Finding meaning in mathematics through its philosophy

An empirical study with 17-year- old Greek students



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Preface

I hereby declare that:

- This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.
- It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the text.

It does not exceed the prescribed word limit for the relevant Degree Committee.

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Table of Contents

Prefaceiv
Acknowledgementsvi
Table of Contents
List of tablesvii
Introduction1
My story1
The story2
The study4
Thesis organisation5
Literature Review
Introduction9
Philosophical (objective) and psychological (subjective) meaning
Philosophical (objective) and psychological (subjective) meaning 10 Meaning of mathematics in philosophy 12 Subjective meaning in the philosophy of mathematics 13 The certainty of platonism 14 Securing foundations for certainty 15 Empiricism 18 Humanism 20
Philosophical (objective) and psychological (subjective) meaning 10 Meaning of mathematics in philosophy 12 Subjective meaning in the philosophy of mathematics 13 The certainty of platonism 14 Securing foundations for certainty 15 Empiricism 18 Humanism 20 Meaning for students 22
Philosophical (objective) and psychological (subjective) meaning 10 Meaning of mathematics in philosophy 12 Subjective meaning in the philosophy of mathematics 13 The certainty of platonism 14 Securing foundations for certainty 15 Empiricism 18 Humanism 20 Meaning for students 22 Philosophical beliefs about mathematics 24
Philosophical (objective) and psychological (subjective) meaning 10 Meaning of mathematics in philosophy 12 Subjective meaning in the philosophy of mathematics 13 The certainty of platonism 14 Securing foundations for certainty 15 Empiricism 18 Humanism 20 Meaning for students 22 Philosophical beliefs about mathematics 24 Psychological, subjective meaning 29
Philosophical (objective) and psychological (subjective) meaning 10 Meaning of mathematics in philosophy 12 Subjective meaning in the philosophy of mathematics 13 The certainty of platonism 14 Securing foundations for certainty 15 Empiricism 18 Humanism 20 Meaning for students 22 Philosophical beliefs about mathematics 24 Psychological, subjective meaning 29 The goal of the research 32

The philosophical background of the study	35
Methods	
Interviews and epistemic beliefs	
Interviews and hermeneutics	
The research	43
Data collection	43
Sample and context	49
Thematic analysis	
Ethical considerations	63
Ontology	67
Introduction	67
The nature of mathematical existence	69
Discovery	69
Invention	72
Summary	76
Mathematics as certain (and immutable)	77
Certainty and immutability	77
Proofs	
Cultural certainty and immutability	79
Traces of uncertainty and change	
Summary	
Mathematics as true	
Correctness	
Mathematical existence	
Cultural truth	

Mathematics as objective	
No disagreement (one answer)	
Proofs	90
Mathematical existence	91
Cultural objectivity	92
Summary	95
Mathematics as rules	95
Summary	98
Concluding remarks	
Epistemology	
Introduction	
Rule-based knowledge	
Rules as necessary (and sufficient)	
Rules as non-binding	
Rules as non-binding	
Rules as non-binding Summary Logically-based knowledge	103 105 105
Rules as non-binding Summary Logically-based knowledge Logic in mathematical reasoning	
Rules as non-binding Summary Logically-based knowledge Logic in mathematical reasoning Cultural logic	
Rules as non-binding Summary Logically-based knowledge Logic in mathematical reasoning Cultural logic Common sense	
Rules as non-binding Summary Logically-based knowledge Logic in mathematical reasoning Cultural logic Common sense Summary	
Rules as non-binding Summary Logically-based knowledge Logic in mathematical reasoning Cultural logic Common sense Summary Empirically-based knowledge	
Rules as non-binding Summary Logically-based knowledge Logic in mathematical reasoning Cultural logic Common sense Summary Empirically-based knowledge The senses	
Rules as non-binding Summary Logically-based knowledge Logic in mathematical reasoning Cultural logic Common sense Summary Empirically-based knowledge The senses Experimentation	
Rules as non-binding Summary Logically-based knowledge Logic in mathematical reasoning Cultural logic Common sense Summary Empirically-based knowledge The senses Experimentation Summary	
Rules as non-binding Summary Logically-based knowledge Logic in mathematical reasoning Cultural logic Cummary Common sense Summary Empirically-based knowledge The senses Experimentation Summary Proof-based knowledge	

Cultural function	
Summary	
Authority-based knowledge	
Mathematics	
Teacher and Book	
Summary	
Concluding remarks	
Subjective Meaning	
Introduction	
Common sense (reasoning)	
Alignment	
Misalignment	
Summary	
Discovery	
Alignment	
Misalignment	
Summary	
Invention	
Alignment	
Misalignment	
Summary	
Certainty	
Alignment	
Misalignment	
Summary	
Subjectivity	

Alignment	
Summary	
Rules	170
Alignment	
Misalignment	
Summary	
Empiricism	
Partial misalignment	
General misalignment	
Summary	
Concluding remarks	
Discussion	
Introduction	
Plurality and cohesiveness of beliefs	
Plurality and cohesiveness of beliefs Plurality of beliefs	
Plurality and cohesiveness of beliefs Plurality of beliefs Cohesiveness of beliefs	
Plurality and cohesiveness of beliefs Plurality of beliefs Cohesiveness of beliefs Mathematical existence	
Plurality and cohesiveness of beliefs Plurality of beliefs Cohesiveness of beliefs Mathematical existence Mathematical reasoning	
Plurality and cohesiveness of beliefs Plurality of beliefs Cohesiveness of beliefs Mathematical existence Mathematical reasoning Logic	
Plurality and cohesiveness of beliefs Plurality of beliefs Cohesiveness of beliefs Mathematical existence Mathematical reasoning Logic Rules	
Plurality and cohesiveness of beliefs Plurality of beliefs Cohesiveness of beliefs Mathematical existence Mathematical reasoning Logic Rules Mathematical certainty	
Plurality and cohesiveness of beliefs Plurality of beliefs Cohesiveness of beliefs Mathematical existence Mathematical reasoning Logic Rules Mathematical certainty Concluding remarks	
Plurality and cohesiveness of beliefs Plurality of beliefs Cohesiveness of beliefs Mathematical existence Mathematical reasoning Logic Rules Mathematical certainty Concluding remarks Conclusions	
Plurality and cohesiveness of beliefs	
Plurality and cohesiveness of beliefs	

Reflexive remarks
References
Appendix 1
An example of objective and subjective meaning in the context of the philosophy of mathematics
Appendix 2
Examples from the application of the interview protocol and the use of dialogue to co-create meaning with the students
Illuminating mathematics' relationship to truth by comparing mathematics with physics and history
Contextualising the subjective meaning of mathematics though the student's conception of truth and its role in life
Negotiating the concept of rule in mathematics
Appendix 3
Thematic analysis
Ontology
Epistemology
Subjective meaning
Discussion

List of tables

Table 1: Themes for Ontology and Epistemology	52
Table 2: Themes for Subjective Meaning	55
Table 3: Themes and subthemes for Ontology	62
Table 4: Themes and subthemes for Epistemology	63
Table 5: Beliefs around which students' stories can be compared	140
Table 6: Combinations of philosophical approaches	203

Introduction

The least of things with a meaning is worth more in life than the greatest of things without it (Carl Jung, 2001, p.67)

This thesis started with a question. What does, or what can, mathematics mean? What can it mean for students, and what does it mean for me? The tools that I had always had for answering that question - ever since I started asking it consciously - were philosophical. In particular, they were related to the idea of mathematical certainty which, as we will see, played and still plays, a key role in the development of the philosophy of mathematics (Hersh, 1997). This led me to my central research problem: whether and how students might attribute meaning to mathematics through ways of thinking that could be seen as effectively philosophical. The next chapter will present a detailed review of the literature that led me to confirm that my research problem amounted to a question that was worth undertaking. Before turning to this task however, it is necessary to set out some introductory comments. These include my own story with mathematics, an overview of the study and the way the thesis has been organised.

My story

Including my own story seems an appropriate way to start, since it explains why the topic of the research was chosen. Furthermore, a major part of the work that follows comprises my attempt to interpret and convey the meaning that the study of mathematics had for the students whom I interviewed for the research. Therefore, since my task has been to represent as fully as possible the meanings that were expressed in the interviews on the basis of my understanding of the students' remarks, I believe it is fair to start by offering my own story, and how I myself have related to mathematics at various stages of my life.

This story should provide an indication of what is to be expected, although it should by no means be compared with any of the stories that the students have shared. This would not be fair, as I occupy a 'privileged' position, which allows me to produce a much more coherent account than the students could. The fact is that during my masters in mathematics education, and even more so during my doctoral research, I was equipped with a set of concepts that allowed me to see my relationship with mathematics under a different light, and clarify many

issues that up to that point had been largely unprocessed. For example, coming across the distinction between discovering mathematics and inventing mathematics, I realised that mathematical certainty was not an isolated issue but deeply related to the ways in which one perceived other aspects of mathematics. By contrast, the students questioned in the research are unlikely to have had the opportunity to consciously question their relationship with mathematics in such depth – and even if they had, they would have lacked the theoretical background that would have allowed them to reach informed answers. This was reflected in the data by the fact that students' accounts were not always wholly consistent.

The story

When I was in school, I was comfortable with all subjects, but mathematics seemed to have a special quality. This tended to become particularly obvious during examination periods. After finishing a test in mathematics I could be sure that I had found the correct answers to the questions and that I would get a good grade. By comparison, after completing a test in literature or the Greek language, I was always worried that I might not have given the right answers, despite the fact that, based on my previous performance, I could rest assured that it would not be a complete disaster. In fact, this tendency of mine to worry about my performance was a general one, and concerned all subjects. But mathematics was the subject with which I felt most comfortable.

Even back in school, I quickly learnt that this was because mathematics examinations included closed questions where the answers were fixed and occasionally were even given as part of the question itself; I only had to reach them, as when I was asked to prove a certain mathematical relationship. Moreover, I had in my toolkit a set of methods which were defined step by step. I still had to choose the correct method and that was not always straightforward but, having chosen a method, I could trust that the steps within the method were correct. I had understood why the method worked when it did and I felt I could depend on it. On the other hand, exams in literature included open-ended questions which required me to interpret the text without having any concrete method on which I could rely; there were guidelines, but these were quite vague, and therefore I could not depend on them. I knew that my answer would be subjective and that it would also be subjectively judged, and this scared me. So mathematics allowed me to be in control of my results and feel a sense of certainty.

Nevertheless, looking back I realise that this is not the whole story. One might expect that other subjects from the positive sciences would make me feel similarly secure, but the fact was that they did not. Somehow, I felt much more comfortable imagining how abstract mathematical concepts would behave than imagining how physical systems would evolve. I am still not sure why this was the case. There could be many factors that contributed to this result. It could simply be that my mind was more inclined to handle abstract concepts rather than concrete systems. I do not believe that only certain people can do mathematics, but it seems to me that some people do have an easier time grasping mathematics. On the other hand, a psychologist might claim that, I felt more at ease with abstract rather than actual objects because for a long period in my life, reality had been deeply hurtful and I was trying to avoid it altogether. Wishing to escape from facts, I would find a haven in the abstract, orderly world of mathematics, where I could forget my chaotic feelings. The fact was that mathematics had a soothing effect on me, and at least until I finished school, I was never troubled by whether mathematics was real or not.

However, I was challenged by this question almost immediately upon entering university. The school environment was highly protective and within it I could easily ignore the 'real world', where I felt unloved and lacked a sense of balance and orientation for life. Arguably, the university environment is not much different, yet it felt a step closer to real life, and it came with an increased sense of responsibility with respect to what I was going to do with my life. The result was that from that moment I gradually stopped deriving pleasure from mathematics. I could probably still feel secure while doing mathematics; it was true that at the university level, I found that the answers to mathematical problems were less obvious than they used to be in school, but I still did not find them tremendously difficult. My problem was not that I could not feel certain within mathematics but that I had no way of transferring this certainty outside of mathematics to real life. There, mathematical reasoning seemed to offer very little if no help. So I started losing interest in mathematics, which increasingly appeared to me to be totally unrelated to life - not on the scientific, technological level, of course - but on that level of practical wisdom which could help me lead a happier life.

Eventually, I almost reached the point of not wanting to have anything to do with mathematics; any evidence of mathematical certainty started to annoy me instead of soothe me. Nevertheless, mathematics had been an important part of my life up until that point and now merely to ignore it, and to turn it to one side, did not feel right. Therefore, instead of abandoning mathematics

completely, I chose to try to clarify my relationship with it. Coming to Cambridge to do my master's degree and thence my doctorate on mathematics education has been part of this process, although I had still not yet realised this when I started the doctorate programme.

At the present moment I have to admit that the research did not help me determine what mathematics is, but it did help me to gain an insight into what mathematics can be for students in particular, and for people in general. As is probably the case with much research in social science, instead of arriving at one right answer I have concluded with a set of alternative and potentially equally valid answers. At least this has helped me to accept my relationship with mathematics as it has developed, and stop wishing to make it right. In fact, my relationship with mathematics no longer depends on whether mathematics leads to certain results or not, or whether it is real or not. I can be content either way, and fortunately I still believe that mathematics is beautiful.

The study

The data that follow reveal that my experiences are not unique. In particular, among the students, Xenofontas and Kleomenis valued, as I had done in school, the certainty of mathematics, enjoying the control which stemmed from dealing with definite facts; Platonas and Agapi believed that mathematical reasoning is applicable to life, and that it could help them with finding solutions to everyday problems, as I had wished it would have; Yerasimos felt that a reasoning which always securely produces the same answer is an absurdity, as I had started thinking in my university years; Foivos and Kosmas expressed an irritation similar to mine with respect to the rigidity of mathematics which cannot capture the variability of life.

Of course, students' experiences covered a wider range and occasionally even opposed mine. Aspasia admired mathematical reasoning and affirmed that if one does not know mathematics, then one cannot reason logically; Kleomenis found freedom in the certainty of mathematics, and he valued it because he believed that it allowed mathematicians to explore infinite possibilities; Ermis perceived mathematics as uncertain as anything else simply because he enjoyed doubting everything in order to learn; Evyenia indicated that mathematics was for her a complete mystery, which provided no solid knowledge, but was merely based on random assumptions and opinions; Ariadni asserted that mathematics, far from offering a way of reasoning, had no logic at all and that if anyone tried to do mathematics based on logic they were bound to make a mistake; Areti declared that mathematics must be correct only because she trusted the education system and her teachers who would not be teaching students something that was wrong.

The data were collected over a period of five months and were generated through in-depth interviews designed and executed within an hermeneutical approach. Since the aim of hermeneutics has been to understand and interpret the meaning initially of texts, but latterly of every meaningful human action (Ricoeur, 1991), I judged that hermeneutics would be an appropriate methodology for grounding research focused on meaning. Using in-depth interviews as a method to access this meaning seemed to follow naturally from the chosen methodology, an approach which puts an emphasis on the place of dialogue in the production of new understanding (Gadamer, 1975).

In the event, I was fortunate to collect a body of rich data which illuminated how students can attribute meaning to mathematics through issues which pertain to mathematical philosophy. I was pleasantly surprised when I realised that even students from the same school, who have more or less undergone the same system of mathematics education, could hold such a diverse range of beliefs about mathematics. What was much less of a surprise - at least for me - was that their beliefs were also loaded with a subjective kind of meaning, reflecting the particular relationships with mathematics that they had developed.

The reason why the lenses through which this research looks at mathematics are philosophical is not simply a product of my own experience. It is also because in a sense each one of us philosophises in order to generate meaning for whatever concerns us (Morgan, & Farsides, 2009). Moreover, there seemed to be a considerable gap in the literature regarding how the philosophy of mathematics could be related to the meaning that students find (or do not find) in mathematics. Empirical studies about how students view issues which pertain to the philosophy of mathematics are scarce (François, & van Bendegem, 2007). Furthermore, no one had yet investigated the correlation between such views and the meaning that students could attribute to mathematics (Kilpatrick, Hoyles, & Skovsmose, 2005a; Vollstedt, 2011).

Thesis organisation

In the following I wish to share with you the fruits of this research. These involve students' beliefs on issues of the philosophy of mathematics and the meaning that they find in mathematics in relation to these beliefs. The relevant literature is examined in detail in the next

chapter. This will include a clarification of the word meaning as employed in this study, distinguishing between objective and subjective aspects of meaning; a review of how philosophers have ascribed meaning to mathematics; an examination of the literature on students' beliefs about mathematics, establishing which among them could be of interest for a study with a philosophical focus; and a presentation of a typology of meaning for mathematics that has been developed by Vollstedt (2011). After the literature review, there follows a description of the methodological background and the methods employed in the study, which as mentioned above, are hermeneutics and in-depth interviews respectively. The philosophical underpinnings of hermeneutics are considered, and the advantages of using interviews for exploring students' beliefs and the meaning that they carry are discussed. In addition to this, details are given about the participants, their school and the Greek educational system, and the principles governing data analysis are set out.

Thereafter the findings of the study are reported. The presentation of the findings is organised around the broad philosophical issues of the ontology and epistemology of mathematics, succeeded by an examination of subjective views of mathematics which may arise from a consideration of these issues. In particular, with respect to ontology, the thesis examines students' views on the following topics: Is mathematics invented or discovered? Is mathematics certain and does it change across time? Is mathematics true and objective? Can mathematical statements be understood as rules? With respect to epistemology, the thesis presents students' understanding of the following: How do mathematical statements, or rules, lead to mathematical knowledge? What is the role of logic, the senses and experience in the production and justification of mathematical knowledge? What is the role of proof in mathematics? Is there any authority - external to the student - involved in the acceptance and justification of mathematical statements? These topics, regarding both ontology and epistemology, were chosen following a critical consideration of the literature and the consequent establishment of areas which have been of interest either within the philosophy of mathematics or within mathematics education. After the presentation of the students' beliefs, and the more or less objective meaning that was attributed to mathematics through them, follows a chapter called 'subjective meaning' which seeks to illustrate how such beliefs could be bearers of meaning at the subjective level. According to the results of this study, the relation between mathematical reasoning and a student's common sense seemed to be an overarching factor which influenced how students attributed negative or positive meaning to mathematics on the subjective level. This factor could also relate to the following issues: mathematics as a discovery or an invention,

mathematics as certain, mathematics as subjective, mathematics as a set of rules, and mathematics as an empirical subject.

A more comprehensive discussion follows the reporting of the findings. In this discussion, I consider the plurality of ideas with respect to the Greek cultural context, and the cohesiveness of students' accounts. I also identify existing interrelations between the various beliefs that the students had been found to hold, and discuss their conceptions in light of the literature, and the various trends that have arisen in the history of the philosophy of mathematics. It is noteworthy - though not surprising - how students tended to combine ideas from multiple philosophical trends without worrying about potential contradictions of the kind that trouble philosophers. However, although on the level of ideas and beliefs, students' accounts might appear to be incoherent, this incoherence could disappear, or at least be explained, by considering the meaning that these beliefs bore for the respective student on the subjective level.

The thesis concludes with remarks regarding the contributions of this study to research in mathematics education. I believe that the meanings that students attributed to mathematics were a centrally important aspect of their relationship with mathematics and that this relationship is of paramount importance (Vollstedt, 2011). As I stated above, there may not be a single answer as to what mathematics may mean for an individual, but there are nevertheless meanings which are more functional than others and which do not present the student with a problem that needs to be solved. Unfortunately, this was not the case for many of the students in this research, as it has not always been the case for myself. Therefore, equipped with the results of this study, I think that I am now in a position to argue strongly that students can, and should be, helped by way of explicit discussion about what mathematics means for them, and that a concentration on philosophical issues could be a useful tool in this direction.

Literature Review

Introduction

This study aims at investigating possible ways through which students search for meaning in mathematics, and suggests that at least some of these ways may be related to, and better understood in the context of, the philosophy of mathematics. Traditionally, the concept of meaning has been studied in the context of the psychology of mathematics education (e.g. Kilpatrick, Hoyles, & Skovsmose, 2005a; Vollstedt, 2011). Nevertheless, as will be shown below, the philosophy of mathematics can be claimed to be an invaluable source of meaning that has been ascribed to mathematics by philosophers and mathematicians over the centuries. Of course, school students cannot be expected to have a sophisticated philosophy of mathematics, since they hardly ever discuss the philosophy of mathematics in the classroom (François, & van Bendegem, 2007). Still it is justified to assume that in their attempt to make sense of mathematics, students may themselves have employed means that may be understood as fundamentally philosophical, that is to say, means that have similarities with those that can be encountered in the history of the philosophy of mathematics. Hence, the current study can best be seen as positioned at the boundary between philosophy of mathematics and the mainstream tradition of the psychology of mathematics education. On the one hand, I approached students as potential philosophers of mathematics, asking their views on issues with which philosophers have grappled before them. On the other hand, students' views were regarded within the wide range of beliefs about mathematics which have been recorded by research in the context of the psychology of mathematics education.

Such a study seems to be lacking from the literature where meaning and the philosophy of mathematics are consistently kept separate. Although there are some studies which have explored what students believe about the nature of mathematics and mathematical knowledge (Op't Eynde, de Corte, & Verschaffel, 2006), these do not proceed to investigate how the students' beliefs influence, or are influenced by, the meaning they attribute to mathematics. Furthermore, studies which are interested in philosophy often make a straightforward assumption that the philosophy which underlies mathematics curricula and teaching is simply passed to the students (François, & van Bendegem, 2007), avoiding an in-depth examination of the meaning that students themselves may find in philosophical topics. Finally, discussion

about meaning has mostly focused on the cognitive understanding of mathematical concepts, ignoring for the most part both the meaning of philosophical concepts and what mathematics might mean for the students (Kilpatrick, Hoyles, & Skovsmose, 2005a). The second issue is handled in Vollstedt's studies (e.g. Vollstedt, 2011). Nevertheless, Vollstedt does not adopt a philosophical perspective, as the current study does. In the next section, I discuss how meaning about mathematics may be understood in terms of an interplay between philosophy and psychology.

Philosophical (objective) and psychological (subjective) meaning

Meaning has been a concept which has been examined in both the fields of philosophy (e.g. (Dummett, 1976; Kaplan, 2011; Wittgenstein, 1953) and psychology (Gendlin, 1962; Noble, 1952; Pennebaker, Mehl, & Niederhoffer, 2003; Wong 2012a). In philosophy, philosophers have aspired to find the more or less objective meaning of a word, a concept or a sentence; that is to say, that which an intelligent person, with sufficient knowledge of the respective language, will understand upon hearing an utterance (Di Sciullo, & Williams, 1987; Dummett, 1976). For instance, regarding words, some kind of objective meaning can be found when we open a dictionary. However, arriving at a purely objective theory of meaning has proven to be a futile endeavour, since human utterances usually carry various subtle, contextualised meanings which are very difficult to capture in a unified manner (Jones, 1947; Putnam, 1975). Furthermore, human utterances tend to have an intentional aspect which renders them highly subjective (Noonan, 1981). For example, one might say 'I am hungry' simply intending to indicate that it is time to prepare a meal, but also in order to blame somebody for not having prepared that meal already. Such an example points towards a subjective, psychological aspect of meaning, where what is important is not only what sentences may mean according to a dictionary definition, but what they mean to a particular human being in a particular temporal, spatial or social context (Gendlin, 1962; Gergen, 2001; Rhoads, 2010). Indeed, it is hard to imagine that any dictionary-derived meaning would associate the phrase 'I am hungry' with notions of accusation and blame, but this could be perfectly valid in the example given above.

Objective and subjective meanings are closely interrelated and cannot be easily separated. Objective meanings seem to arise through the repeated subjective use of words in certain social contexts (Gergen, 2001; Moscovici, 1988; Rhoads, 2010). In the opposite direction, it can be claimed that subjective meanings are limited and shaped by what the society and culture into which an individual is 'thrown' is willing to accept (Geertz, 1973; Merleau-Ponty, 1962). Nevertheless, it can certainly be argued that there are contexts in which either objective or subjective meanings may become more salient (Jahn, & Dunne, 1997). For instance, in the context of philosophy, precedence seems to be given to objective meanings (Douglas, 2004; Jones, 1947); fruitful and scholarly philosophical discussion would be practically impossible without recourse to objective meaning, since each individual philosopher would be trapped into using words in their own idiosyncratic way (Ricketts, 1986; Shanker, 1987). On the other hand, in the context of psychology, where the imperative is to find meaning in, and to make sense of, one's life and experiences, the emphasis appears to shift to the importance of subjective meanings (Wong, 2012a). When individuals narrate their life stories, the subjective meanings they attribute to words and phrases takes precedence over any objective meaning that these may have (Gendlin, 1962; McAdams, 2001).

Nonetheless, even if philosophers commonly pay more attention to objective meanings, this certainly does not mean that philosophy is stripped of all subjective meaning. Meaning pervades all human life (Frankl, 1985; Jung 2001), and it can be claimed that it is this essential need for meaning that gave birth to philosophy in the first place (Smith, 1980). It can therefore be argued that, at least in some sense, philosophical theories operate also as accounts of meaning that carry a subjective import; in this regard, it can be claimed that philosophers may make decisions based on their own values and opinions, advancing their own subjective meaning of the phenomena they study (Holtzman, 2013). As a human science, philosophy may arguably aspire to provide knowledge beyond subjective meaning, namely meaning on the objective level, meaning which potentially could be argued to apply to all people (Hume, 1978). Nevertheless, ongoing disputes within philosophy indicate that this goal remains far from attainment (Russell, 1918). In particular, it is very difficult, if not impossible, to consider that the philosophy of mathematics can claim to be objective when philosophers seem not to be able to agree on the nature of mathematics and mathematical knowledge (Shapiro, 2000).

Similarly, a life story does not exist in isolation from societally objective meanings. Just as philosophers can be said to put forward subjective meanings, it can also be argued that in the process of creating psychological meaning for their lives, all individuals philosophise, producing a worldview which, although subjective, can be shared at a more or less objective level with the rest of the world (Moscovici, 1988; Reichertz, 2004; Rhoads, 2010; Wong, 2012b). This primarily subjective philosophy involves the 'assumptions, beliefs, values and

worldviews that help us make sense of our lives', and has been called the philosophy of life (Tomkins, 1995; Wong, 2012b, p.5). Such a philosophising activity is also relevant for young people, who, as they seek during adolescence to integrate their experiences and create an identity, strive to find meaning for life, society, and themselves in relation to life and society (Habermas & Bluck, 2000; McAdams, 2001).

This thesis aims at combining the objective and subjective meanings that students attribute to mathematics through its philosophy. Bringing together students' beliefs about philosophical issues (as carriers of objective meanings) and students' psychological reactions to these issues (as carriers of subjective meaning), will allow for a more thorough and richer understanding of what mathematics means to them. (For a demonstration of this in the context of philosophy see appendix 1). In the following, I first discuss objective and subjective meanings that have been attributed to mathematics within philosophy, before turning to the literature that considers students.

Meaning of mathematics in philosophy

A variety of philosophical theories have been developed in order to explain the meaning - in more or less objective terms - of mathematics with respect to what mathematics is (ontology), and how mathematical knowledge evolves (epistemology) (Brown, 2008; Shapiro, 2000). Philosophers have shown a great interest in clarifying the mystery of mathematics. This could be attributed firstly to the fact that mathematics lies at the very heart of our desire to satisfy our curiosity with respect to the natural world (Loewenstein, 1994; Shapiro, 1993). As Galileo observed, it appears that 'the book of nature is written in the language of mathematics' (in von Glasersfeld, 1995, p.30). However, the close relationship between mathematics and physics, in which the latter becomes impossible without the former, is quite recent in the history of science (Crosby, 1997; Lindberg, 2007). What seems to have fascinated the human mind even more over the centuries is the apparent certainty of mathematical conclusions; a quality which great thinkers, such as Descartes or Spinoza, had wished to attain in their own philosophical endeavours (Ernest, 1991; Hersh, 1997). Mathematics seems to hold the key to certain knowledge, and this promise carries subjective meaning in the midst of an uncertain world (Quine, 2008; Shapiro, 2000). Certain, indubitable knowledge is trustworthy; something that we can hold to, and to which we can return in order to guide ourselves when we feel lost (Hersh, & John-Steiner, 2011). The human mind appears to need psychological certainty in order to

find subjective meaning in, and deal with, the chaos which life presents (Antonovsky, 1994; Crawford, & Rossiter, 2006; Dewey, 1929; Korotkov, 1998).

The history of philosophy of mathematics therefore seems to revolve around the issue of certainty (Hersh, 1997). Initially, this certainty was taken for granted, and attempts to understand it emerged from the desire to endow other fields of knowledge with the same admirable certainty (e.g. Plato, Descartes, Leibniz). However, by the end of the nineteenth century, certainty had begun to crumble; mathematical conclusions which had been previously regarded as indisputable were revealed to be problematic (Grattan-Guinness, 2000; Russell, 1918). Unwilling to abandon this certainty, mathematicians launched a gigantic effort to secure the foundations of mathematical knowledge, thereby making it absolutely reliable once more. Nevertheless, this dream was crushed (Ernest, 1991; Giaquinto, 2002). Thereafter, a number of philosophers have laboured to rescue whatever had remained of the renowned mathematical certainty, by defending mathematical knowledge as objective (e.g. Putnam, 1975; Resnik, 1981; Shapiro, 2007). Others have decided to abandon or even decry any sense of absolute, indubitable mathematical knowledge, believing that clinging to mathematical certainty is mistaken (Ernest, 1998a; Hersh, 1997). Such philosophers have represented mathematics and mathematical knowledge as socially constructed and dependent (Ascher, 1991; Ernest, 1998a; Lyotard, 1984). This group of philosophers can no longer find subjective meaning in the alleged absolute truths of mathematics, but in the demystification of mathematics itself - conceiving mathematics not as something that transcends humanity, but as something which is intimate to human thought (Ernest, 1991, Hersh, 1997). Below I elaborate further on these phases in the history of the philosophy of mathematics. Before doing so, though, I briefly consider the grounds upon which it becomes sound to view the philosophy of mathematics as a conveyor of subjective meaning, apart from any objective meaning that the philosophers may seek to impart.

Subjective meaning in the philosophy of mathematics

Philosophers of mathematics have not succeeded in providing a unified account of what mathematics means (Colyvan, 2012; Hersh, 1997). In fact, a unified account may not even be possible, since the philosophy of mathematics appears to be metaphysical, at least as far as the problem of mathematical ontology is concerned (Balaguer, 1998; Maddy, 2005; Price, 2009). That is why Dubinsky (2000) proposes that believing in the existence of mathematical objects

which are independent of the mind is a matter of 'personal choice' (p.215). Similarly, Resnik (1981) makes a declaration of faith when he espouses this position despite recognising that it can be problematic (p.529). On such a view, it can be argued that philosophical accounts of the ontology of mathematics reflect the beliefs of this or that philosopher.

Nevertheless, the ontological problem cannot be easily isolated from the epistemological one (Balaguer, 1998). Most philosophers offer accounts for both the epistemology and the ontology of mathematics and they often explain their position about one in terms of the other (Ernest, 1991; Hersh, 1997). But even if epistemology is viewed as separate from ontology, there are epistemological issues where personal choice appears to be pertinent. For instance, determining what is knowable in mathematics involves determining which logic is appropriate for making mathematical inferences (Brouwer, 1913). However, answering this question is not an objective matter (Shapiro, 2007). In sum, philosophers' views on mathematics seem to carry an unavoidable subjectivity. The next sections sketch some major trends and the answers they have produced on the objective meaning of mathematics, but also the potential these answers hold for subjective meaning.

The certainty of platonism

Platonism constitutes the belief that mathematical concepts correspond to abstract objects, i.e. eternal, non-material entities which exist objectively, outside time and space, and independently of human beings (Balaguer, 1998; Brown, 2008). This position has been named after the Greek philosopher Plato, who presented mathematics as dealing with an objective, existing, and eternal reality. This reality was supposed to be accessible to humans through reason, which could somehow grasp these perfect, eternal, abstract entities, while the senses were taken to be able to perceive only the imperfect, material, ever-changing world (Brown, 2008; Shapiro, 2000). Plato's postulated mathematical existence is in perfect accord with the apparent mathematical certainty which cannot be matched in other sectors of human knowledge. This is because within platonism mathematical claims must establish objective, indubitable truths since they refer to eternal objects (Frege, 1964).

In such a context, subjective, psychological meaning can be found in the image of mathematics as a field of knowledge which reveals truths about real objects. Searching for the truth has always been considered important for humans; fundamental human activities, such as science, philosophy and religion, all aim at finding what is true (Gadamer, 1975; Polkinghorne, 2011).

Moreover, platonism also allows for a connection between searching for truth in mathematics and in physics (Balaguer, 2008; Benacerraf, 1973), since within such a tradition, mathematical and physical objects can be viewed as commensurate with respect to their objective existence, and a merging between physical reality and the platonic one appears to be plausible (Penrose, 1999). In this way, platonic mathematics provides truths not only about some abstract entities which may be judged as irrelevant to life (Ernest, 1991), but also about the actual world that we inhabit (Penrose, 1999).

However, in a sense Plato substituted one mystery for another. If we claim that mathematical objects exist, then we have to explain where they are to be found; we have to describe the mathematical reality that platonism postulates, and to indicate the means by which we interact with it and thereby come to know mathematical facts (Maddy, 2005; Resnik, 1995). But it is hard to locate abstract objects that lie out of space and time and are not accessible to our senses (Field, 1988; McGee, 1997). Moreover, one may postulate that humans come to know these objects through logic, but it is hardly clear how this may be possible when we are not even certain about where the purported mathematical objects are located. (Resnik, 1995). Admittedly, such questions were not as problematical in Plato's times or even later, so long as metaphysical explanations were acceptable (Hersh, 1997; Menzel, 1987). Nevertheless, modern science has repudiated metaphysics, and many would now not feel comfortable with assuming the existence of metaphysical entities (Leng, 2005; Passmore, 1966; Price, 2009; Rotman, 1993).

Securing foundations for certainty

The certitude of mathematical knowledge remained more or less undoubted for centuries (Brouwer, 1913; Hersh, 1997). However, when mathematical certainty was put under increasing scrutiny at the beginning of the twentieth century, it proved to be far from an easily defensible claim (Grattan-Guinness, 2000; Kline, 1980). Simple statements that had once seemed self-evident, such as that all objects with a certain property could define a set, were shown to be liable to paradoxes, and could therefore no longer be taken as universal truths (Giaquinto, 2002). Mathematics seemed in danger of losing the meaning (both objective and subjective) that had been attributed to it for centuries, namely that of exemplifying knowledge that was certain (Russell, 1969).

Nevertheless, mathematicians were not ready to abandon certainty (Kline, 1980). On the contrary, they set as one of their fundamental goals the securing of mathematical certainty on solid foundations (Brouwer, 1913; Hilbert, 1902; Russell, 1918). Hilbert envisioned a mathematics which 'no one doubts and where contradictions and paradoxes arise only through our own carelessness' (1983, p.191). In this period, finding meaning (objective or subjective) in mathematics remained equated with finding certainty in mathematics. Furthermore, these attempts were not purely mathematical, but carried philosophical considerations, since what counted as certain was essentially a philosophical issue (Almeder, 1990; Giaquinto, 2002). From this point onwards, three variants of an objective, philosophical meaning for mathematical certainty were developed: logicism, intuitionism, and formalism (Ramsey, 1931).

Logic had always been connected to mathematics, the exemplar of rational thought (Shapiro, 2005; Tiles, 1991). Platonism also postulated that mathematical knowledge is the result of the function of logic. However, with logicism, logic is taken to be not only the means through which mathematics is accessed, but also its fundamental essence (Russell, 1918). According to logicism, all mathematics and mathematical derivations can be paraphrased in versions which include only purely logical expressions and laws of inference (Rayo, 2005). On this basis, the subjective, psychological meaning associated with mathematical certainty is rooted in the subjective, psychological meaning of logic and its certainty (Shapiro, 2007).

Intuitionists maintain that, rather than logic, mathematical knowledge is the result of inherent, intuitive human faculties which allow human beings to comprehend concepts such as time or space (Brouwer, 1913; Hersh, 1997). These intuitions were taken to be universal properties of the human mind (Heyting, 1956). As such, they revealed knowledge that everyone could accept as self-evident certainties (Kline 1980). Therefore, mathematical knowledge could be claimed to be objective and certain. Nevertheless, dependence on human intuition led intuitionists to deny and reject certain parts of contemporary mathematical knowledge which appeared to contradict intuitions (Brouwer, 1913; Ramsey, 1931). One concept that intuitionists would not accept was that of infinity; infinity for them carried no meaning - objective or subjective - since it lay beyond anything that humans could grasp (Benacerraf, & Putnam, 1983b).

For formalists, mathematical certainty is the result neither of logic nor of intuition; it is simply a matter of syntax (Giaquinto, 2002). According to formalism, the subject matter of mathematics is its symbols. These symbols were to be handled according to certain unambiguous syntactic rules (Hilbert, 1983). Whether the symbols and the rules which were used to combine them had any intrinsic meaning was irrelevant to formalists (Ernest, 1998a; Kline, 1980). What mattered was that formalist rules could produce objective, indubitable mathematical propositions exactly because they were free of any meaning (Zach, 2006). This meant that the subjective, psychological meaning of certainty could still apply to these otherwise meaningless propositions.

Each of the three movements sketched above attempted to secure the foundations of mathematics whilst abandoning aspects of certainty, as entailed within a platonic context (Ramsey, 1931). For instance, intuitionism is not objective in the sense that platonism is; it does not refer to objects that exist independently of the mind (Shapiro, 2007). Despite the fact that intuitions may be seen as capable of revealing universal truths, intuitionism is in a sense subjective, since it supposes that mathematical knowledge emerges from the mind (Heyting 1983). Nevertheless, this assumption of intuitionism brings it into a better position than platonism when it comes to explaining how humans can obtain mathematical knowledge. Additionally, intuitionism seems to suggest that mathematics cannot be something that is entirely abstract; intuitionism has an empirical flavour, since knowledge stemming from intuitions amounts, in a sense, to knowledge that can be experienced (Brouwer, 1983). For example, if our understanding of natural numbers depends on our intuitive understanding of time, as Kant assumed (Friedman, 1090), then the way different cultures treat time around the globe must provide empirical evidence that relates to the way humans perceive numbers. In fact, intuitionism has been judged on just these grounds, since there does not seem to be a universal intuition of time (Hersh, 1997).

Moreover, contrary to platonism, logicism and formalism were not really concerned with the issue of truth (Brouwer, 1913; Russell, 1918). The logical rules of logicism and the syntactical rules of formalism may be taken to guarantee that if we start from true propositions (assumptions), the conclusion will be necessarily true (Corcoran, 1994; Curry, 1951). Notwithstanding, the truth of the assumptions is not necessary for producing valid conclusions, and what matters for certainty is validity and not truth (Durand-Guerrier, 2008). Of course, it can be argued that if something is not true, then it can have no valuable psychological (subjective) meaning, even if it is certain, but it can equally well be argued that the security of certainty may carry sufficient psychological (subjective) meaning for humans, even if it rests

upon a lie (Krishnamurti, 1987), and certainty seemed to be far more important for logicists and formalists (Brouwer, 2000).

Empiricism

The attempts to provide secure foundations for mathematics failed; the goal proved to be beyond reach (Tiles, 1991, Kline, 1980). It was no longer possible to assume mathematics as certain and infallible (Lakatos, 1976a). Hence, philosophers were practically obliged to seek the essential meaning of mathematics - both objective and subjective - elsewhere (Ernest, 1991; Hersh, 1997). Instead of granting to mathematics a singular status among other fields of knowledge on the basis of its certainty, philosophers embraced mathematics as an ordinary sector of fallible human knowledge (Ernest, 1994; Lakatos, 1976a). In this view, mathematical activity no longer possesses any transcendental qualities; its subjective, psychological meaning emerges from that fact that is now considered human and thus intimate (Ernest, 1991) In this vein, and one step away from platonism, lie empirical (or quasi-empirical) accounts of mathematics (Hersh, 1997).

Purely empirical accounts of mathematics, such as that put forward by John Stuart Mill, who aimed at grounding all mathematical knowledge in experience and the senses, are somewhat older (Mill, 1851). In the current climate, philosophers prefer a more moderate thesis, according to which mathematical knowledge may be taken to be related to observations, but is also developed independently of them (Kitcher, 1984). However, the crucial supposition on the basis of which mathematics can be called quasi-empirical is not whether mathematical knowledge can be traced back to observational data. What matters for a quasi-empirical theory of mathematics is that mathematical knowledge is taken to evolve in the same way that scientific knowledge does, that is, by testing certain hypotheses (potential theorems) and modifying them until they are capable of explaining all the relevant cases (Lakatos, 1976a). For some mathematicians, this description of the creation of mathematical knowledge makes more sense (on the subjective level) than assuming that mathematics exists 'out there' and that its axioms and their consequences are somehow revealed to them. Such mathematicians claim that doing mathematics is a 'messy' process during which the mathematician can rarely be certain about what is true, and that, as a consequence, axioms appear not at the beginning, but at the end of the process, when the theory has already been developed and has begun to be refined (Hersh, 1997; Lakatos, 1976b).

Empirical and quasi-empirical accounts of mathematics bring mathematics on par with physics (Quine, 1995; Tymoczko, 1991). Absolute certainty is abandoned (Lakatos, 1976a; Putnam, 1975), and in the case of empirical theories, mathematical knowledge is taken to have its origins in human experience through observation and perception (Irvine, 1989). It is not difficult to justify at least some basic mathematics on the basis of experience (Irvine, 1989). Bloor (1994) argues that most people would justify elementary arithmetic statements, such as 2+2 equals 4, by referring to concrete examples involving apples or other objects. From such a basis, mathematical knowledge can be seen to advance, as science does, through idealisations of the patterns offered by our experience (Kalmár, 1967; Kitcher, 1984), and through hypotheses which are falsified and reformulated to rebut the falsification (Lakatos, 1976b). This rooting of mathematics in experience differs from any empirical taste that intuitionism may have. Despite its empirical hues, intuitionism locates the source of mathematical knowledge within the human mind and not in observations of objects external to the mind (Hersh, 1997).

Most importantly, in (quasi-)empirical theories, the criteria for accepting mathematical knowledge become pragmatic, which is to say that mathematical claims or methods are sanctioned by the scientific and mathematical community as long as they produce useful results (Quine, 1951; Putnam, 1975). This implies that mathematical knowledge is not completely objective and value-free, but that when mathematical applications are designed, the respective designer needs to make certain decisions about the initial assumptions that entail the most fruitful conclusions (Bishop, 1988; Skovsmose, 1994).

According to empirical theories, the knowledge generated by both mathematics and physics has the same source. Therefore, such theories bridge the gap between mathematics and physics even better than platonism does, thereby helping us to find meaning - objective or subjective - in the former by resorting to how we find meaning in the latter (Armstrong, 1978; Tymoczko, 1991). Mathematics may be taken to study real objects and to speak about the truth, simply because the science which employs it concerns real objects in the true world (Colyvan, 2001; Resnik, 1995). So, in a sense, empirical theories may be said to resemble platonism, although mathematics is no longer considered to offer certainty (Tymoczko, 1991). With quasi-empirical theories at the very least, mathematical concepts cease to be purely abstract and intangible as they were presented by platonism, logicism and formalism; they are now recognised as open to evaluation on an empirical, pragmatic basis (Kalmár, 1967; Tymoczko, 1991). To this extent, alienating transcendental aspects of mathematics are eliminated, and this may render the study

of mathematics more human, thus enhancing its subjective, psychological meaning (Freire, 1996; Seeman, 1959).

Humanism

Mathematics can be humanised still further (Ernest, 1991). Any sense of a mathematical existence independent of the human mind, which may be preserved in quasi-empirical accounts, disappears if mathematics is seen as a socio-cultural construction (Ernest, 1998a; Hersh, 1997). Instead of being rooted in our sensual experience, such approaches root mathematical knowledge purely on social conventions and agreements, claiming that it continues to change and evolve, even if slowly, as society itself develops (Bloor, 1991; Lyotard, 1984).

A forerunner of this idea, although still dreaming of certainty (Shanker, 1987), was that put forward by Wittgenstein (Ernest, 2004; Phillips, 1977). The certainty Wittgenstein conceived in association with mathematics was essentially a socially constructed one (Ernest, 1998a). He perceived mathematics as a language game which is played according to certain rules (Wittgenstein, 1953). As with formalism, these rules may provide concrete algorithms which can be used to determine beyond any doubt whether a claim is correct or not according to the language employed (McGuinness, 1979). However, Wittgenstein was also interested in the objective, philosophical meaning of such rules, and he located this in the way that the rules were used within a group (Wittgenstein, 1953). This move unavoidably transforms mathematics into a social enterprise, since different groups may decide on different rules for the same language, and thus the rules of the game together with their interpretations are unavoidably contingent, being determined by humans who choose among many other possible rules (Bloor, 1994). According to this view, we eventually may grow accustomed to the rules our society is using, since our education impresses them upon us, and consequently we may forget that they are a human product and that there is nothing necessary or universal about them (Wittgenstein, 1978).

Since Wittgenstein, the idea of mathematics as a socio-cultural construct has been further developed by more recent thinkers (Bloor, 1991; Ernest, 1998a; Hersh, 1997; Restivo, 1992). Such later philosophers may perceive mathematics to resemble a language (Rotman, 1993) and mathematical knowledge to emerge as the product of dialogue between mathematicians, or between teachers and students (Ernest, 1998a; Lakatos, 1976b). Our claims to certainty

regarding mathematical statements can also be the product of dialogue (Rorty, 2009). In this way, mathematicians' claims come to be accepted or rejected by their peers, for instance through peer-reviewed journals (Hersh, 1997). Such views are also often employed by mathematics educators in order to describe mathematical learning in the classroom, where classroom interactions shape the mathematical activities of which the classroom community approves (Voigt, 1994; Yackel, & Cobb 1996). Educators believe that such theories of learning, which depend on a more human picture of mathematics, have the potential to enhance the subjective, psychological meaning students may find in learning mathematics (Cobb, 1994; François, & van Bendegem, 2007).

Apart from the importance of social norms and conventions, theories of learning also stress the active role of the learners in the production and organisation of knowledge (Lerman, 1989; Cobb, 1994; von Glasersfeld, 1995). In the context of philosophy, Kitcher described a notably subjective approach to mathematical knowledge, suggesting that there are psychological factors which cause us to accept or reject this or that mathematical claim or proof (1984). For example, some may be willing to accept claims made by those in authoritative positions, while others may require further evidence. This idea is supported by the literature on personal epistemology which proposes that the reasons to believe or know something vary between individuals (Baxter Magolda, 1992; Hofer, & Pintrich, 1997; King, & Kitchener 1994).

In sum, for modern humanistic accounts of mathematics, certainty is altogether discarded, and any element of objectivity that remains is seen in strictly social terms (Ernest, 1991; Bloor, 1991). Objectivity is reduced to the fact that we have agreed to do mathematics, or reason within it, in a certain way and that this sets the standards for our mathematical activity (Shanker, 1987). These standards are not unique; they were different in the past and they will most probably be different in the future (Ernest, 2004; Lyotard, 1984). They are not even ideal ones, and they can certainly be judged as potentially oppressive (Ernest, 2004; Walkerdine, 1994). However, they are the ones that are currently in force and therefore the ones to which we must adhere, if we wish to communicate any effective mathematical ideas (Bloor, 1994). It may be claimed that while the motto of the foundationalists was to preserve mathematical certainty at any cost because it was essential for the status of mathematics which otherwise would be reduced to mere hypotheses (Ernest, 1991), the motto of humanistic accounts is to reject mathematical certainty at any cost because it is detrimental, falsely presenting mathematics as a rigid non-human structure (Rowlands, Graham, & Berry, 2011).
In the remainder of the chapter, I turn to students and their understanding of mathematics. I discuss the beliefs that students have been found to hold about mathematics in order to determine if any such beliefs could be legitimately classified as philosophical. Then I turn to a discussion of psychological, subjective meaning and particularly of the construct of 'personal meaning' as it has been put forward by Vollstedt (2011). Finally, I discuss the possibility of students using philosophical ideas in order to make sense of mathematics.

Meaning for students

Is there any relationship between the philosophy of mathematics and research in mathematics education? Intuitively, the answer seems to be in the affirmative; however, the issue has not received much attention. It appears self-evident that engaging with mathematics will engender some beliefs about what mathematics is and how mathematical knowledge is produced, i.e. a philosophy of mathematics, and that such beliefs, in turn, will influence further engagement and interaction with mathematics (Ernest, 1991, François, & van Bendegem, 2007; Hofer, & Pintrich, 1997). After all, this is how we make sense not just of mathematics but of life in general; our past experiences supply us with values and beliefs through which we attempt to comprehend future experiences (Cobb, 1986; Habermas & Bluck, 2000; Ormiston, & Schrift, 1990). In essence, philosophising is a meaning-creating activity (Smith, 1980; Gadamer, 1975). So, assuming that students in classrooms struggle to find meaning - objective or subjective for mathematics (Kilpatrick, et al., 2005b), philosophical ideas may contribute towards their efforts. Indeed, such a meaning, emerging from philosophical concepts, will have an objective aspect, but it should also have a subjective, psychological aspect. This is likely to be considerably greater than the subjective meaning attributed to mathematics by philosophers, since contrary to philosophers, students would have no reason to suspend psychological meanings in favour of objective, universal ones.

Students' beliefs about mathematics have been researched principally under the broad term of beliefs, (Leder, Pehkonen, & Törner, 2002). A motive behind such investigations has been the assumption that beliefs may provide access to how students make sense of mathematics (Cobb, 1986). These beliefs in turn are considered to play a significant role in determining students' engagement with mathematics and the subsequent learning of the subject (Callejo, & Vila, 2009; DeBellis and Goldin 2006; Hannula 2006; Ma and Kishor 1997). Nevertheless, the concept of meaning has been largely absent from research concerning beliefs which could be

characterised as philosophical, namely beliefs which concern the nature of mathematics and mathematical knowledge (Ernest, 1991; François, & van Bendegem, 2007; Hofer & Pintrich, 1997; Kilpatrick, Hoyles, & Skovsmose, 2005a). The main foci in this area of research concern the ways in which past experiences, and the beliefs they have induced, affect teachers and the way they teach, or students and the way they learn (Leder, et al., 2002).

Moreover, research and theory about philosophy in mathematics education seem to have focused more on the beliefs of educators and researchers regarding philosophical theses which could form the basis for theories of learning and teaching practices, and much less on students' philosophical beliefs (Ernest, 1994, François, & van Bendegem, 2007). This is precisely why my research focus concentrates on students' beliefs about philosophical issues. In this area of investigation, research has mostly been directed to beliefs about how students learn mathematics, and not to students' beliefs pertaining to the nature of mathematics and mathematical knowledge (Muis, 2004). It is however the case that philosophical considerations have sometimes been studied under the name of epistemic beliefs (Op't Eynde, de Corte, & Verschaffel, 2006; Ruthven & Coe, 1994).

In any event, the available research suggests that students hold beliefs regarding the nature, source and validation of knowledge within specific knowledge domains (Hofer & Pintrich 1997; Muis, Bendixen, & Haerle 2006) - that is, they enjoy philosophical opinions. Moreover, it can be argued that an 'implicit philosophy' about mathematics is conveyed to the students during mathematics instruction (François, 2007; Steiner, 1987). Hence, the legitimate question arises: are students' philosophical considerations related to their attempts to make sense of, or find subjective, psychological meaning in, mathematics?

In the next section, I consider the literature on students' beliefs which could be termed philosophical. Researchers in mathematics education have operationalised many different definitions - some formal and others informal - of beliefs, without having reached a consensus (Leder, et al., 2002). It seems that the definition used on each occasion depends on the context and the audience of the research (McLeod, & McLeod, 2002). Taking into consideration the subject of this research, it seems appropriate to operationalise beliefs as bearers of meaning that is both objective and subjective. Such a definition is able to capture the complexity of the construct of beliefs which lies between the affective, psychological (subjective meaning) and the cognitive or even the social domain (objective meaning) (Goldin, 2002; McLeod, & McLeod, 2002). Subjective meaning appears to fit better with the affectively, psychologically

loaded aspect of beliefs, while objective meaning is shaped by social constraints and may be taken to correspond to what one assumes to know on the basis of one's beliefs. In the context of this research, Schonfeld's understanding of beliefs as 'one's mathematical world view, the perspective with which one approaches mathematics and mathematical tasks' (1985, p.45) seems to provide a useful simile. This 'mathematical world view' to which Schoenfeld refers, can be understood as the meaning - both objective and subjective - that one may attribute to mathematics.

Philosophical beliefs about mathematics

In this section, I focus on the literature which could be claimed to involve the objective, philosophical meaning that students may assign to mathematics. Before summarising what philosophical beliefs are attributed to students according to the literature, it is important to determine which among the range of such beliefs are relevant to this research. There are not many empirical studies on students' beliefs about issues which pertain to the philosophy of mathematics, and most studies assume what the result of certain beliefs would be only on the basis of theoretical arguments (François, & van Bendegem, 2007). Muis (2004) observed that research on students' beliefs about mathematics focuses more on beliefs about learning and less on beliefs about purely philosophical issues. This could be because, on the one hand, learning is inseparable from education and, on the other hand, because the relationship between the philosophy of mathematics and mathematics education is not a straightforward one (Ernest, 1991). The fact is that even studies that explicitly address philosophically related concepts, such as epistemic beliefs, often conclude by speaking about learning attitudes (e.g. Brown et al. 1988; Diaz-Obando, Plasencia-Cruz, & Solano-Alvarado, 2003; Lampert, 1990; Schoenfeld, 1989; Verschaffel, et al., 1999; Yackel, & Cobb, 1996).

Beliefs about learning mathematics are clearly valuable for educational purposes, and they are also a valid philosophical issue in themselves (Ernest, 1991; Leder, et al., 2002; von Glasersfeld, 1995). However, they are not prominent in most philosophical accounts which try to make sense of mathematics, for the simple reason that such accounts are interested in mathematics as a field of knowledge and not in mathematics as a subject to be taught in school (Brown, 2008; Shapiro, 2000). A theory of learning mathematics is not a theory of knowledge of mathematics, although it may be rooted in such a theory (Ernest, 2010; Lerman, 1996; Sfard, 1998). The former concerns principles which may underlie an effective practice of teaching

mathematics, whilst the second directly regards the nature of mathematical knowledge (Hofer, & Pintrich, 1997).

Nevertheless, the way students learn mathematics may influence their appreciation of what mathematics is, and how mathematical knowledge is acquired (François, & van Bendegem, 2007). Researchers have frequently argued that traditional ways of teaching, where the learner is viewed as a vessel wherein knowledge is deposited, promote an image of learning mathematics as a process of memorising and applying rules that are found in textbooks (Carpenter, & Fennema, 1992; Erlwanger, 1973; Garofalo, 1989 Leung, 2001). Such an attitude may reflect a perception of mathematics as a set of rules and procedures which are part of the society's culture, and which the students have to accept as such.

Considering the above, beliefs about learning will not be the focus of this study unless they can be directly connected to an understanding of the nature of mathematics or mathematical knowledge. For example, I will not be interested in beliefs concerning whether learning is supposed to happen quickly, or whether it is the result of continuous effort (see Jehng, Johnson, & Anderson 1993; Schommer 1990). On the other hand, the belief of learning mathematics through applying rules in exercises is of interest, since it indicates that mathematical knowledge emerges through repeated exposure to the use of the rules, and it alludes to Wittgenstein's idea that the philosophical meaning of a rule resides in the way we use it (Wittgenstein, 1953).

With respect to students' beliefs about philosophical issues, the research is limited (François, & van Bendegem, 2007). The category of beliefs which is closer to such issues, and which has been studied often, is epistemic beliefs (Goldin, 2002; Shapiro, 2000). Epistemic beliefs are those concerned with the process of knowledge formation, constituting what is known as an individual's personal epistemology (Hofer, & Pintrich, 2002). So in order to determine beliefs which are relevant to this study, it is necessary to discuss which of the various beliefs that mathematics education researchers have investigated can be said to fall under the rubric of epistemic beliefs. Before doing so however, it is important to consider what epistemic beliefs are, and how they relate to the current study.

Researchers agree that under the category of epistemic beliefs fall beliefs about the certainty, the source and the structure of knowledge, (Op't Eynde, et al., 2006, p.58). Beliefs about certainty – that is, views on how certain we can be about our knowledge – are extremely relevant for this study, since certainty has been at the core of the philosophical discussion about

mathematics (Hersh, 1997). The review of philosophical ideas above also suggests that beliefs about the source of knowledge are equally relevant, since, for example, they may point to logic (logicism), experience (empiricism), existence of mathematical objects (platonism), or social conventions (humanism).

On the other hand, beliefs about the structure of knowledge, i.e. the ways different pieces of mathematical knowledge are, or are not, interconnected, do not seem to have played a crucial role in philosophers' attempts to understand mathematics (Brown, 2008; Hersh, 1997; Shapiro, 2000). Presumably this is because the thinkers who have engaged with the philosophy of mathematics were familiar enough with mathematics to realise that it comprises a complex network of interrelated statements, such as definitions, axioms, theorems and proofs. Of course, students may not have realised that mathematics is not a fragmented field of knowledge, where one topic is unrelated to the other (Schommer, Crouse, & Rhodes 1992). Moreover, they may not be clear about the role that different kinds of mathematical statements, e.g. axioms, definitions, or proofs, play in the network of mathematical knowledge (Amit, & Fried, 2005; Edwards, & Ward, 2004; Hanna, & de Villiers, 2008; Harel, & Sowder, 1998; Moore, 1994). Nevertheless, such issues will be discussed only coincidentally in this thesis, since its foci are topics which belong in the philosophy of mathematics.

There are also other categories of beliefs which have been classified as epistemic by some researchers, though not by all (Hofer, & Pintrich, 1997; Op't Eynde, et al., 2006). Some of these categories are essentially beliefs about learning (Hofer, & Pintrich, 1997), and as such are not pertinent to this study.¹ However, among them, justification of knowledge and attainability of truth (Hofer, 2000) appear to be two categories which are relevant for my research. Justification of mathematical knowledge may be rooted in logic and proofs (logicism), experience (empiricism), mathematical existence (platonism), and social authority of mathematical proofs (humanism). Furthermore, attainability of truth can be connected to objective truth that corresponds to reality (platonism, empiricism), to truth revealed through reason and proofs (logicism, formalism), or to truth as a social construction (humanism).

Regarding mathematics education, drawing on a substantial review of the literature on beliefs, Op't Eynde, et al. (2006) sought to organise these beliefs and clarify which among them could

¹ With the exception stated above.

be called epistemic. These authors proposed that the category of beliefs that relates more to epistemic, or philosophical, beliefs is that concerning mathematics education. According to their classification, beliefs about mathematics education include beliefs about mathematics, beliefs about mathematical learning and problem solving, and beliefs about mathematics teaching (Op't Eynde, et al., 2006). From these, only the first group is strictly relevant to the current research. The relationship between beliefs about learning and epistemic beliefs has been discussed earlier, and a similar relationship exists between beliefs about teaching and epistemic beliefs, since teaching and learning are essentially two sides of the same coin.

The systematisation advanced by Op't Eynde, et al. (2006) also includes two other groups of beliefs: beliefs about the self as a mathematician, which basically comprise motivational beliefs; and beliefs about the mathematics class context, which include beliefs about the role of teacher and students, together with sociomathematical norms. The writers do not relate these groups to epistemic beliefs. Indeed, the former group, although relevant to the experiences that students have in the classroom, and thus relevant to any meaning they attribute to mathematics and mathematical knowledge. The same can be generally claimed about the latter group. However, I wish to exclude sociomathematical norms which may pertain to philosophical concerns. For instance, sociomathematical norms may determine what is considered as an acceptable justification for mathematical claims (Yackel, & Cobb, 1996), a matter that can be linked to the epistemic dimension of justification of knowledge (Hofer, 2000).

What follows is a compilation of examples of beliefs that have been studied in the context of mathematics education and which pertain to philosophy. These examples are drawn from three studies which took issues relevant to philosophy as their focus, and did not simply examine such beliefs coincidentally. The relevant sources here are the studies of Ruthven and Coe (1994), Fleener (1996), and Op't Eynde, et al. (2006). The examples can be organised into: beliefs relating to the certainty or immutability of mathematics; beliefs relating to the truth-status or objectivity of mathematical knowledge; beliefs relating to the source of mathematical knowledge; beliefs relating to the justification of mathematical knowledge; and beliefs relating to the applicability of mathematics. As the earlier discussion indicates, all these are issues which are relevant to the philosophy of mathematics.

• Beliefs relating to the certainty or immutability of mathematics: 'mathematics is continuously evolving, new things are still discovered' (Op't Eynde, et al., 2006; p.66);

'there are some mathematical truths which will never be proven wrong'; '2+2 always equals 4' (Fleener, 1996, p.314); 'the mathematics developed on another planet would be the same as the mathematics we know' (Ruthven, & Coe, 1994, p.103).

- Beliefs relating to the truth-status or objectivity of mathematical knowledge: 'science and math are slowly revealing truths about reality' (Fleener, 1996, p.314); 'there are several ways to find the correct solution of a mathematics problem' (Op't Eynde, de Corte, & Verschaffel, 2006; p.66); 'unlike in most other subjects, in maths there is a clear cut right and wrong'; '"the angles of a triangle add up to a half turn" was true even before any humans recognised it'; 'competent mathematicians would always agree about whether or not a proof is valid' (Ruthven, & Coe, 1994, pp.103,104).
- Beliefs relating to the source of mathematical knowledge: 'mathematical innovations result from scientific inquiry and practical applications'; (Fleener, 1996, p.314); 'if a teacher tells me that something is true then I don't need to check it for myself' (Ruthven, & Coe, 1994, p.104).
- Beliefs relating to the justification of mathematical knowledge: 'once a mathematical result has been proved then you can be certain it is true'; 'if a mathematical relationship is obviously true there is no need to justify or prove it'; 'being shown a proof of a mathematical relationship does not help to understand it fully' (Ruthven, & Coe, 1994, pp.104,105); 'the value of science and math lies in the usefulness in solving practical problems' (Fleener, 1996, p.314).
- Beliefs relating to the applicability of mathematics: 'formal mathematics has little or nothing to do with real thinking or problem solving'; 'mathematics enables man to better understand the world he lives in'; (Op't Eynde, de Corte, & Verschaffel, 2006; pp.64,66). The former is linked to the question of whether mathematics can be claimed to be rooted in logic (Russell, 1918); while the latter is connected with the issue of whether mathematics exists in a way similar, or comparable, to the material world (Colyvan, 2001; Resnik, 1995).

At this point, it is worth noting that studies about beliefs in mathematics have stressed that students generally tend to perceive mathematics as absolute, immutable truths, where the answer to a given problem is unique (Fleener, 1996; Schoenfeld, 1992). These truths are the rules which the students are obliged to follow if they wish to solve mathematical problems (Carpenter, & Fennema, 1992; Erlwanger, 1973; Garofalo, 1989 Leung, 2001). Educators have

attributed such beliefs to the fact that in schools, students mostly encounter closed problems which admit specific answers and can be solved following specific procedures (Skovsmose, 1994). Similarly, within a more philosophical context, such beliefs can be attributed to the fact that students in school come to know only one mathematical system and do not study alternative mathematical theories, such as non-Euclidean geometries (François, 2007; Hersh, 1997). Moreover, these beliefs are strongly associated with the traditional setting of teaching where the students are merely passive receivers of knowledge (Alrø, & Skovsmose, 2002; Carpenter, & Fennema, 1992; Cobb, Wood, Yackel & McNeal, 1992; François, & van Bendegem, 2007; Leung, 2001). Ruthven and Coe (1994) referred to diversity among students' beliefs; however, they attributed this to the fact that the sample considered in their study involved students who had attended less traditional classes. In a more recent study, Op't Eynde, et al. (2006) suggested that plurality of beliefs, ranging from beliefs closer to the absolutistic perspective to beliefs on a par with modern philosophical trends, can be witnessed regardless of the teaching environment. The next section is dedicated more to the subjective, psychological aspect of the study.

Psychological, subjective meaning

Construing subjective, psychological meaning for one's experiences seems to be unavoidable; it seems to be an inherent part of human life (Frankl, 1985). So, young people also go through a process of finding psychological meaning for life, society and themselves, and a substantial part of it is related to school, provided that they follow formal education (Esteban-Guitart, & Moll, 20014). During this process, they are also called upon to find some subjective, psychological meaning for mathematics in relation to society and themselves (Vinner, 2007, Vollstedt, 2011). This is not only because they are taught mathematics in school, and thus mathematics becomes part of their everyday life, but also because mathematics plays a fundamental role in, and is valued by, the contemporary society of which they are a part (Howson, 2005). After all, this is most likely the reason why mathematics is included in most curricula, and why students have to grapple with it (Huckstep, 2007).

However, finding subjective meaning for mathematics does not seem to be a straightforward process (Kilpatrick, et al., 2005b). Students' attempts to find such a meaning in mathematics and their difficulties while doing so become apparent in the so-often asked meta-mathematical question 'Why are we doing this?' (Davis, 2001). In any case, even if they fail to answer this

question, it seems that students have a picture of why mathematics is or is not subjectively meaningful to them and how they are related to it (Diaz-Obando, et al., 2003). In other words, students may not realise the more or less objective meaning that mathematics has for the society, or for the mathematician, but they should know what doing mathematics in school means for them on the subjective, psychological level.

Vollstedt (2011) introduced the construct of 'personal meaning' in mathematics education, suggesting that it is crafted in the context of students' beliefs about mathematics and how mathematics is taught, their goals in school, and their general psychological development. In the terminology employed in this thesis, this is actually a subjective meaning. Vollstedt has also advanced (2011, p.325) the following typology of personal meaning:

- 'fulfilment of social demands', e.g. students may wish to study a subject which requires that they are examined in mathematics, or they may see mathematics as an obligation, and aim at satisfying important people in their lives with good grades and exam results;
- 'active practice of mathematics' which concerns the students' cognitive experiences in the classroom context as active learners of mathematics;
- 'efficient and supportive lesson design' which concerns the students' cognitive experiences in the classroom context as recipients of efficient instruction and support;
- 'emotional-affective development', which concerns the students' affective experiences in the classroom context;
- 'cognitive self-development', e.g. the feeling of development as an autonomous mathematical learner, and the fulfilment of understanding mathematical logic;
- 'relevance of applications', i.e. whether students feel that mathematics relates to everyday life;

• and, 'well-being due to own performance', i.e. satisfaction derived by performing well. At first, Vollstedt's (2011) typology does not seem to be very closely relevant to the philosophy of mathematics. Nevertheless, it can be argued that there are some notable connections between some of the types and philosophical beliefs. These types concern: the students' relationship to mathematical logic (subtype of 'cognitive self-development'); the relevance of mathematics applications to everyday life; and, being an active learner of mathematics. Firstly, the role of logic in mathematics has been a philosophical issue associated with the apparent certainty of both logic and mathematics (Shapiro, 2005). So some students may be attracted by the certainty of mathematical logic, while others may find it alien and struggle to connect it to their human experiences. Secondly, mathematical applicability is related to epistemological considerations about the relationship between mathematics and science (Benacerraf, 1973). Establishing a connection between mathematics and science should help students who value understanding how the material world operates, and are able to use mathematics towards this end, to find subjective meaning in mathematics. Finally, the feeling of being an active learner while doing mathematics can be connected to philosophical discussions about the infallible status of mathematical authority. Some students may experience this authority as absolute, while others may feel free to voice their own opinions and 'play' with mathematics while they learn. In all, it seems that there is potential in philosophical beliefs, which carry a more or less objective meaning, being associated with the subjective, psychological meaning that students attribute to mathematics.

Of course, students are not expected to have a formal philosophical theory about mathematics; they are not professional philosophers or mathematicians (Hofer, & Pintrich, 1997; Ruthven & Coe, 1994). Moreover, philosophical issues are scarcely - if ever - discussed in the classroom, where priority is given to teaching the mathematical content, and not to reflecting on meta-mathematical questions which may seem to be beside the point (Prediger, 2007). Thus, students are unlikely to be aware of the various unresolved controversies which have occupied philosophers for centuries.

Nevertheless, students spend a considerable amount of their time doing mathematics, and it is reasonable to assume that they will have attempted to make sense of their mathematical experiences, even if this is only on a subconscious level (Op't Eynde, et al., 2006). It can be argued that in the process of finding subjective meaning in mathematics students will have developed an implicit, or even explicit, philosophical perspective regarding what mathematics is and how it functions, which would also carry more or less objective meanings. After all, at least in the philosophical context, the hermeneutical task of understanding mathematical practice includes epistemological concerns, and is closely interrelated with ontological, metaphysical aspects of mathematics (Balaguer, 1998).

So as philosophers, students may have considered philosophical issues, though certainly not with the same precision. Moreover, even if they have not entertained such considerations, their experiences should provide them with the raw material which would allow them to form, at least some initial, unrefined answers to philosophical questions. For example, the sociomathematical norms of the classroom would dictate 'what counts as an acceptable mathematical explanation and justification' (Yackel and Cobb, 1996, p.166), thus providing answers to epistemological questions with respect to justification. Essentially, students have the means to construe a 'philosophy' of mathematics, and their beliefs about philosophical issues will unavoidably reflect their endeavour of interpreting, and attributing subjective meaning to, their mathematical experiences, thus bringing together objective (philosophical) and subjective (psychological) aspects of meaning.

In the following I will mostly use the terms 'objective' and 'subjective' meaning instead of 'philosophical' and 'psychological' meaning respectively, because the former contrast with one another more sharply than the latter. However, the term 'objective meaning' will not be as frequently used, because in the context of mathematics education, it makes more sense to talk about the students' beliefs instead of the objective meaning that students attribute to mathematics through their beliefs.

The goal of the research

In the light of the foregoing discussions, my study aims at investigating the following questions:

a) What beliefs do young adults at the end of their schooling (age 17-18) in Greece hold with respect to issues of the philosophy of mathematics?

b) How do their beliefs inform, or are influenced by, what mathematics means for them on a subjective level?

The first question centres upon the objective aspect of meaning that students may attribute to mathematics through philosophical considerations; while the second considers the interplay between such an objective meaning and students' more subjective reactions to mathematics.

The philosophical issues which the study examines have been selected on the basis of the literature, and comprise issues which appear to be prominent either within the philosophy of mathematics or the philosophy of mathematics education. In particular, the study is concerned with the following issues: the ontological debate as to whether mathematics is invented or discovered (Godino, & Batanero, 1998; Tymoczko, 1998; van Moer, 2007); the presentation of mathematics as a unified theory or as comprising multiple systems (Hersh, 1997); the problems of mathematical certainty, immutability, objectivity, and truth (Ernest, 1991; Hersh, 1997); the role of rules in mathematics (Wittgenstein, 1978; Sfard, 2000); the role of logic and

observation in mathematics (Mill, 1851; Russell, 1918, Bloor, 1991); and the role of proof in mathematics (Benacerraf & Putnam, 1983a; Hanna, 1995). These issues are also used as a starting point for discussing students' psychological responses to mathematics, e.g. how they feel about mathematical logic, certainty or rules. In terms of the philosophical issues, the goal will be to report the more or less objective meanings of the students, seeing what beliefs are present in the group as a whole, and how they relate to one another. However, as far as the subjective meanings that students attribute to mathematics are concerned, the focus shifts to also include individual accounts, seeking to capture the uniqueness of the subjective meaning that each student attributed to mathematics. In the next chapter, I discuss the methodology which was employed in order to gather the students' beliefs and elucidate both the objective and subjective meanings that their beliefs had for them.

Methodology

The philosophical background of the study

Hermeneutics seems to be the most appropriate methodological context for a study involving meaning, since making sense of something, or understanding it, is essentially a hermeneutical activity, founded upon the exercise of interpretation. Hermeneutics originally referred to the exegesis of biblical texts. However, in modern philosophy, hermeneutics is envisaged as a general process of understanding by way of interpretation, embracing all aspects of human life (Dilthey, 1972). In fact, within the hermeneutic tradition, understanding is perceived not merely as one among many human activities but as the activity which defines human existence (Gadamer, 1975, 1984). Indeed, as psychological research reveals, understanding and meaning making are essential to human life (Wong, 2012a). Developing meaning through understanding - usually intertwining subjective and objective meanings - allows people to orient themselves in life by creating a coherent account of their experiences which can then act as a guide for future reflection and action (Weinstein, Ryan, & Deci, 2012). Part of the constellation of our experience includes experiences with other people, and the hermeneutical process of understanding can be also applied to these experiences (Martin, & Sugarman, 2001; Risser, 1997). Consequently, hermeneutics, with its emphasis on the interpretation of meaning, can play a central role in research that is concerned with the meaning that humans attribute to their experiences, be it objective or subjective² (Crotty, 1998; Laverty, 2003).

A discussion about hermeneutics leads to a consideration of the tradition of phenomenology, as initiated by Edmund Husserl, and with which it has strong philosophical links (Crotty, 1998). This close relation is particularly evident in the 'phenomenological hermeneutics' developed by Ricoeur and by Gadamer (Jervolino, 1990; Palmer, 1969; Ricoeur, 1975; Thomson, 1981; Tan, Wilson, & Olver 2009). The relationship rests mainly on the fact that both philosophical traditions are based on the concept of intentionality (Bernet, Kern, & Marbach, 1993; Laverty, 2003; Ricoeur, 1975). Intentionality reflects the conviction that consciousness cannot occur without an object - not necessarily a material one - and that although objects may exist independently of consciousness, they cannot have meaning if consciousness is absent (Giorgi,

 $^{^{2}}$ In fact, the term 'meaning' in relation to hermeneutics is used without any qualification as 'subjective' or objective' because in this context it encompasses all the aspects of meaning.

1997; Humphrey, 1992). In other words, intentionality implies an intending towards, a reaching out on the part of the experiencing, sense-making mind, grasping towards an object, emotion, action, or event (Goldie, 2002; Sokolowski, 2000). No understanding can occur without external stimuli, for there can be no pure consciousness in the Cartesian sense. Hence, the notion of intentionality recognises that understanding does not occur in vacuo; it is produced from the interaction between interpreters and objects. In the course of this interaction, the sense-maker and the observed object are mutually implicated, and cannot be perceived in separation from one another (Krishnamurti, 1987; Larkin, Watts, & Clifton, 2006; Valle, King, & Halling, 1989).

A consequence of intentionality is that the art of understanding, and the resultant meaning which emerges from it, become both subjective and objective. On the one hand, intentionality does not allow for utter subjectivism (Mandelbaum, 1979); not every interpretation is acceptable, since validity is limited by the object of the interpretation, which carries some objective meaning (Ricoeur, 1991). On the other hand, intentionality leaves no place for pure objectivism; the mind does not simply mirror reality (Laverty, 2003). As long as the observed object is not approached by a totally empty mind, the mind creates an image that is shaped by our prior preconceptions and conditioning, creating subjective meanings (Gadamer, 1975; Krishnamurti, 1987). Moreover, being born within an ongoing multi-faceted tradition (ethnicity, nationality, social class, family) we unavoidably acquire a complex network of opinions and values through which we prejudge any phenomenon we experience (Merleau-Ponty, 1962; Ortega y Gasset, 1959). So our preconceptions are also endowed with some objective meaning (Geertz, 1973; Reichertz, 2004). In effect, every new situation is being approached through an 'old' pre-understanding that we carry along as part of our culture (objective meaning) and identity (subjective meaning) (Gadamer, 1975; Krishnamurti, 1987; Merleau-Ponty, 1962; Moscovici, 1988).

A central characteristic of phenomenology, which does not necessarily pertain to hermeneutics, is that it assumes an immediate relationship between the observed object and the interpreter (Laverty, 2003; Ricoeur, 1975). Phenomenologists claim that it is possible to bracket, that is, preempt all preconceptions, and reach a 'purer' apprehension of the object (Heron, 1992; Marton, 1986). This assumption complicates the application of phenomenology in research, since it is difficult, if not impossible, to experience the world without any preconceptions (Bell, 2011; Richardson, 2003). Moreover, in the research context, an understanding of the

relationship between the participants (including the researcher) and the subject at hand as immediate is not an accurate one (Fleming, Gaidys & Robb, 2003). An important mediator between the participants and the subject of the research is language (Mead, 1934; Ricoeur, 1975). Indeed language is necessary for research, but it carries preconceptions (Lafont, 2000; Krishnamurti, 1987). By contrast with phenomenology, hermeneutics allows for taking the mediation of language into consideration. Thus, Ricoeur observes how hermeneutical understanding cannot be immediate, but emerges by taking a detour through language (Ricoeur, 1967; Vanhoozer, 1991).

Preconceptions are always present in scientific analysis, and the goal cannot be simply to abolish them through 'bracketing', but rather to consciously introduce them into the process of interpretation (Bell, 2011; Fleming, et al., 2003). In the hermeneutical tradition, Gadamer asserts that preconceptions are a prerequisite of understanding, arguing that although it is possible to become aware of one's prejudices, it is simply impossible to experience the world without them (Gadamer, 1975; Way, 2005). In social research, researchers are not only faced with their own preconceptions regarding the subject they study, but also with the preconceptions that the research participants bring into their understanding of the phenomenon (Tufford & Newman, 2012). So on the one hand, researchers need to strive against imposing their ideas on the participants, while on the other hand, they should be aware that the participants' beliefs are the lenses through which their perception of the world is 'distorted' in an interplay of objective and subjective meanings (Finlay, 2008; Rubin, & Rubin, 2012; Wainwright, 1997). The next section aims at establishing why in-depth interviews would be the most appropriate method in order to pursue the goals of this research.

Methods

In-depth interviews comprise a research method that emerges naturally from the hermeneutical tradition (Fleming, et al., 2003) and which is also the most useful method for an in-depth analysis of epistemic beliefs (Baxter Magolda, 2004) (a category of beliefs which has been broadly studied and fits better the focus of my study than other group of beliefs). Below I review the place of interviews as a data collection method in the tradition of research on epistemic beliefs, where they have facilitated the illumination of individual accounts and their intricacies. Subsequently, I discuss interviews within the broader context of hermeneutics with respect to intentionality, the use of preconceptions, and maintaining a critical stance. The

particular ways in which all these theoretical considerations have been implemented in my research are considered more fully in the next section.

Interviews and epistemic beliefs

The in-depth interview constitutes one of the principal methods which have been employed in research on epistemic beliefs, the rubric under which beliefs about the philosophy of mathematics have been most commonly studied (Hofer, & Pintrich, 1997). In particular, researchers have utilised either interviews which address prominent life experiences through open ended questions (e.g. Baxter Magolda, 1992; Perry, 1970), or more structured interviews aiming at eliciting evaluations of opinions on controversial topics (e.g. the safety of food additives, King & Kitchener, 1994; why criminals return to crime after being released, Kuhn, 1991). In both cases, probes allow researchers to further unfold the intricacies of the participants' epistemic beliefs (Hofer, 2004).

The other method, which is also the most frequently used in research on epistemic beliefs, is the administration of questionnaires (Hofer, 2004). Notwithstanding, questionnaires unavoidably restrict the detailed responses which a participant may offer (Baxter Magolda, 2004; Hofer, 2004). The difficulty of using questionnaires for the current research lies primarily in devising an instrument which would be capable of capturing the fullness of the meanings under investigation, which are of a particularly complex character (Cohen, Manion, & Morrison, 2011; Hofer, 2004). Even where questionnaires include some open-ended questions, they unavoidably focus on the investigation of topics that are based on the theory that governs their formation, thus restricting the range of potential results (Foley, 2012; Groves, Fowler, Couper, Ledowski & Singer, 2009). That is why existing work on epistemic beliefs that has sought to apply questionnaires has given rise to different accounts regarding the organisation of such beliefs (e.g. Buehl, Alexander, & Murphy, 2002; Hofer, 2000; Schommer, 1990).

Of course, questionnaires have been used widely because of the advantages they offer. Above all, they allow for a large sample. A large number of questionnaires can be administered simultaneously, hence providing the researcher with a considerable body of data in a relatively short time (Groves, et al., 2009). Furthermore, it is easier to compare different populations if the same instrument is applied to them. In fact, questionnaires have been widely used in comparative epistemological studies, although there have been concerns for the validity of transferring concepts across cultures (Hofer, 2008). Finally, data from questionnaires allow for

the establishment of generalisations based on statistical analysis, indicative of correlations among epistemic beliefs and other constructs such as learning outcomes (e.g. Jehng, et al., 1993; Schommer, et al., 1992). Despite such advantages, in the case of the current research and its specific objectives, in-depth interview presents itself as a more appropriate tool for investigating the intricacies of the construct of epistemic beliefs. This is still more the case since, even where they may start from the same assumptions, individuals may reach different conclusions regarding the process of knowing (Baxter Magolda, 2004). A questionnaire cannot tackle this problem, since it does not allow for follow up questions which could clarify why a particular answer was given.

A shared shortcoming of both interviews and questionnaires is that they may influence participants' accounts by eliciting information that might not emerge in a natural context (Hofer, 2004; Silverman, 2006). In that light, research conducted in context is likely to yield less contaminated data. However, no method can produce wholly uncontaminated data about individual experiences (Silverman, 2006); no one can access them in their entirety, since they are ultimately private, although they are produced in context, through interaction with the world and its significant symbols (Mead, 1934; Ricoeur, 1976). Moreover, even if obtaining 'pure' data was possible, the process of analysing such data would unavoidably contaminate them, as the researchers would approach them through the lens of their prejudices (Fleming, et al., 2003). Therefore, simply aiming for data which are ultimately unpolluted cannot be the basis on which a research method is chosen.

Interviews and hermeneutics

Data collection

Of course, in disciplined research exercise, the method to be utilised should primarily be a function of the nature of the data which the researcher wishes to obtain (Bickman, & Rog, 2009). According to the principle of intentionality which underlies hermeneutics, the unit of research is the 'object' of the participants' consciousness, or in other words the participants' 'lived experience' (Dilthey, 1977). Gadamer (1975) associates lived experience both with the immediacy of witnessing an event, and with the lasting mark that this event leaves on the witness. Thus, lived experience is not just experience captured in situ, but experience which has been appropriated and integrated into one's identity. This latter aspect of lived experience

is particularly pertinent to beliefs, whose lasting impression is substantial. Hence, it seems that a method which can be appropriate for investigating lived experience and the effects of beliefs would be to invite the person who holds them to share them with the researcher during an interview (Gardner, 2010).

In a sense, one's lived experience can be understood as the 'mark' which our experiences leave on our brain (Changeux, & Ricoeur, 2000). This mark is not something static, but it dynamically develops every time that we revisit it in our memory (Gardner, 2010). In this respect, the interview provides an aid for awakening the contents of memory. This process seems to be appropriately captured by Gadamer's notion of 'fusion of horizons'. According to Gadamer (1975, 2006), new understanding and knowledge emerge through dialogue, where two pre-understandings, or horizons as Gadamer calls them, meet and fuse (Vessey, 2009). On the one hand, this fusion results in the participants re-interpreting their lived experience as they discuss its consequences with the interviewer. On the other hand, the fusion may also lead to the transformation of the interviewer's pre-understanding, as the participants' horizons challenge them to review their preconceptions under a different light. This does not mean that after the fusion of the horizons the interviewer will necessarily hold a different opinion, but it does mean that they will have been placed in a position to re-examine their prejudices in view of new information (Arnswald, 2002).

The concept of fusion of horizons suggests that a productive dialogue-interview is not a passive encounter where the interviewer simply extracts knowledge residing in the participants by posing the right questions (Holstein & Gubrium, 2003). Furthermore, for hermeneutics, the interview itself constitutes a lived experience; it is a meaningful action that takes place between two human beings, and not just a means to produce text. In other words, the interview may leave a mark which may continue to act upon the interviewer and the participants even after the interview's completion; it is a meaningful action that is textually rendered (Ricoeur, 1991). So, as lived experiences are re-processed during the interview, when the horizons of the interviewer and the participant meet, it seems more appropriate to claim that knowledge is co-constructed by the participants and the interviewer (Holstein & Gubrium, 2003; Kvale & Brinkmann, 2009).

Moreover in hermeneutics, understanding is perceived as a function of time (Gadamer, 1975). In particular, after two horizons fuse and the new understanding is integrated and appropriated, the initial horizons (pre-understandings) have changed. Consequently, any new encounter will have a new starting point, bearing the potential to lead to a further more enhanced understanding (Bell, 2011). This suggests that there may be advantages where interviews are conducted in multiple phases (Fleming et al., 2003). Since the interview can leave a mark – hopefully important and productive for the people who engage in it – it may keep returning to their memory. Thus, the interval between two interviews creates a distance, and gives both the interviewer and the participants the time to critically reflect on the fusion that took place during the first interview and appropriate its effects (Earthy & Cronin, 2008). As a result, in the second interview, the interviewer may elaborate and clarify any important issues, while also resolving any contradictions that arose in the first interview.

Data analysis

The concept of 'dialogue' in Gadamer is expanded to include the process of reading: the reader asks and the text 'answers' (Gadamer, 1975). In the context of interviewing, the text is represented by the transcripts of the interviews (Bell, 2011; Fleming, et al., 2003). However, Ricoeur distinguishes between oral and written speech (Ricoeur, 1974). According to his argument, in spoken language there is a certain immediacy. The interlocutors are able to ask one another in order to clarify the meaning of the utterances, and are thus capable of achieving a position from where it may be possible to grasp the speaker's intended meaning – meaning and intention may be brought together in a way that is not possible for a written text (Ricoeur, 1973). The written word introduces a distance between the text and the reader, since the reader is not in the position to request the text to explain itself (Ricoeur, 1976). Nevertheless, when the reader of the text-transcript is also the interviewer, then the distance is diminished, although not entirely eliminated. In this case, the reader has been present in the original dialogue, and has still access to parts of oral speech which cannot be captured by written speech (Gardner, 2010). In other words, during the analysis, I was in a privileged position where I could, to an extent, both immerse myself in, and detach myself from, the dialogue that had taken place during the interview. The former allowed me to relive my and the participants' experience during the interview and report it faithfully, while the latter allowed me to look at the participants' utterances with fresh, potentially less biased, eyes and ponder over implications which might have not been clear during the dialogue.

In fact, an interview results in something more than a transcript. The interview is a meaningful (inter-)action (Ricoeur, 1973). According to Ricoeur, meaningful actions can be said to leave

a 'mark' through their consequences, namely the course of the events which follows the action. Obvious examples here would be major historical events. The mark that the interview - as a meaningful action - leaves also corresponds to the mark made by a text (Ricoeur, 1973). In the same way that an author cannot predict how their text will be interpreted, actors cannot always predict the full range of their actions' consequences. Moreover, the course of the events which follows the action is, in the manner of a text, accessible for all to 'read' and interpret (Ricoeur, 1981). As a result, following the interview, the data to be analysed are not merely the transcript, but the interview per se, an encounter between two individuals (horizons) in a certain physical and social context (Gubrium & Holstein, 2012). In this sense, the researcher cannot but report the interview as their own lived experience.

This lived experience could be reflected in the transcript, using symbols to denote non-verbal behaviour (Potter, & Hepburn, 2005). Nevertheless, an over-loaded transcript may not be useful. A transcription designed to convey the totality of the oral experience to someone who was not present, is practically impossible (Tedlock, 1983), and a hermeneutical analysis would be facilitated if the transcript can be read as a 'pure' text (not a conversation transformed into text). Meaning does not flow as easily in a transcript full of symbols (Smith, Hollway, & Mishler, 2005). Moreover, it is the written text which creates a distance between the researcher and the interview, allowing the former to discover meaning in the latter, without being restricted by the participants' preconceptions about the meaning of their utterances (Ricoeur, 1976). After all, it is a fact that a text may hide additional meaning of which the author was not aware (Ricoeur, 1981). If the researcher needs to revive the immediacy of the interview, then the recording itself would be more helpful than any transcript in augmenting the researcher's memory (Gardner, 2010; Smith, et al., 2005).

Furthermore, even though hermeneutics does not propose a specific method (Gadamer, 1975), philosophers have sketched the process whereby an understanding is reached. Fundamental in this process is the notion of the hermeneutic circle (Geanellos, 1998). The circle does not have a particular starting point, but each time we enter it, we do so equipped with our preconceptions or pre-understandings of the subject at hand. Subsequently, understanding proceeds by moving between the whole of a text and its parts, since understanding the whole requires understanding how its parts are combined, and conversely, understanding a part involves having a picture of the whole (Ast, 1990; Dilthey, 1972; Schleiermacher, Haas, & Wojcik, 1977). A circular movement is also entailed by Gadamer's metaphor of fusing horizons; what appears as new

understanding at the end of a fusion will become the pre-understanding for the next fusion (Gadamer, 1975).

Being in the hermeneutic circle, the first encounter with the data results in what Bell (2011) calls proto-understanding. This emerges before applying any analytical method to the data as an 'almost visceral' (p. 531) impression arising by the initial reading(s). It is the first apprehension of the whole, informed, of course, by the researcher's pre-understanding. This proto-understanding is followed by, and informs, a more rigorous analysis, moving through the hermeneutic circle in Ricoeur's spirit, by means of explanation (analysis) and understanding (synthesis) (Bell, 2011). A thematic analysis should serve to illuminate both the parts and the whole of each interview as well as the whole body of interviews taken together (Fereday, & Muir-Cochrane, 2006).

The research

In this section I describe the methods that were employed for the collection and analysis of the data in accordance to the theoretical considerations that were outlined above, drawing also from the wider literature with respect to conducting interviews.

Data collection

The data were collected through semi-structured in-depth interviews. This method was chosen because a loose interview structure allows the participants to focus on their lived experiences and not on my presuppositions, raising the issues which are important to them (Baxter Magolda, 2004; Polkinghorne, 1989). However, a totally unstructured interview, inviting students to comment on their experiences with mathematics, would not guarantee data which would address all the philosophical issues that were considered above (Bryman, 2015). By contrast, semi-structured interviews would allow the researcher to address certain philosophical issues which are salient in the history of philosophy of mathematics, while also allowing the student to bring to the fore the aspects of these issues which were more important to them on the subjective level. Thus, the semi-structured interview could facilitate access to both objective and subjective meanings that the students held.

As mentioned earlier, since understanding is a function of time, the process of understanding may be facilitated by conducting multiple interviews (Fleming et al., 2003). In accordance with

this, I have conducted two-phase interviews. In the first interview, I discussed – as broadly as possible – the participants' views on the study's topics, while in the second interview, I used preliminary results from the first encounter, in order to clarify and further investigate notable issues. Since time is necessary so that the impression left by the interview both on the interviewer and the participants may develop, the two interviews were conducted at intervals of one week to a month (most commonly two weeks). One week was considered sufficient time for the process of distancing oneself from the interview to have begun. Besides, the initial intention was that the interval between the two interviews would not exceed two weeks, since by then the memory of the interview may have started to fade. However, sometimes this was not possible due to holidays and the school schedule.

Conducting the interview in two phases had additional advantages. In the first place, it facilitated the establishment of a rapport with the participants (Earthy, & Cronin, 2008). This was considered to be important, since a pilot interview had already revealed how trust enabled the participant to express herself with minimum censorship. So creating a climate of confidence made the participants comfortable, and thus resulted in more authentic data (Maxwell, 1992; Rapley, 2004). Furthermore, the authenticity of the accounts was enhanced by the fact that the additional interview gave the students the chance to check and confirm or disconfirm my initial understanding (Creswell, & Miller, 2000). In addition, this helped in promoting confidence between the students and myself, since the former were given some control over the knowledge to be produced, and not the mere acknowledgement that they had contributed towards its production (Kvale, 2006).

However, researchers should be aware of the power imbalance between them and the participants and not misuse any confidence participants may show (Burman, 1997). True equality between the researcher and the participants is scarcely possible (Kirsch, 2005). Even if an interview resembles casual conversation, some of its aspects, both regarding data collection and data analysis, lie by definition under the control of the interviewer. It is the interviewer who determines the topic of the interview and the extent to which the research agenda is revealed to the participants. Most importantly, it is the interviewer who poses the questions and has the final say in reporting the knowledge produced (Briggs, 2002; Kvale, 2006). Although I aimed at decreasing the power imbalance between the participants and myself, in order to sharpen awareness of this power asymmetry I prefer to avoid misleading

egalitarian phrasings such as co-participants, co-researchers, or collaborators (Reason, 1994). I refer to the subjects of my research with the more generic term 'participants' or as students.

The interviews were recorded by means of a laptop with a video camera and an audio recorder. Having a backup system proved valuable, since on a number of occasions one of the alternatives betrayed me. In most cases, the use of the laptop did not make the participants uneasy since it is a commodity with which most young adults are familiar. Laptops are used in many settings which are not associated with being interviewed. Consequently, a laptop can easily blend into the scenery and be ignored. Of course, I was in the position to assure the students that no one else would have access to their data. However, if they still felt uncomfortable, then I simply switched the laptop off.

The interviews took place in the teacher's office during lesson time. Admittedly, this was not the ideal place, because I could not guarantee that there would be no teachers present, although the office was mostly empty during lesson time. In any case, there was no alternative, and students appeared to be comfortable with the arrangement. After all, even if there were teachers inside the office, they were sitting away of us, dedicated to their work, and leaving us to our task. Nevertheless, the office would get crowded during break time, so if the interview went on for more than one lesson, I would allow the students to have break (There were breaks even if small after every lesson). So most interviews were carried out in consecutive, but distinct sessions. The average interview (both the first and the second time that I met with a student) lasted around two lessons (45 minutes each), including the time needed for me to locate the student, get permission to take the student out of the classroom both from them and their teachers, and settle down before starting the interview. In the rest of this section, I try to give a picture of how the interviews proceeded both in terms of their content and in terms of the dynamics between the students and myself.

The broad philosophical themes, as they had been identified from the literature review, that the interviews aimed at covering were: a) whether mathematics exists independently of human beings; b) mathematical certainty, immutability, objectivity, and truth; c) the role of rules in mathematics and in mathematics classrooms; d) the role of logic, and observation in mathematics; and e) the role of proof in mathematics. Apart from questions dedicated to these,³

³ Examples of such questions appear in Appendix 3 with the details of the thematic analysis, located in the sections where ontology and epistemology are discussed.

the interview also included questions whose purpose was to contextualise the students' beliefs in a broader context. These fell under two categories. One group of questions - such as what the student knew about proof by contradiction, and counterexamples, or how could they distinguish between theorems and definitions - were meant to help me realise to what extent students understood mathematical reasoning. This was important in order to contextualise the students' opinions, since, for example, the claim that mathematics is certain can carry different subjective meanings for students who understand mathematics and for students who struggle with mathematics. The former may enjoy this certainty, while the latter may be oppressed by it.⁴ Another group of questions focused on students' beliefs about other subjects and life in general. Firstly, such questions were a valuable aid in elaborating on the students' beliefs about mathematics by contrasting mathematics with other fields of knowledge or experience. This could clarify both the objective and the subjective meaning of their beliefs, since during the comparisons I could discover what the students valued more (Follet, 1995; Gentner, & Namy, 2004; Rittle-Johnson, & Star, 2009). Secondly, they allowed me to situate the students' beliefs about mathematics within a broader context and connect them with the subjective meaning that mathematics had in their lives.⁵ This can be seen as moving in a hermeneutic circle from the whole (life) to the parts (mathematics) and back (Geanellos, 1998; Taylor, 1971). Thirdly, they helped me to establish rapport, since they conveyed that I was interested in the students' experiences in general, and not merely with respect to mathematics, which they might not like (Pitts, & Miller-Day, 2007; West, 1993).⁶

In this spirit, all the interviews opened with me asking about the students' relationship with mathematics. This question would set the tone of the interview, allowing the student to feel that I was genuinely interested in them as individuals, and allowing me to have a first insight into what mathematics meant for them on the subjective level.⁷ Consequently, the philosophical discussion would usually open with the issue of rules (mathematical statements), since this could serve as a stepping stone to introduce many other topics. For instance, having talked

⁴ In fact, this contextualisation proved invaluable for the chapter on subjective meaning.

⁵ Examples of these, and also of questions pertaining to the main issues, can be found in the reporting chapters where the students' quotes are elaborated.

⁶ Therefore I chose to elaborate on them also and not only on questions pertaining to mathematics.

⁷ In fact, I might have started asking them what mathematics meant for them. However, this wording sounded to me more formal and intrusive and so I chose to avoid it, since I wanted to establish an atmosphere that made the students comfortable with me so that the dialogue would flow naturally. Eventually, there was indeed no need to explicitly pose the question what mathematics meant for a student because it had already been answered in the course of the interviews.

about rules and mathematical statements, I could then proceed to enquire whether these statements were true, objective, certain, discovered etc. Moreover, the issue of rules could be the starting point for a fruitful comparison between various areas where rules could appear, e.g. mathematics, other school subjects, the classroom, and life in general. Such comparisons would usually be sustained throughout the interview, and would help to address most of the issues. Finally, in the context of rules, I could also start examining the students' knowledge and understanding of mathematics with respect to the various kinds of mathematical statements (e.g. definitions or theorems).

The conversation in the context of semi-structured interviews allowed the students to introduce properties of mathematics which were salient for them without being guided by me (Galletta, & Cross, 2012; Polkinghorne, 1989). Then, once the discussion had started, it could proceed by means of subjective associations. For instance, during a comparison between rules in mathematics and rules in life, the student could suggest that the former do not change, or that they are stricter. In the first case, the discussion could shift to immutability (e.g. along the path of why mathematical rules have this property, while rules in life do not, and whether this property is desirable both within mathematics and within life). In the second case, the conversation could continue around the issue of rules and the related theme of authority (again by focusing on why mathematics has this property and what happens when one ignores mathematical rules or rules in life).

Apart from any connections made by students, I could also use associations from the literature, in order to investigate on a deeper level the properties which the student had brought to the foreground (Anderson, & Goolishian, 1992; Hands, 2005). For example, if a student was talking passionately about truth, I could ask her about the possibility to verify mathematical conclusions and then about mathematical certainty, or if a student was praising mathematical reasoning, I could ask him about the traits of this reasoning which made it praiseworthy. So eventually, the list of questions which I would have with me at the beginning of the interview served more as a reminder of areas of mathematics which could be of interest, and of certain issues that were important according to the literature (e.g. discovery versus invention, or absolute truth and subjectivity) and could help to re-ignite the conversation after an issue seemed to have been exhausted.

In this context, understanding was facilitated and created through dialogue, as the horizons of the interviewer and the participant met and fused (Gadamer, 1975). The students would bring

to the interview their preconceptions and meanings about mathematics, while I would bring my preconceptions about what philosophers or other students had said about mathematics (Anderson, H., & Goolishian, 1992).⁸ In any case, since the goal of the study was to investigate the students' beliefs, it was the student's horizons, that moved to the centre of the stage during the interview. So, I would follow the student's lead, and reuse their words and terms, trying to understand the associations they made between different concepts (King, & Horrocks, 2010). Moreover, I was careful not to impose my meaning on the students, but to pay attention in the ways that they were expressing themselves in order to elicit what the various concepts that were being utilised in the interview meant for them (Baxter Magolda, 2004; Galletta, & Cross, 2012). So before starting speaking about rules, for example, I would ask the student what that word meant for them and whether they believed that there were indeed any rules in mathematics. Finally, I would regularly acknowledge and validate the students' replies, simply by saying 'okay'. In this way, I would encourage the students to proceed without leading them (Fontana, A., & Frey, 1994).

Nevertheless, instead of simply validating the students' initial responses to a question, by echoing what other students or philosophers had said about mathematics and other related issues,⁹ I could invite the students to elaborate on their answers (Anderson, H., & Goolishian, 1992; Galletta, & Cross, 2012; Gentner, & Namy, 2004). This probing was when my horizon re-entered the interview. For example, if a student claimed that mathematics existed, I could ask the student about the mode of this existence (Hersh, 1997); or if a student suggested that mathematical rules must be followed, I could investigate whether this had to be to the letter and whether nothing could be gained by challenging them (Theodosis used the wording 'to the letter', while Foivos had suggested that mathematical knowledge progresses by challenging current knowledge).

In all, the student's initial response would help me to reach a proto-understanding, which I could then explore further through probes in order to reach a fuller understanding (Bell, 2011;

⁸ I refrained from bringing my views into the interviews unless the students asked, in which case I felt it was fair to answer them (Baxter Magolda, 2004; Reinharz & Chase, 2002), although I would do so only after they had ventured some kind of an answer to the relevant question. In any case, I would make obvious that my answer was just an opinion that seemed plausible to me, unless, of course, the answer concerned factual information, e.g. what an axiom was, or whether axiom-like statements and definitions are supposed to have proofs.

⁹ There was no reason to state a source for my comments though. Naming the source could imply that I was considering to be correct and would have added unwanted authority to my comments (Kvale, & Brinkmann, 2009; Reinharz & Chase, 2002).

Galletta, & Cross, 2012). As Ricoeur (1974) points out, in the context of dialogue, through the probes, I had the opportunity to question the students in order to reduce the gap between what they meant and what I understood. So, in essence, the interviews evolved as the student and I were investigating and elaborating on the meanings of our utterances, trying to resolve any tensions among those meanings (Holstein & Gubrium, 2003; Kvale & Brinkmann, 2009).¹⁰ Appendix 2 is designed to give examples of the issues discussed above, regarding the types of questions used and the use of dialogue; however, examples can also be found in the following chapters where quotes from the interviews are analysed.¹¹

Eventually, during the process of the interviews, my own horizon was broadened (Bell, 2011). This was reflected in the fact that the set of questions which could be used as a guideline for the interviews developed dynamically as the interviews progressed. For example, after Foivos associated rules in mathematics with logic, I decided to ask other students if they thought that logic has rules. In such instances, I realised Lysimachos based logic on the senses ('basically, I believe that the rule of logic are the senses'). However, other students simply discarded the idea of rules in logic suggesting that logic is subjective and everyone has their own rules for their own logic. So eventually, I chose to approach the issue of logic by asking students whether everyone has to follow the same kind of logic or not. The next section seeks to elaborate the research sample and the Greek cultural context.

Sample and context

The interviews were conducted in Greece. The sample of the study consisted of young adults in the last year of secondary education (ages 17-18). This age group was considered to be more appropriate for the purposes of the research, since I could expect that at that age students would be more articulate about their beliefs. After all, at least some kind of philosophical issues, such as truth or objectivity, arise as part of the young adult's identity development (Moshman, 2004). Hence, it can be assumed that by the age of 17-18, students may have contemplated such issues and may have started to integrate them in a coherent narrative (Bruner, 1987; Habermas, & Bluck 2000). Furthermore, at this age students are finishing school; they have

¹⁰ This could happen immediately when a new idea was introduced, but it could also take time. In line with the observation that understanding develops over time (Gadamer, 1975), especially when the idea that was introduced was new for me, I needed to process it before coming back to it either in the same interview or in the second one. ¹¹ Nevertheless, the examples in Appendix 2 are selected to demonstrate the issues which are considered here.

had eleven years of exposure to mathematics classrooms, with one more yet to come. So their opinions may be taken to reflect the generality of their experience as they had been transitioning through the various grades of the Greek educational system, together with the ways they have come to understand mathematics towards the end of their mathematics education in Greece.¹²

In the Greek educational system, after the completion of compulsory education at the age of fifteen, follows what is called lykeio. There are general and technical lykeia. The latter have a vocational orientation, while the former are designed to prepare students for university studies. Lykeio consists of three grades. In the second grade the students are asked to select a track between humanities, sciences, and technology. The track they choose indicates what they wish to study after finishing school. Students spend some time following courses relevant to their chosen track, but there are also general courses which are to be attended by all. Mathematics features among the special courses of the sciences and technology tracks, but also among the general courses. So eventually, all students are required to do mathematics in all grades, although the students who select the sciences or technology track do more.

The Greek curriculum of mathematics seems to have been largely influenced by the emphasis given on proofs (Sdrolias & Triandafillidis, 2008). In essence, the importance of proof in mathematics is an issue of national pride, since it can be claimed that ancient Greece was where mathematical proofs emerged transforming mathematics into a rigorous field of knowledge, particularly through the works of Euclid in geometry (Toumassis, 1990). As a result, geometry occupies an extensive place in the curriculum, while fields as statistics and probability are more or less marginalised. Still, students in the science and technology tracks also cover vectors, elementary number theory, calculus (differentiation and integration), and complex numbers. The fact is that in such a context, empirical aspects of mathematics are mostly downplayed, while emphasis is unavoidably given to the use of logic in supporting mathematical arguments which would appear as certain since they are accompanied by an indubitable proof (Sdrolias & Triandafillidis, 2008).

In any event, the teaching of mathematics that takes place in Greek classrooms can be characterised as traditional (Tzekaki, Kaldrimidou, & Sakonidis, 2002). Teachers 'usually

¹² The last grade of lykeio can indeed be considered as the final grade of schooling, despite the fact that lykeio (ages 15-18) belongs to post-compulsory education. This is because, although the percentage of children who abandon school after compulsory education has recently increased, the vast majority still continues in post-compulsory secondary education (Rouseas, & Vretakou 2008).

teach "from the front" (Sdrolias & Triandafillidis, 2008, p.162). Quite possibly, to this effect contributes the fact that all teachers must teach from the same textbooks which are published by the state (Tzekaki, Kaldrimidou, & Sakonidis, 2002). Therefore it can be expected that the students would attribute a role of authority to their teachers and books. According to the literature, in such traditional contexts, mathematics is widely portrayed as a fixed set of rules concerning abstract entities which exist independently of the human mind (Charalambous, Panaoura, & Philippou, 2009). In other words, the image of mathematics that is put forward in such a traditional context should be one that is in line with the certainty and objectivity of platonism. This image is definitely in accord with the results of the emphasis given on proofs (Sdrolias & Triandafillidis, 2008).

The lykeio I visited was a general one, and most students hoped to go to the university upon completion, though some of them had other plans of their own. The school was selected on grounds of accessibility and willingness to collaborate. It was a state school which more or less accorded with what Greeks would describe as a typical state school. There was an issue regarding whether a single school would be sufficient to provide variation, since it could be expected that students who had experienced broadly similar teaching would hold more or less uniform beliefs about mathematics. Nevertheless, a few interviews were enough to indicate a rich range of beliefs. This can be attributed to the fact that students had divergent past experiences from their elementary (dimotiko) and lower secondary school (gymnasio) education, but also to the fact that each individual experiences the same environment in a different way (Ben-Zeev, Duncan, & Forbes, 2005).

Twenty-eight students volunteered for the research, twelve females and sixteen males. Eleven of them came from the humanities track, while the remainder - seventeen - were from the sciences or technology tracks. In fact, these numbers are in proportion with the numbers of humanities and science/technology students of this grade. I will not differentiate between science and technology students, since they were being taught most subjects, including mathematics, together.

By the time I interviewed the last student, I had already begun to feel that a point of saturation had been reached (Beitin, 2012). In other words, new interviews did not add new ideas concerning issues of the philosophy of mathematics. Of course, each participant had a unique, idiographic story to narrate, which was interesting in its own right, but the study also aimed at

providing a picture of the collective possibilities with respect to philosophical topics. The achievement of such a picture was used as the indicator for bringing the interviews to an end.

Thematic analysis

As mentioned earlier, the analysis of the data was facilitated by a process of thematic analysis (Boyatzis, 1998; Guest, MacQueen, & Namey, 2012). This was focused on three main areas: the philosophical areas of ontology and epistemology, and the psychological area of subjective meaning.¹³ The themes in the first two areas have emerged from the philosophy of mathematics and correspond to issues which were directly addressed during the interviews with the students. These themes appear at the table below. Some of these themes were also associated with subjective meaning. However, the choice of themes with respect to subjective meaning was not guided by the literature but by the students' comments as explained below (Braun, & Clarke, 2006; Fereday, & Muir-Cochrane, 2006).

Ontology	Epistemology
 The nature of mathematical existence Mathematics as a unified theory Mathematics as certain and immutable Mathematics as true Mathematics as objective Mathematics as rules. 	 Rule-based knowledge Logically-based knowledge Empirically-based knowledge Proof-based knowledge Authority-based knowledge

Table 1: Themes for Ontology and Epistemology

Apart from any associations made by the students, each of these themes was primarily investigated through one or more main questions. So the first step during the analysis was to examine the students' answers with respect to these questions and to generate subthemes within each of the main themes, by grouping together similar answers (Boyatzis, 1998). This allowed

¹³ I have also considered as ontological issues that pertain to the nature of mathematical knowledge. At first, these may seem to be more relevant to the epistemology of mathematics. However, in the context of the philosophy of mathematics, it is not always easy to distinguish where mathematics ends and mathematical knowledge begins. That is because if mathematics is considered to be the product of human activity, then the boundaries between it and mathematical knowledge are practically non-existent.

me to form a picture of the kind of remarks that students tended to make regarding a theme and subtheme. Thus, through a subsequent process of constant comparison (Boeije, 2002), I could check whether similar remarks appeared as comments to any question throughout the interviews (not only the main questions of the theme).¹⁴ In essence, this process allowed me to move in the hermeneutic circle (Fleming et al., 2003; Geanellos, 1998) starting from the parts of specific interviews (where the main question was answered), using these to comprehend the totality of the data (formation of themes and subthemes) and returning to parts of specific interviews (through constant comparison) to illuminate them still further. Of course, this does not refer to a single movement along the circle; the whole process involved a continuous moving between parts and whole in order to ensure that the emergent understanding, captured by the themes, remained coherent (Braun, & Clarke, 2006; Fleming et al., 2003)

At the end of the constant comparison, and for each theme and subtheme, I compiled a list of all those students who had commented on it.¹⁵ The subthemes were formulated so that each of them corresponded to one main belief which the students had advanced (Boyatzis, 1998). With respect to ontology, the subthemes were generally homogenous (Patton, 1990), in the sense that the students seemed to express the core belief of the subtheme following more or less the same train of thought - regardless of unavoidable differences in the way that each student presented the belief. Nevertheless, in some cases the reasons that led students to express the belief pertaining to a subtheme were quite distinct. These distinctions will be a part of the subsequent analysis and discussion; however, I did not proceed to divide the subthemes further on this basis. It seemed that instead of making the general picture clearer, further subdivision would only clog it with unnecessary details (Braun, & Clarke, 2006). The only exception is the case of the 'nature of mathematical existence', which because it can pertain to different modes of existence, naturally lends itself to further subdivision (for every different mode of existence).

The picture was slightly different with respect to epistemology, where students had expressed similar opinions following their own distinct train of thought. This seemed to be because the issues of epistemology were more relevant to the students' way of reasoning, affecting how they could come to believe that knowledge - mathematical or not - was produced (Zhang, & Watkins, 2001). Thus, in a sense, the epistemological issues concerned the student in a more

¹⁴ No new subthemes emerged during this process.

¹⁵ These lists can be found in Appendix 3 as footnotes.

intimate way than the ontological ones, which mostly regarded mathematics. However, these individual differences seemed to be much more pertinent to the subjective, psychological meaning of mathematics, and will be developed and elaborated with respect to it. What mattered, from the objective philosophical perspective of meaning, was the core belief that the students advanced, regardless of how exactly they justified it. So the subthemes were still judged to be homogeneous from that perspective (Patton, 1990). The subthemes were again not divided further for the same reasons as in the case of ontology, with the exception of the themes pertaining to logic and empiricism, which were quite broad and had to be divided into more manageable units (Guest, et al., 2012).

In any event, the beliefs which are discussed as part of the themes and subthemes in this thesis are chiefly beliefs which were expressed by a significant number of students. Nevertheless, there are also some cases where a belief is presented, even if only a small number of students advanced it. These are beliefs which echo prominent views in the history of the philosophy of mathematics and/or in the tradition of the mathematics education literature. Thus, these can be regarded as important, since ultimately the students' beliefs were also to be considered in the light of the existent literature (Braun, & Clarke, 2006).

The process of generating themes was different for the concept of subjective meaning, for which the themes were not predetermined by the literature. After reading the transcripts of each interview for the first time, and considering the most important message that each student had communicated to me, I realised that this involved the beliefs which helped them to make sense of what mathematics meant in the subjective context of their life.¹⁶ In a sense, while talking about philosophical issues, the students had also explained why, according to them, mathematics was something to be valued, or something to be largely ignored. This could be called their 'philosophy of mathematics', in much the same way that the beliefs which help humans make sense of their life have been called a 'philosophy of life' (Wong, 2012b, p.5). Such philosophies are primarily bearers of subjective meaning. This was indeed reflected by the fact that, as will be demonstrated in the relevant chapter on meaning, even if students might utilise similar beliefs to make sense of mathematics, they would associate and evaluate them

¹⁶ Interestingly enough, the core belief of this subjective meaning was many times already present in the students' answer to the question about their relationship with mathematics.

in their own unique ways (Szalay, & Deese, 1978). These associations and evaluations constituted the most forceful message that each student had conveyed.

With such a message in mind, it was not difficult to revisit each interview in order to uncover the philosophical beliefs through which the student had expressed it and justified it. These philosophical beliefs belonged to one, or more, of the general themes of ontology or epistemology. Thus, the objective, philosophical meaning of these beliefs could be linked with a subjective, psychological meaning, and the themes for subjective meaning were established (see table below).¹⁷ After the themes arose, I checked, through a process of constant comparison (Boeije, 2002), whether they were relevant to other students too, even if they had not expressed themselves so forcefully about them. This was necessary because sometimes a student's central message exhibited notably salient factors which had overshadowed other relevant, but secondary, themes (Boyatzis, 1998; Braun, & Clarke, 2006). These were themes which would not repeat themselves throughout the interview, but still had the potential to carry subjective meaning. Thus, with respect to subjective meaning, the movement within the hermeneutic circle (Geanellos, 1998) started from contemplating interviews as a whole with respect to their central message. Then, this message was further elucidated by inspecting specific parts of the interviews where it had been conveyed. Finally, the resultant understanding was applied anew to interviews as a whole.

Subjective Meaning	
Common sense	• Subjectivity
• Discovery	• Rules
• Invention	• Empiricism
• Certainty	

Table 2: Themes for Subjective Meaning

However, during this process it became clear that all cases of subjective meaning could be organised effectively with respect to the extent that the students' common sense appeared to be in line with their understanding of mathematics. Students would judge mathematics as

¹⁷ To these, of course, was added a theme regarding the students' future goals which were unavoidably quite salient since the interview was conducted at their last year of schooling just before they entered university or some other professional path. Nevertheless, this theme is not relevant for the purpose of this thesis and will not be further considered for analysis.

valuable (or worthless) because their common sense helped (or did not help) them to understand how mathematics worked, either on the level of content¹⁸ or on a philosophical level. A (mis)alignment in any of these levels could be sufficient to create a positive (or negative) image of mathematics for the respective student.¹⁹ Thus, the themes for subjective meaning were generally divided on this basis.

Finally, having organised individual students under themes and subthemes with respect to ontology, epistemology and subjective meaning, it was possible to compare their beliefs with the views advanced by different philosophical trends. During this process, students were associated with relevant philosophical trends to the extent that their main beliefs reflected some significant aspect of that trend. This categorisation facilitated reviewing the students' beliefs in the light of the relevant literature in the discussion chapter.

In hermeneutics, understanding can fuel a new round of analysis, further movement along the hermeneutic circle from a new starting point, which may lead to new understanding and so on. In theory, moving along the circle is a repetitive procedure which could continue indefinitely. Of course, in practice the circle must close due to time limitations (Fleming et al., 2003). In any case, the movement is progressive, because each time the interpreter re-enters the circle with the benefit of an enhanced understanding, i.e. the result of the previous circle. Hence, interpreting is not a futile attempt. The criterion for exiting the circle is the extent to which the researcher has managed to integrate the parts of the data into a comprehensible whole (Debesay, Nåden, & Slettebø, 2008). This point was reached when I finally produced the three reporting chapters of the study: ontology, epistemology and meaningfulness.

Of course, I cannot claim that my findings represent the participants' views objectively. In any case, I am in the position to faithfully report my lived experience of the interviews where the students' beliefs were given. Moreover, as has been mentioned earlier, hermeneutics is not an utterly subjective endeavour. Ricoeur observes that although a text may admit infinite

¹⁸ 13 students, 46%, seemed able to more or less understand mathematics at the content level: Xenofontas, Lysimachos, Agapi, Aspasia, Kleomenis, Loukianos, Ermis, Danai, Platonas, Diomidis, Filia, Solonas, and Andromachi. 13 students, 46% seemed to have difficulties with understanding mathematics: Foivos, Pelopidas, Polyxeni, Kosmas, Evyenia, Lida, Kleio, Filippos, Yerasimos, Vrasidas, Ariadni, Menelaos, and Afroditi.

¹⁹ In most cases, a (mis)alignment appeared on both levels; however, this was not necessary, since it was possible for a student to accept the way mathematics worked at a philosophical level while they did not understand it at the content level, and vice versa. If the misalignment concerned only one of the levels, then its effects could be mitigated by the alignment on the other level - sometimes significantly. However, since such cases were exceptions, division in subthemes has ignored the levels of (mis)alignment.

interpretations, not all of them are valid (1991). Interpretations are subject to intentionality; they are interpretations of a certain text and as such they are limited by it. The reporting of the findings should contribute towards making transparent the way in which the interpretation the researcher proposes has been reached (Bell, 2011). In view of this, I can claim that the chapters which follow present one of the possible valid interpretations of the data. It can also be claimed that the value of this interpretation lies in the fact that the interpreter is in the position to speak from the point of view of both a mathematician and an educator, while she is also not partial to some specific philosophy of mathematics.

Below, there follow some examples where the organisation of students under themes and subthemes is discussed in detail. There is one example for each of the cases discussed above, though I have included two for the chapter of subjective meaning. I have chosen to include here the themes where the process of thematic analysis had been less straightforward, but in the case of subjective meaning I have also chosen an example which exemplifies better the difference between a (mis)alignment at the content level and a (mis)alignment at a philosophical level. However, the reader will find a detailed analysis for all themes (in all chapters) in Appendix 3. Following the examples, I provide tables with the themes and subthemes for ontology and epistemology. A table regarding subjective meaning was not considered necessary since effectively all the relevant themes are divided into a subtheme which indicated alignment with common sense, and a subtheme which indicated misalignment with common sense.

Mathematics as certain and immutable (ontology)

Certainty can be associated with many other properties that mathematics may be claimed to have. Firstly, certainty guarantees immutability and vice versa, since there is no reason to change something that is certain, while if something is not changing, then we can be certain about it. Secondly, certainty implies correctness, since one can be certain about something only as long as one assumes that this something is correct.²⁰ Finally, certainty can be connected with truth, since one can be certain of true statements. Students were asked with respect to all these issues, i.e. whether they believed that mathematics or mathematical conclusions are correct,

²⁰ In fact, correctness can also be associated with many mathematical properties, i.e. certainty, immutability, truth, and objectivity. However, in the context of the philosophy of mathematics, correctness is usually assumed, and only the other issues are debated. Therefore, I have not included correctness as a separate theme. It is only discussed in relation to the other themes with which it can be related.
true and amenable to change. Subsequently, they could also be asked how certain they were about their remarks on such questions.²¹

Students justified mathematical certainty and immutability by referring to proofs²² and also on cultural grounds.^{23,24} What differed was the degree to which these two justifications blended with one another in an individual student's account. This depended on the extent to which students would succumb to the cultural power of proof, an issue which is addressed in more detail in the section on proofs in the epistemology chapter. The extreme cases concerned students who mostly did not understand proofs and were considered therefore to justify certainty and immutability primarily on cultural grounds,²⁵ and students who could understand proofs, and whose remarks on certainty and immutability were taken to reflect this understanding.²⁶

Cultural immutability or certainty was also suggested independently of proofs.²⁷ Students would suggest that mathematics does not change because our culture is based on it, or that mathematics must be correct because it is presented as such by society and school. To the extent that such students could perceive the respective cultural construct as contingent and not necessary, they would not regard it as absolute. Moreover, traces of uncertainty and change were connected with currently developing mathematical knowledge which was regarded as less stable.²⁸ Finally, lack of absolute certainty could be the result of a natural hesitancy due to idiosyncrasy or lack of expertise. As a result, students would on many occasions hedge their claims about mathematical certainty or immutability.²⁹

²³ This applied to all students, though to different degrees.

²¹ Certainty is not discussed separately from immutability because there were not many comments where students considered certainty without linking it with immutability. On the contrary, the remarks where truth was discussed independently of certainty were more common. Thus, truth figures as a separate theme in my analysis.

²² 16 students, 57%: Foivos, Lysimachos, Agapi, Pelopidas, Kleomenis, Loukianos, Kleio, Danai, Platonas, Diomidis, Filia, Solonas, Theodosis, Andromachi, Vrasidas, and Afroditi.

²⁴ Mathematical certainty and immutability were also connected with mathematical existence. However, this seemed to be mostly the result of mathematical existence implying mathematical truth. So mathematical existence has been retained as a subtheme only for the case of truth, where the connection was stronger.

²⁵ 13 students, 46%: Pelopidas, Polyxeni, Kosmas, Evyenia, Kleio, Areti, Theodosis, Filippos, Yerasimos, Vrasidas, Ariadni, Menelaos, and Afroditi.

²⁶ Seven students, 25%: Lysimachos, Kleomenis, Loukianos, Ermis, Platonas, Solonas, and Andromachi.

²⁷ Seven students, 25%: Lysimachos, Loukianos, Danai, Theodosis, Yerasimos, Vrasidas, and Menelaos.

²⁸ 16 students, 57%: Foivos, Agapi, Aspasia, Kleomenis, Loukianos, Kosmas, Ermis, Evyenia, Lida, Danai, Platonas, Diomidis, Filia, Solonas, Yerasimos, and Menelaos.

²⁹ This applied to all students, though to different degrees, which seemed to be due to idiosyncratic reasons.

Logically-based knowledge (epistemology)

Under this theme, I included remarks which showed what the role that students attributed to logic within the context of mathematics was, and also comments which indicated what the students were referring to when they used the word 'logic'. Students were asked to comment on whether they believed that there was any relationship between logic and mathematics. Moreover, students were asked questions regarding the generation and verification of mathematical knowledge such as: how mathematical rules were produced; how correctness in mathematics was decided; how could they know that mathematical rules were correct/true; and whether mathematical rules could be checked through logic. On all these occasions, students could link logic - mostly voluntarily, but sometimes after probing - with the process through which mathematical knowledge was advanced and validated. Furthermore, students would many times use spontaneously the attribute 'logical(ly)'; while other times they would speak of something as if it was self-evident and needed no further explanation or justification. Such cases were taken to suggest that which each student regarded as compatible with logic, i.e. as common sense.

Students generally connected mathematics with logic - at least at some point during their interviews - by presenting it as a factor of mathematical reasoning.³⁰ Apart from general remarks about logic as the main trait of mathematical reasoning, students also made more specific comments which implicated logic in the process of generating and validating mathematical knowledge. Nevertheless, when students made such claims despite only having a limited understanding of mathematical reasoning, or when they connected mathematics with logic at one point, but denied such an association elsewhere, it felt that they were merely echoing a cultural belief, according to which mathematics was supposed to be logical.³¹ Apart, from this, some students actually suggested that what was logical in mathematics depended on human habits and conventions, thus suggesting that logic in mathematics was a cultural construct.³²

³⁰ The only exception was Ariadni, although this seemed to be a matter of chance, in the sense that if I had asked the questions of the interview in a different sequence, Ariadni could have also initially connected mathematics with logic.

³¹ Ten students, 36%: Pelopidas, Kosmas, Evyenia, Kleio, Areti, Theodosis, Filippos, Yerasimos, Vrasidas, Menelaos, and Afroditi.

³² Four students, 14%: Lysimachos, Loukianos, Yerasimos, Ariadni.

Moreover, there were cases where students pointed towards limitations of logic in the context of mathematics,³³ or even cases when students would deny that mathematics was logical (at some other point in their interviews).³⁴ The main reason for this seemed to be that the way in which they were expected to reason whilst doing mathematics did not always fit with what their experience dictated as logical - in other words, with their common sense. Examples concerned mostly the inability to check at least some mathematical results empirically.³⁵ The fact was that most students did not differentiate between the logic that was used within mathematics and the common sense that was used in everyday life,³⁶ and this could lead them to suggest that mathematics was not (always) logical.

Invention (subjective meaning)

All students who claimed that mathematics was invented seemed to make sense of mathematics by perceiving it as a creation of the human mind.³⁷ The fact was that any kind of misalignment between the students' common sense and mathematics could lead students to portray mathematics as an invention which was alien to them.³⁸ Such students would devalue mathematics, or speak about it with frustration, perhaps indicating that they wished to avoid it. Still, perceiving mathematics as a subjective invention, with which they could potentially disagree, could help some of them to make some sense of its place in their life, since it allowed them to explain why they did not understand it.³⁹ On the contrary, any kind of alignment between the students' common sense and mathematics could lead students to portray mathematics as an invention to which they felt intimately connected.⁴⁰ Such students would

³³ Ten students, 36%: Lysimachos, Aspasia, Pelopidas, Kleomenis, Loukianos, Danai, Platonas, Filia, Solonas, Andromachi.

³⁴ Four students, 14%: Evyenia, Filippos, Yerasimos, and Ariadni.

³⁵ Other cases seemed to be idiosyncratic and so I did not create a subtheme for them. Experience, however, seemed to be a prominent issue. In fact, some students also made comments which suggested that logic stems from empirical data. Such cases have been organised under the theme empirically-based knowledge.

³⁶ 25 students, 89%: Foivos, Xenofontas, Lysimachos, Agapi, Aspasia, Pelopidas, Kleomenis, Loukianos, Kosmas, Ermis, Evyenia, Lida, Kleio, Danai, Platonas, Filia, Areti, Theodosis, Filippos, Yerasimos, Andromachi, Vrasidas, Ariadni, Menelaos, and Afroditi.

³⁷ All the cases were relevant, since the interview would also suggest whether the students judged the invention of mathematics as a valuable one or not. However, for some students this was only a secondary theme. 17 students, 61%: Lysimachos, Pelopidas, Polyxeni, Kleomenis, Loukianos, Kosmas, Evyenia, Lida, Kleio, Danai, Diomidis, Solonas, Filippos, Yerasimos, Vrasidas Ariadni, and Menelaos. These also include students who belonged to the mixed subtheme of 'discovery and invention', but expressed themselves more strongly about invention.

³⁸ Ten students, 36%: Pelopidas, Polyxeni, Kosmas, Evyenia, Kleio, Filippos, Yerasimos, Vrasidas, Ariadni, and Menelaos.

³⁹ Five students, 18%: Evyenia, Filippos, Yerasimos, Vrasidas, and Ariadni.

⁴⁰ Seven students, 25%: Lysimachos, Kleomenis, Loukianos, Lida, Danai, Diomidis, and Solonas.

portray mathematics as an interesting activity which they could enjoy and/or which could generate valuable knowledge for humanity.

Rules (subjective meaning)

Under this theme were gathered students for whom the meaning of mathematics appeared to be influenced by their evaluation of mathematical rules.⁴¹ One group of students suggested that they felt comfortable with mathematical rules, stressing either that rules were useful in general (alignment at a philosophical level),⁴² or that they could use mathematical rules creatively⁴³ (alignment at the content level). Such students could find positive meaning in mathematics as an interesting and valuable set of rules. Another group of students indicated that they felt uncomfortable with mathematical rules, stressing either that rules in general could be too rigid (misalignment at the philosophical level),⁴⁴ or that they could make no use of mathematical rules⁴⁵ (misalignment at the content level). For this group, the subjective meaning of mathematical rules seemed to be a negative one associated with oppression.

Empiricism (Discussion)

This theme involved students who advanced empirical aspects of mathematics by locating mathematics in nature (these students were also taken to associate mathematical existence with mathematical certainty);⁴⁶ or by hinting that empirical reasoning based on experimentation or observation could be relevant to mathematics.⁴⁷ All these beliefs reflected empiricism as a philosophy of mathematics either at the ontological level of mathematical existence or at the epistemological level of production and verification of mathematical knowledge. However, the theme was not applied to students who simply claimed that logic could be connected with the

⁴¹ This theme also corresponded to all students, but this was because of the centrality of the issue of rules in the interview protocol, and because students tended to associate mathematics with rules. On the contrary, the theme of common sense seemed to have been all pervasive in a more structural way, since compatibility or incompatibility with it appeared to unavoidably influence the meaning that students could find in mathematics. For an exposition of this, see the chapter on subjective meaning.

⁴² Six students, 21%: Agapi, Filia, Areti, Theodosis, Menelaos, and Afroditi.

⁴³ 15 students, 54%: Xenofontas, Lysimachos, Agapi, Aspasia, Kleomenis, Loukianos, Ermis Lida, Danai, Platonas, Filia, Diomidis, Solonas, Theodosis and Andromachi.

⁴⁴ Six students, 21%: Foivos, Polyxeni, Filippos, Vrasidas, Ariadni, and Menelaos.

⁴⁵ Nine students, 32%: Pelopidas, Polyxeni, Evyenia, Kleio, Filippos, Yerasimos, Vrasidas, Ariadni, and Menelaos.

⁴⁶ 25%: Foivos, Xenofontas, Ermis, Platonas, Filia, Andromachi, and Afroditi

⁴⁷ 16 students, 57%: Foivos, Xenofontas, Lysimachos, Agapi, Pelopidas, Kleomenis, Loukianos, Kosmas, Ermis, Lida, Platonas, Diomidis, Filia, Theodosis, Filippos, and Afroditi.

senses or that experience was helpful in learning mathematics, unless such students also indicated that the senses or experience were indispensable for logic.

Tables

Below are presented two tables with the themes and subthemes for ontology and epistemology:

Theme	Subthemes
The nature of mathematical existence	 Discovery Empiricist existence Platonic existence Invention Mathematics as immaterial Mathematics as hypotheses Mathematics as an unintelligible invention
Mathematics as certain and immutable	 Certainty and immutability Proofs Cultural certainty and immutability Traces of uncertainty and change
Mathematics as true	 Correctness Mathematical existence. Cultural truth
Mathematics as objective	 Proofs Mathematical existence Cultural objectivity
Mathematics as rules	(no subtheme)

Table 3: Themes and subthemes for Ontology

Theme	Subthemes
Rule-based knowledge	 Rules as necessary (and sufficient) Rules as non-binding
Logically-based knowledge	 Logic in mathematical reasoning a. General remarks b. Generation and validation of mathematical knowledge Cultural Logic a. Limited understanding b. Limitations of logic⁴⁸ c. Logic as a habit Common sense a. Common sense as distinct from mathematical logic b. Limitations of logic 48 c. Common sense as distinct from mathematical logic b. Limitations of logic c. Mathematics as not logical
Empirically-based knowledge	 Senses a. Observation b. Detour though logic Experimentation a. Trial and error b. Applications in practice c. Experience
Proof-based knowledge	 Mathematical function Cultural function
Authority-based knowledge	 Mathematics Teacher and book

Table 4: Themes and subthemes for Epistemology

Ethical considerations

the course of human events ... depends on the coincidence of the will of all who take part in them ... Human dignity ... demands the acceptance of that solution (Tolstoy, War and Peace)

Acknowledging, as Tolstoy urges us to do, that those whom we study are human beings without whom the research would be impossible, an ethical prerequisite for research is posed, that is, that research should be conducted both for the participants' and the researcher's benefit.

⁴⁸ This subtheme could be connected both with common sense and with cultural influences.

Reiman states that ethical research should above all aim at enhancing the participants' freedom (1979). Although such a high goal may not be always realisable, the researcher should aim at providing the participants with the chance to gain something from the research. Such a motivation could also drive the participants to share more authentic data (Talmage, 2012). In terms of my study, I believe that the interviews gave the chance, at least to certain individuals, to illuminate their understanding of mathematics and the reasons for enjoying or loathing it. There were moments where the participants discovered beliefs and opinions which might have gone unnoticed otherwise (Reinharz & Chase, 2002). Moreover, Gadamer (1975) observes that preconceptions are not necessarily perpetuated, but may also be transformed during the process of understanding. Indeed, there were times when some of the students reflected on, and reevaluated, their preconceptions. In all, the participants were in the position to understand better their relationship with mathematics by the end of the interviews, and this could be translated into a benefit, since being aware of this relationship may allow the participants to be more objective when judging mathematics. In turn, this could be proven valuable considering that mathematics permeates numerous aspects of modern life (independently of occupation) (Fischer, 2006).

There was no need to resort to complex technical terms in order to explain my research, so the participants were in the position to readily grasp the aims of the research and give an informed consent (Marzano, 2012). The consent also highlighted anonymity issues and the fact that I would be the only one who would have access to the students' data. Moreover, although consent was given at the beginning of the research, it could be re-negotiated during the research (Finlay, 2012) and the participants were free to stop collaborating at any point. Distress did occasionally emerge with some students who expressed a negative relationship with mathematics. Such students were not always comfortable while talking about mathematics. Nevertheless, their reaction was not judged, and they were not in any way pressed to share more than they wished to. No student chose to withdraw from the research once it had started. Some students only preferred not to be videoed, in which case I respected their wish.

In the next three chapters, the findings of the study are reported. The data in these chapters aim at providing an explanation for the choice of my interpretation. First, ontology is presented, then epistemology, and finally personal meaning for students. It seems natural to discuss ontology prior to epistemology, focusing first on the nature of mathematics, and subsequently to the way we interact with it and gain mathematical knowledge. The meaning that students attribute to mathematics is discussed last, since it is considered in relation to students' beliefs about ontological and epistemological issues.

Ontology

Introduction

This chapter - together with the next one on epistemology - primarily concern what could be called the objective philosophical meaning of the students' utterances. In other words, the chapter aims at presenting what students might have said if they were asked to produce a lemma in a philosophical dictionary about mathematical existence, certainty, truth etc. Nevertheless, the students' beliefs were not purely objective, untainted by any subjective psychological meaning. Although this recognition is not the focus of this chapter, and thus it will not be discussed here in any detail, it becomes evident in cases where the reasons which led the students to express a belief seemed to depend on their subjective relationship with mathematics, i.e. whether they liked and understood mathematics or not.

The reader should bear in mind that these issues are generally not discussed in mathematics classrooms. The observations presented here are therefore impressions that students had formed of their own accord based on their experience with mathematics. Consequently, many such ideas appear unrefined; sometimes there are no proper explanations offered for a belief, and the accounts may not always be internally coherent. However, these views are important in illustrating the raw marks left on students' minds through their interaction with mathematics.

The themes that are discussed here are those that pertain to the ontology of mathematics namely:

- the nature of mathematical existence
- mathematics as certain and immutable
- mathematics as true
- mathematics as objective
- mathematics as a set of rules.

In exploring these themes, I start with the students' beliefs on the purely ontological issue of mathematical existence. This concerns mathematics as a discovery and mathematics as an invention. I then turn to issues which concern the nature of mathematical knowledge. First the

topic of certainty is addressed - and together with it the issue of immutability⁴⁹ - since this has constituted the backbone for the development of the philosophy of mathematics. Here, I present the students' observations on mathematical certainty/immutability on the basis of proof, on cultural certainty/immutability, and on traces of uncertainty and change in mathematics. Subsequently, the students' beliefs about truth and objectivity of mathematics are discussed. Regarding truth, I explain how students associated, or dissociated, truth with proofs and mathematical existence, and how they based truth on cultural grounds. Regarding objectivity, I consider how students believed that there could be no disagreements in mathematics, how objectivity -linked with truth - also pertained to proofs and mathematical existence, and cultural aspects of objectivity together with traces of subjectivity. Finally, a section is dedicated to mathematics as rules, considering whether students associated the term 'rule' with mathematics or not. Discussion of this issue functions as a bridge between the ontology chapter and the epistemology chapter, where rules are considered anew with respect to their function in producing mathematical knowledge.

During the following account, each section is broken into subsections according to the relevant main points (subthemes) that pertain to the respective theme. For example, the section 'nature of mathematical existence' consists of a subsection on 'discovery' and a subsection on 'invention', both further subdivided: the former into 'empiricist existence' and 'platonic existence', and the latter into 'mathematics as immaterial', 'mathematics as hypotheses' and 'mathematics as intelligible invention'. Similarly, the section 'mathematics as certain (and immutable)' comprises four subsections: 'certainty and immutability', 'proofs', 'cultural certainty and immutability', and 'traces of uncertainty and change'. Each subsection (subtheme) opens with a brief overview of the beliefs that students advanced on the relevant issue. Moreover, a summary follows at the end of each main section (theme), drawing together all the subthemes that were presented under it. After the introduction for each subtheme, follows a discussion of relevant extracts from the students' interviews.

The extracts are sometimes lengthy out of necessity, for their inclusion is designed to show both how the student came to express a particular belief in the course of the interview, and also why I reached a particular conclusion during analysis. As a result, a limited selection of quotes was called for here in order to prevent the presentation from becoming too long. In fact, the

⁴⁹ As it was explained in the section on thematic analysis, immutability is discussed together with certainty.

extensiveness of the substantive quotes necessitated the restriction of the cases to be presented to a single example illustrating each distinct belief expressed by the students.⁵⁰ Thus, the number of cases featuring under each subsection depends on the homogeneity of the respective subtheme, that is, on the extent to which the belief pertaining to the subtheme was advanced by different students for similar or for distinct reasons. The choice of the extracts was governed by a number of factors. In the first place, and most importantly, I sought to choose extracts where students had expressed a belief with particular clarity, whilst also - with or without prompts - elaborating on it and possibly justifying it.⁵¹ Secondly, from among such quotes, I preferred those where students had been more articulate or had expressed themselves in more interesting ways (e.g. with metaphors, examples, or simply with emotional force).⁵²

The nature of mathematical existence⁵³

Discovery

Some students applied the idea of discovery to mathematics. Their beliefs were either in line with empiricism, suggesting that since mathematics was employed to describe the natural world it must reside within it; or with platonism, presenting mathematics as a set of abstract entities accessible through reason.

Empiricist existence

Most students who claimed that mathematics was discovered advanced a view of mathematics in line with empiricism. Such students mainly believed that mathematics resided in nature. They were not able to point towards specific occurrences of mathematical objects, but they

⁵⁰ Most subsections are homogenous and pertain to only one belief. However, a few of them group several beliefs together.

⁵¹ A lack of justification was also important, since, for example, it could be indicative of the extent to which the student understood what they were claiming, or the extent to which they were being dogmatic about it.

⁵² This may have resulted in certain students being quoted more than others. However, in the footnotes in Appendix 3, where the details for the thematic analysis are given, a complete list of the students who were inclined towards a particular belief can be found.

⁵³I will not consider in detail the case where students blended discovery and invention, since the justifications they provided in that case are not essentially different from the justifications that other students offered for discovery or invention independently. Nevertheless, such a case will be briefly presented in the discussion chapter, which aims to bring students beliefs' together. In that chapter, the case will be considered in connection with the student's justifications, and therefore will appear under the section where the students' approaches to mathematical reasoning are elaborated, and not under the section on mathematical existence.

were sure that mathematics existed somewhere, embedded in the structure of nature, and generally in the things which mathematics could describe. As a result, they assumed a level of necessity for the mathematical concepts which we use and the statements which hold true for them. In other words, they would suggest that mathematical notions and the propositions which govern them were dictated to us by reality and could not therefore have been different.

The empiricist view of mathematical existence can be illustrated through the example of Ermis. When I presented him with the dilemma of discovery versus invention, Ermis quickly chose the former option and he explained what the two concepts meant correctly. In order to confirm his answer, I checked whether he would agree that mathematics existed before it was discovered. At that point he elaborated on his choice, maintaining that

now, however they're called ... for example instead of [using the symbol] 1, you [may] use [the symbol] a, [but] the way that you'll handle this [number, through] division, multiplication or anything else, all the range of [such] operations ... will be the same, no matter how you call the numbers, the limits and so on.

Essentially, Ermis was sure that mathematics could not be defined differently; we may name things differently, but the way we use the elements of mathematics and relate them to one another through operations did not depend on human choice; it is so necessarily, by virtue of the nature of mathematical objects and relations. For Ermis, this seemed to confirm that mathematics existed independently of humans. However, he had not stated directly whether mathematics existed or not, so after a while I brought the issue back and Ermis indeed confirmed that he believed that mathematics existed, adding that

even complex numbers, which [mathematicians] didn't know, they discovered them ... and it emerged through the discriminant where we say what happens if we have a negative [one]. And I think through physics too, [concerning] the issue of light. In some equations ... some squared variable came up negative, so they said "what's going on now?"

On this basis, Ermis concluded that complex numbers 'already existed, [mathematicians] didn't create them.'⁵⁴ So he held that mathematical knowledge depends on properties of nature (e.g. the behaviour of light), and he interpreted this as a proof that mathematics is discovered.

⁵⁴ A couple of students (Foivos, Xenofontas) excluded complex numbers from this picture, but still insisted that mathematics is discovered.

However, Ermis was not able to demonstrate the mode of mathematical existence. When I asked him in what sense mathematics existed, he simply took mathematical existence for granted and he mentioned how he felt it around him through examples. He admitted that he could not see mathematics, as he could see other things which existed, but he still believed that he could somehow 'sense' its presence in its applications:

as I think of it now - I don't know if [what I think] is right - [but] if mathematics is everywhere, I [can] feel it for instance, through everything around me, either through the computer ... which is a mathematical tool ... the table, in order to be made, it required dimensions, numbers, mathematics ... when you speak with somebody and you say 'give me two loaves of bread, [can] you give me a (one) pen?' You use numbers. ... So somebody can understand [that mathematics exists] like this.

Platonic existence

Contrary to her fellow students who considered mathematics to reside in nature, Andromachi denied any connection between mathematics and nature, but she still asserted that mathematics existed. Moreover, in accordance with the platonic view, she suggested that mathematics was accessible to humans through the faculty of logic. Thus, it seemed that she endorsed a kind of platonism, though she did not refer directly to mathematics as constituting abstract objects or ideas.

Andromachi was the only student who claimed that mathematics exists without connecting this existence to nature. In fact, when I asked her if there was any relationship between mathematics and nature, she answered negatively. Still, she believed that mathematics was discovered and it seemed that her belief could be associated with the platonic ideal of discovering mathematics through logic. Her explanation of mathematical discovery initially consisted in her claiming that 'usually we start from somewhere ... as we saw, in a proof too, you start from somewhere and then you conclude [something].' In other words, it appeared that Andromachi said that mathematical result followed from something which already existed. For me, that remark about how mathematical results emerge from one another was a hint at arguments linked through logic, but I did not wish to put words to this effect into her mouth, so I invited her to consider in what sense mathematical existence was possible. Her response was even vaguer than that of Ermis, simply reaffirming that 'basically, I think that mathematics exists [over] there and we

discover it.' So I asked her how we might discover it, drawing attention to the senses,⁵⁵ but she was not content with this suggestion. She counter-proposed logic as the channel through which humans could access mathematical knowledge, adding that 'we have contact with mathematics through the brain, so it exists.' Thus, having found an answer to her liking regarding my question about how mathematical discovery was possible, she essentially confirmed her fundamental belief that mathematics existed.

Invention

Most students implied that mathematics was invented. Students saw mathematics as an invention for the following reasons: a) mathematics appeared to be immaterial and inaccessible to the senses; b) mathematics was conceived as based on hypotheses or assumptions; c) mathematics was unintelligible and it made more sense to view it as an artefact of the human mind rather than as something that existed.

Mathematics as immaterial

In the case of mathematics being perceived as immaterial, students tended to contrast mathematics with physics. Physics could be said to be discovered - at least to an extent - since students could associate it directly with actual, existing objects of everyday life; while mathematics had to be invented because it primarily concerned abstract, immaterial concepts. Students might recognise that such concepts could also be applied in physics, but they would not infer from this that mathematics existed. They would restrict that claim only to physics, where the relation to the everyday world seemed to be an immediate one and not a result of a detour through a different science.

One of the students who justified his belief that mathematics is invented on the basis of its lack of a material existence was Diomidis. In the first interview, after I explained the difference between discovery and invention, stating that invention is something that is clearly our own construction, Diomidis declared that mathematics is invented.⁵⁶ In the second interview, wishing to understand his previous answer more fully, I opened the subject again. I planned to

⁵⁵ I felt I had to give her some kind of hint because she seemed confused.

⁵⁶ Diomidis did not explain his choice at this point and I did not probe further because he generally seemed very certain in his opinions and his tone would usually suggest that things were pretty obvious and there was nothing more to discuss.

offer the dictionary difference between invention and discovery, but before I did so, Diomidis volunteered an association of invention with something 'found so that we use it in everyday life'. In other words, in his mind, 'invention' appeared to concern something that was created because humans needed it but which would not have existed otherwise. Diomidis applied this idea to mathematics, reaffirming his belief that mathematics is invented. When I prompted him to elaborate on this further, he actually explained why mathematics could not exist, that is, why it could not be discovered, stating 'generally, I don't believe that mathematics exists as a material idea, that is, you can't touch it. It's invented. Yes, it's invented.'

Afterwards, we proceeded to compare mathematics with physics. Contrary to mathematics, Diomidis associated both invention and discovery with physics:

on the one hand, there's a [sic] discovery in physics, and on the other hand, I think there's also invention. First comes the invention, that is, first the idea of physics was created, and then, for example, things were discovered. ... generally physics was invented like mathematics, but because physics is related more to everyday life ... [various] things were discovered, though the invention [of physics].

So it seemed that, for Diomidis, both mathematics and physics were invented, in so far as they were created by humans in order to 'use [them] in everyday life' (see his comment about invention above). Nevertheless, regarding physics, he could not discard the concept of discovery as he was able to do in the case of mathematics. Mathematics appeared to him to concern immaterial notions, but physics was directly implicated in his everyday material life, referring to things which actually existed.

Diomidis was aware that mathematics was related to physics. In fact, later in the interview he suggested that the former was a prerequisite for the latter as we now know it, observing that 'because ... mathematics was invented, physics was also invented ... but if the invention of mathematics didn't exist [sic] ... the invention of physics wouldn't exist as it is now.' However, this observation did not lead him to believe that mathematical concepts could have a claim in existence as physical concepts could. Only the latter were material and available to him through his senses.

Mathematics as hypotheses

Assuming that mathematics did not have to conform to any existing reality, students could also see mathematics as being dependent on human choices or guesswork. Since humans were not restricted by reality while creating mathematics, they were free to postulate concepts and graft theories as they deemed appropriate. Such constructions were seen as intelligible by some of the students, but not all. Students in the first group described the result of this activity as axiomlike statements upon which the rest of mathematics could be founded; while students in the second group tended to see mathematics as a bundle of unsubstantiated hypotheses with no connection to reality.

Kleomenis' belief that mathematics is invented appeared to be associated with his understanding of mathematics as being dependent on assumptions. He knew about axioms as statements 'which [Pythagoras] [sic] had defined as such that they have no proof', and he seemed to consider them as necessary prerequisites for doing mathematics. When I asked him why we accepted such statements without proof, he suggested that 'we can't do otherwise? We can't prove [them]' and then he added that 'if we start from nothing we can't prove something.' In fact, it seemed that Kleomenis attributed the existence of axioms to the very absence of mathematics from the real world, which allowed humans to make such assumptions and define mathematical concepts according to their needs, free from any restraint that reality might dictated. So while comparing mathematics with other sectors of human knowledge, Kleomenis explained that

other sciences, apart from mathematics ... exist. Mathematics doesn't exist in the world ... while the others exist, they demand some real application. And because mathematics is only theoretical, we've defined it so that it's convenient for us.

It seemed that this was why, when I asked Kleomenis whether mathematics was discovered or invented, he suggested, as Diomidis had done, that mathematics 'is eventually invented because it's something we created in order to serve our own conditions [sic].' Later in the interview, he gave the example of geometry which 'was put in order by Euclid ... but then Riemann? Who is it? He didn't like it. He wanted to ... show other things and so he changed it.' This led Kleomenis to the conclusion that 'each time we make [mathematics] as it suits us.'

Evyenia also presented mathematics as consisting of unreal conditionals or hypotheses. While we were discussing truth in mathematics, Evyenia observed that 'in mathematics there's "if this holds then it happens like that." There's only this [kind of statements] ... or 'let. Let, for example, this.' At this point, I remarked that this sounded like making assumptions and Evyenia agreed calling them 'assumptions of the mind'. She also added that 'in the end, by chance, it turns out that what we do through these assumptions is correct', and she stressed again 'by

chance', indicating that for her, it was a mystery how or why mathematical statements were correct.

It seemed that this was the reason why Evyenia later stated that mathematics was invented, and when I asked her - in order to confirm - whether she believed that mathematics was a human creation, she replied 'of course'. She continued with an example from physics, but it appeared that she considered that the same line of reasoning applied to mathematics: 'the other [guy] on whom the apple fell and he discovered [sic] gravity, didn't he do so on his own? [Gravity] didn't exist there on its own and he found it.' As she continued explaining her point, it became clear that she did not actually believe that gravity did not exist before Newton, but that Newton was completely unaware of that existence when he made the invention. So he did not discover gravity; he first made an assumption, he assumed that it existed and then he realised that it was indeed there. In Evyenia's words: 'That is, he didn't know about [gravity] ... First he invented it and then he discovered it ... that is, he thought of it, he had the idea, and then he realised that it exists.' In all, it seemed that Evyenia saw mathematics primarily as an invention because she considered it as a set of weird assumptions and she did not know how else such assumptions could have been reached.

Mathematics as an unintelligible invention

Many students who did not understand mathematics presented it as an invention. It seemed that claiming that mathematics was invented could allow students who had difficulties in comprehending it to justify their lack of understanding. There appeared to be no obligation to understand something which was not real, and was only the product of the human mind. After all, different minds would think differently, so one was not compelled to think in the same ways that mathematicians did, or to find the results of their imagination reasonable.⁵⁷

Ariadni maintained that mathematics was an invention, observing that mathematics was a product of human thought:

It doesn't exist. Because when I say "I discover", I discover a land. While "to invent" [means] that I think, I sit down on my own, I think, I write and I invent [something] ... it's something of the mind.

⁵⁷ See section on invention in the chapter on subjective meaning.

This claim of Ariadni seemed to be related to her difficulty in understanding mathematics. When I asked her if it is possible to have explanations in mathematics, she denied this by asserting that mathematics was only numbers, which did not really exist and so there could be no explanation about them: 'No, no ... because first of all, it isn't something real. It's numbers, so even if [someone] wanted to explain them, they wouldn't be able to. Do you understand?' Moreover, the belief that mathematics was invented allowed her to justify her lack of understanding. She would just not think in the same way that mathematicians did. As she declared when I inquired whether there was any logic behind the rule that a mathematician would make,

for the [person] who made [the rule], of course there's [some] logic ... I simply [can't] find this logic ... I would try to understand why [teachers] say that I have this situation [in mathematics] and I'd ask, I['d] say: "can you explain to me?" "No" they say. "You take this situation as it is. That's the situation that exists." They don't say anything more. But it's reasonable that they wouldn't explain why the is rule [so]. For them to explain why [the person who made the rule] thought about it like this? [It doesn't make sense].'

I wondered why it was reasonable that she should not be given any explanation and she simply replied that 'if they explain it to me, they'll lose the ball. I'm very weird.' So eventually, it seemed that for Ariadni mathematics was an invention made by some minds which reasoned in a particular way, and which some people who reasoned similarly could understand, but she did not see herself belonging to this group of people. She had tried to understand that way of reasoning, but she had not received any intelligible answers, and she had simply accepted that she reasoned in a different, 'weird' way.

Summary

Only one student truly divorced mathematics from nature and appeared to espouse a pure platonic ideal. For the remaining students who maintained that mathematics existed, this belief was accompanied by an empirical flavour, i.e. an assumption that mathematics could be found or felt in the order of the universe. This led such students to perceive mathematical knowledge as pre-determined by the actuality of the world we inhabit. However, most students believed that mathematics was invented. Such students could be divided into two groups. One group which found this invention intelligible and one which did not. Among the second group, some students seemed to advance the belief of invention simply because it would justify their lack of understanding. Otherwise, students would claim that mathematics did not exist mainly because they could not locate it in the material world. In some cases, students also perceived mathematics as either being based on axiom-like statements (students who understood mathematics) or as a set of unreal hypothetical statements (students who had difficulties with mathematics).

Mathematics as certain (and immutable)

Certainty and immutability

Students would often move swiftly between talking about certainty and talking about immutability, without any particular sign that they had changed the subject. It seemed that these two concepts were interchangeable in their minds. Something immutable was something that could not be challenged and something certain was something that would not change.

Platonas appeared to be equating certainty with immutability in his comments. When I suggested the words 'fair' or 'unfair' as attributes for mathematical rules, he rejected them presenting mathematical rules as indubitably correct, that is, as certain:⁵⁸ 'these rules have been checked repeatedly over the history [of mathematics] and they're correct, that is, no one can doubt their existence and what they say.' In a sense, his remark also implied that no one could change these rules and indeed when, a minute later, I enquired about the attribute 'correct' with respect to mathematical rules, he asserted that mathematical rules can be characterised as 'correct, logical, unshakable.' When I prompted him to elaborate on this, he referred to the issue of change even more directly, adding that mathematical rules 'are stable, that is, they don't change. Now you'll say that some change,⁵⁹ but as I said before they've been checked, it has been supported that they're unshakable, their value is permanent.' I tried to learn more about how this had been supported, and this time Platonas connected immutability with certainty, noting that 'over the time, there were several attempts to demolish these [mathematical] laws, but [people] didn't manage to reject them, so that proved that [mathematical rules] are correct, supported [by evidence].'

⁵⁸ It seemed that Platonas implied that the qualities of fairness or unfairness can be associated with doubt, and as such they cannot apply to mathematics.

⁵⁹ The rest of his interview suggested that what he had in his mind was how his beliefs about the properties of real numbers changed when he was introduced to complex numbers. At this point I did not investigate his remark about some statements which had changed further, because he seemed to assign no consequence to them, and because the thread of the discussion led me elsewhere.

Proofs

Students presented proofs as arguments with indubitable, unshakable conclusions. As a result, they would claim that mathematical results were certain and immutable. Such remarks were advanced both by students who generally understood mathematics and students who did not, and showed the cultural power of proofs which students were ready to accept even without inspecting them.⁶⁰

While considering any differences between rules in mathematics and rules in everyday life, Lysimachos offered the immutability of the former in contrast to the changeability of the latter:

Basically, every additional theory built ... in mathematics is proven to be unchanging, that no one can come and demolish it and this becomes obvious as the time goes by; that it's hard [for] a theory in mathematics to be demolished so easily. Now, in everyday life, I think that this happens ... a rule or even a law can be demolished in the context of everyday life, in contrast with mathematics.

Lysimachos regarded this difference as a fortunate one, since 'humans ... change constantly ... they see new things ... broaden their mind ... so if there was something fixed in society ... we wouldn't have such a financial and social development.' On the contrary, he saw no problem with any immutability in mathematics, essentially because mathematical statements, contrary to societal rules, had proofs. For him, a proof meant that the statement at hand was established beyond any doubt, that is, that it was certain; there was no reason to even consider its correctness again; it was more reasonable to use it in order to prove additional propositions. So his further remarks associated immutability with certainty even more directly: 'since something is proven, there's no way that someone will look at what has already been proven, they'll go and search and discover something [new] which could be based on the previous [results].'

The power that Lysimachos attributed to proofs - the power to produce conclusive results became even more evident later. When I asked him whether he could check whether a proven statement holds, he was confused precisely because he saw no reason to verify something that had already been proven. So he questioned me twice in order to make sure that the question concerned the case of something that had been proven. His reaction prompted me to inquire

⁶⁰ This issue will be discussed in detail in the next chapter on epistemology.

whether any check would be necessary at all, and Lysimachos asked me once again 'if it's proven?' When I answered positively, he declared that no check would be necessary 'because, since [something] has been proven, whoever may try to prove that it doesn't hold, they won't succeed, because it's proven that it holds.'

Cultural certainty and immutability

Occasionally students would justify their certainty about the correctness of mathematics by suggesting that something to do with the social status of mathematics meant that it must be correct. Mathematical statements were supported by experts and they were taught in school, so they had to be correct. Moreover, students would note that since the function of our society was based on some particular mathematical rules, it would be difficult to change them. It might not be impossible but it would be impractical, because then we would also need to change our society. Essentially, such comments implied that mathematical certainty or immutability were not inherent qualities of mathematics, but socially crafted ideas. This meant that - at least theoretically - they could be challenged, although, in practice, students tended to endorse the culture in which they were growing up, sometimes even without realising that it could be challenged.

Such a cultural influence could be traced in all students.⁶¹ However, it was much more evident in the interviews with students who generally could not understand mathematics and its proofs. Even when such students would connect certainty or immutability with proofs, it seemed that they were only echoing a cultural belief, according to which proofs were supposed to justify a statement beyond any doubt. For such students, any apparent belief in mathematical certainty or immutability was essentially second-hand.

Danai suggested that the apparent immutability of mathematics could be based on cultural reasons. When I offered the possibility of mathematics changing, she rejected it, maintaining that

No, because everything is based on it ... that is, [software] programmes are based on it ... If this [mathematics that we have] changed, all the rest would have to change too and I don't think that it's possible that something like this will ever happen.

⁶¹ This was mostly because all students seemed to be influenced by the cultural power of proof (see next chapter, but also the case of Vrasidas in this section).

So it seemed that Danai was suggesting that mathematics would not change simply because society would try to avoid such a change, since the way society functioned depended heavily on the mathematics that it used, and thus the cost of a change would be too great. Danai appeared to agree when I offered such an interpretation, commenting that 'I'm not absolute [sic] that [a change] can't occur. I simply don't think that it may occur. I don't know. That is, there's a ten out of a hundred [possibility] that it'll happen.' I suggested to her that this actually sounded as a substantial possibility and she agreed, adding that 'I'm optimistic that there may be minds around here which can [do it].' This confirmed that she did not regard a change in mathematics as inherently impossible, but only as culturally unlikely.

Vrasidas also expressed himself in similar terms to Danai, pointing both towards cultural certainty and towards cultural immutability. I had asked him whether mathematical rules could be characterised as fair or unfair and - perhaps because he did not know what to answer - he initially simply maintained that 'this can be characterised by the very experienced [people] who have specialised in mathematics.' However, wishing to make a comment he added that

because he who wrote [mathematics], now we're all based on it. [It] can't be that he takes it back, that we consider it wrong [or] right. Since, as [teachers] serve it to us, they serve it to us [as] right. They don't say "I say so, but I don't know, it may also be wrong"⁶²

In this remark, Vrasidas suggested that society functions on the basis of mathematical knowledge and so one cannot simply discard that knowledge, whilst also stressing that students in school are given no reason to doubt the mathematical knowledge which is presented to them as correct.

In fact, contrary to Danai, Vrasidas could only justify mathematical certainty or immutability on cultural grounds. That was because he generally did not understand mathematics or its proofs. When I asked him what the verb 'to prove' meant, he stated 'I prove something. I prove, for example that the car moves because it has four wheels ... I can prove that something is happening for a certain reason.' After this, he conceded that if something was proven then that would mean that it was correct and that it could not be challenged; however, he did not appear to be entirely certain about this, and it seemed that he was simply echoing a general, cultural

⁶² It seemed that Vrasidas was changing the subject from whether mathematical rules can be said to be fair to whether they can be said to be correct. This could be the result of the general connotations between what is fair and what is right at least outside of mathematics.

belief, according to which proofs were supposed to secure a conclusion against any doubt. When I later asked him how he could be certain that a proof was indeed correct beyond any doubt, he only noted, as if stating something obvious, that 'since someone has proven [something], why challenge it?' He was aware that whoever had produced the proof might have made a mistake, but he observed that something like this could be established only 'if you could prove the opposite [claim]'. So it appeared that Vrasidas was simply accustomed to accepting proofs as valid without questioning them. He had definitely learned that he could not challenge a proof if he could not prove the opposite, and since he was in not in a position to do so, he had learned to remain silent.

On this basis, it can be claimed that Vrasidas' perception of mathematics as immutable and uncertain was not really genuine, in the sense that it was not the result of his understanding of mathematics, but it was impressed on him by society's and his teachers' attitude towards mathematics and proofs. Indeed, when, in the second interview, I re-used his phrasing and asked him whether the fact that mathematics was 'served' to them as correct meant that it was really correct, he initially replied positively, but then he found himself unable to support this claim, and simply added 'I don't know. I think yes. Why wouldn't it be correct? Has anyone proven the opposite?' Essentially, Vrasidas could not provide a basis for his belief that mathematics was correct, and he only challenged me to prove the opposite on the assumption that I would not be able to do so. Thus, it seemed that his certainty about mathematics was second-hand, a result of him accepting the culture in which he had grown up.

Nevertheless, Vrasidas, did not appear to be aware of the fact that such a culturally crafted certainty was not absolute. Instead he insisted on accepting mathematics as certain. On the contrary, Ariadni - another student who did not understand mathematics and based mathematical certainty primarily on cultural grounds - seemed able to realise that her 'certainty' about mathematics could be challenged. In her interview, we touched on the issue of certainty after discussing the proof for the equality of vertically opposite angles. Then I asked her how certain she could be of the result we had just reached and her reply totally discarded the proof as evidence that could corroborate this statement. She only referred to the social status of mathematics, commenting that 'I'm certain. So many other people are certain. When teachers and scientists are certain about it, I'm also certain.' However, in contrast with Vrasidas, she also wondered 'but okay, if there was no one ...' So Ariadni appeared to realise that her argument about social certainty was not unshakable.

Traces of uncertainty and change

Apart from uncertainty associated with grounding certainty and immutability in cultural reasons (see Danai and Ariadni in the last section), there seemed to be a general tendency among the students for hedging claims to certainty and immutability. The degree of the students' hesitancy seemed to depend on their natural tendency towards uncertainty, but it could be generally attributed to the fact that mathematics was not their expertise, and so they could not be certain about it. Furthermore, some students would diminish the absolute status of certainty or immutability by suggesting that it concerned only old statements and not current advances which could still be mistaken, and was thus amenable to challenge and change. This meant that students could talk about mathematics as a developing entity and not as dead fixed knowledge.

Hedging of claims to mathematical certainty and immutability is evident in many of the above discussed quotes. For example, Lysimachos initially used the phrasing 'no one can come and demolish' a mathematical theory, but he immediately changed this to 'it's hard [for] a theory in mathematics to be demolished so easily.' The former phrasing was indeed absolute, but the latter seemed to imply that demolishing a mathematical theory was not impossible; it was only hard.

Danai referred directly to her lack of expertise to justify her uncertainty. When I asked her if she could be 100% sure about results whose proof she had seen and understood, she replied 'yes, I wouldn't bet my life on it, but yes.' I was surprised because she sounded quite uncertain compared with other students who understood mathematics. When I simply echoed that she would not bet her life on mathematical certainty, she observed: 'No, I wouldn't bet it. I'm not crazy about mathematics, [it's not as if] the only thing I do in my life is to study mathematics and solve exercises so that I could bet my life [on it].' So eventually, it seemed that she would assume that a much greater degree of expertise was needed for anyone who would claim to be certain about mathematics.

Nevertheless, it appeared that Danai's high degree of uncertainty was also a result of her character. During the interview, at one point where I thought she was contradicting herself, she noted 'I'm not a very decisive person.' Later, while she was trying to decide between some choices I had presented to her, she ironically commented again: 'I'm a very decisive person,'

it's obvious.' Indeed, her uncertainty pervaded the whole interview, and apparently also influenced the way she viewed mathematical certainty.

In a different vein, students tended to restrict certainty and/or immutability to old knowledge of mathematics. For instance, when I asked Diomidis if a mathematical rule could be abolished, he noted that something like this could occur only with respect to new mathematical knowledge being developed in universities, and not with respect to the old mathematical propositions which students had been taught in school:

I think that in the mathematics that we do, the mathematics of school, I think no, [a rule could not be abolished]. Because they've been proven and I don't think that someone can abolish them. I think, though, that in the context of university, there ... there's greater margin for somebody to challenge [mathematical rules] and demolish them.

Diomidis' comment seemed to concern both certainty and immutability, since he referred both to abolishing (changing) and challenging (doubting) mathematical rules.

Summary

In all, students would generally portray mathematics as certain and immutable and they would connect these properties with the indubitability of proofs. However, they could also justify certainty and immutability on cultural grounds, which made them appear less absolute even if the student was not aware of this. Moreover, certainty and immutability seemed to primarily involve old mathematical knowledge and not mathematics as a whole entity. Finally, students tended to apply hedges to their claims about certainty and immutability, most probably because they did not consider themselves as experts on the subject.

Mathematics as true

Correctness

Some students seemed to interpret truth as correctness, and implied that mathematical statements were true because they were correct. Such students would essentially attribute their sense of mathematical truth to proofs, presenting proofs as the guarantee of the correctness of mathematical statements. This would also mean that these students implicated mathematical

truth with mathematical certainty insofar as certainty was perceived as the consequence of proofs (see discussion above).

Truth indeed implies correctness; by definition, something wrong cannot be true. However, correctness does not necessarily imply truth. Intuitively it may seem that something correct must be true, and this was probably why many students did not differentiate between the two concepts. Still, correctness may simply point towards validity, and a valid argument may well be false. Indeed, a conclusion may have been correctly inferred on the basis of some specific assumptions, but this does not mean that the assumptions were true. If the assumptions were false, then the conclusion may also be false, even if it is valid. This could be the reason why the students who perceived mathematics as based on axiom-like assumptions stated that mathematics was not necessarily true.

The relationship between truth, correctness, proofs - and even certainty - is clear in Platonas' interview. Platonas had suggested that in life a statement could be partially true, and I asked him if this could also happen within mathematics. He first wanted to clarify whether I was talking about theorems or opinions. Regarding theorems, he asserted that partial truth was impossible, describing theorems as 'proven truth, that is, the theorem expresses the truth.' Only opinions which were not yet proven 'can be [regarded] as true until the opposite is proven, that is, that the [opinion] has no support.' So for Platonas, proven statements were conclusively true. Later, when I asked him about the role of proof in mathematics, Platonas again referred to truth, but he also pointed towards certainty. He stated that '[a proof] is essentially tangible evidence that a proposition which you have assumed is true and no one can challenge it; [proof is] like an argument.' Thus for Platonas, a proof guaranteed that a statement was definitely correct and definitely true.

On the contrary, Lysimachos seemed to distinguish between truth and correctness interpreted as validity. As his comments in the section on certainty and immutability indicated, Lysimachos was sure that mathematics was correct: 'since [something] has been proven, ... it's proven that it holds.' However, immediately after this claim, when I asked him whether a proof convinced him that the relevant result held, his answer was 'no'. I was confused and I asked him to explain why. He responded that 'yes, okay, it may have been proven and we may be using it, but because we can't understand its evidence we may not be sure that it holds.' This was not a very clear explanation, but I did not ask him to elaborate further because it was only a few minutes earlier that Lysimachos had given a similar answer which was easier to

understand. His previous remark was essentially given as a reply to whether there is truth in mathematics. The word 'truth' seemed to puzzle him, so I rephrased the question asking him whether one could say whether a mathematical rule holds or not. Lysimachos replied negatively, stating: 'no, because there are rules, for example, which we can't grasp but we accept ... because we get used to them.' As an example, he offered 'the concept of infinity ... in the sense that we can't grasp the infinite, but we simply use it in exercises and [elsewhere].' So it appeared that although Lysimachos perceived proofs as correct, i.e. valid, arguments he was not always convinced by them, because he could not always connect their conclusions to something which he could understand. Consequently, Lysimachos did not feel that he was in the position to judge whether such arguments, apart from valid, were actually true.

Later, in the second interview, Lysimachos also denied that mathematical rules are necessarily correct and he explained this by referring to 'the initial assumptions'. I again did not ask him to elaborate because in a previous conversation we had had about mathematics being correct, he had already stated that 'basically since it's an assumption, then we can't ever know whether it's correct or wrong.' Despite this, or actually because of this, Lysimachos proceeded to claim that 'so whatever we built on these assumptions will be correct.' This seemed to be a contradiction, but it can be explained if the word 'correct' is interpreted differently in each case. On the one hand, when claiming that whatever is built on the basis of assumptions is correct, Lysimachos could mean that it is valid. On the other hand, while noting that an assumption may be correct or wrong, Lysimachos was most probably implying that the assumption could be true or false. So it appeared that Lysimachos regarded mathematics as valid but not as necessarily true.

Mathematical existence

In general, students who advocated mathematical existence also viewed mathematics as true. Suggesting that mathematical concepts are defined on the basis of what existed in nature, they were indirectly claiming that mathematical statements talked about real objects and aimed to reveal their true properties (see section on empiricist existence, above). However, even such students seemed to be chiefly concerned with truth on the level of human mathematical knowledge and not on the level of pure mathematical existence. Therefore, they seemed to connect truth primarily to mathematical correctness through proof (see the discussion of Platonas above),⁶³ and only indirectly with mathematical existence. However, in some cases the link between mathematical truth and existence appeared also directly on the ontological level as well.

On the contrary, there were some students who dissociated truth from mathematics, claiming that mathematics was unrelated to the actual real world. These were students who could not understand mathematics in general, and as with those who distinguished between correctness and validity, they would also see mathematics as hypotheses (see the example of Evyenia in the section on invention). In effect, such students would not simply note that an hypothesis may not be true, but they would actually claim that mathematical hypotheses were indeed not true, justifying this by explaining how mathematics referred to situations which were unreal and lay beyond human experience. They were unwilling to even consider that such a hypothetical statement could be true; for them, mathematics could not be speaking any truth, since it was not talking about reality.⁶⁴

Xenofontas was one of the students who seemed to be claiming that mathematics was true exactly because he believed that mathematics concerned real, existing objects. When I asked him whether mathematical conclusions were true, he replied positively. Trying to understand his stance better, I also asked him about physics, and there he replied negatively, explaining that physics 'is based on something ideal, something that doesn't exist.' This may sound incongruous, since physics appears to concern existing objects around us. However, the experience of Xenofontas was that physics only approximated reality, it did not really speak about it. Continuing the above quote, he justified his claim with the example of

now, in physics ... we're doing mechanical oscillations. Mechanical oscillations are about a system which has no resistance, [in] which there's no friction. Is it possible for this thing to exist? If it exists, then we can say yes [conclusions about it are true].

So it seemed that Xenofontas believed that mathematical conclusions were true because, in contrast to physics, they concerned the existing reality.

On the other hand, Ariadni dissociated mathematics from truth because she could not connect mathematics to any existing reality. She portrayed mathematics as something that she could

⁶³ In fact, Platonas was a student who believed that mathematics exists 'in the law-based structure of nature', but did not appear to advance this as the main reason for his belief in mathematical truth.

⁶⁴ See also section on empiricism in the chapter on subjective meaning.

not experience, and consequently, as something unreal and not true. Initially, when I asked her if mathematical rules were true, she stated 'ah, I don't know this.' Nevertheless, she tried to reach a conclusion and, remembering how students were asked to choose whether a result was true or not in mathematics exercises, she was inclined to say that mathematical rules were true, in the sense of them being correct. In order to investigate her uncertainty, I asked her whether she believed that physics was true, so that we could compare. At this point, she switched from interpreting 'true' as 'correct' to interpreting 'true' as 'real', and she explained that physics was not about truth 'because it puts you in an imaginary situation.' Then she added, about mathematics this time

I think I understood a bit in what sense you mean ['true']. That it isn't true (real), because simply the rule in mathematics doesn't exist, it isn't something natural. This always made [mathematics] difficult for me. As in physics, it gives you an imaginary situation: "if this existed and if ..." This makes it difficult for me; that there isn't a reality, [something] to see. So the rule in mathematics may not be true.

So ultimately, it appeared that Ariadni dissociated truth from mathematics because she could not connect mathematics with her experience. For her, this meant that mathematics could not be real and thus it could not be true.⁶⁵

Cultural truth

Finally, as with certainty and immutability, there were also cases of cultural justification of truth. Some of the students also presented truth itself as a cultural construct which changed over time depending on the available human knowledge. Apart from this, the arguments they deployed were similar to those regarding certainty and immutability, and seemed to revolve around the fact that mathematics was an established science based on proven arguments and was taught in school. This cultural aspect of the students' comments again implied that the truth they were referring to was not absolute, even if the students themselves were not aware of it.

⁶⁵ To an extent, Lysimachos' suggestion that mathematics was not necessarily true seemed to be also influenced by the fact that at least some mathematical concepts, such as infinity, lay beyond his experience (see section on correctness above). The difference between Lysimachos and Ariadni was that the former generally understood mathematics and for him infinity was one particular concept that he could not grasp, while Ariadni appeared to locate mathematics as a whole out of her experience, and thus out of reality.

Students made comments which pointed towards a cultural justification of truth. For instance, Areti objected when I asked her how she knew that mathematics was true, stating 'what kind of conspiracy theory [would] that be: it isn't true but everyone around us teaches it?' It seemed that it was unreasonable for her to assume that mathematics was untrue when everyone around her behaved as if it was true. Since the society that she lived in valued mathematics as knowledge appropriate to be passed to the next generations, mathematics had to be true.

Kosmas seemed to go a step further than Areti, implying that truth in general was a cultural construct amenable to change as humans and their knowledge evolved. In his first interview, while we were discussing whether mathematical rules could be characterised as right or wrong, Kosmas commented that 'since it's a mathematical rule, it means that it proves something [sic] and that someone has thought it through, and that this thing holds. So until someone negates it, it's correct, it's true. If someone negates it, then it's wrong.' Thus for Kosmas, it seemed that truth in mathematics came along with a potential expiry date, the date when someone would prove that what was considered as true before was actually wrong. This belief was strange to me because I would assume that if something was proven wrong, even if this was at a later stage, this would imply that it had never really been true. That is why I returned to this subject again in the second interview with Kosmas, and as I was trying to remind him what he had said, he again commented that '[mathematical rules] hold until proven otherwise', and when I expressed my puzzlement, he explained that 'it's true, we accept this truth, but many times truth may be overturned as new evidence arise.' I suggested that this could be showing that what we accepted as true was not true, and Kosmas noted that 'yes, but we can't do otherwise, since new evidence hasn't appeared yet; since no one has come to say "guys this is wrong" and to correct it.' Indeed, in such cases, humans tend to treat the statement supported by any current evidence as true, and they act in accordance with it. Kosmas, though, appeared to be claiming something more; he appeared to insist that what was regarded as true in such cases was true indeed, even if only for the time being. I checked to make sure that this was what he meant, and he replied, even before I had the chance to finish my sentence: 'yes, yes, until that moment [when it's overturned] this thing is true.'

Summary

In all, most students presented mathematics as true. Students mostly tended to equate truth with correctness and then they would usually justify it on the basis of proofs. Only a few students

appeared to differentiate between correctness as validity and truth. Moreover, truth seemed to be associated with mathematical existence. Nevertheless, some students did not accept mathematics as true because they interpreted 'true' as 'real' and, contrary to students who believed that mathematics existed, they could not see anything real in mathematics. Finally, students also justified truth on cultural grounds, while some of them even presented truth as a cultural construct which changed according to the available evidence.

Mathematics as objective

No disagreement (one answer)

Generally, students believed that everyone would agree about what was correct in mathematics, and that implied that mathematics was objective, or definitely not subjective. If mathematics had been subjective, then people would be free to bring their personal opinions into mathematical arguments, and this would lead to disagreements as it did in other subjective sectors of life. Students justified this lack of disagreement in mathematics by suggesting that truth in mathematics was absolute, that is, they believed that the correct answer to a mathematical question is uniquely determined, regardless if there might be many ways to reach this answer. For these students, that meant that there was no space for personal subjective preferences in mathematics. Only one student⁶⁶ hinted at open mathematical problems, and suggested that there could be genuine disagreements in mathematics.

When I enquired whether two people could disagree in mathematics, provided that they both understood the statement at hand, Aspasia asserted that

No, because [mathematics] is [about] specific things. It isn't subjective. The truth is one and you can't, you don't introduce your opinion in the way of solving a problem. The solution is one as the truth. Despite the way of reasoning that you [may] follow in order to solve the problem, you'll reach the same result, if the [way] is correct. So you won't disagree; it isn't something subjective.

So, Aspasia argued that mathematical results were not an issue of subjective opinions, and therefore there were no disagreements. Admittedly, she did suggest that personal opinion can

⁶⁶ This student was Lida, but since her case was unique, - it will also be explained more fully in the chapter on subjective meaning - it is omitted from the current discussion.

be involved in choosing among the several ways that a problem could be solved, but she seemed to imply that the correct implementation of any of these ways did not depend on personal choices. After all, as she claimed, mathematical solutions always led to the same answer, to the same unique truth - even if one could choose different ways to reach that answer. Thus, since there could be no plurality of opinions with respect to the result of a problem, there was no reason to disagree over a mathematical statement.

Proofs

In effect, only statements which could not be challenged could be perceived as objective, and thus free of disagreements. Thus - even if not directly - students associated lack of disagreement in mathematics with the presence of proofs, in so far as they believed that proofs guaranteed the correctness of a mathematical statement beyond any doubt.

Solonas was one of the students who referred directly to proofs while he was explaining that truth and correctness in mathematics were not as subjective as truth and correctness in the context of classroom rules. Firstly, regarding correctness he maintained that in classroom

I may consider as correct to start swearing at the [student] next to me ... and for the [student] next to me [this] may be wrong ... [while] in mathematics, something is correct that holds for everyone, not only for one [person] ... the opinion about whether something is correct or wrong is more subjective in the classroom that in mathematics.

Immediately afterwards we compared classroom and mathematical rules with respect to truth, and Solonas again remarked that 'I think that [for] classroom rules [truth] is more subjective than in mathematics. Because in mathematics you prove something and it holds for all, not only for the [person] who proved it.⁶⁷ In other words, one could not disagree about the correctness or truth of a mathematical rule, as they could disagree about what was right and proper behaviour in the classroom. Only the latter was presented by Solonas as an issue of personal opinion. On the contrary, mathematical statements appeared to be accompanied by proofs which could not be influenced by such opinions but were valid for all. This belief seemed to

⁶⁷ Solonas refrained from claiming absolute objectivity for mathematics, but this could be attributed to his awareness that he was far from being an expert. See comments about students not being absolute about mathematical certainty or immutability in the respective section. Solonas' awareness of his lack of expertise was probably even sharper than this of his fellow students since his father was a mathematician and he could realise how much less he knew compared to his parent.

be connected to the general conception of proofs as arguments that secured a result beyond any doubt; only such a result has to be valid for all.

Mathematical existence

As truth, objectivity could be associated with the belief that mathematics existed. If an object existed independently of the human mind, then its qualities could not depend on any personal preferences that a human mind might have. On the contrary, the statements which were true about that object would be determined by its mode of existence, by its nature.

For instance, Ermis suggested that mathematical rules were objective, after explaining that they were not the result of human imagination, but they were formulated on the basis of what humans observed in nature. Effectively, he seemed to be claiming that mathematical rules were objective because they were determined by reality, by 'objects' which actually existed. As such, their correctness was decided by that reality and was independent of anyone's personal wishes. Ermis explained that he accepted mathematical conclusions because

all of them must come from somewhere, that is, from the real world. That is, it can't be that they came to somebody's mind and [that person] put them forward [independently]. Otherwise they would be made up, in the sense that they [would] be from the imagination, ... [But mathematical conclusions] are made through some process [and] there was a need for this process to occur because there was a problem in the physical world.

A few minutes prior to this point, Ermis had suggested that behavioural rules were subjective 'because they're made by people.' In contrast, he seemed to believe that in the process of generating mathematical rules the human factor was not the only one involved. Otherwise mathematical rules would be simply contrived, a product of human imagination; but Ermis believed that they stemmed from reality and nature, which would determine the rules that humans would formulate. Indeed, when I reminded him of his observation about rules of conduct, Ermis appeared to consider mathematical rules to be a different case, and he referred back to this process through which they emerged from the natural world, noting it as the characteristic which distinguished them from rules of conduct. At that point I asked him if mathematical rules were objective or subjective and he asserted that they were objective.

Cultural objectivity

The connection of objectivity with what was true could add to it a cultural element as in the case of mathematical truth. The belief that no one doubted mathematical results, but everyone agreed with them, could simply be a result of cultural influence. Such a culturally induced objectivity was even more evident in cases of some students who occasionally tended to present mathematics as subjective. These were students who had difficulties with mathematics and seemed to be unwilling to endorse its apparent objectivity. This could be because they were able to comprehend the subjectivity of life better than the objectivity of mathematics. In this case, students were likely to assume that it would actually make more sense if mathematics was also subjective.⁶⁸ However, claims to subjectivity could also be the result of a need to justify why such students could not understand mathematics. By presenting it as subjective they were able to defy - at least occasionally - their impression of mathematical statements as objective facts.⁶⁹

A student who seemed to be influenced by cultural beliefs while portraying mathematics as objective was Kosmas. When I asked him whether truth in mathematics was objective,⁷⁰ he replied positively and he justified this by suggesting that 'all of us, who are not [part] of the mathematical community, have accepted that whatever the mathematical community says holds, since we can ...' He appeared to wish to justify this but he could not go on to offer an explanation. Hoping to help him to say more, I invited him to comment on the view of the mathematical community and not only of lay people on the subject. Kosmas maintained that mathematics 'is commonly accepted [by] the mathematical community [too] ... it isn't subjective. Since [the community] accepted [something] and published it, it isn't subjective, it's objective.' Ultimately, he did not explain why the mathematical community accepted a mathematical statement. Earlier in the interview he had admitted that he did not know how that community reached a consensus, though he was sure that they did reach a consensus somehow:

mathematics has been shaped by the mathematical community. For most [statements], I think they say "guys this is so", they all agreed, they raised hands, I don't know what they did, [but they reached the conclusion that] "well guys, this holds, it's over."

⁶⁸ See section on subjectivity in the chapter on subjective meaning.

⁶⁹ See sections on invention and subjectivity in the chapter on subjective meaning.

⁷⁰ Kosmas had already portrayed mathematical knowledge as true.

In any case, as a lay person himself, Kosmas appeared to endorse the consensus which he attributed to the mathematical community, and he expected other lay people to do the same. Thus in a sense, his belief in mathematical objectivity was rooted in cultural grounds. He did not really have the knowledge to judge whether the mathematical community was correct, but he would assume that it was, and he also assumed that other people would make the same assumption. Essentially, it can be claimed that he was influenced by a cultural belief, which endowed mathematicians with authority and their conclusions with objectivity.

On the other hand, although Yerasimos was influenced by the fact that his culture offered mathematics as objective, he also appeared to be suggesting that it was reasonable for mathematics to be subjective as everything else in life was. When I asked him whether two persons could disagree about a life situation while being both right, he seemed to consider this very natural 'since each one [would] be having a different version regarding the thing on which they disagree.⁷¹ After this, I asked the same question with respect to mathematics, and Yerasimos was not so sure but he stated that 'I assume so; reasonably speaking, it can happen.' This declaration was not a result of knowledge, but what seemed to him to be reasonable on the basis of his general experience. His lack of understanding became obvious when he offered as an example that 'two equations may produce the same number, they may produce a different [number].' I noticed that this was not a genuine case of disagreement, since, if there were two equations, then the issue on which the two persons could be disagreeing was not the same. However, Yerasimos simply replied 'I don't know this', pleading ignorance and hinting that he did not have an answer for the question of disagreement within mathematics.

This prompted me to see what Yerasimos believed with respect to whether mathematical problems had specific solutions, since this was a reason offered by other students for the lack of disagreement in mathematics. He did not seem certain about it either, but he was inclined to answer positively: 'I think that yes, [the solutions are specific].' Consequently, he was effectively forced to agree that if this was the case, then a disagreement did not make sense, 'since, if [the solution] is corroborated with mathematical laws, with rules and the like, why would someone disagree?'

⁷¹ This did not necessarily imply that both versions could be correct, but Yerasimos confirmed that this could be the case when I enquired into this.
Nevertheless, later in the interview it became again clear that Yerasimos was not sure about mathematical problems always having the same solution. While we were talking about whether one had a choice about following mathematical rules, he felt the need to ask me: 'But all the ways yield the same result? All the ways?' This implied that although previously he had suggested that mathematics was objective on the basis of assuming that it had specific answers, he was not really sure about that assumption. The reason for this could be that, in fact, he regarded it more reasonable for mathematics to be as subjective as life, and to assume that there could be different answers for the same mathematical problem. This can be supported by the fact that while we were considering the possibility of different mathematical systems, he had asserted that:

an equation may yield the result six. How do we know it that it yields six? ... Because a rule has been made and, according to it, it yields six? There may be another [rule] and [by following that rule] it may yield 46.'

Eventually, it seemed that Yerasimos was more inclined to believe that mathematics was subjective. The impression that mathematical problems always had the same answer was more likely one that had been cultivated within him by his classroom experience, and he would echo it occasionally, but not always.

Filippos also presented mathematics as subjective on certain occasions. This seemed to be the result of his wish to disagree with mathematics since he could not understand it. At one point, while declaring his right to disagree with philosophers, Filippos added: 'that's why I disagree with mathematics.' Nevertheless, he mostly wished than he actually believed that he had this right. When I asked for an example where he could disagree with mathematics, he could not give me one, and he admitted that mathematics was different from philosophy:

in mathematics [you don't disagree] that much because... it will be pointless ... okay, that [person] tells you that "this is a square" and you see it. You'll say "no, it's a triangle?" There's no reason. So I revise and I state that in mathematics you don't disagree, you simply tolerate.

So mathematics had been impressed as objective on Filippos too. He might desired to disagree with it, but he could find no rationale that would allow him to challenge a mathematical claim.

Despite this, Filippos later claimed that it was possible for somebody to continue to believe that they were right after being shown that they were wrong, even in mathematics. We were discussing the possibility of checking the correctness of one's actions in life, and Filippos was explaining that 'in the process [of doing something], you may understand that what I do isn't right and you may stop.' Nevertheless, he also noted that 'it may be that somebody shows you that you make a mistake, and you may continue to believe that you're right.' So I asked him whether this could occur in mathematics. I was expecting that, since he had admitted that he could not disagree with mathematics, he would reply negatively. Yet Filippos replied positively, most likely because again he wished to be able to disagree with mathematics. Eventually, I asked Filippos directly whether he believed that mathematics was objective or subjective and, at this point, he assertively chose the latter option.

Summary

In sum, students overwhelmingly portrayed mathematics as objective. This was mainly because they denied that there could be disagreements about whether a mathematical claim was correct or true. According to them, and as they had learned from the problems they encountered in the classroom, there seemed to be only one correct or true answer to a mathematical question, and this was proven to be indubitable. Apart from this, being connected to truth, objectivity was also justified through proofs, mathematical existence and on cultural grounds. The last case also included students who occasionally portrayed mathematics as subjective either because they found it unnatural that there could be no disagreements in mathematics or because they wished to disagree with something they could not understand.

Mathematics as rules

Most students found the word 'rule' to be appropriate within the context of mathematics. According to their school experience, mathematics seemed to offer them certain guidelines which they could use in order to solve mathematical problems and they were content to call these rules. However, exactly because these were only guidelines suggested by some theory and not orders issued and enforced by some human authority, some students felt that the word 'rule' had overly strict connotations of obligation which were not applicable to mathematics or other school subjects in general. In any case, it was hard to deny that there were certain prescriptions which the students were required to follow, even if these prescriptions were not perceived as restrictive. So even those students who denied that mathematics had rules could

find themselves using the word 'rule' for mathematical statements at some other point in the interview.⁷²

Pelopidas explained that the word 'rule' 'meant that there's a framework of things which I must follow, and which if I go beyond, there may be sanctions.' After this, when I enquired whether there were rules in mathematics, he asserted

of course. [Generally,] in order for some things to apply, there are also some rules. Similarly in mathematics, in order for the formulae and all the like to apply, there are some basic rules which we must follow.

So it seemed that Pelopidas considered that mathematics came with a framework of rules which he had to follow, and that such a framework was necessary if there were to be any kind of guidelines applicable in mathematics. It was the framework which provided him with the formulae which he could trust to apply when he wished to solve mathematical problems.

I asked him if there would be sanctions with respect to mathematical rules too, and Pelopidas initially answered negatively, probably because he was comparing mathematics with life, where a sanction meant jail or a fine, or some form of formal punishment. Then, however, he remembered that school grades could be a form of punishment and he maintained that 'okay there may be sanctions, in the sense of grading, for example, if you're solving an exercise you may make some mistake.' Thus, it appeared that, according to his understanding of the word 'rule', there was no reason to exclude rules from mathematics. Moreover, this answer confirmed that Pelopidas saw rules in mathematics as the guidelines which would determine how to solve an exercise correctly.

On the other hand, Kleio suggested that the word 'rule' was not exactly appropriate for school subjects. She explained that '[rules are] like laws; [they say] what we must do.' Thus she seemed to associate a strong impression of obligation with the word 'rule'. Following this, I initially enquired about mathematics, but Kleio answered that 'I don't know' if it has rules.⁷³ Therefore, I decided to investigate whether she would associate rules with other subjects,

⁷² This could also be because the word 'rule' was used in the interview repeatedly, at least with respect to classrooms and life in general. Nevertheless, even in this case, the fact that the students would eventually use the word 'rule' to refer to mathematical statements indicated that they did not find it wholly irrelevant in the context of mathematics.

⁷³ She had been disengaged from mathematics for a long time and she tended to plead ignorance with respect to it on many occasions during the interview.

hoping that if we could decide what the word 'rule' meant in the context of another subject, then she could also check whether it applied to mathematics. Kleio was not sure if other subjects had rules. She wondered: 'Rules? In what sense though?' Then she added that

[In] ancient Greek, for example, which I know ... I don't know if they're rules, but we must follow [certain things], for example, to [be] in accordance with the grammar. Essentially they're rules, it's just that ...

She could not express what made her feel that the word 'rule' was not exactly appropriate in this context. So I began to suggest that 'it's not exactly in ...' and Kleio completed my sentence: 'in the military sense.' Thus, it seemed that although, she would agree that subjects, such as ancient Greek, had guidelines which she had to follow, she would not attach to these guidelines the same degree of compulsion and obligation that she would attach to rules.

Nevertheless, Kleio had not really discarded the word 'rule' and so I continued using it, wishing to clarify its meaning even more as we also introduced classroom rules into our discussion. Eventually, after inviting her to compare classroom rules with subject rules, Kleio commented that 'I don't see them as rules in ancient Greek.' I asked her what word she would use instead, and she was not sure but in the end she offered the word 'theory'. This was a word that could indeed hint at the existence of a guiding framework, without carrying any connotations of obligation.

Consequently, I returned to mathematics, reminding her how an equation was solved and asking if this method could be called a rule. Having her memory refreshed, Kleio suggested that 'you follow some rules ... that they have to ... change the signs.' Then we considered the Pythagorean Theorem, and Kleio was willing to assume that it was a rule too, though she also asked to check what my opinion was. I had to admit that 'yes, in a sense I would call it ...', I meant to say rule, but she said 'theorem'. So it appeared that in mathematics too, Kleio would refrain from using the word 'rule'. This prompted me to ask about one more case and Kleio suggested that 'everything is in the mind, it [depends on] how one sees it. Some [people] would call it rule; others' The previous discussion would imply that she belonged to that group of people who would not use the word rule; however, she did not make this clear.

Seeking more clarity, I enquired again both about ancient Greek and mathematics. With respect to the former, I received as a reply a clear 'no'; but with respect to the latter, Kleio mumbled 'mathematics is more ... it's numbers; it's ... I don't know. There it's more [like] rules.' After

this she also suggested that 'okay, in ancient Greek there are rules too, let's say [so]. I'm simply not used to calling them so.' Nevertheless, it seemed that she proceeded to make this statement because I would not drop the subject and because she could not really explain what made her feel differently about mathematics. One explanation might result from the fact that she did not like mathematics, and therefore felt it more as an obligation compared to ancient Greek towards which her feelings appeared to be neutral.

Summary

In all, students agreed that mathematics offered guidelines which they had to follow in order to solve mathematical problems. However, some students were not ready to call these guidelines 'rules' because they would not consider them as binding enough. It appeared that for such students, a rule was something that came along with a sense of obligation and compulsion, while mathematical statements understood as guidelines were simply describing how a problem could be solved without forcing anyone to take any action.

Concluding remarks

The above discussion indicates that students had been influenced by the culture of their society and classroom in so far as they presented mathematical statements as rules emerging from indubitable proofs. As a result, students echoed traditional beliefs, portraying mathematics as a set of certain, immutable, true and objective rules, with proof being one of the main bases for their claims.⁷⁴ Nevertheless, against a traditional picture of mathematics, most of them did not believe that mathematics existed, and even when they did, they perceived mathematics as residing in nature (empiricism), and not as a collection of abstract entities (platonism). Furthermore, students did not perceive mathematics as a completely unchanging entity, at least not as far as its present and future were concerned. Moreover, even if students experienced them as absolute because of their culture, certainty, immutability, truth and objectivity were to a great extent based on cultural factors, thus echoing humanism. So many times, mathematics was taken to be unchanging only because humans had agreed so, while it was taken to be certain or true simply because it was a science with proofs. Finally, even though they constituted a

⁷⁴ The extent to which students were affected by their culture with respect to perceiving mathematics as rules or guidelines is more relevant to the next chapter, where the importance that they attributed to such rules is discussed.

minority, some students even denied that mathematics was true, while others implied that mathematics was subjective. A similar pattern seemed to occur with respect to epistemology, which is discussed in the next chapter. Here too, traditional beliefs were espoused, but only to an extent, while more modern beliefs were also present.

Epistemology

Introduction

This chapter, like the previous one, involves the objective philosophical meaning of the students' remarks.⁷⁵ As noted in the previous chapter, these remarks were not entirely objective, and as mentioned in the section on thematic analysis, this was even more the case with respect to epistemological issues. Such issues did not simply concern a separate entity, that of mathematics, but involved the way of reasoning that would lead to the production of mathematical knowledge. As such, they appeared to be strongly influenced by each student's individual way of reasoning, for example the way they understood logic, the senses, or authority. Nevertheless, such individual differences are mentioned here only to contextualise the examples given; it was not possible to cover all the cases without overwhelming the chapter with unnecessary details. Such differences are elaborated further in the next chapter, which discusses the subjective meaning that students attributed to mathematics.

The themes that are discussed here are:

- Rule-based knowledge
- Logically-based knowledge
- Empirically-based knowledge
- Proof-based knowledge
- Authority-based knowledge

Continuing from where we left off at the end of the previous chapter, the discussion starts with rules, this time with respect to the production of mathematical knowledge. In this context, rules are portrayed as necessary, though not always binding. Next follow the issues which may be taken to indicate the origin of mathematical rules. First considered are the issues of logical and empirical knowledge which have been quite prominent in the history of philosophy in general. Logic is handled first, since its link with mathematics has been stronger in the history of the philosophy of mathematics. The section on logic involves its relation to mathematical reasoning, and to common sense and culture, but also considers instances which indicated a

⁷⁵ The beliefs presented here again concern issues which are unlikely to be discussed in the classroom. Hence, students' beliefs mostly reflect the unrefined impressions of mathematics on their minds.

cultural influence on the students' comments. The section on empirical knowledge concerns issues relating to the senses and to experience. This is followed by the topic of proofs as the arguments through which mathematical knowledge is generated and validated. Apart from this mathematical function, the cultural function of proofs is also discussed. This is associated with authority, which is handled in the last section of the chapter with respect both to the authority of mathematics and the authority of the teacher and the book in the classroom.

The chapter sections are organised into subsections corresponding to the different subthemes as in the previous chapter, and again, each main belief is represented by one case. However, as these subthemes are not homogenous it should not be assumed that all students advanced a main belief following a similar kind of reasoning. The quotes were again chosen on the basis of clarity, thoroughness and vividness, but they could not cover the entire range of ways in which the students reasoned. Still, the chapter is complete with respect to the philosophical beliefs which the students discussed, that is, the objective meaning that they could locate in mathematics.

Rule-based knowledge

Rules as necessary (and sufficient)

In line with their experiences in the mathematics classroom, nearly all the students implied that mathematical rules⁷⁶ must be followed if a mathematical question was to be answered correctly. It seemed that in their minds, not following the rules meant making a mistake. Moreover, by stating this, they also seemed to imply that if they followed the rules correctly, they would find the correct answer. In other words, valid mathematical knowledge could be reached if, and only if, one stuck to the rules. Furthermore, following the rules seemed to be a way to learn them, and thus acquire mathematical knowledge.

⁷⁶ As mentioned in the section on rules in the ontology chapter, not all students would use the word 'rule' in the context of mathematics. However, all of them agreed that mathematics provided some kind of guidelines. Here, and in the rest of the thesis, for reasons of convenience, the word 'rule' is used to denote such guidelines.

For instance, Theodosis commented that '[mathematics] has [rules], which ... we must observe to the letter so that we solve the exercises, [solve] anything.' He was the first person to use the phrase 'to the letter', and so I asked him to elaborate on it.⁷⁷ He simply replied that

in order to solve our exercises and to succeed, basically yes, in order to solve an exercise we must be led by these rules that we know and we learn [in] mathematics so that we solve [the exercise]. Because without [the rules] we don't know what our steps would be in order to [solve the exercise].

So Theodosis believed that in order to solve a mathematical problem correctly he had to follow the rules that he had learnt in the classroom and that if he did not do so he would have no way of reaching a solution.

Theodosis had also used the word 'comply' with respect to rules in general in the first interview, and in the second one I asked if the same word should be applied to mathematical rules. Theodosis, felt that it was necessary to comply with mathematical rules as well, 'because if we don't observe these rules, we won't be able to solve an exercise.' I was interested to see the extent to which he felt the need to observe mathematical rules, and I continued asking similar questions, only to receive similar answers. So when I enquired whether it would be wrong not to follow mathematical rules, Theodosis repeated that 'if you don't observe the rules which hold exactly, you may reach other (wrong) results.' In all, it appeared that he believed that mathematical problems could be solved if, and only if, he followed mathematical rules.

Rules as non-binding

Occasionally students maintained that rules were not entirely binding as far as expert mathematicians were concerned. For such experts, the rules appeared to be less binding, and students assumed that an expert could bend the rules in order to find a new result or a new way to solve a problem. However, it seemed that inside the classroom there was no such possibility and the students had to follow the rules they were given.⁷⁸

For example, Lysimachos suggested that 'there are some rules in mathematics which if you don't observe, you can't solve the problems which you have in front of you.' So it seemed that

⁷⁷ Subsequently I would use this phrase with other students to check if they agreed with it.

⁷⁸ Only two students suggested that rules were not necessarily binding even in the classroom. However, these cases are not presented here. Firstly, because they concerned only two students, and secondly, because the arguments were not essentially different from those that concerned experts.

he considered following rules as necessary for solving mathematical problems. However, when I asked him if not solving a problem was the repercussion of not following a rule, Lysimachos noted 'not necessarily, but the rule is like an aid, a help, so that you can solve the problem more easily.' In that instance, it appeared that Lysimachos was contradicting himself, claiming that it was not necessary to observe mathematical rules in order to solve a problem. I asked to check if he really meant that a solution could be found even without following the rules and he replied

yes, but if you don't follow [the rule], theoretically, you'll reach some other theories, in which you'll necessarily have to put some other rules in the game. That is, either way, again you'll reach a rule which you'll have to follow. [Either] this, or you create new rules.

Thus, it appeared that although Lysimachos considered the existence of some kind of mathematical rules necessary in order to reach a conclusion, he did not believe that the rules which one had to follow were fixed. On the contrary, he assumed that there could be flexibility with respect to the set of rules that was observed each time, and that this could lead to the development of new mathematical theories.

Nevertheless, when I enquired how easy it was to generate new rules in the classroom, Lysimachos initially wondered if I was referring to classroom rules instead of mathematical rules; probably because it seemed easier to him that a new rule would be created in this context. I asked him to reply with respect to both, and regarding mathematics he noted that

basically during the lesson, when we enter [sic] a chapter ... the teacher will declare to us that there are some specific things which have been proven and are continually applied, so essentially, these are the rules of mathematics. That's how I comprehend it at least.

This description of what happened in the classroom suggested that Lysimachos did not consider creating new mathematical rules in that context plausible; it was the teacher that was providing the students with rules that they would have to follow and apply. Indeed, when I asked again what would happen if one did not follow these rules, Lysimachos maintained that 'if you don't follow them, the most likely [result] is that you won't be able to continue with mathematics. Something like this; until now at least.' So it seemed that he did not regard it as wise to not follow the rules which were given by the teacher, since this strategy would most probably block one's mathematical development. He did hedge this claim, but this appeared to be mostly because this could change in the future, when presumably he would know more mathematics and he could act as an expert mathematician and not as a student.

Summary

Influenced by the traditional teaching of mathematics which predominates in Greek schools, most students believed that in order to reach the correct answers to the mathematical problems which they were given to solve they had to correctly follow the mathematical rules which they learnt in school. If they did not observe these rules, then this would inevitably lead to a mistake, and would impede their progress in mathematics. However, some students pointed out that this did not necessarily hold for expert mathematicians, who were imagined as being freer to be creative with mathematical rules.

Logically-based knowledge

Logic in mathematical reasoning

General remarks

Many students presented logic as an integral part of mathematical reasoning. This finding could be the result of logic being stressed as the basis for mathematical reasoning in school. In any event, it appeared obvious to many students that doing mathematics implied and required using one's logic. This was most probably the reason why they would occasionally suggest that learning how to solve mathematical problems cultivated one's logical skills not only with respect to mathematics but in general.⁷⁹

For instance, Aspasia indicated logic as the main trait of mathematical reasoning; a trait that, according to her, would be desirable in all cases of reasoning. She firstly referred to mathematical reasoning when I asked her to elaborate on her claim that it would be better if life was as certain and clear as mathematics,⁸⁰ observing that 'basically, if everyone had this way of reasoning (the mathematical one), and they weren't stupid [sic], [life] would be much better.' So I tried to understand why she made this claim, and she explained that

⁷⁹ See also the section on common sense in the chapter on subjective meaning.

⁸⁰ We had been talking about certainty and clarity in various sectors of knowledge, and Aspasia had indicated that in mathematics, one could be more certain than in life or psychology. I asked her what she preferred, having in mind that many students would prefer life to be more uncertain than mathematics, believing that this made life more interesting.

I think that you must think logically; that is, mathematics contributes to this, to thinking [using] logic, and generally to thinking; most people don't think; you [must] use emotion, but first logic.

It seemed that for Aspasia logic was the hallmark of mathematical reasoning, and since she believed that thinking logically instead of emotionally would improve life, she had also asserted that reasoning mathematically would improve life. She repeated the same argument when later I wondered if it would be better for life to be as objective as mathematics: 'yes, certainly. Yes, because when [things] are subjective you bring in selfishness too, you bring other things in too, you advance emotion and not your logic.'

In the second interview, while I was endeavouring to understand how mathematics could help human life,⁸¹ Aspasia returned to the issue of logic and mathematical reasoning, claiming that mathematics cultivated reason. When she maintained that mathematics was helpful even in the context of social sciences, I asked her to give me an example and she explained that

[mathematics] generally cultivates human reason. And it isn't that you'll solve the exercise and the numbers will work; [it isn't] only this. It's that you'll learn the technique and you'll learn how you must think. So this is passed to the person who must think the various issues that concern them, so [it's passed] too out [of mathematics] in life, and so in the social sciences too.

Thus, it appeared that Aspasia believed that by doing mathematics one could learn not only to reason within mathematics, but to reason in general with respect to any situation that life would bring.

Generation and validation of mathematical knowledge

Students also tended to implicate logic in specific aspects of mathematical reasoning. In particular, students could associate logic with the generation, justification or validation of mathematical knowledge, either generally, or within the context of problem solving or proving. So students seemed to suggest that logic could be used to link together mathematical statements in order to produce a new conclusion. Moreover, they would assume that by virtue of such a process, the resultant conclusion was justified as a logical one. Consequently, they could

⁸¹ When she spoke about how mathematics develops as a science in the first interview, Aspasia used the phrase 'it helps everywhere', presumably to show why humans care to advance mathematical knowledge. In the second interview, I wanted to investigate what had prompted her to suggest that mathematics helps everywhere.

suggest that one could confirm the conclusion as correct or true, by using logic in order to check whether the mathematical statements involved had been indeed logically linked.

For example, Agapi implicated logic in the process which would generate true and valid mathematical results. I had invited her to compare certainty about true conclusions in mathematics and in life, when Agapi commented that

in order to say that something is true in mathematics, you follow a route, this, that, with equivalences, with equalities, you know that what you'll reach is valid ... the corresponding [thing] in life is arguments, propositions which through a logical sequence lead you to the conclusion.

I asked if that conclusion would also be valid, and Agapi declared 'yes, it's valid and true.' Though in that instance Agapi did not use the word 'logic' directly for mathematics,⁸² she implied a correspondence between life and mathematics. So it seemed that she was suggesting that in both cases there was a logical route or sequence which one could follow in order to reach a valid result. In other words, she was implying that mathematical claims could be linked to one another in a logical manner that guaranteed that the result was true and valid.

Furthermore, according to Agapi, the result of such a process should also be logical. This became clearer later on various occasions. For instance, when I asked her what would happen if one did not follow what mathematics dictated, she observed that 'if we don't follow [the concepts of mathematics], we can't reach a logical conclusion as we must do in mathematics.' Thus, Agapi believed that mathematical conclusions were logical. In fact, what she claimed was that mathematical conclusions had to be logical. It appeared that Agapi indeed connected logic both with argumentation and with mathematics, and that she assumed that a sequence of mathematical relationships which was logically arranged was bound to lead to a logical result.

Thus, when later I asked her how she could know that a conclusion was correct while solving a problem, she noticed that

if the [result] is logical with respect to a problem, I know that it's correct. You don't have 100% accuracy [sic], but if you don't have a solution, as there is many times below the problem – [you have] the result only - you believe through logic that it's correct.'

⁸² She did so later (see next paragraph).

Agapi referred to the fact that many times, along with a problem she was given the correct result (written below the problem), so she could check her result against that. However, she also asserted that when the correct result was not available to her she could still check her result - though not with absolute certainty⁸³ - through logic. So it seemed that Agapi had in mind that by utilising logic she could check whether all the equivalences and equalities that she had written were indeed logically linked, and thus whether her result was logical, i.e. correct.

Cultural logic

Limited understanding of mathematical reasoning

Some students seemed to be influenced by their culture when they claimed that mathematics was connected with logic. Such students could indicate how their teachers would present mathematics as related to logic. Most importantly though, such students seemed to lack sufficient experience that could allow them to confirm that the process of generating and verifying mathematical knowledge involved logic. They were students who had difficulties with understanding mathematics. In other words, they had difficulties with evaluating and utilising mathematical knowledge. So when, despite this, they would confidently assert that logic was used to comprehend or to solve mathematics, it appeared that they were mostly echoing what their culture had impressed on their minds.

Afroditi's first reaction when I asked her whether logic was related with mathematics was 'yes, I consider this [relationship] very great, because, don't they always say to us in mathematics: "take it a bit logically." I believe that yes, logic has a great relationship [with mathematics].' After this, she stopped momentarily, somewhat uncertain about how to explain the relationship further. It seemed that the explanation that was more readily available to her was practically a cultural one, justifying a relationship between logic and mathematics on the basis that this was how mathematics had been presented to her in the cultural context in which she had grown.

However, Afroditi also claimed that she had experienced this relationship between logic and mathematics. When I asked her if she had seen this relationship in practice or she had concluded it from what she had been hearing, she appeared to reject the second option, noting that

⁸³ See section on traces of uncertainty in the previous chapter concerning how students would hedge claims to mathematical certainty.

if I study something in mathematics, in order to write something, for example, or to study and the like, I won't understand everything as well, and I'll try through logic to understand how the specific [thing] may work ... how something can follow logic and be solved. That is, I think it's an issue of logic.

So essentially, Afroditi seemed to suggest that she used logic whenever she had to write a mathematical solution, or whenever she had to understand the given mathematical theory. In other words, she implied that she would use logic in order to generate and verify mathematical knowledge, and that this was the reason why she had asserted that logic had a special relationship with mathematics.

Nevertheless, the cultural influence on Afroditi's opinion could not be discarded. Considering her experience with mathematics, Afroditi had expressed herself disproportionately strongly when she was asserting the 'great' relationship that logic had with mathematics. As she had explained, she had hardly had any exposure to mathematics in gymnasio (lower secondary school), and she had not been able not understand mathematics when she reached lykeio (upper secondary school). This meant that the opportunities that she had had to understand and solve mathematics, and thus to witness the use of logic in this process, were scarce and rare. Moreover, there would have been cases in the recent past where she would not have been able to use logic in mathematics. On this basis, it was hard to believe that she had sufficient evidence in which to ground such a strong assertion about the relationship between logic and mathematics. It seemed more reasonable that the culture in which she had been taught mathematics had simply reinforced any slight evidence that she had, leading her to declare that logic had a strong relationship with mathematics.

Logic as a habit (and limitations of logic)

Another indication that students were influenced by culture could be seen when they were not consistent in their beliefs and they assumed that logic was related to mathematics at one point, whilst they doubted this at another point; thus, indicating that, for them, mathematics was not absolutely or strictly logical.⁸⁴ In such a case, a student also portrayed mathematical knowledge as a cultural construct, suggesting that mathematical results were not necessarily logical; the

⁸⁴ The ways in which that happened were idiosyncratic, but ultimately revolved around the fact that the belief that mathematics is logical contradicted the student's common sense.

results could as well be irrational, but humans considered them - instead of some others - as logical only because they had become accustomed to them.

The first time I asked Yerasimos whether logic was related with mathematics he replied 'I think yes. I don't know. That's why I say that I think there's a relationship.' Yerasimos' admission of not really knowing whether logic was related with mathematics was in accordance with his lack of understanding of mathematics. However, he proceeded - though not with absolute certainty - to suggest that there was such a relationship, hinting at the operation of a cultural influence.

In order to clarify his answer, I encouraged Yerasimos to state what he believed on the basis of his experience and - though again not with absolute certainty - he responded that 'well yes, mathematics is an issue of logic. One and one make two, that is, there isn't another result. That's why.' Surprisingly, Yerasimos seemed to imply that mathematics was logical because it always yielded the same answer. It was not clear what the connection between these two qualities was but I did not probe him further because he already appeared to have considerable difficulty in explaining why there would be a relationship between mathematics and logic.⁸⁵ This difficulty further confirmed that his claim was a result of cultural influence.

In any case, in the second interview, Yerasimos practically implied that mathematics was not necessarily logical; a belief which was founded in his newly-acquired understanding that mathematics did not always yield the same answer. Admittedly, this understanding had been influenced by my intervention in corroborating that there may be different answers for the same mathematical question. Nevertheless, it had actually been Yerasimos' discontent with the belief that this was not the case that had prompted him to ask about that issue.⁸⁶ Thus, his suggestion that mathematics was not necessarily logical was based on a belief that was in accordance with what he generally believed to be reasonable. To the contrary, the belief that mathematics is logical, which he had initially expressed, was based on the opposite belief which did not appear reasonable to him. It could therefore be claimed that the picture of mathematics as logical did

⁸⁵ A possible explanation was that if mathematical questions always have the same answer, then one cannot disagree with this answer, and as a result this answer must be logical.

⁸⁶ See the section on cultural objectivity in the previous chapter.

not emerge from his own understanding of what was logical, but was impressed upon him by the culture in which he had grown up.⁸⁷

The fact was that when we returned to the issue of logic in the second interview, and, using the same example, I enquired whether it was logical that one and one made two, Yerasimos was not as absolute in his answer as he had been in the first interview. He noted that

with the data that we have, it's logical, but it's not proven [by] anybody what [one and one] makes. We say that [it's logical], since for so many billions of years, it makes [as much], one and one make two; somebody may appear and say that one and one make three.

So, Yerasimos seemed to suggest that there was no proof that one and one made necessarily two and that in the future someone could reasonably claim that the result could be different. In the meantime, humans had assumed that it was logical to maintain that 'one and one made two' because they had become used to this result throughout the history of mathematics.⁸⁸

Furthermore, Yerasimos had appeared to imply that this did not guarantee that the claim for the result of one and one being two was indeed logical. I tried to elucidate the issue by inviting him to compare this with an empirical sentence, such as that the desk in front of us was grey.⁸⁹ Nevertheless, with respect to the example from mathematics, Yerasimos did not go beyond repeating what he had claimed before: 'okay, it seems logical to me too that one and one make two, but ... aren't there mathematicians who say that it makes three; don't they challenge it?' Later though, he did maintain that we did not know if this claim was logical. This happened after I asked him whether the correctness of a rule also implied that the rule was logical. He responded negatively, using again the same example: 'It's what we were saying about one and one. It's correct, but we don't know if it's logical.' At that point, I enquired whether mathematical rules could be irrational, and Yerasimos replied positively, almost divorcing logic from mathematics.

⁸⁷ In other words, the belief of mathematics as logical was not in accordance with his common sense (see next section).

⁸⁸ I asked to confirm if this was the case, and Yerasimos agreed.

⁸⁹ It was indeed grey.

Common sense

In general, students would utilise the word 'logic' and its derivatives in the interviews as they would use them in everyday discourse.⁹⁰ As a result, they could call logical more or less, anything that they would perceive as reasonable on the basis of their experience. In other words, they were referring to what could be called common sense,⁹¹ or at least to their personal understanding of what counted as common sense. Nevertheless, some students suggested a distinction between the kind of logic that was applicable to mathematics and the kind of logic that could be used in everyday life. This seemed to be because they had noticed that there were cases in which mathematics seemed to contradict common sense.

Common sense as distinct from mathematical logic

One of the students who clearly differentiated between common sense and the logic that was used in mathematics was Solonas. When we were discussing logic and I asked him if logic was common for everyone, he basically distinguished common sense from the logic that was involved in specialised mathematics. In particular, he referred to the fact that mathematicians would reason in more than three dimensions, which defied common sense where humans moved in a three-dimensional space:

I think that the greater percentage of people follows that logic, the specific [sic] one (common sense). Now, there are other people, such as those who found different dimensions, who think that this logic isn't unique, and that there are other dimensions too, we're not only in 3D ... so [these people] didn't remain with that common logic and they wanted to go further.

Thus, according to Solonas, most individuals would reason following what could be called common sense, but some amongst them would transcend common sense and would find that there were things which lay beyond it. Such were the mathematicians who had introduced more than three dimensions in mathematical reasoning.

I wished to clarify this concept of common logic, so later I asked whether it could be applied everywhere. Solonas noted that

⁹⁰ In fact, so did I. The interview was supposed to resemble the natural context of everyday discourse and it would be too artificial for me to introduce a distinction unless the student spoke of one.

⁹¹ In Greek, the attribute 'common' in the phrase 'common logic' is very frequently dropped in everyday speech.

common logic is usually about everyday things which one can easily realise. In contrast, that is, things in branches of physics or mathematics may not be perceived by a person who has not engaged [with these branches] at all; while I may understand them, they may not be able to understand.

So, again he maintained that there were special sectors of mathematics which required a special kind of logic, and for which common sense was not sufficient. In fact, if one restricted oneself to common sense, one might not be able to understand these sectors of mathematics. It actually seemed that this inability of common sense to handle special mathematical cases, was what had led Solonas to differentiate between it and a logic that was particular to mathematics.

Limitations of logic

Other students were also aware that some mathematical results contradicted common sense. However, since they would not proceed to differentiate between common sense and logic as used in mathematics, they effectively claimed that some mathematical results contradicted logic, suggesting that cases which were not exactly in accordance with what they would understand as logical were, in a sense, illogical. In essence, such students indicated that there were limitations to the extent to which logic could be implicated in mathematics. Such a claim could collide with their general picture of mathematics as logical, leading them to statements which appeared to be contradictory - if one insisted on not distinguishing between logic within and without mathematics. Thus, a student could suggest that even these seemingly illogical results had to be logical because they were essentially the result of a logical mathematical process.

In contrast with Solonas, Platonas did not differentiate between common sense and the logic used in mathematics. He seemed to suggest that there was something more involved in doing mathematics than everyday logic, but he insisted that there was only one kind of logic. This was why he seemed to be claiming that there were cases where logic was not enough to understand mathematics. So initially, when I asked him whether logic was related with mathematics, he asserted grandiosely 'but essentially mathematics is logic, isn't it? Because' Nevertheless, he did not even start the explanation that he meant to give; he realised that he had exaggerated and stated 'yes, but okay, I believe that after a point, you can't think [about mathematics] logically, that is, you go on only through operations; you can't think [about] it, you can't understand it logically.' When I invited him to elaborate on this, he mentioned

for example, imaginary numbers or something with the velocity of light ... I had heard something, that above the velocity of light the properties of matter change ... okay, you can't think [about] these [things], you reach them through operations.'

Essentially, Platonas was suggesting that there were special sectors of mathematical, or scientific knowledge in general, where logic was of no use because these sectors involved subjects which were beyond what thought and logic could even conceive, let alone comprehend. This reaction was similar to that of Solonas. Nevertheless, by postulating a different kind of logic, which pertained to mathematics only, Solonas had been able to suggest that all mathematics is based on logic. On the contrary, Platonas had to admit that there were certain mathematical results which lay beyond logic, because he assumed that everyday, ordinary logic was all that there is.

In fact, after the above conversation, I asked Platonas if he would reverse his original claim, and state that logic is mathematics. Initially, he appeared to disagree, maintaining that 'okay, not always, because there's logic ... in other issues; for example, the logic that if I don't pay the rent, I'll ...'. However, when I enquired if the logic that was used in these other sectors could contradict that of mathematics, Platonas declared 'no, because logic is one and common; so if you're a rational human being, you'll have the same logic.'

After this, I felt the need to ask why not everyone understood mathematics, if logic was common for all. To reply, Platonas returned to his comment that there were cases where one had to trust the operations. He commented that lack of understanding was

either because [some people] don't spend so much time to understand [mathematics], or mostly, I told you before, you may not understand it, but ... to reach the result through the operations. That is, many times you may not have to understand something, but the logic of operations leads you to it.

In essence, by coining the term 'logic of operations' Platonas re-introduced logic to the totality of mathematics. This logic seemed to be applicable even to results which were incomprehensible according to common logic. Indeed, as Platonas elaborated further on this issue, it became clear that he believed that the logic of operations could guarantee the logic of any result that was reached through it: 'logic through the operations brings you to a result which, since [logic] led you logically to it, this too will be [a] logical result.' However, all this was confusing, because essentially, Platonas had not really differentiated between the logic that was involved in performing mathematical operations and the logic that was used in everyday life, i.e. common sense. As a result, on the one hand, he had suggested that some mathematical results were beyond logic, and on the other hand, he was asserting that these results only appeared to be irrational, but were practically logical.

Mathematics as not logical

The apparent contradiction which could occur by confusing common sense and logic as used in mathematics could also be resolved if one dissociated logic from mathematics altogether. This was what happened when the cases which appeared illogical to a student were far too many. The discordance between mathematics and common sense concerned mostly cases where the students could not apprehend certain (or most) mathematical results empirically, because they lay beyond their experience.⁹² In effect, Platonas had also connected common sense with experience. His examples about mathematical or scientific facts, of which he could not even think, regarded issues which were beyond what he could experience such as imaginary numbers, and velocities greater than those of light.

By contrast with Platonas, Ariadni claimed that logic had no place in mathematics at all. However, again she seemed to be referring to some kind of empirically based common sense. In any event, she initially dissociated logic from physics, while she was comparing it and mathematics, explaining that only the former was connected to the real world:

[physicists] realise some things which happen around us and we can't understand them. But I don't think that always everything is logical ... I can't understand them. That is, I think, I think, I think [about them] and again I don't find them logical. Because physics isn't logical. Certain things aren't logical.

I did not ask her why she considered physics not to be logical. It was apparent that this was because she could not understand physics and our previous discussion had made clear why this was the case. The problem for her was that physics concerned unreal, imaginary situations to which she could not connect, and she had in fact suggested that this concerned both physics and mathematics.⁹³

So, I asked Ariadni if mathematics was logical, to which she responded that '[mathematics] is certainly logical for those who invented it; for me it isn't.' With this remark, she declared that

⁹² See section on empiricism in the chapter on subjective meaning.

⁹³ See section on truth in the ontology chapter.

she could not find any logic in mathematics, though she did suggest that mathematicians would find some logic in it. Wishing to investigate the issue of logic without connecting it to particular individuals too, I also asked whether logic was related with mathematics. Ariadni, in contrast to other students, replied negatively: 'no, it seems to me that this is why I don't like [mathematics]; because it isn't something which I can understand; it's something [that is] very much outside of me.' Effectively, she was still noting that mathematics was not logical for her, but even if this was the basis for her claim, consistent with what she could understand, she asserted that logic was not related with mathematics.⁹⁴ Moreover, she again indicated that this was because mathematics was too far from her, far from her experience, far from her common sense, far from what she could understand. I actually asked her to explain what she meant when she stated that mathematics was 'outside of me'. However, she found it hard to elaborate on this, probably precisely because she could not relate mathematics was 'out of her' because it was not logical: '[mathematics] gives you some knowledge, some operation, some equation, and it also gives you a rule to solve it. I don't find this rule logical.'

Trying to explain, she returned to physics which she could accept as occasionally logical:

In physics you can think about it a bit differently. That is, you think. It's something more logical, [in situations] where it's possible [to think, e.g.] the ball will go towards the earth because there's attraction. You think about it. While the mathematician doesn't think anything; they think numbers; [in mathematics] you only do operations.

So essentially, Ariadni appeared to be claiming that mathematics was even more illogical than physics. The latter could concern situations about which she could reason because they were real, such as a ball falling on earth. On the contrary, she could not find any reality in mathematics; it seemed to her that it concerned only numbers and operations and she could not even imagine that a mathematician could reason about them. Once more, she was connecting logic to experience. In fact, when I asked for an example where physics was irrational, she exactly referred to a situation for which she had no immediate experience: '[physics] may not speak to you about the earth, it may speak to you about the universe, [and] in the universe you don't even know what exists. So you can't think [about it] logically.' In all, it appeared that Ariadni found mathematics, and sometimes even physics, illogical because she found them to

⁹⁴ Later, Ariadni repeated that mathematics must be logical for those who create it, but definitely not for her (see section on invention in the ontology chapter).

be incompatible with her common sense, the logic which she could use to reason about situations which were part of her experience.

Summary

Influenced by the remarks of their teachers, students generally connected logic with mathematics, suggesting that it was a pivotal factor in mathematical reasoning. Some of them associated logic specifically with the process of generating and verifying mathematical knowledge. Nevertheless, on some occasions it seemed that such comments were the result of cultural influence. In fact, occasionally students would comment how their teachers regarded mathematics as logical. Resisting this influence, some students would occasionally suggest that mathematical logic as a result of habit, or because students could not associate mathematics with experience. Such cases were also indicative of the fact that students would not differentiate in their remarks between common sense and mathematical logic - a more general phenomenon relevant for almost all the students. Thus, students could claim that mathematical issues, which they could not understand, were, or at least appeared to be, irrational.

Empirically-based knowledge

The senses

Observation

It seemed that students found it hard to imagine how they could use their senses to interact with mathematics. This was apparent by remarks such as those of Diomidis in the section on invention, and of Ermis in the section about discovery, both considered in the previous chapter, and therefore not discussed further here. However, when students referred to particular mathematical statements, they could implicate the senses in the process of comprehending verifying, and applying such examples. For instance, students might note how one could see when two angles were vertically opposite, or what the sum of two small numbers would be. A

few students went a step further, suggesting that mathematical knowledge could have emerged through observation of our physical surroundings.⁹⁵

There were certain cases where the senses seemed to be relevant to mathematics. For instance, Agapi referred to vision while she was trying to explain why it would be easier for one to unlearn a mistaken result with respect to two and two make four, than to unlearn a bad behaviour. We were discussing the differences between violating a mathematical rule, such as that two and two make four, and a rule of conduct, such as observing a queue, when she remarked that

the person who overtakes me in the queue ... doesn't have respect for the others; no one can [make] them to [behave correctly], if they don't have it inside them [sic] ... that's how they are, that's how they've learned, that how they'll be. Somebody who writes that two and two makes five, they'll learn at some point that it makes four.

When I wondered why she claimed so, Agapi maintained that eventually such a person would come to see what two and two made: 'at some point, they'll learn it. They'll say, they'll see: two apples and two apples; they'll say 'four apples.''

Menelaos gave an example from geometry. He had just declared that the senses were related to mathematics, but he had explained it by stating that 'because you have to use your mind in order to understand [mathematics] and to ...'. He could not find what more to add and it was not clear whether the reference to mind could be interpreted as involving the senses, or it was mostly connected to logic. So I asked about the senses again, and Menelaos replied positively, this time with an example: 'yes, and your vision too, if you see that two angles are vertically opposite, you immediately realise that they're equal.'⁹⁶ I asked him what would happen if he did not know the respective theorem, and he observed that in that case 'I can't be certain of anything.' So Menelaos was not simply suggesting that the angles were equal because they seemed equal, but that one could recognise, through vision, a particular instance as an instance to which the theorem applied and then derive the respective conclusion.

However, Agapi's and Menelaos' remarks seemed to concern particular cases. On the contrary, Foivos' comment suggested that mathematics as a whole was based on observation. He

⁹⁵ In fact, if mathematics is taken to exist in the structure of nature, this appears to be a reasonable conclusion. However, this claim is also compatible with invention, since it is possible to be inspired by something that exists in order to craft something which does not exist.

⁹⁶ We had earlier discussed the proof for the equality of vertically opposite angles.

maintained this when I asked him whether we could have defined mathematics differently.⁹⁷ He was not sure about it, but he believed that this was not possible because mathematics was based on observations about the world around us: 'I don't know, because whatever we've defined, we've defined it on the basis of our universe, on the basis of things which we observe.' At this point, he seemed to be reminded of a question I had asked him previously about the possibility of mathematics being different on another planet of the universe, and he added 'I assume that on another planet, where something else holds, maybe they would speak [about mathematics] somehow differently.' Thus, Foivos appeared to imply that exactly because mathematics was based on observation, it could be defined differently on another planet, where the surroundings to be observed could be different.

Detour through logic

Sometimes students linked the senses to mathematics through the detour of logic; that is, students assumed that there was a link between the senses and logic, and since they also assumed that there was a link between logic and mathematics, it could be claimed that they also implied that the senses could be used while applying logic in mathematics. In particular, students could observe how whatever was perceived through the human senses could be considered logical; that is, how seeing or hearing something - especially repeatedly - could lead one to infer that it was logical for that thing to be happening.⁹⁸ Furthermore, students could note that the senses were necessary for providing the input on which logic could function. Then, they could also note how they used their senses to read a mathematical problem, or to hear their teacher speaking. It could be assumed that such an implication of the senses in mathematical reasoning was minimal, but it was still more than other students would assume.

A student who connected the senses with logic was Lysimachos. When I invited him to compare the two, he appeared to consider them to be inextricably linked, and he was initially somewhat puzzled with my enquiry: 'Logic? Logic, meaning? Simply because they're

⁹⁷ I did so because Foivos had mentioned that mathematics starts from statements without proof and I wished to check if he would consider that there could be alternatives to these statements.

⁹⁸ It must be noted that such inferences are generally valid with respect to common sense. An example in the context of mathematics would be seeing that two plus two equals four, and on that basis, inferring that it is logical for two and two to make four. Students would make such inferences. The student whose case is presented here did not offer a particular example, but explained the link between logic and the senses much more clearly.

connected, that's why.' So instead, I asked him to explain how logic and the senses were connected, and he commented that

I mean that when we see, or we observe something, when we hear something, either because we've seen it again, either because we've become used to it, either because it seems familiar to us anyway, we consider it absolutely logical that [this thing] happens.

In other words, Lysimachos noted that what was perceived through the senses was regarded as logical, especially when previous experience confirmed it.⁹⁹

After I accepted this, Lysimachos also added 'yes, I think that they're interconnected, that we function both with the senses and with logic. And basically, logic comes through the senses.' Thus, it appeared that Lysimachos was not only claiming that sensory inputs were logical, but that logic was essentially governed by sensory inputs. This was not an idea with which I was familiar, so I wished to investigate the issue of logic further, and I went on to enquire whether logic had rules, assuming that such rules could be independent of any sensory input. However, Lysimachos declared that 'basically, I think that the rule of logic is the senses', confirming that he saw logic as the result of the senses. I was still confused, and I referred to some syllogisms which we had discussed earlier, asking if in that case, when he was making his inferences, he had been using his senses. He admitted that he had been using 'logic, but practically I use my senses in order to understand the statements. So I use logic afterwards, after I have already used the senses, in order to reach some conclusions.' Thus, it appeared that Lysimachos could imagine logic functioning independently of the senses, but only after it had received some input from them.

This could be the reason why, when the discussion turned to mathematics, and in particular to algebra - which could be considered less empirical - Lysimachos still commented 'if you don't see [something] how are you going to write it?' Nevertheless, when I asked how he could see an equation, he seemed to be unable to connect this to vision and he asked me 'what do you mean? How you comprehend the equation?' I insisted on using the verb 'to see' and eventually he asked again 'you mean [seeing] through logic?' I observed that I was trying to understand how he would use his senses in the context of algebra, and he replied 'yes, that you use your senses only to see what you're going to write ... to read it, to write, all these, and then you use

⁹⁹ In a way, this is reminiscent of Yerasimos' remarks that one and one make two because that was what we had become accustomed to claiming. In fact, when logic is connected to experience, then it becomes a cultural construct, something that does not hold necessarily, but only as a matter of habit.

logic.' Such a description could lead someone to suggest that the senses were practically not involved in algebraic reasoning. However, the fact was that Lysimachos would not made this conclusion. Even if he had to admit that the contribution of the senses in that context was minimal, he still suggested that without them nothing could even begin.

Experimentation

Trial and error

Some students associated mathematics with experiments by interpreting the latter as trials. Such students suggested that mathematical results emerged through trial and error, i.e. by checking possible formulas against the available data, until the correct one was reached.¹⁰⁰

For example, while she was explaining how a theory would emerge, Afroditi used the word 'experiment', noting that

everything begins from an experiment which you do, either in the [context] of philosophy, or of mathematics, or of sciences, and in all fields; and all [experiments] reach, result in a conclusion, a discovery.

Later, while she was comparing mathematics with physics, she referred to experiments again, without being clear whether she associated them only with physics or with both physics and mathematics: 'Yes, I believe that [physics] is related [to the world], because as mathematics was discovered, so physics was discovered too, through some experiments, through some [people] who engaged with this.'

At that point, having in my mind Afroditi's previous remark as well, I asked her to tell me what an experiment was and then whether there were experiments in mathematics. She suggested that in an experiment

¹⁰⁰ In a sense, this belief is reminiscent of Lakatos' proofs and refutations (1976b) according to which the proof of a theorem gradually evolves as counterexamples are found and addressed; each new version of the proof can be interpreted as a new trial, while the counterexample corresponds to the data that lead to the rejection of this trial. Nevertheless, the trial and error model of the students was far more empirical and arbitrary than that which Lakatos had in mind. Only one student expressed himself clearly in a way that was closer to Lakatos' view: 'when something is to be advanced in a science, somebody says an idea; 500 [people] agree, 600 [people] disagree, and eventually one of the 600 finds something else (a counterexample) ... or they simply all come to agree because one of the 500 proves that [the original idea] holds for additional reasons which had not been found by the first person [who introduced the idea] (bypassing the counterexample) and so it goes on.'

somebody has a theory in their mind; they try to put it in practice and they're not sure of the results which it'll induce. So [they] try their thought out in order to see if it corresponds to reality, and if it can be realised.

So essentially, Afroditi interpreted an experiment as a trial which could allow one to check if one's assumptions were correct by checking whether they were applicable in practice.¹⁰¹ After this definition, it was easy for her to imagine mathematicians making such trials. Using the pythagorean theorem as an example, she described how Pythagoras would have had proceeded through trial and error before his thought matched reality and he reached the correct formula which fitted the available data:

I could say that [mathematics] has experiments. Because, there is, for example, a mathematical relation ... let's say the pythagorean theorem, which you mentioned. In order for the formula of the pythagorean [theorem] to be discovered, somebody, Pythagoras, had some things in his mind, some thoughts. He didn't know if they'd be actually realised. He made an experiment, and he changed the formula, and he changed the theory many times - experiment in quotation marks there, of course - many times in order to reach the specific formula which verifies [the data] and reaches something real and absolute.

During her explanation, Afroditi seemed to realise that this kind of experiment was somewhat different from scientific experiments and that was why she noted that the word should be put in quotation marks. She confirmed this, observing that 'it isn't a real experiment as in chemistry ... it's an experiment, experiment okay, it's just ...' She did not have much experience with any kind of experiments, so it was difficult for her to complete her sentence; however, she had not abandoned the word experiment completely and I offered to call what she had described as a 'thought experiment', a term which she accepted gladly. In all, Afroditi connected mathematics with experiments as trials which eventually led to the correct conclusion.

Applications in practice

Students would suggest that applying mathematical results in practice was some sort of 'experiment'¹⁰² which could corroborate these results. In other words, they would assume that

¹⁰¹ Afroditi's reference to reality suggested that thought was to be checked against empirical data. This was no problem, since Afroditi believed that mathematics existed in nature. However, a similar argument could be advanced, and was advanced by some students, independently of mathematical existence, if thoughts were to be checked against the 'reality' posited by a theory.

¹⁰² They would not use this term directly though.

the fact that mathematics was applicable in science and everyday life indicated that mathematics was correct.

For instance, Lysimachos pointed towards some kind of experimentation when he explained how mathematical results could be confirmed through their applications in science and everyday life. In fact, first, it appeared that Lysimachos was suggesting that assumptions made in physics could be confirmed by mathematics: 'if we prove something mathematically [in physics], then that assumption which we've made in physics will hold.' It was not exactly clear how this would happen, and I wished to confirm that he really meant that conclusions in physics could be checked through mathematics. He agreed and he immediately added 'and mathematics [can be checked] through physics.' I asked him to elaborate on this and he stated that 'since mathematics is used in everyday life, we check the validity of what we've proven.' So it seemed that Lysimachos claimed that the fact that mathematics could be successfully utilised in practice was a confirmation that it was correct.

This belief was repeated again in the second interview, when I asked Lysimachos how the axiom-like statements and the rules which he had to follow while doing mathematics had been determined.¹⁰³ I was principally interested in seeing what he would say about the former which he had presented as assumptions. He suggested that

maybe some of these [assumptions] have been confirmed through experiments and from practice in everyday life; while some others are simply based on previous propositions which hold, and so the next [ones] will hold.

So Lysimachos indicated that axiom-like statements, which lacked a mathematical basis, could be corroborated through their applications; as long as they yielded useful results, they had to be valid. In fact, when I later enquired why we all accept the axioms, Lysimachos responded that

firstly, I don't think that there've been attempts to change what mathematicians have created, I mean the foundations. So since they have results and validity in everyday life, [people] continue to use them without further changes.

In other words, Lysimachos suggested that the foundations of mathematics had not changed precisely because their applicability in everyday life confirmed their validity. In essence, since the remaining mathematical statements were based on these axioms, as Lysimachos had

¹⁰³ Lysimachos had claimed that mathematics was based on such statements.

observed in the previous comment, according to him, all mathematics was corroborated in practice, through its applications.

Experience

A result of experimentation was experience. Students noted how by doing mathematics, that is, by following and applying the rules in order to solve exercises, they could gain experience which could help them to understand better what they had been taught and how they could use it.

For example, Diomidis observed how by experimenting with mathematical rules, students could acquire experience which would elucidate aspects of the rules that had up to that point been vague, and thereby make them clearer. In the first interview, when I asked him if mathematical rules could be challenged, Diomidis observed that he had done this previously, but he had always realised that he was mistaken in doing so: 'at least, I've challenged some [mathematical rules] in the past, but through the challenging, I understood that I wasn't right.' This comment led me to ask in the second interview, how he had understood that he had been mistaken in such cases. Diomidis maintained that

basically, I believe that the first time that you hear a rule, you haven't understood it 100%, but then, the more you see it and observe it better, I believe that you can understand that what you were thinking previously [when challenging the rule] is somewhat wrong. And then, I think that the more you use that rule, you sometimes understand that what you thought at the beginning is wrong.

So essentially, Diomidis was suggesting that understanding mathematics required him to get used to it and to experiment with it, i.e. to apply its rules in exercises and to observe how they could be implemented in practice.

Summary

Students connected the senses with logic, and through it, they appeared also to connect the senses with mathematics. In any case, although students would mostly dissociate the senses from mathematics as a whole, they did implicate them in specific examples of mathematical statements. Moreover, some students suggested that mathematics was based on observation. Similarly, other students maintained that mathematical knowledge was the result of experimentation, through trial and error; or that mathematical results were corroborated

empirically, through their applications. Finally, some students noted how experimenting with mathematical rules in exercises could help them to familiarise themselves with these rules.

Proof-based knowledge

Mathematical function

Students presented proofs as the means by which mathematical knowledge was produced and validated. In particular, students noted how a proof showed the procedure through which a mathematical proposition had been reached, or claimed that a proof guaranteed the relevant mathematical proposition. Such claims could be advanced by any student, regardless of whether they understood mathematics and proofs. Nevertheless, students who understood proofs could also refer to this understanding, suggesting that the proof allowed them to understand how the relevant proposition was reached and why it was justified; in other words, for them, proof was a convincing argument, and not just some arbitrarily enforced claim.

Andromachi explained how proofs could help students to understand how a mathematical statement was established. She even asserted that proofs were the essence of mathematics, presumably because without them, mathematics would be reduced to a mass of arbitrary statements with no justification. So when I asked her about the role of proof in mathematics, she declared that

The proof. Basically, to begin with, I like proofs very much because, essentially, you comprehend something [through the proof], you see its process, so you understand how it was made. So I think that it's very important, and essentially, this thing is mathematics. The proof is mathematics.

At this point, I enquired whether the students did proofs in the classroom, and Andromachi admitted that it depended on the teacher. However, she again asserted how doing proofs was valuable because it helped to understand the rationale behind a mathematical statement:

they do [proofs] for us, but everything depends on the teacher. For example, I have teachers who always do the proof for us, because this is something very good; because [with respect to] something you enter the whole way of reasoning, you comprehend it. But okay, usually they do [proofs], but not all the teachers.

It seemed to me that Andromachi was effectively claiming that without the proof one could not understand why a mathematical statement had been advanced, and that, in that case, one would have to accept it arbitrarily without full comprehension. In order to confirm this, I asked her whether without proofs mathematics would appear to be arbitrary, and she claimed that

Without proofs I'd say that it's a bit as if mathematics is lost. I'd say that, yes, that you learn something and you use it as you do in other [school] subjects; while, in essence, there's a development, in which what you must be able to do is to understand, to see. That is, in order to understand something you must take it from its beginning.

It therefore seemed that without the proof, Andromachi would lose the essence of mathematics, which would become for her as any other school subject, where she had to learn and apply unjustified, arbitrary facts. In contrast to this, for her, mathematics involved the development of validating arguments, i.e. proofs, and she wished to understand the rationale behind these arguments from the beginning to the end. In all, Andromachi not only claimed that proofs showed how and why mathematical propositions were concluded, but that through them, she could understand why these propositions held.

On the contrary, Yerasimos had difficulties with mathematics and it would be hard to claim that proofs helped him to understand mathematics. In fact, when I first asked him about the role of proof in mathematics, he simply responded 'it proves', and he did not seem to have any particular explanation of what that might mean. Wishing to understand his thoughts on the issue better, I rephrased the question, asking him why mathematicians do proofs. He initially replied 'in order to help the world? That is, every basic experiment is also written in mathematics, a thing, you must also write it in mathematics. So ...' He did not finish his sentence but he seemed to imply that whenever something was written in mathematics, it was also accompanied by a proof. Thus, I asked him whether all mathematical statements had proofs or not. However, Yerasimos replied negatively and this prompted me to enquire whether it was an issue for a mathematical statement to not have a proof. He stated that 'it doesn't bother me; it bothers whoever is solving [the problem].' This remark confirmed that Yerasimos did not care about proofs, as he did not care about mathematics. In fact, when I asked him why a lack of proof would bother the person who was solving the problem, he replied 'because they'd be engaged [with the problem] for years, that is to solve [a problem]. In other words, Yerasimos suggested that proofs were important only for those who took mathematics seriously; and he did not consider himself one of these, since he could not understand mathematics.

Still, Yerasimos had not really answered why not having a proof was regarded as a problem. So I asked him what problem would exist if there were no proof. At that point, he responded that 'that is, I can say something ... there may be no proof and I may be simply saying it. Will it be right?' Thus, eventually, Yerasimos maintained that the proof was there to validate a mathematical claim. In order to confirm this, I asked if a proof showed that the result was correct and he replied 'well, yes! The way, basically, it shows the way that [the result] occurred.' In other words, it seemed that, despite not understanding mathematics and proofs, Yerasimos believed that the role of proof in mathematics was to lead to a conclusion and to justify it. He even noted how the production and the validation of mathematical statements were so inextricably linked through proofs, by remarking that 'it's like what we say: what came first, the chicken or the egg? No one knows, no one can prove it.' So in a sense, Yerasimos appeared to suggest that by proving something one simultaneously generated it and validated it. After this, I asked again whether in order to claim something in mathematics, one needed to prove it, and Yerasimos replied positively.¹⁰⁴

Cultural function

As mentioned earlier, even students who did not understand proofs would claim that they were used in order to generate and validate mathematical propositions. This was probably the result of the importance given to proof while learning mathematics in school. In particular, students would observe how in order to claim anything in mathematics they needed a proof for it (see Yerasimos above). In practice, this meant that students could not challenge the mathematical propositions they were being taught, because, of course, they were not in a position to either disprove them, or prove an alternative statement (see also Vrasidas' remarks in the section on cultural certainty in the previous chapter). In all, it appeared that students had learned that a proof validated a mathematical statement regardless of its ability to convince them; the mere fact that there was a proof was enough to guarantee the relevant result. In fact, even students who generally understood mathematics would succumb to this cultural power of proof. Some of them would do so only when they could not do otherwise, that is, when the proof was not available to them. However, other students were content to trust proofs without feeling the need to inspect them, assuming that they could trust mathematicians, and their book and teacher. Thus, apart from the mathematical level, proofs seemed also to function on a cultural level, sanctioning mathematics simply by virtue of them being regarded as indubitable arguments.

¹⁰⁴ It seemed that, although Yerasimos might have heard of some results without proof, these would be the exceptions, the general impression being that mathematical claims had to come with a proof.

The cultural power of proof was definitely impressed on Filippos' mind. Filippos believed that all mathematical statements had a proof, although he could not justify why this had to be the case. When I asked him about the role of proof in mathematics, he noted that 'I haven't understood it. They usually say that everything in mathematics has a proof and that's why you do a proof.' So it seemed that, despite not understanding why proofs were needed, Filippos had learned that any mathematical statement was accompanied by a proof, and that one could not claim anything in mathematics without providing a proof for it. Still, in his mind proofs were practically not necessary. It seemed that instead of helping him to understand, a proof only complicated things which otherwise appeared simple. Thus, later, while we were comparing what was needed to understand human sciences and mathematics, he observed with respect to the latter that

basically in mathematics, the problem is that, theoretically, in my mind, [something] may appear simple and may hold, but on your paper they say to you that you must prove it. And there it doesn't happen. There you can't apply the rules.

After this, I asked him if, to his mind, the proof was necessary, and he replied negatively.

Notwithstanding, Filippos had in mind that proofs where not to be challenged. While discussing what the verb 'to prove' meant, he offered the explanation: 'I perform operations in mathematics so that I reach somewhere.' I asked if there could be a proof outside the context of mathematics¹⁰⁵ and Filippos maintained that 'yes, all life is actions¹⁰⁶ in order to prove who I am, or in order to prove that what I do is right.' After this, I enquired if proving something made it definitively correct and Filippos replied positively. In order to confirm this, I also asked if something proven could not be challenged, and Filippos agreed again. So eventually, the school had offered him proof as an indubitable, yet incomprehensible, argument which sanctioned mathematical results.

Nevertheless, even students such as Platonas, who understood mathematics, seemed to be influenced by the cultural power of proof. As Andromachi, Platonas believed that a proof could help students to understand the relevant claim. Seeing how easily he understood mathematics, at one point I asked him if students in general would understand the mathematics taught in the classroom. Platonas replied positively, stating that 'yes, when you see the proof, based on what

¹⁰⁵ Most students did not restrict proof to mathematics, although they would usually remark that without mathematics proofs were not as absolute.

¹⁰⁶ Interestingly enough, the word for mathematical operations and for actions is the same in the Greek language.

you know, you think that it's true, you believe [it].' Contrary to Filippos though, Platonas did not appear to accept proofs blindly. When, after he had explained how proofs were arguments which verified mathematics,¹⁰⁷ I asked him if he was convinced by such arguments, he remarked 'if they're correctly corroborated.' In other words, Platonas seemed to imply that a proof was not to be trusted without being inspected; first one had to check that the proof was correct and then one could be convinced by it.

Despite this comment, at one point, Platonas also suggested that if he was given a statement without a proof, he would still accept it:¹⁰⁸

Now, okay, sometimes ... the [author] has not put [a proof] in the book, so you learn the rule as it is and you say "okay, since [mathematicians] have said this, it'll be correct, why not learn it?"

This tendency to accept mathematical statements even when the proof was not available could simply be a result of a generalisation on the basis of all the cases where Platonas had seen a proof and he had been convinced by it. This kind of inductive argument is quite common in everyday life. However, as all inductive arguments, it is not absolute. Therefore, Platonas' answer can be taken to indicate that he was affected by the general culture which perceived proofs as correct. After all, his explanation for accepting the statement without a proof did not make use of such an inductive argument, but it did assume that the statement at hand had a proof which had been advanced by some experts. Thus, in a sense, his answer reflected the authority attributed by society to proofs and the mathematicians who produced proofs.

Somewhere in the middle - between the cases of Filippos and Platonas - lay students, such as Agapi, who simply appeared to be content with accepting what they learned in school without feeling the need to see a proof for it. Agapi did not seem to have much trouble with mathematics and its proofs; however, contrary to what Platonas had suggested, she did not seem to consider proof as an absolute prerequisite for understanding mathematics. When I asked her if the students would do proofs in the classroom, she observed 'no, very rarely, and in specific cases', without any sign that this was a problem. Up to that point she had given me the impression that she comprehended mathematics; however, I asked to check whether her apparent disregard for proofs was due to her not understanding them. Confirming my impression she agreed that when

¹⁰⁷ See section on truth in the ontology chapter.

¹⁰⁸ This happened exactly after his comment about proofs helping students to understand mathematics, when I asked him if they usually saw the proofs in the classroom.
they did proofs she would be convinced by the argument. In fact, she appeared to claim that this was why the proof existed, to be the convincing argument: 'yes [I'm convinced]; otherwise the proof wouldn't exist ... this is this and that is that, there's a basis, some data and we reach the result, the theorem.' Still, it was not clear if she regarded proofs as necessary for understanding and accepting a theorem, so I asked her if it would be enough for her to know that there was a proof, or if she would want to see it. Agapi noted that

okay, in everyday life, I think that if [somebody] says [about] something to us that this is true, we take it as true; but okay, some [people] want the proof too. I think that knowing the proof too is more correct.

At this point, I returned to the classroom context to see if she regarded knowing the proof as necessary there. She admitted though that 'it isn't needed, no ... we don't use proofs.' In all, Agapi, seemed to be satisfied with this situation. When a proof was given she would understand and accept it, but if it was not available, she would happily trust that what she was told was correct, using the same kind of trust that humans tend to exhibit in most of their exchanges with other individuals in their everyday lives. For instance, when we ask a stranger for the time, we simply accept their answer as true, assuming that they are not trying to deceive us.

Summary

Students claimed that proofs were the arguments that led to the generation of mathematical knowledge, which they also validated. In that context, students who understood proofs explained how they could use them to understand mathematics. However, even students who did not understand proofs had learned to accept them as correct. This indicated the cultural power of proof as an indubitable argument. This power seemed to influence even students who understood proofs, in so far as they were ready to occasionally, or regularly, accept mathematical propositions without having examined the proof.

Authority-based knowledge

Mathematics

Students generally portrayed mathematics as an authority.¹⁰⁹ It was this authority which sanctioned the mathematical rules, through proofs (see previous section). However, students who could understand proofs would be able, through their understanding, to justify this authority, which they did not have to follow blindly. On the other hand, students who did not understand mathematics seemed to have no other basis for its authority apart from the fact that it was part of their culture,¹¹⁰ where mathematics was presented as a respectable science.

The authority of mathematics was finely paired with understanding in the case of Platonas. When I asked him how that which was correct was decided within mathematics, he responded that 'whether [something] is correct in mathematics is regulated by what is correct in mathematics, that is, if you make a mistake with some of the rules.' Essentially, Platonas appeared to declare that correctness in mathematics was determined by mathematics itself, through the mathematical rules that dictated what was correct. These rules were the authority that had to be respected and handled without making mistakes. However, Platonas did not seem to consider this authority as arbitrary. He believed that mathematical rules were generated on the basis of logic. So when I enquired how mathematical rules were advanced he attributed them to 'educated mathematicians who, through logic and rationalism [sic], manage to reach conclusions which cannot be changed and are taken as rules.' As was mentioned earlier, Platonas also believed that these rules were the result of proofs and that the proofs helped him to understand the rules, presumably by showing him the rationale behind their creation. Consequently, for Platonas, mathematical authority was based on logic and was justified.

On the contrary, students such as Menelaos could not justify the authoritative status which they attributed to mathematics. So when I asked Menelaos whether he could characterise mathematical rules as correct or not, he simply replied that 'reasonably speaking, they'll be correct. I'm not involved [in mathematics], so since I'm not involved, I can't judge whether they're correct or not, but I think that yes [they're correct].' In other words, Menelaos admitted

¹⁰⁹ The case of Ermis, who was the only student who seemed to defy any sense of mathematical authority, is not discussed here, since it is unique. However, it is expounded in the chapter on meaning.

¹¹⁰ This did not necessarily mean that such students had a problem in accepting this authority (see section on rules in the chapter on subjective meaning).

that he had no knowledge that would allow him to decide whether mathematics was correct or not, and yet he claimed that it was, indicating that he perceived it as an authority. When I enquired why he suggested that assuming that mathematics was correct was reasonable, he observed 'because mathematics is a science where the rules must hold. That's why when some [rule] doesn't hold, they create some other to replace it.' So it seemed that Menelaos accepted mathematics as an authority simply because of its status as a science in society; this status implied that mathematics had to be correct, it had to be an authority which society could trust. The mathematicians would take care of this by replacing faulty statements, while he, as a representative member of the general public, could assume that their work was indeed correct.

Teacher and Book

Apart from mathematics, students also attributed authority to their teachers and books.¹¹¹ The teachers were supposed to be knowledgeable with respect to mathematics, while the book was written by experts who were again knowledgeable. In essence though, this kind of authority simply reflected the authority of mathematics. The teachers and the books were correct because they reproduced knowledge which was supposed to be correct either for cultural reasons or because the students actually understood the rationale of mathematics (see the previous section). Nevertheless, students would also occasionally indicate that the authority of the book or the teacher was wrongly intensified by the way that mathematics was taught. One reason was their exams, which forced them to learn whatever they needed for them irrespectively of whether it was correct, or whether they understood it. Another reason was the fact that, in effect, the teachers imposed what was correct. Moreover, it was highly unlikely that an explanation would satisfy all students, and for practical reasons, such as the size of the classroom, the teachers were not able to tailor their teaching.¹¹²

The authority of the book and the teacher became evident during the discussion with Kosmas. When I asked him how what was correct in mathematics was decided, Kosmas firstly referred to the authority of mathematics, suggesting that this was determined by mathematics itself. However, he also mentioned his teacher as a representative of this authority:

¹¹¹ All schools in Greece use the same textbook which is published by the state.

¹¹² This would also be needed if they wished to help students discover mathematical results by themselves.

What is correct in mathematics is firstly determined, that we can refer [to the book or mathematics] to find what is correct. And I believe that with respect to the subject that some teacher teaches, they must know. I can't [imagine] that some student must challenge them; and if they do, always with respect.

Just before, Kosmas had explained that students could challenge their teacher with respect to what counted as correct behaviour in the classroom. However, he suggested that this would not be right with respect to mathematics, where the teacher was supposed to know the subject and to be able to refer to it in order to tell the students what was correct and what was not. After this, I enquired if Kosmas would attribute a role of authority to his teacher with respect to mathematics and he replied positively.

Consequently we turned to the book as a source of authority and Kosmas noted

99%, 99.5% because there's always the mistake. I find it even now, in the biology [book] there are some mistakes. The teacher tells us 'add this', in the history [book] there are some mistakes, the teacher says 'add this'. But 95%, 99.5%, I believe, [the book] is an authority because what it says holds and it's also the material [on which we'll be examined]

He did not refer to his mathematics book specifically, but he did not seem to differentiate between books on different subjects; apart from occasional mistakes, which were a negligible minority, he appeared to consider all his books as authorities for the respective subject.¹¹³ This was firstly because he believed that what the books stated was correct, but secondly also because even if it was not correct, the book constituted the material on which he was to be examined, so he had to learn it as correct. In other words, any authority that Kosmas attributed to the book was magnified by the fact that the book dictated what was correct as far as his exams were concerned.

Since Kosmas had mentioned that the teacher might correct the book, I invited him to compare the book with the teacher as sources of authority. However, Kosmas claimed that the book had a higher authority status because the teacher could not simply declare whatever they wished; they had to follow the book: 'in relation to the teacher, I believe that the book outdoes [them] a bit; that since the book says that, shouldn't the teacher follow it?' I noticed though, how just before he had said something different - that the teacher might correct the book - and he replied 'Oh, yes, but he knows it, he's heard it somewhere, while the book, okay the 0.5 may come

¹¹³ In this grade he was mostly disengaged from mathematics, while he did have to study history and biology for his exams. So this might be why he mentioned these subjects.

from the teacher, but the 99.5 is of the book.' So Kosmas clearly attributed the authority of the teacher to the authority of the subject they had learned, in our case to mathematics.

Nevertheless, he spoke as if the authority of the book came from the book itself, though he did seem to imply that it came from its content, i.e. again the respective subject. In order to confirm, I asked whether books had fallen from the sky. Kosmas, of course, denied this and noted that 'you said [to compare] in relation to the teacher, not in relation to those who made the book.' So it appeared that Kosmas believed that the book was an authority because it was written by experts on the respective subject. In a sense, this implied that the book also drew its authority from the respective subject, which was the work of those experts. After all, Kosmas had no way of differentiating between the authority of mathematics itself and the authority of mathematicians, since he attributed both of them to the same cultural source, i.e. mathematics as a valid science (see Kosmas' remarks in the sections on truth and objectivity in the previous chapter).

If the authority of the book could be augmented by the exams, the authority of the teacher could be intensified by their role in the classroom. This became evident in Kleomenis' answers; he suggested that it was practically impossible for the teacher to explain mathematics appropriately to every single student in the classroom. When I asked him about the extent of the authority of the teacher in the classroom, he wondered 'if they have, or if they should have?' This already implied that he would consider it preferable if the teachers had no authority, but that in fact they did. Indeed, when I invited him to comment on both he stated that

usually yes, especially in lower grades where mathematics is a bit simpler and it's easy [for the teacher] to impose his opinion, because it's simple. While, in reality, it shouldn't [happen] that much. They're indeed those who know, but they should show somehow, they should help the children to understand [mathematics] on their own.

So Kleomenis believed that it would be better if students could see for themselves what was correct in mathematics, while he noted that usually, they were just told what was supposed to be correct. If this was simple they would most probably agree; if not it seemed that they had no other option but to remain silent. At that point though, I could not understand how it was easier for the teacher's authority to be imposed when mathematics was simpler, because I wrongly assumed that if the mathematics was simple and the students understood it, then there was no authority involved. In any case, I mentioned my confusion to Kleomenis, who simply

observed that ultimately the teachers had no other option but to impose their opinion because they had to handle so many students:

it's a bit compulsory that [their] opinion is imposed, because okay, it's hard to not impose yourself, [when you're] alone [with] 20 kids where each one would say their [opinion] and there would be a chaos. I mean it's a bit compulsory.

It could be that Kleomenis could not perceive the situation otherwise, simply because he was so used to it. However, the fact remained that this implied that some students would have opinions or ideas which would not be addressed, and that as a result these students might not understand. I enquired if this was the case, and Kleomenis agreed, adding 'something like this, that is, in an ideal classroom it would be easier [for the teacher] to not impose their opinion and to [teach], for instance, as they say that Socrates taught.' So it appeared that Kleomenis believed that ideally it would be possible for every student to understand, if the teacher did not impose the correct mathematical result, but helped students to find it themselves, as Socrates had done with his pupils. Nevertheless, he considered this very difficult in practice.

Summary

Students perceived mathematics as an authority either because they understood the logic that rendered it correct, or simply because, according to their culture, it was assumed to be correct as a science. Moreover, students noted how this authority was transferred to their books and teachers, while also being magnified by the authoritative way in which it was delivered in the classroom.

Concluding remarks

Traditional beliefs, supported by the way that mathematics is predominantly taught in Greece, were again present. In particular, the students' beliefs seemed to have been influenced by the culture of their society and classroom, where mathematical rules were presented as binding, proofs were seen as absolute, and mathematics was portrayed as logical. As a result, when they did not understand mathematics, they would justify its authority (and the authority of proofs) not on the basis of logic, but on cultural grounds. This however, echoed a humanistic view of mathematics. Moreover, the logic that students referred to was hardly the abstract, deductive logic that traditional philosophy would have in mind; it was much closer to common sense. In particular, some students linked logic to the senses and to habitual experience, giving it an

inductive and even humanistic character. Finally, some students advanced an empirical view of mathematics based on observation and experimentation.

Having reported students' beliefs on the objective meaning of ontological and epistemological matters, I now turn to the ways in which these beliefs either informed, or were informed by, the subjective meaning that students attributed to mathematics on a psychological level.

Subjective Meaning

Introduction

This chapter, in contrast with the two previous ones which mostly conveyed the objective meaning of the students' comments, aims at presenting the subjective meaning that students found in mathematics according to their interviews.¹¹⁴ The fact was that subjective meaning pervaded the interviews from the first to the last, and it appeared to be the most important message that the students were conveying, since this had more value for them than any purely objective meaning that could be ascribed to philosophical beliefs. Talking about the philosophy of mathematics seemed to acquire a deeper level of interest when subjective meaning was involved, and this was probably why students tended to return more often to the beliefs which were psychologically laden for them. These beliefs can be seen to reflect their own subjective 'philosophy of mathematics'.¹¹⁵

Thus it can be said that the interviews were strongly inclined towards a subjective aspect of meaning. The individual stories that were gathered through the interviews were highly diverse. In the same way that everyone's life story is unique because individuals interpret similar experiences differently (Rosenthal, 1993), so were the students' mathematical stories - their 'philosophies of mathematics' - were unique, since each student would make their own subjective associations and evaluations of philosophical beliefs. For example, Kosmas, Ermis and Kleomenis noted that life was unpredictable. However, Ermis associated mathematical reasoning with life on this basis, while Kosmas used the same argument to divorce mathematics seemed predictable to him. Similarly, Agapi, Filia and Afroditi expressed a need for absolute truth - reliable guidelines or rules - which would help them lead their lives. Yet, only Agapi suggested that mathematical reasoning was the way of reasoning which could help her realise what was morally right and wrong in life; only Filia portrayed rules as utterly

¹¹⁴ This chapter is notably longer that the two previous ones, since it deals concurrently with the subjective meaning pertaining to both ontological and epistemological comments. The division between ontology and epistemology may help in order to organise the students' objective meanings, but any such division would more probably weaken this chapter by artificially fragmenting what appeared to hold students' stories together (as will be shown in the next chapter), namely the subjective meaning that they could locate in mathematics while answering philosophical questions.

¹¹⁵ In line with a 'philosophy of life' (Wong, 2012b).

inviolable; and only Afroditi stressed that these guidelines were not externally imposed. Thus it must be noted that the students' stories only describe potential ways in which students might attribute meaning to mathematics, and do not suggest that all students holding similar beliefs would react in the same way.

Nevertheless, the main aim of this chapter is to illustrate how students could find subjective meaning in mathematics through philosophical themes, and to suggest general patterns and trends around these themes. Therefore, the chapter is not structured around the different associations that students made with respect to specific ideas such as the unpredictability of life, but around the themes mentioned in the section on thematic analysis, each of which is exemplified by means of a certain number of students' stories. In this context, since a student's story and a certain idea do not pertain necessarily to only one theme, and since it was necessary to distribute stories across all themes, it is quite common that stories which make use of the same idea in varied ways are organised under different themes. However, in order to facilitate comparison between individual stories, a table is given at the end of the introduction, indicating the main topics which can be used as axes of comparison and the students whose stories relate to each of these topics.

The stories for each (sub)theme have been selected so that, firstly, the corresponding theme is manifested strongly in them, and secondly, they address as many of the important aspects of the (sub)theme as possible. Nevertheless, for reasons of space, additional aspects are sometimes included under other (sub)themes; where this is the case, it is indicated by a footnote at the introduction of a (sub) theme.¹¹⁶ Within a specific story, quotes have been selected either because they were rich in subjective meaning, or because they illustrated how the subjective meaning that the student attributed to mathematics could be linked with the objective meaning of their beliefs. Quotes rich in subjective meaning had a strong psychological aspect, encapsulating the students' wishes or preferences - though sometimes this was only evident in the tone of their voice and not in their words. It appeared that such wishes and preferences propelled students to express certain beliefs and to make certain associations between them on the level of objective meaning.

¹¹⁶ Moreover, the stories presented here do not offer a complete account of the meaning that a student found in mathematics, but focus only to those aspects of this account which are relevant to the respective theme.

The themes which are discussed below are:

- Common sense
- Discovery
- Invention
- Certainty
- Subjectivity
- Rules
- Empiricism

I have chosen to start my account with the theme of common sense, which during the analysis had proven to be central to subjective meaning.¹¹⁷ The remainder of the themes have been arranged according to the sequence in which they appeared in the two previous chapters. Essentially, however, each account that follows can be interpreted as a story in which the respective student's way of reasoning (common sense) aligns or clashes with some aspect of mathematical reasoning either at the content level or at a philosophical one. In order to simplify the following exposition, a (mis)alignment at the content level is simply signalled by a comment concerning the student's ability to understand mathematics; while a (mis)alignment at a philosophical level is signalled by a remark that the student's common sense was (not) in line with a perceived quality of mathematics. In this context, the influence of culture becomes apparent in each case where students perceived mathematical reasoning as certain, abstract, logical, objective, or as a set of rigid rules. However, while some students simply endorsed the beliefs offered by their cultural context, others appeared to react to them, and others even rejected them.

In the following, the attribute 'subjective' is normally dropped when references to meaning are made so that the text is more easily read. The dialogue format that was used in the previous chapters to present quotes is more or less retained, but is in many cases condensed in order to control the length of each story. After all, for this chapter, the dialogue-context from which specific quotes emerged is not as important as the way in which these quotes could be linked in order to paint the picture of the meaning that students found in mathematics. So less

¹¹⁷ See the section of thematic analysis.

emphasis is placed on the context, especially when the quotes that comprise a story have been taken from responses to quite diverse topics.

Belief	Students
One logic, or one truth	Agapi, Afroditi, Polyxeni, Filia, Filippos
Subjectivity	Ariadni, Polyxeni, Lida, Kleio, Filippos, Evyenia
Certainty	Agapi, Kleomenis, Afroditi, Polyxeni, Filia
Unpredictability	Kosmas, Ermis, Loukianos
Rigidity of mathematics or mathematical rules	Agapi, Kosmas, Foivos, Ariadni, Afroditi, Filia, Filippos, Evyenia
Flexibility of rules	Ermis, Kleomenis
Disagree with mathematics	Evyenia, Filippos, Ariadni

Table 5: Beliefs around which students' stories can be compared

Common sense (reasoning)

Alignment

Students tended to attribute positive meaning to mathematics if their common sense was in line with perceived aspects of mathematical reasoning.¹¹⁸ In such cases, they could perceive mathematical reasoning as a useful way of reasoning that could also be applied in their everyday life. Most of these students also understood mathematics, although they did not

¹¹⁸ As the following analysis will show, the aspects varied among the students. However, some central issues were the rigidity of mathematics, mathematical certainty, mathematical abstractness, mathematical objectivity, and the value of rules.

always appreciate its content.¹¹⁹ In any event, their appreciation of mathematical reasoning could lead them to disregard any problems they could have with the content of mathematics.

Agapi

Agapi's common sense appeared to be in line with the image of mathematical reasoning as a way to reach valid, dependable truths. This alignment seemed to be the reason why she compared mathematical reasoning to argumentation and claimed that this was how mathematics was connected with life. Thus, Agapi essentially believed that through mathematical reasoning she could cultivate her reasoning in general. This was of paramount importance to her since she was convinced that by doing so she could advance her self-awareness and improve herself on a moral level. So she valued mathematics immensely, although she made clear that mathematics as content had nothing to do with her life, where her main goal was to be a better person through promoting her self-awareness.

Agapi was convinced that 'the way of reasoning is the same'¹²⁰ in the attempt to find truth in mathematics and in one's personal life. She explained that mathematical reasoning was not so different from using logic without the context of mathematics. The difference was only that the former used mathematical methods while the latter was composed of arguments. So when I asked her to compare rules in mathematics with rules of logic,¹²¹ she commented that

in logic we have the argument, which is sentences that lead us to a conclusion, a true conclusion. So in mathematics too, with the same rationale we reach the conclusion, from something that holds to the conclusion. For me, this is what links logic with mathematics. (See also Agapi's comments in the section on mathematical reasoning in the epistemology chapter.)

The truth which she was seeking would allow her to live a moral life. This concern of hers became apparent when I asked her about the possibility of two individuals being both right while supporting opposing claims. In her answer she used the term 'complete truth' and when I wondered if this was attainable, she replied positively. So I tried to understand what could help to achieve such an end, and Agapi explained how 'moral values help in life, that is, the

¹¹⁹ Examples of students who could both understand and appreciate the content are Ermis (see theme of rules) and Kleomenis (see theme of invention); while an example of a student who appreciated the content but could not understand it is Afroditi (under the theme of certainty)

¹²⁰ With that declaration she ended the discussion about rules in mathematics and rules of logic.

¹²¹ She had already stated that logic had rules.

sense of self-awareness and of truth, [being able] to know what is really right, [what] holds and being able to say so.'

Agapi realised that the truth of moral values was not necessarily independent of humans. In the second interview, I asked Agapi again whether any objective truth could be attained,¹²² but she remarked that 'we define [what is objective truth].' Among such human-made truths, Agapi included moral values.¹²³ She observed that

morality is the same.¹²⁴ Your parents will tell you, not only your parents, society in general, that you must always dress in a dignified way; but whether other [people] will dress differently or they'll come to a place inappropriately dressed, this is their problem, it's their truth, their opinion; but the general truth¹²⁵ is that you must always be dressed in a dignified way.

It seemed that Agapi depended on such human-made 'general truths' to guide her life appropriately. Indeed, when I asked her if it would be better if everyone behaved according to the general truths or not, she asserted that 'it's definitely better that [people] follow the general truth, but diversity will never stop occurring; so we can't do anything about it.' Her tone while saying this was one of resignation to the fact that people would continue to behave wrongly, and it seemed that she herself was determined to avoid this as much as possible.

When I went on to ask her how such truths have come to be legitimised, Agapi stated that 'a general truth is based on ... the several opinions that existed before it and it holds for all ... on the basis of previous theories ... [and through] logical thoughts, a general truth is created.' This reference to theory and logical thoughts reminded me of mathematics, and when I said so Agapi passionately maintained:

¹²² I returned to this issue in the second interview because I had not been convinced about the capacity of moral values to reveal complete truths.

¹²³ I enquired specifically about morality precisely because Agapi had put so much emphasis on it.

¹²⁴ Before this she had given another example concerning the use of the chair: 'we say that we sit on the chair ... this is a truth; it holds for all people.' Agapi recognised that some people 'will step on the chair, others will turn it upside-down ... but [the fact that generally] we sit on the chair is a truth, which we have defined.'

¹²⁵ She had used this term during the first interview. I had asked her how we could know that something was true, and she mentioned that 'there are several kinds of truths ... that is, there's a different version of truth for each society, but general truths are the same for all.' This term intrigued me and I wished to return to it in the second interview.

Yes, this! They're both linked. Because it's the way of reasoning; and that's what [teachers] try to pass on to us all these years. Our teachers constantly tell us: "I want you to learn how to think." Indeed it's a way of reasoning.

So logic and mathematical reasoning were valuable for her because they allowed her to cultivate her reasoning, and could thus help her with her purpose of comprehending the general truths of life.

In fact, Agapi continued to relate that mathematics as a content was not useful in life:

Nowhere you'll need that three x plus five makes I don't know what. In your life, you won't need it anywhere, anywhere. You won't need a derivative anywhere apart from your job, and even there I doubt if it will be needed so much.

However, Agapi's appreciation for mathematical reasoning helped her to disregard the fact that she did not find any connection between the content of mathematics and her life. So when I eventually asked her if there was a relationship between mathematics and life she replied positively, though noting that 'not a quantitative, but a qualitative [one, where] quality [refers] to the way of reasoning, and quantity to the content.'

In all, Agapi found positive meaning in mathematics as a way of reasoning which could be applied with worthwhile results in her life. This meaning was only minimally shadowed by her belief that mathematics was not as useful at the level of content. At least she understood this content, since she could judge mathematical arguments. In any case, the meaning she attributed to mathematics seemed to be based on the fact that her common sense was in line with a view of mathematical reasoning as one which leads to absolute truths.

Misalignment

In the case of a misalignment between the students' common sense and the way they perceived mathematical reasoning, the meaning attributed by them to mathematics would be negative. Such students would devalue mathematical reasoning because they believed that it was irrelevant to the way they would reason in everyday life, or to the way that they would prefer

to reason in everyday life.¹²⁶ Interestingly enough, all these students, to a greater or to a lesser extent, had difficulties understanding mathematics.

Kosmas

Kosmas found it hard to appreciate mathematical reasoning. In fact, he simply could not see that it could be of any use when it came to real-life situations. He believed that it was incompatible with common sense, since the latter involved life, which was unpredictable; this fascinated him. On the other hand, mathematical reasoning seemed to him to be monotonous and boring; it regarded well-defined, predictable cases and it was very precise. Moreover, he could hardly think of any worthwhile application of mathematical reasoning. Ultimately, he perceived mathematical reasoning as purely theoretical, and this had led him to view mathematics as pointless.

Mathematical reasoning was not in accord with Kosmas' preferences; it was too boring, always producing the same results. He almost apologised when he stated that 'now, you've studied mathematics,¹²⁷ but I would find it monotonous that this solution will bring me this; while psychology is something different.'¹²⁸ The unpredictability of psychology seemed to fascinate him. I had asked him about being in control in mathematics and in psychology and he explained excitedly that

a human is an unpredictable [being] ... there are infinite variables which can influence the outcome ... [if there's a patient] they may draw a gun and shoot, they may draw a gun and shoot themselves.

This unpredictability was something that Kosmas appeared to have learned to enjoy, as it was an essential, unavoidable part of life.

Therefore, even regardless of his preferences, Kosmas actually doubted that mathematical reasoning, where one could determine truth and falsehood, could be very useful in life. When I asked if he could decide truth in life reliably, he observed that 'I can try. That's what judges do, find the truth about specific things. That's what philosophers do.' However, he recognised

¹²⁶ For examples of students where the misalignment concerned only their preferred way of reasoning see Filippos (under the theme of rules) and Foivos (under the theme of discovery).

¹²⁷ At this point I encouraged him, indicating that I did not mind if he expressed himself negatively about mathematics.

¹²⁸ Psychology was one of the subjects he wished to study and we occasionally compared it with mathematics.

that there were limitations which did not seem to apply so much to mathematics which 'is specific; it is these theorems, many [theorems] of course, but these theorems, these proofs, these, these, these. Humans are totally unpredictable.' Kosmas did not really believe that mathematics could handle this unpredictability that made life attractive. Indeed, when I asked which situation he preferred more, he replied that the accuracy of mathematics would be

better as far as justice is concerned; now in some other [sectors], I don't know if the one-sided nature [*sic*] of mathematics (which always produces the same answer), could help in our everyday life ... or generally humans in the way they live.

Thus, Kosmas seemed to believe that if mathematical reasoning was applied to life, then this would become one-sided too, and it would lose the variety which made it interesting. Later, while we were talking about accuracy and precision, Kosmas again devalued mathematics and its reasoning: '[mathematics gives] too much information, needless [information], generally, I think that mathematics is magniloquent ... too many things in order to find a simple solution.'

In fact, Kosmas did not appear to consider mathematical reasoning to be a way of reasoning which was closely related to human thought. On another occasion involving a comparison between psychology and mathematics, he noted that

in mathematics some thought is needed, but it doesn't have any relation with humans, it has a relation with how humans think, but the methodology is given from within mathematics. It's a school subject.

In other words, it seemed that, for Kosmas, mathematics was simply another school subject which might demonstrate how people thought, but this was only because it dictated a method for them to follow. On the contrary, later, he described psychology as a science which 'delve[s] ... into the human mind, [and] into how humans think' on their own accord, in everyday situations.

Moreover, Kosmas believed that mathematical reasoning concerned abstract, theoretical concepts which were not directly applicable to life. So apart from divorcing mathematics from life at the level of reasoning, he also did so at the level of content. When I invited him to compare mathematics with physics he maintained that 'physics is nicer; definitely, because mathematics is completely theoretical; completely; while physics has some application both in nature and in our life, generally everywhere.' He could hardly think of any mathematical application; maybe 'in architecture ... but okay, architecture is not an everyday thing, while a

spring ... what else in physics? Friction, an everyday thing.' He could only understand how the logic of basic mathematics could be used in life

up to gymnasio (lower secondary school) some application [of the mathematics we learned] would be needed here and there, but then it goes [too] far ... now I see [what my classmates do], it doesn't have numbers, it has only letters. Where am I going to need this? One application, give me one application in everyday life.¹²⁹

In sum, Kosmas found only negative meaning in mathematics, judging mathematical reasoning (and content) as irrelevant to common sense and life. This was because he perceived mathematical reasoning as too abstract and rigid. However, Kosmas seemed to seek to react to these traits, which he devalued since his common sense was not aligned with them.

Summary

It appeared that if the students' common sense was aligned with their understanding of mathematical reasoning, then they could find positive meaning in mathematics. In particular, such students considered that mathematical reasoning was useful to them because they could also apply it in their lives apart from mathematics. However, if the students' common sense and mathematical reasoning were misaligned, then they struggled to understand what the usefulness of mathematical reasoning could be, and they would attribute a negative meaning to mathematics. In any event, it seemed that the extent to which students valued or understood the content of mathematics could accentuate or mitigate the positive or negative aspect of the respective meaning.

Discovery

Alignment

The belief in the existence of mathematics implied that mathematics was not merely a game of the mind, allowing students to find positive meaning in it as something actual which affected human lives. In the data, the effect of this belief could be further augmented by it being coupled

¹²⁹ Although I was aware of mathematical applications, I felt unable to provide one that Kosmas would judge as relevant to everyday life any more than architecture, so I remained silent. Nevertheless, it can be claimed that if Kosmas had many such examples at his disposal he might not disregard them so easily. This could help improve his esteem for mathematics even if he would not attribute a positive meaning to it.

with a belief echoing Galileo's claim that 'the book of nature is written in the language of mathematics.' In other words, such students believed that mathematics existed and its purpose was to help human beings to understand the world around them. These students experienced mathematics as a means to comprehend how the empirical world functions,¹³⁰ thus portraying mathematics as a more or less empirical field of knowledge. However, the strength with which such a meaning was felt depended on whether the students could relate mathematics to their common sense.

Ermis

Ermis wanted to understand how the world around him worked. He considered this a fundamental human need, and it was definitely an urgent need of his own, which was why he wished to become an astrophysicist. Therefore, believing that mathematics existed and could describe how the world works, he found meaning in it as an invaluable tool and an integral part of his life. Moreover, this positive meaning seemed to be stressed by the fact that his common sense agreed with mathematics,¹³¹ and thus could indeed help him to comprehend the world around him.

For Ermis mathematics was a passion founded on his belief that mathematics existed and manifested itself in nature's function. While we were discussing his relationship with mathematics, he narrated how he perceived mathematics as 'a tool which helps me to understand - how do they say it? My world.' It is important to note that mathematics helped him not merely to explain the function of a world which was extraneous to him, but to appreciate a world to which he felt intimately close, *his* world. How significant it was for Ermis to understand the world became evident when he continued to state that he wished to place such an understanding at the heart of his profession:

[mathematics helps me] to find some solutions about - how to explain this to you now? Anyway, I want to become a physicist, I want to become a cosmologist, an astrophysicist, and so I need mathematics in order to find some solutions through [sic] some problems which are posed to me.

¹³⁰ Both the example in this section and the one in the next section concern such cases because these were those in which the theme manifested more strongly. Nevertheless, as long as the student believed that mathematics was real, they could appreciate its capacity to explain some aspect of reality. So the mechanism of attributing meaning would be the same, even if the student did not associate mathematics with the empirical world.

¹³¹ For more details regarding this convergence see the part of Ermis' story under the theme of rules.

The problems for which Ermis needed mathematics did not seem to arise from an extraneous source, but eventually from his own self, from an inner urge to comprehend what ultimately he recognised as his home, that is, the world in which he lived. Thus, mathematics was indispensable for him. Indeed, he concluded his previous comment by maintaining that 'so mathematics is like a part of my life', and when I probed him to elaborate on this he added 'I cannot take it out of my life; it's something which is my everyday basis.' At that point I enquired if he found mathematics useful, and on this occasion he felt that he should stress that mathematics was more than a tool for him, it was, as he had just declared, a part of his life: 'I like it, I like being engaged with it ... it isn't only like a tool; it's a part of my life, to put it this way.'

It is worth observing that Ermis' passion for mathematics was facilitated by the fact that reasoning mathematically seemed to come naturally to him. After his passionate confession about mathematics being a part of his life, I asked him why he liked mathematics, and he responded that

I'm able to understand the world around me easier through [mathematics] ... if there's a problem, an exercise, anything, and I try to solve it mathematically and I find some solutions, I'll understand easier why it is so, why it gives me this result, than if [somebody] explained it to me theoretically.

In other words, he suggested that there could be other ways to understand the world, but mathematics was the one that worked best for him. In fact, his first response to my question about his relationship with mathematics had pointed towards the same issue, as he explained that

I can express myself more freely through mathematics relative to language. It's a bit strange; it's just that I like writing [solutions to] exercises etc, [mathematical] expressions [and] so, than writing compositions.

So mathematics was not only a tool which Ermis could utilise in order to satisfy his curiosity about the world, it was also the tool that came more naturally to him.

Thus Ermis found positive meaning in mathematics as a tool which allowed him to explore interesting questions about the real world around him. However, it was possible for him to use mathematics for such an exploration only because his common sense rendered it as the most appropriate tool for this end.

Misalignment

In case of a misalignment between the students' common sense and mathematics, students could still value the fact that mathematics referred to something real. However, this effect would be mitigated by the perceived misalignment, while the negative meaning could be accentuated still further to the extent that a student was not able to understand mathematics.

Foivos

Mathematics was not the tool that Foivos would naturally choose in order to understand the world in which he lived. His common sense was not in line with his perceived rigidity of (mathematical) logic, and would urge him to use tools which were not strictly logical, such as the arts or even philosophy. However, he still found some positive meaning in mathematics exactly because he believed that it was manifested both in nature and in human artefacts constructed through the knowledge of nature. In fact, this seemed to be the only reason why he did not reject mathematics as an entirely worthless endeavour.

When I asked Foivos about his relationship with mathematics, he replied that 'I don't understand mathematics much, but okay, everything is based there, so okay, you can't hate it either.' In this sentence, he communicated his conflict about what mathematics meant for him. On the one hand, he suggested that mathematics was important because it was to be found behind every human advance; and on the other hand, he revealed that if this had not been the case he would most probably hate mathematics.

Foivos exhibited a general admiration for the human curiosity which pushed humans to evolve. In his second interview, where we investigated further what could be the basis of 'everything', he explained how

basically, everything is [based on] the curiosity of humans ... generally we wouldn't even exist as we are [now], if we didn't wonder, we wouldn't even exist as we are [now]; we'd [still] be in the jungles.

In this context, he also seemed to value mathematics, and this appreciation appeared to stem from his belief that mathematics existed and explained the way nature functioned. His reaction was telling when I ventured to suggest that mathematics was a human creation. He interrupted me before I had the chance to finish my sentence, asserting: 'but mathematics hasn't been created ...¹³² mathematics is something that exists.' Moreover, Foivos did not really seem to differentiate between mathematics and physics when it came to their purpose, which was to help humans to understand how the world around them worked and how they could use this knowledge in their benefit. So, while he was comparing mathematics with physics, he noted that they

don't have much differences; mathematics is the tool to do physics and whatever mathematical [things] humans advance, they advanced them to explain something, or to create something they want.

The problem was that mathematics would not be Foivos' preferred way to understand the world. I returned to the issue of the basis of 'everything' in the second interview exactly because I was intrigued by him perceiving mathematics in such a way despite not liking it. I asked him if there could be anything else which could replace mathematics as the basis of 'everything'. He seemed confused by this question, so I offered as alternatives philosophy or art, and then he endorsed both options wholeheartedly:

Of course philosophy; basically mathematics and arts, which arts, okay, coincide with philosophy ... okay, philosophers knew years ago all these things that we now say [through mathematics or physics], they simply couldn't prove them. ... mathematics is nothing more than an explanation of what philosophers say.

So practically, Foivos gave precedence to philosophy as the basis of 'everything'; this was where the true inspiration lay. Mathematics seemed to come later, only to prove and confirm what philosophers had already known.

I also asked Foivos with respect to his claim that art coincided with philosophy and he explained that

art is a way of expression while philosophy is a way of explaining what happens. Though this is why they coincide ... with art too you're trying to explain something you feel in another way, that is to say, without speech. So I think that this is a form of philosophy too, only a lighter form and usually simpler.

Interestingly enough, he would not express himself in a similar way regarding mathematics which he considered secondary to philosophy. The fact was that he related art to feelings and

¹³² During the pause in this quote, Foivos mentioned that there were some exceptions too. The first time he had claimed that mathematics was discovered he had referred to complex numbers as an exception. Nevertheless, these exceptions would not change his general belief that mathematics existed.

mathematics to logic, and he seemed to be predisposed more to the former than to the latter.¹³³ So when I asked whether he would trust more to his logic or his senses, Foivos confused the senses with emotions,¹³⁴ and initially suggested that he would trust both, but he eventually admitted 'although, I generally happen to trust a lot my emotions.' The importance that he attributed to emotions became particularly evident when we were discussing clarity in mathematics¹³⁵ and in life, and he noted that 'I can't imagine life without emotions, [and as a result] with absolute precision. Many things that we know today wouldn't exist [in that case].' The result of this clash between his preferred way of reasoning and his appreciation for the contribution of mathematics, as he did in his remark about not hating mathematics because it was the basis of 'everything'.

Thus, on the one hand, Foivos found positive meaning in mathematics as a discovery which explained the actual world around him; and on the other, he found negative meaning in mathematics as a way of reasoning which was not to his taste. In particular, Foivos seemed to wish to react to the image of the rigid logic of mathematics because his common sense was not in line with it.

Summary

It seemed that the belief that mathematics existed could help students find positive meaning in mathematics. On the basis of this belief students could claim that mathematics represented something real, and as such it could help humans to comprehend some part of reality. In the case where students located mathematical existence in nature, this aspect would be the world around them. However, it seemed that, in this context, mathematics had more meaning for students whose common sense was in line with the way they perceived mathematics. Otherwise mathematics could lose some of its positive meaning.

¹³³ Foivos actually wanted to study music.

¹³⁴ The two words share a common root in Greek, where 'αισθήσεις' is the word for the senses, and the word '(συν)άισθημα' can be used to refer to emotions. Another student made the same mistake, and then I started explaining that the question regarded vision, hearing and so on. Nevertheless, some interesting comments could result from the students' confusion.

¹³⁵ Foivos seemed to attribute mathematical clarity to the fact that it lacked emotion and was strictly logical. When I asked if such clarity existed in life he replied positively, but remarked that 'we don't see [that clarity]' and when I invited him to explain why, he mentioned that 'there are other factors [apart from logic] involved; basically emotion.'

Invention

Alignment

Students could find meaning in mathematics by perceiving it as a valuable invention of the human mind. All such students were able to understand mathematics, though it seemed that they would speak even more appreciatively of it if their common sense was also strongly in line with some philosophical aspect of mathematics.¹³⁶ In any case, such students could experience a subjective meaning of ownership for mathematics. Seeing it as an invention, and being able to understand it, essentially meant that when they were doing mathematics, they were in a sense partaking in the process of creating mathematics.

Kleomenis

Kleomenis believed that mathematics was invented because he knew of axioms.¹³⁷ Since these were assumptions without proof that humans had made, mathematics had to be an invention. In fact, he attributed to the invention of mathematics qualities that made it attractive for him, such as its usefulness and the determinism of its reasoning, which is usually associated with the belief that mathematics is discovered. In particular, determinism seemed to be a property of mathematics which agreed with Kleomenis' common sense, and he enjoyed it, since it allowed him to be in control without worrying about unexpected surprises; he could trust mathematical results which were not governed by chance. His sense of being in control was also indicative of his understanding of mathematics, while it suggested that for him, mathematics was not something alien, but an activity of the human mind which he could coown together with all inventors of mathematical knowledge.

It was not difficult to realise how important mathematics was for Kleomenis, by the fact that he had taken time to investigate and make some sense¹³⁸ of the nature of axioms and non-

¹³⁶ An example where this is not the case is that of Solonas, under the theme of empiricism.

¹³⁷ This may seem like a special case since most students did not know of axioms. However, the meaning that Kleomenis found in mathematics did not depend on this knowledge but on his belief that mathematics was invented, a belief he shared with many other students.

¹³⁸ He was not only aware of the existence of different axioms for non-Euclidean geometries, but also of the fact that the space and corresponding lines looked different in such geometries. This also implied that he could understand mathematical concepts.

Euclidean geometries, which were not part of the curriculum.¹³⁹ His understanding of axions seemed to be the reason why he suggested that mathematics was invented. He referred to axioms early in the interview in connection to the definition of the word rule:

[rules] are something that we've said that holds ... they don't exist for a specific reason, in the sense, [that although] they're the result of something [sic], [still] we have [introduced them] so that we help ourselves ... in order to delimit certain things.

I was interested in how this definition applied to mathematics and Kleomenis referred to 'those who [Pythagoras] [sic] had defined as such without proving them.' So Kleomenis suggested that the rules of mathematics were its axioms which he seemed to understand as statements, which although might stem organically from the subject that was studied, were also somewhat arbitrary and depended on what humans wished to accomplish through them. Later, when I asked him why we accepted axioms, he maintained that 'if we don't start from something we can't prove anything.' So in essence, Kleomenis suggested that mathematics had to be based on human-made assumptions, and as such, it had to be an invention.

Indeed, when we later discussed invention and discovery, Kleomenis asserted that '[mathematics] is eventually invented, because it's something which we created in order to serve our own conditions.'¹⁴⁰ This also implied that for Kleomenis mathematics was a markedly significant invention. It had not been invented light-heartedly, but to serve humans. In other words, it was a purposeful invention where new knowledg appeared because it served to tackle new problems. As Kleomenis explained when he was trying to delineate the goal of mathematics, this was why non-Euclidean geometries were developed: 'with the intention to solve some problems which the Euclidean, the regular [sic], ordinary geometry could not solve.'

Moreover, as an invention, mathematics presented Kleomenis with a vast creative space in which he could be the master (this was of course possible only because he understood mathematics). This became apparent when I tried to understand whether he saw any sense of obligation in mathematics.¹⁴¹ Kleomenis appeared confused and he stated that 'there isn't something that we must do in mathematics.' Furthermore, he signalled towards his appreciation

¹³⁹ I was surprised by the extent of his knowledge and I endeavoured to understand how much of it could be coming from the classroom, but as he explained, 'we've not gone deep in other geometries [in the class], we've simply mentioned that there are other geometries.' This agreed with my own experience as a student. ¹⁴⁰ See also the section on invention in the ontology chapter.

¹⁴¹ I did so because he had connected the word 'rule' only with axioms and not with mathematics in general.

of freedom and creativity while we were discussing if he preferred order or chaos, and he suggested that 'a life with absolute order [such as that of mathematics] - apart from the fact that it doesn't exist - would be repetitive and boring, I guess.' Nevertheless, he refused the idea that mathematics was boring, claiming that 'mathematics is infinite, you can go on forever saying things, saying things; so you don't have time to repeat.' In the second interview, we returned to this infinity of mathematics, and Kleomenis again commented that 'in mathematics, the [possible] states are infinite, because one time we choose this, another time we choose the other.' So he perceived mathematics as a field of knowledge which presented him with infinite possibilities of advancing this knowledge. In this infinite space Kleomenis was in charge. Mathematical activity was under his control; contrary to life which could be unpleasantly unpredictable, in mathematics, the choices depended on him. Continuing his last statement, he related how

we choose the state in which we are, [the state] which we examine in mathematics, while in life we don't have this option to choose; even if you reject everything, it's always possible that something will happen for which we aren't able to do anything, that is, an option may be demolished.

Thus, Kleomenis spoke of mathematics as something that was intimately his, something which he could master. In a sense, this comment suggested that he could feel a greater degree of ownership of mathematics than of life, which sometimes could proceed against his will.

Kleomenis seemed to associate this sense of control with a deterministic image of mathematics, which he again attributed to its invented status. In fact, since his common sense seemed to be in line with determinism, he was inclined towards understanding not only mathematics, but also the world, deterministically. He assumed that both in life and in mathematics the same conditions would always lead to the same conclusion. While we were discussing how mathematical conclusions emerged, he commented that 'if the previous [statements] hold, then what we reached holds too.' I asked if there could be any exception, and he replied that

there may be, but it has appeared [in the process of inferring the conclusion], that is, if one, two and three hold, then four will necessarily hold too ... as long as there are no exceptions to the initial [statements] then there's no exception to the result.

So he was sure that, in mathematics, certain conditions led to certain results, and any possible exception had already been taken care of in the process of proving the conclusion. After this, I

asked him if he could say that the situation was similar with respect to life in general, and Kleomenis maintained that

in life, I assume yes; but it must - the specific conditions are somewhat more, [and] more complex, that is, it's harder ... there are many things which are needed in order for something to hold always.

When I asked him if it was possible to determine all the necessary conditions in life, Kleomenis admitted that succeeding in it was practically impossible 'because there's always the factor [of] chance.' However, he still seemed to believe that the world functioned in a deterministic way even if humans were not able to distinguish all the relevant factors. Eventually, when I enquired what made life different from mathematics, Kleomenis explained this difference by resorting to the fact that mathematics was an invented theory: 'mathematics is a theory; say, if life was also theory, I can [sic] assume¹⁴² that today I'll do this and this and this ... while in practice, something may happen.' Thus, if life was a theory, the necessary conditions could be postulated, and chance could be excluded, as it happened in mathematics.

In all, Kleomenis made sense of mathematics as an invention. He even used this belief in order to justify traditional traits of mathematics such as its determinism. Of course, finding this positive meaning in mathematics as a significant invention seemed to be possible for him only because he was able to understand it. This allowed him to feel a sense of ownership of mathematics. Moreover, his appreciation of mathematics appeared to be intensified by the fact that his perceived determinism of mathematics was in accord with his common sense.

Misalignment

Students could also perceive mathematics as an incomprehensible invention which they disowned; mathematics was the concern only of those who practised it, and not theirs. These were students who experienced difficulties with understanding mathematics, though the negative meaning could be intensified by a disagreement between the students' common sense and some philosophical aspect of mathematics.¹⁴³ In any case, by presenting mathematics as an invention, such students could justify any negative issue they had with mathematics; if

 $^{^{142}}$ The verb that he used here in Greek has the same root as the word theory ($\theta\epsilon\omega\rho i\alpha$ is the noun, and $\theta\epsilon\omega\rho \dot{\omega}$ is the verb).

¹⁴³ An exception is the case of Kleio, presented under the theme of subjectivity, where the story is given with no reference to invention, which was a secondary theme for Kleio.

mathematics was the product of an alien mind, then there seemed to be no obligation on the student's part to engage with it, or to understand its content and reasoning. In fact, by locating mathematics in other individuals' minds, some of these students could make sense of mathematics as a subjective invention.¹⁴⁴

Ariadni

Ariadni was not able to understand mathematics. This seemed to be the reason why she asserted that mathematics was invented, and she generally presented it as a creation made by some strange group of people with whom she felt she had nothing in common. On the contrary, feeling alien to their invention, she almost considered mathematicians as enemies who conspired to keep her in ignorance. The negative meaning that she attributed to mathematics as an invention seemed to be aggravated by the fact that her common sense was not in line with her perception of mathematics as a set of rigid rules with no empirical basis.

When we were talking about her relationship with mathematics, Ariadni made clear that she could not understand it: 'I can't understand it at all. Either no one was found to teach it properly to me, or I may not have been attentive enough to understand. In any case, I've never understood it.' So naturally she did not like mathematics, because 'whatever you can't understand, you don't like', as she remarked later when I asked her if she knew why she did not like mathematics. This situation was evidently further intensified by the dependency of her common sense on empiricism.¹⁴⁵ Consequently, Ariadni felt alienated from mathematics and she maintained that 'it's something [that is] very much outside of me', a feeling which she explained by referring to the issue of the rigidity of mathematical rules which she judged as unacceptable:

[mathematics] gives you an operation, an equation, and it gives you a rule with which to solve it. ... Why should I solve it using this way and not solve it with another way of my own? This rule ... restricts you a lot.

Earlier, while Ariadni and I were talking about truth, she had exploded, insisting that: 'it could be true, but again it's a rule which somebody has made; that's what I can't [stand]; somebody

¹⁴⁴ An example of a student who stated that mathematics was invented while clearly portraying it as objective is that of Polyxeni, under the theme of certainty. The story, however, is again given with no reference to invention for the same reason as above.

¹⁴⁵ See the section on logic in the epistemology chapter.

has made a rule and you must observe that rule in order to be correct.' Thus, Ariadni was attributing the mathematical rules which she could not tolerate and understand to the invention of some human mind.

In fact, by perceiving mathematics as a '[creation] of the mind',¹⁴⁶ Ariadni was liberating herself from any obligation towards it; she was distancing herself from it for reasons that she could see as perfectly justifiable; she had no obligation to understand something which was the product of an alien mind. So Ariadni supported this view,¹⁴⁷ and she repeatedly distinguished herself from that group of people who had invented mathematics. Essentially, Ariadni was portraying mathematics as a subjective invention; the mathematicians could have their view, and she would keep her own. Thus, while we were discussing whether proofs were useful, and I asked if a proof could be useful to her, she seemed perplexed that I would even ask, and said 'Proof? No, not at all.' However, when after this I asked her if proofs were useful to mathematics, she replied 'well yes, for normal people.' It was surprising to hear her seemingly claim that she was not normal as compared to people who could do mathematics. It seemed that she was willing to accept that she was not as 'normal', if only to separate herself from that group. Of course, when I protested, asking her if she was not indeed normal, she simply stated that 'I don't like mathematics; for those who like it - there are guys who like mathematics ... in order to find if the solution is correct, they need to find the proof.' In the second interview, when we talked again about proofs, Ariadni also commented that 'proofs aren't logical for me; other people found some logic [in them]; they find logic just so, out of nowhere.' Ultimately, when I wondered if mathematicians could know whether their rules were correct or not, Ariadni again suggested that mathematicians were an alien, unfriendly group which did not even care to include others like her in their closed tribe: '[mathematicians] have created [mathematical rules] in such a way that you're simply left in the darkness; they think that [mathematics] is only for them; for us there must be a correct solution through a simple way, they didn't make this.'

In sum, Ariadni found negative meaning in mathematics as an incomprehensible activity of the human mind which felt completely alien to her. At least by making sense of mathematics as a subjective invention, she was able to justify her inability to understand its logic, while she could

¹⁴⁶ Her explanation of 'invention'.

¹⁴⁷ See the section on invention in the ontology chapter.

also stress her disagreement with the image of mathematics that she attributed to mathematicians. This was an image of mathematics as a set of abstract, fixed and logical rules which was not in line with her common sense.

Summary

Depending on whether they understood mathematics or not, students could find positive or negative meaning in it as an invention of the human mind. In the first case, students could feel a sense of ownership. In the second case, seeing mathematics as an invention could justify the students' lack of understanding, while it also allowed them to disown mathematics and disengage from it. In any event, a (mis)alignment between the students' common sense and the way they perceived mathematics could intensify the students' positive (or negative) experience.

Certainty

Alignment

Students who sought or appreciated certainty could find a haven in mathematics. For them the belief that mathematics could provide absolute answers was a blessing. This seemed to be particularly the case for students who exhibited an absolutistic mindset, believing that it was more or less possible to attain absolute truths in life. Such students appeared content to endorse the apparently rigid absolutism of mathematics since this image agreed with their common sense, and they could find positive meaning in it even if they did not understand mathematics.¹⁴⁸ Of course, as long as a student could understand mathematics, they could find meaning in its certainty even without exhibiting a strong absolutistic mindset.¹⁴⁹

Afroditi

Afroditi generally wished to be sure about herself and this was a state which she could effectively attain since her common sense dictated that there was an absolute truth, allowing her to be certain about her values. She was not much engaged with mathematics, but this

¹⁴⁸ Examples of students who understood mathematics are that of Agapi under the theme of common sense and that of Filia under the theme of rules.

¹⁴⁹ Such an example is that of Kleomenis, under the theme of invention.

seemed to be a matter of coincidence. She related that she was never properly exposed to mathematics when she was younger due to what seemed to be an irresponsible teacher. So she lacked the necessary basis which would allow her to get involved with mathematics later on. In any case, she still appreciated mathematics, and this appeared to be because she found meaning in it as a field of knowledge with the power to produce absolute truths, as the ones which her common sense used to guide her life.

While we were discussing truth, Afroditi declared that 'I think truth is one, one, first [sic], invariant, unique.'¹⁵⁰ We returned to this in the second interview, and I asked her whether she believed that this truth could be found by humans. She replied that

There are some people who support truth, what is truth; some others don't. That is, it's complicated, but truth is to be found everywhere ... for example, a scientist who serves his purpose, that is, to put science in the service of human[ity], who's not influenced by political interests and factors, this human [being] is essentially serving the truth.

So Afroditi suggested that knowing the truth was possible. Moreover, considering the way she spoke about professional scientists, living according to this truth seemed to be important to her. Mathematicians belonged to this group, as her response to my question regarding the attainability of truth within mathematics indicated: 'since we're proving everything, in mathematics everything has a proof, it leads to something true and certain.' Furthermore, when I asked if it would be a problem to not have a proof for a mathematical claim, she stressed the significance of being able to be sure of one's claims: 'I simply think that proof is important ... apart from making you sure if something is correct, basically this, in order to make sure if something is correct.'

Afroditi recognised that many issues were subjective, such as the colour of the sea,¹⁵¹ but she was positive there were also facts that could be trusted. In fact, it appeared that even in issues that were subjective, Afroditi felt the need to have some certain conclusions. So when I asked her if she liked subjectivity or it would be better to know what was right and wrong, she explained that

¹⁵⁰ At this point the conversation was sidetracked because Afroditi mentioned how this was the opinion of Socrates, so I took the opportunity to introduce philosophy into the conversation in order to examine whether she had any knowledge of axioms. I would parallel mathematical systems with philosophical theories and ask whether either of them could be based on different assumptions. This allowed me to talk about axiom-like statements even with students who knew nothing about them.

¹⁵¹ I draw on such examples in order to determine the extent to which students believed, or not, in a unique truth.

I can't [stand it] ... when I support something, and [somebody else] supports something [different], and [I don't manage] to conclude something in the end. This is the worst. I want to ... have a focus in my mind: that this is the right [answer].

This seemed to be the reason why - despite admitting that morality was generally subjective - she still asserted that humans deep down know what is morally correct. Thus, after her previous comment, Afroditi asserted that 'more or less, deep down, all of us know what is correct'; while earlier when I had enquired if morality was objective or subjective, she had observed that

the moral law is subjective for each one, ... that is, each one's measure is defined as one thinks [better]; that is, the measure is not common for all, [the measure] that each one uses for one's morality.

So it seemed that Afroditi sought certainty. This attitude of hers explained why, when I enquired if it would be positive for life to be as certain as mathematics, she answered positively, stating that 'yes, [it'd be] better. Because then we'd have more secure conclusions; we'd know the truth.' So effectively the alignment between Afroditi's common sense and mathematical certainty led her to appreciate it.

It can be assumed that Afroditi could have been a keen fan of mathematics, but the circumstances had not allowed her to acquire a solid mathematical basis. While we were discussing her relationship with mathematics, she explained that

in gymnasio (lower secondary school) ... when we started doing the first important (difficult) mathematics ... we had a teacher ... [who] simply spoke [about irrelevant stuff] during the mathematics lesson ... and probably because of it ... I couldn't manage at the beginning of lykeio (upper secondary school).

So Afroditi was not really in a position to know whether she liked mathematics. However, it seemed that it was easier for her to assume that 'I've never liked mathematics', as that would mean that she had not lost the opportunity to engage with something that could be important for her. She remarked that 'other children ... who had the basis from gymnasio, they might like [mathematics], since they [could] master it.' The fact was that not being able to understand it, she did not like mathematics.

Despite this, Afroditi did hold mathematical knowledge in high esteem. While we were discussing whether mathematics was useful, she claimed that mathematics

is more useful than all the other knowledge we [may] acquire. Through mathematics we measure many things; we measure time, we measure things, we measure length, we measure all things in our everyday life.

It seemed that similar measurements could provide evidence for the truth that she was seeking on several occasions. After all, Afroditi also referred to a sense of measure while she was talking about morality (see above).

In all, Afroditi could find positive meaning in mathematical certainty which agreed with her common-sense understanding of truth. Thus, she was content to internalise an image of mathematics as comprising certain and absolute truths. However, her lack of understanding unavoidably mitigated this positive meaning.

Misalignment

Students could appreciate the certainty which mathematics appeared to offer, even if this image was not exactly in accordance with their common sense. This could be the case if they felt that the rigid, clear-cut rules of mathematics could safeguard them against mistakes. However, such a need to avoid mistakes appeared to be the result of a lack of confidence associated with the fact that such students could not understand mathematics well.¹⁵² This double misalignment between their common sense and mathematics (at the level of content and at a philosophical level concerning certain, absolute truths) resulted in a significant reduction of any positive meaning such students could find in mathematical certainty. In fact, such students appeared to wish to react to the impression they had formed of mathematics as a set of certain, absolute truths.

Polyxeni

The meaning attribute to mathematics by Polyxeni had been shaped by her fear of mistakes. In fact, she wished to be certain that she was correct before she spoke in general, and not only regarding mathematics. As a result, she felt comfortable with perceiving mathematics as a rigid, objective discipline, even though she was not very good at it, and despite the fact that this objectivity was not in line with what her common sense would evaluate as preferable. Still, this

¹⁵² In fact, a lack of confidence could be relevant even for students who more or less understood mathematics. However, such students would still get sufficient opportunities to solve mathematical problems successfully, and thus could enjoy a greater degree of mathematical certainty.

misalignment was apparent in the fact that she seemed to be mostly at ease with totally subjective contexts, such as art, where there was no correct answer, and hence opinions would not be judged at all.

Polyxeni's wish to avoid mistakes was apparent in the interview. She felt quite uncomfortable when I asked her a question for which she did not know the answer, and it required much time before she relaxed. With respect to mathematics, the effect of her fear was that she had come to value the strict, rigid framework that mathematical rules usually offered. While we were comparing the freedom to have one's opinion while doing mathematics and literature, Polyxeni explained that essentially in both subjects she had to make sure that her opinion was in line with what she had been taught, or there would be consequences: 'as in mathematics, in literature, you may have an opinion, but this may not be correct.' I asked in what sense it would be not correct and Polyxeni mumbled that 'it may not be to the liking ...' presumably of her teachers, and added that 'this can really cause you a [problem] regarding grades and the like; while in mathematical rules could safeguard her against cases where she would not know what her teachers would regard as correct. I enquired if this was positive or negative, and she replied that 'it's good for me.' So it seemed that she did not mind that she could not have her own opinion in mathematics, and when I asked if this was indeed the case, she agreed with me.

Unfortunately, at least within mathematics, Polyxeni's need for certainty appeared to be chiefly the result of distressing past experiences. I asked her when her relationship with mathematics became negative, and she related how she had started distancing herself from mathematics since she entered gymnasio (lower secondary school), 'where we didn't have too good a teacher.' I enquired what made that teacher problematic and Polyxeni found it hard to explain, but she maintained that 'it's simply the way he taught, personally he made me feel stressed; even if I knew [something], I couldn't perform.' I asked her if her relationship with mathematics had also been influenced by the fact that it was potentially harder when she entered gymnasio, or if it had been mostly an issue of having the wrong teacher. This was one of the few times that she was certain that 'definitely [mathematics] becomes more difficult in gymnasio, but it was mostly an issue of the teacher.' Later, when I asked her why she would be stressed about mathematics in the past and why she did not just ignore it, as she was doing presently, she again revealed her need to not make mistakes but to be a good student who could

solve mathematics correctly: 'I wanted to be good at mathematics, maybe in order to get a good grade; because I wanted to be able to solve [the problems].'

However, Polyxeni's fear seemed to interfere with her preferences, and her need for certainty did not seem to be so much related to her common sense judging certainty as an ideal. Sensing her fear, I finally asked her what she would prefer if there were no danger of mistakes involved: freedom or having rules as in mathematics, subjectivity or the certainty of objectivity. She admitted that if it were not for mistakes, she would prefer to be free instead of having rules, and she would rather handle subjective than objective situations. So, her appreciation of rigid rules was ultimately against her inner preferences.

This seemed to be the result of the fact that Polyxeni's common sense agreed that there was a unique truth, while it indicated that humans were not able to find it 'because everyone grasps [the truth] in a different way, so one can't be sure that this is it.' While I was trying to investigate if this was a problem, it became apparent that Polyxeni did not consider this negative for subjective fields like the arts: '[arts are] a very subjective sort [of human activity], that is, arts in general are very subjective, and everyone may believe what they [wish].' Interestingly enough, while she was uttering that sentence she seemed much less stressed than she generally was. Nevertheless, when I wished to understand if she liked that subjectivity: 'it depends on the case; in [some cases] subjectivity is needed, in other [cases] objectivity [is needed.]' I asked what the case for mathematics was and Polyxeni answered 'objectivity.' It seemed that in such cases, Polyxeni could not relax and interpret the truth subjectively as she could do in arts. She had to interpret it according to objective guidelines, and she was grateful if at least those guidelines were well-defined as in mathematics.

Indeed, it seemed that the rigid rules of mathematics had allowed Polyxeni - at least through her initial efforts - to cope with her stress and occasionally feel some degree of confidence in her results. At one point, trying to understand how comfortable she felt with the absolute rules of mathematics, I asked her if she felt controlled by mathematics or she felt in control of it. She related how both could be the case; the former when one could follow the rules correctly, the latter when one made a mistake:

you control [mathematics] when you're solving an exercise and you know what you have to do, but it controls you, that is, if you make a mistake at some point in the exercise then you may get the whole exercise wrong.

Still, it seemed likely that Polyxeni felt that the occasions where she was controlled by mathematics were much more frequent than the occasions where she was in control, and they were not sufficient to counterbalance her negative experiences.

Thus, although Polyxeni could find meaning in mathematical certainty, this was not a particularly positive one. This seemed to be both because she did not understand mathematics well and because her common sense did not judge mathematical certainty as an ideal. This ideal was externally imposed on her in fields such as mathematics which were supposed to be objective. As a result, she felt that in such fields, having rigid, clear-cut rules could help her make sense of what was expected from her and avoid mistakes. So Polyxeni had accepted a picture of mathematics as objective and absolute, but she appeared to wish to react to it and set herself free in a subjective space where mistakes would be absent.

Summary

Some students found meaning in mathematics by endorsing an image of mathematics as certain and objective. In particular, an alignment between the students' common sense and mathematics as a field which revealed absolute truths would lead students to find positive meaning in mathematics even if they did not understand it. Nevertheless, generally a positive image of mathematical certainty presupposed that students were able to reach that certainty, that is, that they could understand mathematics and use its rules correctly. If this was not the case, then the meaning that students attributed to mathematical certainty was not particularly positive. In fact, such students did not seem to value certainty; their common sense would judge as preferable subjective environments where there was no need to be certain or correct. Still, in that case students could find meaning in mathematical certainty because it would at least help them to avoid mistakes in the objective context of mathematics. This, however, implied that such students essentially had troubles internalising an image of mathematics as objective and certain.

Subjectivity

Alignment¹⁵³

There were students whose common sense dictated a relativistic mindset, believing that nothing was unambiguously right or wrong, and that contradictory statements could be equally well supported. When such students could sustain a subjective image of mathematics - which was thus aligned with their common sense - they were able to make sense of it since it would fit their general understanding of other aspects of the world. Such a subjective image of mathematics such as open problems (Lida). However, lack of experience with mathematics altogether could have the same effect. Not being able to understand mathematics, some students could resort to their common sense to decide whether it was reasonable for it to be considered as subjective or not (Kleio). However, in this case, the positive impact of this perceived alignment between their common sense and mathematics seemed to be negligible.

Lida

Lida could make sense of mathematics because her common sense was in line with her perceived subjectivity of mathematics, and thus she could see in it the same subjectivity that she found in life. It seemed that somehow she had encountered in the past open problems which admitted more than one correct solution. This picture, being in accord with her understanding of the world, allowed her to connect mathematics to life, and thus find positive meaning in it. Consequently, her image of mathematics as subjective was reinforced, and she would mostly disregard data which did not fit with it.

Lida claimed that in mathematics, as in life, we do not obtain specific answers. We were discussing examples of empirical and moral truths, when I asked her which case would mathematics resemble more and she declared that 'in mathematics there's both black and white, and sometimes [even] grey. That's why mathematics [may] enter life too, because there can be both black and white and grey.' With respect to mathematics, she had given an example, at the point where I had asked whether mathematical problems had specific solutions, and she was

¹⁵³ This is the only subtheme discussed here since the theme of misalignment has been effectively covered under the theme of invention (see the case of Ariadni).
trying to explain this was not necessarily the case. Her example was crude, but it seemed to allude to problems which could have more than one solution: 'if they give you some euros and you must buy ... a quantity, this may be a quantity of 7.5 [units], or a quantity of eight [units], or even a quantity of only seven [units].' It appeared to be the case that the buyer in the problem should make some decisions and that the answer depended on these decisions.

After the example, she continued to relate cases where the whole classroom had reasoned differently from the book, produced a different answer, and both answers were correct:

there's much mathematics that we've done in the classroom, [and] we've been given the solution [by the book], and one solution we have found [ourselves] ... and another solution is [to be found] in the answer [key].

I asked which solution was correct, and she replied, 'they both work fine.' Soon I found the opportunity to ask again about the possibility of different results, since Lida was the only student who had expressed herself in that way. She had just suggested that each exercise required a specific set of operations in order to be solved correctly, and I asked whether these operations would lead to the same result. However, Lida explained that as long as the correct set of operations was performed, the result would be correct independently of anything else: 'independently of whether it'll be the specific one that [the book] requires or something similar, it'll also be correct, as we said before that there are different solutions to some problem.'

This picture did not correspond with her current experiences, but it was one Lida had kept vividly in her mind, probably because it was the one that made most sense to her. When I asked her whether there were mathematical statements without proofs, she recalled how

in gymnasio (lower secondary school), we encountered many exercises which wouldn't be proven [sic]¹⁵⁴ because we had a good mathematics teacher who knew to give us both exercises which were proven and others which were not proven so that we could see the difference. Here we don't have [such] a mathematician, but this is another issue.

¹⁵⁴ Lida was not aware of the terminology 'open problem'. So at this point she referred to such problems as 'exercises which wouldn't be proven' which seemed to describe problems for which the answer cannot be proven as the only correct one. In a sense, lack of a proof implied that many different answers could be equally valid. That would be what happens in life when different people support different opinions and none of them can be proven more correct than the other.

During the interview, Lida seemed to have forgotten the latter type of problem and spoke of mathematics as if it consisted only of problems of the former type. This was the kind of problem that brought mathematics close to her and helped her to find positive meaning in it.

This also appeared to be the type of problem which made mathematics true for Lida. While she compared truth in mathematics and in physics, she explained how she trusted mathematics to be closer to truth than physics, precisely because she knew that mathematical problems could have multiple answers, while she was not aware of such problems in physics. She seemed much readier to accept as true answers those which she had worked herself, rather than answers given by some external authority such as the book:

in mathematics, you do the operations, you do several things, that is, you investigate the issue; while in physics, you do the operations, but you only find the result that the book would have [written] on [it] ... you can't find some other result on your own.

So Lida did not need the certainty of specific answers; she only needed answers which she could feel as hers, and mathematics could provide this for her because of its subjective characteristics.

In sum, Lida could find positive meaning in mathematics as a subjective field of knowledge. This was because this image of mathematics was in line with her common sense and it allowed her to connect mathematics to life. Consequently, she ignored, or reacted to, any experiences which portrayed mathematics as objective.

Kleio

Kleio was disengaged from mathematics. In the past, strong negative experiences had led her to withdraw from any mathematical activity, and it seemed that this had happened before she had had the chance to form an opinion of what mathematics was. Therefore, in a sense, she had to make sense of mathematics through subjectivity. This was because the interview concerned her beliefs on mathematics, and so she tried to adopt some perspective. The readily available solution was to use the perspective dictated by her common sense, the one from which she looked at life in general. This was a highly relativistic perspective, according to which life was, more or less, a matter of subjective opinions.

Kleio had been hurt as a mathematics student. As a result, her reaction to mathematics was very negative. She did not limit herself to stating that she did not like mathematics, she actually said

'I hate it', even before the interview had begun.¹⁵⁵ She could recognise that this was the result of her teachers' behaviour and not something for which the nature of mathematics was responsible. She could not even find a specific quality of mathematics that made her hate it. When I asked her about this she simply answered 'I don't know ... I simply know that I don't like mathematics. It's not that I can't understand mathematics. It's that they have made me hate mathematics.' Hence, when I asked her if she was able to express her personal opinion in the context of mathematics, she simply maintained that 'I didn't have an opinion [about mathematics], and I [still] don't have, [and] I don't want to have [one].' Moreover, often she answered questions about mathematics simply stating 'I don't know.'

Whenever Kleio felt comfortable enough to share more, she tended to see mathematics through her general common-sense worldview. This consisted in an extremely subjective attitude towards almost anything. While we were discussing what could be termed as truth or falsehood, she suggested that almost everything could be refuted. She related the story of 'a philosopher who [during the morning] ... held a bat in his hand and claimed that now it's night since the bat is here.' She concluded her story by maintaining that 'there's no objectivity on this issue.' I asked her if the philosopher's claim was reasonable, and she replied negatively, but she still added that 'he had arguments though.' So although she would recognise that the philosopher's claim was not entirely sound, she would acknowledge to him, and presumably to anyone, the right to their claim as long as they had arguments to support it.

Kleio handled mathematics in the same fashion. She might not have an opinion about it, but the mathematicians would have one, and they would have arguments for it, so she could not challenge them. Again while we were discussing subjectivity and opinions, I enquired whether she could have an opinion about the pythagorean theorem, she replied 'people should ask Pythagoras; [he can] say his opinion.' She refused to have her own opinion and she paralleled this with an inability to oppose the great philosophers: 'essentially, it's like me saying something regarding a view of Socrates.' I wondered if she indeed could not disagree with Socrates and she admitted that she could, but she did not believe that she would be able to support an opposing view with arguments so she considered any disagreement futile: 'I could [claim something opposite], but I wouldn't be able to prove it easily, as Socrates who had proven what he was saying.' When I returned to Pythagoras, it became clear that she could not

¹⁵⁵ She may have been checking that this was alright with me and it would not disqualify her as an interviewee.

perceive any difference between challenging a view of Socrates and a view of Pythagoras: 'that's what I mean, that I wouldn't be able to [oppose] Pythagoras either.' I asked her to confirm if she really believed that it would be equally difficult in both cases and she replied positively. She continued to remark that 'that's why the pythagorean theorem has not been demolished, because no one has emerged to say the opposite of [what] Pythagoras [said].' At that point I wondered if that was the case with Socrates too, and Kleio commented that 'with Socrates, they simply didn't agree. The opposite? They simply don't agree.' So she did not really believe that other philosophers had invalidated Socrates' view, they simply had a different opinion from him. When I enquired if anyone had disagreed with Pythagoras, she simply answered 'I don't know. I haven't looked into it, while about Socrates I know.' Thus ultimately, lacking any experience, Kleio had no way to distinguish between subjectivity in mathematics and in philosophy.

Similarly with that argument of the philosopher who was holding the bat, Kleio appeared to accept mathematics, although without endorsing it. She did not even have to endorse it, since for her a proof did not indicate that something could not be challenged. While we were discussing what the verb 'to prove' meant, and I wondered if something proven could be challenged, she replied positively. When I asked about mathematical statements in particular, she maintained 'yes, can't they be challenged? Yes they may', and after she thought about it somewhat more, she added 'they [can] definitely be challenged, [whether] they [can] be proven? It could be that they can't.' Thus it was apparent that Kleio interpreted proofs in accordance with her subjective view of knowledge.

The fact was that Kleio had already been alienated from mathematics, and due to her strong negative feelings, she refused to attempt to make any sense of it. Subsequently, if she had to make sense of mathematics, then she had to resort to her common sense. This provided her with a subjective outlook on life, which Kleio would also apply to mathematics. Nevertheless, this perceived alignment between her common sense and mathematics as subjective was not sufficient to counteract the negative meaning she attributed to mathematics as a subject with which she had difficulties.

Summary

Perceiving mathematics in subjective terms could be in accord with the students' common sense. In that case, students could make sense of mathematics as a subjective field of

knowledge. This could occur if the students had some subjective experiences with mathematics; or simply because - due to their lack of understanding - students had not had sufficient experiences with mathematics and applied to it the general perspective which they applied to life. It seemed, however, that only in the first case, seeing mathematics as subjective could help students find some positive meaning in it.

Rules

Alignment

Students could find meaning in mathematics as a valuable set of rules. This would be either because their common-sense understanding of rules was in line with their impression of mathematical rules and/or because they could comprehend and use mathematical rules. Simply being able to understand mathematical rules meant that students could find positive meaning in mathematics by interpreting the learning of mathematics as a creative process of playing with mathematical rules. Such students could suggest that mathematical rules were not really restrictive and rigid.¹⁵⁶ On rare occasions, such an experience could be intensified if the student's common sense was in line with a scientific attitude which would comprehend rules as hypotheses to be tested (Ermis).¹⁵⁷ However, in some cases, the students' common sense suggested that rigid rules provided useful guidelines, a necessary prerequisite for guiding one's actions both in mathematics and in life. Such students usually understood mathematical rules too,¹⁵⁸ and were content to internalise a view which presented mathematics as a system of clearly defined rules, since they could find positive meaning in it (Filia).

Ermis

Ermis' sense of ease within mathematical rules stemmed from his general tendency to challenge anything. In particular, his reaction to mathematics seemed to be related to his common sense

¹⁵⁶Many of these students would not agree that the term 'rule' was the most appropriate for the guidelines offered by mathematics. For another example, see the case of Kleomenis under the theme of invention. The remainder of the examples pertaining to rules refer to cases where the student was content to use this term within the context of mathematics.

¹⁵⁷ All other examples which pertain to rules (such as that of Kleomenis under the theme of invention) did not exhibit this scientific attitude.

¹⁵⁸ An example of a student who did not understand mathematics can be found in the case of Afroditi, under the theme of certainty.

which reflected a scientific attitude, according to which he would not take anything for granted, but would investigate everything, always keeping in mind that he could not know everything, and there would always be more to discover. This was also how he perceived mathematics, as a creative space of learning, full of 'rules' waiting to be tested. Of course, he would not have been able to delve into such an investigation if he did not understand mathematics.

It seemed that, for Ermis, nothing was certain. He was always ready to consider possibilities which could overturn the current reality. That is why he was not entirely comfortable with the idea that mathematics had rules. So when I asked him if he could say that there is a sense of 'must' in mathematics, he was not exactly positive. He stated

what can I say to you now? I can't answer to you this now. I can't consider it, because, to speak personally, I'm thinking about mathematics, and [somebody] shows me something, that this must be happening, I'll consider it, I'll analyse it; and maybe not me, somebody else may see that this which [supposedly] must [occur] either it mustn't [or] the opposite may occur [too].

So he was not ready to accept something as a de facto rule, simply because some authority would claim so. He would wish to analyse this claim, and even if he could not refute it, he would still keep in mind that somebody else might be able to refute it. That was why he would always try to (dis)confirm a given theorem, using certain 'cases'. Thus, when I enquired if there were exceptions to mathematical rules, he noted that

if you give me a rule and you tell me that it certainly holds, personally, I'll think, I'll try to think of cases where it doesn't hold, where it may not hold. If I find one, I'll be jumping around with joy.

Of course, he had not really ever managed to negate a mathematical statement; however, what mattered to him was that he could not exclude the possibility, and that he did not judge it as right to exclude the possibility. As he explained, when I wondered if 'not correct' was the same as wrong,

it's too specific, that is, too superficial to assume this, that if something isn't correct then it's wrong. I'd [consider] this superficiality, that is, I'd need to investigate it too, I won't leave everything to chance like this.

This attitude of Ermis appeared to be based on the fact that his common sense had prepared him to anticipate the 'unexpected'. When I enquired if he could check what was wrong and right in life, he simply replied that 'I think that in life, all the possibilities may be there ... I

don't think [life] is unexpected, but I accept that everything may happen.' While he was elaborating on this, he explained that 'I may be run over by a car ... [it's part of] the possibilities that exist when I [walk] in the street'. So it seemed that for him, having a mathematical claim disproved would be part of the programme, as would be being hit by a car.

Ermis' tendency to consider all the possibilities became even more apparent when he explained that he enjoyed being precise. Speaking generally, without carefully stating the conditions under which a statement was true, simply did not feel right to him. That was why, when he was faced with the sentences 'if I press the switch the light goes on' and 'if I press the switch, and there is no blackout, the light goes on', Ermis not only suggested that the second option was more complete, but, contrary to all other students, he replied positively when I asked if it would be positive to talk so precisely in everyday life. His misfortune was that he was aware that

the others would not listen; they aren't not interested. And I've experienced it. For instance, I was with my friends and we were discussing. So one moment I was talking and I was saying: "This [thing] may not be so, and the other may [hold]." Okay, some [friends] were from the humanities track and they ignored me, some were from the technology track and they weren't interested. But there were some of my friends who would discuss this with me ... [to others] it seems too extreme.

I wondered how it seemed to him, and he responded 'normal; that is, what I've said: that life is boring if you accept everything [as it's given]. You must put some imagination too; you must think why this is that way and not the other.' Thus, according to his common sense, he felt impelled to investigate because otherwise life would be dull.

In all, Ermis' common sense seemed to dictate a way of reasoning with rules that was in accord with his view of mathematics and science. This view allowed him to perceive mathematical rules as statements which could be challenged. In turn, this allowed him to find positive meaning in mathematics as a field open to exploration.

Filia

Filia's common sense was in line with her perceived authority of mathematical rules, and rules in general. She described rules as omnipotent and almost inviolable, but she did not appear to be at all oppressed by them. This could be explained by the fact that she perceived rules as necessary guidelines in order to live one's life. As a result, she had no issue with the rigidity of mathematical rules. She actually seemed to welcome it, and considered it as an ideal. The perfect state for her would be if rules in life where as inescapable as those of mathematics and everyone obeyed social rules as scrupulously as they obeyed mathematical rules.

When I enquired what the word 'rule' meant, Filia declared that 'a rule is something inviolable, that we must follow whether we want it or not; we can't violate it, that is, that rule must hold always.' I asked her for an example, and it became apparent that her definition included rules of social behaviour such as 'coming decently dressed to school.' According to her common sense, rules appeared to be omnipotent. Hence she maintained that rules were necessary both for mathematics and for life. In particular for mathematical, she presented rules as absolutely necessary and significant. I wondered if mathematical rules were also inviolable, and she replied positively, explaining 'because otherwise, you won't find the solution that you desire;' and when I wondered if this had to be a problem she confirmed it, stating that 'yes, because you'll have wrong impressions about something.' I asked again why this would be negative, and she insisted again that 'you won't be right.' Regarding life, and the classroom in particular, the importance she attributed to adhering to rules became evident when I asked her if she stood to gain anything by following them. Filia asserted that

in the classroom, it's obvious that you gain [if you follow the rules], because afterwards when you'll enter society - since school is a small, a closed society - you'll be able to behave properly to the other people around you.

Similarly, when I enquired if it could be advantageous to challenge rules sometimes, Filia doubted this: 'it isn't so good to challenge a rule; you must follow them; that is the reason why rules have been made ... so that you don't violate them.'

In fact, when Filia compared school rules and mathematical rules, she offered as a similarity the fact that 'they cannot be violated.' She realised, of course, that somebody could violate a rule, but somehow she interpreted 'should not be violated' as 'cannot be violated', as if she believed that the undesirable consequences that would follow from breaching a rule should be enough to guarantee that the rule was in effect inviolable. I could not restrain myself from commenting that it seemed that rules were violated occasionally, and considering what went on in her school, she had to agree. Clearly though, she did not consider this as an ideal and she observed that this was the case only because the repercussions of breaching a rule were not enforced:

yes, they're violated, regarding them being violated, they're violated, because essentially the repercussions, which I mentioned earlier, are not implemented. So every student thinks that they have the right to violate the rules.

I wondered if something similar could apply to mathematics, but Filia replied negatively. Moreover, later, when I asked her whether a rule of conduct could contradict reality, she initially replied negatively, as if the rule dictated the reality. Only when I reversed the question and asked if reality could be in contrast with the rule she noted 'yes, as we said before about appearance, okay, if somebody is dressed differently, this is a reality.' Then the conversation switched to mathematics, and at some point I again suggested that it might be possible to not follow mathematical rules too, but Filia again claimed that 'no, you follow them.' She also found it hard to imagine that someone could produce something new by challenging mathematical rules. When I asked if it would be negative to defy mathematical rules even for a mathematician, she responded 'yes, because they wouldn't be able to prove something, if they had been asked to prove something through mathematics.'

Nevertheless, Filia did not appear to be oppressed by this state of affairs. On the contrary, it seemed that her common sense suggested that such a state was natural, and still more, positive. After all, when I asked her if rules of conduct could be characterised as right or wrong, she claimed that 'they must be correct.' Similarly when I enquired about truth and fairness, she remarked respectively that 'you can't assume a rule as false because then it wouldn't be a rule' and that 'to consider a rule unfair [would be] because it isn't favourable for us.' Furthermore, while she was commenting on her belief that rules of conduct should be observed to the letter, she stressed that those who did not follow the rules of society '[show] egoism, and they don't care about what people around them do, [but] they're only interested in themselves.' With respect to mathematical rules, she also believed that 'you can't say that a rule is wrong ... I think mathematical rules are always correct'

Thus, Filia conceived rules as truths, and consequently she could trust them and use them as guidelines without worrying herself about what would be the right thing to do. In fact, she believed that life resembled mathematics. So when, after comparing mathematics and literature with respect to freedom of having one's own opinion, I wondered whether life more closely resembled mathematics or literature, Filia declared that '[life is similar] with mathematics, since there are rules.' Moreover, when I asked if this was beneficial, she replied positively, suggesting that otherwise, without following rules, life would be chaotic: 'yes, because if we

don't follow logic, anyone would do whatever they wished, indeed.' Overall, her conception of rules was in line with her general view of truth as absolute: As she commented when I asked her if she believed in a unique truth, '[something] either is [true] or it isn't.'

In sum, Filia had willingly endorsed, and found positive meaning in, an image of mathematics as a rigid set of rules. That was because she perceived rules both within and without mathematics as useful. In other words, her common sense was in line with the way she perceived mathematical rules.

Misalignment

Students could also find negative meaning in mathematics as a set of oppressive rules. All such students had some difficulties with understanding mathematics, while some of them also would not appreciate functioning within a rigid system of rules.¹⁵⁹ This did not necessarily mean that they did not believe that rigid rules were a part of life,¹⁶⁰ but their common sense did not judge this as the preferred state of affairs. In other words, this perceived convergence between common sense and their understanding of mathematical rules actually regarded what they perceived as the common sense of society, not their own common sense.¹⁶¹ Consequently, it would not change their negative view of mathematics, although it might help them to make sense of it as a set of rules which agreed with the way society functioned. Effectively all students under this particular subtheme wished to react - more or less - to an image of mathematics as a strict set of rules.

Filippos

Filippos studied mathematics only because he was forced to do so, because he could not always behave as he wished, and he had to conform with society's demands. Not surprisingly, he did not seem to be very content with this. He generally appeared to not like following strict rules. Although his common sense indicated that this was required in society and that this was how mathematical logic related to life, it also indicated that this was not his preferred way of being.

¹⁵⁹ An example of a student who did not appear to object reasoning with rigid rules is that of Evyenia, under the theme of empiricism.

¹⁶⁰ An example of a student who did not seem to connect the rigidity of mathematics to life is that of Ariadni, presented under the theme of invention.

¹⁶¹ Unfortunately, some of these students seemed to believe that as long as their common sense disagreed with that of the society it was of no consequence.

Nevertheless, there did not seem to be much that he could do either about this or about learning mathematics.

While he was trying to explain that there was one logic which applied to all, Filippos observed that people could have their own opinions, but ultimately they had to operate in accord with the common sense dictated by the society, even if they did not agree with its decrees. So he commented that

my logic [says] that I don't want to do mathematics in school; it's useless for me, beyond [learning that] 1 and 1 make 2; but beyond this, there's a logic [which says] that I must do [mathematics], because okay, I can't be the only one exempted from the whole classroom, there are [students] who may want to engage with mathematics.

I asked him if his own logic was unreasonable, and of course, he denied this; he would not think that he was wrong because his common sense disagreed with what society prescribed, but his common sense again also made him note that 'it isn't possible that I do only what I want.' He returned to the same example when he wished to explain that one's logic could be deceptive precisely because it conflicted with the logic of the society, and he once again commented that the way the school was organised meant that he could not avoid doing mathematics: 'it has become a social stereotype that all [students] must learn whatever [mathematics] we do until the [end of school], and we can't simply know [the basics].' I wondered if such stereotypes had to be correct and he replied negatively, but only 'on a personal level, because almost the whole society accepts them.' Thus he could judge this stereotype as wrong, but on a societal level it was right since most people agreed with it. Hence, when I asked why students should follow mathematical rules if they did not understand them, he replied that 'I also have this question, because I don't understand [mathematics], but okay, until you finish [school] it must be in your life.'

After all, Filippos did not suggest that mathematics was completely unrelated with life. He saw some connection in the fact that certain areas in life were as organised by mathematical principles. However, he was not pleased with this state of affairs. So when I asked him if the school provided a reason to learn mathematics, he remarked that - even though he would rather wish otherwise - the logic of mathematics was applicable to life too, and thus he had a reason to learn it.

they give you a reason, in the sense that okay, everything in life is mathematics; no matter how much I don't want to believe it, it holds; mathematics has a [kind of] logic which from time to time you must follow independently [of whether you like it].

I invited him to elaborate on this, and he explained that

the logic that [we] have to follow from time to time is compatible with mathematics ... when you're at your job and you have to do a task, you follow [some] things from the beginning to the end in in a sequence, the correct sequence, as you do in mathematics.

Still, his common sense did not seem to be at ease with this aspect of the general common sense which required him to follow strict rules. It did not really seem that he was able to internalise the utility of mathematics, but the fact that it existed appeared to justify why he should bear it.

However, Filippos attributed to mathematics and its rules an absolute authority, even stricter than society's authority. Regarding the common sense of society, he was free to differ. As he observed when we returned to the issue of a unique logic in the second interview, he was allowed his own subjective opinion: 'you may have your own [logic]; essentially this is your personal issue ... If in my mind [I consider something] as correct ... no one is going to change my opinion.' On the contrary, regarding the logic of mathematics, not only he, but also society as a whole, had to accept it; mathematical objectivity was imposed on all even if on a subjective level they wished to disagree.¹⁶² When I asked if the logic of mathematical logic] cannot be fought, because it's mathematics, no one doubts it.' I asked if that meant that mathematical logic was correct, but he suggested that this was not an issue of correctness; it was simply that it was not possible to challenge mathematics: '[people] will say [that the logic of mathematics] is wrong, but no one will challenge it.' So for Filippos mathematics and its rules presented him with an inescapable, oppressive authority.

In all, Filippos would recognise that the mathematical way of reasoning was in accord with the general common sense of society, but it was not similarly in accord with his own common sense. As a result, although he could make some sense of mathematics as related to life, he still retained a negative meaning of mathematics as a source of oppression sanctioned by society.

¹⁶² See Filippos' remarks in the section on objectivity in the ontology chapter.

Summary

Provided that they understood mathematics, students could find positive meaning in it as a space for creative exploration and investigation. This experience could be amplified if the student's common sense suggested that mathematical rules were hypotheses to be tested. On the other hand, students whose common sense dictated that rules were to be valued and trusted as correct guiding principles could also find positive meaning in mathematics by endorsing a picture of mathematics as a set of rigid rules. However, students who could not understand mathematics well would attribute negative meaning to it and characterise its rules as oppressive. These students essentially felt the need to react to an image of mathematics as a set of rigid rules. They might recognise that following such rules was part of society's common sense, but still, according to their own common sense, this was not a preferred way of reasoning. So this recognition would not affect their negative image of mathematics.

Empiricism¹⁶³

Partial misalignment

Some students were simply perplexed by specific parts of mathematics which seemed to oppose their experiences. In this case, the perceived misalignment between mathematical reasoning and the student's empirical common sense was relatively small, and therefore it could be relatively easily ignored. In fact, such students understood most of the mathematics which they had encountered, and the reasoning behind it. As a result, this would mostly counteract any negative impact of the few cases which puzzled them, even though they would still make sense of mathematics as something partially mysterious.

Solonas

Solonas generally understood mathematics and its reasoning well, and he was comfortable with most - if not all - of the mathematics that he had learned in school. However, he was aware of certain mathematical issues, such as multi-dimensional spaces and non-Euclidean geometries,

¹⁶³ Students who experienced mathematics as an empirical field of knowledge are discussed under the theme of discovery. However, one of them is Lida, who is used as an example here and who did not believe that mathematics was discovered.

which he found extremely bewildering. His puzzlement seemed to stem from the fact that these issues were not part of his common sense. Moreover, both his examples were connected to the senses: the space around us is only 3-dimensional, while if one draws a line and a point on a paper, it seems obvious that there is only one line which is parallel to the given line and goes through the given point (Euclidean postulate). It was because of such cases that Solonas had divided mathematics into two parts: one that could stem from ordinary common sense and another which required a special type of reasoning and appeared to be unrelated to experience.

Solonas' ease with mathematics and its reasoning became apparent when we discussed linear equations in the context of how he learned mathematics and whether he understood it. He noted that the method which students learn at school 'simply teaches [the student] the steps through which [the equation] will be solved ... [but eventually] you realise on your own that you have to separate known from unknown [quantities].' Hence he admitted that in the classroom the reasoning behind the methodology was not explained, but he believed that by 'hear[ing] the methodology [*sic*] ... you enter into its rationale, so you understand why you're [following] it.' In other words, he effectively claimed that the rationale behind the method was more or less obvious and all students could understand it, even if this was not made explicit. This is a feeling which can be justified only when one has no problems inferring the rationale oneself.

Nevertheless, there were mathematical facts which opposed Solonas' intuition, essentially because they were contradicting what his senses, and his common sense, dictated. This became apparent as we were talking about the function of the various kinds of mathematical statements, and we finally reached axioms.¹⁶⁴ When I asked him why we accepted axioms he suggested that we did so on the basis of common sense: 'yes, right; you can assume [an axiom] to be a conjecture too; but it's something which is - not obvious - acceptable by common sense?' However, when I invited him to explain how both Euclid's postulate and the axioms in non-Euclidean geometries, which negate them, could be both accepted by common sense, he was at a loss and he admitted that 'what can I tell you? This was one of the concerns I had.' As we continued sharing what we knew about non-Euclidean geometries, at some point Solonas explained that

¹⁶⁴ The very fact that he was aware of what axioms were, and of particular examples, testified to his interest in mathematics.

I wonder what was the rationale on the basis of which the first person started referring to other - not surfaces - to other dimensions,¹⁶⁵ and to manage not only to think of them [but also] to say that I'll draw this line ... That is, how can somebody think about [this issue] in that way?'

Thus, Solonas expressed his bewilderment about issues which seemed to him to lie beyond what common sense based on experience would consider, and wondered how mathematicians had gone beyond this barrier of experience to create new ways of thinking. Moreover, it was noteworthy that non-Euclidean geometries and spaces in more than three dimensions seemed to be related in his mind. He was convinced that Euclid's axiom 'holds on a two-dimensional surface, but it doesn't hold in spaces with more dimensions.' So essentially, he was confused about mathematical matters which appeared to lie beyond what his senses could experience, that is, the three-dimensional world in which he lived. We returned to this issue again in the second interview, and it seemed that Solonas had learned to explain Euclid's axiom by common sense, while the other axioms were logical only through 'the experience and the knowledge [mathematicians] have about other [*sic*] dimensions.'

It appeared that it was because of his concerns about issues which contradicted experience that Solonas had distinguished common sense from mathematical logic.¹⁶⁶ Furthermore, he appeared to divide mathematics into two parts: one part that would be based on common sense as such, and a second part such as geometry, which had to be based on axioms. So when I asked him whether there could be mathematics without axioms, he suggested that 'for some things, no; for instance, geometry, I don't think it could exist without some axioms ... [But the theory of linear] equations could have been derived without axioms.' When I wondered where such a theory could have been based he replied 'on rational thought', or in other words, on common sense.

In any case, Solonas seemed to have difficulties accepting aspects of mathematics which he could not directly ground in his common sense and experience. Fortunately for him though, he still understood what he had learned in school,¹⁶⁷ and consequently, he could find positive meaning in mathematics. This meaning could be associated with the themes of 'rules' and

¹⁶⁵ In his mind, Solonas associated non-Euclidean geometries with more than three dimensions.

¹⁶⁶ See section on common sense in the epistemology chapter.

¹⁶⁷ All the examples Solonas mentioned concerned material which had not been part of his curriculum.

'invention'. His level of engagement with mathematics suggested that so far he could use its rules effectively and that it was an invention to which he felt intimate.¹⁶⁸

General misalignment

For other students, the misalignment between the empiricism dictated by their common sense and mathematics was too deep. Such students found it hard to reason with abstract concepts, or without being able to resort to experience, and essentially needed to react to an image of mathematics as abstract. All these students also had difficulties understanding mathematics. As a result, they would portray mathematics negatively as a complete mystery devoid of positive meaning (Evyenia), unless their circumstances had allowed them to focus only on aspects of mathematics which they understood and enjoyed (Lida). In the second case, students could actually find positive meaning in mathematics as an empirical field of knowledge.

Evyenia

Evyenia had a preference for the senses over logic. This could be why she found it hard to understand on which basis she could reason with things outside of her experience; her common sense seemed to be of little use in such cases. Among these she included mathematics. In fact, she denied that mathematics could be judged on any objective basis precisely because she dissociated it from experience. The result was that, for her, mathematics seemed to be a mysterious problem.

Evyenia expressed a preference for the senses and experiential data over logic. When I asked her what she would trust more, her senses or her logic, she suggested that the senses are more trustworthy because 'with my senses, if I used all, all of them, I would reach somewhere. The three of them would be correct, for example, the other two wouldn't be [correct] ... senses are more than logic.' It seemed that for Evyenia this was a matter of quantity - five senses compared to one logic; if logic was wrong there was nothing else through which she could check it, while with the senses she could use one to verify the input of another. Moreover, as she explained when I asked her to compare mathematical rules with rules in life in general, the latter were much easier to learn because they were experienced: 'in your everyday life, because you

¹⁶⁸ Solonas had most probably presented mathematics as an invention precisely because he believed that some parts of it transcended common sense and required the creativity of the human mind.

experience it every day and you hear from several [people] and they tell you; it [comes with] the flow of the day.'

The dependency of Evyenia's common sense on empirical reasoning seemed to be the reason why she regarded mathematics as a problem without an easy solution. While we were discussing rules of logic and mathematics, she commented that 'mathematics by itself is a problem, generally the whole concept [of mathematics is a problem], ... which branches out all over.' This problem seemed to be too complex for her to comprehend. After all, mathematics was too abstract, it did not correspond to her experience, it did not discuss existing things, it only made 'assumptions', 'always using' phrases such as 'let it be' or 'if ...'¹⁶⁹ So she could not use her experience, that is, her common sense, in order to judge what was mathematically logical and what was mathematically wrong, unless she already knew what the right answer was supposed to be according to the rules of mathematics. So, after making her remark about mathematics being a problem, she explained how her experience would allow her to judge that if one said that

"it's logical that you can walk in outer space", [then] this isn't reasonable, it's simply [one's] fantasy ... while in mathematics ... it will be something which you have never seen before in your life, you won't know it ... so it may seem reasonable to you, [even] if one is saying something [mathematically] crazy ... if [one] says that five is less than zero, you may say with your mind that "yes it's right."

The fact was that mathematics did not always agree with her experience. As she observed when we were discussing if reality could contradict mathematical or classroom rules: 'they've told me that a triangle['s angles add up to] 180 degrees, but if it is bent, if it has been broken?' That was probably the reason why Evyenia did not always agree with mathematics in relation to what was a correct way to reason, essentially suggesting that mathematics could be judged subjectively. When I tried to understand whether she believed that the way mathematicians reasoned, according to her book, was correct, she disclosed that, according to her common sense, 'the way I see things, I agree with some of the [stuff in my book], [but] I disagree with some [other stuff, thinking] "what is that [guy] saying here?" So Evyenia had trouble with over-abstract mathematical reasoning, while she had no way to release her frustrations. Thus, when I asked her if mathematics had rules, Evyenia could not help noting 'that it would be

¹⁶⁹ See section on invention in the ontology chapter.

better if mathematics had less and more understandable rules.' If she could understand the rules, that is, if she could relate them to her experience, then mathematics would not be a mysterious problem anymore.

In order to cope with mathematics conceived as a mystery, Evyenia had resorted to the mechanical, rigidly algorithmical aspects of mathematics. She had realised that exercises which merely required her to follow this or that rule were within her grasp, even if they had nothing to do with her experience. These were usually algebraic exercises where she did not need to make logical connections between things that lay out of her experience, but she could simply follow a rule that she knew and the consequent operations, and reach the result without thinking much. So when I asked her about her relationship with mathematics, she eventually commented that 'usually, I like better whatever is more algebraic.' In order to explain this she resorted to examples, contrasting limits with functions:¹⁷⁰

limits place you in a [specific way of] reasoning, they make you think in a different way from functions. ... [with limits,] you do more operations, you don't need to think more [using] your mind and saying that "since this holds ... this happens" ... [instead, the answer] is given to me by the formula [which] I've learned before.

However, it was not the case that Evyenia merely preferred not to think too much. As she disclosed at some point where we were discussing her future plans: 'I like philosophising [about] things on my own;' but these were things of her life, things which she had experienced.

In sum, the gap between Evyenia's common-sense empiricism and mathematics was too deep to ignore. Reacting to an image of mathematics as abstract, she presented mathematics as a mysterious problem. There seemed to be cases where she could grasp mathematical reasoning - cases where she could blindly follow rigid rules - but they were not sufficient to counteract the negative meaning that she found in mathematics.

Lida

Lida's common sense also exhibited empirical tendencies. She considered as 'proper' mathematics only calculations with concrete numbers and not with abstract variables. She also regarded as proper numbers only natural, and possibly rational numbers which could be the

¹⁷⁰ In fact, both her examples belong to algebra, but they do demonstrate what the algebraic way of reasoning meant for Evyenia on the subjective level.

result of a simple division. Nevertheless, she would definitely not treat negative numbers as worthy to be called numbers; they did not represent any real quantity, and they were contrary to common sense and experience. Lida was disengaged from mathematics, but this did not seem to be only the result of her empirical preferences. She was also influenced by the fact that she felt that her teachers did not provide adequate support. Interestingly enough, it appeared to be her current disengagement with mathematics that allowed her to still like mathematics; being disengaged in the present moment, she could still enjoy the memories of a happy past, when she enjoyed playing with numbers and mathematics was an empirical field of knowledge.

Lida appeared to be limited by her empirical approach to mathematics. What she felt comfortable in handling were concrete numbers which corresponded to quantities which she could see; reasoning with abstract variables, or even negative numbers, seemed to be almost irrational for her. Thus, when I asked her how she learned mathematics, she firstly replied 'by chance', but then she added that

I learn mathematics by listening and I perform operations ... but not operations of the kind "alpha times alpha equals beta times gamma times delta times the whole alphabet"; I perform operations with numbers; sometimes [operations] work out with letters too, but basically, it's numbers, numbers; because when you say "mathematics" you mean numbers.

When I enquired what was included in the category of numbers, Lida was ready to include all the natural numbers: 'all numbers, from zero to infinity.' I enquired about negative numbers, but it seemed that these did not represent real quantities for her. She explained that

unless you are a fathead, you can't go to a child and tell them "how much is one plus two? [it's] minus three [*sic*]" ... [If anyone did so,] you'd say "what is the madman talking about" ... there's a game [for toddlers] ... and it has blocks with numbers; the child [who plays,] and who is three to four years old, doesn't see minus one ... it sees one, two, three four, five.

After some thought, Lida conceded the inclusion of rational numbers in the picture, recognising that division may lead to numbers which are not integers: 'it's not impossible that I'll do the operation seven over two.' However, she did not appear willing to consider anything else as rightfully belonging to the category of numbers. The remainder cases would lie outside of what her common sense would be content to consider.

The discrepancy between mathematics and Lida's empirically oriented common sense should have been one of the reasons that she had distanced herself from mathematics. While we were discussing her relationship with mathematics, she explained very vividly that 'we haven't exactly gotten a divorce, but we're separated.' Nevertheless, she would not locate the reason for her disengagement in mathematics per se. Instead she felt that 'the teachers ... don't help me to love mathematics as much as I should, so I don't pay too much attention to it ... teachers dampen your enthusiasm ... they don't inspire me.' This seemed to be a defence which allowed her to still relish earlier times when she was happy doing mathematics. She felt the need to stress that 'it isn't that I don't want mathematics, [or] that I don't like mathematics, after all that's what I want to follow [professionally], accounting ... and [things] like it.' Extending her metaphor about couples she remarked that '[it]'s the [sort of] couple, where although [we're] split, I still love ... mathematics has been my soft spot since I was young.'

Lida was no readier than Evyenia to reason with non-empirical, abstract aspects of mathematics which lay beyond what her common sense could grasp. However, Lida's reaction was to reject the abstract part of mathematics that her instruction in school was trying to enforce on her. Therefore, Lida had been able to retain in her mind a more or less positive image of mathematics as an empirical field of knowledge.

Summary

Students' empirical tendencies suggested a gap between their common sense and abstract ways of mathematical reasoning. As such, they could turn mathematics - or at least certain aspects of it - into an unintelligible mystery which students found hard to grasp. However, if the mystery concerned only a minority of cases, the students could still find a positive meaning in mathematics, aided by the positive influence of all the instances where they were able to understand mathematical reasoning and content. Otherwise, students would struggle to find any positive meaning in mathematics, unless their circumstances had helped them to build some defence mechanisms which allowed them to effectively ignore what they could not understand, and still find positive meaning in mathematics as an empirical field of knowledge. In any event, such students seemed to react - though to different degrees - to what they perceived as overly abstract aspects of mathematics.

Concluding remarks

From the analysis of the research data, students were found to attribute subjective meaning to mathematics through issues which pertained to philosophy. In particular, students could attribute positive or negative meaning to mathematics through their responses to the following philosophical questions: How does mathematics function, and what is its relation to everyday common sense? Is mathematics discovered or invented? Is mathematics certain or can it be subjective? What is the purpose of mathematics' (rigid) rules? Among the above, the compatibility between some perceived aspects of mathematics and the student's common sense seemed to be sufficient to account for all the ways that students were able to find meaning in mathematics through philosophy, although the result also appeared to depend on the extent to which students understood mathematics. Thus, if mathematics, as perceived by the students, could be easily grasped, or could fit neatly within their understanding of common sense, then students could generally find positive meaning in mathematics; otherwise the meaning they attributed to it was more or less negative.

The remainder of the themes essentially indicated specific aspects of mathematics which could appear to be in contrast, or in accord with, a student's common sense (certainty, subjectivity, function of rules, abstractness), or philosophical issues which could be linked to that (mis)alignment (discovery, invention). Apart from the case of subjectivity, the (mis)alignment between the students' common sense and mathematics appeared to mostly concern aspects which were in line with their traditional cultural context, even if the students wished to react to them. However, themes such as invention or empirical discovery suggest that such (mis)alignments took place against a much more complex background which would occasionally clash with traditional aspects of mathematics. This is more fully elaborated in the next chapter which is dedicated to discussing the relationship between the students' various beliefs, and particularly the interrelationship between objective and subjective meanings.

Discussion

Introduction

The aim of this chapter is to bring together the findings of the analysis undertaken thus far, and to offer a consideration of these findings within the context of the existing literature. In doing this, the chapter will serve to bring together the objective and subjective meanings that the students attributed to mathematics by way of philosophical issues. Thus, the objective meanings which have been presented under the themes that have been discussed in the ontology and epistemology chapter will be reviewed in order to ascertain how they relate to one another, and how they might operate in the creation of subjective meaning for mathematics. Not all of these themes will appear as separate sections; instead they are organised under more general headings that serve to bring together particular themes. In this manner, the chapter addresses the topics of mathematical existence; mathematical reasoning (including mainly logic, common sense, and rules, but also observations on the senses, experience, and proof); and mathematical certainty (including mainly immutability, truth and objectivity, but also remarks on proof).

Before embarking on this review, it is important to reflect both on the structure of the data and on their richness, particularly with respect to the Greek cultural context. Hence, this chapter opens with a section concerning the plurality and cohesiveness (with respect to individual accounts) of beliefs. Consequently, and reflecting the order of the chapters in which the findings of the research have been reported, I consider first the ontological issue of mathematical existence, and second the epistemological issue of mathematical reasoning. The topic of certainty is handled last, because it brings together the ontological concerns of mathematical existence, and the epistemological concerns of mathematical reasoning.¹⁷¹ Subjective meaning is not discussed separately, but permeates all the sections, just as it permeated all the interviews. Occasionally, in order to avoid continuous repetition, the term 'subjectively meaningful(-less)' or 'meaningful(-less) on the subjective level' has been used instead of the term 'positive (negative) subjective meaning'.

¹⁷¹ This is because certainty actually concerns the nature of mathematical knowledge and not of mathematics per se. However, as noted in the ontology chapter, the line between mathematics and mathematical knowledge is not always clear, and therefore the line between mathematics, ontology and epistemology is not always clear either.

This chapter makes use chiefly of previously introduced examples. These are briefly mentioned - sometimes simply by giving in a parenthesis the name of the student which they concern. There were a small number of cases where the introduction of new examples was appropriate. These follow the format of the previous chapters, though the reference to the context of the dialogue in these cases is now minimal, since the focus of the current chapter is no longer to discuss quoted extracts from the interviews.

Plurality and cohesiveness of beliefs

In this section, I first discuss the richness of the data, and subsequently, the cohesiveness of separate accounts.¹⁷² The former has been fully exhibited in the data analysis chapters by means of demonstration. The latter is introduced here since it concerns the beliefs of a student as a whole rather than with respect to different topics (Hofer, & Pintrich, 2002; Leder, et al., 2002). Discussing the variety of students' beliefs is necessary since, according to the literature, students who have been taught in a traditional setting might have been expected to share similar beliefs about mathematics (Cobb, et al., 1992; François, & van Bendegem, 2007). In this respect, the diversity of students' beliefs needs to be particularly considered with reference to the Greek cultural context, which can be categorised as a traditional one (Tzekaki, Kaldrimidou, & Sakonidis, 2002).

Plurality of beliefs

The extent of the range of beliefs exhibited by students astonished me, even as I was conducting the interviews. All students came from the same school, and the same region, so it could fairly be expected that they would share broadly similar beliefs about mathematics. More precisely, and following the literature, since the students had been taught in a highly teacher-centred way, they might have been expected to generally concur that mathematics and mathematical knowledge were independent of the student, immutable, and indubitable, whilst ignoring ideas which assumed mathematics as a fallible, ever-changing product of the human mind (Alrø, & Skovsmose, 2002; Carpenter, & Fennema, 1992; Chassapis, 2007; Cobb, et al., 1992; François, & van Bendegem, 2007; Leung, 2001). In other words, it could have been expected that all, or

¹⁷² The data as a whole cannot be cohesive exactly because of the plurality of students' beliefs, i.e. their divergent views on the same topic.

the majority of, students would have perceived mathematics in line with older traditions in the philosophy of mathematics, such as platonism, and foundationalist trends (Ernest, 1991; Hersh, & John-Steiner, 2011). After all, the literature has mainly attributed any diversion from such traditional views to alternative ways of teaching (De Corte, Op't Eynde, & Verschaffel, 2002; Ruthven, & Coe, 1994; Solomon, 2006), which were effectively absent in this case.

However, Op't Eynde, et al. (2006) have found that students coming from similar teaching backgrounds may well hold very diverse beliefs, ranging from beliefs closer to platonism and foundationalism to beliefs closer to humanism. The findings of this study concur with those of Op't Eynde, et al. (2006). On the surface, it may indeed appear that the students entertain the expected beliefs of mathematics as objective and certain. Nevertheless, a more thorough analysis reveals that there are many different shades in students' understanding. It seems that the use of interviews instead of questionnaires to explore the students' beliefs allowed for variations behind seemingly uniform beliefs to come, more readily and more openly, to the surface (Cohen, et al., 2011).

It was, of course, the case that the students were influenced by the culture in which they had been taught mathematics. In line with a traditional method of teaching, where mathematical knowledge is simply transferred from the teacher to the student as correct (Tzekaki, Kaldrimidou, & Sakonidis, 2002), most students did indeed present mathematics as a set of certain, immutable, true, and objective rules. Moreover, the importance accorded to proofs in the Greek context (Sdrolias & Triandafillidis, 2008) was evident in the fact that students had learnt to accept proof as an authority, while they also tended to justify the above mathematical traits on the basis of proof. The influence of cultural context seemed to be even stronger with respect to the subjective meaning that the students found in mathematics. In other words, students would express such a meaning by endorsing, or reacting to, the dominant images of mathematics offered by culture. These images presented mathematics as a set of absolute, objective truths; or as a set of rigid, authoritative rules; or as an abstract field of knowledge (Sdrolias & Triandafillidis, 2008; Tzekaki, Kaldrimidou, & Sakonidis, 2002).

Nevertheless, the students did not restrict themselves to comments that were in line with the traditional, dominant picture of mathematics. The most important deviation concerned beliefs about mathematical existence. In this regard, and in contrast with what the literature would suggest (Charalambous, Panaoura, & Philippou, 2009; Chassapis, 2007), very few students espoused the platonistic view of mathematics as comprising existing abstract objects. The

majority of students suggested that mathematics was invented, while claims to mathematical discovery and existence tended to bear an empirical flavour despite the fact that in Greek classrooms precedence is given to the theoretical, abstract aspect of mathematics (Sdrolias & Triandafillidis, 2008). This could be the reason why so many students also advanced other beliefs which were in line with more modern philosophical trends such as humanism and empiricism. As a result, an apparently traditional belief such as mathematical certainty and immutability, or objectivity, could be justified on partially, or even purely, cultural grounds; that is to say, students could suggest that such traits were cultural constructs, a result of the status of mathematics in their culture. Moreover, students could present mathematics as an empirical science stemming from observation or validated by its utility and applications. Similarly, students could indicate that mathematics was a developing entity, or even that mathematics was a subjective activity, or not necessarily true or logical. Many of the above beliefs were also present in the students' stories of subjective meaning. So ultimately, both the objective meanings (as reflected by students' beliefs) and the subjective meanings (as reflected by students' stories) that students associated with mathematics through its philosophy were not shaped entirely in accordance with the students' cultural context.

This, however, may not be hard to explain. The fact is that, by espousing so-called modern beliefs, most of the students simply seemed to advance what their common sense judged to be reasonable.¹⁷³ It seems reasonable, for example, to assume that research in mathematics continues as in any other fields of human knowledge, and thus mathematics, or at least our knowledge of it, is not static. Similarly, it seems reasonable that the applications of mathematics in real life would add to its validity; such an empirical way of reasoning, whereby a piece of information is judged with respect to how well it fits into an already established general picture is quite common in everyday life. Moreover, for a student who is familiar with authority, represented in the classroom by the teacher, it seems reasonable to attribute the apparent certainty, objectivity and rationality of mathematics to the authoritative status of mathematics as a respected science within the national culture. This was yet more the case where such a student possessed no other means to (in)validate such traits because they did not understand mathematics. Actually, it seems reasonable to believe that something which one is

¹⁷³ Apart from Lida, who seemed to have some experience with open problems, and Kleomenis and Solonas who had some elementary, but still non-negligible, knowledge of axiomatic systems, the remainder of the students had no special knowledge about mathematics

not able to comprehend cannot be entirely objective or logical. It appeared that students who had difficulty with mathematics, although influenced by their culture, felt some fundamental desire to rebel against a completely orderly picture of mathematics, and that the expression of this desire came to the surface during the interview. However, even students who could understand mathematics could feel that some of the above arguments made sense, especially those concerning the development of mathematical knowledge and those which connected mathematical validity to its applications.

Cohesiveness of beliefs

In fact, the findings of the current research add to those of Op't Eynde, et al. (2006), to the extent that students' beliefs were found to range from traditional to modern, not only across the sample, but also within individuals themselves. Yet, apparently contradictory beliefs are not necessarily incompatible (Ruthven, & Coe, 1994). There may be a number of reasons for what appear as manifest contradictions within a student's beliefs. Firstly, inconsistencies may be no more than an issue of an imprecise use of words. For example, Agapi contradicted herself: when I asked her, in the first interview, if a theorem could change, she suggested that 'inventors [could] find something else which cancelled what the first person [who introduced the theorem] has said.' However, in the second interview, she agreed that mathematical knowledge was stable and she insisted that 'old [knowledge] doesn't change.' This contradiction can be explained by the fact that on the first occasion Agapi was using the verb 'cancel' in a very limited way. As became apparent in the second interview, what she actually believed was that new mathematical knowledge may shed light upon old knowledge, but that this served mainly to polish old knowledge by further clarifying the cases to which it was relevant, and not to essentially refute it. Thus, when I reminded her how she had spoken in the first interview, she remarked that: 'the old [knowledge] may be cancelled with respect to when it holds.'

Secondly, seemingly contradictory opinions may result from the fact that, although philosophers have generally conceived mathematical ontology and epistemology as being interrelated, these two issues may be approached independently (Shapiro, 2000). This means that a realist ontology, according to which mathematical objects exist, may be combined with a fallibilistic epistemology (Ernest, 1998a; Lakatos, 1976b). Thus, despite claiming that mathematics was discovered, Foivos explained how mathematical knowledge advanced as new individuals came forward to challenge already established knowledge. After all, ontology

involves what mathematics is, while epistemology concerns the ways in which mathematical knowledge is produced (Shapiro, 2000).¹⁷⁴ The former may posit transcendental entities which exist independently of humans, as does platonism (Field, 1988; McGee, 1997; Menzel, 1987), while the latter may note that the production of mathematical knowledge involves humans, who are inherently fallible (Hersh, 1997; Tymoczko, 1984).

Such distinctions allow us to hold apparently contradictory beliefs by applying them to different contexts, which may occur even within a specific field of knowledge (Furinghetti, & Pehkonen, 2002; Hammer, & Elby, 2002; Wittgenstein, 1953). Thus, there may be contradictory beliefs even within the field of epistemology (or the field of ontology), as long as a certain issue is viewed from different perspectives. This was the case with students who were confronted with the beliefs of a culture which presented mathematics as certain and objective (Cobb, et.al., 1992; François, & van Bendegem, 2007), but found it hard to endorse these beliefs (e.g. stories of Ariadni or Filippos, but also see Yerasimos' comments in the ontology and epistemology chapters). Without understanding, these students could not accept such views on any other ground than that of authority (Hanna, 1995; Rowlands, et al., 2011). For them, certainty could only stem from the fact that the society in which they lived would not warrant them to condemn mathematics (Alrø, & Skovsmose, 2002; Hanna, 1995; Harel, & Rabin, 2010). Thus, they were experiencing mathematics as certain, even though they could not subscribe to the truth of its claims. Their principal way out of this predicament was to distance themselves from mathematics, by claiming that it was essentially the invention of some eccentric people (Solomon, 2006). In doing so, such students could declare that for them mathematics was irrational or subjective, separating themselves from the mathematicians' 'caste', for which mathematics made sense and could be certain. In other words, such students were in the position of holding two opposing beliefs: one that concerned society and its conventional view of mathematics, and one that concerned themselves and their understanding of mathematics.

Thirdly, the students' accounts exhibited significant unifying traits which could bring together apparent inconsistencies. As mentioned in the chapter on subjective meaning, for each student, there were concepts and beliefs which surfaced again and again during the interview. These

¹⁷⁴ As noted in the ontology chapter, the boundary between epistemology and ontology is blurred when mathematics is perceived as an invention of the human mind, but it may still be relevant when mathematics is considered to exist independently of the human mind.

beliefs would mark their accounts while also serving as a link between their various objective meanings. For instance, Kleomenis' account was unified through his use of the word 'theory'.¹⁷⁵ For him, this term indicated a product of the human mind which could be independent of any practical reality. Thus, his concept of theory managed to bring together his seemingly contradictory beliefs, according to which mathematics was both a human invention, and a demonstration of determinism and certainty. He viewed uncertainty and chaos as part of life's actuality, but mathematics was pure invented theory, and as such, it had no obligation to conform to life's actualities. Instead, free from any reality constraints, mathematics could be designed to be precise, and thus, certain (Hersh, & John-Steiner, 2011; Schlimm, 2016). Kleomenis' concept of theory seemed to emerge from a wish for certainty and an admiration for preciseness. Such a wish could sustain the belief that, when they are not bound by reality, humans can generate theories that lay claim to certainty, despite the fact that this opinion could be easily refuted, since human creations seem to be inherently fallible (Ernest, 1991; Tymoczko, 1984).

In the same vein, Evyenia's account was unified through her frustration with the non-empirical nature of mathematics. It was this frustration that led her to perceive mathematics as a mystery and to suggest that mathematics was invented although it existed. It was this frustration that made her eventually claim that logic was not connected to mathematics, although initially she had replied positively to the relevant question. It was this frustration that led her to doubt the reasoning behind the statements in her books, while also commenting 'it would be a bit weird if we said about anyone that they don't reason correctly.' Finally, it was this frustration that led her to prefer mathematical exercises where she did not have to think and reason, although she seemed to enjoy such activities outside the context of mathematics.

It seems that what brought students' stories into a cohesive whole was primarily the subjective meaning, and not the objective meaning, that they attributed to mathematics; it reflected their subjective associations with respect to various aspects of mathematics, and not a purely conceptual analysis of their beliefs (Op't Eynde, et al., 2006). After all, individual beliefs tend to be subjective, and serve to hold together an image with which the individual feels comfortable (Rue, 1994; Snow, Corno, & Jackson III, 1996). That was why students would

¹⁷⁵ A term which he also used in instances which were not included in his story in the chapter on subjective meaning.

return to beliefs which helped them attribute subjective meaning to mathematics. A further example was Filippos' image of mathematics as irrefutable, and yet subjective. This image captured what mathematics meant for him, reflecting his longing to disagree with mathematics, while living in a society where this was practically not allowed. In all, the subjective factor of meaning appeared to be able to hold together beliefs which otherwise would be deemed contradictory.

In fact, it can be argued that subjective meaning allowed for the forming of a cohesive story by bringing together the objective meanings which students advanced with respect to various philosophical issues (Op't Eynde, et al., 2006). In the opposite direction, students' philosophical beliefs and their objective meaning could explain and justify the subjective meaning that mathematics had for them (Ernest, 1991; François, & van Bendegem, 2007). The structure of students' accounts might not be exactly logical, but it can be claimed to be quasilogical (Op't Eynde, et al., 2006; Ruthven, & Coe, 1994), based on both rational (objective) and psychological (subjective) factors.

In conclusion, it is difficult to claim that students generally held a cohesive 'philosophical' theory of mathematics or mathematical knowledge. This, of course, was not unexpected, as students in Greece, like students elsewhere (François, 2007), do not have opportunities to discuss and refine philosophical beliefs in the classroom. On the other hand, it was noteworthy that their accounts did not consist of discrete, unrelated beliefs. Hofer and Pintrich (1997) suggest a continuum, at one end of which lie discrete facts, while formal scientific or philosophical theories can be found on the other end. Students' philosophical accounts seem to be somewhere in between, with some of them being more coherent than others. The analysis that follows demonstrates how students' beliefs on certain issues could be organised in ways in which different concepts and views mingled with one another, with respect to both objective and subjective meanings.

Mathematical existence

This section delineates the interconnection between beliefs relevant to mathematical existence. Students' opinions on this matter were presented in the ontology chapter, and were organised around the discovery-invention dichotomy (Godino, & Batanero, 1998; van Moer, 2007).¹⁷⁶ Nevertheless, here I do not repeat the students' solutions. The focus of this section is on how beliefs relevant to either 'invention' or 'discovery' interacted with other suggestions that the students made, and how the resulting images of what mathematics was could help students to find meaning in mathematics on the subjective level.¹⁷⁷ In this process, the connections between students' beliefs, their cultural context, and philosophical topics are also highlighted. First, I elaborate how the concepts of 'discovery' and 'invention' related to subjective meaning. This leads to a discussion on the educational issue of mathematics as a human activity (Ernest, 1991; Hersh, 1997) and on the philosophical issue of how mathematics can be applicable if it is invented (Benacerraf, 1973), both of which appear to be related to the ontological question of discovery and invention.

The interviews revealed that only just over one third of the students were substantially influenced by the belief that mathematics could exist independently of the human mind, a principle which is supposed to inform a traditional context of teaching (Charalambous, et al., 2009; Chassapis, 2007). Moreover, this belief was mostly associated with the conviction that mathematics described and reflected the structure of nature (Ermis, Foivos). So, mathematical existence was mainly postulated in an empiricist context, as the result of physical existence, (Colyvan, 2001; Resnik, 1995), despite the fact that in the Greek cultural context it is the abstract aspect of mathematics that is chiefly highlighted (Sdrolias & Triandafillidis, 2008). However, some students appeared to be influenced by the platonistic ideal (Andromachi). For them, mathematical existence seemed to involve some kind of abstract entities, possibly accessible through reason (Balaguer, 1998; Burnyeat, 2000).¹⁷⁸ In any case, apart from Andromachi, even these students believed that mathematics could contribute to the understanding of the empirical world. Connecting mathematics with the real world could make mathematics meaningful on the subjective level, especially if students considered understanding how the world works as a valuable goal (Hersh, & John-Steiner, 2011; Loewenstein, 1994; Vollstedt, 2011). Significantly, this was the case with all the students who

¹⁷⁶ This is not necessarily an either/or issue (Wheeler, 1993), as some of the students' answers also indicated.

¹⁷⁷ The numbers discussed here correspond to the discovery and invention themes for subjective meaning. In other words, I do not consider only what the students answered to the discovery-invention question, but also the aspect of this answer which seemed to be more salient in their mind, helping them to attribute subjective meaning to mathematics.

¹⁷⁸ Other than Andromachi, who was given as an example in the chapter on ontology, this applied to students who could not decide whether mathematics existed or not but had the impression that mathematical claims were true independently of the human mind.

advanced this belief, and who thus could appreciate mathematics even if they did not understand it or were not very fond of it (Ermis' and Foivos' stories).

Nevertheless, despite having been taught in a traditional context, the belief of mathematics as an invention was a salient one for more than half of the students. Apart from expressing the general belief that mathematics as a product of the human mind, sometimes students gave more concrete reasons for rejecting mathematical existence, and these echoed philosophers' considerations. For instance, as many others who would distrust metaphysics (Price, 2009; Rotman, 1993), at least one fourth of the students were not comfortable with the existence of more or less metaphysical, immaterial, abstract objects which were not accessible by the senses (Hersh, 1997; Menzel, 1987). Hence, since students could not see and touch numbers, variables, functions etc., in the same obvious way that they could see and touch a table or a chair, they assumed that mathematical concepts must be the creation of the human mind (Diomidis). So it actually seemed that the abstract, non-empirical, picture of mathematics advanced by the students' cultural context did not lead them to postulate the existence of abstract mathematical entities as platonism would do, but rather to assume that mathematics did not exist, and was thus invented. Still, seeing mathematics as independent of empirical experience could cause difficulties with respect to subjective meaning for students who exhibited empirical tendencies (Evyenia's or even Solonas' stories) (Recio, & Godino, 2001; Stenning, & van Lambalgen, 2008).

Furthermore, around one fourth of the students suggested that mathematics was invented, because they conceived mathematics as comprising hypotheses based on assumptions (Russell, 1918). Students who had difficulties with understanding mathematics limited themselves to observing that mathematical statements concerned hypothetical situations (Evyenia). Other students, who were more interested in mathematics, remarked that mathematics was based on axiom-like assumptions (Kleomenis, Lysimachos). As Kleomenis suggested, this is an issue which could be mentioned in Greek classrooms in the context of geometry; however, any reference to axioms and non-Euclidean geometries would be brief and it had not been impressed on the minds of most students who reacted as if they had not heard that term before. In any case, no matter why students saw mathematics as hypotheses, this image suggested that mathematical statements could be seen as conditionals (Russell, 1918; Tiles, 1991). These conditionals could be taken to lack any objective meaning with reference to an empirical reality, and could again be problematical at the level of subjective meaning (Evyenia's and

Ariadni's story). However, they could still retain meaning - at least on the objective level - as logically true (Lysimachos' comments on truth), since for deductive logic it does not matter whether they refer to reality or not (Enderton, & Enderton, 2001; Russell, 1918; Tiles, 1991). Students who understood this could still find positive subjective meaning in mathematics as assumptions. After all, students did not always isolate mathematical assumptions from what occurred in reality. For example, Kleomenis and Lysimachos indicated that they were chosen on the basis of pragmatic criteria regarding what seemed to work and produce useful results (Quine, 1951; Tymoczko, 1991; Skovsmose, 1994). For such students, this connection to reality rendered mathematics subjectively meaningful (Vollstedt, 2011).

The literature also suggests that the belief that mathematics is an invention of the human mind may render mathematics meaningful on a subjective level, by bringing mathematics closer to students; mathematics would be perceived as a human activity and not as something which exists independently of humans (Ernest, 1998a). However, seeing mathematics as an invention was not sufficient to give students ownership of, or accessibility to, this invention (Solomon, 2006). Students were able to feel as co-travellers in the human journey of mathematics (Ernest, 1998a) only to the extent that their common sense was in line with mathematics (mostly at the content level, but also at a philosophical one). Then indeed, the mathematical invention could have for them a positive subjective meaning (Kleomenis' story). On the contrary, students who claimed that mathematics was an invention while experiencing a misalignment between their common sense and mathematics, seemed to be doing so in order to distance themselves from mathematics and not to come nearer to it (Ariadni's story). In fact, it was probably because of this misalignment that such students would wish to react to the objective picture of mathematics that their culture offered (Filippos' comments on objectivity), and thus would suggest that mathematics was the creation of the human mind, a belief which unavoidably renders mathematics a subjective activity (Shapiro, 2007). Nevertheless, it was true that the belief in mathematics as a subjective invention could alleviate some of the oppression students might feel as a consequence of an impression of mathematics as utterly objective (Yerasimos' comments on objectivity and logic).

On the other hand, at least half of the students who assumed that mathematics existed, did not strip it of a human face, as modern philosophers would do (Ernest, 1991; Hersh, 1997).¹⁷⁹ They observed the obvious: that the mathematical knowledge available to us was produced by humans, and not by aliens (Rowlands, et al., 2001). It seemed that such students distinguished between mathematics itself, as a network of interrelated entities (ontology), and our mathematical knowledge, as the part of mathematics that human activity had revealed to people (epistemology) (Shapiro, 2000). The former might be transcendental, but the latter was unavoidably human.¹⁸⁰ In fact, from the perspective of subjective meaning, students had no reason to discard mathematics as a human activity, since this was indeed a belief which could render mathematics subjectively meaningful, as happened in the case of students who saw mathematics as a comprehensible invention (Ernest, 1998a; Snow, et al., 1996). Retaining this belief, in conjunction with the belief that mathematics described nature, could enhance the subjective meaning such students could find in mathematics.

Finally, an issue which had troubled philosophers - but not the students - was the apparent misalignment between the belief that mathematics does not exist, because it is invented, while it is so remarkably applicable in the actual, existing world (Benacerraf, 1973; Resnik, 1981; Shapiro, 2000). Especially students from the technology and science tracks, who constantly used mathematics in physics, usually took mathematical applications for granted and did not wonder how they were possible, even if they believed that mathematics was invented (Kleomenis' story, or Lysimachos' comments on experimentation). Once more, this could be because connecting mathematics to reality applications could positively add to the subjective meaning found in mathematics (Hersh, & John-Steiner, 2011; Loewenstein, 1994; Vollstedt, 2011), and such meaningful images are not easily discarded (Rue, 1994; Snow, et al., 1996). On the contrary, they are likely to survive, even in a context which puts more emphasis on the abstractness of mathematics (Sdrolias & Triandafillidis, 2008). After all, many of these students wished to study fields which would involve mathematical applications, so their goals could increase the subjective meaning of mathematical applications even more (Vollstedt, 2011). However, students who assumed that mathematics was invented - while they did not

¹⁷⁹ This has not been discussed in the ontology chapter since it did not play an integral role in the students' arguments. Nevertheless, it is relevant for the discussion, since it constitutes an important issue in the literature.

¹⁸⁰ See section above.

have many chances to use mathematics in other subjects - did tend to assume, probably under the influence of the abstract image of mathematics advanced by their culture, that mathematics was not really connected, or applicable, to the world around them. These students tended to distinguish between mathematics and physics, suggesting that mathematics was wholly theoretical, and that only physics was useful to understand the world (see Ariadni in the section on truth in the ontology chapter) (Matthews, Adams, & Goos, 2009). This meant that many of mathematics' applications were irrelevant to these students (Kosmas' story), who were thus deprived of a means which might help them find positive meaning in mathematics (Vollstedt, 2011).

In sum, the platonistic view of mathematics which has been associated with traditional teaching (Charalambous, et al., 2009; Chassapis, 2007) was generally not so prevalent among the students. Even when students assumed that mathematics existed, their beliefs echoed an empirical understanding of this existence. This could be because empirical aspects of mathematics seem to carry positive subjective meaning and could thus be given attention even in a context which underplayed them (Sdrolias & Triandafillidis, 2008). In addition to this, the pursuit of mathematical knowledge was seen as a human activity, and more than half of the students claimed that mathematics contributed to students seeing it as an invention, though this belief appeared to also be associated with a need to react to the image of mathematics as objective and indubitable, a result of the emphasis given to proofs in the Greek cultural context (Sdrolias & Triandafillidis, 2008).

Furthermore, the question of mathematical existence seemed to have a rich potential for helping students to find positive subjective meaning in mathematics, but this potential was not always actualised as educators might have hoped (Brown, 1994; Ernest, 1991). Where the students saw mathematics as an invention, what seemed to be important in order for them to perceive this invention as an intimate one was whether they could understand mathematical reasoning and its applications. On the contrary, the belief that mathematics was discovered tended to make students feel intimate to this discovery. Such students valued the capacity of mathematics to help humans in exploring the real world (Loewenstein, 1994).

Mathematical reasoning

This section examines remarks on logic, common sense, experience, the senses, and rules as part of reasoning within mathematics. Logic has played an important role in the philosophy of mathematics (Shapiro, 2005), and especially in traditional approaches which have been considered to influence mathematics education even today, despite the emergence of more modern beliefs (François, & van Bendegem, 2007). Moreover, logic is unavoidably stressed in a cultural context which places emphasis upon proofs (Sdrolias, & Triandafillidis, 2008). Nevertheless, recourse to experience and the senses have been proposed as alternatives to logic by various philosophers (Mill, 1851; Kitcher, 1984); while experience and the senses are closely related to common sense, which is generally important as a tool which could help students when encountering new mathematical concepts (Freudenthal, 1991; Keitel, & Kilpatrick, 2005). The concept of common sense that is employed here concerns that which students see as self-evident truths (Davis, 2006). Finally, rules appear to be an essential part of mathematical reasoning, comprising an element which belongs both to the philosophical tradition (Shanker, 1987), and to mathematics education, where it is especially closely associated with traditional approaches to teaching mathematics, as is the case in Greece (Davis, & Simmt, 2003; Goldin; 2002; McLeod, 1992; Tzekaki, Kaldrimidou, & Sakonidis, 2002; Sfard, 2000).

Rules may be treated as a separate issue. However, it is scarcely possible to discuss logic, common sense, experience, and the senses in isolation. As logic has been the factor which has traditionally played a more eminent role in the philosophy of mathematics, it will occupy the primary focus in what follows. Experience and the senses will mainly be considered to the extent that students believed that they constituted a part of mathematical reasoning. Common sense will be regarded in more detail since it was particularly important for the subjective meaning that students found in mathematics.

Logic

This section is a lengthy one since it includes the students' beliefs on the relationship of logic with all the philosophical approaches and with common sense. The section is therefore divided into three parts. First there is an introductory part summarising the students' beliefs. Then follows a second part which concerns the ways in which logic, as evidenced in the students' accounts, could be interrelated with the various philosophical approaches. Finally, the third

part discusses the relation between logic and common sense and the manner in which this relation influenced the subjective meaning that students could attribute to mathematics.

Introduction

Students' beliefs on logic, and their confusion between mathematical logic and common sense, were considered in the epistemology chapter under the section 'Logical and Empirical'. That section also discussed issues pertaining to the senses and experience. These two issues will be revisited here, expressed as tensions between mathematical reasoning and other more informal forms of reasoning. In this respect, the discussion about logic and common sense will also include remarks on proofs. This topic will not be considered separately, since its influence on both the objective and subjective meanings that students associated with mathematics was mainly funnelled through its association with logic and certainty, which will be discussed later. In this section, I focus on how students' beliefs about logic reflected philosophers' beliefs, the role of common sense, and the potential of students' beliefs with respect to (not) finding positive subjective meaning in mathematics.

Effectively, all students linked logic with mathematics, implying that logic played a role in the formation and verification of mathematical claims. This was a reflection of the fact that logic played an important role in the Greek cultural context of mathematics education, where there is significant emphasis on abstract reasoning and proofs (Sdrolias, & Triandafillidis, 2008). Philosophers and mathematicians would also not deny that logic is an essential factor - though not necessarily the only one - in understanding and creating mathematics (Ernest, 1998a; Floyd, 2004; Pólya, 1945; Restivo, 1992; Weyl, 1946). However, logic was utilised by the students as an all-encompassing term, comprehending anything that common sense could judge as sensible, rational, or reasonable. So essentially, more than three quarters of the students did not distinguish at all between logic as used in mathematics and common sense. By doing so, students brought mathematical reasoning closer to their common sense and were thus able to grasp the culturally advanced picture of mathematics as logical on a subjective level. Indeed, as the chapter on subjective meaning indicated, the relationship between their perceptions of common sense and mathematical logic exerted a strong influence on the subjective meaning they would attribute to mathematics. Nevertheless, regardless of whether logic was understood as common sense, the links that students suggested between it and mathematics covered a vast range of beliefs, and these could be associated with many different philosophical currents.
In practice, students often tended to 'borrow' ideas from more than one philosophical camp, reconciling with some ease divisions which philosophers would assume to be insurmountable. Certainly, it can be claimed that some of these traditions are more closely related than others, e.g. platonism with foundationalist trends (Ernest, 1991), or even all of these with empiricism (Hersh, 1997). However, students' accounts (more than half) went beyond these combinations. As the table below shows, views from formalism, logicism, empiricism, humanism, and intuitionism co-existed in many different combinations¹⁸¹ (drawing from as many as four philosophical traditions), without any obvious pattern. Students did not demur from portraying logic as the essence of mathematics (logicism), while suggesting that logic may emerge, at least to some degree, from empirical observations (empiricism), or social conventions, and habits (humanism), or both (empiricism and humanism). In fact, it can be argued that such combinations served to bring mathematical logic and reasoning closer to students' common sense, thus making mathematics subjectively meaningful. Similarly, students did not object to implications that mathematics consisted of meaningless rules (formalism), which were social conventions (humanism).¹⁸² Again it can be claimed that this would help them to make sense of mathematics on a subjective level - though not necessarily in a positive way; social conventions comprised a concept that students could grasp readily, since they were familiar with it, but it could also be a concept that they did not value on the subjective level since it was imposed on them externally. Such connections indicate that existing philosophical theories could be helpful in understanding students' beliefs, but should by no means be treated as rigid categories according to which the students' beliefs could themselves be categorised. What is important is that the students could combine the objective meaning of beliefs in order to find subjective meaning in mathematics in their own individual ways. The table below shows how many students indicated beliefs that could be associated with different philosophical pairings. For instance, 15 students held beliefs which could be associated both with formalism and with humanism.¹⁸³ The next part is devoted to discussing the interrelations between the various philosophical approaches with respect to logic.

¹⁸¹ Platonism will not be considered, since there were not enough students who expressed purely platonistic beliefs (see thematic analysis).

¹⁸² More combinations are discussed in the following.

¹⁸³ The numbers in the analysis that follows refer back to this table.

	formalism	intuitionism	empiricism	humanism
logicism	10	6	14	13
formalism	-	4	9	15
intuitionism		-	4	5
empiricism			-	12

Table 6: Combinations of philosophical approaches

Philosophical approaches

I start with logicism since this is the philosophical trend in which logic enjoys the most important role. I then consider the other foundationalist approaches, (formalism, and intuitionism), before moving to less traditional views such as empiricism and humanism.

Views that could be associated with pure logicism were practically absent. The idea of logic that students had in mind was not always congruent with that offered by logicism. Confusing logic with common sense, students combined ideas from logicism with elements from other philosophical trends. As the table shows, ten students who perceived mathematics as rooted in logic also indicated some kind of formalism when they suggested that by applying mathematical rules blindly - or by following 'the logic of operations', as Platonas called it - they could reach results which were correct, but in a sense lacked meaning (Hilbert, 1983) because they contradicted logic, or at least common sense, thus appearing to be irrational. In fact, as intuitionists had done, four students differentiated between results which seemed sensible and intuitive, and results which seemed absurd (Brouwer, 1913), although they would not reject the 'non-intuitive' ones.¹⁸⁴ For instance, Solonas considered the theory of linear

¹⁸⁴ Two more students seemed to suggest that all mathematics could be rooted in some logical intuition.

equations as a part of mathematics which was in accordance with human intuition and common sense, while he would reject non-Euclidean geometries as such.

Moreover, even when students trusted logic, this appeared to be on the basis of common sense (Cobb, et al., 1992; Mercier, 2010, Walkerdine, 1994). This indicated that they did not necessarily trust logic because of a conviction that it revealed absolute truths (Burnyeat, 2000), but because logic would point to the most probable conclusion (Stenning, & van Lambalgen, 2008). For instance, Lysimachos remarked how repeated input through the senses would be judged as logical by common sense, and Yerasimos insisted that the same held with respect to repeated cultural ideas. In all, in place of the rationalism posited by logicism or platonism (Hersh, 1997), 14 students portrayed a logic which could have an empirical, observational basis (empiricism) (Mill, 1851; Kitcher, 1984), and 13 students portrayed a logic which could have a social, conventional character (humanism) (Bloor, 1991; Restivo, 1991; Wittgenstein, 1953). Thus, Foivos suggested that mathematics was logical because it was defined on the basis of observations, while Lysimachos indicated that mathematical statements were acceptable on the basis of empirical data gathered through their application. Moreover, Agapi suggested that truths could be man-made, and ultimately such a position was implied by all students who suggested that mathematics was both true and invented.

In any event, the influence of the emphasis given to logic in the students' cultural context was evident in the fact that 68% of the students' accounts could be claimed to exhibit elements of logicism, implying that logic was essential for mathematics, and that it produced, through proofs, reliable results (Russell, 1918). Platonas even used the word 'rationalism' - though not in the philosophical sense - when he spoke of how mathematical statements were produced. Moreover, no matter how students viewed logic, traits of logicism suggested that they found mathematics rational, and thus suggested that they could find some meaning in it (Keitel, & Kilpatrick, 2005). This could be an objective meaning at the level of content, but also a subjective one. In fact, with respect to subjective meaning, the extent to which it would take a positive form appeared to depend on whether students found mathematical reasoning and logic to be compatible with their common sense, either at the level of content or at a philosophical level (Harel, & Sowder, 1998; Keitel, & Kilpatrick, 2005). This actually meant that by attributing to mathematical logic empirical or social traits, and thus interpreting it as common sense, students could enhance the subjective meaning of mathematics. For example, Ermis would enjoy mathematics both because he perceived its reasoning to be in accord with his

scientific attitude and because he could understand its content. On the other hand, Kosmas would attribute a negative meaning to mathematics, because although he would admit that it proceeded on the basis of logic, his common sense judged mathematical logic and reasoning as too rigid to be useful in real life. The fact was that among the students who were associated with logicism, only three would not understand mathematics both at the content and at a philosophical level. Thus, for most of these students mathematics was subjectively meaningful.

Evidence of formalist views of mathematics and its logic, such as handling symbols without any meaning (Hilbert, 1983), were presented by 64% of the students. Formalism was associated with logicism mainly when the students were able to understand why one sequence of symbols led syntactically to the next, that is, when students could understand mathematics at the level of content.¹⁸⁵ To that extent, mathematics was still a rational activity for such students. This seemed to be the reason why Platonas, after suggesting that some mathematical results seemed irrational, reintroduced the concept of logic in mathematics as 'the logic of operations'. As explained earlier, such a rational activity could carry positive subjective meaning (Harel, & Sowder, 1998; Keitel, & Kilpatrick, 2005). In fact, this case seemed to resemble that of empiricism in the chapter concerned with subjective meaning. Insofar as students coul understand the bulk of mathematics that they had been taught, they could ignore formalist experiences easily. Thus Platonas went on to assume that 'the logic of operations' was sufficient to render logical even statements which appeared irrational. As the table illustrates, in cases where formalism was connected with logicism it was also linked to intuitionism, empiricism, and humanism as long as students connected logic with one of these approaches. The four cases of formalism coexisting with intuitionism and the seven out of nine cases where it coexisted with empiricism were also cases where formalism coexisted with logicism.

However, formalism's association with humanism, and occasionally with empiricism, went beyond formalism's connection with logicism. The remaining two cases where formalism coexisted with empiricism and eight of the 15 cases where it coexisted with humanism, concerned students who not only had serious difficulties with understanding how one mathematical sequence of symbols could be derived from another, but also did not attribute a prominent role to logic within mathematics, or even challenged that role altogether - at least on

¹⁸⁵ Only two of the ten cases concerned students who could not understand mathematics, but were still convinced that mathematical propositions were logically linked.

some occasions. In other words, common sense did not help such students to comprehend mathematics at the level of content and possibly at a philosophical level too, and the gap between it and mathematical reasoning was substantial. All these students saw mathematics as a human invention, and it seemed that this allowed them to suggest - even if not explicitly that mathematical reasoning was therefore a subjective product of the human mind, and hence they were justified in not understanding it, or even in disagreeing with it. Ultimately such students implied that the logic of mathematics was just one among other logics (Walkerdine, 1994). Therefore, as Ariadni indicated, mathematics and its proofs could have positive subjective meaning only for those who enjoyed reasoning in line with its logic, but it remained irrational and devoid of any meaning for students like her. Essentially, students like Ariadni did not dissociate logic, but common sense, from mathematics and proofs (Keitel, & Kilpatrick, 2005). After all, common sense was something that they could feel as their own, while logic in mathematics and proofs was simply something alien and incomprehensible. In any case, by depicting mathematics as an invention devoid of any meaning, such students were essentially reacting to the image of mathematics as logical that their culture advanced, and could make some sense of mathematics on the subjective level, even though in a negative way.

Approximately 21% of the students expressed an intuitionistic attitude, implying that at least certain mathematical facts, such as axioms, or simple properties, were intuitively graspable by human common sense (Heyting, 1956). Such facts were considered to be intuitive, because they appeared to be logical in an obvious way; in other words, they were regarded as part of a universal logic, or common sense (Beziau, 2005; Heyting, 1956; Hogan, 2010). So all the remarks relating to intuitionism were offered by students who also hinted at logicism. In fact, the allusion to intuitionism seemed to result simply from the fact that some students had no other way to explain why certain mathematical claims appeared logical to their common sense. For example, with respect to commutativity of addition, it appeared obvious that when one added two numbers, it did not matter which number would be taken first. Thus, while we were trying to clarify whether mathematics was discovered or invented and I wondered if the commutative property of addition existed before humans had formulated it, Theodosis replied positively, explaining that '[it stems] from common sense.'¹⁸⁶ As in previous cases, such

¹⁸⁶ Although Theodosis was grounding this property in common sense instead of nature, as with other students who suggested that mathematics seems to be a discovery, he was essentially claiming that mathematical properties like this could not have been defined differently. See below for the way in which Theodosis also indicated that mathematics could be seen as an invention.

students, seeing common sense in line with mathematics and its reasoning, could also attribute some meaning to it. Moreover, intuitionism was also connected with empiricism and humanism to the effect that students saw logic as being connected with either of these trends.

Empiricist considerations with respect to mathematical reasoning were put forward by 57% of the students. Such considerations concerned the generation of mathematical knowledge on the basis of observations (Resnik, 1981; Kitcher, 1984), and/or the use of pragmatic, empirical criteria for the verification of mathematical knowledge (Colyvan, 2001; Kalmár, 1967; Tymoczko, 1991). For instance, Foivos noted how mathematical concepts arose from observation, while Kleomenis suggested that we accepted certain axioms because they produced useful results. Moreover, Lysimachos explained that using mathematics was justified by its applications, while Afroditi suggested that mathematical claims were judged against empirical data.

Essentially, empiricist beliefs were advanced either by students who believed that mathematics existed in the structure of nature, as did Foivos, (Colyvan, 2001; Resnik, 1995); or by students who implied that mathematics progressed by a hypothetico-deductive model, which tested hypotheses against available empirical data - as in the case of science, as did Lysimachos (Lakatos, 1976a). Mathematics was particularly meaningful on the subjective level for the first group, as indicated in the discussion in the section on mathematical existence (see the stories of Ermis and Foivos). The second group also included students who believed that mathematics was invented. In this case, as was noted in the chapter on subjective meaning, whether or not students could find positive meaning in mathematics seemed to depend on whether they understood mathematical reasoning (mostly, but not only, at the level of content), and whether or not they were interested in being a part of the process of mathematical invention themselves (Solomon, 2006; Vollstedt, 2011). However, all but two of the students who were associated with empiricism also exhibited elements of logicism. Therefore, as was mentioned earlier, such students were likely to find positive subjective meaning in mathematics. The 11 of the 12 cases where empiricism coexisted with humanism seemed to be the result of the cases where both of

them coexisted with logicism - because students attributed both empirical and social traits to logic as common sense - and not an independent connection.¹⁸⁷

Evidence of humanism was present in 75% of the interviewed students. These students indicated that there was a human factor involved in what appeared to be mathematically logical. This human influence concerned either the whole of mathematics or parts of it. In the first case, mathematics appeared as a social construct comprising conventions, or human-made truths. As Yerasimos suggested, these conventions had been agreed upon in the past, and now were taken as logical truths out of habit, thus being very difficult to change (Bloor, 1991; Shanker, 1987; Walkerdine, 1994). Moreover, Kosmas noted that what was considered as mathematically true was not necessarily fixed, but changed as mathematical knowledge advanced (Lyotard, 1984), while Ariadni implied that it was practically subjective, depending on the arguments that someone was ready to accept (Kitcher, 1984; Walkerdine, 1994). In the second case, where human influence was restricted to parts of mathematics, the human factor was located either in axiom-like statements or symbolism. For instance, Kleomenis asserted that axioms were the necessary raw material for mathematical logic, and that axioms had been put forward by humans in order to solve problems that were important for them (Bloor, 1994; Ernest, 1991; Hersh, 1997). Furthermore, Theodosis initially suggested that mathematics was invented because 'somebody thought of making this rule, or this code in order to help us with something.¹⁸⁸ In other words, he indicated that the code of mathematical symbols and practices was unavoidably introduced by humans, (Hersh, 1997; Rotman, 1993).

Mathematics seemed to be highly meaningful on the subjective level for students in the second group, who would combine meaningful elements from both the modern and the traditional philosophical trends, suggesting that mathematics had both human and transcendental elements (Ernest, 1991; Hersh, 1997). Thus, on the one hand, students like Theodosis had access to beliefs which could carry subjective meaning by portraying mathematics as human; while on the other hand, they also had access to beliefs which could carry subjective meaning by

¹⁸⁷ It was, however, possible to suggest that a human invention could be inspired by observation (Kitcher, 1984), even without connecting logic with mathematics, as was indicated by the single case where empiricism coexisted with humanism, but not with logicism.

¹⁸⁸ This statement echoes Kleomenis and Diomidis, who suggested that axioms, or mathematics in general, was invented in order to help humans.

portraying mathematics as universal¹⁸⁹ (Ernest, 1991; Hersh, 1997). Another example is that of Kleomenis, who claimed that mathematics was invented (human factor), but in a way that permitted it to transcend human fallibility and subjectivity, and thus be logical and certain. Nevertheless, among those students who perceived the human factor as encompassing the totality of mathematics, some felt that mathematical reasoning and the associated activity, even if human, was not accessible to them, since it contradicted their common sense (Ariadni's story) (Harel, & Sowder, 1998). For these students the human face of mathematics (Ernest, 1998a) could not result in positive subjective meaning; however, it did seem to have the potential to somewhat alleviate any oppression engendered by the apparent rigidity of mathematics (Ernest, 1991; François, & van Bendegem, 2007; Hersh, 1997). For instance, Yerasimos expressed relief when I told him that it was not necessary for a mathematical question to always have the same answer, but that the answer could depend on different assumptions that we may make. It appeared that by attributing social traits to mathematical logic such students could be helped to make some sense of mathematics on the subjective level, but not in a particularly positive way. Still, this was possibly the only means they had for attributing subjective meaning to mathematical reasoning which was enforced upon them by their society while they did not understand it.

In all, it seemed that, although students were influenced by their cultural context in so far as they stressed the role of logic in mathematical reasoning, they did not perceive this role in the fundamental way that logicism or formalism would suggest, and as the mathematics education literature would imply (Hersh, 1997; François, & van Bendegem, 2007). For students whose common sense was in line with mathematical reasoning, grounding mathematical reasoning in experience (empiricism) or society (humanism) meant bringing it even closer to common sense, and thus could help them to enhance the positive subjective meaning that they found in mathematics (Keitel, & Kilpatrick, 2005). Nevertheless, if students' common sense was not in line with mathematical reasoning and logic, then they seemed to interpret the latter from a humanistic perspective simply because it was enforced on them by their culture. There were hints at the pure, self-reliant logic of logicism or formalism when some students assumed that mathematical operations were taken to carry their own logic (Rayo, 2005), but this seemed to

¹⁸⁹ See above for Theodosis' belief about the commutative property of addition existing prior to its discovery. Theodosis' example was not discussed in the chapter on subjective meaning because both the themes of discovery and invention were secondary themes in his case.

be a device to explain the fact that certain results of mathematical reasoning appeared to contradict students' common sense. A short section concerning common sense follows in order to illuminate more fully the extent to which students found subjective meaning in mathematics through logic as a factor of mathematical reasoning.

Common sense

The students' beliefs about common sense appeared to play a more significant role with respect to the subjective meaning that students attributed to mathematics than did their beliefs about mathematical logic (Harel, & Sowder, 1998; Hofer, & Pintrich, 2002). In fact, students alluded to a breach between common sense and mathematical reasoning whenever they referred to a mathematical concept (level of content) or trait (philosophical level) which they found hard(er) to make sense of and to internalise (Freudenthal, 1991; Keitel, & Kilpatrick, 2005). For example, Lysimachos, who generally understood mathematics well, talked about the unintelligibility of infinity, echoing Brouwer (1913) and other intuitionists; while Yerasimos was perplexed by the apparent objectivity of mathematics and its reasoning until I discarded it for him. In all, mathematics appeared to lose (some of) its positive subjective meaning when students were unable to bridge the perceived gap between mathematical reasoning and common sense, either at the content level or at a philosophical level. That was the case with Evyenia who could hardly find any common point between her purely empirical common sense and mathematical reasoning. A similar gap puzzled Solonas with respect to non-Euclidean geometries, even though otherwise he understood mathematics quite well. As Vollstedt (2011) has indicated, students might find subjective meaning in the pure logic of mathematics, but this seemed to happen only when this logic was in accordance with their understanding of common sense (Harel, & Sowder, 1998). In this case, mathematical reasoning could acquire a strong positive meaning for students, since they would feel that it was relevant to their wider lives (Ernest, 1998b; Freudenthal, 1991; Hofer, & Pintrich, 2002; Mercier, 2010; Prediger, 2007). That was the reason Agapi considered mathematics to be highly meaningful on the subjective level, although she could offer no useful connection between its content and her life.

The perceived overlap between mathematical reasoning and common sense was apparently more substantial among students for whom mathematics came easily (alignment at the level of content). Such students found it easier to disregard any misalignments between their common sense and mathematical reasoning as exceptions, and thus could find positive subjective

meaning in mathematics (see Solonas' story, but also that of Platonas, as mentioned earlier) (Keitel, & Kilpatrick, 2005; Stenning, & van Lambalgen, 2008). Otherwise, if students had difficulty with mathematics, the breach between their common sense and mathematical reasoning at the content level was often accompanied by a breach at a philosophical level, a breach that tended to remain irreparable, causing a significant blow to the subjective meaning that students could attribute to mathematics (Vollstedt, 2011). Thus, Filippos, who struggled with mathematics (content level), could not really find any positive subjective meaning in it or its rigid reasoning, although he admitted that occasionally it appeared to be relevant to life. In fact, his common sense would not appreciate following rigid rules (philosophical level), and would judge such occasions as a source of discomfort. Students who managed to bridge the gap between common sense and mathematical reasoning in a positive way, even though they did not understand mathematics at the level of content, were students who felt at ease with an absolutistic view of life (Kuhn, 1991). Therefore, they could attribute positive subjective meaning to mathematics as a field of absolute truths, as did Afroditi.

An aspect of common sense which appeared to have the potential to influence significantly the subjective meaning that students found in mathematics was inductive, empirical reasoning (Hanna, & de Villiers, 2008; Healy, & Hoyles, 2000; Recio, & Godino, 2001; Stylianides, & Stylianides, 2009). Approximately one third of all the students exhibited a strong preference towards inductive, experiential logic. Some of these students could accept that deductive logic produced arguments that appeared valid, but, as Lysimachos commented, this did not necessarily mean that these arguments were logical or true, as they might have no reference to the respective student's experience (Stenning, & van Lambalgen, 2008). It seemed that a stronger empirical basis for mathematics (Kitcher, 1984) would make mathematics more meaningful on the subjective level for such students. Having considered at some length the issues of logic and common sense, I now turn to the issue of rules as part of mathematical reasoning.

Rules

Students' beliefs on rules were presented in the ontology and epistemology chapter. Here the focus is on students' understanding of rules as a source of authority, since this seemed to be a decisive factor for the subjective meaning that students could attribute to mathematics. Authority issues were also discussed in the epistemology Chapter. The discussion that follows

also involves proof, which was generally considered by students as evidence for the authority of mathematical rules (Amit, & Fried, 2005; Harel, & Rabin, 2010).

The view that mathematical reasoning required an adherence to, and the application of, certain rules or guidelines (Sfard, 2000; Wittgenstein, 1953) was espoused by all students, and was in line with the traditional context in which they had learnt mathematics (Garofalo, 1989; Tzekaki, Kaldrimidou, & Sakonidis, 2002; Schoenfeld, 1992). What differed among them, as the previous section indicated, was their answer to the question of the origin of mathematical rules in logic, in nature, in empirical applications, or in human activity. Nevertheless, the subjective meaning that students would find in mathematics and its rules appeared to also depend on the authority students assigned to rules. In particular, the subjective meaning that mathematics had for students seemed to be affected by the extent to which they were willing to submit to mathematical authority and follow mathematical rules unquestioningly, (Romberg, & Kaput, 1999; Solomon, 2006). Finally, the subjective meaning of rules was also influenced by the extent to which students were capable of using these rules effectively and creatively to solve mathematical problems (Rowlands, & Carson, 2002; Skovsmose; 2000; Yow, 2012).

Not all students could follow rules, which they did not understand, in order to enhance their performance in mathematics (De Corte, et al., 2002). In particular, students whose common sense was not aligned with mathematics at the level of content would mostly find it hard to apply a rule that they did not understand, even if they wished to do so. That is why Menelaos seemed puzzled when I asked him whether he would follow a mathematical rule which he would not understand; he felt that this would be impossible, and eventually replied negatively: 'No. That's why I don't understand mathematical rules, that's why I don't follow them.' Consequently, such students would rarely gain any satisfaction from using their cognitive skills creatively in order to solve a problem (Middleton, & Spanias, 1999; Vollstedt, 2011). They could not apply rules which, for them, were like formalist rules without any meaning (Ernest, 1991; Hersh, 1997). In any case, they would not bother to ask for an explanation, accepting that any explanation would also be incomprehensible for them (Ryan, Pintrich, & Midgley, 2001; Webb, 1991). Thus, Ariadni suggested that she was 'weird' and she would not understand if her teachers tried to explain. As a result, such students would attach negative subjective meaning to mathematics and its rules.

On the other hand, students who mostly understood mathematics, that is, students for whom what counted as common sense overlapped to a greater extent with their teacher's explanations

(Harel, & Sowder, 1998), were capable of following rules, even if they did not comprehend them fully. After all, for these students, incomprehensible rules were the minority, and as Diomidis noted, experience had shown to them that it was quite probable that they would be led to comprehend a rule, which they had not grasped initially, through deploying it in exercises. This behaviour is in accord with Wittgenstein's idea that the meaning of the rule is located in its use (Wittgenstein, 1953). Nevertheless, the understanding to which students referred did not necessarily involve an explanation for the rule. Lysimachos suggested that the rule could simply be regarded as justified because of the fact that it was repeatedly producing useful results in solving exercises (Rigo-Lemini, 2013). In any case, for such students, rules which were left unexplained did not necessarily influence negatively the subjective meaning they could find in mathematics. Since their common sense mostly agreed with mathematics, they could discard any exceptions and appreciate mathematics as subjectively meaningful on the basis of being able to use mathematical rules effectively and even creatively. As discussed above, this meaning would also depend on whether they perceived mathematical rules as originating in logic, nature, experience and/or human activity. However, for students like Solonas, some parts of mathematics would remain a mystery, thus damaging the overall positive subjective meaning they might attribute to mathematics.

Any subjective meaning that mathematical rules retained for students who found it hard to grasp and use the rules was related to that of an oppressive authority (Amit, & Fried, 2005; Rowlands, & Carson, 2002; Skovsmose; 2000). This meaning would be rendered even more negative if the students' common sense was not in line with mathematics at a philosophical level regarding the rigidity of its rules. This negativity could only be partially mitigated by perceiving the rules as invented contraptions. For example, Filippos described how oppressed he felt by the rigidity of mathematical rules. Essentially, he would prefer not to follow mathematical rules at all. Nevertheless, he was expected to use the rules in the classroom in any event. Consequently, such students had no choice but to accept mathematical rules as correct, at least on the surface, since their teachers, along with everyone in wider society, seemed to regard them as correct (Alrø, & Skovsmose, 2002; Rigo-Lemini, 2013). It followed that if students claimed otherwise they would be at best ignored, or at worst mocked (Ryan, et al., 2001; Webb, 1991). Nevertheless, deep down these students did not agree with mathematical rules. That is why Filippos declared that such rules were subjective even though whenever he was forced to examine this claim closer, he was not able to defend it.

However, there were also students who were willing to accept the authority of mathematical rules, without having a justification to support it. Admittedly, there were cases when this was the only option students had, since their teacher would simply announce that a particular statement was proven using higher mathematics (Amit, & Fried, 2005). Nevertheless, half of all the students seemed willing to conform with authority, even if they did not like it. This could be a behaviour they had learned, being taught in a traditional context where most of the information derived from teacher transmission (Alrø, & Skovsmose, 2002). So, and as long as, their teacher confirmed a rule's validity, they did not judge it as necessary to also have the rule proven or explained to them (Rigo-Lemini, 2013; Rowlands, et al., 2011; Alrø, & Skovsmose, 2002). This end was also helped by the cultural power that proof seemed to have as a result of the emphasis given to it within the Greek educational context (Sdrolias & Triandafillidis, 2008). This power was evident regardless of whether students generally understood mathematics and the logic behind it, and indicated an alignment between common sense and mathematics though, as in the case of Filippos, this alignment could concern the common sense of society instead of the student's own common sense, and therefore could be effectively irrelevant for subjective meaning. Nevertheless, if the students understood mathematics (content level), then they could feel successful in the classroom even without explanations and proofs, and this could counteract the negative effect that an externally imposed authority could have on subjective meaning (Vollstedt, 2011).

In particular, students who were content without explanations could find positive subjective meaning in mathematics if they valued the certainty stemming from having straightforward rules to guide them (Hersh, & John-Steiner, 2011; Vollstedt, 2011), and/or if they trusted authority (Rigo-Lemini, 2013), which meant that the apparent rigidity of reasoning within the authority of mathematical rules was in line with their own common sense and not simply with that of the society. Such students could trust either their teachers, who would not lie to them, and/or mathematics as a discipline which comprised proven, and thus correct, statements (Amit, & Fried, 2005; Alrø, & Skovsmose, 2002; Rigo-Lemini, 2013). For instance, Filia's common sense seemed content to reason with rigid rules assuming that once something was given the status of a rule it was inviolable, correct and true. Such students did not need to have seen the proof; it was enough to know, or to assume that it existed. After all, they were just students, they were not mathematicians. On this view, they had no reason to invest more energy in understanding mathematics than that which was necessary for doing well in school (De

Corte, et al., 2002; Vollstedt, 2011). As Agapi noticed, we generally tend to accept what other people tell us as true without being suspicious (Stenning, & van Lambalgen, 2008).

Only students who were quite passionate about mathematics appeared to need something more than their teachers' or mathematics' authority. They needed to take ownership of their knowledge, and internalise any relevant authority, so that it was not external. That was why Andromachi suggested that proofs were essential for understanding mathematics (Hanna, 2000; Rowlands, et.al., 2011). Such students could attribute an authority-free subjective meaning to mathematics (see the stories of Ermis and Kleomenis). It can be claimed that such a meaning was a healthier one, but this would not necessarily imply that it was also a deeper one. Thus, in the same way that Ermis or Kleomenis were passionate about mathematics, Filia seemed to be passionate about the authority of mathematics.

The need for knowing the justification and proof of a rule was an important difference between the students' and Wittgenstein's understanding regarding the meaning of a rule residing in its use (Wittgenstein, 1953). Wittgenstein would consider proof as part of the way a mathematical rule is used, and thus as part of its meaning (Shanker, 1987). However, 75% of the students considered proofs necessary only for mathematics as a science, and not for themselves as learners of mathematics (De Corte, et al., 2002; Rowlands, et al., 2011; Solomon, 2006). After all, understanding proofs was not so significant for their examinations (Basturk, 2010; Vollstedt, 2011). Hence, Ariadni noted that mathematicians are expected to prove their claims, while she also made clear that such proofs were inconsequential to her. What seemed to matter for students was that they knew that mathematical claims had proofs, and that proofs were socially sanctioned. So Vrasidas, for instance, felt that there was no reason to doubt a proven result. In other words, students would trust mathematical authority based on proof to be reliable, because that was the role of proof according to society, and especially according to the Greek context of mathematics education (Amit, M., & Fried, 2005; Sdrolias & Triandafillidis, 2008; Skovsmose, 2000). In fact, trust in social authority seemed to be stronger even in cases where students valued proofs. That was why students, such as Agapi, would regard proofs as informative, but not as necessary.

In all, what seemed to matter, in terms of subjective meaning, was the extent to which the students' common sense could help them to understand mathematical rules either at the level of content, or at a philosophical level regarding following rigid rules. As was discussed above, severe difficulties in understanding mathematical reasoning at the level of content rendered

mathematics consonant with formalism, i.e. a congeries of meaningless rules (Ernest, 1991). In the opposite case, students were able to use the rules more or less effectively and creatively, and this would help them to find positive subjective meaning in mathematics as a set of rules (Vollstedt, 2011). Moreover, if the students' common sense did not value reasoning within an authoritative context of rigid rules, then they also tended to attribute negative subjective meaning to the rigid application of mathematical rules. That was why Foivos was not so fond of mathematics, although he valued it as a means to understand nature. Such students were effectively unwilling to internalise the authority of mathematical rules and proofs that their culture advanced. On the other hand, when the students' common sense valued reasoning with rigid rules, students were content to accept the authority of mathematical rules and proofs and mathematics became subjectively meaningful to them (Rowlands, et al., 2011). In fact, the subjective meaning attributed to mathematical rules could be authority-free only if students were interested in comprehending mathematical proofs, and could therefore be less influenced by the cultural power of proof upon which their context placed a strong emphasis (Hanna, 2000; Rowlands, et.al., 2011; Sdrolias & Triandafillidis, 2008).

Mathematical certainty

This section concerns students' beliefs relating to the certainty of mathematics. These beliefs were presented in the ontology chapter, where certainty was connected with immutability as both its implication and its prerequisite. In fact, both in philosophy and in students' accounts, the certainty of mathematics could be seen as closely intertwined not only with immutability, but also with objectivity and truth (Ernest, 1991; Hersh, 1997). Consequently, although the ontology chapter included separate sections relating to the themes of truth and objectivity, in the current discussion, these topics are handled simultaneously, as if they were effectively synonymous with one another. This is practically unavoidable, since the aim of this chapter, as a whole, is not to present different themes, but to discuss interconnections between different beliefs. In order to facilitate the presentation, certainty has been chosen as the central topic, since it also constitutes the main axis around which philosophical trends have evolved.

Mathematical knowledge appeared indubitable to more than three quarters of all the students. As with philosophers, the apparent mathematical certainty seemed to have left a strong impression on most of the students' minds (Ernest, 1991; Hersh, 1997). This picture is in line with the cultural emphasis given to the indubitability of proofs (Sdrolias & Triandafillidis,

2008), and generally with what is expected in a traditional mathematics classroom, where teaching is restricted to transferring knowledge from the teacher to the student, and thus is taken to imply an indubitable external mathematical reality (Carpenter, & Fennema, 1992; Chassapis, 2007; Cobb, et al., 1992).

However, the roots of students' certainty revealed a different picture. As in the case of logic, students blended traditional ideas about absolute mathematical certainty with modern philosophical views, according to which mathematical certainty was - at least partially - socially constructed (Bloor, 1991). In fact, purely traditional solutions to the problem of certainty were offered only by 39% of all the students. As would be expected, according to the literature (Ernest, 1991; François, 2007; Hersh, 1997), traditional solutions - whether pure or not - were mainly connected with some of the issues discussed earlier, in particular with mathematical existence (32%) (Frege, 1964), and logic and proofs (57%) (Hilbert, 1983; Russell, 2007). The second solution also bore witness to the emphasis placed upon proofs, and thus upon logic, in the Greek cultural context (Sdrolias & Triandafillidis, 2008). In the following, I start with traditional beliefs related to mathematical existence and logic, and I conclude with modern beliefs regarding social certainty.

As with philosophers, students tended to connect the belief that mathematics existed with the conviction that mathematical knowledge concerned objective, eternal truths (Frege, 1964; Hersh,1997). The mere fact of existence could be taken to make something true and objective (Kirkham, 1995; Patterson, 2003). Assuming that mathematical claims reflected reality, students could readily claim that what was mathematically true was necessarily true. For instance, Foivos and Ermis stressed that mathematical concepts were defined on the basis of what existed, so a definition could be expressed in different words, but eventually, all such linguistic formulations would have to point to the same thing. In general, if mathematics existed, the answers to mathematical questions were taken to be dictated by an objective reality, and thus could not have been different from the ones we already have (Shapiro, 2000). In other words, apart from being true they were also regarded as indubitable and thus certain.

In any event, mathematics was rich in subjective meaning for the students who based mathematical certainty on its very existence. Nevertheless, as mentioned earlier, this seemed to be the case precisely because students believed that mathematics existed, and not so much because they perceived it as certain. Certainty and immutability seemed to be coincidental traits which added, or not, to the subjective meaning of mathematics, depending on other factors. So certainty could add to such a meaning when students held a more or less absolutistic view of life (Kuhn, 1991), that is, when their common sense was in line with the perceived absolutism of mathematical reasoning. Students like Filia, or Afroditi¹⁹⁰ felt comfortable with certainty; they actually wished for greater certainty, since they assumed that it was the ideal. According to them, all knowledge was potentially objective, while disagreements appeared to emerge only due to human imperfections. However, the perceived certainty seemed to be subjectively meaningless for students, such as Foivos, who appeared to feel more comfortable with the uncertainty associated with emotions which allowed him to express himself more freely. Foivos believed mathematics to be important because it existed, but he also felt that it was too rigid for his taste.

Apart from mathematical existence, 57% of the students connected mathematical certainty and immutability with logic (Ernest, 1991; Russell, 1969). This view was advanced regardless of whether students had advocated that mathematics comprised a discovery or an invention (Rowlands, et al., 2001). In fact, among students who believed that mathematics pre-existed and was discovered, logic could be pushed in the background, since existence was sufficient to guarantee truth, immutability and subsequently, certainty. Nevertheless, judging by the emphasis put on each topic, for Platonas, who believed that mathematics 'was in the orderliness of nature',¹⁹¹ certainty based on logic appeared to be even more salient than certainty based on existence. Thus, it seemed that instead of implying that mathematics was logically certain because it existed (Frege, 1964), Platonas hinted that mathematics existed because it was logically certain (Brown, 2008; Hersh, 1997).

In any case, logic could also guarantee immutability and certainty independently of mathematical existence. For instance, Kleomenis would vouch for mathematical determinism and the resultant certainty. Students seemed to ground the belief that mathematical certainty was based on logic in a view of logic as producing valid, generalisable arguments, or in other words, proofs, (Corcoran, 1994; Curry, 1951), and as being devoid of any subjective extralogical traits such as emotion (Hardy, 1940). The latter also guaranteed that mathematics was objective. For instance, Agapi compared mathematical arguments and proofs with syllogisms where an inference is reached through purely logical means. Moreover, Foivos attributed

¹⁹⁰ Both had claimed that mathematics was discovered.

¹⁹¹ As he claimed while he was trying to explain how mathematics began.

mathematical clarity and the resultant certainty to the fact that mathematical logic was devoid of that emotion which was responsible for the lack of clarity in life.

For such students, certainty based on logic could be subjectively meaningful to the extent that their common sense was in line with mathematical reasoning at the level of content, or even at a philosophical level (see Kleomenis' story). In fact, an alignment at the philosophical level seemed to be sufficient for students, such as Afroditi, to at least value mathematical certainty even if they could not entirely understand it. Such students appeared to be content to endorse the picture of mathematics as certain and objective in the way it was presented by their culture. Only when a misalignment at the level of content was coupled with a misalignment at a philosophical level did students seem to have trouble enjoying mathematical certainty (see Foivos' story) even if they valued it (see Polyxeni's story). Effectively, such students appeared to wish to react to the apparent objectivity and certainty of mathematics that their culture supported.

In any event, when students understood mathematical reasoning, more or less - having seen the proofs for many of the mathematical propositions that they had encountered - they were convinced that these proofs were logical and correct (Hanna, 2007; Rowlands, et al., 2011). Hence, such students were certain of these propositions, which were essentially compatible with their common sense (Rigo-Lemini, 2013). Nevertheless, as a result of the cultural power of proof in the tradition of Greek mathematics education, their trust in the indubitability and objectivity of mathematical proofs was not restricted to propositions for which they had seen the proof (Amit, & Fried, 2005; Sdrolias & Triandafillidis, 2008). On the contrary, they tended to generalise their experience and assume that all mathematics was trustworthy, in a similar way that some students would inductively proceed to accept a mathematical claim based on a few examples (Stylianides, & Stylianides, 2009). The only possible exceptions could be axioms, as long as students were aware of their function in mathematics. So Lysimachos acknowledged that mathematical truth and certainty based on logic and proofs was not unconditional, but required a fixed axiomatic system (Giaquinto, 2002; Grattan-Guinness, 2000).

Consequently, the issue of truth was not as straightforward when certainty was taken as the result of logic instead of mathematical existence (Russell, 2007). After all, when certainty was associated with mathematical existence, the truth of mathematical statements was entailed by the very existence of mathematical objects and not by the certainty of the statements (Resnik,

1981), while logic could handle with certainty even things which did not exist (Stenning, & van Lambalgen, 2008). In fact, it can be claimed that the initial statements on which mathematics is based are not necessarily true, and indeed students, such as Lysimachos, Kleomenis, indicated that we are unaware of their truth. Such students would conclude, therefore, that the truth of a proven mathematical claim depended on whether the axioms on which the proof was based were taken to be true or not. Nevertheless, students who trusted mathematical logic found it hard to go beyond the apparent truth of mathematical axioms, or the statements which seemed to lie at the foundation of mathematics (in cases where they did not know about axioms). They would allow themselves to accept axiom-like statements as true, because they 'rang' true; they were in line with their common sense, and so they were inclined to accept them as true regardless of proof (Bloor, 1994; Gödel, 1983; Sankey, 2009). That was why Platonas was actually surprised to hear that the obvious - for him - fifth postulate of Euclid had no proof.¹⁹²

Apart from the traditional solutions on certainty, 57% of the students also alluded to the modern view that mathematical certainty was socially constructed, that is to say, that mathematics was certain, immutable or objective because society accepted it as so (Bloor, 1991; Ernest, 1998a). This perspective was virtually absent from students who perceived mathematics as certain because it existed (Hersh, 1997). Nevertheless, such a solution was offered by some students who grounded mathematical certainty in logic while assuming that mathematics was invented (21%). So students like Danai seemed to imply that mathematics was contingently stable and certain, in the sense that it could be done differently if humans would choose to do so, but it could be regarded as certain as long as humans took mathematical conventions for granted. However, such students did not tend to stress the social aspect of certainty, and had no reason to do so, since for them mathematics was only theoretically amenable to change. In practice, it was immutable and certain, since they would find it hard to imagine that anyone would defy the culturally given traditions (Bloor, 1991; Restivo, 1991).

Nevertheless, 36% of the students attributed mathematical certainty only to society without connecting it to logic (Amit, & Fried, 2005; Rowlands, et al., 2011). These were students whose common sense was more or less incompatible with mathematical reasoning both at the content level and at a philosophical one. Thus, not being able to understand mathematical reasoning

¹⁹² Both in the first and in the second interview.

and proofs, such students, confronted with a picture of mathematics being certain and objective - as advanced by their cultural context - had no other means to explain it apart from the fact that it was socially imposed on them (Bloor, 1991; Skovsmose, 1994). As discussed earlier, mathematics was subjectively meaningless for these students. They could make sense of mathematics on the subjective level - though not in a positive way - only to the extent that they believed that the rigid authority of mathematics reflected the rigid authority of society (see Filippos' story). Hence, seemingly inclined to conform to an external authority, some of these students likewise accepted the authority of mathematical certainty, considering it to be in accordance with the usual state of affairs (Freire, 1996; Krishnamurti, 2001; Skovsmose, 1994). For example, Vrasidas saw no reason to challenge a science such as mathematics which was supposed to have proofs. Such students were not really concerned about whether mathematics was rightfully called true or objective. They simply knew that it was called so, and from this they inferred that it should be so. Consequently, these students tended to feel more or less oppressed by mathematics (Restivo, 2011; Torres, 2014) and wished to revolt against the image offered by their cultural context, since they had significant trouble understanding mathematics. Thus, although they would echo what their society and culture wished them to believe, deep down they did not consider mathematics as actually objective or true. On the contrary, at certain points they would portray mathematics as the subjective opinions of some persons. So Evyenia admitted that she did not always agree with what was in her book, while Filippos ultimately declared mathematics to be subjective. It almost seemed that such students would agree with intuitionists that mathematics was based on properties of the mind (Heyting, 1956; Shapiro, 2007), but they would reject the claim that these properties were universal. That was why Ariadni specifically separated herself from people who liked mathematics.

Connected to all reasons for assuming mathematical certainty was proof, a finding which can be attributed to the emphasis placed on proof in the Greek cultural context. Proof seemed to be able to encompass all aspects of certainty. Firstly, proof was directly connected with the belief that certainty was the result of logic, since students who advocated this belief considered proofs to be logical. Hence Platonas explained that proofs, based on logic, led to certain results. Secondly, proof could be associated with certainty on the basis of mathematical existence by implying that mathematical proofs were certain due to that existence which, as Ermis remarked, meant that mathematical concepts were defined in an unequivocal way. Thirdly, proof seemed to be a powerful tool on a cultural level, since, as Vrasidas indicated, it was sanctioned by society as a sound, indubitable argument (Nickson, 1994; Walkerdine, 1994). However, in general (79%), certainty and immutability were not viewed by the students as absolute. Socially constructed certainty is not supposed to be absolute in any case; conventions are not necessities (Ernest, 1991). That was what allowed certain students, such as Filippos, effectively to disagree with them, at least on a personal level. But even students who regarded certainty and immutability as based on logic or mathematical existence were not entirely confident about them. This certainly reflected their own lack of expertise, but students, as Diomidis, also noted that mathematical knowledge was a developing entity. Therefore, effectively, lack of expertise could be assumed for humanity as a whole which did not possess the entirety of potential mathematical knowledge.

In the case of students who assumed that mathematics existed, any occasions of uncertainty could be explained by the distinction between mathematics and mathematical knowledge, which was mentioned earlier in the section on the plurality and cohesiveness of beliefs. In the case of students who saw mathematics as a logical invention, a distinction could be drawn between mathematical logic as an ideal, on a theoretical level, and logic as it was actualised in practice by human beings (Rowlands, et al., 2011). In both cases what was portrayed as objective, certain, and unchanging seemed to be either mathematics or mathematical logic. However, mathematical knowledge was only assumed to approximate these qualities. Although it was considered to be relatively stable, the production of mathematical knowledge was still portrayed as a fallible human activity and as a historically developing entity, where new discoveries could lead to an amendment of older theories (Heeffer, 2007; Jankvist, 2010; Lakatos, 1976b). This seemed to be the reason why, when I asked Agapi if mathematical conclusions were solid and unshakable, she accepted the former but not the latter; while Kosmas insisted that truth had an expiry date.

In all, the degree to which students were ready to suggest that mathematical knowledge was certain, objective, or immutable seemed to be determined by their general epistemic attitude towards certainty, according to their common sense (Muis, et al., 2006). Such an influence seemed to be present irrespective of the particular beliefs which students held about mathematics. In fact, contrary to what mathematics educators might predict or assume (Ernest, 1991; François, & van Bendegem, 2007), the student who was ready to doubt mathematical knowledge more than any other was Ermis, who believed that mathematics existed. Ermis' common sense was permeated by a scientific attitude, willing to doubt everything in order to explore and learn more. So although, he would claim that mathematics itself was immutable,

he suggested that our knowledge of it developed. Hence, he insisted that we should not take anything for granted, because, for example, tomorrow another mathematician may find some cases for which our current understanding may no longer be true. Moreover, students such as Danai, whose common sense suggested that it was difficult to be certain of what was correct at least outside mathematics - transferred some of this uncertainty to mathematics. On the other hand, students, such as Afroditi, whose common sense judged that certainty might be attainable, if it were not for human deficiencies, tended to be less hesitant about the certainty of mathematical knowledge.

Only two students appeared to exclude certainty from mathematics completely, ignoring the influences of their cultural context. These students did so because such an uncertainty rendered mathematics subjectively meaningful to them. One of these students was Ermis who, as explained above, focused entirely on the uncertainty of mathematical knowledge (Ernest, 1991; Lakatos, 1976b). After all, it was this uncertainty which carried for him positive subjective meaning, since it promised him an interesting journey full of surprises (Bruner, 1966; Drengson, 1981). The second student was Lida, who had come to equate mathematics with open mathematical problems, where the answer was not predefined. These were probably the minority of the problems which she had encountered. Notwithstanding, they were the kind of problems that could make mathematics subjectively meaningful for her, since they allowed her to investigate, and they were also in line with the subjectivity dictated by her common sense (Ernest, 1998a). There was a third student, Yerasimos, who eventually banished objectivity and certainty from mathematics completely. However, he proceeded to do so only after I indicated that the same question might have different answers in different mathematical systems. Up to that point, his common sense made it hard for him to believe that mathematics was so different from life and always produced the same answer to a problem, but that was what everyone around him was claiming. When I gave him a way out, he seized it willingly, freeing himself from the subjectively meaningless objectivism of mathematics (Ernest, 1998a; Hersh, 1997).

Ultimately, students seemed to be strongly influenced by their culture as far as their beliefs on certainty, truth and objectivity were concerned. These influences tended to be resilient even when students advanced less traditional solutions for these issues. Nevertheless, any willingness on the part of the students to doubt mathematical certainty and immutability possesses special gravity, since they had been bombarded by cultural images of mathematics as certain, immutable, and objective (Charalambous, et al., 2009; Sdrolias & Triandafillidis,

2008; Tzekaki, Kaldrimidou, & Sakonidis, 2002), whilst unlike professional philosophers, they lacked substantial experience of mathematical claims having been doubted in the past (Heeffer, 2007; Hersh, 1997). Apart from encountering complex numbers in the science or technology track, students had no examples which could sway their certainty, and a single example could be easily discounted. Moreover, complex numbers could be perceived rather as complementing rather than contradicting what students knew about real numbers. This relationship between old and new knowledge could be generalised to include all mathematics.

Moreover, certainty was part of the subjective meaning that students found in mathematics either positively or negatively. Vollstedt (2011) has observed that some students enjoy the certainty of mathematical knowledge and this adds to the personal, subjective meaning that mathematics has for them. This seems to confirm the traditional view of certainty, and is in line with the description of the positive subjective meaning of certainty as a basic psychological need (Antonovsky, 1994; Crawford, & Rossiter, 2006; Korotkov, 1998). Certainty means that mathematical claims can be trusted, (see Agapi's story), that one can be in control when handling mathematics (see Kleomenis' story), and that it is easier to avoid mistakes (Hersh, & John-Steiner, 2011). The latter could be important even for students who had difficulties with mathematics, but had a great need for certainty - although in that case the subjective meaning they would attribute to certainty would not be particularly positive (see Polyxeni's story). On the other hand, when certainty was purely seen as socially constructed then it could become oppressive (Hanna, 2007; Nickson, 1994), even when students accepted it as natural (Freire, 1964; Krishnamurti, 2001). In that case, students could find subjective meaning in mathematics by denying - at least partially - mathematical objectivity and any resultant certainty.

Concluding remarks

The above discussion shows how, in a single grade where students had relatively similar backgrounds, different images of mathematics emerged, correlating with the subjective meaning that individual students attributed to it (Op't Eynde, et al., 2006). Moreover, students' beliefs about mathematics were not restricted to those traditional beliefs which might be expected according to their traditional learning context (Coe & Ruthven, 1994). In fact, many beliefs about mathematics which could be termed as progressive seemed to be in line with what the students' common sense would suggest. Finally, what gave cohesion to students' accounts

was not this or that philosophical concept, but the subjective meaning that mathematics had for them.

Although platonism is the philosophical trend which is associated with traditional learning methods, as are to be found in the Greek cultural context, (Charalambous, et al., 2009; Chassapis, 2007; Cobb, et al., 1992), this was conspicuously absent from the students' accounts. Some students did hold the belief that mathematics existed, but in that case, they mostly located mathematics in the structure of nature, as empiricists would do (Colyvan, 2001; Resnik, 1995). Moreover, contrary to what would be expected (Ernest, 1991; François, & van Bendegem, 2007), this belief did not result in a feeling of alienation from mathematics, but made mathematics subjectively meaningful for students. This was probably why it had been sustained despite the fact that the students' cultural context placed emphasis on abstract aspects of mathematics (Sdrolias & Triandafillidis, 2008). Mathematical existence meant that mathematics involved truths about reality, and this had positive subjective meaning for students who were interested in discovering truths about their world (Hersh, & John-Steiner, 2011; Loewenstein, 1994; Vollstedt, 2011). Moreover, the stress on the abstract aspects of mathematics seemed to actually influence students in favour of the belief that mathematics was invented. This was a belief which was also advanced by all students who were alienated from mathematics. Such a belief allowed them to portray mathematics as essentially subjective (Shapiro, 2007), and thus to justify why they did not understand it and why it was subjectively meaningless for them. Notwithstanding, a belief that mathematics was invented added to the subjective meaning which mathematics held for students whose common sense was in line with mathematical reasoning, and who therefore could perceive themselves as comrades of mathematicians in the journey of mathematics (Hersh, 1997).

Mathematical logic, although stressed due to cultural influences (Sdrolias & Triandafillidis, 2008), was also interpreted in various ways which were not exclusively traditional. Formalist views were present when students did not understand mathematics (Ernest, 1991), but students also grounded mathematical logic either in empirical experience or in social conventions (Kitcher, 1984; Ernest, 1998a). In cases where students could understand mathematics, this seemed to have the potential to enhance the subjective meaning they could find in it, since it would bring mathematical reasoning closer to common sense. Still, students whose common sense was not in line with mathematics (neither at the content nor at a philosophical level), tended to view mathematical logic through a formalist lens and would portray it as subjectively

meaningless; any humanistic traits that they would attribute to it seemed to be merely the result of the fact that this logic was enforced on them by their culture.

Generally, the degree to which students could find positive subjective meaning in mathematics was influenced by the extent to which their common sense was compatible with mathematical reasoning, either at the level of content or at a philosophical level. (Keitel, & Kilpatrick, 2005). In fact, a (mis) alignment usually appeared together at both levels. In any event, a misalignment at the content level could be ignored if the student appeared to be at ease with strong absolutistic views about life (at the philosophical level). Similarly, the rigidity of mathematical logic and its rules would be oppressive only as long as they were not in line with the students' common sense at both levels. Thus, students would attribute negative subjective meaning to mathematics as a set of rules when they did not feel comfortable with authority, or where they were not able to use these rules successfully (Rigo-Lemini, 2013; Vollstedt, 2011). Otherwise, students, also influenced by the cultural power of proof, would not object to the authoritative status of mathematics (Harel, & Rabin, 2010). In fact, the emphasis given to proof in the Greek cultural context (Sdrolias & Triandafillidis, 2008) meant that the subjective meaning of mathematical rules could be authority-free only if the students could actually understand most proofs and did not accept them as indubitable simply because their culture suggested that they were so (Hanna, 1995; Rowlands, et al., 2011).

Finally, certainty was also not based on platonistic beliefs. However, it was strongly evident in the students' accounts (François, & van Bendegem, 2007), although it was hardly ever absolute. The extent to which students were hesitant about mathematical certainty was influenced by their general epistemology (Muis, et al., 2006), that is, by their general attitude towards certainty in life, as dictated by their common sense. Moreover, students did not base certainty only on mathematical existence or on logic and proofs (Frege, 1964; Russell, 1969), but also on wider social conventions (Ernest, 1998a; Hersh, 1997). Associating logic and proofs with certainty, truth and objectivity was quite common since all these issues were advanced simultaneously by the traditional context within which students were taught (Sdrolias & Triandafillidis, 2008; Tzekaki, Kaldrimidou, & Sakonidis, 2002). Nevertheless, the certainty that students attributed to logic was often a culturally constructed one. This was especially the case with students whose common sense was not in line with mathematical reasoning, and therefore had no other way to explain the certainty and objectivity imposed on them by their culture (Bloor, 1991; Skovsmose, 1994). Such students would not find any particularly positive

subjective meaning in mathematical certainty even if they appreciated the fact that it could safeguard them against mistakes. However, in line with what psychology and the philosophy of mathematics suggest (Antonovsky, 1994; Russell, 1969), the image of mathematical certainty could be subjectively meaningful for students whose common sense was in line with mathematical reasoning.

Conclusions

I wish to start by summarising the main intellectual contribution which this study has sought to make, before moving to a consideration of the practical implications which the work indicates. I then offer some observations about the methodological contribution made by the study, concluding with some reflexive remarks regarding how conducting this study has enriched my understanding of mathematics and of doing research.

Intellectual contribution

Regarding the data as a whole, this study showed, in accordance with Op't Eynde, et al. (2006), that students with similar backgrounds may hold quite disparate ideas about the nature and the creation of mathematics and mathematical knowledge. Moreover, the findings indicate that if one examines students' ideas in depth, then the variations that come to the surface are very likely to suggest deviations from the rigid picture of mathematics which is associated with the traditional way that mathematics is usually taught - in Greece, as elsewhere - that is, as a set of fixed, eternal rules that exist independently of humans (Carpenter, & Fennema, 1992; Charalambous, et al., 2009; Chassapis, 2007; Cobb, et al., 1992; François, & van Bendegem, 2007; Tzekaki, Kaldrimidou, & Sakonidis, 2002). Although the students involved in this research were apparently influenced by their cultural context - as seen for example in the emphasis that they would put on proofs (Sdrolias, & Triandafillidis, 2008) - they would also express beliefs which were not in line with this context. This was because their beliefs were more in line with their common sense, helping them to make sense of mathematics on the level of objective meaning, but also allowing them to attribute some subjective meaning to mathematics - whether positive or negative.

For example, despite having been taught in a traditional context (Tzekaki, Kaldrimidou, & Sakonidis, 2002), students advanced humanistic beliefs. Such beliefs could help students whose common sense was not in line with mathematical reasoning to make sense of mathematics. In particular, they would allow the students to imply that mathematics was a subjective invention and its certainty a social construct. Thus on the one hand, such students could explain why they could not agree with mathematics, and on the other hand, they could explain why society presented it as objective. However, humanistic or empiricist beliefs would also be valuable for students whose common sense was in line with mathematical reasoning.

Such beliefs, connecting mathematical reasoning with human culture and experience, would actually bring mathematics closer to what common sense could understand (Ernest, 1998a).

At least within the Greek cultural context within which this study was conducted, issues with a rich potential for subjective meaning included mathematical existence, mathematical certainty and objectivity, and the rigidity of mathematical rules. Contrary to what might be expected according to the literature (Ernest, 1991; Hersh, 1997), the belief that mathematics existed could help even students who did not understand mathematics (in terms of content and/or at a philosophical level) to find some positive subjective meaning in it, while this was not necessarily the case for the belief that mathematics was invented. Moreover, some students valued mathematical certainty as philosophers of the past would have done (Russell, 1918), while for others this certainty was incomprehensible and oppressive (Hanna, 2007; Freire, 1964; Nickson, 1994). Finally, the way students would react to rules depended both on whether they understood mathematics at the level of content, but also on how their common sense evaluated rules at a philosophical level. In all, it seemed that students would be able to find positive subjective meaning in mathematics to the extent that mathematical reasoning was in line with their common sense, either at the level of content or at a philosophical level. Furthermore, students seemed to be willing to endorse the image of mathematics that their cultural context offered, as long as they could attach positive subjective meaning to it, but they would wish to react to this image otherwise.

In any event, it seemed that in order to make sense of mathematics, students tended to combine ideas from different philosophical accounts. However, even if this led to apparent contradictions, the study strongly indicates that a student's account demonstrated the potential for being coherently organised around subjective meaning. So the subjective meaning that students attributed to mathematics could bring together the various objective meanings that their beliefs bore, while this orchestration appeared to be guided by each student's common sense.

These findings indicate that researchers should avoid assuming that the students would simply echo the beliefs that their cultural context would seem to advance (François, & van Bendegem, 2007). Each student interprets this context according to their own common sense and may reach quite disparate conclusions. This suggests that there is a need for more empirical studies examining the beliefs that students hold on philosophical issues (François, & van Bendegem, 2007). Moreover, this research shows that it is important to assess the meaning of such beliefs

not only on an objective level, but also on a subjective level, since it is on this level that the students' beliefs come together to form a functioning 'philosophy of mathematics' (Wong, 2012b).

Practical Implications

This research concurs with other work that has examined the philosophy of mathematics in the context of mathematics education that students could positively benefit from the explicit introduction of philosophical issues in the teaching of mathematics (François, & van Bendegem, 2007). In fact, in light of the findings of this study this becomes of paramount importance since students' beliefs on such issues can also be associated with the subjective meaning they attribute to mathematics. Thus, explicitly discussing such meanings should help students to create a clearer picture not only of what mathematics may be on an objective level, but also of what mathematics means for them on a subjective level. In that context, it is not a particular philosophy of mathematics that should be advanced as the 'correct' one. Rather, students should be encouraged to express their own beliefs while also becoming acquainted with the views of other thinkers. In this way, students would be free to explore what mathematics means to them both on the objective and the subjective levels without any single vision of mathematics being imposed upon them.

The goal should be that a student should be able to find meaning in mathematics in a healthy way. This does not mean that they would necessarily like mathematics and attribute a positive subjective meaning to it, but at least they could be aware of the properties of mathematics that align or misalign with their personal taste, without feeling oppressed by them. A clearer picture of mathematics should contribute to the demystification of mathematics and this should be particularly helpful for students who find mathematics difficult and tend to develop mathanxiety or fear of mathematics (Ashcraft, 2002; Newstead, 1998; Seeger, 2011)

According to the data, one particular issue that seems to be especially pertinent to this end concerns logic or common sense and its relationship with mathematical reasoning. It can be claimed that discussing this topic explicitly would result in students being able to differentiate between mathematical logic and their common sense, and to perceive mathematical logic as simply one kind of logic - among others - which can contribute to the solution of a certain kind of problems. This should allow students to accept more easily results which lie out of their experience and therefore appear to contradict their common sense. Moreover, provided that

mathematical reasoning can be considered in the context of solving real problems (Bartell, 2013), understanding its traits may help students to develop some esteem for this type of reasoning even if it is not according to their taste. On the other hand, students whose common sense is in line with mathematical reasoning could benefit from becoming aware of its limitations, so that they may avoid expecting more from it than it can deliver. This might safeguard students in cases where life presents them with a problem which they fail to solve using mathematical reasoning. In any event, clarifying the relationship between mathematical reasoning and common sense should place students in a better position to handle any misalignments between mathematical reasoning and their common sense without simply being drawn to react negatively to them. Thus, for example, students could interpret a misalignment between their common sense and mathematical logic as an individual difference and not necessarily as a deficiency.

Methodological contribution

Regarding methodology, the current research indicates that a qualitative method which allows for an in-depth investigation of the students' views may help in differentiating between beliefs which on the surface appear the same. For example, students who suggested that mathematics was invented could be divided into two substantially differing groups: a group which enjoyed mathematics and could see themselves as co-creators of mathematical knowledge, and a group who had difficulties with mathematics and perceived its invention as alien to them. Another example would be the view of mathematics as logical. Some students claimed it to be so because they could understand mathematical logic, while others declared it so only because it appeared absurd to do otherwise.

Moreover, an in-depth investigation, such as this, gives the participants the chance to explain and qualify their views at some length. In this way, it can be seen that ideas, which on the surface appear to be connected with the traditional image of mathematics as certain and unchanging, may be revealed to deviate from it, approaching more modern views of mathematics. For example, the idea that mathematics is objective or is rooted in logic, may be linked with the belief that this objectivity or this logic are socially determined. Furthermore, the belief that mathematics cannot change may simply concern what students believe to be more likely to happen, and not something that they consider as an absolute truth. It could be that the strongly traditional views that are generally attributed to students by research (François, & van Bendegem, 2007; Schoenfeld, 1992) are no more than the result of students reporting their beliefs without elaborating them. Moreover, it appears that the separation between two contradictory groups, one espousing traditional, and one more modern, beliefs (Op't Eynde, et al., 2006) may be artificial, since these groups are not necessarily mutually exclusive.

Finally, an in-depth investigation allows researchers to access not only objective, but also subjective meanings that students' beliefs may carry. As I explained earlier, understanding these is important because of the central role that subjective meaning can play in bringing together seemingly contradictory beliefs. In fact, looking back across the process of the research, its planning, execution and analysis, it becomes apparent that the work has been productive in evaluating objective and subjective meanings chiefly because it was designed to explore the students' beliefs in depth.

However, for this objective to become possible, a necessary prerequisite was that the students who participated in the study would feel comfortable and trusting. This becomes evident in light of the relatively few cases where students, though willing to help, felt somehow uncomfortable, either because they were deeply afraid of mathematics, or because they were not sure that their opinions mattered. The latter issue was easily resolved, since during the interview, sooner or later the students realised that I valued their views as important. However, handling the issue of fear of mathematics was not as straightforward. If students simply disliked mathematics, and were willing to talk about it, this posed no problem. Nevertheless, fear could block them and prevent them from elaborating their answers, probably because they wished to avoid exploring something that made them feel insecure. I believe that in such cases, the results would have been still better had I been able to spend more time with these students so that they would eventually feel more at ease with me, even while talking about mathematics.

Therefore, I believe that future investigations should employ in-depth means that will also allow researchers to uncover the subjective and objective meanings that students may attribute to mathematics from a position of greater trust. This could be achieved in the context of an ethnography or an action-research project where the researcher would spend more time in contact with the students (Hatch, 2002; Stringer, 2004). Fruitful directions for such future research include conducting similar research in different cultures and focusing on individual students as case studies. As Vollstedt's study (2011) indicates, there may be differences across different cultures in the subjective, personal meaning that students ascribe to mathematics. Moreover, individual stories may be seen to be of intrinsic interest, since extensive reports on

specific cases should be able to illuminate better how different ideas may combine with one another in creating meaning for mathematics. Because subjective meaning is something ultimately personal, it will be understood more deeply if the research focus shifts more explicitly to personal accounts.

Reflexive remarks

Apart from the substantive conclusions that this research has drawn, it has also been helpful in opening my mind hermeneutically to understanding how students may find meaning in mathematics. I was quite open to possibilities when the research began, since I did not believe that I already possessed any right answer, or indeed that there was a right answer at all. Nevertheless, the data generated views which I could not have easily imagined, because they were not exactly in accord with my own experiences or because they concerned blind spots in my experience. For instance, I was not in the position to indicate how a student could dissociate logic from mathematics. Moreover, I had not before realised how one of the reasons that I had welcomed the idea that mathematics was invented - when I first heard it during my Masters course - was that at the point I was going through a stage in my own intellectual development when I felt impelled by a desire to devalue mathematics, because it appeared of no use with respect to my personal problems. Eventually, in trying to understand and not to judge the students' experiences, I was also able to process my own experience of mathematics without judgement, and my personal conflict with mathematics dissolved. I have stopped wishing that mathematical reasoning could help me in life problems, while I can still appreciate the order and the beauty of that reasoning.

There is another result of the research, in terms of the development of my thought, which belongs to the sphere of methodology. The more I analysed the data, the more I found myself agreeing with Gadamer (1975) that method limits, if not distorts, the truth, constraining it to fit to particular patterns. This was particularly relevant for me when I was reporting the findings. There, the truth of each individual student - although manifestly present - was also inescapably compromised to some degree as I sought to present a general picture with respect to certain themes. I do not at all imply that my findings, as reported in the current study, are not faithful to the data. To the absolute best of my ability and intent, they are. However, in the process of analysis, the students' accounts seem somehow to have lost some of their original vitality and richness. Admittedly, I do not know how systematic research would be possible without

analysing and applying some method, but I believe that researchers should bear in mind that, in the process of their interpretive engagement with their data, the meanings they discern can never reveal a picture that can claim to be complete or finished, and that the hermeneutic circle of understanding never ceases its movement.

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Appendix 1

An example of objective and subjective meaning in the context of the philosophy of mathematics

Below, I try to demonstrate the mutual interplay between objective and subjective meanings in the contexts of philosophy and psychology, using as an example two statements by Bertrand Russell: 'In the whole philosophy of mathematics, which used to be at least as full of doubt as any other part of philosophy, order and certainty have replaced the confusion and hesitation which formerly reigned' (Russell, 2013, p.61); and, 'I wanted certainty in the kind of way in which people want religious faith' (Russell, 1969, p.220). The first statement is taken from the book '*Mysticism and Logic*' which contains various essays of Russell, while the second comes from '*The autobiography of Bertrand Russell*'.

The sentence from Russell's essay is part of a philosophical discourse, supported by various observations about the achievements of philosophers and mathematicians, and it aims at conveying Russell's conviction that, at the point he was writing, the philosophy of mathematics had achieved a significant level of certainty. In that context, it can be argued that Russell is here primarily concerned with the objective, dictionary meaning of the word 'certainty', as opposed to the objective meanings of the words 'confusion' and 'hesitation'. However, the same sentence can be claimed to have a hidden subjective, psychological meaning which stems from the fact that the words 'confusion' and 'hesitation' tend to be associated with negative feelings. By opposing the word 'certainty' to these negative terms, Russell implies that it is something positive, something that it is worth striving for, something that one wishes to achieve. This can be said to be the subjective meaning, since other individuals could equally well associate certainty with dogmatism which usually carries negative connotations.

Russell's psychological meaning of the word 'certainty' is more apparent in the second sentence which is not part of a philosophical discourse, but more part of a confession with respect to the meaning that Russell was hoping to find in his life. The main message that is conveyed from this sentence is that of somebody who feels lost and hopes to combat this feeling by finding certainty, as other people might combat it through religious belief. Thus, in order to understand what Russell says, what is more important is the psychological meaning according to which 'certainty' was for him a desirable, comforting state. Of course, the objective meaning of 'certainty' may still help us to understand why Russell held this subjective meaning.

As the above cases illustrate, although objective meaning appears to be more prominent for philosophical discourse, bringing subjective meaning to an interpretation allows for a richer understanding of what a philosopher might be saying. Moreover, although subjective meaning seems to be more relevant in the context of a confession, objective meaning can further illuminate a writer's remarks. In fact, the picture becomes stronger and more complete if one has at one's disposal both meanings. The two meanings complement each other (Jahn, & Dunne, 1997). On the one hand, the objective meaning found in the philosophical statement clarifies why Russell cherished certainty according to his confession, i.e. because it meant lack of confusion and hesitation. On the other hand, the psychological statement confirms the implication that one could derive from the philosophical statement, i.e. that Russell cherished certainty.

Appendix 2

Examples from the application of the interview protocol and the use of dialogue to co-create meaning with the students

The extracts which appear in this appendix are presented in their fullest form. This seemed to be appropriate, since they aim at giving a taste of the interviews' atmosphere as they were actually conducted, and by doing so, to explain how the discussion between the students and myself proceeded. However, as such, the extracts include all the confusion that arose between the students and myself as we were trying to clarify certain ideas and concepts, e.g. hesitations when we were trying to find the right words to express ourselves. Therefore, they may, at least occasionally, not make perfect sense.

Illuminating mathematics' relationship to truth by comparing mathematics with physics and history

The following extracts illustrate how my proto-understanding evolved with the help of a comparison. Moreover, they indicate how students could make associations between different subjects and how such associations could shape the interview in later stages.

When I asked Xenofontas about truth in mathematics, he replied positively, but he qualified his sentence with the attribute 'almost always' and he explained that he was referring to exceptions. This answer allowed me to form a proto-understanding about the way he viewed truth in relation to mathematics. Xenofontas seemed to believe that one could find truth in mathematics, but that that truth was not absolute. In order to illuminate this proto-understanding I invited him to comment about the same issue with respect to physics and to history. While doing so, the extent to which he was ready to associate truth with mathematics became clearer, as he claimed that mathematics regarded truth to a greater degree than physics but to a lesser degree than historical facts. Moreover, with respect to physics, he associated the relative lack of truth to the fact that physics concerned phenomena and objects which did not exist. By contrast, as he explained later, he believed that mathematics involved existing entities. So again, from this comparison, I could conclude that Xenofontas related mathematical truth with mathematics than within history, but this did not seem to influence his

impression that truth in history could carry a higher degree of certainty than truth in mathematics.

My understanding on that issue was further elucidated in the second interview, when spurred by Xenofontas' association of absolute truths with history, I asked him about which subjects produce specific answers. Indeed, as a result of this, Xenofontas spoke of history again, and although he did not refer specifically to truth, he justified the fact that history had specific answers with more or less the same answer that he had justified that historical facts were true. He also suggested that answers in mathematics were less specific. Thus, it could be said that Xenofontas associated truth with specific answers and he believed that history had an advantage over mathematics when it came to truth exactly because mathematics did not always offer specific answers. So this extract completed the comparison regarding truth in history and mathematics, and allowed me to deepen my understanding with respect to the meanings that Xenofontas attributed to truth in mathematics. These were both objective as far as the content of his beliefs was concerned, but also subjective as far as they depended on his subjective associations and evaluations of what counted as absolute truth.

Eleni: Would you say that mathematical conclusions are true?

Xenofontas: Yes, most times yes, almost always.

Eleni: what do you mean almost:

Xenofontas: As we said here, there are some exceptions. They are a few, but they exist.

Eleni: Hmm. Conclusions in the context of the science of physics, would you say that they are true?

Xenofontas: No. About phy- [...] we're talking about physics right?

Eleni: Yes.

Xenofontas: No, no, not at all. Not at all.

Eleni: Okay.

Xenofontas: Because most [...] almost everything in physics is based on something ideal, something which doesn't exist. For example, ... we're now doing, in physics of the third grade of lykeio (upper secondary school), oscillations in mechanics. Oscillation in mechanics

are about a system which has no friction. There's no friction. Is it possible that this thing exists? If it exists then we can say yes.

Eleni: Would you say that historical facts are true?

Xenofontas: Yes. Yes, they aren't [...] they're something which has happened, something which we've analysed, it has been proven, it can't be changed.

Eleni: Okay, where ...

Xenofontas: Something may come which will demolish it, only this, if we're talking about something mathematical, for example. But history as history, as a science, it can't be changed.

Eleni: Ok, where can you check truth more easily, [...] or anyway, somebody who knows, where can they check truth more easily, in the context of history or in the context of mathematics?

Xenofontas: I think in the context of mathematics. In the context of mathematics, there's something which [...] you'll take a piece of paper, you'll sit down, you'll try to use your head so that you find something and verify it and then verify it again, look at it. While history, for example, has things which have not been proven, they haven't even been written, that is, history, how can we say this? You're based on something which has been written by a colleague of yours many many years ago. Right? You can't know that he wasn't [biased], that [...] it might have been in his interest to write this. Facts have happened as facts, they exist, they are truths. [This] doesn't mean that truths have been written.

Eleni: When you write something in mathematics and you verify it, anyway [...] for example, can you be certain about it?

Xenofontas: Never!

Eleni: Okay.

Xenofontas: If you are not an absolute authority, you can't be certain of anything.

Eleni: In comparison with other sectors, for example, [...] can you be more certain?

Xenofontas: Yes, yes, there are other [...] other sectors, for example [...] as we said physics, which in a sense is based on something ideal, something which doesn't exist so [...] At least in the other (mathematics), you play with integral things, things which you know they exist.

... (second interview)

Eleni: In relation to other subjects [...] where would you say that answers to various questions are more specific, in mathematics or in other subjects?

Xenofontas: Ehm, in other subjects.

Eleni: Such as:

Xenofontas: History.

Eleni: Okay.

Xenofontas: What happened then? This happened.

Eleni: While in mathematics?

Xenofontas: In mathematics, it's a bit more [...] more abstract. That is, there may be exercises where [the answer] is strictly [something]. And there are also some exercises where [the answer] can be either this, or that one and the other one, there may be different solutions.

Eleni: Are these the exercises which you like?

Xenofontas: I like more integral things, just so. If [...] [I like] something to be certain. It is so, [and that's the] end.

Eleni: Okay. And do you like history?

Xenofontas: Yes, specifically, it's my favourite subject.

Contextualising the subjective meaning of mathematics though the student's conception of truth and its role in life.

The following extracts indicate how commenting on mathematics within a broader context concerning an individual student's life could lead to a deeper understanding of what mathematics meant to that student on the subjective, psychological level. Moreover, the extracts again illustrate how the interview could proceed on the basis of student's associations, and how a proto-understanding could develop over time into a fuller understanding through probes inspired by philosophy, or by the responses of other students.

While talking about truth, Agapi introduced the concept of general truths, but she also referred to variations of truth. At that point, my proto-understanding suggested that Agapi allowed for

both objective and subjective truths. In order to clarify this, I wondered if two individuals could disagree while both being right. This led me to realise that although Agapi might acknowledge that there could be subjective understandings of truth, she actually believed in an absolute, complete truth which could be attained. Since she had associated general truths with mathematics, and having in mind that in traditional currents of philosophy mathematics has been considered as a path to such truths, I asked her whether she believed that mathematics could help humans to find general truths. At that point, Agapi brought the conversation back to life, noticing that what was important there was to find moral truths and not mathematical truths. Later, she would explain how she believed that mathematical reasoning could be applied to life in order to lead to truths; however, at this earlier point she felt the need to stress that what mattered in life was not mathematics, but morality. This was not the answer I had expected but, against my preconceptions, I expanded my proto-understanding in order to include moral values in the category of absolute truths. In this way, I followed Agapi's lead and I sought to understand how she associated morality with general, absolute truths. Unfortunately, Agapi was not ready to explain this. It was easier for her to give examples of general truths from the sciences. It seemed we had reached an impasse, and since I could not associate absolute truths with morality either, I dropped the issue for the time being.

Nevertheless, since these issues were obviously important to Agapi, I revisited them in the second interview. This time Agapi, probably also having the chance to think over these issues in between the interviews, was readier to offer examples that concerned life in general, and moral conduct in particular. Thus, my understanding of the connection between morality and general truths deepened. Even more so, in consequence of the way she reacted to my remarks which aimed at introducing a notion of subjectivity into the equation, in line with other students' observations. Agapi again acknowledged subjectivity, but obviously did not value deviations from general truths. I needed some time to process this view which countered my preconceptions about life, but after a while, I reopened the discussion about general truths, trying to clarify what set them apart from simple subjective opinions. Agapi's reply brought mathematics again in the discussion, by associating the process of generating general truths with the process of generating mathematical truths through logical reasoning. Thus eventually Agapi seemed to agree with philosophers who viewed mathematical reasoning as a means to uncover absolute truths, and her associations revealed that this was how she found subjective meaning in mathematics.

Eleni: In life generally, would you say that there are different degrees of truth?

Agapi: Yes, I could say this. We could say that there are general truths, as rules are in mathematics and [...] but there are also different variations of truth, which other times hold and others not.

... Eleni: Can it be, for example, that two people disagree and they're both right?

Agapi: It can't be that they're both right .

Eleni: Okay.

Agapi: This can't happen. That is, it would [...] there would partially be a truth in both of them, but not complete [truth]. So it can't be.

Eleni: Do you think that it's possible to find what you named as complete truth?

Agapi: Yes.

Eleni: Would you think that mathematics could help in this end?

Agapi: In life no. In our own, practical life, it can't be done with mathematics.

Eleni: What would help us in our practical life?

Agapi: Ehm, moral values help in life. That is, the sense of self-awareness and of truth. To know what really holds and to be able to say it. But it must be right indeed. Only this, only this helps. That is, mathematics is practice on paper and it can't hold in our life.

Eleni: And in life, how can you know that something is true indeed?

Agapi: There are several sorts of truth. That is, for some people some other things hold, for some [other people] other [things], that is, there's a different version of truth for [...] for every society, for example, but the general truths are for everyone.

Eleni: General truths such as, I mean can you mention one?

Agapi: The earth revolves [around the sun], for example.

Eleni: Okay.

Agapi: No one can reject this.

Eleni: But this is not a moral truth, is it?

Agapi: No, but it still is true and we use it in our life without [bringing in] mathematics.

... (second interview)

Eleni: I think, if I'm not mistaken, you had said, now I don't remember which word we had used, that there is an objective truth, or in any case one [absolute] truth.

Agapi: Yes, there is.

Eleni: Okay: Can we find it?

Agapi: We define it. Basically, how can I say it? It is as [...] [a] concept, for example.

Eleni: Yes.

Agapi: We say that we sit on the chair. Truth. This is a truth; it holds for all people. Other [people] will step on the chair, other [people] will turn the chair [upside down], other [...] that is, there's this thing, but that we sit on the chair, this is a truth.

Eleni: Okay.

Agapi: Which we have defined.

Eleni: Yes, fine. Ehm, and other things, such as morality, for example.

Agapi: Such as?

Eleni: Morality.

Agapi: Morality [is] the same. Your parents will tell you that [...], not only your parents, the society generally, that you must always be appropriately dressed. But whether other [people] will be dressed differently, or differently, or they'll come inappropriately dressed in a place, this is their problem, it's their truth, their opinion.

Eleni: Okay.

Agapi: But the general truth is that a proper place, you must be properly dressed.

Eleni: Okay. And what is it that makes this general truth hold? For instance, why [should] this general truth that you mentioned be correct, and not another one?

Agapi: Apart from, how do they call it? I can't think of the verb. Not enacted. Apart from being [...] it holds for most people.

Eleni: Okay.
Agapi: It's, and what can I say? It's logic that leads us there. That is, there's something like a value.

Eleni: Okay. And the others, who do not function accordingly, they're not guided by logic?

Agapi: They're not guided by logic. They're guided by their own mind, their own opinion, that they want to go against something [...] this.

Eleni: Okay. And would you say that this is without logic.

Agapi: It is not without logic. It is diversity.

Eleni: Okay.

Agapi: It is their own logic. But they also know that if they go to a play, they shouldn't go in a swimming suit. They know this. It's just their right to do so.

Eleni: Okay and would you say that it would be better if we all converged towards this general truth or is it better that many [people] act in their own way.

Agapi: It is definitely better.

Eleni: Which of the two [is] better?

Agapi: That they follow the general truth. But diversity will never cease to exist. So we can't do anything about it.

... Eleni: You had used (in the last interview), what we also said now (in this interview), the term general truth. Could you tell me what makes a truth a general truth?

Agapi: A general truth is based on [...] on the opinions, on the several opinions which were there before and it holds for all. This [is it]. That is,

Eleni: That is, has it emerged through a path?

Agapi: Yes, on the basis of previous theories, for example, thoughts, logical thoughts, this is created, a general truth.

Eleni: This reminds me a bit of mathematics, the way you put it. Okay.

Agapi: Yes this.

Eleni: Okay.

Agapi: Ah! The two are connected. Because it's the mode of reasoning. And this is what they try to pass to us in mathematics all these years. They continually say to us, our teachers, 'I want you to learn how to think'. Indeed [mathematics] is a way of reasoning. The 'three x plus five' won't be of use to you anywhere that much. In your life, it won't be of use to you anywhere, anywhere. A derivative won't be of use to you anywhere, apart from your job, though even there I doubt whether it'll be of so much use.

Eleni: It depends on the job you'll do.

Agapi: Yes: But it'll teach you that in order to solve something, that in order to solve an equation you'll follow a specific path. You'll do this, you'll separate this, you'll do [...] and you'll reach the result.

Eleni: So would you say that the way of reasoning that mathematics is teaching can be applied in life too?

Agapi: Yes, everywhere. This [...] and this is the corresponding argument. The equation is the argument.

Eleni: And if I asked you what traits has this way of reasoning after all?

Agapi: It has premises, logical statements, that is, things which hold and on the basis of whether all these are valid and true - and in mathematics the same - they reach a logical conclusion which is true, which is logically sound.

Negotiating the concept of rule in mathematics

The following extract demonstrates how I would try to realise the way students understood the concept of 'rule', and how associations emerging from this discussion could lead to further comments.

Foivos would not see rules in mathematics the same way I did. Instead of using it as an umbrella term for mathematical statements, he restricted it to the use of logic in mathematics. I accepted his definition and in the rest of the interview, I would not use the word 'rule' to refer to mathematical statements. Nevertheless, since Foivos suggested that logic had rules, I chose to explore this association further. In such an instance, while comparing theorems and rules of

logic, he suggested an interesting metaphor with respect to how mathematical statements were produced.

Eleni: If I mentioned the word rule, for example, in the context of mathematics, what would you think?

Foivos: Yes, and I was [...] when you mentioned this the other day (when I introduced myself in his classroom), I don't know, I was taken aback a bit. I was thinking. On the one hand, I thought of the straight edge (in Greek the word for 'rule' and 'straight edge' is the same) and the compass ... On the other hand, I thought that "I don't think it's this." So, I don't know, maybe the logic of mathematics? I don't know.

Eleni: Yes, will [...] no, okay, it could be. Now

Foivos: Eventually, what among all these is a 'rule'?

Eleni: I call [rule] something much broader, but we'll see what you think. Here ... I ask which of the following would you characterise as a rule in mathematics.

... Foivos: I think [choice] d (proof by contradiction).

Eleni: At least, according to what you said earlier, I would also expect you to say d. Ehm, fine. The rest [of them]?

Foivos: Ehm, the rest are theorems, they aren't [...] this is more [like] logic, how can I say it.

... Foivos: The broader point for the theorems to exist, is so that you're able to synthesise more, so that you have more complex things.

Eleni: You mean to use them ...

Foivos: So that you can use this and that and that in order to reach the fifth one, for example.

Eleni: In [order to do] this, you use the theorems or the rules of logic?

Foivos: Both. The rules of logic are the synthesis, the theorems are the atomic things ... let's call them 'parts'.

Appendix 3

Thematic analysis

Below are given the details of the organisation of students into themes and subthemes for all the chapters of the study.

Ontology

The nature of mathematical existence

The theme 'nature of mathematical existence' was examined by asking the students whether they believed that mathematics was invented or discovered. To 'discover' implies that the thing being discovered pre-exists: a galaxy can be claimed to exist even if no one has ever observed it, so an astronomer discovers it. Contrary to 'discovering', 'inventing' implies that the invented thing comes into existence as the result of an invention process: books were not printed before Gutenberg invented the printing press.

The answers students offered could be broadly categorised as: mathematics is discovered;¹⁹³ mathematics is invented;¹⁹⁴ and mathematics seems to be both discovered and invented.¹⁹⁵ In the 'discovery' subtheme could be found students who connected mathematics with nature, observing that mathematical concepts and properties are dictated by what exists there¹⁹⁶, and also a student who did not locate mathematics in nature but seemed to hint at the platonic ideal of mathematics comprising abstract entities.¹⁹⁷ In the 'invention' subtheme could be found students who denied the material existence of mathematics,¹⁹⁸ students who presented

¹⁹³ Ten students, 36%: Foivos, Xenofontas, Aspasia, Ermis, Platonas, Filia, Areti, Theodosis, Andromachi, and Afroditi. Three of these students (Aspasia, Platonas, and Theodosis) also suggested that at least some mathematics could have been invented.

¹⁹⁴ 16 students, 57%: Lysimachos, Polyxeni, Pelopidas, Kleomenis, Loukianos, Kosmas, Evyenia, Lida, Kleio, Danai, Diomidis, Filippos, Yerasimos, Vrasidas, Ariadni, Menelaos. Three of these students (Evyenia, Filippos, Yerasimos) believed that at least some mathematics could have been invented.

¹⁹⁵ Six students, 21%: Aspasia, Platonas, Evyenia, Theodosis, Filippos, and Yerasimos.

¹⁹⁶ Nine students, 32%: Foivos, Xenofontas, Aspasia, Ermis, Platonas, Filia, Areti, Theodosis, and Afroditi.

¹⁹⁷ Andromachi.

¹⁹⁸ Nine of the 16 students, 56%: Lysimachos, Kleomenis, Loukianos, Kosmas, Evyenia, Lida, Danai, Diomidis, and Ariadni.

mathematics as being based on hypotheses,¹⁹⁹ and students who simply pictured mathematics as a creation of the human mind because they could not understand it.²⁰⁰ In the mixed subtheme - 'discovery and invention' - belonged students whose comments could be associated both with discovery and with invention. Such students either claimed that mathematics was both invented and discovered, or were not able to decide which case was true, because they would combine beliefs related to discovery with beliefs related to invention.²⁰¹ Nevertheless, even in such cases, it seemed that one of the two trends (discovery or invention) was given precedence in the students' remarks. So students in the mixed theme could also be associated with one of the pure themes of either discovery or invention.²⁰²

Mathematics as certain and immutable

Certainty can be associated with many other properties that mathematics may be claimed to have. Firstly, certainty guarantees immutability and vice versa, since there is no reason to change something that is certain, while if something is not changing, then we can be certain about it. Secondly, certainty implies correctness, since one can be certain about something only as long as one assumes that this something is correct.²⁰³ Finally, certainty can be connected with truth, since one can be certain of true statements. Students were asked with respect to all these issues, i.e. whether they believed that mathematics or mathematical conclusions are correct, true and amenable to change. Subsequently, they could also be asked how certain they were about their remarks on such questions.²⁰⁴

¹⁹⁹ Eight students of the 16 students, 50%: Lysimachos, Kleomenis, Loukianos, Evyenia, Diomidis, Solonas, Yerasimos, and Ariadni.

²⁰⁰ Nine of the 11 students who had difficulties with mathematics and believed that mathematics is invented, 81%: Polyxeni, Kosmas, Evyenia, Kleio, Filippos, Yerasimos, Vrasidas, Ariadni, Menelaos.

²⁰¹ However, their beliefs were not essentially different from the ones advanced by students who were clear about whether mathematics was invented or discovered, so that theme will not be discussed in the ontology chapter. ²⁰² This categorisation is the one considered with respect to subjective meaning.

²⁰³ In fact, correctness can also be associated with many mathematical properties, i.e. certainty, immutability, truth, and objectivity. However, in the context of the philosophy of mathematics, correctness is usually assumed, and only the other issues are debated. Therefore, I have not included correctness as a separate theme. It is only discussed in relation to the other themes with which it can be related.

²⁰⁴ Certainty is not discussed separately from immutability because there were not many comments where students considered certainty without linking it with immutability. On the contrary, the remarks where truth was discussed independently of certainty were more common. Thus, truth figures as a separate theme in my analysis.

Students justified mathematical certainty and immutability by referring to proofs²⁰⁵, and also on cultural grounds.^{206,207} What differed was the degree to which these two justifications blended with one another in an individual student's account. This depended on the extent to which students would succumb to the cultural power of proof, an issue which is addressed in more detail in the section on proofs in the epistemology chapter. The extreme cases concerned students who mostly did not understand proofs and were considered therefore to justify certainty and immutability primarily on cultural grounds,²⁰⁸ and students who could understand proofs, and whose remarks on certainty and immutability were taken to reflect this understanding.²⁰⁹

Cultural immutability or certainty was also suggested independently of proofs.²¹⁰ Students would suggest that mathematics does not change because our culture is based on it, or that mathematics must be correct because it is presented as such by society and school. To the extent that such students could perceive the respective cultural construct as contingent and not necessary, they would not regard it as absolute. Moreover, traces of uncertainty and change were connected with currently developing mathematical knowledge which was regarded as less stable.²¹¹ Finally, lack of absolute certainty could be the result of a natural hesitancy due to idiosyncrasy or lack of expertise. As a result, students would on many occasions hedge their claims about mathematical certainty or immutability.²¹²

Mathematics as true

As mentioned above, students were asked whether they believed that mathematics concerned truth or whether mathematical statements could be called true. When mathematics was

²⁰⁵ 16 students, 57%: Foivos, Lysimachos, Agapi, Pelopidas, Kleomenis, Loukianos, Kleio, Danai, Platonas, Diomidis, Filia, Solonas, Theodosis, Andromachi, Vrasidas, and Afroditi.

²⁰⁶ This applied to all students, though to different degrees.

²⁰⁷ Mathematical certainty and immutability were also connected with mathematical existence. However, this seemed to be mostly the result of mathematical existence implying mathematical truth. So mathematical existence has been retained as a subtheme only for the case of truth, where the connection was stronger and was also manifested in more students.

²⁰⁸ 13 students, 46%: Pelopidas, Polyxeni, Kosmas, Evyenia, Kleio, Areti, Theodosis, Filippos, Yerasimos, Vrasidas, Ariadni, Menelaos, and Afroditi.

²⁰⁹ Seven students, 25%: Lysimachos, Kleomenis, Loukianos, Ermis, Platonas, Solonas, and Andromachi.

²¹⁰ Seven students, 25%: Lysimachos, Loukianos, Danai, Theodosis, Yerasimos, Vrasidas, and Menelaos.

²¹¹ 16 students, 57%: Foivos, Agapi, Aspasia, Kleomenis, Loukianos, Kosmas, Ermis, Evyenia, Lida, Danai, Platonas, Diomidis, Filia, Solonas, Yerasimos, and Menelaos.

²¹² This applied to all students, though to different degrees, which seemed to be due to idiosyncratic reasons.

portrayed as true, this was on the basis of proofs,²¹³ on the basis of mathematical existence,^{214,215} or on the basis of cultural factors.²¹⁶ Students in the first group equated truth with correctness and utilised proofs to justify truth as they did to justify certainty; students in the second group hinted at mathematics as describing the true reality of nature; and students in the third group accepted mathematics as true because it was a science with proofs taught in school. However, there were also students who dissociated truth from mathematics. So there were students who distinguished between truth and correctness and noticed that mathematics may be correct without necessarily being true;²¹⁷ there were students who denied that mathematics exists and has any relation with the true, real experiential world;²¹⁸ and there were students who did not regard cultural truth as absolute.²¹⁹

Mathematics as objective

The issue of objectivity was generally approached by checking whether students believed in the existence of a unique truth, or whether they assumed that each individual could hold their own version of truth. Apart from enquiring directly, students' beliefs on that issue were also assessed by asking them whether it was possible for two individuals to disagree while both being right. If they believed in an objective truth, then they would suggest that when two persons disagree only one of them could be correct. Otherwise, they could claim that both persons could be right. After checking what students believed generally, I asked them what the case in mathematics was.²²⁰

Students generally suggested that disagreements in mathematics were not possible, or, if they were present they were the result of ignorance.²²¹ This belief was usually justified by

²¹³ So comments on truth could also be part of a discussion about correctness. 11 students, 39%: Foivos, Agapi, Loukianos, Kosmas, Lida, Danai, Platonas, Filia, Solonas, Andromachi, and Afroditi.

²¹⁴ In contrast with the theme of certainty and immutability, mathematical existence has been considered as a subtheme for the theme of truth. This is because the link between existence and truth is much more direct. Something which exists is practically by definition real and true independently of the mode of existence. However, only a platonic mode of existence, as an abstract entity, can also justify that it may still be certain and immutable. ²¹⁵ Nine students, 32%: Foivos, Xenofontas, Aspasia, Ermis, Platonas, Filia, Areti, Theodosis, and Afroditi.

²¹⁶ 12 students, 43%: Agapi, Loukianos, Kosmas, Lida, Danai, Filia, Solonas, Areti, Filippos, Vrasidas, Ariadni, and Menelaos.

²¹⁷ Three students, 11%: Lysimachos, Kleomenis, and Yerasimos.

²¹⁸ Two students, 7%: Evyenia and Ariadni.

²¹⁹ Four students, 14%: Agapi, Kosmas, Vrasidas, and Menelaos.

²²⁰ In this context, the words 'objective' and 'subjective' would often appear, though not necessarily.

²²¹ All students except for Lida.

suggesting that mathematical questions had only one correct answer.²²² Apart from this, since in essence objectivity was connected with truth, it was also justified on similar grounds: proofs,²²³ mathematical existence,²²⁴ and culture.²²⁵ Nevertheless, there were traces of subjectivity. In particular, one student specifically stated that mathematical problems may have different solutions.²²⁶ Moreover, despite echoing the cultural belief of mathematics as objective, some students fundamentally could not believe that there really could be no disagreement in mathematics. Such students made remarks which implied that they considered a plurality of opinions more natural on any given subject, or that they wished to disagree with mathematics because they did not always understand it.²²⁷

Mathematics as rules

Students were asked to explain what the word 'rule' meant for them, and consequently whether they believed that mathematics had rules. Most of them accepted this word as an attribute which could apply to the 'guidelines' which they were following while solving exercises.²²⁸ Nevertheless, there were some students who found it somewhat hard to associate the word 'rule' with mathematics, or with school subjects in general.²²⁹ The reason for this seemed to be that for them the word 'rule' bore negative connotations of compulsion, while school subjects appeared to them to simply offer neutral guidelines.

²²² 20 students, 71%: Foivos, Xenofontas, Lysimachos, Aspasia, Pelopidas, Polyxeni, Kleomenis, Kosmas, Ermis, Danai, Platonas, Diomidis, Filia, Areti, Solonas, Theodosis, Andromachi, Vrasidas, Menelaos, and Afroditi.

²²³ So comments on truth could also be part of a discussion about correctness. 11 students, 39%: Foivos, Agapi, Loukianos, Kosmas, Lida, Danai, Platonas, Filia, Solonas, Andromachi, and Afroditi.

²²⁴ Ten students, 36%: Foivos, Xenofontas, Aspasia, Ermis, Platonas, Filia, Areti, Theodosis, Andromachi and Afroditi.

²²⁵ 11 students, 39%: Pelopidas, Kosmas, Evyenia, Kleio, Areti, Filippos, Yerasimos, Vrasidas, Ariadni, Menelaos and Afroditi.

²²⁶ Lida.

²²⁷ Six students, 21%: Evyenia, Kleio, Filippos, Yerasimos Vrasidas, and Ariadni.

²²⁸ 20 students, 71%: Lysimachos, Agapi, Pelopidas, Polyxeni, Loukianos, Kosmas, Ermis, Evyenia, Lida, Danai, Platonas, Filia, Solonas, Theodosis, Filippos, Yerasimos, Vrasidas, Ariadni, Menelaos, and Afroditi.

²²⁹ Eight students, 29%: Foivos, Xenofontas, Aspasia, Kleomenis, Kleio, Diomidis, Areti, and Andromachi.

Overview

The table below indicates the themes and subthemes of ontology based on the foregoing discussion.

Theme	Subthemes
The nature of mathematical existence	 Discovery Empiricist existence Platonic existence Invention Mathematics as immaterial Mathematics as hypotheses Mathematics as an unintelligible invention
Mathematics as certain and immutable	 Certainty and immutability Proofs Cultural certainty and immutability Traces of uncertainty and change
Mathematics as true	 Correctness Mathematical existence. Cultural truth
Mathematics as objective	 Proofs Mathematical existence Cultural objectivity
Mathematics as rules	(no subtheme)

Epistemology

Rule-based knowledge

As was explained in the theme of rules relevant to ontology, students were asked whether mathematics had rules. Consequently, and throughout the interview, they were asked questions related to the observance of these rules, such as: what happens if these rules were not observed?; why they follow these rules?; must these rules be followed to the letter?; did they had something to gain or lose by following the rules?

The students mostly expressed the necessity of following mathematical rules. In other words they suggested that following these rules was a necessary condition for reaching the solution of a mathematical problem.²³⁰ During the analysis, an attempt was made to differentiate between a necessary condition and a sufficient one. However, when the students spoke during the interview, they did not seem to have such a distinction in mind, and admittedly neither did I. The fact is, that in everyday discourse, most people would interpret a sentence stating that 'in order for the machine to operate, you must push the red button' as meaning that if they pushed the red button the machine would operate and that if they did not push the red button the machine would not operate. So a tedious logical analysis of the students' utterances seemed unproductive. It might be possible to see which of the conditionals each student expressed, but in most cases there was no way to check if they restricted themselves only to this conditional or they also implied the reverse one too. Nevertheless, a subtheme for non-binding rules was created, regarding students who maintained that a mathematical problem could be solved even if one had not followed the rules. Some students claimed that this could be the case with respect to expert mathematicians,²³¹ but only two students suggested that this could be possible in the classroom context too.²³²

Logically-based knowledge

Under this theme, I included remarks which showed what was the role that students attributed to logic within the context of mathematics, and also comments which indicated what the students were referring to when they used the word 'logic'. Students were asked to comment on whether they believed that there was any relationship between logic and mathematics. Moreover, students were asked questions regarding the generation and verification of mathematical knowledge such as: how mathematical rules were produced; how correctness in mathematics was decided; how could they know that mathematical rules were correct/true; and whether mathematical rules could be checked through logic. On all these occasions, students could link logic - mostly voluntarily, but sometimes after probing - with the process through which mathematical knowledge was advanced and validated. Furthermore, students would

²³⁰ 26 students, 93%: Foivos, Xenofontas, Lysimachos, Agapi, Pelopidas, Polyxeni, Kleomenis, Loukianos, Kosmas, Evyenia, Lida, Kleio, Danai, Platonas, Diomidis, Filia, Solonas, Areti, Theodosis, Filippos, Yerasimos, Andromachi, Vrasidas, Ariadni, Menelaos, and Afroditi.

²³¹ Nine students, 32%: Foivos, Lysimachos, Kleomenis, Kosmas, Platonas, Yerasimos, Filippos, Andromachi, and Vrasidas.

²³² Two students, 7%: Aspasia and Ermis.

many times use spontaneously the attribute 'logical(ly)'; while other times they would speak of something as if it were self-evident and needed no further explanation or justification. Such cases were taken to suggest that which each student regarded as compatible with logic, i.e. as common sense.

Students generally connected mathematics with logic - at least at some point during their interviews - by presenting it as a factor of mathematical reasoning.²³³ Apart from general remarks about logic as the main trait of mathematical reasoning, students also made more specific comments which implicated logic in the process of generating and validating mathematical knowledge. Nevertheless, when students made such claims despite only having a limited understanding of mathematical reasoning, or when they connected mathematics with logic at one point, but denied such an association elsewhere, it felt that they were merely echoing a cultural belief, according to which mathematics was supposed to be logical.²³⁴ Apart, from this, some students actually suggested that what was logical in mathematics depended on human habits and conventions, thus suggesting that logic in mathematics was a cultural construct.²³⁵

Moreover, there were cases where students pointed towards limitations of logic in the context of mathematics,²³⁶ or even cases when students would deny that mathematics was logical (at some other point in their interviews).²³⁷ The main reason for this seemed to be that the way in which they were expected to reason whilst doing mathematics did not always fit with what their experience dictated as logical - in other words, with their common sense. Examples concerned mostly the inability to check at least some mathematical results empirically.²³⁸ The fact was that most students did not differentiate between the logic that was used within

²³³ The only exception was Ariadni, although this seemed to be a matter of chance, in the sense that if I had asked the questions of the interview in a different sequence, Ariadni could have also initially connected mathematics with logic.

²³⁴ Ten students, 36%: Pelopidas, Kosmas, Evyenia, Kleio, Areti, Theodosis, Filippos, Yerasimos, Vrasidas, Menelaos, and Afroditi.

²³⁵ Four students, 14%: Lysimachos, Loukianos, Yerasimos, Ariadni.

²³⁶ Ten students, 36%: Lysimachos, Aspasia, Pelopidas, Kleomenis, Loukianos, Danai, Platonas, Filia, Solonas, Andromachi.

²³⁷ Four students, 14%: Evyenia, Filippos, Yerasimos, and Ariadni.

²³⁸ Other cases seemed to be idiosyncratic and so I did not create a subtheme for them. Experience, however, seemed to be a prominent issue. In fact, some students also made comments which suggested that logic stems from empirical data. Such cases have been organised under the theme 'empirically-based knowledge'.

mathematics and the common sense that was used in everyday life,²³⁹ and this could lead them to suggest that mathematics was not (always) logical.

Empirically-based knowledge

Students' remarks which pointed towards empiricism were connected to the senses or to experimentation. With respect to the senses, students could be asked whether they believed that the senses were associated with mathematics, or with logic; or they could be invited to compare the senses with logic. However, students could make remarks related to the senses in many other cases. In particular, discussing what could be accepted as a proof or where mathematics originated were issues that could be seen as involving the senses. Moreover, discussing mathematical existence could lead to comments about whether mathematics, as other things which exist, could be perceived through the senses. Finally, any case that could lead students to comment on whether they could understand specific mathematical statements, or mathematics in general, could result in them referring to the senses, since that which was accessible through the senses was something which they could understand readily.²⁴⁰ With respect to experimentation, students could be asked if there were experiments in mathematics. Moreover, the idea of experimentation could arise in the context of following, learning, producing or verifying mathematical rules.

Many students made remarks which suggested that mathematics was not accessible through the senses.²⁴¹ However, students would also suggest that - at least some - mathematical results could be based on observation.²⁴² Moreover, students seemed to relate the senses with mathematics - at least indirectly - in so far as they related the senses to logic while assuming that logic was related to mathematics.²⁴³ Such students would indicate that the senses provided

²³⁹ 25 students, 89%: Foivos, Xenofontas, Lysimachos, Agapi, Aspasia, Pelopidas, Kleomenis, Loukianos, Kosmas, Ermis, Evyenia, Lida, Kleio, Danai, Platonas, Filia, Areti, Theodosis, Filippos, Yerasimos, Andromachi, Vrasidas, Ariadni, Menelaos, and Afroditi.

²⁴⁰ This could happen in practically any context. For example, if a student did not understand mathematics they could comment on it while comparing mathematical rules with rules in life.

²⁴¹ 15 students, 54%: Aspasia, Kleomenis, Loukianos, Ermis, Evyenia, Lida, Danai, Diomidis, Filia, Solonas, Areti, Theodosis, Filippos, Andromachi, and Ariadni.

²⁴² 17 students, 61%: Foivos, Xenofontas, Agapi, Pelopidas, Kleomenis, Ermis, Lida, Diomidis, Filia, Areti, Theodosis, Filippos, Andromachi, Vrasidas, Ariadni, Menelaos and Afroditi. However, only 5 students, 18%, made this claims mathematics in general and not for particular statements: Foivos, Xenofontas, Filia, Theodosis, and Filippos.

²⁴³ 16 students, 46%: Xenofontas, Lysimachos, Pelopidas, Kosmas, Kleio, Diomidis, Filia, Solonas, Areti, Theodosis, Filippos, Vrasidas, Ariadni, Menelaos, and Afroditi.

the raw material on which logic functioned, or that statements which could be verified by the senses were logical. Regarding experimentation, students commented on the following: how engaging/experimenting with mathematics could lead to experience which could help them to better understand and handle mathematical issues;²⁴⁴ how mathematical results could be found and validated through a method of trial and error;²⁴⁵ and how experimenting could allow one to corroborate mathematical results, by checking whether these results were applicable in everyday life, or in accordance with available scientific data.²⁴⁶

Proof-based knowledge

Students were asked what the role of proof was in mathematics. Moreover, axiom-like statements and their provability might be discussed, or students could be asked whether all mathematical statements had proofs. However, remarks about proof could emerge while discussing any of the ontological issues of certainty, immutability, truth and objectivity (see explanations of these themes above).

Apart from ontological issues, students connected proofs with mathematical epistemology, i.e. the generation and validation of mathematical knowledge, suggesting that a proof was the process through which mathematical results emerged,²⁴⁷ or through which a mathematical statement could be confirmed as correct.²⁴⁸ Nevertheless, apart from these mathematical functions of proof, students' comments also pointed towards a cultural function of proof: that of an authority.²⁴⁹ Such comments suggested that proofs were correct simply by virtue of being

²⁴⁴ Eight students, 29%: Lysimachos, Pelopidas, Kleomenis, Ermis, Platonas, Diomidis, and Solonas, Areti, and Theodosis.

²⁴⁵ Seven students, 25%: Kosmas, Lida, Diomidis, Filia, Theodosis, Filippos, and Afroditi.

²⁴⁶ Nine students, 32%: Lysimachos, Kleomenis, Loukianos, Ermis, Danai, Platonas, Filia, Theodosis, and Menelaos.

²⁴⁷ 16 students, 57%: Xenofontas, Lysimachos, Pelopidas, Kleomenis, Loukianos, Danai, Diomidis, Filia, Solonas, Areti, Theodosis, Filippos, Yerasimos, Andromachi, Vrasidas, and Afroditi.

²⁴⁸ 16 students, 57%: Foivos, Agapi, Pelopidas, Loukianos, Danai, Platonas, Diomidis, Filia, Solonas, Areti, Theodosis, Yerasimos, Andromachi, Ariadni, Menelaos, and Afroditi.

²⁴⁹ Practically all students though to different degrees, since students who understood proofs did not have to rely on their cultural authority as often, though sometimes they would. Students who did not understand proofs were 13 students, 46%: Pelopidas, Polyxeni, Kosmas, Evyenia, Kleio, Areti, Theodosis, Filippos, Yerasimos, Vrasidas, Ariadni, Menelaos, and Afroditi. Students who mostly understood proofs were seven students, 25%: Lysimachos, Kleomenis, Loukianos, Ermis, Platonas, Solonas, and Andromachi. The remainder eight students, 29%, seemed to be content without proofs even if they could understand them: Foivos, Xenofontas, Agapi, Aspasia, Lida, Danai, Diomidis, and Filia.

proofs, indicating that proofs could not be challenged even if one did not understand them, or that proofs were correct even if one had not checked them.

Authority-based knowledge

In the context of the discussion about rules, students could be asked whether they would follow a rule that they might not understand, and whether it was possible to challenge mathematical rules. Both these questions could lead to comments about authority which could not be challenged but could be trusted blindly. Similarly, all questions that revolved around whether mathematics was correct or even true, and how this could be known, could give rise to comments about the authority that granted mathematical correctness or truth. Finally, students were also asked directly whether they attributed a role of authority to their teacher or to their book.

The students perceived as sources of authority mainly mathematics itself,²⁵⁰ but also their teacher²⁵¹ or book.²⁵² Students' remarks were taken to point towards mathematics as an authority whenever they suggested that mathematics formed a body of statements which were to be trusted and/or which could not be challenged.²⁵³ In all cases, this authority seemed to be connected with the concept of proof, which was taken to sanction mathematics (see previous section on proof-based knowledge). However, students who did not understand proofs, and mathematics in general, were simply obliged to accept this authority without understanding why they had to do so;²⁵⁴ while students who mostly understood proofs had a way of justifying the authority of mathematics - even if they did not feel the need to validate proofs.²⁵⁵ It was generally the mathematical authority which the teacher and the book were taken to represent; that is, students would note how the teacher or the book would not state something arbitrary,

²⁵⁰ Only one student, Ermis, openly denied mathematical authority.

²⁵¹ 15 students, 56%: Xenofontas, Lysimachos, Agapi, Pelopidas, Kleomenis., Loukianos, Kosmas, Evyenia, Platonas, Diomidis, Areti, Filippos, Vrasidas, Menelaos, and Afroditi. Filia was the only student who regarded her teachers' past mistakes as too grave to allow her to trust them.

²⁵² 19 students, 68%: Xenofontas, Agapi, Aspasia, Polyxeni, Kosmas, Evyenia, Kleio, Danai, Diomidis, Filia, Solonas, Theodosis, Filippos, Yerasimos, Andromachi, Vrasidas, Ariadni, Menelaos, and Afroditi. Lysimachos, and Areti were the only students who were openly reluctant about trusting their mathematics book.

²⁵³ This could be for cultural reasons (see explanations about the themes on certainty, truth and objectivity).

²⁵⁴ 12 students, 43%: Pelopidas, Polyxeni, Kosmas, Evyenia, Kleio, Areti, Filippos, Yerasimos, Vrasidas, Ariadni, Menelaos, and Afroditi.

²⁵⁵ 16 students, 57%. Among them, seven considered proofs valuable for understanding mathematics: Lysimachos, Kleomenis, Loukianos, Ermis, Platonas, and Andromachi; while eight seemed to be content even without inspecting proofs: Foivos, Xenofontas, Agapi, Aspasia, Lida, Danai, Diomidis, Filia, and Theodosis.

but something that was supposed to be correct within mathematics. However, students would also note how the teacher's or the book's authority was intensified by the way that mathematics was being taught.

Overview

The table below indicates the themes and subthemes of epistemology based on the previous discussion.

Theme	Subthemes
Rule-based knowledge	 Rules as necessary (and sufficient) Rules as non-binding
Logically-based knowledge	 Logic in mathematical reasoning a. General remarks b. Generation and validation of mathematical knowledge Cultural Logic a. Limited understanding b. Limitations of logic²⁵⁶ c. Logic as a habit Common sense a. Common sense as distinct from mathematical logic b. Limitations of logic256 c. Mathematics as not logical
Empirically-based knowledge	 Senses a. Observation b. Detour though logic Experimentation a. Trial and error b. Applications in practice c. Experience
Proof-based knowledge	 Mathematical function Cultural function
Authority-based knowledge	 Mathematics Teacher and book

²⁵⁶ This subtheme could be connected both with common sense and with cultural influences.

Subjective meaning

Common sense (reasoning)

Under this theme were organised students who seemed to find positive or negative meaning by the manner in which they reacted to the mathematical way of reasoning. In the first case, students made remarks which suggested that the mathematical way of reasoning was applicable to life (demonstrating an alignment between common sense and mathematics).²⁵⁷ This alignment could be in terms of solving problems, following rules, finding absolute truths, the law of cause and effect, not taking anything for granted, or even subjectivity. In the second case, students suggested either that mathematical reasoning was irrelevant to life²⁵⁸ or that it was not their preferred way of reasoning²⁵⁹ (demonstrating a misalignment between common sense and mathematics). This misalignment could concern solving problems, the rigidness of mathematical rules or logic, the apparent objectivity of mathematics, or abstract aspects of mathematics.

Discovery

All students who claimed that mathematics was discovered²⁶⁰ seemed to suggest that mathematical knowledge was more or less valuable because it concerned real existing entities. That meant that such students could find positive subjective meaning in mathematics as something real. This was particularly evident in students who located mathematics in nature and would express themselves appreciatively about the fact that mathematical knowledge could help them, or humans in general, to understand the world they inhabited. However, students could, or would be willing to, use mathematics to explore real aspects of the world only if there

²⁵⁷ Nine students, 32%: Agapi, Aspasia, Kleomenis, Ermis, Lida, Platonas, Solonas, Andromachi, and Afroditi.

²⁵⁸ Four students, 14%: Pelopidas, Kosmas, Evyenia, and Ariadni.

²⁵⁹ Five students, 18%: Foivos, Polyxeni, Filippos, Vrasidas, and Menelaos.

²⁶⁰ All the cases were considered relevant, since the interview would also indicate the extent to which students valued mathematical discovery. In some cases though, this was only a secondary theme. Ten students, 36%: Foivos, Xenofontas, Aspasia, Ermis, Platonas, Filia, Areti, Theodosis, Andromachi, and Afroditi. These also included students who belonged to the mixed subtheme of 'discovery and invention', but expressed themselves more strongly about discovery.

was an alignment between their common sense and mathematical reasoning at the content level, or at a philosophical level.

Invention

All students who claimed that mathematics was invented seemed to make sense of mathematics by perceiving it as a creation of the human mind.²⁶¹ The fact was that any kind of misalignment between the students' common sense and mathematics could lead students to portray mathematics as an invention which was alien to them.²⁶² Such students would devalue mathematics, or speak about it with frustration, perhaps indicating that they wished to avoid it. Still, perceiving mathematics as a subjective invention, with which they could potentially disagree, could help some of them to make some sense of its place in their life, since it allowed them to explain why they did not understand it.²⁶³ On the contrary, any kind of alignment between the students' common sense and mathematics could lead students to portray mathematics as an invention to which they felt intimately connected.²⁶⁴ Such students would portray mathematics as an interesting activity which they could enjoy and/or which could generate valuable knowledge for humanity.

Certainty

Under this theme were gathered students whose comments indicated that they valued certainty.²⁶⁵ Such students could find positive meaning in mathematics as a field of knowledge which led to certain conclusions. They could indicate that they believed in absolute truths, but they could also simply suggest that they enjoyed being in control while doing mathematics, or that mathematical certainty could help them to avoid mistakes. Nevertheless, the extent to which students could appreciate mathematical certainty depended on whether their common

²⁶¹ Again all the cases were relevant, since the interview would also suggest whether the students judged the invention of mathematics as a valuable one or not. However, for some students this was only a secondary theme. 17 students, 61%: Lysimachos, Pelopidas, Polyxeni, Kleomenis, Loukianos, Kosmas, Evyenia, Lida, Kleio, Danai, Diomidis, Solonas, Filippos, Yerasimos, Vrasidas Ariadni, and Menelaos. These also included students who belonged to the mixed subtheme of 'discovery and invention', but expressed themselves more strongly about invention.

²⁶² Ten students, 36%: Pelopidas, Polyxeni, Kosmas, Evyenia, Kleio, Filippos, Yerasimos, Vrasidas, Ariadni, and Menelaos.

²⁶³ Five students, 18%: Evyenia, Filippos, Yerasimos, Vrasidas, and Ariadni.

²⁶⁴ Seven students, 25%: Lysimachos, Kleomenis, Loukianos, Lida, Danai, Diomidis, and Solonas.

²⁶⁵ Ten students, 36: Xenofontas, Agapi, Aspasia, Polyxeni, Kleomenis, Danai, Platonas, Filia, Areti, and Afroditi.

sense was aligned with mathematics. Thus, students who understood mathematical reasoning at the level of content felt that they could more or less safely enjoy certainty within mathematics.²⁶⁶ Moreover, students who believed that absolute truth was - up to a point - attainable generally in life (alignment at a philosophical level),²⁶⁷ could find positive meaning in believing that mathematics revealed absolute truths.²⁶⁸

Subjectivity

This theme was associated with students who either perceived, or wished, mathematics to be subjective.²⁶⁹ The first case involved students who did not understand mathematical reasoning at the level of content, and thus wished to disagree with it. This case effectively coincided with that of students perceiving mathematics as a subjective invention. The second case concerned students whose common sense judged that it was reasonable for mathematics and its reasoning to be as subjective as life.²⁷⁰ Such students could have had experienced subjectivity in mathematics (e.g. in the form of open problems), or could be simply applying their general worldview to mathematics. In any event, this alignment between their common sense and their understanding of mathematical reasoning at a philosophical level could help them to find a place for mathematics within their general picture of life and thus make sense of it as a subjective field of knowledge.

Rules

Under this theme were gathered students for whom the meaning of mathematics appeared to be influenced by their evaluation of mathematical rules.²⁷¹ One group of students suggested that they felt comfortable with mathematical rules, stressing either that rules were useful in general (alignment at a philosophical level),²⁷² or that they could use mathematical rules

²⁶⁶ Eight students, 29%: Xenofontas, Agapi, Aspasia, Kleomenis, Danai, Platonas, Filia, Areti.

²⁶⁷ Seven students, 25%: Xenofontas, Agapi, Aspasia, Platonas, Filia, Areti, and Afroditi.

²⁶⁸ Afroditi.

²⁶⁹ Seven students, 25%: Evyenia, Lida, Kleio, Filippos, Yerasimos, Vrasidas, Ariadni.

²⁷⁰ Four students, 14%: Lida, Kleio, Yerasimos, Vrasidas. Among them, only Lida could point to experiences in the classroom which validated her belief.

²⁷¹ This theme also corresponded to all students, but this was because of the centrality of the issue of rules in the interview protocol, and because students tended to associate mathematics with rules. On the contrary, the theme of common sense seemed to have been all-pervasive in a more structural way, since compatibility or incompatibility with it appeared to unavoidably influence the meaning that students could find in mathematics. For an exposition of this, see the chapter on subjective meaning.

²⁷² Six students, 21%: Agapi, Filia, Areti, Theodosis, Menelaos, and Afroditi.

creatively²⁷³ (alignment at the content level). Such students could find positive meaning in mathematics as an interesting and valuable set of rules. Another group of students indicated that they felt uncomfortable with mathematical rules, stressing either that rules in general could be too rigid (misalignment at the philosophical level),²⁷⁴ or that they could make no use of mathematical rules²⁷⁵ (misalignment at the content level). For this group, the subjective meaning of mathematical rules seemed to be a negative one associated with oppression.

Empiricism

This theme concerned students who made sense of mathematics as an empirical field,²⁷⁶ but also students who expressed discomfort at not being able to make sense of non-empirical aspects of mathematics which contradicted their common sense.²⁷⁷ The first case, however, involved mostly students who believed that mathematics was discovered and is effectively analysed under the theme of discovery.²⁷⁸ In the second case, the discomfort could concern specific issues, such as infinity or multiple dimensions,²⁷⁹ but it could also concern substantial parts of mathematics as being too abstract and therefore unrelated to experience in general.²⁸⁰ The subjective meaning found by such students in mathematics would be negatively influenced by the fact that some aspects of mathematics remained for them a mystery which transcended their experience. Nevertheless, such negative influence depended on the extent of the misalignment between a student's common sense and mathematics.

²⁷³ 15 students, 54%: Xenofontas, Lysimachos, Agapi, Aspasia, Kleomenis, Loukianos, Ermis Lida, Danai, Platonas, Filia, Diomidis, Solonas, Theodosis and Andromachi.

²⁷⁴ Six students, 21%: Foivos, Polyxeni, Filippos, Vrasidas, Ariadni, and Menelaos.

²⁷⁵ Nine students, 32%: Pelopidas, Polyxeni, Evyenia, Kleio, Filippos, Yerasimos, Vrasidas, Ariadni, and Menelaos.

²⁷⁶ Seven students, 21%: Foivos, Xenofontas, Ermis, Lida, Platonas, Filia and Afroditi.

²⁷⁷ Six students, 21%: Lysimachos, Pelopidas, Evyenia, Lida, Solonas, Ariadni. These were the students who expressed a discomfort. However, their remarks could be relevant to other students who also seemed to feel particularly comfortable with empirically-based knowledge.

²⁷⁸ Lida was the only student in that group who believed that mathematics was invented and she was also part of the second group. Her case is discussed in the chapter on subjective meaning.

²⁷⁹ Two students, 7%: Lysimachos, Solonas.

²⁸⁰ Four students, 14%: Pelopidas, Evyenia, Lida, Ariadni.

Overview

Theme	Subthemes
Common Sense (Reasoning)	 Alignment²⁸¹ Misalignment
Discovery	 Alignment Misalignment
Invention	 Alignment Misalignment
Certainty	 Alignment Misalignment
Subjectivity	 Alignment Misalignment
Rules	 Alignment Misalignment
Empiricism	 Partial misalignment General misalignment

Discussion

Platonism

Under platonism were chiefly gathered students who believed that mathematics was discovered, without connecting this discovery to nature and the empirical world.²⁸² Apart from such students, with this theme were also connected - if more loosely - students who could not decide whether mathematics was discovered or not, but were under the impression that correctness in mathematics was decided independently of humans.²⁸³ Such students could be seen as at least hinting at the possibility of the existence of mathematical concepts as abstract

²⁸¹ All these attributes are with respect to common sense

²⁸² Andromachi.

²⁸³ Aspasia and Theodosis.

entities, which constitutes the main tenet of platonism. On that basis, it was also assumed that such students connected mathematical certainty with mathematical existence. It could have also been assumed that students who dissociated mathematics from experience sought to suggest that in mathematics one reasoned with abstract ideas. However, all these students suggested that mathematics was invented, and were categorised under formalism because they were essentially suggesting that these abstract ideas had no meaning.

Logicism

Students were taken to echo logicism insofar as they consistently connected logic with the production and validation of mathematical knowledge, or simply presented logic as a central factor of mathematical activity.²⁸⁴ Apart from the students' utterances, the centrality of logic in mathematics was also judged by the zest with which they would talk about it. Insofar as their remarks were not simply general, such students were also taken to associate mathematical certainty with proofs and logic.²⁸⁵

Formalism

This theme concerned two groups of students.²⁸⁶ One group more or less suggested that mathematics was nothing more than empty symbols devoid of meaning - because they could not understand it.²⁸⁷ The second group suggested that occasionally one could find a correct solution which common sense could not grasp simply by following syntactically the operations of mathematics.²⁸⁸ Both groups were taken to hint towards an image of mathematical statements as potentially meaningless collections of symbols, which accorded with the main belief underlying formalism.

²⁸⁴ 19 students, 68%: Foivos, Xenofontas, Lysimachos, Agapi, Aspasia, Polyxeni, Kleomenis, Loukianos, Kosmas, Ermis, Lida, Danai, Platonas, Diomidis, Filia, Solonas, Theodosis, Andromachi and Afroditi.

²⁸⁵ 16 students, 57%: Foivos, Lysimachos, Agapi, Aspasia, Kleomenis, Loukianos, Ermis, Lida, Danai, Platonas, Diomidis, Filia, Solonas, Theodosis, Andromachi, Afroditi.

²⁸⁶ 18 students, 64%: Lysimachos, Aspasia, Pelopidas, Polyxeni, Kleomenis, Loukianos, Kosmas, Evyenia, Kleio, Danai, Platonas, Filia, Solonas, Filippos, Yerasimos, Vrasidas, Ariadni, and Menelaos.

²⁸⁷ Ten students, 36%: Pelopidas, Polyxeni, Kosmas, Evyenia, Kleio, Filippos, Yerasimos, Vrasidas, Ariadni, and Menelaos.

²⁸⁸ Eight students, 29%: Lysimachos, Aspasia, Kleomenis, Loukianos, Danai, Platonas, Filia, and Solonas.

Intuitionism

Under this theme were organised students who hinted that mathematical results could be accepted because - or to the extent to which - they seemed to be effortlessly in accord with human intuition, or in other words, common sense.²⁸⁹ Although such students would not reject any mathematics that did not fit this pattern, they did seem to echo the intuitionists' criterion for what made sense of mathematics.

Empiricism

This theme involved students who advanced empirical aspects of mathematics by locating mathematics in nature (these students were also taken to associate mathematical existence with mathematical certainty);²⁹⁰ or by hinting that empirical reasoning based on experimentation or observation could be relevant to mathematics.²⁹¹ All these beliefs reflected empiricism as a philosophy of mathematics either at the ontological level of mathematical existence or at the epistemological level of production and verification of mathematical knowledge. However, the theme was not applied to students who simply claimed that logic could be connected with the senses or that experience was helpful in learning mathematics, unless such students also indicated that the senses or experience were indispensable for logic.

Humanism

This theme was applied to all students who suggested that mathematics was invented, but also to those students who, whilst suggesting that mathematics was discovered, stressed the involvement of humans in the production of mathematical knowledge (for example, in the choice of mathematical symbols). Moreover, the theme included students who appeared to justify one or more traits of mathematics, such as certainty, immutability or truth, on cultural grounds, since the image of mathematics as a human activity, grounded in cultural conventions, comprises the main tenet of humanism.²⁹² However, only this last group of students was

²⁸⁹ Six students, 21%: Foivos, Lysimachos, Aspasia, Kleomenis, Solonas, and Theodosis.

²⁹⁰ 25%: Foivos, Xenofontas, Ermis, Platonas, Filia, Andromachi, and Afroditi

²⁹¹ 16 students, 57%: Foivos, Xenofontas, Lysimachos, Agapi, Pelopidas, Kleomenis, Loukianos, Kosmas, Ermis, Lida, Platonas, Diomidis, Filia, Theodosis, Filippos, and Afroditi.

²⁹² 21 students, 75%: Foivos, Lysimachos, Agapi, Pelopidas, Polyxeni, Kleomenis, Loukianos, Kosmas, Evyenia, Lida, Kleio, Danai, Platonas, Diomidis, Solonas, Theodosis, Filippos, Yerasimos, Vrasidas, Ariadni, and Menelaos.

connected with cultural certainty. Students who claimed that mathematics was invented but based certainty on logic and proofs were taken to offer a traditional answer to the problem of certainty, despite attributing cultural traits to mathematical reasoning.