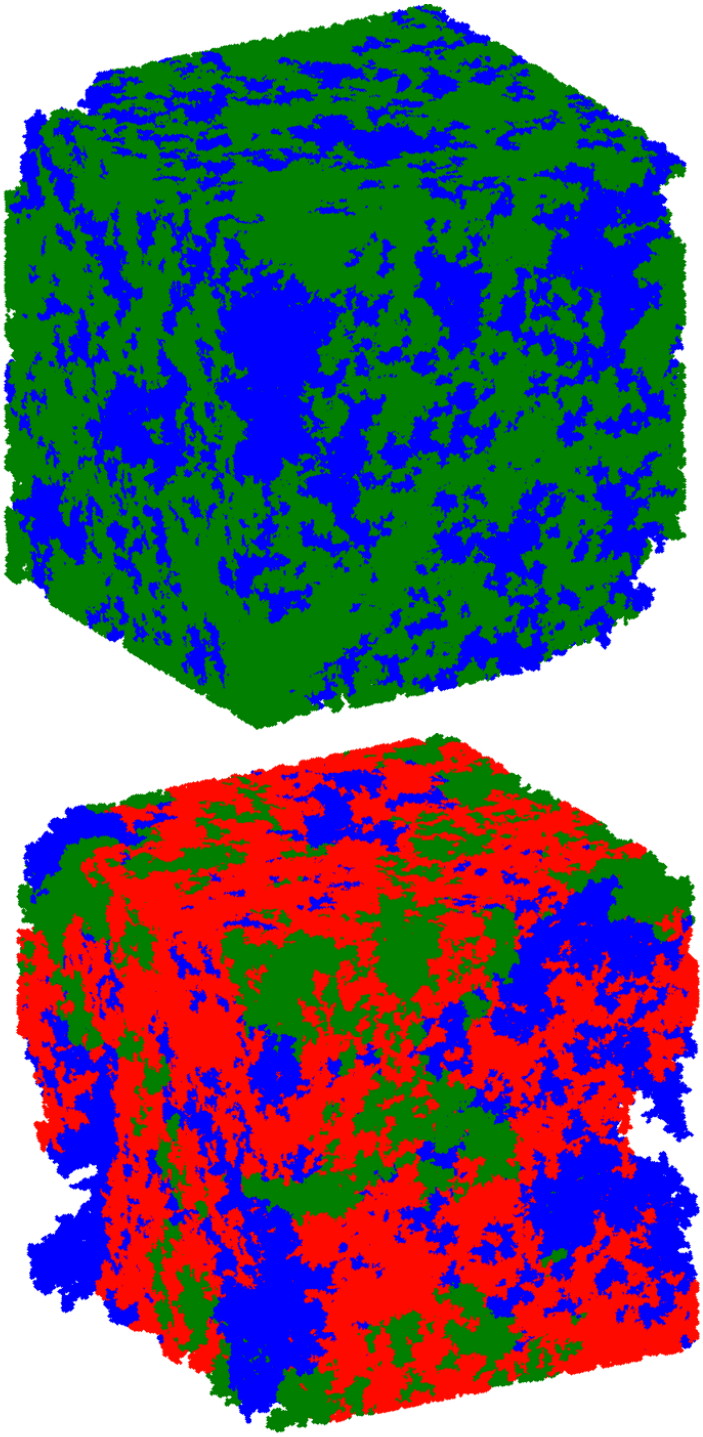


Figure 3.1.4: 3 dimensional PCF
The two images show realisations of PCF on a 256 by 256 by 256 cubic lattice with $\alpha = 1$. Since this is now a 3 dimensional model we have $\alpha < \alpha_c$ and thus we are in the super-critical regime. The first image has the largest warm cluster freezing at $t = 0.38$ – this is shown in blue. The green cluster is then a secondary macroscopic cluster which has formed subsequently and wrapped around the first. The second image is the result of the largest warm infinite cluster freezing at $t = 0.365$ (blue) and the new largest warm cluster freezing at $t = 0.4$ (green). The subsequent red cluster then leaves us with 3 intertwined macroscopic clusters.

higher. For further discussion on criticality and the number of infinite clusters the reader is directed to the open problems in Section 3.1.4.

For videos that show how the PCF process evolves in time the interested reader is also directed towards the following Youtube play-list, [Mot12]:

`https://www.youtube.com/playlist?list=PLIpM_wOXrr6XBh9n5b0uRlE-bv_UVktSD`

On a tree It turns out that PCF on a tree can be related to bond percolation on the tree via a suitable time rescaling. From this the behaviour of the model on a tree can be well understood, and in particular the distribution of component sizes can be calculated explicitly.

Theorem 3.1.4. *Consider PCF on the rooted binary tree $\mathbf{T}_2$. There is a critical rate of freezing $\alpha_c = 1$ such that for $\alpha < \alpha_c$ infinite clusters form almost surely, but for $\alpha \geq \alpha_c$ all clusters are almost surely finite. Moreover writing $P_k(\alpha)$ for the probability that the root cluster has size k when it freezes then*

- *if* $\alpha < 1$ $\quad P_k(\alpha) \sim C_\alpha\, k^{-2}$ $\quad$ *for some constant* C_α,
- *if* $\alpha = 1$ $\quad P_k(1) \sim C\, k^{-\frac{7}{4}}$ $\quad$ *for some constant* C,
- *if* $\alpha > 1$ $\quad$ *then* $P_k(\alpha)$ *decays exponentially in* k *at a rate dependent on* α.

More generally on a rooted d-ary tree we have $\alpha_c = d - 1$ and at $\alpha = \alpha_c$ we have $P_k(\alpha_c) \sim C_d\, k^{-\left(2-\frac{1}{2d}\right)}$. For each d we know that critical percolation on the d-ary tree has $\mathbb{P}_{p_c}(\text{the root cluster has size } k) \asymp k^{-\frac{3}{2}}$, and that for critical percolation on $\mathbb{Z}^d$ we also have $\mathbb{P}_{p_c}(\text{the cluster at the origin has size } k) \asymp k^{-\frac{3}{2}}$ whenever d is larger than the *upper critical dimension* (believed to be 6). Therefore the dependence of the PCF model on the degree of the graph is rather surprising; and suggests that in contrast to other models there is no upper critical dimension for PCF on $\mathbb{Z}^d$.

On a complete graph Heuristically we can understand PCF on the complete graph $\mathbf{K}_n$ (as $n \longrightarrow \infty$) by comparing with PCF on the n-ary tree and rescaling time so that edges open at rate $\dfrac{1}{n}$. Our proof then carries over to the mean field case providing a method for rigorously confirming the results of Ben–Naim and Krapivsky, [BNK05a] and [BNK05b]. See Remark 3.4.3 for more details.

3.1.3 Variants of the model

The PCF process presented here is based upon bond percolation. It is equally possible to consider a model of site percolation with constant freezing. In such a model we would open each $v \in \mathcal{V}$ independently at rate 1, and freeze each open cluster independently at rate α. Once a cluster had frozen all adjacent vertices would remain closed forever. Theorem 3.1.1 and Theorem 3.1.4 also hold in this case by following essentially the same proofs.

3.1.4 Open problems

The PCF process leaves us with a wealth of open problems. Some of these are presented here in the hope of persuading the reader of the richness of the model.

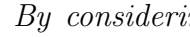
Monotonicity Intuitively one would expect that increasing the rate of freezing would lead to clusters freezing more quickly and thus being smaller when they do so. This intuition might lead us to hypothesise that for $0 < \alpha < \beta$ we have

$$\mu_{\mathbf{G},\beta} \leq_{\text{st}} \mu_{\mathbf{G},\alpha}. \tag{3.1.1}$$

Stochastic ordering is a relatively strong condition and so it could be the case that stochastic ordering does not hold, but that a weaker monotonicity condition does. However, we do not know of any finite graphs on which (3.1.1) fails.

Such a monotonicity condition is of particular interest because it would imply the existence of a critical value α_c.

A related question is whether or not an *FKG inequality* holds for the model. Whilst the following example reveals that that the so-called *FKG lattice condition* does not hold even in the case that $\mathbf{G}$ is a line segment with 4 vertices; we have been unable to find a finite graph on which the FKG inequality itself does not hold. It is worthwhile noting that a BK type inequality does not hold for this model on even the most simple graphs; therefore any attempts to adapt the lace expansion to this model would require more powerful tools.

Example 3.1.5. *Let* $\mathbf{G} =$  . *By considering the possible orders in which edges can open or clusters freeze we can calculate the probability of each of the 8*

possible PCF configurations.

$$\mu_{\mathbf{G},\alpha}(\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet) = \frac{6}{(1+2\alpha)(2+3\alpha)(3+4\alpha)}$$

$$\mu_{\mathbf{G},\alpha}(\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet\quad\bullet) = \mu_{\mathbf{G},\alpha}(\bullet\quad\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet) = \frac{8\alpha}{(1+2\alpha)(2+3\alpha)(3+4\alpha)}$$

$$\mu_{\mathbf{G},\alpha}(\bullet\!\!-\!\!\bullet\quad\bullet\!\!-\!\!\bullet) = \frac{6\alpha}{(1+2\alpha)(2+3\alpha)(3+4\alpha)}$$

$$\mu_{\mathbf{G},\alpha}(\bullet\!\!-\!\!\bullet\quad\bullet\quad\bullet) = \mu_{\mathbf{G},\alpha}(\bullet\quad\bullet\quad\bullet\!\!-\!\!\bullet) = \frac{3\alpha+12\alpha^2}{(1+2\alpha)(2+3\alpha)(3+4\alpha)}$$

$$\mu_{\mathbf{G},\alpha}(\bullet\quad\bullet\!\!-\!\!\bullet\quad\bullet) = \frac{\alpha+12\alpha^2}{(1+2\alpha)(2+3\alpha)(3+4\alpha)}$$

$$\mu_{\mathbf{G},\alpha}(\bullet\quad\bullet\quad\bullet\quad\bullet) = \frac{10\alpha^2+24\alpha^3}{(1+2\alpha)(2+3\alpha)(3+4\alpha)}.$$

Here a line denotes an open edge and a space denotes a closed one. Now let $\omega_1 = \{\bullet\!\!-\!\!\bullet\quad\bullet\quad\bullet\}$ *and* $\omega_2 = \{\bullet\quad\bullet\quad\bullet\!\!-\!\!\bullet\}$*. Then*

$$\mu_{\mathbf{G},\alpha}(\omega_1 \vee \omega_2)\,\mu_{\mathbf{G},\alpha}(\omega_1 \wedge \omega_2) = \frac{60\alpha^3+144\alpha^4}{(1+2\alpha)^2(2+3\alpha)^2(3+4\alpha)^2}$$

$$< \mu_{\mathbf{G},\alpha}(\omega_1)\,\mu_{\mathbf{G},\alpha}(\omega_2) = \frac{9\alpha^2+72\alpha^3+144\alpha^4}{(1+2\alpha)^2(2+3\alpha)^2(3+4\alpha)^2},$$

for all $\alpha > 0$*. Thus the FKG lattice condition does not hold for* $\mu_{\mathbf{G},\alpha}$ *for any* $\alpha > 0$*. However, a quick check shows that for any pair of increasing events* A *and* B *we have*

$$\mu_{\mathbf{G},\alpha}(A) \times \mu_{\mathbf{G},\alpha}(B) \leq \mu_{\mathbf{G},\alpha}(A \cap B),$$

and so (for this choice of $\mathbf{G}$ *at least) the FKG inequality itself does hold.*

Existence of infinite clusters on $\mathbb{Z}^d$ Proposition 3.1.3 tells us that when α is sufficiently large then PCF on $\mathbb{Z}^d$ does not produce infinite clusters. However, simulations (and Figure 3.1.3) suggest that when α is small then an infinite cluster will form. The following intuition suggests why this should indeed be the case:

Since an edge is joined to at most 2 clusters the probability it is open at time t is at least $p_{\alpha,c}(t) = \frac{1}{1+2\alpha}(1 - \mathrm{e}^{-(1+2\alpha)t})$. Now let $\alpha > 0$ be small, and suppose we run PCF until some time t, where t is sufficiently large for $p_{\alpha,c}(t) > p_c$. Here p_c is the critical value for bond percolation on $\mathbb{Z}^d$. Thus if the edges were independent then $\mathbb{Z}^d$ would contain some infinite cluster. Now observe that at this time at most αt of the vertices in the graph will be frozen and so – provided α is sufficiently small – the number of vertices which have been removed from the infinite percolation cluster (by freezing) are not sufficient to

partition the cluster into finite pieces. Therefore an infinite cluster will remain.

Difficulties in extending this intuition to a proof arise from the fact that the freezing of vertices is not independent, and large clusters can freeze potentially creating barriers which stop warm clusters from growing. Moreover we should not expect to have good control on the size of these frozen clusters since simulations show that their size has a polynomial tail. The problem therefore is to find an alternative method for showing that an infinite PCF cluster must exist.

Uniqueness of infinite clusters for $d \geq 2$ During the construction of an infinite volume PCF process it is shown – in Proposition 3.2.13 – that any warm infinite cluster is necessarily unique. Ben–Naim and Krapivsky showed that this is also true in the mean field case, but that it is also possible for an infinite cluster to freeze and allow subsequent infinite clusters to form. The geometry of the plane makes it very unlikely that could happen with PCF on $\mathbb{Z}^2$ – since any infinite cluster is likely to partition the space into finite pieces. But there is no such problem in higher dimension, and both simulations (see Figure 3.1.4) and the following heuristic argument suggest that in dimension $d \geq 3$ it is possible for subsequent infinite clusters to form after previous infinite clusters have frozen.

Let $d \geq 3$ and consider rate α PCF on $\mathbb{Z}^d$. Observe that if we could set $\alpha = 0$ then the PCF process would become a percolation process where edges open independently at rate 1. A result of Campanino and Russo tells us that $p_c(d) < \frac{1}{2}$ for $d \geq 3$, [CR85]. From this we see that without freezing there would be some critical time $t_c < \log 2$ at which an infinite cluster will first appear. Therefore if we have $0 < \alpha \ll 1$ it seems reasonable to suppose that the time $t_{\alpha,c}$ at which an infinite cluster appears in the rate α PCF process is close to t_c. Since $\alpha > 0$ there is then a positive probability that this infinite cluster will freeze shortly after forming. Thus provided α is sufficiently small there is a positive probability that an infinite cluster will form and freeze before $t = \log 2$. At this time the probability that a given edge will be closed and warm is close to $\frac{1}{2}$, so since these edge events will be almost independent there must be an infinite connected set of closed and warm edges in which a new infinite cluster can form. Thus at some later time T we will have two infinite clusters (one warm and one frozen).

This leads to the following conjecture.

Conjecture 3.1.6. *Let $d \geq 3$, then provided $\alpha > 0$ is sufficiently small there is a positive probability that the final distribution of the rate α PCF process on $\mathbb{Z}^d$ will contain 2 or more infinite clusters.*

One can show that there can only ever be a finite number of infinite clusters in the final distribution. However, this conjecture still leads us to ask how many infinite clusters is it possible to have, and whether or not the maximum number of clusters is controlled by α. Observe that the possibility of having arbitrarily many clusters would imply the possibility of having clusters of arbitrarily low density. Therefore asking questions about the maximum number of infinite clusters is similar to asking if the percolation function $\theta(p)$ is continuous for higher dimensional lattices. The work of Hara and Slade, [HS90], tells us how $\theta(p)$ behaves when the dimension d is sufficiently large, but when we have $3 \leq d < 19$ the continuity of $\theta(p)$ remains an open problem.

Influence of boundary conditions In this chapter we use warm boundary conditions to construct an infinite volume limit of the model with free boundary conditions. We ask therefore what influence the boundary conditions have on the model, and in particular if a infinite volume version of the model where clusters freeze as soon as they touch the boundary exists.

Existence of the model on a general graph This chapter shows that PCF can be constructed on any countable tree or amenable vertex transitive graph. We ask therefore if the process can be defined on a more general class of graphs, or indeed on any graph.

3.2 Constructing PCF

The construction of PCF relies upon various auxiliary processes and measures. In order to make the notation as comprehensible as possible a table of notation is included below.

Process	Symbol	Measure	Mapping
PCF process	$(\eta^t)_{t\in[0,\infty]}$	$\mu = \mu_{\mathbf{G},\alpha}$	$\psi = \psi_{\mathbf{G}} : \Pi \longrightarrow \Omega \times [0,\infty]$
warm PCF process	$(\zeta^t)_{t\in[0,\infty]}$	$\nu = \nu_{\mathbf{G},\alpha}$	$\phi = \phi_{\mathbf{G}} : \Pi \longrightarrow \Omega \times [0,\infty]$
liminf of PCF	$(\tilde{\eta}^t)_{t\in[0,\infty]}$	$\tilde{\nu} = \tilde{\mu}_{\mathbf{G},\alpha}$	$\tilde{\psi} = \tilde{\psi}_{\mathbf{G}} : \Pi \longrightarrow \Omega \times [0,\infty]$

We now start our construction with a formal definition for the percolation with constant freezing process.

Definition 3.2.1 (PCF on a finite graph)**.**

Let $\alpha > 0$. Given a finite subgraph $\mathbf{H} = (\mathcal{V}_\mathbf{H}, \mathcal{E}_\mathbf{H})$ of $\mathbf{G} = (\mathcal{V}, \mathcal{E})$, we consider PCF as a Markov process on $\mathbf{H}$ where an edge can be either *open* (state 1) or *closed* (state 0), and a vertex can be either *warm* (state w) or *frozen* (state f). Thus our state space is $\Omega_\mathbf{H} = \{w, f\}^{\mathcal{V}_\mathbf{H}} \times \{0,1\}^{\mathcal{E}_\mathbf{H}}$. Initially each edge is closed, and each vertex is warm. Therefore $\eta^0 = (w, \ldots, w; 0, \ldots, 0)$.

Given a configuration η and vertices $v, w \in \mathcal{V}_\mathbf{H}$, we write $v \overset{\eta}{\longleftrightarrow} w$ if there is a path of open edges in η connecting v to w. A *cluster* is then defined to be a maximal set of connected vertices, and in particular the cluster containing the vertex $v \in \mathcal{V}_\mathbf{H}$ is given by $C_v(\eta) = \{w \in \mathcal{V} : w \overset{\eta}{\longleftrightarrow} v\}$. Now for each edge $e \in \mathcal{E}_\mathbf{H}$ define η^e by

$$\eta^e(x) = \begin{cases} 1 & x = e \\ \eta(x) & x \neq e, \end{cases} \tag{3.2.1}$$

for each edge and vertex $x \in \mathcal{V} \cup \mathcal{E}$; and for each cluster $C \subseteq \mathcal{V}$ define η_C by

$$\eta_\mathrm{C}(x) = \begin{cases} f & x \in C \\ \eta(x) & x \notin C. \end{cases} \tag{3.2.2}$$

The generator $G : \Omega_\mathbf{H} \times \Omega_\mathbf{H} \longrightarrow \mathbb{R}$ for the PCF process is now defined by

$$\begin{aligned} G(\eta, \eta^e) &= 1 && \text{for all } \eta \text{ and } e = v_1v_2 \text{ such that } \eta(v_1) = \eta(v_2) = w \\ G(\eta, \eta_\mathrm{C}) &= \alpha && \text{for all clusters } C \text{ of } \eta \\ G(\eta, \theta) &= 0 && \text{otherwise.} \end{aligned} \tag{3.2.3}$$

Definition 3.2.2 (PCF measure)**.** The PCF process described in Definition 3.2.1 induces

a measure $\mu_{\mathbf{H},\alpha} = (\mu^t_{\mathbf{H},\alpha} : t \in [0,\infty))$ on $\Omega_{\mathbf{H}} \times [0,\infty)$. Here $\mu^t_{\mathbf{H},\alpha}$ is the measure on $\Omega_{\mathbf{H}} = \{w,f\}^{\mathcal{V}_{\mathbf{H}}} \times \{0,1\}^{\mathcal{E}_{\mathbf{H}}}$ induced by the possible configurations of the PCF process at time t.

Since the PCF process on $\mathbf{H}$ can make only finitely many jumps, it will reach some final distribution in finite time. Therefore as $t \longrightarrow \infty$ the measures $\mu^t_{\mathbf{H},\alpha}$ will converge to $\mu^\infty_{\mathbf{H},\alpha}$ – the measure of the final distribution. By including $\mu^\infty_{\mathbf{H},\alpha}$ we extend $\mu_{\mathbf{H},\alpha}$ to the compact space $\Omega_{\mathbf{H}} \times [0,\infty]$, and call $\mu_{\mathbf{H},\alpha} = (\mu^t_{\mathbf{H},\alpha} : t \in [0,\infty])$ the *PCF measure* of $\mathbf{H} \subseteq \mathbf{G}$.

In order for us to compare the PCF process on other subgraphs it will be convenient for us to extend the process $(\eta^t)_{t\in[0,\infty]}$ to the whole of $\mathbf{G}$ by working on the state space $\Omega = \{w,f\}^{\mathcal{V}} \times \{0,1\}^{\mathcal{E}}$ and letting $\eta^t(v) = f$ for all t and each $v \in \mathcal{V} \setminus \mathcal{V}_{\mathbf{H}}$ and $\eta^t(e) = 0$ for all t and each $e \in \mathcal{E} \setminus \mathcal{E}_{\mathbf{H}}$. The measure $\mu_{\mathbf{H},\alpha}$ is then extended to $\Omega \times [0,\infty]$ in the obvious way. Intuitively one might find it helpful to think of this as PCF with free boundary conditions.

Observe that if we choose an enumeration for the vertices of $\mathbf{G}$, $\mathcal{V} = \{v_1, v_2, \ldots\}$, then we can identify each cluster by asking which is the vertex of lowest index (or *highest priority*) that it contains. By doing so we can control the freezing of clusters by attaching independent rate α exponential clocks X_v to each vertex $v \in \mathcal{V}_{\mathbf{H}}$. The exponential clocks are only needed by the vertices of highest priority; however, we attach them to each $v \in \mathcal{V}_{\mathbf{H}}$ to enable us to construct a coupling between different models later on. We can also control the opening of edges with exponential rate 1 clocks X_e attached to each edge $e \in \mathcal{E}_{\mathbf{H}}$. This leads to the following algorithm for simulating PCF on a finite graph.

3.2.1 An Algorithm for PCF

Algorithm 3.2.3 (PCF on a finite subgraph)**.** Let $\alpha > 0$, and enumerate the vertices of $\mathcal{V} = \{v_1, v_2, \ldots\}$. Then given a finite subgraph $\mathbf{H} = (\mathcal{V}_{\mathbf{H}}, \mathcal{E}_{\mathbf{H}}) \subseteq \mathbf{G}$ we get $\mathcal{V}_{\mathbf{H}} = \{v_{i_1}, \ldots, v_{i_n}\}$. Now label the vertices of $\mathbf{H}$ by $\ell^0(v_{i_k}) = i_k$ for $1 \le k \le n$. These labels will evolve in time and satisfy

$$\ell^t(v_i) = \inf\{j : v_j \in C_{v_i}(\eta^t)\} = \inf\{j : v_j \xleftrightarrow{\eta^t} v_i\} \in \{i_1, \ldots, i_n\}. \tag{3.2.4}$$

for each $i \in \{i_1, \ldots, i_n\}$. Thus $\ell^t(v_i)$ can be used to denote which cluster the vertex v_i is in at time t.

Let $\{X_v\}_{v\in\mathcal{V}_\mathbf{H}}$ be independent Exp(α) random variables and let $\{X_e\}_{e\in\mathcal{E}_\mathbf{H}}$ be independent Exp(1) random variables. Our càdlàg PCF process is now described by its evolution. At time $t=0$ set $\eta^0=\{w\}^\mathcal{V}\times\{0\}^\mathcal{E}$, and for each $x\in\mathcal{V}\cup\mathcal{E}$ at time $t=X_x$

(a) if $t=X_{v_i}$ then set $C=\{v_k\in\mathcal{V}:\ell^{t^-}(v_k)=i\}$ and let $\eta^t=(\eta^{t^-})_\mathrm{C}$.

(b) if $t=X_e$ with $e=v_iv_j\in\mathcal{E}$, then if $\eta^{t^-}(v_i)=\eta^{t^-}(v_j)=w$ set $\eta^t=(\eta^{t^-})^e$.

To ensure that (3.2.4) continues to hold we then set $\ell^t(v_k)=\min\{\ell^{t^-}(v_i),\ell^{t^-}(v_j)\}$ for all v_k with $\ell^{t^-}(v_k)\in\{\ell^{t^-}(v_i),\ell^{t^-}(v_j)\}$.

Algorithm 3.2.3 can be thought of as a map taking a realisation of the exponential clocks to a PCF process,

$$\begin{aligned}\psi_\mathbf{H}:[0,\infty)^{\mathcal{V}_\mathbf{H}}\times[0,\infty)^{\mathcal{E}_\mathbf{H}}&\longrightarrow\Omega_\mathbf{H}\times[0,\infty]\\(s_v)_{v\in\mathcal{V}_\mathbf{H}}\times(s_e)_{e\in\mathcal{E}_\mathbf{H}}&\longmapsto(\eta^t)_{t\in[0,\infty]}.\end{aligned}\tag{3.2.5}$$

This mapping makes it clear that a PCF event $A\subseteq\Omega_\mathbf{H}$ is measurable with respect to $\mathcal{F}_\mathbf{H}=\sigma(\{X_v\}_{v\in\mathcal{V}_\mathbf{H}};\{X_e\}_{e\in\mathcal{E}_\mathbf{H}})$. Note that here an explicit choice of enumeration of the vertices is required. However, since the X_v are independent (and thus interchangeable) it is clear that this choice is arbitrary and does not affect the law of the process itself. Therefore we see that this algorithm gives a process satisfying (3.2.3), and so as $\mathbf{H}$ is finite this must be the unique process that satisfies Definition 3.2.1.

3.2.2 The warm PCF process

Given an amenable vertex transitive graph $\mathbf{G}$ we begin the construction of the PCF process on $\mathbf{G}$ by first constructing an auxiliary process which obeys a suitable monotonicity condition. We consider a finite subgraph $\mathbf{H}\subseteq\mathbf{G}$ for now and define *warm PCF* as follows.

Definition 3.2.4 (Warm PCF on a finite subgraph)**.** Suppose $\mathbf{H}=(\mathcal{V}_\mathbf{H},\mathcal{E}_\mathbf{H})$ is a finite subgraph of $\mathbf{G}=(\mathcal{V},\mathcal{E})$, we say that $v\in\mathcal{V}_\mathbf{H}$ is a *boundary vertex* if it meets an edge $e=vw$ in $\mathcal{E}_\mathbf{G}$ which is not in $\mathcal{E}_\mathbf{H}$. We denote the set of boundary vertices by $\partial\mathbf{H}$. Now given a configuration $\zeta\in\Omega_\mathbf{H}=\{w,f\}^{\mathcal{V}_\mathbf{H}}\times\{0,1\}^{\mathcal{E}_\mathbf{H}}$ we say that C is a *boundary cluster* of ζ if C is a cluster of ζ with $C\cap\partial\mathbf{H}\neq\emptyset$. We say that a cluster $C\subseteq\mathcal{V}_\mathbf{H}$ of ζ which is not a boundary cluster is an *interior cluster*.

We now set our initial configuration to $\zeta^0 = \{w\}^{\mathcal{V}_\mathbf{H}} \times \{0\}^{\mathcal{E}_\mathbf{H}}$, and define the generator G for the warm PCF process on $\mathbf{H}$ to be

$$\begin{aligned} G(\zeta, \zeta^e) &= 1 \qquad \text{for all } \zeta \text{ and } e = v_1 v_2 \text{ such that } \zeta(v_1) = \zeta(v_2) = w \\ G(\zeta, \zeta_{\mathrm{C}}) &= \alpha \qquad \text{for all interior clusters } C \text{ of } \zeta \\ G(\zeta, \theta) &= 0 \qquad \text{otherwise.} \end{aligned} \tag{3.2.6}$$

Therefore boundary clusters will remain warm forever. Intuitively one might like to think of this as PCF with warm wired boundary conditions.

It will be useful for us to be able to compare warm PCF on two subgraphs $\mathbf{H}_1$ and $\mathbf{H}_2$. To make sense of this we extend warm PCF on $\mathbf{H}$ to a process on $\mathbf{G}$ by working with the state space $\Omega = \{w, f\}^{\mathcal{V}} \times \{0, 1\}^{\mathcal{E}}$ and setting our initial configuration to $\zeta^0 = \{w\}^{\mathcal{V}} \times \{0\}^{\mathcal{E}_\mathbf{H}} \times \{1\}^{\mathcal{E}\setminus\mathcal{E}_\mathbf{H}}$. Since G only acts on a finite number of edges and vertices we see that (3.2.6) gives a well defined process.

Warm PCF differs from PCF in that the transition $\zeta \longrightarrow \zeta_{\mathrm{C}}$ only occurs for interior clusters in (3.2.6), while in PCF this transition can take place with all clusters (see (3.2.3)). This means that the warm PCF process stochastically dominates the PCF process – a notion we shall make rigorous in Lemma 3.2.8.

Definition 3.2.5 (Warm PCF measure)**.** In the same way as described in Definition 3.2.2 the warm PCF process induces a measure $\nu_{\mathbf{H},\alpha} = (\nu^t_{\mathbf{H},\alpha} : t \in [0, \infty])$ on $\Omega \times [0, \infty]$. This is the *warm PCF measure* of $\mathbf{H} \subseteq \mathbf{G}$.

We also remark that Algorithm 3.2.3 can be adapted to warm PCF by replacing *(a)* with

(a) if* $t = X_{v_i}$ *then set* $C = \{v_k \in \mathcal{V} : \ell(v_k) = i\}$ *and provided* $C \cap \partial\mathbf{H} = \emptyset$ *let* $\zeta^t = (\zeta^{t^-})_{\mathrm{C}}$.

We shall denote this modified algorithm as Algorithm 3.2.3*.

Again this algorithm can be thought of as giving a function $\phi_\mathbf{H}$ taking a realisation of $\{X_v\}_{v\in\mathcal{V}_\mathbf{H}}$ and $\{X_e\}_{e\in\mathcal{E}_\mathbf{H}}$ to a warm PCF process $(\zeta^t)_{t\in[0,\infty]}$. This then tells us that a warm PCF event $A \subseteq \Omega$ is also measurable with respect to $\mathcal{F} = \sigma(\{X_v\}_{v\in\mathcal{V}}; \{X_e\}_{e\in\mathcal{E}})$.

Definition 3.2.6 (Coupling)**.** Given a countable graph $\mathbf{G} = (\mathcal{V}, \mathcal{E})$ and a collection of finite subgraphs $\{\mathbf{H}_i\}_{i\in I}$ we can use Algorithm 3.2.3 and Algorithm 3.2.3* to couple the PCF and warm PCF processes on each of the $\mathbf{H}_i$. To do this we fix an enumeration of the vertices of $\mathbf{G}$, and assign an $\mathrm{Exp}(\alpha)$ random variable X_v to each $v \in \mathcal{V}$ and

an Exp(1) random variable X_e to each $e \in \mathcal{E}$. We shall use $x \in \mathbf{G}$ as shorthand for $x \in \mathcal{V} \cup \mathcal{E}$, and denote the probability space for the random variables $\{X_x\}_{x \in \mathbf{G}}$ by $(\Pi, \lambda_{\mathbf{G},\alpha})$. Now for each $i \in I$ Algorithm 3.2.3 and Algorithm 3.2.3* give us coupled mappings $\psi_i : (\Pi, \lambda_{\mathbf{G},\alpha}) \longrightarrow (\Omega \times [0,\infty], \mu_{\mathbf{H}_i,\alpha})$ and $\phi_i : (\Pi, \lambda_{\mathbf{G},\alpha}) \longrightarrow (\Omega \times [0,\infty], \nu_{\mathbf{H}_i,\alpha})$ from $\{X_x\}_{x \in \mathbf{G}}$ to a PCF process on $\mathbf{H}_i$ and a warm PCF process on $\mathbf{H}_i$.

The key reason for introducing the warm PCF process is that it obeys a monotonicity condition. Given two configurations $\eta, \zeta \in \Omega$ we say $\eta \leq \zeta$ if and only if $\zeta(e) = 1$ for all $e \in \mathcal{E}$ with $\eta(e) = 1$ and $\zeta(v) = w$ for all $v \in \mathcal{V}$ with $\eta(v) = w$. That is, ζ has more open edges and warmer vertices than η. From this partial ordering of Ω we define $A \subseteq \Omega$ to be an *increasing event* if and only if $\eta \in A$ and $\eta \leq \zeta$ implies $\zeta \in A$. Whence we get a *stochastic ordering* by saying that if μ and ν are two measures on Ω then $\mu \leq_{\mathrm{st}} \nu$ if and only if $\mu(A) \leq \nu(A)$ for all measurable increasing events A.

To extend this notion of stochastic ordering to processes on $\mathbf{G}$ we consider trajectories $(\eta^t)_{t \in [0,\infty]}$ and $(\zeta^t)_{t \in [0,\infty]}$, and say that $(\eta^t) \leq (\zeta^t)$ if and only if $\eta^t \leq \zeta^t$ for all $t \in [0,\infty]$. An event $A \subseteq \Omega \times [0,\infty]$ is then an *increasing process event* if and only if $(\eta^t) \in A$ and $(\eta^t) \leq (\zeta^t)$ implies $(\zeta^t) \in A$. Therefore given two process P and Q with respective measures $\mu = (\mu^t)_{t \in [0,\infty]}$ and $\nu = (\nu^t)_{t \in [0,\infty]}$, we say that Q *stochastically dominates* P if and only if $\mu(A) \leq \nu(A)$ for all measurable increasing process events A, and write $\mu \leq_{\mathrm{st}} \nu$.

Lemma 3.2.7 (Monotonicity for warm PCF)**.** *Suppose* $\mathbf{H}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ *and* $\mathbf{H}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ *are finite subgraphs of* $\mathbf{G}$ *with* $\mathbf{H}_1 \subseteq \mathbf{H}_2$*, then the rate* α *warm PCF process on* $\mathbf{H}_1$ *stochastically dominates the rate* α *warm PCF process on* $\mathbf{H}_2$*.*

Proof. Here we work with the processes extended to $\mathbf{G} = (\mathcal{V}, \mathcal{E})$, and use the notion of coupling from Definition 3.2.6. Let $\{X_v\}_{v \in \mathcal{V}}$ and $\{X_e\}_{e \in \mathcal{E}}$ be as in Definition 3.2.6 and let $(\zeta_1^t)_{t \in [0,\infty]}$ and $(\zeta_2^t)_{t \in [0,\infty]}$ be the trajectories associated with warm PCF on $\mathbf{H}_1$ and $\mathbf{H}_2$ respectively. It now suffices to show that the coupling gives $\zeta_1^t \geq \zeta_2^t$ for all $t \in [0,\infty]$.

For the purposes of our proof we shall show further that if C is a warm interior cluster of ζ_1^t then it is also a warm interior cluster of ζ_2^t. Since $\mathbf{H}_1 \subseteq \mathbf{H}_2$ this is clear at $t = 0$, and because $\mathcal{E}_1 \subseteq \mathcal{E}_2$ we also have $\zeta_1^0 = \{w\}^{\mathcal{V}} \times \{0\}^{\mathcal{E}_1} \times \{1\}^{\mathcal{E} \setminus \mathcal{E}_1} \geq \{w\}^{\mathcal{V}} \times \{0\}^{\mathcal{E}_2} \times \{1\}^{\mathcal{E} \setminus \mathcal{E}_2} = \zeta_2^0$.

Now because (ζ_1^t) and (ζ_2^t) can only change at discrete times $t = X_x$, where $x \in \mathcal{V}_2 \cup \mathcal{E}_2$, it suffices for us to show for each $t \in \{X_x : x \in \mathcal{V}_2 \cup \mathcal{E}_2\}$ that if our hypothesis holds and we have $\zeta_1^{t^-} \geq \zeta_2^{t^-}$ at time t^- then the hypothesis also holds at time t. There are several cases.

(a) If $t = X_v$ and v is the vertex of highest priority in a warm interior cluster C of $\zeta_1^{t^-}$ then C must also be a warm interior cluster of $\zeta_2^{t^-}$ with v again being the vertex of highest priority. So we have $\zeta_1^t = (\zeta_1^{t^-})_{\mathrm{C}} \geq (\zeta_2^{t^-})_{\mathrm{C}} = \zeta_2^t$, and C is no longer a warm interior cluster of ζ_1^t or ζ_2^t.

If v is a vertex of highest priority in a warm interior cluster C of $\zeta_2^{t^-}$ but not in $\zeta_1^{t^-}$ then we get $\zeta_1^t = \zeta_1^{t^-} \geq \zeta_2^{t^-} \geq (\zeta_2^{t^-})_{\mathrm{C}} = \zeta_2^t$. Moreover, C remains warm in ζ_1^t but not in ζ_2^t.

If v is not the vertex of highest priority in some warm interior cluster of either $\zeta_1^{t^-}$ or $\zeta_2^{t^-}$ then ζ_1^t and ζ_2^t remain unchanged.

(b) If $t = X_e$ with $e = v_i v_j$, then if $\zeta_2^{t^-}(v_i) = \zeta_2^{t^-}(v_j) = w$ we must also have $\zeta_1^{t^-}(v_i) = \zeta_1^{t^-}(v_j) = w$, since $\zeta_1^{t^-} \geq \zeta_2^{t^-}$ and so any vertex which is warm in $\zeta_2^{t^-}$ must also be warm in $\zeta_1^{t^-}$. Therefore if e is open in ζ_2^t then it must also be open in ζ_1^t, and so we see that $\zeta_1^t \geq \zeta_2^t$.

If $C_{v_i} = C_{v_j}$ then there is no change in the clusters, so we assume C_{v_i} and C_{v_j} are disjoint. If C_{v_i} and C_{v_j} are warm interior clusters of ζ_1 then they are also warm interior clusters of ζ_2 and so $C_{v_i} \cup C_{v_j}$ is a warm interior cluster of both ζ_1 and ζ_2. If one of C_{v_i} or C_{v_j} is not a warm interior clusters of $\zeta_1^{t^-}$, then it must be a must be a boundary cluster of $\zeta_1^{t^-}$ and so C_{v_i} and C_{v_j} will be (contained in) boundary clusters of ζ_1^t, and so our condition on the clusters is satisfied.

If $\zeta_1^{t^-}(v_i) = \zeta_1^{t^-}(v_j) = w$ but $\zeta_2^{t^-}(v_i) = f$ or $\zeta_2^{t^-}(v_j) = f$ then we have $\zeta_1^t = (\zeta_1^{t^-})^e \geq \zeta_1^{t^-} \geq \zeta_2^{t^-} = \zeta_2^t$. Moreover, one of C_{v_i} or C_{v_j} must have been a boundary cluster of $\zeta_1^{t^-}$ and so C_{v_i} and C_{v_j} will be (contained in) boundary clusters of ζ_1^t, and so again the condition on the clusters is satisfied.

If $\zeta_1^{t^-}(v_i) = f$ or $\zeta_1^{t^-}(v_j) = f$ then $\zeta_2^{t^-}(v_i) = f$ or $\zeta_2^{t^-}(v_j) = f$ and so ζ_1^t and ζ_2^t remain unchanged, and the condition remains satisfied. □

Note that no such monotonicity condition need exist when we compare PCF on finite subgraphs $\mathbf{H}_1 \subseteq \mathbf{H}_2$. However, we do obtain monotonicity when we compare the warm PCF process to the PCF process on a finite subgraph $\mathbf{H} \subseteq \mathbf{G}$.

Lemma 3.2.8 (Stochastic domination of PCF by warm PCF)**.** *Suppose $\mathbf{H} \subseteq \mathbf{G}$ is a finite subgraph, then the rate α warm PCF process on $\mathbf{H}$ stochastically dominates the rate α PCF process on $\mathbf{H}$. Furthermore, if we fix a realisation of $\{X_x\}_{x \in \mathbf{G}}$ as $(s_x)_{x \in \mathbf{G}} \in \Pi$ and let $(\eta^t)_{t \in [0,\infty]} = \psi_{\mathbf{H}}((s_x)_{x \in \mathbf{G}})$ and $(\zeta^t)_{t \in [0,\infty]} = \phi_{\mathbf{H}}((s_x)_{x \in \mathbf{G}})$ be coupled PCF and warm PCF processes, then at all $t \in [0, \infty]$ any interior cluster C of ζ^t is also a cluster of ζ^t.*

Proof. This is proved in the same way as Lemma 3.2.7 by using the coupling of Definition 3.2.6. The details are omitted. □

Given $\alpha > 0$ we can now construct the unique infinite volume warm PCF process on an amenable vertex transitive graph $\mathbf{G}$. Let $\mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \ldots$ be any exhaustion of $\mathbf{G}$ with each $\mathbf{G}_n$ finite, and for each n consider the measure ν_n induced by warm PCF on $\mathbf{G}_n$. By monotonicity we have that $\nu_1, \nu_2, \ldots$ converges to some measure ν.

To check the limit ν does not depend on the exhaustion suppose $\mathbf{H}_1 \subseteq \mathbf{H}_2 \subseteq \ldots$ is another exhaustion of $\mathbf{G}$ with each $\mathbf{H}_n$ finite, and let ν_n^* be the induced warm PCF measure for each n. Again the ν_n^* must converge to some measure ν^*. Now since $\{\mathbf{G}_n : n \in \mathbb{N}\}$ and $\{\mathbf{H}_n : n \in \mathbb{N}\}$ are both exhaustions of $\mathbf{G}$ there exists subsequences $(i_k)_{k\geq 1}$ and $(j_k)_{k\geq 1}$ such that $\mathbf{G}_{i_1} \subseteq \mathbf{H}_{j_1} \subseteq \mathbf{G}_{i_2} \subseteq \mathbf{H}_{j_2} \subseteq \ldots$, and from this we see that $\nu_{i_1} \geq_{\text{st}} \nu_{j_1}^* \geq_{\text{st}} \nu_{i_2} \geq_{\text{st}} \nu_{j_2}^* \geq_{\text{st}} \ldots$ and deduce that this sequence converges. Since this sequence contains a subsequence of both (ν_n) and (ν_n^*) we must have that $\nu = \nu^*$. Therefore we define this unique limiting $\nu = \nu_{\mathbf{G},\alpha}$ to be the *infinite volume warm PCF measure* on $\mathbf{G}$.

Because of the way ν has been constructed by taking an exhaustion $\mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \ldots$ of $\mathbf{G}$ it is clear that for any finite $\mathbf{\Lambda} \subseteq \mathbf{G}$ the restriction of warm PCF on $\mathbf{G}_n$ to $\mathbf{\Lambda}$ will converge to a unique limiting process as $n \longrightarrow \infty$. This limiting process is the restriction of ν to $\mathbf{\Lambda}$. Therefore ν defines an infinite volume warm PCF measure in the sense of local limits.

Suppose $\{X_v\}_{v\in\mathcal{V}}$ and $\{X_e\}_{e\in\mathcal{E}}$ are independent exponential random variables at respective rates α and 1, and denote the product space of these by Π with measure $\lambda = \lambda_{\mathbf{G},\alpha}$. In Definition 3.2.6 we introduced functions $\phi_n : (\Pi, \lambda) \longrightarrow (\Omega \times [0,\infty], \nu_n)$ given by Algorithm 3.2.3* for each finite $\mathbf{G}_n$. We have now seen that the ϕ_n have an almost sure limit as $n \longrightarrow \infty$ which we define to be $\phi_{\mathbf{G}} = \phi : (\Pi, \lambda) \longrightarrow (\Omega \times [0,\infty], \nu)$.

Lemma 3.2.9 (Translation invariance)**.** *Suppose that $f : \mathbf{G} \longrightarrow \mathbf{G}$ is an isomorphism, and that A is measurable with respect to the σ-algebra of ν, then $\nu(A) = \nu(f(A))$. Thus if $\mathbf{G}$ is vertex transitive then ν is translation invariant.*

Proof. Consider a fixed exhaustion of $\mathbf{G}$, $\mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \ldots$ say, and let $\nu_1, \nu_2, \ldots$ be the induced warm PCF measures. Now given any isomorphism f of $\mathbf{G}$ let $\nu_1^*, \nu_2^*, \ldots$ be the measures induced by warm PCF on $f(\mathbf{G}_1), f(\mathbf{G}_2), \ldots$. For any measurable event A we have that $\nu_n(A) = \nu_n^*(f(A))$ for all n. Therefore we get

$$\nu(A) = \lim_{n\to\infty} \nu_n(A) = \lim_{n\to\infty} \nu_n^*(f(A)) = \nu(f(A)), \tag{3.2.7}$$

as required. □

Observe that up to now we have only required for $\mathbf{G}$ to be countable – in order for the exhaustion $\mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \dots$ to exist; and for $\mathbf{G}$ to be locally finite – in order for us to make sense of boundary vertices. Thus we have proved the following.

Proposition 3.2.10. *For every locally finite countable graph $\mathbf{G}$ and every fixed rate of freezing $\alpha > 0$, there exists an infinite volume warm PCF process on $\mathbf{G}$ in the sense of local limits. Moreover in the case that $\mathbf{G}$ is vertex transitive then ν is also translation invariant.*

3.2.3 Proof of Theorem 3.1.1

Whilst this auxiliary process is interesting in its own right, it differs from the infinite volume PCF process of Theorem 3.1.1 in that an infinite cluster will never freeze. This is because an infinite cluster must have been a boundary cluster of each $\mathbf{G}_n$. Our strategy therefore is to attach an exponential clock to each warm infinite cluster – making it freeze at rate α. To do this we must first show that there are only finitely many warm infinite clusters at any given time T. This is where it is necessary to use the fact that $\mathbf{G}$ is amenable. The content of Proposition 3.2.13 below is to show that any warm infinite cluster is necessarily unique. Once we know this we can complete our construction by freezing the unique warm infinite cluster (if it exists) at times $0 < T_1 < T_2 < \dots$. Here the T_i will be chosen to have independent rate α exponential increments. The way we freeze the warm infinite cluster at time T_i is defined below.

Definition 3.2.11 (Freezing the infinite cluster at times $\mathcal{T} = \{0 < T_1 < T_2 < \dots\}$)**.** Let $0 < T_1 < T_2 < \dots$ be a fixed sequence of times, and let $\mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \dots$ be an exhaustion of $\mathbf{G}$ with each $\mathbf{G}_n$ finite. Now use the random variables $\{X_e\}_{e\in\mathcal{E}}$ and $\{X_v\}_{v\in\mathcal{V}}$ to run coupled warm PCF process on each $\mathbf{G}_n$ – giving a trajectories $(\zeta_n^t)_{t\in[0,\infty]}$ – and let $(\zeta^t)_{t\in[0,\infty]}$ be the limiting warm PCF process on $\mathbf{G}$.

At time T_1 set $C_{\infty,1} = \{v \in \mathcal{V} : v$ is contained in an infinite cluster of $\zeta^{T_1^-}$ and $\zeta^{T_1^-}(v) = w\}$, and for each n let $C_{n,\infty,1} = C_{\infty,1} \cap \mathbf{G}_n$. Note that if no infinite cluster exists at time T_1 then $C_{\infty,1} = \emptyset$. We now change each warm PCF process to have $\zeta_n^{T_1} = (\zeta_n^{T_1^-})_{C_{n,\infty,1}}$, before continuing to use Algorithm 3.2.3* to evolve ζ_n^t for $t > T_1$. Denote these new processes by $(\zeta_{n,T_1}^t)_{t\in[0,\infty]}$.

Suppose $m \leq n$, then since $\mathbf{G}_m \subseteq \mathbf{G}_n$ we have $C_{m,\infty,1} \subseteq C_{n,\infty,1}$, and so because $\zeta_n^{T_1^-} \leq \zeta_m^{T_1^-}$

we must also have $\zeta_n^{T_1} = (\zeta_n^{T_1^-})_{C_{n,\infty,1}} \leq (\zeta_m^{T_1^-})_{C_{m,\infty,1}} = \zeta_m^{T_1}$. Moreover, since freezing $C_{n,\infty,1}$ and $C_{m,\infty,1}$ does not change the warm interior clusters of $\zeta_n^{T_1}$ or $\zeta_m^{T_1}$ the proof of Lemma 3.2.7 tells us that $\zeta_{m,T_1}^t \geq \zeta_{n,T_1}^t$ for all t.

Because this monotonicity property still holds we can repeat the argument given above to see that $(\zeta_{n,T_1}^t)_{t\in[0,\infty]} \longrightarrow (\zeta_{T_1}^t)_{t\in[0,\infty]}$ as $n \longrightarrow \infty$. This in turn gives us a new measure ν_{T_1}.

Now we repeat this modification at times $T_2, T_3, \ldots$ by setting $C_{\infty,j+1} = \{v \in \mathcal{V} : v$ is contained in an infinite cluster of $\zeta_{T_1,\ldots,T_j}^{T_{j+1}^-}$ and $\zeta_{T_1,\ldots,T_j}^{T_{j+1}^-}(v) = w\}$ and $C_{n,\infty,j+1} = C_{\infty,j+1} \cap \mathbf{G}_n$ for each n. Doing so we obtain a sequence of new measures ν_{T_1,T_2}, ν_{T_1,T_2,T_3}, etc. Since $\nu_{T_1,\ldots,T_j}$ and $\nu_{T_1,\ldots,T_k n}$ are equal when restricted to $t \in [0, T_{\min\{j,k\}}]$, if we make the further assumption that $T_j \longrightarrow \infty$ as $j \longrightarrow \infty$ then these measures must converge to some final measure $\nu_{\mathcal{T}}$.

Note that for each $\mathcal{T} = \{0 < T_1 < T_2 < \ldots\}$ with $T_j \longrightarrow \infty$ we can combine Algorithm 3.2.3* with the procedure above to obtain a function $\phi_{\mathcal{T}} : (\Pi, \lambda) \longrightarrow (\Omega \times [0,\infty], \nu_{\mathcal{T}})$ taking a sequence $(s_x)_{x\in\mathbf{G}} \in \Pi$ to a process $(\zeta_{\mathcal{T}}^t)_{t\in[0,\infty]} \in \Omega \times [0,\infty]$.

Remark 3.2.12. If $\mathbf{G}$ is vertex transitive then for fixed $\mathcal{T} = \{0 < T_1 < T_2, \ldots\}$ with $T_j \longrightarrow \infty$ the measure $\nu_{\mathcal{T}}$ is again translation invariant. This can be seen inductively since the set $C_{\infty,1}$ is determined by the translation invariant measure ν^{T_1}, and so by applying the argument of Lemma 3.2.9 to $\mathbf{G} \setminus C_{\infty,1}$ we see that ν_{T_1} is translation invariant for $t > T_1$ (as well as for $0 \leq t \leq T_1$). Now suppose we know that $\nu_{T_1,\ldots,T_j}$ is translation invariant, then because $C_{\infty,j+1}$ is determined by $\nu_{T_1,\ldots,T_j}^{T_j+1}$ it must be a statistically translation invariant set, and so by applying Lemma 3.2.9 to $\mathbf{G} \setminus C_{\infty,1} \cup \ldots \cup C_{\infty,j+1}$ we see that $\nu_{T_1,\ldots,T_{j+1}}$ is again a translation invariant measure.

Suppose multiple disjoint warm infinite clusters were present at T_j^-, then the process of Definition 3.2.11 would lead to them all freezing at T_j and so their freezing would not be independent. However, we shall now show that any warm infinite cluster is unique almost surely, and thus the independence of freezing is maintained.

Proposition 3.2.13 (Uniqueness of the warm infinite clusters)**.** *Let $\alpha > 0$, and suppose $\mathbf{G}$ is an amenable vertex transitive graph. Consider the infinite volume warm PCF process on $\mathbf{G}$ at rate α, then at any time $T \geq 0$ we have that ν^T–almost every $\zeta \in \Omega$ contains at most one infinite cluster.*

Moreover, if we modify the process so that any warm infinite cluster freezes at fixed times

$0 < T_1 < \ldots < T_k$, as in Definition 3.2.11, then at any time $T \geq T_k$ we have that $\nu^T_{T_1,\ldots,T_k}$–almost every warm infinite cluster is unique.

Remark 3.2.14. Theorem 1.1.9 due to Burton and Keane, [BK89, Theorem 2], says that for any measure μ on Ω which has the so-called *finite energy property* a configuration ζ has at most one infinite cluster μ–almost surely. Whilst it is possible to show that ν^T meets the hypotheses of [BK89, Theorem 2], the modified measures $\nu^T_{T_1,\ldots,T_k}$ do not (changing the state of one vertex from warm to frozen could have infinite effect). The proof of this proposition deals with this problem by using only events for which a finite energy condition does hold.

Proof of Proposition 3.2.13. Observe that the first claim is a special case of the second with $k = 0$, so we set $k \geq 0$ and let $\mathcal{T} = \{0 = T_0 < T_1 < \ldots < T_k\}$ be fixed. The claim is trivial if $T = T_k$, so we also fix $T > T_k$. Given a configuration ζ define $N(\zeta) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ to be the number of warm infinite clusters of ζ. If $\nu^T_{\mathcal{T}}$ was ergodic then we would know that N is equal to some constant $\tilde{N}$ $\nu^T_{\mathcal{T}}$-almost surely. Our plan of attack would then be to show that

1. If $\tilde{N} \in \{2, 3, \ldots\}$ then $\nu^T_{\mathcal{T}}(\mathbf{A}_1) > 0$, where $\mathbf{A}_1$ is an event whose positive probability would contradict $N = \tilde{N}$ $\nu^T_{\mathcal{T}}$–almost surely.

2. If $\tilde{N} = \infty$ then $\nu^T_{\mathcal{T}}(\mathbf{A}_2) > 0$, where $\mathbf{A}_2$ is another event whose positive probability would again lead to a contradiction.

We would then be able to conclude that $N(\zeta) \in \{0, 1\}$ $\nu^T_{\mathcal{T}}$–almost surely.

However, since we do not know that $\nu^T_{\mathcal{T}}$ is ergodic, we have to rely on the Ergodic Decomposition Theorem, recall Theorem 1.1.6. This says that a translation invariant measure (such as $\nu^T_{\mathcal{T}}$) can be decomposed into ergodic measures. More precisely, for any translation invariant measure space Ω there exists a measurable map from Ω to the space of ergodic measures on Ω, $m : \Omega \longrightarrow \mathscr{E}(\Omega)$, such that

$$\mu(A) = \int_{x \in \Omega} m_x(A) \, \mathrm{d}\mu(x), \tag{3.2.8}$$

for all translation invariant measures μ and all measurable A. The following lemma allows us to relate the changes in energy of m_x and $\nu^T_{\mathcal{T}}$ from the modification of a finite set of edges and vertices.

Lemma 3.2.15. *Suppose $\mathbf{\Lambda} \subseteq \mathbf{G}$ is a finite set of edges and vertices, and $A \in \Omega$ is an event depending only on $\mathbf{\Lambda}$. Write $\mathcal{F}_{\mathbf{G}\setminus\mathbf{\Lambda}}$ for the σ-algebra generated by the edges and*

vertices of $\mathbf{G} \setminus \mathbf{\Lambda}$. *Then for* $\nu_{\mathcal{T}}^T$*–almost all* m_x *in the ergodic decomposition of* $\nu_{\mathcal{T}}^T$ *and each* $A \in \mathcal{F}_{\mathbf{\Lambda}}$ *we have that*

$$m_x(A \,|\, \mathcal{F}_{\mathbf{G}\setminus\mathbf{\Lambda}}) = \nu_{\mathcal{T}}^T(A \,|\, \mathcal{F}_{\mathbf{G}\setminus\mathbf{\Lambda}}) \qquad m_x\text{–almost everywhere.}$$

Proof. The result follows from a careful adaptation of the proof of [GKN92, Lemma 1]. □

Remark 3.2.16. One might be tempted to try and overcome the lack of ergodicity by working with the product space (Π, λ) rather than $(\Omega, \nu_{\mathcal{T}})$. However, whilst the event $\{$there are k warm infinite clusters at time $T\}$ is clearly tail measurable with respect to Ω, we do not know that its pre-image $\phi_{\mathcal{T}}^{-1}(\{$there are k warm infinite clusters at time $T\})$ is tail measurable with respect to Π. Therefore whilst we shall use (Π, λ) later on for making explicit calculations – such as in (3.2.14) – it is necessary for us begin the proof by working with $(\Omega, \nu_{\mathcal{T}})$.

Step 1 For contradiction suppose that $\nu_{\mathcal{T}}^T(N(\zeta) \in \{2, 3, \ldots\}) > 0$, then there must be some $k \in \{2, 3, \ldots\}$ with $\nu_{\mathcal{T}}^T(N(\zeta) = k) > 0$. For each $v \in \mathcal{V}$ let $d(0, v)$ be the length of the shortest path from 0 to v, and set $\mathbf{\Lambda}_r = \{v \in \mathcal{V} : d(0, v) \leq r\}$. We now define

$$A_r = \{\zeta : N(\zeta) = k \text{ and } \zeta \text{ contains at least 2 warm infinite clusters intersecting } \mathbf{\Lambda}_r\}.$$

Now $\nu_{\mathcal{T}}^T(A_r) \longrightarrow \nu_{\mathcal{T}}^T(N(\zeta) = k) > 0$ as $r \longrightarrow \infty$, therefore we can fix $R \in \mathbb{N}$ with $\nu_{\mathcal{T}}^T(A_R) > 0$. For a given configuration ζ let

$$\begin{aligned}\mathbf{\Lambda}_R^*(\zeta) &= \{v \in \mathcal{V} : v \text{ is in } \mathbf{\Lambda}_R \text{ or in a finite cluster of } \zeta \text{ intersecting } \mathbf{\Lambda}_R\}\\ \bar{\mathbf{\Lambda}}_R(\zeta) &= \mathbf{\Lambda}_R^* \cup \{e \in \mathcal{E} : e \text{ meets } \mathbf{\Lambda}_R^*(\zeta)\}.\end{aligned}$$

Observe that $\nu_{\mathcal{T}}^T(A_R \cap \{\bar{\mathbf{\Lambda}}_R(\zeta) \subseteq \mathbf{\Lambda}_\rho\}) \longrightarrow \nu_{\mathcal{T}}^T(A_R)$ as $\rho \longrightarrow \infty$, and therefore $\nu_{\mathcal{T}}^T(A_R \cap \{\bar{\mathbf{\Lambda}}_R(\zeta) \subseteq \mathbf{\Lambda}_\rho\}) > 0$ for some $\rho \in \mathbb{N}$. Moreover, because $\mathbf{\Lambda}_\rho$ is finite there must be some subset $\mathbf{\Lambda} \subseteq \mathbf{\Lambda}_\rho$ with $\nu_{\mathcal{T}}^T(A_R \cap \{\bar{\mathbf{\Lambda}}_R(\zeta) = \mathbf{\Lambda}\}) > 0$. These sets are shown in Figure 3.2.1. We now fix such a $\mathbf{\Lambda}$ and write $A_{R,\mathbf{\Lambda}} = A_R \cap \{\bar{\mathbf{\Lambda}}_R(\zeta) = \mathbf{\Lambda}\}$.

Because $\nu_{\mathcal{T}}^T(A_{R,\mathbf{\Lambda}}) > 0$ there must be a positively measurable set of $x \in \Omega$ such that m_x is in the ergodic decomposition of $\nu_{\mathcal{T}}^T$ with $m_x(A_{R,\mathbf{\Lambda}}) > 0$. We fix such an m_x, and note that since m_x is ergodic and translation invariant it must have $N(\zeta) = k$, m_x–almost surely. To get our contradiction we shall let $\mathbf{A}_1$ be the event that $N(\zeta) < k$; we claim that

$$m_x(\mathbf{A}_1) > 0. \tag{3.2.9}$$

Proof of Claim. To estimate $m_x(\mathbf{A}_1)$ we define two more events:

$$B_{R,\mathbf{\Lambda}} = \{\tilde{\zeta} \in \Omega : \text{there is some } \zeta \in A_{R,\mathbf{\Lambda}} \text{ with } \tilde{\zeta}(x) = \zeta(x) \text{ for all } x \in \mathbf{G} \setminus \mathbf{\Lambda}\}, \quad (3.2.10)$$

$$O_{R,\mathbf{\Lambda}} = \{\zeta \in \Omega : \text{all vertices in } \mathbf{\Lambda} \text{ are warm, all edges with two ends in } \mathbf{\Lambda} \text{ are open and all edges with only one end in } \mathbf{\Lambda} \text{ are closed}\}. \quad (3.2.11)$$

Note that $B_{R,\mathbf{\Lambda}} \in \mathcal{F}_{\mathbf{G}\setminus\mathbf{\Lambda}}$ and $O_{R,\mathbf{\Lambda}} \in \mathcal{F}_{\mathbf{\Lambda}}$.

Suppose $\tilde{\zeta} \in B_{R,\mathbf{\Lambda}} \cap O_{R,\mathbf{\Lambda}}$, then we can find $\zeta \in A_{R,\mathbf{\Lambda}}$ with $\tilde{\zeta}(x) = \zeta(x)$ for all $x \in \mathbf{G} \setminus \mathbf{\Lambda}$. Now ζ has k warm infinite clusters, at least two of which meet $\mathbf{\Lambda}_R$. Therefore since all the edges of $\mathbf{\Lambda}_R$ are open and warm, $\tilde{\zeta}$ will have at least two of the infinite clusters of ζ joined together. Moreover, since ζ and $\tilde{\zeta}$ are equal outside $\mathbf{\Lambda}$ no new infinite clusters are created. Thus $N(\tilde{\zeta}) < k$, and so $B_{R,\mathbf{\Lambda}} \cap O_{R,\mathbf{\Lambda}} \subseteq \mathbf{A}_1$. Because $B_{R,\mathbf{\Lambda}} \in \mathcal{F}_{\mathbf{G}\setminus\mathbf{\Lambda}}$ we can apply Lemma 3.2.15 to see that

$$\begin{aligned} m_x(\mathbf{A}_1) \geq m_x(B_{R,\mathbf{\Lambda}} \cap O_{R,\mathbf{\Lambda}}) &= \mathbb{E}(m_x(\mathbb{1}_B\, \mathbb{1}_O | \mathcal{F}_{\mathbf{G}\setminus\mathbf{\Lambda}})) \\ &= \mathbb{E}(\mathbb{1}_B\, m_x(\mathbb{1}_O | \mathcal{F}_{\mathbf{G}\setminus\mathbf{\Lambda}})) \\ &= \mathbb{E}(\mathbb{1}_B\, \nu_{\mathcal{T}}^T(\mathbb{1}_O | \mathcal{F}_{\mathbf{G}\setminus\mathbf{\Lambda}})). \end{aligned} \quad (3.2.12)$$

where $B = B_{R,\mathbf{\Lambda}}$ and $O = O_{R,\mathbf{\Lambda}}$. Since $m_x(B_{R,\mathbf{\Lambda}}) > 0$ we can prove the claim by showing that $\nu_{\mathcal{T}}^T(O_{R,\mathbf{\Lambda}} | \zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}) > 0$ for each feasible configuration $\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}$ of the edges and vertices of $\mathbf{G} \setminus \mathbf{\Lambda}$.

Let $\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}$ be fixed. We now switch to working with the product space (Π, λ). Using the function defined in Definition 3.2.11, $\phi_{\mathcal{T}} : (\Pi, \lambda) \longrightarrow (\Omega \times [0,\infty], \nu_{\mathcal{T}}^T)$, we have

$$\nu_{\mathcal{T}}^T(O_{R,\mathbf{\Lambda}} | \zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}) = \lambda(\phi_{\mathcal{T}}^{-1}(O_{R,\mathbf{\Lambda}}) | \phi_{\mathcal{T}}^{-1}(\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}})),$$

where $\phi_{\mathcal{T}}^{-1}(\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}) = \{(s_x)_{x\in\mathbf{G}} : \phi_{\mathcal{T}}((s_x)_{x\in\mathbf{G}})^T(x) = \zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}(x) \text{ for all } x \in \mathbf{G} \setminus \mathbf{\Lambda}\}$. From this we define $U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}$ to be the set of $(t_x)_{x\in\mathbf{G}}$ for which there exists $(s_x)_{x\in\mathbf{G}} \in \phi_{\mathcal{T}}^{-1}(\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}})$ with $t_x = s_x$ for all $x \in \mathbf{G} \setminus \mathbf{\Lambda}$, and note that $U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}} \supseteq \phi_{\mathcal{T}}^{-1}(\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}})$. Without loss of generality we may assume that the vertices of $\mathbf{\Lambda}$ have highest priority in the algorithm used to construct $\phi_{\mathcal{T}}$. Now define

$$\begin{aligned} U^O_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}} = U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}} &\cap \{(s_x) : s_v > T \text{ for all } v \in \mathbf{\Lambda}\} \\ &\cap \{(s_x) : s_e > T \text{ for all } e \text{ with one end in } \mathbf{\Lambda}\} \\ &\cap \{(s_x) : s_e < T \text{ for all } e \text{ with two ends in } \mathbf{\Lambda}\}. \end{aligned} \quad (3.2.13)$$

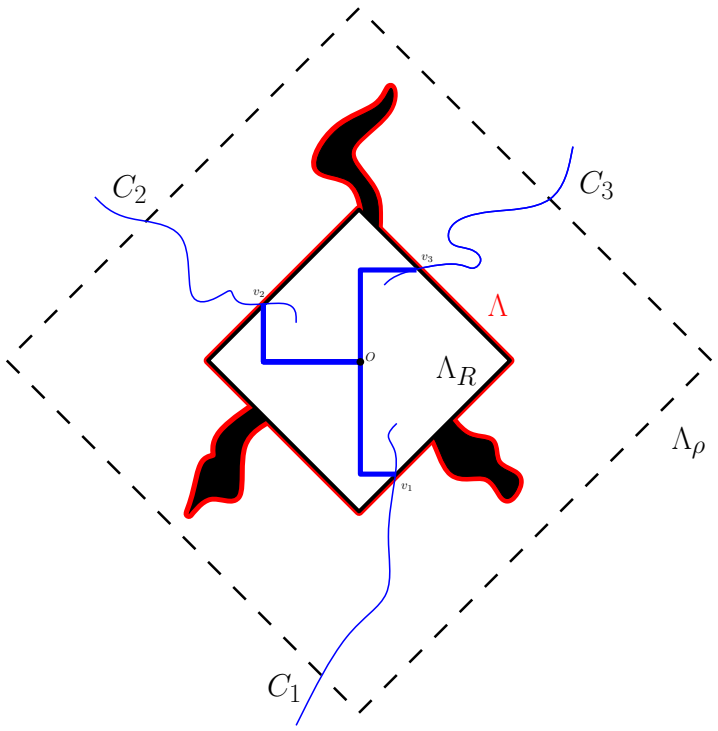


Figure 3.2.1: The diagram shows a box $\mathbf{\Lambda}_R$ intersecting three infinite clusters. $\mathbf{\Lambda}$ (in red) is formed with the addition of the finite clusters (and their boundary edges) which meet $\mathbf{\Lambda}_R$. Note that ρ has been chosen so that $\mathbf{\Lambda} \subseteq \mathbf{\Lambda}_\rho$ with positive probability.
Also shown the three paths P_1, P_2, P_3 from v_1, v_2, v_3 to O. In the second part of the proof we ensure that $P = P_1 \cup P_2 \cup P_3$ is open so that the origin becomes a trifurcation point.

Note that each of the events on the right hand side are independent of $U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}$. We now include two lemmas which allow us to analyse $\phi_{\mathcal{T}}((s_x)_{x\in\mathbf{G}})$ for $(s_x)_{x\in\mathbf{G}} \in U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}$.

Lemma 3.2.17. *Set $T > 0$ and let $E \subseteq \mathcal{E}$ and $V \subseteq \mathcal{V}$. Suppose that $(s_x)_{x\in\mathbf{G}} \in \Pi$ is such that setting $\phi((s_x)_{x\in\mathbf{G}}) = \zeta$ gives $\zeta^T(e) = 0$ for all $e \in E$ and $\zeta^T(v) = w$ for all $v \in V$. If $(\tilde{s}_x)_{x\in\mathbf{G}}$ satisfies $\tilde{s}_x = s_x$ for all $x \in \mathbf{G} \setminus (E \cup V)$, and $\tilde{s}_x > T$ for all $x \in E \cup V$, then writing $\phi((\tilde{s}_x)_{x\in\mathbf{G}}) = \tilde{\zeta}$ gives $\tilde{\zeta}^T = \zeta^T$ (and indeed $\tilde{\zeta}^t = \zeta^t$ for all $t \in [0, T]$).*

Moreover, if $\mathcal{T}$ is fixed then the result still holds when ϕ is replaced by $\phi_{\mathcal{T}}$.

Proof. Since ϕ (respectively $\phi_{\mathcal{T}}$) is a limit of the ϕ_n (respectively $\phi_{n,\mathcal{T}}$) it suffices to consider the algorithm for warm PCF on one of the $\mathbf{G}_n$. Suppose $e \in \mathcal{E}_n$ with $s_e < T$ then since $\zeta^T(e) = 0$ the transition at time s_e is $\zeta^{s_e^-} \longrightarrow \zeta^{s_e} = \zeta^{s_e^-}$. Similarly if $v \in \mathcal{V}_n$ with $s_v < T$ then since $\zeta^T(v) = w$ the transition at time s_v is also $\zeta^{s_v^-} \longrightarrow \zeta^{s_v} = \zeta^{s_v^-}$. Therefore since $\tilde{\zeta}$ does not change at s_x we have that $\tilde{\zeta}^{s_x^-} = \zeta^{s_x^-}$ implies $\tilde{\zeta}^{s_x} = \zeta^{s_x}$, and so the lemma is proved by induction. □

Lemma 3.2.18. *Set $T > 0$, let $V \subseteq \mathcal{V}$, and suppose our enumeration of $\mathcal{V}$ is such that if $v_i \in V$ and $v_j \in \mathcal{V} \setminus V$ then $i < j$ – i.e. V has higher priority than $\mathcal{V} \setminus V$. Now given*

$(s_x)_{x\in\mathbf{G}} \in \Pi$ let $(\tilde{s}_x)_{x\in\mathbf{G}}$ satisfy $s_x = \tilde{s}_x$ for all $x \in \mathbf{G}\setminus V$ and $\tilde{s}_x > T$ for all $x \in V$. If we write $\phi((s_x)_{x\in\mathbf{G}}) = \zeta$ and $\phi((\tilde{s}_x)_{x\in\mathbf{G}}) = \tilde{\zeta}$ then for all $t \in [0,T]$ we have

(a) $\tilde{\zeta}^t \geq \zeta^t$,

(b) for each $e \in \mathcal{E}$ with $\zeta^t(e) = 0$ and $\tilde{\zeta}^t(e) = 1$ there exists an open path in $\tilde{\zeta}^t$ from e to V.

Moreover, if $\mathcal{T} = \{0 < T_1 < T_2 < \ldots\}$ is fixed then the result still holds if we replace ϕ by $\phi_{\mathcal{T}}$.

Proof. We consider our exhaustion $\mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \ldots$ of $\mathbf{G}$. Since ϕ is a limit of the ϕ_n, to show the result it suffices to show that the result holds for each ϕ_n. Now consider the algorithm for warm PCF on $\mathbf{G}_n$. If a cluster C intersects V then its freezing time is controlled by a vertex $v \in V$, therefore since $s_v > T$ for all $v \in V$ we can treat each $v \in V$ as a boundary vertex up until time T. *(a)* then follows by using the coupling argument of Lemma 3.2.7 with $\mathbf{H}_1 = \mathbf{G}_n \setminus V$ and $\mathbf{H}_2 = \mathbf{G}_n$.

Suppose an edge $e \in \mathcal{E}$ has $\zeta^t(e) = 0$ and $\tilde{\zeta}^t(e) = 1$ for some $t \in [0,T]$, then since $\zeta^t(e) = 0$ we must have $e \in \mathbf{G}_n$, and as $\tilde{\zeta}^t(e) = 1$ we must have $s_e = \tilde{s}_e < t$. Now there can only be finitely many such edges (as $\mathbf{G}_n$ is finite), $\{e_1, \ldots, e_k\}$ say, and so to prove *(b)* we can consider what must happen at each of the times s_{e_i} in order.

At a time $t = \min\{s_{e_1}, \ldots, s_{e_k}\}$ the edges of ζ^{t^-} and $\tilde{\zeta}^{t^-}$ are equal and thus so are the clusters, but then ζ has the transition $\tilde{\zeta}^{t^-} \longrightarrow \tilde{\zeta}^t = (\tilde{\zeta}^{t^-})^e$ whilst ζ is unchanged. Thus one of the end points of e must have $\zeta^t(v) = f$ and $\tilde{\zeta}^t(v) = w$. The only way this can happen is if the freezing of v is controlled by a vertex in V and so there must be a path from e to V. At subsequent times $t \in \{s_{e_1}, \ldots, s_{e_k}\}$ we must also have some vertex v with $\zeta^t(v) = f$ and $\tilde{\zeta}^t(v) = w$. This can only happen if either the freezing of v is controlled by some vertex in V or if the freezing of v was controlled by different vertices in ζ and $\tilde{\zeta}$. The first case implies that there is an open path from e to V, and the second case implies that C_v must differ in ζ and $\tilde{\zeta}$. For C_v to differ, v must be joined to some edge e' with $\zeta^t(e') = 0$ and $\tilde{\zeta}^t(e') = 1$. Therefore $e' \in \{e_1, \ldots, e_k\}$ with $s_{e'} < s_e$, and so we can apply induction to see that there is an open path from e to V (through e').

The result for $\phi_{\mathcal{T}}$ follows by the same argument. □

Suppose $(t_x)_{x\in\mathbf{G}} \in U^O_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}$ and find $(s_x)_{x\in\mathbf{G}} \in \phi_{\mathcal{T}}^{-1}(\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}})$ with $t_x = s_x$ for all $x \in \mathbf{G}\setminus\mathbf{\Lambda}$. Write $\zeta = \phi_{\mathcal{T}}((s_x)_{x\in\mathbf{G}})$ and $\tilde{\zeta} = \phi_{\mathcal{T}}((t_x)_{x\in\mathbf{G}})$, and define $E = \{e \in \mathbf{\Lambda} : e \text{ has exactly one end in } \mathbf{\Lambda}\}$ and $V = \{v \in \partial\mathbf{\Lambda}_R : v \text{ is in a warm infinite cluster of } \zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}\}$. Suppose

$(\tau_x)_{x\in\mathbf{G}}$ has $\tau_x = t_x$ for all $x \in E \cup V$ and $\tau_x = s_x$ for all $x \in \mathbf{G} \setminus E \cup V$, then by applying Lemma 3.2.17 we see that $\phi_{\mathcal{T}}((\tau_x)_{x\in\mathbf{G}}) = \zeta$. Now since the edges on the boundary of $\mathbf{\Lambda}$ are closed in $\phi_{\mathcal{T}}((\tau_x)_{x\in\mathbf{G}})$ we can use Lemma 3.2.18 with $(\tau_x)_{x\in\mathbf{G}}$ and $(t_x)_{x\in\mathbf{G}}$ to deduce that $\tilde{\zeta}(x) = \zeta(x)$ for all $x \in \mathbf{G} \setminus \mathbf{\Lambda}$. A simple check reveals that $\tilde{\zeta}^T \in O_{R,\mathbf{\Lambda}}$, therefore $U^O_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}} \subseteq \phi_{\mathcal{T}}^{-1}(O_{R,\mathbf{\Lambda}}) \cap U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}$. We can now estimate $\nu^T_{\mathcal{T}}(O_{R,\mathbf{\Lambda}}|\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}})$,

$$\begin{aligned}\nu^T_{\mathcal{T}}(O_{R,\mathbf{\Lambda}}|\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}) &= \lambda(\phi_{\mathcal{T}}^{-1}(O_{R,\mathbf{\Lambda}})|\phi_{\mathcal{T}}^{-1}(\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}})) \\ &\geq \lambda(U^O_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}|U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}) \\ &= (\mathrm{e}^{-\alpha T})^{|\mathbf{\Lambda}\cap\mathcal{V}|} \times (\mathrm{e}^{-T})^{|\mathbf{\Lambda}\cap\mathcal{E}\setminus E|} \times (1-\mathrm{e}^{-T})^{|E|} > 0. \end{aligned} \tag{3.2.14}$$

Thus we indeed have $m_x(\mathbf{A}_1) > 0$, and so (3.2.9) is proved. □

Step 2 We now move on to showing that $N(\zeta) \neq \infty$ $\nu^T_{\mathcal{T}}$–almost surely. To do this we recall the notion of *trifurcation points*. Given a configuration ζ we say a vertex $v \in \mathcal{V}$ is a trifurcation point if v is contained in some infinite component C and removing v (and its adjoining edges) from C leaves three disjoint infinite clusters C_1, C_2 and C_3 with $C_1 \cup C_2 \cup C_3 \cup \{v\} = C$. By following the argument of Burton and Keane [BK89] we see that any translation invariant measure μ on an amenable vertex transitive graph $\mathbf{G}$ can not have $\mu(0 \text{ is a trifurcation point}) > 0$.

The argument of [BK89] (which is repeated in Theorem 1.1.9) is based on the observation that if there are n trifurcation points in $\mathbf{\Lambda}_r$ then $\partial\mathbf{\Lambda}_r = \mathbf{\Lambda}_r \setminus \mathbf{\Lambda}_{r-1}$ must intersect infinite clusters in at least $n+2$ places. This is illustrated in Figure 1.1.2. Since μ is translation invariant then for a given a measurable function f the Ergodic Theorem tells us that μ-almost surely we have that

$$\frac{1}{|\mathbf{\Lambda}_r|}\sum_{v\in\mathbf{\Lambda}_r} f(T_v(\eta)) \longrightarrow \mathbb{E}(f(\eta)) \quad \text{as} \quad r \longrightarrow \infty.$$

Here $T_v : \mathbf{G} \longrightarrow \mathbf{G}$ denotes the translation taking v to 0. Therefore by setting f to be the indicator of the event $\{0 \text{ is a trifurcation point}\}$ we see that since $\mathbb{E}(f) = \mu(0 \text{ is a trifurcation point}) > 0$, then there must exist $c > 0$ such that

$$\mu(\mathbf{\Lambda}_r \text{ contains at least } c|\mathbf{\Lambda}_r| \text{ trifurcation points}) \longrightarrow 1 \quad \text{as} \quad r \longrightarrow \infty. \tag{3.2.15}$$

Because $\mathbf{G}$ is amenable then $|\partial\mathbf{\Lambda}_r|/|\mathbf{\Lambda}_r| \longrightarrow 0$ as $r \longrightarrow \infty$. This means that $|\partial\mathbf{\Lambda}_r| < c|\mathbf{\Lambda}_r|$ for sufficiently large r, and so (3.2.15) is impossible. Therefore we let $\mathbf{A}_2$ be the event that 0 is a trifurcation point, and get a contradiction by showing that it occurs with positive

probability.

Claim 3.2.19. *If $\nu_{\mathcal{T}}(N(\zeta)=\infty)>0$ then $\nu_{\mathcal{T}}(\mathbf{A}_2)>0$.*

Proof. We prove the claim in a similar way to before: Find a finite $\mathbf{\Lambda}$ where a suitable configuration on $\mathbf{G}\setminus\mathbf{\Lambda}$ occurs with positive probability, and modify the edges of $\mathbf{\Lambda}$ with only a finite cost. However, unlike before no ergodicity assumption is required. We start by defining A_r^∞ in a similar way to A_r, but this time insisting that $\mathbf{\Lambda}_r$ meets at least three warm infinite clusters.

$$A_r^\infty=\{\zeta : N(\zeta)=\infty \text{ and } \zeta \text{ contains at least 3 warm infinite clusters intersecting } \mathbf{\Lambda}_r\}.$$

Since $\nu_{\mathcal{T}}(N(\zeta)=\infty)>0$ then there exists some $R\in\mathbb{N}$ with $\nu_{\mathcal{T}}(A_R^\infty)>0$. As before we can now find a finite $\mathbf{\Lambda}\supseteq\mathbf{\Lambda}_R$ for which $A_{R,\mathbf{\Lambda}}^\infty=A_R^\infty\cap\{\bar{\mathbf{\Lambda}}_R(\zeta)=\mathbf{\Lambda}\}$ has $\nu_{\mathcal{T}}(A_{R,\mathbf{\Lambda}}^\infty)>0$. Given three distinct vertices $v_1,v_2,v_3\in\partial\mathbf{\Lambda}_R$ we define a further event

$$A_{R,\mathbf{\Lambda},v_1,v_2,v_3}^\infty=A_{R,\mathbf{\Lambda}}^\infty\cap\{\zeta : v_1,v_2,v_3 \text{ are in disjoint warm infinite clusters of } \zeta\}. \quad (3.2.16)$$

There are only finitely many distinct 3-sets $\{v_1,v_2,v_3\}$, and so with positive probability there exists $v_1,v_2,v_3\in\partial\mathbf{\Lambda}_R$ with $\nu_{\mathcal{T}}(A_{R,\mathbf{\Lambda},v_1,v_2,v_3}^\infty)>0$. Fix such a 3-set and let $P_1,P_2,P_3\subseteq\mathbf{\Lambda}_R$ be disjoint paths from v_1,v_2,v_3 to 0, with $P=P_1\cup P_2\cup P_3$. See Figure 3.2.1. We now define analogues to $B_{R,\mathbf{\Lambda}}$ and $O_{R,\mathbf{\Lambda}}$.

$$B_{R,\mathbf{\Lambda},v_1,v_2,v_3}=\{\tilde{\zeta}\in\Omega : \text{there is some } \zeta\in A_{R,\mathbf{\Lambda},v_1,v_2,v_3} \text{ with } \tilde{\zeta}(x)=\zeta(x) \text{ for all } x\in\mathbf{G}\setminus\mathbf{\Lambda}\}, \quad (3.2.17)$$

$$\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3}=\{\zeta\in\Omega : \text{all vertices in } \mathbf{\Lambda} \text{ are warm, all edges } e\in\mathbf{\Lambda}\setminus P \text{ are closed and all edges } e\in P \text{ are open}\}. \quad (3.2.18)$$

Note that since $A_{R,\mathbf{\Lambda},v_1,v_2,v_3}^\infty\subseteq B_{R,\mathbf{\Lambda},v_1,v_2,v_3}$ we have $\nu_{\mathcal{T}}(B_{R,\mathbf{\Lambda},v_1,v_2,v_3})>0$. Moreover, $B_{R,\mathbf{\Lambda},v_1,v_2,v_3}$ depends only on the edges and vertices of $\mathbf{G}\setminus\mathbf{\Lambda}$ and $\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3}$ depends only on the edges and vertices of $\mathbf{\Lambda}$.

Suppose $\tilde{\zeta}\in B_{R,\mathbf{\Lambda},v_1,v_2,v_3}\cap\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3}$, then we can find ζ in $A_{R,\mathbf{\Lambda},v_1,v_2,v_3}^\infty$ with $\zeta(x)=\tilde{\zeta}(x)$ for all $x\in\mathbf{G}\setminus\mathbf{\Lambda}$. Now since ζ contains three disjoint infinite clusters C_1,C_2,C_3 meeting v_1,v_2,v_3, and because v_1,v_2,v_3 are joined to 0 by open paths in $\tilde{\zeta}$, then $\tilde{\zeta}$ must contain an infinite cluster C with a trifurcation point at 0. This means $\mathbf{A}_2\supseteq B_{R,\mathbf{\Lambda},v_1,v_2,v_3}\cap\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3}$, and so

$$\nu_{\mathcal{T}}(\mathbf{A}_2)\geq\nu_{\mathcal{T}}(\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3}|B_{R,\mathbf{\Lambda},v_1,v_2,v_3})\times\nu_{\mathcal{T}}(B_{R,\mathbf{\Lambda},v_1,v_2,v_3}).$$

Therefore to prove the claim it suffices to show that $\nu_{\mathcal{T}}(\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3}|\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}) > 0$ for each feasible configuration $\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}$. To do this we again switch to working with the product space (Π,λ). Using our measure preserving function $\phi_{\mathcal{T}} : (\Pi,\lambda) \longrightarrow (\Omega \times [0,\infty], \nu_{\mathcal{T}}^T)$ we have

$$\nu_{\mathcal{T}}^T(\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3}|\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}) = \lambda(\phi_{\mathcal{T}}^{-1}(\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3})|\phi_{\mathcal{T}}^{-1}(\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}})).$$

Define $U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}$ to be the set of $(t_x)_{x\in\mathbf{G}}$ for which there exists $(s_x)_{x\in\mathbf{G}} \in \phi_{\mathcal{T}}^{-1}(\zeta)$ with $t_x = s_x$ for all $x \in \mathbf{G}\setminus\mathbf{\Lambda}$, and let

$$\begin{aligned} U^{\Delta}_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}} = U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}} &\cap \{(s_x) : s_v > T \text{ for all } v \in \mathbf{\Lambda}\} \cap \{(s_x) : s_e < T \text{ for all } e \text{ in } P\} \\ &\cap \{(s_x) : s_e > T \text{ for all } e \in \mathbf{\Lambda}\setminus P\}. \end{aligned}$$

Using Lemma 3.2.17 and Lemma 3.2.18 we can check that $U^{\Delta}_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}} \subseteq \phi_{\mathcal{T}}^{-1}(\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3}) \cap U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}$, and so we can estimate $\nu_{\mathcal{T}}(\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3}|\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}})$ as follows.

$$\begin{aligned} \nu_{\mathcal{T}}(\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3}|\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}) &= \lambda(\phi_{\mathcal{T}}^{-1}(\Delta_{R,\mathbf{\Lambda},v_1,v_2,v_3})|\phi_{\mathcal{T}}^{-1}(\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}})) \\ &\geq \lambda(U^{\Delta}_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}|U_{\zeta_{\mathbf{G}\setminus\mathbf{\Lambda}}}) \\ &= (\mathrm{e}^{-\alpha T})^{|\mathbf{\Lambda}\cap\mathcal{V}|} \times (\mathrm{e}^{-T})^{|P|} \times (1-\mathrm{e}^{-T})^{|\mathbf{\Lambda}\cap\mathcal{E}\setminus P|} > 0. \end{aligned} \tag{3.2.19}$$

Thus we can conclude that $\nu_{\mathcal{T}}(\mathbf{A}_2) > 0$. □

As both $\nu_{\mathcal{T}}(N(\zeta) \in \{2,3,\ldots\}) > 0$ and $\nu_{\mathcal{T}}(N(\zeta) = \infty) > 0$ lead to a contradiction, we must have $N(\zeta) \in \{0,1\}$ $\nu_{\mathcal{T}}$–almost surely. □

In Definition 3.2.11 we freeze each of the infinite clusters present some time T_i. However, since we have shown that a warm infinite cluster is necessarily unique, then freezing all warm infinite clusters at T_i will result in at most one infinite cluster freezing. Therefore we let $X_{\infty,1}, X_{\infty,2}, \ldots$ be a sequence of independent Exp(α) random variables, set $T_i = \sum_{j=1}^{i} X_{\infty,j}$ for $i = 1,2,\ldots$, denote $\mathcal{T} = \{T_1, T_2, \ldots\}$, and then define $\mu = \nu_{\mathcal{T}}$.

It is easy to see that the process corresponding to μ satisfies the transition rates (3.2.3) of Definition 3.2.1. Indeed each warm edge e will be contained in $\mathbf{G}_n$ for n sufficiently large, and so will open at rate 1. Likewise each warm cluster C is either finite and thus an interior cluster of some $\mathbf{G}_n$ meaning it freezes at rate α or it is infinite meaning its freezing is controlled by some $X_{\infty,j}$ – again at rate α. We shall now conclude the proof of Theorem 3.1.1 by showing that the measure μ corresponds to PCF on $\mathbf{G}$ in the sense of local limits.

Theorem 3.2.20. *Fix $\alpha > 0$ and let $\mathbf{G}$ be an amenable vertex transitive graph. Then if $\mathbf{\Lambda} \subseteq \mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq ...$ (each $\mathbf{G}_n$ finite) is an exhaustion of $\mathbf{G}$ with PCF measures $\mu_1, \mu_2, \ldots$ then $\mu_n|_{\mathbf{\Lambda}} \longrightarrow \mu|_{\mathbf{\Lambda}}$ as $n \longrightarrow \infty$. Thus $\mu_n \longrightarrow \mu$ in the sense of local limits. Here $.|_{\mathbf{\Lambda}}$ denotes the restriction of a measure to the subgraph $\mathbf{\Lambda}$ and $\mu = \nu_{\mathcal{T}}$ is as defined above.*

Proof. Since Ω is a product of discrete spaces and thus compact the obvious approach is to use Prokhorov's Theorem to extract a convergent subsequence $(\mu_{n_k})_{k\geq 1}$, let $\mu^* = \lim_{k\to\infty} \mu_{n_k}$ and compare $\mu^*|_{\mathbf{\Lambda}}$ to $\mu|_{\mathbf{\Lambda}}$. However, since μ^* depends on the subsequence $(n_k)_{k\geq 1}$ the is no a priori reason why this limit should be unique. We get around this problem by constructing a third measure $\tilde{\mu}$ with $\tilde{\mu} \leq_{\text{st}} \mu^*$ for all $(n_k)_{k\geq 1}$. By showing that in fact $\tilde{\mu} = \mu^*$ we can then complete the proof by using features of both μ^* and $\tilde{\mu}$ to compare $\mu^*|_{\mathbf{\Lambda}}$ with $\mu|_{\mathbf{\Lambda}}$.

Given $(s_x)_{x\in\mathbf{G}} \in \Pi$ and a finite subgraph $\mathbf{H} \subseteq \mathbf{G}$, Algorithm 3.2.3 gives us a function $\psi_{\mathbf{H}}$ taking $(s_x)_{x\in\mathbf{G}}$ to a PCF process on $\mathbf{H}$, $(\eta^t_{\mathbf{H}})_{t\in[0,\infty]}$ say. We now define $\tilde{\psi}_{\mathbf{G}_n}$ to be the infimum of $\psi_{\mathbf{H}}$ taken over all finite $\mathbf{H}$ with $\mathbf{G}_n \subseteq \mathbf{H} \subseteq \mathbf{G}$. Therefore

$$\tilde{\psi}_{\mathbf{G}_n}((s_x)_{x\in\mathbf{G}})^t(e) = \begin{cases} 1 & \text{if } \eta^t_{\mathbf{H}}(e) = 1 \text{ for all finite } \mathbf{H} \supseteq \mathbf{G}_n \\ 0 & \text{otherwise} \end{cases}, \tag{3.2.20}$$

$$\tilde{\psi}_{\mathbf{G}_n}((s_x)_{x\in\mathbf{G}})^t(v) = \begin{cases} w & \text{if } \eta^t_{\mathbf{H}}(v) = w \text{ for all finite } \mathbf{H} \supseteq \mathbf{G}_n \\ f & \text{otherwise} \end{cases}. \tag{3.2.21}$$

If we write $\tilde{\psi}_{\mathbf{G}_n}((s_x)_{x\in\mathbf{G}}) = (\tilde{\eta}^t_{\mathbf{G}_n})_{t\in[0,\infty]}$ then it is clear that $\tilde{\eta}^t_{\mathbf{G}_n} \leq \eta^t_{\mathbf{G}_n}$ for all t. Moreover if $\mathbf{G}_m \subseteq \mathbf{G}_n$ then since $\{\mathbf{H} : \mathbf{G}_m \subseteq \mathbf{H} \text{ finite}\} \supseteq \{\mathbf{H} : \mathbf{G}_n \subseteq \mathbf{H} \text{ finite}\}$ we see that $\tilde{\eta}^t_{\mathbf{G}_m} \leq \tilde{\eta}^t_{\mathbf{G}_n}$ for all t. Because of this monotonicity $(\tilde{\eta}^t_{\mathbf{G}_n})_{t\in[0,\infty]}$ must converge to some process $(\tilde{\eta}^t)_{t\in[0,\infty]}$ as $n \longrightarrow \infty$. Therefore if we define $\tilde{\mu}_{\mathbf{G}_n,\alpha}$ to be the measure associated with $(\tilde{\eta}^t_{\mathbf{G}_n})$ and $\tilde{\mu}$ to be the measure associated with $(\tilde{\eta}^t)$, then we see that the measures $\tilde{\mu}_{\mathbf{G}_n,\alpha}$ converge to $\tilde{\mu}$.

It is easy to see from the construction that the limit $\tilde{\mu}$ does not depend on $\mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \ldots$, and that $\tilde{\mu}$ is translation invariant. This is because for each $e \in \mathcal{E}$ we have that

$$\begin{aligned} \tilde{\eta}^t(e) = 0 &\Leftrightarrow \text{ for all } n \text{ there exists a finite } \mathbf{H} \supseteq \mathbf{G}_n \text{ such that } \eta^t_{\mathbf{H}}(e) = 0 \\ &\Leftrightarrow \text{ for all } n \text{ there exists a finite } \mathbf{H} \supseteq \mathbf{H}_n \text{ such that } \eta^t_{\mathbf{H}}(e) = 0, \end{aligned}$$

where $\mathbf{H}_1 \subseteq \mathbf{H}_2 \subseteq \ldots$ is any other exhaustion of $\mathbf{G}$ with $\mathbf{H}_n$ finite for all n. Likewise $\tilde{\eta}^t(v)$ does not depend on the exhaustion for each $v \in \mathcal{V}$. To see that translation invariance also holds let $\mathbf{H}_n = f(\mathbf{G}_n)$ where f is any isomorphism of $\mathbf{G}$ and follow the argument of

(3.2.7). Since $\tilde{\mu}$ is translation invariant the proof of Proposition 3.2.13 applies to $\tilde{\mu}$ and so we see that for all t any warm infinite cluster of $\tilde{\eta}^t$ is unique almost surely. Therefore by keeping $(s_x)_{x\in\mathbf{G}} \in \Pi$ fixed we can use $(\tilde{\eta}^t)_{t\in[0\infty]}$ to define a coupled sequence of random freezing times $0 < T_1 < T_2 < \dots$ as follows.

Definition 3.2.21 (Coupled freezing times)**.** Let $(s_x)_{x\in\mathbf{G}} \in \Pi$ be fixed and suppose $(\tilde{\eta}^t)_{t\in[0,\infty]}$ is as defined above. For each time t let $\ell(t) = \inf\{k : v_k \text{ is in a warm infinite cluster of } \tilde{\eta}^t\}$ with the convention that $\ell(t)$ is infinite if no warm infinite cluster exists at t. Now suppose $X_{\infty,1}, X_{\infty,2}, \dots$ are independent $\mathrm{Exp}(\alpha)$ random variables and define

$$T_1 = \begin{cases} X_{\infty,1} & \text{if } \ell(X_{\infty,1}) = \infty \\ \inf\{t : s_{v_{\ell(t)}} = t\} & \text{otherwise} \end{cases} . \tag{3.2.22}$$

Observe that if $\ell(X_{1,\infty}) < \infty$ then $v_{\ell(X_{1,\infty})}$ must have joined the infinite cluster at time $t < X_{1,\infty}$. Moreover because $v_{\ell(X_{1,\infty})}$ must have been warm at t then $s_{v_{\ell(t)}} > t$ (and $s_{v_{\ell(t)}} - t \sim \mathrm{Exp}(\alpha)$). Therefore $\{t : s_{v_{\ell(t)}} = t\} \neq \emptyset$ and so the memoryless property tells us that $T_1 \sim \mathrm{Exp}(\alpha)$.

$T_2, T_3, \dots$ are now defined inductively. Suppose we have defined $T_1, \dots, T_j$, then define

$$T_{j+1} = \begin{cases} T_j + X_{\infty,j+1} & \text{if } \ell(T_j + X_{\infty,j+1}) = \infty \\ \inf\{t : s_{v_{\ell(t)}} = t\} & \text{otherwise} \end{cases} , \tag{3.2.23}$$

and note that we again have $T_{j+1} - T_j \sim \mathrm{Exp}(\alpha)$ from the memoryless property.

Now consider the warm PCF process $(\zeta^t)_{t\in[0,\infty]} = \phi((s_x)_{x\in\mathbf{G}})$, and let $\tau = \inf\{t : \zeta^t$ contains an infinite cluster$\}$. If $T_1 = s_v$ for some v, then $T_1 \geq \tau$ and v is in an infinite cluster of $\tilde{\eta}$. Lemma 3.2.8 tells us that $\tilde{\eta} \leq \zeta$ and so v must also be in an infinite cluster of of ζ (which can not freeze). Therefore conditioning on (η^t) gives no further information about s_v. Thus the law of T_1 given (η^t) is

$$\begin{aligned} \mathbb{P}(T_1 \in .\,|(\eta^t)) &= \mathbb{P}(T_1 \in .\,|(\eta^t); X_{\infty,1} < \tau) \times \mathbb{P}(X_{\infty,1} < \tau|(\eta^t)) \\ &\quad + \mathbb{P}(T_1 \in .\,|(\eta^t); X_{\infty,1} \geq \tau) \times \mathbb{P}(X_{\infty,1} \geq \tau|(\eta^t)) \\ &= \mathbb{P}(X_{\infty,1} \in .\,|X_{\infty,1} < \tau) \times \mathbb{P}(X_{\infty,1} < \tau) \\ &\quad + \mathbb{P}(s_v \in .\,|s_v \geq \tau) \times \mathbb{P}(X_{\infty,1} \geq \tau) \\ &= \mathbb{P}(X_{\infty,1} \in .\,), \end{aligned} \tag{3.2.24}$$

and so T_1 is independent of (η_t).

Claim 3.2.22. *For $t \in [0, T_1)$ we have $\tilde{\eta}^t = \zeta^t$.*

Proof. Consider the corresponding processes on $\mathbf{G}_n$. Lemma 3.2.7 and Lemma 3.2.8 combine to tell us that if C is an interior cluster of ζ_n^t then C is also an interior cluster of $\zeta_{\mathbf{H}}^t$ for all finite $\mathbf{H} \supseteq \mathbf{G}_n$, and so C must also be a cluster of $\tilde{\eta}_n^t$. Since $\mathbf{G}_n \longrightarrow \mathbf{G}$ we must therefore have that if C is a finite cluster of ζ^t then C is also a finite cluster of $\tilde{\eta}^t$. Conversely if C is an interior cluster of $\tilde{\eta}_n^t$ then C is also an interior cluster of η_n^t and ζ_n^t. Therefore any finite cluster of $\tilde{\eta}^t$ is also a finite cluster of ζ^t.

Now suppose some edge $e \in \mathcal{E}$ has $\tilde{\eta}^t(e) \neq \zeta^t(e)$, then e must be in an infinite cluster of ζ^t with $\zeta^t(e) = 1$ and $\tilde{\eta}^t(e) = 0$. Since $\zeta^t(e) = 1$ we must have $s_e < t$, therefore there is some time $s < s_e$ such that if $e = v_i v_j$ then either v_i or v_j has frozen in $\tilde{\eta}^s$ but not in ζ^s. Suppose it is v_i, then because we know that the finite clusters of $\tilde{\eta}^s$ and ζ^s are equal we must then have v_i in an infinite cluster of $\tilde{\eta}^s$. However, this is impossible since no infinite cluster freezes in either model before time T_1. Thus $\tilde{\eta}^t = \zeta^t$ for all $t \in [0, T_1)$. □

We now consider $(\zeta_{T_1}^t)_{t\in[0,\infty]} = \phi_{T_1}((s_x)_{x\in\mathbf{G}})$. We know that $\zeta_{T_1}^{T_1^-} = \tilde{\eta}^{T_1^-}$, and so because any infinite cluster of $\tilde{\eta}$ freezes at T_1 then we also have $\zeta_{T_1}^{T_1} = \tilde{\eta}^{T_1}$. By following the argument above and considering only the edges and vertices in $\mathbf{G}\backslash\{\text{frozen infinite cluster}\}$ we can extend this to $\tilde{\eta}^t = \zeta_{T_1}^t$ for all $t \in [0, T_2)$. Moreover since $T_2 = T_1 + X_{\infty,2}$ or $T_2 = s_v$ for some v in a warm infinite cluster of ζ_{T_1}, we can repeat the argument of (3.2.24) to see that T_2 is independent of $(\zeta_{T_1}^t)$.

By iterating we have that $\zeta_{T_1,\ldots,T_j}^t = \tilde{\eta}^t$ for all $t \in [0, T_{j+1})$. Furthermore T_j is independent of $(\zeta_{T_1,\ldots,T_k}^t)$ for all $j > k$. Therefore if we set $\mathcal{T} = \{0 < T_1 < T_2 < \ldots\}$ with the T_j defined by Definition 3.2.21 then $\tilde{\eta}^t = \zeta_{\mathcal{T}}^t$ for all t. Because $\mathcal{T}$ has independent $\text{Exp}(\alpha)$ increments and is independent of $\tilde{\eta}^t$, then the law of $\zeta_{\mathcal{T}}^t$ must be μ, and so $\tilde{\mu} = \mu^*$.

It now only remains to compare $\mu|_{\mathbf{\Lambda}}$ and $\mu^*|_{\mathbf{\Lambda}}$. We do this by showing that for the coupled processes (η_n^t) and $(\zeta_{\mathcal{T}}^t)$ we have $\eta_n^t|_{\mathbf{\Lambda}} = \zeta_{\mathcal{T}}^t|_{\mathbf{\Lambda}}$ for all t and all n sufficiently large. If this is true then it also follows that $\mu_n|_{\mathbf{\Lambda}} \longrightarrow \mu|_{\mathbf{\Lambda}}$ as required.

Suppose some edge or vertex $x \in \mathbf{\Lambda}$ is in a finite cluster of $\zeta_{\mathcal{T}}^\infty$, then x is in some interior cluster of ζ_n^∞ for n sufficiently large and so by Lemma 3.2.8 we know that $\eta_n^t(x) = \zeta_{\mathcal{T}}^t(x)$ for all t and all n sufficiently large. Now suppose instead that $x \in \mathbf{\Lambda}$ is in an infinite cluster of $\tilde{\eta}^\infty = \zeta_{\mathcal{T}}^\infty$, and that this infinite cluster froze at time T_k. Let $v = v_{\ell(T_k)}$ then at time T_k- there must have been some warm path from x to v in $\tilde{\eta}^{T_k^-}$, and so for n sufficiently large this path must also be present in $\tilde{\eta}_n^{T_k^-}$ and $\eta_n^{T_k^-} \geq \tilde{\eta}_n^{T_k^-}$. Since $\eta_n^{T_k^-} \leq \zeta_{n,\mathcal{T}}^{T_k}$

we know that η_n does not have x joined to a vertex of higher priority than v, and thus x will freeze in η_n at time T_k. Since the state of an edge e is determined only by s_e and the freezing times of the adjacent vertices we must again have $\eta_n^t(x) = \zeta_{\mathcal{T}}^t(x)$ for all t and all n sufficiently large. Therefore, since $\mathbf{\Lambda}$ is finite, when n is sufficiently large we must have $\eta_n^t(x) = \zeta_{\mathcal{T}}^t(x)$ for all $t \in [0, \infty]$ and $x \in \mathbf{\Lambda}$. This completes the proofs of Theorem 3.2.20 and Theorem 3.1.1. □

3.2.4 PCF on a infinite tree

We have shown that there is an infinite volume PCF process on every amenable vertex transitive graph $\mathbf{G}$. The theorem below now tells us that an infinite volume PCF process also exist on any countably infinite tree $\mathbf{T}$. Whether or not the PCF process can be defined on any graph remains open – see Section 3.1.4.

Theorem 3.2.23. *For every countable tree $\mathbf{T} = (\mathcal{V}, \mathcal{E})$, and every fixed rate of freezing $\alpha > 0$, there exists an infinite volume PCF process on $\mathbf{T}$. This induces the PCF measure $\mu_{\mathbf{T},\alpha}$ on $\{0,1\}^{\mathcal{E}}$.*

Proof. Given a (countably) infinite tree, $\mathbf{T} = (\mathcal{V}, \mathcal{E})$, we can assign some vertex $v_0 \in \mathcal{V}$ to be the root. It is then possible to label the remaining vertices $\mathcal{V} \backslash \{v_0\} = \{v_1, v_2, \ldots\}$ in such a way that for each v_j the unique path from v_0, $v_0 v_{i_1} \ldots v_{i_k} v_j$ say, has $i_1 < \ldots < i_k < j$. Suppose we have such a labelling and let $v_0 \in \mathbf{G}_1 \subseteq \mathbf{G}_2 \subseteq \ldots$ be an exhaustion of $\mathbf{T}$ with each $\mathbf{G}_n$ connected and finite.

By now assigning independent $\mathrm{Exp}(\alpha)$ random variables X_v to each $v \in \mathcal{V}$ and independent $\mathrm{Exp}(1)$ random variables X_e to each $e \in \mathcal{E}$, we can use the coupling of Definition 3.2.6 to construct coupled processes on each of the $\mathbf{G}_n$ – $(\eta_n^t)_{t\in[0,\infty]}$ with measures $\mu_{n,\alpha} = (\mu_{n,\alpha}^t : t \in [0, \infty])$ say. Now given n, the opening of any edge $e = v_i v_j \in \mathbf{G}_n$ is determined only by X_e and the states of v_i and v_j. Moreover, the state of a vertex $v \in \mathbf{G}_n$ is determined by the $X_{v'}$ – where v' is the vertex of highest priority in the cluster containing v. Note that our choice of enumeration means we must have $v' \in \{v_0, v_{i_1}, \ldots, v_{i_k}, v\}$ and so we need only know that states of the edges $v_0 v_{i_1}, v_{i_1} v_{i_2}, \ldots, v_{i_k} v$. Since all of these edges and vertices must be in $\mathbf{G}_n$ then if we consider PCF on $\mathbf{G}_m \supseteq \mathbf{G}_n$ for some $m > n$, then transitions of each of the edges and vertices of $\mathbf{G}_n$ remain unchanged. Therefore we see that $\eta_m^t|_{\mathbf{G}_n} = \eta_n^t$ for all t, and so $\mu_{m,\alpha}|_{\mathbf{G}_n} = \mu_{n,\alpha}$, implying that $\mu_{n,\alpha}$ converges as $n \longrightarrow \infty$. The limit, $\mu_{\mathbf{T},\alpha}$, is the infinite volume rate α PCF measure on $\mathbf{T}$. □

3.3 Proof of Proposition 3.1.3

On the cubic lattice $\mathbb{Z}^d = (\mathcal{V}, \mathcal{E})$ we say a vertex $v \in \mathcal{V}$ is $\star$-*open* in the PCF model if at least one of its incident edges is open, otherwise we say that v is $\star$-*closed*. Observe that if a PCF configuration η contains an infinite cluster of open edges then it must also contain an infinite $\star$-open cluster of vertices.

Now consider the site percolation model on $\mathbb{Z}^d$ (where vertices are open with probability p and closed with probability $1-p$ independently of each other). It is well know that there exists a critical value $0 < p_c^{\text{site}} < 1$ such that if $p < p_c^{\text{site}}$ then all open clusters are almost surely finite – see e.g. [Gri99] for details. Therefore to prove Proposition 3.1.3 it suffices to show that when $\alpha > 0$ is sufficiently large the PCF measure of $\star$-open vertices is stochastically dominated by some site percolation measure $\mathbb{P}_p$ with $p < p_c^{\text{site}}$. This will be a consequence of the lemma below.

Lemma 3.3.1. *Set $d \geq 2$, and consider rate α PCF on $\mathbb{Z}^d$. Let $v \in \mathcal{V}$, and A be any event which is measurable with respect to the $\star$-configuration of $\mathcal{V} \setminus \{v\}$, then*

$$\mu_{\mathbb{Z}^d,\alpha}(v \text{ is } \star\text{-open} \,|\, A) \leq f_d(\alpha),$$

where $f(\alpha) \longrightarrow 0$ as $\alpha \longrightarrow \infty$.

Proof. We use a coupling argument. To each vertex $v \in \mathcal{V}$ we associate an independent $\text{Exp}(\alpha)$ random variable X_v and $2d$ independent $\Gamma(\frac{1}{2}, 1)$ random variables $(Y_{v,1} \dots Y_{v,2d})$. Now at each edge $e = v_1 v_2 \in \mathcal{E}$ choose unique $Y_{v_1,i}$ and $Y_{v_2,j}$ from v_1 and v_2 and set $X_e = Y_{v_1,i} + Y_{v_2,j}$. Note that since X_e is the sum of two independent $\Gamma(\frac{1}{2}, 1)$ random variables we have $X_e \sim \text{Exp}(1)$. We can now use these random variables to drive PCF on $\mathbb{Z}^d$ (as in Algorithm 3.2.3).

Now a vertex v is $\star$-open in the PCF model only if an adjoining edge opens before v freezes. For this we require

$$X_v > \min\{X_e : e \text{ is adjacent to } v\} > \min\{Y_{v,1} \dots Y_{v,2d}\}.$$

Therefore we are done by setting

$$f_d(\alpha) = \mathbb{P}(X_v > \min\{Y_{v,1} \dots Y_{v,2d}\}) \longrightarrow 0 \quad \text{as} \quad \alpha \longrightarrow \infty. \qquad \square$$

Now if we set $\alpha^\star = \sup\{\alpha : f_d(\alpha) \geq p_c(d)\}$ then for $\alpha > \alpha^\star$ the measure of $\star$-open vertices is stochastically dominated by $\mathbb{P}_p$ with $p < p_c$ and so all the clusters in the PCF configuration η on $\mathbb{Z}^d$ are finite almost surely.

Observe that this proof only relies on $\mathbb{Z}^d$ having $p_c^{\text{site}} > 0$, and $\mathbb{Z}^d$ having maximum degree $2d < \infty$. Therefore this proof allows Proposition 3.1.3 to be extended to any amenable vertex transitive graph $\mathbb{L}^d$ where the degree of every vertex is bounded by some $\Delta < \infty$.

Remark 3.3.2. For $d = 2$ we have $p_c^{\text{site}} \approx 0.59$ and so a back of the envelope calculation gives $\alpha^\star \approx 20$. This upper bound for α_c far exceeds the estimate of $\alpha_c \approx 0.55$ found in Section 3.5.

3.4 The results on a tree

This section begins with a restatement of Theorem 3.1.4 in greater generality.

Theorem 3.4.1. *Let $\alpha > 0$ be fixed and consider a rate α PCF process on the rooted d-ary tree $\mathbf{T}_d$ – with measure $\mu_{\mathbf{T}_d,\alpha}$. Let A_k be the event that the root cluster has size k (k vertices) in the final distribution. Then*

- *if $\alpha < d-1$ we have $\mu_{\mathbf{T}_d,\alpha}(A_k) \sim C_{d,\alpha}\, k^{-2}$, for some fixed constant $C_{d,\alpha}$.*
- *If $\alpha = d-1$ we have $\mu_{\mathbf{T}_d,\alpha}(A_k) \sim C_d\, k^{-\left(2-\frac{1}{2d}\right)}$, for some fixed constant C_d.*
- *If $\alpha > d-1$ then $\mu_{\mathbf{T}_d,\alpha}(A_k)$ decays exponentially in k.*

Before embarking upon the proof of Theorem 3.4.1 we remark that this result is also valid for unrooted d-ary trees. Moreover, with only minor modifications to the proof, one can show that the result can be extended to any tree $\mathbf{T}$ where the limit

$$d = \lim_{n\to\infty} \sqrt[n]{|\partial \mathbf{\Lambda_n}|}$$

exists. Here $|\partial \Lambda_n|$ gives the number of vertices at distance n from the root. In this case the d plays the same role as in Theorem 3.4.1.

3.4.1 Time rescaling

As mentioned in Section 3.1, our understanding of PCF on the tree comes through viewing it as time rescaled percolation. The following proposition makes sense of this.

Proposition 3.4.2. *Suppose we have a rate α PCF process on a tree $\mathbf{T}$. Let $W_v(t)$ be the event that $\eta_t(v) = w$ – i.e. that the vertex v is still warm at time t. Suppose C is any finite connected component containing v, then if we condition on $W_v(t)$ the probabilities of $\{C_v(\eta_t) \subseteq C\}$ and $\{C_v(\eta_t) = C\}$ are given by*

$$\mu^t_{\mathbf{T},\alpha}(\{C_v(\eta_t) \subseteq C\} | W_v(t)) = p^{|C|}, \tag{3.4.1}$$

$$\text{and} \quad \mu^t_{\mathbf{T},\alpha}(\{C_v(\eta_t) = C\} | W_v(t)) = p^{|C|}(1-p)^{|\partial C|}. \tag{3.4.2}$$

Where

$$p = \frac{1 - \mathrm{e}^{-(1+\alpha)t}}{1+\alpha}. \tag{3.4.3}$$

Proof. Label the edges of C, $e_1, e_2, \ldots, e_{|C|}$ in such a way that the labels of any path away from v are increasing. Then (using '$e = vw$ warm' to mean both v and w are warm) we get

$$\begin{aligned}\mu^t_{\mathbf{T},\alpha}\left(\{C_v(\eta_t) \subseteq C\} \,|\, W_v(t)\right) = {} & \mu^t_{\mathbf{T},\alpha}\left(e_1 \text{ open at } t \,|\, v \text{ warm}\right) \\ & \times \mu^t_{\mathbf{T},\alpha}\left(e_2 \text{ open at } t \,|\, v, e_1 \text{ warm}\right) \\ & \times \cdots \times \mu^t_{\mathbf{T},\alpha}\left(e_{|C|} \text{ open at } t \,|\, v, e_1, \ldots, e_{|C|-1} \text{ warm}\right),\end{aligned}$$

and

$$\begin{aligned}\mu^t_{\mathbf{T},\alpha}\left(\{C_v(\eta_t) = C\} \,|\, W_v(t)\right) = {} & \mu^t_{\mathbf{T},\alpha}\left(C_v(\eta_t) \subseteq C\right) \\ & \times \mu^t_{\mathbf{T},\alpha}\left(\text{boundary edges closed} \,|\, C_v(\eta_t) \text{ warm}\right).\end{aligned}$$

Our choice of labelling means that if v is warm at t and $e_1, \ldots, e_{j-1}$ are open then the end of e_j closest to v must also be warm at time t. Thus

$$\begin{aligned}\mu^t_{\mathbf{T},\alpha}\left(e_j \text{ open at } t \,|\, v, e_1, \ldots, e_{j-1} \text{ warm}\right) &= \mu^t_{\mathbf{T},\alpha}\begin{pmatrix} e_j \text{ opens before } t \text{ and before} \\ \text{its other end freezes}\end{pmatrix} \\ &= \int_0^t \mathrm{e}^{-\alpha s}\mathrm{e}^{-s}\,\mathrm{d}s \\ &= \frac{1-\mathrm{e}^{-(1+\alpha)t}}{1+\alpha}. \qquad (3.4.4)\end{aligned}$$

Since we are working on a tree each of the boundary edges is independent and so we get

$$\mu^t_{\mathbf{T},\alpha}\left(\text{boundary closed} \,|\, C_v(\eta_t) \text{ warm}\right) = \mu^t_{\mathbf{T},\alpha}\left(\text{boundary edge closed} \,|\, C_v(\eta_t) \text{ warm}\right)^{|\partial C|},$$

where

$$\mu^t_{\mathbf{T},\alpha}\left(\text{boundary edge closed} \,|\, C_v(\eta_t) \text{ warm}\right) = 1 - \frac{1-\mathrm{e}^{-(1+\alpha)t}}{1+\alpha} = \frac{\alpha + \mathrm{e}^{-(1+\alpha)t}}{1+\alpha}. \qquad (3.4.5)$$

The result then follows. □

To understand the sense in which Proposition 3.4.2 gives a time rescaling, observe that if $\alpha = 0$ then we have a bond percolation process where edges open independently at rate 1. Let $\mathbb{P}_\tau$ denote the measure of this process. In this case the probability that a given edge is open at time τ is $\mathbb{P}_\tau(e \text{ open at } \tau) = 1 - \mathrm{e}^{-\tau}$. Comparing this with (3.4.3) we see

that

$$\mu^t_{\mathbf{T},\alpha}(\{C_v(\eta_t) \subseteq C\}|W_v(t)) = \mathbb{P}_\tau(\{C_v(\tau) \subseteq C\}),$$
$$\text{and} \quad \mu^t_{\mathbf{T},\alpha}(\{C_v(\eta_t) = C\}|W_v(t)) = \mathbb{P}_\tau(\{C_v(\tau) = C\}),$$

where

$$\tau = -\log\left(\frac{\alpha + \mathrm{e}^{-(1+\alpha)t}}{1+\alpha}\right). \tag{3.4.6}$$

Note that the distribution of infinite components is entirely determined by the events $C_v(\eta_t) \subseteq C$ for finite C. Therefore the time rescaling of Proposition 3.4.2 applies equally to finite and infinite clusters. Because there is always a positive probability that a given cluster $C_v(\eta_t)$ is warm, we deduce from the proposition that running PCF on $\mathbf{T}$ to time t yields infinite clusters if and only if the bond percolation on $\mathbf{T}$ yields infinite clusters at time τ (where τ is given by (3.4.6)).

Remark 3.4.3. In [BNK05a] and [BNK05b], Ben-Naim and Krapivsky work on the complete graph $\mathbf{K}_n$, opening edges at a rate $\dfrac{1}{n}$ and allowing vertices to freeze at rate α. By using a Smoluchowski equation they obtain a result which is similar to Proposition 3.4.2 for the $n \longrightarrow \infty$ limit. Indeed, in this setting it is shown that – conditionally on a cluster being warm – a cluster's size at time t is equal in distribution to that of a percolation cluster at time

$$\tau = \frac{1 - \mathrm{e}^{-\alpha t}}{\alpha}.$$

In fact it is possible for us to deduce this result from Proposition 3.4.2. We start with the fact that for large n small clusters are cycle free with high probability, and therefore for $0 \leq t < 1$ they grow in the same way as cluster on an n-ary tree. See [Dur07, Chapter 2] for details. Now because we are opening edges on $\mathbf{K}_n$ at rate $\dfrac{1}{n}$, we need to rescale time to $\dfrac{t}{n}$ and the rate of freezing to $n\alpha$. Putting this in equation (3.4.3) we get

$$\tau = np = n\,\frac{1 - \mathrm{e}^{-(1+n\alpha)\frac{t}{n}}}{1+n\alpha} \longrightarrow \frac{1 - \mathrm{e}^{-\alpha t}}{\alpha} \quad \text{as} \quad n \longrightarrow \infty.$$

Therefore we see that for finite clusters, mean field PCF can be thought of as the limiting case of PCF on a n-ary tree as $n \longrightarrow \infty$.

3.4.2 The critical phenomenon

Consider an infinite tree $\mathbf{T}$ with a distinguished vertex v, suppose also that $\mathbf{T}$ satisfies the *exponential volume growth property*

$$d = \liminf_{n\to\infty} \sqrt[n]{|\Gamma^n(v)|} > 1, \tag{3.4.7}$$

where $|\Gamma^n(v)|$ is the number of vertices at distance n from v. Then if we run percolation (without freezing) on $\mathbf{T}$ until time τ, there is a strictly positive probability that $C_v(\tau)$ is infinite ($\mathbb{P}_\tau(\{|C_v(\tau)| = \infty\}) > 0$) if and only if

$$\mathbb{P}(e \text{ open at } \tau) = 1 - \mathrm{e}^{-\tau} > \frac{1}{d}.$$

This follows from considering $C_v(\tau)$ as having grown from a branching process. For details see e.g. Section 2.1 of [Dur07]. By combining this with the time rescaling of Section 3.4.1 we can prove the following.

Proposition 3.4.4. *Consider rate α PCF running on an infinite tree $\mathbf{T}$ with distinguished vertex v. Suppose $\mathbf{T}$ has growth rate $d = \liminf_{n\to\infty} \sqrt[n]{|\Gamma^n(v)|} > 1$. Then the critical rate of freezing for the existence of infinite clusters is $\alpha_c = d - 1$. That is*

$$\mu_{\mathbf{T},\alpha}(\{|C_v(\infty)| = \infty\}) \begin{cases} = 0 & \text{for } \alpha \geq \alpha_c \\ > 0 & \text{for } \alpha < \alpha_c \end{cases}. \tag{3.4.8}$$

Moreover, in the case $\alpha < \alpha_c$, the critical time for the emergence of infinite clusters is

$$t_c = \frac{1}{1+\alpha} \log\left(\frac{d}{1+\alpha}\right). \tag{3.4.9}$$

Proof. From Proposition 3.4.2 we know that if we condition on $W_v(t)$, then the distribution of the structure of $C_v(t)$ is equal to that of percolation cluster where edges are open with probability $p = \frac{1}{1+\alpha}\left(1 - \mathrm{e}^{-(1+\alpha)t}\right)$. Let $\mathbb{P}_p(\{|C_v| = \infty\})$ be the probability that the cluster containing v is infinite in the percolation model. Then given $\alpha < \alpha_c = d - 1$ for all $t > t_c$ we have $p > \frac{1}{d}$, implying $\mathbb{P}_p(\{|C_v| = \infty\}) > 0$. Therefore, since $\mu^t_{\mathbf{T},\alpha}(W_v(t)) = \mathrm{e}^{-\alpha t} > 0$ for all t, we get

$$0 < \mu^t_{\mathbf{T},\alpha}(W_v(t)) \times \mathbb{P}_p(\{|C_v| = \infty\}) \leq \mu^t_{\mathbf{T},\alpha}(\{|C_v(\eta_t)| = \infty\}) \leq \mu_{\mathbf{T},\alpha}(\{|C_v| = \infty\}).$$

However, if $t < t_c$, then $p < \frac{1}{d}$ and thus $\mu^t_{\mathbf{T},\alpha}(\{|C_v(\eta_t)| = \infty\}) \leq \mathbb{P}_p(\{|C_v| = \infty\}) = 0$. Hence the time t_c is critical.

Conversely suppose $\alpha \geq \alpha_c$. If C_v is infinite in the final distribution, then there must be some finite time t at which $C_v(\eta_t)$ is infinite. But

$$\mu^t_{\mathbf{T},\alpha}(\{|C_v(\eta_t)| = \infty\}) \leq \mathbb{P}_p(\{|C_v| = \infty\}) = 0, \tag{3.4.10}$$

for all t, and so $\mu_{\mathbf{T},\alpha}(\{|C_v| = \infty\}) = 0$. □

From now on we shall restrict our focus to d-ary trees $\mathbf{T}_d$. Observe that a d-ary tree has $|\Gamma^n(v)| = (d+1)d^{n-1}$, and a rooted d-ary tree has $|\Gamma^n(v)| = (d+1)d^{n-1}$. Thus they both have exponential growth at rate d.

Corollary 3.4.5. *Suppose we run rate α PCF on the (possibly rooted) d-ary tree $\mathbf{T}_d$. Then*

- *if $\alpha < d-1$, then at time $t > t_c = \frac{1}{1+\alpha}\log\left(\frac{d}{1+\alpha}\right)$, $\mathbf{T}_d$ will contain infinite components almost surely.*
- *if $\alpha \geq d-1$, then $\mathbf{T}_d$ will have no infinite component with probability 1.*

3.4.3 The final distribution – proof of Theorem 3.4.1

Heuristically, the reason that we have clusters of algebraic size in the super-critical regime is because our system must pass though a critical time

$$t_c = \frac{1}{1+\alpha}\log\left(\frac{d}{1+\alpha}\right), \tag{3.4.11}$$

at which large clusters exist, but have not yet merged to become infinite. Some of these large clusters can then freeze at or near this critical time leaving us with large finite clusters in the final distribution. However, the process is only sufficiently close to this critical time for a short period, meaning that there are fewer large clusters. Consequentially the exponent of the cluster sizes is larger than the value of $\frac{3}{2}$ we see for clusters in critical bond percolation.

In the case of critical PCF the system spends a much more significant amount of time approaching criticality, but the clusters never become infinite. Therefore there are more

large clusters to freeze, thus explaining why we have a lower exponent than in the super-critical case. To see why the exponent should depend on the dimension we look at the time rescaling of equation (3.4.6), and the rate at which it reaches its maximum value

$$\tau_{\max} - \tau = \frac{1}{1+\alpha} - \frac{1-\mathrm{e}^{-(1+\alpha)t}}{1+\alpha} = \frac{\mathrm{e}^{-(1+\alpha)t}}{1+\alpha}.$$

Noting that $d = 1 + \alpha_c$ we see that the degree has a dramatic difference on the rate at which the system approach criticality, and therefore it is not surprising that the algebraic exponent depends on the dimension.

We can now prove Theorem 3.4.1 by calculating the probabilities $\mu_{\mathbf{T}_d,\alpha}(A_k)$ explicitly. To do this we integrate the probability $\mu^t_{\mathbf{T},\alpha}(\{C_v(\eta_t) = C\} \,|\, W_v(t))$ with respect to $\alpha\mathrm{e}^{-\alpha t}\mathrm{d}t$. Here $\alpha\mathrm{e}^{-\alpha t}\mathrm{d}t$ is the probability measure of the freezing time of v. Thus we get

$$\begin{aligned}\mu_{\mathbf{T},\alpha}(\{C_v(\infty) = C\}) &= \int_0^\infty \mu^t_{\mathbf{T},\alpha}(\{C_v(\eta_t) = C\} \,|\, W_v(t))\, \alpha\mathrm{e}^{-\alpha t}\mathrm{d}t \\ &= \int_0^\infty p^{|C|}(1-p)^{|\partial C|}\, \alpha\mathrm{e}^{-\alpha t}\mathrm{d}t,\end{aligned}$$

where $p = \dfrac{1}{1+\alpha}\left(1 + \mathrm{e}^{-(1+\alpha)t}\right)$. By rewriting this as an integral in terms of p, and then making the substitution $\tilde{p} = (1+\alpha)p$ we get

$$\mu_{\mathbf{T},\alpha}(\{C_v(\infty) = C\}) = \alpha \int_0^{\frac{1}{1+\alpha}} p^{|C|}(1-p)^{|\partial C|}(1-(1+\alpha)p)^{-\frac{1}{1+\alpha}}\,\mathrm{d}p \tag{3.4.12}$$

$$= \alpha \int_0^1 \left(\frac{\tilde{p}}{1+\alpha}\right)^{|C|} \left(1 - \frac{\tilde{p}}{1+\alpha}\right)^{|\partial C|} (1-\tilde{p})^{-\frac{1}{1+\alpha}}\, \frac{\mathrm{d}\tilde{p}}{1+\alpha}. \tag{3.4.13}$$

Finally, we recall Euler's hypergeometric transform

$$\int_0^1 \frac{s^{a-1}(1-s)^{c-a-1}}{(1-sz)^b}\,\mathrm{d}s = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n}\frac{z^n}{n!} = {}_2F_1\left[\begin{matrix}a, b\\ c\end{matrix}; z\right]. \tag{3.4.14}$$

Here $(a)_n$ denote the (rising) Pochhammer symbol, $(a)_n = a\cdot(a+1)\cdot(a+2)\cdot\ldots\cdot(a+n-1)$, and ${}_2F_1$ denotes the hypergeometic function. Using this (3.4.13) becomes

$$\mu_{\mathbf{T},\alpha}(\{C_v(\infty) = C\}) = \frac{\alpha}{1+\alpha}\frac{1}{(1+\alpha)^{|C|}}\,{}_2F_1\left[\begin{matrix}|C|+1, -|\partial C|\\ |C| + \frac{2+\alpha}{1+\alpha}\end{matrix}; \frac{1}{1+\alpha}\right]. \tag{3.4.15}$$

Observe also that the probability $\mu_{\mathbf{T},\alpha}(\{C_v(\infty) \subseteq C\})$ can also be calculated using the same formulae by setting $|\partial C| = 0$. Whilst it is nice to have a closed form for $\mu_{\mathbf{T},\alpha}(\{C_v = C\})$ – equation (3.4.15) – the hypergeometric function ${}_2F_1$ can often prove impenetrable,

and so for practical purposes we shall use (3.4.12).

To give explicit results on the rooted d-ary tree $\mathbf{T}_d$ we shall need the following two lemmas. We have chosen to focus on the rooted tree in order to simplify calculations (especially Lemma 3.4.7). However, the corresponding results for the unrooted case prove to be very similar, and the main result of this section, Theorem 3.4.1, holds for both cases.

Lemma 3.4.6. *Consider the rooted d-ary tree. Suppose $C \subseteq \mathbf{T}_d$ is a connected component of size k – i.e. with k vertices. Then $|C| = k - 1$ and $|\partial C| = (d-1)k + 1$, and thus*

$$\mu_{\mathbf{T}_d,\alpha}(\{C_v(\infty) = C\}) = \alpha \int_0^{\frac{1}{1+\alpha}} p^{k-1}(1-p)^{(d-1)k+1}(1-(1+\alpha)p)^{-\frac{1}{1+\alpha}}\,\mathrm{d}p. \tag{3.4.16}$$

Lemma 3.4.7. *The number of trees on k vertices which are rooted sub-graphs of the d-ary tree $\mathbf{T}_d$ is given by*

$$N_k^d = \frac{1}{k}\binom{d\,k}{k-1}.$$

The proofs of these lemmas are mainly combinatorial. For details see e.g. [Kla70].

Proof. We can now give a formal proof of Theorem 3.4.1. Throughout this proof we shall use $C_{d,\alpha}$, C_d and C_α to represent constants whose values can change from line to line. It is possible to calculate their values explicitly, but not enlightening to do so. Using Stirling's formula one can show

$$N_k^d = \frac{1}{k}\binom{d\,k}{k-1} \sim C_d \left(\frac{d^d}{(d-1)^{d-1}}\right)^k \frac{1}{k^{\frac{3}{2}}}. \tag{3.4.17}$$

Therefore by combining equations (3.4.16) and (3.4.17) we get

$$\begin{aligned}\mu_{\mathbf{T}_d,\alpha}(A_k) &= \frac{1}{k}\binom{d\,k}{k-1}\alpha \int_0^{\frac{1}{1+\alpha}} p^{k-1}(1-p)^{(d-1)k+1}\,(1-(1+\alpha)p)^{-\frac{1}{1+\alpha}}\,\mathrm{d}p \\ &\sim C_{d,\alpha}\frac{1}{k^{\frac{3}{2}}}\int_0^{\frac{1}{1+\alpha}} f^{k-1}(x)g_\alpha(x)\,\mathrm{d}x \end{aligned} \tag{3.4.18}$$

where $f(x) = \dfrac{d^d}{(d-1)^{d-1}}x(1-x)^{d-1}$, $g_\alpha(x) = \dfrac{(1-x)^d}{(1-(1+\alpha)x)^{\frac{1}{1+\alpha}}}$.

Observe now that $f(x)$ is positive on $[0,1]$, and attains a unique maximal value of 1 at $x = \frac{1}{d}$. We also note that g_α is integrable on $[0,1]$ (for $\alpha > 0$) and is continuous everywhere except $x = \dfrac{1}{1+\alpha}$. We can now consider the different cases.

$\underline{\alpha > d-1:}$ Bounding $f(x)$ by its supremum we have

$$\begin{aligned}\mu_{\mathbf{T}_d,\alpha}(A_k) &\lesssim C_{d,\alpha}\frac{1}{k^{\frac{3}{2}}}\int_0^{\frac{1}{1+\alpha}} \sup_{x\in\left[0,\frac{1}{1+\alpha}\right]} f^{k-1}(x)g_\alpha(x)\,\mathrm{d}x \\ &= C_{d,\alpha}\frac{1}{k^{\frac{3}{2}}}\left(\sup_{x\in\left[0,\frac{1}{1+\alpha}\right]} f(x)\right)^{k-1}\int_0^{\frac{1}{1+\alpha}} g_\alpha(x)\,\mathrm{d}x, \end{aligned} \tag{3.4.19}$$

which decays exponentially in k since $\sup\{f(x) : 0 \le x \le \frac{1}{1+\alpha}\} < 1$.

$\underline{\alpha < d-1:}$ Because the singularity of $g_\alpha(x)$ occurs at $x = \frac{1}{1+\alpha}$ and not at the maximum of $f^{k-1}(x)$ then it is relatively easy to verify that

$$\mu_{\mathbf{T}_d,\alpha}(A_k) \sim C_{d,\alpha}\frac{1}{k^{\frac{3}{2}}}\left[\int_0^1 f^{k-1}(x)\,\mathrm{d}x + \mathcal{O}\left(\frac{1}{k}\right)\right].$$

This integral can now be evaluated explicitly in terms of the Beta function as

$$\int_0^1 f^{k-1}(x)\,\mathrm{d}x = \left(\frac{d^d}{(d-1)^{d-1}}\right)^{k-1}\mathrm{B}(k,(d-1)k-d+2) \sim C_d\frac{1}{k^{\frac{1}{2}}}. \tag{3.4.20}$$

From which the result follows.

$\underline{\alpha = d-1:}$ In this case the singularity of g_α coincides with the maximum of f, and so its contribution becomes significant. By expanding f about $x = \frac{1}{d}$ we get

$$f(x) = 1 - \frac{d^3}{2(d-1)}\left(x-\tfrac{1}{d}\right)^2 + \mathcal{O}\left(\left(x-\tfrac{1}{d}\right)^3\right) = \tilde{f}(x) + \mathcal{O}\left(\left(x-\tfrac{1}{d}\right)^3\right). \tag{3.4.21}$$

Therefore

$$\begin{aligned}\mu_{\mathbf{T}_d,\alpha}(A_k) &\sim C_d\left[\frac{1}{k^{\frac{3}{2}}}\int_0^{\frac{1}{d}} \tilde{f}^{k-1}(x)\frac{(1-x)^d}{(1-dx)^{\frac{1}{d}}}\,\mathrm{d}x + \mathcal{O}\left(\frac{1}{k}\right)\right] \\ &= C_d\left[\frac{1}{k^{\frac{3}{2}}}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)\right)\int_0^1 (1-y)^{k-1}y^{-\frac{1}{2d}}\frac{\mathrm{d}y}{\sqrt{y}} + \mathcal{O}\left(\frac{1}{k}\right)\right]. \end{aligned} \tag{3.4.22}$$

Where the second line follows from setting $y = 1-\tilde{f}(x)$ and approximating $(1-x)^d$ by $\left(1-\frac{1}{d}\right)^d$ – its value at $x = \frac{1}{d}$. The integral can then be calculated explicitly as

$$\int_0^1 (1-y)^{k-1}y^{-\frac{1}{2d}}\frac{\mathrm{d}y}{\sqrt{y}} = \mathrm{B}\left(k,\frac{1}{2}-\frac{1}{2d}\right) \sim C_d\frac{1}{k^{\frac{1}{2}-\frac{1}{2d}}},$$

and so the result follows. □

3.5 Simulations of PCF on $\mathbb{Z}^2$

Using Algorithm 3.2.3 we can write a fairly efficient program to simulate PCF on a finite graph. In this section we use Monte Carlo simulations of PCF on an n by m square grid in order to give us insight into PCF on $\mathbb{Z}^2$. We begin by using simulations to estimate the critical value α_c.

3.5.1 Estimating α_c

It is known (e.g. see [Gri99]) that for bond percolation on an n by $n+1$ square grid, the probability that there exists a left–right crossing, C_{LR}, satisfies

$$\mathbb{P}_p(C_{LR}) \begin{cases} = \dfrac{1}{2} & \text{for } p = p_c = \frac{1}{2} \\ \xrightarrow[n\to\infty]{} 0 & \text{for } p < p_c = \frac{1}{2} \\ \xrightarrow[n\to\infty]{} 1 & \text{for } p > p_c = \frac{1}{2} \end{cases} .$$

Therefore, the existence of a left–right crossing of a n by $n+1$ grid would seem to be a reasonable indicator for the existence of an infinite component for PCF on $\mathbb{Z}^2$. Figure 3.5.1 shows a Monte Carlo simulation of how the probability of a left–right crossing varies with α for various values of n. We focus on values of α between 0.45 and 0.65 since this is where the transition between sub-critical and super-critical PCF occurs.

Observe that for each n, we get $\mu_\alpha(C_{LR}) = \frac{1}{2}$ when $\alpha \approx 0.55$, and therefore $\alpha_c \approx 0.55$ would seem to be a reasonable estimate for the critical rate of freezing on $\mathbb{Z}^2$. Moreover, we see that as n increases the transition becomes sharper – i.e. changes in α have a greater effect on the existence of a left–right crossing, as we would expect.

3.5.2 The cluster size distribution

It is known (for a reference see e.g. [Gri99]) that in sub-critical bond percolation on $\mathbb{Z}^d$ the distribution of the size of the clusters decays exponentially, and in super-critical bond percolation on $\mathbb{Z}^d$ the size of the finite clusters decays sub-exponentially. This means to

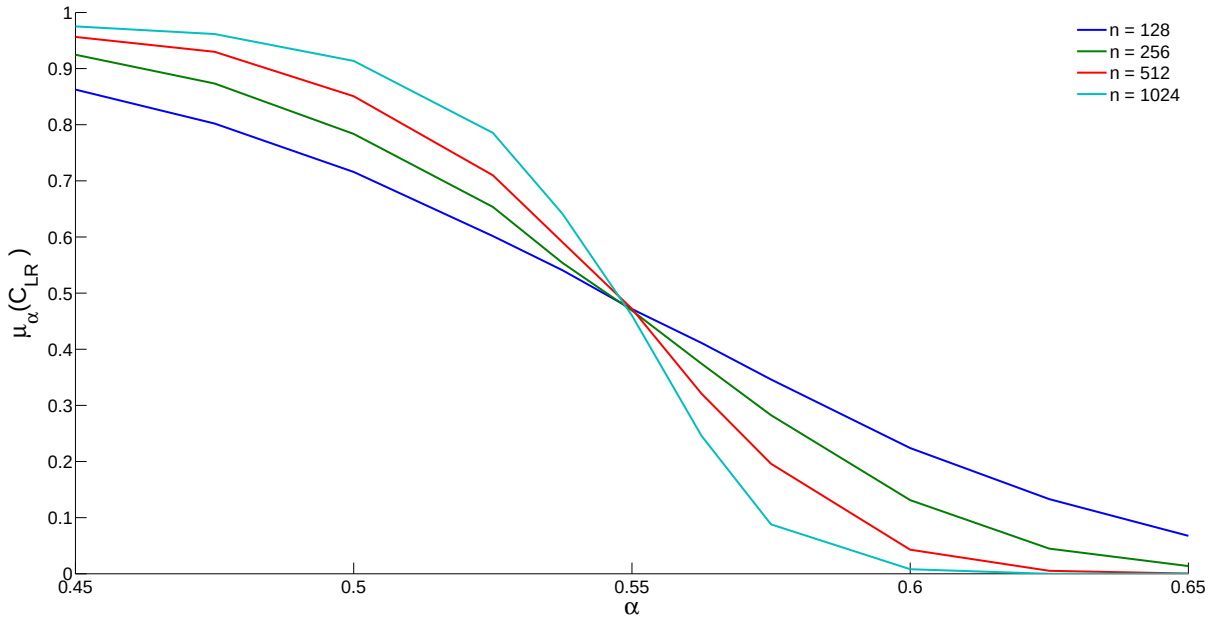


Figure 3.5.1: Monte Carlo simulations for the probability of a left–right crossing of a n by $n+1$ square grid when $n = 128$, 256, 512 and 1024. For each n at least 2500 simulations have been used for each data point.

say that for $0 < p < p_c$ or $p_c < p < 1$ there exits a $\lambda(p) > 0$ such that

$$\mathbb{P}_p(\{|C_0| = k\}) \prec \begin{cases} \mathrm{e}^{-\lambda(p)k} & \text{if } p < p_c \\ \mathrm{e}^{-\lambda(p)k^{\frac{d-1}{d}}} & \text{if } p > p_c \end{cases}.$$

Thus it is only at criticality, $p = p_c$, that large finite clusters are likely to exist. In this case it is believed that

$$\mathbb{P}_{p_c}(\{|C_0| = k\}) \asymp k^{-\gamma(d)},$$

for some constant $\gamma(d)$ which depends only on the dimension. It is thought that in 2 dimensions the exponent is $\gamma(2) = \frac{96}{91} \approx 1.055$, and that $\gamma(d)$ attains a mean field value of $\frac{3}{2}$ for dimension $d > 6$ (see [Gri10]).

A similar result is true for bond percolation on a d-ary tree where

$$\mathbb{P}_p(\{|C_0| = k\}) \begin{cases} \prec \mathrm{e}^{-\lambda(p)k} & \text{for } p \neq p_c \\ \asymp k^{-\frac{3}{2}} & \text{for } p = p_c \end{cases}.$$

However, as we discovered in Section 3.4, running PCF on a d-ary tree results in somewhat

different behaviour, and we have a power law distribution for the cluster size in both the critical and super-critical regimes. Indeed Theorem 3.4.1 tells us that

$$\mu_{\mathbf{T}_d,\alpha}(\{|C_0| = k\}) \begin{cases} \prec \mathrm{e}^{-\lambda(\alpha)k} & \text{for } \alpha > \alpha_c \\ \asymp k^{-\left(2-\frac{1}{2d}\right)} & \text{for } \alpha = \alpha_c \\ \asymp k^{-2} & \text{for } \alpha < \alpha_c \end{cases},$$

also demonstrating that the exponents in the critical and super-critical regimes differ.

This happens because in a super-critical regime the process must pass through some critical time t_c (depending on $\alpha < \alpha_c$) at which an infinite cluster will first appear. Since it is possible for large clusters to freeze at this time rather than merge into an infinite cluster we are left with large clusters in our final distribution. We account for the larger exponent in the super-critical case from the fact that the process only briefly passes through criticality, whereas for $\alpha = \alpha_c$ the length of time the process spends near criticality is much more significant, and so more large finite clusters will freeze. Super-critical PCF on $\mathbb{Z}^d$ must also pass through a critical time, and therefore it seems reasonable to expect that PCF on $\mathbb{Z}^d$ will behave in a similar way.

In the case of $\mathbb{Z}^2$, the shape of the super-critical cluster of Figure 3.1.3 (with its large voids) goes some way to showing that super-critical PCF contains large finite clusters, and so we investigate this further using Monte Carlo simulations on a finite square grid. Figure 3.5.2 presents our results as a log–log histogram.

The curves for $\alpha = 1.00$ and $\alpha = 0.70$ correspond to sub-critical PCF, $\alpha = 0.55$ corresponds to near critical PCF, and the values $\alpha = 0.40$ and $\alpha = 0.20$ correspond to super-critical PCF. Thus we see strong numerical evidence that the cluster size distribution of PCF on $\mathbb{Z}^2$ behaves in a similar way to that of PCF on a d-ary tree:

- The plots for $\alpha = 1.00$ and $\alpha = 0.70$ both curve downwards corresponding to exponential decay in the size of the components. Moreover this rate of decay is faster when the rate of freezing is faster as we would expect.
- The log-log plot for the near-critical $\alpha = 0.55$ appears to give a straight line, suggesting that the component size has a power law distribution. Also note that the gradient is slightly less than -1 and thus the exponent for $\mu_\alpha(\{|C_0| = k\})$ is slightly larger than 1 – as is also the case in bond percolation.
- The plots for $\alpha = 0.20$ and $\alpha = 0.40$ also follow straight lines, demonstrating that the cluster size in the super-critical case also obeys a power law distribution. We

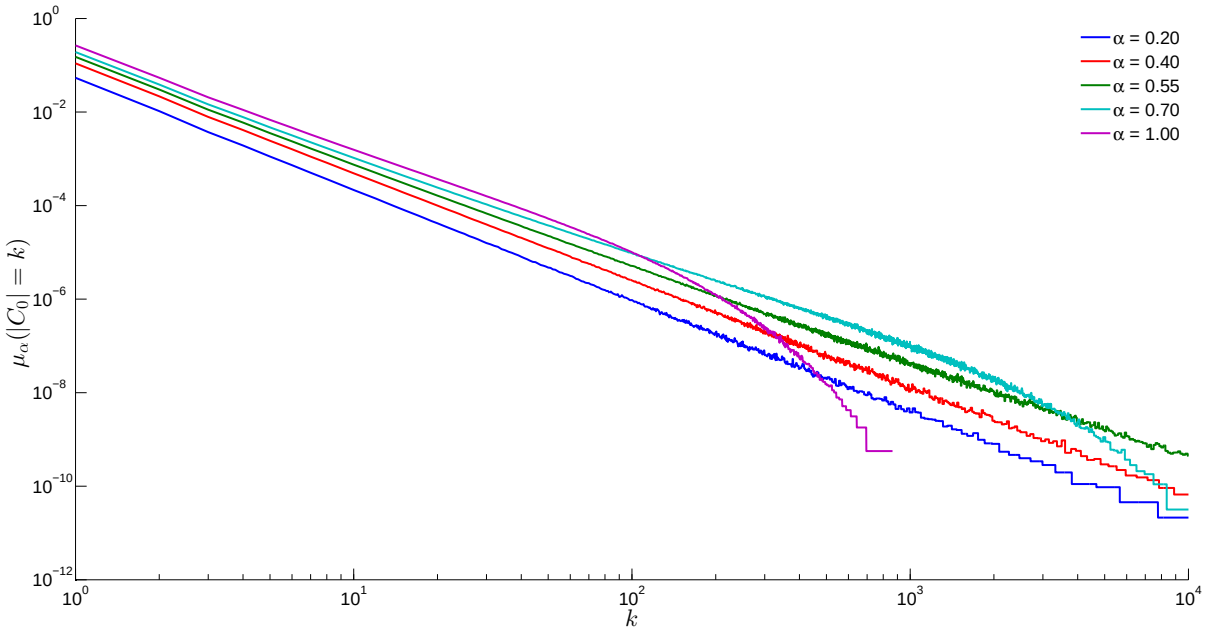


Figure 3.5.2: Histogram plot with log–log axis of the cluster size distribution when running PCF on a 1024 by 1024 square grid at rates $\alpha = 0.20, 0.40, 0.55, 0.70$ and 1.00. Note that the data has been compiled from 1000 simulations for each value of α, and the widths of the bars are set so that each represents at least 100 clusters.

notice further that the lines for $\alpha = 0.20$ and $\alpha = 0.40$ appear to have almost the same gradient as each other, but are both slightly steeper than in the near-critical regime. This suggests that we again have a larger exponent for $\mu_\alpha(\{|C_0| = k\})$ in the super-critical regime than at criticality.

We now recall that the decay exponent of PCF on a d-ary tree at criticality is $2 - \frac{1}{2d}$, since this depends on the dimension and is never constant for finite d it therefore seems reasonable to assume that critical PCF on $\mathbb{Z}^d$ will never reach a mean-field value. However, in the super-critical regime we still have a fixed exponent (of 2) on a tree, and therefore we would expect that in sufficiently large dimension super-critical PCF on $\mathbb{Z}^d$ will also exhibit mean field behaviour. Our observations therefore lead us to make the following conjecture.

Conjecture 3.5.1. *Suppose we run PCF on $\mathbb{Z}^d$ and let C_0 be the cluster containing the origin in the final distribution, then its size obeys*

$$\mu_{\mathbb{Z}^d,\alpha}(\{|C_0| = k\}) \begin{cases} \prec \mathrm{e}^{-\lambda(\alpha,d)k} & \text{for } \alpha > \alpha_c \\ \asymp k^{-\gamma(d)} & \text{for } \alpha = \alpha_c \\ \asymp k^{-\delta(d)} & \text{for } \alpha < \alpha_c \end{cases}.$$

Where $\gamma(d)$ never attains a mean field value but satisfies $\gamma(d) < \delta(d) \leq 2$ for all d, and δ is such that $\delta(d) = 2$ for d sufficiently large.

Chapter 4

A universal exponent for Brownian entropic repulsion

In this chapter we investigate the extent to which the phenomenon of Brownian entropic repulsion is universal. Consider a Brownian motion conditioned on the event $\mathcal{E}$ – that its local time is bounded everywhere by 1. This event has probability zero and so must be approximated by events of positive probability. We prove that several natural quantities, in particular the speed of the process, are highly sensitive to the approximation procedure, and hence are not universal. However, we also propose an exponent κ – which measures the strength of the entropic repulsion by evaluating the probability that a particular point comes close to violating the condition $\mathcal{E}$. We show that $\kappa = 3$ for several natural approximations of $\mathcal{E}$, and conjecture that $\kappa = 3$ is universal in a sense that we make precise.

This chapter is based on work appearing in [Mot14a].

4.1 Introduction

Suppose $(W_t)_{t\geq 0}$ is a Brownian motion conditioned to have bounded local time, $L_x(t) \leq 1$ for all $x \in \mathbb{R}$ and all $t \geq 0$ say. Under this conditioning W_t has a self avoiding nature, and so intuitively one would expect W_t to escape to infinity with positive speed. Moreover, we would expect this speed to be at least equal to 1 since that is precisely what it means to spend less than 1 unit of local time at a given level. Because it is relatively expensive for a Brownian motion to have positive velocity we might expect the limiting speed to be 1. However, since the local time of a Brownian motion can fluctuate wildly, the effect of *entropic repulsion* comes into play, and so the easiest way for the process to meet the global constraint $L_x(t) \leq 1$ is for it to have an average local time which is significantly less than 1. This means that the speed of the process must be strictly greater than 1.

In [BB10, Theorem 2] Benjamini and Berestycki make this argument precise in the following way. Set $\tau_a = \inf\{t : W_t \geq a\}$, and let

$$\mathcal{E}_a^* = \{L_x(t) \leq 1 \text{ for all } x \in \mathbb{R} \text{ and all } t \leq \tau_a\} = \{L_x(\tau_a) \leq 1 \text{ for all } x \in \mathbb{R}\}, \qquad (4.1.1)$$

then the conditioned Wiener measures $\mathbb{W}(\cdot \,|\, \mathcal{E}_a^*)$ converge weakly to a measure $\mathbb{Q}^*$ as $a \longrightarrow \infty$. Moreover, $\mathbb{Q}^*$-almost surely we have

$$\lim_{t\to\infty} \frac{W_t}{t} = \gamma^* = \frac{3}{1 - 2j_0^{-2}} = 4.586\ldots, \qquad (4.1.2)$$

where j_0 is the first zero of the Bessel function of the first kind, $\mathcal{J}_0(x)$.

It is clear that $\bigcap_{a>0} \mathcal{E}_a^* = \{L_x(t) \leq 1 \text{ for all } x \text{ and } t\}$. However, the event $\mathcal{E} = \{L_x(t) \leq 1$ for all $x \in \mathbb{R}$ and $t \geq 0\}$ can be realised as a limit of events with positive $\mathbb{W}$-probability in many other ways. Indeed, one could argue that it is more natural to fix T and condition on the events

$$\tilde{\mathcal{E}}_T^\bullet = \{L_x(t) \leq 1 \text{ for all } x \in \mathbb{R} \text{ and all } t \leq T\} = \{L_x(T) \leq 1 \text{ for all } x \in \mathbb{R}\}. \qquad (4.1.3)$$

By doing this we also get $\bigcap_{T>0} \tilde{\mathcal{E}}_T^\bullet = \{L_x(t) \leq 1 \text{ for all } x \text{ and } t\} = \mathcal{E}$.

Conditionally on $\mathcal{E}_a^*$ we have $W_{\tau_a} = a > 0$, and so $W_t \longrightarrow +\infty$, $\mathbb{Q}^*$-almost surely. However, when we condition on $\tilde{\mathcal{E}}_T^\bullet$ there is no preferred direction for W_t. For simplicity

we now restrict our attention to the case $W_T \geq 0$, and therefore we replace $\tilde{\mathcal{E}}_T^\bullet$ by

$$\mathcal{E}_T^\bullet = \{L_x(T) \leq 1 \text{ for all } x \in \mathbb{R} \text{ and } W_T \geq 0\}. \tag{4.1.4}$$

Having done this we can now prove the following.

Theorem 4.1.1. *The conditioned Wiener measures $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ converge weakly to a measure $\mathbb{Q}^\bullet$ as $T \longrightarrow \infty$. Moreover, $\mathbb{Q}^\bullet$-almost surely we have*

$$\lim_{t\to\infty} \frac{W_t}{t} = \gamma^\bullet, \tag{4.1.5}$$

where $1 < \gamma^\bullet < \gamma^*$.

Remark 4.1.2. By using symmetry we can reconstruct $\mathbb{W}(\cdot \,|\, \tilde{\mathcal{E}}_T^\bullet)$ from $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$. Therefore from Theorem 4.1.1 we can also deduce that the limit $\tilde{\mathbb{Q}}^\bullet = \lim_{T\to\infty} \mathbb{W}(\cdot \,|\, \tilde{\mathcal{E}}_T^\bullet)$ exists and that $\lim_{t\to\infty} \frac{W_t}{t} \in \{-\gamma^\bullet, \gamma^\bullet\}$ $\tilde{\mathbb{Q}}^\bullet$-almost surely.

Theorem 4.1.1 shows us that the limiting speed of the process is sensitive to the particular way that we condition on $\mathcal{E}$. Therefore it is clear that for a general set of events $\{\mathcal{E}'_a\}_{a>0}$, with $\mathcal{E} = \bigcap_a \mathcal{E}'_a$, no limiting process need exist. However, in Section 4.5 we shall suggest a general framework where – provided v is not too small – it is possible to use stopping times to construct a sequence of events of positive probability, $\{\mathcal{E}_a^v\}_{a>0}$, such that $\bigcap_{a>0} \mathcal{E}_a^v = \mathcal{E}$ and where the weak limit

$$\mathbb{Q}^v = \lim_{a\to\infty} \mathbb{W}(\cdot \,|\, \mathcal{E}_a^v) \tag{4.1.6}$$

exists. In each of theses cases we conjecture that $\lim_{t\to\infty} \frac{W_t}{t} = v$ in $\mathbb{Q}^v$-probability.

A calculation of the speed $v = \lim_{t\to\infty} \frac{W_t}{t}$ might perhaps seem like the most natural way of measuring the entropic repulsion phenomenon. However, we see from Theorem 4.1.1 that the value of v is highly sensitive to the approximation of $\mathcal{E}$ by events of positive probability. Therefore, in order to find a more universal way of quantifying Brownian entropic repulsion, we shall also consider how likely it is for $L_x(\infty) = \lim_{T\to\infty} L_x(T)$ – the local time at level x – to be close to 1. We can then prove the following.

Theorem 4.1.3. *There exists constants C^* and $C^\bullet$ such that*

$$\lim_{x\to\infty} \mathbb{Q}^*(L_x(\infty) > 1-\varepsilon) \sim C^*\varepsilon^3 \quad \textit{and} \quad \lim_{x\to\infty} \mathbb{Q}^\bullet(L_x(\infty) > 1-\varepsilon) \sim C^\bullet\varepsilon^3 \tag{4.1.7}$$

as $\varepsilon \longrightarrow 0$.

Similar behaviour is also present in the general framework of Section 4.5, and therefore we conjecture that this is a universal property of Brownian entropic repulsion.

4.1.1 Measuring the entropic repulsion of the Gaussian free field

In [BDG01] and [BDZ95] the authors consider the Gaussian free field, ϕ, with a hard wall at 0. It is shown that if the field is conditioned to be positive on some open set $D \subseteq (\mathbb{Z}/N\mathbb{Z})^d$, then the value of the field on D is typically of order $\log N$. This behaviour occurs because of the effect of entropic repulsion. In fact the easiest way for the field to satisfy the condition $\phi_x \geq 0$ for all $x \in D$ is for the field to experience a global shift of a size equal to the size of the largest fluctuations.

Since this paper shows that it is possible to measure the strength of Brownian entropic repulsion by looking at how likely it is for $L_x(\infty)$ to be close to 1, we also ask if there is anything similar that can be said for the Gaussian free field. In particular we pose the following question.

Question 1. Can we find scaling functions $f(N)$ and $g(N)$ such that

$$\frac{1}{f(N)}\,\mathbb{P}\left(\phi_y < \varepsilon\, g(N) \,|\, \phi_x > 0 \text{ for all } x \in D\right) \tag{4.1.8}$$

converges to a non-trivial function of ε as $N \longrightarrow \infty$? If so, then how does this function behave as $\varepsilon \longrightarrow 0$?

There is also a natural comparison between our work and the Brownian excursion. Therefore we also ask the following.

Question 2. Let $(B_t)_{t\in[0,1]}$ be a standard Brownian excursion with law $\mathbb{B}$, and suppose that $0 < x < 1$ is fixed. What can we say about $\mathbb{B}(B_x < \varepsilon)$? In particular is there a constant C_x (depending on x) such that $\mathbb{B}(B_x < \varepsilon) \sim C_x\,\varepsilon^3$ as $\varepsilon \longrightarrow 0$? If so then how does C_x depend on x?

4.1.2 Outline of the chapter

Section 4.2 is devoted to preliminary lemmas. In particular we find integral forms for the Donsker–Varadhan rate functions of BESQ^2 and BESQ^0 processes, and then use these to

find unique minimising measures of the local time process. From these calculations it is then possible to identify the limiting speed $\gamma^\bullet$.

Having done this preliminary work, the proofs of Theorem 4.1.1 and Theorem 4.1.3 are then contained in Section 4.3 and Section 4.4 respectively. It is worth noting that, once the limiting measure $\mathbb{Q}$ has been constructed, the proof of Theorem 4.1.3 is relatively simple. On the other hand, computing the limiting speed in Theorem 4.1.1 proves to be a much more arduous task.

The chapter closes with a discussion of how our results could be extended to a more general setting, see Section 4.5.

4.2 Calculating the rate function for the occupation measure of a square Bessel process

Recall that Theorem 2.3.3 tells us that the local time of a Brownian motion can be described in terms of conditioned BESQ^2 and BESQ^0 processes. If $(Y_x)_{x\geq 0}$ is such a process then its occupation measure at time T is defined by

$$L((Y_x), T, \cdot) = \frac{1}{T}\int_0^T \mathbb{1}_{\{Y_x \in \cdot\}}\,\mathrm{d}x.$$

The Donsker–Varadhan Theorem, Theorem 2.4.1, gives us a powerful tool for estimating the large deviations of the occupation measure in terms of a rate function I, see (2.4.2). From now on we shall use I_2 and I_0 to denote the Donsker–Varadhan rate functions for the occupation measures of BESQ^2 and BESQ^0 processes. We shall also use the abuse of notation $\mathbb{E}(\mu) = \int x\,\mathrm{d}\mu(x)$ to denote the expectation of the identity with respect to the measure μ. This section is then devoted to the proofs of the following three lemmas.

Lemma 4.2.1. *There is a unique probability measure μ^* with* $\mathrm{support}(\mu^*) \subseteq [0,1]$ *which minimises $I_2(\mu)$ over all measures μ with* $\mathrm{support}(\mu) \subseteq [0,1]$. *Furthermore, μ^* is such that $\mathbb{E}(\mu^*) = (\gamma^*)^{-1} = \frac{1}{3}(1 - 2j_0^{-2}) < 1$ and $I_2(\mu^*) = \frac{1}{2}j_0^2 > 0$. Here j_0 denotes the first zero of the Bessel function $\mathcal{J}_0$.*

Lemma 4.2.2. *There is a unique probability measure μ° with* $\mathrm{support}(\mu^\circ) \subseteq (0,1]$ *which minimises $\mathbb{E}(\mu)^{-1}I_0(\mu)$ over all measures μ with* $\mathrm{support}(\mu) \subseteq (0,1]$. *Furthermore, by explicit calculation we can show that μ° is defined by* $\dfrac{\mathrm{d}\mu^\circ}{\mathrm{d}x} = \dfrac{1}{Z}\dfrac{1}{x}\sin(\pi x)^2$ *where* $Z = \displaystyle\int_0^1 \frac{1}{x}\sin(\pi x)^2\,\mathrm{d}x$; *and that $\mathbb{E}(\mu^\circ)^{-1}I_2(\mu^\circ) = 2\pi^2$.*

Lemma 4.2.3. *There is a unique probability measure $\mu^\bullet$ with* $\mathrm{support}(\mu^\bullet) \subseteq [0,1]$ *which minimises $\mathbb{E}(\mu)^{-1}I_2(\mu)$ over all measures μ with* $\mathrm{support}(\mu) \subseteq [0,1]$. *Furthermore, $\mu^\bullet$ satisfies $\mathbb{E}(\mu^\bullet)^{-1}I_2(\mu^\bullet) < 2\pi^2$ and $\mathbb{E}(\mu^*) < \mathbb{E}(\mu^\bullet) < 1$. Here μ^* is as defined by Lemma 4.2.1.*

At this point we set $\gamma^\bullet = \mathbb{E}(\mu^\bullet)^{-1}$, $\gamma^* = \mathbb{E}(\mu^*)^{-1}$ and $\Gamma^\bullet = \mathbb{E}(\mu^\bullet)^{-1}I_2(\mu^\bullet)$. We also make the following definition.

Definition 4.2.4. Define $J : [0,1] \longrightarrow [0,\infty) \cup \{\infty\}$ by

$$J(\alpha) = \inf\left\{I_2(\mu) : \mathrm{support}(\mu) \subseteq [0,1] \text{ and } \mathbb{E}(\mu) = \alpha\right\}. \tag{4.2.1}$$

The proofs of the three lemmas are mainly calculation, and therefore can be omitted in a first reading of the chapter.

4.2.1 The rate function of BESQ^2 and BESQ^0 processes

In order to prove Lemma 4.2.1, Lemma 4.2.2 and Lemma 4.2.3 we shall first use (2.4.5) and (2.4.7) to find integral forms of I_2 and I_0. Since the square Bessel process of dimension d satisfies (2.3.2) then its infinitesimal generator is given by

$$\mathcal{L}_d f(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(4x\,\frac{\mathrm{d}}{\mathrm{d}x}\right) f(x) + (d-2)\frac{\mathrm{d}}{\mathrm{d}x} f(x). \tag{4.2.2}$$

When $d = 2$ then $\mathcal{L}_d$ is self-adjoint with respect to Lebesgue measure. Therefore we can use (2.4.5) to write down the Donsker–Varadhan rate function I_2 as

$$I_2(\mu) = \begin{cases} \left\|\sqrt{-\mathcal{L}}g\right\|_2^2, & \text{where } \dfrac{\mathrm{d}\mu}{\mathrm{d}x} \text{ exists and } g = \sqrt{\dfrac{\mathrm{d}\mu}{\mathrm{d}x}} \in \mathcal{D}_2 \\ \infty, & \text{otherwise} \end{cases} . \tag{4.2.3}$$

Here $\mathcal{D}_2$ denotes the domain of $\mathcal{L}_2$. Since $g \in \mathcal{D}_2$ implies that $g(x) \longrightarrow 0$ as $x \longrightarrow \infty$, we can integrate by parts to get

$$\begin{aligned} \left\|\sqrt{-\mathcal{L}_2}g\right\|_2^2 = \langle g, -\mathcal{L}_2 g\rangle_2 &= -\frac{1}{2}\int_0^\infty g(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(4x\frac{\mathrm{d}}{\mathrm{d}x}\right) g(x)\,\mathrm{d}x \\ &= \int_0^\infty 2x\left(\frac{\mathrm{d}}{\mathrm{d}x}g(x)\right)^2 \mathrm{d}x. \end{aligned} \tag{4.2.4}$$

However, when $d = 0$ then $\mathcal{L}_0$ is no longer self-adjoint and so we have to perform a change of measure and use equation (2.4.7) in order to calculate the rate function. We have the drift term $b(x) = -2$ and so solving $b(x) = a(x)\dfrac{\mathrm{d}}{\mathrm{d}x}Q(x)$ gives

$$Q(x) = \int^x -\frac{2}{4y}\,\mathrm{d}y = -\frac{1}{2}\log x + c.$$

Now set $\mathrm{d}\mu_{\mathrm{rev}} = \mathrm{e}^{2Q(x)}\mathrm{d}x = \mathrm{e}^{2c}\dfrac{1}{x}\mathrm{d}x$ for $x > 0$. Note that it does not matter that we have an unknown constant of integration since this cancels later. Suppose μ is a probability

measure supported on $(0,\infty)$ for which $g = \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}x}}$ exists, we then have

$$h(x) = \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}\mu_{\mathrm{rev}}}}(x) = \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}\mu_{\mathrm{rev}}}}(x) = g(x)\sqrt{\frac{\mathrm{d}x}{\mathrm{d}\mu_{\mathrm{rev}}}}(x) = g(x)\mathrm{e}^{-c}\sqrt{x}.$$

Therefore from (2.4.7) we get

$$\left\|\sqrt{-\mathcal{L}_0}h\right\|^2_{2,\mu_{\mathrm{rev}}} = \mathrm{e}^{2c}\frac{1}{2}\int_0^\infty 4x\left(\frac{\mathrm{d}}{\mathrm{d}x}h(x)\right)^2\frac{1}{x}\,\mathrm{d}x = \int_0^\infty 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}g(x)\right)^2\,\mathrm{d}x.$$

Hence if we write $\mathcal{D}_0$ for the domain of $\mathcal{L}_0$ then for each μ supported on $(0,\infty)$ we have

$$I_0(\mu) = \begin{cases} \displaystyle\int_0^\infty 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}g(x)\right)^2\,\mathrm{d}x, & \text{if } \dfrac{\mathrm{d}\mu}{\mathrm{d}x} \text{ exists and } g = \sqrt{\dfrac{\mathrm{d}\mu}{\mathrm{d}x}} \in \mathcal{D}_0 \\ \infty, & \text{otherwise} \end{cases}. \tag{4.2.5}$$

Remark 4.2.5. $Y_x = 0$ is an absorbing state for a BESQ^0 process and therefore we shall only be interested in occupation measures supported on $(0,\infty)$.

4.2.2 Proof of Lemma 4.2.1

Since (4.2.4) gives us an explicit form for $I_2(\mu)$, then it is clear that minimising $I_2(\mu)$ over $\{\mu : \mathrm{support}(\mu) \subseteq [0,1]\}$ is equivalent to finding a $g \in C^1([0,1])$ with $\|g\|_2 = 1$ and

$$\int_0^1 2x\left(\frac{\mathrm{d}}{\mathrm{d}x}g(x)\right)^2\,\mathrm{d}x = \inf\left\{\int_0^1 2x\left(\frac{\mathrm{d}}{\mathrm{d}x}h(x)\right)^2\,\mathrm{d}x : h \in C^1([0,1]) \text{ and } \|h\|_2 = 1\right\}. \tag{4.2.6}$$

Such a problem can be solved using the Euler–Lagrange equation. The interested reader is directed to [Boa06, Chapter 9] for an overview of the calculus of variations. In this case the additional constraint that $\|g\|_2 = 1$ can be included by adding the Lagrangian multiplier

$$\lambda\left(\int_0^1 g(x)^2\,\mathrm{d}x - 1\right). \tag{4.2.7}$$

Thus we have

$$\frac{\partial \mathcal{F}}{\partial g} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \mathcal{F}}{\partial g'}\right) = 0, \tag{4.2.8}$$

where $\mathcal{F}[x, g(x), g'(x), \lambda] = 2x\, g'(x)^2 + \lambda(g(x)^2 - 1)$, and so we get

$$2x\frac{\mathrm{d}^2}{\mathrm{d}x^2}g(x) + 2\frac{\mathrm{d}}{\mathrm{d}x}g(x) - \lambda g(x) = 0. \tag{4.2.9}$$

The solutions of this are precisely the eigenfunctions of $\mathcal{L}_2$ (with eigenvalue λ). We also have the additional constraints that $g(x) \geq 0$ for $x \in [0,1]$, because $g(x)$ is the positive square root of the Radon–Nikodym derivative of μ, and that $g(1) = 0$, because $g(x)$ must be continuous on $[0, \infty)$. Therefore there is a unique solution

$$g(x) = \begin{cases} \dfrac{\mathcal{J}_0(j_0\sqrt{x})}{\mathcal{J}_1(j_0)} & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{4.2.10}$$

Here $\mathcal{J}_n(x)$ are the Bessel functions of the first kind, and j_0 is the first zero of $\mathcal{J}_0(x)$. Thus by defining μ^* by $\dfrac{\mathrm{d}\mu^*}{\mathrm{d}x} = g(x)^2$ we get the unique minimiser of I_2 over $\{\mu : \text{support}(\mu) \subseteq [0,1]\}$.

By calculation we can now check that

$$\mathbb{E}(\mu^*) = \int_0^1 x\, \mathrm{d}\mu(x) = \int_0^1 x \left(\frac{\mathcal{J}_0(j_0\sqrt{x})}{\mathcal{J}_1(j_0)}\right)^2 \mathrm{d}x = \frac{1}{3}(1 - 2j_0^{-2}) = \frac{1}{\gamma^*},$$

and

$$I_2(\mu^*) = \int_0^1 2x \left(\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathcal{J}_0(j_0\sqrt{x})}{\mathcal{J}_1(j_0)}\right)^2 \mathrm{d}x = \frac{1}{2}j_0^2,$$

as claimed. □

Remark 4.2.6. By using Lemma 4.2.1 and the Donsker–Varadhan Theorem it is now possible to show the following.

Let $(Y_x)_{x\geq 0}$ be a BES$Q^2(y)$ process for some $0 \leq y < 1$, and let $\mathbb{Y}_y^T$ be the law of $(Y_x)_{x\geq 0}$ conditioned on the event $\{Y_x \leq 1 \text{ for all } x \leq T\}$. If we use $\mathbb{E}_{\mathbb{Y}_y^T}(X)$ to denote the expectation under this measure of a random variable $X \geq 0$. Then

$$\lim_{t\to\infty}\lim_{T\to\infty} \mathbb{E}_{\mathbb{Y}_y^T}(Y_t) = \frac{1}{\gamma^*}. \tag{4.2.11}$$

Therefore this method provides an alternative way of proving [BB10, Lemma 9] – a central component in the proof of ballistic behaviour in Benjamini and Berestycki's paper.

4.2.3 Proof of Lemma 4.2.2

We may assume that $g(x) = \sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}x}}(x)$ exists and is piecewise differentiable on $(0,1]$ and that $\lim_{x\to 0} g(x) = 0$ and $g(1) = 0$. If not then $I_0(\mu) = \infty$. From (4.2.5) we then have

$$I_0(\mu) = \int_0^1 2\left(\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{x}g(x)\right)^2 \mathrm{d}x \quad \text{and} \quad \mathbb{E}(\mu) = \int_0^1 xg(x)^2\,\mathrm{d}x.$$

Letting $h(x) = \sqrt{x}g(x)$ this becomes

$$I_0(\mu) = 2\int_0^1 h'(x)^2\,\mathrm{d}x \quad \text{and} \quad \mathbb{E}(\mu) = \int_0^1 h(x)^2\,\mathrm{d}x.$$

We now claim that if $f : [0,1] \longrightarrow \mathbb{R}$ is a C^1 function such that $f(0) = f(1) = 0$ then

$$\pi^2 \int_0^1 |f(x)|^2\,\mathrm{d}x \leq \int_0^1 |f'(x)|^2\,\mathrm{d}x, \tag{4.2.12}$$

with equality if and only if $f(x) = c\sin(\pi x)$ for some $c \neq 0$. To prove this we note that because f is periodic and C^1 then it can be written as a Fourier series,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \quad \text{and} \quad f'(x) = \sum_{n=1}^{\infty} n\pi a_n \sin(n\pi x).$$

Parseval's identity now tells us that

$$\int_0^1 |f(x)|^2\,\mathrm{d}x = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \quad \text{and} \quad \int_0^1 |f'(x)|^2\,\mathrm{d}x = \frac{1}{2}\pi^2\sum_{n=1}^{\infty} n^2 a_n^2,$$

from which (4.2.12) is clear. Furthermore we can only have equality when $a_1 \neq 0$ and $a_n = 0$ for all $n > 1$, implying that $f(x) = c\sin(\pi x)$ for some $c \neq 0$.

The lemma now follows since $g(0)$ is finite implying that $h(0) = 0$. Therefore

$$I_0(\mu) = 2\int_0^1 h'(x)^2\,\mathrm{d}x \geq 2\pi^2 \int_0^1 h(x)^2\,\mathrm{d}x = 2\pi^2\mathbb{E}(\mu),$$

with equality if and only if $\frac{\mathrm{d}\mu}{\mathrm{d}x} = c^2\frac{1}{x}\sin(\pi x)^2$. Here the requirement that $\int_0^1 \frac{\mathrm{d}\mu}{\mathrm{d}x}\,\mathrm{d}x = 1$

tells us that $c^2 = \frac{1}{Z} = \left(\int_0^1 \frac{1}{x} \sin(\pi x)^2 \, \mathrm{d}x \right)^{-1}$. □

Remark 4.2.7. The inequality given by (4.2.12) is a special case of the Poincaré inequality, and is sometimes known as Wirtinger's inequality.

4.2.4 Proof of Lemma 4.2.3

Lemma 4.2.3 is proved by studying the properties of the function J. See Definition 4.2.4.

Lemma 4.2.8. *J has the following properties.*

1. *$J(\alpha) < \infty$ if and only if $\alpha \in (0,1)$.*
2. *For each $\alpha \in [0,1]$ there is a unique μ_α with $\mathrm{support}(\mu_\alpha) \subseteq [0,1]$, $\mathbb{E}(\mu_\alpha) = \alpha$ and $I_2(\mu_\alpha) = J(\alpha)$.*
3. *J is strictly convex on $[0,1]$.*
4. *J is continuously differentiable on $(0,1)$.*
5. *J has a unique minimum at $\alpha = (\gamma^*)^{-1}$, with $J\left((\gamma^*)^{-1}\right) > 0$.*
6. *$J(\alpha) \longrightarrow \infty$ as $\alpha \longrightarrow 1$.*
7. *The function $vJ\left(v^{-1}\right)$ has a unique minimum at $v = \gamma^\bullet$, with $1 < \gamma^\bullet < \gamma^*$. What is more this minimum satisfies $\gamma^\bullet J\left((\gamma^\bullet)^{-1}\right) = \Gamma^\bullet < 2\pi^2$.*

Proof. We shall prove these claims sequentially.

1. The only probability measures supported on $[0,1]$ with expectation 0 or 1 are the point masses δ_0 and δ_1. Neither of these have a Radon–Nikodym derivative so we have $J(0) = I_2(\delta_0) = \infty$ and $J(1) = I_2(\delta_1) = \infty$. To show $J(\alpha) < \infty$ for each $\alpha \in (0,1)$ if suffices to find a measure μ_α with $\mathrm{support}(\mu_\alpha) \subseteq [0,1]$, $\mathbb{E}(\mu_\alpha) = \alpha$ and $I_2(\mu_\alpha) < \infty$.

   For a fixed α let $\tilde{\alpha} = \min\{\alpha, 1-\alpha\}$ and define $g_\alpha : \mathbb{R} \longrightarrow [0,\infty)$ to be the piecewise linear function with $g_\alpha(x) = 0$ for $x \leq \alpha - \tilde{\alpha}$ and $x \geq \alpha + \tilde{\alpha}$, and $g_\alpha(\alpha) = \sqrt{\frac{3}{2\tilde{\alpha}}}$. From this we can define the probability measure μ_α by $\frac{\mathrm{d}\mu_\alpha}{\mathrm{d}x} = g_\alpha^2$. This measure satisfies our conditions since $\mathrm{support}(\mu_\alpha) = [\alpha - \tilde{\alpha}, \alpha + \tilde{\alpha}] \subseteq [0,1]$, $\mathbb{E}(\mu_\alpha) = \alpha$ and

$$I_2(\mu_\alpha) = \int_0^1 2x \left(\frac{\mathrm{d}}{\mathrm{d}x} g_\alpha(x) \right)^2 \mathrm{d}x \leq 2 \left(\frac{1}{\tilde{\alpha}} \sqrt{\frac{3}{2\tilde{\alpha}}} \right)^2 < \infty,$$

as required.

2. The claim is trivial for $\alpha = 0$ or $\alpha = 1$, so fix $\alpha \in (0,1)$ and find a sequence of measures $\{\mu_{\alpha,j}\}_{j\geq 1}$ with $\text{support}(\mu_{\alpha,j}) \subseteq [0,1]$, $\mathbb{E}(\mu_{\alpha,j}) = \alpha$ for each j, and such that $I_2(\mu_{\alpha,j}) \longrightarrow J(\alpha)$ as $j \longrightarrow \infty$. Because $[0,1]$ is compact we can take a weakly convergent subsequence $\mu_{\alpha,j_k} \longrightarrow \mu_\alpha$ say. Since I_2 is lower semi-continuous we must have $I_2(\mu_\alpha) \leq J(\alpha)$. Therefore because we must also have $\text{support}(\mu_\alpha) \subseteq [0,1]$ and $\mathbb{E}(\mu_\alpha) = \alpha$, then in fact $I_2(\mu_\alpha) = J(\alpha)$. Thus it follows that for each $\alpha \in [0,1]$ the infimum of $I_2(\mu)$ over the set

$$\{\mu \in \mathcal{P}(\mathbb{R}) : \text{support}\{\mu\} \subseteq [0,1] \text{ and } \mathbb{E}(\mu) = \alpha\}$$

is attained. Now to show uniqueness we start with the claim that for any two probability measures μ_1 and μ_2 which are supported on $[0,1]$, and for any $\lambda, \tilde{\lambda} \in (0,1)$ with $\lambda + \tilde{\lambda} = 1$ we have

$$I_2(\lambda\mu_1 + \tilde{\lambda}\mu_2) \leq \lambda I_2(\mu_1) + \tilde{\lambda} I_2(\mu_2), \tag{4.2.13}$$

with equality if and only if $\mu_1 = \mu_2$.

To proves this we first assume that μ_1 and μ_2 have respective Radon–Nikodym derivatives m_1 and m_2, otherwise the right hand side is infinite, and then use the form of I_2 given by (4.2.4) to calculate explicitly.

$$\begin{aligned} I_2(\lambda\mu_1 + \tilde{\lambda}\mu_2) &= \int_0^1 2x \left(\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{\lambda m_1(x) + \tilde{\lambda} m_2(x)} \right)^2 \mathrm{d}x \\ &= \int_0^1 \frac{x}{2} \frac{\left(\lambda m_1'(x) + \tilde{\lambda} m_2'(x)\right)^2}{\lambda m_1(x) + \tilde{\lambda} m_2(x)} \, \mathrm{d}x, \end{aligned} \tag{4.2.14}$$

and

$$\begin{aligned} \lambda I_2(\mu_1) + \tilde{\lambda} I_2(\mu_2) &= \int_0^1 2x \left(\lambda \left(\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{m_1(x)} \right)^2 + \tilde{\lambda} \left(\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{m_2(x)} \right)^2 \right) \mathrm{d}x \\ &= \int_0^1 \frac{x}{2} \frac{\lambda m_2(x) m_1'(x)^2 + \tilde{\lambda} m_1(x) m_2'(x)^2}{m_1(x) m_2(x)} \, \mathrm{d}x. \end{aligned} \tag{4.2.15}$$

By subtracting (4.2.14) from (4.2.15) we get

$$\lambda I_2(\mu_1) + \tilde{\lambda} I_2(\mu_2) - I_2(\lambda\mu_1 + \tilde{\lambda}\mu_2) = \int_0^1 \frac{x}{2} \frac{\lambda\tilde{\lambda}\Big(m_2(x)m_1'(x) - m_1(x)m_2'(x)\Big)^2}{m_1(x)m_2(x)\Big(\lambda m_1(x) + \tilde{\lambda} m_2(x)\Big)}\,\mathrm{d}x, \tag{4.2.16}$$

which is greater than or equal to 0 with equality if and only if $m_2(x)m_1'(x) - m_1(x)m_2'(x) = 0$ for all $x \in [0,1]$. This would imply that m_2 is a constant multiple of m_1 and thus $\mu_1 = \mu_2$. Therefore the claim follows.

Having shown this, now suppose that μ_1 and μ_2 are two measures with $\mathbb{E}(\mu_1) = \mathbb{E}(\mu_2) = \alpha$ and $I_2(\mu_1) = I_2(\mu_2) = J(\alpha)$. Then because $\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ also has $\mathbb{E}\left(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2\right) = \alpha$, we then know that

$$J(\alpha) \leq I_2\left(\tfrac{1}{2}\mu_1 + \tfrac{1}{2}\mu_2\right) \leq \tfrac{1}{2}I_2(\mu_1) + \tfrac{1}{2}I_2(\mu_2) = J(\alpha).$$

Hence we must have equality throughout, implying that $\mu_1 = \mu_2$.

3. Let $0 \leq \alpha_1 < \alpha_2 \leq 1$. Then we need to show that $J(\lambda\alpha_1 + \tilde{\lambda}\alpha_2) < \lambda J(\alpha_1) + \tilde{\lambda} J(\alpha_2)$ for each $\lambda, \tilde{\lambda} \in (0,1)$ with $\lambda + \tilde{\lambda} = 1$. To do this we first find μ_1 and μ_2 with $I_2(\mu_1) = J(\alpha_1)$, $I_2(\mu_2) = J(\alpha_2)$, $\mathbb{E}(\mu_1) = \alpha_1$ and $\mathbb{E}(\mu_2) = \alpha_2$. Now $\mathbb{E}(\lambda\mu_1 + \tilde{\lambda}\mu_2) = \lambda\alpha_1 + \tilde{\lambda}\alpha_2$, and so from (4.2.13) we know that

$$J(\lambda\alpha_1 + \tilde{\lambda}\alpha_2) \leq I_2(\lambda\mu_1 + \tilde{\lambda}\mu_2) < \lambda I_2(\mu_1) + \tilde{\lambda} I_2(\mu_2) = \lambda J(\alpha_1) + \tilde{\lambda} J(\alpha_2).$$

The second inequality is strict since μ_1 and μ_2 have different means and therefore are not equal. As this holds for each $\lambda, \tilde{\lambda} \in (0,1)$ with $\lambda + \tilde{\lambda} = 1$ and for each $0 \leq \alpha_1 < \alpha_2 \leq 1$ the claim is proved.

4. Because J is convex and finite for each $\alpha \in (0,1)$, then it must also be continuous on $(0,1)$. Moreover, convexity implies that J has left and right derivatives $\partial_- J$ and $\partial_+ J$. In order to prove that J is differentiable it suffices to show that these are always equal. We shall do this by showing that at each point $\alpha \in (0,1)$ there exists a neighbourhood $\mathcal{N}_\alpha$ on which we can construct a differentiable function f_α, with $f_\alpha(\alpha) = J(\alpha)$ and $f_\alpha(\beta) \geq J(\beta)$ for all $\beta \in \mathcal{N}_\alpha$. Having done this we must then have $\partial_- J(\alpha) \geq f_\alpha'(\alpha)$ and $\partial_+ J(\alpha) \leq f_\alpha'(\alpha)$. However, because J is convex we also have $\partial_- J(\alpha) \leq \partial_+ J(\alpha)$, and so $\partial_- J(\alpha) = \partial_+ J(\alpha) = f_\alpha'(\alpha)$. Whence we see that J is differentiable at α with $J'(\alpha) = f_\alpha'(\alpha)$. Since the convexity of J also implies that $J'(\alpha)$ is monotonically increasing then it must also be the case that J' is continuous.

Given a fixed $\alpha \in (0,1)$ we now find the unique probability measure μ_α supported

on $[0,1]$ with $\mathbb{E}(\mu_\alpha) = \alpha$ and such that $I_2(\mu_\alpha) = J(\alpha)$. Because $I_2(\mu_\alpha) < \infty$ then we know μ_α has a Radon–Nikodym derivative, $\dfrac{\mathrm{d}\mu_\alpha}{\mathrm{d}x} = m_\alpha(x)$ say. For each $\xi \in (-1,1)$ we also define a bijection $\varphi_\xi : [0,1] \longrightarrow [0,1]$ by $\varphi_\xi(x) = x^{1+\xi}$. Using these we can then define a collection of measures $\{\mu_{\alpha,\xi}\}_{\xi\in(-1,1)}$ by $\mu_{\alpha,\xi}(A) = \mu_\alpha(\varphi_\xi^{-1}(A))$ for each measurable set $A \subseteq [0,1]$. Note that each $\mu_{\alpha,\xi}$ has a Radon-Nikodym derivative given by $\dfrac{\mathrm{d}\mu_{\alpha,\xi}}{\mathrm{d}x}(x) = \dfrac{1}{(\varphi_\xi^{-1})'(x)} m_\alpha(\varphi_\xi^{-1}(x))$.

Now observe that $\mathbb{E}(\mu_{\alpha,\xi})$ is a continuously differentiable function of ξ, and that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\mathbb{E}(\mu_{\alpha,\xi}) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\xi} x^{\frac{1}{1+\xi}}\, \mathrm{d}\mu_\alpha(x) = \int_0^1 \frac{1}{(1+\xi)^2} x^{\frac{1}{1+\xi}} \log\frac{1}{x}\, \mathrm{d}\mu_\alpha(x).$$

Since $\left.\dfrac{\mathrm{d}}{\mathrm{d}\xi}\mathbb{E}(\mu_{\alpha,\xi})\right|_{\xi=0} > 0$, then there is some neighbourhood $\mathcal{N}_\alpha$ of α on which a continuously differentiable inverse to $\xi \longmapsto \mathbb{E}(\mu_{\alpha,\xi})$ exists. We call this inverse $i_\alpha : \mathcal{N}_\alpha \longrightarrow (-1,1)$, and define $f_\alpha : \mathcal{N}_\alpha \longrightarrow \mathbb{R} \cup \{\infty\}$ by $f_\alpha(\beta) = I_2(\mu_{\alpha,i_\alpha(\beta)})$. Now observe that $f_\alpha(\alpha) = I_2(\mu_{\alpha,0}) = J(\alpha)$, and because $\mathbb{E}(\mu_{\alpha,i_\alpha(\beta)}) = \beta$ then $f_\alpha(\beta) = I_2(\mu_{\alpha,i_\alpha(\beta)}) \geq J(\beta)$. Moreover, for $\beta \in \mathcal{N}_\alpha$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\beta} f_\alpha(\beta) = \frac{\mathrm{d}}{\mathrm{d}\beta} I_2(\mu_{\alpha,i_\alpha(\beta)}) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\beta}\, 2x \left(\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{\frac{m_\alpha(\varphi^{-1}_{i_\alpha(\beta)}(x))}{(\varphi^{-1}_{i_\alpha(\beta)})'(x)}} \right)^2 \mathrm{d}x.$$

Thus f_α is differentiable on $\mathcal{N}_\alpha$, and so we can conclude that J is differentiable at α. Since α was arbitrary it then follows that J is continuously differentiable on $(0,1)$ as claimed.

5. Because of the way that J is defined then this is an immediate consequence of Lemma 4.2.1.

6. Fix $\varepsilon > 0$ and suppose we have a probability measure μ with $\mathrm{support}(\mu) \subseteq [0,1]$ and $\mathbb{E}(\mu) \geq 1 - \varepsilon^2$. Since $\mathbb{E}(\mu) \leq (1-\varepsilon)\mu([0,1-\varepsilon]) + \mu((1-\varepsilon,1])$ we must have $\mu([1-\varepsilon,1]) \geq 1-\varepsilon$. We can now use this fact to give a lower bound for $I_2(\mu)$.

   For each $\delta > 0$ define $u_{\varepsilon,\delta} : [0,1] \to (0,\infty)$ by

$$u_{\varepsilon,\delta}(x) = \begin{cases} 1+\delta & \text{for } 0 \leq x \leq 1-\varepsilon \\ \sin\left(\dfrac{\pi}{2\varepsilon}(1-x)\right) + \delta & \text{for } 1-\varepsilon < x \leq 1 \end{cases},$$

   and now recall that the infinitesimal generator for a BESQ^2 process is given by

$$\mathcal{L}_2 f(x) = 2x\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + 2\frac{\mathrm{d}}{\mathrm{d}x} f(x).$$

Now observe that if we set $u_\varepsilon(x) = \lim_{\delta\to 0} u_{\varepsilon,\delta}(x)$ then

$$\frac{\mathcal{L}_2 u_\varepsilon(x)}{u_\varepsilon(x)} \leq \begin{cases} 0 & \text{for } 0 \leq x \leq 1-\varepsilon \\ -2(1-\varepsilon)\left(\frac{\pi}{2\varepsilon}\right)^2 & \text{for } 1-\varepsilon < x \leq 1 \end{cases}.$$

Therefore because each $u_{\varepsilon,\delta}$ is in the domain of $\mathcal{L}_2$ and each $u_{\varepsilon,\delta}$ is strictly positive on $[0,1]$ we can use (2.4.2) to get that

$$\begin{aligned} I_2(\mu) &= -\inf_{u\in,\mathcal{D},u>0} \int_0^1 \frac{\mathcal{L}_2(u)}{u} \mathrm{d}\mu \geq \lim_{\delta\to 0} -\int_0^1 \frac{\mathcal{L}_2(u_{\varepsilon,\delta})}{u_{\varepsilon,\delta}} \mathrm{d}\mu \\ &\geq \int_{1-\varepsilon}^1 2(1-\varepsilon)\left(\frac{\pi}{2\varepsilon}\right)^2 \mathrm{d}\mu(x) \geq 2(1-\varepsilon)^2\left(\frac{\pi}{2\varepsilon}\right)^2. \end{aligned}$$

Because this holds for each μ with $\mathrm{support}(\mu) \subseteq [0,1]$ and $\mathbb{E}(\mu) \geq 1-\varepsilon^2$ we must have $J(1-\varepsilon^2) \geq \frac{\pi^2}{2}\frac{(1-\varepsilon)^2}{\varepsilon^2}$ for each $\varepsilon > 0$. It therefore follows that $J(\alpha) \longrightarrow \infty$ as $\alpha \longrightarrow 1$.

7. Suppose that $\alpha^{-1}J(\alpha)$ has minima at distinct points $0 \leq \alpha_1 < \alpha_2 \leq 1$, and set $\tilde{\alpha} = \frac{1}{2}(\alpha_1+\alpha_2)$. Now because J is strictly convex we have

$$\begin{aligned} \frac{1}{\tilde{\alpha}} J(\tilde{\alpha}) &= \frac{2}{\alpha_1+\alpha_2} J\left(\frac{\alpha_1+\alpha_2}{2}\right) < \frac{1}{\alpha_1+\alpha_2}(J(\alpha_1)+J(\alpha_2)) \\ &= \frac{1}{\alpha_1+\alpha_2}\left(J(\alpha_1)+\frac{\alpha_2}{\alpha_1}J(\alpha_1)\right) = \frac{1}{\alpha_1}J(\alpha_1), \end{aligned}$$

contradicting the fact that α_1 is a minimum. Therefore the map $v \longmapsto vJ\left(v^{-1}\right)$ must attain a unique minimum at some point $1 \leq \gamma^\bullet \leq \infty$. Furthermore, since claim 6 tells us that $J(\alpha) \longrightarrow \infty$ as $\alpha \nearrow 1$ then we know that $vJ\left(v^{-1}\right) \longrightarrow \infty$ as $v \searrow 1$ implying that $\gamma^\bullet > 1$.

To check that $\gamma^\bullet < \gamma^*$ we now use the fact that J is continuously differentiable (claim 4) and so

$$\frac{\mathrm{d}}{\mathrm{d}v} vJ\left(v^{-1}\right) = J\left(v^{-1}\right) + v\frac{\mathrm{d}}{\mathrm{d}v}J\left(v^{-1}\right) = J\left(v^{-1}\right) - \frac{1}{v}J'\left(v^{-1}\right).$$

Because J is convex then we know that $\frac{\mathrm{d}}{\mathrm{d}v}J\left(v^{-1}\right)$ is increasing in v. Therefore, as γ^* is the minimum of J, it follows that $\frac{\mathrm{d}}{\mathrm{d}v}vJ\left(v^{-1}\right)$ is increasing for $v \geq \gamma^*$. Evaluating at γ^* gives

$$\frac{\mathrm{d}}{\mathrm{d}v} vJ\left(v^{-1}\right)\Big|_{v=\gamma^*} = J\left((\gamma^*)^{-1}\right) - \frac{1}{\gamma^*}J'\left((\gamma^*)^{-1}\right) = J\left((\gamma^*)^{-1}\right) > 0.$$

Here $J'\left((\gamma^*)^{-1}\right) = 0$ because $(\gamma^*)^{-1}$ is the unique minimum of J. Thus the derivative of $vJ\left(v^{-1}\right)$ is strictly positive for $v \geq \gamma^*$, and so its minimum must occur at a point $\gamma^\bullet < \gamma^*$.

Finally, to bound $\gamma^\bullet J\left((\gamma^\bullet)^{-1}\right)$ we recall the explicit values for γ^* and $J\left((\gamma^*)^{-1}\right)$ from Lemma 4.2.1. From these we get

$$\gamma^\bullet J\left((\gamma^\bullet)^{-1}\right) < \gamma^* J\left((\gamma^*)^{-1}\right) = \frac{3}{1-2j_0^{-2}}\frac{1}{2}j_0^2 < 2\pi^2.$$

Here j_0 denotes the first zero of the Bessel function $\mathcal{J}_0$, and $\dfrac{3}{1-2j_0^{-2}}\dfrac{1}{2}j_0^2 \approx 13.26$. □

To prove Lemma 4.2.3 we now observe that if we put $\mu^\bullet = \mu_\alpha$ with $\alpha = (\gamma^\bullet)^{-1}$ then for each μ supported on $[0,1]$ we have

$$\mathbb{E}(\mu)^{-1} I_2(\mu) \geq \mathbb{E}(\mu)^{-1} J(\mathbb{E}(\mu)) \geq \gamma^\bullet J\left((\gamma^\bullet)^{-1}\right) = \mathbb{E}(\mu^\bullet)^{-1} I_2(\mu^\bullet) = \Gamma^\bullet,$$

with equality if and only if $\mu = \mu^\bullet$. □

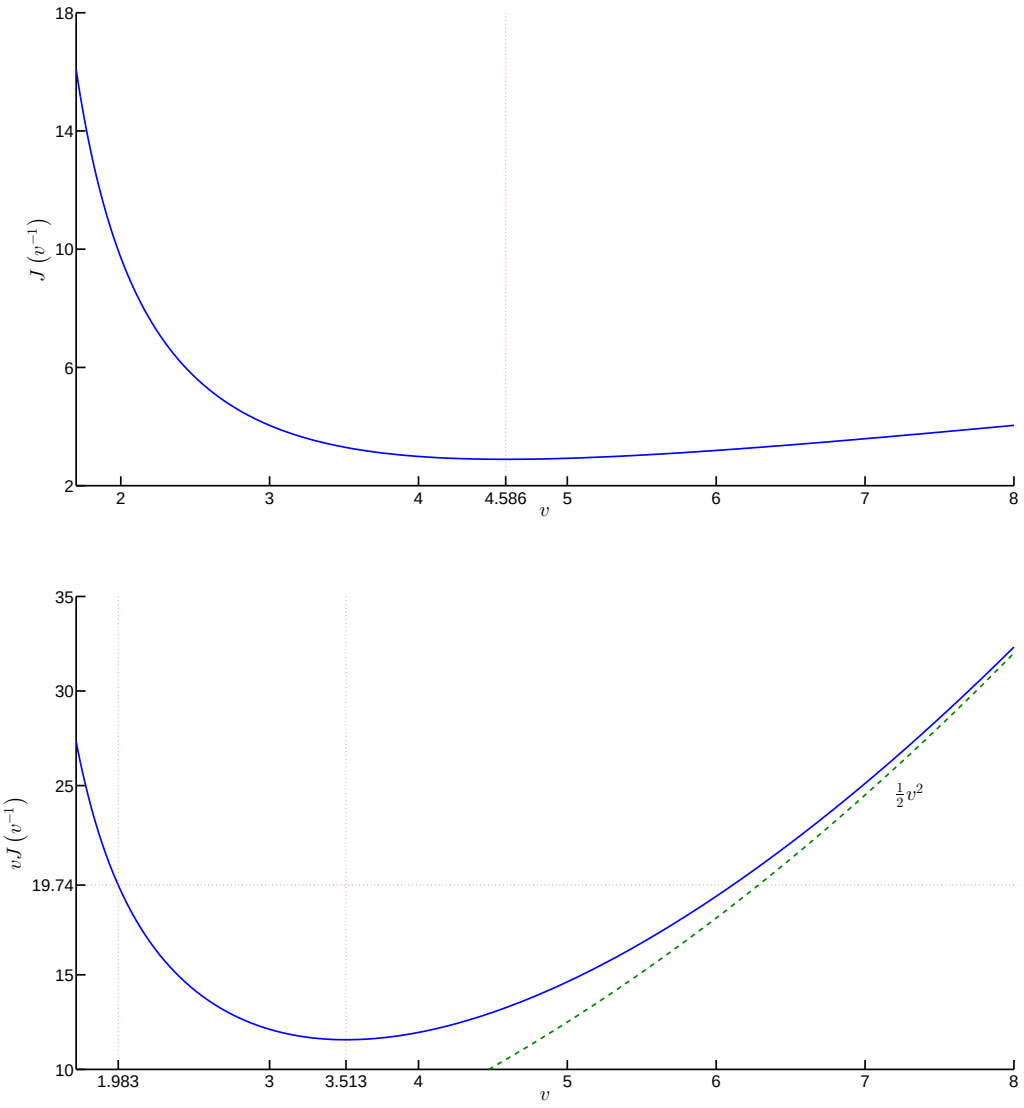


Figure 4.2.1: Plots of $J(v^{-1})$ and $vJ(v^{-1})$ against v. Although we can not find a closed form for $J(\alpha)$, by using the by solving the Euler–Lagrange equations it is possible to plot $J(\alpha)$ by numerical methods. Observe that $J(v^{-1})$ has a unique minimum at $\gamma^* \approx 4.586$ and that the unique minimum of $vJ(v^{-1})$ is attained at $\gamma^\bullet \approx 3.513 < \gamma^*$. Note also that the smallest root of $vJ(v^{-1}) = 2\pi^2 \approx 19.74$ is at $\gamma^\circ \approx 1.983$. This value represents the minimal velocity for which we believe Conjecture 4.5.2 holds.

In Section 4.5 we describe $vJ(v^{-1})$ as being the exponential cost of a Brownian motion spending a unit of time with ballistic rate v and maintaining $L_x(t) \leq 1$. Because (4.5.2) shows that the unit time cost for a Brownian motion to have ballistic rate v is $\frac{1}{2}v^2$, then this must be a lower bound for $vJ(v^{-1})$.

4.3 Proof of Theorem 4.1.1

The proof of Theorem 4.1.1 divides into two halves. In the first we find estimates for $\mathbb{W}(W_t \,|\, \mathcal{E}_T^\bullet)$ – from which we can deduce that the limiting process is ballistic. The second half then deals with the weak convergence of $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ as $T \longrightarrow \infty$.

4.3.1 Ballistic behaviour

Lemma 4.2.3 tells us that there is a unique measure $\mu^\bullet$ which minimises $\mathbb{E}(\mu)^{-1} I_2(\mu)$ over all probability measures supported on $[0,1]$. We know that $\mathbb{E}(\mu^\bullet)^{-1} = \gamma^\bullet$ where $1 < \gamma^\bullet < \gamma^*$, and so our aim is to use this fact to show that for each $\varepsilon > 0$ there is a $T_\varepsilon > 0$ such that for all $T_\varepsilon \leq t \leq T$ we have

$$\mathbb{W}\left(\left|\frac{W_t}{t} - \gamma^\bullet\right| \geq \varepsilon \,|\, \mathcal{E}_T^\bullet\right) < \varepsilon. \tag{4.3.1}$$

Provided the limit $\mathbb{Q}^\bullet = \lim_{T\to\infty} \mathbb{W}(\cdot \,|\, \mathcal{E}_T)$ exists – which we will show in Section 4.3.2 – we can then deduce that

$$\lim_{t\to\infty} \frac{W_t}{t} = \gamma^\bullet \tag{4.3.2}$$

in $\mathbb{Q}^\bullet$-probability. Since the left hand side of (4.3.1) can be written as

$$\mathbb{W}\left(\left|\frac{W_t}{t} - \gamma^\bullet\right| \geq \varepsilon \,|\, \mathcal{E}_T^\bullet\right) = \frac{\mathbb{W}(\mathcal{E}_T^\bullet \cap \{|W_t - \gamma^\bullet t| \geq \varepsilon t\})}{\mathbb{W}(\mathcal{E}_T^\bullet)}, \tag{4.3.3}$$

then our goal is to show that the ratio between $\mathbb{W}(\mathcal{E}_T^\bullet \cap \{|W_t - \gamma^\bullet t| \geq \varepsilon t\})$ and $\mathbb{W}(\mathcal{E}_T^\bullet)$ is small when t and T are large. For this we rely on the Ray–Knight Theorems and the calculations of Section 4.2. Recall that in Section 2.3 we defined $S_T^- = \int_0^T \mathbb{1}_{\{W_s<0\}}\,\mathrm{d}s$ and $S_T^+ = \int_0^T \mathbb{1}_{\{W_s>W_T\}}\,\mathrm{d}s$. Theorem 2.3.3 tells us that the local time profile $(L_x(T))_{x\in\mathbb{R}}$ can be described by two BESQ^0 processes, with integrals S_T^- and S_T^+, joined to a BESQ^2 bridge with integral $T - S_T^- - S_T^+$. More specifically, when we condition on $\{W_T \geq 0\}$ then the 5-tuple $(W_T, L_{W_T}(T), L_0(T), S_T^-, S_T^+)$ admits a density on $[0,\infty)^5$, and when we condition on $(W_T, L_{W_T}(T), L_0(T), S_T^-, S_T^+) = (a, b, c, s^-, s^+)$ we have

- $(Y_x^-)_{x\geq 0} = L_{-x}(T)$ is a BES$Q^0(c)$ process conditioned to have integral equal to s^-.
- $(Y_x)_{0\leq x\leq W_T} = L_x(T)$ is a BES$Q_a^2(c,b)$ bridge conditioned to have integral equal to $T - s^- - s^+$.

- $(Y_x^+)_{x\geq 0} = L_{a+x}(T)$ is a BES$Q^0(b)$ process conditioned to have integral equal to s^+.

Now observe that $\mathcal{E}_T^\bullet = \{L_x(T) \leq 1 \text{ for all } x \in \mathbb{R}\}$ is exactly the event that Y^-, Y and Y^+ are all bounded above by 1. Therefore, if we use $\mathbb{Y}_c^0$ to denote the law of a BES$Q^0(c)$ process and $\mathbb{Y}_{a,c,b}^2$ to denote the law of a BES$Q_a^2(c,b)$ bridge, then the first step to controlling (4.3.3) is to understand how the probabilities

$$Q_0(c,s) = \mathbb{Y}_c^0\left(Y_x \leq 1 \text{ for all } x \geq 0 \,|\, \textstyle\int_0^\infty Y_x \,\mathrm{d}x = s\right) \tag{4.3.4}$$

$$\text{and} \quad Q_2(a,c,b,s) = \mathbb{Y}_{a,c,b}^2\left(Y_x \leq 1 \text{ for all } 0 \leq x \leq a \,|\, \textstyle\int_0^a Y_x \,\mathrm{d}x = s\right) \tag{4.3.5}$$

depend on $c, b \in [0,1)$ and $a, s \geq 0$. Before estimating these it will be useful for us to construct an auxiliary process which is parametrised by the integral of $(Y_x)_{x\geq 0}$.

Definition 4.3.1 (Auxiliary process)**.** Suppose we are given a BES$Q^d(c)$ process $(Y_x)_{x\geq 0}$ with integral $S = \int_0^\infty Y_x \,\mathrm{d}x \in [0,\infty) \cup \{\infty\}$. For each $0 \leq t \leq S$ we set $\rho(t) = \inf\left\{u : \int_0^u Y_x \,\mathrm{d}x = t\right\}$, then $\rho : [0,S) \longrightarrow [0,\infty)$ is a strictly increasing differentiable function. Using ρ we define the coupled auxiliary process $(Z_t)_{t\geq 0}$ by $Z_t = Y_{\rho(t\wedge S)}$, and denote its law by $\mathbb{Z}_c^d$.

Lemma 4.3.2. *$(Z_t)_{t\geq 0}$ is a Markov process whose infinitesimal generator, $\tilde{\mathcal{L}}_d$, is given by*

$$\tilde{\mathcal{L}}_d f(x) = \frac{1}{x}\mathcal{L}_d f(x) = 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(x) + d\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}f(x). \tag{4.3.6}$$

Here $\mathcal{L}_d$ denotes the generator of the BES$Q^d(c)$ *process $(Y_x)_{x\geq 0}$. Recall (4.2.2).*

Proof. We can only have $S < \infty$ if $d = 0$. In this situation we have $Z_t = Y_{\rho(S)} = 0$ for all $t \geq S$, and so from now on we will assume that $0 \leq t < S$. Because ρ is an increasing function of t, then $(Z_t)_{t\geq 0}$ must inherit the Markov property from $(Y_x)_{x\geq 0}$. To show (4.3.6) we observe that $\displaystyle\int_0^{\rho(t)} Y_x \,\mathrm{d}x = t$, and differentiate both sides with respect to t to get

$$Y_{\rho(t)}\frac{\mathrm{d}\rho}{\mathrm{d}t}(t) = 1 \quad \Longrightarrow \quad \frac{\mathrm{d}\rho}{\mathrm{d}t}(t) = \frac{1}{Z_t}.$$

Since $\rho(0) = 0$ we can approximate $\rho(t)$ near 0 by $\rho(t) \sim t\,\rho'(0)$. Therefore making the substitution $s = t\,\rho'(0)$ gives

$$\begin{aligned}\tilde{\mathcal{L}}_d f(x) &= \lim_{t\to 0}\frac{\mathbb{E}^x\{f(Z_t)\} - f(x)}{t} = \lim_{t\to 0}\frac{\mathbb{E}^x\{f(Y_{\rho(t)})\} - f(x)}{t} \\ &= \lim_{t\to 0}\frac{\mathbb{E}^x\{f(Y_{t\,\rho'(0)})\} - f(x)}{t} = \rho'(0)\lim_{s\to 0}\frac{\mathbb{E}^x\{f(Y_s)\} - f(x)}{s} = \frac{1}{x}\mathcal{L}_d f(x). \qquad \square\end{aligned}$$

If I_d is the Donsker–Varadhan rate function of the occupation measure of $(Y_x)_{x\geq 0}$, then we can deduce $\tilde{I}_d$, the rate function of the occupation measure of $(Z_t)_{t\geq 0}$, as follows: Assume that a probability measure μ has $0 < \int_0^\infty x^{-1}\,\mathrm{d}\mu(x) < \infty$ (else $\tilde{I}_d(\mu) = \infty$). Because the work of Donsker and Varadhan, [DV75a], tells us that $\tilde{I}_d(\mu)$ is finite if and only if μ has a continuous Radon–Nikodym derivative $\dfrac{\mathrm{d}\mu}{\mathrm{d}x}$ in the domain of $\tilde{\mathcal{L}}_d$, then we may assume that $\dfrac{\mathrm{d}\mu}{\mathrm{d}x}$ exists. Now for each such μ we can construct a tilted measure $\psi(\mu)$, with Radon–Nikodym derivative given by

$$\frac{\mathrm{d}\psi(\mu)}{\mathrm{d}x}(x) = \frac{1}{\int_0^\infty x^{-1}\,\mathrm{d}\mu(x)}\frac{1}{x}\frac{\mathrm{d}\mu}{\mathrm{d}x}(x). \tag{4.3.7}$$

One can then check that $\dfrac{\mathrm{d}\psi(\mu)}{\mathrm{d}x}$ is in the domain of $\mathcal{L}_d$ and that

$$\mathbb{E}(\psi(\mu)) = \int_0^\infty x\,\mathrm{d}\psi(\mu)(x) = \frac{1}{\int_0^\infty x^{-1}\,\mathrm{d}\mu(x)}.$$

By recalling the definition of the Donsker–Varadhan rate function given by (2.4.8), we can now find $\tilde{I}_d(\mu)$ is terms of $\psi(\mu)$.

$$\begin{aligned}\tilde{I}_d(\mu) &= -\inf_{u\in C^2(0,\infty),u>0}\int_0^\infty \frac{1}{x}\frac{\mathcal{L}_d(u(x))}{u(x)}\frac{\mathrm{d}\mu}{\mathrm{d}x}(x)\,\mathrm{d}x \\ &= -\inf_{u\in C^2(0,\infty),u>0}\int_0^\infty \frac{\mathcal{L}_d(u(x))}{u(x)}\frac{1}{\mathbb{E}(\psi(\mu))}\frac{\mathrm{d}\psi(\mu)}{\mathrm{d}x}(x)\,\mathrm{d}x = \frac{1}{\mathbb{E}(\psi(\mu))}I_d(\psi(\mu)).\end{aligned} \tag{4.3.8}$$

Using the auxiliary process gives a useful tool for adapting the Donsker–Varadhan Theorem to the occupation measure of a square Bessel process stopped at the random time $\rho(s)$.

Lemma 4.3.3. *Suppose that $(Y_x)_{x\geq 0}$ is a* BESQ$^d(c)$ *process with law $\mathbb{Y}_c^d$. We then have*

$$\limsup_{s\to\infty}\frac{1}{s}\log\mathbb{Y}_c^d(L((Y_x),\rho(s),\cdot)\in C) \leq -\inf_{\mu\in C}\mathbb{E}(\mu)^{-1}I_d(\mu) \tag{4.3.9}$$

$$\liminf_{S\to\infty}\frac{1}{s}\log\mathbb{Y}_c^d(L((Y_x),\rho(s),\cdot)\in O) \geq -\inf_{\mu\in O}\mathbb{E}(\mu)^{-1}I_d(\mu) \tag{4.3.10}$$

for each closed set $C\subseteq\mathcal{P}(\mathbb{R})$, and each open set $O\subseteq\mathcal{P}(\mathbb{R})$ such that $c\in\mathrm{support}(\mu)$ for each $\mu\in O$.

Remark 4.3.4. It was noted in Remark 2.4.2 that the Donsker–Varadhan Theorem continues to hold when the start-point, Y_0, is a real valued random variable. Since Lemma 4.3.3 is a consequence of Theorem 2.4.1, then (4.3.9) and (4.3.10) will also hold when c is random. In this case we substitute $\mathbb{Y}_c^d(L((Y_x),\rho(s),\cdot)\in\cdot)$ with $\mathbb{E}^c\big\{\mathbb{Y}_c^d(L((Y_x),\rho(s),\cdot)\in$

$\cdot)\big\}$, and replace the condition that $\big\{c \in \text{support}(\mu) \text{ for each } \mu \in O\big\}$ with the condition that $\big\{\text{support}(c) \subseteq \text{support}(\mu) \text{ for each } \mu \in O\big\}$.

Proof of Lemma 4.3.3. Fix $s > 0$, then given $(Y_x)_{0\le x\le \rho(s)}$ let $(Z_t)_{0\le t\le s}$ be the auxiliary process to Y. Suppose $\mu = L((Y_x), \rho(s), \cdot)$ is the occupation measure of $(Y_x)_{0\le x\le\rho(s)}$, and use $\varphi(\mu) = L((Z_t), s, \cdot)$ to denote the occupation measure of $(Z_t)_{0\le t\le s}$. Since Y and Z are both continuous Markov processes then μ and $\varphi(\mu)$ have Radon–Nikodym derivatives almost surely. Furthermore, by using $\dfrac{\mathrm{d}\rho(t)}{\mathrm{d}t} = \dfrac{1}{Y_{\rho(t)}}$ we can check that

$$\frac{\mathrm{d}\varphi(\mu)}{\mathrm{d}x}(x) = \frac{1}{\int_0^\infty x\,\mathrm{d}\mu(x)} x\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) = \frac{1}{\mathbb{E}(\mu)} x\frac{\mathrm{d}\mu}{\mathrm{d}x}(x). \tag{4.3.11}$$

Observe that (4.3.11) gives a continuous inverse to (4.3.7), and thus $\mu \longmapsto \varphi(\mu)$ is a bicontinuous bijection with inverse $\varphi(\mu) \longmapsto \psi(\varphi(\mu)) = \mu$. As φ is continuous then $\varphi(C)$ is also a closed set, and so we can use the Donsker–Varadhan Theorem and the natural coupling between Y and Z to get

$$\begin{aligned}\limsup_{s\to\infty} \frac{1}{s}\log \mathbb{Y}_c^d\big(L((Y_x),\rho(s),\cdot)\in C\big) &= \limsup_{s\to\infty}\frac{1}{s}\log\mathbb{Z}_c^d\big(L((Z_x),s,\cdot)\in\varphi(C)\big)\\ \le -\inf_{\mu\in\varphi(C)}\tilde{I}_d(\mu) &= -\inf_{\mu\in\varphi(C)}\mathbb{E}(\psi(\mu))^{-1}I_d(\psi(\mu)) = -\inf_{\mu\in C}\mathbb{E}(\mu)^{-1}I_d(\mu).\end{aligned}$$

Similarly when O is open then so too is $\varphi(O)$ and therefore

$$\begin{aligned}\liminf_{s\to\infty} \frac{1}{s}\log \mathbb{Y}_c^d\big(L((Y_x),\rho(s),\cdot)\in O\big) &= \liminf_{s\to\infty}\frac{1}{s}\log\mathbb{Z}_c^d\big(L((Z_x),s,\cdot)\in\varphi(O)\big)\\ \ge -\inf_{\mu\in\varphi(O)}\tilde{I}_d(\mu) &= -\inf_{\mu\in\varphi(O)}\mathbb{E}(\psi(\mu))^{-1}I_d(\psi(\mu)) = -\inf_{\mu\in O}\mathbb{E}(\mu)^{-1}I_d(\mu).\end{aligned}$$ □

Because it is clear that $Y_x \le 1$ for all x if and only if $Z_t = Y_{\rho(t)} \le 1$ for all t, then we can use the auxiliary process to generalise (4.3.5) to the case where the bridge length a is not fixed. From (4.3.7) we know that

$$\frac{\mathrm{d}L(Y,a,\cdot)}{\mathrm{d}x}(x) = \frac{1}{x}\frac{1}{\int x^{-1}\,\mathrm{d}L(Z,s,x)}\frac{\mathrm{d}L(Z,s,\cdot)}{\mathrm{d}x}(x).$$

Therefore from the condition that $\int_0^a Y_x\,\mathrm{d}x = s$ we get

$$\begin{aligned}&\int_0^a Y_x\,\mathrm{d}x = a\int x\,\mathrm{d}L(Y,a,x) = \frac{a}{\int x^{-1}\,\mathrm{d}L(Z,s,x)}\int\frac{\mathrm{d}L(Z,s,x)}{\mathrm{d}x} = s \qquad (4.3.12)\\ &\implies s\int x^{-1}\,\mathrm{d}L(Z,s,x) = \int_0^s\frac{1}{Z_t}\,\mathrm{d}t = a.\end{aligned}$$

This allows us to rewrite (4.3.5) in terms of the auxiliary process $(Z_t)_{0\le t\le s}$ as

$$Q_2(a,c,b,s) = \mathbb{Z}^2_{s,c,b}\left(Z_t \le 1 \text{ for all } 0\le t\le s \,|\, \textstyle\int_0^s Z_t^{-1}\,\mathrm{d}t = a\right).$$

Here $\mathbb{Z}^2_{s,c,b}$ denotes $\mathbb{Z}^2_c(\cdot\,|\,Z_s = b) = \lim_{\varepsilon\to\infty}\mathbb{Z}^2_c(\cdot\,|\,|Z_s - b|<\varepsilon)$. $Q_2(a,c,b,s)$ can now be generalised by letting $A\subseteq[0,\infty)$ be a measurable set and defining

$$\tilde{Q}_2(A,c,b,s) = \mathbb{Z}^2_{s,c,b}\left(Z_t \le 1 \text{ for all } 0\le t\le s \text{ and } \textstyle\int_0^s Z_t^{-1}\,\mathrm{d}t \in A\right). \tag{4.3.13}$$

$$\tilde{Q}_2(c,b,s) = \tilde{Q}_2([0,\infty),c,b,s) = \mathbb{Z}^2_{s,c,b}\left(Z_t \le 1 \text{ for all } 0\le t\le s\right). \tag{4.3.14}$$

Hence if we suppose that $(Y_x)_{0\le x\le \mathrm{len}}$ is a BES$Q^2_{\mathrm{len}}(c,b)$ bridge of undetermined length, which is conditioned to have $\int_0^{\mathrm{len}} Y_x\,\mathrm{d}x = s$, then the probability that its length len is in A and that $Y_x\le 1$ for all $0\le x\le \mathrm{len}$ is given by $\tilde{Q}_2(A,c,b,s)$. Here it is assumed that $c,b\in[0,1)$, $s>0$ and $A\subseteq[0,\infty)$ are all fixed.

Remark 4.3.5. As we know that $\mathcal{E}^\bullet_T = \{L_{-x}(T)\le 1 \text{ for all } x\ge 0\}\cap\{L_x(T)\le 1 \text{ for all } 0\le x\le W_T\}\cap\{L_{W_T+x}(T)\le 1 \text{ for all } x\ge 0\}$, then when we condition on $L_0(T)=c$, $L_{W_T}(T)=b$, $S^- = \int_0^\infty L_{-x}(T)\,\mathrm{d}x = s^-$ and $S^+ = \int_0^\infty L_{W_T+x}(T)\,\mathrm{d}x = s^+$ we get

$$\mathbb{W}(\mathcal{E}^\bullet_T\,|\,c,b,s^-,s^+) = Q_0(c,s^-)\times\tilde{Q}_2(c,b,T-s^--s^+)\times Q_0(b,s^+). \tag{4.3.15}$$

$\mathbb{W}(\mathcal{E}^\bullet_T)$ can now be found by integrating (4.3.15) with respect to the joint law of $L_0(T)$, $L_{W_T}(T)$, S^- and S^+.

Lemma 4.3.6. *$Q_0(c,s)$ is decreasing as a function of both $c\in(0,1)$ and $s>0$. Furthermore, for each fixed $c\in(0,1)$ we have*

$$\lim_{s\to\infty}\frac{1}{s}\log Q_0(c,s) = -2\pi^2. \tag{4.3.16}$$

Lemma 4.3.7. *If we assume that c and b are random variables supported on $[0,1)$, that $V_C\subseteq[1,\infty)$ is closed, and use sV_C to denote $\{a : s^{-1}a\in V_C\}$, then*

$$\limsup_{s\to\infty}\frac{1}{s}\log\tilde{Q}_2(sV_C,c,b,s) \le -\inf_{v\in V_C} vJ\left(v^{-1}\right), \tag{4.3.17}$$

where J is given by Definition 4.2.4. Conversely, if $V_O\subseteq[1,\infty)$ is an open set then

$$\liminf_{s\to\infty}\frac{1}{s}\log\tilde{Q}_2(sV_O,c,b,s) \ge -\inf_{v\in V_O} vJ\left(v^{-1}\right). \tag{4.3.18}$$

Proof of Lemma 4.3.6. Since $(Y_x)_{x\ge0}$ is a BES$Q^0(c)$ process conditioned on $\int_0^\infty Y_x\,\mathrm{d}x = s$, then the auxiliary process $(Z_t)_{0\le t\le s}$ has starting point $Z_0 = c$ and first hits 0 at $t=s$.

We have already noted that $Y_x \leq 1$ for all $x \geq 0$ if and only if $Z_t \leq 1$ for all $0 \leq t \leq s$. Therefore by coupling two BESQ^0 processes Y and $\tilde{Y}$ with respective starting points $0 < c \leq \tilde{c} < 1$ in such a way that their auxiliary process Z and $\tilde{Z}$ satisfy $Z_t \leq \tilde{Z}_t$ for all $0 \leq t \leq s$, it is easy to deduce that

$$Q_0(c,s) = \mathbb{Y}_c^0\left(Y_x \leq 1 \,|\, \textstyle\int_0^\infty Y_x\,\mathrm{d}x = s\right) \geq \mathbb{Y}_{\tilde{c}}^0\left(\tilde{Y}_x \leq 1 \,|\, \textstyle\int_0^\infty \tilde{Y}_x\,\mathrm{d}x = s\right) = Q_0(\tilde{c},s).$$

Thus $Q_0(c,s)$ is decreasing as a function of $c \in (0,1)$. The monotonicity of $Q_0(c,s)$ as a function of s follows via a similar coupling argument.

Now define $E_C = \{\mu : \text{support}(\mu) \subseteq (0,1]\}$ and $E_O = \{\mu : \text{support}(\mu) \subseteq (0,1)\}$, and note that $E_C \subseteq \mathcal{P}((0,\infty))$ is weakly closed and $E_O \subseteq \mathcal{P}((0,\infty))$ is weakly open. If we fix $c \in (0,1)$ and let $s \geq 1$, then we can use equation (2.3.4) to get the bounds

$$\frac{\mathrm{e}^{-1}}{\sqrt{8\pi}}\frac{c}{s^{\frac{3}{2}}} \leq \mathbb{Y}_c^0\left(\textstyle\int_0^\infty Y_x\,\mathrm{d}x \in [s, s-1]\right) \quad \text{and} \quad \mathbb{Y}_c^0\left(\textstyle\int_0^\infty Y_x\,\mathrm{d}x \in [s, s+1]\right) \leq \frac{c}{s^{\frac{3}{2}}}$$

for all fixed $c \in (0,1)$ and all $s \geq 1$. Therefore because $Q_0(c,s)$ is decreasing as a function of s we have

$$\begin{aligned}
Q_0(c,s) &\leq \mathbb{Y}_c^0\left(Y_x \leq 1 \text{ for all } x \geq 0 \,|\, \textstyle\int_0^\infty Y_x\,\mathrm{d}x \in [s-1,s]\right) \\
&= \frac{\mathbb{Y}_c^0\left(Y_x \leq 1 \text{ for all } x \geq 0 \text{ and } \int_0^\infty Y_x\,\mathrm{d}x \in [s-1,s]\right)}{\mathbb{Y}_c^0\left(\int_0^\infty Y_x\,\mathrm{d}x \in [s-1,s]\right)} \\
&\leq \sqrt{8\pi}\mathrm{e}\, s^{\frac{3}{2}} c^{-1}\, \mathbb{Y}_c^0\left(Y_x \leq 1 \text{ for all } x \geq 0 \text{ and } \textstyle\int_0^\infty Y_x\,\mathrm{d}x \geq s-1\right) \\
&= \sqrt{8\pi}\mathrm{e}\, s^{\frac{3}{2}} c^{-1}\, \mathbb{Y}_c^0\big(L((Y_x), \rho(s-1), \cdot) \in E_C\big),
\end{aligned}$$

and similarly

$$\begin{aligned}
Q_0(c,s) &\geq \mathbb{Y}_c^0\left(Y_x \leq 1 \text{ for all } x \geq 0 \,|\, \textstyle\int_0^\infty Y_x\,\mathrm{d}x \in [s,s+1]\right) \\
&\geq s^{\frac{3}{2}} c^{-1}\, \mathbb{Y}_c^0\big(L((Y_x), \rho(s), \cdot) \in E_O\big).
\end{aligned}$$

As we know that $\lim_{s\to\infty} s^{-1} \log s^{\frac{3}{2}} c^{-1} = 0$, then we can now get an upper and lower bound

for $\lim_{s\to\infty} \frac{1}{s} \log Q_0(c,s)$ by using Lemma 4.3.3.

$$\begin{aligned}
\limsup_{s\to\infty} \frac{1}{s} \log Q_0(c,s) &\leq \limsup_{s\to\infty} \frac{1}{s} \log \mathbb{Y}_0^c(L((Y_x),\rho(s-1),\cdot) \in E_C) \\
&\leq -\inf_{\mu\in E_C} \frac{1}{\mathbb{E}(\mu)} I_0(\mu) = -2\pi^2, \\
\liminf_{s\to\infty} \frac{1}{s} \log Q_0(c,s) &\geq \liminf_{s\to\infty} \frac{1}{s} \log \mathbb{Y}_0^c(L((Y_x),\rho(s),\cdot) \in E_O) \\
&\geq -\inf_{\mu\in E_O} \frac{1}{\mathbb{E}(\mu)} I_0(\mu) = -2\pi^2.
\end{aligned}$$

Here the equality $\inf_{\mu\in E_C} \frac{1}{\mathbb{E}(\mu)} I_0(\mu) = \inf_{\mu\in E_O} \frac{1}{\mathbb{E}(\mu)} I_0(\mu) = 2\pi^2$ comes from Lemma 4.2.2. □

Proof of Lemma 4.3.7. Let $E_{V_C} = \{\mu : \text{support}(\mu) \subseteq [0,1] \text{ and } \int_0^\infty x^{-1}\,\mathrm{d}\mu(x) \in V_C\}$ and note that if $L((Z_t),s,\cdot) \in E_{V_C}$ then we must have $Z_t \leq 1$ for all $0 \leq t \leq s$. Furthermore, because we also have that $\int_0^s Z_t^{-1}\,\mathrm{d}t = s\int_0^\infty x^{-1}\,\mathrm{d}L((Z_t),s,x)$ then we see that the probability being estimated in (4.3.17) is exactly $\mathbb{Z}^2_{s,c,b}(L((Z_t),s,\cdot) \in E_{V_C})$. Since E_{V_C} is closed with respect to the weak topology, then our aim is to estimate this probability by using the Donsker–Varadhan Theorem. However, in order to do this we must first check that $\mathbb{Z}^2_{s,c,b}(L((Z_t),s,\cdot) \in E_{V_C}) = \lim_{\varepsilon\to 0} \mathbb{Z}^2_c(L((Z_t),s,\cdot) \in E_{V_C} \,|\, |Z_s - b| < \varepsilon)$ is comparable with $\mathbb{Z}^2_c(L((Z_t),s,\cdot) \in E_{V_C})$.

By following the methods of Pinsky, [Pin85b], we know that when we condition on the occupation measure of a diffusion being contained within a collection of measures of uniformly bounded support (such as when we condition on $\{L((Z_t),s,\cdot) \in E_{V_C}\}_{s\geq 1}$) then the distribution of the end point, Z_s, will converge to an absolutely continuous random variable as $s \longrightarrow \infty$. Therefore there must exists a $K_1 < \infty$ (depending on V_C) such that

$$\mathbb{Z}^2_c\big(|Z_s - b| < \varepsilon \,|\, L((Z_t),s,\cdot) \in E_{V_C}\big) < \varepsilon\, K_1 \tag{4.3.19}$$

for all $s \geq 1$ and all $b \in [0,1]$. By inspecting the infinitesimal generator of Z, as given by (4.3.6), we see that $(Z_s)_{s\geq 0}$ is equal in law to a (rescaled) Bessel process. Therefore there must be constants $0 < k_2 < K_2 < \infty$ such that

$$k_2 < \frac{s^{\frac{3}{2}}}{\varepsilon(b+\varepsilon)^2} \mathbb{Z}^2_c(|Z_s - b| < \varepsilon) < K_2, \tag{4.3.20}$$

for all $\varepsilon > 0$, $b \in [0,1)$ and $s \geq 1$. Hence we can deduce that

$$\lim_{\varepsilon \to 0} \mathbb{Z}_c^2(L((Z_t), s, \cdot) \in E_{V_C} \,|\, |Z_s - b| < \varepsilon) \leq \frac{K_1}{k_2} \frac{s^{\frac{3}{2}}}{b^2} \mathbb{Z}_c^2 \left(L((Z_t), s, \cdot) \in E_{V_C}\right). \tag{4.3.21}$$

Since $\lim_{s \to \infty} \frac{1}{s} \log \frac{K_1}{k_2} \frac{s^{\frac{3}{2}}}{b^2} = 0$, then by applying the Donsker–Varadhan Theorem we get

$$\begin{aligned}
\limsup_{s \to \infty} \frac{1}{s} \log \tilde{Q}_2(sV_C, c, b, s) &\leq \limsup_{s \to \infty} \frac{1}{s} \log \mathbb{Z}_c^2 \left(L((Z_t), s, \cdot) \in E_{V_C}\right) \\
&\leq - \inf_{\mu \in E_{V_C}} \tilde{I}_2(\mu) = - \inf_{\mu \in E_{V_C}} \mathbb{E}(\psi(\mu))^{-1} I_2(\psi(\mu)) \\
&= - \inf_{\mu \in \varphi(E_{V_C})} \mathbb{E}(\mu)^{-1} I_2(\mu).
\end{aligned}$$

Where ψ and φ are given by (4.3.7) and (4.3.11). It is easy to verify that $\varphi(E_{V_C}) = \{\mu : \text{support}(\mu) \subseteq [0,1] \text{ and } \mathbb{E}(\mu)^{-1} \in V_C\}$, and so

$$\begin{aligned}
- \inf_{\mu \in \varphi(E_{V_C})} \mathbb{E}(\mu)^{-1} I_2(\mu) &= - \inf_{v \in V_C} v \, \inf\{I_2(\mu) : \text{support}(\mu) \subseteq [0,1] \text{ and } \mathbb{E}(\mu)^{-1} = v\} \\
&= - \inf_{v \in V_C} v J\left(v^{-1}\right),
\end{aligned}$$

proving (4.3.17).

Now set $E_{V_O} = \left\{\mu : \text{support}(\mu) \subseteq [0,1) \text{ and } \mathbb{E}(\mu)^{-1} \in V_O\right\}$ and note that if $L((Z_t), s, \cdot) \in E_{V_O}$ then $(Z_t)_{t \geq 0}$ must satisfy $Z_t \leq 1$ for all $0 \leq t \leq s$ and $s^{-1} \int_0^s Z_t^{-1} \, \mathrm{d}t \in V_O$. In order to show (4.3.18) we also note that for each $b \in [0,1)$ we can find a k_3 (depending on b and V_O) such that

$$\varepsilon \, k_3 < \mathbb{Z}_c^2\big(|Z_s - b| < \varepsilon \,|\, L((Z_t), s, \cdot) \in E_{V_O}\big) \tag{4.3.22}$$

for all $s \geq 1$ and all ε sufficiently small. By combining this with (4.3.20) we get

$$\lim_{\varepsilon \to 0} \mathbb{Z}_c^2(L((Z_t), s, \cdot) \in E_{V_O} \,|\, |Z_s - b| < \varepsilon) \geq \frac{k_3}{K_2} \frac{s^{\frac{3}{2}}}{b^2} \mathbb{Z}_c^2 \left(L((Z_t), s, \cdot) \in E_{V_O}\right). \tag{4.3.23}$$

Therefore, as E_{V_O} is open with respect to the weak topology, (4.3.18) follows by applying

the Donsker–Varadhan Theorem.

$$\begin{aligned}\liminf_{s\to\infty}\frac{1}{s}\log\tilde{Q}_2(sV_O,c,b,s) &\geq \liminf_{s\to\infty}\frac{1}{s}\log\mathbb{Z}_c^2\left(L((Z_t),s,\cdot)\in E_{V_O}\right)\\ &\geq -\inf_{\mu\in E_{V_O}}\tilde{I}_2(\mu) = -\inf_{\mu\in E_{V_O}}\mathbb{E}(\psi(\mu))^{-1}I_2(\psi(\mu))\\ &= -\inf_{\mu\in\varphi(E_{V_O})}\mathbb{E}(\mu)^{-1}I_2(\mu) = -\inf_{v\in V_O} vJ\left(v^{-1}\right). \qquad \square\end{aligned}$$

At this point we can now show that for each $\varepsilon > 0$ there exists a T_ε such that for all $T \geq T_\varepsilon$ we have

$$\frac{\mathbb{W}(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\})}{\mathbb{W}(\mathcal{E}_T^\bullet)} < \varepsilon. \tag{4.3.24}$$

This is precisely (4.3.3) in the case where $t = T$. To do this we need an upper bound for $\mathbb{W}(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\})$ and a lower bound for $\mathbb{W}(\mathcal{E}_T^\bullet)$, and hence it suffices to prove the following two claims.

Claim 4.3.8. *We have*

$$\liminf_{T\to\infty}\frac{1}{T}\log\mathbb{W}(\mathcal{E}_T^\bullet) \geq -\Gamma^\bullet,$$

where $\Gamma^\bullet = \gamma^\bullet J\left((\gamma^\bullet)^{-1}\right)$ *is the minimal value of* $vJ\left(v^{-1}\right)$. *See page 88.*

Claim 4.3.9. *For each* $\varepsilon > 0$ *there exists* $k > 0$ *such that*

$$\limsup_{T\to\infty}\frac{1}{T}\log\mathbb{W}(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\}) \leq -\Gamma^\bullet - k.$$

Remark 4.3.10. By combining Claim 4.3.8 and Claim 4.3.9 we get

$$\limsup_{T\to\infty}\frac{1}{T}\log\frac{\mathbb{W}(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\})}{\mathbb{W}(\mathcal{E}_T^\bullet)} \leq -k,$$

and so (4.3.24) follows immediately.

Proof of Claim 4.3.8. Suppose we use $f_T(\cdot,\cdot)$ to denote the joint probability distribution of $L_0(T)$ and $L_{W_T}(T)$. Observe that if we fix $T = 1$ then, because $f_1(\cdot,\cdot)$ is continuous and strictly positive on $[0,\infty)^2$, by compactness there must exist $0 < k_1 < K_1 < \infty$ such that $k_1 \leq f_1(c,b) \leq K_1$ for all $c, b \in [0,1]$. We also note that the scaling property of Brownian motion implies that $f_T(c,b) = T^{-1}f_1\left(cT^{-\frac{1}{2}}, bT^{-\frac{1}{2}}\right)$ for all $T > 0$ and $c, b \in [0,\infty)$.

Therefore it must follow that

$$\frac{k_1}{T} \leq f_T(c,b) \leq \frac{K_1}{T} \tag{4.3.25}$$

for all $c, b \in [0,1]$ and all $T \geq 1$. We can now observe that

$$\begin{aligned}
\mathbb{W}(\mathcal{E}_T^\bullet) &= \int_{[0,\infty)^2} \mathbb{W}(\mathcal{E}_T^\bullet \,|\, L_0(T) = c, L_{W_T}(t) = b) f_T(c,b)\,\mathrm{d}c\,\mathrm{d}b \\
&\geq \frac{k_1}{T} \int_{\left[\frac{1}{4},\frac{3}{4}\right]^2} \mathbb{W}(\mathcal{E}_T^\bullet \,|\, L_0(T) = c, L_{W_T}(T) = b)\,\mathrm{d}c\,\mathrm{d}b \\
&\geq \frac{k_1}{4T} \inf_{c,b \in \left[\frac{1}{4},\frac{3}{4}\right]} \mathbb{W}(\mathcal{E}_T^\bullet \,|\, L_0(T) = c, L_{W_T} = b).
\end{aligned} \tag{4.3.26}$$

From now on let $\hat{c}$ and $\hat{b}$ be the values of $c, b \in \left[\frac{1}{4},\frac{3}{4}\right]$ which minimise (4.3.26), and use $g_{T,\hat{c},\hat{b}}(\cdot,\cdot)$ to denote the joint probability density of $S^- = \int_0^\infty L_{-x}(T)\,\mathrm{d}x$ and $S^+ = \int_0^\infty L_{W_T+x}(T)\,\mathrm{d}x$ with respect to $\mathbb{W}(\cdot \,|\, L_0(T) = \hat{c}, L_{W_T}(T) = \hat{b})$. Brownian scaling also tells us that $g_{T,\hat{c}\sqrt{T},\hat{b}\sqrt{T}}(s^-,s^+) = T^{-2} g_{T,\hat{c},\hat{b}}(s^- T^{-1}, s^+ T^{-1})$, therefore, since decreasing $\hat{c}$ and $\hat{b}$ would only make it more likely for S^- and S^+ to be small, there must be some $k_2 > 0$ such that

$$\frac{k_2}{T^2} < \mathbb{W}(S^- \leq 1 \text{ and } S^+ \leq 1 \,|\, L_0(T) = \hat{c}, L_{W_T}(t) = \hat{b}). \tag{4.3.27}$$

By combining this with (4.3.26), and recalling that (4.3.4) and (4.3.14) give the probabilities of $(L_{-x}(T))_{x\geq 0}$, $(L_{W_T+x}(T))_{x\geq 0}$ and $(L_x(T))_{0\leq x\leq W_T}$ being bounded above by 1, we get

$$\begin{aligned}
\mathbb{W}(\mathcal{E}_T^\bullet) &\geq \frac{k_1 T}{4k_2} \mathbb{W}(\mathcal{E}_T^\bullet \,|\, L_0(T) = \hat{c}, L_{W_T}(T) = \hat{b}, S^- \leq 1 \text{ and } S^+ \leq 1) \\
&\geq \frac{k_1 T}{4k_2} \inf_{0\leq s^-\leq 1} Q_0(\hat{c}, s^-) \times \inf_{0\leq s^+\leq 1} Q_0(\hat{b}, s^+) \times \inf_{0\leq s\leq T} \tilde{Q}_2(\hat{c},\hat{b},s).
\end{aligned}$$

Since Lemma 4.3.6 tells us that $Q_0(c,s)$ is decreasing as a function of both c and s then we must have $\inf_{0\leq s^-\leq 1} Q_0(\hat{c}, s^-)$, $\inf_{0\leq s^+\leq 1} Q_0(\hat{b}, s^+) \geq Q_0\left(\frac{3}{4}, 1\right) > 0$. Therefore when we take the logarithm the only term to contribute is $\inf_{0\leq s\leq T} \tilde{Q}_2(\hat{c},\hat{b},s)$. We can now complete the proof by using Lemma 4.3.7.

$$\liminf_{T\to\infty} \frac{1}{T} \log \mathbb{W}(\mathcal{E}_T^\bullet) \geq \liminf_{T\to\infty} \frac{1}{T} \log \tilde{Q}_2(\hat{c},\hat{b},T) \geq -\inf_{v\in[0,\infty)} vJ\left(v^{-1}\right) = -\Gamma^\bullet. \qquad \square$$

Proof of Claim 4.3.9. Suppose we have conditioned on the event $\{W_T \geq 0\}$, and use $(L_x(T))_{x\in\mathbb{R}}$ to denote the local time profile of $(W_t)_{t\geq 0}$ at time T. Recall that $S =$

$\int_0^{W_T} L_x(T)\,\mathrm{d}x$, then in order to estimate $\mathbb{W}(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\})$ it will be useful for us to let $\eta = \eta(\varepsilon) > 0$ be a positive constant – which we shall determine later – and consider the cases $S > T(1-\eta)$ and $S \leq T(1-\eta)$ separately. By the Law of Total Probability we have

$$\begin{aligned}\mathbb{W}\left(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\}\right) =& \mathbb{W}\left(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\} \cap \{S > T\,(1-\eta)\}\right) \\ &+ \mathbb{W}\left(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\} \cap \{S \leq T\,(1-\eta)\}\right),\end{aligned}$$

and therefore we need to show that both of these terms are sufficiently small.

If we condition on $L_0(T) = c$, $L_{W_T}(T) = b$ and $\int_0^{W_T} L_x(T)\,\mathrm{d}x = s$ then $(Z_t)_{t\geq 0}$, the auxiliary process to $(L_x(T))_{0\leq x\leq W_T}$, has $Z_0 = c$, $Z_s = b$ and $\int_0^s Z_t^{-1}\,\mathrm{d}t = W_T$. Therefore if $T(1-\eta) < s \leq T$ and $|W_T - \gamma_T^\bullet| \geq \varepsilon T$ then

$$\left|\frac{W_T}{T} - \gamma^\bullet\right| = \left|\frac{1}{T}\int_0^s Z_t^{-1}\,\mathrm{d}t - \gamma^\bullet\right| > \varepsilon \quad \Longrightarrow \quad \left|\frac{1}{s}\int_0^s Z_t^{-1}\,\mathrm{d}t - \gamma^\bullet\right| > \frac{\varepsilon}{2} \tag{4.3.28}$$

provided s is sufficiently close to T, i.e. provided η is sufficiently small. Assume that η is small enough for (4.3.28) to hold and set $V_O = \left\{v \in [0,\infty) : |v - \gamma^\bullet| > \frac{1}{2}\varepsilon\right\}$. Since the event $\mathcal{E}_T^\bullet$ implies that $Z_t \leq 1$ for all $0 \leq t \leq s$, then we can use Lemma 4.3.7 to estimate $\mathbb{W}\left(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\} \cap \{S > T\,(1-\eta)\}\right)$.

$$\begin{aligned}&\limsup_{T\to\infty} \frac{1}{T}\log \mathbb{W}\left(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\} \cap \{S > T\,(1-\eta)\}\right) \\ &\quad\leq \limsup_{T\to\infty} \frac{1}{T}\log\left(\sup_{\substack{T(1-\eta)<s<T \\ b,c\in[0,1)}} \tilde{Q}_2(sV_O, c, b, s)\right) \leq -(1-\eta)\inf_{v\in V_O} vJ\left(v^{-1}\right).\end{aligned}$$

From Lemma 4.2.8 we also know that $vJ\left(v^{-1}\right)$ attains a unique minimum value of $\Gamma^\bullet$ at $v = \gamma^\bullet$. Therefore, by continuity, there must exist some $\eta > 0$ and some $k_1 > 0$ such that

$$-(1-\eta)\inf_{v\in V_O} vJ\left(v^{-1}\right) \leq -\Gamma^\bullet - k_1.$$

Hence we have controlled the first term. We now fix such a value of η and move on to considering the case where $0 \leq S \leq T\,(1-\eta)$. Recall that we defined $S^- = \int_{-\infty}^0 L_x(T)\,\mathrm{d}x$ and $S^+ = \int_{W_T}^\infty L_x(T)\,\mathrm{d}x$, and that when we condition on $L_0(T) = c$, $L_{W_T}(T) = b$, $S^- = s^-$ and $S^+ = s^+$ then $(L_{-x}(T))_{x\geq 0}$ and $(L_{W_T+x}(T))_{x\geq 0}$ are equal in law to BES$Q^0(c)$ and BES$Q^0(b)$ processes conditioned to have integrals equal to s^- and s^+ respectively. The probabilities that these processes are bounded above by 1 are given by $Q_0(c, s^-)$ and

$Q_0(b, s^+)$. Thus since we know that $S^- + S + S^+ = T$ we get

$$\mathbb{W}(\mathcal{E}_T^\bullet \cap \{|W_T - \gamma^\bullet T| \geq \varepsilon T\} \cap \{S \leq T(1-\eta)\}) \leq \mathbb{W}(\mathcal{E}_T^\bullet \cap \{S \leq T(1-\eta)\})$$
$$\leq \sup_{\substack{0\leq s\leq T(1-\eta)\\ s^-+s+s^+=T}} \left(\sup_{c\in[0,1)} Q_0(c, s^-) \times \sup_{b\in[0,1)} Q_0(b, s^+) \times \sup_{c,b\in[0,1)} \tilde{Q}_2([0,\infty), c, b, s) \right).$$

From Lemma 4.3.6 and Lemma 4.3.7 we know that

$$\limsup_{s\to\infty} \frac{1}{s} \log \sup_{c,b\in[0,1)} \tilde{Q}_2([0,\infty), c, b, s) \leq -\Gamma^\bullet \quad \text{and} \quad \limsup_{s\to\infty} \frac{1}{s} \log \sup_{c\in[0,1)} Q_0(c, s) - 2\pi^2.$$

Therefore, because $\Gamma^\bullet < 2\pi^2$, we get

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{W}(\mathcal{E}_T^\bullet \cap \{S \leq T(1-\eta)\}) \leq \sup_{0\leq\beta\leq 1-\eta} -\beta\,\Gamma^\bullet - (1-\beta)\,2\pi^2 = -\Gamma^\bullet - k_2$$

for some $k_2 > 0$. The claim is now proved by setting $k = \min\{k_1, k_2\}$. □

Having shown that (4.3.24) holds for all $T \geq T_\varepsilon$ we can now show (4.3.3) by noting that

$$\frac{\mathbb{W}(\mathcal{E}_T^\bullet \cap \{|W_t - \gamma^\bullet t| \geq \varepsilon t\})}{\mathbb{W}(\mathcal{E}_T^\bullet)}$$
$$= \frac{\mathbb{W}(\mathcal{E}_T^\bullet \,|\, \mathcal{E}_t^\bullet \cap \{|W_t - \gamma^\bullet t| \geq \varepsilon t\})}{\mathbb{W}(\mathcal{E}_T^\bullet \,|\, \mathcal{E}_t^\bullet)} \frac{\mathbb{W}(\mathcal{E}_t^\bullet \cap \{|W_t - \gamma^\bullet t| \geq \varepsilon t\})}{\mathbb{W}(\mathcal{E}_t^\bullet)},$$

and then checking that both $\mathbb{W}(\mathcal{E}_T^\bullet \,|\, \mathcal{E}_t^\bullet \cap \{W_t - \gamma^\bullet t| \geq \varepsilon t\})$ and $\mathbb{W}(\mathcal{E}_T^\bullet \,|\, \mathcal{E}_t^\bullet)$ are comparable with $\mathbb{W}(\mathcal{E}_{T-t}^\bullet)$. This is the content of the following two lemmas.

Lemma 4.3.11. *Let $t \geq 0$ and suppose A is any event which is measurable with respect to $\mathcal{F}_t$. We must then have $\mathbb{W}(\mathcal{E}_T^\bullet \,|\, A) \leq \mathbb{W}(\mathcal{E}_{T-t}^\bullet)$ for all $T \geq t$.*

Lemma 4.3.12. *There exists $\eta > 0$ such that*

$$\eta\, \mathbb{W}(\mathcal{E}_{T-t}^\bullet) \leq \mathbb{W}(\mathcal{E}_T^\bullet \,|\, \mathcal{E}_t^\bullet) \tag{4.3.29}$$

for all $0 \leq t \leq T$.

Remark 4.3.13. Having proved these two lemmas we then have that

$$\frac{\mathbb{W}(\mathcal{E}_T^\bullet \cap \{|W_t - \gamma^\bullet t| \geq \varepsilon t\})}{\mathbb{W}(\mathcal{E}_T^\bullet)} \leq \frac{1}{\eta} \frac{\mathbb{W}(\mathcal{E}_t^\bullet \cap \{|W_t - \gamma^\bullet t| \geq \varepsilon t\})}{\mathbb{W}(\mathcal{E}_t^\bullet)}. \tag{4.3.30}$$

Because we can conclude from Claim 4.3.9 and Claim 4.3.8 that the ratio on the right hand side becomes arbitrarily small as $t \longrightarrow \infty$, then it follows that we can always find

a T_ε such that for all $T_\varepsilon \le t \le T$ we have

$$\mathbb{W}\left(\left|\frac{W_t}{t} - \gamma^\bullet\right| \ge \varepsilon \,|\, \mathcal{E}_T^\bullet\right) < \varepsilon,$$

proving (4.3.1). Therefore, once we have proved Lemma 4.3.11 and Lemma 4.3.12, we will have shown that the limiting process is ballistic with speed $\gamma^\bullet$. □

Proof of Lemma 4.3.11. If $(W_s)_{s\ge 0}$ is a Brownian motion and $t > 0$ is fixed, then we know that $(W_{s+t} - W_t)_{s\ge 0}$ is also a Brownian motion which is independent of $(W_s)_{0\le s\le t}$. Therefore $(L_{W_t+x}(T) - L_{W_t+x}(t))_{x\in\mathbb{R}}$ is independent of $(L_x(t))_{x\in\mathbb{R}}$ and is equal in law to $(L_x(T-t))_{x\in\mathbb{R}}$. Now if we are given $(W_s)_{0\le s\le t}$ for some $(W_s)_{s\ge 0}$ in A then the event $(W_s)_{s\ge 0} \in \mathcal{E}_T^\bullet$ is precisely the event that $L_{W_T+x}(T) - L_{W_T+x}(t) \le 1 - L_{W_T+x}(t)$ for all $x \in \mathbb{R}$. Because $1 - L_{W_T+x}(t) \le 1$ for all $x \in \mathbb{R}$ then we get the upper bound

$$\mathbb{W}(\mathcal{E}_T^\bullet \,|\, A) \le \mathbb{W}(L_{W_T+x}(T) - L_{W_T+x}(t) \le 1 \text{ for all } x \in \mathbb{R}) = \mathbb{W}(\mathcal{E}_{T-t}^\bullet). \qquad \square$$

The proof of Lemma 4.3.12 is somewhat more tricky and uses the following.

Lemma 4.3.14. *Given a Brownian motion $(W_t)_{t\ge 0}$ and a fixed $T > 0$ we let $S^- = \int_0^\infty L_{-x}(T)\,\mathrm{d}x$ denote the amount of time that $(W_t)_{0\le t\le T}$ spends below 0. Suppose we condition $(W_t)_{t\ge 0}$ on $\mathcal{E}_T^\bullet$, then S^- has an exponential tail whose exponent does not depend on T. In other words there are universal constants $\xi > 0$ and $\Xi < \infty$ such that for each $T > 0$ and all $a \ge 0$ we have*

$$\mathbb{W}\big(S^- = \textstyle\int_0^\infty L_{-x}(T)\,\mathrm{d}x > a \,|\, \mathcal{E}_T^\bullet\big) < \Xi\, \mathrm{e}^{-\xi a}. \tag{4.3.31}$$

Proof. Recall that $\tilde{Q}_2(c,b,s)$ gives us the probability that the auxiliary process $(Z_t)_{t\ge 0}$, conditioned to have $Z_0 = c$ and $Z_s = b$, satisfies $Z_t \le 1$ for all $0 \le t \le s$. Lemma 4.3.7 allows us to understand how $\tilde{Q}_2(c,b,s)$ behaves as $s \longrightarrow \infty$, but we would also like to understand how $\tilde{Q}_2(c,b,s+\Delta s)$ relates to $\tilde{Q}_2(c,b,s)$. Consider the measure

$$\nu_{S,\Delta S,c,b} = \mathbb{Z}_c^2(Z_s \in \cdot \,|\, Z_t \le 1 \text{ for all } 0 \le t \le s + \Delta s \text{ and } Z_{s+\Delta s} = b).$$

By standard coupling arguments one can show that the measure $\nu = \lim_{c,b\to 1} \nu_{1,1,c,b}$ stochastically dominates $\nu_{S,\Delta S,c,b}$ for all $c, b \in [0,1)$ and each $s, \Delta s \ge 1$. It is also simple to use coupling arguments to show that $\tilde{Q}_2(c,b,s)$ is a decreasing function of c, b and s. Therefore by using the fact that $(Z_s)_{s\ge 0}$ is a Markov process and applying Chebyshev's sum

inequality we get

$$\begin{aligned}\tilde{Q}_2(c,b,s+\Delta s) &= \int_0^1 \tilde{Q}_2(c,y,\Delta s)\tilde{Q}_2(y,b,s)\,\mathrm{d}\nu_{s,\Delta s,c,b}(y)\\ &\geq \int_0^1 \tilde{Q}_2(c,y,\Delta s)\tilde{Q}_2(y,b,s)\,\mathrm{d}\nu(y)\\ &\geq \int_0^1 \tilde{Q}_2(c,y,\Delta s)\,\mathrm{d}\nu(y) \times \int_0^1 \tilde{Q}_2(y,b,s)\,\mathrm{d}\nu(y)\\ &\geq k_1\,\tilde{Q}_2(c,b,s) \times \int_0^1 \tilde{Q}_2(c,y,\Delta s)\,\mathrm{d}\nu(y),\end{aligned}$$

for some k_1 which is independent of c, b and s. Since Lemma 4.3.7 tells us how $\tilde{Q}_2(c,y,\Delta s)$ behaves (for large Δs), therefore we can deduce that

$$\liminf_{\Delta s\to\infty}\frac{1}{\Delta s}\inf_{\substack{s\geq 1\\ c,b\in[0,1)}}\log\frac{\tilde{Q}_2(c,b,s+\Delta s)}{\tilde{Q}_2(c,b,s)} \geq -\inf_{v\in[0,\infty)} vJ\left(v^{-1}\right) = -\Gamma^{\bullet}. \tag{4.3.32}$$

Now suppose we condition on $L_0(T)=c$, $L_{W_T}(T)=b$ and $(L_{W_T+x}(T))_{x\geq 0}=(Y_x^+)_{x\geq 0}$ for some $c,b\in[0,1]$ and some $(Y_x^+)_{x\geq 0}$ with $Y_x^+\leq 1$ for all $x\geq 0$ and $\int_0^\infty Y_x^+\,\mathrm{d}x = s^+$, $0\leq s^+\leq T$. Use $\mu_{T,s^+,c,b}$ to denote the measure of $\mathbb{W}\big(S^- = \int_0^\infty L_{-x}(T)\,\mathrm{d}x \in \cdot \,|\, W_T > 0, L_0(T)=c, L_{W_T}(T)=b \text{ and } S^+=\int_0^\infty L_{W_T+x}(T)\,\mathrm{d}x = s^+\big)$, then by arguing in the same way as we did for (4.3.27) we can find a $k_2>0$ such that

$$\mu_{T,s^+,c,b}([0,1]) > \frac{k_2}{T} \tag{4.3.33}$$

for all $c,b\in[0,1)$ and all $0\leq s^+\leq T$. We then have

$$\begin{aligned}\mathbb{W}(\mathcal{E}_T^{\bullet}\,|\,c,b,Y^+) &= \int_0^{T-s^+} Q_0(s,c)\,\tilde{Q}_2(T-s^+-s,c,b)\,\mathrm{d}\mu_{T,s^+,c,b}(s)\\ &\geq \frac{k_2}{T}\inf_{s\in[0,1]} Q_0(s,c)\,\tilde{Q}_2(T-s^+,c,b),\end{aligned}$$

and

$$\begin{aligned}\mathbb{W}(\mathcal{E}_T^{\bullet}\cap\{S^->a\}\,|\,c,b,Y^+) &= \int_a^{T-S^+} Q_0(s,c)\,\tilde{Q}_2(T-s^+-s,c,b)\,\mathrm{d}\mu_{T,s^+,c,b}(s)\\ &\leq \sup_{s>a} Q_0(s,c)\,\tilde{Q}_2(T-s^+-a,c,b).\end{aligned}$$

Lemma 4.3.6 tells us that $\limsup_{s\to\infty}\frac{1}{s}\log\sup_{c\in[0,1)} Q_0(s,c)\leq -2\pi^2$, and so by combining this

with (4.3.32) we deduce that there is some $\Xi < \infty$ such that

$$\begin{aligned}\frac{\mathbb{W}(\mathcal{E}_T^\bullet \cap \{S^- > a\} \,|\, c, b, Y^+)}{\mathbb{W}(\mathcal{E}_T^\bullet \,|\, c, b, Y^+)} &\leq \frac{T}{k_2} \frac{\sup_{s>a} Q_0(s,c)}{\inf_{s\in[0,1]} Q_0(s,c)} \frac{\tilde{Q}_2(T - s^+ - a, c, b)}{\tilde{Q}_2(T - s^+, c, b)} \\ &\leq \Xi \frac{\exp\{-(2\pi^2 - 1)a\}}{\exp\{-(\Gamma^\bullet + 1)a\}} = \Xi \, \mathrm{e}^{-\xi a}\end{aligned} \tag{4.3.34}$$

for all $c, b \in [0, 1)$, Y^+, $T \geq 1$ and all $a \geq 0$. Here $\xi = 2\pi^2 - \Gamma^\bullet - 2 > 0$. (4.3.31) now follows by integrating over c, b and Y^+. □

Proof of Lemma 4.3.12. Let $m_t = \inf\{W_s : 0 \leq s \leq t\}$, $M_t = \sup\{W_s : 0 \leq s \leq t\}$, and define the events

$$\begin{aligned}\mathcal{E}_t^{\bullet -} &= \{m_t \geq -1, L_x(t) \leq \tfrac{1}{2} \text{ for all } x \in [-1, 1] \text{ and } L_x(t) \leq 1 \text{ for all } x \in \mathbb{R}\} \\ \mathcal{E}_t^{\bullet +} &= \{M_t - W_t \leq 1, L_x(t) \leq \tfrac{1}{2} \text{ for all } x - W_t \in [-1, 1] \text{ and } L_x(t) \leq 1 \text{ for all } x \in \mathbb{R}\}.\end{aligned}$$

Observe that if $(W_s)_{0\leq s\leq t} \in \mathcal{E}_t^{\bullet +}$ and $(W_{t+s} - W_t)_{s\geq 0} \in \mathcal{E}_{T-t}^{\bullet -}$ then $(W_s)_{s\geq 0} \in \mathcal{E}_T^\bullet$. Therefore since $(W_s)_{0\leq s\leq t}$ and $(W_{t+s} - W_t)_{s\geq 0}$ are independent we get

$$\mathbb{W}(\mathcal{E}_T^\bullet \,|\, \mathcal{E}_t^\bullet) = \frac{\mathbb{W}(\mathcal{E}_T^\bullet)}{\mathbb{W}(\mathcal{E}_t^\bullet)} \geq \frac{\mathbb{W}(\mathcal{E}_t^{\bullet +}) \mathbb{W}(\mathcal{E}_{T-t}^{\bullet -})}{\mathbb{W}(\mathcal{E}_t^\bullet)},$$

and so if we can find a $k \geq 0$ such that

$$\mathbb{W}(\mathcal{E}_t^{\bullet -}) = \mathbb{W}(\mathcal{E}_t^{\bullet +}) > k \, \mathbb{W}(\mathcal{E}_t^\bullet) \tag{4.3.35}$$

for all $t \geq 0$, then we would have shown (4.3.29) with $\eta = k^2$.

From Lemma 4.3.14 we know that when we condition on $\mathcal{E}_T^\bullet$ then S^- and (by symmetry) S^+ must both be (uniformly) small. Therefore, since $S = T - s^- - s^+$, there must be some universal constant K_1 such that

$$\mathbb{W}\big(S = \textstyle\int_0^{W_T} L_x(T) \, \mathrm{d}x < T - K_1 \,|\, \mathcal{E}_T^\bullet, L_0(T) = c, L_{W_T}(T) = b\big) \geq \frac{1}{2}$$

for all $T \geq 1$ and $c, b \in [0, 1)$. From (4.3.32) we also know that there must be some universal K_2 such that

$$\sup_{T - K_1 \leq s \leq T} \tilde{Q}_2(c, b, s) < K_2 \, \tilde{Q}_2(c, b, T)$$

for all $T \geq 1$ and $c, b \in [0, 1)$. Now write $f_T(\cdot, \cdot)$ for the join density of $L_0(T)$ and $L_{W_T}(T)$, and recall that $\tilde{Q}_2(c, b, s)$ represent the probability of $\{L_x(T) \leq 1 \text{ for all } 0 \leq x \leq W_T\}$

conditionally on $L_0(T) = c$, $L_{W_T}(T) = b$ and $\int_0^{W_T} L_x(T)\,\mathrm{d}x = s$. We must then have

$$\begin{aligned}\mathbb{W}(\mathcal{E}_T^\bullet) &\le 2K_2 \int_0^1 \int_0^1 \tilde{Q}_2(c,b,T) f_T(c,b)\,\mathrm{d}c\,\mathrm{d}b \\ &\le 32K_2 \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \tilde{Q}_2(c,b,T) f_T(c,b)\,\mathrm{d}c\,\mathrm{d}b. \end{aligned} \tag{4.3.36}$$

Here the second inequality follows because both $\tilde{Q}_2(c,b,T)$ and $f_T(c,b)$ are decreasing as functions of c and b.

On the other hand, since one can check that the proof of Lemma 4.3.14 also holds when we condition on $\mathcal{E}_T^{\bullet-}$ rather than $\mathcal{E}_T^\bullet$, we see that there must also be a universal constant K_3 such that

$$\mathbb{W}(S^- < K_3 \text{ and } S^+ < K_3 \,|\, \mathcal{E}_T^{\bullet-}, L_0(T) = c, L_{W_T}(T) = b) \ge \frac{1}{2}$$

for all $T \ge 1$, $c, b \in \left[0, \frac{1}{4}\right)$. Therefore if we are given $c, b \in \left[0, \frac{1}{4}\right)$ we can lower bound $\mathbb{W}(\mathcal{E}_T^{\bullet-} \,|\, L_0(T) = c, L_{W_T}(T) = b)$ by

$$\begin{aligned}\mathbb{W}(\mathcal{E}_T^{\bullet-} \,|\, L_0(T) = c, L_{W_T}(T) = b) &\ge \frac{1}{2}\,\mathbb{W}(\mathcal{E}_T^{\bullet-} \,|\, L_0(T) = c, L_{W_T}(T) = b \text{ and } S^-, S^+ < K_3) \\ &\ge \frac{1}{2} \inf_{0\le s\le K_3} \mathbb{Y}_c^0\big(Y_x \le \tfrac{1}{2} \,|\, \textstyle\int_0^\infty Y_x\,\mathrm{d}x = s\big) \times \inf_{0\le s\le K_3} \mathbb{Y}_b^0\big(Y_x \le 1 \,|\, \textstyle\int_0^\infty Y_x\,\mathrm{d}x s\big) \times \tilde{Q}_2(c,b,T).\end{aligned}$$

We can now find constants $k_4, k_5 > 0$ such that

$$\begin{aligned}&\inf_{c\in\left[0,\frac{1}{4}\right)} \inf_{0\le s\le K_3} \mathbb{Y}_c^0\big(Y_x^- \le \tfrac{1}{2} \text{ for all } x \ge 0 \,|\, \textstyle\int_0^\infty Y_x\,\mathrm{d}x = s\big) \ge k_4 \\ \text{and} \quad &\inf_{c\in\left[0,\frac{1}{4}\right)} \inf_{0\le s\le K_3} \mathbb{Y}_c^0\big(Y_x^- \le 1 \text{ for all } x \ge 0 \,|\, \textstyle\int_0^\infty Y_x\,\mathrm{d}x = s\big) \ge k_5,\end{aligned}$$

Therefore by integrating and comparing with (4.3.36) we get

$$\begin{aligned}\mathbb{W}(\mathcal{E}_T^{\bullet-}) &\ge \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \mathbb{W}(\mathcal{E}_T^{\bullet-} \,|\, L_0(T) = c, L_{W_T}(T) = b) f_T(c,b)\,\mathrm{d}c\,\mathrm{d}b \\ &\ge \frac{k_4 k_5}{2} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \tilde{Q}_2(c,b,T) f_T(c,b)\,\mathrm{d}c\,\mathrm{d}b \ge \frac{k_4 k_5}{64 K_2}\,\mathbb{W}(\mathcal{E}_T^\bullet).\end{aligned}$$

Hence we have proved (4.3.29) with $\eta = \left(\dfrac{k_4 k_5}{64 K_2}\right)^2$. □

4.3.2 The weak convergence of $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$

To prove that $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ has a unique weak limit as $T \longrightarrow \infty$ we shall follow the techniques developed by Benjamini and Berestycki in [BB10]. By doing so we shall in fact prove that the measures $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ form a Cauchy sequence with respect to the total variation distance on sets restricted to $\mathcal{F}_R$ (for each fixed $R > 0$). It turns out that this is a stronger condition than that of weak convergence, and so we start by presenting a lemma which shows that controlling the total variation distance (for each fixed $R > 0$) does indeed imply weak convergence in the Skorokhod topology. It will then suffice to prove that $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ is a Cauchy sequence in this sense.

Given $R > 0$ and probability measures $\mathbb{P}$ and $\mathbb{Q}$ on $(\Omega, \mathcal{F}_R)$ we define the total variation metric, d_R, by

$$d_R(\mathbb{P}, \mathbb{Q}) = \sup_{A \in \mathcal{F}_R} |\mathbb{P}(A) - \mathbb{Q}(A)|.$$

We now have the following lemma from [BB10, Lemma 6].

Lemma 4.3.15. *Let $\{\mathbb{P}_T\}_{T>0}$ be a sequence of probability measures on $\mathcal{F}$ which satisfy the following two conditions.*

Condition 1 *For every $R > 0$ the restrictions of $\mathbb{P}_T$ to $\mathcal{F}_R$ form a Cauchy sequence for the distance d_R. I.e. for every $\varepsilon > 0$ there exists $T_{R,\varepsilon}$ such that for all $T, T' \geq T_{R,\varepsilon}$ we have $d_R(\mathbb{P}_T, \mathbb{P}_{T'}) < \varepsilon$.*

Condition 2 *For each fixed $R > 0$*

$$\lim_{k \to \infty} \lim_{T \to \infty} \mathbb{P}_T \left(\sup_{0 \leq t \leq R} |W_t| \geq k \right) = 0. \tag{4.3.37}$$

Then there exists a unique probability measure $\mathbb{P}$ such that $\mathbb{P}_T \longrightarrow \mathbb{P}$ weakly in the Skorokhod topology as $T \longrightarrow \infty$.

To prove that $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ converges to a unique weak limit it now suffices to show that $\{\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)\}_{T>0}$ satisfies Condition 1 and Condition 2. The second condition is required to show that W_t does not escape to infinity in finite time – and thus the limit of $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ is non-trivial. Since we already have the tools to show that $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ satisfies Condition 2 then this is where we shall start.

Proof that $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ satisfies Condition 2 Suppose that $R > 0$ is fixed, then for each $\varepsilon > 0$ we need to find an M such that

$$\mathbb{W}\left(\sup_{0\le t\le R} |W_t| \ge M \,|\, \mathcal{E}_T^\bullet\right) \le \varepsilon \tag{4.3.38}$$

whenever $T \ge R$. From Lemma 4.3.11 and Lemma 4.3.12 we see that there is a universal constant $\eta > 0$ such that if A is $\mathcal{F}_R$-measurable then

$$\mathbb{W}(A \,|\, \mathcal{E}_T^\bullet) \le \frac{1}{\eta}\, \mathbb{W}(A \,|\, \mathcal{E}_R^\bullet) \tag{4.3.39}$$

for all $T \ge R$. $\sup_{0\le t\le R} |W_t|$ has a Gaussian tail and so we can find an M such that

$$\mathbb{W}\left(\sup_{0\le t\le R} |W_t| \ge M\right) < \varepsilon\, \eta\, \mathbb{W}(\mathcal{E}_R^\bullet).$$

Hence

$$\mathbb{W}\left(\sup_{0\le t\le R} |W_t| \ge M \,|\, \mathcal{E}_T^\bullet\right) \le \frac{1}{\eta}\, \mathbb{W}\left(\sup_{0\le t\le R} |W_t| \ge M \,|\, \mathcal{E}_R^\bullet\right) \le \frac{1}{\eta} \frac{\mathbb{W}\left(\sup |W_t| \ge M\right)}{\mathbb{W}(\mathcal{E}_R^\bullet)} < \varepsilon$$

for all $T \ge R$. □

Proof that $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ satisfies Condition 1 Fix $R > 0$ and $\varepsilon > 0$. Our aim is to find a constant $T_{R,\varepsilon}$ such that

$$|\mathbb{W}(A \,|\, \mathcal{E}_T^\bullet) - \mathbb{W}(A \,|\, \mathcal{E}_{T'}^\bullet)| < \varepsilon, \tag{4.3.40}$$

for every $A \in \mathcal{F}_R$ and all $T, T' \ge T_{R,\varepsilon}$.

As a first step we shall decompose the event A into the disjoint union of well behaved and badly behaved parts, $A = (A \cap B_{R,\varepsilon}) \sqcup (A \setminus B_{R,\varepsilon})$. Provided we can show the probability of $B_{R,\varepsilon} \in \mathcal{F}_R$ is small, $\mathbb{W}(B_{R,\varepsilon} \,|\, \mathcal{E}_T^\bullet) < \frac{1}{2}\varepsilon$ for all $T \ge R$ say, then we can use the identity

$$\mathbb{W}(A \,|\, \mathcal{E}_T^\bullet) = \mathbb{W}(A \,|\, \mathcal{E}_{T'}^\bullet) \frac{\mathbb{W}(\mathcal{E}_{T'}^\bullet \,|\, A \cap \mathcal{E}_T^\bullet)}{\mathbb{W}(\mathcal{E}_{T'}^\bullet \,|\, \mathcal{E}_T^\bullet)}. \tag{4.3.41}$$

to show (4.3.40). In particular if we have

$$\left|1 - \frac{\mathbb{W}(\mathcal{E}_{T'}^\bullet \,|\, A \cap \mathcal{E}_T^\bullet)}{\mathbb{W}(\mathcal{E}_{T'}^\bullet \,|\, \mathcal{E}_T^\bullet)}\right| < \frac{1}{2}\,\varepsilon, \tag{4.3.42}$$

for all $A \in \mathcal{F}_R$ with $A \cap B_{R,\varepsilon} = \emptyset$ and all $T, T' \ge T_{R,\varepsilon}$, then the result is proved.

Claim 4.3.16. *Recall that we defined* $m_R = \inf\{W_t : 0 \le t \le R\}$ *and* $M_R = \sup\{W_t : 0 \le t \le R\}$. *Suppose we set* $\ell_{\max}(R) = \sup\{L_x(R) : x \in \mathbb{R}\}$, *then for each* $\varepsilon > 0$ *we can find* $M > 0$ *and* $k < 1$ *(depending on* R *but not on* T*) for which the bad event*

$$B_{R,\varepsilon} = \{m_R \le -M\} \cup \{M_R \ge M\} \cup \{\ell_{\max}(R) \ge k\} \tag{4.3.43}$$

has probability $\mathbb{W}(B_{R,\varepsilon} \,|\, \mathcal{E}_T^\bullet) < \frac{1}{2}\varepsilon$ *for all* $T \ge R$.

Proof of Claim. From (4.3.38) we know that when M is sufficiently large we have

$$\mathbb{W}(\{m_R \le -M\} \cup \{M_R \ge M\} \,|\, \mathcal{E}_T^\bullet) = \mathbb{W}\left(\sup_{0\le t\le R} |W_t| \ge M \,|\, \mathcal{E}_T^\bullet\right) < \frac{1}{4}\,\varepsilon$$

for all $T \ge R$. Having fixed such a value of M we now observe that when we condition on $\mathcal{E}_R^\bullet$ then then $\ell_{\max}(R)$ is a random variable supported on $[0, 1)$. From Lemma 4.3.11 and Lemma 4.3.12 of Section 4.3.1 we have

$$\begin{aligned}\mathbb{W}(\ell_{\max}(R) \ge y \,|\, \mathcal{E}_T^\bullet) &= \mathbb{W}(\ell_{\max}(R) \ge y \,|\, \mathcal{E}_R^\bullet)\frac{\mathbb{W}(\mathcal{E}_T^\bullet \,|\, \mathcal{E}_R^\bullet \cap \{\ell_{\max}(R) \ge y\})}{\mathbb{W}(\mathcal{E}_T^\bullet \,|\, \mathcal{E}_R^\bullet)} \\ &\le \frac{1}{\eta}\,\mathbb{W}(\ell_{\max}(R) \ge y \,|\, \mathcal{E}_R^\bullet),\end{aligned}$$

where $\eta > 0$ is independent of $T \ge R$. Therefore there must exist $k \in [0, 1)$ such that $\mathbb{W}(\ell_{\max}(R) \ge k \,|\, \mathcal{E}_T^\bullet) \le \frac{1}{4}\varepsilon$ for all $T \ge R$. Summing these probabilities completes the proof. □

By removing the behaviour of the bad event $B_{R,\varepsilon}$ we can ensure that the Brownian motion does not become trapped to the left of M_R, and therefore at large times the effect of the event A becomes negligible. We now aim to couple two processes $(W_t)_{t\ge0}$ and $(\tilde{W}_t)_{t\ge0}$ with respective laws $\mathbb{W}(\cdot \,|\, A \cap \mathcal{E}_T^\bullet)$ and $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$ in such a way that their local times agree on a large region to the right of M_R. More explicitly, suppose we are given two processes $(W_t)_{t\ge0}$ and $(\tilde{W}_t)_{t\ge0}$ with laws $\mathbb{W}(\cdot \,|\, A \cap \mathcal{E}_T^\bullet)$ and $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$. Let their respective local time profiles be $(L_x(t))_{x\in\mathbb{R},t\ge0}$ and $(\tilde{L}_x(t))_{x\in\mathbb{R},t\ge0}$, then we seek a pair of levels $X, \tilde{X} \in \mathbb{R}$ such that

$$\begin{aligned} X > M_R = \sup\{W_t : 0 \le t \le R\}, \quad \tilde{X} > 0, \quad L_X(T) = \tilde{L}_{\tilde{X}}(T) \\ \text{and} \quad \int_{-\infty}^{X} L_x(T)\,\mathrm{d}x = \int_{-\infty}^{\tilde{X}} \tilde{L}_x(T)\,\mathrm{d}x. \end{aligned} \tag{4.3.44}$$

Our coupling will then replace $(\tilde{W}_t)_{t\ge0}$ with a new process $(\hat{W}_t)_{t\ge0}$ whose local time

$(\hat{L}_x(t))_{x\in\mathbb{R},t\geq 0}$ is equal in law to $(\tilde{L}_x(t))_{x\in\mathbb{R},t\geq 0}$ and is such that

$$\hat{L}_x(T) = \begin{cases} \tilde{L}_x(T) & \text{for all } x \leq \tilde{X} \\ L_{X-\tilde{X}+x}(T) & \text{for all } x \geq \tilde{X} \end{cases}. \tag{4.3.45}$$

Definition 4.3.17 (Coupling)**.** Fix $A \in \mathcal{F}_R$ with $A \cap B_{R,\varepsilon} = \emptyset$, and let $T \geq R$. Suppose $(W_t)_{t\geq 0}$ is a process with law $\mathbb{W}(\cdot \,|\, A \cap \mathcal{E}_T^\bullet)$ and $(\tilde{W}_t)_{t\geq 0}$ is an independent process with law $\mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$. Define the random variables X, $\tilde{X}$ and Δ_T by

$$X = \inf\left\{z > M_R : \exists\, \tilde{z} > 0 \text{ with } L_z(T) = L_{\tilde{z}}(T) \text{ and } \textstyle\int_{-\infty}^{z} L_x(T)\,\mathrm{d}x = \int_{-\infty}^{\tilde{z}} \tilde{L}_x(T)\,\mathrm{d}x\right\}$$
$$\tilde{X} = \inf\left\{\tilde{z} > 0 : \textstyle\int_{-\infty}^{\tilde{z}} \tilde{L}_x(T)\,\mathrm{d}x = \int_{-\infty}^{X} L_x(T)\,\mathrm{d}x\right\}$$
$$\Delta_T = \begin{cases} W_T - X & \text{if } W_T > X \text{ and } \tilde{W}_T > \tilde{X} \\ -\infty & \text{otherwise} \end{cases}.$$

We now define a new process $(\hat{W}_t)_{t\geq 0}$ by considering two cases.

1. If $X \geq W_T$ or $\tilde{X} \geq \tilde{W}_T$ then set $(\hat{W}_t)_{t\geq 0} = (\tilde{W}_t)_{t\geq 0}$.
2. Otherwise we have $X < W_T$ and $\tilde{X} < \tilde{W}_T$. It is known from Itô's theory of excursions that a Brownian motion started at 0 can be decomposed into a sequence of positive and negative excursions from 0, and that the excursions are indexed by the local time at level 0. For a reference to excursion theory see [RY99, Chapter XII]. Therefore, because the local times at level 0 of $(W_t - X)_{0\leq t\leq T}$ and $(\tilde{W}_t - \tilde{X})_{0\leq t\leq T}$ are equal at time T, then we can form $(\hat{W}_t - \tilde{X})_{0\leq t\leq T}$ by taking the negative excursions of $(\tilde{W}_t - \tilde{X})_{0\leq t\leq T}$ and combining with the positive excursions of $(W_t - X)_{0\leq t\leq T}$. By doing this we ensure that $\hat{L}_x(T) = L_x(T)$ for all $x \leq \tilde{X}$ and $\hat{L}_x(T) = L_{X-\tilde{X}+x}(T)$ for all $x \geq \tilde{X}$, and so (4.3.45) is satisfied.

Write $\mathbb{J}_{T,A}$ for the joint law of $(W_t)_{0\leq t\leq T}$ and $(\hat{W}_t)_{0\leq t\leq T}$. It is clear from our construction that the first marginal satisfies

$$\mathbb{J}_{T,A}^{(1)}\big((L_x(T))_{x\in\mathbb{R}} \in \cdot\big) = \mathbb{W}\big((L_x(T))_{x\in\mathbb{R}} \in \cdot \,|\, A \cap \mathcal{E}_T^\bullet\big). \tag{4.3.46}$$

Our construction also gives

$$\mathbb{J}_{T,A}^{(2)}\big((\hat{L}_{\tilde{X}-x}(T))_{x\geq 0} \in \cdot\big) = \mathbb{W}\big((\tilde{L}_{\tilde{X}-x}(T))_{x\geq 0} \in \cdot \,|\, \mathcal{E}_T^\bullet\big).$$

Therefore we now fix $(\hat{L}_{\tilde{X}-x}(T))_{x\geq 0} = (Y_x)_{x\geq 0}$ and consider

$$\begin{aligned}&\mathbb{J}^{(2)}_{T,A}\big((\hat{L}_{\tilde{X}+x}(T))_{x\geq 0} \in \cdot \,|\, (\hat{L}_{\tilde{X}-x}(T))_{x\geq 0} = (Y_x)_{x\geq 0}\big)\\ &\qquad = \mathbb{W}\big((L_{X+x}(T))_{x\in\mathbb{R}} \in \cdot \,|\, A\cap \mathcal{E}^{\bullet}_T, (L_{X-x}(T))_{x\geq 0} = (Y_x)_{x\geq 0}\big).\end{aligned}$$

From Theorem 2.3.3, and the fact that a square Bessel process is Markovian, we see that when we condition on $W_T > X$ then the distribution of $(L_{X+x}(T))_{x\geq 0}$ depends only on $L_X(T)$ and $\int_{-\infty}^{X} L_x(T)\,\mathrm{d}x$. We also observe that because A is $\mathcal{F}_R$ measurable and $M_R < X$ then, as far as the distribution of $(L_{X+x}(T))_{x\in\mathbb{R}}$ is concerned, conditioning on $\{A\cap\mathcal{E}^{\bullet}_T, (L_{X-x}(T))_{x\geq 0} = (Y_x)_{x\geq 0}\}$ is equivalent to conditioning on $\{\mathcal{E}^{\bullet}_T, (L_{X-x}(T))_{x\geq 0} = (Y_x)_{x\geq 0}\}$. Hence if we know that $(L_{X-x}(T))_{x\geq 0}$ has $L_X(T) = y$ and $\int_{-\infty}^{X} L_x(T)\,\mathrm{d}x = r$ then

$$\begin{aligned}&\mathbb{J}^{(2)}_{T,A}\big((\hat{L}_{\tilde{X}+x}(T))_{x\geq 0} \in \cdot \,|\, (\hat{L}_{\tilde{X}-x}(T))_{x\geq 0} = (Y_x)_{x\geq 0}\big)\\ &\qquad = \mathbb{W}\big((L_{X+x}(T))_{x\in\mathbb{R}} \in \cdot \,|\, A\cap\mathcal{E}^{\bullet}_T, L_X(T) = y, \textstyle\int_{-\infty}^{X} L_x(T)\,\mathrm{d}x = r\big)\\ &\qquad = \mathbb{W}\big((\tilde{L}_{\tilde{X}+x}(T))_{x\in\mathbb{R}} \in \cdot \,|\, \mathcal{E}^{\bullet}_T, \tilde{L}_{\tilde{X}}(T) = y, \textstyle\int_{-\infty}^{\tilde{X}} \tilde{L}_x(T)\,\mathrm{d}x = r\big)\\ &\qquad = \mathbb{W}\big((\tilde{L}_{\tilde{X}+x}(T))_{x\in\mathbb{R}} \in \cdot \,|\, \mathcal{E}^{\bullet}_T, (\tilde{L}_{\tilde{X}-x}(T))_{x\geq 0} = (Y_x)_{x\geq 0}\big).\end{aligned}$$

Since this holds for all choices of $(Y_x)_{x\geq 0}$ then by interating we can conclude that

$$\mathbb{J}^{(2)}_{T,A}\big((\tilde{L}_x(T))_{x\in\mathbb{R}} \in \cdot\big) = \mathbb{W}\big((\tilde{L}_x(T))_{x\in\mathbb{R}} \in \cdot \,|\, \mathcal{E}^{\bullet}_T\big), \tag{4.3.47}$$

as desired.

Having constructed this coupling our next step is to show that Δ_T is large with high $\mathbb{J}_{T,A}$-probability. This would mean that there is a large region of space on which the local time profiles of $(W_t)_{0\leq t\leq T}$ and $(\hat{W}_t)_{0\leq t\leq T}$ agree. Later we shall use this property to show that the ratio between $\mathbb{W}(\mathcal{E}^{\bullet}_{T'} \,|\, (W_t)_{0\leq t\leq T})$ and $\mathbb{W}(\mathcal{E}^{\bullet}_{T'} \,|\, (\hat{W}_t)_{0\leq t\leq T})$ becomes arbitrarily close to 1 as $T, T' \longrightarrow \infty$.

Claim 4.3.18. *Given $A \in \mathcal{F}_R$ with $A\cap B_{R,\varepsilon} = \emptyset$, let $\{\mathbb{J}_{T,A}\}_{T\geq R}$ be the sequence of joint laws defined by Definition 4.3.17. We then have that*

$$\mathbb{J}_{T,A}\left(\Delta_T \geq \tfrac{1}{2}T\right) \longrightarrow 1 \quad \textit{as} \quad T\longrightarrow\infty. \tag{4.3.48}$$

What is more, this convergence is uniform over all events $A \in \mathcal{F}_R$ with $A\cap B_{R,\varepsilon} = \emptyset$.

Proof of Claim. Given $(W_t)_{t\geq 0} \in A \cap \mathcal{E}_T^\bullet$ and $(\tilde{W}_t)_{t\geq 0} \in \mathcal{E}_T^\bullet$ we define

$$\Sigma_{R,T} = \max\left\{\int_{-\infty}^{M_R} L_x(T)\,\mathrm{d}x, \int_{-\infty}^{0} \tilde{L}_x(T)\,\mathrm{d}x\right\}.$$

Now let $\delta > 0$. Our first task is to show that we can always find a K_1 (depending on R, but not on T or A such that

$$\mathbb{J}_{T,A}(\Sigma_{R,T} \geq K_1) < \frac{1}{3}\,\delta. \tag{4.3.49}$$

Let M be as given by Claim 4.3.16, and define $r = \int_{-\infty}^{-M} L_x(T)\,\mathrm{d}x$. Because $A \cap B_{R,\varepsilon} = \emptyset$ then we must have $M_R \leq M$. Therefore as $L_x(T) \leq 1$ for all $x \in \mathbb{R}$ it must follow that $\int_{-\infty}^{M_R} L_x(T)\,\mathrm{d}x \leq 2M + r$. Now suppose we condition on $(W_t)_{0\leq t\leq R} = (V_t)_{0\leq t\leq R}$ and $(L_x(T))_{-M\leq x\leq M} = (Y_x)_{-M\leq x\leq M}$. If we assume that $\sup_{0\leq t\leq R} |V_t| \leq M$ then we can see from Theorem 2.3.3 that the law of $(L_x(T))_{x\in\mathbb{R}\setminus[-M,M]}$ depends only on the values of Y_{-M}, Y_M and $\int_{-M}^{M} Y_x\,\mathrm{d}x$ (and not on $(V_t)_{0\leq t\leq R}$). Therefore, if we also condition on $(L_{W_T+x}(T))_{x\geq 0} = (Y_x^+)_{x\geq 0}$ with $\int_0^\infty Y_x^+\,\mathrm{d}x = s^+$, then we can follow the argument of Lemma 4.3.14 and replace (4.3.34) by

$$\begin{aligned}
&\frac{\mathbb{W}(\{r > a\} \cap \mathcal{E}_T^\bullet \,|\, V, Y, Y^+)}{\mathbb{W}(\mathcal{E}_T^\bullet \,|\, V, Y, Y^+)} \\
&\quad\leq \frac{\mathbb{W}(\{r > a\} \cap \{L_x(T) \leq 1 \text{ for all } x \in \mathbb{R}\setminus[-M,M]\} \,|\, V, Y, Y^+)}{\mathbb{W}(\mathcal{E}_T^\bullet \,|\, V, Y, Y^+)} \\
&\quad\leq \frac{T}{k_2} \sup_{s>a} \sup_{c\in[0,1)} \frac{\sup_{s>a} Q_0(c,s)}{\inf_{s\in[0,1]} Q_0(c,s)} \sup_{c,b\in[0,1)} \frac{\tilde{Q}_2(T - 2M - s^+ - a, c, b)}{\tilde{Q}_2(T - s^+, c, b)}.
\end{aligned}$$

Since our assumption on the event A implies that $\sup_{0\leq t\leq R} |W_t| \leq M$ we can now integrate over A and deduce that there exists a universal constant $\Xi \leq K_2 < \infty$ such that

$$\mathbb{W}(\{r > a\} \cap \mathcal{E}_T^\bullet \,|\, A) \leq K_2\,\mathrm{e}^{-\xi a}\,\mathbb{W}(\mathcal{E}_T^\bullet),$$

for all $A \in \mathcal{F}_R$ with $A \cap B_{R,\varepsilon} = \emptyset$, all $T \geq R$ and all $a \geq 0$. Here $\xi > 0$ is as given by Lemma 4.3.14. On the other hand, our assumption that $A \cap B_{R,\varepsilon} = \emptyset$ means that $L_x(R) \leq k < 1$ for all $x \in \mathbb{R}$. Therefore if we let τ_{M+1} be the first hitting time of $M + 1$ (and assume $k \geq \frac{1}{2}$), then the intersection of the events

$$\begin{aligned}
A \cap \{&L_x(\tau_{M+1}) - L_x(R) \leq 1 - k\,\forall\, x \in \mathbb{R}\} \cap \{L_x(T) - L_x(\tau_{M+1}) \leq 1\,\forall\, x \geq M + 1, \\
&L_x(T) - L_x(\tau_{M+1}) \leq \tfrac{1}{2}\,\forall\, M < x < M + 1 \text{ and } L_x(T) - L_x(\tau_{M+1}) = 0 \text{ otherwise}\}
\end{aligned}$$

is contained within $\mathcal{E}_T^\bullet$. The probability of $\{L_x(\tau_{M+1}) - L_x(R) \leq 1 - k \text{ for all } x \in \mathbb{R}\}$

is minimised when W_R is minimal. Therefore, since our conditions on A imply that $W_R \geq -M$, we find that there must be a positive quantity $\zeta(M,k) > 0$ such that

$$\begin{aligned}\mathbb{W}(L_x(\tau_{M+1}) - L_x(R) \leq 1-k \text{ for all } x \in \mathbb{R} \,|\, A) \\ \geq \mathbb{W}(L_x(\tau_{2M+1}) \leq 1-k \text{ for all } x \in \mathbb{R} \,|\, A) \geq \zeta(M,k) > 0.\end{aligned}$$

Note that since M and k depend only on R and ε then so too does $\zeta(M,k)$. The probability of the third event is equal to $\mathbb{W}(\mathcal{E}^{\bullet -}_{T-\tau_{M+1}})$, where $\mathcal{E}^{\bullet -}_{T-\tau_{M+1}}$ is defined in the proof of Lemma 4.3.12. As we know that $\mathbb{W}(\mathcal{E}^{\bullet -}_{T-\tau_{M+1}}) \geq \mathbb{W}(\mathcal{E}^{\bullet -}_{T}) \geq \eta\, \mathbb{W}(\mathcal{E}^{\bullet}_{T})$ then we get

$$\mathbb{W}(\{r > a\} \,|\, A \cap \mathcal{E}^{\bullet}_T) = \frac{\mathbb{W}(\{r > a\} \cap \mathcal{E}^{\bullet}_T) \,|\, A)}{\mathbb{W}(\mathcal{E}^{\bullet}_T \,|\, A)} \leq \frac{K_2\, \mathrm{e}^{-\xi a}\, \mathbb{W}(\mathcal{E}^{\bullet}_T)}{\zeta(M,k)\, \eta\, \mathbb{W}(\mathcal{E}^{\bullet}_T)} \leq K_3\, \mathrm{e}^{-\xi a},$$

for some $K_3 < \infty$ which depends only on R and ε. From this we conclude that there must exist a $K_4 < \infty$ such that

$$\mathbb{J}^{(1)}_{T,A}\left(\textstyle\int_{-\infty}^{M_R} L_x(T)\,\mathrm{d}x \geq K_4\right) < \frac{1}{6}\,\delta,$$

For all $A \in \mathcal{F}_R$ with $A \cap B_{R,\varepsilon} = \emptyset$ and all $T \geq R$. Lemma 4.3.14 also tells us that we can find a K_5 such that

$$\mathbb{J}^{(2)}_{T,A}\left(\textstyle\int_{-\infty}^{0} \tilde{L}_x(T)\,\mathrm{d}x \geq K_5\right) < \frac{1}{6}\,\delta,$$

for all A and $T \geq R$. Thus we can conclude that (4.3.49) holds for $K_1 = \max\{K_4, K_5\}$.

Using symmetry we can see that $S^+ = \int_{W_T}^{\infty} L_x(T)\,\mathrm{d}x$ is equal in law to $S^- = \int_{-\infty}^{0} L_x(T)\,\mathrm{d}x$. Therefore it is a simple corollary of Lemma 4.3.14 to find a $K_6 < \infty$ such that

$$\mathbb{J}^{(1)}_{T,A}(S^+ > K_6) < \frac{1}{6}\,\delta \quad \text{and} \quad \mathbb{J}^{(2)}_{T,A}(\tilde{S}^+ > K_6) < \frac{1}{6}\,\delta$$

for all A and $T \geq R$. By combining this with 4.3.49) we see that

$$\mathbb{J}_{T,A}\left(\Sigma_{R,T} < \tfrac{1}{6}T \text{ and } \min\left\{\textstyle\int_{-\infty}^{W_T} L_x(T)\,\mathrm{d}x, \int_{-\infty}^{\tilde{W}_T} \tilde{L}_x(T)\,\mathrm{d}x\right\} > \tfrac{5}{6}T\right) \leq \frac{2}{3}\,\delta \qquad (4.3.50)$$

for all A and all $T \geq \max\{R, 6K_1, 6K_6\}$.

Recall that in Definition 4.3.1 we constructed $(Z_t)_{t\geq 0}$ as an axillary process to $(L_x(T))_{x\geq 0}$ and that $(Z_t)_{t\geq 0}$ satisfies $Z_t = L_{\rho(t)}(T)$, where $\rho(t)$ is such that $\int_0^{\rho(t)} L_x(T)\,\mathrm{d}x = t$. Let $(\tilde{Z}_t)_{t\geq 0}$ be the axillary process to $(\tilde{L}_x(T))_{x\geq 0}$, $S^- = \int_{-\infty}^{0} L_x(T)\,\mathrm{d}x$ and $\tilde{S}^- = \int_{-\infty}^{0} \tilde{L}_x(T)\,\mathrm{d}x$. Suppose we can find a $t \geq \Sigma_{R,T}$ such that $Z_{t-S^-} = \tilde{Z}_{t-\tilde{S}^-}$. If we put $X = \rho(t - S^-)$

and $\tilde{X} = \tilde{\rho}(t - \tilde{S}^-)$ then we can check that $X > M_R$, $\tilde{X} > 0$, $L_X(T) = \tilde{L}_{\tilde{X}}(T)$ and $\int_{-\infty}^{X} L_x(T)\,\mathrm{d}x = \int_{-\infty}^{\tilde{X}} \tilde{L}_x(T)\,\mathrm{d}x$. Thus X and $\tilde{X}$ satisfy (4.3.44).

If we assume that $\Sigma_{R,T} < \frac{1}{6}T$ and $\min\left\{\int_{-\infty}^{W_T} L_x(T)\,\mathrm{d}x, \int_{-\infty}^{\tilde{W}_T} \tilde{L}_x(T)\,\mathrm{d}x\right\} > \frac{5}{6}T$, then $(Z_{t-S^-})_{\Sigma_{R,T}\le t\le \frac{7}{8}T}$ and $(\tilde{Z}_{t-\tilde{S}^-})_{\Sigma_{R,T}\le t\le \frac{5}{6}T}$ are both Markov processes which are conditioned to stay in $[0,1]$, and with infinitesimal generator given by (4.3.6). For a fixed $\Sigma_{R,T} \le t \le \frac{5}{6}T - 1$ consider $\mathbb{J}_{T,A}(Z_{t-S^-+s} = \tilde{Z}_{t-\tilde{S}^-+s}$ for some $s \in [0,1])$. This probability will depend on Z_{t-S^-} and $\tilde{Z}_{t-\tilde{S}^-}$, but must be positive for all $(Z_{t-S^-}, \tilde{Z}_{t-\tilde{S}^-}) \in [0,1]^2$. Therefore, by compactness, we find a $k_7 > 0$ such that

$$\mathbb{J}_{T,A}(Z_{t+s} = \tilde{Z}_{t+s} \text{ for some } s \in [0,1]) \ge k_7$$

for all possible t, Z_{t-S^-} and $\tilde{Z}_{t-\tilde{S}^-}$. If K_7 is large enough then $(1-k_7)^{K_7} < \frac{1}{3}\delta$. Therefore we can use that that Z and $\tilde{Z}$ are Markovian to deduce that

$$\mathbb{J}_{T,A}\left(\text{there exists } \Sigma_{R,T} < t < \tfrac{1}{3}T \text{ such that } Z_{t-S^-} = \tilde{Z}_{t-\tilde{S}^-}\right) > 1 - \frac{1}{3}\,\delta, \tag{4.3.51}$$

for all $T \ge 6K_7$. If we assume that the events given by (4.3.50) and (4.3.51) both hold then we must have $\tilde{W}_T > \tilde{X}$ and $\int_X^{W_T} L_x(T)\,\mathrm{d}x \ge \frac{1}{2}T$. Since $L_x(T) \le 1$ for all $x \in \mathbb{R}$ we must then also have $\Delta_T = W_T - X > \frac{1}{2}T$ and so

$$\mathbb{J}_{T,A}\left(\Delta_T \ge \tfrac{1}{2}T\right) > 1 - \delta$$

for all $A \in \mathcal{F}_R$ with $A \cap B_{R,\varepsilon} = \emptyset$ and all $T \ge \max\{R, 6K_1, 6K_6, 6K_7\}$. As δ was arbitrary then (4.3.48) is now proved. □

We now have the tools we need to estimate (4.3.42). Given $T \le T'$, define $m_{T,T'} = \inf\{W_t - W_T : T \le t \le T'\}$, and observe that we can partition the event $\mathcal{E}^\bullet_{T'}$ as $\mathcal{E}^\bullet_{T'} = (\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} \ge -\Delta_T\}) \sqcup (\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} < -\Delta_T\})$. Therefore (4.3.42) becomes

$$\begin{aligned}\left|1 - \frac{\mathbb{W}(\mathcal{E}^\bullet_{T'} \mid A \cap \mathcal{E}^\bullet_T)}{\mathbb{W}(\mathcal{E}^\bullet_{T'} \mid \mathcal{E}^\bullet_T)}\right| &= \left|1 - \frac{\mathbb{J}^{(1)}_{T,A}(\mathcal{E}^\bullet_{T'})}{\mathbb{J}^{(2)}_{T,A}(\mathcal{E}^\bullet_{T'})}\right| \\ &= \left|1 - \frac{\mathbb{J}^{(1)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} \ge -\Delta_T\}) + \mathbb{J}^{(1)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} < -\Delta_T\})}{\mathbb{J}^{(2)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} \ge -\Delta_T\}) + \mathbb{J}^{(2)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} < -\Delta_T\})}\right|. \end{aligned} \tag{4.3.52}$$

We now claim that $\mathbb{J}^{(1)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} \ge -\Delta_T\})$ and $\mathbb{J}^{(2)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} \ge -\Delta_T\})$ are equal. Indeed, suppose we are given $\Delta_T > 0$ and $(W_t)_{0\le t\le T} \in A \cap \mathcal{E}^\bullet_T$ (or $(\hat{W}_t)_{0\le t\le T}\mathcal{E}^\bullet_T$). The probability of the event $\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} \ge -\Delta_T\}$ is then exactly equal to the probability

that an independent Brownian motion $(\bar{W}_t)_{0\leq t\leq T'-T}$ has $\bar{L}_x(T'-T) \leq 1 - L_{W_t+x}(T)$ (or $\bar{L}_x(T'-T) \leq 1 - \hat{L}_{\hat{W}_t+x}(T)$) for all $x > -\Delta_T$ and $\bar{L}_x(T'-T) = 0$ otherwise. Because of the way that $\mathbb{J}_{T,A}$ was constructed in Definition 4.3.17 we know that the distribution of $(L_{W_T+x}(T))_{x\geq-\Delta_T}$ with respect to $\mathbb{J}^{(1)}_{T,A}$ is equal to the distribution of $(\hat{L}_{\hat{W}_T+x}(T))_{x\geq-\Delta_T}$ with respect to $\mathbb{J}^{(2)}_{T,A}$. Thus the equality of $\mathbb{J}^{(1)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} < \Delta_T\})$ and $\mathbb{J}^{(2)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} < \Delta_T\})$ must hold. From equation (4.3.52) we now get

$$\begin{aligned}\left|1 - \frac{\mathbb{W}(\mathcal{E}^\bullet_{T'} \,|\, A \cap \mathcal{E}^\bullet_T)}{\mathbb{W}(\mathcal{E}^\bullet_{T'} \,|\, \mathcal{E}^\bullet_T)}\right| &\leq \left|\frac{\mathbb{J}^{(1)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} < -\Delta_T\})}{\mathbb{J}^{(1)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} \geq -\Delta_T\})} - \frac{\mathbb{J}^{(2)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} < -\Delta_T\})}{\mathbb{J}^{(2)}_{T,A}(\mathcal{E}^\bullet_{T'} \cap \{m_{T,T'} \geq -\Delta_T\})}\right| \\ &\leq \left|\mathbb{J}^{(1)}_{T,A}(m_{T,T'} < -\Delta_T \,|\, \mathcal{E}^\bullet_{T'}) - \mathbb{J}^{(2)}_{T,A}(m_{T,T'} < -\Delta_T \,|\, \mathcal{E}^\bullet_{T'})\right|.\end{aligned}$$

By using Lemma 4.3.14 it can be shown that the tail of $m_{T,T'}$ can be uniformly bounded over all $R \leq T \leq T'$. Since Claim 4.3.18 tells us that Δ_T becomes arbitrarily large as $T \longrightarrow \infty$ we deduce that $\mathbb{J}^{(1)}_{T,A}(m_{T,T'} < -\Delta_T \,|\, \mathcal{E}^\bullet_{T'})$ and $\mathbb{J}^{(2)}_{T,A}(m_{T,T'} < -\Delta_T \,|\, \mathcal{E}^\bullet_{T'})$ must converge to 0 as $T \longrightarrow \infty$. Therefore we can find a $T_{R,\varepsilon}$ such that $\mathbb{J}^{(1)}_{T,A}(m_{T,T'} < -\Delta_T \,|\, \mathcal{E}^\bullet_{T'}) < \frac{1}{4}\varepsilon$ and $\mathbb{J}^{(2)}_{T,A}(m_{T,T'} < -\Delta_T \,|\, \mathcal{E}^\bullet_{T'}) < \frac{1}{4}\varepsilon$ for all $A \in \mathcal{F}_R$ and all $T_{R,\varepsilon} \leq T \leq T'$, and so (4.3.42) is satisfied. □

Since both Condition 1 and Condition 2 are satisfied then Lemma 4.3.15 tells us that $\mathbb{W}(\cdot \,|\, \mathcal{E}^\bullet_T)$ must converge weakly as $T \longrightarrow \infty$, and so Theorem 4.1.1 is proved. □

4.4 Proof of Theorem 4.1.3

In Benjamini and Berestycki's paper, [BB10], it is shown that when we condition on $\mathcal{E}_a^*$ then $(L_x(\tau_a))_{x\geq 0}$ converges to a process $(L_x(\infty))_{x\geq 0}$ with stationary distribution μ^* as $a \longrightarrow \infty$. As a first step to proving Theorem 4.1.3 we shall now show that $(L_x(\infty))_{x\geq 0}$ also has a stationary distribution, $\mu^\bullet$, with respect to $\mathbb{Q}^\bullet = \lim_{T\to\infty} \mathbb{W}(\cdot \,|\, \mathcal{E}_T^\bullet)$. Having done this we then complete the proof using calculations of Radon–Nikodym derivatives.

4.4.1 The stationary distribution of $(L_x(\infty))_{x\geq 0}$

As a consequence of Lemma 4.2.3 we know that there is a unique measure $\mu^\bullet \in E_C = \{\mu \in \mathcal{P}(\mathbb{R}) : \text{support}(\mu) \subseteq [0,1]\}$ which minimises $\mathbb{E}(\mu)^{-1} I_2(\mu)$ over all $\mu \in E_C$. By using the Donsker–Varadhan Theorem we shall now show that $\mu^\bullet$ gives the stationary distribution of $(L_x(\infty))_{x\geq 0} = \lim_{T\to\infty} (L_x(T))_{x\geq 0}$ with respect to $\mathbb{Q}^\bullet$.

Lemma 4.4.1. *Let $\mu^\bullet$ be as defined above, suppose $(Y_x)_{x\geq 0}$ is a* BES$Q^2(y)$ *process for some $c \in [0,1)$, and recall that we defined $\rho(s) = \inf\{u : \int_0^u Y_x \,\mathrm{d}x \geq s\}$. If we now condition on the event $\{Y_x \leq 1$ for all $0 \leq x \leq \rho(s)\}$ then $L((Y_x), \rho(s), \cdot)$ converges in $\mathbb{Y}_c$-probability to $\mu^\bullet$ as $s \longrightarrow \infty$.*

Proof. Let $U \subseteq \mathcal{P}(\mathbb{R})$ be any open set (with respect to the weak topology) containing $\mu^\bullet$. It now suffices to prove that

$$\begin{aligned}\liminf_{s\to\infty} \frac{1}{s} \log \mathbb{Y}_c(L((Y_x), \rho(c), \cdot) \in U \cap E_C) \\ - \limsup_{s\to\infty} \frac{1}{s} \log \mathbb{Y}_c(L((Y_x), \rho(s), \cdot) \in U^{\mathrm{C}} \cap E_C) > 0. \end{aligned} \tag{4.4.1}$$

This would imply that the ratio between $\mathbb{Y}(L((Y_x), \rho(s), \cdot) \in U \,|\, Y_x \leq 1$ for all $x \in [0, \rho(s)])$ and $\mathbb{Y}(L((Y_x), \rho(s), \cdot) \in U^{\mathrm{C}} \,|\, Y_x \leq 1$ for all $x \in [0, \rho(s)])$ tends to infinity as $s \longrightarrow \infty$. Since this holds for arbitrary $U \ni \mu^\bullet$ then $L((Y_x), \rho(s), \cdot)$ must converge to $\mu^\bullet$ in $\mathbb{Q}^\bullet$-probability.

Define $E_O = \{\mu \in \mathcal{P}(\mathbb{R}) : \text{support}(\mu) \subseteq [0,1)\} \subseteq E_C$. Since both E_O and $U \cap E_O$ are open and so we can apply Lemma 4.3.3 and Lemma 4.2.3 to get

$$\begin{aligned}&\liminf_{s\to\infty} \frac{1}{s} \log \mathbb{Y}_c(L((Y_x), \rho(s), \cdot) \in U \cap E_C) \\ &\quad \geq \liminf_{s\to\infty} \frac{1}{s} \log \mathbb{Y}_c(L((Y_x), \rho(s), \cdot) \in U \cap E_O) \geq - \inf_{\mu \in E_O \cap U} \frac{I_2(\mu)}{\mathbb{E}(\mu)} = -\frac{I_2(\mu^\bullet)}{\mathbb{E}(\mu^\bullet)}. \end{aligned} \tag{4.4.2}$$

Likewise $E_C \cap U^{\mathrm{C}}$ is closed and so from Lemma 4.3.3 and Lemma 4.2.3 we have

$$\limsup_{s\to\infty} \frac{1}{s} \log \mathbb{Y}_c(L((Y_x), \rho(s), \cdot) \in U^{\mathrm{C}} \cap E_C) \leq - \inf_{\mu \in E_C \cap U^{\mathrm{C}}} \frac{I_2(\mu)}{\mathbb{E}(\mu)} < -\frac{I_2(\mu^\bullet)}{\mathbb{E}(\mu^\bullet)}. \tag{4.4.3}$$

Here there is a strict inequality because the minimiser of $\mathbb{E}(\mu)^{-1} I_2(\mu)$ over $\mu \in E_C$ is unique. By combining (4.4.2) and (4.4.3) we can deduce (4.4.1) and so the lemma is proved. □

Now use $(Z_t)_{t\geq 0}$ to denote the auxiliary process to $(Y_x)_{x\geq 0}$. From the results of [Pin85b] we know that if we condition on $Z_t \leq 1$ for all $0 \leq t \leq s$ then as $s \longrightarrow \infty$ $(Z_t)_{t\geq 0}$ will converge to some stationary process. Consequentially if we condition on $Y_x \leq 1$ for all $0 \leq x \leq \rho(s)$ then $(Y_x)_{x\geq 0}$ must also converge to some stationary process as $s \longrightarrow \infty$. Furthermore, because we know that the occupation measure of $(Y_x)_{x\geq 0}$ converges to $\mu^\bullet$, then $\mu^\bullet$ must also be the limiting stationary distribution of $(Y_x)_{x\geq 0}$.

In our proof of ballisticity (Section 4.3.1) we showed that if $(L_x(T))_{x\in\mathbb{R}}$ is the local time of a Brownian motion, $(W_t)_{t\geq 0}$, conditioned on $\mathcal{E}_T^\bullet$ then $(L_x(T))_{0\leq x\leq W_T}$ is equal in law to $(Y_x)_{0\leq x\leq W_T}$. Since $W_T \longrightarrow \infty$ as $T \longrightarrow \infty$ it must follow that, with respect to the law $\mathbb{Q}^\bullet$, $(L_x(\infty))_{x\geq 0}$ has an invariant distribution $\mu^\bullet$.

4.4.2 The tails of μ^* and $\mu^\bullet$

Theorem 4.1.3 now follows fairly easily by showing that there exists constants C^* and $C^\bullet$ with

$$\mu^*((1-\varepsilon, 1]) \sim C^* \varepsilon^3 \quad \text{and} \quad \mu^\bullet((1-\varepsilon, 1]) \sim C^\bullet \varepsilon^3, \tag{4.4.4}$$

as $\varepsilon \longrightarrow 0$. This is the content of the lemma below.

Lemma 4.4.2. *Let $\alpha \in (0,1)$, and suppose μ_α is the unique probability measure supported on $[0,1]$ which has $\mathbb{E}(\mu_\alpha) = \alpha$ and $I_2(\mu_\alpha) = J(\alpha)$, then there exists C_α such that*

$$\mu_\alpha((1-\varepsilon, 1]) \sim C_\alpha \varepsilon^3, \tag{4.4.5}$$

as $\varepsilon \longrightarrow 0$.

Proof. We prove this by analysing the Radon–Nikodym derivative of μ_α. The measure μ_α is defined to be the minimiser of $I_2(\mu)$ over $\{\mu : \text{support}(\mu) \subseteq [0,1] \text{ and } \mathbb{E}(\mu) = \alpha\}$.

Therefore we can use the integral form for I_2 given by (4.2.4), and observe that minimising $I_2(\mu)$ over $\{\mu : \text{support}(\mu) \subseteq [0,1] \text{ and } \mathbb{E}(\mu) = \alpha\}$ is equivalent to finding a $g_\alpha \in C^1([0,1])$ with $\|g_\alpha\|_2 = 1$, $\|\sqrt{x}g_\alpha(x)\|_2 = \alpha$ and

$$\int_0^1 2x \left(\frac{\mathrm{d}}{\mathrm{d}x} g_\alpha(x)\right)^2 \mathrm{d}x = \inf\left\{\int_0^1 2x \left(\frac{\mathrm{d}}{\mathrm{d}x} g(x)\right)^2 \mathrm{d}x : \|g\|_2 = 1 \text{ and } \|\sqrt{x}g(x)\|_2 = \alpha\right\}. \tag{4.4.6}$$

As in the proof of Lemma 4.2.1 we can do this using the Euler–Lagrange equation. To include the twin constraints $\|g\|_2 = 1$ and $\|\sqrt{x}g(x)\|_2 = \alpha$ we must also include the Lagrangian multipliers

$$\lambda\left(\int_0^1 g(x)^2 \,\mathrm{d}x - 1\right) \quad \text{and} \quad \nu\left(\int_0^1 x g(x)^2 \,\mathrm{d}x - \alpha\right). \tag{4.4.7}$$

Therefore we get $\mathcal{F}[x, g(x), g'(x), \lambda, \nu] = 2x\, g'(x)^2 - 2\lambda(g(x)^2 - 1) - 2\nu(xg(x)^2 - \alpha)$. Putting this into the Euler–Lagrange equation

$$\frac{\partial \mathcal{F}}{\partial g} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \mathcal{F}}{\partial g'}\right) = 0$$

now gives

$$x\frac{\mathrm{d}^2}{\mathrm{d}x^2} g(x) + \frac{\mathrm{d}}{\mathrm{d}x} g(x) - (\lambda + \nu x) g(x) = 0, \tag{4.4.8}$$

for $x \in [0,1]$. The particular λ and ν will depend on α. However, even without knowing these we can now deduce that g_α is twice differentiable on $(0,1)$ – since it satisfies (4.4.8), and that $g'_\alpha(1) \neq 0$. This second claim follows because the Radon-Nikodym derivative of μ must be continuous on $[0,\infty)$ (else $I_2(\mu) = \infty$) and therefore $g_\alpha(1) = 0$. By inspecting (4.4.8) we see that if we also had $g'_\alpha(1) = 0$, then g_α would be the trivial solution, $g_\alpha(x) = 0$. Since this can not be the case we must therefore have $g'_\alpha \neq 0$.

By taking the Taylor expansion of $g_\alpha(x)$ at $x = 1$ we now have

$$\begin{aligned} \mu_\alpha((1-\varepsilon, 1]) &= \int_{1-\varepsilon}^1 \frac{\mathrm{d}\mu_\alpha}{\mathrm{d}x}\,\mathrm{d}x = \int_{1-\varepsilon}^1 (g_\alpha(x))^2 \,\mathrm{d}x = \int_0^\varepsilon \left(g'_\alpha(1)x + \mathcal{O}\left(x^2\right)\right)^2 \mathrm{d}x \\ &= \frac{1}{3} g'_\alpha(1)^2 \varepsilon^3 + \mathcal{O}\left(\varepsilon^4\right) \sim C_\alpha \varepsilon^3, \end{aligned} \tag{4.4.9}$$

as required. □

Because we know that $\mu^* = \mu_{\alpha^*}$ with $\alpha^* = (\gamma^*)^{-1}$ and $\mu^\bullet = \mu_{\alpha^\bullet}$ with $\alpha^\bullet = (\gamma^\bullet)^{-1}$, then this completes the proof of Theorem 4.1.3. □

Remark 4.4.3. For this proof it was not necessary to try and solve (4.4.8) explicitly. However, doing so provides a good way of calculating the function J numerically. See Definition 4.2.4 and Figure 4.2.1. Solving (4.4.8) would also enable us to compute C^* and $C^\bullet$.

4.5 A general framework

We conclude this chapter by considering a more general framework in which a Brownian motion $(W_t)_{t\geq 0}$ can be conditioned to have its local time bounded by 1. In this section we give three conjectures about the existence and behaviour of these limiting processes, and give non-rigorous explanations of why we believe these conjectures to be true.

Consider $(W_t, t)_{t\geq 0}$ as a process on $\mathbb{R}\times[0,\infty)$, and let $U \subseteq \mathbb{R}\times[0,\infty)$ be an open set containing $(0,0)$. For each $a \in (0,\infty)$ we write $aU = \{(x,t) : (a^{-1}x, a^{-1}t) \in U\}$, and define τ_a^U by $\tau_a^U = \inf\{t \geq 0 : (W_t, t) \notin aU\}$. From this we then obtain a collection of events

$$\mathcal{E}_a^U = \{L_x(\tau_a^U) \leq 1 \text{ for all } x \in \mathbb{R}\}. \tag{4.5.1}$$

Observe that since U is open then $\bigcup_{a>0} aU = \mathbb{R}\times[0,\infty)$, and thus $\bigcap_{a>0}\mathcal{E}_a^U = \{L_x(t) \leq 1$ for all x and $t\} = \mathcal{E}$. Note also that for $U = \{(x,t) \in \mathbb{R}\times[0,\infty) : x < 1\}$ this definition gives $\mathcal{E}_a^U = \mathcal{E}_a^*$, where $\mathcal{E}^*$ is given by (4.1.1), and when $U = \{(x,t) \in \mathbb{R}\times[0,\infty) : t < 1$ and $x \geq 0\}$ then $\mathcal{E}_a^U = \tilde{\mathcal{E}}_a^\bullet$, where $\tilde{\mathcal{E}}_a^\bullet$ is given by (4.1.3). We believe that, subject to certain conditions on the set U, the measures $\mathbb{W}(\cdot \,|\, \mathcal{E}_a^U)$ will weakly converge as $a \longrightarrow \infty$. Furthermore the behaviour of the limiting process with measure $\mathbb{Q}^U = \lim_{a\to\infty} \mathbb{W}(\cdot \,|\, \mathcal{E}_a^U)$, can be deduced from the set U.

4.5.1 Understanding $\mathbb{W}(\cdot \,|\, \mathcal{E}_a^U)$ via the theory of large deviations

One of the main principles of the theory of large deviations is that when we condition on a process satisfying a sequence of increasingly (exponentially) unlikely events, then – in the limit – the path taken by the process is the one which is least unlikely. Suppose an event Θ has $\mathbb{W}(\Theta) = \mathrm{e}^{-\vartheta}$, then we say the *cost* of $(W_t)_{t\geq 0}$ satisfying Θ is ϑ. Now consider a set $U \subseteq \mathbb{R}\times[0,\infty)$. Using the language of cost we can then say that if we condition on $\mathcal{E}_a^U$ then – in the limit – the path taken by $a^{-1}(W_t)_{t\geq_0}$ will be the one which is least *expensive*.

It is well known that

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbb{W}(W_T \geq vT) = -\frac{v^2}{2}, \tag{4.5.2}$$

and so it is clearly costly for a Brownian motion to be ballistic with a high speed. However, in order to satisfy $L_x(\tau_a^U) \leq 1$ for all $x \in \mathbb{R}$, a Brownian motion must travel at a speed of at least 1. Furthermore, the faster a Brownian motion travels, the easier it is for $(W_t)_{t\geq 0}$ to satisfy the condition on its local time. Therefore we end up with a trade-off between the cost of travelling quickly and the cost of satisfying $L_x(\tau_a^U) \leq 1$.

When a Brownian motion is ballistic then we can estimate its speed though a point x by calculating $\lim_{T\to\infty} \mathbb{E}(L_x(T))^{-1}$. Since $L_x(T)$ can be described in terms of a BESQ^2 process, then the cost of a Brownian motion travelling a unit distance at speed v, whilst ensuring its local time is bounded by 1, is given by $J(v^{-1})$. Recall Definition 4.2.4.

In the case where $U = aU^* = \{(x,t) : x < a\}$, $(W_t)_{t\geq 0}$ simply has to travel a units of distance whilst maintaining a bounded local time. Therefore $(W_t)_{t\geq 0}$ will travel at a speed for which the cost $J(v^{-1})$ is minimal. Lemma 4.2.8 tells us that $J(v^{-1})$ is minimised when $v = \gamma^*$, and thus we can use the theory of large deviations to confirm the result of Benjamini and Berestycki, [BB10]. However, in the case where $U = aU^\bullet = \{(x,t) : t < a\}$ then the limiting process has a speed which is least expensive per unit time. Because a particle travelling at speed v will cover v units of distance in each unit of time, then we can deduce that the unit time cost of a satisfying $L_x(t) \leq 1$ is $vJ(v^{-1})$. Lemma 4.2.8 also tells us that $vJ(v^{-1})$ has a unique minimum at $\gamma^\bullet < \gamma^*$, and so we see why the limiting measures $\mathbb{Q}^*$ and $\mathbb{Q}^\bullet$ should be different.

Now for a general set U there may be many possible paths P for $a^{-1}(W_t, t)_{t\geq 0}$ to take as it leaves U, and for each of these there will be a cost for $a^{-1}(W_t, t)$ to stay close to P and maintain $L_x(\tau_a^U) \leq 1$. However, if we know that there is is unique least expensive path, then conditionally on $\mathcal{E}_a^U$ the limiting route taken by $a^{-1}(W_t, t)_{t\geq 0}$ will converge to this least expensive path as $a \longrightarrow \infty$.

4.5.2 The cost of $a^{-1}(W_t, t)_{t\geq 0}$ following a path

Assume that a path P can be parametrised on an interval $[0, T]$ by $(f(t), t)$ for some piecewise differentiable increasing function f. The cost of $a^{-1}(W_t, t)_{t\geq 0}$ staying close to P will then be asymptotically equal to $a \times \text{cost}(P)$, where

$$\text{cost}(P) = \int_0^T J\left(\frac{1}{f'(t)}\right) \, \mathrm{d}t. \tag{4.5.3}$$

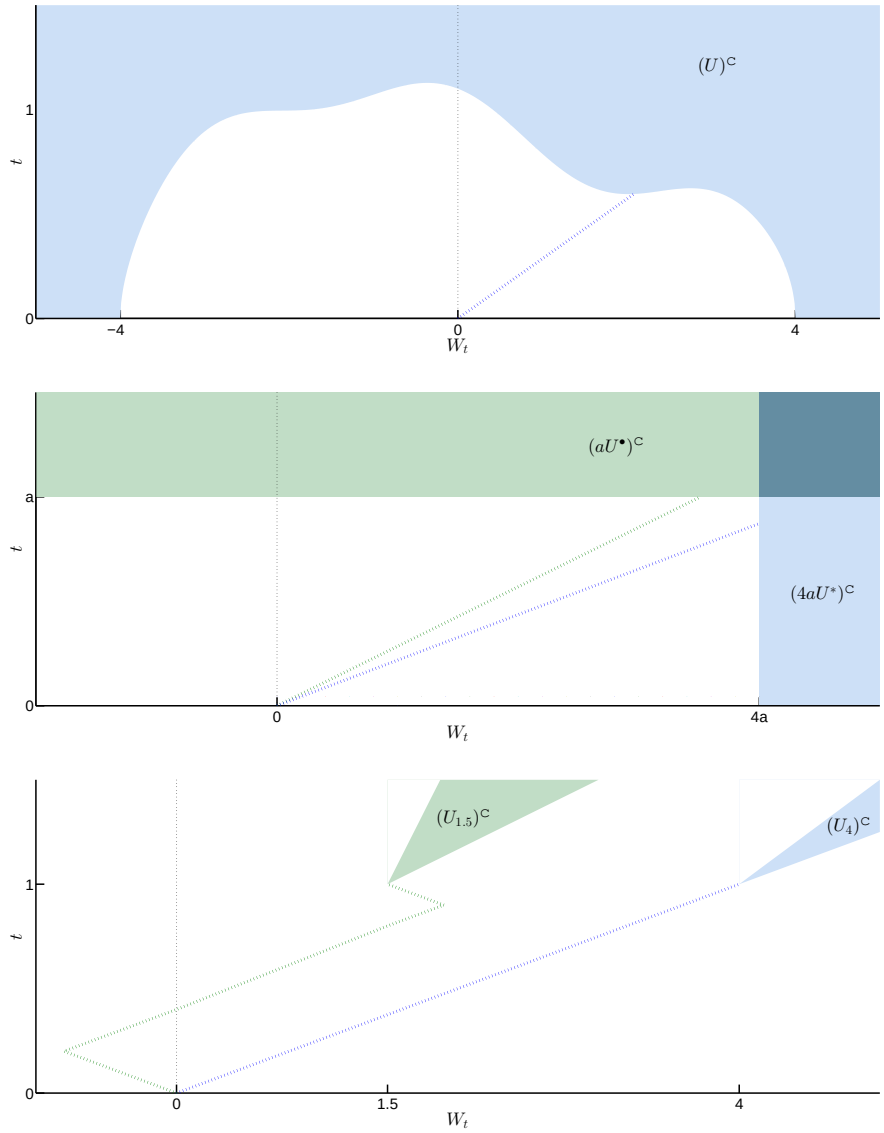


Figure 4.5.1: The first graph shows the process $(W_t, t)_{t\geq 0}$ taking the path P as it leaves a set U. This need not necessarily be the shortest, but will be the path with lowest cost. What is more, if the least expensive path P is unique then it must be a straight line. Obviously the gradient of the line will depend on the set U, and this is why different conditionings can lead to Brownian motion with bounded local time having different ballistic rates.

The second graph shows the most likely paths taken by $(W_t, t)_{t\geq 0}$ as it leaves $aU^{\bullet}$ (green path) and $4aU^{*}$ (blue path). Since (W_t, t) is able to leave $4aU^{*}$ sooner by travelling at a quicker speed we see that $\mathbb{W}(\cdot \,|\, \mathcal{E}_a^{*})$ and $\mathbb{W}(\cdot \,|\, \mathcal{E}_a^{\bullet})$ will converge to processes with different ballistic rates.

The third graph shows that for a given v we can set U_v to be $\mathbb{R} \times [0, \infty)$ minus a wedge with its point at $(v, 1)$. Provided v is sufficiently large this will give a unique less expensive path for $(W_t, t)_{t\geq 0}$. However, if v is too small then it ends up being easier for W_t to go back on itself than to go forwards at a slow speed. Therefore there can be no unique least expensive path, and so the measures $\mathbb{W}(\cdot \,|\, \mathcal{E}_a^{U_v})$ need not converge.

Note that because J is convex then $\text{cost}(P)$ can only be minimised when P is a straight line.

In the case where P is a path whose first parameter is not strictly increasing (or decreasing) then $(W_t)_{t\geq 0}$ must revisit regions where it has already been. From Lemma 4.3.6 we can deduce that

$$\limsup_{T\to\infty} \frac{1}{T} \log \mathbb{W}\left(W_T \in [-1,1] \text{ and } L_x(T) \leq 1 \text{ for all } x \in \mathbb{R}\right) \leq -2\pi^2, \tag{4.5.4}$$

and from this we get a lower bound on the cost of $a^{-1}(W_t, t)_{t\geq 0}$ following a path which comes back on itself. Using (4.5.4) as lower bound it is possible to show that for certain U there is a unique path P from $(0,0)$ to ∂U which minimises $\text{cost}(P)$. We now make the following conjecture.

Conjecture 4.5.1. *Suppose $(0,0) \in U \subseteq \mathbb{R} \times [0,\infty)$ is an open set, and assume that there is a unique path P from $(0,0)$ to ∂U with $\text{cost}(P) < \text{cost}(\tilde{P})$ for all other paths $\tilde{P}$ from $(0,0)$ to ∂U. Then*

- *P can be parametrised by (vt, t) for some constant v.*
- *$\mathbb{W}(\cdot \,|\, \mathcal{E}_a^U)$ converges to a weak limit, $\mathbb{Q}^U$, as $a \longrightarrow \infty$.*
- *$\mathbb{Q}^U$ is such that $\dfrac{W_t}{t} \longrightarrow v$ in $\mathbb{Q}^U$-probability.*

Following on from Conjecture 4.5.1 we now ask for which values of v can we find a set U_v such that $\mathbb{Q}^{U_v}$ exists and has ballistic rate v. For simplicity we shall restrict our attention to the case where $v \geq 0$.

Clearly it is not possible for us to have $v < 1$ as this would imply that $L_x(T) > 1$ for some x and t. What is more, Lemma 4.2.8 tells us that $J\left(v^{-1}\right) \longrightarrow \infty$ as $v \searrow 1$ and so we see that it is very expensive for $(W_t)_{t\geq 0}$ to maintain $L_x(t) \leq 1$ whilst travelling slowly. Consequentially, if we want $(W_t)_{t\geq 0}$ to be near vT at time T and satisfy $L_x(T) \leq 1$ for all $x \in \mathbb{R}$, then when v is small the least expensive way for this to happen is if $(W_t)_{t\geq 0}$ goes off at some speed greater than v and then changes direction in order to come back to vT.

From (4.5.4) we can see that the asymptotic cost of $(W_t)_{t\geq 0}$ returning to the interval $[0, vT]$ after λT units of time is at least $2\pi^2\lambda T$. In fact without too much difficulty one can show that this lower bound is sharp.

We also know that the cost of $(W_t)_{0\leq t\leq T}$ spending $(1-\lambda)T$ units of time in the inter-

val $[0, vT]$ whilst maintaining $L_x(T) \leq 1$ is asymptotically equal to $vTJ\left((1-\lambda)v^{-1}\right)$. Therefore we see that staying close to $(vt, t)_{0\leq t\leq T}$ proves to be the least expensive way for $(W_t, t)_{t\geq 0}$ to end up near (vT, T) if and only if

$$vJ\left(v^{-1}\right) < \lambda 2\pi^2 + (1-\lambda)vJ\left((1-\lambda)v^{-1}\right) \tag{4.5.5}$$

for all $0 < \lambda \leq 1$. Although we do not include details, it can be shown by studying the properties of J that there is a critical value $1 < \gamma^\circ < \gamma^\bullet$, equal to the minimal root of $vJ\left(v^{-1}\right) = 2\pi^2$, such that (4.5.5) is satisfied for all $v > \gamma^\circ$ and for no $v < \gamma^\circ$. See Figure 4.2.1 and Figure 4.5.1.

Conjecture 4.5.2. *There exists $1 < \gamma^\circ < \gamma^\bullet$ such that for each $v > \gamma^\circ$ there is an open set U_v for which $\mathbb{Q}^{U_v} = \lim_{a\to\infty} \mathbb{W}(\cdot \,|\, \mathcal{E}_a^{U_v})$ exists and is such that*

$$\lim_{t\to\infty} \frac{W_t}{t} = v \quad \textit{in } \mathbb{Q}^{U_v}\textit{-probability.} \tag{4.5.6}$$

What is more, for each $v < \gamma^\circ$ there is no open set U_v for which (4.5.6) holds.

4.5.3 The universal exponent

Suppose the measure $\mathbb{Q}^v = \lim_{a\to\infty} \mathbb{W}(\cdot \,|\, \mathcal{E}_a^{U_v})$ has $\lim_{t\to\infty} \frac{W_t}{t} = v$ in $\mathbb{Q}^v$-probability, for some $v > \gamma^\circ$. Provided we knew that $(L_x(\tau_a^{U_v}))_{x\geq 0}$ converged to a stationary process as $a \longrightarrow \infty$, then because the occupation measure of $(L_x(\infty))_{x\geq 0}$ must converge to a measure which minimises $I_2(\mu)$ over all $\mu \in \{\mu \in \mathcal{P}(\mathbb{R}) : \text{support}(\mu) \subseteq [0,1] \text{ and } \mathbb{E}(\mu) = v^{-1}\}$, we would be able to deduce that the stationary measure of $(L_x(\infty))_{x\geq 0}$ with respect to $\mathbb{Q}^v$ is $\mu_{v^{-1}}$. Lemma 4.4.2 then tells us that there is a constant $C_v > 0$ for which we have

$$\mu_{v^{-1}}((1-\varepsilon, 1]) \sim C_v \varepsilon^3, \tag{4.5.7}$$

and so we can make the following conjecture.

Conjecture 4.5.3. *For each $v > \gamma^\circ$ and each measure $\mathbb{Q}^{U_v} = \lim_{a\to\infty} \mathbb{W}(\cdot \,|\, \mathcal{E}_a^{U_v})$ there exists a constant $C_v > 0$ with*

$$\lim_{x\to\infty} \mathbb{Q}^v(L_x(\infty) > 1-\varepsilon) \sim C_v \varepsilon^3. \tag{4.5.8}$$

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